

# On operator estimates in homogenization of nonlocal operators of convolution type

A. Piatnitski <sup>a,b,\*</sup>, V. Sloushch <sup>c</sup>, T. Suslina <sup>c</sup>, E. Zhizhina <sup>b,a</sup>

<sup>a</sup> The Arctic university of Norway, campus Narvik, P.O. Box 385, 8505 Narvik, Norway

<sup>b</sup> Institute of Information Transmission Problems of RAS, Bolshoi Karetny, 19, 127051 Moscow, Russia

<sup>c</sup> St. Petersburg State University, Universitetskaya nab. 7/9, 199034 St. Petersburg, Russia

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## Abstract

The paper studies a bounded symmetric operator  $\mathbb{A}_\varepsilon$  in  $L_2(\mathbb{R}^d)$  with

$$(\mathbb{A}_\varepsilon u)(x) = \varepsilon^{-d-2} \int_{\mathbb{R}^d} a((x-y)/\varepsilon) \mu(x/\varepsilon, y/\varepsilon) (u(x) - u(y)) dy;$$

here  $\varepsilon$  is a small positive parameter. It is assumed that  $a(x)$  is a non-negative  $L_1(\mathbb{R}^d)$  function such that  $a(-x) = a(x)$  and the moments  $M_k = \int_{\mathbb{R}^d} |x|^k a(x) dx$ ,  $k = 1, 2, 3$ , are finite. It is also assumed that  $\mu(x, y)$  is  $\mathbb{Z}^d$ -periodic both in  $x$  and  $y$  function such that  $\mu(x, y) = \mu(y, x)$  and  $0 < \mu_- \leq \mu(x, y) \leq \mu_+ < \infty$ . Our goal is to study the limit behaviour of the resolvent  $(\mathbb{A}_\varepsilon + I)^{-1}$ , as  $\varepsilon \rightarrow 0$ . We show that, as  $\varepsilon \rightarrow 0$ , the operator  $(\mathbb{A}_\varepsilon + I)^{-1}$  converges in the operator norm in  $L_2(\mathbb{R}^d)$  to the resolvent  $(\mathbb{A}^0 + I)^{-1}$  of the effective operator  $\mathbb{A}^0$  being a second order elliptic differential operator with constant coefficients of the form  $\mathbb{A}^0 = -\operatorname{div} g^0 \nabla$ . We then obtain sharp in order estimates of the rate of convergence.

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\* Corresponding author.

E-mail addresses: [apiatnitski@gmail.com](mailto:apiatnitski@gmail.com) (A. Piatnitski), [v.sloushch@spbu.ru](mailto:v.sloushch@spbu.ru) (V. Sloushch), [t.suslina@spbu.ru](mailto:t.suslina@spbu.ru) (T. Suslina), [elena.jjjina@gmail.com](mailto:elena.jjjina@gmail.com) (E. Zhizhina).

## 0. Introduction

### 0.1. Motivation: contact model

Evolution processes in the models of mathematical biology and population dynamics are often described in terms of parabolic equation of the form  $\partial_t u = -Au$  with a nonlocal convolution type operator  $A$ , the nonlocality of  $A$  reflects the fact that the interaction in these models is nonlocal. The kernel of  $A$  has a multiplier  $a(x - y)$  that specifies the intensity of interaction depending on the distance and determines the localization properties of  $A$ .

One of the models of this type that has been considered in the existing literature is the so-called *contact model* in  $\mathbb{R}^d$ , see [32–34]. This model relies on a continuous time Markov process that belongs to the class of death and birth processes and is defined on the space  $\Gamma$  of infinite locally finite configurations  $\gamma \subset \mathbb{R}^d$ . The behaviour of the process is determined by the intensity of birth and death. On the one hand, each point  $x \in \gamma$  can produce an offspring  $y$  with the intensity  $a(x - y)$ , independently of other points, and we assume that  $\int_{\mathbb{R}^d} a(z) dz = 1$ . On the other hand, each point of  $\gamma$  has a random life time, and the intensity of death is  $m(x) > 0$ . In the general case the intensities of birth and death might depend on the position in the space. The infinitesimal generator of such dynamics reads

$$LF(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x - y) (F(\gamma \cup y) - F(\gamma)) dy + \sum_{x \in \gamma} m(x) (F(\gamma \setminus x) - F(\gamma)).$$

The case of constant death intensity,  $m(x) \equiv \kappa$ , has been studied in details in [32], the so-called critical regime  $m(x) \equiv 1$  being of special interest here. In this regime the process has a family of invariant measures.

The contact model has a remarkable property: the equation for the first correlation function describing the density of configuration is closed and not coupled with the equations for higher order correlation functions. Notice that for the second and higher order correlation functions the corresponding evolutions have a complicated hierarchical structure that involves lower order correlation functions. The evolution equation for the first correlation function reads

$$\frac{\partial u}{\partial t} = -Au, \quad u = u(t, x), \quad x \in \mathbb{R}^d, \quad t \geq 0, \quad u(0, x) = u_0(x) \geq 0, \tag{0.1}$$

where

$$Au(x) = m(x)u(x) - \int_{\mathbb{R}^d} a(x - y)u(y) dy.$$

If  $m(x) \equiv 1$ , the operator  $A$  takes the form

$$Au(x) = u(x) - \int_{\mathbb{R}^d} a(x - y)u(y) dy = \int_{\mathbb{R}^d} a(x - y)(u(x) - u(y)) dy. \tag{0.2}$$

The operator  $A$  with a kernel  $a(x - y)$  that depends only on the difference  $x - y$  is a proper tool for describing space homogeneous media. In order to model the processes in space inhomogeneous media it is natural to consider operators with more general kernels of the form

$a(x - y)\mu(x, y)$ . These kernels depend both on the difference  $x - y$  and on the starting and end points of jumps, the function  $\mu(x, y)$  representing the local characteristics of the environment. In particular, for the contact model in a periodic medium the operator  $A$  in evolution equation (0.1) for the first correlation function should be replaced with an operator

$$\mathbb{A}u(x) = \int_{\mathbb{R}^d} a(x - y)\mu(x, y)(u(x) - u(y))dy \tag{0.3}$$

with a periodic function  $\mu(x, y)$ .

The study of large time behaviour of solutions to nonlocal evolution equations can be reduced to proper homogenization problems for the corresponding nonlocal operators. To illustrate this we consider a parabolic equation  $\partial_t u = -Au$ , where  $A$  can be both second order elliptic differential operator and nonlocal convolution type operator. The natural way to avoid growing time intervals is to multiply the temporal variable by a small positive parameter that we call  $\varepsilon^2$ . Then, to preserve the structure of the equation, one has to multiply the spatial variable by  $\varepsilon$ , this change of variables being called the diffusive scaling. In a non-homogeneous medium this leads to a family of equations that depends on a small parameter  $\varepsilon$ , and we naturally arrive at the homogenization problem.

While the homogenization problems for differential operators have been actively studied for quite a long period, see for instance the monographs [1], [2] and [29] and the bibliography therein, similar problems for nonlocal convolution type operators with integrable kernels have not been considered in the existing literature till recently. For the first time the periodic homogenization problem for such operators was investigated in [44], where it was shown that, under natural moment and coerciveness conditions, the limit operator is a second order elliptic differential operator with constant coefficients and a positive definite effective matrix. The homogenization procedure relied on the corrector techniques, however the justification of convergence required new methods adapted to studying the nonlocal operators. It was also proved in [44] that the corresponding jump Markov process satisfies the invariance principle in the Skorokhod topology. Similar homogenization problems in random media have been considered in [46]. In the case of non-symmetric convolution type operators homogenization problem has been addressed in [45], it was shown that for the corresponding parabolic equations homogenization takes place in moving coordinates. Convolution type operators in periodically and randomly perforated domains have been studied in [11] and [12] by the variational methods.

A number of interesting homogenization results for the Levy type nonlocal operators with non-integrable kernels have been obtained in the recent works [47], [48], [13]. However, the properties of these operators differ a lot from the properties of operators studied in this work.

### 0.2. Estimates for the rate of convergence in homogenization theory

Essential interest in obtaining various estimates for the rate of convergence in the mathematical homogenization theory arose shortly after emergence of this theory itself. This interest was stimulated by important applications of this theory in such disciplines as composite materials, photonic crystals, porous media, network constructions and many others. In addition, obtaining these estimates turned out to be an interesting mathematical challenge.

For periodic media the first results in this direction have been obtained in [30,31] and [69]. The works [30,31] dealt with boundary value problems for elliptic operators with oscillating

coefficients, for these problems  $L_2$  estimates for the rate of convergence were proved. Then in [69] similar estimates have been obtained for elliptic operators in periodically perforated domains including spectral problems with the Steklov condition on the perforation border.

Later on many authors contributed to this topic, a large number of interesting results on the rate of convergence for solutions to elliptic and evolution differential equations and systems of equations have been obtained both in periodic and locally periodic environments. Among others, the elasticity system and, in some particular cases, the Maxwell system have been investigated. We refer here only to some books [1,2,29], where these questions have been addressed.

For homogenization problems in random statistically homogeneous environments the first estimates for the rate of convergence were obtained in [72]. Further progress in this field was achieved in the recent works [22,23].

Notice however that in all the above mentioned papers the estimates for the rate of convergence were obtained in the strong topology, not the operator topology, the constants in these estimates depended on the regularity of the right-hand sides.

In the pioneer works of M. Birman and T. Suslina [3–5] a new approach to homogenization problems in periodic media has been developed, it relies on a version of the spectral method. With the help of this method for a wide class of homogenization problems estimates in the operator topology were justified.

In order to clarify this method, let us consider a scalar elliptic operator of the form  $A_\varepsilon = -\operatorname{div} g(x/\varepsilon)\nabla$ ,  $\varepsilon > 0$ , in  $L_2(\mathbb{R}^d)$  with a positive definite bounded  $\mathbb{Z}^d$ -periodic matrix  $g(\cdot)$ . Classical result states that for any  $F \in L_2(\mathbb{R}^d)$ , as  $\varepsilon \rightarrow 0$ , a solution  $u_\varepsilon$  of the equation

$$(A_\varepsilon u_\varepsilon)(x) + u_\varepsilon(x) = F(x), \quad x \in \mathbb{R}^d,$$

converges in  $L_2(\mathbb{R}^d)$  to a solution  $u_0$  of the homogenized equation that reads

$$(A^0 u_0)(x) + u_0(x) = F(x), \quad x \in \mathbb{R}^d;$$

here  $A^0 = -\operatorname{div} g_{\text{hom}}\nabla$  is the so-called effective operator and  $g_{\text{hom}}$  is a constant positive definite matrix. Moreover, the estimate

$$\|u_\varepsilon - u_0\|_{L_2(\mathbb{R}^d)} \leq C(F)\varepsilon$$

holds true. In [3] a stronger estimate has been obtained. Namely, it was shown that

$$\|u_\varepsilon - u_0\|_{L_2(\mathbb{R}^d)} \leq C\varepsilon\|F\|_{L_2(\mathbb{R}^d)}$$

with a constant  $C$  that does not depend on  $F$ . This result can be reformulated in terms of operator convergence: as  $\varepsilon \rightarrow 0$ , the resolvent  $(A_\varepsilon + I)^{-1}$  converges in the operator norm in  $L_2(\mathbb{R}^d)$  to  $(A^0 + I)^{-1}$ , and the following order-sharp estimate is fulfilled:

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon. \tag{0.4}$$

In the homogenization theory the inequalities of this type are called operator estimates of approximation error. Then in [4] a more advanced uniform operator approximation of the resolvent  $(A_\varepsilon + I)^{-1}$  in  $L_2(\mathbb{R}^d)$  has been found, the estimate derived in this work takes into account the

corrector and provides the precision of order  $O(\varepsilon^2)$ . As was shown in [5] similar approximation yields the precision of order  $O(\varepsilon)$  in the space of operators acting from  $L_2(\mathbb{R}^d)$  to  $H^1(\mathbb{R}^d)$ .

Analogous results for parabolic semigroups  $e^{-A\varepsilon t}$ ,  $t > 0$ , have been obtained in [55,57,70,58]. It should be emphasized that the above mentioned results on the estimates in operator topology are valid for a wide class of second order matrix elliptic operators.

The spectral method used in the cited collection of works relies on scaling transformation, Floquet-Bloch theory and analytic perturbation theory.

We illustrate this method by explaining the main ideas of obtaining the estimate in (0.4). Making the scaling transformation we conclude that estimate (0.4) is equivalent to the inequality

$$\|(A + \varepsilon^2 I)^{-1} - (A^0 + \varepsilon^2 I)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon^{-1}, \tag{0.5}$$

where  $A = -\operatorname{div} g(x)\nabla = D^*g(x)D$ ,  $D = -i\nabla$ . Applying then the unitary Gelfand transform we represent the operator  $A$  as a direct integral with respect to the operators  $A(\xi)$  acting in  $L_2(\Omega)$  and depending on a parameter  $\xi \in \tilde{\Omega}$  (quasi-momentum); here  $\Omega = [0, 1)^d$  is the periodicity cell  $\mathbb{R}^d/\mathbb{Z}^d$ , and  $\tilde{\Omega} = [-\pi, \pi)^d$  is the cell of the dual lattice. The operator  $A(\xi)$  is of the form  $A(\xi) = (D + \xi)^*g(x)(D + \xi)$  and equipped with the periodic boundary conditions. Estimate (0.5) is a consequence of the following one:

$$\|(A(\xi) + \varepsilon^2 I)^{-1} - (A^0(\xi) + \varepsilon^2 I)^{-1}\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C\varepsilon^{-1}, \quad \xi \in \tilde{\Omega}.$$

The family of operators  $\{A(\xi)\}$  is analytic, and all the operators of this family have a compact resolvent. Thus the methods of analytic perturbation theory apply, and the resolvent  $(A(\xi) + \varepsilon^2 I)^{-1}$  can be approximated in terms of the spectral characteristics of this operator family at the spectral edge. In particular, the effective matrix is proportional with the coefficient  $\frac{1}{2}$  to the Hessian of the first eigenvalue  $\lambda_1(\xi)$  of the operator  $A(\xi)$  at  $\xi = 0$ . Therefore, the homogenization can be treated as a spectral threshold effect at the edge of the spectrum of an elliptic operator.

Later on with the help of the spectral method various operator estimates for the rate of convergence were obtained for a wide class of elliptic and evolution equations with periodic coefficients, among them are non-stationary Schrödinger-type and hyperbolic equations, see [6,36,61,17–19], stationary and non-stationary Maxwell system, see [54,56,20], and many others.

Another approach to estimating the operator norm of the discrepancy in the homogenization procedure, the so-called “shift method”, was developed by V. Zhikov and S. Pastukhova in [73–75], see also the survey paper [76] and references therein. This approach is based on the analysis of the first approximation of the solution and on introducing an additional variable that takes on values in  $\Omega$ . It allows to deal with homogenization of boundary-value problems in bounded domains in  $\mathbb{R}^d$ .

The works [73,74] studied homogenization problems both for elliptic operators defined in the whole  $\mathbb{R}^d$  and for boundary-value problems with the Dirichlet and Neumann boundary conditions. It was shown that the discrepancy admits the estimate of order  $O(\varepsilon^{1/2})$  in the uniform operator topology in  $L_2$  and, in the presence of correctors, in the operator norm  $L_2 \rightarrow H^1$ . The estimates are not as good as in the case of the whole space due to the influence of the domain boundary. Similar results for the Dirichlet and Neumann problems for a scalar operator in a bounded domain have been obtained in [24,25] by means of the unfolding method. In [25] for the first time an estimate of the sharp order  $O(\varepsilon)$  was justified in the operator norm in  $L_2$ .

For elliptic systems similar estimates have been proved in [28] and [43,59,60], the case of initial-boundary problems for the corresponding parabolic equations has been treated in [37,21], and Maxwell system in a bounded domain was considered in [64,65].

In recent years further progress has been achieved in this topic. We quote here [26], where operator estimates were obtained for the Stokes system, [71,35,39,40,53,38,67,68] that dealt with elliptic operators of higher order in  $\mathbb{R}^d$ , and [62,63,66], where the higher order elliptic operators in a bounded domain were considered.

The works [7,41,42,49–51] focused on operator estimates for operators with locally periodic coefficients, the high contrast problems were investigated in [14,16,15]. Similar questions can be raised when boundary homogenization problems are considered. Some of them have been addressed in [9,10,8].

However, it should be emphasized that in all these papers the operator estimates have been obtained for differential operators. To our best knowledge the case of nonlocal operators has not been examined in the existing literature.

### 0.3. Problem setup. Main result

The present work studies convolution type operators that read

$$\mathbb{A}_\varepsilon u(x) = \frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} a\left(\frac{x-y}{\varepsilon}\right) \mu\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) (u(x) - u(y)) dy, \quad x \in \mathbb{R}^d, \quad u \in L_2(\mathbb{R}^d), \quad (0.6)$$

here  $\varepsilon$  is a small positive parameter,  $a(x)$  is an even non-negative function such that  $a(\cdot) \in L_1(\mathbb{R}^d)$  and  $\|a\|_{L_1} > 0$ ,  $\mu(x, y)$  is a  $\mathbb{Z}^d$  periodic in  $x$  and  $y$  function such that  $\mu(x, y) = \mu(y, x)$  and  $0 < C_1 \leq \mu(x, y) \leq C_2$  for some constants  $C_1$  and  $C_2$ . Under these conditions the operator  $\mathbb{A}_\varepsilon$  is bounded, self-adjoint and non-negative in  $L_2(\mathbb{R}^d)$ . We assume furthermore that the moments  $M_k = \int_{\mathbb{R}^d} |x|^k a(x) dx$  are finite for  $k = 1, 2, 3$ .

As was mentioned above, the homogenization problem for operators defined in (0.6) was solved in [44]. It was shown that the resolvent  $(\mathbb{A}_\varepsilon + I)^{-1}$  converges, as  $\varepsilon \rightarrow 0$ , in the strong operator topology to the resolvent  $(\mathbb{A}^0 + I)^{-1}$  of the effective operator being the second order elliptic operator of the form  $\mathbb{A}^0 = -\operatorname{div} g^0 \nabla$ . It is interesting to observe that although the operators  $\mathbb{A}_\varepsilon$  are bounded and nonlocal for any  $\varepsilon > 0$ , the limit operator is unbounded and local.

Although the estimates for the rate of convergence in the just mentioned homogenization problem seem not to be presented in the mathematical literature, for sufficiently regular right-hand sides such estimates can be obtained with the help of asymptotic expansions constructed in [44].

Our goal here is to prove the convergence of the resolvent  $(\mathbb{A}_\varepsilon + I)^{-1}$  to the resolvent  $(\mathbb{A}^0 + I)^{-1}$  in the operator norm topology in  $L_2(\mathbb{R}^d)$  and to derive order-sharp estimates for the discrepancy.

In order to formulate the main result of this paper, we first recall the definition of the effective matrix  $g^0$ . Consider an auxiliary cell problem that reads: find a  $\mathbb{Z}^d$ -periodic vector-valued solution of the equation

$$\begin{cases} \int_{\mathbb{R}^d} a(x-y) \mu(x, y) (v(x) - v(y)) dy = \int_{\mathbb{R}^d} a(x-y) \mu(x, y) (x-y) dy, \\ \int_{\Omega} v(y) dy = 0; \end{cases} \quad (0.7)$$

here  $\Omega := [0, 1)^d$  is a periodicity cell. This problem has a unique solution. The effective matrix  $g^0 = \frac{1}{2}\{g_{ij}\}_{i,j=1,\dots,d}$  is defined by

$$g_{ij} = \int_{\Omega} dx \int_{\mathbb{R}^d} dy \left( (x_i - y_i)(x_j - y_j) - v_j(x)(x_i - y_i) - v_i(x)(x_j - y_j) \right) a(x - y) \mu(x, y),$$

$$i, j = 1, \dots, d.$$

It turns out that the matrix  $g^0$  is positive definite, see [44]. The domain of the homogenized operator coincides with the Sobolev space  $H^2(\mathbb{R}^d)$ . Our main result is Theorem 3.1 which states that the following estimate holds true:

$$\|(\mathbb{A}_\varepsilon + I)^{-1} - (\mathbb{A}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(a, \mu)\varepsilon, \quad \varepsilon > 0. \tag{0.8}$$

This estimate is sharp in order, and the constant  $C(a, \mu)$  can be calculated explicitly.

#### 0.4. The methods

To approximate the resolvent of operator (0.6) and justify the formulated above result we modify and adapt the spectral method discussed in Section 0.2.

The first two steps, the scaling transformation and the unitary Gelfand transform allowing to represent the operator in (0.3) as a direct integral with respect to the family  $\mathbb{A}(\xi)$ , remain unchanged. However, in contrast to the case of differential operators, the family  $\mathbb{A}(\xi)$  depending on the parameter  $\xi \in \widetilde{\Omega}$  need not be analytic in  $\xi$ , thus the methods of analytic perturbation theory do not apply to this family. Instead, our approach relies on a  $C^3$  regularity of this family which is granted by the existence of the first three moments of function  $a(z)$ .

Let us explain some details of this approach. It is known that, in order to approximate the resolvent  $(\mathbb{A}(\xi) + \varepsilon^2 I)^{-1}$  for small  $\varepsilon > 0$ , it suffices to construct the asymptotics of the operator-function  $\mathbb{A}(\xi)F(\xi)$ ,  $\xi \rightarrow 0$ , where  $F(\xi)$  is a spectral projector of the operator  $\mathbb{A}(\xi)$  that corresponds to some neighbourhood of zero.

In the previous works that dealt with the differential operators the asymptotics of the operator  $\mathbb{A}(\xi)F(\xi)$  for small  $|\xi|$  relied on the asymptotics of the principal eigenvalue  $\lambda_1(\xi)$  of operator  $A(\xi)$ . In the present paper, in addition to this classical method, we provide two alternative ways of computing the asymptotics of  $\mathbb{A}(\xi)F(\xi)$  as  $\xi \rightarrow 0$  which are not based on constructing the asymptotics of  $\lambda_1(\xi)$ . One of them that looks promising for applications in the homogenization theory relies on computing the coefficients of the asymptotics of  $\mathbb{A}(\xi)F(\xi)$ ,  $\xi \rightarrow 0$ , by means of integrating the resolvent  $(\mathbb{A}(0) - \zeta I)^{-1}$  as well as some operator-functions of this resolvent, over appropriate contours, see Section 4.2.

As in the case of differential operators, the key role in the studied homogenization problem is played by the spectral characteristics of operator  $\mathbb{A}$  near the lower edge of its spectrum (the so-called threshold characteristics). Thus, the homogenization of nonlocal operator can also be considered as a threshold effect at the spectrum edge.

### 0.5. Plan of the paper

The paper consists of Introduction, three sections and Appendix. In Section 1 we introduce operator  $\mathbb{A}$ , represent this operator as a direct integral over the family  $\mathbb{A}(\xi)$  and obtain lower bounds for the quadratic form of the operator  $\mathbb{A}(\xi)$ .

In Section 2 the threshold characteristics of the operator family  $\mathbb{A}(\xi)$  are studied in the vicinity of the spectrum bottom. Then we approximate the resolvent  $(\mathbb{A}(\xi) + \varepsilon^2 I)^{-1}$  for small  $\varepsilon$ .

Finally, in Section 3 we complete the proof of the main result of this work.

The Appendix contains a number of auxiliary statements.

### 0.6. Notation

The norm in a normed space  $X$  is denoted by  $\|\cdot\|_X$ , the index  $X$  is dropped if this does not lead to an ambiguity. If  $X$  and  $Y$  are linear normed space, the norm of a linear bounded operator  $T : X \rightarrow Y$  is denoted  $\|T\|_{X \rightarrow Y}$  or just  $\|T\|$ . The notation  $\mathcal{B}(X)$  is used for the space of bounded linear operators in a normed space  $X$ . Given a collection of vectors  $F \subset X$ , we denote its span by  $\mathcal{L}\{F\}$ . An open ball in a normed space centered at  $x_0$  of radius  $r > 0$  is denoted by  $B_r(x_0)$ .

The spectrum and the essential spectrum of a self-adjoint operator  $\mathbb{A}$  in a Hilbert space  $\mathfrak{H}$  are denoted by  $\sigma(\mathbb{A})$  and  $\sigma_e(\mathbb{A})$ , respectively.

For a Lebesgue measurable set  $\mathcal{O} \subset \mathbb{R}^d$  of positive Lebesgue measure we denote by  $L_p(\mathcal{O})$ ,  $1 \leq p \leq \infty$ , the standard  $L_p$  space of functions defined on  $\mathcal{O}$ . The inner product in  $L_2(\mathcal{O})$  is denoted by  $(\cdot, \cdot)_{L_2(\mathcal{O})}$ , while for the scalar product in  $\mathbb{R}^d$  and  $\mathbb{C}^d$  the notation  $\langle \cdot, \cdot \rangle$  is used. Symbol  $\mathcal{S}(\mathbb{R}^d)$  stands for the Schwartz class of functions in  $\mathbb{R}^d$ . The characteristic function of a set  $E \subset \mathbb{R}^d$  is denoted by  $\mathbf{1}_E$ .

## 1. Nonlocal Schrödinger operator: representation as a direct integral and estimates

### 1.1. Operator $\mathbb{A}(a, \mu)$

Given real-valued functions  $a \in L_1(\mathbb{R}^d)$  and  $\mu \in L_\infty(\mathbb{R}^d \times \mathbb{R}^d)$  we define a nonlocal Schrödinger operator  $\mathbb{A} = \mathbb{A}(a, \mu)$  in the space  $L_2(\mathbb{R}^d)$ ,  $d \geq 1$ , by

$$\mathbb{A}(a, \mu)u(x) := \int_{\mathbb{R}^d} a(x - y)\mu(x, y)(u(x) - u(y)) dy, \quad x \in \mathbb{R}^d.$$

Rearranging the expression on the right-hand side, one has  $\mathbb{A}(a, \mu) = p(x; a, \mu) - \mathbb{B}(a, \mu)$  with

$$p(x; a, \mu) := \int_{\mathbb{R}^d} a(x - y)\mu(x, y) dy, \quad x \in \mathbb{R}^d,$$

$$\mathbb{B}(a, \mu)u(x) := \int_{\mathbb{R}^d} a(x - y)\mu(x, y)u(y) dy, \quad x \in \mathbb{R}^d;$$

here and in what follows we identify the function  $p(\cdot; a, \mu)$  and the operator of multiplication by this function. According to the Schur test, see [27, Theorem 5.2], the operator  $\mathbb{B}(a, \mu) : L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)$  is bounded in  $L_2(\mathbb{R}^d)$  and its norm satisfies the upper bound  $\|\mathbb{B}(a, \mu)\| \leq$

$\|\mu\|_{L_\infty}\|a\|_{L_1}$ . For the reader convenience we prove this statement in the Appendix, see Lemma 4.1. The potential  $p(x; a, \mu)$  is also bounded and admits the estimate  $\|p(\cdot; a, \mu)\|_{L_\infty} \leq \|\mu\|_{L_\infty}\|a\|_{L_1}$ . Therefore, the operator  $\mathbb{A}(a, \mu) : L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)$  is bounded. It is convenient to introduce, in addition to  $\mathbb{A}(a, \mu)$ , an auxiliary operator  $\mathbb{A}_0(a) := \mathbb{A}(a, \mu_0)$  with  $\mu_0 \equiv 1$ . Letting  $\mathbb{B}_0(a) := \mathbb{B}(a, \mu_0)$  and  $p_0(x; a) := p(x; a, \mu_0)$  we observe that  $\mathbb{B}_0(a)$  is a convolution operator with the kernel  $a(\cdot)$ , and the potential  $p_0(x; a)$  is constant.

From now on we assume that the functions  $a$  and  $\mu$  possess the following properties:

$$a \in L_1(\mathbb{R}^d), \text{ meas}\{x \in \mathbb{R}^d : a(x) \neq 0\} > 0, \quad a(x) \geq 0, \quad a(-x) = a(x), \quad x \in \mathbb{R}^d; \tag{1.1}$$

$$0 < \mu_- \leq \mu(x, y) \leq \mu_+ < +\infty, \quad \mu(x, y) = \mu(y, x), \quad x, y \in \mathbb{R}^d; \tag{1.2}$$

$$\mu(x + m, y + n) = \mu(x, y), \quad x, y \in \mathbb{R}^d, \quad m, n \in \mathbb{Z}^d; \tag{1.3}$$

$$M_k(a) := \int_{\mathbb{R}^d} |x|^k a(x) dx < +\infty, \quad k = 1, 2, 3. \tag{1.4}$$

Since the coefficients  $a(\cdot)$  and  $\mu(\cdot)$  are real-valued, the potential  $p(x) = p(x; a, \mu)$  is also real. Moreover, under conditions (1.1) and (1.2) the potential  $p(x; a, \mu)$  satisfies the estimate

$$\mu_- \|a\|_{L_1(\mathbb{R}^d)} \leq p(x) \leq \mu_+ \|a\|_{L_1(\mathbb{R}^d)}, \quad x \in \mathbb{R}^d, \tag{1.5}$$

and the operator  $\mathbb{B}(a, \mu)$  is bounded and self-adjoint. Thus  $\mathbb{A}(a, \mu)$  is bounded and self-adjoint as well.

### 1.2. Quadratic form of operator $\mathbb{A}(a, \mu)$

Under assumptions (1.1) and (1.2) the quadratic form of operator  $\mathbb{A}(a, \mu)$  admits the following representation (see, for instance, [33]):

$$(\mathbb{A}(a, \mu)u, u) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dx dy a(x - y) \mu(x, y) |u(x) - u(y)|^2, \quad u \in L_2(\mathbb{R}^d). \tag{1.6}$$

Indeed, from the relations  $a(x - y) = a(y - x)$  and  $\mu(x, y) = \mu(y, x)$  we obtain

$$\begin{aligned} (\mathbb{A}(a, \mu)u, u) &= \int_{\mathbb{R}^d} dx \overline{u(x)} \int_{\mathbb{R}^d} dy a(x - y) \mu(x, y) (u(x) - u(y)) = \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dx dy a(x - y) \mu(x, y) |u(x) - u(y)|^2 + \int_{\mathbb{R}^d} dy \overline{u(y)} \int_{\mathbb{R}^d} dx a(x - y) \mu(x, y) (u(x) - u(y)) = \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dx dy a(x - y) \mu(x, y) |u(x) - u(y)|^2 - (\mathbb{A}(a, \mu)u, u), \quad u \in L_2(\mathbb{R}^d). \end{aligned}$$

This yields (1.6). Considering representation (1.6) we conclude that the operator  $\mathbb{A}(a, \mu)$  is non-negative and the estimates

$$\mu_-(\mathbb{A}_0(a)u, u) \leq (\mathbb{A}(a, \mu)u, u) \leq \mu_+(\mathbb{A}_0(a)u, u), \quad u \in L_2(\mathbb{R}^d), \tag{1.7}$$

hold true. Since  $\mathbb{B}_0(a)$  is a convolution operator, in the space of Fourier images of functions from  $L_2(\mathbb{R}^d)$  it acts as the operator of multiplication by the function  $\widehat{a}(\xi)$ ,  $\xi \in \mathbb{R}^d$ , with

$$\widehat{a}(\xi) := \int_{\mathbb{R}^d} e^{-i(\xi, z)} a(z) dz, \quad \xi \in \mathbb{R}^d.$$

Notice that due to (1.1) the function  $\widehat{a}(\xi)$  is continuous and tends to zero at infinity. Consequently, the operator  $\mathbb{A}_0(a)$  is unitary equivalent to the operator of multiplication by the function  $\widehat{a}(0) - \widehat{a}(\xi)$ , and  $\lambda_0 = 0$  belongs to the spectrum of  $\mathbb{A}_0(a)$ . Since  $\mathbb{A}_0(a)$  is a non-negative operator,  $\lambda_0$  is its spectral edge. In view of estimates (1.7) the point  $\lambda_0 = 0$  is also the lower edge of the spectrum of operator  $\mathbb{A}(a, \mu)$ .

### 1.3. Representation of $\mathbb{A}(a, \mu)$ as a direct integral

Due to conditions (1.1)–(1.3) the operator of multiplication by  $p(x; a, \mu)$  and the operator  $\mathbb{B}(a, \mu)$  commute with the operators  $S_n$  defined by

$$S_n u(x) = u(x + n), \quad x \in \mathbb{R}^d, \quad n \in \mathbb{Z}^d.$$

So does  $\mathbb{A}(a, \mu)$ . Thus  $\mathbb{A}(a, \mu)$  and  $\mathbb{B}(a, \mu)$  are periodic operators with a periodicity lattice  $\mathbb{Z}^d$ . Denote by  $\Omega := [0, 1)^d$  the corresponding periodicity cell and by  $\widetilde{\Omega} := [-\pi, \pi)^d$  the periodicity cell of the dual lattice  $(2\pi\mathbb{Z})^d$ .

Let us recall the definition of the Gelfand transform, see, for example, [52] or [3, Chapter 2]. We call it  $\mathcal{G}$ . For functions  $u$  from the Schwartz class  $\mathcal{S}(\mathbb{R}^d)$  it is defined as follows:

$$\mathcal{G}u(\xi, x) := (2\pi)^{-d/2} \sum_{n \in \mathbb{Z}^d} u(x + n) e^{-i(\xi, x+n)}, \quad \xi \in \widetilde{\Omega}, \quad x \in \Omega, \quad u \in \mathcal{S}(\mathbb{R}^d).$$

Since  $\|\mathcal{G}u\|_{L_2(\widetilde{\Omega} \times \Omega)} = \|u\|_{L_2(\mathbb{R}^d)}$ , then  $\mathcal{G}$  can be extended by continuity to a unitary mapping from  $L_2(\mathbb{R}^d)$  to  $\int_{\widetilde{\Omega}} \oplus L_2(\Omega) d\xi = L_2(\widetilde{\Omega} \times \Omega)$ .

In what follows we assume that for any  $v \in \int_{\widetilde{\Omega}} \oplus L_2(\Omega) d\xi$  and any  $\xi \in \widetilde{\Omega}$  the function  $v(\xi, \cdot)$  is extended periodically from the cell  $\Omega$  to the whole  $\mathbb{R}^d$ . Since the operators  $\mathbb{A}(a, \mu)$  and  $\mathbb{B}(a, \mu)$  are periodic, the Gelfand transform partially diagonalizes them. More precisely, the operators  $\mathcal{G}\mathbb{A}(a, \mu)\mathcal{G}^*$  and  $\mathcal{G}\mathbb{B}(a, \mu)\mathcal{G}^*$  take the form

$$\begin{aligned} \mathcal{G}\mathbb{A}(a, \mu)\mathcal{G}^* u(\xi, x) &= \mathbb{A}(\xi, a, \mu)u(\xi, x), \\ \mathcal{G}\mathbb{B}(a, \mu)\mathcal{G}^* u(\xi, x) &= \mathbb{B}(\xi, a, \mu)u(\xi, x), \quad \xi \in \widetilde{\Omega}, \quad x \in \Omega; \end{aligned} \tag{1.8}$$

here the operators  $\mathbb{A}(\xi, a, \mu)$  and  $\mathbb{B}(\xi, a, \mu)$  are bounded self-adjoint operators in  $L_2(\Omega)$  that are defined by

$$\begin{aligned} \mathbb{A}(\xi, a, \mu) &= p(x; a, \mu) - \mathbb{B}(\xi, a, \mu), \\ \mathbb{B}(\xi, a, \mu)u(x) &= \int_{\Omega} \widetilde{a}(\xi, x - y)\mu(x, y)u(y) dy, \quad u \in L_2(\Omega); \end{aligned}$$

$$\tilde{a}(\xi, z) := \sum_{n \in \mathbb{Z}^d} a(z+n)e^{-i\langle \xi, z+n \rangle}, \quad \xi \in \tilde{\Omega}, \quad z \in \mathbb{R}^d.$$

We emphasize that  $p(x; a, \mu)$  does not depend on  $\xi$  and satisfies the relation

$$p(x; a, \mu) = \int_{\Omega} \tilde{a}(0, x-y)\mu(x, y) dy. \tag{1.9}$$

In view of  $\mathbb{Z}^d$ -periodicity of function  $\tilde{a}(\xi, \cdot)$  the estimate  $\int_{\Omega} |\tilde{a}(\xi, z)| dz \leq \|a\|_{L_1}$  implies that  $\tilde{a}(\xi, \cdot) \in L_{1,loc}(\mathbb{R}^d)$  and

$$\|\tilde{a}(\xi, \cdot)\|_{L_1([-1,1]^d)} \leq 2^d \|a\|_{L_1(\mathbb{R}^d)}, \quad \xi \in \tilde{\Omega}.$$

Therefore, by Corollary 4.2, the operator  $\mathbb{B}(\xi, a, \mu)$  is compact, and the following estimate holds

$$\|\mathbb{B}(\xi, a, \mu)\| \leq \mu_+ \int_{[-1,1]^d} |\tilde{a}(\xi, z)| dz \leq 2^d \mu_+ \|a\|_{L_1(\mathbb{R}^d)}, \quad \xi \in \tilde{\Omega}.$$

We conclude that the essential spectrum of the operator  $\mathbb{A}(\xi, a, \mu)$  coincides with  $\text{ess-Ran } p(\cdot; a, \mu)$  and, due to the lower bound in (1.5), for any  $\xi \in \tilde{\Omega}$  the spectrum of  $\mathbb{A}(\xi, a, \mu)$  in the interval  $(-\infty, \mu_- \|a\|_{L_1})$  is discrete. The goal of this section is to estimate the lower edge of the spectrum of operator  $\mathbb{A}(\xi, a, \mu)$  for all  $\xi$ .

The next statement provides a representation of the quadratic form of the operator  $\mathbb{A}(\xi, a, \mu)$  that will be convenient for the further analysis.

**Lemma 1.1.** *Under conditions (1.1)–(1.3) for any  $\xi \in \tilde{\Omega}$  the following relation holds:*

$$(\mathbb{A}(\xi, a, \mu)u, u) = \frac{1}{2} \int_{\Omega} dx \int_{\mathbb{R}^d} dy a(x-y)\mu(x, y) |e^{i\langle \xi, x \rangle} u(x) - e^{i\langle \xi, y \rangle} u(y)|^2, \quad u \in L_2(\Omega); \tag{1.10}$$

it is assumed here that the function  $u \in L_2(\Omega)$  is extended periodically to the whole  $\mathbb{R}^d$ .

**Proof.** By (1.9) for any  $x \in \Omega$  we have

$$\mathbb{A}(\xi, a, \mu)u(x) = \int_{\Omega} (\tilde{a}(0, x-y)u(x) - \tilde{a}(\xi, x-y)u(y)) \mu(x, y) dy, \quad u \in L_2(\Omega). \tag{1.11}$$

Identifying a function  $u \in L_2(\Omega)$  with its  $\mathbb{Z}^d$ -periodic extension to the whole  $\mathbb{R}^d$ , one can rearrange (1.11) as follows:

$$\mathbb{A}(\xi, a, \mu)u(x) = \sum_{n \in \mathbb{Z}^d} \int_{\Omega} (a(x-y+n)u(x) - a(x-y+n)e^{-i\langle \xi, x-y+n \rangle} u(y)) \mu(x, y) dy =$$

$$\begin{aligned}
 &= \int_{\mathbb{R}^d} a(x - y)\mu(x, y)(u(x) - e^{-i\langle \xi, x-y \rangle} u(y)) dy = \\
 &= \int_{\mathbb{R}^d} a(x - y)\mu(x, y)e^{-i\langle \xi, x \rangle} (e^{i\langle \xi, x \rangle} u(x) - e^{i\langle \xi, y \rangle} u(y)) dy, \quad x \in \Omega, \quad u \in L_2(\Omega). \quad (1.12)
 \end{aligned}$$

Then the quadratic form of the operator  $\mathbb{A}(\xi, a, \mu)$  takes the form

$$\begin{aligned}
 (\mathbb{A}(\xi, a, \mu)u, u) &= \int_{\Omega} dx \int_{\mathbb{R}^d} dy a(x - y)\mu(x, y)e^{-i\langle \xi, x \rangle} \overline{u(x)} (e^{i\langle \xi, x \rangle} u(x) - e^{i\langle \xi, y \rangle} u(y)) = \\
 &= \int_{\Omega} dx \int_{\mathbb{R}^d} dy a(x - y)\mu(x, y) |e^{i\langle \xi, x \rangle} u(x) - e^{i\langle \xi, y \rangle} u(y)|^2 + J[u], \quad u \in L_2(\Omega), \quad (1.13)
 \end{aligned}$$

where the functional  $J[u]$  is given by

$$J[u] := \int_{\Omega} dx \int_{\mathbb{R}^d} dy a(x - y)\mu(x, y)e^{-i\langle \xi, y \rangle} \overline{u(y)} (e^{i\langle \xi, x \rangle} u(x) - e^{i\langle \xi, y \rangle} u(y)), \quad u \in L_2(\Omega). \quad (1.14)$$

Exchanging the variables  $x$  and  $y$  and considering conditions (1.1), (1.2) we conclude that

$$\begin{aligned}
 J[u] &= - \int_{\mathbb{R}^d} dx \int_{\Omega} dy a(x - y)\mu(x, y)e^{-i\langle \xi, x \rangle} \overline{u(x)} (e^{i\langle \xi, x \rangle} u(x) - e^{i\langle \xi, y \rangle} u(y)) = \\
 &= - \sum_{n \in \mathbb{Z}^d} \int_{\Omega} dx \int_{\Omega} dy a(x - y + n)\mu(x, y)e^{-i\langle \xi, x+n \rangle} \overline{u(x)} (e^{i\langle \xi, x+n \rangle} u(x) - e^{i\langle \xi, y \rangle} u(y)) = \\
 &= - \sum_{n \in \mathbb{Z}^d} \int_{\Omega} dx \int_{\Omega} dy a(x - y + n)\mu(x, y)e^{-i\langle \xi, x \rangle} \overline{u(x)} (e^{i\langle \xi, x \rangle} u(x) - e^{i\langle \xi, y-n \rangle} u(y)) = \\
 &= - \int_{\Omega} dx \overline{u(x)} \int_{\Omega} dy \mu(x, y) (\tilde{a}(0, x - y)u(x) - \tilde{a}(\xi, x - y)u(y)) = \\
 &= -(\mathbb{A}(\xi, a, \mu)u, u), \quad u \in L_2(\Omega). \quad (1.15)
 \end{aligned}$$

Now (1.10) follows from (1.13) and (1.15).  $\square$

#### 1.4. Estimates of the quadratic form of operator $\mathbb{A}(\xi, a, \mu)$

As above it is convenient to consider the case  $\mu = \mu_0 \equiv 1$  separately. Recalling the notation  $\mathbb{A}_0(\xi, a) := \mathbb{A}(\xi, a, \mu_0)$ ,  $\mathbb{B}_0(\xi, a) := \mathbb{B}(\xi, a, \mu_0)$ , from the relations in (1.2) and (1.10) we derive the estimates

$$\mu_-(\mathbb{A}_0(\xi, a)u, u) \leq (\mathbb{A}(\xi, a, \mu)u, u) \leq \mu_+(\mathbb{A}_0(\xi, a)u, u), \quad u \in L_2(\Omega), \quad \xi \in \tilde{\Omega}. \quad (1.16)$$

The operators  $\mathbb{A}_0(\xi, a)$ ,  $\xi \in \tilde{\Omega}$ , are diagonalized by means of the unitary discrete Fourier transform  $F : L_2(\Omega) \rightarrow \ell_2(\mathbb{Z}^d)$  defined as follows:

$$Fu(n) = \int_{\Omega} u(x)e^{-2\pi i\langle n,x \rangle} dx, \quad n \in \mathbb{Z}^d, \quad u \in L_2(\Omega);$$

$$F^*v(x) = \sum_{n \in \mathbb{Z}^d} v_n e^{2\pi i\langle n,x \rangle}, \quad x \in \Omega, \quad v = \{v_n\}_{n \in \mathbb{Z}^d} \in \ell_2(\mathbb{Z}^d).$$

We have

$$\mathbb{A}_0(\xi, a) = F^*[\hat{a}(0) - \hat{a}(2\pi n + \xi)]F, \quad \hat{a}(k) := \int_{\mathbb{R}^d} a(x)e^{-i\langle k,x \rangle} dx, \quad k \in \mathbb{R}^d; \tag{1.17}$$

here  $[f(n)]$ ,  $f(n) = \hat{a}(0) - \hat{a}(2\pi n + \xi)$ , stands for the operator of multiplication by the function  $f(n)$  in the space  $\ell_2(\mathbb{Z}^d)$ . Thus, the symbol of the operator  $\mathbb{A}_0(\xi, a)$  is a sequence  $\{\hat{A}_n(\xi)\}_{n \in \mathbb{Z}^d}$  with

$$\hat{A}_n(\xi) = \hat{A}(\xi + 2\pi n) = \hat{a}(0) - \hat{a}(2\pi n + \xi) = \int_{\mathbb{R}^d} (1 - \cos(\langle z, \xi + 2\pi n \rangle))a(z) dz, \quad n \in \mathbb{Z}^d, \quad \xi \in \tilde{\Omega}.$$

Here we have used the fact that  $a(z)$  is an even function and therefore the integral  $\int_{\mathbb{R}^d} \sin(\langle z, \xi + 2\pi n \rangle)a(z) dz$  is equal to zero.

**Lemma 1.2.** *Under conditions (1.1)–(1.3) the point  $\lambda_0 = 0$  is a simple eigenvalue of the operator  $\mathbb{A}(0, a, \mu)$ . Moreover,  $\text{Ker } \mathbb{A}(0, a, \mu) = \mathcal{L}\{\mathbf{1}_{\Omega}\}$ .*

**Proof.** Since the set of  $z \in \mathbb{R}^d$  such that  $a(z) > 0$  has a positive Lebesgue measure, the quantities  $\hat{A}(2\pi n) = \int_{\mathbb{R}^d} 2 \sin^2(\langle z, \pi n \rangle)a(z) dz$  are not equal to zero for  $n \in \mathbb{Z}^d \setminus \{0\}$ ; on the other hand,  $\hat{A}(0) = 0$ . Therefore,  $\text{Ker } \mathbb{A}_0(0, a) = \mathcal{L}\{\mathbf{1}_{\Omega}\}$ , and, by (1.16),  $\text{Ker } \mathbb{A}(0, a, \mu) = \mathcal{L}\{\mathbf{1}_{\Omega}\}$ .  $\square$

In order to estimate the quadratic form of the operator  $\mathbb{A}_0(\xi, a)$  for all  $\xi \in \tilde{\Omega}$ , we carry out the detailed analysis of the symbol of the operator  $\mathbb{A}_0(\xi, a)$ ,  $\xi \in \tilde{\Omega}$ . Under condition (1.1) the function

$$\hat{A}(y) = \int_{\mathbb{R}^d} (1 - \cos(\langle z, y \rangle))a(z) dz = 2 \int_{\mathbb{R}^d} \sin^2\left(\frac{\langle z, y \rangle}{2}\right)a(z) dz, \quad y \in \mathbb{R}^d, \tag{1.18}$$

depends continuously on  $y \in \mathbb{R}^d$  and, by the Riemann–Lebesgue lemma, converges to  $\|a\|_{L_1} > 0$  as  $|y| \rightarrow \infty$ . Furthermore, it is easy to check that  $\hat{A}(y) \neq 0$  if  $y \neq 0$ . Consequently, the estimate

$$\min_{|y| \geq r} \hat{A}(y) =: \mathcal{A}_r(a) > 0, \quad r > 0, \tag{1.19}$$

holds. Under condition (1.4) the function

$$\int_{\mathbb{R}^d} a(z)\langle z, y \rangle^2 dz =: M_a(y) \tag{1.20}$$

is continuous in  $y \in \mathbb{R}^d$  and not equal to zero if  $y \neq 0$ . This yields the following inequality:

$$\min_{|\theta|=1} M_a(\theta) =: \mathcal{M}(a) > 0. \tag{1.21}$$

Below, in Section 4.3, under conditions (1.1)–(1.4) and an additional assumption  $a \in L_2(\mathbb{R}^d)$ , we will obtain explicit expressions for  $\mathcal{A}_r(a)$ ,  $r > 0$ , and  $\mathcal{M}(a)$  in terms of  $\|a\|_{L_1}$ ,  $\|a\|_{L_2}$  and  $M_k(a)$  with  $k = 1, 2, 3$ .

**Lemma 1.3.** *Let conditions (1.1)–(1.4) be fulfilled. Then*

$$\hat{A}(\xi + 2\pi n) \geq C(a)|\xi|^2, \quad \xi \in \tilde{\Omega}, \quad n \in \mathbb{Z}^d; \tag{1.22}$$

$$\hat{A}(\xi + 2\pi n) \geq \mathcal{A}_\pi(a), \quad \xi \in \tilde{\Omega}, \quad n \in \mathbb{Z}^d \setminus \{0\}, \tag{1.23}$$

where

$$C(a) := \min \left\{ \frac{1}{4} \mathcal{M}(a), \mathcal{A}_{r(a)}(a) \pi^{-2} d^{-1}, \mathcal{A}_\pi(a) \pi^{-2} d^{-1} \right\} > 0, \tag{1.24}$$

the quantities  $\mathcal{A}_r(a)$ ,  $r > 0$ , and  $\mathcal{M}(a)$  are defined in (1.19) and (1.21), respectively, and  $r(a) := \frac{3}{2} \mathcal{M}(a) M_3^{-1}(a)$ .

**Proof.** If conditions (1.1), (1.4) hold, then the function  $\hat{A}(y)$ ,  $y \in \mathbb{R}^d$ , is three times continuously differentiable, and

$$\hat{A}(0) = 0, \quad \nabla \hat{A}(0) = 0, \quad \langle (H\hat{A})(0)y, y \rangle := \sum_{i,j=1}^d \partial_i \partial_j \hat{A}(0) y_i y_j = \int_{\mathbb{R}^d} \langle z, y \rangle^2 a(z) dz, \quad y \in \mathbb{R}^d; \tag{1.25}$$

$$\partial_i \partial_j \partial_k \hat{A}(y) = - \int_{\mathbb{R}^d} z_i y_j z_k \sin(\langle z, y \rangle) a(z) dz, \quad y \in \mathbb{R}^d. \tag{1.26}$$

Combining condition (1.4) with relations (1.21) and (1.25) we obtain the estimates

$$\mathcal{M}(a)|y|^2 \leq \langle (H\hat{A})(0)y, y \rangle \leq M_2(a)|y|^2, \quad y \in \mathbb{R}^d. \tag{1.27}$$

The following inequality is a consequence of (1.4) and (1.26):

$$\left| \sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d y_i y_j y_k \partial_i \partial_j \partial_k \hat{A}(y_0) \right| \leq M_3(a)|y|^3, \quad y, y_0 \in \mathbb{R}^d. \tag{1.28}$$

By the Hadamard lemma (see Lemma 4.3 below) and due to relations (1.25) and (1.28), we have

$$\hat{A}(y) = \frac{1}{2} \langle (H\hat{A})(0)y, y \rangle + R(y), \quad |R(y)| \leq \frac{1}{6} M_3(a) |y|^3, \quad y \in \mathbb{R}^d. \tag{1.29}$$

The second relation in (1.29) implies that  $|R(y)| \leq \frac{1}{4} \mathcal{M}(a) |y|^2$  for  $|y| \leq r(a) := \frac{3}{2} \mathcal{M}(a) M_3^{-1}(a)$ . Therefore, considering (1.27) and (1.29) we obtain the following lower bound:

$$\hat{A}(y) \geq \frac{1}{4} \mathcal{M}(a) |y|^2, \quad |y| \leq r(a). \tag{1.30}$$

Observe that

$$\begin{aligned} \xi \in \tilde{\Omega} &\implies |\xi| \leq \pi \sqrt{d}; \\ \xi \in \tilde{\Omega}, \quad n \in \mathbb{Z}^d &\implies |\xi + 2\pi n| \geq |\xi|; \\ \xi \in \tilde{\Omega}, \quad n \in \mathbb{Z}^d \setminus \{0\} &\implies |\xi + 2\pi n| \geq \pi. \end{aligned} \tag{1.31}$$

Combining these relations with (1.19) and (1.30) yields

$$\begin{aligned} \hat{A}(\xi + 2\pi n) &\geq \frac{1}{4} \mathcal{M}(a) |\xi|^2, \quad n = 0, \quad \xi \in \tilde{\Omega}, \quad |\xi| \leq r(a); \\ \hat{A}(\xi + 2\pi n) &\geq \mathcal{A}_{r(a)}(a) \geq \mathcal{A}_{r(a)}(a) \pi^{-2} d^{-1} |\xi|^2, \quad n = 0, \quad \xi \in \tilde{\Omega}, \quad |\xi| \geq r(a); \\ \hat{A}(\xi + 2\pi n) &\geq \mathcal{A}_{\pi}(a) \geq \mathcal{A}_{\pi}(a) \pi^{-2} d^{-1} |\xi|^2, \quad n \in \mathbb{Z}^d \setminus \{0\}, \quad \xi \in \tilde{\Omega}. \end{aligned}$$

Finally, we conclude that inequalities (1.22) and (1.23) hold true.  $\square$

The following statement is a consequence of relations (1.16)–(1.17) and Lemmata 1.2, 1.3:

**Proposition 1.4.** *Assume that conditions (1.1)–(1.4) hold. Then we have*

$$(\mathbb{A}(\xi, a, \mu)u, u) \geq \mu_- C(a) |\xi|^2 \|u\|_{L_2(\Omega)}^2, \quad u \in L_2(\Omega), \quad \xi \in \tilde{\Omega}; \tag{1.32}$$

$$(\mathbb{A}(\xi, a, \mu)u, u) \geq \mu_- \mathcal{A}_{\pi}(a) \|u\|_{L_2(\Omega)}^2, \quad u \in L_2(\Omega) \ominus \mathcal{L}\{\mathbf{1}_{\Omega}\}, \quad \xi \in \tilde{\Omega}. \tag{1.33}$$

We will also need

**Proposition 1.5.** *Let conditions (1.1)–(1.4) be fulfilled. Then the following estimate holds:*

$$\|\mathbb{A}(\xi, a, \mu) - \mathbb{A}(\eta, a, \mu)\| \leq \mu_+ 2^d M_1(a) |\xi - \eta|, \quad \xi, \eta \in \tilde{\Omega}. \tag{1.34}$$

**Proof.** Since  $\mathbb{A}(\xi, a, \mu) - \mathbb{A}(\eta, a, \mu)$  is an integral operator in  $L_2(\Omega)$  with the kernel  $(\tilde{a}(\eta, x - y) - \tilde{a}(\xi, x - y))\mu(x, y)$ ,  $x, y \in \Omega$ , then, by the Schur test (see Lemma 4.1), we obtain

$$\begin{aligned} \|\mathbb{A}(\xi, a, \mu) - \mathbb{A}(\eta, a, \mu)\| &\leq \mu_+ \int_{[-1,1]^d} |\tilde{a}(\eta, z) - \tilde{a}(\xi, z)| dz = \mu_+ 2^d \int_{\Omega} |\tilde{a}(\eta, z) - \tilde{a}(\xi, z)| dz \leq \\ &\leq \mu_+ 2^d \int_{\mathbb{R}^d} a(z) |e^{-i(\eta, z)} - e^{-i(\xi, z)}| dz = \mu_+ 2^d \int_{\mathbb{R}^d} a(z) |e^{i(\xi - \eta, z)} - 1| dz \leq \\ &\leq \mu_+ 2^d |\xi - \eta| \int_{\mathbb{R}^d} a(z) |z| dz = \mu_+ 2^d M_1(a) |\xi - \eta|, \quad \xi, \eta \in \tilde{\Omega}. \quad \square \end{aligned}$$

## 2. Threshold characteristics of nonlocal Schrödinger operator in the vicinity of the spectrum bottom

### 2.1. The spectral edge of the operator $\mathbb{A}(\xi, a, \mu)$

According to Lemma 1.2, under conditions (1.1)–(1.3) the lower edge of the spectrum of operator  $\mathbb{A}(0, a, \mu)$  which is defined as  $\min\{\lambda \in \mathbb{R} : \lambda \in \sigma(\mathbb{A}(0, a, \mu))\}$ , is a simple eigenvalue  $\lambda_0 = 0$ . Letting  $d_0(a, \mu) = \text{dist}(\lambda_0, \sigma(\mathbb{A}(0, a, \mu)) \setminus \{\lambda_0\})$  and assuming that conditions (1.1)–(1.4) are fulfilled, we derive from estimate (1.33) the following inequality:

$$d_0(a, \mu) \geq \mu_- \mathcal{A}_\pi(a). \tag{2.1}$$

For brevity we use the short notation  $d_0 := d_0(a, \mu)$ . By the perturbation theory arguments, we deduce from estimate (1.34) that for all  $\xi \in \tilde{\Omega}$  such that  $|\xi| \leq 3^{-1} 2^{-d} M_1(a)^{-1} \mu_+^{-1} \mu_- \mathcal{A}_\pi(a)$  the relations

$$\text{rank } E_{\mathbb{A}(\xi, a, \mu)}[0, d_0/3] = 1, \quad \sigma(\mathbb{A}(\xi, a, \mu)) \cap (d_0/3; 2d_0/3) = \emptyset$$

hold. Denote  $\delta_0(a, \mu) = \min\{1, 3^{-1} 2^{-d} M_1(a)^{-1} \mu_+^{-1} \mu_- \mathcal{A}_\pi(a)\}$ . Observe that  $\delta_0(a, \mu) > 0$ . Combining the above statements we arrive at

**Proposition 2.1.** *Let conditions (1.1)–(1.4) be satisfied. Then for all  $|\xi|$  with  $|\xi| \leq \delta_0(a, \mu)$  the spectrum of the operator  $\mathbb{A}(\xi) = \mathbb{A}(\xi, a, \mu)$  in the interval  $[0, d_0/3]$  consists of just one simple eigenvalue, while the interval  $(d_0/3, 2d_0/3)$  belongs to the resolvent set of  $\mathbb{A}(\xi, a, \mu)$ .*

Under conditions (1.1)–(1.4) the operator-function  $\mathbb{A}(\xi, a, \mu)$  is three times continuously differentiable in the variable  $\xi$  with respect to the operator norm of  $\mathcal{B}(L_2(\Omega))$ , moreover, the following relations hold

$$\begin{aligned} \partial^\alpha \mathbb{A}(\xi, a, \mu)u(x) &= \int_{\Omega} \tilde{a}_\alpha(\xi, x - y) \mu(x, y) u(y) dy, \quad x \in \Omega, \quad u \in L_2(\Omega); \\ \tilde{a}_\alpha(\xi, z) &= (-1)(-i)^{|\alpha|} \sum_{n \in \mathbb{Z}^d} (z + n)^\alpha a(z + n) e^{-i(\xi, z + n)}, \quad \alpha \in \mathbb{Z}_+^d, \quad |\alpha| \leq 3. \end{aligned} \tag{2.2}$$

By the Hadamard lemma (see Lemma 4.3 below), the operator-function  $\mathbb{A}(\cdot, a, \mu)$  admits an expansion

$$\begin{aligned} \mathbb{A}(\xi, a, \mu) &= \mathbb{A}(0, a, \mu) + \sum_{i=1}^d \partial_i \mathbb{A}(0, a, \mu) \xi_i + \\ &+ \frac{1}{2} \sum_{i,j=1}^d \partial_i \partial_j \mathbb{A}(0, a, \mu) \xi_i \xi_j + \mathbb{K}(\xi, a, \mu), \quad |\xi| \leq \delta_0(a, \mu), \end{aligned} \tag{2.3}$$

with

$$\mathbb{K}(\xi, a, \mu) = \sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d \xi_i \xi_j \xi_k \int_0^1 ds_1 s_1^2 \int_0^1 ds_2 s_2^2 \int_0^1 ds_3 \partial_i \partial_j \partial_k \mathbb{A}(s_1 s_2 s_3 \xi).$$

Applying the Schur test one can easily show that

$$\left\| \sum_{i=1}^d \partial_i \mathbb{A}(0, a, \mu) \xi_i \right\| \leq |\xi| \mu_+ M_1(a), \quad |\xi| \leq \delta_0(a, \mu); \tag{2.4}$$

$$\left\| \frac{1}{2} \sum_{i,j=1}^d \partial_i \partial_j \mathbb{A}(0, a, \mu) \xi_i \xi_j \right\| \leq |\xi|^2 \frac{1}{2} \mu_+ M_2(a), \quad |\xi| \leq \delta_0(a, \mu); \tag{2.5}$$

$$\|\mathbb{K}(\xi, a, \mu)\| \leq |\xi|^3 \frac{1}{6} \mu_+ M_3(a), \quad |\xi| \leq \delta_0(a, \mu). \tag{2.6}$$

We introduce some more notation:  $F(\xi)$  denotes the spectral projector of the operator  $\mathbb{A}(\xi, a, \mu)$  that corresponds to the interval  $[0, d_0/3]$ , the symbol  $\mathfrak{N}$  stands for the kernel  $\text{Ker } \mathbb{A}(0, a, \mu) = \mathcal{L}\{\mathbf{1}_\Omega\}$ . We then denote by  $P$  the orthogonal projector on  $\mathfrak{N}$  and set  $P^\perp := I - P$ . Let  $\Gamma$  be a contour on the complex plane that is equidistant to the interval  $[0, d_0/3]$  and passes through the middle point of the interval  $(d_0/3, 2d_0/3)$ . By the Riesz formula that reads

$$F(\xi) = \frac{-1}{2\pi i} \oint_{\Gamma} (\mathbb{A}(\xi, a, \mu) - \zeta I)^{-1} d\zeta, \quad |\xi| \leq \delta_0(a, \mu), \tag{2.7}$$

the operator-functions  $F(\xi)$  and  $\mathbb{A}(\xi, a, \mu)F(\xi)$ ,  $|\xi| \leq \delta_0(a, \mu)$ , are three times continuously differentiable. Therefore, we have

$$F(\xi) = P + \sum_{i=1}^d F_i \xi_i + \frac{1}{2} \sum_{i,j=1}^d F_{ij} \xi_i \xi_j + O(|\xi|^3), \quad |\xi| \rightarrow 0; \tag{2.8}$$

$$\mathbb{A}(\xi, a, \mu)F(\xi) = G_0 + \sum_{i=1}^d G_i \xi_i + \frac{1}{2} \sum_{i,j=1}^d G_{ij} \xi_i \xi_j + O(|\xi|^3), \quad |\xi| \rightarrow 0. \tag{2.9}$$

The main goal of this section is to determine the coefficients of expansions (2.8)–(2.9) and to estimate the remainders.

2.2. Computing the coefficients of the expansions of  $F(\xi)$  and  $\mathbb{A}(\xi)F(\xi)$

From (2.3), (2.6) and (2.8) we derive the relations

$$G_0 = \mathbb{A}(0, a, \mu)P = 0; \tag{2.10}$$

$$G_i = \partial_i \mathbb{A}(0, a, \mu)P + \mathbb{A}(0, a, \mu)F_i, \quad i = 1, \dots, d; \tag{2.11}$$

$$G_{ij} = \partial_i \partial_j \mathbb{A}(0, a, \mu)P + \partial_i \mathbb{A}(0, a, \mu)F_j + \partial_j \mathbb{A}(0, a, \mu)F_i + \mathbb{A}(0, a, \mu)F_{ij}, \quad i, j = 1, \dots, d. \tag{2.12}$$

Since the operators  $\mathbb{A}(\xi, a, \mu)F(\xi)$  are self-adjoint, the second relation implies that the operators  $G_i, i = 1, \dots, d$ , are also self-adjoint. Consequently, taking into account the inequality  $(\mathbb{A}(\xi, a, \mu)F(\xi)u, u) \geq 0, u \in L_2(\Omega), |\xi| \leq \delta_0(a, \mu)$ , the equality  $\mathbb{A}(0, a, \mu)F(0) = 0$  and the definition of  $G_i$  in (2.11), we conclude that

$$G_i = \partial_i \mathbb{A}(0, a, \mu)P + \mathbb{A}(0, a, \mu)F_i = 0, \quad i = 1, \dots, d. \tag{2.13}$$

Combining (2.9), (2.10) and (2.13) yields the following asymptotic representation:

$$\mathbb{A}(\xi, a, \mu)F(\xi) = \frac{1}{2} \sum_{i,j=1}^d G_{ij} \xi_i \xi_j + O(|\xi|^3), \quad |\xi| \rightarrow 0. \tag{2.14}$$

Since  $F(\xi)$  is a spectral projector, we have  $F^2(\xi) = F(\xi)$  and thus, recalling that  $P$  is the orthogonal projection on  $\mathcal{L}\{\mathbf{1}_\Omega\}$ , we obtain

$$F_i P + P F_i = F_i, \quad i = 1, \dots, d. \tag{2.15}$$

The latter relation implies that  $P^\perp F_i P^\perp = 0$  and  $P F_i P = 2P F_i P$ . Therefore,  $P F_i P = 0$  and, by (2.15),

$$F_i = P^\perp F_i P + P F_i P^\perp, \quad i = 1, \dots, d. \tag{2.16}$$

From the latter inequality and (2.13) we deduce that

$$\begin{aligned} \partial_i \mathbb{A}(0, a, \mu)P &= -\mathbb{A}(0, a, \mu)F_i = -P^\perp \mathbb{A}(0, a, \mu)P^\perp F_i = -P^\perp \mathbb{A}(0, a, \mu)P^\perp F_i P, \\ &i = 1, \dots, d. \end{aligned}$$

Finally, we arrive at the following relations:

$$\partial_i \mathbb{A}(0, a, \mu)P = P^\perp \partial_i \mathbb{A}(0, a, \mu)P, \tag{2.17}$$

$$P^\perp F_i P = -P^\perp \mathbb{A}(0, a, \mu)^{-1} P^\perp \partial_i \mathbb{A}(0, a, \mu)P, \quad i = 1, \dots, d; \tag{2.18}$$

here the notation  $\mathbb{A}(0, a, \mu)^{-1}$  is used for the operator that is inverse to the operator  $\mathbb{A}(0, a, \mu)|_{\mathfrak{N}^\perp} : \mathfrak{N}^\perp \mapsto \mathfrak{N}^\perp$ . Notice that this operator is well defined.

Next, considering (2.14), the asymptotic formula  $F(\xi) = P + O(|\xi|)$  and the properties of the operator  $\mathbb{A}(\xi)F(\xi) = (\mathbb{A}(\xi)F(\xi))F(\xi) = F(\xi)(\mathbb{A}(\xi)F(\xi))$  one has  $G_{ij} = PG_{ij}P$ ,  $i, j = 1, \dots, d$ . Consequently, in view of (2.16), relation (2.12) can be rearranged as follows:

$$G_{ij} = P\partial_i\partial_j\mathbb{A}(0, a, \mu)P + P\partial_i\mathbb{A}(0, a, \mu)P^\perp F_j P + P\partial_j\mathbb{A}(0, a, \mu)P^\perp F_i P, \quad i, j = 1, \dots, d. \tag{2.19}$$

Substituting the right-hand side of (2.18) for  $P^\perp F_j P$  in (2.19) yields

$$G_{ij} = P\partial_i\partial_j\mathbb{A}(0, a, \mu)P - P\partial_i\mathbb{A}(0, a, \mu)P^\perp\mathbb{A}(0, a, \mu)^{-1}P^\perp\partial_j\mathbb{A}(0, a, \mu)P - P\partial_j\mathbb{A}(0, a, \mu)P^\perp\mathbb{A}(0, a, \mu)^{-1}P^\perp\partial_i\mathbb{A}(0, a, \mu)P, \quad i, j = 1, \dots, d. \tag{2.20}$$

Since  $P = (\cdot, \mathbf{1}_\Omega)\mathbf{1}_\Omega$  is a first rank operator,  $G_{ij}$  admits a representation  $G_{ij} = g_{ij}P$  with the coefficients  $g_{ij} \in \mathbb{C}$ ,  $i, j = 1, \dots, d$ , that can be computed with the help of formula (2.20). Indeed, from the relation  $P = (\cdot, \mathbf{1}_\Omega)\mathbf{1}_\Omega$  we obtain

$$\partial_j\mathbb{A}(0, a, \mu)P = \mathbf{i}(\cdot, \mathbf{1}_\Omega)w_j \quad \text{with} \quad \mathbf{i}w_j = \partial_j\mathbb{A}(0, a, \mu)\mathbf{1}_\Omega, \quad j = 1, \dots, d. \tag{2.21}$$

Due to (2.2) the following equality holds:

$$\begin{aligned} w_j(x) &= \overline{w_j(x)} = \int_\Omega \sum_{n \in \mathbb{Z}^d} (x_j - y_j + n_j)a(x - y + n)\mu(x, y) dy = \\ &= \int_{\mathbb{R}^d} (x_j - y_j)a(x - y)\mu(x, y) dy, \quad x \in \Omega, \quad j = 1, \dots, d. \end{aligned} \tag{2.22}$$

From (2.17) and (2.21) it follows that, for all  $j = 1, \dots, d$ ,

$$P^\perp\mathbb{A}(0, a, \mu)^{-1}P^\perp\partial_j\mathbb{A}(0, a, \mu)P = \mathbf{i}(\cdot, \mathbf{1}_\Omega)v_j, \quad v_j = P^\perp\mathbb{A}(0, a, \mu)^{-1}P^\perp w_j. \tag{2.23}$$

Notice that the functions  $v_j \in \overline{v_j} \in L_2(\Omega)$ ,  $j = 1, \dots, d$ , satisfy cell problems on  $\Omega$  that read

$$\begin{cases} \int_\Omega \tilde{a}(0, x - y)\mu(x, y)(v_j(x) - v_j(y)) dy = w_j(x), & x \in \Omega, \\ \int_\Omega v_j(x) dx = 0, \end{cases} \quad j = 1, \dots, d.$$

Identifying the functions  $v_j \in L_2(\Omega)$ ,  $j = 1, \dots, d$ , with their periodic extensions one can rewrite these cell problems as follows:

$$\begin{cases} \int_{\mathbb{R}^d} a(x - y)\mu(x, y)(v_j(x) - v_j(y)) dy = \int_{\mathbb{R}^d} a(x - y)\mu(x, y)(x_j - y_j) dy, & x \in \Omega, \\ \int_\Omega v_j(x) dx = 0. \end{cases} \tag{2.24}$$

Making the rearrangements similar to those in (1.6) we conclude that the problems in (2.24) have a unique solution. Then, by (2.21) and (2.23), we have

$$P \partial_i \mathbb{A}(0, a, \mu) P^\perp \mathbb{A}(0, a, \mu)^{-1} P^\perp \partial_j \mathbb{A}(0, a, \mu) P = (v_j, w_i) P, \quad i, j = 1, \dots, d, \quad (2.25)$$

where the functions  $w_j \in L_2(\Omega)$ ,  $j = 1, \dots, d$  are defined in (2.22), and the functions  $v_j \in L_2(\Omega)$ ,  $j = 1, \dots, d$ , satisfy auxiliary problems (2.24) on the periodicity cell  $\Omega$ . In the same way one can derive the relations

$$P \partial_j \partial_i \mathbb{A}(0, a, \mu) P = (w_{ij}, \mathbf{1}_\Omega) P, \quad i, j = 1, \dots, d. \quad (2.26)$$

Here  $w_{ij} = \overline{w_{ij}} = \partial_i \partial_j \mathbb{A}(0, a, \mu) \mathbf{1}_\Omega \in L_2(\Omega)$ , that is

$$\begin{aligned} w_{ij}(x) &= \int_{\Omega} \sum_{n \in \mathbb{Z}^d} (x_i - y_i + n_i)(x_j - y_j + n_j) a(x - y + n) \mu(x, y) dy = \\ &= \int_{\mathbb{R}^d} (x_i - y_i)(x_j - y_j) a(x - y) \mu(x, y) dy, \quad x \in \Omega, \quad i, j = 1, \dots, d. \end{aligned} \quad (2.27)$$

Finally, combining the above relations (2.20), (2.22) and (2.25)–(2.27) we conclude that

$$\begin{aligned} g_{ij} &= (w_{ij}, \mathbf{1}_\Omega) - (v_j, w_i) - (v_i, w_j) = \\ &= \int_{\Omega} dx \int_{\mathbb{R}^d} dy ((x_i - y_i)(x_j - y_j) - v_j(x)(x_i - y_i) - v_i(x)(x_j - y_j)) a(x - y) \mu(x, y), \\ & \hspace{20em} i, j = 1, \dots, d. \end{aligned} \quad (2.28)$$

Here the functions  $v_j(x)$ ,  $x \in \Omega$ ,  $j = 1, \dots, d$ , are defined as solutions of problems (2.24).

It should be noted that there are other methods of computing the coefficients of Taylor expansions for the functions  $F(\xi)$  and  $\mathbb{A}(\xi)F(\xi)$ . Two of them are described in Section 4.2.

### 2.3. Approximation of the operators $F(\xi)$ and $\mathbb{A}(\xi)F(\xi)$

The precision of the above constructed approximations of the operator-functions  $F(\xi)$  and  $\mathbb{A}(\xi, a, \mu)F(\xi)$  for  $|\xi| \leq \delta_0(a, \mu)$  is estimated in the next statement.

**Theorem 2.2.** *Let conditions (1.1)–(1.4) be fulfilled. Then the following estimates hold:*

$$\|F(\xi) - P\| \leq C_1(a, \mu)|\xi|, \quad |\xi| \leq \delta_0(a, \mu); \quad (2.29)$$

$$\left\| \mathbb{A}(\xi, a, \mu)F(\xi) - \frac{1}{2} \sum_{i,j=1}^d g_{ij} \xi_i \xi_j P \right\| \leq C_2(a, \mu)|\xi|^3, \quad |\xi| \leq \delta_0(a, \mu). \quad (2.30)$$

The constants  $C_1(a, \mu)$  and  $C_2(a, \mu)$  defined in (2.34) and (2.42), respectively, depend only on  $d, \mu_-, \mu_+, M_1(a), M_2(a), M_3(a), \mathcal{A}_\pi(a)$ .

**Proof.** By the Riesz formula (2.7), the difference  $F(\xi) - P$  admits the representation

$$F(\xi) - P = \frac{-1}{2\pi i} \oint_{\Gamma} \left( (\mathbb{A}(\xi, a, \mu) - \zeta I)^{-1} - (\mathbb{A}(0, a, \mu) - \zeta I)^{-1} \right) d\zeta, \quad |\xi| \leq \delta_0(a, \mu). \tag{2.31}$$

The term  $(\mathbb{A}(\xi, a, \mu) - \zeta I)^{-1} - (\mathbb{A}(0, a, \mu) - \zeta I)^{-1}$  satisfies the resolvent identity that reads

$$\begin{aligned} & (\mathbb{A}(\xi, a, \mu) - \zeta I)^{-1} - (\mathbb{A}(0, a, \mu) - \zeta I)^{-1} = \\ & = (\mathbb{A}(\xi, a, \mu) - \zeta I)^{-1} (\mathbb{A}(0, a, \mu) - \mathbb{A}(\xi, a, \mu)) (\mathbb{A}(0, a, \mu) - \zeta I)^{-1}, \\ & \quad \quad \quad |\xi| \leq \delta_0(a, \mu), \quad \zeta \in \Gamma. \end{aligned} \tag{2.32}$$

It remains to notice that the length of the contour  $\Gamma$  is equal to  $\frac{\pi+2}{3}d_0$  and that for all  $\zeta \in \Gamma$  both resolvents satisfy the estimates

$$\|(\mathbb{A}(\xi, a, \mu) - \zeta I)^{-1}\| \leq 6d_0^{-1}, \quad \|(\mathbb{A}(0, a, \mu) - \zeta I)^{-1}\| \leq 6d_0^{-1}. \tag{2.33}$$

Now, as a consequence of (1.34) and (2.31)–(2.33), we obtain (2.29) with

$$C_1(a, \mu) := 6(\pi + 2)\pi^{-1}2^d d_0^{-1} \mu_+ M_1(a). \tag{2.34}$$

Denote

$$\begin{aligned} R(\xi, \zeta) & := (\mathbb{A}(\xi, a, \mu) - \zeta I)^{-1}, \quad |\xi| \leq \delta_0(a, \mu), \quad \zeta \in \Gamma; \\ R_0(\zeta) & := R(0, \zeta), \quad \zeta \in \Gamma; \\ \Delta \mathbb{A}(\xi) & := \mathbb{A}(\xi, a, \mu) - \mathbb{A}(0, a, \mu), \quad |\xi| \leq \delta_0(a, \mu). \end{aligned}$$

Iterating (2.32) we derive the relation

$$\begin{aligned} R(\xi, \zeta) & = R_0(\zeta) - R_0(\zeta) \Delta \mathbb{A}(\xi) R_0(\zeta) + \\ & \quad + R_0(\zeta) \Delta \mathbb{A}(\xi) R_0(\zeta) \Delta \mathbb{A}(\xi) R_0(\zeta) - \\ & \quad - R(\xi, \zeta) \Delta \mathbb{A}(\xi) R_0(\zeta) \Delta \mathbb{A}(\xi) R_0(\zeta) \Delta \mathbb{A}(\xi) R_0(\zeta), \quad |\xi| \leq \delta_0(a, \mu), \quad \zeta \in \Gamma. \end{aligned} \tag{2.35}$$

From (2.3) and (2.35) it follows that

$$\begin{aligned} R(\xi, \zeta) & = R_0(\zeta) - R_0(\zeta) \sum_{i=1}^d \partial_i \mathbb{A}(0, a, \mu) \xi_i R_0(\zeta) - R_0(\zeta) \frac{1}{2} \sum_{i,j=1}^d \partial_i \partial_j \mathbb{A}(0, a, \mu) \xi_i \xi_j R_0(\zeta) + \\ & \quad + R_0(\zeta) \sum_{i=1}^d \partial_i \mathbb{A}(0, a, \mu) \xi_i R_0(\zeta) \sum_{j=1}^d \partial_j \mathbb{A}(0, a, \mu) \xi_j R_0(\zeta) - R_0(\zeta) \mathbb{K}(\xi, a, \mu) R_0(\zeta) + \end{aligned}$$

$$\begin{aligned}
 &+ R_0(\zeta) \sum_{i=1}^d \partial_i \mathbb{A}(0, a, \mu) \xi_i R_0(\zeta) \Delta_2 \mathbb{A}(\xi) R_0(\zeta) + R_0(\zeta) \Delta_2 \mathbb{A}(\xi) R_0(\zeta) \Delta \mathbb{A}(\xi) R_0(\zeta) - \\
 &- R(\xi, \zeta) \Delta \mathbb{A}(\xi) R_0(\zeta) \Delta \mathbb{A}(\xi) R_0(\zeta) \Delta \mathbb{A}(\xi) R_0(\zeta), \quad |\xi| \leq \delta_0(a, \mu), \quad \zeta \in \Gamma. \quad (2.36)
 \end{aligned}$$

Here

$$\Delta_2 \mathbb{A}(\xi) := \frac{1}{2} \sum_{i,j=1}^d \partial_i \partial_j \mathbb{A}(0, a, \mu) \xi_i \xi_j + \mathbb{K}(\xi, a, \mu).$$

Taking into account (2.36) and the relation

$$\mathbb{A}(\xi, a, \mu) F(\xi) = \frac{-1}{2\pi i} \oint_{\Gamma} (\mathbb{A}(\xi, a, \mu) - \zeta I)^{-1} \zeta d\zeta, \quad |\xi| \leq \delta_0(a, \mu), \quad (2.37)$$

we obtain the following representations for the coefficients of the expansion in (2.9):

$$G_0 = \frac{-1}{2\pi i} \oint_{\Gamma} R_0(\zeta) \zeta d\zeta; \quad (2.38)$$

$$G_i = \frac{1}{2\pi i} \oint_{\Gamma} R_0(\zeta) \partial_i \mathbb{A}(0, a, \mu) R_0(\zeta) \zeta d\zeta, \quad i = 1, \dots, d; \quad (2.39)$$

$$\begin{aligned}
 G_{ij} &= \frac{1}{2\pi i} \oint_{\Gamma} R_0(\zeta) \partial_i \partial_j \mathbb{A}(0, a, \mu) R_0(\zeta) \zeta d\zeta - \\
 &- \frac{1}{2\pi i} \oint_{\Gamma} R_0(\zeta) \left( \partial_i \mathbb{A}(0, a, \mu) R_0(\zeta) \partial_j \mathbb{A}(0, a, \mu) + \partial_j \mathbb{A}(0, a, \mu) R_0(\zeta) \partial_i \mathbb{A}(0, a, \mu) \right) R_0(\zeta) \zeta d\zeta, \\
 & \quad i, j = 1, \dots, d. \quad (2.40)
 \end{aligned}$$

Due to (2.10) and (2.13) we have  $G_0 = 0, G_i = 0, i = 1, \dots, d$ ; then the representation (2.9) takes the form (2.14) with  $G_{ij} = g_{ij} P$  and  $g_{ij} \in \mathbb{C}$  introduced in (2.28). In view of (2.36) and (2.37) the discrepancy in the expansion (2.14) admits the following representation:

$$\begin{aligned}
 \mathbb{A}(\xi, a, \mu) F(\xi) - \frac{1}{2} \sum_{i,j=1}^d G_{ij} \xi_i \xi_j &= \frac{1}{2\pi i} \oint_{\Gamma} R_0(\zeta) \mathbb{K}(\xi, a, \mu) R_0(\zeta) \zeta d\zeta - \\
 &- \frac{1}{2\pi i} \oint_{\Gamma} R_0(\zeta) \sum_{i=1}^d \partial_i \mathbb{A}(0, a, \mu) \xi_i R_0(\zeta) \Delta_2 \mathbb{A}(\xi) R_0(\zeta) \zeta d\zeta - \\
 &- \frac{1}{2\pi i} \oint_{\Gamma} R_0(\zeta) \Delta_2 \mathbb{A}(\xi) R_0(\zeta) \Delta \mathbb{A}(\xi) R_0(\zeta) \zeta d\zeta +
 \end{aligned}$$

$$+ \frac{1}{2\pi i} \oint_{\Gamma} R(\xi, \zeta) \Delta \mathbb{A}(\xi) R_0(\zeta) \Delta \mathbb{A}(\xi) R_0(\zeta) \Delta \mathbb{A}(\xi) R_0(\zeta) \Delta \mathbb{A}(\xi) R_0(\zeta) \zeta \, d\zeta, \quad |\xi| \leq \delta_0(a, \mu). \quad (2.41)$$

Since  $|\zeta| \leq d_0/2$  for  $\zeta \in \Gamma$ , the relations in (2.41) and estimates (2.4)–(2.6), (2.33) imply (2.30) with  $C_2(a, \mu)$  given by the formula

$$C_2(a, \mu) := \frac{\pi + 2}{2\pi} \left( \mu_+ M_3(a) + \frac{\mu_+^2}{d_0} (3M_2(a) + M_3(a)) (12M_1(a) + 3M_2(a) + M_3(a)) + \frac{\mu_+^3}{d_0^2} (6M_1(a) + 3M_2(a) + M_3(a))^3 \right). \quad \square \quad (2.42)$$

#### 2.4. Approximation of the resolvent of operator $\mathbb{A}(\xi)$

In this section our goal is to approximate the resolvent of the operator  $\mathbb{A}(\xi, a, \mu)$  in the vicinity of the lower edge of its spectrum. By (1.32) we have

$$(\mathbb{A}(\xi, a, \mu)F(\xi)u, u) \geq \mu_- C(a) |\xi|^2 (F(\xi)u, u), \quad u \in L_2(\Omega), \quad |\xi| \leq \delta_0(a, \mu). \quad (2.43)$$

Substituting  $P + O(|\xi|)$  for  $F(\xi)$  in (2.43) and considering (2.14) we derive the relation

$$\frac{1}{2} \sum_{i,j=1}^d g_{ij} \xi_i \xi_j \|Pu\|^2 \geq \mu_- C(a) |\xi|^2 \|Pu\|^2 + O(|\xi|^3), \quad |\xi| \rightarrow 0.$$

We then divide this relation by  $|\xi|^2$ , set  $u = \mathbf{1}_\Omega$  and send  $\xi$  to zero in order to obtain the following inequality:

$$\frac{1}{2} \sum_{i,j=1}^d g_{ij} \theta_i \theta_j \geq \mu_- C(a), \quad \theta \in \mathbb{S}^{d-1},$$

which finally yields

$$\frac{1}{2} \sum_{i,j=1}^d g_{ij} \xi_i \xi_j \geq \mu_- C(a) |\xi|^2, \quad \xi \in \mathbb{R}^d. \quad (2.44)$$

**Theorem 2.3.** Assume that conditions (1.1)–(1.4) hold. Then the following estimate is valid:

$$\left\| (\mathbb{A}(\xi, a, \mu) + \varepsilon^2 I)^{-1} - \left( \frac{1}{2} \sum_{i,j=1}^d g_{ij} \xi_i \xi_j + \varepsilon^2 \right)^{-1} P \right\| \leq S(a, \mu) \varepsilon^{-1}, \quad \varepsilon > 0, \quad |\xi| \leq \delta_0(a, \mu). \quad (2.45)$$

Here the constant  $S(a, \mu)$  is given by the formula

$$S(a, \mu) := \frac{2C_1(a, \mu)}{(\mu_- C(a))^{1/2}} + \frac{C_2(a, \mu)}{(\mu_- C(a))^{3/2}} + (2d_0/3)^{-1/2}$$

and, in view of (1.24), (2.1), (2.34) and (2.42), this constant depends only on  $d, \mu_-, \mu_+, M_1(a), M_2(a), M_3(a), \mathcal{M}(a), \mathcal{A}_\pi(a)$  and  $\mathcal{A}_r(a)$ .

**Proof.** Combining an evident inequality

$$\|(\mathbb{A}(\xi, a, \mu) + \varepsilon^2 I)^{-1}(I - F(\xi))\| \leq (2d_0/3)^{-1}, \quad |\xi| \leq \delta_0(a, \mu), \tag{2.46}$$

with (1.32) and (2.44) results in the following estimates:

$$\|(\mathbb{A}(\xi, a, \mu)F(\xi) + \varepsilon^2 I)^{-1}F(\xi)\| \leq (\mu_- C(a)|\xi|^2 + \varepsilon^2)^{-1}, \quad \varepsilon > 0, \quad |\xi| \leq \delta_0(a, \mu); \tag{2.47}$$

$$\left\| \left( \frac{1}{2} \sum_{i,j=1}^d g_{ij} \xi_i \xi_j + \varepsilon^2 \right)^{-1} P \right\| \leq (\mu_- C(a)|\xi|^2 + \varepsilon^2)^{-1}, \quad \varepsilon > 0, \quad |\xi| \leq \delta_0(a, \mu). \tag{2.48}$$

Since  $(\mathbb{A}(\xi, a, \mu) + \varepsilon^2 I)^{-1}F(\xi) = F(\xi)(\mathbb{A}(\xi, a, \mu) + \varepsilon^2 I)^{-1}F(\xi)$ , after straightforward rearrangements we obtain

$$\begin{aligned} & (\mathbb{A}(\xi, a, \mu) + \varepsilon^2 I)^{-1}F(\xi) - \left( \frac{1}{2} \sum_{i,j=1}^d g_{ij} \xi_i \xi_j + \varepsilon^2 \right)^{-1} P = \\ & = F(\xi)(\mathbb{A}(\xi, a, \mu) + \varepsilon^2 I)^{-1}(F(\xi) - P) + (F(\xi) - P) \left( \frac{1}{2} \sum_{i,j=1}^d g_{ij} \xi_i \xi_j + \varepsilon^2 \right)^{-1} P + \\ & + F(\xi)(\mathbb{A}(\xi, a, \mu) + \varepsilon^2 I)^{-1} \left( \frac{1}{2} \sum_{i,j=1}^d g_{ij} \xi_i \xi_j P - \mathbb{A}(\xi, a, \mu)F(\xi) \right) \left( \frac{1}{2} \sum_{i,j=1}^d g_{ij} \xi_i \xi_j + \varepsilon^2 \right)^{-1} P, \\ & \varepsilon > 0, \quad |\xi| \leq \delta_0(a, \mu). \end{aligned} \tag{2.49}$$

Now (2.45) follows from (2.29), (2.30) and (2.46)–(2.49).  $\square$

### 3. Homogenization of convolution type operators

We assume that conditions (1.1)–(1.4) hold and consider a family of nonlocal operators in  $L_2(\mathbb{R}^d)$  which are defined as follows:

$$\mathbb{A}_\varepsilon u(x) := \varepsilon^{-d-2} \int_{\mathbb{R}^d} a((x - y)/\varepsilon) \mu(x/\varepsilon, y/\varepsilon) (u(x) - u(y)) dy, \quad x \in \mathbb{R}^d, \quad u \in L_2(\mathbb{R}^d), \quad \varepsilon > 0.$$

Recalling the definition of the coefficients  $g_{ij}$  in (2.28) we introduce the so called *effective operator* which is a symmetric second order elliptic operator with constant coefficients in  $\mathbb{R}^d$  that reads

$$\mathbb{A}^0 := \frac{1}{2} \sum_{i,j=1}^d g_{ij} D_i D_j = -\operatorname{div} g^0 \nabla.$$

A  $(d \times d)$ -matrix  $g^0$  with the entries  $\frac{1}{2}g_{ij}$ ,  $i, j = 1, \dots, d$ , is called the *effective matrix*. Due to estimate (2.44) this matrix is positive definite. Our main result is given by

**Theorem 3.1.** *Let conditions (1.1)–(1.4) be satisfied. Then the following estimate holds:*

$$\|(\mathbb{A}_\varepsilon + I)^{-1} - (\mathbb{A}^0 + I)^{-1}\| \leq C(a, \mu)\varepsilon, \quad \varepsilon > 0, \tag{3.1}$$

with the constant  $C(a, \mu)$  defined by the formula

$$C(a, \mu) := \frac{2}{\sqrt{\mu - C(a)} \delta_0(a, \mu)} + S(a, \mu),$$

where  $C(a)$  is introduced in (1.24), while  $\delta_0(a, \mu)$  and  $S(a, \mu)$  are defined in Section 2.1 and in Theorem 2.3, respectively. The constant  $C(a, \mu)$  depends only on  $\mu_-, \mu_+, d, M_1(a), M_2(a), M_3(a), \mathcal{M}(a), \mathcal{A}_\pi(a)$  and  $\mathcal{A}_{r(a)}(a)$ .

**Proof.** Consider a family of scaling transformations of the form

$$T_\varepsilon u(x) := \varepsilon^{d/2} u(\varepsilon x), \quad x \in \mathbb{R}^d, \quad u \in L_2(\mathbb{R}^d), \quad \varepsilon > 0.$$

Notice that  $T_\varepsilon$  are unitary operators and that  $T_\varepsilon \mathbb{A}_\varepsilon T_\varepsilon^* = \varepsilon^{-2} \mathbb{A}(a, \mu)$ . Consequently, by (1.8) we have

$$\begin{aligned} (\mathbb{A}_\varepsilon + I)^{-1} &= T_\varepsilon^* \varepsilon^2 (\mathbb{A} + \varepsilon^2 I)^{-1} T_\varepsilon = T_\varepsilon^* \mathcal{G}^* [\varepsilon^2 (\mathbb{A}(\xi, a, \mu) + \varepsilon^2 I)^{-1}] \mathcal{G} T_\varepsilon = \\ &= T_\varepsilon^* \mathcal{G}^* [\varepsilon^2 \mathbf{1}_{|\xi| \leq \delta_0(a, \mu)} (\mathbb{A}(\xi, a, \mu) + \varepsilon^2 I)^{-1}] \mathcal{G} T_\varepsilon + \\ &\quad + T_\varepsilon^* \mathcal{G}^* [\varepsilon^2 \mathbf{1}_{|\xi| > \delta_0(a, \mu)} (\mathbb{A}(\xi, a, \mu) + \varepsilon^2 I)^{-1}] \mathcal{G} T_\varepsilon, \quad \varepsilon > 0. \end{aligned} \tag{3.2}$$

Estimate (1.32) and the relations in (3.2) imply the inequality

$$\|(\mathbb{A}_\varepsilon + I)^{-1} - T_\varepsilon^* \mathcal{G}^* [\varepsilon^2 \mathbf{1}_{|\xi| \leq \delta_0(a, \mu)} (\mathbb{A}(\xi, a, \mu) + \varepsilon^2 I)^{-1}] \mathcal{G} T_\varepsilon\| \leq \frac{\varepsilon}{\sqrt{\mu - C(a)} \delta_0(a, \mu)}, \quad \varepsilon > 0. \tag{3.3}$$

It also follows from estimates (2.45) and (3.3) that

$$\begin{aligned} \|(\mathbb{A}_\varepsilon + I)^{-1} - T_\varepsilon^* \mathcal{G}^* [\varepsilon^2 \mathbf{1}_{|\xi| \leq \delta_0(a, \mu)} \left(\frac{1}{2} \sum_{i,j=1}^d g_{ij} \xi_i \xi_j + \varepsilon^2\right)^{-1} P] \mathcal{G} T_\varepsilon\| &\leq \\ &\leq \frac{\varepsilon}{\sqrt{\mu - C(a)} \delta_0(a, \mu)} + S(a, \mu)\varepsilon, \quad \varepsilon > 0. \end{aligned} \tag{3.4}$$

As was shown for example in [52] or [3], for any  $H \in L_\infty(\tilde{\Omega})$  the relation  $\mathcal{G}^*[H(\xi)P]\mathcal{G} = \Phi^*[\mathbf{1}_{\tilde{\Omega}}(\xi)H(\xi)]\Phi$  holds, where the symbol  $\Phi$  stands for the Fourier transform. It remains to note that estimate (3.1) is a direct consequence of (2.44) and (3.4).  $\square$

We conclude this section with two almost obvious remarks.

**Remark 3.2.** Theorem 3.1 remains valid if, in condition (1.3), the cubic lattice  $\mathbb{Z}^d$  is replaced with an arbitrary periodic lattice  $\Gamma$  in  $\mathbb{R}^d$ . Under such replacement, the constant  $C$  in estimate (3.1) depends not only on the functions  $a$  and  $\mu$ , but also on the lattice parameters. The coefficients of the homogenized operator  $\mathbb{A}^0$  are still determined by relations (2.24) and (2.28), where  $\Omega$  is the cell of the lattice  $\Gamma$ .

**Remark 3.3.** Suppose that, in Theorem 3.1, condition (1.4) is modified as follows:

$$M_k(a) := \int_{\mathbb{R}^d} |x|^k a(x) dx < \infty, \text{ for } k = 1, 2, k_0$$

with  $k_0$  that satisfies the inequalities  $2 < k_0 < 3$ . Then instead of (3.1) we have

$$\|(\mathbb{A}_\varepsilon + I)^{-1} - (\mathbb{A}^0 + I)^{-1}\| \leq C\varepsilon^{k_0-2}, \quad \varepsilon > 0.$$

Here the constant  $C$  depends on  $a$ ,  $\mu$  and  $k_0$ .

## 4. Appendix

### 4.1. Schur test and Hadamard lemma

In this section for the reader convenience we recall the statements of two classical results. We begin with a simple version of the Schur test.

**Lemma 4.1 (Schur test).** *Let  $(\mathcal{X}, d\rho)$  and  $(\mathcal{Y}, d\tau)$  be separable measurable spaces with  $\sigma$ -finite measures, and assume that  $\mathbb{B} : L_2(\mathcal{Y}, d\tau) \rightarrow L_2(\mathcal{X}, d\rho)$  is a linear integral operator with a kernel  $b(x, y)$ ,  $x \in \mathcal{X}$ ,  $y \in \mathcal{Y}$ . Assume furthermore that the following conditions hold:*

$$\alpha := \rho\text{-ess sup}_{x \in \mathcal{X}} \int_{\mathcal{Y}} |b(x, y)| d\tau(y) < +\infty,$$

$$\beta := \tau\text{-ess sup}_{y \in \mathcal{Y}} \int_{\mathcal{X}} |b(x, y)| d\rho(x) < +\infty.$$

Then the operator  $\mathbb{B}$  is bounded and satisfies the estimate  $\|\mathbb{B}\| \leq (\alpha\beta)^{1/2}$ .

Recalling the notation  $\Omega := [0, 1]^d$  and letting  $\Delta := [-1, 1]^d$  we derive from the latter theorem the following result:

**Corollary 4.2.** *For any functions  $\varphi \in L_1(\Delta)$  and  $\psi \in L_\infty(\Omega \times \Omega)$  the operator  $\mathbb{B} : L_2(\Omega) \rightarrow L_2(\Omega)$  given by*

$$\mathbb{B}u(x) := \int_{\Omega} \varphi(x - y)\psi(x, y)u(y)dy, \quad x \in \Omega, \quad u \in L_2(\Omega),$$

is compact and satisfies the estimate

$$\|\mathbb{B}\| \leq \|\varphi\|_{L_1(\Delta)} \|\psi\|_{L_\infty(\Omega \times \Omega)}. \tag{4.1}$$

**Proof.** By Lemma 4.1 the operator  $\mathbb{B}$  is bounded, and estimate (4.1) holds. To show that  $\mathbb{B}$  is compact we choose a sequence of functions  $\{\varphi_n\}_{n=1}^\infty \subset C_0^\infty(\Delta)$  such that  $\|\varphi_n - \varphi\|_{L_1(\Delta)} \rightarrow 0$  as  $n \rightarrow \infty$ . Denote by  $\mathbb{B}_n$  the operators with the kernels  $\varphi_n(x - y)\psi(x, y)$ ,  $x, y \in \Omega$ . By (4.1) we have

$$\|\mathbb{B}_n - \mathbb{B}\| \leq \|\varphi_n - \varphi\|_{L_1(\Delta)} \|\psi\|_{L_\infty(\Omega \times \Omega)} \rightarrow 0, \quad n \rightarrow \infty.$$

Since for each  $n \in \mathbb{N}$  the kernel  $\varphi_n(x - y)\psi(x, y)$ ,  $x, y \in \Omega$ , is bounded and  $\Omega$  is a compact set, then  $\mathbb{B}_n$  are Hilbert-Schmidt operators. Therefore,  $\mathbb{B}$  is compact.  $\square$

Next, we formulate a version of the Hadamard lemma which is used in this work.

**Lemma 4.3 (Hadamard lemma).** *Let a function  $f(x)$ ,  $x \in \mathbb{R}^d$ , be three times continuously differentiable. Then it admits the following representation:*

$$\begin{aligned} f(y) = f(0) &+ \sum_{i=1}^d y_i \partial_i f(0) + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d y_i y_j \partial_i \partial_j f(0) + \\ &+ \sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d y_i y_j y_k \int_0^1 ds_1 s_1^2 \int_0^1 ds_2 s_2 \int_0^1 ds_3 \partial_i \partial_j \partial_k f(s_1 s_2 s_3 y), \quad y \in \mathbb{R}^d. \end{aligned}$$

4.2. Other methods of determining the coefficients in (2.9)

We call the method of computing the coefficients in (2.9) that was used in Section 2.2 the first method. In this section we describe two other methods.

**Second method.** For sufficiently small  $\xi \in \mathbb{R}^d$  we define

$$\varphi(\xi) := F(\xi)\mathbf{1}_\Omega, \quad e(\xi) := \varphi(\xi)/\|\varphi(\xi)\|, \quad \lambda(\xi) := (\mathbb{A}(\xi, a, \mu)e(\xi), e(\xi)).$$

Then  $\lambda(\xi)$  is the lowest eigenvalue of the operator  $\mathbb{A}(\xi, a, \mu)$ , and  $e(\xi)$  is the corresponding normalized eigenfunction. The eigenvalue  $\lambda(\xi)$  is simple. Therefore, by the Riesz formula, see (2.7), the functions  $\varphi(\xi)$ ,  $e(\xi)$  and  $\lambda(\xi)$  are  $C^3$  smooth functions of  $\xi$  in some neighbourhood of zero, and the Taylor formula yields

$$\lambda(\xi) = \lambda_0 + \sum_{i=1}^d \lambda_i \xi_i + \frac{1}{2} \sum_{i,j=1}^d \lambda_{ij} \xi_i \xi_j + O(|\xi|^3), \quad \xi \rightarrow 0; \tag{4.2}$$

$$e(\xi) = e_0 + \sum_{i=1}^d e_i \xi_i + \frac{1}{2} \sum_{i,j=1}^d e_{ij} \xi_i \xi_j + O(|\xi|^3), \quad \xi \rightarrow 0. \tag{4.3}$$

Since  $\lambda(0) = 0$  and  $\lambda(\xi) \geq 0$  in the vicinity of zero, then  $\lambda_i = 0$ ,  $i = 1, \dots, d$ . We also have  $e_0 = e(0) = \mathbf{1}_\Omega$ . Combining (4.2), (4.3) and the equality

$$\mathbb{A}(\xi, a, \mu)F(\xi) = \lambda(\xi)(\cdot, e(\xi))e(\xi),$$

we obtain the following formulas for the coefficients in (2.9):

$$G_0 = 0, \quad G_i = 0, \quad i = 1, \dots, d, \quad G_{ij} = \lambda_{ij}(\cdot, \mathbf{1}_\Omega)\mathbf{1}_\Omega = \lambda_{ij}P, \quad i, j = 1, \dots, d. \quad (4.4)$$

It remains to compute  $\lambda_{ij}$ . Inserting the expansions (2.3), (4.2) and (4.3) in the relation

$$\mathbb{A}(\xi, a, \mu)e(\xi) = \lambda(\xi)e(\xi),$$

and comparing the coefficients of the first and the second order terms in the resulting relation we conclude that

$$\mathbb{A}(0, a, \mu)e_i + \partial_i \mathbb{A}(0, a, \mu)\mathbf{1}_\Omega = 0, \quad i = 1, \dots, d; \quad (4.5)$$

$$\mathbb{A}(0, a, \mu)e_{ij} + \partial_i \mathbb{A}(0, a, \mu)e_j + \partial_j \mathbb{A}(0, a, \mu)e_i + \partial_i \partial_j \mathbb{A}(0, a, \mu)\mathbf{1}_\Omega = \lambda_{ij}\mathbf{1}_\Omega, \quad i, j = 1, \dots, d. \quad (4.6)$$

From (4.5) we deduce that

$$\partial_i \mathbb{A}(0, a, \mu)\mathbf{1}_\Omega = P^\perp \partial_i \mathbb{A}(0, a, \mu)\mathbf{1}_\Omega, \quad i = 1, \dots, d; \quad (4.7)$$

$$P^\perp e_i = -P^\perp \mathbb{A}(0, a, \mu)^{-1} P^\perp \partial_i \mathbb{A}(0, a, \mu)\mathbf{1}_\Omega, \quad (4.8)$$

and (4.6) yields the relation

$$\begin{aligned} \lambda_{ij} = (\lambda_{ij}\mathbf{1}_\Omega, \mathbf{1}_\Omega) &= (\mathbb{A}(0, a, \mu)e_{ij}, \mathbf{1}_\Omega) + (\partial_i \mathbb{A}(0, a, \mu)e_j, \mathbf{1}_\Omega) + (\partial_j \mathbb{A}(0, a, \mu)e_i, \mathbf{1}_\Omega) + \\ &+ (\partial_i \partial_j \mathbb{A}(0, a, \mu)\mathbf{1}_\Omega, \mathbf{1}_\Omega), \quad i, j = 1, \dots, d. \end{aligned} \quad (4.9)$$

Considering (4.7) and the equality  $(\mathbb{A}(0, a, \mu)e_{ij}, \mathbf{1}_\Omega) = (e_{ij}, \mathbb{A}(0, a, \mu)\mathbf{1}_\Omega) = 0$  we obtain

$$\begin{aligned} (\partial_i \mathbb{A}(0, a, \mu)e_j, \mathbf{1}_\Omega) &= (e_j, \partial_i \mathbb{A}(0, a, \mu)\mathbf{1}_\Omega) = (P^\perp e_j, \partial_i \mathbb{A}(0, a, \mu)\mathbf{1}_\Omega) = \\ &= (\partial_i \mathbb{A}(0, a, \mu)P^\perp e_j, \mathbf{1}_\Omega), \quad i, j = 1, \dots, d. \end{aligned}$$

Finally, this relation combined with (4.8) and (4.9) implies that

$$\begin{aligned} \lambda_{ij} &= ((\partial_i \partial_j \mathbb{A}(0, a, \mu) - \partial_i \mathbb{A}(0, a, \mu)P^\perp \mathbb{A}(0, a, \mu)^{-1} P^\perp \partial_j \mathbb{A}(0, a, \mu))\mathbf{1}_\Omega, \mathbf{1}_\Omega) - \\ &- (\partial_j \mathbb{A}(0, a, \mu)P^\perp \mathbb{A}(0, a, \mu)^{-1} P^\perp \partial_i \mathbb{A}(0, a, \mu)\mathbf{1}_\Omega, \mathbf{1}_\Omega), \quad i, j = 1, \dots, d, \end{aligned} \quad (4.10)$$

and (2.20) follows from (4.10) and (2.20).

**Third method.** The first method described above is not efficient for obtaining the higher order terms of the asymptotics of the operator  $\mathbb{A}(\xi, a, \mu)F(\xi)$  and thus for obtaining a more precise approximation of the resolvent  $(\mathbb{A}_\varepsilon + I)^{-1}$ . The second method fails to work in the case of a system of equations if the lowest eigenvalue of the operator  $\mathbb{A}(0, a, \mu)$  is not simple.

Here we consider the third method of computing the coefficients in (2.9) that does not have the mentioned demerits. The coefficients of expansion (2.9) are given by the contour integrals

(2.38)–(2.40). The computation is based on the following representation of the resolvent of operator  $\mathbb{A}(0, a, \mu)$ :

$$R_0(\zeta) = R_0(\zeta)P + R_0(\zeta)P^\perp = -\frac{1}{\zeta}P + R_0(\zeta)P^\perp, \quad \zeta \in \Gamma. \tag{4.11}$$

We substitute the right-hand side of (4.11) for  $R_0(\zeta)$  in the contour integrals in (2.38) and (2.39) and use the fact that the operator-function  $R_0^\perp(\zeta) := R_0(\zeta)P^\perp$  is holomorphic inside the contour  $\Gamma$ . This yields the relations  $G_0 = 0$  and

$$\begin{aligned} G_i &= \frac{-1}{2\pi i} \oint_{\Gamma} \left(-\frac{1}{\zeta}P + R_0^\perp(\zeta)\right) \partial_i \mathbb{A}(0, a, \mu) \left(-\frac{1}{\zeta}P + R_0^\perp(\zeta)\right) \zeta d\zeta = \\ &= \frac{-1}{2\pi i} \oint_{\Gamma} \left(-\frac{1}{\zeta}P\right) \partial_i \mathbb{A}(0, a, \mu) \left(-\frac{1}{\zeta}P\right) \zeta d\zeta = -P \partial_i \mathbb{A}(0, a, \mu) P, \quad i = 1, \dots, d. \end{aligned}$$

As an immediate consequence of (2.13) we have

$$P \partial_i \mathbb{A}(0, a, \mu) P = 0, \quad i = 1, \dots, d. \tag{4.12}$$

We insert expansion (4.11) in (2.40), take into account the fact that the operator-function  $R_0^\perp(\zeta)$  is holomorphic inside the contour  $\Gamma$ , and use the equality

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{\zeta} R_0^\perp(\zeta) d\zeta = P^\perp \mathbb{A}(0, a, \mu)^{-1} P^\perp$$

and (4.12). This leads to the desired representation (2.20).

### 4.3. Estimates of the constants $\mathcal{A}_r(a)$ and $\mathcal{M}(a)$

This section focuses on estimates of the constants  $\mathcal{A}_r(a)$ ,  $r > 0$ , and  $\mathcal{M}(a)$  defined in (1.18)–(1.21) in terms of  $\|a\|_{L_1}$ ,  $\|a\|_{L_2}$  and  $M_k(a)$ ,  $k = 1, 2, 3$ . Here we assume that conditions (1.1)–(1.4) are fulfilled and that in addition  $a \in L_2(\mathbb{R}^d)$ . The following upper bounds are trivial:

$$\mathcal{A}_r(a) \leq 2\|a\|_{L_1}, \quad r > 0; \tag{4.13}$$

$$\mathcal{M}(a) \leq M_2(a). \tag{4.14}$$

We turn to the lower bounds.

**Lemma 4.4.** *Let conditions (1.1) and (1.4) hold. Then the following lower bound is valid:*

$$\int_{|z| \leq \rho} a(z) dz \geq \frac{7}{8} \|a\|_{L_1}, \quad \rho \geq \rho_0(a) := 2M_3^{1/3}(a) \|a\|_{L_1}^{-1/3}. \tag{4.15}$$

**Proof.** The desired estimate (4.15) follows from the inequalities

$$\int_{|z|>\rho} a(z) dz \leq \rho^{-3} \int_{|z|>\rho} |z|^3 a(z) dz \leq \rho^{-3} M_3(a), \quad \rho > 0. \quad \square$$

Define a set  $\Pi_r^\theta(a) := \{z \in \mathbb{R}^d : |z| \leq \rho_0(a), \ | \langle z, \theta \rangle | \leq r\}$ ,  $r > 0$ ,  $\theta \in \mathbb{S}^{d-1}$ . It is straightforward to check that

$$\text{meas } \Pi_r^\theta(a) \leq 2r \kappa_{d-1} \rho_0^{d-1}(a), \tag{4.16}$$

where  $\kappa_{d-1}$  is the volume of a unit ball in  $\mathbb{R}^{d-1}$ .

**Lemma 4.5.** *Let conditions (1.1) and (1.4) be satisfied, and assume that  $a \in L_2(\mathbb{R}^d)$ . Then the following lower bound holds:*

$$\int_{B_{\rho_0(a)}(0) \setminus \Pi_r^\theta(a)} a(z) dz \geq \frac{1}{2} \|a\|_{L_1}, \quad |\theta| = 1, \quad r \in (0, r_0(a)],$$

$$r_0(a) := \left(\frac{3}{8} \|a\|_{L_1}\right)^2 \left(2\kappa_{d-1} \rho_0^{d-1}(a) \|a\|_{L_2}^2\right)^{-1}. \tag{4.17}$$

Here the constant  $\rho_0(a)$  has been defined in (4.15).

**Proof.** By (4.16) and the Cauchy-Schwartz inequality, we obtain the estimates

$$\int_{\Pi_r^\theta(a)} a(z) dz \leq \|a\|_{L_2} \left(\text{meas } \Pi_r^\theta(a)\right)^{1/2} \leq \left(2r \kappa_{d-1} \rho_0^{d-1}(a)\right)^{1/2} \|a\|_{L_2}, \quad r > 0, \quad |\theta| = 1. \tag{4.18}$$

Now (4.17) is a consequence of (4.15) and (4.18).  $\square$

The next statement gives a lower bound for the constant  $\mathcal{M}(a)$ .

**Lemma 4.6.** *Under the conditions of Lemma 4.5 we have*

$$\mathcal{M}(a) \geq \frac{1}{2} \|a\|_{L_1} r_0(a)^2, \tag{4.19}$$

where  $r_0(a)$  has been defined in (4.17).

**Proof.** By (4.17) for any  $\theta \in \mathbb{S}^{d-1}$  the following estimates hold:

$$M_a(\theta) = \int_{\mathbb{R}^d} a(z) \langle z, \theta \rangle^2 dz \geq \int_{B_{\rho_0(a)}(0) \setminus \Pi_{r_0(a)}^\theta(a)} a(z) \langle z, \theta \rangle^2 dz \geq \frac{1}{2} \|a\|_{L_1} r_0^2(a).$$

Combining these estimates with (1.21) yields (4.19).  $\square$

**Lemma 4.7.** *Let the conditions of Lemma 4.5 hold. Then the function  $\hat{A}(y)$  defined in (1.18) admits the estimate*

$$\hat{A}(y) \geq \frac{1}{8} \|a\|_{L_1} r_0^2(a) |y|^2, \quad |y| \leq (2\rho_0(a))^{-1};$$

here  $\rho_0(a)$  and  $r_0(a)$  have been introduced in (4.15) and (4.17), respectively.

**Proof.** Consider the set of  $y \in \mathbb{R}^d$  such that  $0 < |y| \leq (2\rho_0(a))^{-1}$ , and denote by  $\theta$  the vector  $y/|y|$ . Taking into account an elementary estimate  $1 - \cos t \geq \frac{1}{4}t^2$ ,  $|t| \leq 1/2$ , by Lemma 4.5, we obtain

$$\begin{aligned} \hat{A}(y) &= \int_{\mathbb{R}^d} (1 - \cos\langle z, y \rangle) a(z) dz \geq \int_{B_{\rho_0(a)}} (1 - \cos\langle z, y \rangle) a(z) dz \geq \frac{1}{4} \int_{B_{\rho_0(a)}} \langle z, y \rangle^2 a(z) dz \geq \\ &\geq \frac{1}{4} |y|^2 \int_{B_{\rho_0(a)} \setminus \Pi_{r_0(a)}^\theta(a)} \langle z, \theta \rangle^2 a(z) dz \geq \frac{1}{8} |y|^2 \|a\|_{L_1} r_0^2(a). \quad \square \end{aligned}$$

Introducing the notation

$$Q_\tau^y(a) := \{z \in B_{\rho_0(a)}(0) : 1 - \cos\langle z, y \rangle \leq \tau\}, \quad |y| \geq (2\rho_0(a))^{-1}, \quad \tau \in (0, 1/2],$$

one can easily check that the following relation is valid:

$$\begin{aligned} Q_\tau^y(a) &= \bigcup_{n \in \mathbb{Z}} Q_{\tau,n}^y(a), \\ Q_{\tau,n}^y(a) &:= \{z \in B_{\rho_0(a)}(0) : 2\pi n - \arccos(1 - \tau) \leq \langle z, y \rangle \leq 2\pi n + \arccos(1 - \tau)\}, \\ &|y| \geq (2\rho_0(a))^{-1}, \quad \tau \in (0, 1/2]. \quad (4.20) \end{aligned}$$

**Lemma 4.8.** *Assume that conditions (1.1) and (1.4) are fulfilled. Then the following estimates take place:*

$$\#\{n \in \mathbb{Z} : Q_{\tau,n}^y(a) \neq \emptyset\} \leq \pi^{-1} |y| \rho_0(a) + 2, \quad |y| \geq (2\rho_0(a))^{-1}, \quad \tau \in (0, 1/2]; \quad (4.21)$$

$$\text{meas } Q_{\tau,n}^y(a) \leq 2|y|^{-1} \arccos(1 - \tau) \chi_{d-1} \rho_0^{d-1}(a), \quad |y| \geq (2\rho_0(a))^{-1}, \quad \tau \in (0, 1/2]. \quad (4.22)$$

**Proof.** By the definition of  $Q_{\tau,n}^y(a)$ , for any vector  $z \in Q_{\tau,n}^y(a)$  we have

$$\begin{cases} (2n - 1)\pi \leq \langle z, y \rangle \leq (2n + 1)\pi, \\ -|y| \rho_0(a) \leq \langle z, y \rangle \leq |y| \rho_0(a). \end{cases}$$

Consequently, if the set  $Q_{\tau,n}^y(a)$  is not empty, then the following inequalities hold:

$$\begin{cases} -|y| \rho_0(a) \leq (2n + 1)\pi, \\ (2n - 1)\pi \leq |y| \rho_0(a). \end{cases}$$

Thus for such sets  $Q_{\tau,n}^y(a)$  we obtain the estimate  $|n| \leq (2\pi)^{-1}|y|\rho_0(a) + 1/2$ , which in turn implies (4.21).

In order to estimate the measure of  $Q_{\tau,n}^y(a)$ , one can choose an orthonormal basis in  $\mathbb{R}^d$  in such a way that the first coordinate vector is directed along the vector  $y$ . Under this choice, for any  $z \in Q_{\tau,n}^y(a)$ , we obtain the inequalities

$$2\pi n - \arccos(1 - \tau) \leq z_1|y| \leq 2\pi n + \arccos(1 - \tau), \quad z_2^2 + \dots + z_d^2 \leq \rho_0^2(a).$$

This yields (4.22).  $\square$

From (4.20)–(4.22) we deduce

**Lemma 4.9.** *Under conditions (1.1) and (1.4) the following estimate is satisfied:*

$$\text{meas } Q_{\tau}^y(a) \leq \arccos(1 - \tau)N(a),$$

$$\text{with } N(a) := \left(\frac{2}{\pi} + 8\right)\chi_{d-1}\rho_0^d(a), \quad |y| \geq (2\rho_0(a))^{-1}, \quad \tau \in (0, 1/2]; \quad (4.23)$$

here the quantity  $\rho_0(a)$  is defined in (4.15).

Using the Cauchy-Schwartz inequality we derive from (4.23) that

$$\int_{Q_{\tau}^y(a)} a(z) dz \leq \|a\|_{L_2} \left(\arccos(1 - \tau)N(a)\right)^{1/2}, \quad |y| \geq (2\rho_0(a))^{-1}, \quad \tau \in (0, 1/2]. \quad (4.24)$$

Now our aim is to construct a semi-open interval  $(0, \tau_0(a)] \subset (0, 1/2]$  such that for all  $\tau \in (0, \tau_0(a)]$  the right-hand side in (4.24) does not exceed  $\frac{3}{8}\|a\|_{L_1}$ . To this end we can choose

$$\tau_0(a) := \begin{cases} 1/2, & \text{if } \left(\frac{3}{8}\|a\|_{L_1}\|a\|_{L_2}^{-1}\right)^2 > \pi N(a); \\ \min \left\{ 1/2, 1 - \cos \left( N(a)^{-1} \left( \frac{3}{8}\|a\|_{L_1}\|a\|_{L_2}^{-1} \right)^2 \right) \right\}, & \text{otherwise.} \end{cases} \quad (4.25)$$

Considering (4.15), (4.24) and (4.25) we obtain

**Lemma 4.10.** *Under the assumptions of Lemma 4.5 the following estimate holds:*

$$\int_{B_{\rho_0(a)}(0) \setminus Q_{\tau}^y(a)} a(z) dz \geq \frac{1}{2}\|a\|_{L_1}, \quad |y| \geq (2\rho_0(a))^{-1}, \quad 0 < \tau \leq \tau_0(a); \quad (4.26)$$

here  $\rho_0(a)$  and  $N(a)$  are defined in (4.15) and (4.23), respectively.

As a consequence of the latter statement we have

**Lemma 4.11.** *Under the conditions of Lemma 4.5 the following estimate is satisfied:*

$$\hat{A}(y) \geq \frac{1}{2} \|a\|_{L_1} \tau_0(a), \quad |y| \geq (2\rho_0(a))^{-1};$$

here the constants  $\rho_0(a)$  and  $\tau_0(a)$  are introduced in (4.15) and (4.25), respectively.

Finally, recalling the definition of  $r_0(a)$  in (4.17) and that of  $\tau_0(a)$  in (4.25), and taking into account (4.13) and (4.14), by Lemmas 4.6, 4.7 and 4.11, we obtain

**Proposition 4.12.** *Let conditions (1.1) and (1.4) be fulfilled, and assume that  $a \in L_2(\mathbb{R}^d)$ . Then the following estimates hold:*

$$\begin{aligned} \frac{1}{2} \|a\|_{L_1} r_0(a)^2 &\leq \mathcal{M}(a) \leq M_2(a); \\ \min\left\{\frac{1}{8} \|a\|_{L_1} r_0(a)^2 r^2, \frac{1}{2} \|a\|_{L_1} \tau_0(a)\right\} &\leq \mathcal{A}_r(a) \leq 2\|a\|_{L_1}, \quad r > 0. \end{aligned}$$

**Remark 4.13.** If the assumptions of Proposition 4.12 hold, then  $\mathcal{M}(a)$  can be estimated from below by a positive constant that depends only on  $d$ ,  $\|a\|_{L_1}$ ,  $\|a\|_{L_2}$  and  $M_3(a)$ , while  $\mathcal{A}_r(a)$  admits a lower bound by a positive constant that depends on the same parameters and  $r$ . Furthermore, under the same assumptions, the constant  $\mathcal{C}(a, \mu)$  introduced in Theorem 3.1 satisfies the estimate  $\mathcal{C}(a, \mu) \geq \tilde{C} > 0$ , where  $\tilde{C}$  depends only on  $\mu_-, \mu_+, d, \|a\|_{L_1}, \|a\|_{L_2}$  and  $M_j(a), j = 1, 2, 3$ .

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### References

- [1] N.S. Bakhvalov, G.P. Panasenko, Homogenisation: Averaging Processes in Periodic Media, Mathematics and Its Applications, vol. 36, Springer, Dordrecht, 1989.
- [2] A. Bensoussan, J.-L. Lions, G. Papanicolaou, Asymptotic Analysis for Periodic Structures, Stud. Math. Appl., vol. 5, North-Holland Publishing Co., Amsterdam–New York, 1978.
- [3] M.Sh. Birman, T.A. Suslina, Second order periodic differential operators. Threshold properties and homogenization, Algebra Anal. 15 (5) (2003) 1–108 (in Russian); English transl.: St. Petersburg Math. J. 15 (5) (2004) 639–714.
- [4] M.Sh. Birman, T.A. Suslina, Homogenization with corrector term for periodic elliptic differential operators, Algebra Anal. 17 (6) (2005) 1–104 (in Russian); English transl.: St. Petersburg Math. J. 17 (6) (2006) 897–973.
- [5] M.Sh. Birman, T.A. Suslina, Homogenization with corrector for periodic differential operators. Approximation of solutions in the Sobolev class  $H^1(\mathbb{R}^d)$ , Algebra Anal. 18 (6) (2006) 1–130 (in Russian); English transl.: St. Petersburg Math. J. 18 (6) (2007) 857–955.
- [6] M.Sh. Birman, T.A. Suslina, Operator error estimates in the homogenization problem for nonstationary periodic equations, Algebra Anal. 20 (6) (2008) 30–107 (in Russian); English transl.: St. Petersburg Math. J. 20 (6) (2009) 873–928.
- [7] D.I. Borisov, Asymptotics of solutions of elliptic systems with rapidly oscillating coefficients, Algebra Anal. 20 (2) (2008) 19–42 (in Russian); English transl.: St. Petersburg Math. J. 20 (2) (2009) 175–191.

- [8] D.I. Borisov, G. Cardone, T. Durante, Homogenization and norm resolvent convergence for elliptic operators in a strip perforated along a curve, *Proc. R. Soc. Edinb., Sect. A, Math.* 146 (6) (2016) 1115–1158.
- [9] D.I. Borisov, G. Cardone, L. Faella, C. Perugia, Uniform resolvent convergence for strip with fast oscillating boundary, *J. Differ. Equ.* 255 (12) (2013) 4378–4402.
- [10] D.I. Borisov, T.F. Sharapov, On the resolvent of multidimensional operators with frequently alternating boundary conditions with the Robin homogenized condition, *J. Math. Sci. (N.Y.)* 213 (4) (2016) 461–503.
- [11] A. Braides, A. Piatnitski, Homogenization of quadratic convolution energies in periodically perforated domains, *Adv. Calc. Var.* (2022), <https://doi.org/10.1515/acv-2019-0083>.
- [12] A. Braides, A. Piatnitski, Homogenization of random convolution energies, *J. Lond. Math. Soc.* 104 (2) (2021) 295–319.
- [13] X. Chen, Z.-Q. Chen, T. Kumagai, J. Wang, Periodic homogenization of non-symmetric Lévy type processes, *Ann. Probab.* 49 (6) (2021) 2874–2921.
- [14] K.D. Cherednichenko, S. Cooper, Resolvent estimates for high-contrast elliptic problems with periodic coefficients, *Arch. Ration. Mech. Anal.* 219 (2016) 1061–1086.
- [15] K. Cherednichenko, Y. Ershova, A. Kiselev, Effective behavior of critical-contrast PDEs: microresonances, frequency conversion, and time dispersive properties, *Commun. Math. Phys.* 375 (2020) 1833–1884.
- [16] K.D. Cherednichenko, A.V. Kiselev, Norm-resolvent convergence of one-dimensional high-contrast periodic problems to a Kronig–Penney dipole-type model, *Commun. Math. Phys.* 349 (2017) 441–480.
- [17] M.A. Dorodnyi, Operator error estimates for homogenization of the nonstationary Schrödinger-type equations: sharpness of the results, *Appl. Anal.* (2021), <https://doi.org/10.1080/00036811.2021.1901886>.
- [18] M.A. Dorodnyi, T.A. Suslina, Spectral approach to homogenization of hyperbolic equations with periodic coefficients, *J. Differ. Equ.* 264 (12) (2018) 7463–7522.
- [19] M.A. Dorodnyi, T.A. Suslina, Homogenization of hyperbolic equations with periodic coefficients in  $\mathbb{R}^d$ : sharpness of the results, *Algebra Anal.* 32 (4) (2020) 3–136 (in Russian); English transl.: *St. Petersburg Math. J.* 32 (4) (2021) 605–703.
- [20] M.A. Dorodnyi, T.A. Suslina, Homogenization of a non-stationary periodic Maxwell system in the case of constant permeability, *J. Differ. Equ.* 307 (2022) 348–388.
- [21] J. Geng, Z. Shen, Convergence rates in parabolic homogenization with time-dependent coefficients, *J. Funct. Anal.* 272 (2017) 2092–2113.
- [22] A. Gloria, F. Otto, An optimal variance estimate in stochastic homogenization of discrete elliptic equations, *Ann. Probab.* 39 (3) (2011) 779–856.
- [23] A. Gloria, F. Otto, Quantitative results on the corrector equation in stochastic homogenization, *J. Eur. Math. Soc.* 19 (11) (2017) 3489–3548.
- [24] G. Griso, Error estimate and unfolding for periodic homogenization, *Asymptot. Anal.* 40 (3/4) (2004) 269–286.
- [25] G. Griso, Interior error estimate for periodic homogenization, *Anal. Appl.* 4 (1) (2006) 61–79.
- [26] S. Gu, Convergence rates in homogenization of Stokes system, *J. Differ. Equ.* 260 (7) (2016) 5796–5815.
- [27] P. Halmos, V. Sunder, *Bounded Integral Operators on  $L^2$  Spaces*, Results in Mathematics and Related Areas, vol. 96, Springer-Verlag, Berlin, 1978.
- [28] C.E. Kenig, F. Lin, Z. Shen, Convergence rates in  $L^2$  for elliptic homogenization problems, *Arch. Ration. Mech. Anal.* 203 (3) (2012) 1009–1036.
- [29] V.V. Jikov, S.M. Kozlov, O.A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, 1994.
- [30] S. Kesavan, Homogenization of elliptic eigenvalue problems: part I, *Appl. Math. Optim.* 5 (2) (1979) 153–167.
- [31] S. Kesavan, Homogenization of elliptic eigenvalue problems: part II, *Appl. Math. Optim.* 5 (3) (1979) 197–216.
- [32] Yu. Kondratiev, O. Kutoviy, S. Pirogov, Correlation functions and invariant measures in continuous contact model, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 11 (2) (2008) 231–258.
- [33] Yu. Kondratiev, S. Molchanov, S. Pirogov, E. Zhizhina, On ground state of some non local Schrödinger operators, *Appl. Anal.* 96 (8) (2017) 1390–1400.
- [34] Yu. Kondratiev, S. Pirogov, E. Zhizhina, A quasispecies continuous contact model in a critical regime, *J. Stat. Phys.* 163 (2) (2016) 357–373.
- [35] A.A. Kukushkin, T.A. Suslina, Homogenization of high order elliptic operators with periodic coefficients, *Algebra Anal.* 28 (1) (2016) 89–149 (in Russian); English transl.: *St. Petersburg Math. J.* 28 (1) (2017) 65–108.
- [36] Yu.M. Meshkova, On operator error estimates for homogenization of hyperbolic systems with periodic coefficients, *J. Spectr. Theory* 11 (2) (2021) 587–660.
- [37] Yu.M. Meshkova, T.A. Suslina, Homogenization of initial boundary-value problems for parabolic systems with periodic coefficients, *Appl. Anal.* 95 (8) (2016) 1736–1775.

- [38] A.A. Miloslova, T.A. Suslina, Homogenization of higher-order parabolic equations with periodic coefficients, *Sovrem. Mat. Fundam. Napravl.* 67 (1) (2021) 130–191, <https://doi.org/10.22363/2413-3639-2021-67-1-130-191>.
- [39] S.E. Pastukhova, Estimates in homogenization of higher-order elliptic operators, *Appl. Anal.* 95 (2) (2016) 1–18.
- [40] S.E. Pastukhova, Approximation of resolvents in homogenization of fourth-order elliptic operators, *Mat. Sb.* 212 (1) (2021) 119–142 (in Russian); English transl.: *Sb. Math.* 212 (1) (2021) 111–134.
- [41] S.E. Pastukhova, R.N. Tikhomirov, Operator estimates in reiterated and locally periodic homogenization, *Dokl. Akad. Nauk* 415 (3) (2007) 304–309 (in Russian); English transl.: *Dokl. Math.* 76 (1) (2007) 548–553.
- [42] S.E. Pastukhova, R.N. Tikhomirov, Estimates for locally periodic and reiterated homogenization: parabolic equations, *Dokl. Akad. Nauk* 428 (2) (2009) 166–170 (in Russian); English transl.: *Dokl. Math.* 80 (2) (2009) 674–678.
- [43] M.A. Pakhnin, T.A. Suslina, Operator error estimates for homogenization of the elliptic Dirichlet problem in a bounded domain, *Algebra Anal.* 24 (6) (2012) 139–177 (in Russian); English transl.: *St. Petersburg Math. J.* 24 (6) (2013) 949–976.
- [44] A. Piatnitski, E. Zhizhina, Periodic homogenization of nonlocal operators with a convolution-type kernel, *SIAM J. Math. Anal.* 49 (1) (2017) 64–81.
- [45] A. Piatnitski, E. Zhizhina, Homogenization of biased convolution type operators, *Asymptot. Anal.* 115 (3–4) (2019) 241–262.
- [46] A. Piatnitski, E. Zhizhina, Stochastic homogenization of convolution type operators, *J. Math. Pures Appl.* 134 (2020) 36–71.
- [47] N. Sandrić, Homogenization of periodic diffusion with small jumps, *J. Math. Anal. Appl.* 435 (2016) 551–577, no. 1.
- [48] R. Schwab, Periodic homogenization for nonlinear integro-differential equations, *SIAM J. Math. Anal.* 42 (6) (2010) 2652–2680.
- [49] N.N. Senik, On the homogenization of nonselfadjoint locally periodic elliptic operators, *Funkc. Anal. Prilozh.* 51 (2) (2017) 92–96; English transl.: *Funct. Anal. Appl.* 51 (2) (2017) 152–156.
- [50] N.N. Senik, On the homogenization of locally periodic elliptic and parabolic operators, *Funkc. Anal. Prilozh.* 54 (1) (2020) 68–72; English transl.: *Funct. Anal. Appl.* 54 (1) (2020) 68–72.
- [51] N.N. Senik, Homogenization for locally periodic elliptic operators, *J. Math. Anal. Appl.* 505 (2) (2022), <https://doi.org/10.1016/j.jmaa.2021.125581>.
- [52] M.M. Skriyanov, Geometric and arithmetic methods in the spectral theory of multidimensional periodic operators, *Tr. Mat. Inst. Steklova* 171 (1985) 3–122 (in Russian); English transl.: *Proc. Steklov Inst. Math.* 171 (1987) 1–121.
- [53] V.A. Sloushch, T.A. Suslina, Homogenization of the fourth-order elliptic operator with periodic coefficients with correctors taken into account, *Funkc. Anal. Prilozh.* 54 (3) (2020) 94–99; English transl.: *Funct. Anal. Appl.* 54 (3) (2020) 224–228.
- [54] T.A. Suslina, Homogenization of a stationary periodic Maxwell system, *Algebra Anal.* 16 (5) (2004) 162–244 (in Russian); English transl.: *St. Petersburg Math. J.* 16 (5) (2005) 863–922.
- [55] T.A. Suslina, On homogenization of periodic parabolic systems, *Funkc. Anal. Prilozh.* 38 (4) (2004) 86–90; English transl.: *Funct. Anal. Appl.* 38 (4) (2004) 309–312.
- [56] T.A. Suslina, Homogenization with corrector for a stationary periodic Maxwell system, *Algebra Anal.* 19 (3) (2007) 183–235 (in Russian); English transl.: *St. Petersburg Math. J.* 19 (3) (2008) 455–494.
- [57] T.A. Suslina, Homogenization of a periodic parabolic Cauchy problem, *Transl. Am. Math. Soc.* (2) 220 (2007) 201–233.
- [58] T.A. Suslina, Homogenization of a periodic parabolic Cauchy problem in the Sobolev space  $H^1(\mathbb{R}^d)$ , *Math. Model. Nat. Phenom.* 5 (4) (2010) 390–447.
- [59] T.A. Suslina, Homogenization of the Dirichlet problem for elliptic systems:  $L_2$ -operator error estimates, *Matematika* 59 (2) (2013) 463–476.
- [60] T.A. Suslina, Homogenization of the Neumann problem for elliptic systems with periodic coefficients, *SIAM J. Math. Anal.* 45 (6) (2013) 3453–3493.
- [61] T.A. Suslina, Spectral approach to homogenization of nonstationary Schrödinger-type equations, *J. Math. Anal. Appl.* 446 (2) (2017) 1466–1523.
- [62] T.A. Suslina, Homogenization of the Dirichlet problem for higher-order elliptic equations with periodic coefficients, *Algebra Anal.* 29 (2) (2017) 139–192 (in Russian); English transl.: *St. Petersburg Math. J.* 29 (2) (2018) 325–362.
- [63] T.A. Suslina, Homogenization of the Neumann problem for higher order elliptic equations with periodic coefficients, *Complex Var. Elliptic Equ.* 63 (7–8) (2018) 1185–1215.
- [64] T.A. Suslina, Homogenization of a stationary periodic Maxwell system in a bounded domain in the case of constant magnetic permeability, *Algebra Anal.* 30 (3) (2018) 169–209 (in Russian); English transl.: *St. Petersburg Math. J.* 30 (3) (2019) 515–544.

- [65] T.A. Suslina, Homogenization of the stationary Maxwell system with periodic coefficients in a bounded domain, *Arch. Ration. Mech. Anal.* 234 (2) (2019) 453–507.
- [66] T.A. Suslina, Homogenization of higher-order parabolic systems in a bounded domain, *Appl. Anal.* 98 (1–2) (2019) 3–31.
- [67] T.A. Suslina, Homogenization of the higher-order Schrödinger-type equations with periodic coefficients, in: Pavel Exner, Rupert L. Frank, Fritz Gesztesy, Helge Holden, Timo Weidl (Eds.), *Partial Differential Equations, Spectral Theory, and Mathematical Physics. The Ari Laptev Anniversary Volume*, European Mathematical Society, EMS Publishing House GmbH, 2021.
- [68] T.A. Suslina, Homogenization of the higher-order hyperbolic equations with periodic coefficients, *Lobachevskii J. Math.* 42 (14) (2021) 3518–3542.
- [69] M. Vanninathan, Homogenization of eigenvalue problems in perforated domains, *Proc. Indian Acad. Sci. Math. Sci.* 90 (3) (1981) 239–271.
- [70] E.S. Vasilevskaya, A periodic parabolic Cauchy problem: homogenization with corrector, *Algebra Anal.* 21 (1) (2009) 3–60 (in Russian); English transl.: *St. Petersburg Math. J.* 21 (1) (2010) 1–41.
- [71] N.A. Veniaminov, Homogenization of periodic differential operators of high order, *Algebra Anal.* 21 (5) (2010) 69–103 (in Russian); English transl.: *St. Petersburg Math. J.* 21 (5) (2011) 751–775.
- [72] V.V. Yurinski (Yurinskij), On the homogenization of boundary value problems with random coefficients, *Sib. Math. J.* 21 (1981) 470–482.
- [73] V.V. Zhikov, On operator estimates in homogenization theory, *Dokl. Akad. Nauk* 403 (3) (2005) 305–308 (in Russian); English transl.: *Dokl. Math.* 72 (1) (2005) 534–538.
- [74] V.V. Zhikov, S.E. Pastukhova, On operator estimates for some problems in homogenization theory, *Russ. J. Math. Phys.* 12 (4) (2005) 515–524.
- [75] V.V. Zhikov, S.E. Pastukhova, Estimates of homogenization for a parabolic equation with periodic coefficients, *Russ. J. Math. Phys.* 13 (2) (2006) 224–237.
- [76] V.V. Zhikov, S.E. Pastukhova, Operator estimates in homogenization theory, *Usp. Mat. Nauk* 71 (3) (2016) 27–122 (in Russian); English transl.: *Russ. Math. Surv.* 71 (3) (2016) 417–511.