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Navier–Stokes–Fourier system with phase transitions

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Abstract We consider the Navier–Stokes–Fourier ($\mathcal{NSF}$) system for a class of compressible fluids that exhibit a *gas-liquid* phase transition at low temperatures. For the initial-boundary value problem corresponding to thermally insulated end-points that are held at a constant pressure, we establish the existence and uniqueness of temporally global classical solutions. A novel feature of the analysis presented here is the derivation of *uniform* point-wise apriori estimates on the specific volume, which refines the non-uniform estimates framework developed in Watson (Arch Ration Mech Anal 153:1–37, 2000).

Keywords Navier–Stokes–Fourier system · A priori estimates · Thermoviscoelasticity · Phase transitions

1 Introduction

This article is concerned with an initial-boundary value problem $\mathcal{IBVP}$ for a compressible viscous heat-conducting fluid that exhibits a gas-to-liquid phase transition at sufficiently low temperatures. The governing equations, which encode the balance of mass, momentum and energy, take the form of a modified Navier–Stokes–Fourier system, which may be written in dimensionless Lagrangian form as

$$\begin{aligned} \xi_t &= \nu_x, \\ \nu_t &= \partial_x \left(-\frac{\vartheta}{\xi} - \frac{\xi}{\xi^{q+1}+1} + \mu \frac{\nu_x}{\xi} \right), \\ c_v \vartheta_t &= -\frac{\vartheta}{\xi}\nu_x + \mu \frac{{\nu_x}^2}{\xi} + \kappa \partial_x \left(\frac{\vartheta_x}{\xi} \right), \end{aligned} \tag{$\mathcal{G}$}$$

where the *specific volume* $\xi(x,t)$, velocity $\nu(x,t)$, and *absolute temperature* $\vartheta(x,t)$ are scalar functions of material point $x \in [0,1]$ and time $t \in [0,T)$ (where $0 < T \le \infty$), while the subscript x and t, as well as ∂_x and ∂_t, denote partial derivatives with respect to x and t respectively: note that here the bounded *reference configuration* has been scaled to $[0,1]$. Last, the coefficients of viscosity μ and heat-conductivity κ are both assumed to be positive constants, $\mu, \kappa > 0$, as is the *specific heat* $c_v > 0$, while the *asymptotic exponent* $q > 1$.

We also impose two additional global physical constraints on the Navier–Stokes–Fourier system ($\mathcal{G}$). First, the *Third Law of Thermodynamics* demands that absolute zero is never attained

$$\vartheta(x,t) > 0\,. \tag{$\mathcal{T}$}$$

Second, to preclude the interpenetration of matter, we assume that the *Eulerian mass density*

$$\rho := \frac{1}{\xi}$$

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is pointwise finite, or equivalently that the specific volume is point-wise bounded away from zero,

$$\xi(x,t) > 0. \tag{$\mathcal{F}$}$$

To obtain a closed problem, we take thermo-mechanical boundary conditions associated with a prescribed constant pressure $P_0 > 0$ acting at the boundary of the material body (*the outer-pressure problem*), namely

$$\mathcal{S}(0,t) = -P_0 = \mathcal{S}(1,t), \tag{$\mathcal{M}$}$$

where the *stress* $\mathcal{S}$ is given by

$$\mathcal{S} = -\frac{\vartheta}{\xi} - \frac{\xi}{\xi^{q+1}+1} + \mu\frac{\nu_x}{\xi}. \tag{1}$$

We also impose the Neumann condition on ϑ

$$\vartheta_x(0,t) = 0 = \vartheta_x(1,t), \tag{$\mathcal{N}$}$$

which encodes thermal insulation at the endpoints of the material body. Last, we prescribe the initial velocity $\nu_0(x)$, temperature $\vartheta_0(x)$ and specific volume $\xi_0(x)$ at time $t = 0$, namely

$$\begin{aligned} \xi(x,0) &= \xi_0(x), \\ \nu(x,0) &= \nu_0(x), \\ \vartheta(x,0) &= \vartheta_0(x). \end{aligned} \tag{$\mathcal{IC}$}$$

Remark 1 The key analytical challenge to establishing a global existence and uniquess theory of classical solutions to a Navier–Stokes–Fourier system in one-space dimensions, such as for our elastogas $\mathcal{IBVP}$, is the demonstration of a pointwise *a priori* estimate on the specific volume. Namely, one needs to prove that for every finite-time $T > 0$, any classical solution (ξ, ν, ϑ) on $[0,1] \times [0,T)$ is necessarily uniformly bounded away from 0 and ∞: i.e., there exists $\underline{\xi}, \overline{\xi} \in (0,\infty)$ such that for all $(x,t) \in [0,1] \times [0,T)$

$$0 < \underline{\xi} \le \xi(x,t) \le \overline{\xi} < \infty. \tag{2}$$

Given that by assumption any such classical solution meets the physical constraint ($\mathcal{F}$) on its domain of definition, here $[0,1]\times[0,T)$, the key issue being resolved by such an *a priori* estimate is that the specific volume cannot approach 0 (*no interpenetration of matter*) nor can it go to infinity (no *creation of a vacuum*) as time t approaches T from below ($t \nearrow T$). Note also that if one only wishes to establish the existence of temporally global solutions these bounds need not be independent of T. However, if one also wishes to study the asymptotic behaviour of solutions, one generally needs a uniform a priori estimate on the specific volume.

2 Results

Theorem 1 (Uniform a priori estimate on ξ) *If (ξ, ν, ϑ) is a temporally global classical solution to the initial-boundary value problem ($\mathcal{IBVP}$) given by ($\mathcal{G}$), ($\mathcal{IC}$) ($\mathcal{M}$), ($\mathcal{N}$) with the physical constraints ($\mathcal{T}$) and ($\mathcal{F}$), then there exists $\underline{\xi}, \overline{\xi} \in (0,\infty)$ such that for all $(x,t) \in [0,1] \times [0,\infty)$*

$$0 < \underline{\xi} \le \xi(x,t) \le \overline{\xi} < \infty. \tag{3}$$

The proof of Theorem 1, which is presented in Sect. 4, 5, 6, involves a non-trivial refinement of the general theoretical approach developed by Watson [1,2] for initial-boundary value problems in 1-D *thermoviscoelasticity* [3], which itself was inspired by Kazhikhov and Shelukin's seminal analysis of the Navier–Stokes–Fourier system for a viscous heat-conducting ideal gas [4]. We note that Theorem 1 extends previous results on the outer-pressure problem for a viscous heat-conducting ideal gas [5] to the *elastogas* setting.

Remark 2 The Navier–Stokes–Fourier system in one-space dimension may be viewed as a special case of the equations of thermoviscoelasticity [1,2] [3, p. 32]. In this more general setting, the role of specific volume ξ of the compressible fluid is replaced by the *deformation gradient* χ_x of the thermoelastic material. In this wider context, the key analytical challenge to well-posedness of the $\mathcal{IBVP}$ is likewise to establish a pointwise a priori estimate on the deformation gradient χ_x of classical solutions [1,2,6], namely

$$0 < \underline{\xi} \le \chi_x(x,t) \le \overline{\xi} < \infty.$$

Given Theorem 1, one may then employ the general analytical framework of [1,2] to formulate a complete existence and uniqueness theory for the $\mathcal{IBVP}$ of Theorem 1 in terms of the spaces of Holder continuous functions

$$C^{2+\alpha}[0,1] \quad \text{and} \quad C^{\alpha,\frac{\alpha}{2}}\big([0,1]\times[0,\infty)\big).$$

which arise naturally in the theory of parabolic partial differential equations [7].

Theorem 2 (Global Existence and Uniqueness) *Consider the initial-boundary value problem given by* ($\mathcal{G}$), ($\mathcal{F}$), ($\mathcal{T}$), ($\mathcal{M}$), ($\mathcal{N}$) *and* ($\mathcal{IC}$). *Let* $\alpha > 0$ *and let the initial data* $\xi_0,\ \nu_0,\ \theta_0 \in C^{2+\alpha}[0,1]$ *satisfy the physical constraints* ($\mathcal{F}$) *and* ($\mathcal{T}$), *and be compatible with the boundary conditions* ($\mathcal{M}$) *and* ($\mathcal{N}$). *Then there exists a unique classical solution* (ξ,ν,ϑ) *on* $[0,1]\times[0,\infty)$ *with the regularity*

$$\xi \in C^{1+\alpha,1+\frac{\alpha}{2}}\Big([0,1]\times[0,\infty)\Big),$$
$$\nu,\vartheta \in C^{2+\alpha,1+\frac{\alpha}{2}}\Big([0,1]\times[0,\infty)\Big),$$

and, furthermore, for which there exists $\underline{\xi},\overline{\xi} \in (0,\infty)$ *such that*

$$0 < \underline{\xi} \le \xi(x,t) \le \overline{\xi} < \infty. \tag{4}$$

3 The ideal elastogas

We now elucidate the thermodynamic structure of the compressible fluid that underlies the Navier–Stokes–Fourier system ($\mathcal{G}$), since it will play a key in the proof of Theorem 1, and hence Theorem 2.

The *equilibrium pressure law* $\widehat{\mathcal{P}}(\xi,\theta)$ of an *ideal elastogas* takes the dimensionless form,

$$\widehat{\mathcal{P}}(\xi,\theta) = \frac{\theta}{\xi} + \frac{\xi}{\xi^{q+1}+1}, \tag{$\mathcal{W}_1$}$$

with *asymptotic exponent* $q > 1$, while its *specific internal energy law* $\hat{e}(\xi,\theta)$ is given by

$$\hat{e}(\xi,\theta) := c_v\theta + \widehat{W}(\xi), \tag{$\mathcal{W}_2$}$$

with a constant *specific heat* $c_v > 0$ and a purely elastic stored-energy $\widehat{W}(\xi)$ given by

$$\widehat{W}(\xi) := \int_\xi^{+\infty} \frac{u}{u^{q+1}+1}\,\mathrm{d}u. \tag{5}$$

Remark 3 The definite integral appearing in (5) is well defined since the integrand is both continuous on $[0,\infty)$ and absolutely integrable on $[0,\infty)$: the latter point follows upon noting the asymptotic property

$$\frac{u}{u^{q+1}+1} \sim \frac{1}{u^q} \quad \text{as } u \nearrow \infty$$

and the assumption $q > 1$.

The *entropy law of the ideal elastogas* $\hat{\eta}(\xi,\theta)$ is defined to be the unique solution of the system of partial differential equations

$$\begin{aligned}\partial_\xi \hat{\eta}(\xi,\theta) &= \partial_\theta \widehat{\mathcal{P}}(\xi,\theta),\\ \partial_\theta \hat{\eta}(\xi,\theta) &= \frac{1}{\theta}\partial_\theta \hat{e}(\xi,\theta),\end{aligned} \tag{6}$$

where ∂_ξ and ∂_θ denote partial derivatives with respect to ξ and θ respectively, and which additionally satisfies the algebraic constraint[1]

$$\hat{\eta}(1,1) = 0.$$

By direct calculation, one readily finds

$$\hat{\eta}(\xi,\theta) = c_v \ln\theta + \ln\xi. \tag{$\mathcal{W}_3$}$$

Remark 4 Note that the system of partial differential equations of the form (6) will generally admit a solution $\hat{\eta}(\xi,\theta)$ provided the pressure law $\widehat{\mathcal{P}}(\xi,\theta)$ and internal energy law $\hat{e}(\xi,\theta)$ satisfy the *thermodynamic compatibility relation*

$$\partial_\xi \hat{e}(\xi,\theta) = \theta\partial_\theta \widehat{\mathcal{P}}(\xi,\theta) - \widehat{\mathcal{P}}(\xi,\theta). \tag{7}$$

In this case, one also obtains the 1-form relation [8]

$$\mathrm{d}\hat{e} = \theta\mathrm{d}\hat{\eta} - \widehat{\mathcal{P}}\mathrm{d}\xi, \tag{8}$$

which matches the classical thermodynamic relation between internal energy, temperature, entropy, pressure and (specific) volume [9]. Indeed, it is readily shown that (8) is logically equivalent to (6) and (7). Given that the pressure law $\widehat{\mathcal{P}}(\xi,\theta)$ and internal energy law $\hat{e}(\xi,\theta)$ of the elastogas do indeed satisfy (7), it is thereby thermodynamically justified to identify the constitutive law appearing in ($\mathcal{W}_3$) as an entropy.

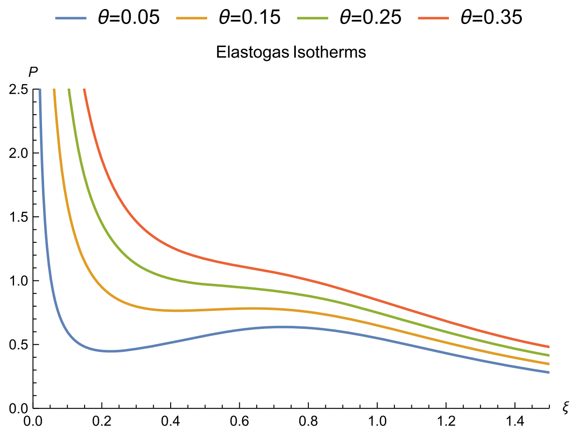

Fig. 1 Plot of the isotherms $P = \widehat{\mathcal{P}}(\xi, \theta)$ for $(\mathcal{W}_1)$ with $q = 3$ for the temperatures $\theta = 0.05, 0.1, 0.25, 0.35$

We plot in Fig. 1 a representative sequence of isotherms $P=\hat{\mathcal{P}}(\xi, \theta)$ for the ideal elastogas to illustrate that its isotherms are monotone decreasing in ξ above a certain *critical temperature* θ_c, but non-monotone in ξ below θ_c. This breaking of monotonicity in the isotherms of the ideal elastogas pressure law $(\mathcal{W}_1)$ *"reveal and define a phase transition"* [10, p. 234] in exactly the same manner as for the van der Waals equation of state [11]: a classical model for a gas-liquid phase transition. The *critical isotherm* $P = \hat{\mathcal{P}}(\xi, \theta_c)$ is itself monotone decreasing in ξ while also possessing a unique point of horizontal inflection at ξ_c. It follows that the *critical state* (ξ_c, θ_c) is the unique solution of the coupled algebraic equations

$$\begin{aligned} &\partial_\xi \widehat{\mathcal{P}}(\xi, \theta) = 0, \\ &\partial_\xi^2 \widehat{\mathcal{P}}(\xi, \theta) = 0, \end{aligned} \tag{9}$$

with $\xi_c > 0$ and $\theta_c > 0$. By direct calculation one finds the following explicit formulae for the critical volume

$$\xi_c = \left(\frac{q(q+5) - \sqrt{(q^2(q+5)^2 - 8(q^2 - q)}}{2(q^2 - q)} \right)^{\frac{1}{q+1}},$$

and critical temperature

$$\theta_c = \frac{\xi_c^2 \left(1 - q\xi_c^{q+1}\right)}{\left(1 + \xi_c^{q+1}\right)^2} \tag{10}$$

Remark 5 The pressure law $(\mathcal{W}_1)$ with its associated phase-transition at low temperature mimics key aspects of the famous *Lennard-Jones fluid*, which serves as a model for liquid helium at low densities. The pressure P of a Lennard-Jones fluid [12, pp. 100–104] is related to the specific volume ξ and temperature θ by

$$P = R\left(\frac{\theta}{\xi} + \frac{B(\theta)}{\xi^2} + \frac{C(\theta)}{\xi^3} + \cdots\right), \tag{11}$$

where R is the universal gas constant, and $B(\theta)$ and $C(\theta)$ are the temperature-dependent second- and third-*virial coefficients*, respectively. The second virial coefficient of Helium is empirically determined to be positive above $\theta_c \sim 23\ K$ and increasingly negative below [12, p. 103], thus opening the door to a gas-liquid phase-transition for sufficiently low temperatures. The pressure law $(\mathcal{W}_1)$ also connects in spirit to certain quantum mechanically inspired hydrodynamic models for nuclear matter [13]. One may also view $(\mathcal{W}_1)$ within the wider context of thermoelasticity with phase transitions: e.g., thermoelastic models of TiNi *shape memory alloys* [14], which undergo an *austenite-to-martensite phase transition* [15] below a critical temperature.

4 Energy and entropy bounds

We may recast the second and third equations of $(\mathcal{G})$ as the conservation of momentum and energy of a linearly viscous, Fourier heat-conducting ideal elastogas, namely

$$\partial_t v = \partial_x \mathcal{S}, \tag{12}$$

$$\partial_t \left(e + \frac{1}{2}v^2\right) = \partial_x \left(\mathcal{S}v - q\right), \tag{13}$$

where the *stress field* $\mathcal{S}$ is given by

$$\mathcal{S} = -\widehat{\mathcal{P}}(\xi, \vartheta) + \mu \frac{v_x}{\xi}, \tag{14}$$

the *heat flux* q by,

$$q := -\kappa \frac{\vartheta_x}{\xi}, \tag{15}$$

and the *internal-energy field* $e(x, t)$ by

$$e := \hat{e}(\xi, \vartheta) = c_v \vartheta + \widehat{W}(\xi). \tag{16}$$

Last, recalling the definition of the entropy response function $(\mathcal{W}_3)$, and then defining the associated entropy field $\eta(x, t)$ of a solution $(\mathcal{G})$ by

$$\eta := \hat{\eta}(\xi, \vartheta) = c_v \ln \vartheta + \ln \xi,$$

[1] The algebraic constraint merely fixes an arbitrary constant.

one also furthermore obtains by direct calculation the *entropy identity*

$$\partial_t \eta = \mu \frac{{\nu_x}^2}{\xi \vartheta} + \kappa \frac{{\vartheta_x}^2}{\xi \vartheta^2} + \kappa \partial_x \left(\frac{\vartheta_x}{\vartheta} \right). \tag{17}$$

The balance laws of momentum (12) and energy (13), in combination with the entropy identity (17), play a crucial role in the analysis that follows.

Here we begin the proof of the pointwise apriori estimate of Theorem 1 by first establishing some basic integral estimates of the internal energy, kinetic energy and entropy. In what follows, we suppose (ξ, ν, ϑ) is a globally defined classical solution of the elastogas $\mathcal{IBVP}$. To minimise notational clutter, we adopt the convention of letting $\lambda > 0$ and $\Lambda > 0$ denote generic "small" and "large" positive constants that depend at most on the initial-data ($\mathcal{IC}$), the outer-pressure P_0, and the parameters of ($\mathcal{G}$), namely c_v, μ and κ. In particular, we will generally use the same symbol λ and Λ within a sequence of inequalities, even though the precise constants will generically differ through such inequalities, provided that doing so does not cause any ambiguity. On those rare occasions when different constants need to be carefully distinguished, we will do so either by carefully tracking the algebraic relationships between such constants or by introducing indexed symbols λ_i or Λ_i.

First, note that the elastic stored-energy response $\widehat{W}(\xi)$ (5) is strictly positive and asymptotic to zero at infinity: i.e.,

$$0 < \widehat{W}(\xi) < \infty \quad \text{and} \quad \lim_{\xi \to \infty} \widehat{W}(\xi) = 0.$$

Recalling the positivity of the temperature field ϑ ($\mathcal{T}$), one thus immediately finds that the internal energy field $e = \hat{e}(\xi, \vartheta)$ (16) is pointwise bounded below by 0:

$$e(x,t) > 0. \tag{18}$$

Integrating the conservation of mass equation $\xi_t = \nu_x$ over the region $[0, 1] \times [0, t]$, we find

$$\int_0^1 (\xi(x,t) - \xi_0(x))\, \mathrm{d}x = \int_0^t (\nu(1,\tau) - \nu(0,\tau))\, \mathrm{d}\tau. \tag{19}$$

Now integrating the balance law for energy (13) over the region $[0, 1] \times [0, t]$, and utilising the zero-flux boundary condition ($\mathcal{N}$), the outer-pressure boundary condition ($\mathcal{M}$) and (19), we thereby find the *effective conservation of energy*:

$$\int_0^1 \left(e + P_0 \xi + \frac{1}{2} \nu^2 \right) \mathrm{d}x = E_0, \tag{20}$$

where the initial-data sets the constant $E_0 \geq 0$:

$$E_0 = \int_0^1 \left(\hat{e}(\xi_0, \vartheta_0) + P_0 \xi_0 + \frac{1}{2} \nu_0^2 \right) \mathrm{d}x.$$

We will now show how global bounds on both the total internal energy and kinetic energy, and the largest spatial extent of the solution, which is encoded in $\int_0^1 \xi \, \mathrm{d}x$, naturally follow from (20).

Lemma 1 *(Energy bounds)*

$$(i) \quad \int_0^1 e(x,t)\mathrm{d}x \leq \Lambda.$$

$$(ii) \quad \int_0^1 \nu^2(x,t)\mathrm{d}x \leq \Lambda.$$

$$(iii) \quad \int_0^1 \xi(x,t)\mathrm{d}x \leq \Lambda.$$

$$(iv) \quad \int_0^1 \vartheta(x,t)\mathrm{d}x \leq \Lambda.$$

Proof Noting the positivity of the constant $P_0 > 0$, the specific internal energy $e > 0$, the specific volume $\xi > 0$ and the kinetic energy term $\frac{1}{2}\nu^2$, we see from (20) that (i) and (ii) immediately follow. Similarly, we find

$$\int_0^1 \xi(x,t)\mathrm{d}x \leq \frac{E_0}{P_0} \leq \Lambda. \tag{21}$$

Last, from ($\mathcal{W}_2$), the positivity of $\widehat{W}$ and (i), we conclude

$$\int_0^1 \vartheta(x,t)\mathrm{d}x \leq \frac{1}{c_v} \int_0^1 e(x,t)\mathrm{d}x \leq \Lambda. \tag{22}$$

□

Turning now to the entropy identity (17), we integrate it over $[0, 1] \times [0, t]$ and utilise the zero-flux boundary condition ($\mathcal{N}$) to find the *entropy-production identity*

$$\int_0^t \int_0^1 \left(\mu \frac{{\nu_x}^2}{\xi \vartheta} + \kappa \frac{{\vartheta_x}^2}{\xi \vartheta^2} \right) \mathrm{d}x\mathrm{d}\tau = \int_0^1 \eta(x,t)\mathrm{d}x - \mathcal{N}_0, \tag{23}$$

with the initial total entropy $\mathcal{N}_0$ being determined by the initial data:

$$\mathcal{N}_0 = \int_0^1 \hat{\eta}(\xi_0(x), \vartheta_0(x))\mathrm{d}x\,.$$

Lemma 2 *(Entropy and entropy production bounds)*

$$(i) \quad \int_0^1 \eta(x,t)\mathrm{d}x \leq \Lambda.$$

$$(ii) \quad 0 < \lambda \leq \int_0^1 \vartheta(x,t)\mathrm{d}x.$$

$$(iii) \quad 0 < \lambda \leq \int_0^1 \xi(x,t)\mathrm{d}x.$$

$$(iv) \quad \int_0^t \int_0^1 \frac{{\vartheta_x}^2}{\xi\vartheta^2}(x,\tau)\mathrm{d}x\mathrm{d}\tau \leq \Lambda\,.$$

Proof Integrating ($\mathcal{W}_3$) over the reference domain [0, 1] and then utilising Jensen's inequality for concave functions and incorporating Lemma 1 (iii) and (iv), we obtain

$$\begin{aligned}\int_0^1 \eta\mathrm{d}x &= \int_0^1 (c_v \ln\vartheta + \ln\xi)\,\mathrm{d}x \\ &\leq c_v \ln\left(\int_0^1 \vartheta\mathrm{d}x\right) + \ln\left(\int_0^1 \xi\mathrm{d}x\right) \leq \Lambda.\end{aligned} \tag{24}$$

Since (23) guarantees the total entropy $\int_0^1 \eta(x,t)\mathrm{d}x$ is bounded below by $\mathcal{N}_0$, we also have

$$\mathcal{N}_0 \leq c_v \ln\left(\int_0^1 \vartheta\mathrm{d}x\right) + \ln\left(\int_0^1 \xi\mathrm{d}x\right), \tag{25}$$

which, when taken together with Lemma 1 (iii) and (iv), allows one to immediately deduce (ii) and (iii). Last, by utilising the global bound on the total entropy (i) within the entropy production identity (23), we arrive at the bound on the entropy production due to heat conduction (iv). □

5 Temperature estimates

We now turn to the proof of Theorem 1. The approach taken to obtain this pointwise a priori estimate on the specific volume involves a refinement of the general theoretical approach developed in [1,2], which itself was inspired by [4].

Since ϑ and ξ are continuous on $[0,1]\times[0,\infty)$ - $\vartheta,\xi \in C\big([0,1]\times[0,\infty),\mathbb{R}\big)$ - we may define their spatial-maximum functions $\vartheta_m : [0,\infty] \to (0,\infty)$ and $\xi_m : [0,\infty] \to (0,\infty)$ by

$$\begin{aligned}\vartheta_m(t) &:= \max_{x\in[0,1]} \vartheta(x,t), \\ \xi_m(t) &:= \max_{x\in[0,1]} \xi(x,t).\end{aligned} \tag{26}$$

Note that both ϑ_m and ξ_m are necessarily continuous functions: $\vartheta_m, \xi_m \in C\big([0,\infty),(0,\infty)\big)$.

We first formulate a preliminary analysis lemma that links pointwise control of the temperature at $(x,t) \in [0,1]\times[0,\infty)$ to the total entropy production induced by heat conduction at time t and the maximum specific volume at t.

Lemma 3 *(Temperature bounds)*

$$\begin{aligned}&(i) \quad \vartheta_m(t) \leq \Lambda\left(1 + \xi_m(t)\int_0^1 \frac{{\vartheta_x}^2}{\xi\vartheta^2}(x,t)\,\mathrm{d}x\right). \\ &(ii) \quad \lambda - \Lambda\int_0^1 \frac{{\vartheta_x}^2}{\vartheta^2}(x,t)\mathrm{d}x \leq \vartheta(x,t)\,.\end{aligned} \tag{27}$$

Proof By applying the standard Sobolev inequality associated with the embedding $W^{1,1}(0,1) \hookrightarrow C[0,1]$ to $\vartheta^{1/2}$, and then utilising Jensen's inequality, Lemma 1 (iv), and the Cauchy-Schwarz inequality with a judicious choice of pairing, we obtain

$$\begin{aligned}\vartheta_m^{1/2}(t) &\leq \int_0^1 \vartheta^{1/2}(x,t)\mathrm{d}x + \frac{1}{2}\int_0^1 \frac{|\vartheta_x|}{\vartheta^{1/2}}(x,t)\mathrm{d}x \\ &\leq \left(\int_0^1 \vartheta\mathrm{d}x\right)^{1/2} + \int_0^1 (\xi\vartheta)^{1/2}\frac{|\vartheta_x|}{\xi^{1/2}\vartheta}\mathrm{d}x \\ &\leq \Lambda + \left(\int_0^1 \xi\vartheta\,\mathrm{d}x\right)^{1/2}\left(\int_0^1 \frac{{\vartheta_x}^2}{\xi\vartheta^2}\mathrm{d}x\right)^{1/2} \\ &\leq \Lambda\left[1 + \xi_m^{1/2}(t)\left(\int_0^1 \frac{{\vartheta_x}^2}{\xi\vartheta^2}\mathrm{d}x\right)^{1/2}\right].\end{aligned}$$

Taking squares of the first and last term in the above inequality, which preserves the inequality since $\vartheta_m(t) > 0$, and then utilising the elementary algebraic inequality $(a+b)^2 \leq 2(a^2+b^2)$, we immediately arrive at (i).

From the fundamental theorem of calculus to applied to $\vartheta^{1/2}$, we naturally find that for any $x, y \in [0,1]$

$$\vartheta^{1/2}(x,t) \geq \vartheta^{1/2}(y,t) - \frac{1}{2}\int_0^1 \frac{|\vartheta_x|}{\vartheta^{1/2}}\mathrm{d}x. \tag{28}$$

Utilising the Cauchy-Schwarz inequality and Lemma 1 (iv), we also deduce

$$\begin{aligned}\int_0^1 \frac{|\vartheta_x|}{\vartheta^{1/2}}\mathrm{d}x &= \int_0^1 \vartheta^{1/2}\frac{|\vartheta_x|}{\vartheta}\mathrm{d}x \\ &\leq \left(\int_0^1 \vartheta\,\mathrm{d}x\right)^{1/2}\left(\int_0^1 \frac{{\vartheta_x}^2}{\vartheta^2}\mathrm{d}x\right)^{1/2} \\ &\leq \Lambda\left(\int_0^1 \frac{{\vartheta_x}^2}{\vartheta^2}\mathrm{d}x\right)^{1/2}.\end{aligned} \tag{29}$$

From (28) and (29) we immediately find

$$\vartheta^{1/2}(x,t) \geq \vartheta^{1/2}(y,t) - \frac{\Lambda}{2}\left(\int_0^1 \frac{{\vartheta_x}^2}{\vartheta^2}(x,t)\mathrm{d}x\right)^{1/2}. \tag{30}$$

Applying the elementary algebraic implication

$$a \geq b - c \quad \text{and} \quad b \geq 0 \implies a^2 \geq \frac{1}{2}b^2 - c^2, \qquad (31)$$

to (30), we arrive at

$$\vartheta(x,t) \geq \frac{1}{2}\vartheta(y,t) - \frac{\Lambda^2}{4}\int_0^1 \frac{{\vartheta_x}^2}{\vartheta^2}(x,t)\mathrm{d}x. \qquad (32)$$

Now integrating the both sides of the inequality (32) with respect to y over the interval $[0, 1]$, and then recalling Lemma 2 (ii), we arrive at (ii). $\square$

6 Proof of Theorem 1

Noting the conservation-of-mass relation $\xi_t = v_x$, we see that the viscous contribution to the stress involves a total time derivative, namely

$$\frac{v_x}{\xi} = \frac{\xi_t}{\xi} = \partial_t (\ln \xi). \qquad (33)$$

Now integrating the balance of momentum equation ($\mathcal{G}$) over $[0, x] \times [0, t]$, and utilising the outer-pressure boundary condition ($\mathcal{M}$) and (33), we then find

$$\mu \ln \xi(x,t) - Y(x,t) = b(x,t), \qquad (34)$$

where

$$Y(x,t) = \int_0^t \left[\widehat{\mathcal{P}}\big(\xi(x,\tau), \vartheta(x,\tau)\big) - P_0\right]\mathrm{d}\tau, \qquad (35)$$

$$b(x,t) = \int_0^x \big(v(r,t) - v_0(r)\big)\mathrm{d}x + \mu \ln \xi_0(x). \qquad (36)$$

By utilising the bound on the kinetic energy provided by Lemma 1 (ii), we find, after a simple application of the Cauchy-Schwarz inequality, that

$$\left|\int_0^1 v\mathrm{d}x\right| \leq \int_0^1 |v|\mathrm{d}x \leq \left(\int_0^1 v^2\mathrm{d}x\right)^{\frac{1}{2}} \leq \Lambda. \qquad (37)$$

It follows that the function b appearing in (36) is bounded on $[0, 1] \times [0, \infty)$: i.e,

$$\|b\|_\infty := \max_{(x,t)\in[0,1]\times[0,\infty)} |b(x,t)| \leq \Lambda. \qquad (38)$$

Note that we may re-write (34) as

$$\xi(x,t) = e^{\frac{1}{\mu}b(x,t)}e^{\frac{1}{\mu}Y(x,t)}. \qquad (39)$$

Differentiating (35) with respect to t, and then inserting (39), we arrive at

$$\partial_t Y = \widehat{\mathcal{P}}\left(e^{\frac{1}{\mu}b}e^{\frac{1}{\mu}Y}, \vartheta\right) - P_0. \qquad (40)$$

Remark 6 It is worth noting that (34) and (40) mirror [2, Eqns. 3.20 and 3.21]. Now the general analytical approach developed in [1,2], which as previously noted was inspired by [4], is sufficient to obtain a priori bounds on the *deformation gradient* (specific volume) for a broad class of *thermoviscolastic materials* - linearly viscous, Fourier heat conducting thermoelastic materials - and a wide range of physically natural boundary conditions. However those bounds are perforce non-uniform due to the generality treated there, and it is precisely that shortcoming that we resolve in this article.

Given that the equilibrium pressure law $\widehat{\mathcal{P}}(\xi, \theta)$ ($\mathcal{W}_1$) is algebraically bounded above and below as follows

$$\frac{\theta}{\xi} \leq \widehat{\mathcal{P}}(\xi,\theta) \leq \frac{1+\theta}{\xi}, \qquad (41)$$

we therefore deduce from (40) and (41) that

$$\partial_t Y \leq (1+\vartheta)e^{-\frac{1}{\mu}b}e^{-\frac{1}{\mu}Y} - P_0 \qquad (42)$$

and

$$\partial_t Y \geq \vartheta e^{-\frac{1}{\mu}b}e^{-\frac{1}{\mu}Y} - P_0. \qquad (43)$$

Recalling (38), we readily obtain

$$0 < \lambda = e^{-\frac{1}{\mu}\|b\|_\infty} \leq e^{-\frac{1}{\mu}b} \leq e^{\frac{1}{\mu}\|b\|_\infty} = \Lambda < \infty. \qquad (44)$$

Combining (44) with the inequalities (42) and (43), we naturally find

$$\lambda\vartheta e^{-\frac{1}{\mu}Y} \leq \partial_t Y + P_0 \leq \Lambda(1+\vartheta)e^{-\frac{1}{\mu}Y}, \qquad (45)$$

or equivalently

$$\frac{\lambda}{\mu}\vartheta \leq \partial_t\left(e^{\frac{1}{\mu}Y}\right) + \frac{P_0}{\mu}e^{\frac{1}{\mu}Y} \leq \frac{\Lambda}{\mu}(1+\vartheta). \qquad (46)$$

Multiplying the above inequality by $e^{\frac{P_0}{\mu}t}$ and noting the appearance of a total time derivative in the middle term, we then find that

$$\frac{\lambda}{\mu}\vartheta e^{\frac{P_0}{\mu}t} \leq \partial_t\left(e^{\frac{P_0}{\mu}t}e^{\frac{1}{\mu}Y}\right) \leq \frac{\Lambda}{\mu}(1+\vartheta)e^{\frac{P_0}{\mu}t}, \qquad (47)$$

or equivalently

$$\lambda\vartheta e^{\alpha t} \leq \partial_t\left(e^{\alpha t}e^{\frac{1}{\mu}Y}\right) \leq \Lambda(1+\vartheta)e^{\alpha t}, \qquad (48)$$

where

$$\alpha := \frac{P_0}{\mu} > 0,$$

and we have replaced $\frac{\lambda}{\mu}$ and $\frac{\Lambda}{\mu}$ with our generic symbols for a small constant $\lambda > 0$ and a large constant $0 < \Lambda$, as per our convention.

Integrating the right inequality appearing in (48) over the time interval $[0, t]$, and then rearranging terms and calculating a definite integral, we find

$$\begin{aligned} e^{\frac{1}{\mu}Y} &\leq e^{-\alpha t} + \Lambda \int_0^t (1+\vartheta) e^{-\alpha(t-\tau)}\,\mathrm{d}\tau \\ &\leq 1 + \Lambda \int_0^t e^{-\alpha(t-\tau)}\,\mathrm{d}\tau + \Lambda \int_0^t \vartheta e^{-\alpha(t-\tau)}\mathrm{d}\tau \\ &\leq 1 + \frac{\Lambda}{\alpha}\left(1-e^{-\alpha t}\right) + \Lambda \int_0^t \vartheta e^{-\alpha(t-\tau)}\mathrm{d}\tau \\ &\leq \Lambda\left(1 + \int_0^t \vartheta_m(\tau) e^{-\alpha(t-\tau)}\mathrm{d}\tau\right). \end{aligned} \tag{49}$$

Multiplying through (49) by $e^{\frac{1}{\mu}b}$ and recalling (39) and (44), we thus obtain

$$\xi(x,t) \leq \Lambda\left(1 + \int_0^t \vartheta_m(\tau) e^{-\alpha(t-\tau)}\mathrm{d}\tau\right), \tag{50}$$

and hence

$$\xi_m(t) \leq \Lambda\left(1 + \int_0^t \vartheta_m(\tau) e^{-\alpha(t-\tau)}\mathrm{d}\tau\right). \tag{51}$$

The estimate for the maximum temperature $\vartheta_m(t)$ appearing in Lemma 3 (i) may be utilised to estimate the second term on the right-hand side of (51) as follows. First, introduce the shorthand $\mathcal{D}(t)$ for the positive thermal entropy production term, namely

$$\mathcal{D}(t) := \int_0^1 \frac{{\vartheta_x}^2}{\xi\vartheta^2}(x,t)\,\mathrm{d}x \geq 0. \tag{52}$$

We now calculate

$$\begin{aligned} &\int_0^t \vartheta_m(\tau) e^{-\alpha(t-\tau)}\mathrm{d}\tau \\ &\leq \Lambda\left(\int_0^t \left(1+\xi_m(\tau)\mathcal{D}(\tau)\right) e^{-\alpha(t-\tau)}\mathrm{d}\tau\right) \\ &\leq \Lambda\left(\int_0^t e^{-\alpha(t-\tau)}\mathrm{d}\tau + \int_0^t \xi_m(\tau)\mathcal{D}(\tau)\mathrm{d}\tau\right) \\ &\leq \frac{\Lambda}{\alpha}\left(1-e^{-\alpha t}\right) + \Lambda\int_0^t \xi_m(\tau)\mathcal{D}(\tau)\mathrm{d}\tau \\ &\leq \Lambda\left(1 + \int_0^t \xi_m(\tau)\mathcal{D}(\tau)\mathrm{d}\tau\right). \end{aligned} \tag{53}$$

Extracting the resulting estimate between the first and last term of (53) and injecting it into (51), we thus obtain

$$\xi_m(t) \leq \Lambda\left(1 + \int_0^t \xi_m(\tau)\mathcal{D}(\tau)\mathrm{d}\tau\right). \tag{54}$$

Recalling that $\xi_m : [0,\infty) \to (0,\infty)$ is a continuous function and $\mathcal{D}(t) \geq 0$, which thereby permits one to apply Gronwall's inequality to (54), we conclude

$$\xi_m(t) \leq \Lambda e^{\Lambda \int_0^t \mathcal{D}(\tau)\mathrm{d}\tau}. \tag{55}$$

Now utilising the bound of the total thermal entropy production given by Lemma 2 (iv), namely

$$\int_0^\infty \mathcal{D}(t)\mathrm{d}t = \int_0^\infty \int_0^1 \frac{{\vartheta_x}^2}{\xi\vartheta^2}(x,t)\,\mathrm{d}x\mathrm{d}t < \infty, \tag{56}$$

we conclude from (55) that there exists $\overline{\xi} > 0$ that uniformly bounds the specific volume from above, namely

$$\xi(x,t) \leq \overline{\xi} \qquad \forall\,(x,t) \in [0,1]\times[0,\infty). \tag{57}$$

Turning now to the demonstration of a uniform lower bound for $\xi(x,t)$, we will draw on an idea that originated with Nagasawa [16]: see also [17]. We first replace the technical lemma appearing in [16, Lemma 3.1] with a more direct result that follows from the Lebesgue dominated convergence theorem, namely the following Lemma.

Lemma 4 *Let $\alpha > 0$ and $g : [0,\infty) \to \mathbb{R}$ be a continuous and absolutely integrable function: i.e.,*

$$\int_0^\infty |g(\tau)|\,d\tau < \infty.$$

It then follows that

$$\lim_{t\to\infty} \int_0^t g(\tau) e^{-\alpha(t-\tau)} d\tau = 0. \tag{58}$$

Proof For each $t \in (0,\infty)$, we introduce the indicator function $\mathfrak{I}_{[0,t]} : [0,\infty) \to \mathbb{R}$ defined by

$$\mathfrak{I}_{[0,t]}(\tau) = \begin{cases} 1 & \tau \in [0,t] \\ 0 & \tau \in (t,\infty) \end{cases} \tag{59}$$

Note that for all $t \in (0,\infty)$

$$\int_0^t g(\tau) e^{-\alpha(t-\tau)}\mathrm{d}\tau = \int_0^\infty g(\tau) e^{-\alpha(t-\tau)} \mathfrak{I}_{[0,t]}(\tau)\mathrm{d}\tau. \tag{60}$$

Note that the t-dependent integrand on the right-hand side of (60), namely $g(\tau)e^{-\alpha(t-\tau)}\mathfrak{I}_{[0,t]}(\tau)$, is *dominated uniformly in t* by the L^1 function g:

$$\left| g(\tau) e^{-\alpha(t-\tau)} \mathfrak{I}_{[0,t]}(\tau) \right| \leq |g(\tau)| \qquad \forall\, t,\tau \in [0,\infty). \tag{61}$$

Furthermore, this integrand converges pointwise to zero as $t \to \infty$: i.e., for all $\tau \in [0,\infty)$

$$\lim_{t\to\infty} g(\tau) e^{-\alpha(t-\tau)} \mathfrak{I}_{[0,t]}(\tau) = 0. \tag{62}$$

Recalling *Lebesgue's dominated convergence theorem* and noting (61) and (62), we may therefore deduce

$$\begin{aligned} &\lim_{t\to\infty} \int_0^\infty g(\tau) e^{-\alpha(t-\tau)} \mathfrak{I}_{[0,t]}(\tau)\mathrm{d}\tau \\ &= \int_0^\infty \lim_{t\to\infty} \left[g(\tau) e^{-\alpha(t-\tau)} \mathfrak{I}_{[0,t]}(\tau) \right] \mathrm{d}\tau = 0, \end{aligned} \tag{63}$$

from which, upon noting (60), we may then conclude (58). □

We may now utilise Lemma 4 to establish the following global lower bound.

Lemma 5 *With* $\alpha = \frac{P_0}{\mu} > 0$, *there exists* $\lambda > 0$ *such that*

$$\int_0^t \vartheta(x,\tau)e^{-\alpha(t-\tau)}\mathrm{d}\tau \geq \lambda > 0$$

Proof Recalling the spatially uniform point-wise lower-bound on the temperature given in Lemma 3 (ii), we obtain a corresponding lower-bound for the minimum temperature, namely for all $t \in [0,\infty)$

$$\min_{x\in[0,1]} \vartheta(x,\tau) \geq \lambda - \Lambda \int_0^1 \frac{{\vartheta_x}^2}{\vartheta^2}(x,\tau)\mathrm{d}x. \tag{64}$$

For notational convenience, we introduce the function $f : [0,\infty) \to \mathbb{R}$ defined by

$$f(t) := \int_0^1 \frac{{\vartheta_x}^2}{\vartheta^2}(x,t)\mathrm{d}x \qquad \forall \tau \in [0,\infty). \tag{65}$$

Now combining (56) and (57), we find

$$\int_0^\infty \int_0^1 \frac{{\vartheta_x}^2}{\vartheta^2}(x,t)\,\mathrm{d}x\mathrm{d}t < \infty, \tag{66}$$

and hence

$$f \in L^1(0,\infty). \tag{67}$$

Multiplying each sides of the inequality (64) by $e^{-\alpha(t-\tau)}$, before integrating in τ over the time-interval $[0,t]$, and then explicitly calculating a definite integral, we obtain

$$\begin{aligned} &\int_0^t \min_{x\in[0,1]} \vartheta(x,\tau)e^{-\alpha(t-\tau)}\mathrm{d}\tau \\ &\quad \geq \lambda \int_0^t e^{-\alpha(t-\tau)}\mathrm{d}\tau - \Lambda \int_0^t f(\tau)\,e^{-\alpha(t-\tau)}\mathrm{d}\tau \\ &\quad = \frac{\lambda}{\alpha}\left(1-e^{-\alpha t}\right) - \Lambda \int_0^t f(\tau)\,e^{-\alpha(t-\tau)}\mathrm{d}\tau. \end{aligned} \tag{68}$$

Recalling Lemma 4 and (67), we may now take the limit inferior as $t \to \infty$ ($\liminf_{t\to\infty}$) of the inequality appearing in (68) to arrive at

$$\liminf_{t\to\infty} \int_0^t \min_{x\in[0,1]} \vartheta(x,\tau)e^{-\alpha(t-\tau)}\mathrm{d}\tau \geq \frac{\lambda}{\alpha} > 0. \tag{69}$$

Given (69), we may now choose $\lambda_1 > 0$ and $\tilde{T} > 0$ such that

$$\int_0^t \min_{x\in[0,1]} \vartheta(x,\tau)e^{-\alpha(t-\tau)}\mathrm{d}\tau \geq \lambda_1, \quad \forall t \in [\tilde{T},\infty). \tag{70}$$

Noting that

$$\int_0^t \min_{x\in[0,1]} \vartheta(x,\tau)e^{-\alpha(t-\tau)}\mathrm{d}\tau$$

is a positive and continuous function of $t \in [0,\infty)$, we may identify its necessarily positive minimum $\lambda_2 > 0$ on the compact set $[0,\tilde{T}]$, i.e.,

$$\lambda_2 := \min_{t\in[0,\tilde{T}]} \left(\int_0^t \min_{x\in[0,1]} \vartheta(x,\tau)e^{-\alpha(t-\tau)}\mathrm{d}\tau\right) > 0. \tag{71}$$

Setting $\lambda = \min\{\lambda_1,\lambda_2\} > 0$, we may combine (70) and (71) to deduce the global lower bound

$$\int_0^t \min_{x\in[0,1]} \vartheta(x,\tau)e^{-\alpha(t-\tau)}\mathrm{d}\tau \geq \lambda, \quad \forall t \in [0,\infty),$$

from which the Lemma immediately follows. □

Returning to the lower inequality appearing in (48), we find, upon integrating over the time interval $[0,t]$, and then rearranging terms that

$$e^{\frac{1}{\mu}Y} \geq e^{-\alpha t} + \lambda \int_0^t \vartheta(x,\tau)e^{-\alpha(t-\tau)}\,\mathrm{d}\tau. \tag{72}$$

Utilising Lemma 5 in (72), we readily obtain

$$e^{\frac{1}{\mu}Y} \geq \lambda > 0. \tag{73}$$

Now recalling (39) and (44), we have

$$\xi = e^{\frac{1}{\mu}b}e^{\frac{1}{\mu}Y} \geq e^{-\frac{1}{\mu}\|b\|_\infty}e^{\frac{1}{\mu}Y} \geq \lambda e^{\frac{1}{\mu}Y}. \tag{74}$$

Combining the inequalities (73) and (74), we now see that there exists a constant $\underline{\xi} > 0$ that uniformly bounds the specific volume from below, namely, for all $(x,t) \in [0,1]\times[0,\infty)$

$$\xi(x,t) \geq \underline{\xi} > 0,$$

which concludes the proof of Theorem 1.

Declaration

Conflict of Interest The authors declare that they have no conflict of interest.

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