FRACTAL SOBOLEV SYSTEMS OF FUNCTIONS ASSOCIATED WITH ORTHONORMAL SYSTEMS OF FUNCTIONS

Abstract. The paper introduces the uniform boundedness of the Sobolev orthonormal systems of functions associated with uniformly bounded orthonormal complete systems of continuous functions and also defines the α-fractal Sobolev system corresponding to Sobolev Orthonormal System. An approximation related result similar to Weierstrass theorem is derived. It has been shown that the set of α-fractal versions of Sobolev sums is dense and complete in the weighted Sobolev space $\mathcal{W}_\rho^{r,2}(I)$. A Schauder basis and a Riesz basis of fractal type for the space $\mathcal{W}_\rho^{r,2}(I)$ are found. The Fourier-Sobolev expansion of an α-fractal function f^α corresponding to a certain set of interpolation points is presented. Also the convergence of $f^\alpha$with respect to uniform norm and Sobolev norm are established.

1. Introduction

Fractal approximation has been used to describe geometrical structure of the objects which have irregular and complex characteristics in nature such as surface of broken stone, lightning, clouds, mountain ranges, coastlines, price graphs, smoke etc. The geometrical structures and the properties of such irregular objects were first addressed by Benoit B. Mandelbrot and he coined it as fractal theory [1].

Fractal interpolation functions whose graphs are fractals have been broadly used in approximation theory, interpolation theory, financial series, computer graphics, signal processing and many on. Barnsley [2], constructed the fractal interpolation function (FIF) using Hutchinson's operator [3] on an iterated function system (IFS), whose attractor is the graph of a continuous function interpolating a certain data set.

Navascués [7, 8], defined an α-fractal interpolation function f^α as a fractal version of a continuous function $f \in \mathcal{C}(I)$ on a compact interval I of $\mathbb{R}$. The function f^α is continuous but non differentiable in nature.

The theory and application of α-fractal interpolation function f^α have been extensively studied by Navascués [8, 10]. In [4], several properties of the operator $\mathcal{F}^\alpha : \mathcal{C}(I) \to \mathcal{C}(I)$ defined by $f \to f^\alpha$ have been analysed and also been extended to more general spaces like $\mathcal{L}^p$-space $\mathcal{L}^p(I)(1 \leq p < \infty)$, r-smooth function space $\mathcal{C}^r(I)$ and the Sobolev space $\mathcal{W}^{r,p}(I)$.

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In [11], Sharapudinov defined the Sobolev orthonormal system $\{\varphi_{r,k}\}_{k=0}^{\infty}$ associated with an orthonormal system $\{\varphi_k\}_{k=0}^{\infty}$ of functions defined on $I=[a,b]$ with weight function $\rho(x)$. Some particular cases of the Sobolev orthonormal systems of the form $\{\varphi_{r,k}\}_{k=0}^{\infty}$ generated by the classical Jacobi, Legendre, Chebyshev and Haar orthonormal systems were considered in [11]-[15]. Results on the uniform convergence of Fourier-Sobolev series can be found in [11].

In [19], the fractal Jacobi system is defined and the convergence of Fourier-Jacobi expansion for an affine FIF as well as non-affine FIF is briefly discussed. In the same paper, a fractal version of a classical result namely Weierstrass theorem is found and also it is proved that the fractal Jacobi system forms a Schauder basis for a space of weighted square integrable functions.

In the present paper, it is shown that the Sobolev orthonormal system of functions is uniformly bounded. Mainly, the α-fractal Sobolev system is defined here and an approximation based result similar to Weierstrass theorem is derived. It is proved that the set of α-fractal versions of Sobolev sums is dense in the weighted Sobolev space $\mathcal{W}_\rho^{r,2}(I)$ and the α-fractal Sobolev system is complete in $\mathcal{W}_\rho^{r,2}(I)$. Also this paper shows that the α-fractal Sobolev system forms a Schauder basis and also a Riesz basis for $\mathcal{W}_\rho^{r,2}(I)$. Some results on convergence of Fourier-Sobolev expansion of an α-fractal function corresponding to a certain data set, are established.

2. Definitions and Notations

2.1. Weighted Sobolev Space.

Consider a closed compact interval $I=[a,b]$. Let $\rho:\mathbb{R}\to(0,\infty)$ be any continuous weight function. For $1\le p<\infty$, a weighted $\mathcal{L}^p$-space is defined as $\mathcal{L}_\rho^p(I):=\{f:I\to\mathbb{R}; f$ is measurable and $\|f\|_{\mathcal{L}_\rho^p(I)}<\infty\}$, where

$$\|f\|_{\mathcal{L}_\rho^p(I)} := \left[\int_I |f(x)|^p \rho(x)dx\right]^{\frac{1}{p}}, 1\le p<\infty.$$

For sup-norm and $\mathcal{L}_\rho^p$-norm, we have, if $f\in\mathcal{L}^\infty(I)$

$$\|f\|_{\mathcal{L}_\rho^p(I)} \le \|f\|_\infty \left[\int_I \rho(x)dx\right]^{\frac{1}{p}},$$

where sup-norm $\|f\|_\infty = ess\sup\{|f(x)|: x\in I\}$.
When $\rho(x)=1$, then $\mathcal{L}_\rho^p(I)=\mathcal{L}^p(I)$ and write $\mathcal{L}^1(I)=\mathcal{L}(I)$.

The weighted Sobolev space [11] is defined as $\mathcal{W}_\rho^{r,2}(I):=\{f:I\to\mathbb{R}; f$ is $(r-1)$-times continuously differentiable on I such that $f^{(r-1)}$ is absolutely continuous on I and $f^{(r)}\in\mathcal{L}_\rho^2(I)\}$. The inner product in $\mathcal{W}_\rho^{r,2}(I)$ is defined by

$$\langle f,g\rangle = \sum_{j=0}^{r-1} f^{(j)}(a)g^{(j)}(a) + \int_I f^{(r)}(x)g^{(r)}(x)\rho(x)dx, \qquad (2.1)$$

for all $f, g \in \mathcal{W}_\rho^{r,2}(I)$, if $\frac{1}{\rho(x)} \in \mathcal{L}(I)$. The norm is defined as

$$\|f\|_{\mathcal{W}_\rho^{r,2}(I)} = \left[\sum_{j=0}^{r-1} \left| f^{(j)}(a) \right|^2 + \int_I \left| f^{(r)}(x) \right|^2 \rho(x)dx \right]^{\frac{1}{2}},$$

for all $f \in \mathcal{W}_\rho^{r,2}(I)$.

This makes $\mathcal{W}_\rho^{r,2}(I)$ a Banach space. $\mathcal{W}_\rho^{r,2}(I)$ is also a Hilbert space with the inner product (2.1). When $\rho(x) = 1$, then $\mathcal{W}_\rho^{r,2}(I) = \mathcal{W}^{r,2}(I)$. It is needed to mention that for any compact interval I, $\mathcal{C}^r(I)$ is dense in $\mathcal{W}_\rho^{r,2}(I)$ (in view of Theorem 2, Page-251 in [21]).

2.2. **Sobolev Orthonormal System.**

Let $\{\varphi_k\}_{k=0}^{\infty}$ be an orthonormal system of functions on a compact interval $I = [a, b]$ with the weight function $\rho(x)$. That is,

$$\langle \varphi_k, \varphi_m \rangle = \int_I \varphi_k(x)\varphi_m(x)\rho(x)dx = \begin{cases} 1; \ k = m. \\ 0; \ k \neq m. \end{cases} \tag{2.2}$$

From (2.2), $\int_I |\varphi_k(x)|^2 \rho(x)dx = 1$. Therefore $\varphi_k \in \mathcal{L}_\rho^2(I)$, $k = 0, 1, 2, ...$ By including the condition that $\varphi_k \in \mathcal{L}(I), k = 0, 1, 2, ...$, one can define the following functions generated by the system $\{\varphi_k\}_{k=0}^{\infty}$ (see [11]). Let $r \in \mathbb{N}$,

$$\varphi_{r,r+k}(x) = \frac{1}{(r-1)!} \int_a^x (x-t)^{r-1} \varphi_k(t)dt; \ k = 0, 1, 2, ... \tag{2.3}$$

and

$$\varphi_{r,k}(x) = \frac{(x-a)^k}{k!}; \ k = 0, 1, 2, ...(r-1). \tag{2.4}$$

It follows from (2.3) and (2.4) that for a.e. $x \in I$,

$$\varphi_{r,k}^{(j)}(x) = \begin{cases} \varphi_{r-j,k-j}(x) \quad if \ 0 \leq j \leq r-1, \ r \leq k. \\ \varphi_{k-r}(x) \quad if \ \ j = r \leq k. \\ \varphi_{r-j,k-j}(x) \quad if \ j \leq k < r. \\ 0 \quad if \ \ k < j \leq r. \end{cases} \tag{2.5}$$

In [11], it is shown that $\{\varphi_{r,k}\}_{k=0}^{\infty}$ is an orthonormal system in $\mathcal{W}_\rho^{r,2}(I)$ and also complete in $\mathcal{W}_\rho^{r,2}(I)$ with respect to (2.1) if $\{\varphi_k\}_{k=0}^{\infty}$ is complete in $\mathcal{L}_\rho^2(I)$ and if $\frac{1}{\rho(x)} \in \mathcal{L}(I)$.

The system $\{\varphi_{r,k}\}_{k=0}^{\infty}$ is known as Sobolev orthonormal system generated by the orthonormal system $\{\varphi_k\}_{k=0}^{\infty}$. If the initial system $\{\varphi_k\}_{k=0}^{\infty}$ is of continuous functions, then from (2.5), each $\varphi_{r,k} \in \mathcal{C}^r(I)$.

The Fourier series of $f \in \mathcal{W}_\rho^{r,2}(I)$ with respect to the system $\{\varphi_{r,k}\}_{k=0}^{\infty}$ is of a mixed form, which is given by (see [11])

$$f(x) \sim \sum_{k=0}^{r-1} f^{(k)}(a) \frac{(x-a)^k}{k!} + \sum_{k=r}^{\infty} f_{r,k}(f)\varphi_{r,k}(x), \tag{2.6}$$

where

$$f_{r,k} = f_{r,k}(f) = \int_a^b f^{(r)}(t)\varphi_{r,k}^{(r)}(t)\rho(t)dt = \int_a^b f^{(r)}(t)\varphi_{k-r}(t)\rho(t)dt. \quad (2.7)$$

The series (2.6) converges to f uniformly also if $\frac{1}{\rho(x)} \in \mathcal{L}(I)$ (see Theorem 22, [11]).

The n-th partial sum of (2.6) is defined by

$$Y_{r,n}(f,x) = \sum_{k=0}^{r-1} f^{(k)}(a)\frac{(x-a)^k}{k!} + \sum_{k=r}^{n} f_{r,k}(f)\varphi_{r,k}(x); \text{ for } x \in I.$$

2.3. **Fractal Interpolation Function.**

Let $N \geq 2$ be an integer. Consider a set of interpolation points $D = \{(x_i, y_i) \in I \times \mathbb{R} : i = 0, 1, ..., N\}$, where $\Delta : x_0 < x_1 < ... < x_N$ is a partition of the closed interval $I = [x_0, x_N]$. Set $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, ..., N$. Let $L_i : I \to I_i$, $i = 1, 2, ..., N$, be contraction homeomorphisms such that

$$L_i(x_0) = x_{i-1},\ L_i(x_N) = x_i, \quad (2.8)$$

$$|L_i(x) - L_i(y)| \leq a_i|x - y|, \quad (2.9)$$

for all x, y in I and $0 \leq a_i < 1$.

Again, let $F_i : I \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, ..., N$ be given continuous functions such that

$$F_i(x_0, y_0) = y_{i-1}, F_i(x_N, y_N) = y_i, \quad (2.10)$$

$$|F_i(x, y) - F_i(x, y')| \leq |\alpha_i||y - y'|, \quad (2.11)$$

for all x in I and for all y, y' in $\mathbb{R}$ and for some $0 \leq |\alpha_i| < 1, i = 1, 2, ..., N$.

Let

$$\mathcal{C}^*(I) = \{f \in \mathcal{C}(I) : f(x_0) = y_0, f(x_N) = y_N\},$$

$$\mathcal{C}^{**}(I) = \{f \in \mathcal{C}(I) : f(x_i) = y_i;\ i = 0, 1, 2, ..., N\}.$$

The Read-Bajraktarvic (RB) operator $T : \mathcal{C}^*(I) \to \mathcal{C}^{**}(I)$ is defined by (see [6])

$$(Tf)(x) = F_i(L_i^{-1}(x), f(L_i^{-1}(x)));\ x \in I_i, i = 1, 2, ..., N.$$

is a contraction with contractivity factor $|\alpha|_\infty = \max\{|\alpha_i| : i = 1, 2, ..., N\}$. Due to Banach fixed point theorem, T has a unique fixed point g (say). Furthermore, g interpolates the data set D and satisfies the fixed point functional equation

$$g(x) = F_i(L_i^{-1}(x), g(L_i^{-1}(x)));\ x \in I_i, i = 1, 2, ..., N. \quad (2.12)$$

Define mappings $W_i : I \times \mathbb{R} \to I_i \times \mathbb{R};\ i = 1, 2, ..., N$ by $W_i(x, y) = (L_i(x), F_i(x, y))$, for all $(x, y) \in I \times \mathbb{R}$. Then

$$W = \langle I \times \mathbb{R}; W_i(x, y) : i = 1, 2, ..., N \rangle,$$

constitutes an IFS. Barnsley [2], proved that this IFS W has a unique attractor G, where G is the graph of a continuous function $g : I \to R$ interpolating the points of D. This function g is called a fractal interpolation function (FIF) and it is the unique function satisfying (2.12).

The most studied FIFs are defined by the iterated mappings

$$L_i(x) = a_i x + e_i,\ i = 1, 2, ..., N, \tag{2.13}$$

where $a_i = \frac{x_i - x_{i-1}}{x_N - x_0}$, $e_i = \frac{x_N x_{i-1} - x_0 x_i}{x_N - x_0}$.

$$F_i(x, y) = \alpha_i y + q_i(x),\ i = 1, 2, ..., N, \tag{2.14}$$

where $q_i(x)$s are suitable continuous functions on I for which F_i satisfies (2.10) and (2.11). For each $i = 1, 2, ..., N$, α_i is a free parameter with $|\alpha_i| < 1$ and is called a vertical scaling factor of the transformation W_i. Then the vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is called the scale vector of this IFS. If $q_i(x)$s are affine maps, then the corresponding FIF is known as affine FIF (AFIF).

2.4. α-Fractal Interpolation Function.

Let $f \in \mathcal{C}(I)$. Consider the IFS defined by the iterated mappings in (2.13) and (2.14), where $q_i(x) = f(L_i(x)) - \alpha_i b(x)$, i.e.,

$$F_i(x, y) = \alpha_i y + f(L_i(x)) - \alpha_i b(x),\ i = 1, 2, ..., N \tag{2.15}$$

and $b \in \mathcal{C}(I)$, known as base function that follow $b(x_0) = f(x_0)$ and $b(x_N) = f(x_N)$. Let f^α be the continuous function whose graph is the attractor of the IFS defined by (2.13) and (2.15). Then, the function f^α is called the α-fractal function of f with respect to the base function b and the partition Δ. From (2.12), f^α satisfies the fixed point equation

$$f^\alpha(x) = f(x) + \alpha_i (f^\alpha - b)(L_i^{-1}(x)), \tag{2.16}$$

for all $x \in I_i$, $i = 1, 2, ..., N$. From (2.16), it is easy to deduce that

$$\|f^\alpha - f\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|f - b\|_\infty . \tag{2.17}$$

For $\alpha = 0$, the fractal function is same as the classical one. More discussion about α-fractal function for different choices of b can be found in [7]-[9]. Take

$$b = Lf,$$

where $L : \mathcal{C}(I) \to \mathcal{C}(I)$ is a linear and bounded operator with respect to the uniform norm on $\mathcal{C}(I)$ such that $Lf(x_0) = f(x_0)$ and $Lf(x_N) = f(x_N)$. Then from (2.17), for any $f \in \mathcal{C}(I)$ and its fractal function f^α satisfies

$$\|f^\alpha - f\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I_d - L\|_\infty \|f\|_\infty , \tag{2.18}$$

where I_d is identity operator and $\|I_d - L\|_\infty$ represents the corresponding operator norm as well.

In the following, the fractal function of any function from the space $\mathcal{C}^r(I)$ is discussed.

Theorem 2.1. [2]*(Barnsley et al.) For a given data set $x_0 < x_1 < ... < x_N$, let $L_i(x) = a_i x + e_i$ be such that it satisfies (2.8), (2.9) and $F_i(x,y) = \alpha_i y + q_i(x)$ satisfy (2.10), (2.11) for $i = 1,2,...,N$. Suppose for some $r > 0$, $|\alpha_i| < s a_i^r$, $0 < s < 1$ and $q_i \in \mathcal{C}^r[x_0, x_N]$, $i = 1,2,...,N$. Let $F_{i,k}(x,y) = \frac{\alpha_i y + q_i^{(k)}(x)}{a_i^k}$, $y_{0,k} = \frac{q_1^{(k)}(x_0)}{a_1^k - \alpha_1}$, $y_{N,k} = \frac{q_N^{(k)}(x_N)}{a_N^k - \alpha_N}$, $k = 1,2,...,r$. If $F_{i-1,k}(x_N, y_{N,k}) = F_{i,k}(x_0, y_{0,k})$ for $i = 1,2,...,N$ and $k = 1,2,...,r$, then $\{L_i(x), F_i(x,y)\}_{i=1}^N$ determines a FIF $f \in \mathcal{C}^r[x_0, x_N]$ and $f^{(k)}$ is the FIF determined by $\{L_i(x), F_{i,k}(x,y)\}_{i=1}^N$ for $k = 1,2,...,r$.*

In view of the Theorem 2.1, P. Viswanathan established α-fractal version of r-smooth functions with scaling functions $\alpha_i(x)$ in place of constant scaling factors α_i.

Theorem 2.2. [4, 5]*(Viswanathan et al.) Let $f \in \mathcal{C}^r(I)$, $r \in \mathbb{N}$. Suppose that $\Delta = \{x_0, x_1, ..., x_N\}$ is a partition of $I = [x_0, x_N]$ satisfying $x_0 < x_1 < ... < x_N$, $I_i = [x_{i-1}, x_i]$ for $i = 1,2,...,N$ and $L_i : I \to I_i$ are affine maps $L_i(x) = a_i x + e_i$ satisfying $L_i(x_0) = x_{i-1}$, $L_i(x_N) = x_i$ for $i = 1,2,...,N$. Suppose that r-times continuously differentiable scaling functions and base function are selected so that*

$$\|\alpha_i\|_{\mathcal{C}^r(I)} \leq (\frac{a_i}{2})^r,$$

$$b^{(j)}(x_0) = f^{(j)}(x_0), b^{(j)}(x_N) = f^{(j)}(x_N);\ j = 0,1,2,...,r.$$

Then the RB operator defined by

$$(Tg)(x) = f(x) + \alpha_i(L_i^{-1}(x))(g - b)(L_i^{-1}(x));\ x \in I_i,\ i = 1,2,...,N,$$

is a contraction on the complete metric space

$$\mathcal{C}_f^r(I) := \{g \in \mathcal{C}^r(I) : g^{(j)}(x_0) = f^{(j)}(x_0), g^{(j)}(x_N) = f^{(j)}(x_N); j = 0,1,...,r\}.$$

and the corresponding fractal function f^α is r-smooth. Furthermore, the derivative $(f^\alpha)^{(j)}$ of its unique fixed point f^α satisfies the self-referential equation

$$(f^\alpha)^{(j)}(x) = f^{(j)}(x) + a_i^{-j}\left[\sum_{m=0}^{j}\binom{j}{m}\alpha_i^{j-m}(L_i^{-1}(x))(f^\alpha - b)^{(m)}(L_i^{-1}(x))\right];$$

$x \in I_i, i = 1,2,...,N$, $j = 0,1,2,...,r$ and consequently, f^α agrees with f at the knot points up to the r-th derivative.

If we consider again constant scale factors, for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ with $|\alpha|_\infty \leq (\frac{a_i}{2})^r$; $i = 1,2,...,N$, the α-fractal function f^α of $f \in \mathcal{C}^r(I)$ satisfies the self-referential equation

$$f^\alpha(x) = f(x) + \alpha_i(f^\alpha - b)(L_i^{-1}(x)) \tag{2.19}$$

and for $j = 0,1,2,...,r$

$$(f^\alpha)^{(j)}(x_i) = f^{(j)}(x_i);\ i = 0,1,2,...,N. \tag{2.20}$$

It follows from (2.19) that

$$\|f^{\alpha} - f\|_{\mathcal{C}^r(I)} \leq \delta \|f^{\alpha} - b\|_{\mathcal{C}^r(I)}, \tag{2.21}$$

$$\text{i.e., } \|f^{\alpha} - f\|_{\mathcal{C}^r(I)} \leq \frac{\delta}{1-\delta} \|f - b\|_{\mathcal{C}^r(I)}, \tag{2.22}$$

where $\delta = \frac{|\alpha|_{\infty}}{\min\{a_i^r;\, i=1,...,N\}}$ and $\|f\|_{\mathcal{C}^r(I)} := \max\{\left\|f^{(j)}\right\|_{\infty} : j = 0,1,2,...,r\}$. In [4], it is seen that for fixed scale vector and various choice of base functions, the operator $\mathcal{F}^{\alpha} : \mathcal{C}^r(I) \to \mathcal{C}^r(I)$ defined by

$$\mathcal{F}^{\alpha}(f) = f^{\alpha}, \text{ for all } f \in \mathcal{C}^r(I)$$

is linear and bounded. Let $b = Lf$, where $L : \mathcal{C}^r(I) \to \mathcal{C}^r(I)$ is a linear and bounded operator with respect to the norm on $\mathcal{C}^r(I)$ such that $Lf^{(j)}(x_0) = f^{(j)}(x_0)$ and $Lf^{(j)}(x_N) = f^{(j)}(x_N)$; $j = 0,1,2,...,r$. Then (2.22) becomes

$$\|f^{\alpha} - f\|_{\mathcal{C}^r(I)} \leq \frac{\delta}{1-\delta} \|I_d - L\|_{\mathcal{C}^r(I)} \|f\|_{\mathcal{C}^r(I)}. \tag{2.23}$$

3. Uniform Bound for Sobolev Orthonormal System

We assume that $\{\varphi_k\}_{k=0}^{\infty}$ is any uniformly bounded orthonormal and complete system of continuous functions on a compact interval $I = [a,b]$ with respect to the weight function ρ in $\mathcal{L}_{\rho}^2(I)$. Then for any $k \in \mathbb{N} \cup \{0\}$, $\|\varphi_k\|_{\infty} \leq B$, where B is a positive real. For $k = 0,1,2,...,(r-1)$ and for all $x \in I$, $|\varphi_{r,k}(x)| \leq \frac{(b-a)^k}{k!}$. Again, for $k = r,(r+1),...$ and for all $x \in I$,

$$|\varphi_{r,k}(x)| \leq \frac{1}{(r-1)!} \int_a^x |x-t|^{r-1} |\varphi_{k-r}(t)|\, dt.$$

For $x,t \in I$, $|x-t|^{r-1} |\varphi_{k-r}(t)| \leq B(b-a)^{r-1}$ and consequently, for $k = r,(r+1),...$ and for all $x \in I$,

$$|\varphi_{r,k}(x)| \leq \frac{B(b-a)^r}{(r-1)!}.$$

Thus for any $k \in \mathbb{N} \cup \{0\}$ and for all $x \in I$, $|\varphi_{r,k}(x)| \leq D$, where $D = \max\{1, (b-a), \frac{(b-a)^2}{2!}..., \frac{(b-a)^{r-1}}{(r-1)!}, \frac{B(b-a)^r}{(r-1)!}\}$. Then

$$\|\varphi_{r,k}\|_{\infty} \leq D.$$

From (2.5), for any $k \in \mathbb{N} \cup \{0\}$ and for all $x \in I$, for $j = 0,1,...,r$,

$$\left|\varphi_{r,k}^{(j)}(x)\right| \leq \mu = \max\{B,D\}.$$

Therefore

$$\|\varphi_{r,k}\|_{\mathcal{C}^r(I)} = \sup\{\left\|\varphi_{r,k}^{(j)}\right\|_{\infty} : j = 0,1,...,r\} \leq \mu.$$

4. α- Fractal Sobolev System

Let us take any compact interval $I=[a,b]$ with partition $\Delta: a=x_0<x_1<...<x_N=b$. Let $\varphi_{r,k},\ k\in\mathbb{N}\cup\{0\}$ be any function from the Sobolev orthonormal system $\{\varphi_{r,k}\}_{k=0}^{\infty}$ generated by the orthonormal and complete system of continuous functions $\{\varphi_k\}_{k=0}^{\infty}$. As each $\varphi_{r,k}\in\mathcal{C}^r(I)$, then the α-fractal function of $\varphi_{r,k}$ exists in $\mathcal{C}^r(I)$ (see Theorem 2.2). Denote $\varphi_{r,k}^{\alpha}$ as the α-fractal function of $\varphi_{r,k}$, then it satisfies the self-referential equation

$$\varphi_{r,k}^{\alpha}(x)=\varphi_{r,k}(x)+\alpha_i(\varphi_{r,k}^{\alpha}-b_{r,k})(L_i^{-1}(x));\ x\in I_i,\ i=1,2,...,N,$$

where $b_{r,k}\in\mathcal{C}^r(I)$ is a base function satisfying

$$\varphi_{r,k}^{(j)}(x_0)=b_{r,k}^{(j)}(x_0),\ \varphi_{r,k}^{(j)}(x_N)=b_{r,k}^{(j)}(x_N); j=0,1,2,...,r.$$

We say that the system $\{\varphi_{r,k}^{\alpha}\}_{k=0}^{\infty}$ is the α-Fractal Sobolev System generated by the system $\{\varphi_k\}_{k=0}^{\infty}$. In view of (2.20), it follows that for $j=0,1,2,...,r$,

$$(\varphi_{r,k}^{\alpha})^{(j)}(x_i)=\varphi_{r,k}^{(j)}(x_i);\ i=0,1,2,...,N. \tag{4.1}$$

Let us assume throughout the paper that the base function $b_{r,k}$ be linearly related with $\varphi_{r,k}$ that is $b_{r,k}=L\varphi_{r,k}$, where $L:\mathcal{C}^r(I)\to\mathcal{C}^r(I)$ is a linear and bounded operator with respect to the norm on $\mathcal{C}^r(I)$ such that

$$L\varphi_{r,k}^{(j)}(x_0)=\varphi_{r,k}^{(j)}(x_0), L\varphi_{r,k}^{(j)}(x_N)=\varphi_{r,k}^{(j)}(x_N); j=0,1,..,r.$$

Lemma 4.1. *Suppose that $\varphi_{r,k}^{\alpha}$ is the corresponding α-fractal function of $\varphi_{r,k}$. Then*

$$\left\|\varphi_{r,k}^{\alpha}-\varphi_{r,k}\right\|_{\mathcal{W}_{\rho}^{r,2}(I)}\leq\frac{\delta}{1-\delta}\left\|I_d-L\right\|_{\mathcal{C}^r(I)}\left\|\varphi_{r,k}\right\|_{\mathcal{C}^r(I)}\left[\int_I\rho(x)dx\right]^{\frac{1}{2}},$$

where $\delta=\frac{|\alpha|_{\infty}}{\min\{a_i^r;\ i=1,...,N\}}$.

Proof. In view of (2.23) and using (4.1),

$$\begin{aligned}
\left\|\varphi^{\alpha}_{r,k}-\varphi_{r,k}\right\|_{\mathcal{W}^{r,2}_{\rho}(I)} &= \left[\begin{array}{c} \sum_{j=0}^{r-1}\left|(\varphi^{\alpha}_{r,k}-\varphi_{r,k})^{(j)}(a)\right|^2 \\ +\int_I \left|(\varphi^{\alpha}_{r,k}-\varphi_{r,k})^{(r)}(x)\right|^2 \rho(x)dx \end{array}\right]^{\frac{1}{2}} \\
&= \left[\int_I \left|(\varphi^{\alpha}_{r,k}-\varphi_{r,k})^{(r)}(x)\right|^2 \rho(x)dx\right]^{\frac{1}{2}} \\
&= \left\|(\varphi^{\alpha}_{r,k}-\varphi_{r,k})^{(r)}\right\|_{\mathcal{L}^2_{\rho}(I)} \\
&\leq \left\|(\varphi^{\alpha}_{r,k}-\varphi_{r,k})^{(r)}\right\|_{\infty}\left[\int_I \rho(x)dx\right]^{\frac{1}{2}} \\
&\leq \left\|\varphi^{\alpha}_{r,k}-\varphi_{r,k}\right\|_{\mathcal{C}^r(I)}\left[\int_I \rho(x)dx\right]^{\frac{1}{2}} \\
&\leq \frac{\delta}{1-\delta}\left\|I_d-L\right\|_{\mathcal{C}^r(I)}\left\|\varphi_{r,k}\right\|_{\mathcal{C}^r(I)}\left[\int_I \rho(x)dx\right]^{\frac{1}{2}}.
\end{aligned}$$

□

The following definitions can be found in [18].

Definition 4.1. *A sequence $\{u_n\}_{n\in\Lambda}$ in a normed linear space V is called total in V if the class of all finite linear combinations $\sum_{n\in\Lambda} a_n u_n$ is dense in V.*

Definition 4.2. *A sequence $\{u_n\}_{n\in\Lambda}$ in a Hilbert space $\mathcal{H}$ is called complete in $\mathcal{H}$ if the only element of $\mathcal{H}$ which is orthogonal to every u_n is the null element, that is*

$$\langle f, u_n\rangle = 0 \text{ for all } n\in\Lambda \Rightarrow f = \mathbf{0}.$$

For any sequence in a Hilbert space, we have the following proposition.

Proposition 4.1. [18] *If $\{u_n\}_{n\in\Lambda}$ is any sequence in a Hilbert space $\mathcal{H}$, may be orthogonal or not. Then the followings are equivalent:*
(a) $\{u_n\}_{n\in\Lambda}$ is complete.
(b) $\{u_n\}_{n\in\Lambda}$ is total.

In the following theorem, it is shown that any function from Sobolev space can be approximated by an α-fractal Sobolev sum.

Theorem 4.1. *Suppose that f is any function in $\mathcal{W}^{r,2}_{\rho}(I)$, $r\in\mathbb{N}$. Consider the data set $D=\{(x_i,f(x_i))\in I\times\mathbb{R};\ i=0,1,...,N\}$, where $N\geq 2$ is an integer and $\Delta: x_0<x_1<...<x_N$ is a partition of the closed interval $I=[x_0,x_N]$. For every $\varepsilon>0$ and for any partition Δ of I and for linear bounded operator $L:\mathcal{C}^r(I)\to\mathcal{C}^r(I)$, there exists an α-fractal Sobolev sum $\Phi^{\alpha}_r(x)=\sum_{m=1}^{M} f_{r,k_m}\varphi^{\alpha}_{r,k_m}(x)$ with $\alpha\neq 0$ such that*

$$\|f-\Phi^{\alpha}_r\|_{\mathcal{W}^{r,2}_{\rho}(I)}<\varepsilon.$$

Proof. Let $\varepsilon > 0$ be given and f be any function in $\mathcal{W}_\rho^{r,2}(I)$. Then there exists $g \in \mathcal{C}^r(I)$ such that

$$\|f - g\|_{\mathcal{W}_\rho^{r,2}(I)} < \frac{\varepsilon}{3}. \tag{4.2}$$

By completeness of $\{\varphi_{r,k}\}_{k=0}^{\infty}$ in $\mathcal{W}_\rho^{r,2}(I)$, the Proposition 4.1 implies that $\{\varphi_{r,k}\}_{k=0}^{\infty}$ is total. Therefore, for any $\varepsilon > 0$, there exists $M \in \mathbb{N}$ and Sobolev sum $\Phi_r(x) = \sum_{m=1}^{M} f_{r,k_m}\varphi_{r,k_m}(x)$ such that

$$\|g - \Phi_r\|_{\mathcal{W}_\rho^{r,2}(I)} < \frac{\varepsilon}{3}. \tag{4.3}$$

Since $\Phi_r(x) = \sum_{m=1}^{M} f_{r,k_m}\varphi_{r,k_m}(x) \in \mathcal{C}^r(I)$, the linearity of $\mathcal{F}^\alpha$ gives

$$\begin{aligned} \Phi_r^\alpha &= \mathcal{F}^\alpha(\Phi_r) \\ &= \mathcal{F}^\alpha\left(\sum_{m=1}^{M} f_{r,k_m}\varphi_{r,k_m}\right) \\ &= \sum_{m=1}^{M} f_{r,k_m}\varphi_{r,k_m}^\alpha. \end{aligned}$$

By the same way of Lemma (4.1),

$$\begin{aligned} \|\Phi_r^\alpha - \Phi_r\|_{\mathcal{W}_\rho^{r,2}(I)} &\le \frac{\delta}{1-\delta}\|I_d - L\|_{\mathcal{C}^r(I)}\|\Phi_r\|_{\mathcal{C}^r(I)}\left[\int_I \rho(x)dx\right]^{\frac{1}{2}} \\ &= \frac{\delta}{1-\delta}\lambda, \end{aligned}$$

where $\delta = \frac{|\alpha|_\infty}{\min\{a_i^r;\, i=1,\ldots,N\}}$ and $\lambda = \|I_d - L\|_{\mathcal{C}^r(I)}\|\Phi_r\|_{\mathcal{C}^r(I)}\left[\int_I \rho(x)dx\right]^{\frac{1}{2}}$.

Therefore

$$\|\Phi_r^\alpha - \Phi_r\|_{\mathcal{W}_\rho^{r,2}(I)} < \frac{\varepsilon}{3}, \tag{4.4}$$

whenever

$$\delta < \frac{\varepsilon}{\varepsilon + 3\lambda}.$$

By (4.2), (4.3) and (4.4)

$$\begin{aligned} \|f - \Phi_r^\alpha\|_{\mathcal{W}_\rho^{r,2}(I)} &\le \|f - g\|_{\mathcal{W}_\rho^{r,2}(I)} + \|g - \Phi_r\|_{\mathcal{W}_\rho^{r,2}(I)} + \|\Phi_r - \Phi_r^\alpha\|_{\mathcal{W}_\rho^{r,2}(I)} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This completes the proof. □

Remark 4.1. *From the Theorem* 4.1*, it is clear that the set of* α*-fractal versions of Sobolev sums is dense and complete in* $\mathcal{W}_\rho^{r,2}(I)$*.*

The following definitions of Schauder basis and basis constant can be read in [16].

Definition 4.3. *Let X be a real normed space and let $\{x_n\}_{n=0}^{\infty}$ be a non zero sequence in X. We say that $\{x_n\}_{n=0}^{\infty}$ is a Schauder basis for X, if for each $x \in X$, there is a unique sequence of scalars $\{a_n\}_{n=0}^{\infty}$ such that $x = \sum_{n=0}^{\infty} a_n x_n$, where the series converges in norm to x. We define a sequence of linear maps $\{P_n\}_{n=0}^{\infty}$ on X by $P_n x = \sum_{i=0}^{n} a_i x_i$, where $x = \sum_{i=0}^{\infty} a_i x_i$. The map P_n is a projection onto span$\{x_i : 0 \le i \le n\}$. In addition, since $\{x_n\}_{n=0}^{\infty}$ is a Schauder basis, it follows that $P_n x \to x$ in norm as $n \to \infty$ for each $x \in X$ and P_n is continuous. Moreover, $K = \sup_n \|P_n\| < \infty$. The number K is called the basis constant of the basis $\{x_n\}_{n=0}^{\infty}$.*

Theorem 4.2. [17]*Every basis $(\{x_n\}, \{a_n\})$ for a Banach space X is a Schauder basis for X. In fact, the coefficients functionals a_n are continuous linear functionals on X which satisfy*

$$1 \le \|a_n\| \, \|x_n\| \le 2K,$$

where K is the basis constant.

4.1. **Expansion in Terms of Fractal Sobolev System.**

Let $\{\alpha^k\}_{k=0}^{\infty}$ be any sequence of scale vectors such that

$$\mu^* = \sum_{k=0}^{\infty} \frac{|\alpha^k|_\infty}{|a|_0 - |\alpha^k|_\infty} < \infty,$$

and $|a|_0 > |\alpha^k|_\infty, k = 0, 1, 2, ...,$ where $|a|_0 = \min\{a_i^r;\ i = 1, ..., N\}$ and assume that L is linear bounded with respect to the $\mathcal{C}^r(I)$-norm.

Theorem 4.3. *Let $\{\varphi_{r,k}^{\alpha}\}_{k=0}^{\infty}$ be α-fractal Sobolev system of the Sobolev orthonormal system $\{\varphi_{r,k}\}_{k=0}^{\infty}$ generated by any uniformly bounded complete orthonormal system of continuous functions $\{\varphi_k\}_{k=0}^{\infty}$. Then the operator $T : span(\{\varphi_{r,k}\}_{k=0}^{\infty}) \to span(\{\varphi_{r,k}^{\alpha^k}\}_{k=0}^{\infty})$ defined by*

$$T\left(\sum_{m=1}^{M} f_{r,k_m} \varphi_{r,k_m}(x)\right) = \sum_{m=1}^{M} f_{r,k_m} \varphi_{r,k_m}^{\alpha^{k_m}}(x)$$

is linear and bounded.

Proof. The linearity is straight forward. To show, the boundedness of T :

$$\begin{aligned} \left\| \sum_{m=1}^{M} f_{r,k_m} \varphi_{r,k_m}^{\alpha^{k_m}} \right\|_{\mathcal{W}_\rho^{r,2}(I)} \le & \left\| \sum_{m=1}^{M} f_{r,k_m} \left(\varphi_{r,k_m}^{\alpha^{k_m}} - \varphi_{r,k_m} \right) \right\|_{\mathcal{W}_\rho^{r,2}(I)} \\ & + \left\| \sum_{m=1}^{M} f_{r,k_m} \varphi_{r,k_m} \right\|_{\mathcal{W}_\rho^{r,2}(I)} . \end{aligned} \tag{4.5}$$

Since $\{\varphi_{r,k}\}_{k=0}^{\infty}$ is an orthonormal basis for $\mathcal{W}_\rho^{r,2}(I)$, $f_{r,k}$ is a bounded linear functional on $\mathcal{W}_\rho^{r,2}(I)$. Therefore

$$|f_{r,k}(f)| \le \|f_{r,k}\|_2 \, \|f\|_{\mathcal{W}_\rho^{r,2}(I)}, \tag{4.6}$$

where $\|.\|_2$ represents the norm as functional operator. Treating f_{r,k_m} as the k_m-th Fourier-Sobolev coefficient of $\sum_{m=1}^{M} f_{r,k_m}\varphi_{r,k_m}(x)$ and using (4.6), the first term of inequality (4.5) becomes

$$\begin{aligned}
&\left\|\sum_{m=1}^{M} f_{r,k_m}\left(\varphi_{r,k_m}^{\alpha^{k_m}}-\varphi_{r,k_m}\right)\right\|_{\mathcal{W}_\rho^{r,2}(I)}\\
\leq\ &\sum_{m=1}^{M}\left\|f_{r,k_m}\left(\varphi_{r,k_m}^{\alpha^{k_m}}-\varphi_{r,k_m}\right)\right\|_{\mathcal{W}_\rho^{r,2}(I)}\\
\leq\ &\sum_{m=1}^{M}\|f_{r,k_m}\|_2\left\|\sum_{m=1}^{M} f_{r,k_m}\varphi_{r,k_m}\right\|_{\mathcal{W}_\rho^{r,2}(I)}\left\|\varphi_{r,k_m}^{\alpha^{k_m}}-\varphi_{r,k_m}\right\|_{\mathcal{W}_\rho^{r,2}(I)}\\
=\ &\left\|\sum_{m=1}^{M} f_{r,k_m}\varphi_{r,k_m}\right\|_{\mathcal{W}_\rho^{r,2}(I)}\sum_{m=1}^{M}\|f_{r,k_m}\|_2\left\|\varphi_{r,k_m}^{\alpha^{k_m}}-\varphi_{r,k_m}\right\|_{\mathcal{W}_\rho^{r,2}(I)}. \quad (4.7)
\end{aligned}$$

Using Lemma (4.1), we have for $m=1,...,M$,

$$\left\|\varphi_{r,k_m}^{\alpha^{k_m}}-\varphi_{r,k_m}\right\|_{\mathcal{W}_\rho^{r,2}(I)} \leq \frac{\delta^{k_m}}{1-\delta^{k_m}}\|I_d-L\|_{\mathcal{C}^r(I)}\|\varphi_{r,k_m}\|_{\mathcal{C}^r(I)}\left[\int_I \rho(x)dx\right]^{\frac{1}{2}},$$

where $\delta^{k_m}=\dfrac{|\alpha^{k_m}|_\infty}{|a|_0}$.

Therefore, (4.7) becomes

$$\begin{aligned}
&\left\|\sum_{m=1}^{M} f_{r,k_m}\left(\varphi_{r,k_m}^{\alpha^{k_m}}-\varphi_{r,k_m}\right)\right\|_{\mathcal{W}_\rho^{r,2}(I)}\\
\leq\ &\left\|\sum_{m=1}^{M} f_{r,k_m}\varphi_{r,k_m}\right\|_{\mathcal{W}_\rho^{r,2}(I)}\sum_{m=1}^{M}\|f_{r,k_m}\|_2\\
&\times\frac{\delta^{k_m}}{1-\delta^{k_m}}\|I_d-L\|_{\mathcal{C}^r(I)}\|\varphi_{r,k_m}\|_{\mathcal{C}^r(I)}\left[\int_I \rho(x)dx\right]^{\frac{1}{2}}. \quad (4.8)
\end{aligned}$$

As a complete orthonormal basis, $\{\varphi_{r,k}\}_{k=0}^{\infty}$ is a Schauder basis for $\mathcal{W}_\rho^{r,2}(I)$, then the following inequality hold (see Theorem 4.2),

$$1 \leq \|f_{r,k_m}\|_2\|\varphi_{r,k_m}\|_{\mathcal{W}_\rho^{r,2}(I)} \leq 2K, \quad (4.9)$$

where K is the basis constant. But $\|\varphi_{r,k_m}\|_{\mathcal{W}_\rho^{r,2}(I)}=1$ and since $\{\varphi_{r,k}\}_{k=0}^{\infty}$ is orthonormal in the Hilbert space $\mathcal{W}_\rho^{r,2}(I)$, then the basis constant $K=1$ (see [22]), then from (4.9)

$$1 \leq \|f_{r,k_m}\|_2 \leq 2. \quad (4.10)$$

Again, for any $k\in\mathbb{N}\cup\{0\}$, (see Section 3)

$$\|\varphi_{r,k}\|_{\mathcal{C}^r(I)} \leq \mu.$$

Therefore

$$\|f_{r,k_m}\|_2 \|\varphi_{r,k_m}\|_{\mathcal{C}^r(I)} \leq 2\mu.$$

Then (4.8) becomes

$$\begin{aligned} & \left\| \sum_{m=1}^{M} f_{r,k_m} \left(\varphi_{r,k_m}^{\alpha^{k_m}} - \varphi_{r,k_m} \right) \right\|_{\mathcal{W}_\rho^{r,2}(I)} \\ \leq & \left\| \sum_{m=1}^{M} f_{r,k_m} \varphi_{r,k_m} \right\|_{\mathcal{W}_\rho^{r,2}(I)} \\ & \times 2\mu \|I_d - L\|_{\mathcal{C}^r(I)} \left[\int_I \rho(x) dx \right]^{\frac{1}{2}} \sum_{m=1}^{M} \frac{\delta^{k_m}}{1 - \delta^{k_m}}. \end{aligned} \tag{4.11}$$

By given conditions on scale vectors, we have $\sum_{m=1}^{M} \frac{\delta^{k_m}}{1-\delta^{k_m}} = \sum_{m=1}^{M} \frac{|\alpha^{k_m}|_\infty}{|a|_0 - |\alpha^{k_m}|_\infty} \leq \mu^*$. Therefore, it follows from (4.11) that

$$\begin{aligned} & \left\| \sum_{m=1}^{M} f_{r,k_m} \left(\varphi_{r,k_m}^{\alpha^{k_m}} - \varphi_{r,k_m} \right) \right\|_{\mathcal{W}_\rho^{r,2}(I)} \\ \leq & \; 2\mu\mu^*\sigma \left\| \sum_{m=1}^{M} f_{r,k_m} \varphi_{r,k_m} \right\|_{\mathcal{W}_\rho^{r,2}(I)}, \end{aligned}$$

where $\sigma = \|I_d - L\|_{\mathcal{C}^r(I)} \left[\int_I \rho(x) dx \right]^{\frac{1}{2}}$. Therefore, from (4.5)

$$\left\| \sum_{m=1}^{M} f_{r,k_m} \varphi_{r,k_m}^{\alpha^{k_m}} \right\|_{\mathcal{W}_\rho^{r,2}(I)} \leq (1 + 2\mu\mu^*\sigma) \left\| \sum_{m=1}^{M} f_{r,k_m} \varphi_{r,k_m} \right\|_{\mathcal{W}_\rho^{r,2}(I)}.$$

Hence T is a linear and bounded operator. □

Lemma 4.2. [19]*(Linear and Bounded Operator Theorem) Let X be a normed linear space, Y be a Banach space and $U : X \to Y$ be a linear and bounded operator. If X is dense in X', then U can be extended to X' preserving the norm of U.*

Theorem 4.4. *The map $\overline{T} : \mathcal{W}_\rho^{r,2}(I) \to \mathcal{W}_\rho^{r,2}(I)$ given by*

$$\overline{T}(f) = \sum_{k=0}^{\infty} f_{r,k}(f) \varphi_{r,k}^{\alpha^k}(x),$$

where

$$f(x) = \sum_{k=0}^{\infty} f_{r,k}(f) \varphi_{r,k}(x),$$

is well defined, linear and continuous.

Proof. Since $\{\varphi_{r,k}\}_{k=0}^{\infty}$ is complete in $\mathcal{W}_\rho^{r,2}(I)$, by the Proposition 4.1, it is total in $\mathcal{W}_\rho^{r,2}(I)$ and therefore, *span*$(\{\varphi_{r,k}\}_{k=0}^{\infty})$ is dense in $\mathcal{W}_\rho^{r,2}(I)$. Then by Lemma 4.2, the operator defined in the Theorm 4.3 can be extended to

$$\overline{T} : \mathcal{W}_\rho^{r,2}(I) \to \mathcal{W}_\rho^{r,2}(I),$$

with $\|\overline{T}\|_{\mathcal{W}_\rho^{r,2}(I)} = \|T\|_{\mathcal{W}_\rho^{r,2}(I)}$.

The linearity and boundedness of $\overline{T}$ imply that $\overline{T}(f) = \sum_{k=0}^{\infty} f_{r,k}(f)\varphi_{r,k}^{\alpha^k}(x)$ whenever $f(x) = \sum_{k=0}^{\infty} f_{r,k}(f)\varphi_{r,k}(x)$. ☐

4.2. Schauder Basis for $\mathcal{W}_\rho^{r,2}(I)$.

In the present section, it is proved that the fractal Sobolev system forms a Schauder basis for $\mathcal{W}_\rho^{r,2}(I)$ under some conditions on scale vectors and the hypothesis given in the Theorem 4.3.

Theorem 4.5. *Suppose that the sequence $\{\alpha^k\}_{k=0}^{\infty}$ of scale vectors are such that*

$$|a|_0 > \left|\alpha^0\right|_\infty \geq \left|\alpha^1\right|_\infty \geq \left|\alpha^2\right|_\infty \geq ...,$$

and

$$\sum_{k=0}^{\infty} \frac{\left|\alpha^k\right|_\infty}{|a|_0 - |\alpha^k|_\infty} < \infty.$$

If $2\mu\mu^* \|I_d - L\|_{\mathcal{C}^r(I)} \left[\int_I \rho(x)dx\right]^{\frac{1}{2}} < 1$, then $\{\varphi_{r,k}^{\alpha^k}\}_{k=0}^{\infty}$ is a Schauder basis for $\mathcal{W}_\rho^{r,2}(I)$.

Proof. To prove, let us consider the operator $V : \mathcal{W}_\rho^{r,2}(I) \to \mathcal{W}_\rho^{r,2}(I)$ defined by

$$V(f) = \sum_{k=0}^{\infty} f_{r,k}(f)\left(\varphi_{r,k} - \varphi_{r,k}^{\alpha^k}\right). \tag{4.12}$$

Now

$$\begin{aligned}
\overline{T}(f(x)) &= \overline{T}\left(\sum_{k=0}^{\infty} f_{r,k}(f)\varphi_{r,k}(x)\right) \\
&= \sum_{k=0}^{\infty} f_{r,k}(f)\varphi_{r,k}^{\alpha^k}(x) \\
&= f(x) - \sum_{k=0}^{\infty} f_{r,k}(f)\left(\varphi_{r,k} - \varphi_{r,k}^{\alpha^k}\right)(x) \\
&= f(x) - V(f)(x).
\end{aligned}$$

This implies that $\overline{T} = I_d - V$.

Using the equations similar to the proof of Theorem 4.3 we have,

$$\begin{aligned}\|V(f)\|_{\mathcal{W}_\rho^{r,2}(I)} &\leq \sum_{k=0}^{\infty} |f_{r,k}(f)| \left\|\varphi_{r,k} - \varphi_{r,k}^{\alpha^k}\right\|_{\mathcal{W}_\rho^{r,2}(I)} \\ &\leq \sum_{k=0}^{\infty} \|f_{r,k}\|_2 \|f\|_{\mathcal{W}_\rho^{r,2}(I)} \left\|\varphi_{r,k} - \varphi_{r,k}^{\alpha^k}\right\|_{\mathcal{W}_\rho^{r,2}(I)} \\ &= \|f\|_{\mathcal{W}_\rho^{r,2}(I)} \sum_{k=0}^{\infty} \|f_{r,k}\|_2 \left\|\varphi_{r,k} - \varphi_{r,k}^{\alpha^k}\right\|_{\mathcal{W}_\rho^{r,2}(I)} \\ &\leq 2\mu\mu^*\sigma \|f\|_{\mathcal{W}_\rho^{r,2}(I)},\end{aligned}$$

where $\sigma = \|I_d - L\|_{\mathcal{C}^r(I)} \left[\int_I \rho(x)dx\right]^{\frac{1}{2}}$ and consequently

$$\left\|I_d - \overline{T}\right\|_{\mathcal{W}_\rho^{r,2}(I)} = \|V\|_{\mathcal{W}_\rho^{r,2}(I)} \leq 2\mu\mu^*\sigma < 1.$$

Hence $\overline{T}$ is a continuous isomorphism, it maps Schauder basis onto Schauder basis. Therefore $\{\varphi_{r,k}^{\alpha^k}\}_{k=0}^{\infty}$ forms a Schauder basis for $\mathcal{W}_\rho^{r,2}(I)$. □

4.3. Riesz Basis for $\mathcal{W}_\rho^{r,2}(I)$.

In this section, it is shown that the α-fractal Sobolev system is a Riesz basis for $\mathcal{W}_\rho^{r,2}(I)$ under some suitable conditions on scale vectors.

Definition 4.4. [20]*A sequence* $\{x_k\}_{k=0}^{\infty}$ *in a Hilbert space* $\mathcal{H}$ *is called a frame for* $\mathcal{H}$ *if the inequalities*

$$A\|x\|^2 \leq \sum_k |\langle x, x_k\rangle|^2 \leq B\|x\|^2$$

hold for some positive constants A *and* B*, and for all* $x \in \mathcal{H}$*. The constants* A *and* B *are called a lower and an upper frame bound, respectively. A frame is said to be exact if it ceases to be a frame when an element is deleted.*

Definition 4.5. [20]*A sequence* $\{x_k\}_{k=0}^{\infty}$ *in a Hilbert space* $\mathcal{H}$ *is called a Riesz basis for* $\mathcal{H}$ *if it is complete and the inequalities*

$$A\sum_k |a_k|^2 \leq \left\|\sum_k a_k x_k\right\|^2 \leq B\sum_k |a_k|^2$$

hold for some positive constants A *and* B*, and for every sequence* $\{a_k\}_{k=0}^{\infty}$ *in* $\ell^2(I)$*. The constants* A *and* B *are called a lower and an upper Riesz bound, respectively.*

Since the Sobolev system $\{\varphi_{r,k}\}_{k=0}^{\infty}$ is complete and orthonormal in $\mathcal{W}_\rho^{r,2}(I)$, then it is a Riesz basis with $A = B = 1$, as for any orthonormal basis $\{x_k\}_{k=0}^{\infty}$, we have

$$\left\|\sum_k a_k x_k\right\|^2 = \sum_k |a_k|^2.$$

It is well known that a sequence in a Hilbert space is a Riesz basis if and only if it is an exact frame [20]. For instance, the Sobolev orthonormal system $\{\varphi_{r,k}\}_{k=0}^{\infty}$ is an exact frame.

Theorem 4.6. *Suppose that the sequence $\{\alpha^k\}_{k=0}^{\infty}$ of scale vectors are such that*

$$|a|_0 > \left|\alpha^0\right|_\infty \geq \left|\alpha^1\right|_\infty \geq \left|\alpha^2\right|_\infty \geq ...,$$

and

$$\sum_{k=0}^{\infty} \frac{\left|\alpha^k\right|_\infty}{|a|_0 - |\alpha^k|_\infty} < \infty.$$

If $2\mu\mu^* \left\|I_d - L\right\|_{\mathcal{C}^r(I)} \left[\int_I \rho(x)dx\right]^{\frac{1}{2}} < 1$, then $\{\varphi_{r,k}^{\alpha^k}\}_{k=0}^{\infty}$ is a Riesz basis for $\mathcal{W}_\rho^{r,2}(I)$.

Proof. In the proof of Theorem 4.5, it is shown that the operator $\overline{T}$ defined in Theorem 4.4, is a continuous isomorphism. A continuous isomorphism maps Riesz basis onto Riesz basis and consequently $\{\varphi_{r,k}^{\alpha^k}\}_{k=0}^{\infty}$ is a Riesz basis for $\mathcal{W}_\rho^{r,2}(I)$. □

Remark 4.2. *From the Theorem 4.6, it concludes that $\{\varphi_{r,k}^{\alpha^k}\}_{k=0}^{\infty}$ is an exact frame.*

5. Convergence of Fourier-Sobolev Expansion

In this section, the convergence of Fourier-Sobolev series of an α-fractal function corresponding to certain data set, is introduced. To establish the main results the following theorem on convergence of the Fourier-Sobolev series is needed.

Theorem 5.1. [11] *Suppose that $\frac{1}{\rho(x)} \in \mathcal{L}(I)$ and the system $\{\varphi_k\}_{k=0}^{\infty}$ is complete in $\mathcal{L}_\rho^2(I)$ and orthonormal with weight $\rho(x)$ on I. Let $\{\varphi_{r,k}\}_{k=0}^{\infty}$ be the Sobolev orthonormal system in $\mathcal{W}_\rho^{r,2}(I)$ with respect to inner product (2.1), generated by the system $\{\varphi_k\}_{k=0}^{\infty}$ according to (2.3) and (2.4). Suppose that $f \in \mathcal{W}_\rho^{r,2}(I)$. Then the Fourier-Sobolev series (2.6) converges to f uniformly with respect to $x \in I$.*

Lemma 5.1. *Suppose that $\frac{1}{\rho(x)} \in \mathcal{L}(I)$ and the system of functions $\{\varphi_k\}_{k=0}^{\infty}$ is complete in $\mathcal{L}_\rho^2(I)$ and orthonormal with weight $\rho(x)$ on I. Let $\{\varphi_{r,k}\}_{k=0}^{\infty}$ be the Sobolev orthonormal system in $\mathcal{W}_\rho^{r,2}(I)$ with respect to inner product (2.1), generated by the system $\{\varphi_k\}_{k=0}^{\infty}$ according to (2.3) and (2.4). Suppose that $f \in \mathcal{W}_\rho^{r,2}(I)$. Then r-th derivative of the Fourier-Sobolev series (2.6) converges to $f^{(r)}$ with $\mathcal{L}_\rho^2(I)$-norm on I.*

Proof. The nth partial sum of Fourier series of $f^{(r)} \in \mathcal{L}^2_\rho(I)$ in the system $\{\varphi_k\}_{k=0}^{\infty}$ is given by

$$S_n(f^{(r)}) = S_n(f^{(r)}, x) = \sum_{k=0}^{n} f_{r,k+r}\varphi_k(x),$$

where the coefficients $f_{r,k+r}$; $k = 0, 1, 2, ...,$ are defined in (2.7). Then

$$\left\| f^{(r)} - S_n(f^{(r)}) \right\|_{\mathcal{L}^2_\rho(I)} \to 0 \text{ as } n \to \infty. \tag{5.1}$$

The $(r+n)$th partial sum of the series (2.6) is

$$Y_{r,r+n}(f, x) = \sum_{k=0}^{r-1} f^{(k)}(a)\frac{(x-a)^k}{k!} + \sum_{k=r}^{r+n} f_{r,k}\varphi_{r,k}(x) \tag{5.2}$$

and Taylor's formula with integral remainder gives us

$$f(x) = \sum_{k=0}^{r-1} f^{(k)}(a)\frac{(x-a)^k}{k!} + \frac{1}{(r-1)!}\int_a^x (x-t)^{r-1} f^{(r)}(t)dt. \tag{5.3}$$

Using (2.3), we have from (5.2) and (5.3),

$$\begin{aligned} f(x) - Y_{r,r+n}(f, x) &= \frac{1}{(r-1)!}\int_a^x (x-t)^{r-1} f^{(r)}(t)dt - \sum_{k=r}^{r+n} f_{r,k}\varphi_{r,k}(x) \\ &= \frac{1}{(r-1)!}\int_a^x (x-t)^{r-1} f^{(r)}(t)dt \\ &\quad - \frac{1}{(r-1)!}\int_a^x (x-t)^{r-1} \sum_{k=r}^{r+n} f_{r,k}\varphi_{k-r}(t)dt \\ &= \frac{1}{(r-1)!}\int_a^x (x-t)^{r-1} \left[f^{(r)}(t) - \sum_{k=r}^{r+n} f_{r,k}\varphi_{k-r}(t) \right] dt \\ &= \frac{1}{(r-1)!}\int_a^x (x-t)^{r-1} \left[f^{(r)}(t) - S_n(f^{(r)}, t) \right] dt. \end{aligned} \tag{5.4}$$

Applying Leibniz's rule for differentiation under the integral sign on (5.4), for $j = 0, 1, 2, ..., (r-1)$,

$$\begin{aligned} & f^{(j)}(x) - Y^{(j)}_{r,r+n}(f, x) \\ = & \frac{1}{\{(r-1)-j\}!}\int_a^x (x-t)^{(r-1)-j} \left[f^{(r)}(t) - S_n(f^{(r)}, t) \right] dt \end{aligned} \tag{5.5}$$

and

$$f^{(r)}(x) - Y^{(r)}_{r,r+n}(f, x) = f^{(r)}(x) - S_n(f^{(r)}, x). \tag{5.6}$$

Using (5.5) and Hölder's inequality, for all $x \in I$ and for $j = 0, 1, 2, ..., (r-1)$,

$$\begin{aligned} \left| f^{(j)}(x) - Y^{(j)}_{r,r+n}(f,x) \right| &\leq \frac{1}{\{(r-1)-j\}!} \left(\int_a^b \frac{|x-t|^{2\{(r-j)-1\}}}{\rho(t)} dt \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_a^b \left| f^{(r)}(t) - S_n(f^{(r)},t) \right|^2 \rho(t) dt \right)^{\frac{1}{2}} \\ &= \frac{1}{\{(r-1)-j\}!} \left(\int_a^b \frac{|x-t|^{2\{(r-j)-1\}}}{\rho(t)} dt \right)^{\frac{1}{2}} \\ &\quad \times \left\| f^{(r)} - S_n(f^{(r)}) \right\|_{\mathcal{L}^2_\rho(I)}. \end{aligned} \tag{5.7}$$

Therefore, by using (5.1), from (5.7), for $j = 0, 1, 2, ..., (r-1)$,

$$\left\| f^{(j)} - Y^{(j)}_{r,r+n}(f) \right\|_\infty \to 0 \text{ as } n \to \infty.$$

Therefore, for $j = 0, 1, 2, ..., (r-1)$,

$$\begin{aligned} \left\| f^{(j)} - Y^{(j)}_{r,r+n}(f) \right\|_{\mathcal{L}^2_\rho(I)} &\leq \left\| f^{(j)} - Y^{(j)}_{r,r+n}(f) \right\|_\infty \left[\int_I \rho(x) dx \right]^{\frac{1}{2}} \\ &\to 0 \text{ as } n \to \infty \end{aligned} \tag{5.8}$$

and from (5.6),

$$\begin{aligned} \left\| f^{(r)} - Y^{(r)}_{r,r+n}(f) \right\|_{\mathcal{L}^2_\rho(I)} &= \left\| f^{(r)} - S_n(f^{(r)}) \right\|_{\mathcal{L}^2_\rho(I)} \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{5.9}$$

Using (5.8) and (5.9), the result follows. □

Theorem 5.2. *Suppose that $\frac{1}{\rho(x)} \in \mathcal{L}(I)$. Let $g \in \mathcal{C}^r(I)$ be the original function providing the data $\{(x_i, y_i)\}_{i=0}^N$ with constant step $h = x_i - x_{i-1}$. Let f be the α-fractal function of g with scale vector α_h such that $|\alpha_h|_\infty < (\frac{a_i}{2})^r h$; $i = 1, 2, ..., N$, defined in the Theorem 2.2. Then the Fourier-Sobolev expansion of f converges to g in weighted Sobolev norm as $h \to 0$ and $n \to \infty$ on I.*

Proof. Suppose that $g^\alpha = f$ is the α-fractal function of g. For convenience, write $Y_{r,n}(f,x) = Y_{r,n}(f)$.

From the Lemma 5.1,

$$\left\| f^{(r)} - Y^{(r)}_{r,r+n}(f) \right\|_{\mathcal{L}^2_\rho(I)} \to 0 \text{ as } n \to \infty. \tag{5.10}$$

From (5.5), for $j = 0, 1, 2, ..., (r-1)$,

$$(f - Y_{r,r+n}(f))^{(j)}(a) = 0. \tag{5.11}$$

Now, using (5.10) and (5.11)

$$\begin{aligned}
\|f - Y_{r,r+n}(f)\|_{\mathcal{W}^{r,2}_\rho(I)} &= \begin{bmatrix} \sum_{j=0}^{r-1} \left|(f - Y_{r,r+n}(f))^{(j)}(a)\right|^2 + \\ \int_I \left|(f - Y_{r,r+n}(f))^{(r)}(x)\right|^2 \rho(x)dx \end{bmatrix}^{\frac{1}{2}} \\
&= \left[\int_I \left|(f - Y_{r,r+n}(f))^{(r)}(x)\right|^2 \rho(x)dx\right]^{\frac{1}{2}} \\
&= \left\|(f - Y_{r,r+n}(f))^{(r)}\right\|_{\mathcal{L}^2_\rho(I)} \\
&\to 0 \text{ as } n \to \infty. \qquad (5.12)
\end{aligned}$$

Now, in view of (2.23), we have

$$\begin{aligned}
\|f - g\|_{\mathcal{W}^{r,2}_\rho(I)} &= \|g^\alpha - g\|_{\mathcal{W}^{r,2}_\rho(I)} \\
&= \begin{bmatrix} \sum_{j=0}^{r-1} \left|(g^\alpha - g)^{(j)}(a)\right|^2 + \\ \int_I \left|(g^\alpha - g)^{(r)}(x)\right|^2 \rho(x)dx \end{bmatrix}^{\frac{1}{2}} \\
&= \left[\int_I \left|(g^\alpha - g)^{(r)}(x)\right|^2 \rho(x)dx\right]^{\frac{1}{2}} \\
&= \left\|(g^\alpha - g)^{(r)}\right\|_{\mathcal{L}^2_\rho(I)} \\
&\le \left\|(g^\alpha - g)^{(r)}\right\|_\infty \left[\int_I \rho(x)dx\right]^{\frac{1}{2}} \\
&\le \|g^\alpha - g\|_{\mathcal{C}^r(I)} \left[\int_I \rho(x)dx\right]^{\frac{1}{2}} \\
&\le \frac{\delta}{1-\delta} \|I_d - L\|_{\mathcal{C}^r(I)} \|g\|_{\mathcal{C}^r(I)} \left[\int_I \rho(x)dx\right]^{\frac{1}{2}}, \qquad (5.13)
\end{aligned}$$

where $\delta = \frac{|\alpha_h|_\infty}{|a|_0}$. Since $|\alpha_h|_\infty < (\frac{a_i}{2})^r h$; $i = 1, 2, ..., N$, it follows from (5.13) that

$$\|f - g\|_{\mathcal{W}^{r,2}_\rho(I)} \to 0 \text{ as } h \to 0. \qquad (5.14)$$

Therefore

$$\begin{aligned}
\|Y_{r,n}(f) - g\|_{\mathcal{W}^{r,2}_\rho(I)} &= \|Y_{r,n}(f) - f + f - g\|_{\mathcal{W}^{r,2}_\rho(I)} \\
&\le \|Y_{r,n}(f) - f\|_{\mathcal{W}^{r,2}_\rho(I)} + \|f - g\|_{\mathcal{W}^{r,2}_\rho(I)}. \qquad (5.15)
\end{aligned}$$

Using (5.12) and (5.14), the result follows from (5.15). □

Theorem 5.3. *Suppose that* $\frac{1}{\rho(x)} \in \mathcal{L}(I)$. *Let* $g \in \mathcal{C}^r(I)$ *be the original function providing the data* $\{(x_i, y_i)\}_{i=0}^N$ *with constant step* $h = x_i - x_{i-1}$. *Let* f *be the* α*-fractal function of* g *with scale vector* α_h *such that* $|\alpha_h|_\infty < (\frac{a_i}{2})^r h$; $i = 1, 2, ..., N$, *defined in the Theorem 2.2. Then the Fourier-Sobolev expansion of* f *converges to* g *uniformly as* $h \to 0$ *and* $n \to \infty$ *on* I.

Proof. Let $g^\alpha = f$. Then from the Theorem 5.1,

$$\|Y_{r,n}(f) - f\|_\infty \to 0 \text{ as } n \to \infty \text{ on } I. \tag{5.16}$$

With the help of (2.18),

$$\begin{aligned} \|f-g\|_\infty &= \|g^\alpha - g\|_\infty \\ &\leq \frac{|\alpha_h|_\infty}{1-|\alpha_h|_\infty} \|I_d - L\|_\infty \|g\|_\infty . \end{aligned} \tag{5.17}$$

Since $|\alpha_h|_\infty < (\frac{a_i}{2})^r h$; $i = 1, 2, ..., N$, it follows from (5.17) that

$$\|f-g\|_\infty \to 0 \text{ as } h \to 0. \tag{5.18}$$

Using (5.16) and (5.18), the result follows from the following inequality

$$\|Y_{r,n}(f) - g\|_\infty \leq \|Y_{r,n}(f) - f\|_\infty + \|f-g\|_\infty .$$

□

Theorem 5.4. *Suppose that* $\frac{1}{\rho(x)} \in \mathcal{L}(I)$. *Let* $g \in \mathcal{C}^r(I)$ *be the original function providing the data* $\{(x_i, y_i)\}_{i=0}^N$ *with constant step* $h = x_i - x_{i-1}$. *Let* f *be the* α*-fractal function of* g *with scale vector* α_h *such that* $|\alpha_h|_\infty < (\frac{a_i}{2})^r h$; $i = 1, 2, ..., N$, *defined in the Theorem 2.2. Then the fractal analogue* $Y_{r,n}^\alpha(f)$ *corresponding to n-th partial sum* $Y_{r,n}(f)$ *converges to* g *in weighted Sobolev norm as* $h \to 0$ *and* $n \to \infty$ *on* I.

Proof. Now,

$$\begin{aligned} \left\|Y_{r,n}^\alpha(f) - g\right\|_{\mathcal{W}_\rho^{r,2}(I)} &\leq \left\|Y_{r,n}^\alpha(f) - Y_{r,n}(f)\right\|_{\mathcal{W}_\rho^{r,2}(I)} + \|Y_{r,n}(f) - f\|_{\mathcal{W}_\rho^{r,2}(I)} \\ &\quad + \|f-g\|_{\mathcal{W}_\rho^{r,2}(I)} . \end{aligned} \tag{5.19}$$

In view of (2.23), we have

$$\begin{aligned} \left\|Y_{r,n}^\alpha(f) - Y_{r,n}(f)\right\|_{\mathcal{W}_\rho^{r,2}(I)} &\leq \left\|Y_{r,n}^\alpha(f) - Y_{r,n}(f)\right\|_{\mathcal{C}^r(I)} \left[\int_I \rho(x)dx\right]^{\frac{1}{2}} \\ &\leq \frac{\delta}{1-\delta} \|I_d - L\|_{\mathcal{C}^r(I)} \|Y_{r,n}(f)\|_{\mathcal{C}^r(I)} \\ &\quad \times \left[\int_I \rho(x)dx\right]^{\frac{1}{2}}, \end{aligned} \tag{5.20}$$

where $\delta = \frac{|\alpha_h|_\infty}{|a|_0}$. Since $|\alpha_h|_\infty < (\frac{a_i}{2})^r h$; $i = 1, 2, ..., N$, it follows from (5.20) that

$$\left\|Y_{r,n}^\alpha(f) - Y_{r,n}(f)\right\|_{\mathcal{W}_\rho^{r,2}(I)} \to 0 \text{ as } h \to 0. \tag{5.21}$$

With the help of (5.12), (5.14) and (5.21), the inequality (5.19) completes the proof. □

Theorem 5.5. *Suppose that* $\frac{1}{\rho(x)} \in \mathcal{L}(I)$. *Let* $g \in \mathcal{C}^r(I)$ *be the original function providing the data* $\{(x_i, y_i)\}_{i=0}^N$ *with constant step* $h = x_i - x_{i-1}$. *Let* f *be the* α*-fractal function of* g *with scale vector* α_h *such that* $|\alpha_h|_\infty < (\frac{a_i}{2})^r h$;

$i = 1, 2, ..., N$, defined in the Theorem 2.2. Then the fractal analogue $Y^{\alpha}_{r,n}(f)$ corresponding to n-th partial sum $Y_{r,n}(f)$ converges to g uniformly as $h \to 0$ and $n \to \infty$ on I.

Proof. Similar to the proof of Theorem 5.3. □

References

[1] B.B. Mandelbrot: *The Fractal Geometry of Nature*, W.H. Freeman and Company, New York (1983).

[2] M.F. Barnsley: *Fractal Everywhere*, 2nd edn. Academic Press, USA (1993).

[3] J.E. Hutchinson: *Fractals and self-similarity*, Indiana Univ. Math. J. **30**, 713-747 (1981).

[4] P. Viswanathan and M.A. Navascués: *A Fractal Operator on Some Standard Spaces of Functions*, Proceedings of the Edinburgh Mathematical Society, vol-**60(3)**, August-2017, pp. 771-786.

[5] P. Viswanathan, M.A. Navascués, A.K.B. Chand: *Associate fractal functions in $\mathcal{L}^p$ spaces and in one-sided uniform approximation*, J. Math. Anal. Appl. **433(2)** 862-876 (2016).

[6] P. Massopust: *Fractal Functions, Fractal Surfaces, and Wavelets*, Academic Press, New York (1995).

[7] M.A. Navascués: *Fractal Polynomial Interpolation*, Z. Analysis Anwend. **25**(2) (2005), 401-418.

[8] M.A. Navascués: *Fractal Trigonometric Approximation*, Electron Trans. Numer. Anal. **20**, 64-74 (2005).

[9] M.A. Navascués: *Non-smooth polynomials*, Int. J. Math. Anal. **1**, 159-174 (2007).

[10] M.A. Navascués: *Reconstruction of sampled signals with fractal functions*, Acta Appl. Math. **110**(3), 1199-1210 (2010).

[11] I. I. Sharapudinov: *Sobolev-orthogonal systems of functions associated with an orthogonal system*, Izvestiya: Mathematics 82:1 212-244 (2018).

[12] I. I. Sharapudinov: *Approximation of functions of variable smoothness by Fourier Legendre sums*, Mat. Sb. 191:5 (2000), 143-160; English transl., Sb. Math. 191:5 (2000), 759-777.

[13] I. I. Sharapudinov: *Approximation properties of mixed series in terms of Legendre polynomials on the classes Wr*, Mat. Sb. 197:3 (2006), 135-154; English transl., Sb. Math. 197:3 (2006), 433-452.

[14] I. I. Sharapudinov: *Approximation properties of the Vallee-Poussin means of partial sums of a mixed series of Legendre polynomials*, Mat. Zametki 84:3 (2008), 452-471; English transl., Math. Notes 84:3 (2008), 417-434.

[15] I. I. Sharapudinov and T. I. Sharapudinov: *Mixed series of Jacobi and Chebyshev polynomials and their discretization*, Mat. Zametki 88:1 (2010), 116-147; English transl., Math. Notes 88:1 (2010), 112-139.

[16] N.L. Carothers: *A Short Course on Banach Space Theory*, London Mathematical Society Student Texts, Cambridge (2004).

[17] C. Heil: *A Basis Theory Primer*, In: Appl. Numer Harm. Analysis. Birkhauser, Boston (2011).

[18] J.R. Higgins: *Completeness and Basis Properties of Sets of Special Functions*, University Press, Cambridge (1977).

[19] M.N. Akhtar, M.G.P. Prasad, M.A. Navascués: *Fractal Jacobi Systems and Convergence of Fourier-Jacobi Expansions of Fractal Interpolation Functions*, Mediterr. J. Math., Dec 2016, Volume 13, Issue 6, pp 3965–3984.

[20] Y. Ha, H. Ryu, I. Shin: *Angle criteria for frame sequences and frames containing a Riesz basis*, J. Math. Anal. Appl. 347 (2008) 90-95.

[21] Lawrence C. Evans: *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society (1998).

[22] Ivan Singer: *Bases in Banach Space*, Vol. 1, Springer-Verlag Berlin, Heidelberg, New York (1970).