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Research article

On the generalized spectrum of bounded linear operators in Banach spaces

Jue Feng*, **Xiaoli Li and Kaicheng Fu**

School of Mathematical Sciences, Yangzhou University, Yangzhou 225009, Jiangsu, China

* **Correspondence:** Email: jfeng@yzu.edu.cn.

Abstract: Utilizing the stability characterizations of generalized inverses, we investigate the generalized resolvent of linear operators in Banach spaces. We first prove that the local analyticity of the generalized resolvent is equivalent to the continuity and the local boundedness of generalized inverse functions. We also prove that several properties of the classical spectrum remain true in the case of the generalized one. Finally, we elaborate on the reason why we use the generalized inverse rather than the Moore-Penrose inverse or the group inverse to define the generalized resolvent.

Keywords: generalized inverse; generalized resolvent; generalized spectrum; stability characterization
Mathematics Subject Classification: 47A10, 47A55

1. Introduction and preliminaries

Let X be a Banach space and $B(X)$ denote the Banach space of all bounded linear operators from X into itself. The identity operator will be denoted by I. For any $T \in B(X)$, we denote by $N(T)$ and $R(T)$ the null space and the range of T, respectively.

The resolvent set $\rho(T)$ of $T \in B(X)$ is, by definition,

$$\rho(T) = \{\lambda \in \mathbb{C} : \ T_\lambda = T - \lambda I \text{ is invertible in } B(X).\}.$$

And, its resolvent $R(\lambda) = (T - \lambda I)^{-1}$ is an analytic function on $\rho(T)$ since it satisfies the resolvent identity:

$$R(\lambda) - R(\mu) = (\lambda - \mu)R(\lambda)R(\mu), \quad \forall \lambda, \mu \in \rho(T).$$

The spectrum $\sigma(T)$ is the complement of $\rho(T)$ in $\mathbb{C}$. As we all know, the spectral theory plays a fundamental role in functional analysis. If T_λ is not invertible in $B(X)$, we can consider its generalized inverse. Recall that $T \in B(X)$ is generalized invertible if there exists an operator $S \in B(X)$ such that $TST = T$ and $STS = S$. We also say that such S is a generalized inverse of T, which is always denoted by T^+. If T has a bounded generalized inverse T^+, then, from [1], we know that both TT^+ and T^+T are projectors on X and

$$X = N(T) \oplus R(T^+) = N(T^+) \oplus R(T). \tag{1.1}$$

If X is a Hilbert space and the direct sum decompositions in (1.1) are orthogonal, the corresponding generalized inverse is the Moore-Penrose inverse. Recall that the operator $T^\dagger \in B(X)$ is said to be the Moore-Penrose inverse of T if $T^\dagger$ satisfies

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger \quad \text{and} \quad (T^\dagger T)^* = T^\dagger T,$$

where T^* denotes the adjoint operator of T.

If the operator $T^\sharp \in B(X)$ satisfies

$$TT^\sharp T = T, \quad T^\sharp TT^\sharp = T^\sharp \quad \text{and} \quad TT^\sharp = T^\sharp T,$$

then $T^\sharp$ is called the group inverse of T. If $T^\sharp$ is the group inverse of T, then $N(T^\sharp) = N(T)$, $R(T^\sharp) = R(T)$ and $X = N(T^\sharp) \oplus R(T^\sharp)$ [1].

If, as the definition of $\rho(T)$, the generalized resolvent set is defined by

$$\rho_g(T) = \{\lambda \in \mathbb{C} : T_\lambda = T - \lambda I \text{ is generalized invertible in } B(X).\},$$

we can find that such $\rho_g(T)$ is meaningless in the case of matrices, since every matrix is generalized invertible and $\rho_g(T) = \mathbb{C}$. To define reasonably the generalized resolvent set, we should add some additional conditions.

Definition 1.1. Let U be an open set in the complex plane $\mathbb{C}$; the function

$$U \ni \lambda \to R_g(\lambda) \in B(X)$$

is said to be a generalized resolvent of $T_\lambda = T - \lambda I$ on U if

(1) for all $\lambda \in U$,

$$(T - \lambda I)R_g(\lambda)(T - \lambda I) = T - \lambda I;$$

(2) for all $\lambda \in U$,

$$R_g(\lambda)(T - \lambda I)R_g(\lambda) = R_g(\lambda);$$

(3) for all λ and μ in U,

$$R_g(\lambda) - R_g(\mu) = (\lambda - \mu)R_g(\lambda)R_g(\mu).$$

The conditions (1) and (2) say that $R_g(\lambda)$ is a generalized inverse of T_λ. While the equality in (3) is an analogue of the classical resolvent identity, we refer to it as the generalized resolvent identity, which assures that $R_g(\lambda)$ is locally analytic. In [2], Shubin points out that there exists a continuous generalized inverse function (satisfying (1) and (2) but not possibly (3)) meromorphic in the Fredholm domain $\rho_\phi(T) = \{\lambda \in C : T - \lambda I \text{ is Fredholm}\}$. And, it remains an open problem whether or not this can be done while also satisfying (3), i.e., it is not known whether generalized resolvents always exist. Many authors have been interested in the existence of the generalized resolvents and the property of the corresponding spectrum in [3–13].

Definition 1.2. *The generalized resolvent set is*

$$\rho_g(T) = \{\lambda \in \mathbb{C} : \text{ There exists an open set } U \subset \mathbb{C},\ \lambda \in U \text{ and } T_\lambda \text{ has a generalized resolvent on } U.\}$$

and the generalized spectrum $\sigma_g(T)$ *is the complement of* $\rho_g(T)$ *in* $\mathbb{C}$; *the generalized spectral radius is*

$$r_{\sigma_g}(T) = \sup\{|\lambda| : \lambda \in \sigma_g(T)\}.$$

In this paper, we utilize the stability characterization of generalized inverses to investigate the properties of the generalized resolvent set in Banach spaces. We also introduce two sets

$$\rho_g^1(T) = \{\lambda \in \mathbb{C} : \text{ There is a } \delta > 0, \text{ such that for all } \mu \text{ satisfying } |\mu - \lambda| < \delta, (T - \mu I)^+ \text{ exists and } (T - \mu I)^+ \to (T - \lambda I)^+ \text{ as } \mu \to \lambda.\ \}$$

and

$$\rho_g^2(T) = \{\lambda \in \mathbb{C} : \text{ There are } M > 0 \text{ and } \delta > 0, \text{ such that for all } \mu \text{ satisfying } |\mu - \lambda| < \delta,\ (T - \mu I)^+ \text{ exists and } \|(T - \mu I)^+\| \leq M.\ \},$$

and prove that they are identical to $\rho_g(T)$. Based on this result, we discuss the relationship between the resolvent set and the generalized resolvent set, as well as the spectrum and the generalized spectrum. We also prove that several properties of the classical spectrum remain true in the case of the generalized one. Finally, we explain why we use the generalized inverse rather than the Moore-Penrose inverse or the group inverse to define the generalized resolvent.

2. Main results

We start with the following lemma, which is preparation for the proofs of our main results.

Lemma 2.1. (1) *If* $R_g(\lambda)$ *and* $R_g(\mu)$ *satisfy the generalized resolvent identity:*

$$R_g(\lambda) - R_g(\mu) = (\lambda - \mu)R_g(\lambda)R_g(\mu),$$

then

$$N\left(R_g(\lambda)\right) = N\left(R_g(\mu)\right)$$

and

$$R\left(R_g(\lambda)\right) = R\left(R_g(\mu)\right).$$

(2) *Let* $P_\lambda = T_\lambda R_g(\lambda)$ *and* $Q_\lambda = R_g(\lambda)T_\lambda$; *then,* P_λ *and* Q_λ *are projectors with*

$$P_\lambda P_\mu = P_\lambda \quad \text{and} \quad Q_\lambda Q_\mu = Q_\mu, \quad \lambda, \mu \in U.$$

(3) *The resolvent set is included in the generalized resolvent set, i.e.* $\rho(T) \subset \rho_g(T)$, *the generalized resolvent set* $\rho_g(T)$ *is open in* $\mathbb{C}$ *and the generalized resolvent* $R_g(\lambda)$ *is locally analytic on* $\rho_g(T)$.

Proof. (1) We exchange λ with μ in the generalized resolvent identity and obtain

$$R_g(\lambda)R_g(\mu) = R_g(\mu)R_g(\lambda)$$

and so

$$\begin{aligned} R_g(\lambda) &= R_g(\mu) + (\lambda-\mu)R_g(\lambda)R_g(\mu) = \left[I + (\lambda-\mu)R_g(\lambda)\right]R_g(\mu) \\ &= R_g(\mu) + (\lambda-\mu)R_g(\mu)R_g(\lambda) = R_g(\mu)\left[I + (\lambda-\mu)R_g(\lambda)\right]. \end{aligned}$$

Then, $N\left(R_g(\mu)\right) \subset N\left(R_g(\lambda)\right)$ and $R\left(R_g(\lambda)\right) \subset R\left(R_g(\mu)\right)$. Thus, exchanging λ with μ again, we can get

$$N\left(R_g(\lambda)\right) = N\left(R_g(\mu)\right), \quad R\left(R_g(\lambda)\right) = R\left(R_g(\mu)\right).$$

(2) Obviously, P_λ and Q_λ are projectors on X. Noting that

$$R(I-P_\mu) = N(P_\mu) = N(R_g(\mu)) = N(R_g(\lambda)) = N(P_\lambda)$$

and

$$R(Q_\mu) = R(R_g(\mu)) = R(R_g(\lambda)) = R(Q_\lambda) = N(I-Q_\lambda),$$

we have $P_\lambda P_\mu = P_\lambda$ and $Q_\lambda Q_\mu = Q_\mu$.

(3) Obviously, $\rho(T) \subset \rho_g(T)$. It follows from the definition of the generalized resolvent that the set $\rho_g(T)$ is open. Since

$$R_g(\lambda) = \left[I + (\lambda-\mu)R_g(\lambda)\right]R_g(\mu),$$

we can see that the operator $I + (\lambda-\mu)R_g(\lambda)$ is invertible for all μ satisfying $|\mu-\lambda|\|R_g(\lambda)\| < 1$. So,

$$R_g(\mu) = [I + (\lambda-\mu)R_g(\lambda)]^{-1}R_g(\lambda).$$

Hence, $\lim_{\mu\to\lambda} R_g(\mu) = R_g(\lambda)$ and

$$\lim_{\mu\to\lambda} \frac{R_g(\mu) - R_g(\lambda)}{\mu-\lambda} = \lim_{\mu\to\lambda} R_g(\lambda)R_g(\mu) = R_g^2(\lambda).$$

Therefore, $R_g(\lambda)$ is locally analytic on $\rho_g(T)$ and $\left[R_g(\lambda)\right]' = R_g^2(\lambda)$. □

Theorem 2.2. *Let X be a Banach space and $T \in B(X)$; then,*

$$\rho_g(T) = \rho_g^1(T) = \rho_g^2(T).$$

Proof. From Lemma 2.1, we can easily see that $\rho_g(T) \subset \rho_g^1(T) \subset \rho_g^2(T)$. To complete the proof, we need show that $\rho_g^2(T) \subset \rho_g(T)$. In fact, for any $\lambda \in \rho_g^2(T)$, we can find $M > 0$ and $\delta > 0$, such that, for all μ satisfying $|\mu-\lambda| < \delta$, T_μ^+ exists and $\|T_\mu^+\| \le M$.

Step 1. We first prove that there exists $\delta_1 < \delta$,

$$R(T_\mu) \cap N(T_\lambda^+) = \{0\}$$

for all $\mu \in \{\mu \in \mathbb{C} : |\mu - \lambda| < \delta_1\}$. In fact, if $N(T_\lambda^+) = \{0\}$, obviously, $R(T_\mu) \cap N(T_\lambda^+) = \{0\}$. We can assume $N(T_\lambda^+) \neq \{0\}$; then, $I - T_\lambda T_\lambda^+ \neq 0$. Let

$$\delta_1 = \min\{(M\|I - T_\lambda T_\lambda^+\|)^{-1},\ \|T_\lambda^+\|^{-1},\ \frac{1}{2}\delta\} < \delta,$$

and consider $\mu \in \mathbb{C}$ such that $|\mu - \lambda| < \delta_1$, Then, for any $y_\mu \in R(T_\mu) \cap N(T_\lambda^+)$, we can get

$$\begin{aligned} & |\mu - \lambda| M \|I - T_\lambda T_\lambda^+\| \|y_\mu\| \\ \geq\ & |\mu - \lambda| \|I - T_\lambda T_\lambda^+\| \|T_\mu^+\| \|y_\mu\| \\ \geq\ & \|(I - T_\lambda T_\lambda^+)(T_\lambda - T_\mu) T_\mu^+ y_\mu\| \\ =\ & \|(I - T_\lambda T_\lambda^+) T_\mu T_\mu^+ y_\mu\| \\ =\ & \|(I - T_\lambda T_\lambda^+) y_\mu\| \\ =\ & \|y_\mu\|. \end{aligned}$$

Hence $y_\mu = 0$. This implies $R(T_\mu) \cap N(T_\lambda^+) = \{0\}$.

Step 2. We shall prove that

$$B_\mu = [I + (\mu - \lambda) T_\lambda^+]^{-1} T_\lambda^+ : X \to X$$

is the generalized resolvent of T_λ on $U = \{\mu \in \mathbb{C} : |\mu - \lambda| < \delta_1\}$. First, by $\|(\mu - \lambda) T_\lambda^+\| < 1$ and the Banach theorem, we can see that $I + (\mu - \lambda) T_\lambda^+$ is invertible and so B_μ is well defined. Second, from the equivalences between (1) and (3) in [14, Theorem 1.1], it follows that B_μ is a generalized inverse of T_λ with $N(B_\mu) = N(T_\lambda^+)$ and $R(B_\mu) = R(T_\lambda^+)$. Third, we shall show that

$$B_\mu - B_\nu = (\mu - \nu) B_\mu B_\nu, \quad \forall\, \mu,\ \nu \in U.$$

Define $P_\mu = T_\mu B_\mu$ and $Q_\mu = B_\mu T_\mu$; then, P_μ and Q_μ are projectors from X onto $R(T_\mu)$ and $R(B_\mu) = B(T_\lambda^+)$, respectively. Hence

$$R(I - P_\nu) = N(P_\nu) = N(B_\nu) = N(T_\lambda^+) = N(B_\mu) = N(P_\mu)$$

and

$$R(Q_\nu) = R(B_\nu) = R(T_\lambda^+) = R(B_\mu) = R(Q_\mu) = N(I - Q_\mu).$$

Thus, we can conclude

$$P_\mu P_\nu = P_\mu \quad \text{and} \quad Q_\mu Q_\nu = Q_\nu, \quad \forall\, \mu,\ \nu \in U.$$

Therefore,

$$\begin{aligned} (\mu - \nu) B_\mu B_\nu &= B_\mu (T_\nu - T_\mu) B_\nu = B_\mu P_\nu - Q_\mu B_\nu \\ &= B_\mu P_\mu P_\nu - Q_\mu Q_\nu B_\nu = B_\mu P_\mu - Q_\nu B_\nu \\ &= B_\mu - B_\nu. \end{aligned}$$

So, B_μ is the generalized resolvent of T_λ on U, which means $\lambda \in \rho_g(T)$. □

Remark 2.3. According to Shubin, there exists a continuous generalized inverse function but not an analytic generalized resolvent [2]. From Theorem 2.1, we can see that if there exists a continuous or locally bounded generalized inverse function, then we can find a relevant analytic generalized resolvent.

Lemma 2.4. (1) *Let U and V be two open sets in $\mathbb{C}$ such that the generalized resolvent identity holds on U and V. If $U \cap V \neq \emptyset$, then the generalized resolvent identity holds on $U \cup V$, i.e.,*

$$R_g(\lambda) - R_g(\mu) = (\lambda - \mu)R_g(\lambda)R_g(\mu), \quad \forall \lambda, \mu \in U \cup V.$$

(2) *Let U be a convex open set in $\rho_g(T)$; then,*

$$R_g(\lambda) - R_g(\mu) = (\lambda - \mu)R_g(\lambda)R_g(\mu), \quad \forall \lambda, \mu \in U.$$

(3) *If U is a convex open set in $\rho_g(T)$ and $\rho(T) \cap U \neq \emptyset$, then*

$$U \subset \rho(T).$$

Proof. (1) For all $\lambda, \mu \in U \cup V$, if $\lambda, \mu \in U$ or $\lambda, \mu \in V$, then the generalized resolvent identity holds. It is sufficient to prove that

$$R_g(\lambda) - R_g(\mu) = (\lambda - \mu)R_g(\lambda)R_g(\mu)$$

holds for $\lambda \in U$ and $\mu \in V$. Let $\nu \in U \cap V$; then,

$$R_g(\lambda) - R_g(\nu) = (\lambda - \nu)R_g(\lambda)R_g(\nu)$$

and

$$R_g(\mu) - R_g(\nu) = (\mu - \nu)R_g(\mu)R_g(\nu).$$

By Lemma 2.1, $P_\lambda P_\nu = P_\lambda$, $Q_\lambda Q_\nu = Q_\nu$, $P_\nu P_\mu = P_\nu$ and $Q_\nu Q_\mu = Q_\mu$. Hence,

$$\begin{aligned}
(\lambda - \mu)R_g(\lambda)R_g(\mu) &= R_g(\lambda)(T_\mu - T_\lambda)R_g(\mu) \\
&= R_g(\lambda)P_\mu - Q_\lambda R_g(\mu) \\
&= R_g(\lambda)P_\lambda P_\mu - Q_\lambda Q_\mu R_g(\mu) \\
&= R_g(\lambda)P_\lambda P_\nu P_\mu - Q_\lambda Q_\nu Q_\mu R_g(\mu) \\
&= R_g(\lambda)P_\lambda P_\nu - Q_\nu Q_\mu R_g(\mu) \\
&= R_g(\lambda)P_\lambda - Q_\mu R_g(\mu) \\
&= R_g(\lambda) - R_g(\mu).
\end{aligned}$$

(2) For all $\lambda, \mu \in U$, the segment $[\lambda, \mu] \subset U$. Then for any $\omega \in [\lambda, \mu]$, there exists a neighborhood $U(\omega) \subset \rho_g(T)$ such that the generalized resolvent identity holds on $U(\omega)$. It follows from the finite covering theorem that we can find $\omega_1, \omega_2, \cdots, \omega_n \in [\lambda, \mu]$, $n \in N$, such that $[\lambda, \mu] \subset \bigcup_{i=1}^{n} U(\omega_i)$. Hence, by (1), we have

$$R_g(\lambda) - R_g(\mu) = (\lambda - \mu)R_g(\lambda)R_g(\mu).$$

(3) Let $\mu \in \rho(T) \cap U$; then, for all $\lambda \in U$, by Lemma 2.1,

$$N\left(R_g(\lambda)\right) = N(R(\mu)) = \{0\} \quad \text{and} \quad R\left(R_g(\lambda)\right) = R(R(\mu)) = X.$$

This implies that $R_g(\lambda)$ is invertible, and so $\lambda \in \rho(T)$. □

Theorem 2.5. *Let X be a Banach space and $T \in B(X)$; then, the generalized spectrum $\sigma_g(T)$ is a nonempty bounded closed subset in $\mathbb{C}$.*

Proof. Since $\rho_g(T)$ is open, $\sigma_g(T) = \mathbb{C}\backslash\rho_g(T)$ is closed. If $|\lambda| > \|T\|$, then, by the Banach's theorem, $T - \lambda I = \lambda\left(\frac{1}{\lambda}T - I\right)$ is invertible and its inverse $(T - \lambda I)^{-1}$ is bounded. Hence

$$\{\lambda \in \mathbb{C} : |\lambda| > \|T\|\} \subset \rho(T) \subset \rho_g(T).$$

So $\sigma_g(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$ and $\sigma_g(T)$ is bounded. Finally, we prove that $\sigma_g(T)$ is nonempty. In fact, if $\sigma_g(T) = \emptyset$, then $\rho_g(T) = \mathbb{C}$. By (3) in Lemma 2.2 and $\{\lambda \in \mathbb{C} : |\lambda| > \|T\|\} \subset \rho(T)$, we can get $\rho(T) = \mathbb{C}$. This is a contradiction with $\sigma(T) \neq \emptyset$. □

Proposition 2.6. *Let X be a Banach space and $T \in B(X)$; then,*
(1) $\partial\sigma(T) \subset \sigma_g(T) \subset \sigma(T)$;
(2) $\sigma(T)\backslash\sigma_g(T) = \sigma(T) \cap \rho_g(T)$ *is open in* $\mathbb{C}$;
(3) $\rho_g(T) = \rho(T) \cup \left[\sigma(T)\backslash\sigma_g(T)\right]$.

Proof. (1) It follows from $\rho(T) \subset \rho_g(T)$ that $\sigma_g(T) \subset \sigma(T)$. Now we shall show that $\partial\sigma(T) \subset \sigma_g(T)$. If there is a $\lambda \in \partial\sigma(T)$ and $\lambda \notin \sigma_g(T)$, then $\lambda \in \rho_g(T)$ and we can find a neighborhood $U(\lambda) \subset \rho_g(T)$. Noting that $\lambda \in \partial\sigma(T)$, we can see that $U(\lambda) \cap \rho(T) \neq \emptyset$. It follows from Lemma 2.2 that $U(\lambda) \subset \rho(T)$, which is contradictory with $\lambda \in \partial\sigma(T)$.

(2) Since $\partial\sigma(T) \subset \sigma_g(T)$, we have

$$\begin{aligned} \sigma(T)\backslash\sigma_g(T) &= \sigma(T) \cap [\sigma_g(T)]^c = \sigma(T) \cap \rho_g(T) \\ &= [\sigma(T)\backslash\partial\sigma(T)] \cap \rho_g(T) = [\sigma(T)]^\circ \cap \rho_g(T) \end{aligned}$$

and it is an open set.

(3)

$$\begin{aligned} \rho_g(T) &= \rho_g(T) \cap [\rho(T) \cup \sigma(T)] = [\rho_g(T) \cap \rho(T)] \cup [\rho_g(T) \cap \sigma(T)] \\ &= \rho(T) \cup \left[[\sigma_g(T)]^c \cap \sigma(T)\right] = \rho(T) \cup \left[\sigma(T)\backslash\sigma_g(T)\right]. \end{aligned}$$

□

Example 2.7. Let T be the right translation operator on l^2, i.e.,

$$T : x = (x_1, x_2, x_3, \cdots, x_n, \cdots) \mapsto (0, x_1, x_2, x_3, \cdots, x_n, \cdots).$$

Then T is a Fredholm operator with

$$\rho(T) = \{\lambda \in \mathbb{C} : |\lambda| > 1\} \quad \textit{and} \quad \sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

Noting that the nullity $n(T_\lambda) = \dim N(T_\lambda) \equiv 0$ and the defect $d(T_\lambda) = \mathrm{codim} R(T_\lambda) \equiv 1$ on $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$, by Theorem 1.2 in [14] and the proof of Theorem 2.1, we know that $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \rho_g(T)$. Since $R(T_\lambda)$ is not closed for λ satisfying $|\lambda| = 1$, T_λ is not generalized invertible and so

$$\sigma_g(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

Thus

$$\rho_g(T) = \{\lambda \in \mathbb{C} : |\lambda| \neq 1\}.$$

Corollary 2.8. *Let X be a Banach space and $T \in B(X)$; then, the generalized spectral radius is just equal to the spectral radius, i.e.,*

$$r_{\sigma_g}(T) = r_\sigma(T).$$

Proof. By $\sigma_g(T) \subset \sigma(T)$, we have $r_{\sigma_g}(T) \leq r_\sigma(T)$. Since $\sigma(T)$ is bounded and closed, we can find $\lambda_0 \in \partial\sigma(T)$ such that $|\lambda_0| = r_\sigma(T)$. By Proposition 2.1, $\lambda_0 \in \sigma_g(T)$ and then

$$r_{\sigma_g}(T) = \sup\left\{|\lambda| : \lambda \in \sigma_g(T)\right\} \geq |\lambda_0|.$$

Hence $r_{\sigma_g}(T) \geq r_\sigma(T)$ and so $r_{\sigma_g}(T) = r_\sigma(T)$. □

At the end, we shall explain why we use the generalized inverse rather than two of the most important unique generalized inverses (the Moore-Penrose inverse and the group inverse [1, 15]) to define the generalized resolvent.

Theorem 2.9. *Let $T \in B(X)$. Then, the Moore-Penrose inverse $T_\lambda^\dagger$ or the group inverse $T_\lambda^\sharp$ of $T_\lambda = T - \lambda I$ is the analytic generalized resolvent on U if and only if*

$$N(T_\lambda) = \{0\} \quad \textit{and} \quad R(T_\lambda) = X.$$

In this case, T_λ is invertible, the Moore-Penrose inverse or the group inverse is the inverse and the generalized resolvent is exactly its classical resolvent.

Proof. It suffices to prove the necessity. We first claim that for all $\lambda, \mu \in U$, $N(T_\lambda) = N(T_\mu)$ and $R(T_\lambda) = R(T_\mu)$. In fact, if the Moore-Penrose inverse $T_\lambda^\dagger$ is the generalized resolvent on U, then, by Lemma 2.1, we have

$$R(T_\lambda^\dagger) = R(T_\mu^\dagger) \quad \text{and} \quad N(T_\lambda^\dagger) = N(T_\mu^\dagger).$$

Hence,

$$N(T_\lambda) = [R(T_\lambda^\dagger)]^\perp = [R(T_\mu^\dagger)]^\perp = N(T_\mu) \quad \text{and} \quad R(T_\lambda) = [N(T_\lambda^\dagger)]^\perp = [N(T_\mu^\dagger)]^\perp = R(T_\mu).$$

If the group inverse $T_\lambda^\sharp$ is the generalized resolvent on U, then

$$N(T_\lambda^\sharp) = N(T_\mu^\sharp) \quad \text{and} \quad R(T_\lambda^\sharp) = R(T_\mu^\sharp).$$

Hence,

$$N(T_\lambda) = N(T_\lambda^\sharp) = N(T_\mu^\sharp) = N(T_\mu) \quad \text{and} \quad R(T_\lambda) = R(T_\lambda^\sharp) = R(T_\mu^\sharp) = R(T_\mu).$$

Now, we prove that $N(T_\lambda) = \{0\}$ and $R(T_\lambda) = X$. For all $x \in N(T_\lambda)$, then $T_\lambda x = T_\mu x = 0$, i.e., $Tx = \lambda x$ and $Tx = \mu x$. So, $x = 0$. This means that $N(T_\lambda) = \{0\}$. For any $y \in X$, $T_\mu y \in R(T_\lambda)$ and there is an $x \in X$, such that $T_\lambda x = T_\mu y$. Then,

$$y = \frac{1}{\lambda - \mu} T_\lambda(x - y) \in R(T_\lambda).$$

We can conclude that T_λ is invertible and the generalized resolvent is exactly its classical resolvent. □

3. Conclusions

In this paper, we have proved that the existence of the analytic generalized resolvents of the linear operators in Banach spaces is equivalent to the continuity and local boundedness of generalized inverse functions. Based on the properties of generalized resolvents, we have shown that the generalized spectrum is a nonempty bounded closed subset. Moreover, the relationship between the resolvent set and the generalized resolvent set, as well as that between the spectrum and the generalized spectrum has been given. An interesting example is given to illustrate our results. Finally, we explain why we use the generalized inverse rather than the Moore-Penrose inverse or the group inverse to define the generalized resolvent.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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