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THE GENERAL THEORY OF ELASTIC STABILITY AT THE END OF THE 19th CENTURY*

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This paper reviews the research on the theory of elastic stability published at the end of the 19th century, with emphasis on the work by G. H. Bryan in Cambridge. The state of the studies on structural stability previous to Bryan is reviewed, and two lines of work are identified: one is a general stability of rigid bodies and the other is a collection of case studies of elastic stability. Bryan's theory is discussed next, presenting his arguments based on first energy principles, which led him to strong conclusions. The importance of the word "general" and the idea of having solved the problem in each case are explained. The impact of the contributions made by Bryan, together with the critiques that this generated, is also discussed.

Keywords: Elastic stability; energy formulation; general theory; history of stability.

1. Introduction

Literature on the stability of elastic structures has grown since the early 1970s, together with the development of a school considering discrete structural systems at University College London, and another one on continuous systems headed by researchers at Harvard University. Both schools were rooted in the work of Koiter in The Netherlands.[1] However, the roots of the work of Koiter himself tend to be forgotten and will soon vanish unless an attempt is made to recover those early sources and highlight their importance.[2] This paper traces the origins of the general theory to the contributions in the last part of the 19th century.

The development of a general theory in the field of elastic stability seems to have been a concern shared by several authors. The title of the first book by Thompson and Hunt[3] was precisely "*A General Theory of Elastic Stability*," and this was a claim made by several other researchers at different times (see e.g., Southwell[4]).

*An earlier version of this paper was presented at the ICSSD meeting to honor the 60th anniversary of Prof. J. N. Reddy, Orlando, FL, June 2005.

The word "general" here stands in opposition to a collection of solutions to specific problems without having a general framework of analysis. The reader may find that this theory started in 1973 with the book mentioned above, or perhaps a couple of decades earlier. This work attempts to show that the origins of the general theory were established by a researcher at Cambridge University, George H. Bryan, in 1888.

2. The State of the Theory of Stability Before Bryan

There were two venues that enriched the solution of stability problem in the second half of the 19th century: theoretical studies on the stability of equilibrium of rigid bodies and stability considerations for specific elastic structures.

The first venue can be exemplified by the work of Larmor,[5] who studied a solid body resting on a rigid surface, where the problem can be solved by purely geometric considerations. Larmor emphasized the critical equilibrium condition. His problem was the identification of a "curve of stability" in a floating body. No elasticity was involved in this approach.

An illustration of the second venue is provided by the work of A. G. Greenhill, who carried out analytical research on the stability of elastic columns within the Euler tradition. In his 1881 paper, Greenhill considered the equilibrium and stability of a clamped-free column under its own weight, in order to find the length in which the structure "becomes unstable and flexure begins" (Greenhill,[6] p. 5). Assuming a circular cross section, Greenhill applied the differential equations to investigate the stability of a tree (a problem suggested to him by Dr. Asa Gray, Professor of Botany at Harvard University in Cambridge, Massachusetts). His results showed that for a solid cylinder of pine with 6 in. in diameter, the critical height was 89.45 ft, but for a conical pole with 20 in. in diameter, the critical length would be 300 ft. Evidence was available at his time of such trees growing up to 221 ft.

A second paper was read by Greenhill at the Institution of Mechanical Engineers (IME), in which he considered a pole under both axial load and torsion at the end, a subject of interest for the design of large vessels. The actual formulation of the problem was reported in an appendix, whereas the main text contained the main equation and worked examples. The discussion that follows the paper covers the next 16 pages in the journal, and provides a good illustration of the spirit from the times of the engineering societies in England. One of the examples worked out by the author had shown that a pole of 22 in. in diameter could have a height of 371 ft before becoming unstable under the considered loading system. This produced some concern for the practicing engineers present at the meeting, who suggested that the formulas should be applied "to a few actual ordinary examples of broken propeller shafts, and calculate whether they ought to have broken or not" (Greenhill,[7] p. 211). The results showed that the most important effect was due to the axial load (governed by Euler's equation), whereas the twist could be neglected in most practical applications. Professor W. C. Unwin raised the point that practicing engineers would use Euler's formula with a factor of safety of five, although such a factor was poorly determined. Other comments included "Professor Greenhill had apologized to some extent for bringing so theoretical a paper before them; but he thought there was some use in it, if only to have it discussed; for unfortunately theoretical utterances were apt to lead people rather astray, unless their bearing upon practice was fully considered" (Greenhill,[7] p. 213). The author closed his intervention saying that "he felt much flattered that practical men of eminence should have offered such valuable remarks on the theory which he had put forward" (Greenhill,[7] p. 224). The final remarks by the president of the IME were that "as a general rule the papers put before the Institution were intensely practical; but it was certainly very desirable that practice should be refined by theory, and that theory should be strengthened by practice" (Greenhill,[7] p. 225).

The episode attempts to illustrate the spirit of the times before Bryan made his contribution, at least how the engineering community undervalued the significance of theoretical work. The work of A. G. Greenhill is not mentioned in the book by Timoshenko,[8] although his contributions are crucial to reconstruct the history of structural stability.

3. Life of Bryan (Not the Monty Python Movie)

George Hartley Bryan was born at Cambridge, England, on March 1, 1864 (Fig. 1). He was the son of a professor who died when he was still very young. He lived in several countries and learned French, Italian, and German, before settling in England to start his academic career. At 22, he was outstanding in mathematics and was awarded a fellowship at Peterhouse, one of the Cambridge colleges, between

Fig. 1. George Hartley Bryan. Reproduced with permission of the Royal Society, London.

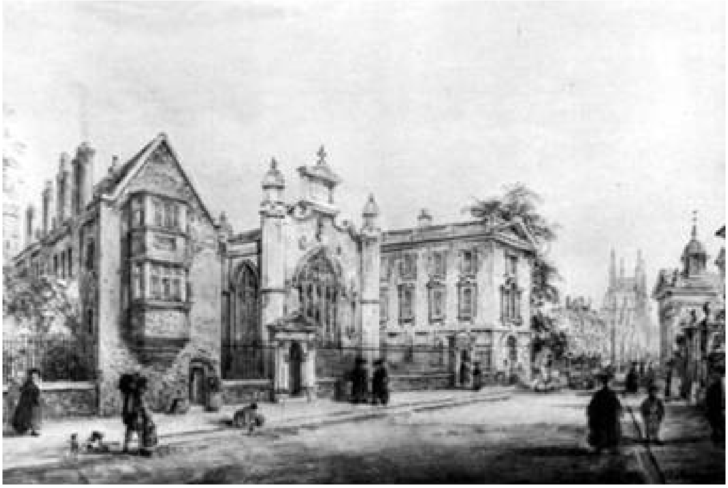

Fig. 2. A painting of the mid-19th century showing Peterhouse, the college in Cambridge where Bryan worked and produced his theory of elastic stability.

1889 and 1895 (Fig. 2). During this period at Peterhouse, he attempted the development of a new theory of stability, a subject that had not been explored by others before him. His obituary described him as a "very eccentric individual with a keenness for 'unsolved problems' in dynamics and hydrodynamics."[9] The spirit of researchers at that time was that if an apple had been bitten before, then it was not a good apple. Naturally, Bryan found this apple intact, as if it were waiting to be eaten.

In 1895 (at age 31), Bryan obtained a position at University College of North Wales as Professor of Pure and Applied Mathematics, where he worked for 31 years until his retirement in 1926. As he moved from Cambridge to Bangor in Wales, his interests shifted from the stability of elastic structures to the stability of flight. This was a timely move, because the dynamics of flight (including the longitudinal and lateral stability of flights) was an emerging field much needed in the aviation industry. Bryan became an authority in this field, and his book[10] soon became the standard reference.

Bryan received several awards during his career, including being elected fellow of the Royal Society in 1895, and president of the Mathematical Association. He died in 1928, at age 64.

4. A Summary of Bryan's Contribution

Bryan published three papers about the new theory of stability. In his first paper,[11] published when he was only 22 years old, he considered Kirchhoff's postulate about the existence of a unique solution of problems in elasticity ("... there is one and only one state of strain in which the body can be in equilibrium, and that equilibrium is essentially stable...").[12] He contrasted this with the early findings by Euler on the loss of stability of a thin shaft under an axial load and with more recent work on instability of a shaft by Greenhill,[6,7] in which more than one equilibrium solution seemed possible.

The question posed by Bryan was: How can uniqueness and instability be both true and part of one theory? He stated his goal in this form: "It therefore appeared to me that it would be worth while to give a general investigation of the circumstances under which an elastic system can be in unstable equilibrium for other than rigid body displacements of the various bodies forming the system" (Bryan,[11] p. 199). The first paper had two main parts, one for three-dimensional elastic bodies and another part devoted to wires, plates, and shells.

Our author developed the general theory using the total potential energy of the system, in much the same way as we use it nowadays: Equilibrium is given by the first variation and the system becomes unstable if the second variation is negative for some variations. A few years later, Love pointed out a shortcoming in Kirchhoff developments: "Kirchhoff's proof depended on the variation of the energy, and he considered only first variations. In the cases we have mentioned, a small displacement really changes the character of the surface-tractions, making it necessary to consider second variations" (Love,[13] p. 23).

Following the notation used in his times, the elastic potential Φ was written in terms of the strain components (e, f, g, a, b, c) in the form

$$\Phi = \frac{1}{2}(m+n)(e+f+g)^2 + \frac{1}{2}n(a^2+b^2+c^2-4\,fg-4\,ge-4\,ef) \tag{1}$$

where m and n are the elastic constants of Thomson and Tait, respectively. The "whole potential energy," W, of the isotropic body would be

$$W = \iiint \Phi\,dx\,dy\,dz + \iiint \rho V\,dx\,dy\,dz + \int \Psi\,dS, \tag{2}$$

where ρ is the density, V stands for the potential of the body forces, and Ψ for the surface tractions acting on the boundary S. The condition of equilibrium was found by allowing small variations in the displacements $(\delta u, \delta v, \delta w)$, leading to the condition

$$\delta W = 0. \tag{3}$$

Next, Bryan used the condition that equilibrium would be stable if W was a true minimum, and unstable if W was a maximum or minimax. For that, he derived the second variation of the energy, $\delta^2 W$, and identified that equilibrium would be unstable only if

$$\delta^2 W = \iiint \delta^2\Phi dx\,dy\,dz + \iiint \rho\,\delta^2 V\,dx\,dy\,dz + \int \delta^2\Psi\,dS < 0, \tag{4}$$

for some variation in the displacements.

Bryan retained quadratic terms in the strain energy Φ and found that the resulting expression of $\delta^2\Phi$ would always be positive because Φ was a homogeneous quadratic function of the strain components. Notice that he never explicitly defined the strain−displacement relations of his problem (i.e., the kinematic equations), and he only speculated about the relative values of strains and displacements.

Therefore, he inferred that the only possibility of having a negative second variation of W was if the load terms became negative and larger than the strain energy terms. For this to occur, he identified that the load potential depends on variations of displacements, whereas the strain energy depends on variations of strains. For the second and third terms in Eq. (4) to be larger than the first, "the displacement must in general be such that the strain variations … are infinitely small compared with the displacement variations" (Bryan,[11] p. 201). His argument leads to the conclusion that under small strains, equilibrium of solid bodies should always be stable, provided rigid body motions are not allowed.

In the second part, he derived the stability of a column in a way close to Euler's problem, and explained the instability from general observations that, under pure bending, a wire has deflections that resemble rigid body motions, and thus may become unstable. Finally, Bryan generalized some conclusions: For a thin member, equilibrium occurs with bending and it may be unstable; however, the thickness of a component under compression or tension may become much larger, in which case the system will be stable.

A further prediction is advanced: "Gauss has proved that a thin shell in the form of a closed surface cannot be deformed by pure bending unaccompanied by extension or compression of the surface. Such a shell is, therefore, essentially stable" (Bryan,[11] p. 210).

The second and third stability papers written by Bryan during his years at Cambridge were applications of his general theory. Bryan stresses again that thinness and flexibility are necessary qualifications for a "collapse through instability of equilibrium." In his second paper,[14] the energy formulation was applied to Euler's column and to a ring under uniform pressure. The ring problem had been previously linked with the work of Euler by Fairnbairn[15] and Unwin,[16] and was relevant to predict the buckling of a pipe under pressure (known as the "problem of the boiler flue"). Here, Bryan devotes some lines to the mode shape of a circular tube under pressure, and makes the point that "when the tube collapses through instability, it does not necessarily follow that it will break … on this hypothesis, if the pressure be removed, the tube will return to the circular form, which it would not do if the material gave way at any point" (Bryan,[14] p. 292).

The third paper[17] concentrated on plates simply supported at the edges under in-plane loads, for which he derived a series solution. This paper focused more on the methodology of analysis than on the behavior of the plate, but still he was able to make a few points about the mechanisms of deformation that occurred in the plate. Regarding combined loads (tension in one direction and compression in the other), Bryan stated that this "may be easily illustrated by wetting a sheet of paper in the middle, and then stretching it over two parallel rulers. The moisture causes the surface of the paper to expand and wrinkle, and if the rulers are pulled apart with increasing force, the wrinkles will become finer and closer" (Bryan,[17] p. 61). The significance and implications of his results were transferred to the sides of a ship, the general problem being "to determine the number of corrugations produced in the buckling of a finite rectangular plate under the influence of both thrusts" (Bryan,[17] p. 63).

To reflect on the work of Bryan, one may identify the "hard core" of the research program started by Bryan, in the following postulates:

(1) Instability occurs only in structural members "which are capable of being deformed by pure bending or twisting," and not in those which work under uniform state of compression. The structural components that satisfy this condition were identified by Bryan to be thin wires, plates, and shells.
(2) Under small strains, equilibrium of an elastic solid in which all dimensions have comparable size is essentially stable.

Bryan excelled in producing heuristic arguments to support his claims. He used both mathematics and heuristics to convince the reader. No attempt was made by Bryan to validate his results with experiments; he rather validates his theoretical developments with the simple column problem already solved by Euler.

The three papers by Bryan, altogether cite eight references, or six authors: Lord Rayleigh's book, *Theory of Sound*, two papers by Kirchhoff, the two papers by Greenhill discussed in the previous section, a paper by Love, one by Unwin, and he mentioned Euler's name, although no bibliographical information was given in this last case. The references were incomplete, so that most information about title, year of publication, and pages of the papers is missing. Notice that this was not uncommon at the times of Bryan: referencing the work of others was frequently inaccurate, and sometimes it was done in an approximate way.

5. The Recognition of the Work of Bryan

The famous books on elasticity by Todhunter and Pearson[18] did not cover their contemporaries, therefore that Bryan was not included as a reference. However, the first edition of the book by Love clearly appreciated the work of Bryan: "Now Mr. Bryan has shown that there are only two cases of possible instability, (1) where nearly rigid-body displacements are possible with very small strains,… and (2) where one of the dimensions of a body is small in comparison with another, as in a thin rod or plate. He proceeded by taking the second variation of the energy-function, and he pointed out that, as in every case, the system tends to take up the position in which the potential energy is least, modes involving flexure will be taken by a thin rod or plate under thrust whenever such modes are possible. Mr. Bryan has given several interesting applications of his theory which we shall consider in our last chapter" (Love,[13] p. 23).

A most important reference is found 25 years later, in a paper by Southwell, a fellow of Trinity College in Cambridge. Southwell[4] acknowledged that Bryan's paper "has become the foundation of the theory in its existing form, Bryan has brought these isolated problems for the first time within the range of a single generalization."

Other researchers knew and acknowledged the work of Bryan. Reissner,[19] working 37 years after Bryan, attempted to improve his treatment by making use of the energy criterion. As many as 40 years after Bryan, Biezeno and Hencky[20] gave only a marginal cite to Bryan and shifted the origin to Southwell: "The first who in a concrete manner tackles the general problem is Southwell." Biezeno returned to the differential formulation, but looks also at the virtual work displacements. As many as 45 years after Bryan, Trefftz[21] returned to the energy criterion of stability, but did not even mention Bryan (also ignoring Southwell).

Koiter[1] gave credit back to Bryan some 57 years later: "Bryan seems to have been the first to attempt developing a general theory of stability." Southwell was also recognized.

Finally, Timoshenko reminded that "an important paper dealing with the general theory of elastic stability was published by G. H. Bryan … He shows that Kirchhoff's theorem on the uniqueness of solutions of equations on the theory of elasticity holds only if all the dimensions of the body are of the same order" (Timoshenko,[8] p. 299).

6. The Critique of the Work of Bryan

Several researchers used the work of Bryan as a reference and commented critically on his formulation and conclusions. Southwell[4] identified three main deficiencies in Bryan's work:

(1) Some of the conclusions are scarcely warranted;
(2) The theory assumes that the strains prior to collapse must be low so that the material remains elastic (a question of limiting the claims of Bryan's theory); and
(3) The methods are only approximations.

Nowadays, the second and third objections would not make uncomfortable to most researchers, and it is the first objection what would remain at stage. Southwell found that his results for the pipe under pressure agreed with Bryan's results on the simplified version of the problem. So, although critical of Bryan, Southwell was still using his work as a source for comparison.

The notion of equilibrium path is not present in Bryan, but it is clear in the work of Southwell. Furthermore, imperfection sensitivity is discussed by Southwell, a topic not envisaged by Bryan. Finally, Bryan stated that future work would be done in new applications, which he never published. Southwell, on the other hand, opened the field to further inquiry and was aware that new theoretical developments had to be made.

The first to fully explain the limitations of Bryan was Koiter, who devoted one full paragraph of his introduction. "His calculation of the elastic energy … takes only terms quadratic in the displacements into account. The second variation of this energy is then of the same form as the energy itself, and is positive like it … so that instability of such cases would be excluded. This conclusion conflicts with experience, as will be clear from the example of a straight bar loaded at its ends and undergoing a prescribed compression; if the compression is chosen great enough, this equilibrium state will be unstable" (Koiter,[1] p. 1).

Therefore, the source of instability in the formulation presented by Bryan was not to be found in the relative value of the load terms versus the strain energy terms in the second variation. There are many cases in which the second variation of the load terms vanish (a situation assumed by Kirchhoff, identified by Bryan, and more recently used to characterize a specialized system.[2] Instability is rather in a negative contribution of the strain energy term itself. And this is only possible if not only just quadratic but also higher order terms are included in the strain energy. The weak assumption in his chain of reasoning was that $\delta^2\Phi$ is essentially positive, a claim which can only hold under small displacements and strains.

7. Conclusions

As many as 120 years after the publication of Bryan's first paper, we here acknowledge his pioneering work. The greatest contribution of Bryan in the field of elastic stability may be the recognition of the need to formulate a general theory for the stability of elastic structures, from which results could be computed as special cases. A few aspects should be stressed about Bryan's contributions toward writing such a theory.

First, the new theory had to be consistent with the development of generalizations which were the prevailing trend of his times, and which was already taking place in the theory of elasticity. Second, Bryan made an effort not just to formulate a theory, but also to understand the underlying mechanisms of buckling, leading to a physical approach coupled with his mathematical formulation. Third, he used energy and variational methods to obtain stability in elastic problems, pretty much the same form that we use today. Fourth, he derived solutions to specific problems from the general theory, for the conditions of critical states.

Bryan developed his theory to such a point that he could make predictions; this was a rich theory, because he could advance beyond what was known at his time. Perhaps, Bryan placed heuristics before anything else and was courageous to produce strong statements which were speculations, but which he regarded as predictions based on theory. Some of his assumptions were later proven to be incorrect; however, this should not obscure the enormous contribution that Bryan made to science and engineering, by opening a new field and allowing others to correct his limitations in order to extend the theory in almost every aspect of theory and practice.

Finally, a new paradigm did not develop at the end of the 19th century, and this school of thought was displaced from the scientific discussions for decades, giving way to a more practically oriented paradigm which aimed at solving specific problems.

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