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Non-Stationary α-Fractal Surfaces

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Abstract. In this paper, we define non-stationary fractal interpolation surfaces on a rectangular domain and give some upper bounds for their fractal dimension. Next, we define a fractal operator associated with the non-stationary fractal surfaces, and study some properties of it. In particular, we hint at the existence of a Schauder basis consisting of non-stationary fractal functions.

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1. Introduction

Motivated by the seminal work [1] of Barnsley on Fractal Interpolation Functions (FIFs), Navascués [23] introduced a family of fractal interpolation functions f^α known as α-fractal functions corresponding to a continuous function f on closed and bounded interval of $\mathbb{R}$. The $\alpha-$fractal function f^α approximates and interpolates f simultaneously. As it is evident from some works of Navascués and her collaborators [8,9,14,22–24,28,30,32] that there are enormous applications of (stationary) fractal functions in different areas of mathematics.

The univariate FIFs are studied more than the bivariate case. In [19], Massopust introduced a construction of fractal surfaces on a triangular domain from coplanar data points on the boundary of the domain. Chand and Kapoor [8] presented a hidden variable FISs on rectangular domains with some conditions but it is observed by Xu and Ruan [26] that their construction is not feasible. In [10], Dalla obtained FISs on rectangular grids for collinear interpolation points on the boundary. Bouboulis and Dalla [5] obtained a continuous fractal surface using coplanar data points. In [21], Metzler and Yun generalised the construction given by Malysz [18] by using a vertical scaling function instead of constant scaling factor. Ruan and Xu [26] introduced a general framework to construct fractal interpolation surfaces

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on a rectangular domain without any conditions on data points. We refer the reader to, albeit incomplete, list of references, [5,6,19,21,24,26] for some constructions of fractal surfaces.

Fractal dimension has been at the heart of fractal geometry. Estimating the fractal dimension for many sets may require a lot of work, sometimes it is too difficult to compute, see, for instance, [13]. We encourage the non-fractalist reader to consult [28] for a better starting in the field of the fractal dimension of bivariate function. In [18], Malysz computed the box dimension of fractal interpolation surfaces. In [15], Kong et al. obtained the box dimension of bilinear fractal interpolation surfaces using the variation of the fractal function over small subrectangles. In [29,30], Verma and Viswanathan introduced and studied the bivariate α-fractal functions from the fractal geometry, operator theoretic and constrained approximation point of view. In [14], Jha et al. obtained some bounds of the box dimension of α-fractal function. They also focus on some approximation and smoothness properties of bivariate α-fractal function.

Recently, Liang and Ruan [17] construct recurrent fractal interpolation surfaces (RFISs) on rectangular grids and introduce bilinear RFISs without any restrictions on interpolating data and vertical scaling factors. Further, they compute the box dimension of bilinear RIFSs with suitable conditions. In [24], Navascu'es et al. launch a new construction method of fractal surfaces on a rectangular domain from a given germ function f and a base function b under the condition that f and b take the same values on the boundary of the rectangular domain.

The first connection between stationary subdivision schemes and self-similar fractals is originated in [27]. Motivated by this work, Levin et al. [16] study trajectories of contraction mappings, which they consider a generalization of Banach fixed point theorem. Using this notion, they establish a nice relation between non-stationary subdivision schemes and fractals. Further, they compared their results with $V-$ variable fractals and superfractals, see, [3]. Following the previous work, Dyn et al. [12] attempted to connect more general types of subdivision schemes to sequence of function systems. Recently, Massopust [20] introduces the concept of non-stationary FIFs by considering a sequence of *Read-Bajraktarević* (RB) operators. He also shows with some examples that non-stationary version of fractal interpolation functions have greater flexibility than stationary case. Now, our aim here is to construct and study non-stationary fractal $\alpha-$fractal surfaces.

We should emphasize that the non-stationary version is presented not merely as generalization of the stationary case; however with an eye towards broadening non-stationary fractal surfaces to the region of constrained approximation.

The content of the paper is as follows. In the second section, we target to develop the non-stationary fractal surfaces on a rectangular domain. In third, we deal with the dimension of the non-stationary FISs using oscillation spaces and Hölder space. In the last section, we define a non-stationary fractal operator, and establish some properties of it.

2. Non-Stationary α-Fractal Surfaces

In this section, we construct a non-stationary bivariate α-fractal function on rectangular grids; for details on stationary bivariate α-fractal function, the reader is referred to [26,30].

Let $I = [a, b]$ and $J = [c, d]$. Define $\square = I \times J$. Let a continuous function $f : \square \to \mathbb{R}$ be given. Define a net Δ by

$$\Delta := \{(x_i, y_j) \in \mathbb{R}^2 : a = x_0 < x_1 < \cdots < x_N = b; c = y_0 < y_1 < \cdots < y_M = d\}.$$

We will use the following notation: $\Sigma_N = \{1, 2, \ldots, N\}$, $\Sigma_{N,0} = \{0, 1, \ldots N\}$, $\partial\Sigma_{N,0} = \{0, N\}$ and $\mathrm{int}\Sigma_{N,0} = \{1, 2, \ldots, N-1\}$.

Here we should note that we are working with a set $D = \{(x_i, y_j, z_{ij}) : i \in \Sigma_{N,0},\ j \in \Sigma_{M,0}\}$ of three-dimensional data points where the date set is originated from the function f, that is, $z_{ij} = f(x_i, y_j)$. Let $s = \{s_k\}_{k\in\mathbb{N}}$ be a sequence of continuous functions $s_k \in \mathcal{C}(\square, \mathbb{R})$ satisfying $s_k \neq f$, $\|s\|_\infty := \sup_{k\in\mathbb{N}} \|s_k\|_\infty < \infty$, and

$$s_k(x_i, y_j) = f(x_i, y_j), \quad \forall\ (i, j) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0}.$$

Let $\alpha = \{\alpha_k\}_{k\in\mathbb{N}}$ be a sequence of continuous functions $\alpha_k \in \mathcal{C}(\square, \mathbb{R})$ such that

$$\|\alpha\|_\infty := \sup\{\|\alpha_k\|_\infty : k \in \mathbb{N}\} < 1.$$

We define affine functions $u_i : I \to I_i := [x_{i-1}, x_i]$ and $v_j : J \to J_j := [y_{j-1}, y_j]$ as follows:

$$u_i(x) = a_i x + b_i, \quad v_j(y) = c_j y + d_j,$$

where constants involved are suitably determined by the following set of equations:

$$\begin{aligned}
&u_i(x_0) = x_{i-1}, \quad u_i(x_N) = x_i, \text{ if } i \text{ is odd},\\
&u_i(x_0) = x_i, \quad u_i(x_N) = x_{i-1}, \text{ if } i \text{ is even},\\
&v_j(y_0) = y_{j-1}, \quad v_j(y_M) = y_j, \text{ if j is odd, and}\\
&v_j(y_0) = y_j, \quad v_j(y_N) = y_{j-1}, \text{ if j is even.}
\end{aligned} \tag{2.1}$$

Set $K = \square \times \mathbb{R}$ and define $F_{ij,k} : K \to \mathbb{R}$ by

$$F_{ij,k}(\boldsymbol{x}, z) = \alpha_k\big(P_{ij}(\boldsymbol{x})\big)z + f\big(P_{ij}(\boldsymbol{x})\big) - \alpha_k\big(P_{ij}(\boldsymbol{x})\big)s_k(\boldsymbol{x}),$$

where $\boldsymbol{x} = (x, y)$ and $P_{ij}(\boldsymbol{x}) := (u_i(x), v_j(y)))$. For each $(i, j) \in \Sigma_N \times \Sigma_M$, we define $W_{ij,k} : K \to \square_{ij} \times \mathbb{R}$ by

$$W_{ij,k}(\boldsymbol{x}, z) = \big(P_{ij}(\boldsymbol{x}), F_{ij,k}(\boldsymbol{x}, z)\big),$$

where $\square_{ij} := I_i \times J_j$. Now, we have a sequence of IFSs $\mathcal{I}_k := \big\{K, W_{ij,k} : (i, j) \in \Sigma_N \times \Sigma_M\big\}$.

Let us mention two examples for such function $s_k \in \mathcal{C}(\square, \mathbb{R})$.

(1) $s_k(\boldsymbol{x}) = f(\boldsymbol{x})t_k(\boldsymbol{x})$, where $t_k \in \mathcal{C}(\square, \mathbb{R})$ is a fixed non-constant function such that $t_k(x_i, y_j) = 1, \quad \forall\ (i, j) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0}$.

(2) $s_k(\boldsymbol{x}) = (f \circ t_k)(\boldsymbol{x})$, where $t_k \in \mathcal{C}(\square, \square)$ is a fixed map such that $t_k \neq Id$, the identity map, and $t_k(x_i, y_j) = (x_i, y_j), \quad \forall\ (i, j) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0}$.

Let X be a metric space and $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of Lipschitz maps on X. We define backward procedures as follows:

$$\Psi_k := T_1 \circ T_2 \circ \ldots T_k.$$

Definition 2.1. Two sequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ in a metric space (X,d) are said to be asymptotically similar if $d(x_k,y_k)\to 0$ as $k\to\infty$.

Example 2.2. Let $X=[0,1]$ and $x_k=z^k,\ y_k=z^{2k}$ for some $z\in X$. Then $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ are asymptotically similar.

Remark 2.3. If $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ are asymptotically similar then

$$\lim_{k\to\infty} x_k = x \iff \lim_{k\to\infty} y_k = x.$$

Proposition 2.4. ([16], Proposition 3.4) *Let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of Lipschitz maps on a complete metric space X such that T_k has Lipschitz constant c_k. If $\lim_{k\to\infty}\prod_{i=1}^k c_i=0$, then $\{\Psi_k(x)\}$, $\{\Psi_k(y)\}$ are asymptotically similar for all $x,y\in X$.*

Example 2.5. Let us consider $X=[0,1]$ and $T_k: X\to X$ defined by $T_k(x)=\frac{x}{k}$. Here $c_k=\frac{1}{k}$ and $\lim_{k\to\infty}\prod_{i=1}^k\frac{1}{i}=0$. Then all the trajectories are asymptotically similar.

Proposition 2.6. *Let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of Lipschitz maps on a complete metric space X. If there exists $x\in X$ such that the sequence $\{d(x,T_k(x))\}$ is bounded, and $\sum_{k=1}^\infty\prod_{i=1}^k c_i<\infty$, then the sequence $\{\Psi_k(x)\}$ converges for all $x\in X$ to a unique limit $\overline{x}$.*

Proof. For $m,k\in\mathbb{N}$, $m>k$, and $x\in X$ satisfying the conditions of the statement:

$$\begin{aligned} d(\Psi_m(x),\Psi_k(x)) \leq\ & d(\Psi_m(x),\Psi_{m-1}(x)) + d(\Psi_{m-1}(x),\Psi_{m-2}(x)) + \\ & \cdots + d(\Psi_{k+2}(x),\Psi_{k+1}(x)) + d(\Psi_{k+1}(x),\Psi_k(x)) \\ \leq & \Big(\prod_{i=1}^{m-1} c_i\Big) d(T_m(x),x) + \cdots + \Big(\prod_{i=1}^{k} c_i\Big) d(T_{k+1}(x),x). \end{aligned}$$

Since the sequence $\{d(x,T_i(x))\}$ is bounded, that is, there exists $M>0$ such that $\sup_{i\in\mathbb{N}} d(x,T_i(x))\leq M$, then $d(\Psi_m(x),\Psi_k(x))\leq M(S_{m-1}-S_{k-1})$, where S_j represents j-th partial sum of the series $\sum_{k=1}^\infty\prod_{i=1}^k c_i$. This further implies that $\{\Psi_k(x)\}$ is Cauchy and hence convergent. Using the previous proposition, all the trajectories $\{\Psi_k(y)\}$ are convergent to the same limit for all $y\in X$. □

From the above proposition, it is immediate to deduce the following result.

Corollary 2.7. *If X is a complete and bounded metric space, and $\{T_k\}$ is a sequence of Lipschitz mappings on X with $\sum_{k=1}^\infty\prod_{i=1}^k c_i<\infty$, then all the trajectories $\{\Psi_k(x)\}$ converge to the same limit.*

Let us recall the following notation:

$$\alpha := \{\alpha_k\}_{k\in\mathbb{N}} \text{ and } s := \{s_k\}_{k\in\mathbb{N}}.$$

Now, let $\mathcal{C}_f(\square) := \big\{g \in \mathcal{C}(\square) : g(x_i, y_j) = z_{ij},\ \forall\ (i,j) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0}\big\}$.

Proposition 2.8. *The set $\mathcal{C}_f(\square)$ is a complete and convex subset of $\mathcal{C}(\square)$ with respect to the uniform (or supremum) distance.*

Proof. Let us consider a convergent sequence of elements $f_m \in \mathcal{C}_f(\square) \subset C(\square)$ and $\lim f_m = g \in \mathcal{C}(\square)$ with respect to the supremum norm. The uniform convergence implies pointwise convergence and $g(x_i, y_j) = \lim_m f_m(x_i, y_j)$ $= \lim_m z_{ij} = z_{ij}$. Besides g must be a continuous function, therefore $g \in \mathcal{C}_f(\square)$. Consequently, $\mathcal{C}_f(\square)$ is a closed set of the complete space $\mathcal{C}(\square)$ and thus complete. The convexity is a straightforward consequence of the definition. □

For $k \in \mathbb{N}$, we define a sequence of RB operators $T^{\alpha_k} : \mathcal{C}_f(\square) \to \mathcal{C}_f(\square)$ by

$$(T^{\alpha_k} g)(\boldsymbol{x}) = F_{ij,k}(Q_{ij}(\boldsymbol{x}), g(Q_{ij}(\boldsymbol{x}))\ \forall \boldsymbol{x} \in \square_{ij},\ (i,j) \in \Sigma_N \times \Sigma_M,$$

where $\boldsymbol{x} = (x, y)$ and $Q_{ij}(\boldsymbol{x}) := (u_i^{-1}(x), v_j^{-1}(y))$. For completeness, we shall hint at the well-definedness of T^{α_k}.

Proposition 2.9. *The above $T^{\alpha_k} : \mathcal{C}_f(\square) \to \mathcal{C}_f(\square)$ is well-defined for each $k \in \mathbb{N}$.*

Proof. Let us denote the boundary of a set X by $\partial(X)$. If $(x, y) \in (x_{i-1}, x_i)\times (y_{j-1}, y_j)$, then $T^{\alpha_k} g(x, y)$ takes exactly one value.
If $(x, y) \in \partial(I_i \times J_j)$, then we have at least one of the following:

(1) $(x, y) \in \partial(I_i \times J_j) \cap \partial(I_{i+1} \times J_j)$
(2) $(x, y) \in \partial(I_i \times J_j) \cap \partial(I_i \times J_{j+1})$
(3) $(x, y) \in \partial(I_i \times J_j) \cap \partial(I_{i-1} \times J_j)$
(4) $(x, y) \in \partial(I_i \times J_j) \cap \partial(I_i \times J_{j-1})$.

We show the first case, as the others will follow similarly. Assuming (x, y) $= (x_i, y)$ as a point in $\partial(I_i \times J_j)$ we get

$$(T^{\alpha_k} g)(x, y) = f(x, y) + \alpha_k(x, y)\Big(g\big(u_i^{-1}(x), v_j^{-1}(y)\big) - s_k\big(u_i^{-1}(x), v_j^{-1}(y)\big)\Big) \tag{2.2}$$

and treating $(x, y) = (x_i, y)$ as a point in $\partial(I_{i+1} \times J_j)$

$$(T^{\alpha_k} g)(x, y) = f(x, y) + \alpha_k(x, y)\Big(g\big(u_{i+1}^{-1}(x), v_j^{-1}(y)\big) - s_k\big(u_{i+1}^{-1}(x), v_j^{-1}(y)\big)\Big). \tag{2.3}$$

If i is an even number, then $u_i^{-1}(x_i) = u_{i+1}^{-1}(x_i) = x_0$, otherwise $u_i^{-1}(x_i) = u_{i+1}^{-1}(x_i) = x_N$, This with Eqs. (2.2) and (2.3) yield the same value for $T^{\alpha_k} g(x_i, y)$. Hence, $T^{\alpha_k} g$ is well-defined. □

We further note that T^{α_k} is a contraction map, that is,

$$\|T^{\alpha_k} g - T^{\alpha_k} h\|_\infty \le \|\alpha_k\|_\infty \|g - h\|_\infty.$$

Now, we are ready to prove the non-stationary version of [[30], Theorem 3.1].

Theorem 2.10. *Let us consider the sequence of operators $\{T^{\alpha_k}\}$ on $\mathcal{C}_f(\square)$ defined above with the conditions described. Then for every $g \in \mathcal{C}_f(\square)$ the sequence $\{T^{\alpha_1} \circ T^{\alpha_2} \circ \cdots \circ T^{\alpha_k} g\}$ converges to a map f_s^A of $\mathcal{C}_f(\square)$.*

Proof. Let us consider $g \in \mathcal{C}_f(\square)$. We first check that $\{\|T^{\alpha_k} g - g\|_\infty\}$ is bounded. Applying the definition of T^{α_k},

$$\begin{aligned}\|T^{\alpha_k} g - g\|_\infty &\leq \|f\|_\infty + \|g\|_\infty + \|\alpha_k\|_\infty (\|g\|_\infty + \|s_k\|_\infty) \\ &= (1 + \|\alpha\|_\infty)\|g\|_\infty + \|f\|_\infty + \|\alpha\|_\infty \|s\|_\infty,\end{aligned}$$

the bound does not depend on k. Applying Proposition 2.6, there exists $f_s^A \in \mathcal{C}_f(\square)$ such that $f_s^A = \lim_{k\to\infty} T^{\alpha_1} \circ T^{\alpha_2} \circ \cdots \circ T^{\alpha_k} g$ for any $g \in \mathcal{C}_f(\square)$. $\square$

Remark 2.11. Recall [30, Theorem 3.2] that if $\|\alpha\|_\infty < (1+\|Id-L\|)^{-1}$, then the associated (stationary) fractal operator $\mathcal{F}^\alpha_{\Delta,L} : \mathcal{C}(\square) \to \mathcal{C}(\square)$ defined by $\mathcal{F}^\alpha_{\Delta,L} = f^\alpha_{\Delta,L}$, where $f^\alpha_{\Delta,L}$ is the (stationary) α-fractal function corresponding to the scaling function $\alpha : \square \to \mathbb{R}$ and base function Lf satisfying the self-referential equation:

$$\begin{aligned}f^\alpha_{\Delta,L}(x,y) = f(x,y) + \alpha(x,y) f^\alpha_{\Delta,L}\big(u_i^{-1}(x), v_j^{-1}(y)\big)\\ -\alpha(x,y)(Lf)\big(\big(u_i^{-1}(x), v_j^{-1}(y)\big),\end{aligned}$$

is a topological automorphism. See [30] for notation and more details. The mentioned result convinces us that every function in $\mathcal{C}(\square)$ is a stationary α-fractal function corresponding to some parameters. But, it should be noted [2] that the construction of an IFS for a given function is really a difficult task.

Definition 2.12. The function f_s^A is the non-stationary fractal surface with respect to f, α, s and the net Δ. Note that to make distinction between stationary and non-stationary α-fractal functions, we are denoting the non-stationary α-fractal function by f_s^A, where A represents the sequence α. This may be a slight abuse of notation.

Remark 2.13. Since each T^{α_k} is a contraction, there exists a unique function f_k^α, known as (stationary) $\alpha-$fractal function corresponding to T^{α_k}, such that $T^{\alpha_k}(f_k^\alpha) = f_k^\alpha$. Being the fixed point of the RB-operator T^{α_k} [30], f_k^α satisfies the functional equation:

$$f_k^\alpha(\boldsymbol{x}) = F_{ij,k}\Big(Q_{ij}(\boldsymbol{x}), f_k^\alpha(Q_{ij}(\boldsymbol{x}))\Big) \quad \forall\, \boldsymbol{x} \in \square_{ij},$$

where $\boldsymbol{x} = (x,y)$ and $Q_{ij}(\boldsymbol{x}) := (u_i^{-1}(x), v_j^{-1}(y))$. That is, for all $(i,j) \in \Sigma_N \times \Sigma_M$ and $\boldsymbol{x} \in \square_{ij}$, we have

$$f_k^\alpha(\boldsymbol{x}) = f(\boldsymbol{x}) + \alpha_k(\boldsymbol{x}) f_k^\alpha(Q_{ij}(\boldsymbol{x})) - \alpha_k(\boldsymbol{x}) s_k(Q_{ij}(\boldsymbol{x})). \tag{2.4}$$

Define $W^{\alpha_k}(B) := \cup_{(i,j)\in\Sigma_N\times\Sigma_M} W_{ij,k}(B)$ for $B \subseteq \square \times \mathbb{R}$.

Proposition 2.14. *Let us consider the IFS $\mathcal{I}_k := \big\{K, W_{ij,k} : (i,j) \in \Sigma_N \times \Sigma_M\big\}$. Let $h : \square \to \mathbb{R}$ be a continuous function interpolating the data $\{(x_i, y_j, f(x_i, y_j)) : i \in \Sigma_{N,0},\ j \in \Sigma_{M,0}\}$, then if G_h is the graph of h, $W^{\alpha_k}(G_h) = G_g$ where G_g is the graph of a continuous function $g : \square \to \mathbb{R}$ interpolating* *the data, then $T^{\alpha_k}h = g$ where T^{α_k} is the operator defined as previously. Moreover, if $g = T^{\alpha_k}h$ then $W^{\alpha_k}(G_h) = G_g$.*

Proof. Let G_h be the graph of a continuous function $h : \square \to \mathbb{R}$. Let $(\boldsymbol{x}, h(\boldsymbol{x})) \in G_h$ then

$$W_{ij,k}(\boldsymbol{x}, h(\boldsymbol{x})) = (P_{ij}(\boldsymbol{x}), F_{ij,k}(\boldsymbol{x}, h(\boldsymbol{x})))$$

If $P_{ij}(\boldsymbol{x}) = \widetilde{\boldsymbol{x}}$, then

$$W_{ij,k}(\boldsymbol{x}, h(\boldsymbol{x})) = (\widetilde{\boldsymbol{x}}, F_{ij,k}(Q_{ij}(\widetilde{\boldsymbol{x}}), h(Q_{ij}(\widetilde{\boldsymbol{x}}))))$$

and $\widetilde{\boldsymbol{x}} \in \square_{ij}$. Each $W_{ij,k}$ transforms the graph G_h in the graph of a continuous map $g_{ij} : \square_{ij} \to \mathbb{R}$ defined by

$$g_{ij}(\widetilde{\boldsymbol{x}}) = F_{ij,k}(Q_{ij}(\widetilde{\boldsymbol{x}}), h(Q_{ij}(\widetilde{\boldsymbol{x}})))$$

The values of g_{ij} at the boundaries of $\square_{ij}$ are coinciding. In this case, we can link the surfaces g_{ij} to compose a continuous function g such that $g(\boldsymbol{x}) = g_{ij}(\boldsymbol{x})$ for $\boldsymbol{x} \in \square_{ij}$. The definition of g agrees with the image of h by the operator T^{α_k} so that $T^{\alpha_k}h = g$. The proof can be followed in inverse sense. $\square$

The graph of the function attractor satisfies the self-referential equation

$$G_{f_k^\alpha} = \bigcup_{(i,j)\in\Sigma} G_{f_{k,ij}^\alpha} \tag{2.5}$$

where $f_{k,ij}^\alpha$ is the restriction of f_k^α to $\square_{ij}$.
Let us consider the set of graphs of the continuous functions belonging to $\mathcal{C}_f(\square)$, $\mathcal{G} = \{G_h\}_{h\in\mathcal{C}_f(\square)}$. In this collection of sets it seems natural to define the metric

$$d(G_h, G_g) = \|h - g\|_\infty \tag{2.6}$$

The set $\mathcal{G}$ is a complete metric space due to the completitude of $\mathcal{C}_f(\square)$. We can define the map

$$\widetilde{F} : \mathcal{G} \to \mathcal{C}_f(\square) \tag{2.7}$$

such that $\widetilde{F}(G_h) = h$. $\widetilde{F}$ is an isomorphism of metric spaces, in particular is an isometric embedding and it preserves the fractal dimension (see [3]). $\widetilde{F}$ also preserves the topological invariants. The topology of $\mathcal{G}$ agrees with the identification topology induced by $\widetilde{F}$ ([3]). The next result can be easily proved.

Theorem 2.15. *If $S : \mathcal{C}_f(\square) \to \mathcal{C}_f(\square)$ is a contractive transformation on $(\mathcal{C}_f(\square), \|\cdot\|_\infty)$ with contractivity factor r, the map $\widehat{S} : \mathcal{G} \to \mathcal{G}$ such that $\widehat{S}(G_h) = G_{S(h)}$ is a contraction with the same factor.*

We could have followed the inverse way and define the H-metric on the space of functions (see [25]), given by

$$d(h, g) = d_H(G_h, G_g)$$

where G_h, G_g are the graphs of h and g, respectively, and d_H is the Hausdorff distance (for sets) between them, but the uniform distance seems more intuitive.
We can consider that $(\mathcal{G}, W^{\alpha_k})$ is a dynamical system with an attracting fixed point $G_{f_k^\alpha}$.

Proposition 2.16. *The attractor of a finite composition $W^{\alpha_1} \circ W^{\alpha_2} \circ \cdots W^{\alpha_k}$ is the graph of a function $g_k : \square \to \mathbb{R}$ interpolating the data.*

Proof. Let us consider $T^{\alpha_1} \circ T^{\alpha_2} \circ \cdots \circ T^{\alpha_k} : \mathcal{C}_f(\square) \to \mathcal{C}_f(\square)$. Since $\mathcal{C}_f(\square)$ is complete and the mapping is a contraction then it admits a fixed point $g_k \in \mathcal{C}_f(\square)$. The corresponding attractor of $W^{\alpha_1} \circ W^{\alpha_2} \circ \cdots W^{\alpha_k}$ is the graph of this function. $\square$

Remark. The function f_s^A is the uniform limit of the maps g_k defined in the Proposition 2.16.

3. Fractal Dimension of Non-Stationary α-Fractal Surfaces

Next we aim to compute the fractal dimension of the graph of a non-stationary fractal interpolation function using the concept of oscillation and Hölder spaces. We refer the reader to [7,11] for oscillation spaces. Fractal dimension is a tool which tries to distinguish fractals. We denote by $\underline{\dim}_B(C), \overline{\dim}_B(C)$ and $\dim_H(C)$ the lower box dimension, upper box dimension and Hausdorff dimension of a set C respectively. The reader can consult [13] for definitions and some properties of fractal dimensions.

Let $Q \subset [0,1] \times [0,1] =: I^2$ dyadic square so that $Q = \left[\frac{i}{2^m}, \frac{i+1}{2^m}\right] \times \left[\frac{j}{2^m}, \frac{j+1}{2^m}\right]$ for some integers $m \geq 0$ and $0 \leq i, j < \frac{1}{2^m}$. For a continuous function $f : I^2 \to \mathbb{R}$ we define oscillation of f over Q as follows:

$$\begin{aligned} R_f(Q) &= \sup_{\boldsymbol{x}, \boldsymbol{y} \in Q} |f(\boldsymbol{x}) - f(\boldsymbol{y})| \\ &= \sup_{\boldsymbol{x} \in Q} f(\boldsymbol{x}) - \inf_{\boldsymbol{x} \in Q} f(\boldsymbol{x}), \end{aligned}$$

and total oscillation of order m,

$$Osc(m, f) = \sum_{|Q| = 2^{-m}} R_f(Q)$$

where the sum ranges over all dyadic squares $Q \subset I^2$ of side-length $|Q| = \frac{1}{2^m}$. For a given $0 < \beta \leq 1$, we define oscillation space $\mathcal{V}^\beta(I^2)$ by

$$\mathcal{V}^\beta(I^2) = \left\{ f \in \mathcal{C}(I^2) : \sup_{m \in \mathbb{N}} \frac{Osc(m, f)}{2^{m(2-\beta)}} < \infty \right\}.$$

One can define

$$\mathcal{V}^{\beta-}(I^2) = \{ f \in \mathcal{C}(I^2) : f \in \mathcal{V}^{\beta-\epsilon}(I^2) \ \forall \ \epsilon > 0 \}$$

and

$$\mathcal{V}^{\beta+}(I^2) = \{ f \in \mathcal{C}(I^2) : f \notin \mathcal{V}^{\beta+\epsilon}(I^2) \ \forall \ \epsilon > 0 \}.$$

Theorem 3.1. ([7], Theorem 4.1) *Let f be a real-valued continuous function defined on I^2, we have*

$$\overline{\dim}_B(Graph(f)) \le 3-\gamma \iff f \in \mathcal{V}^{\gamma-}(I^2) \text{ if } 0<\gamma\le 1$$

and

$$\overline{\dim}_B(Graph(f)) \ge 3-\gamma \iff f \in \mathcal{V}^{\gamma+}(I^2) \text{ if } 0\le\gamma< 1.$$

Now, it is easy to check that $\|f\|_{\mathcal{V}^\beta} := \|f\|_\infty + \sup_{m\in\mathbb{N}} \frac{Osc(m,f)}{2^{m(2-\beta)}}$ is a norm on $\mathcal{V}^\beta(I^2)$. Let us note the following result.

Theorem 3.2. *The space $(\mathcal{V}^\beta(I^2), \|.\|_{\mathcal{V}^\beta})$ is a Banach space.*

Proof. Proof follows on the similar lines of [31, Theorem 3.6]. □

Lemma 3.3. *Let $N = M = 2^n$ for some $n \in \mathbb{N}$ and $|u_j(I)| = |v_j(I)| = \frac{1}{2^n}$ for all $j \in \Sigma_N$. Then for $m > k$, we have*

$$Osc(m, g) = N^2 Osc(m-n, f),$$

where $g(\boldsymbol{x}) := f(Q_{ij}(\boldsymbol{x}))$ for $\boldsymbol{x} \in \square_{ij}$.

Proof. We have

$$\begin{aligned} Osc(m,g) &= \sum_{|Q|=2^{-m}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |g(\boldsymbol{x}) - g(\boldsymbol{y})| \\ &= \sum_{|Q|=2^{-m}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |f(Q_{ij}(\boldsymbol{x})) - f(Q_{ij}(\boldsymbol{y}))| \\ &= \sum_{i,j\in\Sigma_N} \sum_{|Q|=2^{-(m-n)}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |f(\boldsymbol{x}) - f(\boldsymbol{y})| \\ &= N^2 Osc(m-n, f), \end{aligned} \tag{3.1}$$

completing the proof. □

For the upcoming theorem, let us introduce the following notation: $\|\alpha\|_\infty := \sup_{k\in\mathbb{N}} \|\alpha_k\|_\infty < 1$ and $Osc(m,\alpha) := \sup_{k\in\mathbb{N}} Osc(m,\alpha_k) < \infty$. Let $\mathcal{V}_f^\beta(I^2) := \{g \in \mathcal{V}^\beta(I^2) : g(x_i,y_j) = f(x_i,y_j),\ \forall\ (i,j) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0}\}$. We observe that the space $\mathcal{V}_f^\beta(I^2)$ is a closed subset of $\mathcal{V}^\beta(I^2)$. It follows that $\mathcal{V}_f^\beta(I^2)$ is a complete metric space with respect to the metric induced by norm $\|.\|_{\mathcal{V}^\beta}$.

Now, we are well-equipped to establish the next result.

Theorem 3.4. *Let $f, s_k, \alpha_k \in \mathcal{V}^\beta(I^2)$ be such that $s_k(x_i,y_j) = f(x_i,y_j)$, $\forall(i,j)$ $\in \partial\Sigma_{N,0}\times\partial\Sigma_{N,0}$. Further, we assume that $|u_j(I)| = |v_j(I)| = \frac{1}{2^n}$ for some $n \in \mathbb{N}$ and for all $j \in \Sigma_N$ with $N = M = 2^n$. If $\max\Big\{\|\alpha\|_\infty + N^2 \sup_{m\in\mathbb{N}} \frac{Osc(m,\alpha)}{2^{m(2-\beta)}}, \frac{N^2\|\alpha\|_\infty}{2^{n(2-\beta)}}\Big\} < 1$, then for any $g \in \mathcal{V}_f^\beta(I^2)$ the sequence $\{\Psi_k(g)\}$ converges in norm $\|.\|_{\mathcal{V}^\beta}$ to a map $f_s^A \in \mathcal{V}^\beta(I^2)$. Furthermore, we have $2 \le \dim_H\big(G_{f_s^A}\big) \le \underline{\dim}_B\big(G_{f_s^A}\big) \le \overline{\dim}_B\big(G_{f_s^A}\big) \le 3-\beta$.*

Proof. It is well-known [13] that

$$2 \leq \dim_H \left(G_{f_s^A}\right) \leq \underline{\dim}_B\left(G_{f_s^A}\right) \leq \overline{\dim}_B\left(G_{f_s^A}\right).$$

Now, we continue by defining a sequence of mappings $T_k : \mathcal{V}_f^\beta(I^2) \to \mathcal{V}_f^\beta(I^2)$ by

$$(T_k g)(\boldsymbol{x}) = f(\boldsymbol{x}) + \alpha_k(\boldsymbol{x})\ (g - s_k)(Q_{ij}(\boldsymbol{x})),$$

for all $\boldsymbol{x} \in \Box_{ij}$, $(i,j) \in \Sigma_N \times \Sigma_N$, where $\boldsymbol{x} = (x,y)$ and $Q_{ij}(\boldsymbol{x}) := (u_i^{-1}(x), v_j^{-1}(y))$.

Using Lemma 3.3, for $g, h \in \mathcal{V}_f^\beta(I^2)$ we have

$$\begin{aligned}
&\|T_k g - T_k h\|_{\mathcal{V}^\beta}\\
&\quad= \|T_k g - T_k h\|_\infty + \sup_{m\in\mathbb{N}} \frac{Osc(m, T_k g - T_k h)}{2^{m(2-\beta)}}\\
&\quad\leq \|\alpha\|_\infty \|g-h\|_\infty + \sum_{(i,j)\in\Sigma_N\times\Sigma_N} \frac{\|\alpha_k\|_\infty}{2^{n(2-\beta)}} \sup_{m\in\mathbb{N}} \frac{Osc(m-n, g-h)}{2^{(m-n)(2-\beta)}}\\
&\qquad+ \sum_{(i,j)\in\Sigma_N\times\Sigma_N} \|g-h\|_\infty \sup_{m\in\mathbb{N}} \frac{Osc(m,\alpha_k)}{2^{m(2-\beta)}}\\
&\quad\leq \left(\|\alpha_k\|_\infty + \sum_{(i,j)\in\Sigma_N\times\Sigma_N} \sup_{m\in\mathbb{N}} \frac{Osc(m,\alpha_k)}{2^{m(2-\beta)}}\right)\|g-h\|_\infty\\
&\qquad+ \Big(\sum_{(i,j)\in\Sigma_N\times\Sigma_N} \frac{\|\alpha_k\|_\infty}{2^{n(2-\beta)}}\Big) \sup_{m\in\mathbb{N}, m>n} \frac{Osc(m-n, g-h)}{2^{(m-n)(2-\beta)}}\\
&\quad\leq \max\left\{\|\alpha_k\|_\infty + N^2 \sup_{m\in\mathbb{N}} \frac{Osc(m,\alpha_k)}{2^{m(2-\beta)}}, \frac{N^2\|\alpha\|_\infty}{2^{n(2-\beta)}}\right\}\|g-h\|_{\mathcal{V}^\beta}.
\end{aligned}$$

From the condition taken, it follows that each T_k is a contraction map on $\mathcal{V}_f^\sigma(I^2)$. Note also that the sequence $\{\|T_k g - g\|_{\mathcal{V}^\beta}\}$ is bounded.

On applying Proposition 2.6, the backward trajectories $\Psi_k := T_1 \circ T_2 \circ \cdots \circ T_k$ of (T_k) converge for every $g \in \mathcal{V}_f^\beta(I^2)$ to a unique attractor $f_s^A \in \mathcal{V}_f^\beta(I^2)$. With the help of Theorem 3.1, we obtain $\overline{\dim}_B\left(G_{f_s^A}\right) \leq 3-\beta$. This completes the proof. $\Box$

Remark 3.5. In [11], Deliu and Jawerth showed that the oscillation spaces are refinements to Hölder spaces. They also expressed the fractal dimension in terms of different classical oscillation measures and in terms of wavelet expansions by comparing the oscillation spaces to certain Besov spaces. In [14,28,29], box dimensions of (stationary) α-fractal functions are estimated using Hölder spaces and variation method, however, the case of oscillation spaces are not discussed. Our result may generalize the results available in [15,17], because bilinear FIS is a particular case of (stationary) α-fractal function, see, for instance, [30, Remark 2.1]. It should be noted that in [15,17], the exact value of box dimensions of bilinear FISs and bilinear RFISs are computed under certain conditions, however, we provide lower and upper bounds of dimensions of non-stationary fractal function under less restrictive conditions, see, [30, Remark 2.1].

A function $f : \square \to \mathbb{R}$ is said to be Hölder continuous with exponent σ if

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \le k_f \|\boldsymbol{x} - \boldsymbol{y}\|_2^\sigma,\ \forall\ \boldsymbol{x}, \boldsymbol{y} \in \square,$$

and for some $k_f > 0$.
For Hölder continuous functions f with exponent σ, let us define σth Hölder seminorm as

$$[f]_\sigma = \sup_{\boldsymbol{x} \neq \boldsymbol{y}} \frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|}{\|\boldsymbol{x} - \boldsymbol{y}\|_2^\sigma}$$

and consider the Hölder space

$$\mathcal{H}^\sigma(\square) := \{g : I \times J \to \mathbb{R} :\ \text{g is Hölder continuous with exponent } \sigma\}.$$

The space $\mathcal{H}^\sigma(\square)$ is a Banach space when endowed with the norm $\|g\|_\sigma := \|g\|_\infty + [g]_\sigma$.

Remark 3.6. If $f \in \mathcal{H}^{\sigma_0}(\square)$ then $f \in \mathcal{H}^\sigma(\square)$ for each $0 < \sigma < \sigma_0$.

Let us define $\mathcal{H}_f^\sigma(\square) = \{g \in \mathcal{H}^\sigma(\square) : g(x_i, y_j) = f(x_i, y_j),\ \forall\ (i, j) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0}\}$. One can check that the space $\mathcal{H}_f^\sigma(\square)$ is a closed subset of $\mathcal{H}^\sigma(\square)$. It follows that $\mathcal{H}_f^\sigma(\square)$ equipped with the obvious metric is complete.

Now, we are ready to prove the next result.

Theorem 3.7. *Let f and α_k be Hölder continuous with exponent σ_1 and σ_2 respectively for every $k \in \mathbb{N}$. Let s_k be Hölder continuous with exponent σ_3 satisfying $s_k(x_i, y_j) = f(x_i, y_j),\ \forall\ (i, j) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0},\ k \in \mathbb{N}$. If $\max\left\{\|\alpha\|_\sigma, \frac{\|\alpha\|_\infty}{(\min\{|a_i|,|c_j|\})^\sigma}\right\} < 1$ then for any $g \in \mathcal{H}_f^\sigma(\square)$ the sequence $\{\Psi_k(g)\}$ converges in norm $\|.\|_\sigma$ to a map $f_s^A \in \mathcal{H}^\sigma(\square)$, where $\sigma = \min\{\sigma_1, \sigma_2, \sigma_3\}$ and $\|\alpha\|_\sigma = \sup\{\|\alpha_k\|_\sigma : k \in \mathbb{N}\}$. Furthermore, $2 \le \dim_H\left(G_{f_s^A}\right) \le \underline{\dim}_B\left(G_{f_s^A}\right) \le \overline{\dim}_B\left(G_{f_s^A}\right) \le 3 - \sigma$.*

Proof. Note [13] that

$$2 \le \dim_H\left(G_{f_s^A}\right) \le \underline{\dim}_B\left(G_{f_s^A}\right) \le \overline{\dim}_B\left(G_{f_s^A}\right).$$

From Remark 3.6, we say that f, α_k and s_k are elements of $\mathcal{H}^\sigma(\square)$, where $\sigma = \min\{\sigma_1, \sigma_2, \sigma_3\}$. Now we proceed by defining a sequence of mappings $T_k : \mathcal{H}_f^\sigma(\square) \to \mathcal{H}_f^\sigma(\square)$ by

$$(T_k g)(\boldsymbol{x}) = f(\boldsymbol{x}) + \alpha_k(\boldsymbol{x})\ (g - s_k)(Q_{ij}(\boldsymbol{x}))$$

for all $\boldsymbol{x} \in \square_{ij}$, $(i,j) \in \Sigma_N \times \Sigma_M$, where $\boldsymbol{x} = (x,y)$ and $Q_{ij}(\boldsymbol{x}) := (u_i^{-1}(x), v_j^{-1}(y))$. Then we have

$$\begin{aligned}
\big[(T_k g)\big]_\sigma = & \max_{(i,j)\in\Sigma_N\times\Sigma_M} \sup_{\boldsymbol{x}\neq\boldsymbol{y},\boldsymbol{x},\boldsymbol{y}\in\square_{ij}} \frac{|T_k g(\boldsymbol{x}) - T_k g(\boldsymbol{y})|}{\|\boldsymbol{x}-\boldsymbol{y}\|_2^\sigma} \\
\leq & \max_{(i,j)\in\Sigma_N\times\Sigma_M} \sup_{\boldsymbol{x}\neq\boldsymbol{y},\boldsymbol{x},\boldsymbol{y}\in\square_{ij}} \Bigg[\frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|}{\|\boldsymbol{x}-\boldsymbol{y}\|_2^\sigma} \\
& + \frac{|\alpha_k(\boldsymbol{x})| \Big|(g-s_k)(Q_{ij}(\boldsymbol{x})) - (g-s_k)(Q_{ij}(\boldsymbol{y}))\Big|}{\|\boldsymbol{x}-\boldsymbol{y}\|_2^\sigma} \\
& + \frac{\Big|(g-s_k)(Q_{ij}(\boldsymbol{y}))\Big| |\alpha_k(\boldsymbol{x}) - \alpha_k(\boldsymbol{y})|}{\|\boldsymbol{x}-\boldsymbol{y}\|_2^\sigma} \Bigg] \\
\leq & \ [f]_\sigma + \frac{\|\alpha\|_\infty}{(\min\{|a_i|,|c_j|\})^\sigma} \big([g]_\sigma + [s_k]_\sigma\big) + \|g - s_k\|_\infty [\alpha]_\sigma,
\end{aligned}$$

this immediately show that T_k is well-defined. Let $g, h \in \mathcal{H}_f^\sigma(\square)$. We have

$$\begin{aligned}
\|T_k g - T_k h\|_\sigma &= \|T_k g - T_k h\|_\infty + [T_k g - T_k h]_\sigma \\
&\leq \|\alpha_k\|_\infty \|g-h\|_\infty \\
&\quad + \frac{\|\alpha_k\|_\infty}{(\min\{|a_i|,|c_j|\})^\sigma} [g-h]_\sigma + [\alpha_k]_\sigma \|g-h\|_\infty \qquad (3.2) \\
&\leq \max\left\{ \|\alpha\|_\sigma, \frac{\|\alpha\|_\infty}{(\min\{|a_i|,|c_j|\})^\sigma} \right\} \|g-h\|_\sigma.
\end{aligned}$$

This yields that each T_k is a contraction mapping on $\mathcal{H}_f^\sigma(\square)$.

Note also that the sequence $\{\|T_k g - g\|_\sigma\}$ is bounded. Using Proposition 2.6, the backward trajectories $\Psi_k := T_1 \circ T_2 \circ \cdots \circ T_k$ of (T_k) converge for every $g \in \mathcal{H}_f^\sigma(\square)$ to a unique attractor $f_s^A \in \mathcal{H}_f^\sigma(\square)$. Since $f_s^A \in \mathcal{H}_f^\sigma(\square)$, one deduces $\overline{\dim}_B\big(G_{f_s^A}\big) \leq 3 - \sigma$. This proves the result. $\square$

Remark 3.8. The above theorem can be compared with [28, Theorems 5.2.7 - 5.2.9]. To be precise, if we choose $s_k = s$ and $\alpha_k = \alpha$ for all $k \in \mathbb{N}$ then the above theorem will reduce to [28, Theorems 5.2.7 - 5.2.9], and further, it can be compared to [14, Theorem 4.2 (i)], where the box dimension of the stationary bivariate fractal function is estimated, that is, $\overline{\dim}\big(G_{f_s^A}\big) \leq 3 - \sigma$ under the condition:

$$\sum_{i=1}^{N} \sum_{j=1}^{M} \sup\{\alpha(x,y) : (x,y) \in \square_{ij}\} \leq 1,$$

with the help of method of variation over small subrectangles. However, we here use the dimensional property of Hölder space to obtain the bound: $\overline{\dim}_B\big(G_{f_s^A}\big) \leq 3 - \sigma$. Further, we emphasize on the fact that both of the above techniques have been applied in [28] for obtaining different bounds on the box dimension of stationary bivariate fractal function under different conditions.

4. Associated Fractal Operator

Let us recall that $\|\alpha\|_\infty = \sup_{k\in\mathbb{N}} \|\alpha_k\|_\infty < 1$ and $\|s\|_\infty = \sup_{k\in\mathbb{N}} \|s_k\|_\infty$. In view of Theorem 2.10 and Sect. 2, we are now well-equipped to prove the next theorem.

Theorem 4.1. *The non-stationary fractal function f_s^A satisfies the following properties.:*

(1) If A is the null sequence then $f_s^A = f$.

(2) The perturbation error is given by:

$$\|f_s^A - f\|_\infty \leq \sum_{k=1}^{\infty} \|\alpha\|_\infty^k \|f - s_k\|_\infty \leq \frac{\|\alpha\|_\infty C_{f,s}}{1-\|\alpha\|_\infty},$$

where $C_{f,s} = \sup_{k\in\mathbb{N}}\{\|f - s_k\|_\infty\}$.

(3) If the sequence $\{\alpha^m = \{\alpha_k^m\}\}_{m\in\mathbb{N}}$ converges to the null sequence in sup-norm then $f_s^{A^m}$ converges to f.

(4) If the sequence $\{s^m = \{s_k^m\}\}_{m\in\mathbb{N}}$ is such that $C_{f,s^m} \to 0$ as $m \to \infty$, then $f_{s^m}^A$ converges to f.

Proof. (1) Due to Eq. (2.2),

$$T^{\alpha_k} g(\boldsymbol{x}) = f(\boldsymbol{x}) + \alpha_k(\boldsymbol{x}) g(Q_{ij}(\boldsymbol{x})) - \alpha_k(\boldsymbol{x}) s_k(Q_{ij}(\boldsymbol{x})),\ \forall\, \boldsymbol{x} \in \square_{ij}, \quad (i,j) \in \Sigma_N \times \Sigma_M,$$

where $\boldsymbol{x} = (x,y)$ and $Q_{ij}(\boldsymbol{x}) := (u_i^{-1}(x), v_j^{-1}(y))$. Let A be a null sequence. That is, $\alpha_k = 0$ for all $k \in \mathbb{N}$. Then we get $T^{\alpha_k} g(\boldsymbol{x}) = f(\boldsymbol{x})$ for all $\boldsymbol{x} \in \square$. This on induction provides

$$T^{\alpha_1} \circ T^{\alpha_2} \circ \cdots \circ T^{\alpha_k} g(\boldsymbol{x}) = f(\boldsymbol{x}),\ \forall\, \boldsymbol{x} \in \square.$$

On taking the limit $k \to \infty$, we have

$$f_s^A(\boldsymbol{x}) = \lim_{k\to\infty} T^{\alpha_1} \circ T^{\alpha_2} \circ \cdots \circ T^{\alpha_k} g(\boldsymbol{x}) = f(\boldsymbol{x}),\ \forall\, \boldsymbol{x} \in \square,$$

proving the assertion.

(2) By definition of RB operators T^{α_k}, we have

$$\begin{aligned} &T^{\alpha_1} \circ T^{\alpha_2} \circ \cdots \circ T^{\alpha_k} f(\boldsymbol{x}) - f(\boldsymbol{x}) \\ &\quad = \alpha_1(\boldsymbol{x}) \Big(T^{\alpha_2} \circ T^{\alpha_3} \circ \cdots \circ T^{\alpha_k} f - s_1 \Big)(Q_{ij}(\boldsymbol{x})), \forall\, \boldsymbol{x} \in \square_{ij}. \end{aligned}$$

Inductively, we get

$$T^{\alpha_1} \circ T^{\alpha_2} \circ \cdots \circ T^{\alpha_k} f(\boldsymbol{x}) - f(\boldsymbol{x}) = \sum_{l=1}^{k} \alpha_1(\boldsymbol{x}) \ldots \alpha_l(Q_{ij}^l(\boldsymbol{x})) \big(f - s_l\big)(Q_{ij}^l(\boldsymbol{x})), \quad \forall\, \boldsymbol{x} \in \square_{ij},$$

where Q_{ij}^l is a suitable finite composition of mappings Q_{ij}. Now,

$$\|T^{\alpha_1} \circ T^{\alpha_2} \circ \cdots \circ T^{\alpha_k} f - f\|_\infty \leq \sum_{l=1}^{k} \|\alpha\|_\infty^l \|f - s_l\|_\infty.$$

As $k \to \infty$, we have

$$\|f_s^A - f\|_\infty \le \sum_{l=1}^{\infty} \|\alpha\|_\infty^l \|f - s_l\|_\infty \le \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \sup_{k\in\mathbb{N}} \|f - s_k\|_\infty.$$

This completes the proof of item (2).

(3) From the proof of item (2), we get

$$\|f_s^{A^m} - f\|_\infty \le \frac{\|\alpha^m\|_\infty C_{f,s}}{1 - \|\alpha^m\|_\infty}.$$

Since $\|\alpha^m\|_\infty = \sup_{k\in\mathbb{N}} \|\alpha_k^m - 0\|_\infty \to 0$ as $m \to \infty$, we have the required result.

(4) The proof follows on similar lines of the previous item.

□

Remark 4.2. In [32–34], Vijender introduced and studied the (univariate stationary) Bernstein fractal functions by considering the base function as Bernstein polynomial $B_n(f)$ of the generating function f in the $\alpha-$fractal function. Our result mentioned above can be treated as a non-stationary generalization of his result, and we believe that the above result will find many applications similar to Vijender's work.

By considering $s_k = L_k(f)$, where $L_k : \mathcal{C}(\square) \to \mathcal{C}(\square)$ is a bounded linear operator such that $L_k g(x_i, y_j) = g(x_i, y_j),\ \forall\ (i,j) \in \partial\Sigma_{N,0} \times \partial\Sigma_{M,0},\ k \in \mathbb{N}$, and $\|L\|_\infty := \sup_{k\in\mathbb{N}} \|L_k\| < \infty$.

Now, we define an operator so-called non-stationary fractal operator $\mathcal{F}_s^A : \mathcal{C}(\square) \to \mathcal{C}(\square)$ defined as:

$$\mathcal{F}_s^A(f) = f_s^A,$$

where f_s^A is the non-stationary fractal function corresponding to f and fixed sequences (α_k) and (L_k).

Theorem 4.3. *The fractal operator $\mathcal{F}_s^A : \mathcal{C}(\square) \to \mathcal{C}(\square)$ is a bounded linear operator where s_k is taken as above.*

Proof. Let $f, g \in \mathcal{C}(\square)$ and $\beta, \gamma \in \mathbb{R}$. In view of the proof of item (2) in Theorem 4.1, we write

$$(\beta f)_s^A(\boldsymbol{x}) = \beta f(\boldsymbol{x}) + \lim_{k\to\infty} \sum_{l=1}^{k} \alpha_1(\boldsymbol{x}) \dots \alpha_l(Q_{ij}^l(\boldsymbol{x}))\big(\beta f - L_l(\beta f)\big)(Q_{ij}^l(\boldsymbol{x})),$$

and

$$(\gamma g)_s^A(\boldsymbol{x}) = \gamma g(\boldsymbol{x}) + \lim_{k\to\infty} \sum_{l=1}^{k} \alpha_1(\boldsymbol{x}) \dots \alpha_l(Q_{ij}^l(\boldsymbol{x}))\big(\gamma g - L_l(\gamma g)\big)(Q_{ij}^l(\boldsymbol{x})).$$

Using linearity of L_k, the above equations give

$$(\beta f)_s^A(\boldsymbol{x}) + (\gamma g)_s^A(\boldsymbol{x}) = (\beta f + \gamma g)(\boldsymbol{x}) + \lim_{k\to\infty} \sum_{l=1}^{k} \alpha_1(\boldsymbol{x}) \dots$$
$$\alpha_l(Q_{ij}^l(\boldsymbol{x}))\big(\beta f + \gamma g - L_l(\beta f + \gamma g)\big)(Q_{ij}^l(\boldsymbol{x})).$$

Now, since

$$(\beta f+\gamma g)_s^A(\boldsymbol{x}) = (\beta f+\gamma g)(\boldsymbol{x}) + \lim_{k\to\infty}\sum_{l=1}^{k}\alpha_1(\boldsymbol{x})\ldots \\ \alpha_l(Q_{ij}^l(\boldsymbol{x}))\big(\beta f+\gamma g - L_l(\beta f+\gamma g)\big)(Q_{ij}^l(\boldsymbol{x})),$$

we deduce that

$$(\beta f+\gamma g)_s^A = (\beta f)_s^A + (\gamma g)_s^A,$$

that is, $\mathcal{F}_s^A$ is a linear operator. Again by item (2) of Theorem 4.1, we get

$$\|\mathcal{F}_s^A(f)\|_\infty - \|f\|_\infty \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\sup_{k\in\mathbb{N}}\|f-L_kf\|_\infty \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\,C_L\|f\|_\infty,$$

where $C_L=\sup_{k\in\mathbb{N}}\|I-L_k\|$. Equivalently,

$$\|\mathcal{F}_s^A(f)\|_\infty \le \Big(1+\frac{C_L\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\Big)\|f\|_\infty,$$

establishing the result. □

Remark 4.4. We can deduce [14, Theorem 3.2] from the above theorem by choosing $L_k=B_{m,n}$ and $\alpha_k=\alpha$ for all $k\in\mathbb{N}$ and for some $m,n\in\mathbb{N}$, where $B_{m,n}$ denotes the (bivarate) Bernstein polynomial operator of order (m,n).

Remark 4.5. Using item-(2) in Theorem 4.1, we have

$$\|\mathcal{F}_s^A(f)-f\|_\infty \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}C_L\|f\|_\infty,$$

where $C_L=\sup_{k\in\mathbb{N}}\|I-L_k\|$. From this, we deduce the following result: If $\|\alpha\|_\infty<\frac{1}{1+C_L}$ then $\mathcal{F}_s^A$ is a topological isomorphism. The topological isomorphism of $\mathcal{F}_s^A$ can be used to construct a Schauder basis of $\mathcal{C}(\square)$ consisting of nonstationary fractal functions. Since $\mathcal{F}_s^A$ is a topological automorphism for $\|\alpha\|_\infty<\frac{1}{1+C_L}$, we have

$$\bigcup_{A,s}\mathcal{F}_s^A(\mathcal{C}(\square))=\mathcal{C}(\square),$$

where the union is taken over all possible A and s.

Remark 4.6. In view of the above results, one can prove the non-stationary version of [30, Theorems 3.2, 4.1, 4.4].

In the next result we show the existence of a non-trivial closed invariant subspace for the non-stationary fractal operator. The proof of this next theorem is given by modifying and adapting some standard techniques present in the literature on invariant subspace problem; see, for instance, [4, 28].

Theorem 4.7. *There exists a non-trivial closed invariant subspace for the fractal operator* $\mathcal{F}_s^A:\mathcal{C}(\square)\to\mathcal{C}(\square)$.

Proof. We begin by taking a non-zero continuous function $f : \square \to \mathbb{R}$ such that $f(x_i, y_j) = 0$ for every $(x_i, y_j) \in \Delta$. Denote by $(\mathcal{F}_s^A)^r$ the r-fold composition of $\mathcal{F}_s^A$ with itself, and $(\mathcal{F}_s^A)^0 := f$. Now, we construct the $\mathcal{F}_s^A$-cyclic subspace generated by f, that is,

$$Y_f = \operatorname{span}\{f, \mathcal{F}_s^A(f), (\mathcal{F}_s^A)^2(f), \dots\}.$$

It is obvious that $Y_f \neq \{0\}$ and $\mathcal{F}_s^A(Y_f) \subseteq Y_f$. Let $g \in Y_f$. Using the definition of Y_f, there exist constants $t_i \in \mathbb{R}$ and $r_i \in \mathbb{N} \cup \{0\}$ such that

$$g = t_1(\mathcal{F}_s^A)^{r_1}(f) + t_2(\mathcal{F}_s^A)^{r_2}(f) + \dots + t_m(\mathcal{F}_s^A)^{r_m}(f).$$

With the help of the interpolatory property of the fractal operator, we obtain

$$f(x_i, y_j) = (\mathcal{F}_s^A(f))(x_i, y_j)$$

for every $(x_i, y_j) \in \Delta$. This implies that $g(x_i, y_j) = 0$, $\forall\ (x_i, y_j) \in \Delta$. Assume that $Y = \overline{Y_f}$. We immediately establish that $\mathcal{F}_s^A(Y) \subseteq Y$, hence that Y is a closed invariant subspace of $\mathcal{F}_s^A$.

To prove Y is a nontrivial closed invariant subspace of $\mathcal{F}_s^A$, we proceed as follows. Let $h \in Y$. Then there exists a sequence $(h_n)_{n \in \mathbb{N}} \subset Y_f$ such that $h_n \to h$ uniformly. Since uniform convergence implies pointwise convergence, we deduce that $h(x_i, y_j) = 0$, $\forall\ (x_i, y_j) \in \Delta$. From which we conclude that a continuous function that is nonzero at some points in Δ does not belong to Y. In particular, $Y \neq \mathcal{C}(\square)$, completing the proof. $\square$

Remark 4.8. Let us consider a set of data on a grid $\{(x_i, y_j, t_{ij}) : i \in \Sigma_{N,0},\ j \in \Sigma_{M,0}\}$. The set of continuous interpolants of these data is a convex, closed invariant set with respect to the transformation $\mathcal{F}_s^A$. If $t_{ij} = 0$ for all i, j, then it is also an invariant linear subspace.

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Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article. There is no data associated with this paper.

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