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Representable Functions on the Unit Ball of a Banach Space

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Abstract. In this paper we treat the problem of integral representation of analytic functions over the unit ball of a complex Banach space X using the theory of abstract Wiener spaces. We define the class of representable functions on the unit ball of X and prove that this set of functions is related with the classes of integral $k-$homogeneous polynomials, integral holomorphic functions and also with the set of L^p-representable functions on a Banach space.

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1. Introduction

This note continues recent work on generalizations of the Cauchy integral formula to infinite-dimensional Banach spaces. Given a Banach space X, known integral expressions for homogeneous polynomials and holomorphic functions all involve integration over X', the dual of X, rather than X itself. For example, an integral polynomial [4] over X is defined, integrating over the unit ball of X' respect to a regular Borel measure μ, as

$$P(x) = \int_{B_{X'}} \gamma(x)^k \, d\mu(\gamma).$$

In [2], an integral holomorphic function over B_X°, is defined in a similar way,

$$f(x) = \int_{B_{X'}} \frac{1}{1-\gamma(x)} \, d\mu(\gamma).$$

The measure μ is said to represent the function f. In [11], the authors proved two generalizations of the one-dimensional Cauchy integral formula, valid for

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some classes of holomorphic functions on infinite dimensional Banach spaces. These formulae are of the type:

$$f(z) = \int_{X'} K(z,\gamma)\,\hat{f}(\gamma)\,dW(\gamma),$$

where W is a Wiener measure on X', K is an integral kernel and $\hat{f}$ is a transformation of f involving the covariance operator A of the measure.

Working with the kernel $K(z,\gamma) = e^{z(\gamma)}$, the sets of L^p−representable functions were introduced for $p > 1$ in [9]. Also, it was proved that these functions are related to the Hardy-type space $\mathcal{H}^2(B_H)$, defined in [8], for a suitable Hilbert space H. The object of this paper is to study the class of holomorphic functions on the unit ball of X which can be represented using the alternative kernel $K(z,\gamma) = \left(1 - \frac{\gamma(z)}{\|\gamma\|_0}\right)^{-1}$. In Section 2 we summarize the relevant material related to the integral formula: complex valued random variables, Gaussian measure on a separable Banach space X and Gross' theorem. In Section 3 we define the set of w^*−representable functions on the unit ball of X'' and representable functions on the unit ball of X. Finally, in Sections 4 and 5, we study relations between these analytic functions and the set of L^p−representable and integral holomorphic functions.

2. Definitions and general results

We begin by recalling a few properties of holomorphic functions, for a fuller account of the theory we refer the reader to [3]. Given a Banach space X, the space of holomorphic functions from X to $\mathbb{C}$ is denoted by $\mathcal{H}(X)$. If $f = \sum_k f_k$ is the Taylor series expansion of a holomorphic function in infinite dimensions, then it converges uniformly in some neighborhood around the point of expansion. The radius of uniform convergence may be calculated as $r = \left(\limsup \|f_k\|^{1/k}\right)^{-1}$, where $\|f_k\|$ is the norm of the k-homogeneous polynomial f_k. The space of holomorphic functions whose Taylor series have infinite radius of uniform convergence is denoted $\mathcal{H}_b(X)$. Such functions are bounded on bounded sets, and are said to be of bounded type. Let B_X° the open unit ball in X, the space of holomorphic functions of bounded type on B_X° is defined by:

$$\mathcal{H}_b(B_X^\circ) = \left\{ f \in \mathcal{H}(B_X^\circ) : \|f\|_\rho := \sup_{x \in B(0,\rho)} |f(x)| < \infty, \text{ for all } \rho < 1 \right\}.$$

A complex-valued Gaussian random variable, with mean m and variance σ^2 is one whose density function $\delta : \mathbb{C} \longrightarrow \mathbb{R}$ is defined by

$$\delta(w) = \frac{1}{\pi\sigma^2}\, e^{-|w-m|^2/\sigma^2}.$$

If we consider $\mathbb{C}^n$ endowed with the standard Gaussian measure, the projections to the coordinates are complex-valued Gaussian random variables with mean zero and variance one.

2.1. Wiener measure on separable Banach spaces

We refer the reader to [7] for a detailed exposition of Wiener measures on real Banach spaces. We need to extend this construction to the complex setting but, considering the real underlying structure, there are not significative changes on the techniques and ideas involved in the real case. Therefore, we omit many of the proofs of the following statements.

Given a separable Hilbert space H, if P is a finite-rank orthogonal projection in H, a cylinder in H is a set of the form $C = \{x \in H : Px \in \Delta\}$, where Δ is a Borel subset of PH. We will denote by Γ the Gaussian cylinder measure defined on cylinder sets by

$$\Gamma(C) = \frac{1}{\pi^n} \int_\Delta e^{-|w|^2} dw,$$

where n is the complex dimension of PH and the integral is with respect to Lebesgue measure. In this way we obtain a finitely additive measure Γ, which is not σ-additive. However, integrals of cylinder functions (i.e. $F : H \to \mathbb{C}$ of the form $F = h \circ P$, for some measurable function $h : PH \to \mathbb{C}$) may be defined by setting:

$$\int_C F\, d\Gamma = \int_\Delta h\, d\Gamma_n,$$

where Γ_n is standard n-dimensional Gaussian measure. For example, any continuous linear functional $\phi \in H'$ is a cylinder function. Moreover, it is a complex-valued Gaussian random variable with mean 0 and variance $\|\phi\|^2$.

A norm on H is called *measurable* if given any real number $\varepsilon > 0$, there exists a finite-rank orthogonal projection P_ε such that, for every finite-rank orthogonal projection P verifying $P \perp P_\varepsilon$, we have

$$\Gamma\{x \in H : \|Px\| > \varepsilon\} < \varepsilon.$$

One way to construct a *measurable* norm is given by an injective Hilbert-Schmidt operator S defined on H, let $\|\cdot\|_S = \langle S(\,\cdot\,), S(\,\cdot\,)\rangle^{\frac{1}{2}}$. Upon completing $(H, \|\cdot\|)$ one obtains a Banach space X. This abstract setting generalizes Wiener's construction of a measure on the continuous functions and so, $\iota : H \hookrightarrow X$ is called an abstract Wiener space. The inclusion ι is continuous and dense. Given $x'_1, \ldots, x'_n \in X'$ and a Borel set $\Delta \subset \mathbb{C}^n$, one defines cylinder sets in X

$$C_X = \{x \in X : (x'_1(x), \ldots, x'_n(x)) \in \Delta\}.$$

Note that $C = \iota^{-1}(C_X)$ is a cylinder set in H and so, it is possible to define

$$\widetilde{\Gamma}(C_X) = \Gamma(C) \quad \text{for } C = \iota^{-1}(C_X).$$

This function $\widetilde{\Gamma}$ has a unique σ-additive extension to a measure W (called Wiener measure) on the Borel σ-algebra $\mathcal{B}$ of X. The natural question is if every separable Banach space can be constructed in this fashion. The real version of the following theorem can be found in [6]. Here, we present the sketch of the proof for the complex case because it will be necessary for fixing notation.

Theorem 2.1 (Gross). *If X is a separable complex Banach space, then there is a Hilbert space H and a continuous injection $\imath : H \hookrightarrow X$ such that $(\imath, H, X)$ is an abstract Wiener space. Furthermore: there is a smaller abstract Wiener space $(\imath_0, H, H_0)$, such that $H \hookrightarrow H_0 \hookrightarrow X$, and an increasing sequence of finite-rank orthogonal projections $\{p_n\}$ converging to the identity in H; these extend to P_n on H_0 where they converge to the identity as well. Also, $W(H_0) = 1$.*

Proof. Since X is separable we can take an increasing sequence of subspaces $\{F_n\}_{n\in\mathbb{N}}$ and a sequence $\{w_k\}_{k\in\mathbb{N}} \subset X$ such that

- $\dim(F_n) = n$.
- $F = \bigcup_{n=1}^{\infty} F_n$ is dense in X.
- The set $\{w_1, w_2, \ldots, w_n\}$ is a basis of F_n for all $n \in \mathbb{N}$.

As in the real setting, there exists a sequence of positive real numbers $\{\alpha_j\}_{j\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, and for any vector $(\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{C}^n$, then

$$\sum_{j=1}^{n} |\beta_j|^2 \leq 1 \Rightarrow \left\| \sum_{j=1}^{n} \beta_j \alpha_j w_j \right\|_X < 1.$$

We may now define an inner product $\langle \cdot, \cdot \rangle_0$ on F such that $\{\alpha_n w_n\}_{n\in\mathbb{N}}$ is an orthonormal set. Given $x, y \in F$, there exist $n, m \in \mathbb{N}$, and complex numbers $\{a_j\}_{j=1}^n$, $\{b_j\}_{j=1}^m$ such that $x = \sum_{j=1}^n a_j w_j$ and $y = \sum_{j=1}^m b_j w_j$. The inner product is defined by

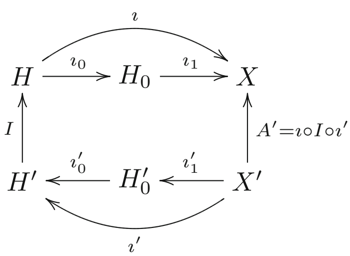
$$\langle x, y \rangle_0 = \sum_{j\geq 1} \alpha_j^{-2}\, a_j \overline{b_j},$$

and if we let $|x|_0 = \langle x, x \rangle_0^{1/2}$, then we have $\|x\|_X < |x|_0$ for all $x \in F$.

Let us denote H_0 the completion of $(F, |\cdot|_0) \subset X$. Since the natural inclusion $\imath_1 : H_0 \to X$ is injective, $\imath_1' : X' \to H_0'$ has dense range, so we can take an orthonormal basis $\{u_n'\}_{n\in\mathbb{N}}$ on H_0' such that $u_n' \in \imath_1'(X')$ for all $n \in \mathbb{N}$. Fix the basis $\{u_n\}_{n\in\mathbb{N}} \subset H_0$, such that $\{u_n'\}_{n\in\mathbb{N}}$ is its dual basis.

Also, H_0 has a real Hilbert space structure, which will be denoted $H_{0,\mathbb{R}}$. Its inner product is given by $\langle x, y \rangle_{0,\mathbb{R}} = \Re\langle x, y \rangle_0$, and it has the orthonormal basis $\{u_n, iu_n\}_{n\in\mathbb{N}}$.

Let $I : H' \to H$ the operator obtained from the Riesz representation theorem, the following diagram will be useful.



Summarizing, there is a sequence $\{z_n\}_{n\in\mathbb{N}} \subset X'$ such that $\{\imath'(z_n)\}_{n\in\mathbb{N}}$ is an orthonormal basis for H', dual to $\{I \circ \imath'(z_n)\}_{n\in\mathbb{N}}$, these basis will be denoted $\{e'_n\}_{n\in\mathbb{N}}$ and $\{e_n\}_{n\in\mathbb{N}}$ respectively. Since $\imath_0 : H \hookrightarrow H_0$ is dense, it follows that $\{\imath_0(e_n)\}_{n\in\mathbb{N}}$ is a complete orthogonal set in H_0. Moreover, $\|\imath_0(e_n)\|_0 = \lambda_n$ for all $n \in \mathbb{N}$. It is easy to see that the subspaces

$$S_n = \left[\frac{\imath_0(e_1)}{\lambda_1}, \frac{\imath_0(e_2)}{\lambda_2}, \ldots, \frac{\imath_0(e_n)}{\lambda_n}\right] \subset H_0,$$

are such that the sequence of operators $\{P_n\}_{n\in\mathbb{N}}$, defined as the orthogonal projections onto S_n, converges in the strong operator topology to the identity on H_0. □

The following results give us useful criterions for integrability of measurable functions on $(X, \mathcal{B}, W)$.

Theorem 2.2 (Fernique, [5]**).** *If X is a separable Banach space and W is a Wiener measure defined on its Borel subsets, then there exists $\varepsilon > 0$ such that the integral $\int_X e^{\varepsilon\|\gamma\|^2}\, dW(\gamma)$ is finite.*

Corollary 2.3. *Given an abstract Wiener space $(\imath, H, X)$, then for any $k \geq 0$ the function $N_k(\gamma) = \|\gamma\|^k$ is integrable.*

3. Representable functions on the unit ball of a Banach space

Now, we turn our attention to the integral formula proved in [11]:

$$f(z) = \int_{X'} \frac{1}{1 - \frac{z(\gamma)}{\|\gamma\|_0}}\, \hat{f}(\gamma)\, dW(\gamma), \tag{3.1}$$

here, W is a Wiener measure on X' and $\hat{f}$ is a transformation of f involving the covariance operator A of the measure. This representation holds, for example, for holomorphic functions of $A-$harmonic type. We are interested in study the set of functions for which an integral representation similar to (3.1) holds. Also, we want to be able to apply this formula to holomorphic functions on the unit ball of a Banach space, even if it is not a dual space.

We begin by doing the following observation: since $\imath_1 : H_0 \to X$ is a norm one continuous inclusion and $W(\imath_1(H_0)) = 1$, then the function

$$K_z : B_X^\circ \to \mathbb{C}$$

defined by

$$K_z(\gamma) = \frac{1}{1 - z\left(\frac{\gamma}{\|\gamma\|_0}\right)},$$

belongs to $L^\infty(W)$ for all $z \in B_{X'}^\circ$.

Definition 3.1. A function $f : B_{X'}^\circ \to \mathbb{C}$ is said to be w^*-representable on the unit ball of X', if there exists $g \in L^1(W)$ such that

$$f(z) = \int_X \frac{1}{1 - z\left(\frac{\gamma}{\|\gamma\|_0}\right)}\, \overline{g(\gamma)}\, dW(\gamma) \quad \text{for all } z \in B_{X'}^\circ.$$

The set of w^*-representable functions on the ball of X' will be denoted by $\mathcal{R}_*(B^\circ_{X'})$.

Proposition 3.2. *If $f \in \mathcal{R}_*(B^\circ_{X'})$, then $f \in \mathcal{H}_b(B^\circ_{X'})$. Moreover, there exists a unique $F \in \mathcal{H}_b(B^\circ_{H'_0})$ such that $F \circ \imath'_1(z) = f(z)$.*

Proof. Let $g \in L^1(W)$ such that

$$f(z) = \int_X \frac{1}{1 - z\left(\frac{\gamma}{\|\gamma\|_0}\right)} \overline{g(\gamma)}\, dW(\gamma) \quad \text{for all } z \in B^\circ_{X'}.$$

Fix $z \in B^\circ_{X'}$, for $\gamma \in \imath_1(H_0)$ we have

$$\left| \sum_{k=0}^{\infty} \frac{z(\gamma)^k}{\|\gamma\|_0^k} \right| \leq \sum_{k=0}^{\infty} \|\imath'_1(z)\|^k_{H'_0} = \frac{1}{1 - \|\imath'_1(z)\|_{H'_0}},$$

it follows that the series $\sum_{k=0}^{\infty} \frac{z(\gamma)^k}{\|\gamma\|_0^k}$ converges to $\left[1 - z\left(\frac{\gamma}{\|\gamma\|_0}\right)\right]^{-1}$ in $L^\infty(W)$. Since $g \in L^1(W)$, by continuity, we can write

$$f(z) = \sum_{k=0}^{\infty} \int_X \frac{z(\gamma)^k}{\|\gamma\|_0^k} \overline{g(\gamma)}\, dW(\gamma).$$

So, if $f = \sum_k f_k$ is the Taylor series expansion of f at 0, then we have that

$$f_k(z) = \int_X \frac{z(\gamma)^k}{\|\gamma\|_0^k} \overline{g(\gamma)}\, dW(\gamma),$$

and we obtain the following upper bound:

$$|f_k(z)| = \left| \int_X \frac{z(\gamma)^k}{\|\gamma\|_0^k} \overline{g(\gamma)}\, dW(\gamma) \right| \leq \|g\|_1 \, \|\imath'_1(z)\|^k_{H'_0}.$$

As $\imath'_1(X')$ is dense in H'_0, since $|f_k(z)| \leq \|g\|_1 \, \|\imath'_1(z)\|^k_{H'_0}$, we conclude that there is a unique continuous extension of f_k to H'_0, and if F_k denotes this extension, then $F(x) = \sum_{k \geq 0} F_k(x)$ extends f. Since $\|F_k\| \leq \|g\|_1$, and consequently $\limsup_{k\to\infty} \sqrt[k]{\|F_k\|} \leq 1$, we obtain that $F \in \mathcal{H}_b(B^\circ_{H'_0})$. □

If we need to work on a Banach space X which is not a dual space, we can use the Aron-Berner extension of the functions to the bidual X'' of X. The Aron-Berner construction may be seen in [1], [3] and [12]. We need to recall only the following facts: given f, if $\sum_k f_k$ is its Taylor series expansion, then each $k-$homogeneous polynomial $f_k : X \to \mathbb{C}$ may be canonically extended to the bidual: $AB(f_k) : X'' \to \mathbb{C}$, and the Aron-Berner extension of f is defined to be $AB(f) = \sum_k AB(f_k)$.

Let X a Banach space and suppose that X' is separable. Given $f \in \mathcal{H}(B^\circ_X)$, if $AB(f)$ is w^*-representable on the ball of X'', then there exists $g \in L^1(W)$ such that

$$AB(f)(\zeta) = \int_{X'} \frac{1}{1 - \frac{\zeta(\gamma)}{\|\gamma\|_0}} \overline{g(\gamma)}\, dW(\gamma) \text{ for all } \zeta \in B^\circ_{X''}.$$

Let $J : X \to X''$ the canonical inclusion of X into its bidual, by definition:

$$f(z) = AB(f)(Jz) = \int_{X'} \frac{1}{1 - \frac{(Jz)(\gamma)}{\|\gamma\|_0}} \overline{g(\gamma)}\, dW(\gamma)$$
$$= \int_{X'} \frac{1}{1 - \frac{\gamma(z)}{\|\gamma\|_0}} \overline{g(\gamma)}\, dW(\gamma).$$

From this, we may introduce the following definition.

Definition 3.3. Suppose X has a separable dual. A function $f : B_X^\circ \to \mathbb{C}$ is said to be representable on the unit ball of X, if there exists $g \in L^1(W)$ such that

$$f(z) = \int_{X'} \frac{1}{1 - \frac{\gamma(z)}{\|\gamma\|_0}} \overline{g(\gamma)}\, dW(\gamma) \quad \text{for all } z \in B_X^\circ.$$

The set of representable functions on the ball of X will be denoted $\mathcal{R}(B_X^\circ)$.

Clearly, as we have seen, given any $f \in \mathcal{H}(B_X^\circ)$, if the holomorphic function $AB(f)$ is w^*−representable we have that $f \in \mathcal{R}(B_X^\circ)$. One may ask whether the converse holds. For answer it, we begin recalling the following result from [12].

Theorem 3.4. *If $Q \in \mathcal{P}({}^kX'')$ and $Q|_X = P$, then $Q = AB(P)$ if and only if*
(a) for each $x \in X, DQ(x)$ is w^*-continuous, and
(b) for each $z \in X''$ and $(x_\alpha) \subset X$ converging w^* to z, we have

$$DQ(z)(x_\alpha) \to DQ(z)(z).$$

Now, we are able to prove the following.

Theorem 3.5. *Suppose X has a separable dual. The holomorphic function $f : B_X^\circ \to \mathbb{C}$ is representable on the ball of X if and only if $AB(f)$ is w^*−representable on the ball of X''.*

Proof. $\Leftarrow$) From the considerations done before Definition 3.3, we already know that if $AB(f)$ is w^*−representable on the ball of X'', then f is representable on the ball of X.

$\Rightarrow$) It is sufficient to show that the Aron-Berner extension of the polynomial

$$f_k(z) = \int_{X'} \frac{\gamma(z)^k}{\|\gamma\|_0} \overline{g(\gamma)}\, dW(\gamma)$$

is

$$Q_k(\zeta) = \int_{X'} \frac{\zeta(\gamma)^k}{\|\gamma\|_0} \overline{g(\gamma)}\, dW(\gamma).$$

We will prove that $Q_k(\zeta)$ has w^*-sequentially continuous first-order differentials and, by a classical result, then it has w^*-continuous first-order differentials, therefore (a) and (b) are satisfied.

Fix $\zeta \in X''$,

$$DQ_k(\zeta)(\,\cdot\,) : X'' \to \mathbb{C}$$

$$DQ_k(\zeta)(\xi) = k\int_{X'} \frac{\xi(\gamma)}{\|\gamma\|_0}\,\frac{\zeta(\gamma)^{k-1}}{\|\gamma\|_0^{k-1}}\,\overline{g(\gamma)}\,dW(\gamma).$$

If $\{\xi_n\}_{n\in\mathbb{N}} \subset X''$ is a sequence such that $\xi_n \xrightarrow{w^*} \xi$, then $M = \sup_n \|\xi_n\|_{X''} < \infty$ and

$$\frac{\xi_n(\gamma)}{\|\gamma\|_0}\,\frac{\zeta(\gamma)^{k-1}}{\|\gamma\|_0^{k-1}}\,\overline{g(\gamma)} \underset{n\to\infty}{\longrightarrow} \frac{\xi(\gamma)}{\|\gamma\|_0}\,\frac{\zeta(\gamma)^{k-1}}{\|\gamma\|_0^{k-1}}\,\overline{g(\gamma)} \qquad \text{for } \gamma \in X'.$$

As $\|\xi\|_{X''} \le \liminf_{n\in\mathbb{N}} \|\xi_n\|_{X''} \le M$, we have

$$\left|\frac{\xi_n(\gamma)}{\|\gamma\|_0}\,\frac{\zeta(\gamma)^{k-1}}{\|\gamma\|_0^{k-1}}\,\overline{g(\gamma)}\right| \le M^k\,\|\zeta\|^{k-1}\left(\frac{\|\gamma\|}{\|\gamma\|_0}\right)^k |g| \le M^k\,\|\zeta\|^{k-1}\,|g| \in L^1(W).$$

Thus, applying the Lebesgue dominated convergence theorem,

$$k\int_{X'} \frac{\xi_n(\gamma)}{\|\gamma\|_0}\,\frac{\zeta(\gamma)^{k-1}}{\|\gamma\|_0^{k-1}}\,\overline{g(\gamma)}\,dW(\gamma) \underset{n\to\infty}{\longrightarrow} k\int_{X'} \frac{\xi(\gamma)}{\|\gamma\|_0}\,\frac{\zeta(\gamma)^{k-1}}{\|\gamma\|_0^{k-1}}\,\overline{g(\gamma)}\,dW(\gamma).$$

Hence, $DQ_k(\zeta)(\xi_n) \underset{n\to\infty}{\longrightarrow} DQ_k(\zeta)(\xi)$. From this, we conclude that

$$\begin{aligned} AB(f)(\zeta) &= \sum_{k=0}^{\infty} AB(f_k)(\zeta) = \sum_{k=0}^{\infty} Q_k(\zeta) = \sum_{k=0}^{\infty} \int_{X'} \frac{\zeta(\gamma)^k}{\|\gamma\|_0}\,\overline{g(\gamma)}\,dW(\gamma) \\ &= \int_{X'} \frac{1}{1-\frac{\zeta(\gamma)}{\|\gamma\|_0}}\,\overline{g(\gamma)}\,dW(\gamma). \end{aligned}$$

Therefore $AB(f) \in \mathcal{R}_*(B^\circ_{X''})$. □

4. L^p-representability vs. $\mathcal{R}(B^\circ_X)$

In this section we show the relation between L^p-representable functions and the space $\mathcal{R}_*(B^\circ_{X''})$. We begin by recalling the notion of L^p-representability, for more details we refer the reader to [9]. For this, suppose we are given a separable Banach space X and a Wiener measure on its Borel set constructed as in Theorem 2.1.

Definition 4.1. Let $p > 1$. A function $f \in \mathcal{H}(X')$ is called L^p-representable if

$$f(z) = \int_X e^{z(\gamma)}\overline{g(\gamma)}\,dW(\gamma) \text{ with } g \in L^p(W).$$

For different values of $p > 1$, the set of L^p-representable functions varies (see Theorem 4.1, [9]). However, the set of L^p-representable polynomials does not change as is stated in the next theorem.

Theorem 4.2 (Theorem 3.1, [9]**).** *If $f = \sum f_k$ is an L^p-representable function for $1 < p$, then f_k is L^2-representable for all $k \ge 0$.*

Let ν_n be the measure on $\mathbb{C}^n$ defined as a product of one-dimensional Gaussian measures on $\mathbb{C}$, each of them with mean 0 and variance $\lambda_k^2 > 0$ for $k = 1, \dots, n$. This measure is absolutely continuous respect to Lebesgue measure and its density function is

$$\delta_n(\omega_1, \omega_2, \dots, \omega_n) = \frac{1}{\pi^n \lambda_1^2 \lambda_2^2 \cdots \lambda_n^2}\, e^{-\sum_{k=1}^n |\omega_k|^2/\lambda_k^2}.$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ we will employ the standard notations: $|\alpha| = \alpha_1 + \cdots + \alpha_n,\ \alpha! = \alpha_1! \cdots \alpha_n!$ and, given $\omega \in \mathbb{C}^n$, we write ω^α for $\omega_1^{\alpha_1} \cdots \omega_n^{\alpha_n}$. A norm $N : \mathbb{C}^n \to \mathbb{R}$ is said to be absolute if

$$N(w_1, w_2, \dots, w_n) = N(|w_1|, |w_2|, \dots, |w_n|) \ \text{ for all } (\omega_1, \dots, \omega_n) \in \mathbb{C}^n.$$

The proof of the following Lemma is standard, so we leave it to the reader.

Lemma 4.3 (Lema 3.3.2, [10]**).** *If N is any absolute norm, then*

$$\int_{\mathbb{C}^n} \omega^\alpha\, \overline{\omega}^\beta\, d\nu_n(\omega) = \int_{\mathbb{C}^n} \omega^\alpha\, \overline{\omega}^\beta N(\omega)^{|\alpha|-|\beta|}\, d\nu_n(\omega) = \delta_{\alpha\beta}\, \alpha! \prod_{k=1}^n \lambda_k^{2\alpha_k}.$$

Given the sequence $\{z_n\}_{n\in\mathbb{N}}$ from the end of the proof of Theorem 2.1 and a multi-index $\alpha,\ |\alpha| = k$, to simplify notation, we write $z^\alpha(\gamma) := \prod_{n=1}^\infty z_n(\gamma)^{\alpha_n}$. From Lemma 2.4 in [9], we know that the following equality holds:

$$\overline{\operatorname{span}\{e^{z(\gamma)}\}_{z\in X'}}^{\|\cdot\|_2} = \overline{\operatorname{span}\{z^\alpha(\gamma)\}_{z\in X',\ |\alpha|=k,\ k\geq 0}}^{\|\cdot\|_2}$$

where $\|\cdot\|_2$ denotes the norm on $L^2(W)$.

We are ready to show a link between these two notions of representability. From now on, we assume that X is a Banach space which has a separable dual.

Proposition 4.4. *Let $P \in \mathcal{P}(^k X'')$. Assume that P is L^p-representable for some $p > 1$, then P is w^*-representable on the ball of X''.*

Proof. According to Theorem 4.2, it follows that any L^p-representable polynomial is also L^2-representable, so it is sufficient to prove that those polynomials which are L^2-represented by functions in $\operatorname{span}\big\{\{z^\alpha/\sqrt{\alpha!}\}_{|\alpha|=k}\big\}$ are also w^*-representable on the ball of X''. Given $l \in \mathbb{N}_0$, and a multi-index α with $|\alpha| = l$, we let

$$p_\alpha(z) = \int_{X'} e^{z(\gamma)}\, \frac{\overline{z^\alpha(\gamma)}}{\sqrt{\alpha!}}\, dW(\gamma) = \int_{X'} \sum_{k=0}^\infty \frac{z(\gamma)^k}{k!}\, \frac{\overline{z^\alpha(\gamma)}}{\sqrt{\alpha!}}\, dW(\gamma)$$

$$= \sum_{k=0}^\infty \int_{X'} \frac{z(\gamma)^k}{k!}\, \frac{\overline{z^\alpha(\gamma)}}{\sqrt{\alpha!}}\, dW(\gamma) = \int_{X'} \frac{z(\gamma)^l}{l!}\, \frac{\overline{z^\alpha(\gamma)}}{\sqrt{\alpha!}}\, dW(\gamma).$$

We claim that

$$\int_{X'} \frac{z(\gamma)^k}{\|\gamma\|_0^k}\, \frac{\|\gamma\|_0^l}{k!}\, \frac{\overline{z^\alpha(\gamma)}}{\sqrt{\alpha!}}\, dW(\gamma) = \delta_{kl}\, p_\alpha(z) = \int_{X'} \frac{z(\gamma)^k}{\|\gamma\|_0^k}\, \frac{\|\gamma\|_0^l}{l!}\, \frac{\overline{z^\alpha(\gamma)}}{\sqrt{\alpha!}}\, dW(\gamma),$$

and then, summing over $k \in \mathbb{N}_0$, we have that

$$p_\alpha(z) = \int_{X'} \frac{1}{1 - \frac{z(\gamma)}{\|\gamma\|_0}} \frac{\|\gamma\|_0^l \, \overline{z^\alpha(\gamma)}}{l! \sqrt{\alpha!}} \, dW(\gamma).$$

The proof of our claim stems from considering the sequence of orthogonal projectors $\{P_n\}_{n\in\mathbb{N}}$ onto the subspaces $S_n = \left[\frac{\imath_0(e_1)}{\lambda_1}, \frac{\imath_0(e_2)}{\lambda_2}, \ldots, \frac{\imath_0(e_n)}{\lambda_n}\right]$ as in Theorem 2.1, and applying the Lebesgue dominated convergence theorem,

$$\frac{z(P_n\gamma)^k}{\|P_n\gamma\|_0^k} \, \|P_n\gamma\|_0^l \, \frac{\overline{z^\alpha(P_n\gamma)}}{\sqrt{\alpha!}} \underset{n\to\infty}{\longrightarrow} \frac{z(\gamma)^k}{\|\gamma\|_0^k} \, \|\gamma\|_0^l \, \frac{\overline{z^\alpha(\gamma)}}{\sqrt{\alpha!}} \quad \text{for all } \gamma \in H_0,$$

and so almost everywhere. Also, we have the bound

$$\begin{aligned} \left| \frac{z(P_n\gamma)^k}{\|P_n\gamma\|_0^k} \, \|P_n\gamma\|_0^l \, \frac{\overline{z^\alpha(P_n\gamma)}}{\sqrt{\alpha!}} \right| &\le \|z\|^k \, \frac{\|z^\alpha\|_{\mathcal{P}(^l X)}}{\sqrt{\alpha!}} \, \|P_n\gamma\|_0^{2l} \\ &\le \|z\|^k \, \frac{\|z^\alpha\|_{\mathcal{P}(^l X)}}{\sqrt{\alpha!}} \, \|\gamma\|_0^{2l} \end{aligned}$$

which is integrable by Fernique's theorem. Therefore,

$$\int_{X'} \frac{z(\gamma)^k}{\|\gamma\|_0^k} \, \|\gamma\|_0^l \, \frac{\overline{z^\alpha(\gamma)}}{l! \sqrt{\alpha!}} \, dW(\gamma) = \lim_{n\to\infty} \int_{X'} \frac{z(P_n\gamma)^k}{\|P_n\gamma\|_0^k} \, \|P_n\gamma\|_0^l \, \frac{\overline{z^\alpha(P_n\gamma)}}{l! \sqrt{\alpha!}} \, dW(\gamma).$$

Our next objective is to evaluate these integrals. Since they are cylinder functions, we can compute them on the finite dimensional subspaces $P_n H_0$ with respect to the measures induced by W. These measures, noted by $P_n W$, are defined on Borel sets $\Delta \subset P_n H_0$, as $P_n W(\Delta) = W(P_n^{-1}(\Delta))$.

Letting $v_r = \lambda_r^{-1} \imath_0(e_r)$ for $1 \le r \le n$, we can fix a basis $\{v_1, v_2, \ldots, v_n\}$ of S_n. Coordinates are given by its dual basis, so $P_n W$ is the product of n one-dimensional Gaussian measure on $\mathbb{C}$, each of them with mean 0 and variance $\|\lambda_r \, \imath_0'(\imath_1'(z_r))\|_{H'}^2 = \|\lambda_r \, e_r'\|_{H'}^2 = \lambda_r^2$. Hence, if $(w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$ are the coordinates of $\gamma \in S_n$, the density function of $P_n W$ is

$$\delta_n(w_1, w_2, \ldots, w_n) = \frac{1}{\pi^n \, \lambda_1^2 \lambda_2^2 \cdots \lambda_n^2} \, e^{-\sum_{r=0}^n |w_r|^2/\lambda_r^2}.$$

We may now integrate:

$$\int_{X'} \frac{z(P_n\gamma)^k}{\|P_n\gamma\|_0^k} \, \|P_n\gamma\|_0^l \, \frac{\overline{z^\alpha(P_n\gamma)}}{l! \sqrt{\alpha!}} \, dW(\gamma) = \int_{P_n H_0} \frac{z(\gamma)^k}{\|\gamma\|_0^k} \, \|\gamma\|_0^l \, \frac{\overline{z^\alpha(\gamma)}}{l! \sqrt{\alpha!}} \, dP_n W(\gamma).$$

Writing the sub-integral expression in coordinates:

$$\frac{\left(\displaystyle\sum_{r=0}^n w_r \, z(v_r)\right)^k}{\left(\displaystyle\sum_{r=0}^k |w_r|^2\right)^{k/2}} \left(\sum_{r=0}^k |w_r|^2\right)^{l/2} \prod_{s=1}^\infty \overline{z_s \left(\sum_{r=0}^n w_r \, v_r\right)^{\alpha_s}} \, \frac{\delta_n(w_1, w_2, \ldots, w_n)}{\sqrt{\alpha!}}.$$

Since $z_s(v_r) = \delta_{rs}/\lambda_s$, using the multinomial formula, we obtain

$$\left[\sum_{|\beta|=k} \frac{k!}{\beta!}\left(\prod_{r=1}^{n} w_r^{\beta_r} z(v_r)^{\beta_r}\right)\right]\left[\prod_{s=1}^{n}\overline{\left[\frac{w_s}{\lambda_s}\right]^{\alpha_s}}\right]\left(\sum_{r=0}^{k}|w_r|^2\right)^{\frac{l-k}{2}} \frac{\delta_n(w_1,\ldots,w_n)}{\sqrt{\alpha!}}.$$

Summarizing, we must compute

$$\sum_{|\beta|=k}\frac{k!}{l!\sqrt{\alpha!}}\prod_{r=1}^{n}\frac{z(v_r)^{\beta_r}}{\lambda_r^{\alpha_r}\,\beta_r!}\int_{\mathbb{C}^n} w^\beta\,\overline{w}^\alpha\left[\sum_{r=0}^{k}|w_r|^2\right]^{\frac{l-k}{2}}\delta_n(w_1,\ldots,w_n)\,dw_1\,\ldots dw_n.$$

From Lemma 4.3, we have

$$\int_{\mathbb{C}^n} w^\beta\,\overline{w}^\alpha\left(\sum_{r=0}^{k}|w_r|^2\right)^{\frac{l-k}{2}}\delta_n(w_1,\ldots,w_n)\,dw_1\,\ldots dw_n = \delta_{\alpha\beta}\,\alpha!\prod_{r=1}^{n}\lambda_r^{2\alpha_r}.$$

Hence, for $|\alpha| = l$ and $k \neq l$, we have $\delta_{\alpha\beta} = 0$ for all multi-index β such that $|\beta| = k$. From this we conclude that

$$\frac{1}{k!}\int_{X'}\frac{z(\gamma)^k}{\|\gamma\|_0^k}\,\|\gamma\|_0^l\,\frac{\overline{z^\alpha(\gamma)}}{\sqrt{\alpha!}}\,dW(\gamma) = \delta_{kl}\,\frac{1}{k!}\int_{X'} z(\gamma)^k\,\frac{\overline{z^\alpha(\gamma)}}{\sqrt{\alpha!}}\,dW(\gamma) = \delta_{kl}\,p_\alpha(z),$$

therefore, summing over $k \in \mathbb{N}_0$,

$$p_\alpha(z) = \int_{X'}\frac{1}{1-\frac{z(\gamma)}{\|\gamma\|_0}}\,\frac{\|\gamma\|_0^l\,\overline{z^\alpha(\gamma)}}{l!\,\sqrt{\alpha!}}\,dW(\gamma).$$

Since $\frac{\|\gamma\|_0^l}{l!}$ and $\frac{z^\alpha(\gamma)}{\sqrt{\alpha!}}$ are L^2-functions, their product is in $L^1(W)$ and it follows that p_α is w^*-representable on the ball of X''. □

Finally, we are able to prove the analogous theorem with holomorphic functions replacing $k-$homogeneous polynomials.

Theorem 4.5. *Let $f \in \mathcal{H}(X'')$. Assume that f is L^p-representable for some $p > 1$, then f is w^*-representable on the ball of X''.*

Proof. Given $p > 1$, we can choose $\rho > 0$ such that for any L^p-representable function f, there exists an L^2-representable function v satisfying (Theorem 3.2, [9])

$$f(z) = v(\rho z) \quad \text{for all } z \in X''.$$

Since v is L^2-representable, there exist a sequence $\{g_k\}_{k\in\mathbb{N}_0} \subset L^2(W)$ such that

$$v_k(z) = \int_{X'} e^{z(\gamma)}\,\overline{g_k(\gamma)}\,dW(\gamma), \quad v(z) = \int_{X'} e^{z(\gamma)}\,\overline{\sum_{k\geq 0} g_k(\gamma)}\,dW(\gamma),$$

with $\sum_{k\geq 0} g_k$ convergent in $L^2(W)$. Using Lemma 2.4 from [9], we know that for $k \in \mathbb{N}_0$, there exist complex numbers $\{a_\alpha\}_{|\alpha|=k}$ with $\sum_{|\alpha|=k}|a_\alpha|^2 < \infty$, such that $g_k(\cdot) = \sum_{|\alpha|=k} a_\alpha \frac{z^\alpha(\cdot)}{\sqrt{\alpha!}}$, where the convergence is in $L^2(W)$. Therefore, since $e^{z(\gamma)} \in L^2(W)$,

$$v_k(z) = \int_{X'} e^{z(\gamma)} \overline{\sum_{|\alpha|=k} a_\alpha \frac{z^\alpha(\gamma)}{\sqrt{\alpha!}}} \, dW(\gamma) = \sum_{|\alpha|=k} \overline{a_\alpha} \int_{X'} e^{z(\gamma)} \overline{\frac{z^\alpha(\gamma)}{\sqrt{\alpha!}}} \, dW(\gamma).$$

According to Theorem 4.4, we can write:

$$v_k(z) = \sum_{|\alpha|=k} \overline{a_\alpha} \int_{X'} \frac{1}{1 - \frac{z(\gamma)}{\|\gamma\|_0}} \frac{\|\gamma\|_0^k}{k!} \overline{\frac{z^\alpha(\gamma)}{\sqrt{\alpha!}}} \, dW(\gamma).$$

Since the function $\frac{1}{1 - \frac{z(\gamma)}{\|\gamma\|_0}} \frac{\|\gamma\|_0^k}{k!} \in L^2(W)$, we can now proceed analogously and write:

$$\begin{aligned} v_k(z) &= \sum_{|\alpha|=k} \overline{a_\alpha} \int_{X'} \frac{1}{1 - \frac{z(\gamma)}{\|\gamma\|_0}} \frac{\|\gamma\|_0^k}{k!} \overline{\frac{z^\alpha(\gamma)}{\sqrt{\alpha!}}} \, dW(\gamma) \\ &= \int_{X'} \frac{1}{1 - \frac{z(\gamma)}{\|\gamma\|_0}} \frac{\|\gamma\|_0^k}{k!} \left[\sum_{|\alpha|=k} \overline{a_\alpha} \overline{\frac{z^\alpha(\gamma)}{\sqrt{\alpha!}}} \right] dW(\gamma) \\ &= \int_{X'} \frac{1}{1 - \frac{z(\gamma)}{\|\gamma\|_0}} \frac{\|\gamma\|_0^k}{k!} \overline{g_k(\gamma)} \, dW(\gamma). \end{aligned}$$

From this, we conclude that

$$f(z) = \sum_{k=0}^{\infty} v_k(\rho z) = \sum_{k=0}^{\infty} \rho^k \, v_k(z) = \sum_{k=0}^{\infty} \int_{X'} \frac{1}{1 - \frac{z(\gamma)}{\|\gamma\|_0}} \frac{\rho^k \, \|\gamma\|_0^k}{k!} \overline{g_k(\gamma)} \, dW(\gamma).$$

Our next claim is that if we define the sequence $h_N(\gamma) = \sum_{k=0}^{N} \frac{\rho^k \, \|\gamma\|_0^k}{k!} \overline{g_k(\gamma)}$, then $(h_N)_N$ is a Cauchy sequence in $L^1(W)$. Fix $\varepsilon > 0$, we need to estimate $\|h_M - h_N\|_1$. Suppose $M > N \geq 0$,

$$\begin{aligned} \|h_M - h_N\|_1 &= \int_{X'} \left| \sum_{k=N+1}^{M} \frac{\rho^k \, \|\gamma\|_0^k}{k!} \overline{g_k(\gamma)} \right| dW(\gamma) \\ &\leq \int_{X'} \sum_{k=N+1}^{M} \frac{\rho^k \, \|\gamma\|_0^k}{k!} \, |\, g_k(\gamma) \,| \, dW(\gamma) \\ &\leq \frac{1}{2} \int_{X'} \sum_{k=N+1}^{M} \left(\frac{\rho^{2k} \, \|\gamma\|_0^{2k}}{k!^2} + |\, g_k(\gamma) \,|^2 \right) dW(\gamma) \\ &\leq \frac{1}{2} \int_{X'} \sum_{k=N+1}^{M} \frac{\rho^{2k} \, \|\gamma\|_0^{2k}}{k!^2} dW(\gamma) + \frac{1}{2} \left\| \sum_{k=N+1}^{M} g_k \right\|_2^2 \end{aligned}$$

$$\leq \frac{1}{2}\int_{X'}\sum_{k=N+1}^{M}\frac{\|\gamma\|_0^k}{k!}e^{\rho^2\|\gamma\|_0}\,dW(\gamma)+\frac{1}{2}\left\|\sum_{k=N+1}^{M}g_k\right\|_2^2$$

$$\leq \frac{1}{2}\left\|e^{\rho^2\|\gamma\|_0}\right\|_2\left\|\sum_{k=N+1}^{M}\frac{\|\gamma\|_0^k}{k!}\right\|_2+\frac{1}{2}\left\|\sum_{k=N+1}^{M}g_k\right\|_2^2.$$

From Fernique's theorem, we know that $\left\|e^{\rho^2\|\gamma\|_0}\right\|_2<\infty$, and since we have $\sum_{k=0}^{\infty}\frac{\|\gamma\|_0^k}{k!}\to e^{\|\gamma\|_0}$ in $L^2(W)$, there exists $K_1>0$ such that the first term is smaller than $\varepsilon/2$ for any $M,N>K_1$. Also, from the L^2-convergence of $\sum_{k\geq 0}g_k$, we conclude that there exists $K_2>0$ such that the second term is smaller than $\varepsilon/2$ for $M,N>K_2$. Hence, for $K>\max\{K_1,K_2\}$, we have $\|h_M-h_N\|_1<\varepsilon$ for all $M,N>K$. By completeness, there exists $h\in L^1(W)$ such that

$$f(z)=\sum_{k=0}^{\infty}\int_{X'}\frac{1}{1-\frac{z(\gamma)}{\|\gamma\|_0}}\frac{\rho^k\,\|\gamma\|_0^k}{k!}\overline{g_k(\gamma)}\,dW(\gamma)=\int_{X'}\frac{1}{1-\frac{z(\gamma)}{\|\gamma\|_0}}\overline{h(\gamma)}\,dW(\gamma)$$

and so f is w^*-representable on the ball of X''. □

Theorem 4.6. (Theorem 5.1, [9]) *Let X a Banach space. If X' is separable, then the function $f\in\mathcal{H}(X)$ has an integral representation*

$$f(z)=\int_{X'}e^{\gamma(z)}\overline{g(\gamma)}\;dW(\gamma)\quad \text{for } g\in L^p(W),\ 1<p$$

if and only if $AB(f)$ is L^p-representable.

Combining these results, we have the following corollary.

Corollary 4.7. *Let X a Banach space which has a separable dual. If $f\in\mathcal{H}(X)$ has an integral representation*

$$f(z)=\int_{X'}e^{\gamma(z)}\overline{g(\gamma)}\;dW(\gamma)\quad \text{for } g\in L^p(W)\quad\text{and } 1<p,$$

then $f\in\mathcal{R}(B_X^\circ)$.

Proof. If f has such a representation, then $AB(f)$ is L^p-representable (Theorem 4.6), from Theorem 4.5 we deduce that $AB(f)$ is w^*-representable on the ball of X'' and, from Theorem 3.5, we conclude that $f\in\mathcal{R}(B_X^\circ)$. □

5. $\mathcal{R}(B_X^\circ)$ vs. Integral holomorphic functions

This section is devoted to show that those functions which are representable on the unit ball of a Banach space are integral holomorphic functions in the sense of [2]. The space of $k-$homogeneous integral polynomials over X is denoted by $P_I({}^kX)$.

Theorem 5.1. *If $f=\sum_{k=0}^{\infty}f_k$ is a representable function on the unit ball of X, then $f_k\in\mathcal{P}_I({}^kX)$ for all $k\geq 0$.*

Proof. Let $g \in L^1(W)$ such that

$$f(z) = \int_{X'} \frac{1}{1 - \frac{\gamma(z)}{\|\gamma\|_0}} \overline{g(\gamma)}\, dW(\gamma).$$

According to the proof of Proposition 3.2, we have

$$f(z) = \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} \int_{X'} \left(\frac{\gamma(z)}{\|\gamma\|_0}\right)^k \overline{g(\gamma)}\, dW(\gamma).$$

Let us consider the linear functional

$$\mathcal{T} : \mathcal{C}(B_{X'}, w^*) \to \mathbb{C}$$

$$\mathcal{T}(\varphi) = \int_{X'} \varphi\left(\frac{\gamma}{\|\gamma\|_0}\right) \overline{g(\gamma)}\, dW(\gamma).$$

Since $\left\|\gamma \|\gamma\|_0^{-1}\right\| \leq 1$ a.e., the function $\varphi\left(\gamma \|\gamma\|_0^{-1}\right)$ is well defined. Moreover, it is Borel measurable because φ is w^*-continuous . Also, we have the bounds:

$$|\mathcal{T}(\varphi)| \leq \int_{X'} \left|\varphi\left(\frac{\gamma}{\|\gamma\|_0}\right)\right| \left|\overline{g(\gamma)}\right| dW(\gamma) \leq \|\varphi\|_\infty \|g\|_1.$$

From this, we conclude that $\mathcal{T}$ is continuous and there exists a regular measure μ defined on Borel sets of $(B_{X'}, w^*)$ such that

$$\mathcal{T}(\varphi) = \int_{B_{X'}} \varphi(\gamma)\, d\mu(\gamma).$$

For $z \in X$ and $k \in \mathbb{N}_0$, we consider the w^*-continuous function $\widehat{z}(\gamma)^k = \gamma(z)^k$. Let us compute

$$\mathcal{T}(\widehat{z}(\,\cdot\,)^k) = \int_{B_{X'}} \widehat{z}(\gamma)^k\, d\mu(\gamma) = \int_{B_{X'}} \gamma(z)^k\, d\mu(\gamma).$$

On the other hand,

$$\mathcal{T}(\widehat{z}(\,\cdot\,)^k) = \int_{X'} \left[\widehat{z}\left(\frac{\gamma}{\|\gamma\|_0}\right)\right]^k \overline{g(\gamma)}\, dW(\gamma)$$

$$= \int_{X'} \left(\frac{\gamma(z)}{\|\gamma\|_0}\right)^k \overline{g(\gamma)}\, dW(\gamma) = f_k(z).$$

From this, we conclude that there exists a regular Borel measure μ defined on $(B_{X'}, w^*)$ such that

$$f_k(z) = \int_{B_{X'}} \gamma(z)^k\, d\mu(\gamma) \quad \text{for all } z \in X,$$

and so $f_k \in \mathcal{P}_I({}^kX)$. □

Corollary 5.2. *If $f = \sum_{k=0}^{\infty} f_k$ is a representable function on the unit ball of X, then f is an integral holomorphic function in the sense of* [2].

Proof. In the last theorem the measure μ is not dependent of $k \in \mathbb{N}_0$. Since we can use it to represent all the polynomials involved in the Taylor series expansion of f, and the geometric series is normally convergent for all $z \in B_X^\circ$,

$$f(z) = \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} \int_{B_{X'}} \gamma(z)^k \, d\mu(\gamma) = \int_{B_{X'}} \frac{1}{1-\gamma(z)} \, d\mu(\gamma). \qquad \square$$

Remark 5.3. Unfortunately, unless we are working in a finite dimensional setting, given a Banach space X with a separable dual there are many integral holomorphic functions over B_X° which are not representable on the unit ball of X. For example, it is well known that any linear functional is an integral polynomial but we will show that there exist linear functionals which are not in $\mathcal{R}(B_X^\circ)$.

Given $\gamma_0 \in X'$, we know from Theorem 3.5 that $\gamma_0 \in \mathcal{R}(B_X^\circ)$ if and only if $AB(\gamma_0) \in \mathcal{R}_*(B_{X''}^\circ)$. Suppose that γ_0 is representable on the unit ball of X, then we conclude from Proposition 3.2 that there exist $\Phi : H_0' \to \mathbb{C}$ such that

$$\Phi \circ \imath_1'(z) = AB(\gamma_0)(z) = \widehat{\gamma_0}(z) \quad \text{ for all } z \in X'',$$

in other words, $\imath_1''(\Phi) = \widehat{\gamma_0}$. Let us check the following commutative diagram, where J_{H_0} and $J_{X'}$ are the canonical inclusions into the biduals,

$$\begin{array}{ccc} H_0 & \xrightarrow{\ \imath_1\ } & X' \\ {\scriptstyle J_{H_0}}\downarrow & & \downarrow{\scriptstyle J_{X'}} \\ H_0'' & \xrightarrow[\ \imath_1''\ ]{} & X''' \end{array}$$

Since H_0 is reflexive and $\imath_1$ is not surjective,

$$\imath_1''(H_0'') = \imath_1'' \circ J_{H_0}(H_0) = J_{X'} \circ \imath_1(H_0) \subsetneq J_{X'}(X').$$

From this, we deduce that there are many linear functionals $\gamma \in X'$ for which $\widehat{\gamma} \notin \imath_1''(H_0')$. Therefore, there are many linear functionals $\gamma \in X'$ which are not representable on the unit ball of X.

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