

**LANGUAGE, GESTURES AND TOUCHSCREEN
DRAGGING IN SCHOOL CALCULUS:
BILINGUALS' LINGUISTIC AND NON-LINGUISTIC
COMMUNICATION**

by

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Dissertation Submitted in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

in the
Mathematics Education Program
Faculty of Education

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SIMON FRASER UNIVERSITY

Spring 2016

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Abstract

This research study concerns patterns of bilingual high-school learners' communication when they interact with a touchscreen-based dynamic geometry environment (DGE) during calculus discussion and exploration. Specifically, three research questions were proposed for the study, addressing respectively: (1) the interplay between linguistic and non-linguistic communication, (2) the mathematical competence demonstrated in the students' activity and (3) the role of the technology for facilitating calculus thinking. Using a participationist lens and the theoretical framing of thinking-as-communicating, I provide qualitative video analyses of six pairs of participants' communication by focusing on their word use, gestures and touchscreen-dragging actions with DGEs during their mathematical activity. The goal of this study is to identify bilingual learners' competence during pair-work on mathematical tasks with touchscreen-based DGEs.

In Part I of the study, I compared two pairs of participants' thinking in response to two types of visual mediators: "static" (as those found in textbook diagrams) and "dynamic" (as exploited by the use of DGEs). The analysis provides evidence that the participants utilised different modes—utterances, gestures and touchscreen-dragging—of communication. In particular, touchscreen-dragging emerged as a form of gesture for communicating dynamic and temporal calculus relationships. In addition, the students communicated the fundamental calculus ideas differently when prompted by different types of visual mediators. In Part II, I provide analyses of communication involving four pairs of participants while exploring the area-accumulating functions with a touchscreen-based DGE. Findings resonate with Part I: the students relied on gestures and touchscreen-dragging as non-linguistic features of the mathematical discourse in order to communicate dynamic aspects of calculus. Moreover, by adopting a non-deficit model and examining the interplay among word use, gesture and touchscreen-dragging with DGEs, it was possible to identify bilingual learners' competence in mathematical communication.

This study underscores the importance of considering bilingual learners' non-linguistic forms of communication for understanding their mathematical thinking. It also presents implications for teaching dynamic aspects of functions and calculus, by arguing for a multimodal view of communication to capture the use of gestures and touchscreen-dragging in mathematical communication. Furthermore, it allowed me to identify new forms of communication mobilised in dynamic, touchscreen environments.

Keywords: Thinking-as-communicating; bilingual learners; non-deficit model; high school calculus; dynamic geometry environments; touchscreen-dragging

To my grandmothers, two of my greatest teachers.

Acknowledgements

I would like to thank my senior supervisor, Dr. Nathalie Sinclair, for her continual support and guidance in this four-and-a-half-year journey through my PhD studies. Especially, I am grateful for her tremendous mentorship on conducting research and writing, as well as the abundant opportunities that she has provided me to do research in the field, from the very beginning of my studies to the concluding stage of my thesis. She has pushed my boundaries and enlightened my path in becoming the researcher that I am today.

I would like to thank also all the faculty members of the Mathematics Education department at Simon Fraser University for their teaching and advice on both personal and scholarly levels, in particular, Dr. David Pimm for his time and support as a member of both my comprehensive exam committee and my supervisory committee.

Finally yet importantly, I thank my friends and family for being there and providing their unconditional support throughout this endeavour. A special thanks go to the anonymous students who participated in the study. Without their passionate collaboration, this study would have been impossible.

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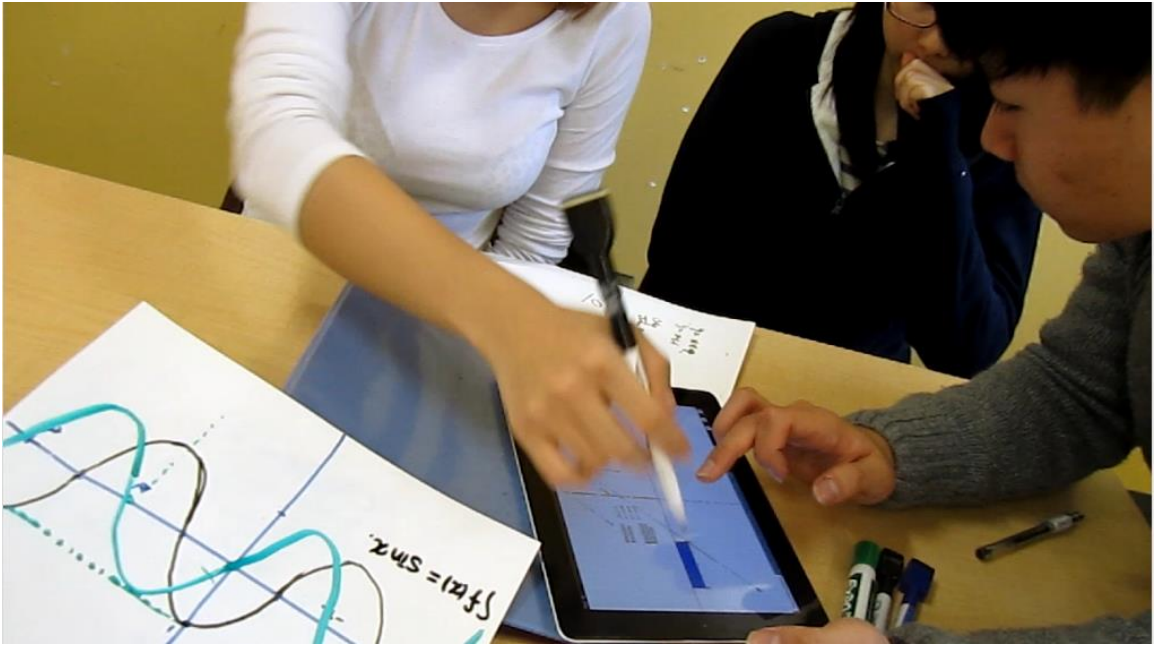
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Chapter 1. Introduction

In 2011–12, one in four (23.8 %) of public school students spoke a primary language at home other than English. Almost double the number of ELL students (135,651) live in families where the primary language spoken at home is other than English. (BCTF, 2012, pp. 11-12)

The context of my research interest emerged at a young age. I was born and raised in a Chinese family in Hong Kong, a British colony at the time. I completed kindergarten and primary school in a bilingual setting like the vast majority of ordinary pupils in Hong Kong. In this setting, nearly all pupils' home language was Cantonese but English was the language of instruction in selected subjects. At some schools this included Mathematics but that was not the case at my school. Therefore, from kindergarten to grade 6, I was a learner of mathematics in my home language.

The historical "Handover" in 1997 meant that the sovereignty of Hong Kong was to be handed over to the Republic of China, and this impacted not only me and my family, but also families and communities all over the world. At age 13, my family and I immigrated to Vancouver, Canada a few years before the Handover. As many Vancouverites would attest, tens of thousands of families, mine included, migrated to the city from the 1990s onwards to begin a new chapter of their lives. This influx of immigrants led to a dramatic shift in student demographics which set the stage for a new outlook in education in the Canadian province of British Columbia. I still remember what a regular school day was like back then in grade seven, when I had to attend "ESL" (English as a Second Language) classes, and times when even mathematics was a new "language" to me because of the limited English I understood at the time. I also remember that I did not like talking with peers at school because it was difficult for me to communicate verbally and that one of my peers made fun of my English pronunciation. When permitted, I would prefer to communicate in written form or to use my body

language. As I write this memoir, I feel that these experiences would resonate with other immigrant and international students as well.

What I did not know back then was that I would become a school teacher and mathematics educator eventually. This year, the year of 2016, marks the ten-year milestone of my teaching career. Since the very first day of teaching, I learned that besides facing the challenges of everyday teaching, my role as a mathematics teacher demands that I grapple with the complexity of teaching in multilingual mathematics classrooms. The linguistic diversity has increased even more since I was an English language learner (ELL) in the 1990s due to globalisation and an ever increasing intra- and inter-national movement. The “2012 BC Education Facts,” published by the British Columbia Teachers’ Federation (BCTF, 2012), provides a glimpse of the context of teaching mathematics in multilingual classrooms in BC currently:

- In 2011-12, one in four (23.8 %) of public school students spoke a primary language at home other than English.
- Almost double this number of ELL students (135,651) live in families where the primary language spoken at home is other than English, an increase of 16,874 students since 2001-02 and 8,676 students since 2007-08.
- Overall, enrolment of non-resident students increased by 271 students since 2007-08 (when it was 9,512 students) to 9,783 students in 2011-12. (pp. 11-12)

As I alluded to previously, the outlook of education in British Columbia responded to the changing needs of the student population in terms of language and cultural diversity. The Integrated Resource Package (IRP), a set of provincial curriculum documents prepared in consultation with both the Western and Northern Canadian Protocol (WNCP) and the United States National Council of Teaching Mathematics (NCTM) both expressed the need to address mathematics learning in a non-native language regardless of a lack of proficiency in the language of instruction. At the same time, there was an increasing emphasis on communication as an essential process for mathematics learning: “communicating to learn mathematics and learning to communicate mathematically” (NCTM, 2000, p. 60). This was easier said than done in a multilingual mathematics classroom. Speaking from my own experience as a teacher of

mathematics, there are multiple facets of challenges in facilitating communication for learners from different mathematical and linguistic backgrounds. The number of languages that my students can speak ranges from five to ten in any given class of up to thirty students: Korean, Mandarin, Cantonese, Portuguese, Spanish, Farsi, Italian, Russian and Urdu. While challenging, I have also very much enjoyed teaching in this setting by embracing the diversity of student linguistic and cultural backgrounds.

In 2011, I began my PhD studies with an interest in addressing some of the issues around the teaching and learning of mathematics in a multilingual context. Taking on the role of a mathematics education researcher, I read the work of prominent researchers in the field (Moschkovich, Setati and Adler) who have devoted efforts on the issue of linguistic diversity in mathematics education research. While Moschkovich's work is in the context of high school mathematics classrooms in the United States, where a large proportion of learners were Spanish-speaking, Setati and Adler conduct their research in South African mathematics classrooms, where up to eleven national languages are spoken in any given class. Upon a more thorough literature review, I learned that the issue of linguistic diversity in mathematics education must be studied in social, cultural, and political contexts. This made me intrigued about my own classroom context from a research perspective. Under the guidance of my supervisors, I completed a directed studies course in which I delved into reading of important research in the field, some of which has shaped my theoretical underpinnings and contributed greatly to my thesis.

The experience of reading the literature gave me the "language" I needed to understand my experience both as a teacher and learner in a multilingual mathematics classroom. Three ideas have stuck with me since then and changed my view on bilingual learners ever after. This first one is an analogy used by Setati (2005). She recalls Grosjean's (1985) analogy from the domain of athletics to explain the unnecessary dichotomy about home language and English language teaching. Setati argues that, like high hurdlers who blend high jumping and sprinting, multilingual learners blend multiple language competencies. In this sense, restricting multilingual learners to the use of only one language while learning is like making a hurdler compete with a sprinter in athletics, an analogy that I echo personally from my own experience.

The second idea that I appreciated very much was Moschkovich's sociocultural view of bilingual learners. Moschkovich (2002, 2007a) acknowledges that learning is situated in human interactions and sociocultural contexts. Her line of inquiry centres on bilingual learners' mathematical communication, the resources they use and the competence they show in mathematical activities. Moschkovich's work was transformative as it challenged me to consider a model that supports learning in a multilingual context. She argues against a deficit model which focuses on what learners do not know and cannot do. In fact, she turns the question around to look for *how* we can learn to see more of the expertise bilingual learners bring and the resources they use in the mathematics classroom. To begin, she chose to use the term "bilingual¹" or "multilingual" learners instead of "English language learners" because it focuses on what students know and can do (speak two or more languages) instead of what they do not know (English). In this research, I use the term "bilingual learners" for the same reason, and I reserve the term "multilingual" to describe the classroom contexts in which learners come from a diverse language backgrounds and often do not share the same home language.

Thirdly, in discussing the relationship between home language and learning, Moschkovich (2010, 2011) recommends researchers to avoid the deficit models of learners and their communities. In particular, she reminds us to exercise caution when comparing monolingual with bilingual learners. One must not assume that monolingual learners have an advantage over bilinguals, or that monolinguals are the norm because of their proficiency in the language of instruction: "Any time we use monolingual learners (or classrooms) as the norm, we are imposing a deficit model on bilingual learners. Bilinguals learning mathematics need to be described and understood on their own terms and not only by comparison to monolinguals" (Moschkovich, 2010, p. 11).

The message suggested by Setati and Moschkovich is powerful: bilingual learners blend multiple competencies in mathematical activities, and their learning of mathematics need to be understood on its own terms. Despite their work, I have found a paucity of research

¹ Moschkovich (2007c) provides the definition of bilinguals, as that used by Valdés-Fallis (1978): "the product of a specific linguistic community that uses one of its languages for certain functions and the other for other functions or situations" (p. 4).

that shares an understanding of this line of research. Since the start of my research, I have been frequently asked the questions, “how are bilingual learners different,” and “how do they compare with monolingual learners.” In response, I explain that my research is not about how bilingual learners are *different* from monolinguals, but about uncovering their competence and the resources they use to communicate effectively in mathematical activities. This line of work is much needed for achieving equity in mathematics education, especially given the lack of understanding shared in the mathematics education community in this regard.

Upon synthesising the literature, some of which mentioned above, I was able to shape my research direction. In response to Moschkovich, I am interested in focusing my research on seeing more of the expertise and resources that bilingual learners bring to the mathematics classroom. I also make the same theoretical assumptions as Setati, that bilingual learners blend multiple competencies in mathematics learning, and they are not comparable with monolinguals. As I explained, the classroom context is quite particular in British Columbia because of its history, where my research is situated. As such, my study aims to contribute to the understanding of linguistic diversity in mathematics education from both an equity (global) point of view as well as from a Canadian/British Columbian (local) perspective. In the next section, I continue to describe the context of my emerging research with respect to the study of calculus and the use of dynamic geometry technology.

1.1. Calculus communication in dynamic geometry environments

I became interested in bilingual learners’ communication and their interactions with dynamic geometry environment (DGEs) in the learning of calculus during my fifth year of teaching, in 2007. During 2007-2009 and 2011-2016, I taught a single-variable calculus (differential and integral) course in a culturally diverse high school in Metro Vancouver, British Columbia, Canada. Traditionally, the number of bilingual learners enrolled in mathematics classes is high; this number is even greater in my calculus class of which the majority of students were non-native English speakers. Looking back, I realise that my calculus class was a perfect site for inquiring about my target group of

bilingual learners since none of these students had learned calculus before they came to Canada. This makes learning calculus a singular experience for them because they had to learn calculus concepts that were new to them and in a non-native language as opposed to, for example, pre-calculus which some of them had learned in their home country.

Teaching calculus for the first time in 2007, I noticed that many students had difficulties grasping calculus concepts when they are represented graphically. They often focused too heavily on algebraic manipulation and were reluctant to build on multiple representations of concepts. When asked to communicate their solutions to a problem, they struggled to justify their strategies conceptually, which demonstrated their preference of procedural and algebraic knowledge over conceptual and geometrical understanding. Their reluctance to make connections between algebraic with graphical representations of calculus led to difficulties with dealing with graphs of derivative functions and area-accumulating functions. Both of these functions can be constructed dynamically, yet they thought of graphs as static objects rather than a dynamic process. Hence, when asked to construct the derivative or area-accumulating function, they struggle with graphing the slope of the tangent or the accumulated area under curve as a function.

I suspected that traditional calculus lectures and static textbooks contributed to students' struggles with graphical representation and dynamic notions of calculus. Much of the study of calculus involves change in a dynamic and continuous sense, which could be difficult to capture via a static medium like paper. I analysed the textbook that I used at the time, *Calculus Transcendental* (Stewart, 2008), to look into the way "change" was portrayed in the static medium of textbooks (Figure 1a & b). I also reviewed relevant literature and found that students' struggles with building conceptual and geometrical knowledge in calculus were recurring trends in mathematics education research.

(a)	(b)
"The relationship between secant and tangent to illustrate the definition of derivative" (Stewart, 2008).	"The pink region illustrates the accumulating area, $g(x)$, as x varies" (Stewart, 2008).

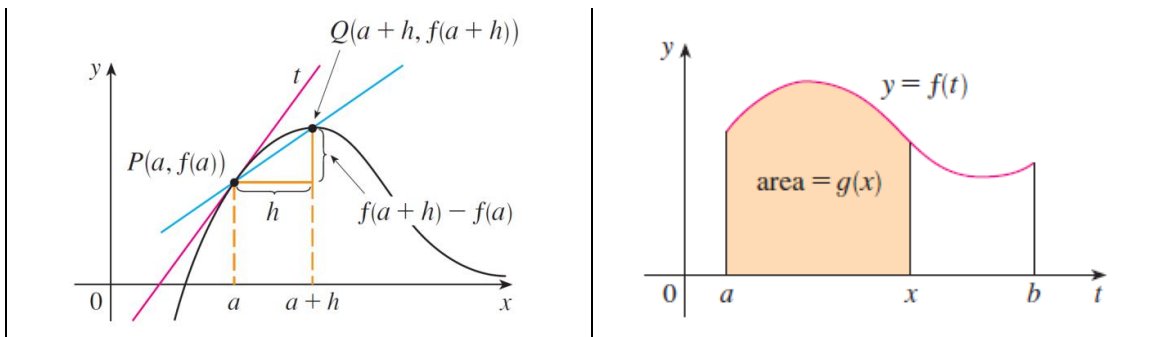


Figure 1. Typical diagrams found in calculus textbooks

My literature review on students' difficulties with the study of calculus revealed two trends of importance to my research: calculus students have significant difficulties in making connections between algebraic with graphical representations of calculus concepts and in dealing with simultaneous change of variables in calculus relationships (Tall & Vinner, 1981; Tall, 1986; Graham & Ferrini-Mundy, 1989; Thompson, 1994; Ubuz, 2007). For example, although students are able to work with limits and continuity in a dynamic sense that is strongly tied with motions, they are sometimes unable to communicate what it means to compute the limit of a function algebraically (Graham & Ferrini-Mundy, 1989). On the other hand, research shows that while students can obtain the derivative and anti-derivative functions algebraically, they have significant difficulties coping with graphs of derivative functions and especially with area-accumulating functions in both high school and college (Tall, 1986; Ubuz, 2007). Thompson (1994) found that visual understanding of the simultaneous change of all three variables, x , $f(x)$, and $\int_a^x f(t)dt$ in Fundamental Theorem of Calculus (FTC) to be a challenge for students. Berry and Nyman (2003) confirmed students' algebraic symbolic view of calculus and the fact that they find it difficult to make connections between the graphs of a derived function and the function itself. Furthermore, they indicated that students' thinking about the links between the graph of a function and its derived function was enhanced by asking students to "walk" these graphs as if they were displacement—time graphs. Their study suggests that these activities help students to extend their understanding of calculus concepts from a symbolic representation to a graphical representation and to what they termed a "physical feel".

In 2008, I conducted a small-scale research study as part of my Masters studies in order to investigate more deeply into how this “physical feel” may be communicated by calculus students. I analysed two pairs of bilingual students’ communication about “limit” using the lens of embodied cognition (Lakoff & Núñez, 2000; Núñez, 2000; Núñez, Edwards & Matos, 1999). Amongst my findings, I was most interested in students’ multimodal communication incorporating gestures and diagrams as well as the informal language they used to talk about calculus. I also found encouraging evidence that bilingual learners utilised non-verbal modes to communicate conceptually and dynamically, and I wanted to discover more ways to support this communication both as a classroom teacher and as a researcher.

In 2012, the College Board adopted a new Advanced Placement Calculus curriculum that included the regular use of technology by students and teachers “to reinforce the relationships among the multiple representations of functions, to confirm written work, to implement experimentation, and to assist in interpreting results” (College Board, 2012, p. 6). This change of curriculum allowed me to explore the use of dynamic geometry technology in my calculus teaching in 2012-2013. I designed lessons incorporating exploratory activities in which students interacted with pre-made sketches presented on touchscreen-based DGEs (Figure 2a & b), and I facilitated opportunities for my students to communicate about calculus in pairs or in groups during and after the activities. I found that my students talked quite differently compared with when I taught the course four years ago without the technology. For example, I was surprised to hear that my students referred to the study of calculus as “a study of change” on their written reflections during the middle of the school year. I also observed that their communication about geometrical and dynamic notions of calculus improved over the course of the school year. They expressed significant calculus ideas in multimodal ways incorporating language, gestures and diagrams during and after the exploratory activities. In addition, I began to attend to my own communication while I was teaching calculus with the aid of DGEs. I was intrigued by the language I used, and found it most interesting that I was constantly moving my hands and body for pointing, referencing and enacting the movement presented in the DGEs. I speculated that this change in the way my students and I talked about calculus was afforded by the use of DGEs.

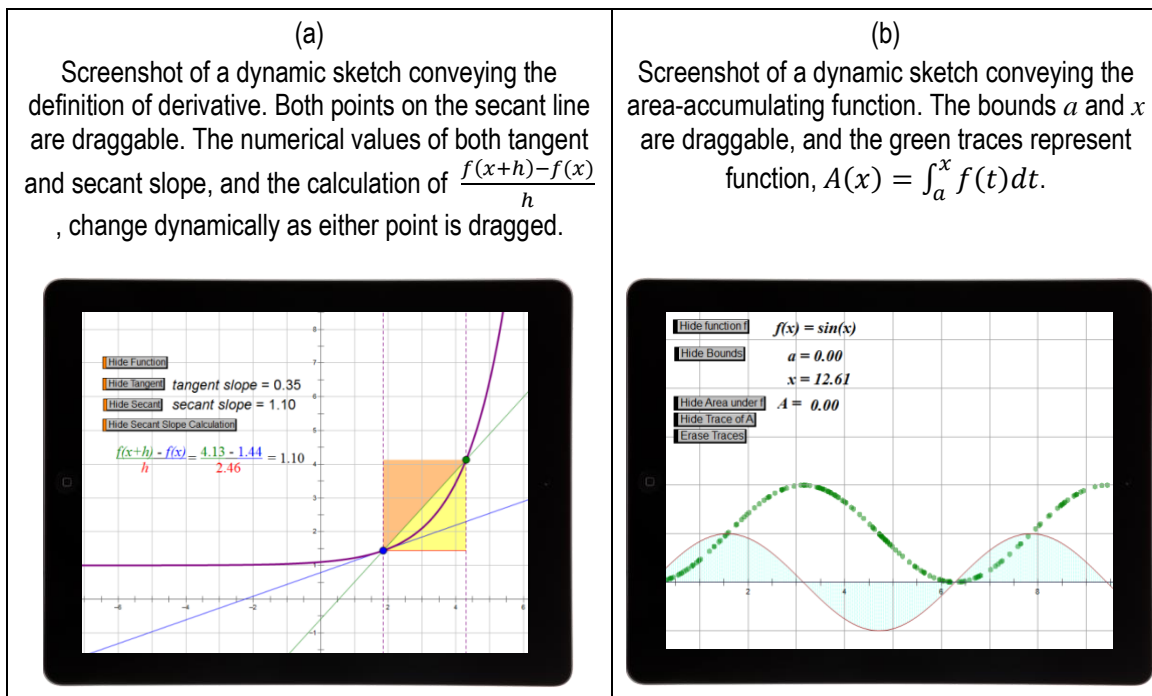


Figure 2. Screenshots of two dynamic sketches for conveying calculus relationships

Given this experience, I have become interested in the multimodal ways that calculus students communicate with the use of touchscreen-based DGEs. In particular, my interest about bilingual learners' has led me towards some potential research questions related to bilingual learners' patterns of communication during pair-work on mathematical activities with the use of touchscreen-based DGEs. I propose to address both *linguistic and non-linguistic modes* of bilingual learners' calculus communication and the *mathematical content* communicated during pair-work on mathematical activities with the use of touchscreen-based DGEs. I also aim to examine the *role of touchscreen-based DGEs* for facilitating this communication.

1.2. Meta-commentary

Throughout my journey of conducting research, I have been frequently questioned about how communication patterns compare between monolingual and bilingual learners, or how bilingualism stood out in the study. I would like to make it clear my rationale for studying bilingual learners' communication in the study. First, as I mentioned, it is one of my goals to highlight bilingual learners' competence in

mathematical communication and to challenge a deficit model of bilingual learners. This can only be achieved by studying bilingual learners' communication on its own terms and without assuming monolinguals as the norm. Secondly, studying bilingual learners' communication patterns allows me to address communication patterns of learners from all linguistic backgrounds, including monolinguals. This is because bilingual learners are a subset of learners from all linguistic backgrounds, and therefore, certain aspects of bilingual learners' communication may be characteristic of communication common to all learners. On the other hand, I also intend to highlight certain aspects of bilingual learners' communication in terms of the significance of bilingualism, for example, what my findings suggest about bilingual learners. Hence, the study enables me to highlight different aspects of bilingual learners' communication as common to all learners and as specific to bilingual learners.

In this chapter, I have used an autobiographic approach to introduce my research interest in relation to my upbringing as a bilingual learner and experience as mathematics teacher in Canada. I recalled some of the moments from my life that have brought me to this very research project. I consider this project significant on two levels. On a personal level, I am passionate about teaching bilingual learners, teaching with DGEs and teaching calculus; this project has given me much to learn about myself, my teaching, my students and mathematics, besides pure enjoyment. On a broader level, as a researcher, I believe that this study will contribute to mathematics education research towards bilingual learners' communication and the role of multimodality in mathematical thinking and learning, particularly with the use of touchscreen-based DGEs, for learners from all linguistic backgrounds.

This dissertation is organised into nine chapters including this Introduction which I present as Chapter 1. In Chapter 2, I review existing literature on the teaching and learning of mathematics in a multilingual context as well as the teaching and learning of calculus with the use of digital technology. In Chapter 3, I focus on theories related to the role of communication in mathematical thinking and learning, drawing on Sfard's communicational framework as the basis of my theoretical framework. In Chapter 4, I describe my research methodology, including the participants, tasks used and my methods of data collection and data analysis in both Parts I and II of my study. This is

followed by Chapter 5, where I describe the textbook diagrams and design of the dynamic sketches used in both parts of the study. I present my results in Chapter 6 and 7. Chapter 6 consists of analyses from Part I of the study, in which two pairs of participants communicated about various topics in calculus when prompted by some textbook diagrams and dynamic sketches presented on touchscreen-based DGEs. Chapter 7 consists of analyses from Part II of the study, in which four pairs of participants communicated about a chosen calculus topic during an exploratory activity using touchscreen-based DGEs. In Chapter 8, I highlight some recurrent themes from the results of the study, such as the significance of bilingualism and the relationship between modes of communication, mathematical thinking and type of visual mediator used. I also extend the results in developing a framework for understanding pairs of bilingual learners' communication during exploratory activities with touchscreen-based DGEs. In Chapter 9, I respond to the research questions posed in Chapter 1, and I conclude with comments on the study's contribution to research, my own reflection of the study and implications for future research in the field.

Chapter 2. Literature Review

As Gee [1999] would put it, students are essentially learning how to act, interact, think, value, talk, write and read in mathematically appropriate ways with appropriate props in the appropriate places. If they are learning mathematics in a language that is not their home language, then their task is even more demanding, because they have to learn to do all of the above in a new language that they are still learning. (Setati-Phakeng & Moschkovich, 2013, p. 126)

Calculus is the study of change. It is different from math because it is dynamic and involves the concept of time. (Quotation from a student from my calculus class in 2013)

The two quotes above reflect my role and challenges as a mathematics educator. Throughout my experience of teaching mathematics in Canada, I recognise the language demands placed on bilingual learners when communication with their teachers and peers in their home language is not available. At the same time, the study of calculus is much more than manipulation of symbols and solving algebraic equations; it is a study of change and involves the concept of time. Because of the nature of this study, communication and visualisation are important processes in the learning of calculus. In this chapter, I review and discuss literature related to three aspects of my study: (1) teaching and learning mathematics in multilingual contexts, (2) students' difficulties in the learning of calculus and (3) the learning of calculus in technology-enhanced environments. My goal is to incorporate this literature review to highlight areas of intersection among bilingual learners, calculus thinking and DGE-based learning for shaping my research and research contribution to each of these areas.

2.1. Teaching and learning of mathematics in multilingual contexts

Language plays a key role in teaching and learning mathematics for both monolingual and bilingual students. Vygotsky (1978) proposes that language is a tool for human thought and mediation of scientific concepts; hence, teaching and learning mathematics requires special attention to the use of language. In a multilingual context, issues about language, teaching and learning become more complex because the mediation of scientific concepts takes place within more than one language system. In addition, this situation is complicated by the notion that language is cultural and political. In this section, I begin by examining contemporary studies that address the role of learners' home language in the learning of mathematics. Then, I review the literature related to the complexities around teaching and learning mathematics in multilingual contexts in order to shed light on what it means to engage in mathematical communication in a multilingual mathematics classroom. I end by reviewing literature relevant to factors that support mathematics learning for bilingual learners.

2.1.1. The role of learners' home language

Cummins (1978) hypothesises that students need a high degree of proficiency in at least one language in order to make satisfactory progress at school. He also predicts that students with strength in two or more languages will outperform their peers, while those without a high degree of proficiency in any language will underachieve. Cummins' hypotheses have been demonstrated by research in the field of mathematics education in different parts of the world. For example, Clarkson's (2007) study with Australian students found a strong link between poor mathematical performance and low proficiency in all languages. According to Clarkson, this finding may explain some minority groups' underperformance in mathematics. There is also some evidence that students with strengths in two languages do better in mathematics than other students.

In his influential study, Dawe (1983) examines the relationship between bilingual learners' abilities to reason mathematically and their home language (L1) and English (L2) competence. His findings draw on Cummins' (1978) theory of linguistic

interdependence, which asserts that a cognitively and academically beneficial form of bilingualism can only be achieved on the basis of adequately developed L1 skills. Dawe analyses his data with instruments intended to tap into variables such as socioeconomic status, mathematical reasoning, cognitive and language competence (L1 and L2) from a large sample of research participants aged 11-14 years: 53 Punjabi, 50 Mirpuri, 50 Italian and 50 Jamaican bilinguals and 167 English monolinguals. According to his hypothesis, it is predicted that “when age, sex, intelligence, schooling background are controlled, bilingual learning mathematics in English as a second language who perform highly on a test of deductive reasoning, will also be those children whose L1 is most developed” (p. 336). He also compared the level of mathematical reasoning with the “language distance” of the participants’ home language from English, in the order of: English (for monolinguals), Italian, Punjabi, Mirpuri and Jamaican; his findings refuted his own hypothesis as English monolinguals did not outperform each bilingual group.

In addition, his finding supports Cummins’ Threshold Hypothesis for both *upper* and *lower* thresholds: in particular, he observes a highly significant correlation for Mirpuri bilinguals between high levels in both L1 and L2 and high scores of mathematical reasoning. Furthermore, Italian bilinguals with native-like language competence in one of L1 and L2 are correlated with moderate scores of mathematical reasoning. Of great interest and much surprise to the author is the finding that Mirpuri children can reason deductively in English as a second language at a higher mean level than their English peers: “That this competence has been achieved at no expense to their L1 is an important contribution to the validity of the Threshold Hypothesis” (p. 336).

In my opinion, both Dawe’s and Cummins’ theoretical assumptions fail to underscore mathematics learning as situated in social contexts. By analyzing learners’ collective performance on his test instruments, these contemporary studies suggest that mathematics is a type of *a priori* knowledge and that mathematical knowledge can be *acquired* unambiguously. According to Dawe, social interaction and culture are *factors* that play a role in mathematics learning and these factors can be *controlled* to study mathematical reasoning quantitatively. I argue that this assumption is problematic because learners from different cultures have different levels of access to written testing. Carraher et al. (1985), in their ground-breaking study of Brazilian child street

vendors, demonstrate that paper-and-pencil test results may not reflect true competence of learners' mathematical reasoning. In other words, Carraher et al. reveal that learning mathematics is highly situated in social contexts, and true competence of mathematical reasoning must account for sociocultural aspects of the setting. The interplay between culture and learning is self-evident in Dawe's study when he himself admits in his footnote that Jamaicans were not included in his analysis, because these "children expressed feelings of inferiority about Creole language and many claimed to speak 'English' only" (p. 351).

When defining bilingualism, Dawe models bilingual learners' language background by using a two dimensional grid, placing levels of L1 and L2 competence on the axes from low, medium to high. This way of characterizing bilingual learners oversimplifies the complexity of language and learners' interdependency between L1 and L2. This also exemplifies Dawe's view that patterns of bilingualism are prescribed across different cultures rather than closely tied to *identity* of one's culture. In the next sub-section, I highlight the important role of social and cultural norms in mathematics learning. It is also argued by d'Ambrosio (1985) that patterns of reasoning are closely tied to mathematical practices of cultural groups, an ethno-mathematical observation.

In summary, neither the complementary roles nor the complexities around home language and second language learning have been fully captured in contemporary studies about bilingual mathematics learners. This leads to the false assumption that mathematical reasoning can be compared between monolingual and bilingual learners, by means of standardised testing. I argue that a fuller range of social, cultural and political aspects, such as teacher's language use and background, classroom culture and norms, and curriculum policy should be considered in the research before conclusions about bilinguals' mathematical reasoning are drawn. Some of these aspects of teaching and learning in multilingual mathematics classrooms are highlighted in studies reviewed in the next sub-section.

2.1.2. Complexities of teaching and learning mathematics in multilingual contexts

A growing number of research studies in the field of linguistic diversity in mathematics education have refuted the assumption that students' home languages are irrelevant and should be ignored; rather, processes such as code switching are complex language practices that should be examined thoroughly before conclusions can be drawn. For example, tensions around language use has been well documented in Adler's research located in South Africa. Although the multilingual situation in South Africa is rather different from that found in Canadian classrooms, some of the key issues that arose are relevant for my study.

Adler (2001) highlights several dilemmas (she calls them: code switching, mediation and transparency) of teaching in multilingual mathematics classrooms. The dilemma of code-switching refers to the tension that teachers face between developing students' mathematical understanding in their native tongue and students' competence in English or mathematical English. She analyses a teaching episode in which a mathematics teacher was delivering a lesson on linear inequalities. Teaching in English, the teacher found it difficult to explain the meaning of the mathematical symbol " $<$ " or " \leq ", because her students struggled to understand the lesson in English. Adler draws on Walkerdine (1998) who challenges the common sense assumption that a familiar opposite of "more" is an unproblematic "less". This means that a student who is familiar with "no more" in their everyday discourse may still have difficulties interpreting "less" in the mathematical discourse. In addition, "at most" is a scientific concept in a Vygotskian perspective and thus is problematic because "shifting into everyday might well not be sufficient to attach the appropriate new conceptual meaning" (Adler, 2001, p. 28), even if the teacher code-switches to students' home language. All this is complicated by the fact that there is no direct translation for the term "at most" in the students' home language.

The dilemma of mediation involves the tension between validating diverse learner meanings and intervening so as to work with learners to develop their mathematical communicative competence. In Lave and Wenger's (1991) terms, it arises along the boundary between *talking within* and *talking about* mathematics. Engaging in group-work tasks, students are said to be talking *within* their mathematical practice,

while they are said to be talking *about* their mathematical ideas when communicating them back to their teacher or peers. The dilemma of mediation was felt by a teacher participant, who had to mediate the curriculum through her students' talk within and about mathematics. Since mathematics is a scientific concept, "it is neither a natural development of an everyday concept nor a matter of negotiation, but is acquired through instruction" (Adler, 2001, p. 31). Herein lies the teacher's tension—to provide opportunities for her students to participate in mathematical discussion but also to grant epistemic access of mathematical and curriculum knowledge.

Adler also identifies the dilemma of transparency: tensions between using an implicit or explicit language focus in multilingual classrooms, where students often do not have the vocabulary of mathematical terms and are less familiar with school discourse. Using Lave and Wenger's (1991) concept of *transparency*, language in the classroom must be both *visible* (explicit) and *invisible* (implicit). Analysing a teacher's lesson on trigonometric ratios, Adler illustrates how the teacher felt that her lesson had "gone much too long" because she had spent too much time on explicit language teaching and defining trigonometric ratios. Although there seemed to be a common understanding about the way students described trigonometric ratios, the teacher's pursuit to correct her students' explicit mathematical language was an example of making the language visible. However, by focusing on the explicit language, "the question as to the meaning of 'trigonometry' had disappeared" (p. 31). Drawing on Vygotsky and Lave and Wenger, Adler proposes that language is a tool of human thought, and learning mathematics involves learning to use appropriate resources to talk about mathematics. This means more than using the correct vocabulary but also acting appropriately in a community of practice.

Similarly, Moschkovich (2002, 2007a) argues that language should not be seen merely as a set of vocabulary items. If learning mathematics is seen solely as acquiring vocabulary, instructions would be mainly about teaching vocabulary terms and increasing learners' reading comprehension of mathematics texts and problems. In a bilingual context, learning mathematics would be mainly about translating words from one language to another and from words to mathematical symbols. With this view, learners are assessed by their fluency of using mathematical terms; therefore, many

students will appear less competent. Moschkovich (2002) cautions that, “if we focus on a student’s failure to use a technical term, we might miss how a student constructs meaning for mathematical terms or uses multiple resources, such as gestures, objects, or everyday experiences” (p. 193).

Moschkovich examines another view of teaching and learning mathematics, one that draws on the notion of the mathematics register due to Halliday (1978). Because the mathematics register comprises a set of meanings associated with different words, there may be multiple meanings for the same term. Under this view, learners of mathematics are essentially learning to use different meanings appropriately in different situations: “Emphasizing multiple meanings shifts the focus [...] from learning words with single meanings to understanding multiple meanings, and learning vocabulary to using language appropriately in different situations” (Moschkovich, 2002, p. 195). Confusion of meaning may occur when moving between the mathematics to the everyday register. However, if the multiple-meaning perspective is used to emphasize obstacles that bilingual learners face, it can lead to the conception that multiple meanings make mathematics more confusing and mathematics learning more difficult.

Barwell (2014) explains the tensions around teaching and learning mathematics in multilingual classrooms using Bakhtin’s (1981) characterization of centripetal and centrifugal language forces. According to Bakhtin, these two types of force are present in any utterance: centripetal forces represent the drive for unitary language, standardisation and linguistic hegemony; centrifugal forces represent the presence of heteroglossia, stratification and decentralisation. Barwell uses this theoretical perspective to examine a second language mathematics classroom in Canada, one in which the students are almost all speakers of Cree, one of the original languages of Canada. His analysis highlights three situations in which the tension between centripetal and centrifugal forces is particularly salient: the students’ use of Cree; working on mathematical word problems; and producing mathematical explanations.

Planas (2011) and Setati (2008) reveal several challenges posed in the classroom by students who do not use their home languages and teachers who do not promote their use. In the contexts of Catalonia, Spain and of South Africa, they contend

that while the language policies were created to address social inclusion, in practice they unintentionally increase the gap in access to classroom participation and learning opportunities for different language groups of students. In the same line of work, Planas and Setati (2014) draw on Ruiz's (1984) three perspectives on language to show how policy documents in both countries are framed by the *language-as-right* perspective, how teachers' thinking about the use of languages in their multilingual classrooms is influenced by the *language-as-problem* perspective, and how their language choices in practice are shaped by the *language-as-resource* perspective. They argue that if priority is given to language-as-resource rather than language-as-problem and/or language-as-right, we will be closer to reducing some of the unequal conditions of learning mathematics in multilingual classrooms. Their studies show that language is inherently political in multilingual classrooms and that policy documents which reflect language as human right is problematic in a pedagogical and didactical sense.

As I alluded to in the introduction, Moschkovich (2011) recommends avoiding the deficit models of learners and their communities. She reminds us to exercise caution when comparing monolingual with bilingual learners. One must not assume that monolingual learners have an advantage over bilinguals or that monolinguals are the *norm* because of their relative proficiency in the language of instruction. It is vital to consider that, "any time we use monolingual learners (or classrooms) as the norm, we are imposing a deficit model on bilingual learners. Bilinguals learning mathematics need to be described and understood on their own terms and not only by comparison to monolinguals" (p. 11). In addition, Moschkovich advises researchers to avoid coming to superficial conclusions about language and mathematics cognition. For example, regardless of what our personal experiences may tell us about code-switching, empirical research in sociolinguistics has not yet shown evidence that code switching is related to deficiency in learning. Rather, as the studies in the next sub-section show, encouraging students to use their home languages in the mathematics discussion maybe beneficial.

2.1.3. Factors that support mathematics learning for bilingual learners

Empirical studies in mathematics education have offered practical responses to the teaching and learning of mathematics in multilingual settings. For example, Barwell (2008) investigates how bilingual students engage in interpreting and discussing mathematical problems. Drawing on Gerofsky's (2004) *word problems as a genre*, Barwell examines mathematical word problems as a particular genre that is artificial, where real-world thinking is not required. In his study, he observed two students (one native speaker and one bilingual learner) working together on the task of creating and refining a word problem. He found that when creating a word problem, students have a similar prototypical word problem regardless of their language ability. He also found that the rich discussion amongst the students allowed for growth in vocabulary in both mathematics and everyday discourse: "The task allowed students to draw on personal experience of the world, promoted explicit attention to and discussion of the form of word problems, and led to a productive interaction between language learning and mathematical thinking" (p. 11).

In a different study, Barwell (2003) shows that bilingual learners find word problems less perplexing if they are able to relate them to their own experiences. Moreover, students need the opportunities to discuss problems in order to make sense of them. This is not to suggest that teachers must design problems based on the lives of their students; rather, tasks can be designed to allow learners to bring their experiences and interests to mathematics. In his study, Cynthia, a bilingual learner, was able to incorporate aspects of her daily life into the word problem and, in the process, develop a better understanding of word problems and mathematics. Barwell's studies show that, although bilingual learners do not have strong verbal and written language skills, their perceptions of word problems do not differ from native language learners. Furthermore, the pairing between a native speaker and a bilingual learner seemed to help both in developing their mathematics discourse. The students discussed, raised questions, and revised their word problems and were able to solve them at the end.

Moschkovich (2009) highlights the teacher's role in working with bilingual learners. She suggests that a focus on mathematical meaning rather than on language

per se supports mathematical thinking and learning of bilingual learners. She draws on and develops the idea of *revoicing*, where a teacher allows students to develop their understanding through questioning and conversation without evaluating their correctness, providing an example where two bilingual learners and their teacher were interpreting graphs and scales. In the episode, the teacher was able to unfold students' different interpretations of scales on the axes and draw on these interpretations to develop a better understanding of scaling. Moschkovich suggests that multiple interpretations can serve as resources for instruction in bilingual classrooms and provides recommendations for instructional strategies such as revoicing to support bilingual learners in mathematics classrooms.

Chval and Khisty (2008) provide a case study showing the effective use of students' writing by bilingual Latino students in the US. Their work was different from the previous two studies that I have discussed in that students' writing were collected throughout the course of a school year; therefore, the authors were able to analyse the progress of bilingual learners on their written work longitudinally. In their longitudinal study, they observed and gathered students' written reflections. The teacher used a variety of strategies to work with her students' written solutions to particular mathematics problems, including giving feedback on their written work and providing opportunities for multiple revisions of it. As a result, students came to clarify their ideas and became increasingly fluent in their written communication of mathematics. With the teacher's persistence and determination to build on students' writing, they noticed a positive change on the students' mathematical understanding, reflected by their written work as well as performance on standardized testing.

In a different study, Khisty (1995) compared the mathematics culture of two different English-Spanish bilingual, second grade classrooms. In one, the teacher controlled discussion through the use of repetition and choral responses; Khisty argues that this approach depersonalizes mathematics for the students. In the other classroom, mathematics was negotiated through discussion, challenge and debate. This environment enabled students to explain their ideas and to draw on previous experience to make sense of the mathematics. Khisty suggests that the culture of the second

classroom led students to making mathematical meanings through interaction among the students and between the teacher and students.

The studies reviewed in this sub-section highlight the importance of providing opportunities for bilingual learners to engage in meaningful mathematical discussions without evaluating the correctness of their use of the verbal or written language. They also share the belief that bilingual learners draw on different resources, including their previous experience and non-language resources such as gestures and diagrams to support their mathematics learning. Most significantly, the researchers in these studies took on a non-deficit view of bilingual learners, which enabled them to identify bilingual learners' competence in different kinds of mathematical tasks. My study has been influenced strongly theoretically and methodologically by this line of research, and I discuss in more details how my theoretical underpinnings and methodology are based upon these research in Chapters 3 and 4.

2.2. Students' difficulties in the learning of calculus

Besides inquiring in the area of linguistic diversity, my study also addresses calculus learning in the high school level. In order to gain a deeper understanding of high school students' learning of calculus, I have delved into classic studies offering insights into obstacles to learning various calculus concepts, such as graphical interpretations of the derivative, limit, rate of change, differentiation and the Fundamental Theorem of Calculus (FTC). Overall, the majority of studies examining students' calculus difficulties were conducted at the undergraduate level. Therefore, I include studies at both the undergraduate and the high school level in this section.

Tall and Vinner (1981) traced students' difficulties in the study of limits and continuity as due to a dichotomy between dynamic and static notions of the concepts. They characterized the problem as students having non-coherent concept images and concept definition. For example, as students think of the idea of limit, they are likely to evoke concept images of a process which includes a dynamic feeling of motion, as in "the limit of $f(x)$ as x approaches a is L ". However, the formal definition of a limit is static: x and $f(x)$ values do not move. Therefore, the dynamic and embodied element in the

informal limit definition causes students' difficulties in moving towards a more formal understanding of limit.

Studies have shown that dynamic limit definition, which includes definite feeling of motion, is strong in students (Williams, 1991; Tall, 1980). Even after students are exposed to formal limit instruction, they continue to hold a dynamic view of limit. These studies contend that the dynamic conception is easy to grasp and natural to develop for students because of its embodied nature. Graham and Ferrini-Mundy (1989) approached the problem from students' representation of limits. They showed that students' algebraic understanding of limit is independent of their graphical understanding. When students are asked to evaluate limits of the form $\lim_{x \rightarrow a} f(x)$ they are quite successful, but when asked for a geometric interpretation, students showed very little understanding. In one interview, a student explained that limit problems were simply functions to be evaluated and that the graph cannot help them find an answer. Orton (1983) suggests that many of the difficulties encountered by students in dealing with other concepts (continuity, differentiability, integration) can be related to their difficulties with limits

In her exploration of undergraduate students' understanding of the FTC, Thomas (1995) attained that students who were unable to deal with the area-accumulating function (a function defined by means of a definite integral with a variable endpoint) were unsuccessful in solving problems related to the FTC. Thompson (1994) found visual understanding of the simultaneous change of all three variables, x , $f(x)$, and $\int_a^x f(t)dt$ in FTC to be a challenge for students. Selden, Selden and Mason (1994) approached student understanding with a problem-solving framework. They found that students failed to use calculus strategies when dealing with non-routine problems, and even students who were able to perform well on routine calculus problems had difficulties with non-routine problems.

Similarly, a number of studies reported students' difficulties in creating a graphical representation of a function's rate of change function (Tall, 1986; Ubuz, 2007). Berry and Nyman (2003) confirmed the students' algebraic symbolic view of calculus and the fact that they find it difficult to make connections between the graph of a derived

function and that of the function itself. Furthermore, they indicated that students' thinking about the links between the graph of a function and its derived function was enhanced by asking students to "walk" these graphs as if they were displacement-time graphs. Their study suggests that these activities help students to extend their understanding of calculus concepts from a symbolic representation to a graphical representation and to what they termed a "physical feel". Castillo-Garsow (2010) attributes students' difficulties in understanding derivative as a rate with their inability to establish meaning for ratios. Weber et al. (2012) conjectured that students' difficulties with function notation, their struggles to connect algebraic with graphical representations of functions, and understanding of rate of change may explain their struggles to think about derivative as a function. They also contended that, "the definition of derivative, as it was found in the contemporary calculus books we surveyed, failed to convey mental imagery that would support students in constructing the derivative function" (p. 278).

I have found resonance with many of the studies discussed on students' difficulties learning calculus in my experience teaching calculus in the high school level. Like Thomas's (1995) result, I find that students have significant difficulties relating algebraic and graphical representations of calculus concepts, and this leads to their difficulties with dealing with graphs of derivative functions and area-accumulating functions. In general, students lack experiences working with covariation and functions in a dynamic sense in their junior level school math courses; they think of graphs as static objects rather than a dynamic process. Hence, when asked to construct the derivative of an area-accumulating function, they struggle with tracing the slope of tangent or the accumulative area under a curve as a function.

The above studies reveal that calculus students have significant difficulties with connecting algebraic to graphical representations of calculus concepts and in dealing with simultaneous change of variables in calculus relationships. Within the North American curriculum, students are seldom given opportunities to work with covariation and functions in a dynamic sense in their pre-calculus years. Traditional methods for teaching functions and their transformations are inadequate, as functions are commonly manipulated algebraically at the object level without making use of the idea of continuous change. The tendency to think of functions and graphs as static objects

rather than a dynamic process may contribute to their struggles in the learning of calculus. According to Weber et al.'s (2012) studies, calculus textbooks have also contributed to students' struggles, for static discourse like that in textbooks are ineffective for showing dynamic mathematical relationships. In Chapter 5, I examine how the static medium of paper, like textbooks, is used to convey calculus ideas. I pay special attention to the use of words, symbols and visual representations used in textbooks in anticipation of my study about students' communication patterns with visual representations.

2.3. The learning of calculus in a technology-enhanced environment

The inventions of computer algebra systems, graphing technology and dynamic mathematics software have impacted the teaching and learning of calculus greatly in the past twenty years. As I mentioned in the Introduction, the College Board revised the AP Calculus AB and BC curriculum to include the regular use of technology by students and teachers. Speaking from my experience, I found that my lessons have become more focused on mathematical communication, reasoning, and making connections between multiple representations when teaching calculus in a technology-enhanced environment. The various technological tools that I use to enhance my calculus lessons include graphing calculators, motion sensors and graphing tools, and dynamic applets. Literature on these technological tools to support student learning is discussed in this section. This review is aimed at responding to some of the challenges and difficulties in the learning of calculus as reported in the last section. Again, I include studies on both college and high school level learners in the review.

Studies from 1980s focus on a computer aided learning environment for promoting students' understanding of calculus. Hsiao (1984) showed that using the computer as a tool for performing the procedures of calculus and algebra can encourage students to concentrate on the underlying concepts. Students in the study undertaken by Heid (1988) stated that they enjoyed computer work because it freed them from the manipulative work and gave them confidence in results which were based on their reasoning. It also allowed them to focus more attention on the global aspects of problem

solving. Heid's study showed that students using a computer-aided environment understood the concepts as well as, and in most cases better than the students in the comparison class. Mathews (1989) discusses the use of a computer algebra system, muMATH, to verify the chain rule. Using the computer to produce symbols is expected to help students understand the truth value of the rule. Although the utilisation of computational technology can potentially aid students to better understand calculus concepts in a symbolic sense, it does little help for students who struggle to build relationship between symbolic and geometrical representations of calculus.

Studies in the 1990's and 2000's investigated the use of graphing and dynamic technology to enhance the learning of calculus visually. A number of pieces of research investigated the teaching of functions from a graphical point of view using graphing and computer environments (Confrey & Smith, 1994; Schwarz & Bruckheimer, 1990; Cuoco, 1994). These studies offer evidence supporting students overcoming difficulties with functions at the high school level by using particular technology, such as *Function Probe*, *Triple Representation Model*, and *Logo*. The environments allowed student to have control over a function by switching between representations, exploring covariation, and changing individual parameters. Falcade, Laborde & Mariotti (2007) showed how the DGE *Cabri-Géomètre* (Baulac *et al.*, 1988) could help high school students grasp the notion of function; they focused on the affordances of the *Trace* tool as a semiotic mediator that could introduce the twofold meaning of trajectory, both global and pointwise. This study is relevant to my research because it was noted in a previous section that students have difficulties interpreting derivative and area-accumulating functions which involve both global and pointwise trajectories.

A longitudinal study was completed at the high school level to investigate the effect of introducing derivative using *Graphic Calculus* on student learning (Tall, 1985). The technology was used to magnify graphs, allowing students to see graphs as "less curved" under high magnification. It was suggested that this enabled students to distinguish between continuity of a graph (one which will "pull flat" under high magnification) and differentiability (which involves graphs that are "locally straight"). Lagrange (1999) reported on a study of 11th grade students learning about functions with advanced calculators (TI92). He pointed out that a symbolic-graphic calculator could

enhance the understanding of calculus concepts in terms of numerical and graphical representations before appearing in symbolic form. The calculator acted as a mediator in the learning process; furthermore, the technical constraints in the calculators could potentially be exploited by teachers to mediate mathematical meanings.

Robutti and Ferrara (2002) introduced motion graphs with motion sensors that records displacement over time which can then generate a space-time graph. They concluded that the technology facilitated transitions between static and dynamic interpretations of the space-time graphs. Using the classroom connectivity of TI-83 calculators, Nemirovsky (2003) reported students using a water wheel connected to real-time graphing software to draw on learning acceleration through perceptual-motor activity. Arzarello and Paola (2003) designed a teaching experiment involving students to move with respect to a motion sensor so that the calculator would reproduce a graphic that is as close to the one drawn at the blackboard by the teacher as possible. They argued that this embodied activity using motion sensors is effective for introducing functions and their first and second derivatives within the same experience field. In contrast, traditional teaching activities for calculus introduced these aspects at separate times.

Recognising the intuitive versus formal approach to various topics in calculus, Ferrara, Pratt and Robutti (2006) reviewed a number of studies using dynamic geometry technology for enhancing the teaching of derivatives and integrals. In one study, a DGE was used to plot a function along with its derivative together dynamically on the same screen. As one student put it, "I never understood what it meant to say that the derivative of $\sin(x)$ is $\cos(x)$ until I saw it grow on the computer" (p. 261). As the student saw the derivative function "grow" dynamically on the screen, the idea of derivative functions was mediated in an embodied way through the dynamic images. Moreover, students seemed to be more comfortable when regarding the computer as an authority compared to the teacher: "They seem far more willing to discuss conceptual difficulties thrown up by the computer than they would difficulties in understanding a teacher's explanation" (p. 261).

More recent studies continue to provide insight into the multi-representational aspects of learning calculus in a technology-enhanced environment. These studies offer

responses to, as the literature suggests, the difficulties that students experience when learning graphs of derivatives, area-accumulating functions and the FTC and when making connections among different representations. Thompson et al. (2013) emphasise an approach that allows students to explore variation and covariation in a technological environment before leading up to the study of FTC. They designed a digital environment that simulate the “bottle problem” in which water accumulates in a bottle and the students are asked to graph the volume of water in the bottle as a function of its height. The authors argue that this approach help students build a reflexive relationship between concepts of accumulation and rate of change, one that could only be made possible with the use of technology.

Similarly, Yerushalmy and Swidan (2012) used a semiotic lens to observe students’ use of dynamic and multi-representation environment for learning the concept of accumulation graph. The artifact was designed to support exploration using dynamic and multiple representations of an area-accumulation function. With an interface that allows interactive changes of parameters and direct manipulation of graphic objects, the graph of the area-accumulation function can simultaneously be drawn directly below the given function. They found that the zeros of the accumulation graph and use of colour coding for positive and negative areas served pivotal roles in the process of semiotic mediation.

Hong and Thomas (2013) examined the design of a curriculum where students use digital technology to develop a more balanced dual view of calculus ideas as both process and concept. Their results call for a teaching approach incorporating frequent use of dynamic geometry technology and the graphing calculators can encourage versatile embodied and inter-representational thinking. Further, using a calculator to display between numerical and graphical representations and engagement can support students in constructing derived functions and the development of local or interval thinking.

Evolution in digital technology has affected our thinking, learning and modes of interactions with mathematics. In particular, the use of graphing and dynamic geometry technologies have offered new ways of doing, representing and exploring mathematics.

The literature reviewed in this section exemplifies the positive effect of exploiting the dynamic nature of DGEs to support calculus thinking and learning. Another aspect of digital technology that was shown to be effective for calculus learning is the feedback provided numerically and graphically, which enabled learners to respond to the environment accordingly without the presence of the teacher's evaluative comments. Related to this, students seemed to be more willing to discuss conceptual difficulties when interacting with technology. This is important for my study, as I am interested in bilingual learners' development of their mathematical thinking when interacting with DGEs in pairs. On the other hand, research about the effect of touchscreen-based DGEs is very limited. My study contributes to this area, as I hypothesise that a touchscreen-based DGE may offer additional affordances by providing tactile and kinesthetic modes of interaction—hence, further facilitate bilingual learners' communication in calculus.

2.4. Summary

Literature examining learners' home language, the complexities of teaching and learning in multilingual contexts, students' difficulties in the learning of calculus and factors that support mathematics learning for both bilingual learners and calculus learners, including the use dynamic and graphing technology, have been reviewed in this chapter. I began by discussing significant issues surrounding home language and bilingualism in multilingual learning environments. This review highlights the complexities in teaching and learning mathematics in multilingual contexts and the importance of avoiding superficial conclusions about language and mathematics cognition.

I also examined the research literature in relation to learning calculus. Studies consistently show that a dichotomy exists between graphical and algebraic representation of calculus, and students have difficulties working with dynamic elements of calculus robustly beyond a physical and embodied level. Moreover, calculus students find it difficult to make sense of the simultaneous change in the variables in the graphs of derivative functions and area-accumulating functions. These struggles may be attributed to traditional textbooks and teaching methods that seldom capture the dynamicity in calculus by visual means.

Technology advances have impacted the learning of calculus on many levels. Computer-algebra systems may ease calculations and algebraic manipulations, while motion sensors and motion graphs can introduce the idea of rate of change in embodied ways. Although there is growing evidence suggesting that DGE-based learning can support calculus thinking and communication, studies on the effect of the touchscreen interface combined with DGE affordances are limited.

The literature reviewed in relation to calculus learning so far has mostly adopted cognitive theories of learning. This line of work rarely considers *sociocultural* aspects of learning, that is, learning as a social activity. Furthermore, a cognitive lens is usually associated with a dualistic view of learning calculus, suggesting that thinking and doing are dual processes with the former controlling the latter. As I am informed by my review about bilingual learning as situated in social, cultural and political context (Section 2.1), I have chosen a sociocultural and non-dualistic perspective for studying calculus thinking. This view, as I discuss in detail in Chapter 3, suggests that learning calculus is not merely developing mental schema in the head, but it is a discursive activity that occurs in social interactions.

Chapter 3. Theoretical Framework

Thought is not an incorporeal process which lends life and sense to speaking, and which it would be possible to detach from speaking. (Wittgenstein, 1953, p. 109)

"I think. Therefore, I am." René Descartes

Contemporary theories of learning calculus shared an acquisitionist and dualistic view of learning. These works were influenced by the legacy of Plato and Descartes who view learning as occurring in the *mind* separately from bodily experience (see quote above). For example, the theory of APOS by Dubinsky (1991) views learning mathematics as acquiring mathematical knowledge cognitively. And, although the theory of embodied cognition considers how calculus may be understood cognitively through an individual's bodily-based metaphors, it pays little attention to learning as arising through social interaction in communicative settings.

By contrast, my theoretical framework has been influenced strongly by participationism, a view that considers learning as inherently social and highly situated. I take on this participationist view of learning, in the sense that mathematical learning occurs in social contexts and is situated in mathematical activities. Through participating in mathematical activities, the learner changes her way of acting and talking about mathematics—a change in her mathematical discourse. This view that learning mathematics is a discursive activity is adopted in Moschkovich's work with bilingual Spanish American students. She also defines the term mathematical Discourse practice, which I find especially useful for focussing on bilingual learners' mathematical communicative competence. In this chapter, I begin by describing Moschkovich's sociocultural view of bilingual learners which has influenced me greatly in my research.

In addition, bilingual learners draw on linguistic and non-linguistic modes of communication in the development of their mathematical discourse. These resources

are multimodal and embodied, such as gesturing and interacting with visual representations. Thus, my theoretical framework must consider different kinds of multimodal communication and their roles in mathematical thinking. In the latter part of chapter, I describe Sfard's (2008) communicational approach for theorising the role of language, gestures and visual representations in mathematical thinking and learning. Her non-dualistic theory of thinking and learning also reflects a participationist view which is complementary to Moschkovich. I discuss relevant constructs of Sfard and the basis of which I have chosen it as a theoretical framework for my study.

3.1. Moschkovich's sociocultural view of bilingual learners

Moschkovich (2007a) describes three views of bilingual mathematics learners and examines how these views impact instruction. The first perspective emphasises acquiring *vocabulary*, the second emphasises *multiple meaning*, and the third emphasises participation in *mathematical Discourse practices*. She questions the efficacy of the first two perspectives for understanding bilingual mathematics learners because they focus on what learners *don't know* or *can't do* (see Section 2.1.2). In contrast, the third perspective, *the sociocultural view*, focuses on bilingual learners' competences and the resources they use in communication.

The sociocultural view sees learning as participating in *mathematical Discourse practices*. Where most literature distinguishes *school mathematics* discourse from *everyday* discourses, Moschkovich (2007b) suggests that the mutual inclusiveness of the conventional terms maybe problematic, and rather, uses the term *mathematical Discourse practices* to distinguish between practices. To define *mathematical Discourse practices*, Moschkovich draws on Gee's notion of Discourse. By a *Discourse*, with a capital "D", Gee means a socially accepted association among ways of being, acting and using language at certain times and places, so as to assume particular "recognisable" identities. Within Discourses, a human being is "but one actant among many with things, expressions, places, technologies, and the 'natural world' (Pickering, 1992)" (Gee, 1994, p.36). Thus, Gee argues for a view of learning as induction into Discourses (ways of being), not just discourses (ways of using words).

Gee's definition of Discourse provides a platform to study wider ways of using language, including "symbolic expressions, and 'artifacts', of thinking, feeling, believing, valuing and acting that can be used to identify oneself as a member of a socially meaningful group... or to signal (that one is playing a socially meaningful role" (Gee, 1996, p. 131). From this perspective, learning mathematics is a discursive activity (Forman, 1996) that involves following classroom socio-mathematical norms (Cobb et al., 1993) and using multiple material, linguistic and social resources.

Discourse practice is social, cultural, discursive and also cognitive because they involve thinking, signs, and tools. For example, simply knowing a list of technical language about bikes will not ensure successful interaction in a biker bar (Gee, 1996). Practices that are shared by members who belong in the community of bikers illustrate *Discourse practices*. To understand what *Mathematical Discourse practices* are, Moschkovich (2007a) suggests that one begins by looking for the kinds of *Discourse practices* that emerge in mathematics classroom activities. These mathematical practices are specific to particular mathematical ideas and are mutually shared by the teacher and students as norms of the classroom community. For example, "in general, abstracting, generalising, searching for certainty, and being precise, explicit, brief, and logical are highly valued activities across different mathematical communities" (p. 10). According to Moschkovich, imagining (for example, infinity to zero), visualising, hypothesising and predicting are also valued mathematical Discourse practices.

Moschkovich (2007a) illustrates *Mathematical Discourse practices* with examples of Spanish American students' communication during mathematical activities. In one example, she examines mathematical discussion between two bilingual learners. She illustrates that Alicia, who was asked by her teacher to describe a pattern, used gestures and her native language to explain what she meant. Although Alicia did not have the vocabulary of *rectangle*, *length*, and *width*, her "non-language resources" revealed that she was appropriately describing patterns and making comparisons between the perimeter, length, and width of a rectangle. Her study highlights bilingual learners' mathematical Discourse practices even without using the right vocabulary: "even a student who is missing vocabulary may be proficient in using mathematical constructions or presenting clear arguments" (p. 207).

In another example, Moschkovich (2007a) shows how bilingual learners' home language can serve as resources for mathematics learning. She analysed the transcript of two bilingual learners who engaged in a mathematical discussion about the steepness of linear functions. The students did not struggle with the vocabulary, but rather used their home language, Spanish, and English interchangeably to negotiate the meaning of *steep* and *less steep*. In addition, they used their everyday experience of “*x*-axis is the ground” as resources. The students were also actively participating in valued mathematical Discourse practices, in particular, stating an assumption and making connection to support claims, evident in “because look, let’s say that this is the ground [...]” (p. 205).

The sociocultural view of bilingual learners is a suitable approach for supporting teaching and learning in multilingual mathematics classrooms. The idea of mathematical Discourse practices shifts the focus from bilingual learners' deficiencies to competencies in a multilingual learning environment. It reflects aspects of Lave and Wenger's theory of situated learning that learning is situated in practice. This participationist lens is highly relevant for my purpose of studying bilingual learners' communication in mathematical activities. On the other hand, my research demands that I study bilingual learners' mathematical communication in their non-native language—they do so by utilising linguistic and non-linguistic modes of communication. Hence, I adopted Sfard communicational theory as a complementary theoretical framework for addressing the role of language, gestures and visual representations in communication.

3.2. Sfard's communicational theory: thinking as communicating

The *learning as participation* perspective establishes a strong link between mathematics learning and communication; it is a framework for conceptualising learning in its social dimensions (Lave & Wenger, 1991; Wenger, 1998). This perspective suggests that learning is located neither in the heads nor outside of the individual, but in the relationship between a person and a social world. Sfard's communicational framework (2008) is based upon the social dimensions of learning and highlights the communicative aspects of thinking and learning. For Sfard, thinking and communicating

are two parts of the same entity. This non-dualistic approach denies thinking as a purely cognitive phenomenon as well as the thinking-communicating dichotomy. It has roots in the work of Lev Vygotsky, who claimed that speech and thought is inseparable, and that studying thought (or meaning) and words as separate entities is like trying to understand water by investigating hydrogen and oxygen separately (Vygotsky, 1978). Sfard also drew on Wittgenstein (1953) who rejected the idea of *pure thought*: “Thought is not an incorporeal process which lends life and sense to speaking, and which it would be possible to detach from speaking” (Wittgenstein, 1953, p. 109).

Sfard redefines thinking as an “*individualised version of (interpersonal) communicating*” (p. 81). The term *commognition* stresses the fact that cognition (intrapersonal communication) and interpersonal communication are manifestations of the same phenomenon. This perspective is offered as a way to avoid the quandaries facing paradigms that treat learning as *acquisition*. Rather than merely a cognitive phenomenon, mathematics learning involves individualising or developing one’s mathematical communication. This notion is useful for addressing the problem about transfer, the development of numerical thinking, and the process of abstracting from arithmetic to algebra. For example, in terms of development from arithmetic to algebra, it is a case of engaging the discourse about arithmetic at the object-level so that it can be used in the discourse about algebra. Engaging in object-level discourses requires distilling processes—such as counting a set of objects and ending on the word five—into discursively constructed objects—such as the number five. Sfard argues that this act of objectification is central to the development of human thought and of mathematics.

Sfard proposes four features of the mathematical discourse, *word use, visual mediator, routines, and narratives*, which could be used to analyse mathematical thinking and changes in thinking. *Word use* is a main feature in mathematical discourse; it is “an-all important matter because [...] it is what the user is able to say about (and thus to see in) the world” (p. 133). As a student engages in a mathematical problem, her mathematical discourse is not limited to the vocabulary she uses. For example, her hand-drawn diagram and gestures can be taken as forms of *visual mediator* to complement word use. A visual mediator is a visual realisation of the object of a discourse. Visual mediators include primary objects that pre-exist the discourse (such as

triangles) and artifacts created especially for the sake of communication (such as written symbols). *Routines* are meta-rules defining a discursive pattern that repeats itself in certain types of situations. In learning situations, teachers may use certain words or gestures repeatedly to model a discursive pattern, such as looking for similarities and what it means to be “the same”. *Narratives* are a series of utterance, spoken or written, that are framed as a description of objects, of relations between objects, or processes with or by objects, and are subject to endorsement or rejection, that is, to being labeled as “true” or “false”.

Sfard conceptualises learning mathematics as a change in one’s mathematical discourse. This can be observed through one’s word use, visual mediator, routines, and narratives (Table 1). *Incommensurable discourses* arise as inconsistent uses of words, visual mediators, routines or narratives are communicated. Some examples of potential points of incommensurable discourse are shown in

Table 2.

Table 1. Examples of a mathematical discourse about “quadratic functions”

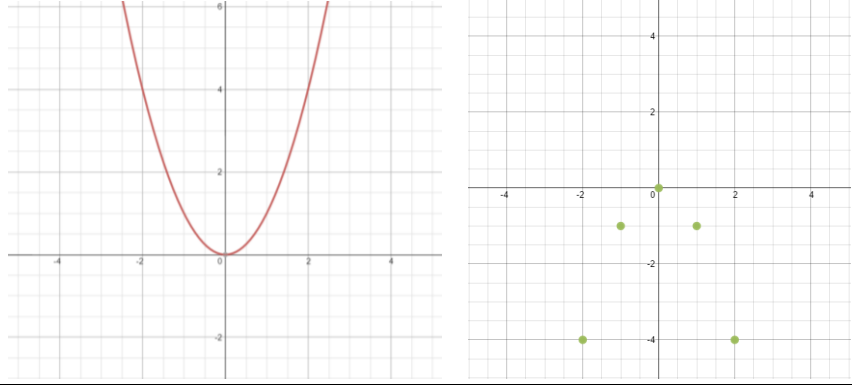
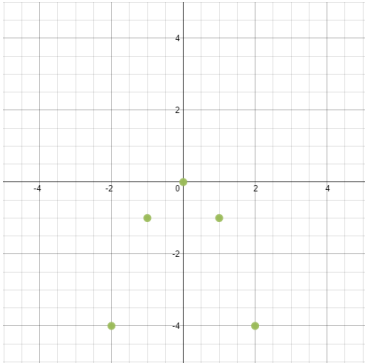
Word use	A quadratic function is a polynomial function of degree 2. It has a graph that is “U-shape”, and it can be drawn without lifting your pencil.
Visual mediator	
Routine	To sketch the graph of any quadratic function, use a table of values to calculate $f(x)$ for integral values of x .

Table 2. Examples of potential points of incommensurable discourse

Discourse 1	Discourse 2	Potential points of incommensurable discourse between Discourse 1 and 2
	A quadratic function has a graph that is “U-shape”	The phrase “U-shape” is inconsistent with the visual mediator shown at left which opens down.
	It can be drawn “without lifting your pencil”	The phrase “without lifting your pencil” is inconsistent with the visual mediator shown at left.
	To sketch the graph of any quadratic function, use a table of values to calculate $f(x)$ for integral values of x .	Although the visual mediator is consistent with the routine of graphing functions using a table of values for integral values of x , it is inconsistent with the phrase “without lifting your pencil”.

Communicating in incommensurable discourses may lead to *commognitive conflicts*—the encounter between interlocutors who use the same mathematical signifier in different ways or perform the same mathematical tasks according to different rules. An effective way to examine the coherence in an interlocutors’ use of words and symbols is to try find out their realisations of those signifiers. Signifiers are words or symbols that function as nouns in a discourse, like “quadratic functions”. A realisation is an association endorsed about the signifier through a discursive transition. Realisations can take many forms: visual (written words or symbols, iconic, concrete, gestural) and vocal (spoken words). For this reason, a signifier like “ $f(x)=x^2$ ” may be realised in a number of ways, such as algebraically as a rule, visually as graph, or as a table of values. Each mode of realisation “entails a particular combination of verbal actions, visual scanning and physical manipulations [...] Whereas operating on symbols is a version of the inherently linguistic activity of reasoning, iconic and concrete procedures require a relatively small number of verbalisation” (p. 156). Often the realisation of one signifier may be necessary for realisations of other ones; this recursive structure is necessary for expanding mathematics as a discipline.

3.2.1. Utterance, gestures and mathematical thinking

Mathematical communication involves transitions from *signifiers* to other entities called *realisations*, which can be observed through one's use of words, visual mediators, routines, and endorsed narratives. Furthermore, Sfard (2009) examines the relationship between talking, gesturing and mathematical thinking. She explains that language and gestures should not be counterpoised one to another, since language is any symbolic system used in communication, and gestures are "the actual communication" (p. 194). Instead, she suggests that the proper verbal counterpart of gesture would be utterance, a communicational act that is aural-mediated.

According to Sfard (2009), utterances and gestures inhabit different modalities and serve different functions in the cognitive process. *Recursivity* is a linguistic property offered by utterances. The unlimited possibility to expand linguistically allows human to work in meta-discourse, or thinking about thinking. On the other hand, gestural communication ensures all interlocutors "speak about the same mathematical object" (p. 197). Gestures are essential for effective mathematical communication: "Using gestures to make interlocutors' realizing procedures public is an effective way to help all the participants to interpret mathematical signifiers in the same way and thus to play with the same objects" (p. 198). Gestures can be realised actually when the signifier is present, or virtually when the signifier is imagined. Sfard (2009) illustrates how a student uses "cutting", "splitting", and "slicing" gestures to realise the signifier "fraction". Since these gestures were performed in the air, where the signifier "fraction" is imagined, they provide an instance of virtual realization. Therefore, the same signifier "fraction" may be realised differently with different kinds of gesture or word use.

In relation to my study, it is anticipated that students would make use of gestures and dragging when interacting with DGEs. For example, a student may repeatedly use her arm to signify slope when comparing slopes of different line segments. This is an example of the student using gestures as a routine to look for what is "the same". The same can be said of the use of dragging to compare the slopes of tangent at different points of the function on DGEs. Hence, gestures and dragging can be taken as both a routine for defining a discursive pattern and a visual mediator of the students'

mathematical discourse in this study. This communication can be interpersonal when it is directed to another student or intrapersonal when it is directed to oneself.

3.2.2. Saming, reification and encapsulation

There are three mechanisms for the production of *compound discursive objects*, which results in greater range and depth of the realization: *saming*, *reification*, and *encapsulation*. The process of *saming* can be seen as the act of calling different things the same name. *Reification* involves replacement of talk about processes with talk about objects. Finally, *encapsulation* is the act of assigning a noun or pronoun (signifier) to a *specific set* of discursive objects, so that some of the stories about the members of this set that have, so far, been told in plural may now be told in singular. Table 3 shows an example of each process where the signifier “quadratic function” is realised.

Table 3. Examples of saming, reification and encapsulation, and how each add depth to the realisation of quadratic functions.

	Example	How it adds depth to the realisation
Saming	Using the words “quadratic function” and the word “parabola” interchangeably as if they are the “same”.	A quadratic functions are graphs that are parabolic in shape.
Reification	Using a pronoun to talk about the ordered pair $(x, f(x))$, where $f(x)$ is a quadratic function.	A quadratic function is the ordered pair $(x, f(x))$; it is obtained by evaluating $f(x)$ for different values of x .
Encapsulation	Referring the function $y=f(x)$ as an object or gesturing the shape of the $f(x)$ as a whole, where $f(x)$ is a quadratic function.	A quadratic function is the set of all ordered pairs $(x, f(x))$ satisfying the equation $y=f(x)$, or the set of all points on the graph of $y=f(x)$.

3.2.3. Temporality, gestures and DGE

Sfard (2008) argues that mathematics could not progress as a discipline without the process of objectifying actions into nouns. On the other hand, she cautions us that removing the temporality and context of processes hides important details. Although Sfard considers gestures and diagrams as different forms of visual mediators, she does not distinguish between static and dynamic visual mediation. For example, in the case of diagrams, the mediation by a hand-drawn diagram is different from that by a textbook diagram because temporality is conveyed in the act of drawing the diagram. This is also

true for gestures and even more so for DGEs which readily mediates temporality and dynamism, something that Sfard does not adequately address in her theory. This may explain why Sfard cautions the removal of temporality in the development of mathematical discourse.

The temporal functions of gestures have not been widely examined in gesture studies. Leading gesture specialist David McNeil's (1992) categorization of gestures into deictic, iconic, metaphoric, and beat, broadly characterizes the type of functions served by gestures. For example, deictic gestures serve as pointing devices, while metaphoric gestures serve to represent the mathematical objects themselves. Although useful for identifying the general functions of gestures, they do not distinguish between the static and dynamic nature of gestures, in particular, when gestures are used to convey temporal relationship. For example, when a person makes a metaphoric gesture to realize the signifier, *a linear function*, it could be of static nature, with the arm or hand enacting the function, or of dynamic nature, with the hand or finger tracing the motion of the function's path. In the latter case, the gestures communicate temporal relationships of the linear function as opposed to the shape of the linear function statically.

A few studies have shown that temporality can be evoked by the use of dynamic visual mediators like gestures. Núñez (2003) studied how mathematicians use hand gestures as a way to express dynamic thinking of functions, continuity, and other abstract mathematical ideas. He suggests that the gestures and the linguistic expressions used tell a very different conceptual story about the mathematicians' thinking. In his analysis, he shows that "these mathematicians are referring to fundamental dynamic aspects of the mathematical ideas they are talking about" (p. 177). Furthermore, these mathematicians say "approaching," "tending to," "going farther and farther," to express a sense of motion, while producing metaphoric gestures tracing the trajectory of the point or particle with their fingers. Sinclair and Gol Tabaghi (2010) also examine motion in gestures, in particular, mathematician's hand gestures depicting movement of vectors, providing evidence of time and motion-based conceptualization of vectors. These two studies point to the dynamic and temporal aspects of mathematicians' thinking; they also reveal that mobile hand movements are important features this type of mathematical thinking.

On the role of DGEs for evoking temporality, DGEs enable learners to observe and manipulate visual objects that are moving and changing over time. Because of its dynamic nature, visual mediations by DGEs are significantly different from those by textbook diagrams. Visual mediations by static visual representations evoke images of static mathematical objects such as triangles or artifacts such as a number line. In contrast, visual mediation by DGEs may evoke mathematical relationships and properties due to its potential to represent mathematical objects of an invariant property continuously. For example, a student may realise new mathematical properties by dragging a vertex of a triangle or a point on the number line dynamically. With respect to the study of calculus, concepts like graphs of derivative functions can be evoked readily on a dynamic sketch by utilising the *Dragging and Trace Tool*.

Sinclair and Yurita (2008) drew on Sfard's communicational approach to investigate the impact of using DGEs on mathematical thinking by identifying changes in teachers' discourse in a grade ten geometry class. They found significant differences in the ways that the teachers talk about geometric objects when moving from a static to dynamic environment. For example, the teacher no longer relied on comparing a given static shape to a definition, but they began to use the *Dragging* tool to show whether properties of a given quadrilateral can be "broken". Their study shows how DGEs may change the way teachers use visual artifacts and geometric reasoning.

In a similar way, my study uses Sfard's communicational framework for examine changes in bilingual learners' mathematical discourse during their interactions with touchscreen-based DGEs. I highlight the role of the dynamic visual mediator, the DGE, and I focus on the use of gestures also as a dynamic visual mediator for conveying temporal relationship. In other words, I use a combination of utterances, gestures and touchscreen-dragging on DGE for studying a *multimodal* mathematical discourse. In doing so, I extend Sfard's communicational theory which does not acknowledge temporality in the mathematical discourse. The distinction between dynamic and static visual mediators is my effort to "bring back" temporality in the mathematical discourse.

3.3. Summary

In this chapter, I illustrated how Moschkovich's (2007a) sociocultural view of bilingual learners can empower bilingual learners by focussing on their mathematical Discourse practices in mathematical activities. This learning-as-participation perspective forms the theoretical basis of my study on bilingual learners' communication. On the role of utterance, gestures and diagrams in mathematical thinking, I complement Moschkovich's sociocultural view of bilingual learners with Sfard's communicational theory. The relationship among utterance, gestures and diagrams is worth exploring in my research to better understand bilingual learners' communication. While gestures studies have provided insights into the multidimensional nature of mathematical thinking, much of this work address the role of gestures as independent from cognitive processes. With this dualist approach, gestures are external acts that represent the mathematical thought from within and embodied acts that make cognitive processes explicit. In contrast, Sfard's (2008) communicational approach rejects the dichotomy between gestures (or speech) and thought. Moreover, Sfard (2009) suggests that gestures and utterances complement each other by serving different functions in communication. For these reasons, I find Sfard's communicational framework useful for studying bilingual learners' communication. In particular, I extend Sfard's theory by making a distinction between dynamic and static visual mediators in order to "bring back" temporality in the mathematical discourse.

Chapter 4. Methods

The bilingual is not the sum of two complete or incomplete monolinguals; rather, he or she has a unique and specific linguistic configuration. (Grosjean, 1985, p. 19)

Over the course of my research study, I have come to understand the importance of a three-way connection among research question, methodology and data analysis. I understand that it is essential for my research questions, as informed the literature review and theoretical framework laid out in Chapters 2 and 3 respectively, to guide the methodological decisions that I make in the study. For this reason, I begin this chapter by stating the research questions that have guided me through my methodological design. Then, I describe the details and rationale of the methods that I undertook in the study.

4.1. Research questions

My study investigates bilingual learners' patterns of communication during pair-work mathematical activities with the use of touchscreen-based DGEs. In particular, I address the following three research questions:

1. How do bilingual learners utilise linguistic and non-linguistic modes of communication during pair-work on mathematical activities with the use of touchscreen-based DGEs?
2. What kinds of mathematical discourse practices do bilingual learners engage in, and what kinds of calculus ideas are communicated during pair-work on mathematical activities with the use of touchscreen-based DGE?
3. What is the role of technology for facilitating bilingual learners' communication during pair-work on mathematical activities with the use of touchscreen-based DGEs?

The first of my three research questions addresses both *linguistic and non-linguistic modes* of bilingual learners' calculus communication during pair-work on mathematical activities with the use of touchscreen-based DGEs. It explores the interplay between linguistic and non-linguistic modes of communication in mathematical thinking and learning. Although some research has shed light on bilingual learners' non-linguistic forms of communication, such as gestures and diagrams (Gutierrez, Sengupta-Irving, & Dieckmann, 2007; Moschkovich, 2007a, 2009), this work has not addressed the use of digital technologies, DGEs in particular—which have been shown to facilitate student communication by providing visual and dynamic modes of interaction (Ferrara, Pratt & Robutti, 2006; Falcade, Laborde & Mariotti, 2007).

The second question concerns the mathematical content—mathematical Discourse practices and the calculus ideas—communicated by bilingual learners as they interact with touchscreen-based DGEs in pairs. More details about mathematical Discourse practices have been discussed in Chapters 3; essentially, they are practices that are shared and valued in the given mathematics community. This question allows me to uncover bilingual learners' competence in mathematical communication, by investigating their use of language, gestures, touchscreen-dragging and the DGE as part of the emergence of mathematical thought and participation as members of their mathematics classroom community.

The third question examines the role of touchscreen-based DGEs for facilitating calculus communication. Although numerous studies have discussed the effect of DGE-mediated learning of calculus concepts (Yoon, Thomas, & Dreyfus, 2011; Yerushalmy & Swidan, 2012; Hong & Thomas, 2013), research on the effect of touchscreen-based DGEs is limited. It is hypothesised that a touchscreen-based DGE may offer additional affordances by providing tactile and kinesthetic modes of interaction—hence, further facilitate bilingual learners' communication in calculus.

4.2. The Participants

From May 2013 to May 2014, I undertook a two-part research study involving twelve participants who were bilingual calculus learners. The participants were grade 12

students enrolled in two sections of the Advanced Placement² (AP) Calculus course in a culturally diverse high school in Western Canada. Four students were enrolled in a section of AP Calculus during the school year of 2012-2013; they participated in Part I of the study in May 2013 and were graduates of 2013. The other eight students were enrolled in a section of AP Calculus during the school year of 2013-2014; they participated in Part II of the study in January 2014 and were graduates of 2014. The participants were selected for their bilingual background; all of them declared that they were born outside of Canada and spoke a language other than English at home. The class size for each section was 23 (in 2012-2013) and 26 (in 2013-2014) respectively, with roughly one-half bilingual learners in each section.

A questionnaire was given to each participant for collecting some demographic information such as gender, age, and home country, as well as more information about their linguistic and educational background before leaving their home country for Canada. Table 4 shows the results of the brief survey. Of the twelve participants, seven were male, and five female. Their ages ranged from 17 to 19 in the year of 2013. Seven of the twelve participants left their home country at the age of 16; hence, these students had only lived in Canada for two to three years. The students' home countries consist of four different geographical locations in Asia, namely China (7 students), Korea (2 students), Hong Kong (2 students) and Taiwan (1 student), which meant that each participant shared the same home language (Mandarin, Korean, or Cantonese) with at least one other participant. With the exception of the two students from Hong Kong, who had some experience studying mathematics in English, none of the students had ever studied mathematics under English instruction before leaving their home countries. In terms of experience with mathematics, none of the participants had studied calculus in their countries of origin.

² The curriculum and rigour of this course is equivalent to a typical single-variable calculus course in a North American university.

Table 4. Demographics of the participants

Name and Gender	Grade and age in September 2013	Left home country at age of	Language spoken at home	Experience studying math in English in home country	Experience studying calculus in home country
School Year 2012-2013					
Ana; F	Grade 12; 19	China; 16	Mandarin	No	No
Tammy; F	Grade 12; 19	China; 16	Mandarin	No	No
Melissa; F	Grade 12; 18	China; 15	Mandarin	No	No
Yee; M	Grade 12; 19	China; 16	Mandarin	No	No
School Year 2013-2014					
Jay; M	Grade 12; 17	Korea; 15	Korean	No	No
Katie; F	Grade 12; 17	Korea; 15	Korean	No	No
Sam; M	Grade 12; 18	China; 16	Mandarin	No	No
Mario; M	Grade 12; 17	Hong Kong; 14	Cantonese	Some	No
Larry; M	Grade 12; 17	Hong Kong; 10	Cantonese	Some	No
Ivy; F	Grade 12; 17	China; 12	Mandarin	No	No
Huang; M	Grade 12; 18	China; 16	Mandarin	No	No
George; M	Grade 12; 17	Taiwan; 14	Mandarin	No	No

4.3. The Study

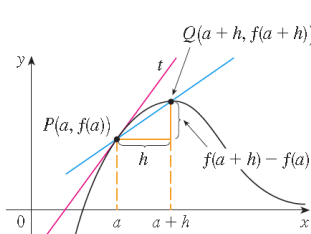
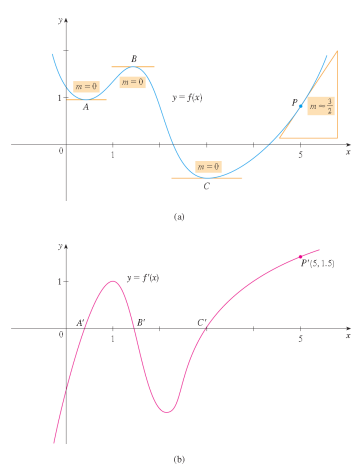
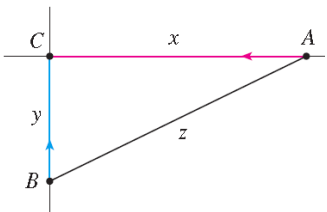
The study was designed to address bilingual learners' patterns of communication in a touchscreen, dynamic calculus environment. In order to address their communication about calculus, the participants needed to have a certain degree of experience working with touchscreen-based DGEs before the study. Therefore, I fostered a dynamic calculus learning environment in the participants' regular calculus classes before the study took place. From 2012-2014, I taught three sections of AP Calculus where the use of touchscreen-based DGEs was consistently incorporated into lessons for exploring calculus ideas. All year long, I emphasised the importance of multiple representations of functions (using a table of values, graphical representation, and algebraic representation). During the lessons where DGEs were incorporated, I invited students to explore the pre-designed dynamic sketches in pairs for roughly ten

minutes before leading a whole class discussion about the exploratory activities. Upon exploring concepts with the DGEs, I would formally introduce the concepts, provide examples to be solved algebraically on paper, and ask students to interpret their solutions geometrically or graphically on the DGEs.

Prior to the study, I had taught the same course, AP Calculus, for two full years and had been teaching mathematics at the school for eight years. Given this experience, I made sure that my “teacher” role in the study was consistent with my past experience teaching calculus, so that my “researcher” role would not interfere with my “teacher” role in the regular classroom, to my best ability (Ainley, 1999). At the beginning of the year, I informed all students in AP Calculus that I was undertaking research about calculus students’ communication and asked for their voluntary participation after school hours. I ensured all potential participants were informed: (1) that their participation was entirely voluntary, (2) that my regular teaching would not be affected in any way, and (3) that in no way would any form of their regular calculus classroom learning be affected by their participation or lack of participation in the study. I conducted the study during non-class time in the participants’ regular calculus classrooms. The rationale for choosing this setting was to make the participants feel as natural as possible by participating in a physical environment that they were used to.

The study consists of two parts. Part I of the study took place at the end of the course, in May of the school year 2012-2013. The main purpose of this part of the study was to investigate the participants’ communication patterns as prompted by two different types of visual mediators, static and dynamic, with respect to the study’s research questions. Since the students had just completed a year-long AP Calculus course, where key concepts in calculus were taught using a class set of touchscreen-based DGEs, they were experienced at exploring and discussing calculus concepts through geometrical, dynamic sketches in pairs at the time of study. These concepts included the definition of a derivative, derivative functions, related rates, and the Fundamental Theorem of Calculus. This particular setting in Part I allowed me to compare the participants’ use of linguistic and non-linguistic resources in communication about the same calculus concept but facilitated by different types of visual mediators.

The four participants from Part I of the study were divided into two pairs: Ana and Tammy had been regular classroom partners, while Melissa and Yee had sat in proximity to each other but were not regular classroom partners. These pairings were intended to foster the kinds of student-pair communication that would occur in the students' regular calculus classroom. The pairs were asked to discuss ten different diagrams—five textbook diagrams shown in PDF form and then five dynamic diagrams presented in an iPad-based DGE application, *Sketchpad Explorer* (Jackiw, 2011). The five textbook diagrams (some of which are discussed in Chapter 5), were taken from the students' regular calculus textbook (Stewart, 2008). The five dynamic sketches were minimally adapted from the ones that the students had used in class during the school year. For the purpose of comparing patterns of communication, each of the five static diagrams had a corresponding dynamic sketch that involved the same target concept.

<p>a) Page 1: Definition of derivative</p> 	<p>b) Page 2: Derivative functions</p> 
<p>c) Page 3: Related rates</p> 	<p>d) Page 4: Linear approximation</p>

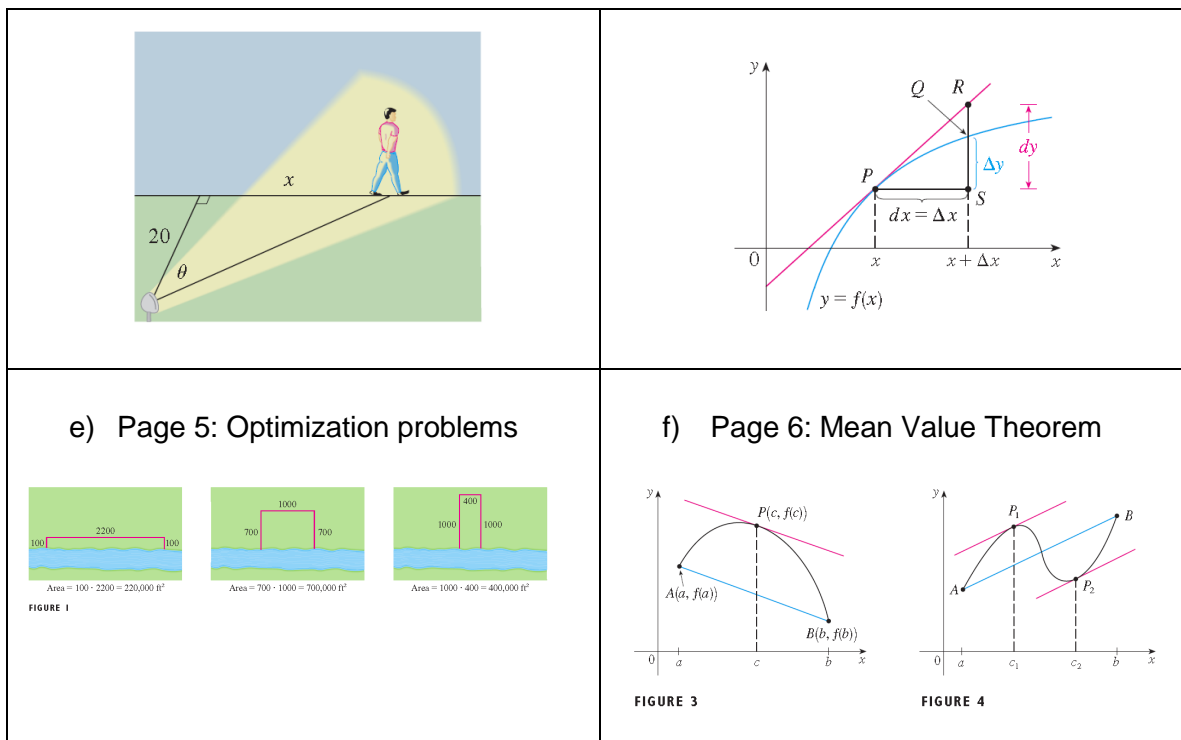


Figure 3. Textbook diagrams used³ for each target concept in Part I of the study

Part II of the study took place after school hours, in January of the school year of 2013-2014. The main purpose of Part II differed from that of Part I in that it addressed the participants' patterns of communication while exploring a calculus idea that they had not yet learned. Therefore, it was necessary for this part of the study to take place during the middle of the school year, before the target concept was taught in class. The target calculus concept, the area-accumulating function, was chosen for three reasons. First, the function $\int_a^x f(t)dt$ could be represented geometrically, and it was possible for one to explore the change in $\int_a^x f(t)dt$ as 'x' varies without knowing the corresponding symbols. This can be achieved by thinking of the change of $\int_a^x f(t)dt$ as area-accumulation. Secondly, the timing of the introduction of the concept was appropriate because the students would have had some experience with learning calculus in dynamic environments before the time of study. In particular, they would have used a similar dynamic sketch for exploring derivative functions by interpreting derivative as

³ See Appendix A for larger images for each diagram.

tangent slope of a graph geometrically. This experience was similar to exploring area-accumulating function, since both explorations required learners to interpret change (tangent slope and area-accumulation) geometrically and identify a covariance between 'x' and the function ($f'(x)$ and $A(x)$). Finally, the area-accumulating function was chosen as a response to the literature review, which pointed out that the simultaneous change of the variables 'x', $f(x)$, and $A(x)$ was difficult to grasp among calculus learners.

The task used in Part II of the study invited the students to discuss a single sketch presented in *SketchExplorer* that they had not previously seen. Table 5 shows the timeline of which the task took place in relation to the course. As the table shows, the concept of area-accumulating functions was new to the students at the time of study. In January 2014, the students had just completed the differential calculus component of the course and one lesson on indefinite integrals. For the purpose of examining students' routines during an exploratory activity using touchscreen-based DGEs, they were given a sketch containing five pages all related to the concept of area-accumulating functions. The participants were asked to "explore the pages, talk about what you see, what concepts may be involved" in each page of the sketch and then to move onto the "Try" page of the sketch where a problem was posed. They were asked to solve the problem on a dry-erase whiteboard, and they were told that I, as the teacher-researcher, would check in with the students from time to time to make sure that they could ask questions related to technical aspects of the sketch. Like Part I, the eight participants from Part II of the study were assigned into pairs on the basis that they were regular partners during assigned pair-work activities in class and identified as motivated and comfortable working with each other.

Table 5. Schedule of lessons when DGEs were incorporated in 2013-14 (for Part II of study)

Name of dynamic sketches used	Month and year that the sketch was used during lesson	Month and year that the sketch was used in the study
Definition of derivative	Oct 2013	
Derivative functions	Oct 2013	
Derivative of polynomial functions	Nov 2013	
Chain rule	Nov 2013	

Related rates	Nov 2013	
Linear approximation	Nov 2013	
Optimization problems	Jan2014	
Mean Value Theorem	Jan 2014	
Area-accumulating functions	Feb 2014	Jan 2014
Volume by revolution	Feb 2014	
Slope fields	Mar 2014	
Derivative of inverse functions	Apr 2014	

Table 6 summarizes the participant pairings in the study (for both Parts I and II). It is worth mentioning that, as shown in the table, a new pairing was formed on the same day that “Jay and Katie”, and “Larry and Ivy” completed the task. Specifically, the pairs Jay/Katie and Larry/Ivy were asked to switch partners after their initial discussion of the sketch so that they could begin a discussion with a new partner. The rationale for this was to compare patterns of communication for different pairings, namely, communication between the new pairs “Jay and Larry” and “Ivy and Katie” with that of “Jay and Katie” and “Larry and Ivy”.

Table 6 Summary of the study’s timeline and participant pairings.

Part I of the study	Part II of the study	
May 2013 Discussing various target concepts	Jan 2014 Discussing area-accumulating functions	
Ana and Tammy	Sam and Mario	
Melissa and Yee	Huang and George	
	Larry and Ivy (same day)	Larry and Jay (same day)
	Jay and Katie (same day)	Ivy and Katie (same day)

4.4. Data Collection Process

For both Parts I and II of the study, a digital camera with video-recording function was placed at an angle in front of the students’ desks where they were discussing and

interacting with the iPads. The video-recording function was turned on right when I began to the task to the students and demonstrate some technical aspects of the DGE, after which the students would start discussing (see Appendix B for a transcript of my introduction and instructions at the start of a session). In particular, it was stressed to the student participants in Part II, that the activity was not about finding the right or wrong answer, but more about exploring the concept, looking for patterns, and communicating what they saw with each other.

For most sessions, there were other students present while the task took place. For example, there were simultaneous video-recording in different parts of the classroom going on in some sessions. This was intended to make the participants not feel that they were “under the spotlight”. I, the teacher-researcher, was occasionally present in the room to ask if the students have any technical questions and to ensure that they were carrying out the task. I was out of the classroom at other times to give the students the sense that I was not listening directly to their conversation in real time. A total of 170 minutes and 12 seconds of video data was collected in the study. Table 7 shows a breakdown of the video data collected from each pair.

Table 7. A breakdown of the video data collected from each pair in the study.

Part I of the study	Part II of the study	
May 2013 Discussing various target concepts	Jan 2014 Discussing area-accumulating functions	
Ana and Tammy = 21:23	Sam and Mario = 29:57	
Melissa and Yee = 22:05	Huang and George = 29:59	
	Larry and Ivy (same day) = 28:55	Larry and Jay (same day) = 03:18
	Jay and Katie (same day) = 27:25	Ivy and Katie (same day) = 07:10
Total length of video collected = 170:12		

4.5. Data Analysis Process

The data analysis process consisted of three phases: Phase 1, transcribing the data; Phase 2, reviewing the transcript; and Phase 3, analysing the transcript, each described in detail in the sub-sections below.

4.5.1. Phase 1: Transcribing the data

During Phase 1, transcribing the data, I selected about 75% of the video data, transcribed the words spoken and made note of the gesturing and dragging actions that took place in the video data using the software Nvivo. Transcribing this portion of data was sufficient for me to identify common patterns of communication as well as unique features of communication for particular student pairs. In line with my research questions, it was important for me to transcribe the video data in terms of the student pairs' utterances, gestures and touchscreen-dragging actions with the DGE during the task. According to Arzarello (2006), this allows for a synchronic analysis for examining the inter-relationships between words spoken, gestures and diagrams at a certain point in time. Therefore, I organised the transcript to highlight the interplay between words spoken, gestures and dragging actions within the student pairs' communication. Unlike conventional transcripts which informs only "who spoke what", I introduced two columns, the "gesturer" and "dragger" columns, in the transcript in order to track "who gestured" and "who dragged" simultaneously. Screenshots of certain gesturing and dragging actions were taken and included in the transcript. In addition, I used underlining of the transcript to record which words were spoken while a dragging and gesturing action was performed simultaneously by one of the students. In doing the above, my rationale is to value each of the three actions: speaking, gesturing, and touchscreen-dragging on the DGE in the data analysis as significant forms of communication for the bilingual learners'.

To enhance readability, punctuation was added to the transcript. The first priority for placing punctuation was to reflect different kinds of pauses in speech; it was not a priority to place punctuation for grammatical purposes. A comma (,) was used for very short pauses; a period (.) was also used for short pauses, in particular those pauses

which seem to mark a termination of one complete sentence or thought; and finally three consecutive periods (...) were used to denote longer pauses.

The transcript is further enhanced by being divided into time-stamped “turns”. A “turn” is defined as a timespan in which a certain utterance, gesturing or dragging action took place. Table 8 shows a sample transcript which contains three turns or completed actions that took place between 00:00.0 and 00:05.0. As the “time” column shows, three non-overlapping actions took place. The first turn was an utterance by Melissa, “you want try” (Turn 1). This was followed by Yee’s dragging on the touchscreen DGE (Turn 2), and then Melissa’s dragging (Turn 3), both of which were completed without any speech. Hence, if no more than one action (speaking, gesturing and dragging) was performed simultaneously, each turn contains one completed action by one person. A change of turn means that either the mode of communication has changed by the same person, or a different person.

Table 8. Sample transcript

Turn	Time	What was said	Speaker	Gesturer	Dragger
1	00:00.0-00:05.0	You wanna try?	M		
2	00:05.1-00:10.0	<no speech>	Nil		Y
3	00:10.1-00:15.0	<no speech>	Nil		M

If more than one action (speaking, gesturing or dragging) were performed simultaneously, the longer of the completed actions was used to define a turn. In most cases, the longer action is a speaking action (for example, when a person is speaking and gesturing at the same time, or speaking and dragging at the same time, the speaking took longer than the gesturing or dragging). For instance, Table 9 shows another sample transcript between Melissa and Yee’s discussion of a DGE sketch in a 20 second interval. Overlapping speech between two people is noted in the “time” column. As seen in Turn 1 and 2, Yee began talking before Melissa completed her utterance. Then, while Melissa continued to drag on the touchscreen DGE, Yee spoke and gestured at the same time (Turn 3). Finally, in Turn 4, both Melissa and Yee

dragged on the same touchscreen DGE by exploiting the multi-touch capabilities on the iPad. The advantage of transcribing in this way is that simultaneous speaking, gesturing and dragging actions can be noted, and analysis can be made as to what kinds of linguistic and non-linguistic resources were used synchronically in communication.

Table 9. Another sample transcript

Turn	Time	What was said	Speaker	Gesturer	Dragger
1	0:00.0-0:05.0	What do you think?	M		M
2	0:04.0-0:10.0	It looks like it's going like <u>this</u> .	Y	<u>Y</u>	M
3	0:10.0-0:15.0	I don't understand this.	Y		Y
4	0:15.0-0:20.0	<no speech>	Nil		M/Y

4.5.2. Phase 2: Reviewing and selecting the data

After transcribing the video data using the scheme discussed above, I reviewed and made notes on the transcripts during Phase 2. During the reviewing process, it became apparent that certain words, gestures and dragging actions were used repeatedly by one or more participants. Therefore, I began to make notes on these in the transcripts. Adopting a discursive analysis approach, the repeated use of different semiotic resources constitute a semiotic domain (e.g., oral or written language, images, equations, symbols, sounds, gestures, graphs, artifacts, and so forth) in communication (Gee, 1999). To be fluent in a specific discourse not only means to use a set of shared semiotic domain in the community of practice but also to understand the “design grammar” or “patterns in terms of which materials in the domain are combined to communicate complex meanings” (Gee, 2008, p. 138). Attending to the repeated use of certain resources in communication is helpful for determining these patterns or *routines* in communication (Sfard, 2008).

For the purpose of investigating the *interplay* between linguistic and non-linguistic modes of communication, I needed to observe the way that linguistic and non-linguistic resources were used simultaneously and in succession. This was achieved by attending

to turn-taking; instances of simultaneous speaking, dragging and gesturing actions (either by the same person or by different persons); and dragging or gesturing actions without accompanying speech. The list below summarises my “units for analysis” during the reviewing phase. Together, they make up the student pairs’ “patterns of discourse” within mathematical activities using touchscreen-based DGEs (Figure 4).

1. Particular words or phrases that were used repeatedly in the situation.
2. Particular dragging and gesturing actions that were used repeatedly in the situation.
3. Turn-taking: modes of communication (speaking, dragging, gesturing) when a new turn began.
4. Particular words that were used simultaneously with dragging or gesturing actions by one or two persons
5. Dragging or gesturing without accompanying speech by one or two persons.
6. Recurrent sentence structures within a certain time span.
7. Recurrent mode of communication (speaking, dragging, gesturing) within a certain time span.

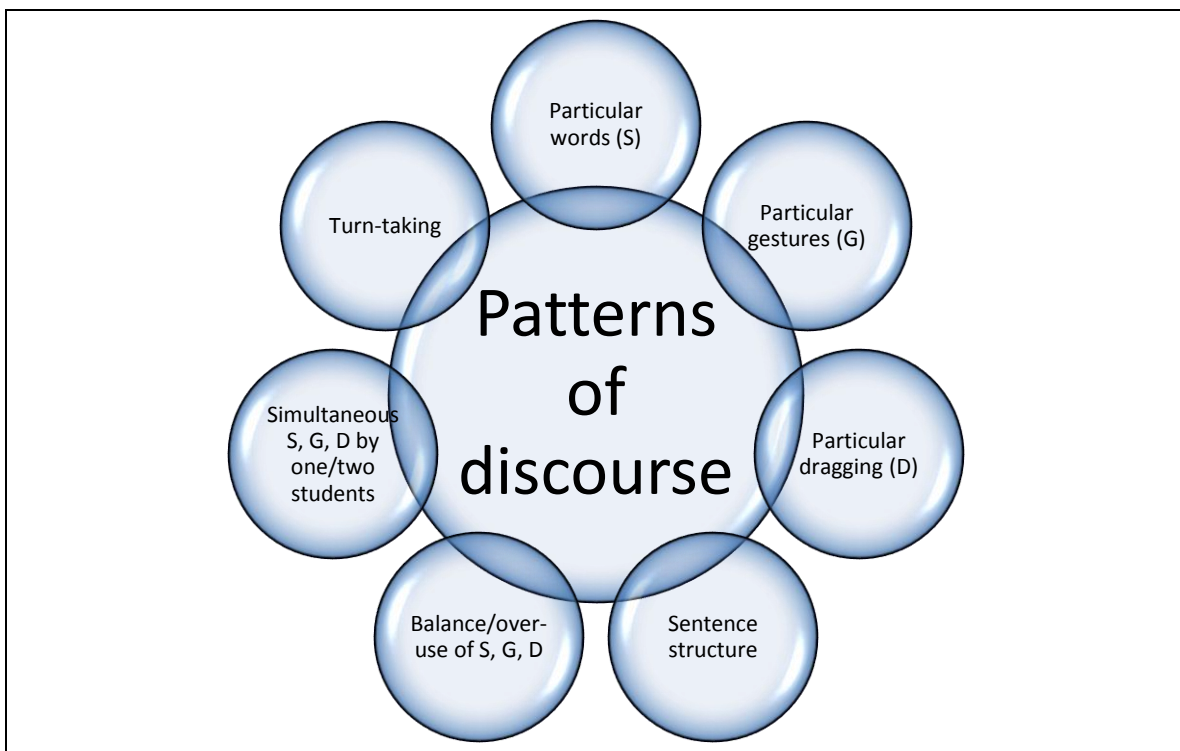


Figure 4. Student pairs’ “patterns of discourse” within mathematical activities using touchscreen-based DGEs

Besides identifying the above “units for analysis” in the transcripts, I made note of different characteristics of speaking, gesturing and dragging actions, including gaze,

tones of voice, timespan of a dragging or gesturing action, and “gesture space” in terms of the speaker’s as well as her interlocutors (Goodwin, 2000). These features can be useful in the next phase for unfolding the *meaning* of specific word use, gesturing and dragging actions in the communication. For example, a soft tone of voice while the speaker is dragging simultaneously may suggest that the speaker is uncertain about what she is seeing in the sketch, and gestures that are performed in a space not oriented to another person may suggest that the gesturer is communicating intrapersonally as opposed to interpersonally. Although McNeil (1992) defines “gesture space” with reference to the body of the gesturer, I have chosen to expand the notion of gesture space using a “multi-party participatory framework” (Goodwin, 2000, p. 89) to gain insight into when a gesture is utilised in intrapersonal communication or in interpersonal communication. Hence, in my review of the transcript, I made note of these features of verbal and non-verbal communication for further analysis in the next phase.

At this point, all the transcribed data had been reviewed with notes made on them. Before moving onto Phase 3, I selected a number of transcripts from each pair in preparation for data analysis. The basis of this selection was to reflect the students’ overall communication patterns, in terms of their general “patterns of discourse” and how they change over time. Certain characteristics of communication were also selected to highlight unique aspects of particular student pairs’ communication. A consideration for selection was that of the mathematical contents communicated by the students, especially as they involve dynamic and static notions of calculus. The selection criteria also included how well the data could be used to answer the proposed research questions. For example, when multiple sets of transcripts answered the same research question, only one or two of them would be selected in order to be non-redundant. At the end of the selection process, roughly 5000 words of transcript data were selected for detailed analyses.

4.5.3. Phase 3: Analysing the transcript

The data analysis phase considered Moschkovich’s perspective on how different methodological views may help reveal or undermine bilingual learners’ competence in mathematical communication. Moschkovich’s (2007a) questions the efficacy of two

traditional perspectives for understanding bilingual mathematics learners. The *vocabulary* perspective views the acquisition of vocabulary as a central component of learning mathematics for bilingual learners. The *multiple meaning* perspective focusses on learning to use different meanings appropriately in different situations. Both perspectives focus on what learners do not know or cannot do. Challenging these deficit models, Moschkovich calls for the *sociocultural* view, which focusses on describing the *resources* that bilingual learners use to communicate mathematically (see Section 3.1). As it is vital for my research to consider the resources that bilingual learners utilise in communication, I adopted the sociocultural view of bilingual learners in my data analysis process.

Blending Sfard (2008, 2009) and Moschkovich (2007a), I created three guiding questions to drive the data analysis process:

1. What are the situated meanings of the words, phrases, visual mediators and routines used in the mathematical activity?
2. How do students utilise multimodal communication (speaking, gesturing, dragging)? What signifiers are realised, and how are they realised?
3. What mathematical Discourse practices are demonstrated in the activity, and how are they demonstrated?

The idea of semiotic bundle (Arzarello, 2006) was useful for my pursuit of the above guiding questions. While traditional linguistic studies have used the terms “speech-gestures match” and “mismatch” for distinguishing speech-gesture that appear to convey the same or different meaning, I found these terms not useful for my study because they do not highlight the way speech and gesture *complement* each other as a *growing* set of resources in the learning process. Arzarello proposes the term semiotic bundle as being made of deeply intertwined sign systems, like gesture and language. The key is that a semiotic bundle must be considered as a unitary system but not as a juxtaposition of semiotic sets. He suggests that two kinds of analyses, synchronic and diachronic, are needed to fully understand a semiotic activity and to show a growing semiotic bundle. *Synchronic analysis* enables the study of relationship among different semiotic sets activated simultaneously. *Diachronic analysis* studies the same phenomenon in successive moments. In his study, he shows that a group of young

children first used a variety of gestures to develop understanding of the problem. This synchronic analysis provides evidence that “through gesturing, children make the problem more tangible” (p. 291). Then, they moved to a new semiotic set in using a written signs (diachronic analysis) as the students begin to write and draw on their papers. At last, the students used a multimodal approach integrating speech, gestures and written representation to develop a working rule and then a global rule for solving the problem.

I see both synchronic and diachronic analyses as suitable for addressing my research questions: a synchronic lens enables me to analyse the interrelationships between linguistic and non-linguistic modes of communication, while a diachronic analysis allows for an investigation of how this multimodal communication, namely the use of words, gestures and touchscreen dragging, *change* over time. Using Sfard’s framework that conceptualises learning as a *change* of communication, this diachronic analysis allows me to address bilinguals’ learning as they explored calculus with DGEs. By performing these analyses, my goal is to show how the sociocultural view examining bilingual learners’ utterance, gestures and touchscreen-dragging may uncover their competence in mathematical activities and communication.

4.6. Summary

The methods of one’s research are extremely important in the pursuit of a research study; it is the apparatus of which one uses to carry out her research questions. In this chapter, I discussed the various methodological components that were relevant to my study: the participants, the study, data collection process and a three-phase data analysis process. Qualitative analysis of data, in Chapter 6 and 7, is based on the methodology described in this chapter.

Chapter 5. Diagrams and Sketches

As mentioned, I identified two kinds of visual mediators or visual mediation, static and dynamic, as an element of my theoretical consideration. This distinction enables me to study mathematical thinking as prompted by textbook diagrams (static visual mediators) and the use of touchscreen-based DGEs (dynamic visual mediators). Since my study aims to address the communication patterns that arise in these two settings, I need to understand the affordances of both the textbook diagrams and the dynamic sketches used in the study. I devote this chapter for examining some of the textbook diagrams and dynamic sketches used in the study as an extension of my methodology.

5.1. Review of textbook diagrams

In this section, I discuss the textbook diagrams chosen from Stewart (2008), some of which used in Part I of my study. *Calculus* by James Stewart is one of the most popular calculus textbooks in first year university calculus courses in Canada. Reviewing the diagrams published in this textbook serves to inform me how the study of calculus is viewed in the professional mathematics and mathematics education community. I have been using Stewart's (2008) *Calculus: Early Transcendental*, 6E since 2008 and am using a newer version of the same textbook since 2013. Over the school year, I have incorporated theorems, proofs, examples and diagrams from the textbook into my lessons. In the following detailed review, I am interested in the way a paper medium, in the form of a textbook, expresses a sense of change in the study of calculus.

One of the characteristics of diagrams used in all textbooks is the use of symbols to label mathematical objects. In Stewart (2008), the use of symbols ranges from naming the objects such as point A or line L , to more complex ideas such as the value of a function, $f(x)$, or derivative of a function, $f'(x)$. Colours are commonly used to differentiate between different mathematical objects, and dotted lines act as visual cues, as seen in

Figure 5(a & b). In Figure 5(a), the red line t is the tangent to the function f at point P , and the blue line is the secant of f through points P and Q . In Figure 5(b), the shaded pink region illustrates the area under $y=f(t)$ from $t=a$ to $t=x$, which generates a function of ‘ x ’, as denoted by $g(x)$, the area-accumulating function. The two diagrams respectively introduce the idea of a derivative and the area-accumulating function which are central to the study of calculus.

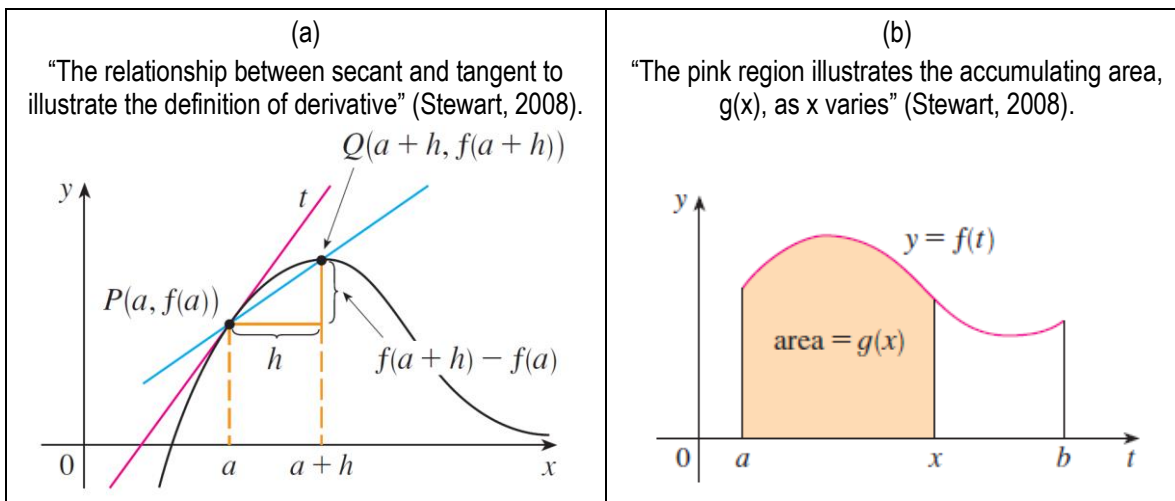
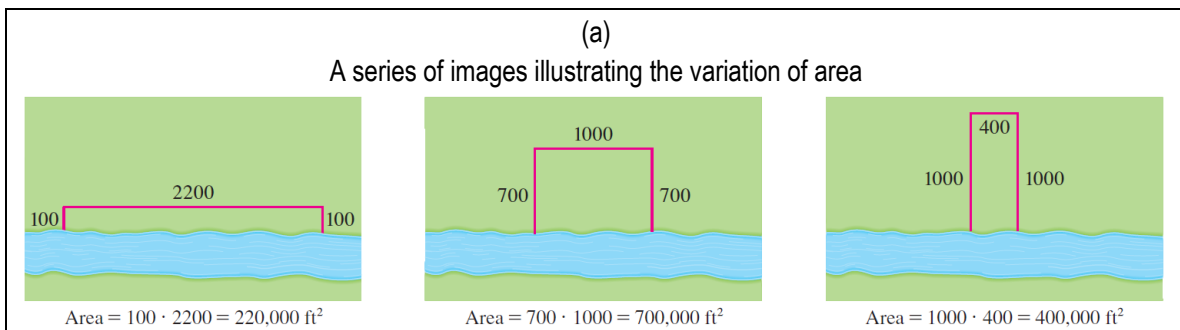


Figure 5. Typical diagrams found in calculus textbooks

Both figures are insufficient for illustrating the change of variables ‘ h ’ and ‘ x ’ dynamically as $h \rightarrow 0$ or as ‘ x ’ varies. In the absence of a visual effect of “change”, both diagrams are captioned with words to describe a sense of change. Words such as “approaches” and “becomes” were used to introduce movement and a sense of change along with the diagram in Figure 5(a): “Notice that as x approaches a , h approaches 0 [...] and so the expression for the slope of the tangent line in Definition 1 becomes $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ ” (Stewart, 2008, p. 145). The word “notice” seems an interesting choice of word in a static diagram like this one, as the readers cannot “notice” anything since nothing in the diagram suggest that any variables are changing. Similarly, the diagram in Figure 5(b) is complemented with words to suggest the varying of ‘ x ’, as in: “If we then let x vary, the number $\int_a^x f(t)dt$ also varies and defines a function of x denoted by $g(x)$ ” (Stewart, 2008, p. 380). The conditional statement that begins with “if” suggests that there is a state of change, and along with the verb “vary”, seem to be inviting the readers to imagine the movement of ‘ x ’ much like the combination “notice...

approaches” in the previous diagram. Since both diagrams do not communicate dynamic properties, it is suggested that the diagrams are not the main focus here; it is in words that the students would find the important information.

Besides the use of words, the use of arrows and a series of images are incorporated to convey a sense of change in a static medium of paper. Figure 6(a) shows a series of images illustrating the classic optimisation problem: maximize a rectangular area with a fixed perimeter. Using three examples of area equals to $220,000\text{ft}^2$, $700,000\text{ft}^2$, and $400,000\text{ft}^2$, the idea of varying area with a fixed perimeter is suggested. Also, Figure 6(b) shows a series of images depicting “the secant *lines* approaching the tangent line” (Stewart, 2008, p. 82). It is interesting to note the plural form in secant *lines*, since in a dynamic approach, there is only *one* secant line approaching the tangent line. Therefore, the series of images convey changes *discretely* but not continuously. In contrast, arrows can be used to denote continuous change, as found in Figure 6(c). The arrows indicated on both the blue and red lines denote the change of the respective distances. For example, as Point *A* moves to the left and Point *B* moves upward, the distance of ‘*x*’ and ‘*y*’ vary, but the distance ‘*z*’ stays constant, as no arrows were indicated on *z*. Although arrows are used conventionally to signify continuous movement, the readers must still visualise the movement dynamically in their imagination.



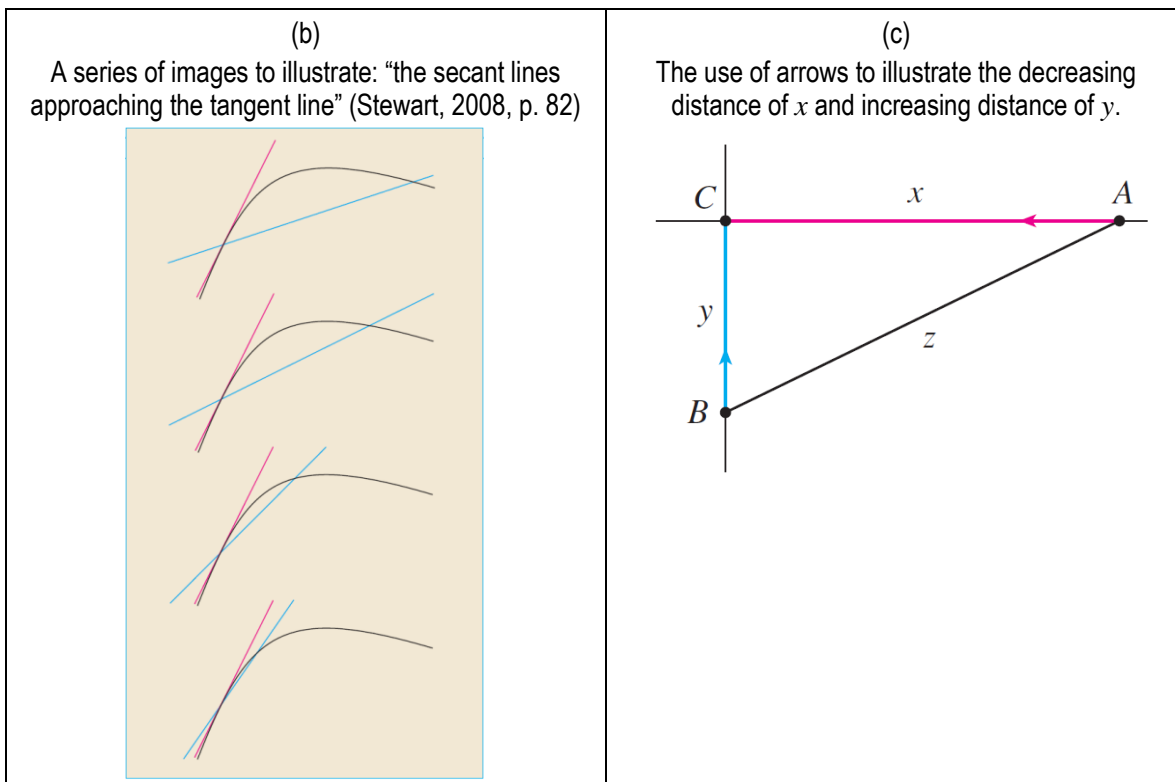


Figure 6. Typical diagrams found in calculus textbooks, where a sense of change is conveyed through the use of arrows and series of images

Finally, pointwise and global trajectories are communicated statically using two graphs, one directly above the other, along with visual cues to suggest the same value on the x -axis. Figure 7 conveys the idea of plotting the slope of tangent at different points of the graph above, $y=f(x)=\sin(x)$, to obtain the graph of its derivative, $y=f'(x)$. However, the dynamism of constructing the derivative function is lost in the diagram, as seen in the caption, “by *measuring slopes* at points on the sine curve, we get strong visual evidence that the derivative of the sine function is the cosine function” (Stewart, 2008, p. 172). “Measuring” the “slopes” at points (note the plural form in slopes again) on the sine curve implies that the process of obtaining the graph of $y=f'(x)$ is again a discrete one. Had the process been a dynamic one, the tangent slope (singular form) would be observed as ‘ x ’ varies in order to obtain the graph of a derivative function instead.

“By measuring slopes at points on the sine curve, we get strong visual evidence that the derivative of the sine function is the cosine function.” (Stewart, 2008, p. 172)

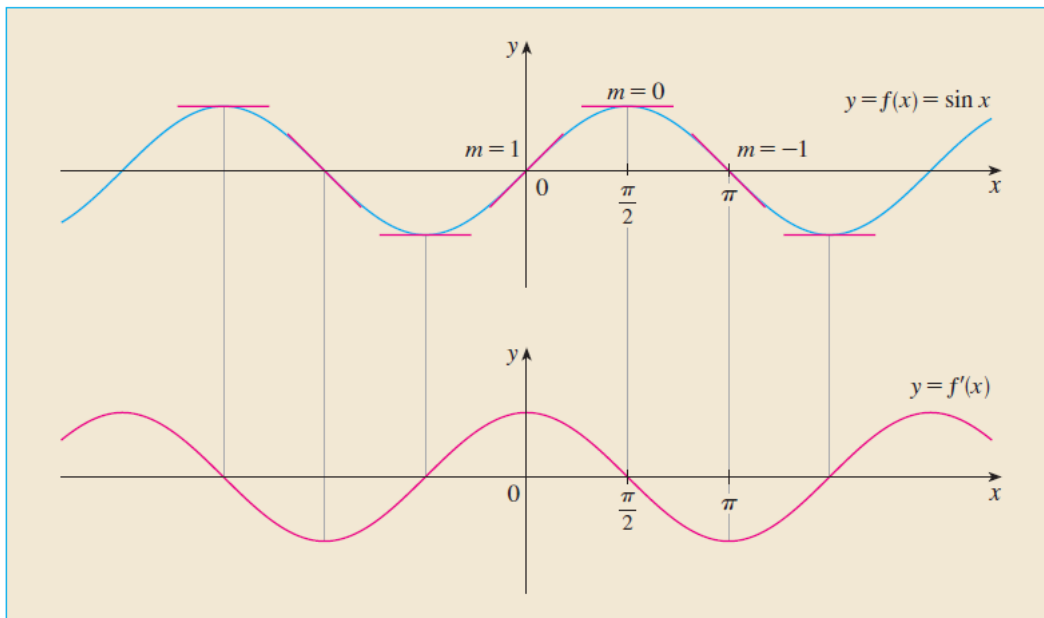


Figure 7. A diagram conveying that the derivative of the sine function is the cosine function

Weber et.al (2012) conjectured that students’ difficulties with function notation, their struggles to connect algebraic with graphical representations of functions, and understanding of rate of change may explain their struggles to think about derivative as a function. They also contended that traditional calculus textbooks do not support the thinking and learning of certain calculus concepts. In this section, I reviewed some textbook conventions for conveying a sense of change in the study of calculus. The analysis supports Weber et al.’s claim that traditional textbooks do not sufficiently provide mental imagery that would support development of dynamic aspects of calculus, such as constructing the derivative function. While students still struggle with calculus, recent studies have shown positive effect on calculus learning made possible by technology. In particular, the introduction of DGEs has given rise to new ways of doing and representing calculus. In the next section, I review the design of two dynamic sketches used in the study in order to contrast the different types of visual mediation, static and dynamic, in the learning of calculus.

5.2. Design of sketches

My study aims to address bilingual learners' communication about calculus concepts in a touchscreen and dynamic environment. With this aim, the dynamic sketches used in the study were designed to highlight dynamic aspects of calculus, exploit touchscreen dragging and connect algebraic to geometric representations of calculus. In this section, I detail the designs of two dynamic sketches used in the study to illustrate the general affordances of the technology. The first sketch described was one of the five sketches used in Part I of the study; it was designed for the learning of the definition of a derivative. The second sketch described was used in Part II of the study for the learning of area-accumulating functions. To examine the role of touchscreen-dragging in mathematical thinking, the iPad application, *SketchExplorer* (Jackiw, 2011), was used to present the sketches that I originally designed with the computer program *Geometer's Sketchpad* (Jackiw, 2001).

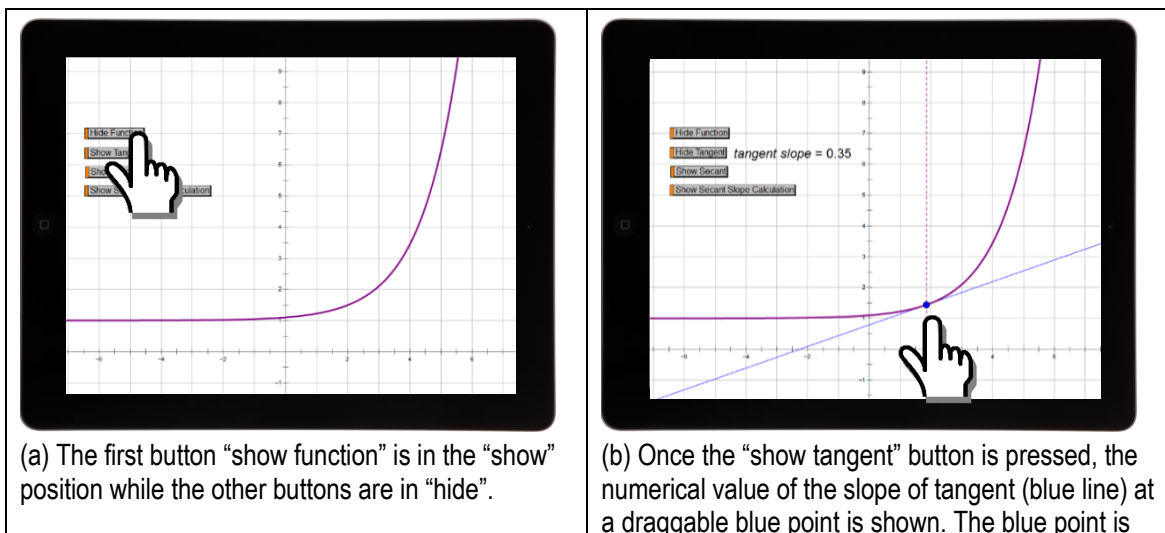
5.2.1. “The definition of a derivative” sketch

The design of this sketch mainly features two functionalities offered by Geometer's Sketchpad: the *Hide/Show* button and the *Dragging* Tool. These functionalities have the potential to evoke mathematical relationships that would have been difficult to capture in static diagrams. The *Hide/Show* button allows different mathematical objects, texts, and numerical calculations to be shown or hidden when pressed. For example, Figure 8(a) shows the screen of a sketch when the first button “show function” is activated. The capability to show or hide this function with the press of a button enhances the effect of seeing the function as a reified mathematical object.

The *Dragging* Tool can be combined with the *Hide/Show* button to effectively communicate the relationship between objects effectively. Figure 8(b) shows the screen of a sketch when the “show tangent” and the “show function” buttons are both activated. Now, as the user drags the point of tangency along the graph dynamically, the slope of the tangent at different points on the function changes. Performing this kind of “guided dragging” (Arzarello et al., 2002) enables the user to attend to the variance of the tangent slope. Furthermore, by assigning the “show tangent” as a second button in the

sketch, the tangent line can be seen as another reified mathematical object. However, one realises that it is a “child” object dependent upon the function when the user figures out that dragging is restricted to only points along the graph.

Figure 8(c) and Figure 8(d) illustrate how a dynamic sketch may connect symbolic with geometrical representations of calculus concepts. Upon activating the “show secant line” button in Figure 8(c), the green point can then be dragged along the graph and the corresponding numerical values of the *secant slope* is displayed. At the same time, the values of the *secant slope* are represented geometrically by two triangles conveying rise and run. Therefore, dragging actions produce a simultaneous change to the numerical value, the rise, and run of the slope triangles. Finally, the last button, when activated, shows the numerical value of the *secant slope* calculation (see Figure 8(d)). The simultaneous change of all the variables, $f(x+h)$, $f(x)$ and h , as well as the continuous change of the rise/run triangle, provide a strong visual mediation connecting geometrical, algebraic, and numerical representations of the definition of derivative. As one drags the green point dynamically along the graph, the numerical calculations of the secant slope and the rise/run triangles change corresponding. Meanwhile, the use of colour enhances the visual effect, since the same colour is assigned to the mathematical object and its symbolic equivalent. For example, both ‘ h ’ and the distance conveying change of ‘ x ’ are coloured red in Figure 8(d). In summary, the design of the sketch is intended to convey the definition of derivative geometrically, numerically and algebraically by exploiting the dragging tool of the DGE.



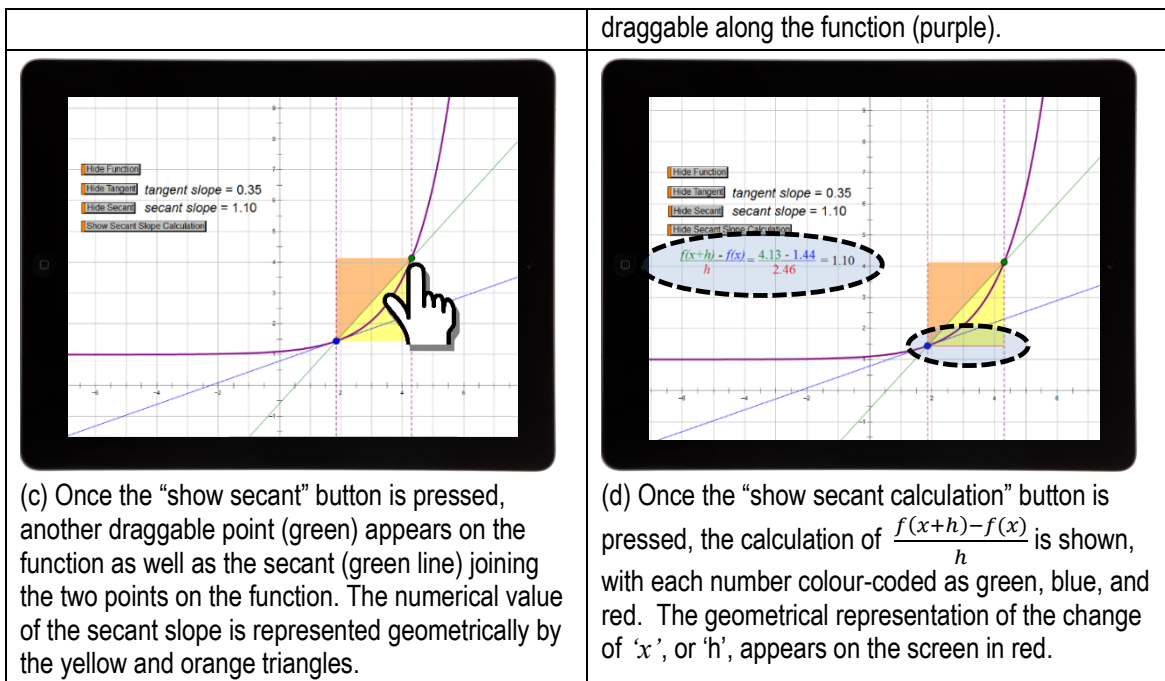


Figure 8. The “definition of a derivative” sketch

5.2.2. The “Area-accumulating function” sketch

The design of the sketch mainly features three functionalities offered by *The Geometer’s Sketchpad* (Jackiw, 2001): the *Hide/Show* button, the *Dragging* tool, and the *Trace* tool. With the exception of the last page, the “Try” page, the first four pages of the sketch all contain the same *Hide/Show* buttons to allow the objects, “Function f”, “Bounds”, “Area under f” and the “Trace of A” to be shown or hidden conveniently. Each page displays a different function when the “Show function f” button is activated: a constant function on Page 1 (Figure 9a), a linear function on Page 2, a quadratic function on Page 3, and the sine function on the Page 4 (Figure 9b). After showing the functions, the student-pairs may explore the “area under the functions” “area under the functions” ($A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$) both numerically (A) and geometrically by dragging the points ‘a’ or ‘x’. For example, Figure 9a shows that the area under the function “ $f(x)=1$ ” is “ $A=4.70$ ” when the bounds are set to “ $a=0$ ” and “ $x=4.70$ ”.

Like the previous sketch, the *Dragging* tool can be combined with the *Hide/Show* button to effectively communicate the relationship between objects. As the user drags the points ‘a’ or ‘x’ along the x -axis continuously, the value of ‘A’ and the shaded area

change correspondingly. Performing this kind of dragging actions mediates a functional dependency between variables by enabling the user to distinguish between what is independent (x) and dependent (A) (see Falcade, Laborde & Mariotti, 2007). The present sketch also illustrates how the dynamicity of dragging may connect numerical with geometrical representations of calculus concepts, since dragging ‘ a ’ or ‘ x ’ simultaneously change the *numerical* value as well as the *geometrical* representation of ‘ A ’. As one drags ‘ x ’ dynamically along the x -axis, the simultaneous change of all the variables, ‘ x ’, $f(x)$ and ‘ A ’ are strongly mediated by visual means.

The *Trace* tool can be used to generate a set of “green traces”, which represent the graph of the area-accumulating function $A(x) = \int_a^x f(t)dt$ for the chosen ‘ a ’ and $f(x)$. When the button “*Show Trace of A*” is pressed, a green point appears on the page at (x, A) . This point is not draggable which implies that it is not an independent object. More importantly, the green point leaves behind traces of its previous positions as one drags ‘ x ’ along the x -axis, creating green traces in the shape of the corresponding area-accumulating function. Since the only way to vary the green point is to drag ‘ x ’ (or ‘ a ’ which would result in a vertical translation the green point), this conveys the idea that the graph of the area under ‘ f ’ is dependent on ‘ x ’, and hence *area is a function of ‘ x ’*.

As the students had not encountered the function $A(x) = \int_a^x f(t)dt$ in their regular classroom, the goal of this sketch was to introduce the idea of *area as a function*, and this can be achieved when the students are able to relate the “green traces” as the graph of “area under f ” from ‘ a ’ to ‘ x ’. It is anticipated that when $f(x)$ is below the x -axis, the students would find it difficult to interpret the “area” bounded by $f(x)$ and the x -axis as “negative” since they had yet to learn that “area accumulation” is meant by $A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$. To facilitate students’ exploration of “area accumulation” when $f(x) > 0$ and when $f(x) < 0$, the areas were coloured-coded differently (Figure 9b).

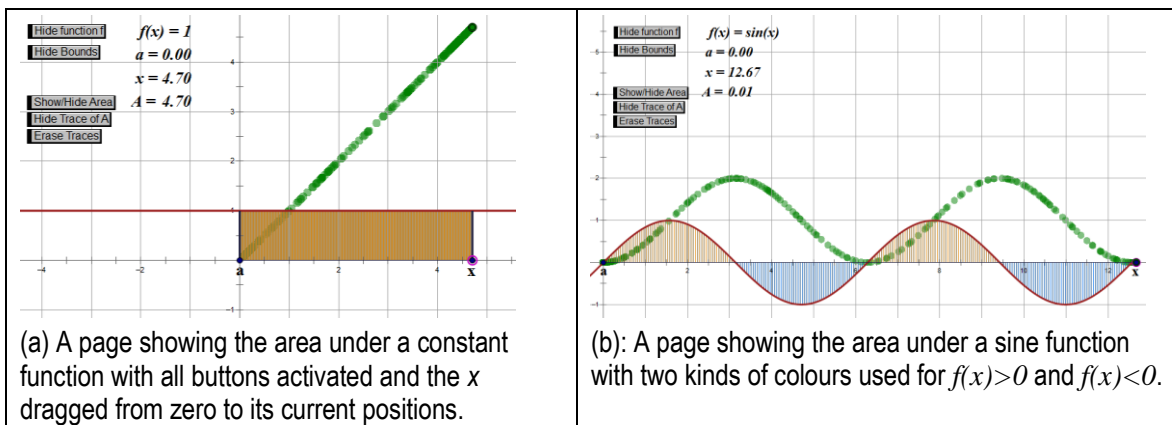


Figure 9. The area-accumulating function sketch

In this chapter, I first reviewed some textbook conventions for conveying calculus relationships. This review supports the claim that due to the static nature of the paper medium, textbooks may insufficiently convey the idea of functions as processes, as well as covariation and continuous change in a dynamic sense. While the discourse of textbooks may have contributed to students' difficulties in the learning of calculus, I am more interested the competence that students demonstrate in the mathematical activities, and how the use of technology may support their communication during mathematical activities. My literature review has suggested positive effect on calculus learning made possible by dynamic and multi-representational digital technology. Hence, in the latter part of this chapter, I discussed the design of two dynamic sketches to illustrate the general affordances of sketches used in the study. As described in the sketch designs, DGE capabilities fill the gap among numerical, algebraic and graphical representations of functions and support dynamic thinking by producing a seemingly limitless table of values for an algebraic expression in the act of continuous dragging. In summary, this chapter informs me the affordances of paper- and digital-based visual representations in anticipation for my analysis about bilingual learners' calculus communication with or without DGEs.

Chapter 6. Analysis of calculus communication across static and dynamic environments (Part I of study)

Oh I got it... This is how to find the approximate value of a point by knowing one point and its derivative right? And its slope... So from P, we can know that the slope at P, point P and we can find the function, and now we input another value which is x plus delta x , and we get the values r ... (Participant Yee discussing a static diagram in Part I of the study)

As the latter falls, we can see that x ... is increasing and y is decreasing, but z remains constant. (Participant Yee discussing a dynamic sketch in Part I of the study)

This chapter includes analyses of the participants' discussions about the static diagrams and dynamic sketches presented to them in Part I of the study, drawing on the communicational theoretical framework as overviewed in Chapter 3. Part I of the study aims at comparing calculus students' communication as it is facilitated by two environments: the first is a static environment as found in traditional textbooks, and the second is a dynamic environment as exploited by the use of *SketchExplorer*, a touchscreen-based DGE application. Following the methodology as informed in Chapter 4, I analysed the participant pairs' communication by attending to their patterns of discourse consisting of particular words, gestures and touchscreen-dragging, as well as the interplay between the three modes of communication, such as simultaneous use of speech, or gesturing and dragging by one or two participants. This analysis requires a transcript annotating simultaneously and sequentially the exact use of speech, gesturing and dragging. Hence, I included transcripts along with the analyses to capture the inter-relationship within the three modes of communication at each turn and the relationship between turns. At the end of this chapter, I include a summary of the participants' discourse during Part I of the study.

As I mentioned in Chapter 4, two pairs of students participated in Part I of the study: Ana and Tammy, and Melissa and Yee. Each pair discussed what were shown to them on the iPad, which were five static diagrams, followed by five dynamic sketches of the same target calculus concepts as conveyed in the static diagrams. The order of the analysis reflects the chronological order of data collection.

6.1. Ana and Tammy

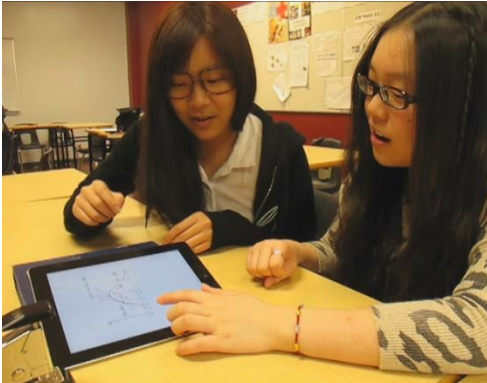
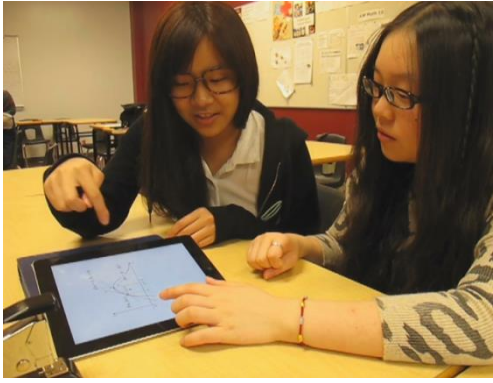
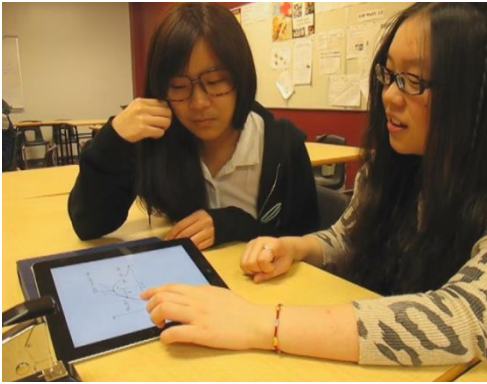
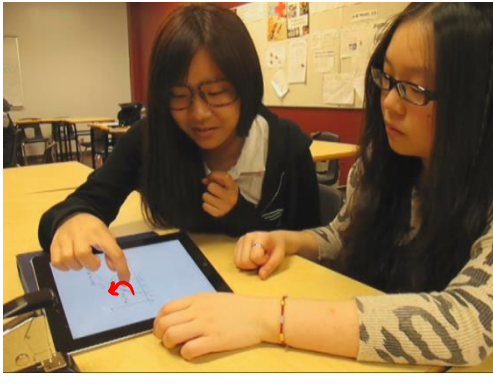
This section includes detailed analyses of three episodes of Ana and Tammy's engagement with the task when they were given a static diagram and then a dynamic sketch related to the definition of a derivative. Each episode begins with a transcript⁴ followed by an analysis. The episodes were chosen to characterise and contrast patterns of communication as demonstrated by the student pair. The use of gestures was prevalent in Ana and Tammy's discourses with static diagrams and with DGE respectively, but different types of gestures were observed in each environment, including the emergence of dragging as a form of gestural communication. This analysis grounds the work of further analysis in the sections to follow.

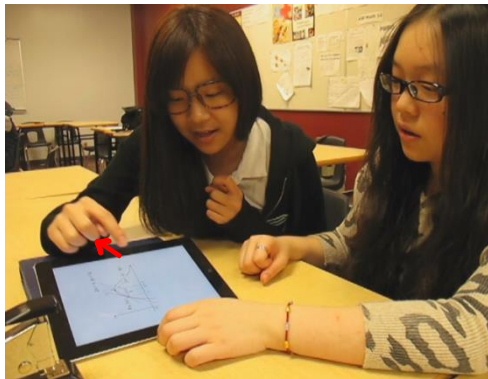
6.1.1. Defining and stating mathematical objects as prompted by static diagrams

Turn	Timespan	What was said <what was done>	S -er	G -er	D -er
1	0:17.0 - 0:20.9	<u>Ok, so the red line is...</u>	T	<u>I</u>	
2	0:20.9 - 0:25.7	The tangent line of the <u>function</u> .	A	<u>A</u>	
3	0:25.7 - 0:38.0	And the <u>blue line is secant line</u> .	T	<u>I</u>	
4	0:38.0 - 0:45.7	Um.. From <u>here to here</u> , the h is the um <u>change of x</u> .	A	<u>A</u> , <u>A</u>	
5	0:45.7 - 0:47.4	And this <u>distance</u> is the y , change of y .	T	<u>I</u>	
6	0:47.4 - 0:56.6	Ya change of y . And how to calculate the slope of the <u>secant line</u> , is... Change of x divided by... Change of y	A	<u>A</u>	

⁴ Transcript convention: **S-er** = Speaker, **D-er** = Dragger, **G-er** = Gesturer; Underlined or double-underlined transcript = utterance spoken simultaneously with dragging or gesturing; Question mark (?) = utterances spoken with a high intonation at the end (see Chapter 4 for details of the transcription process)

		divided by change of x .			
7	0:56.6 - 1:00.8	To get $f'(x)$.	T		
8	1:00.9 - 1:20.5	For the red line, the sec... the secant line is... 预测 [to approximate], 预测 [to approximate] 那个 [that] tangent line.	A	<u>I</u>	
9	1:20.5 - 1:28.0	We also can use the tangent line to get the slope.	T		
10	1:28.0 - 1:44.5	Oh I think there is a function, like f , a plus h minus f , a divided by... f , x ...	A	<u>A</u>	
11	1:43.0 - 1:43.5	<u>f, a</u>	T	<u>I</u>	
12	1:44.5 - 1:48.4	f , a , oh, divided by h .	A		
13	1:47.0 - 1:47.4	Ya.	T		
14	1:48.4 - 1:49.8	And that's the function of...	A		
15	1:49.8 - 1:52.5	<u>Limit, limit.</u>	T	<u>I</u>	
16	1:51.3 - 1:57.9	Ah <u>limit, limit</u> , that's the function of tangent line.	A	<u>A</u>	

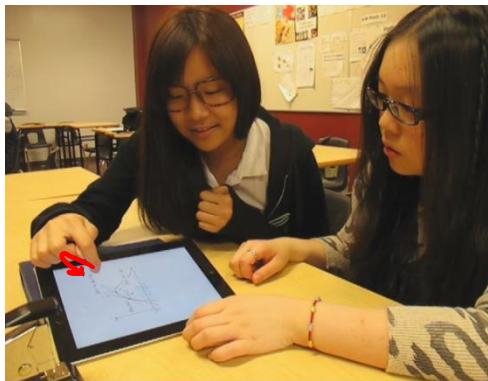
<p>(a) T: Ok, so the red line is...</p> 	<p>(b) A: The tangent line of the function.</p> 
<p>(b) T: And the blue line is secant line.</p> 	<p>(d) A: Um.. From here to here,</p> 
<p>(e) A: h is the <u>change of x</u>.</p>	<p>(f) T: And this distance is the y, change of y.</p>



(g) A: And how to calculate the slope of the secant line, is...



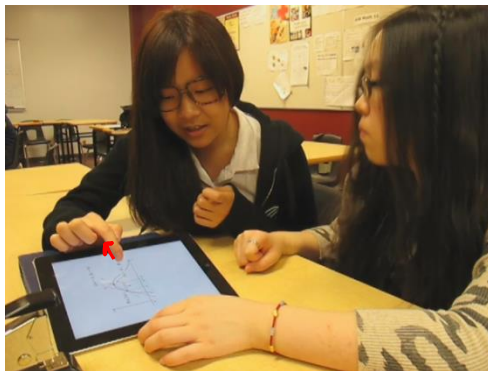
(h) A: For the red line, the secant line is... 预测 [to approximate], 预测 [to approximate] 那个 [that] tangent line.



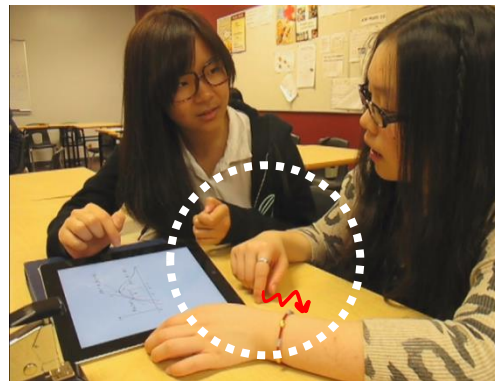
(i) A: Oh I think there is a function, like f, a plus h minus f, a divided by... f, x ...



(j) T: f, a



(k) T: Limit, limit.



(l) A: Ah limit, limit, that's the function of tangent line.

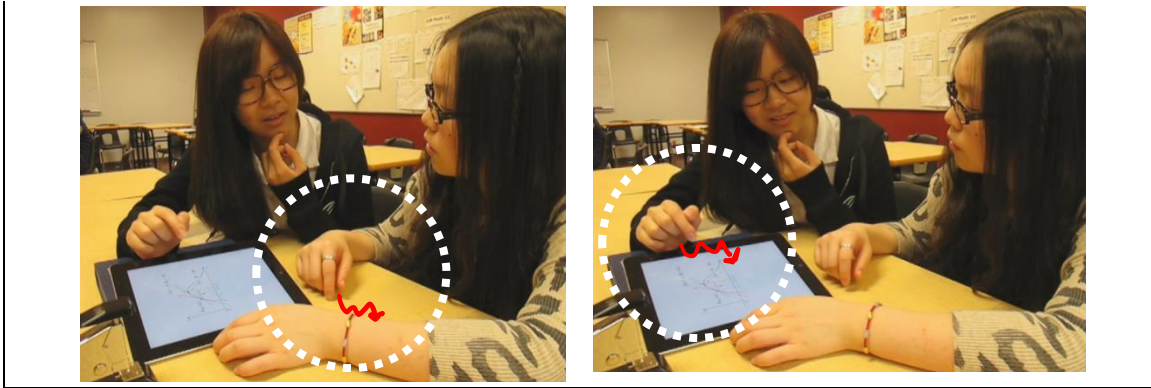


Figure 10. Selected snapshots of Ana and Tammy’s gestures used while discussing the definition of a derivative with a static diagram

The transcript above shows that Ana and Tammy were mainly engaging in two kinds of mathematical Discourse practices, namely that of defining and that of stating the mathematical objects shown in the static diagram. From Turn 1 to Turn 6, the students took turns to define the red line, the blue line, the change of ‘ x ’ and the change of ‘ y ’ respectively. The use of gestures was present as the students identified each of the four mathematical objects during this part of their discussion. Of the first six gestures performed, five of them were deictic pointing gestures while the speaker also referred to the particular objects in their utterances. For example, Tammy used her left index finger to point to the tangent line while she uttered “the red line is...” (Turn 1). Then, Ana completed Tammy’s statement with “the tangent line of the function” (Turn 2) accompanied by a similar pointing gesture with her right index finger pointing towards the tangent line.

These gestures seemed significant in the early part of this episode not only because they were frequently used by the two students, but also in terms of the interplay between gestures and word use in communication. With the use of deictic gestures like the ones found within the first six turns, word use was transformed: deictic words like “*this distance*” (Turn 5) and “from *here* to *here*” (Turn 4) appear in the students’ utterances. Using deictic words, the speakers no longer needed to refer to the mathematical objects by describing them verbally, but they could use deictic gestures along with pronouns and locative nouns to replace the descriptions completely. Therefore, these gestures could significantly reduce the number of words needed to refer to the mathematical objects, as found in Tammy’s “*this distance is the change of y* ”

(Turn 5) and Ana's "from *here* to *here* is the change of x " (Turn 4). As Sfard explains, gestures help ensure that the interlocutors speak about the same mathematical objects. Significantly for the bilingual learners, gestures serve a complementary function to language in communication. As seen in this episode, Ana and Tammy were able to use a combination of utterances and gestures to communicate effectively about the mathematical objects. In particular, word use was transformed with the presence of deictic gestures.

By contrast, the gesture performed by Ana in Turn 4 characterises a different type from the other ones noted above. When speaking about the "change of x ", Ana moved her index finger laterally from left to right. It could be said that this gesture is deictic for pointing to the mathematical object, but it also served to enact the mathematical idea of "change of x ". The communicational function of the gesture seems to be that of communicating the temporal aspects of "change of x ". Performing this gesture by tracing a path of a point communicates a sense of change, in this case, the "change of x ". Outside of this episode, it was observed that both Ana and Tammy occasionally used this type of gesture to enact "change of x ", "tangent line", and "secant line" etc. and to convey a sense of temporality in the objects (see Núñez, 2003).

The diagram, a static visual mediator, may have influenced Ana and Tammy's thinking about calculus. The students' mathematical discourse, as observed in their word use and gestures, reflected a static way of thinking about the definition of a derivative. The students resorted to the verb "is" four times in this episode, each time followed by nouns "tangent line" (Turn 2), "secant line" (Turn 3), "change of x " (Turn 4), and "change of y " (Turn 6) for naming each of the visual mediators shown in the static diagram. These verb-noun combinations generate statements that are static in nature since no actions are taken in the statements. During the first six turns, the only instance where dynamism was conveyed was when Ana moved her index finger while uttering "change of x " in Turn 4. When speaking about the "change of x ", Ana moved her index finger laterally from left to right (Figure 10e). It could be said that this gesture was deictic, pointing to the mathematical object, but it also served to enact the mathematical idea of "change of x ". The communicational function of this gesture seemed to be that of communicating the temporal aspects of "change of x ". Other than that, all of the gestures in this part of their

discussion were for deictic purposes only and static, as opposed to dynamic and performative in the sense of conveying temporality in mathematics (Núñez, 2003). These word use and gestures were highly relevant to the students' particular mathematical Discourse practices in the moment, that of defining and stating mathematical objects, which are static in nature as well. The unchangeable mathematical objects evoked by the static visual mediator may have facilitated this form of communication.

Beginning in Turn 6, the students moved to a discussion about what mathematical relationships were suggested in the diagram. Ana began by talking about "how to calculate the slope of the secant line," followed by describing the calculation of, "change of y divide by change of x ". Tammy agreed and added that the quotient Ana was referring to was called $f'(x)$ (Turn 7). It is evident in Turns 6 and 7 that the students were communicating about a procedural understanding of derivative as tangent slope. Their discourse suggests that the tangent slope was a quantity that could be "calculated" by means of performing a mathematical operation.

From Turn 9 to Turn 16, the students continued to develop a formula for the definition of a derivative. Prompted by Tammy's suggestion that "we can use the tangent line to get the slope" (Turn 9), Ana attempted to provide a formula for finding the tangent slope (Turn 10). This formula included symbols that were labelled in the diagram, such as $f(a+h)$, $f(a)$, ' x ' and ' h '. At the end of the conversation, Tammy introduced the word "limit" in the formula (Turn 15), and Ana concurred: "ah limit, limit, that's the function of the tangent line" (Turn 16). They seemed satisfied with the formula and ended the conversation on that note.

Analysis of verb use during Turns 6 to 16 shows that the students continued to think about calculus in a static sense. Although they did not resort to the "is-noun" combination noted earlier, Tammy used the words "to get" twice to suggest that both $f'(x)$ and the tangent slope can somehow be obtained. This is a procedural way of thinking about derivative, in contrast to a dynamic way of thinking about it by letting $h \rightarrow 0$. Although Tammy mentioned that the secant slope can be "used" "to get" the tangent slope, she did not explain how. In other words, she did not communicate the limiting process for obtaining the tangent slope. Similarly, when the idea of a limit was

introduced by Tammy, Ana acknowledged it, but they did not fully explain what a limit was, and why it appeared in the formula. Therefore, the geometrical representations of the secant and tangent slope were never explicitly communicated by either student. Hence, the students' discourse reflected a static and symbolic representation of slope of a tangent, as opposed to a dynamic and geometrical one.

Related to this, the mathematical Discourse practices demonstrated in this part of the episode were computing, calculating and formulating. A sociocultural view would see both students as utilising multiple resources in mathematical communication, including their home language, gestures and the static diagram available to them. For example, although Ana did not know the English word for “approximate”, she was able to use her home language “预测” (Turn 10) to communicate with Tammy what she meant. She also misused the word “function” three times in the episode, by saying “function” for tangent slope (Turn 12, 16, 18) when she really meant “formula”. Although Ana did not use the correct English word “formula”, she demonstrated her competence of formulating the method of solving for the secant slope as “change of y over change of x ”. In contrast to this sociocultural view, the vocabulary perspective would have focused on Ana's failure to use the correct English word in communication.

The sociocultural view provides a lens to see that the students were successfully formulating, computing and calculating using a variety of resources in the given static environment. For example, gestures played an important role in their communication. Ana and Tammy continued to use deictic gestures (Figure 10f and Figure 10h) and gestures to enact mathematical objects (Figure 10g). In addition, both students performed a type of gesture involving the movement of the hand, imitating the scribing of the words in their utterances—I hence call these “scribing gestures” (Figure 10i to l). The first “scribing gesture” was performed by Ana while she attempted to give a formula for tangent slope. As she uttered, “divided by” in her attempted formula, she gestured a straight line as if she were writing down the line in the quotient of a/b (Turn 12; Figure 10i). Tammy responded to this formula by adding “ f, a ” as she performed a similar “scribing gesture” (Turn 13; Figure 10j), and then again when she said the word “limit” (Turn 17; Figure 10k). Finally, Ana responded with a “scribing gesture” as she acknowledged Tammy and uttered “ah, limit, limit” (Turn 18; Figure 10l). It is

hypothesised that these gestures were not found in the first half of the episode because the content of the discussion did not involve symbols and notations, whereas the discussion surrounded an algebraic formula for tangent slope in the latter half. The presence of “scribing gestures” suggests that the students thought of tangent slope in a procedural way that involved symbols and formulae.

6.1.2. Dragsturing as prompted by dynamic sketches

Below, I provide a detailed analysis of Ana and Tammy’s discussion about a dynamic sketch relating to the definition of a derivative. For the purpose of identifying themes, the episode is further divided into two parts: Turns 1 to 5 are analysed in this section, and Turns 6 to 14 are analysed in the next section.

Turn	Timespan	What was said <what was done>	S -er	G -er	D -er
1	10:31.9 - 10:41.0	From zero to positive, <u>the slope is...</u> <T drags ‘x’>	T		<u>I</u>
2	10:41.0 - 10:42.5	<u>The tangent line is increasing.</u>	A		
3	10:42.5 - 10:48.2	<u>Tangent line is increasing.</u> And from here to zero, it’s decreasing.	T		
4	10:48.2 - 10:51.7	<u>And at zero, the tangent line is zero.</u> <A drags ‘x’>	A		<u>A</u>
5	10:50.3 – 10:50.6	<u>Zero.</u>	T		

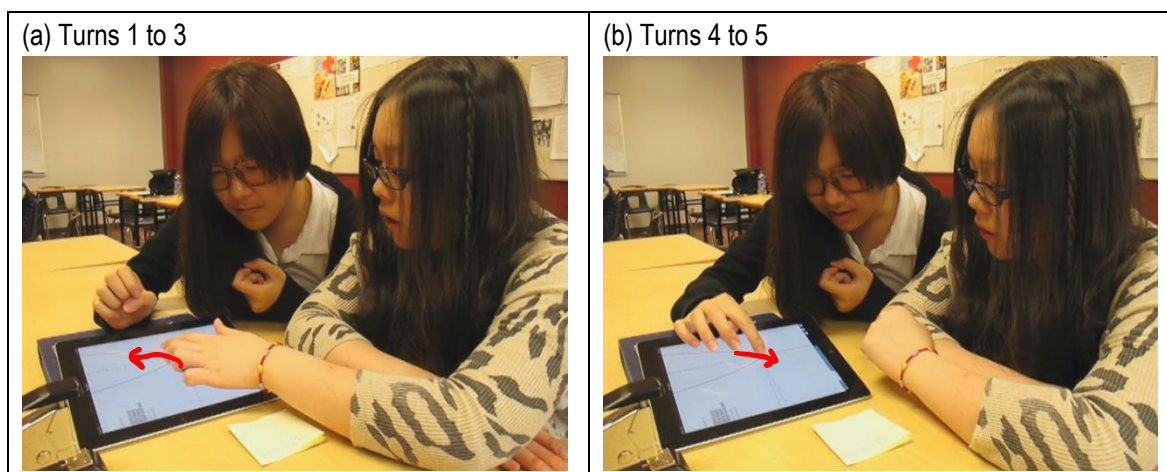


Figure 11. Selected snapshots of Ana and Tammy’s dragging while discussing the definition of a derivative with a dynamic sketch (Turns 1 to 5)

When the students opened the sketch, two buttons were already in the “show” position; therefore, the graph of a parabola, $f(x)=x^2$ and its tangent line at a given point appeared on the sketch. As seen in the transcript, Ana and Tammy explored the dynamic sketch using the dragging modality. In the first exchange, Tammy’s utterance, “tangent line is increasing” (Turn 1) was accompanied by dragging the point of tangency from left to right (Figure 11a), although technically it was the tangent *slope* that was increasing and not the tangent line. Following that, Ana seemed to be mimicking Tammy’s utterance-dragging combination with her utterance, “the [slope of the] tangent line is zero” (Turn 4) while dragging the point of tangency back towards the vertex (Figure 11b). These were two of five series of dragging actions observed that spanned up to five seconds within the first fourteen turns of their discussion with a dynamic sketch.

My analysis suggests that these two dragging actions were not merely dragging but also gestural communication—to communicate the dynamic features and properties in the sketch at the very moment of dragging. Recall that when faced with a static environment, Ana and Tammy frequently used static, deictic gestures to refer to different mathematical objects. In contrast, such static, deictic gestures were not observed in the dynamic environment. Instead, the students’ gestures were blended within their dragging actions as they spoke about the change in the slope of the tangent. To illustrate why the dragging actions were also considered gestures, it would be possible to imagine a static environment where the dragging modality is not available. If a speaker moves his/her finger along a graph while referring to the tangent slope as “increasing” or “decreasing”, this action can be considered a kind of dynamic gesture for communicating the idea, “as x varies along this graph”. In the present episode, the dynamic environment allowed the dragging with one finger on the touchscreen and the gesturing with the index finger to blend together as one action. The importance here is that the dragging/gesturing action is one action subsuming both dragging and gesturing characteristics, in that it causes the point to be moved on the screen (dragging), and it fulfills a communicational function also (Sfard’s definition of gestures). Hence, I refer to this action as “dragsturing”.

Although I have named this action dragsturing, my purpose for naming is not solely to objectify an action into a noun, but to present the dual functions of dragging and

gesturing in the dragsturing action for analysing the students' thinking-communicating process. For example, the analysis of the students' dragsturing in the episode suggests that they were thinking dynamically and mathematically about the tangents to a curve. Furthermore, this analysis addresses the role of a touchscreen, dynamic environment for facilitating this form of communication.

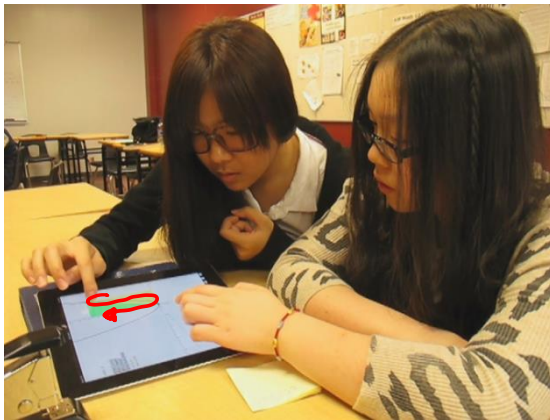
I now turn to a synchronic analysis of the students' word use and dragsturing actions in the episode. During the first exchange, Tammy used the phrases "is increasing" and "is decreasing" to describe the tangent slope. Her utterances were accompanied by her dragsturing, which seemed to be mimicked by Ana in the next turn. The use of the present continuous tense "is [verb]-ing" was a change from their previous discussion over a static diagram, where the students used the verb form "is [noun]" four times when discussing the same topic. The word use "is increasing" and "is decreasing" were accompanied by *dragsturing* to *communicate* the change of tangent slope as the point was being *dragged*. Thus, in the present episode, dragging and gesturing transformed the way Ana and Tammy communicated about the tangent slope. The verb forms suggest that "something is happening" at the very moment. This analysis is made possible by studying the interplay among dragsturing, word use and touchscreen-based diagrams in the students' mathematical discourse.

The frequent use of dragsturing in this episode suggests that dragging is a significant mode of communication for the students. They used dragsturing, accompanied by utterances, to talk about the variance of tangent slopes, and this was facilitated by the dynamic visual mediator. The design of the sketch played a role, since the draggable point was also the point of tangency of the function, which was a geometrical object. Hence, dragging the point has a dual meaning of changing the x -coordinate numerically as well as physically moving the point of tangency visually. This may have initiated the blending of dragging and gesturing about the movement of the point of tangency. In summary, the dynamic environment, touchscreen technology, and the design of the sketch which include the exploitation of the dragging tool all played a role in the students' discourse about dynamic features of calculus.

6.1.3. Comparing, predicting and generalising as prompted by dynamic sketches

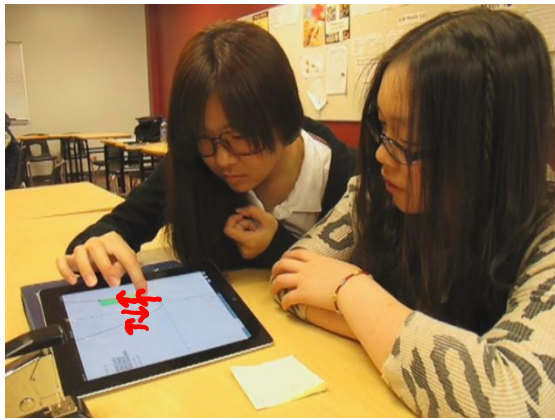
Turn	Timespan	What was said <what was done>	S -er	G -er	D -er
6	10:51.7 - 10:53.7	Do you want to use the buttons? Try the buttons.	O		
7	10:53.7 - 10:55.2	What do you mean the buttons?	A		
8	10:56.2 - 11:01.0	These buttons. So there is, the show, the hide means you already showed them. So try the last two buttons.	O		
9	11:01.0 - 11:04.6	<T presses "show secant" button> Secant.	A		
10	11:04.5 - 11:11.1	<A presses "show secant calculation" button> Hm.	T		
11	11:11.1 - 11:26.9	For... if you want to get the secant line, you have to find two points to, ah, to calculate the change of y and change of x.	T		<u>A</u> <u>A</u>
12	11:26.9 - 11:46.1	I think when the two points get closer, the tangent line is... there is less different between the tangent line and secant line.	A	<u>A</u>	<u>A</u>
13	11:46.1 - 11:50.4	And... they will be together.	T		<u>I</u>
14	11:47.2 - 11:52.9	And if... there are the same point, they will be the same, the two lines.	A		

(a) T: if you want to get the secant line, you have to find two points to, ah, to



(c) A: the tangent line

(b) T: calculate the change of y and change of x.
A: I think when the two points get closer



(d) T: And they will be together.

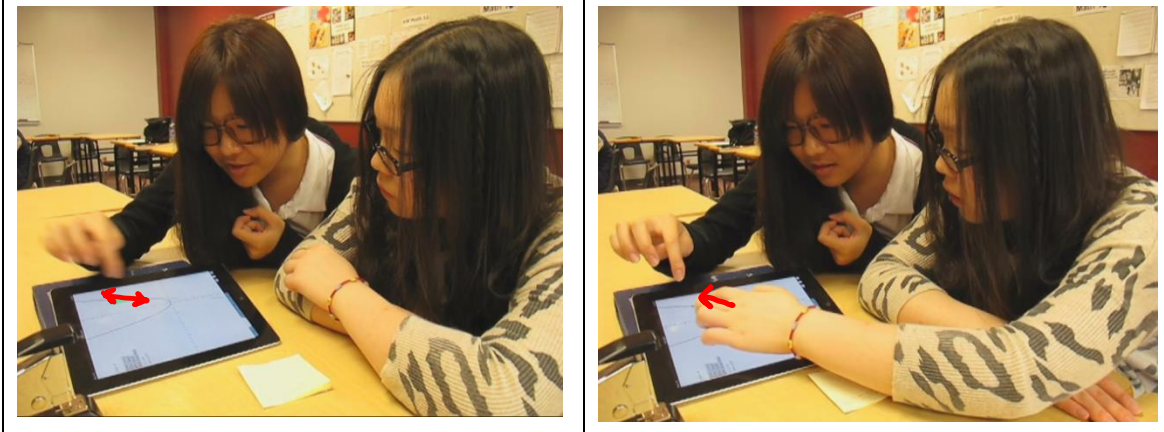


Figure 12. Selected snapshots of Ana and Tammy’s dragging while discussing the definition of a derivative with a dynamic sketch (Turns 6 to 14)

Prompted by my suggestion to “try the buttons” (Turn 6), Ana and Tammy began to explore the other two *Hide/Show* buttons and continued to utilise the dragging modality. As the episode unfolded, they moved from discussing procedures to talking conceptually about the definition of a derivative. This could be observed through the evolution of the different mathematical Discourse practices they engaged in. Upon *exploring* the change of tangent slope from Turn 1 to 5 of the episode, Tammy suggested that “if you want to get the secant line... You have to find two points to, to calculate the change of y and change of x ” (Turn 11). At this point, Tammy’s mathematical Discourse practice was focused on *calculating*.

However, the students’ talk did not end with a formula as had been observed in the static environment; Tammy’s *calculating* was followed by Ana’s *comparing*, evident in her word use “closer” and “less different” (Turn 12) for describing the state of the two lines when one approaches the other. Her comparing led to *predicting* and *generalising* about the tangent line in Tammy’s “the two points *will be* together” (Turn 13) and Ana’s “they *will be* the same, the two lines” (Turn 14). The use of future tense in “will be” in both statements indicates that both students had moved from a procedural and algebraic way of thinking about derivative to generalising about the derivative geometrically and conceptually. Tammy’s *dragsturing* (Figure 12d) at the end to bring the secant line towards the tangent line can be taken as her *verifying* that the two slopes will eventually be the same.

In contrast to the sociocultural view, the vocabulary perspective would criticise Ana and Tammy for incorrectly stating that the “*tangent line* is increasing... and from here to zero, it’s decreasing,” (Turn 2 and 3) in the earlier part of the episode when it was really the *tangent slope* that was changing. Likewise, the multiple meaning perspective would point to Ana’s inability to grasp the meaning of “function” later in the episode. Hence, neither perspective would view Ana and Tammy as engaging in valued mathematical Discourse practices like *comparing*, *predicting* and *generalising*.

Since gestures were taken as communicational acts in Sfard’s terms, it was interesting to observe that the students incorporated gestures for responding to each other. For example, while Tammy talked about the two points on the secant line, Ana was *dragsturing* the points on the secant line around, which seemed to be responding to Tammy’s utterance. Then the two exchanged roles; when Ana suggested that the secant line would get “closer” to the tangent line, Tammy seemed to have responded by *dragsturing* to bring the lines “together”. These gesture—utterance correspondences were noted in the analysis of other pairs of bilingual learners’ conversational patterns involving dynamic sketches as well.

6.2. Melissa and Yee

The previous analysis of Ana and Tammy’s discussion focused on their engagement in a task on one particular topic and compared their communication across two environments. I performed the analysis of my second student pair’s, Melissa and Yee, communication differently. I found their communication interesting in terms of particular features relative to the mathematical ideas that were being discussed in the given environment. Therefore, I summarise some key findings about this student pair’s communication about various mathematical ideas in this section. In addition, I highlight particular words, gestures and dragging actions that were not observed in Ana and Tammy’s analysis.

6.2.1. The interplay among posture, word use, gestures and dragging

Overall, Melissa and Yee's posture evolved over the course of the task. At the beginning of the task, both students had one arm placed on the desk and the other elbow placed against the desk to support their jaws (Figure 13a). They remained in this position for twelve seconds before Melissa moved her elbow and turned it into an arms-crossed position (Figure 13b). She did not move her arms or body for 01:04, and this lack of body movement was consistently observed for both students throughout their discussion about the given static diagrams. In particular, minimal pointing gestures with the index finger were present during the discussion. The infrequent number of gestures observed may be linked to the student pair's word use. Compared with Ana and Tammy, Melissa and Yee's word use was more precise in the sense that fewer deictic words and pronouns were used. For example, in talking about the diagram related to the definition of the derivative, Yee explained that, "as a approaching zero, the slope of the secant line is really closer and closer to the slope of the tangent line" (00:54). In this utterance, both the secant and tangent lines were clearly stated. In contrast, recall that in a similar discussion about the definition of derivative, Ana said, "and if... there are the same point, they will be the same, the two lines." In this utterance, Ana used the pronoun "they" and the phrase "the two lines" to refer to the tangent and secant lines without naming them precisely. Although this type of communication may seem like a lack of reference from an outsider, Ana and Tammy did not seem troubled by it, perhaps because deictic gestures were generally used to accompany speech. In general, Melissa and Yee's utterances were much more descriptive, and they used fewer gestures, particularly deictic gestures, to accompany speech. This meant that, unlike Ana and Tammy, Melissa and Yee were relying more on descriptive word use to compensate for their irregular use of deictic words and gestures.

After about thirteen minutes of discussing the static diagrams, the students moved on to discussing the dynamic sketches. The students' postures had changed during this time, as seen in Figure 13(c). The snapshot in Figure 13(c) shows that both students had released their arms as jaw support and were interacting with the sketch with their fingers. There was a change in the way they moved their hands and bodies in a dynamic environment, especially for Melissa, who had been very inactive with her

hands and body throughout the first twelve minutes. Melissa's dragging on the touchscreen DGE can be taken as a form of non-verbal communication; therefore, it can be said that the dynamic, touchscreen environment facilitated Melissa's participation in the discussion in ways that the static environment did not.

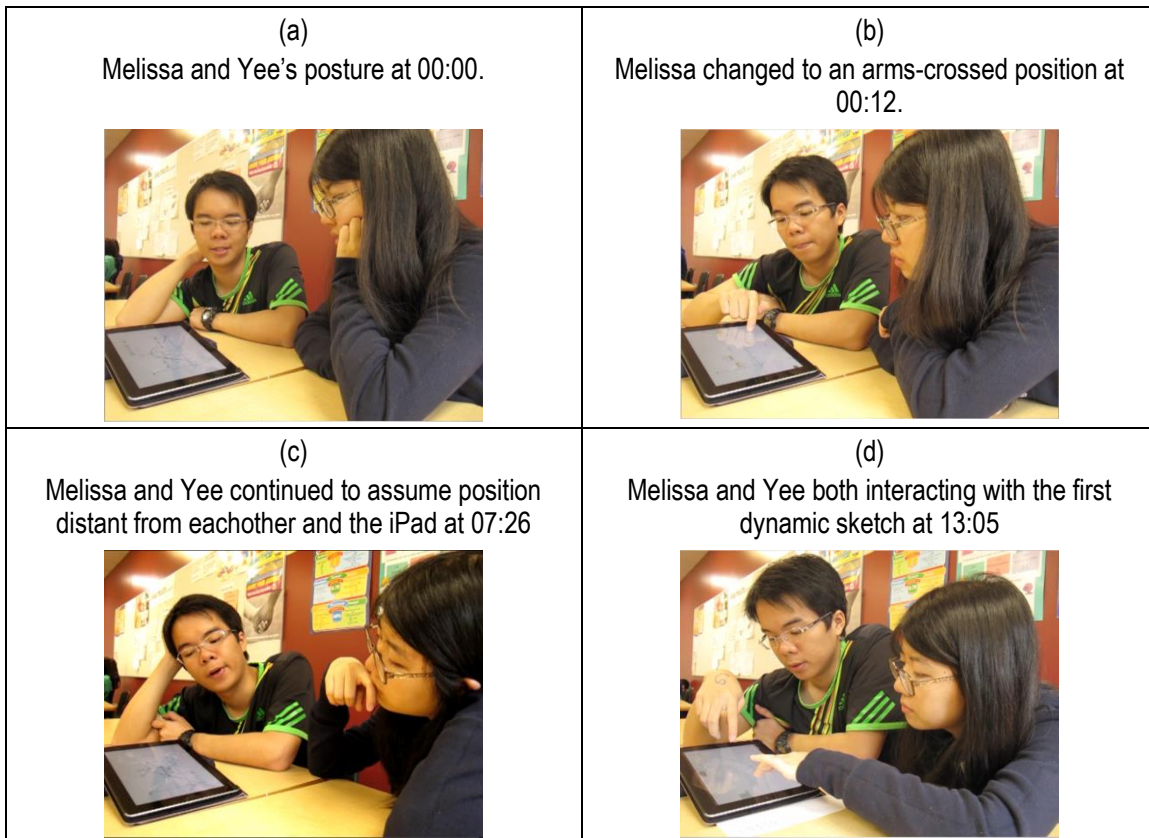


Figure 13. Melissa and Yee's change of posture during the first 13 minutes

6.2.2. Melissa: Self-repairing speech

Melissa was exploring the static diagram related to the definition of a derivative when, at 0:35, she used a series of self-repairs moves to communicate about the tangent and secant lines. While she continued to cross her arms and did not make use of any gestures, she uttered:

And the slope of the secant line is uhm... 'a' plus 'h' over 'f', 'a' plus 'h', the, we can, uhm... uhm... if we, if 'h' is approaching zero, then we can get the tangent line. (0:35)

The utterance contained series of self-repairing speech, and the hesitation sound of “uhm” appeared three times. Recall that the students only used gestures occasionally while discussing the static diagrams. Furthermore, the absence of the dragging modality meant that the students needed to use speech as a primary mode of mathematical communication. As evident in her self-repairing speech, using words alone in communication was difficult for Melissa. It is hypothesised that a multimodal communication incorporating speech, gesturing and dragging might have helped her communicate mathematically in a non-native language.

As mentioned, Melissa had been participating quietly in the task. She had not talked very much other than at time 00:35, when she used a series of self-repairing speech in her communication. Then, at the fifteen minute mark of their discussion, Melissa spoke before Yee for the first time while the two were exploring the dynamic sketch related to the idea of derivative functions. She also spoke in complete sentences in the utterance:

If we drag the x , on the function of x , we get the tangent slope, and it's always y equals three, and uhm here, the⁵ tangent line, the tangent line is the same as the x -values⁶, and then the y -value is the secant, ah the tangent slope, so it's always three because it's on the same function, and the function has the same slope. (15:05)

The speech above was very different from that noted earlier, for three reasons. First, the sentences were long and complete, and there was little self repair. Secondly, Melissa continued to drag over her speech, and she gestured as she spoke “same” and “function”. Thirdly, there were connectives (“so” and “and”) and conjunctions (“if”) that suggest Melissa was trying to connect ideas in a coherent way. The dynamism pertaining to Melissa’s dragging actions may have facilitated these patterns of communication, since she was able to see that the tangent slope was “always” equal to three and the linear function “has the same slope” as she dragged ‘ x ’. The present analysis suggests that Melissa was participating more actively within a DGE than when the pair was given static diagrams.

⁵ Double-underlined transcript denote words spoken simultaneously with dragging

⁶ Underlined transcript denote words spoken simultaneously with gestures

6.2.3. Yee: Other gestures

Melissa and Yee used similar kinds of gestures, such as deictic and “scribing” ones as those noted in Ana and Tammy’s communication. Besides these, two other interesting use of gestures were observed in Yee’s communication when he talked about “linear approximation” in a static environment. At 06:47, he used his right index finger and thumb to form the shape of a “C” while he said “and we get the values r , and actually the actual value is pretty close right” with a high intonation (Figure 14a). This is what I call a “measuring gesture” for conveying distance or proximity between objects. Then, as he continued to speak, he brought the tips of his index and thumb together to touch each other, hence using another “measuring gesture”. However, since the distance between the two points were “pretty close”, the distance between his fingers was also reduced in his gesture (Figure 14b).

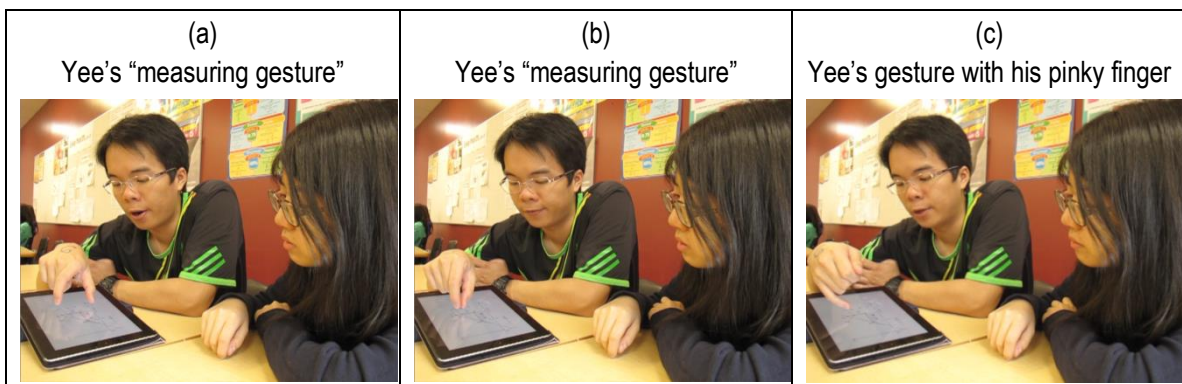


Figure 14. Yee’s gestures

At 06:40, Yee continued to use a combination of words and gesture to communicate the approximation of “ Δy ” using “ dy ”. In saying that “ dy and change of y are a little bit different”, he used his right pinky finger to point to the distance between the labelled “ dy ” and “ Δy ” (Figure 14c). This gesture resembled the kinds of deictic pointing gestures that he had used before, but it was also different in that the pinky was used for pointing instead of the index finger. The change of finger for pointing suggested that Yee was communicating something slightly different here. More than just pointing, he may be thinking about a degree of precision with this gesture to complement his word use “a little bit different”. The use of pinky was observed again when Yee talked about a different diagram, that of Newton’s Method. With the use of his pinky for gesturing, he

commented that, “the point is getting closer and closer” (11:30) when Newton’s Method was performed to approximate the root of an equation recursively.

It could be argued that both the gesture with a pinky finger and the “measuring gesture” were performed because it was not possible to move the objects close together physically in a static diagram; therefore, these gestures were used to convey the idea of bringing something close to another in the process of approximation. For example, in the diagram related to Newton’s Method, the approximate roots x_1, x_2, \dots, x_n were explicitly shown in the static diagram. Hence, it could be difficult to express the action of obtaining x_{n+1} recursively using the diagram. Perhaps, the pinky finger was utilized deictically (instead of the index finger) to convey a level of precision around approximating the root of an equation.

A little later, Yee made a scribing gesture when he spoke, “As x is approaching, x approaching x , n , and then like, the x , n is like closer to the r ” (12:05). This observation was similar to the ones found in Ana and Tammy’s discussion, and it was useful for examining Yee’s thinking at the moment. In particular, his scribing gestures suggested that he was thinking algebraically, perhaps the Newton’s Method formula, and not geometrically.

6.2.4. Yee: Communicating “change” in different environments

It can be observed that the static visual mediator had occasioned certain kinds of discourse on “change” for Melissa and Yee. There is evidence suggesting that the students were thinking about change statically and discretely when prompted by the given static diagrams. As mentioned, the students’ discourse seldom contained deictic words, “this, that, here” etc. Without consistent usage of deictic words, the students’ discourse lacked reference to the diagrams; rather, their discourse seemed to be focused on some mathematical concepts that existed outside of the diagrams. This was evident in the students’ word use and gestures as they discussed the diagrams. On three occasions, Yee discussed mathematics as if it existed outside of the diagrams with his mathematical theorem-like talks. When discussing the static diagrams related to “derivative functions”, “linear approximation”, “Mean Value Theorem”, Yee’s talk

resembled the genre of mathematical theorems in the sense that he stated the givens statically and then deduced the results formally.

One important characteristic of mathematical theorems is the absence of temporality. The majority of Yee’s verb use, in the form of “is-noun” or “is-adjective,” implied a timeless sense of calculus, as opposed to verb forms that describe a *process* or *dynamic* relationships in calculus. Table 10 shows some of his theorem-like talk when discussing the static diagrams. In terms of modal verbs, he said that “we can know” in two of his utterances, which implies that there were some concepts that the diagrams were intended to convey. Occasionally, he also included the conditions of which the statements would hold true, such as, “So like *when* the graph of the function is decreasing, *we can know* that its derivative is less than zero. And *when* the graph of a function is increasing, the derivative is always greater than zero: (03:10). The word “when” conveyed one static moment. Moreover, the phrase “when [...], we can know” suggest that his mathematical Discourse practice was of stating a calculus relationship as a timeless story.

Table 10. Yee’s theorem-like discourse as timeless stories

(a)	“So like when the graph of the function is decreasing, we can know that its derivative is <u>less than zero</u> . And <u>when the graph</u> of a function is increasing, the derivative is always greater than zero” (03:10).
(b)	“Oh I got it... This is how to find the approximate value <u>of a point by knowing one point</u> and its derivative right? and its slope... <u>So from P</u> , we can know that the slope at P, point P <u>and we can find the function</u> , and now we input another value <u>which is x plus delta x</u> , and <u>we get the values r...</u> ” (06:18)
(c)	“I think it’s mean value theorem... ya should be mean value. So we have a function, <u>you know a and b</u> , and you <u>draw a line across it</u> and you get a <u>secant, secant function, secant line function</u> , and there must, if the <u>function is continuous</u> , so there must be a point p, which its <u>slope is equal to the slope of the secant line a,b</u> . so for this here, at least one, it can be two, for <i>figure 4</i> , there is actually two points, P1 and P2, they both have the same slope as A, the <u>secant line a,b</u> . So there must be at least one point” (10:00).

In addition, the static and dynamic visual mediators might have occasioned different ways of communicating “change”. The analysis of the word use “as”, which presents dynamic qualities, supports this claim. Yee’s first use of the word “as” was during the discussion of derivative functions in a static environment. Interestingly, he did not finish his sentence after beginning with the words, “so, as...” (02:38); rather, he self-

repaired his speech and finished his sentence with the utterance: “from zero to a, from negative infinity to a, the graph is decreasing, right? So the graph of the derivative is under zero, less than zero...” (02:40) In other words, Yee’s sentence began with “as”, but he immediately changed this way of talking about derivative functions by suggesting an interval for which the function is decreasing. Having begun with “as...”, he could have finished his sentence with something like, “as x increases, the tangent slope remains negative,” which would convey continuous change, but he did not. Instead, he stated a property of the function, namely “the graph is decreasing” over the interval $(0, -\infty)$. The notion that “the graph is decreasing” is discrete and static in nature, for it requires that $f(b) < f(a)$ for all $a < b$. Although the verb itself ends in “-ing”, a “decreasing” function does not necessarily imply a sense of motion or continuous change, as it is only necessary to provide an interval for which $f(b) < f(a)$ for all a and b in the interval where $b > a$.

Unlike in a static environment, Yee did finish his sentence beginning with “as...” in a dynamic environment. When he pressed the action button to animate the falling of the ladder in the DGE, he clearly communicated a sense of continuous change: “as the ladder falls, we can see that $x...$ is increasing and y is decreasing, but z remains constant.” (17:10). By “ $x...$ is increasing and y is decreasing”, Yee was referring to the change of distance from the two ends of the ladder to the wall and ground respectively. The change communicated here was continuous which was visually mediated by the DGE. Numerically, the change was conveyed by the numerical values of ‘ x ’, ‘ y ’, and ‘ z ’ (rounded to 2 decimal places) shown in the sketch (Figure 15). Besides communicating continuous change of ‘ x ’ and ‘ y ’, Yee also communicated the invariance of ‘ z ’. Although it may sound trivial that ‘ z ’, the length of the ladder, was invariant, it was not trivial to comment on the invariance and variance of the three variables, ‘ x ’, ‘ y ’ and ‘ z ’, as well as the simultaneous change of them as the ladder fell. This way of thinking about the variables is necessary for the learning of related rates, which was the intended learning target of the sketch.

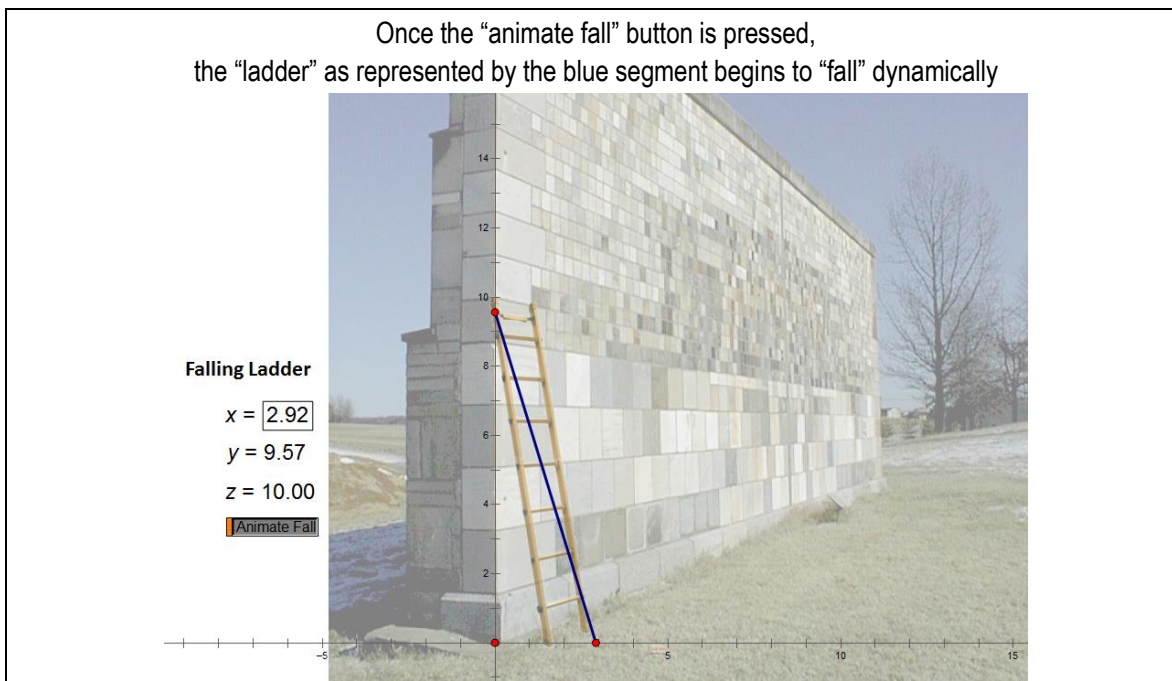


Figure 15. The “falling ladder” sketch

One example where Yee’s word use conveyed discrete change was in his discussion of optimising area in a static diagram. As seen in Figure 16, the diagram consisted of three figures depicting the enclosure of a rectangular area that borders a river geometrically and numerically. Given the diagram, Yee did not communicate the variance of area and the dimensions of the enclosure in a dynamic and continuous sense; rather, he described a discrete change of the enclosed area. He used numerical values to reason why “you can’t have a very long side” of enclosure (Table 10a). There was no indication of continuous change in his verb use.

In contrast, Yee used different words to describe the dimensions of the box that would optimise volume when the problem was posed in a dynamic sketch. He talked of the height of the box as “greater and greater” and the volume as getting “smaller and smaller” (Table 10b). Although the sketch showed numerical values of the dimensions (Figure 17), he did not provide any numerical examples as he had with a static diagram. Moreover, functional dependency was also noted in his use of “if... then...” statements. The functional dependency involved here was that the volume of the box was dependent on the height of the box: as the height increases, the volume decreases. All of these

supported the claim that Yee’s discourse about continuous change was occasioned by the dynamic visual mediator.

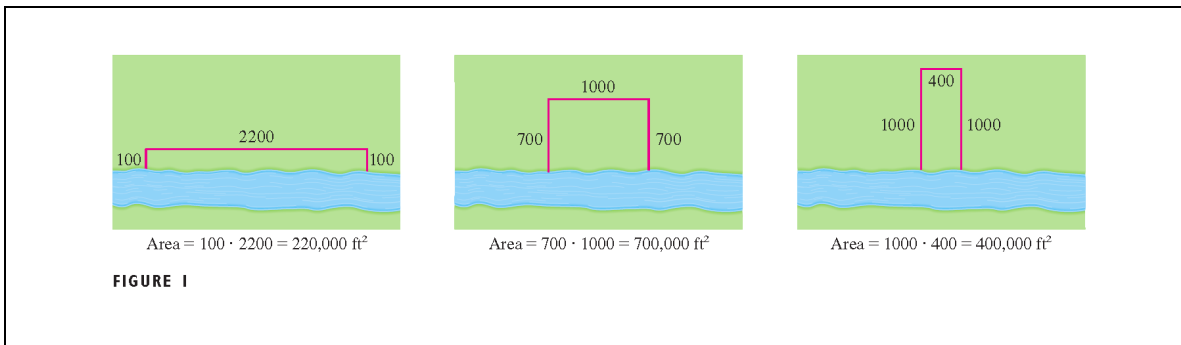


Figure 16. A static diagram illustrating optimization of area

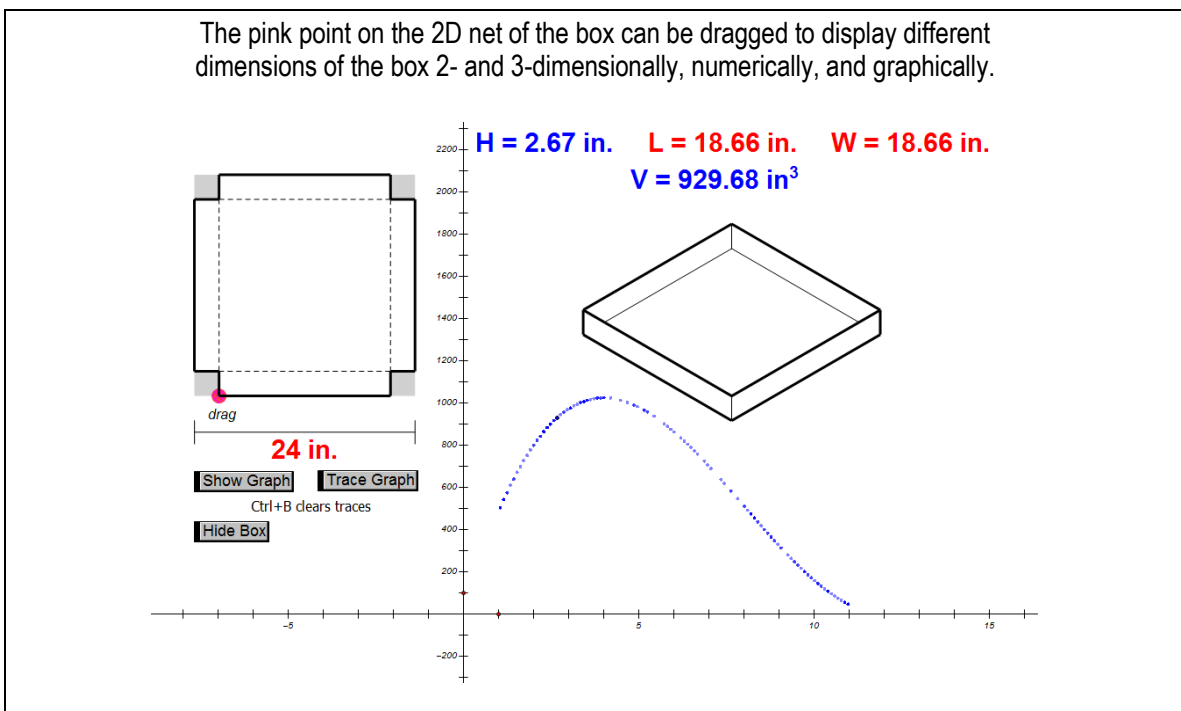


Figure 17. A dynamic sketch conveying volume optimization.

Table 11 Highlights of Yee’s transcript when he discussed the optimization of area (a) and volume (b).

(a) With static diagram	“[be]cause you can’t have a <u>very long one side and a very small two side, to maximize area</u> . It has to be, both of them has to be like a big number, in order to have a bigger area. I think that’s the point, like for this one both of them have <u>1200 that’s pretty long</u> , but the other two sides are only 100, so the area is actually not that big right?” (09:30)
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(b) With DGE	“If the <u>length</u> , the height of the <u>box is greater and greater</u> , the volume is smaller and smaller, you actually can see.” (20:15)
	“So now <u>the height is decreasing</u> , <u>the length is increasing</u> , the volume is actually getting bigger and bigger, but if the, <u>if the length continues increasing</u> , the <u>volume decrease</u> again, so there is one point like <u>here... remains</u> the greatest volume. I think that’s it.” (21:30)

6.2.5. Communicating variance and invariance through dragging

Overall, the use of dragging was prevalent throughout Melissa and Yee’s discourse in the dynamic environment. It was used as a routine extensively to explore and describe the dynamic relationship shown in the sketch. The consistent use of dragging changed the students’ discourse about calculus: they did not use theorem-like talk to explain what the sketch was intended to say, but rather, they communicated the variance, covariance and invariance that were implicated in the sketch through words, gestures and dragging. In some cases, dragging was used to explore the continuous change of variables, after which the students described the dynamic relationships about the variables verbally. Conversely, in other cases, dragging was used to verify a certain relationship after the students had initially hypothesised the relationship. In either cases, dragging routine gave rise to verb forms that imply motion, such as “become” (14:13) and “getting closer and closer” (14:19), which were not observed in the static environment. Figure 18 shows snapshots of some of Melissa and Yee’s dragging routine.

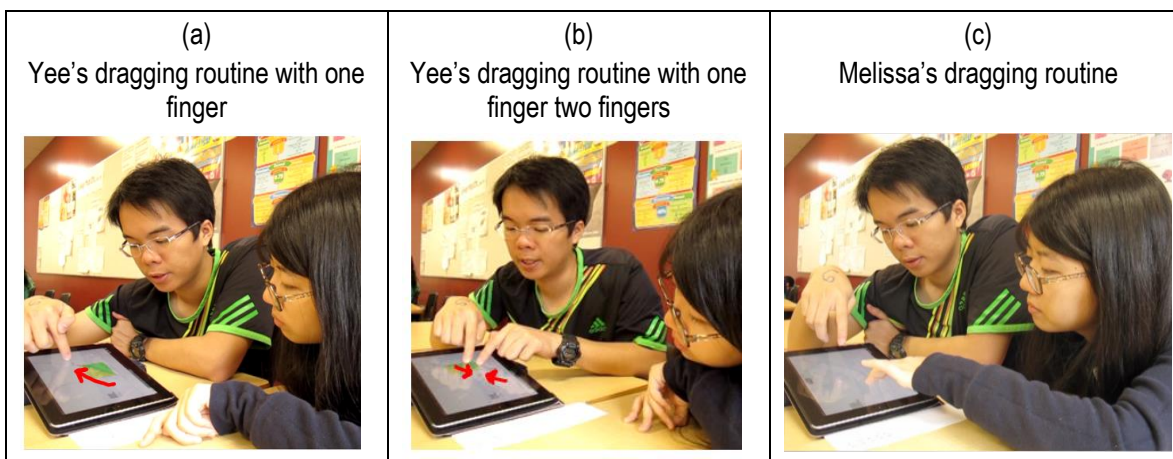


Figure 18. Melissa and Yee’s dragging routines

Dragging also gave rise to different mathematical Discourse practices in a dynamic environment. Recall that the students had often been *stating* static calculus ideas with theorem-like statements previously. In contrast, there were significantly fewer *stating* but more *comparing* and *reasoning* practices observed in the dynamic environment. For example, consider Yee's utterance:

If we have a linear function, the slope doesn't change. It remains the same because in this case f' , x' , is three, x' , minus, two, and the derivative of f' is always three, it's a constant. (16:05)

The words “doesn't change” and “always” suggest that Yee had been observing the change of the derivative of the function $f(x)=3x-2$. In order to say that the derivative is “always” three, one needs to compare the derivative of $f(x)$ across different values of x . This was achieved through Yee's dragging of x' . Besides *comparing*, Yee was also *reasoning* about why “the slope doesn't change”. As seen in the word “because”, he was reasoning that the derivative was always three, a constant, hence the slope does not change.

In my previous analysis, I showed that Melissa also used dragging to communicate, in full sentences, the invariance of the tangent slope of $f(x)=3x-2$.

If we drag the x' , on the function of x' , we get the tangent slope, and it's always y' equals three, and uhm here, the tangent line, the tangent line is the same as the x -values, and then the y -value is the secant, ah the tangent slope, so it's always three because it's on the same function, and the function has the same slope. (15:05)

What Melissa was communicating here was more than invariance of $f'(x)$: she was communicating the covariance of two functions, $f(x)$ and $f'(x)$. In the first part of her utterance, she was referring to “the function of x ”, and that its tangent slope was always equal to three. Then, she mentioned that, “the tangent line is the same as the x -values”, which I interpret as her comparing the x -values of the points $(x, f(x))$, $(x, f'(x))$, since they would be “the same”. Finally, she suggested that the “ y -value is the [...] tangent slope,” which shows that she was attending to the mapping of $(x, f'(x))$ on $y=f'(x)$. Hence, Melissa was referring to the simultaneous change in both $f(x)$ and $f'(x)$.

Related to variance and invariance was the idea of the general and particular. The contrast between Yee's discourse on the Mean Value Theorem across environments was illuminating in this respect. Recall that Yee's utterance resembled the genre of a mathematical theorem in a static environment, and he used the phrase, "we can *know* that", which implied that there was some knowledge to be acquired cognitively through perceiving the diagram. With DGE, however, Yee's discourse differed in several ways. First, he uttered "we can *see* that" as opposed to "we can *know* that". The transcript showed that Yee used the exact wording "we can see that" twice (both in the dynamic environment) and "we can know that" twice (both in the static environment) in the task. It was likely that the difference in verb use was due to the fact that dragging in DGE allowed one to "see" the change visually. This was significant because it shows that Yee's word use was connected with his dragging routine, and that he was making reference to the sketch directly. It was as if mathematics was *happening* right in front of him that he could "see", and it was no longer some knowledge to be acquired or to "know".

Secondly, Yee was able to use dragging to convey both generality and particularity in his communication. His dragging and word use complemented each other to convey generality, as he said, "we can choose random two points, a , b in this case these two points, a and b ," while he dragged both ' a ' and ' b ' back and forth. In contrast, the idea of choosing two random points ' a ' and ' b ' was never communicated in a static environment. In fact, he used the verb "know" to introduce ' a ' and ' b ' in the static diagrams: "So we have a function, you know a and b ". Later, Yee also dragged the point ' c ' on the function to locate $x=c$ such that $f'(c)$ would be equivalent to the secant slope through $f(a)$ and $f(b)$. His previous dragging of ' a ' and ' b ' to "choose two random" points, combined with the dragging of ' c ' conveyed a sense of variance and invariance here, in that no matter which two points ' a ' and ' b ' he chose, he could always find a point ' c ' such that $f'(c) = [f(b) - f(a)] / [b - a]$. In other words, the theorem works for all chosen ' a ' and ' b '. Finally, he described a particular example of the MVT in the utterance, "which is 0.79 in this case, right". By illustrating particularity, he was also implying generality because he had said that the particular ' a ' and ' b ' were chosen randomly. This means that dragging is an important mode of communication about calculus ideas, and more importantly, the calculus ideas were situated in the dynamic environment. The static

diagram occasioned Yee's talk of the MVT as a theorem (Table 12a), but the DGE occasioned Yee's talk of MVT in terms of variance and invariance (Table 12b). The dynamism of the sketch and the draggable points 'a' and 'b' may have facilitated this way of talking.

Table 12 Yee discussing the Mean Value Theorem across two environments

	(a) With static diagram	(b) With DGE
Defining the interval (a,b)	So we have a function, you know <u>a</u> and <u>b</u> ,	We can <u>choose random two points, a, b in this case these two points, a and b,</u>
Drawing a secant line through f(a) and f(b)	and you <u>draw a line across it</u> and you get a <u>secant, secant function, secant line function</u> , and there must,	and we draw a <u>secant line</u> , so we can see that <u>there is a point, here here, the slope at c is pretty close to the slope at a, b. If I can draw...</u> yes, exactly the same one. So which means,
Stating the conditions and the results	if the <u>function is continuous</u> , so there must be a point <u>p</u> , which its <u>slope is equal to the slope of the secant line a, b.</u>	if a function is continuous from <u>a to b</u> , there must be <u>at least one point that the slope of c is equal to the slope of a, b,</u>
Verifying from the static diagram or dynamic sketch	so for this here, at least one, it can be two, for figure four, there is actually two points, P1 and P2, they both have the same slope as A, the <u>secant line a, b.</u> so there must be at least one point.	which is 0.79 in this case right? And that's called mean value theorem?

To revisit, Melissa and Yee began their discussion with physical postures somewhat distant both from each other and from the static diagrams. Their posture changed when engaging with the dynamic sketches, as they leaned towards the iPad and became more active with their hands and fingers. The change in posture signified a shift in the way they were attending, which may have contributed—in and of itself—to the change in discourse. They seemed more interested in the dynamic sketches than in the static diagrams and to make sense of them. Significantly, the fundamental mathematical ideas were communicated different in each environment.

6.3. Summary

In this chapter, I analysed both the overall and the specific features pertaining to two participant pairs' calculus communication in two different environments. From Ana and Tammy's engagement with the task, I focused on comparing particular words and

gestures (which includes dragging) that were used in communication with a given environment. For example, the use of verb forms accompanied by deictic gestures and scribing gestures as the students discussed over static diagrams revealed that they were thinking about the derivative statically (through naming mathematical objects) and procedurally (that derivative can be expressed as a formula). By contrast, the students used dragsturing to communicate the change of tangent slope in a dynamic environment, where their verb forms changed from “is [noun]” to “is [verb]-ing” (such as “is increasing”). The dynamism present in the dynamic sketch led to the students’ comparing, predicting and generalising practices about the change of tangent slope.

The analysis of Ana and Tammy points to a distinctive feature of students’ communication with touchscreen-based DGE—dragsturing. This blended action of dragging and gesturing is further extended in the analysis of Melissa and Yee’s communication. While talking about the Mean Value Theorem, Yee utilised dragsturing to communicate the variance of ‘ a ’ and ‘ b ’ in the interval (a,b) by dragging both points back and forth. He then dragged the point ‘ c ’ on the function so that he “can see” the conclusion of the theorem. In the absence of a dynamic visual mediator, Melissa and Yee used different words, gestures and dragging in order to communicate a sense of change. Specific features of their communication included the “measuring gesture” which changed sizes and the pinky finger for conveying precision.

Adopting a thinking-as-communicating approach, Part I of my study shows the importance of studying the interplay between linguistic and non-linguistic modes of communication for understanding mathematical thinking. It points to an expanded view of the mathematical discourse that includes gestures, diagrams and touchscreen-dragging, especially to communicate dynamic aspects of mathematical ideas. It also provides powerful evidence of the participants’ different discourses, or mathematical thinking, when prompted by two different kinds of visual mediators. I discuss further the results of Part I of the study in Chapter 8. Part II of the study (reported in Chapter 7) is grounded in the results of Part I in two respects: that patterns of discourse is situated within the activity and the visual mediators used, and that the use of a DGE is instrumental for facilitating dynamic ways of thinking about calculus.

Chapter 7. Analysis of communication during calculus exploration (Part II of study)

Ok, let's predict. This, the graph is going to be a... negative cosine. (Participant Mario discussing with his partner Sam during Part II of the study)

This graph is the derivative of this graph [...]. This graph is the derivative of this graph [...]. So, the sine graph is the derivative of negative cosine 'x' right? (Participant Katie discussing with her partner Ivy during Part II of the study)

This chapter includes analyses of data gathered for Part II of the study, where the participants discussed a dynamic sketch that they had not seen before. I continue to draw on the communicational theoretical framework as overviewed in Chapter 3. Part II of the study aims to examine students' communication with the use of touchscreen-based DGEs for exploring calculus ideas, namely, the area-accumulating function. Following the methodology as informed in Chapter 4, I analysed the participant pairs' developing discourse around area-accumulating functions by attending to their patterns of discourse consisting of particular words, gestures and dragging actions, as well as the interplay among the three modes of communication, such as simultaneous use of speech, gesturing and dragging by one or two participants. Special attention was paid towards the participants' change of discourse, by observing how their use of speech, gesturing or dragging evolved over time. Similar to Chapter 6, I include transcripts along with the analyses to capture the inter-relationships among the three modes of communication in each turn and their relationships. At the end of this chapter, I provide a summary or meta-analysis of the participants' discourse during Part II of the study.

Four pairs of participants participated in Part II of the study. All participants were in the middle of a year-long calculus course when the study took place. They had just finished the differential calculus component of the course and one lesson on the

indefinite integral. This setting allowed me to examine their “exploratory talk” about the sketch related to the area-accumulating function, a concept that they had yet to learn in class. Each pair of participants discussed what they saw in a dynamic sketch designed with five pages, where the last page was a “Try” page that posed a problem for the students to be solved on a mini-whiteboard. The order of the analysis reflects the chronological order of data collection.

7.1. Huang and George

In this section, I analyse Huang and George’s communication as they engaged in the task. In particular, I focus my analysis on the first six minutes from a total of thirty minutes of data collected on the student pair. I chose to analyse this data because the students used their dragging routine prevalently during this interval; therefore, this analysis may shed light on students’ early discourse when interacting with a DGE for exploring calculus ideas. Then, I end with an analysis of a 50-second episode during the “whiteboard” part of their discussion, to illustrate their change of discourse over the course of their exploration, as well as an overall summary of their use of speech, gestures and dragging during the task.

7.1.1. Dragging as non-verbal communication

Overall, the limited number of words spoken was a unique characteristic of Huang and George’s communication. During the first ten minutes of interacting with the dynamic sketch, the pair spoke a total of 141 words, a rate of 14.1 words per minute. Although the word count increased to 260 and 212 words respectively during the next two ten-minute intervals, their use of verbal communication could be considered quite limited throughout their discussion. This observation may suggest that Huang and George were using other forms of communication for exploring the sketch related to the area-accumulating function. Indeed, my analysis shows that Huang and George were communicating non-verbally, through dragging and gesturing. The use of dragging was prevalent in the pair’s first ten minutes of exploration, where I observed 61 turns in the transcript where dragging was present during this interval, 33 of which were present without accompanying speech. Since there were a total of 113 turns that appeared in the

first ten minutes of transcript, this equated to 61/113 (54%) of the turns that involved dragging, of which 33/113 (29%) were unaccompanied by any speech.

The following is the first 17 turns taken from the transcript between Huang and George. As seen in the transcript, 11 dragging turns were observed in the span of 2 minutes and 12 seconds, 6 of which were dragging actions not accompanied by speech.

Turn	Timespan	What was said <what was done>	S -er	D -er	G -er
1	0:00.0 - 0:10.4	Let's try this one, George.	H		
2	0:10.4 - 0:17.4	<George pressed the first three buttons on the sketch and dragged 'x' intermittingly>	NIL	G	
3	0:17.4 - 0:49.3	<George dragged 'a' and then 'x' back and forth>	NIL	G	
4	0:49.2 - 1:00.3	<Huang dragged 'x' from right to left and back>	NIL	<u>H</u>	
5	1:00.3 - 1:01.0	So...	G		
6	1:01.0 - 1:02.9	<u>What's 'a' for.</u>	H		
7	1:02.9 - 1:06.9	'x'...move to the left...	G		
8	1:06.9 - 1:09.3	<silence>	NIL		
9	1:09.3 - 1:12.1	<u>This one...</u> <George pointed towards the iPad>	G		<u>G</u>
10	1:12.0 - 1:23.5	<Huang hovered his finger over the iPad>	NIL		H
11	1:23.4 - 1:25.1	<u>Is this slope?</u> <George pointed towards the iPad; Huang dragged 'x' >	G	<u>H</u>	<u>G</u>
12	1:25.1 - 1:26.4	Yeah...	H		
13	1:26.4 - 1:34.9	<George dragged 'x' and then 'a' back and forth>	NIL	<u>G</u>	
14	1:34.9 - 1:37.6	<u>What's</u> <inaudible>...	H		
15	1:37.6 - 1:39.0	<George continued to drag 'x' back and forth>	NIL	<u>G</u>	
16	1:39.0 - 1:44.4	<u>Oh...K...</u>	G		
17	1:44.4 - 2:12.2	<George continued to drag 'x' back and forth>	NIL		

Frequent dragging turns were observed in George and Huang's communication in the above transcript, some accompanied and some unaccompanied by speech. There were eight turns that contained no speaker, which meant that one or both students were dragging and/or gesturing during these non-speaking turns. In terms of speech, the transcript shows that the maximum number of words spoken in a single utterance was five, said by Huang in Turn 1 ("Let's try this one George") and George in Turn 7 ("x... move to the left"). Moreover, Turn 1 ("Let's try this one George") and Turn 6 ("What's 'A'")

for”), both uttered by Huang, were the only two complete sentences spoken, as all other utterances were incomplete sentences containing only one or two words (Turn 5, 9, 12 and 16). These incomplete utterances suggest incomplete thought as they were missing either the subject or the predicate. In Sfard’s terms, the students had yet to develop their mathematical discourses fully around area-accumulating functions at this stage of their exploration.

Although not much verbal communication was observed, it can be said that the student pair was communicating by means other than speech. The transcript shows that Huang and George exchanged dragging immediately when they began exploring the sketch. From Turn 2 to Turn 3, George dragged ‘ a ’ and ‘ x ’ for 28 seconds—unaccompanied by speech—upon Huang’s suggestion of “let’s try this one, George” (Turn 1). He first dragged ‘ x ’ intermittingly, then dragged ‘ a ’ back and forth from left to right, changed his dragging direction three times before switching from dragging ‘ a ’ to ‘ x ’. He dragged ‘ x ’ back and forth in a similar fashion, this time changing direction ten times. By changing directions, he was able to change the colour that was shaded under $f(x)=1$ from orange to blue twice. All these frequent changes of direction and object being dragged occurred within 28 seconds. After George’s dragging in Turn 3, Huang took up the dragger role and began to drag ‘ x ’ for another 11 seconds in Turn 4. His dragging routine differed from George’s significantly in that the pace was slow and steady, whereas George’s was quick and sporadic. In fact, it took Huang 10 seconds just to drag ‘ x ’ for about 5 units on the iPad from right to left and another second to drag it back to its original position. It was unclear what the students were thinking respectively while they were dragging, as they did not speak concurrently. However, based on their different dragging routines, it is hypothesised that their mathematical thinking and attention to the sketch might be quite different. In other words, the analysis of words spoken may suggest that the students did not participate verbally, but this does not mean that they were not thinking mathematically. Rather, the analysis of dragging routines taken up by the students adds another dimension, namely their mathematical Discourse practices, in that they were exploring the sketch in *different* ways through dragging.

From Turns 9 to 11, Huang and George exchanged verbally and non-verbally with gestures as they continued to explore the sketch. By then, the “show Trace of A”

button had been pressed and the graph was showing a set of linear green traces. George pointed at the green traces in Turn 9 without speaking and then again in Turn 11 while saying, “Is this slope?” This shows that he was attending to the set of green traces. His partner, Huang, also gestured in Turn 10. His gesture was interesting, as he was using his right middle finger (the finger that he used previously for dragging) to hover over the area of the screen in proximity to the draggable points ‘a’ and ‘x’ (Figure 19a). He circulated his finger around the two points for 4 seconds before placing it on ‘x’ and dragged it around in Turn 10. His hovering of the “dragging finger” around the draggable point ‘x’ suggests that he was thinking about that part of the screen with the draggable points.

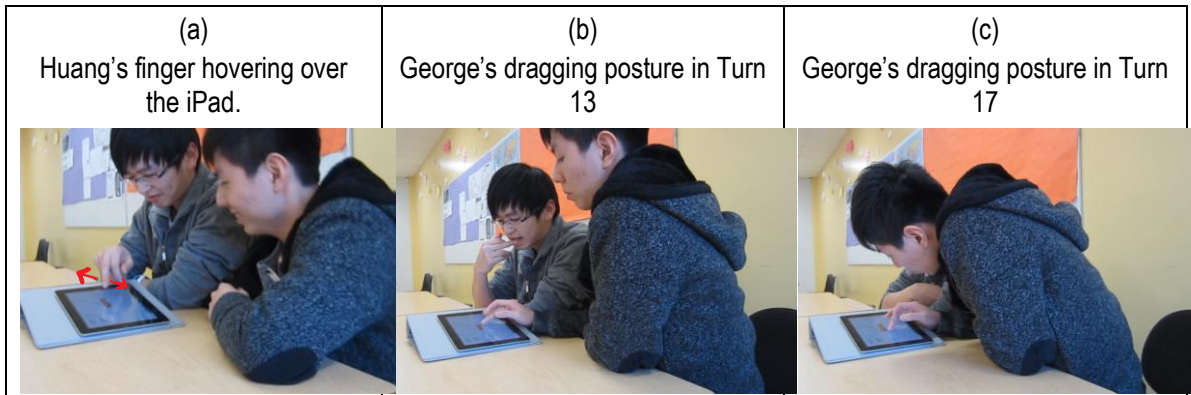


Figure 19. Selected snapshots of Huang and George’s dragging (Turns 1 to 17)

The last five turns of the transcript were dominated by George’s dragging. He dragged ‘x’, ‘a’, and then ‘x’ again back and forth for a total of 34 seconds. Very few words were spoken during this span, but it was observed that George’s posture changed over this span. He began with an upright position (Figure 19b) and later assumed to a leaning position towards to the iPad (Figure 19c). This change of posture may be indicative of a change of George’s attention and focus, although he did not speak during this time. In summary, Huang’s hovering gesture and George’s dragging posture were examples of non-verbal features of communication. More importantly, these forms of communication were situated in their dragging routines, in the sense that they existed only in the presence of the dragging modality and the DGE. As seen in the analysis, attending to these non-verbal communication help inform the students’ attention to the sketch as well as possible changes in thinking.

The students communicated in a similar fashion as shown above for the first 5 minutes and began to make some progress in the development of their mathematical discourse verbally at the five-minute mark. The following is the transcript of a 50-second episode of Huang and George’s discussion at the 5-minute mark.

Turn	Timespan	What was said <what was done>	S -er	D -er	G -er
53	4:56.3 - 4:58.0	<George dragged ‘a’ continuously>	NIL	<u>G</u>	
54	4:58.0 - 5:01.5	<u>S...s...</u>	H		<u>H</u>
55	5:01.5 - 5:08.8	<u>Moving ‘a’ only shift...vertical...</u>	G		
56	5:08.8 - 5:12.8	<inaudible>	H		<u>H</u>
57	5:12.8 - 5:14.5	<u>Hmm...</u>	G		
58	5:14.5 - 5:16.9	<u>Area's negative there.</u> <Huang pointed to the iPad>	H		
59	5:16.9 - 5:18.1	<u>Area...</u>	G		
60	5:18.1 - 5:24.5	<silence>	NIL		
61	5:24.5 - 5:27.9	<u>The area of here is...</u>	H		<u>H</u>
62	5:27.9 - 5:32.1	<Huang dragged ‘x’ and George dragged ‘a’ on the iPad>	NIL	<u>G/H</u>	
63	5:32.1 - 5:37.9	<u>Wait. What are you doing, George?</u> <Huang and George continued to drag simultaneously on the iPad>	H		
64	5:37.9 - 5:46.4	<Huang dragged ‘x’>	NIL	<u>H</u>	

Three observations are worthy of note in the above transcript. First, the word “area” appeared at 05:14 for the first time ever since the beginning of the students’ exploration. As seen in the above transcript, it was said by Huang and then immediately responded by George in Turns 58 and 59. Moments before that, George had been dragging ‘a’ continuously back and forth in Turn 53. Subsequently, with the iPad displaying a set of vertical green traces, George described the “vertical” movement of the green traces using the words “moving” and “shift” in his utterance in Turn 55. The use of active verbs accompanied by the word “vertical” shows that George was thinking about the behaviour of the green trace geometrically and dynamically. Then, at one point during George’s dragging of ‘a’, the net area under $f(x)$ became negative. At this very moment, Huang gestured with his right index finger towards the centre of the iPad and said, “Area's negative there” (Turn 58). The “S-er”, “G-er”, “D-er” columns clearly show that three modes of communication were simultaneously used by two students during this turn (see also Figure 20a). This suggests that the students were coordinating with

each other as a pair, since Huang was able to talk and gesture about the sketch as it was being altered by George's dragging. The observation that George replied to Huang with the word "area" (Turn 59) and then again "the area here is..." (Turn 61) also suggests that both students had begun to develop their verbal communication around "area". Although they had yet to communicate verbally the functional dependency between x and $A(x)$, their verbal communication about "area" simultaneously with dragging implicitly suggest that they were attending to "area" as a function of dragging x .

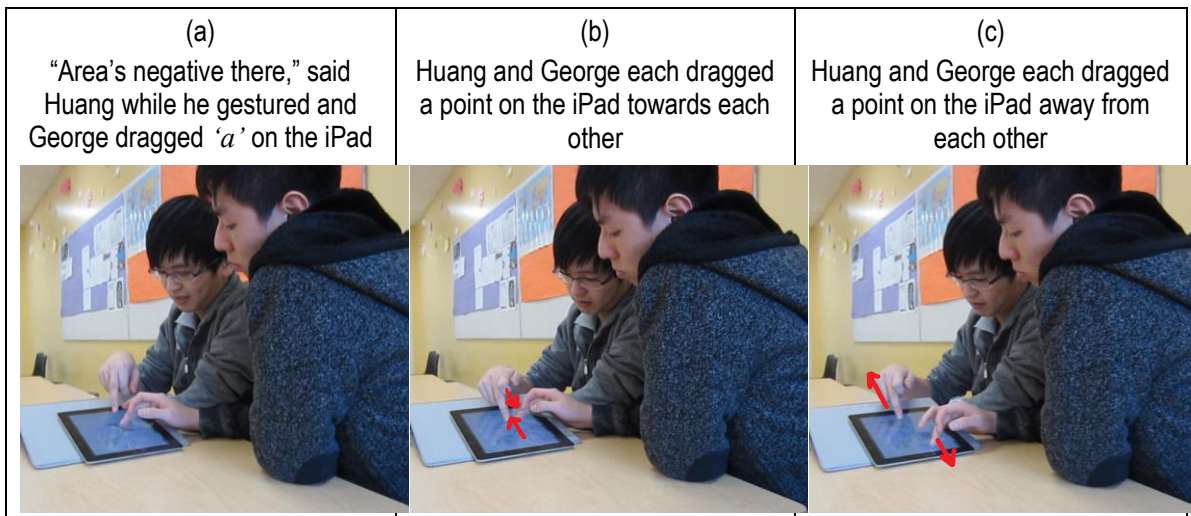


Figure 20. Selected snapshots of Huang and George's dragging (Turns 58 and 62)

Dragging continued to be a significant mode of communication for the student pair between Turns 60 and 64. My second observation of interest occurred in Turn 62. In Turn 61, George had been dragging a , when Huang appeared to "join in" by dragging x alongside with George. This resulted, in Turn 62, in two students simultaneously dragging on the iPad for 5 seconds. During this span, Huang and George dragged x and a respectively, first towards each other, then crossing over each other, and finally away from each other, as seen in Figure 20(b & c). Prior to this, the students had been dragging with one finger at a time, and so this observation was a first of its kind. In other words, the students were exploiting the multi-touch functionality of the iPad for the first time. Not only that, they were doing so by coordinating with each other's finger, which supports the claim that there was mutual communication going on even in the absence of speech.

Thirdly, a significant utterance was observed in Turn 63. After 5 seconds of dragging 'a' and 'x' simultaneously and silently by the two students, Huang broke silence with his question, "What are you doing, George?" Since both students had already been dragging for 5 seconds, it seemed unlikely that Huang had just realised that George was also dragging alongside him, and so this analysis did not regard his question, "what are you doing", in a literal sense. The situated meaning of this question seems to be that Huang was in disagreement with what George was doing. There might have been a change in his thinking from 5 seconds ago, since he had not raised this question back then. This suggests that there existed a commognitive conflict in the students' discourse, even though they were not actually speaking. In Sfard's terms, a commognitive conflict occurs when there are two or more conflicting discourses between the interlocutors. It seems that a commognitive conflict, as reflected in the students' dragging routines, led to Huang's question about what George was doing. Upon hearing Huang's question, George refrained from dragging and, in so doing, let Huang be the sole dragger on the iPad (Turn 64). George's response could be taken as his attempt to resolve the commognitive conflict in the students' dragging routines, by voluntarily refraining from actively participating in the discourse and stop dragging. Hence, in Turn 64, George watched still as Huang dragged 'x' for another 9 seconds, twice changing his direction of dragging. The present analysis shows that commognitive conflicts could be present in non-verbal communication, demonstrated by students' dragging routines. Importantly, the analysis shows that George's means to resolve the commognitive conflict was also non-verbal. Therefore, it can be said that the students had relied on non-verbal communication for developing their mathematical discourses during this episode.

7.1.2. Communicating "change" and the area-accumulating function during the paper and pencil task

I now turn to an analysis of the students' communication seven minutes before the end of their exploration. Huang and George had already explored all the pages of the sketch and just begun attempting the "Try" page, where the problem of sketching the area-accumulating function given $f(x)=\cos(x)$, $a=0$ was posed to them. They were asked to solve the problem on a mini whiteboard. The students reached the "Try" page about

10 minutes before, but had only begun to engage in this whiteboard task at about 23:00 (from here, I have used “whiteboard” and “paper-and-pencil” interchangeably to emphasise the static nature of the task). A 50-second episode was chosen and analysed below during the students’ solving process. The significance of this episode was that the students were communicating with considerably less dragging, more speech and more gesturing.

The following transcript illustrates the change of discourse in terms of Huang and George’s use of speech, dragging and gesturing before and after 20:00. Right before the start of the transcript, the students had just begun actively engaging with the paper-and-pencil task. Huang had offered to draw the area-accumulating function for the given function, $f(x)=\cos(x)$ with $a=0$. Seeing that the students seemed to have finished, I approached the pair and asked them to explain to me what they had drawn, as shown in Turn 274:

Turn	Timespan	What was said <what was done>	S -er	D -er	G -er
274	22:33.1 - 22:35.3	Can you explain, can you explain, Huang?	O		
275	22:35.3 - 22:36.9	<inaudible>	G		
276	22:36.9 - 22:57.1	Uh...like the...at first, like, the...uh, the rate of the area changes...uh, is biggest and then it's decreasing...and so... Uh...it's like concave down. And like, after the...	H		<u>H</u> , <u>H</u>
277	22:57.1 - 22:58.1	After...	O		
278	22:58.1 - 23:00.3	Oh, after the...inflection point...	G		
279	23:00.1 - 23:06.6	...this point the area, negative, so there become, became like, decreasing...	H		H
280	23:06.6 - 23:11.4	So <u>this is this?</u> <u>This is this</u> , right?	O		<u>O</u>
281	23:11.3 - 23:12.0	Ah, yeah.	H		O
282	23:12.0 - 23:15.8	And then it's going down, okay. Okay.	O		O
283	23:15.7 - 23:20.7	It's like the...<takes pen> sine function. <refines drawing>	H		
284	23:20.6 - 23:21.3	Is it.	O		
285	23:21.3 - 23:24.2	It's, like, always goes up and down. <Huang uses pen to trace the shape of the green graph on whiteboard>	H	G	

At Turn 276, Huang communicated the “change” of “area” for the first time in their exploration. In fact, his utterance, “the rate of the area changes” suggests that he was actually talking about the rate at which area was changing under the function $f(x)=\cos(x)$,

as opposed to the change of area itself. What Huang was communicating here was a more sophisticated idea than the change of area; in order for him to be able to identify rate of change of area, he needed to have a well-developed discourse around the change of area. Indeed, Huang's comment about the rate as "biggest" and then "decreasing" were both accurately stated mathematically, when matched to the area accumulation under $y=\cos(x)$ with $a=0$ and $a<x<\pi$. Although he did not use any gestures to accompany this part of his speech, he did gesture deictically at the beginning of his utterance by pointing towards $a=0$, which suggests that he was referring to the area accumulation under $y=\cos(x)$ with $a=0$. Similarly, in the rest of Turn 276, Huang stated the rate of change of area accurately with his speech complemented by a deictic gesture. Right before saying that, "ah... it's like concave down," Huang used his index finger to point towards the points $x=0$ and then $x=\pi/2$. What he seemed to be inferring was that the area-accumulating function ought to be concave down from $x=0$ to $x=\pi/2$. Moreover, the meaning of concave down matched his previous utterance about the rate of change as decreasing. All of these support the claim that Huang was communicating a sophisticated idea: that the rate of change of area affects whether the graph of the area-accumulating function is increasing or decreasing. In addition, George's response about an "inflection point" in Turn 278 was also commensurable with Huang's as well.

In Turn 279, Huang continued to communicate about the shape of the area-function. He said, "this point, the area, negative, so there become, became like, decreasing..." while he used a "measuring gesture" to form a "C" shape with his thumb and index finger as if he was "measuring" the distance between $x=\pi/2$ and $x=\pi$ on the graph of $f(x)=\cos(x)$ (Figure 22). In mathematical terms, since the graph of $\cos(x)$ was below the x -axis from $(\pi/2, \pi)$, the accumulated area begins to decrease and so does the graph of $A(x) = \int_a^x \cos t \, dt$. This seemed to be precisely what Huang was communicating in his utterance in Turn 279. His use of "so" suggests that he was disclosing a causal relationship. Although he did not communicate in full sentences verbally, his gesture helped him state the interval of $(\pi/2, \pi)$ of which the function was "decreasing" according to him. In general, I observed that Huang did not speak in full sentences very much and used some self-repair speech in his communication, as can be seen in both Turn 276 and 279. However, Huang was able to communicate the reason why the area-accumulating function ought to be decreasing with the present analysis. Furthermore,

the analysis points out that Huang was engaging in a valued mathematical Discourse practice, that of reasoning mathematically.



Figure 21 Huang used a “measuring gesture” while uttering “so there become, became like, decreasing...”

While Huang continued to reason about the shape of his hand-drawn area-accumulating function, he picked up a pen and used it to refine his drawing (Figure 23) in Turn 283. He then used his pen as a “pointing device” to trace the shape of the area-function with his pen in Turn 285 while talking, “It’s, like, always goes up and down”. At the same time, George crossed his right arm over Huang’s right arm (which was his writing arm), while he was still drawing (Figure 22). He was doing so in order to reach the iPad that was at Huang’s right, and once he reached it, he started to drag ‘ x ’ on the iPad as Huang continued to draw more periods of the area-accumulating function. The two students continued this seemingly awkward position crossing over each other’s arm, as shown in Figure 22, for 16 seconds. In particular, George seemed to be checking something on the iPad after hearing Huang’s explanation about the shape of the area-accumulating function. A minute later, George said, “sine and cosine, similar”, in reference to the similarity of the “range” of the two functions, and he went on to alter the range of the existing graph of the area-accumulating function originally drawn by Huang. The one that Huang had drawn at this point did not dip down below the x -axis, and the parts that George had added indeed correctly represented the graph of the area-accumulating function for $f(x)=\cos(x)$ and $a=0$ (Figure 23). Perhaps he saw that the range of the area-accumulating function on the iPad was two units long, and so the area-accumulating function for $f(x)=\cos(x)$ should also be similar. Unlike Huang, George did

not communicate the change of area or its rate of change in the entire task, so it seemed that George's discourse of area differed from Huang's, which was dynamic in nature. Despite this, the two students together created the correct graphical representation of the area-accumulating function after roughly 30 minutes of engaging in the task, and this evidenced their learning about area-accumulating functions and change of discourse over the course of their exploration.



Figure 22. Huang drew on the whiteboard while George checks with the iPad

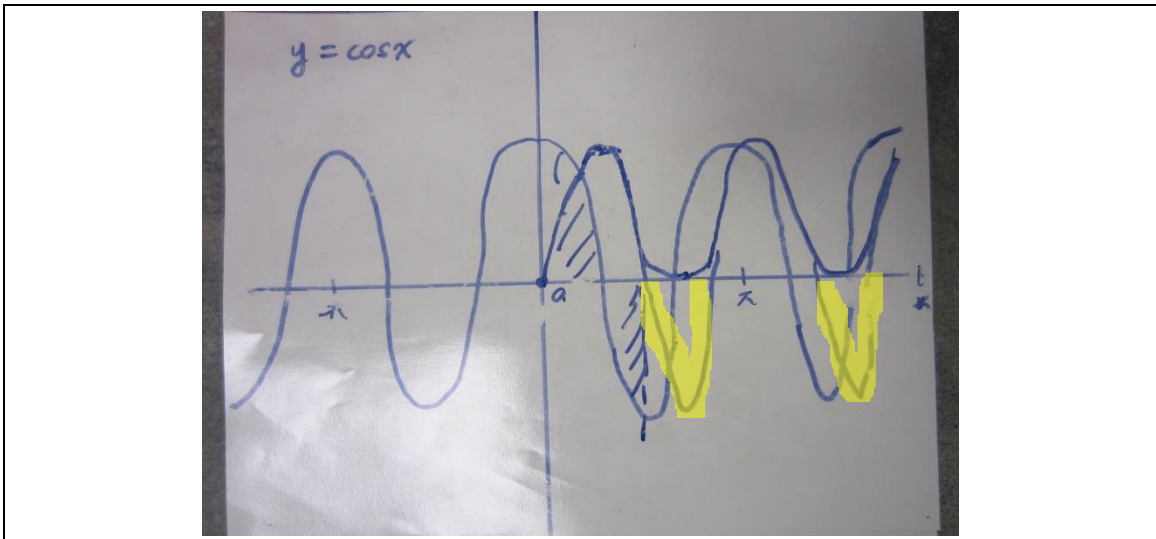


Figure 23. Huang and George's final solution for the paper-and-pencil task (The part highlighted yellow was added by George)

7.1.3. Summary of Huang and George's speech, gestures and dragging

Overall, the number of dragging turns stayed steady within the first twenty minutes of the task, at 61 and 57 for each 10-minute intervals respectively. However, this number dropped during the last ten minutes of the students' 30-minute long engagement with the task. In addition, the number of words increased by 119 words (nearly doubled, from 141 to 260) from the first to the second 10-minute interval. Although the increased number of words was still considered low for an average conversation (26 words per minute), it shows that the students were developing their verbal discourses while they continued to use dragging for exploring the sketch. On the other hand, a considerable decrease of the number of dragging turns was observed between the second to the final 10-minute intervals, from 57 to 29 dragging turns. This happened when the number of words stayed about the same, from 260 to 212 words. It was also observed that, during the final 10-minute, the students were mainly engaged with the paper-and-pencil task. By contrast, between 10:00 and 20:00, they seldom looked at the whiteboard and were mainly gazing at the iPad. They only began to tackle the whiteboard task and to gaze away from the iPad at the 23:00 mark. With the above observations, I completed a multiple line graph involving the students' combined number of words spoken, dragging turns and gesturing turns at different times during their task.

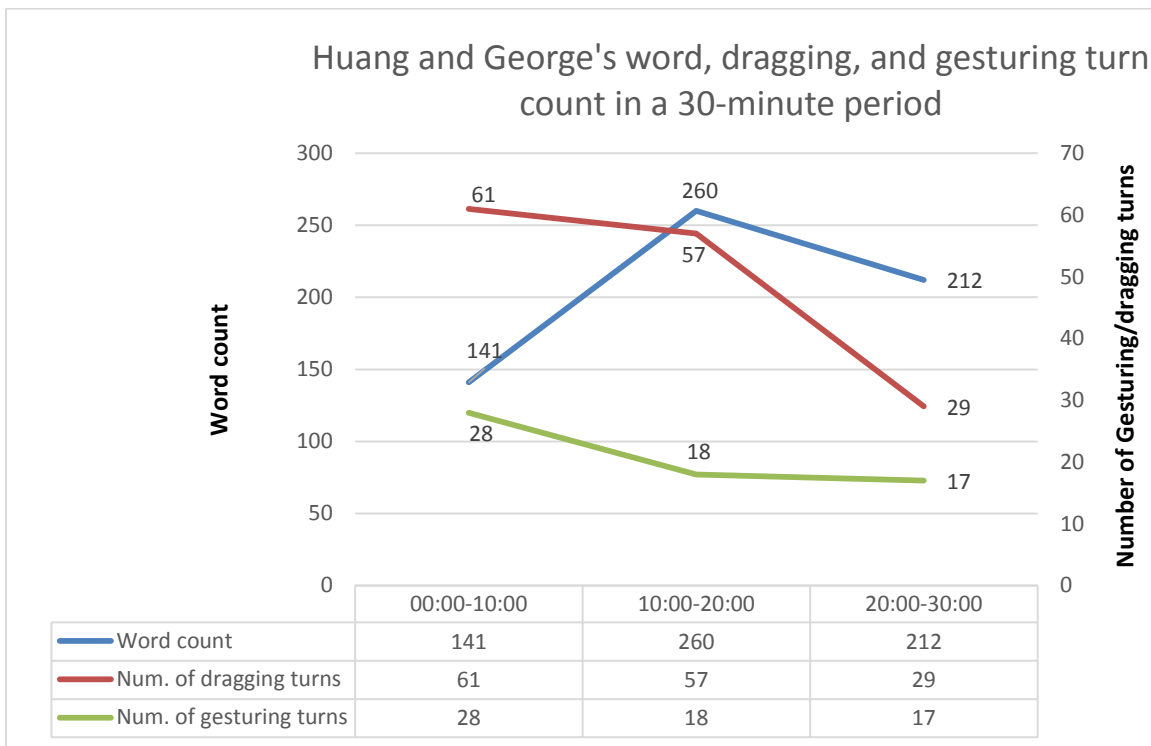


Figure 24. Huang and George's word, dragging, and gesturing turn count in a 30-minute period

The data as represented in the multiple line graphs (Figure 24) about the students' communication over their exploration with a DGE evokes some interesting patterns. The 00:00, 10:00, and 20:00 mark of the students' exploration roughly marked the different foci in the student pair's exploration. Namely, from 00:00 to 20:00, the students were focussed on the iPad, and from 20:00 to 30:00, they were focussed on the paper-and-pencil task. The line graph shows that the word count increased over the first ten minutes, after which both word and dragging turn counts decreased. These patterns seemed to have been facilitated by the kinds of activities that the students were engaged in: DGE-based for the first twenty minutes, and paper-and-pencil-based thereafter.

In summary, the student pair, Huang and George showed a change of discourse in the task exploring the area-accumulating function. The students' word use was limited throughout the task, but they were able to communicate "change" and the shape of the area-accumulating function through a combination of speech, dragging and gesturing. Without such an analysis, it would be difficult to claim that Huang and George had

developed their mathematical discourses whatsoever for the duration of the task. Within the first twenty minutes, the students used the dragging modality extensively for developing their mathematical discourse, which included noticing and resolving their commognitive conflicts. During the last ten minutes, the students were focused on completing the paper-and-pencil part of the task, and the use of dragging in communication decreased. In particular, the students engaged in valued mathematical Discourse practices such as reasoning (with gestures) and checking their work (with dragging). The increased use of speech suggest that the students might have developed their mathematical discourses verbally. As shown in the analysis, there was a change in their utterances, from uttering one or two words at a time, to reasoning about the “rate of change” of area as “decreasing” using a combination of words, gestures and dragging.

7.2. Larry and Ivy

In the previous section, I analysed Huang and George’s exploratory talk around the concept of area-accumulating function. Huang and George’s communication was characterised by a low number of words spoken throughout the task. By contrast, Larry and Ivy’s communication featured a high rate of words spoken. Larry and Ivy were the most experienced of all the participants in being schooled in an English-speaking environment. Also, the two students seemed quite comfortable working together, since I observed that they smiled and conversed naturally with each other during the task.

In this section, I summarise Larry and Ivy’s communication during the task in response to my research questions. As some of their linguistic and non-linguistic resources for communication were similar to other pairs, I have chosen to focus mainly on the ones that were particular to them in order to add depth to the data analysis. Finally, I completed an analysis of word, dragging and gesturing turn counts like the one seen in Huang and George’s communication. This analysis provides insight in terms of similarities and differences between the student pairs’ communication. While I am not interested in comparing the student pairs, I intend to make a case for studying the students’ speech, dragging and gestures for understanding their mathematical thinking and learning.

7.2.1. From coherent to incoherent discourses

The student pair, Larry and Ivy, together produced 585 words during their first 10 minutes of exploration. During this span, dragging and gesturing were also observed consistently, as exemplified by the transcript below.

Turn	Timespan	What was said <what was done>	S -er	D -er	G -er
15	2:15.8 - 2:24.9	So, A is probably the <u>dotted line, right?</u> Oh, so like it's like <u>from here, to here.</u> <Larry drags 'x' >	L	<u>L</u>	
16	2:24.8 - 2:48.4	Oh. Oh, <u>so that's the area.</u> <Larry continues to drags 'x' >	I		<u>I</u>
17	2:27.7 - 2:59.0	<Ivy drags 'x' > 'Cuz... <u>one and one...one times one is one... one times two is two...one times three is three...so that's...</u> I guess that slope is the area of that. Shaded. <Ivy taps Larry's arm> Try the next one. 'Kay.	I	<u>I</u>	<u>I</u>
18	2:59.0 - 3:08.9	<u>So we move this...</u> area will be <u>over here.</u> <Inaudible>	L	<u>L</u>	<u>L</u>

The students noticed and verbalised the effect of dragging 'a' and 'x' quite early; the first word "area" appeared in the transcript at 2:24 (Turn 16). At Turn 15, Larry had suggested that 'A' was "probably the dotted line," while dragging 'x' around. This could be interpreted as Larry observing that dragging 'x' would change the value of 'A', and that it was represented by the "dotted line" in green. Moreover, the use of the hedge word "probably" suggests a degree of uncertainty in his thinking. Then, while Larry was still dragging, Ivy responded, "Oh. Oh, so that's area." Her use of pronoun "that" seemed to be referring to Larry's "dotted line" in his previous utterance. If this analysis is valid, then Ivy was communicating the dotted line as area. In addition, Ivy's hand gestures supports the analysis, as she used her left middle finger to point towards the middle of the iPad screen, suggesting that she was referring to something on the screen (Figure 25a). As this observation appeared quite early in their exploration, it shows that the students' verbal discourses had progressed within only three minutes of interacting with the sketch.

At Turn 17, Ivy took on the role of dragging and used a series of gestures while speaking of "area" numerically. She talked of three different moments of calculating area under $f(x)=1$ from $a=0$ to 'x' as 'x' was dragged. She first dragged 'x' to $x=0$. Then, while she said "one times one is one," she used her middle finger to point towards the

draggable point 'x', which was at (1,0) at the time, followed by pointing towards the point (1,1) (Figure 25b). Finally, she dragged 'x' continuously from $x=1$ to $x=3$ and uttered, "one times two is two, one times three is three" (Figure 25c). It was clear that Ivy was stating how the area of the rectangles could be calculated numerically. The timing of her utterance corresponded to the very state of the rectangle as 'x' was dragged continuously, showing that she was coordinating her dragging and speech. She added the word "shaded" at the end of, "I guess that slope is the area of that," to clarify that she was talking about the shaded area. The fact that Ivy also used a hedge "I guess" shows that both Larry and Ivy were not very certain about their interpretations. Their limited experience exploring the sketch was a plausible reason for the observed degree of uncertainty in their communication.

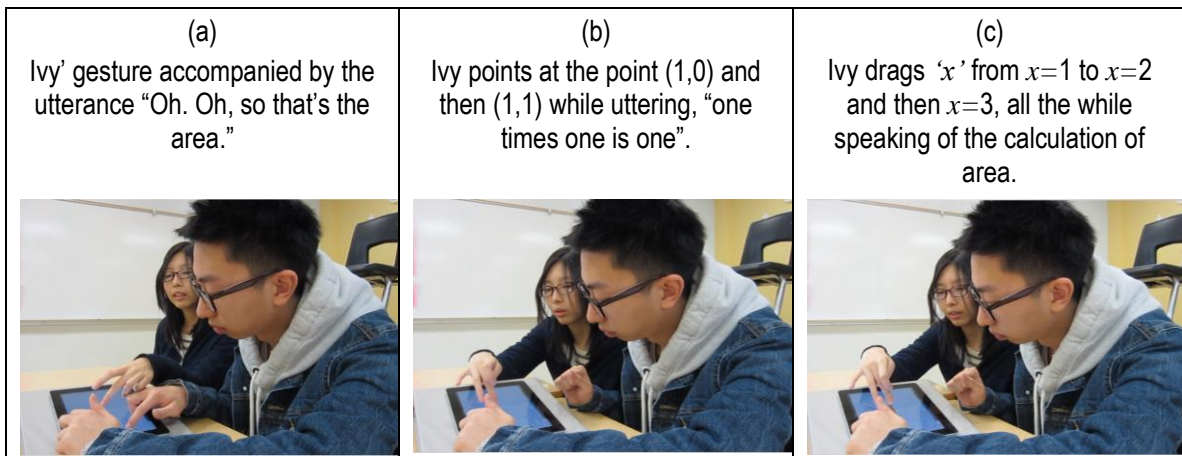


Figure 25. Selected snapshots of Ivy's gestures and dragging (Turns 15 to 18)

Ivy ended her utterance with "Try the next one. 'Kay". The significance of this utterance was that Ivy asked Larry to turn the page, and that Larry agreed to do so. This suggests that the two students were "on the same page" both literally and figuratively, figuratively in the sense that they were both ready to move on to the next page. As shown in the forthcoming analysis, they were not "on the same page" a little later.

A minute later, while they were on Page 2, which showed the function $f(x)=x$, the students noticed that the sign of 'x' and 'y' affected the sign of the "area". Turns 23 to 25 briefly illustrates the students' discourse about the sign of the "area".

Turn	Timespan	What was said <what was done>	S-er	D-er	G-er
23	3:22.5 - 3:30.9	<u>Mmm... so yeah, the area becomes negative because that's a negative 'x', right?</u>	I	<u>L</u>	I
24	3:30.9 - 3:32.0	Mmm.	L		
25	3:31.9 - 3:36.6	So <u>when it's at zero</u> , it'd be zero.	L	<u>L</u>	

The students talked about “area”, in particular, the sign of “area” in the above transcript. At Turn 23, Ivy’s speech accompanied Larry’s dragging of ‘x’ when she said, “the area becomes negative because that’s a negative ‘x’, right?” In this utterance, Ivy used the conjunction “because” to suggest a causal relationship between the sign of the area and the sign of ‘x’. Perhaps, she had noticed that when the sign of ‘x’ was negative, the area under the function $f(x)=x$ from $a=0$ to ‘x’ was negative. The use of “because” for reasoning why “the area becomes negative” was quite early, considering it was only three minutes into the students’ exploration. Her reasoning suggested that, at this point, Ivy had already assigned the green trace as a representation of area. Earlier, Ivy had told Larry to activate the “Show trace of A” button as soon as the page was opened by Larry. It seems that Ivy had known what the trace of ‘A’ meant and so had purposely asked that the trace of ‘A’ be shown on the sketch.

Turn 24 and 25 showed Larry’s response to Ivy. He first stopped dragging at Turn 24 and said, “Mmm.” Then, he resumed dragging of ‘x’, dragged it towards $x=0$, and said, “So when it’s at zero, it’d be zero.” There were two “it” in this utterance, and the syntax suggests that the two were meant to be different objects. It was very likely that Larry was naming the two mathematical objects that Ivy had talked about previously with the pronoun, “it”. This would mean that he was communicating the idea that, “when [‘x’] is at zero, [the area] would be zero”. Note that his dragging of ‘x’ back to zero shows that his speech and dragging were complementary. More importantly, the time of his speech came before he dragged, which suggests that he might be using dragging to *verify* his utterance, as opposed to using dragging to *explore* relationships. This analysis, as well as the previous one on Ivy’s speech/dragging, shows that the timing of speech and dragging could help reveal students’ thinking as well as mathematical Discourse practices.

Recall that Larry and Ivy were in agreement in the beginning about when to turn to the next page. However, the two were not in agreement when Larry wanted to turn from Page 3 (with $f(x)=x^2$) to Page 4 (with $f(x)=\sin(x)$). At 04:28, Larry was ready to turn the page when he said “Kay, and then there’s sine,” as he placed his finger on the page tab “Area under sine.” Apparently, Ivy was not ready to move on, as she said, “wait, wait”. This meant that she was hoping to stay on the page and perhaps to find out something that Larry did not know. Moments before this, Larry and Ivy had agreed that the green traces on this page were of “third degree”. Perhaps Larry was satisfied with this discovery, but Ivy was not. In fact, it was observed that their thinking began to take different turns from this point forward. Larry and Ivy began to develop incoherent discourses after Ivy’s “wait, wait”. More specifically, Ivy seemed interested in finding invariance across all pages, but Larry did not seem interested in pursuing it. Ivy’s discourse continued to grow in terms of word use and gestures, especially evident between 10:00 to 20:00, but Larry’s discourse did not change much after the 5-minute mark.

Turn	Timespan	What was said <what was done>	S-er	D-er	G-er
39	4:24.4 - 4:29.4	I guess. 'Kay, <u>and</u> then there's sine.	L	L	I
40	4:28.9 - 4:30.3	Wait, wait.	I		

The students were on the “try” page at the 06:30 mark of their exploration. They seemed to have a good grasp of all the pages, evident in their talk of area and the sign of area with respect to ‘x’ and ‘y’. However, they did not talk much of how “area is changing” in their discourses. They communicated the shape of the green traces, like “quadratic”, “third degree” etc., but they did not talk about area as *increasing* or *decreasing*. Recall that Larry talked about “zero” as the point where there is no area earlier. At about 05:25, Ivy also made a similar comment, as in, “So this basically cancels out the... negative... gets you an area of zero.” This suggests that Ivy was thinking of the “area of zero” as the point where the positive and negative areas “cancel” out. These were static ways of thinking about the zero of the area-accumulating function as opposed to something like “the area decreases and goes back down to zero,” which would pertain dynamic qualities.

During the paper-and-pencil part of the task, the students produced the graph of $y=\cos(x)$ and its corresponding area-accumulating function as shown in Figure 26. The students went back and forth between the iPad and the whiteboard, and Ivy used gestures frequently to enact the shape of the area-accumulating function. From four choices of colours, they chose the green marker to draw the area-accumulating function, which matched the colour of the green traces on the iPad.

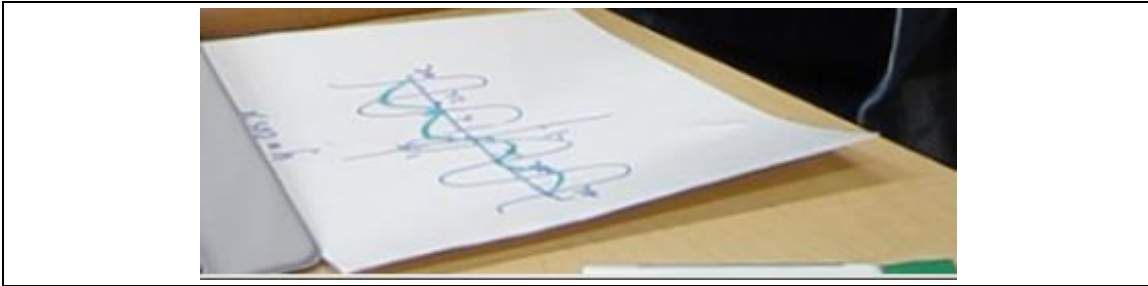


Figure 26. Larry and Ivy’s final solution for the paper-and-pencil task

The paper-and-pencil task seemed to have made the students more aware of the green trace. Both students questioned “how far” they needed to “go” when they began to draw an increasing and then decreasing function with their markers. These questions were likely initiated by the nature of the task: since they had to draw the graph physically, they wondered how far up or down to draw; in other words, they were forced to pay attention to the distance between the green point and the x -axis, which had not been noticed before. Perhaps with the technology, everything was drawn out for them, and all they did was to drag, and so they did not pay attention to it as much before. This meant that mathematical thinking is situated in the activity and the tools used. With a DGE, the students noticed the shape of the green point and explored dynamic relationships through dragging. However, the paper-and-pencil task achieved something different, as the question of “how far do I go” arose in this context. The two tasks involving different use of tools were complementary for developing the students’ discourses around area-accumulating functions, since it was important to know not only what to draw (increasing or decreasing), but how to draw (how high or low).

After drawing their area-accumulating function, Ivy noticed something “strange”, but she did not go back to change the drawing, and Larry seemed satisfied with it. The

students high-fived each other and echoed “yeah” to mark the end of their activity (Figure 27).



Figure 27. The students’ high-fived each other as Ivy said, “Yeah, we got it!” and Larry echoed “Yeah!”

7.2.2. Summary of Larry and Ivy’s speech, dragging, and gestures

Overall, Larry and Ivy’s verbal communication was clearly more developed than Huang and George’s. The total word count over their 30-minute exploration was 1862 words, more than triple the number of words spoken by Huang and George (613 words). In terms of the change of words spoken over time, a similar pattern was observed in the two pairs of students, as Larry and Ivy produced the peak number of words from 10:00 to 20:00, by more than 200 words. A similar pattern was also observed with dragging turns. The highest number of dragging turns occurred in the first 10-minute, at 47 turns; it then decreases to 23 turns and 6 turns respectively during the next two 10-minute intervals (Figure 28). This suggests that the students were mainly communicating by means other than dragging during the last 10-minute, in which they were focussed on the paper-and-pencil task. It could also be argued that, not only did the nature of the task changed, the students’ overall pattern of communication had changed, since they used a lot more words and a lot less dragging in their discourse later on. Also, it is noted that acts of drawing, which occurred mainly in the last 10-minute, were not included in the count for analysis. Hence, the students were doing something else with their hands during this interval—drawing on the mini-whiteboard with their markers—at times when acts of dragging were not present.

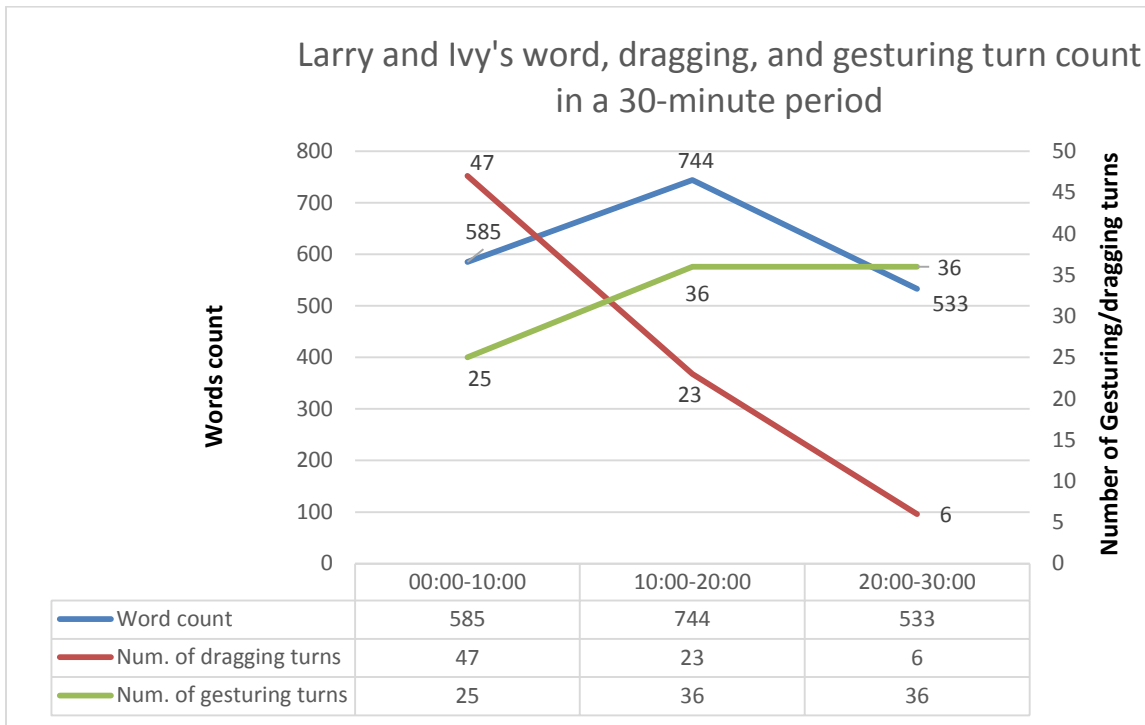


Figure 28 Larry and Ivy's word, dragging, and gesturing turn count in a 30-minute period.

An interesting observation can be made about the relationship between the number of dragging and gesturing turns over the course of Larry and Ivy's exploration. There seems to be an inverse proportional relationship between the two modes of communication: as the number of dragging turns decreased, the number of gesturing turns increased. This observation differed from the one in Huang and George's communication. For Huang and George, the number of dragging turns consistently exceeded gesturing turns in each of the intervals, by 33, 39 and 12 respectively. This shows that their use of the dragging and gesturing as modes of communication remained about the same in the exploration. With Larry and Ivy, their decreasing use of dragging was complemented by an increasing use of gesturing. These gestures included deictic gestures that accompanied deictic words as well as gestures that conveyed temporality through enacting the movement of the green traces. The increasing use of gestures by the students suggests that they were using gestures as visual mediators and routines in communication. It seemed that for the students, purely talking about it was not enough; they needed some form of visual mediator to complement their speech.

Significantly, this analysis provides evidence that having a relatively developed verbal discourse does not necessarily reduce the number of gestures used in communication.

7.3. Jay and Katie

Jay and Katie were both Korean-born students who had been studying in an English-speaking environment for two years. In the regular classroom setting, they were described by the classroom teacher (myself) as students who often used their home language (Korean) for discussing calculus ideas with each other. They were also described as a pair of students who seldom participated in classroom discussions, such as volunteering to speak after a question was raised or raising questions in front of the class. From all the participant pairings in Part II of my study, Jay and Katie were the only pair who were from the same home country (George and Huang shared a home language of Mandarin, but they were from different countries, Taiwan and China respectively). Given these backgrounds and experience, it was not surprising to observe that Jay and Katie communicated in their home language during the task. Indeed, the students spoke in Korean for the majority of task. They used some English words occasionally in their utterances during the task, and they spoke in English whenever I approached to interact with them.

7.3.1. Using a home language as a resource in communication

Table 13 shows the English words that were spoken in the midst of a discussion in Korean between Jay and Katie in the order they were first spoken. As the table shows, some words were in the mathematics register (derivative, y -axis, quadratic, sine, cosine, antiderivative) and some were in the everyday register.

Table 13 Words that were code-switched in Jay and Katie’s discussion.

Words or phrases that were code-switched (Korean to English)
area, derivative, trace, okay, next thing, three times three equals, y -axis, purpose, quadratic, sine, under, cosine, we didn’t go left, right, minus, does it have a trick, negative, antiderivative

The following transcript⁷ shows the first time that the word “area” appeared in the transcript. The student pair, Jay and Katie, had already turned the first page, which showed the function $f(x)=1$, to the second page showing $f(x)=x$. While they were on the first page, the word “area” did not appear, but they talked about “multiplication” and “width”, which suggest that they were noticing something about the rectangular region under $f(x)=1$ from ‘a’ to ‘x’.

Turn	Timespan	What was said <what was done>	S -er	D -er	G -er
16	1:20.8 - 1:25.8	What is this dot?	K	K	
17	1:24.0 - 1:25.8	[Area. Area.] ⁸	J		
18	1:25.8 - 1:28.5	Are you sure it's the area?	K		
19	1:28.5 - 1:35.7	Look... <Jay drags 'x' > [Three, times three, equals nine right?]	J	J	
20	1:35.7 - 1:41.4	It's a triangle, <u>so halve it</u> . What is the area? It's <u>four point five</u> .	J		<u>J</u>
21	1:41.4 - 1:42.4	Hm.	K		
22	1:42.4 - 1:52.0	Look. <u>Two times two</u> , area two. <u>One, one</u> , should be zero point five.	J	<u>J</u>	
23	1:52.0 - 1:55.2	One, one, is zero point five. Oh ya it is.	K		

The transcript above shows that Jay and Katie were using numerical values and the visual mediators as shown on the sketch to explore the relationship between ‘x’, $f(x)$, and $A(x)$. Their discussion could be considered highly valued in the mathematics community because it resembled two interlocutors actively engaging in advancing mathematics. It began with Katie’s dragging of ‘x’ and her questioning of “what is this dot” (Turn 16). The question was a good one for advancing the students’ mathematics discourse, and it arose likely because Katie noticed that her dragging had made the green point move in a parabolic path. Jay responded to Katie’s question by the word “area”, spoken in English (Turn 17).

⁷ Note: the video data underwent two rounds of translation/transcription to ensure validity of the process. In each round, I asked a Korean-Canadian to translate the Korean words spoken in the video into English while I transcribe the data.

⁸ Since the participants spoke in English occasionally, it was necessary to differentiate the language spoken after the translation. The actual English words spoken by the participants were written in squared brackets “[]”.

Katie prompted Jay to explain his reasoning in Turn 18, and so Jay took over the role of the dragger and began explaining in Turns 19, 20 and 22. His dragging was purposeful, as he dragged ‘ x ’ to $x=3$ to describe the product of ‘ x ’ and $f(x)$ in his English utterance, “three times three equals nine right”. Then, he changed his mode from dragging to gesturing while his discourse reflected a change from numerical to geometrical approach to area. Specifically, he used his left index finger to point towards the iPad screen near $x=3$ (the angle of the camera did not capture exactly where his finger was pointing to) during his talk of, “It’s a triangle, so halve it. What is the area? It’s four point five” (Figure 29a). In this utterance, Jay provided the formula for calculating the area of a triangle and computed the area of the triangle that was located under his finger. Katie acknowledged the calculation, and Jay continued to reason that the green dot referred to “area”, by giving two more numerical verifications in Turn 22. His verification combined dragging and speech, as he spoke about the area of the triangle at that moment in time, in between his dragging from $x=3$ to $x=2$ (Figure 29b), and then from $x=2$ to $x=1$ (Figure 29c). Katie acknowledged Jay’s explanation again in Turn 23, where this time her utterance reflected a similar structure as Jay’s utterance. A little later on, she continued to use this “when” sentence structure in Turn 28: “When it’s five... when it’s five, yes, twelve point five”. This shows that the two students had gained an understanding of the green point as “area” under the function from ‘ a ’ to ‘ x ’. They did so with a routine similar to Larry and Ivy, since both pairs of students used dragging and the numerical values displayed on the DGE to verify their conjectures. However, Jay and Katie were on the page containing the function $f(x)=x$, whereas Larry and Ivy were exploring the page containing $f(x)=1$.

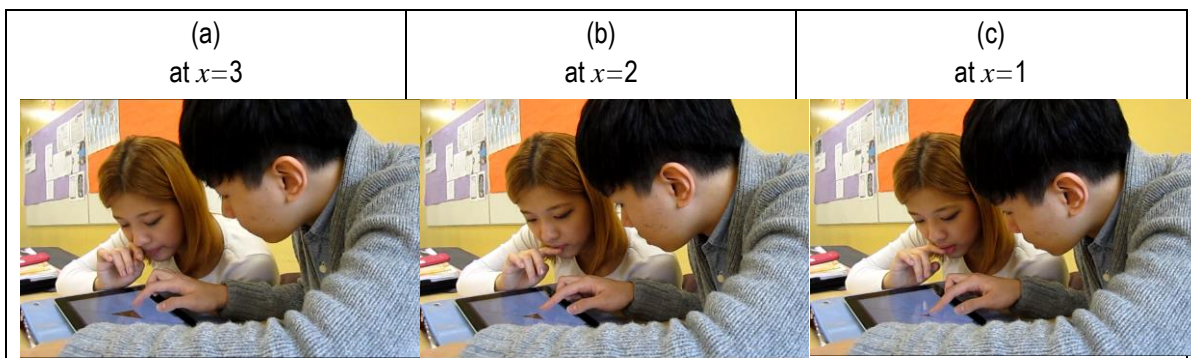


Figure 29. Jay combined dragging and speech when explaining to Katie the calculation of area of three different triangles

Jay and Katie were on a faster pace, compared with other pairs, in terms of moving their focus from the DGE to the whiteboard. They struggled for a while with Page 3 of the sketch at the beginning because they could not find a way to calculate area under a quadratic function geometrically, as well as Page 4 of the sketch when they encountered “negative area”. These were moments of commognitive conflict since their talk did not agree with what was shown on the sketch, but they were able to resolve the conflicts by observing a consistency in the sketch through dragging. About twenty minutes after their initial exploration, they had already completed the paper-and-pencil task of drawing of the area-accumulating function given $y=\cos(x)$, $a=0$. They completed the task by using a shortcut that no other pair had done. When I asked the pair to explain their drawing, Jay uttered in English:

What we did was since the cosine graph is, like shifted to, left or right, half π ... we get the same as cosine graph, we move ‘a’... to half pi, pi over two. And we use the graph provided to get the area. (20:36.0-21:02.9)

What Jay was referring to above was that they had used the sketch containing the page $y=\cos(x)$, shifted ‘a’ from $a=0$ to $a=\pi/2$, and then used the green traces obtained from dragging ‘x’ as a guide for their area-accumulating function on the “Try” page. This was mathematically correct since the accumulation of area under $y=\sin(x)$ from $a=\pi/2$ is identical to the accumulation of area under $y=\cos(x)$ from $a=0$. As their calculus teacher and a researcher, I wanted to find out more about the students’ realisation about this area-accumulating function; hence I prompted them to “explain why”. Katie provided a 40-second explanation incorporating speech and 11 counts of gestures, followed by a 20-second explanation incorporating speech and 6 counts of gestures. The transcript below illustrates her communication during this span.

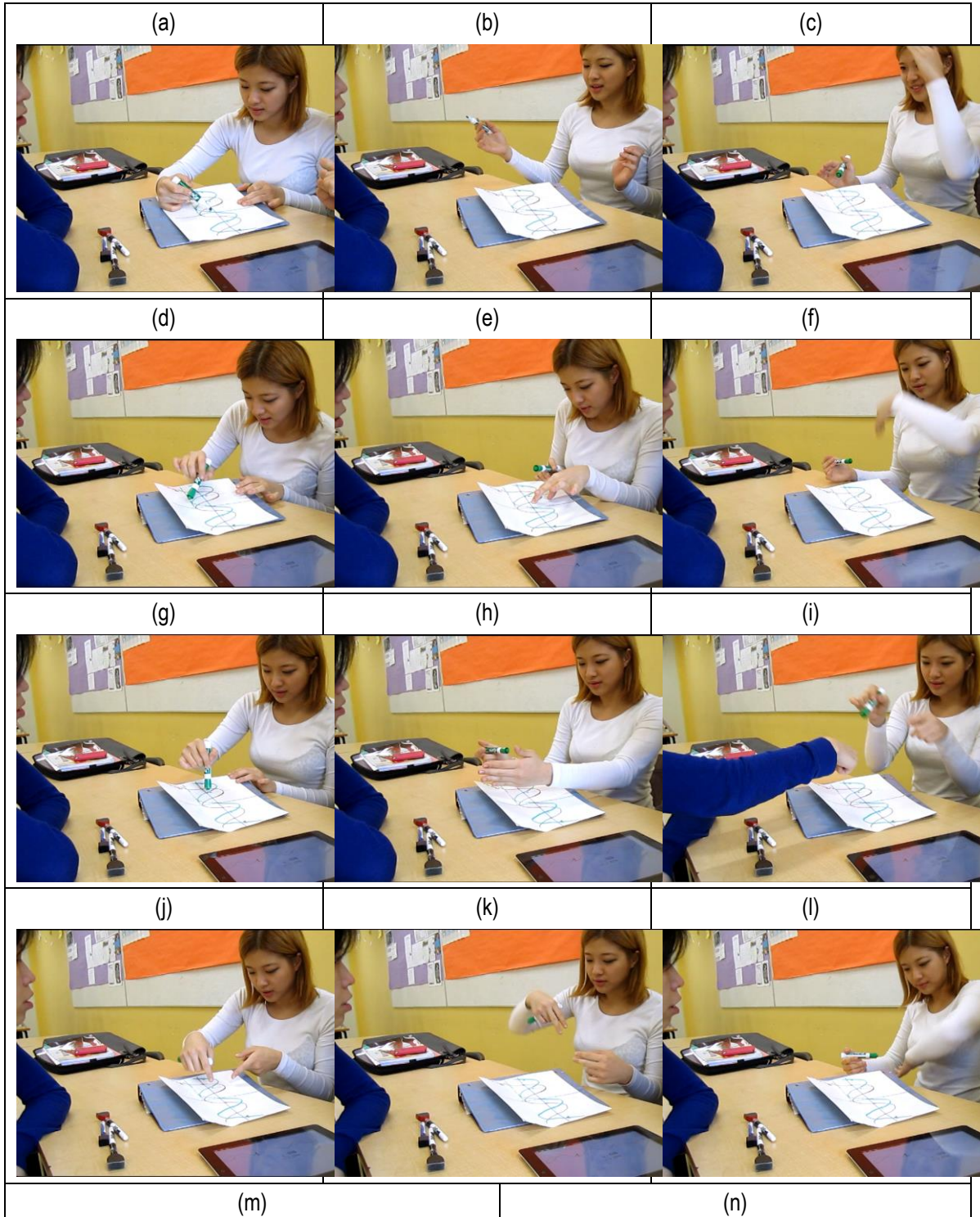
Turn	Timespan	What was said <what was done>	S-er	D-er	G-er
10	21:02.9 - 21:18.3	Ok, makes sense. Do you wanna explain... Can you explain why the green is like that though? Instead of just... <u>using the, the thing</u> , is there a reason, can you explain why does <u>it goes up and down</u> ?	O		<u>O</u>
11	21:18.3 - 21:57.2	When the cosine graph is at π , the area of <u>these, area, equals this area right</u> ? And they, somewhat <u>cancel</u> ? Each other? So the area becomes <u>zero</u> . The area of <u>this graph</u> ... kind of go <u>like this right</u> ? And <u>so it looks like this</u> . And this	K		<u>K</u>

		part, at the same rate, it <u>goes down</u> . So it looks <u>like this</u> . <u>Symmetrical</u> . And it's <u>like this</u> .			
12	21:57.2 - 22:00.3	You <u>got this part</u> , can you explain <u>this part</u> ?	O		<u>O</u>
13	22:00.3 - 22:15.8	And, <u>this</u> . For <u>this part</u> , the area <u>decreases</u> right? <u>At this rate</u> . So... And it keeps... <u>increasing</u> after.	K		<u>K</u>
14	22:15.8 - 22:17.0	After what? After...	O		
15	22:17.0 - 22:20.9	<u>After this point</u> . So...	K		<u>K</u>

At Turns 11 and 13 alone, Katie gestured fourteen times while also speaking. Of the fourteen gestures, nine of them were used as visual mediators or routines that accompanied the word “this”. These gestures communicated significant mathematical ideas, such as positive and negative areas (Figure 30a), the change of area (Figure 30c & f) at different parts of $f(x)=\sin(x)$ (Figure 30e), and the periodic behaviour of the area-accumulating function (Figure 30i). Other gestures communicated the “cancelling” of positive and negative area (Figure 30b), the symmetrical nature of sinusoidal functions (Figure 30h), and the location of which the area changed from increasing to decreasing (Figure 30j). These gestures contained so much information that it would be impossible to interpret what Katie was communicating without looking at her gestures. By examining her speech and gestures synchronously, it was possible to see Katie’s reasoning of the shape of the area-accumulating function in her discourse. First, she explained that the area “becomes” zero as $x=\pi$. Then, she gestured that the area in the interval $(0, \pi/2)$ “go like this,” and pointed to the shape of the area-accumulating function with her whiteboard pen and said, “And so it looks like this”. Having explained the interval $(0, \pi)$, she moved on to explaining the interval $(\pi/2, \pi)$ with a similar combination of speech and gesture, adding that “at the same rate”, the area “goes down”. Katie’s word use “becomes”, “go like this”, and “go down”, accompanied by her hand gestures enacting the movement of area, suggest a dynamic and temporal realisation of the area-accumulating function.

Sensing that she was finished with her explanation, I asked her to continue explaining her drawing in the interval $(\pi, 2\pi)$ at Turn 12. Upon my request, Katie explained that the area decreased, and then it would start increasing “after this point” while pointing to the point $(0, 3\pi/2)$ on the cosine graph. In summary, she was able to relate the accumulation of positive and negative area as ‘ x ’ changes from $x=0$ to $x=2\pi$. She also described the shape of the area-accumulating function as the “rate” of which

area was increasing and decreasing in lament terms. This suggests that she was thinking about *how* area was changing, for example, that area was not changing at a constant rate but at a varying rate, as shown in the movement of her gestures.



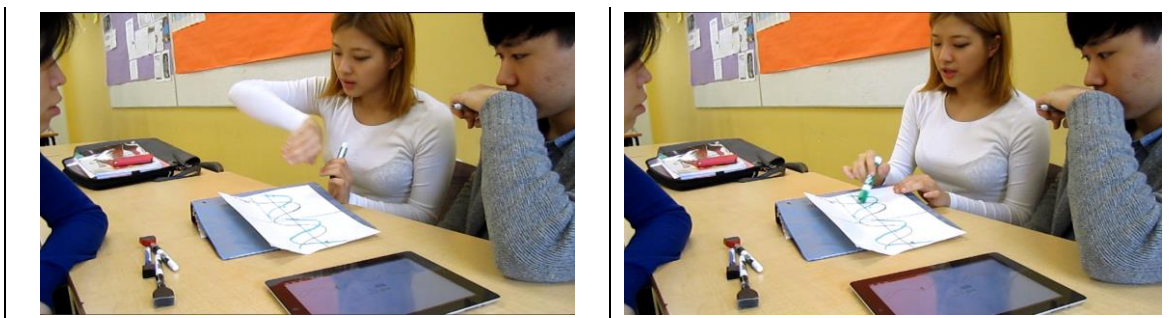


Figure 30. Snapshots of Katie's gestures during Turns 10 to 15

It looked like the two students had finished the task, when Jay became excited about a new discovery. Katie was already standing and talking on the phone after I had thanked the two for their participation, but Jay kept his hands and eyes on the iPad. He silently turned the pages over to the next upon dragging 'x' back and forth rapidly on each page. After 25 seconds of doing so, he snapped his fingers three times which drew Katie's attention.

Turn	Timespan	What was said <what was done>	S-er	D-er	G-er
37	23:29.0 - 23:55.0	<Jay drags 'x' back and forth rapidly on different pages>	NIL	J	
38	23:55.0 - 24:07.5	<Jay snaps his fingers three times> <u>Look</u> . If this is <u>degree one</u> , then the [area] is degree two. <Jay turns to Page 3>	J	<u>J</u>	<u>J</u>
39	24:06.1 - 24:07.0	That's true.	K		
40	24:07.5 - 24:14.5	<Jay drags 'x' > <u>If you find the [derivative] of degree three, then you get degree two.</u> <Jay turns to Page 4>	J	<u>J</u>	
41	24:14.5 - 24:19.2	[d]?	K		
42	24:15.5 - 24:23.4	If you take the [derivative] of something then you get [sine].	J		
43	24:18.4 - 24:31.6	[Derivative] of something. Then if this graph is [negative cosine] then this is right.	K		
44	24:19.2 - 24:30.0	Derivative of something, <u>it is [negative cosine].</u>	J	<u>J</u>	
45	24:31.6 - 24:38.3	<u>It is [negative cosine], right! Right!</u>	K	<u>J</u>	
46	24:38.3 - 24:49.5	<Katie grabs the whiteboard> If you take the [derivative] of something, you get [cosine x], that is [negative sine x], no no [sine x] it is. <Jay turns to the "Try" page>	K		
46	24:49.5 - 24:51.5	<u>Yes...</u> <Katie drags 'x' rapidly>	K	<u>K</u>	
47	24:51.5 - 24:57.0	Antiderivative. <Jay and Katie high-fived each other>	K		J/K

The discussion above shows Jay and Katie's new discovery about the area-accumulating function and their excitement over this discovery. At Turn 37, Jay snapped

his fingers as if he had noticed something significant. His rapid dragging of 'x' during this time suggests that he was no longer observing the discrete location of each of the green traces, but he was likely attending to the final shape of the green traces, perhaps because dragging rapidly allowed him to see the shape sooner. This dragging routine suggests that Jay was thinking of the set of green traces as one—the graph of the area-accumulating function. This analysis highlights Jay's "rapid dragging" routine as evidence of his encapsulation of the set of points $(x, A(x))$ into a singular object, the area-accumulating function, $A(x)$. Using this "rapid dragging" routine, he identified the relationship between $f(x)$ and the area-accumulating function $A(x)$, in that the former was the derivative of the latter. At Turns 38, 40 and 42, he verified this newly discovered relationship with Katie, using gestures to specify $f(x)$ and "rapid dragging" to trace the shape of $A(x)$.

It was observed that Katie was quick to react to the relationship proposed by Jay. At Turn 43, she used this relationship between $f(x)$ and $A(x)$ to predict that $A(x)$ needed to be "negative cosine x " for the relationship to hold true. She said this before Jay dragged 'x' rapidly to reveal the final shape of the area-accumulating function, and she was quite excited to see that her prediction was correct upon Jay's dragging, exclaiming, "it is [negative cosine], right! Right!" From there, Katie picked up the whiteboard showing two graphs in black and green and used the same relationship to predict the shape of $A(x)$, given $f(x)=\cos(x)$ and $a=0$ (Figure 31). Without saying the word antiderivative, she successfully communicated the idea that the antiderivative of cosine was sine in her utterance, "If you take the [derivative] of something, you get [cosine x]" and "[sine x] it is" (Turn 46). In terms of mathematical Discourse practices, Katie was actively predicting and verifying the relationship about the two graphs that Jay had found. This shows that Katie had also developed her discourse by encapsulating the set of points $(x, A(x))$ into an object—the area-accumulating function. The students high-fived each other at the end of the selected episode which, again, marked their excitement over what they had found.

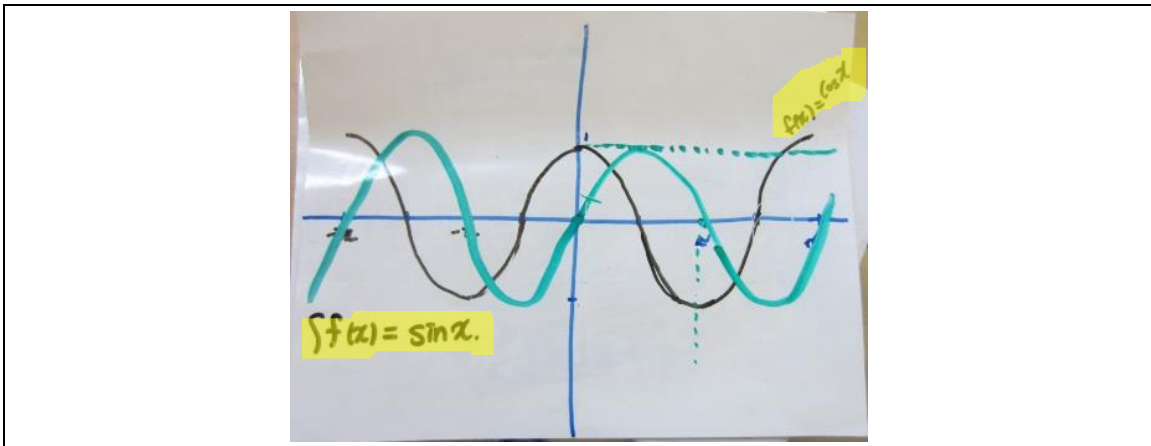


Figure 31. The graph of the area-function drawn by Jay and Katie on the whiteboard; the highlighted part was added a few minutes after Turn 15

7.4. Katie and Ivy

I had initially arranged for Jay and Katie, and Larry and Ivy to work in pairs for the task on the day that the four students came to participate in the task. After about 30 minutes of working in the task with their original partner, both pairs were finished with the task, and so I asked both pairs to switch partners so that they could “chat about” what they had explored with their new partner.

7.4.1. Communicating different realisations about area-accumulating functions

The following is a 95-second episode taken from Katie and Ivy’s discussion about what they had “found” during their explorations with their respective partners.

Turn	Timespan	What was said <what was done>	S-er	D-er	G-er
1	7:34.3-7:35.6	Did you find it?	K		
2	7:38.6-7:40.1	Yeah... What did you find	I		
3	7:41.5-7:48.9	<u>This graph</u> is the derivative of <u>this graph</u> . <Katie taps “Page 2”> <u>This graph</u> is the derivative of <u>this graph</u> .	K		<u>K</u>
4	7:48.9-7:49.5	Oh	R		
5	7:49.5-7:57.3	<u>So, the sine graph is the derivative of negative cosine ‘x’ right?</u> <Katie drags ‘x’>	K	<u>K</u>	

6	8:00.0-8:02.6	Oh. Ok. That explains it.	I		
7	8:02.6-8:13.7	<Katie taps the “Try” page> And the... cosine graph... is the derivative of <u>sine graph</u> right? So... it's like this. The sine graph.	K		<u>K</u>
8	8:14.9-8:20.1	Ya. Ok. <Katie chuckles>	I		
9	8:20.5-8:48.6	Well, we were, we were actually looking at area? So ‘x’ and ‘y’. And then <Ivy taps “Page 1”> So here, both ‘x’ and ‘y’ are <u>positive, the area is...</u> <Ivy drags ‘x’> <u>going up</u> . And then, ‘x’ is negative <Ivy drags ‘x’>, but ‘y’ is <u>positive</u> , so the area is negative. <Ivy taps “Page 2”> It works the same for all of these. <u>That’s how we graphed ours</u> . <Ivy drags ‘x’> <u>If they are not... like the same sign, then it’s going down</u> . <Katie nods>	I	<u>I</u>	<u>I</u>

At the start of the transcript, Ivy and Katie took turns to discuss what they found in their previous exploration. Although both students said that they had “found” something, their discourse indicated two very different realizations about area-accumulating functions. Katie began by providing two examples; she used “Page 1” and “Page 2” of the sketch to state the relationship between $f(x)$ and the green traces, that the former is the derivative of the latter. She also used gestures to realize both functions geometrically, while she uttered, “*this* graph is the derivative of *this* graph” on both pages (Figure 32a & b). In Sfard’s terms, her gestures were instances of *actual realization*, as they were performed in the presence of the signifiers. In her third example, she turned to “Page 4” of the sketch which showed the sine function. Unlike her previous communication incorporating gestures, here she used dragging to show the movement of the green traces. Since Katie and her partner had previously set $a=\pi/2$, the green traces were in the shape of: $A(x) = \int_{\pi/2}^x \sin t dt = -\cos x$. Hence, Katie asks Ivy rhetorically, “So, the sine graph is the derivative of negative cosine ‘x’ right?”

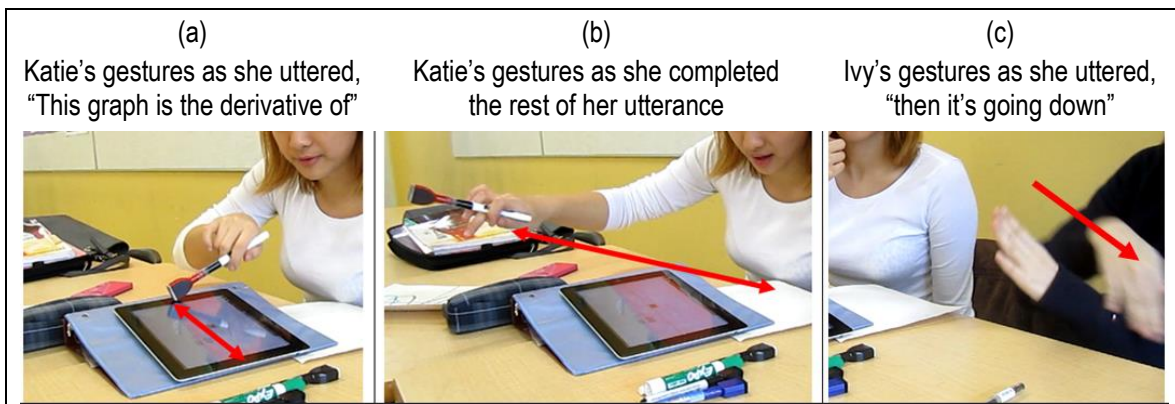


Figure 32. Selected snapshots of Katie and Ivy’s gestures (Turns 1 to 9)

It can be observed that Ivy did not see what Katie saw in the sketch. What she had “found” seemed to be something different, evident in her statement, “Oh, that explains it,” and the expression “oh” in two occasions. After a long pause at the end of Katie’s last remark, Ivy explained that she was “actually looking at area” followed by a gesture in which the thumb and index finger mimicked the act of measuring something horizontal (as she uttered ‘ x ’) and then something vertical (as she uttered ‘ y ’). She referred to the green point as “going up” on “Page 1” as she dragged ‘ x ’. The verb “going up” suggests that she was thinking of area-accumulation as the process of plotting the ordered pairs $(x, A(x))$. Although she saw $(x, A(x))$ as a reified, discursive object, she did not see the set of all $(x, A(x))$ —the graph of $A(x)$ —as a compound discursive object. In contrast, Katie’s use of the singular form “this graph” suggests that her realization of area-accumulating function was the set of all ordered pairs. She was referring to the “graph” as a new discursive object through *encapsulating* all ordered pairs $(x, A(x))$. Later, Ivy used the process of *saming* to show that her reasoning of the sign of ‘ x ’ and ‘ y ’ “works the *same* for all of these. That’s how we graphed ours”. She ended with, “If they are not... like the same sign, then it’s going down” (Figure 32c). Altogether, Ivy had explained how to determine the movement of the green trace by looking at the sign of ‘ x ’ and ‘ y ’. This seemed to be what she meant by the *same* for all the pages.

After this discussion, I came to check in with the new pair Katie and Ivy. I prompted the students to explore the meaning of ‘ a ’, since they had not talked about it yet. The students explored the sketch silently for about three minutes and did not seem to make any progress. Jay (who was Katie’s original partner) joined the discussion with Katie, Ivy and myself. The students were exploring the effect of dragging ‘ a ’ on “Page 1” when I asked the students to try dragging ‘ a ’ and then try dragging ‘ x ’. At this point, the sketch was showing a set of vertical green traces (obtained by their previous dragging of ‘ a ’) and two sets of parallel green traces (obtained by their previous and most recent dragging of ‘ x ’). Figure 33 shows a sample screenshot of the sketch at this point of their discussion.

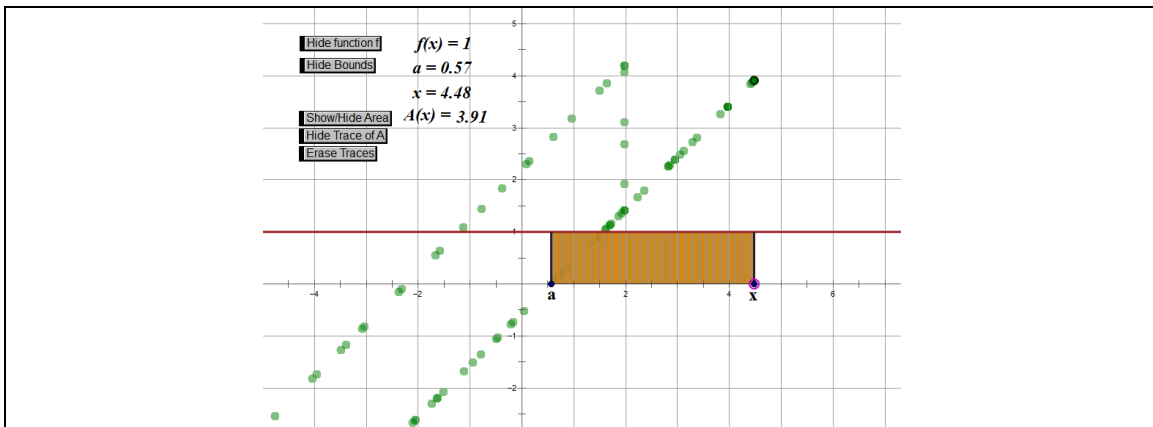


Figure 33. Sample screenshot of a sketch showing two sets of green traces, both representing the antiderivatives of $f(x)=1$

Katie and Ivy noticed that there were two sets of parallel green traces. I wanted to prompt the students to see that no matter where they set 'a', the graph of $A(x) = \int_a^x 1 dt = x + C$. In other words, the two sets of green traces are both antiderivatives of 'f' that differ by a constant 'C'. Then, the following discussion unfolded.

Turn	Timespan	What was said <what was done>	S-er	D-er	G-er
1	3:08.8-3:19.3	Ok. So... If they are parallel... Let's see, if they are <u>parallel</u> , does that still... Like your original one was ' <u>a=0</u> ', right? What, what would that look like? Does it look similar to <u>that one</u> ? <u>Those two</u> ?	O		<u>O</u>
2	3:21.5-3:24.3	Ya, <u>it looks the same</u> .	I		<u>I</u>
3	3:24.3-3:25.7	It's the same... same what? Same...	O		
4	3:26.1-3:27.2	Oh...	K		
5	3:29.1-3:31.3	So they are all going to be like <u>that</u> .	O		<u>O</u>
6	3:31.1-3:36.9	They are going to be like <u>this</u> , but the <u>y...intercept</u> gonna be different.	K		<u>K</u>
7	3:35.9-3:50.0	Ok. Ya. Do you guys agree? They are all gonna be like... And so... what about the derivative thing? Does it still work? Or not? You said the, the derivative thing.	O		
8	3:49.9-3:57.6	It still works. Except that... <long pause> It does.	K		
9	3:57.7-4:01.9	It does? So you mean <u>these ones right</u> ? The derivative of <u>these ones</u> will still be...	O		<u>O</u>
10	4:02.5-4:11.7	'Cause, for <u>these, these two graphs</u> , the y-intercept is ' <u>c</u> ' right? Constant. They will be <u>cancelled</u> .	K		<u>K</u>
11	4:19.0-4:22.8	Do you want to say more? Tell them what you think.	O		
12	4:22.1-4:43.4	Like for <u>this graph</u> , it will be ' <u>y=x+4</u> ' or something like that. And this graph, ' <u>y=x+8</u> ' something like that right? And the derivative	K		<u>K</u>

		of those graphs will be 'y=1' right? 'Cause, constant will be cancelled out.			
13	4:45.3-4:48.2	Constant will be what? Constant will be what?	O		
14	4:47.5-4:51.6	Zero. <K writes the numeral '0' on the whiteboard>	J/K		
15	4:54.5-5:03.5	<u>So do I set 'a' somewhere else? Ya right.</u> <Jay drags 'a' then 'x' slowly>	J	↓	

In the above transcript, I asked the students to recall what the green traces would have looked like with the “original one”, where $a=0$. Ivy responded, “it looks the same”, and Katie added that, “they are going to be like that”. Using her hand gestures to visually mediate slope, Katie was able to explain that the two graphs would be parallel, “but the y ...intercept gonna be different.” I revoiced Katie’s comment and asked the students to think about whether “the derivative thing” would “still work”. Katie said it would still work because, “These two graphs, the y -intercept is ‘ c ’ right? Constant. They will be cancelled.” This was a turning point in the discussion for Jay and Katie, as they both seemed to have found that the relationship that they previously found continued to hold true for all values of ‘ a ’. After this transcript, they continued to explain to Ivy by writing equations $y=x+4$ and $y=x+6$ on the whiteboards. For example, Jay explained that the red line was the “ dy/dx ”, followed by Katie’s “always the same”.

The episode shows the importance of the process of encapsulation for the learning of antiderivatives. Throughout the discussion, my realization of the area-accumulating function was one where the set of ordered pairs $(x, A(x))$ was encapsulated into a graph as a whole. I frequently asked the students to compare the two sets of parallel green traces and the “original one”, $y=x$. Jay and Katie took much time in order to shift their discourse. In the beginning, they did not attend to the shape of the green traces. When prompted, Katie began to talk of the two sets of green traces as “they”. The use of the pronoun in the plural form suggests that Katie was thinking of the green traces as an encapsulation of two sets of $(x, A(x))$. This seemed to allow her to compare the shapes of the two “graphs” and use *saming* to refer to the “graphs” algebraically, by writing equations like $y=x+4$ and $y=x+8$ on the whiteboard.

On the other hand, Ivy only made one comment in the discussion: “it looks the same”. It is possible that her realization of area-accumulating functions influenced her

participation in the discussion. The previous analysis shows that Ivy’s realization was in relation to discrete ordered pairs $(x, A(x))$. These ordered pairs were reified in the sense that they were “timeless” stories about relations between objects. However, she did not see the graph of $A(x)$ as a singular object by encapsulating the set of all ordered pairs $(x, A(x))$. Unlike Katie, who frequently mentioned “this graph”, Ivy never used the word “graph” as a noun to signify the area-accumulating function. Her only use of the word “graph” was in “that’s how we *graphed* ours”—a verb. Not able to see area-accumulating function at the object level may have affected both Ivy’s learning and her participation in the discussion with Jay, Katie and myself.

7.5. Sam and Mario

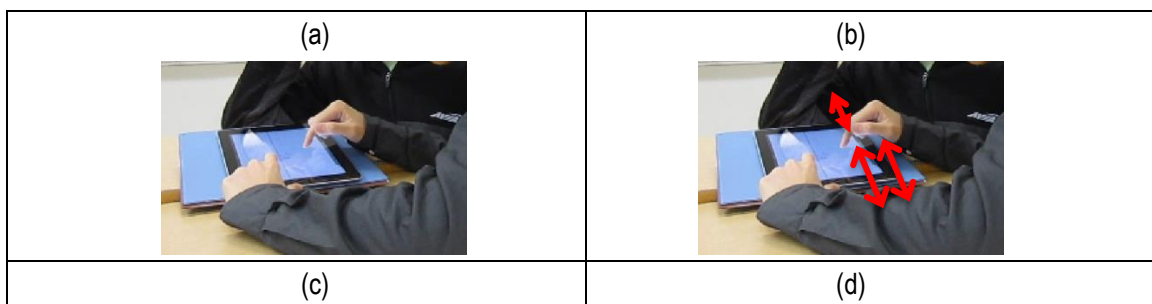
Like Larry and Ivy, Sam and Mario did not share a home language, but in contrary to Larry and Ivy, they had the least experience of studying in an English-language environment. Hence, it could be said that Sam and Mario had the least linguistic resources available to them, since English was the only language they shared in common and code-switching to their home languages was not possible. In this section, I provide a detailed analysis of the use of speech, gestures and touchscreen-dragging in their developing discourse around area-accumulating functions. In particular, I focus my analysis on the first ten minutes from a total of thirty minutes of data collected on the student pair. The chosen data is further divided into three sections for identifying themes in each.

7.5.1. Questioning and communicating mathematically through dragging

The excerpt below revolves around Sam and Mario’s first two-minute interaction with the sketch conveying area-accumulating functions. At the start, all buttons were in the *Hide* position.

Turn	Timespan	What was said <what was done>	S-er	D-er	G-er
1	0:00.0 - 0:03.0	Show function? <Sam presses the <i>Show Function f</i> button>	S		
2	0:03.0 - 0:07.3	Just show everything. <Sam presses the <i>Show Bounds</i> button>	M		

3	0:07.4 - 0:18.1	So basically, <u>we have two x-values</u> , here is one, and then. <Sam presses the <i>Show Area under f</i> button>	S		<u>S</u>
4	0:18.1 - 0:21.4	Ok, it's an area	S		
5	0:21.6 - 0:25.0	<u>What, what?</u> <Sam tries to drag the green point and the page moves>	S	<u>S</u>	
6	0:30.0 - 0:34.5	What's <i>Trace of A</i> ?	M		
7	0:34.5 - 0:36.3	<u>How do we drag this trace?</u> <Sam tries to drag the green point>	S	<u>S</u>	
8	0:36.3 - 0:39.0	Are we supposed to?	M		
9	0:39.1 - 0:46.8	<u>What? Oh, oh, it's this one. Ok, makes sense.</u> <Sam drags x horizontally>	S	<u>S</u>	
10	0:46.8 - 0:50.0	<u>Hm? What the?</u> <Sam drags a horizontally>	S		
11	0:50.0 - 0:53.0	<u>Is that the area?</u>	M		
12	0:53.0 - 1:03.4	<u>Wait, I, I, I don't get this. You see? As we drag this... the... the area becomes...</u>	S		
13	1:11.0 - 1:13.2	<u>What the?</u> <Sam drags x from one side of a to the other side>	S		
14	1:11.0 - 1:13.2	Get <u>this...</u>	M		<u>M</u>
15	1:13.3 - 1:16.7	Ok, I was actually shocked.	S		
16	1:13.3 - 1:16.7	<u>Get this to... ah...</u> <Mario drags x horizontally>	M	<u>M</u>	
17	1:16.7 - 1:26.0	<u>You see how this one moves? So it's like the area.</u>	S	<u>M</u>	<u>S</u>
18	1:26.0 - 1:32.1	<u>Oh... What?</u> <Sam drags a horizontally>	S	<u>S</u>	
19	1:31.5 - 1:34.9	What? I don't understand this.	S		
20	1:34.9 - 1:38.0	<u>Let's drag it down. You can't drag it down.</u> <Mario tries to drag the green point down>	M	<u>M</u>	
21	1:38.0 - 1:38.6	What is this?	S		
22	1:38.7 - 1:41.8	No you can't. You can only go like...	S		<u>S</u>
23	1:41.8 - 1:46.0	<u>I don't understand. Do you understand this?</u>	S	<u>S</u>	
24	1:46.0 - 1:46.7	<u>No.</u> <M shake head>	M		



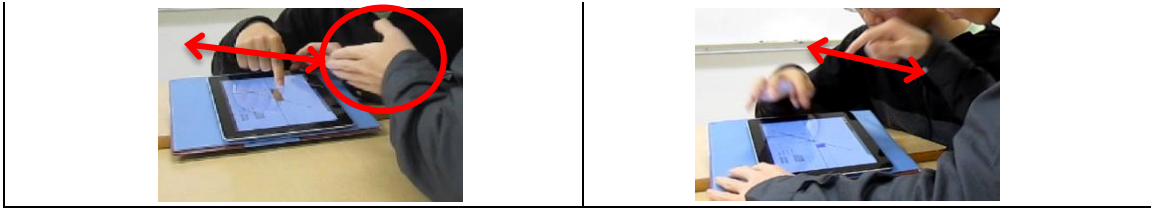


Figure 34. Selected snapshots of Sam and Mario’s gestures and dragging during Turns 1 to 24

The episode above highlights the way Sam and Mario made use of gestures and dragging to question and communicate mathematically. The students seemed unsure about what to do with the points ‘ a ’, ‘ x ’ and the green point initially. They questioned the functions of the DGE with question markers “what” (eight times) and “how” (once) in the first two minutes of the episode. Most of these questions were formulated as Sam used the dragging modality to investigate the behaviour of different points. For example, in Turn 5, Sam asked “what” repeatedly as he tried to drag the green point which was not draggable, and the whole page was moved incidentally. Although he had acknowledged that he had previously pressed a button which showed “an area” (Turn 4), he had not realised that the green point had plotted the area in terms of ‘ x ’, evident in his questions, “what’s trace of A ” (Turn 6) and “how do we drag this trace” (Turn 7). Then, upon dragging ‘ x ’ around, he finally concluded, “oh, oh, it’s this one. Ok, makes sense,” (Turn 9). At this point, his *commognitive conflict* seemed to be resolved perhaps because his dragging of ‘ x ’ made the green point move; hence he realised the green point was a dependent non-draggable object. However, it appeared that he remained unsure about what the green point meant, stating that he did not yet “understand” in Turn 23.

From Turn 9 to 18, Sam and Mario took turns dragsturing in a conversation-like manner, beginning with Sam’s dragsturing, which spanned 30 seconds, from Turn 9 to 13 (Figure 34b). During this occurrence, Sam dragged ‘ x ’, then ‘ a ’ and finally ‘ x ’ again. Observing the students’ word use and dragging actions, it seemed that both students made some progress in their learning about area-accumulating functions during this span. For example, in Turn 11, Mario asked, “Is that *the* area” as Sam dragged a around to leave some green traces that were vertical. This was the second time the word “area” appeared in the transcript, and it was used differently from the first usage in Turn 4. The word “area” was first used when Sam pressed the “Show Area under f ” button and

uttered, “it’s *an* area” (indirect article “an”). Since Sam simply pressed the button and did not use any dragging to change the area, it is suggested that his realisation of area was static in Turn 4. In contrast, Mario’s utterance seemed to suggest that he might be thinking about area in a dynamic sense. This is indicated by the way Mario referred to the green point as “*the area*” (direct article “the”) while the shaded area was changing dynamically upon Sam’s dragging.

In Turn 12, Sam uttered, “As we drag this, the area *becomes...*” while he dragged ‘x’. This utterance-dragging combination suggests that Sam was also thinking about area as having dynamic qualities. It shows how dragging mediated the way Sam thought of the area as becoming. The use of “as...becomes” implied something was happening, in particular, the area was changing as ‘x’ was dragged. Furthermore, Sam’s statement structure resembled an “if... then...” statement structure which called upon a causal or functional relationship between ‘x’ and the area. It was interesting to note that Sam never finished his sentence after uttering “becomes”. Since Sam mentioned that he did not “understand” in the last part of the excerpt, it is speculated that Sam did not finish his sentence because he had yet to *realise*, in a Sfardian sense, the simultaneous change in the variables despite noticing the area is changing. Similarly, Mario used a hedge word in his utterance, “it’s *like* the area,” suggesting a degree of uncertainty about whether or not the green traces meant the area.

Different draggers and speakers were observed concurrently in the episode. In Turn 17, as Mario dragged ‘x’ back and forth, Sam was responding verbally and simultaneously, “You see how this one moves? So it’s like the area.” A similar exchange was also noted in Turn 11, where Sam was the dragger, as Mario spoke, “Is that the area?” These two instances where the dragger and speaker were different people seemed effective for creating a mutual and simultaneous communication. Although it may seem impolite and unconventional for one student to “talk over” another student, the presence of “talking over someone else’s dragging” was not an issue here. Indeed, Sam’s utterance did not interfere with Mario’s dragging and vice versa; rather, from the way one talked about area while the other was dragging, they seemed to have made significant progress as a result of this concurrent communication.

Also observed was the consistent use of gestures in mathematical communication. Namely, Sam used three types of gestures, which in Sfard's terms, functioned quite differently in each usage. In Turn 3, Sam used a pointing gesture as he talked about the bounds to make sure both interlocutors *spoke about to talk about the same mathematical object* (Figure 34a). In Turn 17, he used his hand to signify the linear pattern of the green traces, an instance of *actual realisation* (Figure 34c). Finally, in Turn 22, he flipped his right index finger left and right (Figure 34d) while uttering, "No you can't. You can only go like", which was another actual realisation of the possible movement of the green point. Moreover, this gesture was not accompanied by any speech, which suggests that Sam relied on gestures as a visual mediator in his mathematical discourse to communicate in the absence of word use.

7.5.2. Exploring variance and invariance and conjecturing

Immediately following the first excerpt, the students continued to explore the sketch for another 2 minutes and 30 seconds, as seen in the transcript below.

Turn	Timespan	What was said <what was done>	S -er	D -er	G -er
26	1:46.7 - 1:52.3	Erase trace. <M presses <i>Erase Trace</i> button>	M	--	--
27	1:46.7 - 1:52.3	<u>The area.</u>	S	<u>M</u>	--
28	2:02.2 - 2:06.1	<u>Is it how the area is changing?</u> <S dragged x back and forth>	S	<u>S</u>	--
29	2:06.1 - 2:11.5	<no speech>	--		--
30	2:11.5 - 2:17.6	<no speech>	--	<u>M</u>	--
31	2:17.8 - 2:21.0	<u>Are we supposed to learn something?</u>	M		--
32	2:17.8 - 2:21.0	This is pretty hard. <S opens the 2 nd page of the sketch and immediately pressed the first two Hide/Show buttons>	S		--
33	2:28.5 - 2:33.0	Oh this one is a little bit...	S	--	--
34	2:28.5 - 2:33.0	Same thing.	M	--	--
35	2:33.0 - 2:36.8	Same thing, but then, <u>a little bit different.</u> <S uses his right index finger to drag x around and then uses his left index finger to press the <i>Show Area under f</i> and <i>Show Trace of A</i> buttons>	S	<u>S</u>	--
36	2:36.8 - 2:42.8	<u>Are we supposed to move <i>Trace of A</i>?</u> <S drags x and then a>	M	<u>S</u>	--
37	2:42.8 - 2:49.8	<u>I think we are supposed to move</u> <inaudible>.	M		--
38	2:49.8 - 2:56.2	<u>The thing is, no matter how you move, <u>this one</u>, if you move the <u>a</u>, it's, it's always goes like this.</u>	S		<u>S, S</u>

39	2:56.2 - 3:02.2	And then, <u>if you move this one... oh, wait a sec.</u> <S drags x>	S	<u>S</u>	--
40	3:02.2 - 3:09.5	<u>This is actually, the</u> derivative of the graph, function.	S		<u>S</u>
41	3:09.8 - 3:14.0	No, I don't know...	M	--	--
42	3:14.0 - 3:27.7	You see here, <u>this is probably x,</u> x squared, right? And we have a line here, that line, is probably the derivative of, of <u>x squared.</u>	S	--	<u>S</u>
43	3:27.7 - 3:32.5	Is it? It's not.	S	--	--
44	3:27.7 - 3:32.5	<M nods head>	--	--	--
45	3:32.5 - 3:37.3	<u>Is this x squared?</u>	M	--	<u>M</u>
46	3:32.5 - 3:37.3	Oh ya, let me see.	S	--	--
47	3:37.3 - 3:44.0	<u>This is one, two.</u>	S	--	<u>S</u>
48	3:44.0 - 3:45.8	x squared divided by two.	M	--	--
49	3:45.8 - 3:48.7	<u>The line is basically,</u> y equals to x right?	S	--	<u>S</u>
50	3:48.7 - 3:51.8	y equals x.	M	--	--
51	3:48.7 - 3:51.8	Ya, see? y equals x.	S	--	--
52	3:51.8 - 3:57.6	So the... graph thing we get is...	S	--	--
53	3:51.8 - 3:57.6	x squared divided by two, <u>this graph.</u>	M	--	<u>M</u>
54	3:57.6 - 4:04.4	<u>One over two, no, x squared over two.</u>	S	--	<u>S</u>
55	4:04.4 - 4:05.0	Ya.	M	--	--

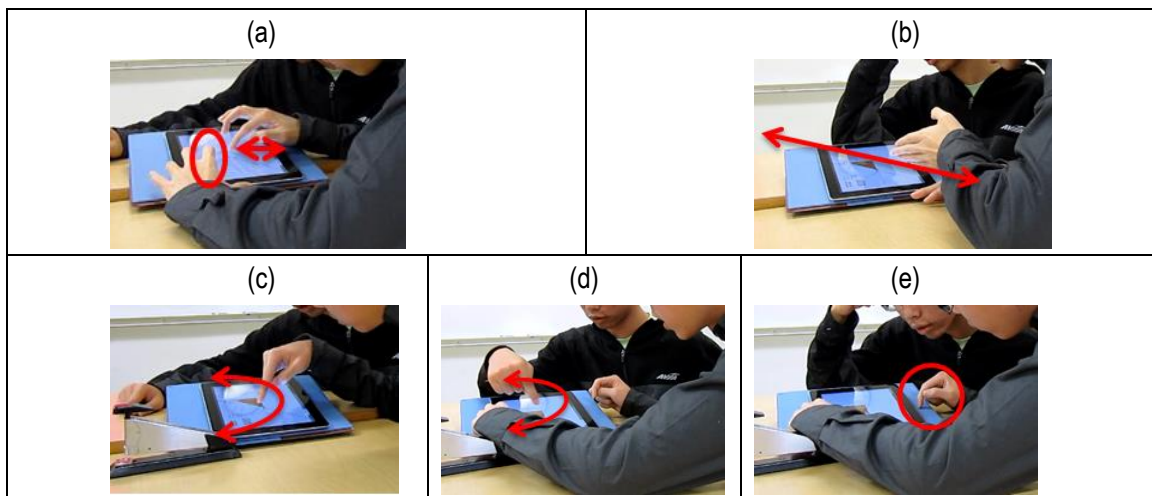


Figure 35. Selected snapshots of Sam and Mario's gestures and dragging during Turns 26 to 55

Like the first excerpt, Sam and Mario consistently utilised gestures and dragging in different parts of this excerpt to communicate mathematically. The dragger column of

the transcript clearly shows that in the first half of the transcript (Turn 27-39), the students' communication was dominated by dragging, with or without speech. For example, in Turn 28, Sam was dragging while he asked the question, "Is that how the area is changing?" Unlike the first excerpt, Sam was able to describe exactly that *the area is changing* with no hedge words in this excerpt. He continued to drag for a span of 9 seconds without speech before letting Mario also tried dragging for another 6 seconds without speech (Turn 29-30). The switching of draggers suggest that both students were engaged in some interpersonal or intrapersonal mathematical communication while dragging. This was also interesting because it seemed as though, for Mario, it was not enough to see Sam dragged; he had to do it himself too. If speech was analysed alone, some important analyses about the students' thinking in between speech would have been missed.

Through word use, gestures and dragging, the students demonstrated valued mathematical Discourse practices such as *exploring* and *conjecturing*. My analysis of this excerpt shows that the students paid more attention to the green traces compared to the first excerpt. For example, they used words like "it" and "this" to refer to the green trace and verbs to describe its motions throughout this excerpt. The student pair used the words "move" five times, and "goes" and "changing" once each to talk about the state of the sketch as they dragged either 'a' or 'x'. This shows that the student-pair moved from questioning the technology to exploring and describing the dynamism shown in the sketch. Thus, it can be said that the students had shifted their discourse from the *act of dragging* earlier to the *meaning of dragging*. In addition, the students used the *Hide/Show* buttons and dragged points 'a' and 'x' purposefully without struggling with their functions as they did previously. This can be observed in Turn 35 (Figure 35a) when Sam used his right index finger to drag 'x' around and then his left index finger to press the *Show Trace of A* button. It seemed like Sam was checking if 'a' was draggable before he pressed the next button. Sam's coordination of two fingers from different hands to interact with the *Dragging* tool and the *Hide/Show* button rather seamlessly suggest that Sam had begun to use the technology meaningfully for *exploring* the meaning of the sketch.

A little after this discussion, Sam and Mario performed a series of hand gestures from Turn 40 to 54. Initiated by his own dragging of x , Sam remarked that, “oh, wait a sec. This is actually, the derivative of the graph, function” (Turn 40) while he used a hand gesture to signify the shape of a linear function (Figure 35b). To restate what he had said, he then used his index finger and traced a “U” shape in the air as he continued to conjecture that the line was “probably the derivative of x , x -squared” (Turn 42, Figure 35c). Mario responded with a similar “U” shape gesture as he asked, “is this x -squared” (Turn 45, Figure 35d). These gestures and word use pairings provide evidence that the students were engaging in *conjecturing* about the shape of the green traces. In addition, they helped identify Sam and Mario’s competence in the mathematical activity.

As the students conjectured about the relationships between the two graphs, their mathematical discourse become more developed in both in geometric and algebraic terms. This can be evidenced through their word use accompanied by different kinds of gestures. One kind of gestures was a kind of hand gestures mimicking the geometrical shape of the functions, as used by both students on three occasions. In Turn 40, Sam aligned his fingers and palm together and gestured a line in the air, as he uttered, “*This is actually*, the derivative of the graph, function.” Since his gesture was performed simultaneously with the utterance “this is actually”, it is suggested that Sam was referring to the linear function. In addition, Sam and Mario both used similar hand gestures to trace a “U” in the air to refer to the parabolic green traces on the page. These gestures mimicking the shape of functions revealed the students’ geometrical realisation of derivatives, that the “line” (Turn 49) was the derivative of the parabolic “graph thing” (Turn 52). Besides talking in geometrical terms, the students mentioned some algebraic expressions such as “ x -squared” and “ y equals to x ” as well in the same discussion about the relationships of the two functions. With respect to gestures, Sam used a kind of *scribing gesture* in which his index finger enacted a pen as if he was writing something on the table (Figure 35e), while he said, “one over two, no, x -squared over two” (Turn 54). This analysis supports the claim that the students were thinking about derivatives algebraically, in the sense that $y=x$ is the derivative of $y=x^2/2$.

It was noted that the deictic word “this” was used extensively, appearing five times in the last part of the episode. Using deictic words, the speakers no longer needed

to refer to the mathematical objects by describing them verbally, but they could use deictic words along with different gestures to replace the descriptions completely. This was found in Sam's "this is actually, the derivative" (Turn 40), "no matter how you move, this one always" (Turn 38), and "this is probably x , x -squared" (Turn 42). As Sfard explains, gestures help ensure that the interlocutors speak about the same mathematical objects. Significantly for Sam and Mario, gestures served complementary functions to speech in communication. In this episode, the two students were able to use a combination of utterances and gestures to communicate the relationship of the mathematical objects effectively.

Through word use, gestures and dragging, the students realised dynamic, geometrical and algebraic notions of calculus. Moreover, the *Trace* tool and shaded area gave feedback about the relationship of the green traces and the area under f , which enabled the students to *conjecture* the possible relationship between the two graphs as one being the derivative of the other. However, at this point, the students' language still contained the hedge word "probably" (Turn 42).

7.5.3. Verifying conjectures and communicating as "one"

The final except was taken forty seconds after the end of the previous except. Sam and Mario continued to use the DGE for exploring calculus ideas.

Turn	Timespan	What was said <what was done>	S-er	D-er	G-er
70	4:47.8 - 4:56.3	Let's try a different page	S	--	--
71	4:56.3 - 5:07.0	Show area. <S presses the first three buttons and then drags x around>	S	--	--
72	5:07.0 - 5:09.8	Show trace. <S presses the <i>Show Trace of A</i> button>	S	--	--
73	5:09.8 - 5:16.0	<u>You see? It's basically...</u> <S drags x >	S	<u>S</u>	--
74	5:16.0 - 5:17.2	<u>The derivative.</u>	M		--
75	5:17.2 - 5:32.5	<u>Ya. So the graph we have here is specifically the derivative of the... what we just graphed here, like the function here is basically the derivative of what we just graphed here.</u>	S		<u>S</u>
76	5:32.5 - 5:34.8	<u>What do you think?</u>	S		--
77	5:34.8 - 5:38.5	<u>Is that the same thing as before?</u>	M		<u>M</u>
78	5:38.4 - 5:43.2	Why is there something to do with area?	S	--	--

79	5:43.2 - 5:51.0	'Cause, <u>these</u> are, represent like the <u>total</u> area, represented.	S	--	<u>S</u> , <u>S</u>
80	5:51.0 - 5:58.1	<u>And then if we move... Go straight down.</u> Oh I see. <S drags a >	S	<u>S</u>	--
81	5:58.1 - 6:07.2	<u>Which one are we supposed to move?</u> <M drags x >	M	<u>M</u>	--
82	5:58.1 - 6:07.2	<u>I think both.</u>	S		--
83	6:07.2 - 6:28.9	<u>What?</u> <S drags a then x >	S	<u>S</u>	--
84	6:28.9 - 6:34.4	Why it's something to do with area?	S	--	--
85	6:33.4 - 6:34.4	<M performs <u>scribing gesture without speech</u> >	--	--	<u>M</u>
86	6:34.4 - 6:38.4	I wanna go back to the first one.	M	--	--
87	6:34.4 - 6:38.4	Ok.	S	--	--
88	6:38.4 - 6:43.2	Erase trace. <M presses <i>Erase Trace</i> button>	M	--	--
89	6:43.2 - 6:54.6	<u>So f.</u> <M drags x then a >	S	<u>M</u>	--
90	6:54.6 - 7:10.0	<u>So basically the area, so the area is, is what you want, oh the area? Is the integral of this line?</u>	S		--
91	7:10.1 - 7:16.1	<u>What?</u>	M		--
92	7:16.1 - 7:19.5	Move x . Try to move x . <S drags x >	S	--	--
93	7:19.5 - 7:38.0	Ok, ok, <u>when x is, when x is here</u> , it's at 2. 2 times 1 is... is 2. And then you see <u>here</u> , that's 2.	S	<u>S</u>	<u>S</u>
94	7:23.5 - 7:30.4	Ya, it's the original.	M	--	--
95	7:38.0 - 7:40.4	Now let's look at this one. <M opens the last page of sketch and presses all but the "Show Trace of A: button">	M	--	--
96	7:40.4 - 7:49.9	What the? We have another one?	S	--	--
97	7:49.9 - 8:02.0	<u>Ok, let's predict. This, the graph is going to be a... negative cosine.</u>	M	--	<u>M</u>
98	8:03.0 - 8:04.5	Negative cosine, ya.	S	--	--
99	8:04.5 - 8:08.7	<M drags x > No, you got to show trace. <M presses the <i>Show Trace of A</i> button>	S	<u>M</u>	--
100	8:08.7 - 8:13.8	<M drags x back and forth rapidly without speech>	--	<u>M</u>	--
101	8:13.7 - 8:24.0	Why, why is it?	S	--	--
102	8:31.7 - 8:40.0	<u>I don't think we are supposed to move this.</u> It just makes a weird graph. <S drags a >	M	<u>S</u>	--
103	9:05.0 - 9:09.5	Ok, let's see, erase trace.	M	--	--
104	9:23.4 - 9:27.5	<u>Oh, I think I got it.</u> <S drags x >	M	<u>S</u>	--

(a)	(b)
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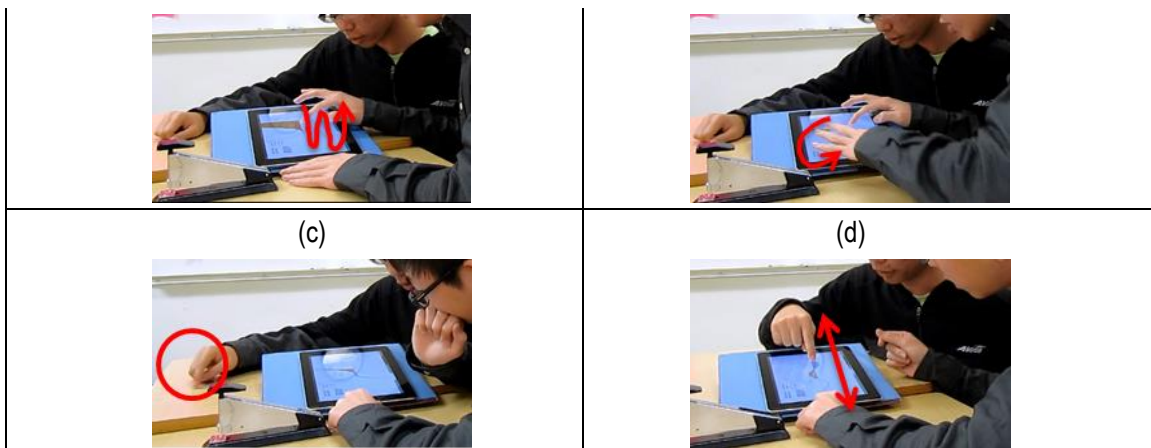


Figure 36. Selected snapshots of Sam and Mario's gestures and dragging during Turns 70 to 104

The students' discourse around area-accumulating functions became increasingly developed in this excerpt. First, the questions posed in this excerpt were markedly different from the first two excerpts. Recall that during the first excerpt, the students asked repeatedly, "what", at times without finishing their questions. The previous analysis showed that these questions reflected a degree of uncertainty about "what" the sketch meant to them. In contrast, Sam asked three questions that began with "why" in final excerpt. He asked, "Why is there something to do with area?" in Turns 78 and 84, as well as "why is it?" in Turn 101. By asking these "why" questions, Sam seemed to be looking for the reason as to why the relationship that they found about the two graphs were related to the area under a function. This was a valid question considering that Sam had yet to learn the idea of "definite integral as area" in his class. Regardless, asking "why" implies investigating the reasons of something that is clearly existential. In this case, Sam seemed to be investigating the reason why the *area* under a function had something to do with its antiderivative.

From Turn 73 to 77, Sam performed a prolonged dragging action (Figure 36a), while the two students exchanged comments verbally back and forth. In particular, by far the longest spoken sentence was observed in Turn 75, spoken by Sam while he was simultaneously dragging:

Ya. So the graph we have *here* is *specifically* the derivative of the... what we just graphed *here*, like the function *here* is *basically* the derivative of what we just graphed *here*. (05:10-05:39)

The sentence was very rich in a multimodal sense because it was spoken while the speaker was dragging, and gestures were used simultaneously as the speaker uttered, “the function here” (Figure 36b). Some interesting word uses were also observed. For instance, the word “here” was used four times, and the words “specifically” and “basically” each once. In line with a previous analysis of the use of deictic words, the use of locative noun “here” accompanied by gestures allowed the speakers to talk about the same mathematical object. Although Sam used the same word “here” four times, he actually meant to refer to two different mathematical objects, the function and its derivative. Perhaps this was why Sam complemented his utterances with gestures to specify the objects he was talking about. Secondly, the contrasting use between “specifically” and “basically” by Sam was also fascinating. Since Sam used the word “basically” quite frequently throughout the task, his word use “specifically” as opposed to “basically” in this sentence drew attention to the analysis. Consistent with his usage of “basically” in other parts of the transcript, it seemed that Sam used the word to suggest a generality or invariance that exists outside of the sketch. In contrast, it is speculated that he used “specifically” to refer to the particular “graph” that was the derivative of another on a specific page of the sketch. According to this speculation, Sam was able to talk about area-accumulating functions both in its generality and particularity, which is a highly valued practice in the mathematics community.

Although up to this point, the analysis seems as though Sam was more engaged in the task, one can see that Mario was also an active participant through careful observations of his utterances, gestures and dragging. Throughout the three excerpts, the two students used the pronoun “we” extensively, with few occasions where “I” or “you” were used to differentiate between the speaker’s own intention from that of the collective. For example, in Turn 75, which was described above as a significant moment in the students’ discussion, the word “we” appeared three times in a single utterance. In addition, the word “let’s” [let us] was used three times by Mario, in “*let’s* look at this one” (Turn 92), “*let’s* predict” (Turn 97), and “*let’s* see, erase trace” (Turn 103). The most significant of the three was, “*let’s predict*”, since it led the students to predicting and verifying conjectures, from what was exploring and occasionally conjecturing in previous two excerpts.

“Predicting” the shape of the graph of area under the sine curve marked a significant change in the students’ discourse around area-accumulating functions. At first Mario suggested, “Let’s look at this one,” as he opened the last page of the sketch and pressed all but the “Show Trace of A” button. Upon Sam’s acknowledgement, “we have another one” (Turn 96), Mario responded, “Let’s predict. This, the graph is going to be a [...] negative cosine.” The tone of Mario’s utterance was firm and the use of present continuous tense “is going to be” confirms that he was predicting the shape of the graph of area under the sine curve. Furthermore, the statement contained no hedge words and so the degree of certainty was much higher than previous statements with “like” and “probably”. It can be argued that Mario did not press the “Show Trace of A” button intentionally when he first opened the page because he wanted to predict the shape of the green traces all along. Upon both students’ agreement that the equation of the green traces should be “negative cosine”, Mario began to drag ‘x’ back and forth rapidly (Figure 36d) which left behind a set of green traces in the shape of $A(x) = \int_0^x \sin(t) dt = -\cos(x) + 1$. His rapid dragging action had never been done before prior to this moment, as all dragging actions performed by the two students had been steady but not rapid. As discussed before with Jay’s rapid dragging, the green points could be traced more quickly in this way, and this seemed to be Mario’s intention behind the action also. More importantly, Jay and Mario’s rapid dragging seemed to be a case of them trying to advance the process of tracing, therefore, *encapsulating* the ordered pairs $(x, A(x))$ into a discursive object—the area-accumulating function (see Sfard, 2008).

Besides his dragging and leading the discussion towards predicting the shape of the graph of area under sine, Mario also showed that he was fully engaged in the task with his word use and gestures. For example, he finished Sam’s sentence, “this is basically...” with the word “derivative” in Turn 74. He also performed another “scribing gesture” (Figure 36c) when Sam questioned “why it has to do with area” (Turn 84). This “scribing gesture” can be taken as his non-verbal response to Sam or his own intrapersonal communication. In either case, it can be shown that Mario was thinking-communicating mathematically and not as disengaged as it might seem, with the present analysis incorporating word use, gestures and dragging.

As mentioned, the two students took 30 minutes in total to complete the task. A few minutes after the third analysed excerpt, Sam mentioned that “it is the integral, but with the ‘c’, because [...]” Further analysis beyond this excerpt shows that the students appeared to have achieved beyond the learning outcome of the activity. However, for the scope of this dissertation, I have decided to focus my analysis on the interplay between linguistic and non-linguistic features of the students’ discourse with touchscreen-based DGE. As stated, the goal of the study is to uncover bilingual learners’ competence in mathematical communication, which current analysis establishes.

7.6. Summary

In this chapter, I described the participants’ emerging discourse about the area-accumulating function as they interacted with the dynamic sketch in pairs in Part II of the study. The four participant pairs developed their mathematical discourse quite differently: Huang and George communicated with very little word use, especially during the early parts of their interactions with the dynamic sketch. Instead, they communicated non-linguistically, by utilising dragging and gesturing as significant modes of communication, which over time, allowed them to develop their verbal discourse around “area” and eventually the “rate of change of area” later on. While Huang and George had the least experience studying in an English environment, Larry and Ivy were the most experienced English learners in the study. Their verbal discourse were more developed, and they began a discussion around “area” very early in the task. The pair attended to the sign of the green traces through utilising the dragging modality and communicated some consistencies across pages through words and gestures. However, their discourse began to diverge as Ivy pursued to explore the relationship underlying the sketch, when Larry did not seem interested to pursue such exploration.

Jay and Katie, who share the same home language of Korean, used a combination of Korean and English to communicate the change of area geometrically, numerically and algebraically. While on the page showing $f(x)=x$ and with $a=0$, Jay dragged ‘x’ from $x=3$, to $x=2$ and finally to $x=1$, while describing the area of the triangle that is half of the square of ‘x’ at each x -value. Upon accurately disclosing the change of area discretely, they encapsulated the set of green traces as the graph of the

antiderivative of $f(x)$, as facilitated by Jay's rapid dragging. This realisation of the area-accumulating function was communicated when Katie and Ivy were asked to pair up and discuss what they had found with their previous partner.

The analysis of Sam and Mario's discussion shows the use of speech, gestures and dragging for participating in different mathematical Discourse practices during their interaction with the dynamic sketch. Their verbal discourse around area-accumulating function became increasingly sophisticated. At first, they questioned about the sketch and communicated mainly through dragging; then, they explored variances and invariances as well as conjectured the relationships found. Finally they verified their conjectures by way of rapid dragging.

My analysis of the participants' linguistic and non-linguistic modes of communication, patterns of how their mathematical discourse was developed, and the coherence of their discourse led me to proposing a framework with three "lenses" for understanding student pairs' mathematical discourse when communicating with touchscreen-based DGEs. In Chapter 8, I discuss further this framework, its implications for teaching and learning, and some reflections on the bilingual aspects of the study.

Chapter 8. Discussion

I move; therefore, I am. (Seitz, 1993)

In this chapter, I extend the results from the study as discussed in Chapters 6 and 7. The chapter is organised into three themes, which have emerged from the results of the study: (1) the relationship among modes of communication, mathematical thinking and types of visual mediator; (2) common features of student-pairs' mathematical discourse during exploratory activities with the use of touchscreen-based DGEs; and (3) the significance of non-linguistic communication for bilingual learners during mathematical activities with the use of touchscreen-based DGEs. For each theme, I draw on the results of the study to compare, contrast and extend student pairs' patterns of communication during exploratory activities with touchscreen-based DGEs. The purpose of this chapter is to synthesise the results coherently and constructively in order to inform future work in the areas of non-linguistic communication, mathematical thinking with DGEs and mathematical learning for bilingual students.

Given the different aims of Part I and Part II of my study, the first theme described in this chapter is more relevant to the results obtained from Part I (Section 8.1) while the second theme is more relevant to Part II (Section 8.2) respectively. On the other hand, bilingual aspects are an overarching theme of my study; therefore, both Parts I and II will be drawn on in extending the research on bilingual learners in Section 8.3.

8.1. Relationship between modes of communication, mathematical thinking and types of the visual mediator

The analysis of Part I of the study provides strong evidence that the participants utilised different modes—speech, gesture and dragging—in their mathematical

communication, and they also communicated fundamental calculus ideas differently when prompted by different types (static and dynamic) of visual mediators. Given these findings, I argue that both the modes of communication and the kinds of mathematical ideas communicated are situated in the visual mediator used in the activity. On the other hand, it is possible to see an interplay between modes of communication and mathematical thinking. This leads to the conclusion that all three characteristics of communication at the centre of the study—modes of communication, mathematical thinking and types of visual mediator—are deeply interrelated.

Gestures were prevalent but took on different roles in communication with different types of visual mediator. With a static visual mediator, the students mainly communicated with utterances accompanied by deictic gestures and occasionally moved their fingers to gesture a sense of change such as “change of x ”. In addition, Yee changed the size of his “measuring gestures” to convey a change of distance, and he used his pinky finger when he talked about a level of precision related to the diagram, perhaps because he could not physically alter the objects shown on the static diagram. This communication routine evolved in the presence of the dragging modality over a dynamic visual mediator. A new form of gesture emerged in the touchscreen-dragging action with DGEs which fulfilled the dual function of dragging and gesturing. The presence of dragging transformed the use of speech. As shown in the analysis, Ana and Tammy resorted to verbs in present continuous tense (is [verb]-ing) to communicate that something was happening while they used dragging to change the tangent slope. This was a change of verb-form from their earlier discussions around the static diagrams, where the students used the “is [noun]” form to communicate a static sense of calculus. As the students became more active with their hands, their postures evolved from sitting at a distance from the iPad to leaning in closer towards it. Their change in posture signified a shift in the way they were attending, which may have contributed—in and of itself—to the change in their mathematical discourse. They seemed more interested in the dynamic sketches than in the static diagrams and seemed to make more sense of them.

Indeed, it was observed that the students communicated different mathematical content in the presence of different types of visual mediator. With the static diagrams,

the students communicated about calculus procedurally and statically by defining mathematical objects, developing a formula and communicating in a theorem-like discourse. The use of “scribing gestures” and utterances such as “we can know” and “we know a and b ” supports the claim that they were thinking about calculus procedurally and statically. They communicated change discretely, by using numerical examples or referring to static diagrams that convey change in separate moments. With dynamic sketches, the students used dragging as a form of communication, accompanied by speech, to *dragsture* the variance of tangent slopes. In particular, Yee’s discourse changed from “we can know” to “we can see,” when he began to drag on the touchscreen-based DGE. The touchscreen-dragging affordance facilitated this change of communication, by enabling Yee to “see” the dynamic relationships unfold on the screen as he was dragging. The design of the sketch also played a role, since the draggable point was also the point of tangency on the graph of the function, which was a geometrical object. Hence, dragging the point has a dual effect of changing the x -coordinate numerically as well as moving the point of tangency physically. This may have initiated the blending of dragging and gesturing the movement of the point of tangency. Also, in line with Falcade, Laborde and Mariotti (2007), Melissa may have exploited the functionalities of the *Dragging* and *Trace Tool* to communicate covariance of a function and its derivative function geometrically and dynamically. In summary, the dynamic environment, touchscreen technology and the design of the sketch which includes the exploitation of the dragging tool all played a role in the students’ discourse about dynamic features of calculus.

Figure 37 illustrates the relationship between modes of communication, mathematical thinking and types of visual mediator as elaborated above. At the centre of the diagram is the activity: student-pair mathematical communication with a visual mediator. Using a communicational approach, the activity at the centre can be understood in terms of three aspects of communication: mode of communication, mathematical content communicated (or mathematical thinking) and type of visual mediator used. Furthermore, the double-headed arrows indicate that these three aspects must not be studied in isolation, but as deeply related sets that constitute the activity. This means that one cannot make sense of student pairs’ communication unless all three aspects of communication are considered. For example, to make sense of a

certain word spoken or gesture performed by a student fully, it is important to consider the mathematical idea communicated and which visual mediator was used.

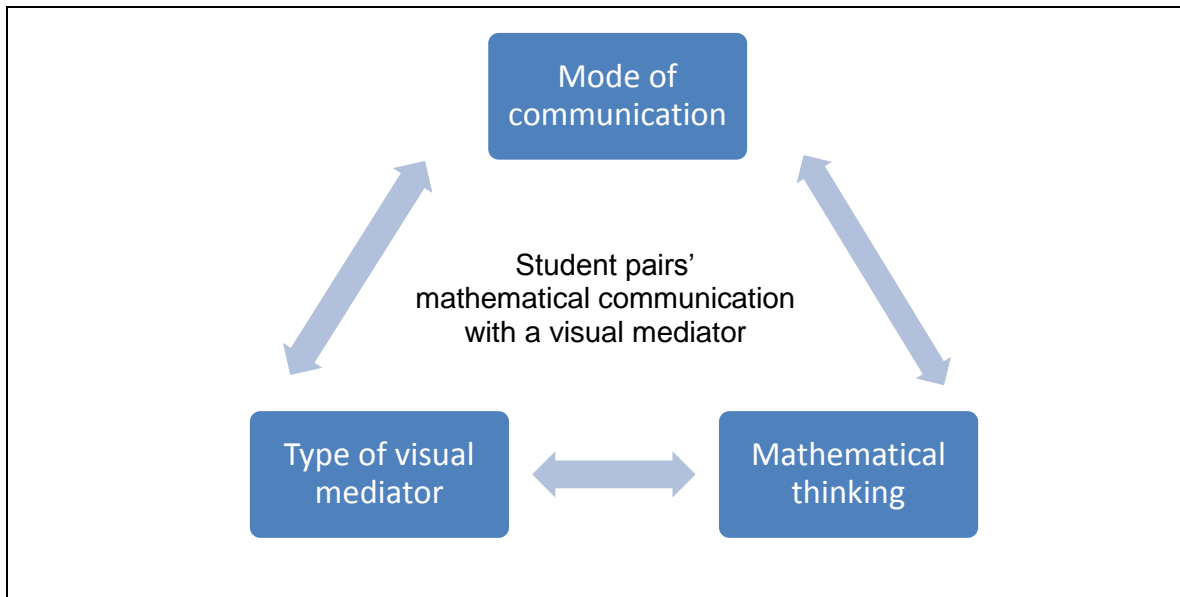


Figure 37 Mode of communication, mathematical thinking, and type of visual mediator as deeply interrelated sets

As an extension, the results from Part I of the study point to an expanded view of mathematical discourse that includes gestures and touchscreen-dragging for understanding mathematical thinking with digital tools, especially in terms of the dynamic aspects of mathematical ideas. Sfard's communicational framework—which defines gestures as communicational acts—was especially useful for understanding the kinds of communicational functions of the gestures (and dragsturing) used by the participants. The consistent use of dragsturing by the participants to complement their own speech or as a response to their partner's further contributes to current literature of the dragging practices in DGEs. In particular, Arzarello et al. (2002) identified different types of dragging used by students according to their different purposes during the solution process of open problems. In a similar way, my analysis shows that my participants performed different types of dragging (different pace of dragging, dragging with or without accompanying speech) to engage in mathematical communication and in different mathematical Discourse practices. I also identified a new type of dragging to shed light on the process of encapsulation, which I termed “rapid dragging”.

Another extension can be made in relation to what counts as a “turn” from a conversational perspective. My study shows that new conversational patterns were introduced by the student pairs in the presence of *dragsturing*, where gesture-gesture and gesture-utterance sequences were observed in the conversation. This observation suggests that student pairs make use of gestures to respond to each other in mathematical communication. Also in the study, I observed one person *dragsturing* simultaneously to the other speaking; this allowed two students to communicate concurrently without interfering with each other in the speech channel—communicating at the same time without speaking at the same time. These new kinds of conversational patterns redefine the notion of “turn-taking” in conversation analysis. Firstly, it has been shown that a “turn” is not only marked by a change of speaker, but also potentially by a change of dragger or gesturer. Secondly, my analysis refutes the view that concurrent communication (when two people are not “turn-taking”, in the sense that one speaks after the other finishes) interrupts the flow of the conversation. Therefore, a “turn” can be reconceptualised as any overlapping or non-overlapping speech, gestures or dragging actions performed in communication by different interlocutors. This complex view of turn-taking also raises implication for turn-taking in other digital environments such as online conversation.

Finally, this study contributes to the refinement of Sfard’s notion of *visual mediator* by distinguishing two kinds of visual mediation, *dynamic and static*. This distinction was important for this study because of the potential for dynamic visual mediators such as gestures and DGEs to evoke temporal and mathematical relations (Ng & Sinclair, 2013), particular for the study of calculus (Núñez, 2003). It also helped guide my analysis in terms of how “change” was conveyed in students’ discourse, by identifying differences in deictic gestures and gestures (or dragging) that conveyed temporal relationships. Thus, I call for more studies to consider extending the notion of visual mediators and routines to include gestures and touchscreen-dragging. As touchscreen learning technologies continue to enhance the digital experience of learners, the blending of gestures and touchscreen-dragging for providing a haptic and mobile environment in mathematics learning is worthy of further examination (see also Sinclair & de Freitas, in press).

It has been shown that different content and modes of calculus communication can be facilitated by different visual mediators. Several implications can be identified in relation to supporting student (monolingual and bilingual) communication in calculus. First, in terms of fostering multiple representations in calculus, my study suggests that providing opportunities for students to learn both with static diagrams and with dynamic sketches may help students develop their mathematical discourse, such as numerical, algebraic and geometrical discourse. Second, the use of DGEs was shown to be instrumental in facilitating dynamic and temporal—motion-based and action-oriented—thinking in calculus. It is important to bear in mind that, at the time of Part I of the study, the participants had previously learned the target calculus concepts with the use of touchscreen-based DGEs in their regular classrooms. Despite having learned the topics with the *same* dynamic sketches used in the study, the participants employed different discourse patterns when prompted by two different kinds of visual mediators. This identifies important implications for classroom teaching, since it shows that mathematical thinking is not located in the heads but in the activity and in the kinds of visual mediators used (Chen & Herbst, 2012). In order to develop and assess certain aspects of mathematical discourse in the study of functions and calculus, I argue that providing situations for students to communicate these ideas in both static and dynamic environments, as well as adopting a multimodal view of communication, is beneficial.

8.2. A framework for understanding student-pair communication during exploratory activities with touchscreen-based DGEs

Besides shedding light on a multimodal view of communication, the results of this study also provide a deeper understanding of student-pair mathematical discourse pattern during exploratory activities with touchscreen-based DGEs. From the analysis detailed in Chapters 7, it became apparent that there were similarities in the way the student pairs developed their mathematical discourse during the activity. By similarities, I do not simply mean particular words, gestures or dragging used by different pairs of students. More broadly, I am also referring to certain characteristics in the student pairs' discourse, such as how words, gestures and dragging were used discursively: (1) to *engage initially in the development*, *engage actively in the development* and *engage in a*

“*formal*” mathematical discourse; (2) to participate *individually* or *collectively*; and (3) to engage in processes such as *saming*, *reification* and *encapsulation* to add range to the realisations of a signifier.

The next three sub-sections delve into the categories mentioned above in more depth and discuss relations among them. In the first sub-section, I describe characteristics of three patterns of communication in the development of the mathematical discourse and examine implications for teaching and learning. In the second sub-section, I elaborate on the detection of student pairs’ participation in the mathematical discourse, in terms of whether or not they were participating individually or collectively. In third one, I illustrate, with examples drawn from the study, insights gained on teaching calculus with regards to three Sfardian processes for creating new discursive objects in their mathematical discourse, namely *saming*, *reification* and *encapsulation*.

8.2.1. Engage initially in the development, engage actively in the development and engage in a formal mathematical discourse

All participants of Part II of the study engaged in some development of their mathematical discourse by the end of the exploratory activity. They did not begin to communicate as they did at the end, and some of the pairs engaged in a more developed mathematical discourse than others at the end in the sense that some communicated more mathematical meanings than others. This suggests that all student-pairs’ mathematical discourse went through some changes during the course of their exploration. Figure 38 describes three patterns of communication found to be recurrent in student-pairs’ development of the mathematical discourse in Part II of the study: an initial discourse pattern dominated by questioning, in particular, about the functionality of the DGE; a discourse pattern mainly focused on making sense of what mathematical relations are shown on the DGE; and a “formal” mathematical discourse (MD) in the sense that it has been adopted as “mathematical” and norms in the classroom. Since the participants in Part II had not learned the concept nor seen the sketch before, the student pairs were more likely to engage in each one of these development in the course of their exploration. Hence, I draw on results from Part II of the study for a more thorough discussion about each category in the following paragraphs.

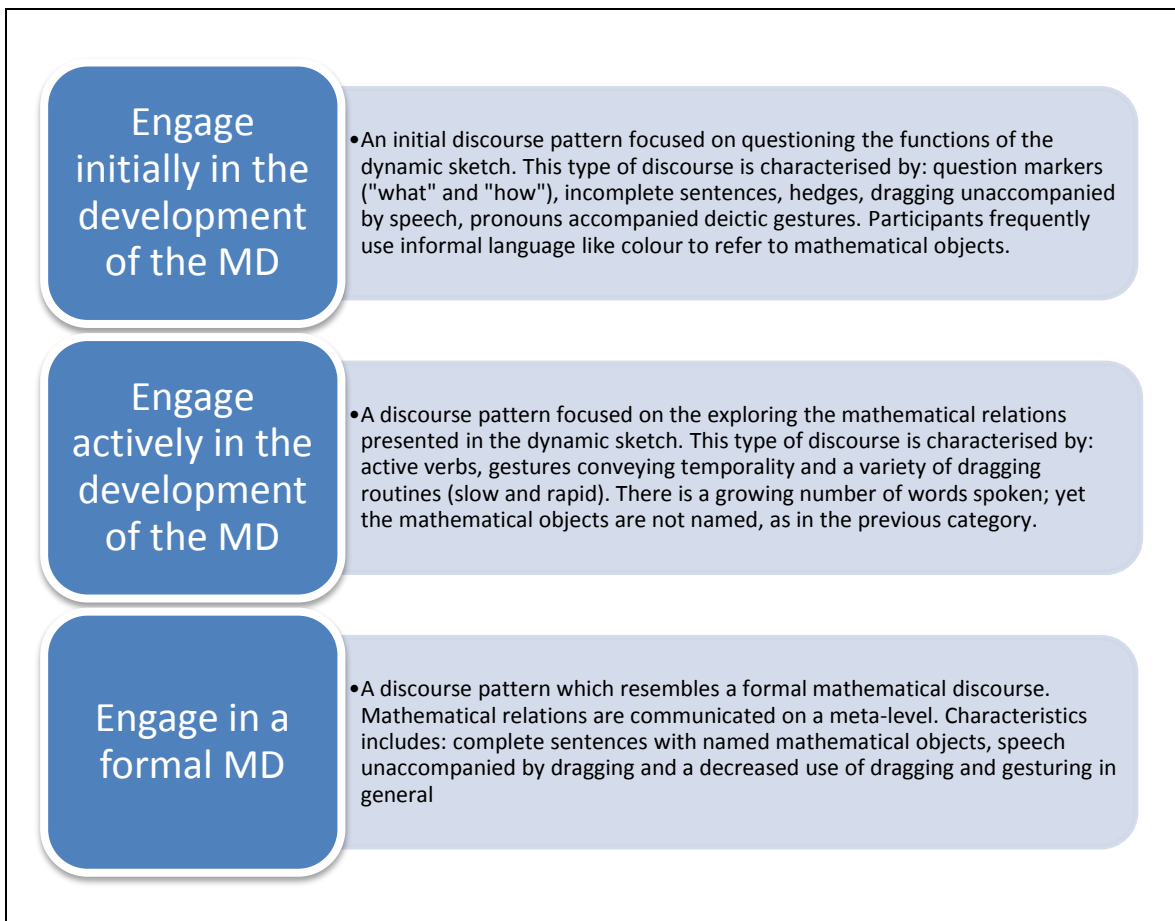


Figure 38. Three patterns of communication in the development of the mathematical discourse (MD)

The type, “engage initially in the development of the mathematical discourse” was found to be recurrent among student-pairs who had just begun their exploration with touchscreen-based DGEs. When engaging initially in the development of the mathematical discourse, the students used a combination of speech, gestures and dragging to gain familiarity with the dynamic sketch. Since it was new to them, their initial exploration involved much questioning about the functions and objects shown in the dynamic sketch, which was characterised by the use of question markers “what” and “how” accompanied by gestures and dragging. The analysis shows that all four student pairs used the question marker “what” for questioning something in the sketch, such as “this point”, “Trace of A”, “this dot”, “ a ”, etc. during their early exploration. Some of these questions were asked while a student was dragging some draggable object, suggesting that dragging may have prompted them to question something that they had not noticed

before in the sketch. In terms of speech, this discourse type was characterised by a low number of words used, since the student pairs had yet to develop their spoken mathematical discourse. Related to this, the sentence structures in their speech were short or incomplete. For example, when Huang and George engaged initially in the development of their mathematical discourse, all but one of their utterances in the first 17 turns were missing a subject or a predicate. This indicates incomplete thought on the part of the student-pair: perhaps as they were still trying to learn the many functionality of the sketch. The use of hedges was also characteristic of this type of discourse which can indicate uncertainty in the student pair's thinking.

Given the novelty of the sketch, student-pairs who were engaging initially in the development of their mathematical discourse had yet to develop a range of vocabulary for communicating the objects shown in the sketch. Hence, the use of mathematical terms was rare, and conversely, they frequently used pronouns "this" and "it" to replace the name of mathematical objects, which gave rise to deictic gestures for pointing towards those objects. This interplay between speech and gestures is similar to the interplay between speech and dragging discussed above, where dragging prompted students to question about the sketch. Moreover, I hypothesise that the number of gestures and dragging turns in this type of discourse was high due to an under-developed verbal discourse and the exploratory nature of this part of their activity.

As the student pairs gained familiarity with the sketch, they moved to a second category of communication in the development of the mathematical discourse, that is, they "engaged actively in the development of their mathematical discourse". The main difference between these two categories was in the foci of the communication: the former was focused on the functions of the dynamic sketch, while the latter was focused on the mathematical relations presented by the sketch. In other words, the student pairs progressed from talking about *how* to use the technology to *what* mathematical meaning was presented *by* the technology. With the aim to make sense of the sketch mathematically, the student pairs engaged in valued mathematical Discourse practices such as comparing, conjecturing, reasoning, verifying and predicting, and they also actively explored the variance and invariance presented in the sketch (Figure 39).

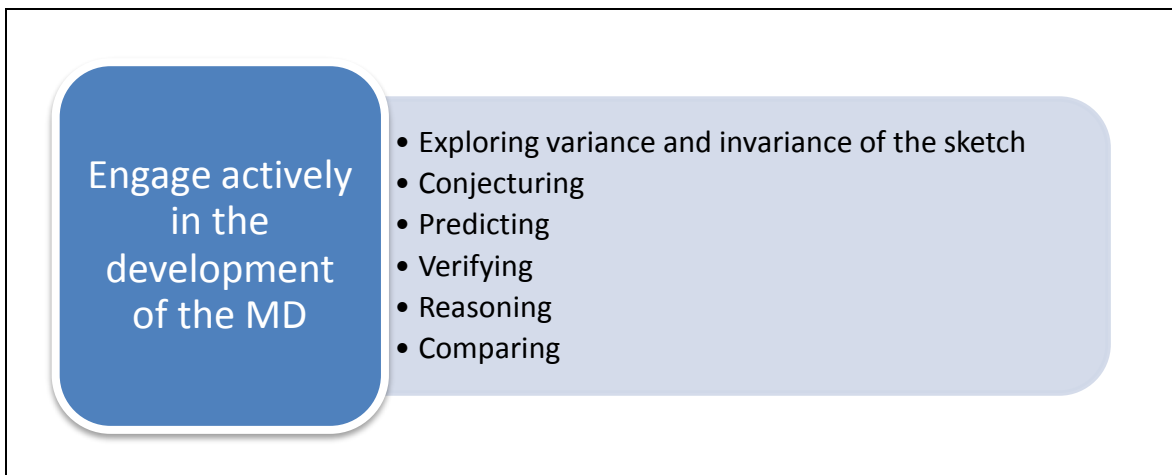


Figure 39. Mathematical Discourse practices demonstrated when student pairs “engage actively in the development of the mathematical discourse”

The student pairs in the study engaged actively in the development of their mathematical discourse by integrating different modes of communication, by means of speech, gestures and dragging. Like their initial engagement, the student pairs continued to use pronouns accompanied with deictic gestures and touchscreen-dragging for exploring the sketch. However, they no longer focused on the functions of the technology and instead focused on describing the dynamic relationships presented during their exploration. As such, this type of discourse was characterised by verbs in present continuous tense ([verb]+ing), gestures that conveyed temporal relationships, different time and directions of dragging, and the use of non-mathematical terms to compare, conjecture, predict, verify and reason mathematically.

The use of non-mathematical terms is interesting because the students were able to engage actively in the development of their mathematical discourse using everyday language rather than technical mathematical terms. For example, the participants shared a mutual understanding of what “this” or “it” meant, even without stating what they meant explicitly. This was exemplified in one of Sam’s utterance, in which he used the pronouns “it” and “this” three times in a single utterance, each time referring to a different mathematical object. Although it might seem problematic to refer to different objects with the same pronoun, the student pairs were able to talk like this and even made progress through this kind of talk. The use of gestures was crucial in helping the interlocutors “speak about the same mathematical object” (Sfard, 2009) in

this regard. This may explain why the occurrence of gestures and dragging remained high even when students engaged more actively in the development of their mathematical discourse. Moreover, the student pairs began to develop their verbal discourse, and this resulted in an increase in the number of words spoken from their initial engagement, as observed in the student pairs' (Huang and George; Larry and Ivy) communication during the second 10-minute interval of their engagement with the task.

The third type of communication in the development of their mathematical discourse is “engage in a ‘formal’ mathematical discourse”. This does not mean that the student pairs were no longer engaged in “developing” their mathematical discourse, since their mathematical discourse was always in development or expanding. Recall that when student pairs were engaged actively in the development of their mathematical discourse, they consistently used active verbs to predict, conjecture, verify, etc. By contrast, when engaging in a “formal” mathematical discourse, they used fewer active verbs and more verbs in the form of “to be”, such as “this is the derivative of this” (Katie) and “but y is positive, so the area is negative” (Ivy). These statements were typical of an objectified discourse, in the sense that the dynamic relationships that were explored previously had been objectified from actions into timeless facts to be asserted or declared. Another characteristic of engagement in a “formal” mathematical discourse is the use of mathematical terms, such as derivative, integral, quadratic, degree, etc. Perhaps, the student pairs were aware that I would check-in with them at the end of their exploration, and so they were expected to come up with some explanations that were deemed mathematically acceptable.

This means that the student pairs understood what it meant to provide a mathematically sound explanation or argument when engaging in a “formal” mathematical discourse. By mathematically acceptable, I do not mean that the students were expected to generate rigorous proofs or provide a high quality mathematical argument. What they considered as “mathematically acceptable” was situated within the classroom community, in particular, modelled by their previous experience participating in regular classroom discussions before the study. For example, it seems that, for these students, a developed or “formal” mathematical discourse should contain mathematical

terms and statements about relationships between mathematical objects, such as the functional relationship between x and $f(x)$.

While modelling a “formal” discourse was important, it is also worth mentioning that I provided opportunities for the student pairs to engage in both object-level and meta-level discourse. The task was designed such that the student pairs were given opportunities to “talk within” mathematics at an object level, and after that, opportunities to report back to me or “talk about” mathematics on a meta-level. Talking within mathematics enabled the students to explore the mathematical relationships presented in the sketch without the constraints of communicating “formally”, whereas talking about mathematics helped the students to formalise and objectify the relationships that were developed previously. Therefore, I claim that providing situations for both types of talk is helpful for fostering the development of mathematical discourse.

To summarise, this sub-section describes one dimension for understanding student pairs’ patterns of communication in the development of the mathematical discourse during exploratory activities with touchscreen-based DGEs. In particular, three categories of communication, *engage initially in the development*, *engage actively in the development*, and *engage in a “formal” mathematical discourse*, have been proposed. It is important to note that, although the regions representing the categories are shown as non-overlapping regions with equal area in Figure 38, it is not intended to suggest a clear distinction between categories nor an equal distribution of time spent in each category during student-pair exploratory activities with touchscreen-based DGEs. Perhaps, a continuum would be more effective for conveying a gradual development; however, for the purpose of showing relations between other categories in later sections, I have chosen to use two-dimensional regions to represent the categories.

In terms of implications for classroom teaching and learning, the framework proposed in this section suggests that opportunities for student pairs or groups to “talk within” and “talk about” mathematics may allow student-pairs to engage actively in the development of the mathematical discourse as well as engage in a “formal” mathematical discourse. A familiarity with or understanding of the functionalities offered by the DGE is needed before student pairs can engage actively in their development of

the mathematical discourse. Because many valued mathematical Discourse practices are demonstrated in this type of communication, I suggest teachers should foster active engagement in the development of the mathematical discourse during exploratory activities and to make use of the categories developed in this section to assess student-pairs' level of engagement in order to provide appropriate feedback in the process. For example, if a student pair is questioning and hedging during the exploratory activity, formative feedback and prompts to help students familiarise certain features of the sketch may prove to be constructive.

8.2.2. Individual or collective participation in the development of mathematical discourse

My analysis shows that while the student pairs' communication was mostly coherent and in sync, at times they exemplified incoherent or diverging discourses during the exploratory activity. Figure 40 describes two levels of participation: individual or collective development of mathematical discourse. The two types of participation can be found while student pairs engage initially in the development, engage actively in the development or engage in a "formal" mathematical discourse; Figure 41 illustrates the mutual inclusiveness of types of participation and patterns of communication in the development of the mathematical discourse.

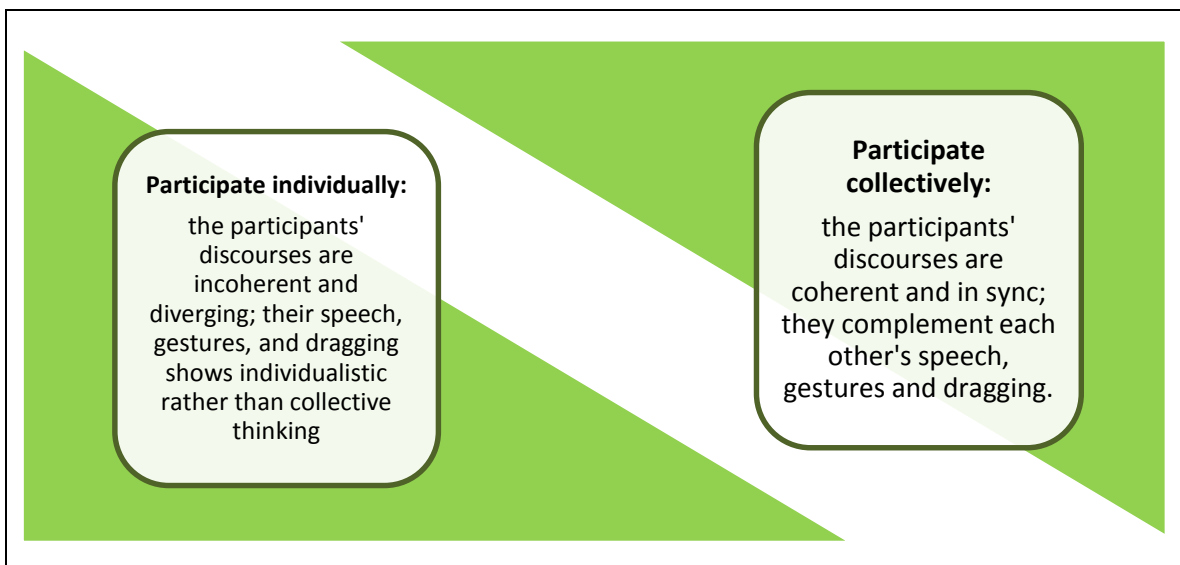


Figure 40. Two types of participation in the development of mathematical discourse

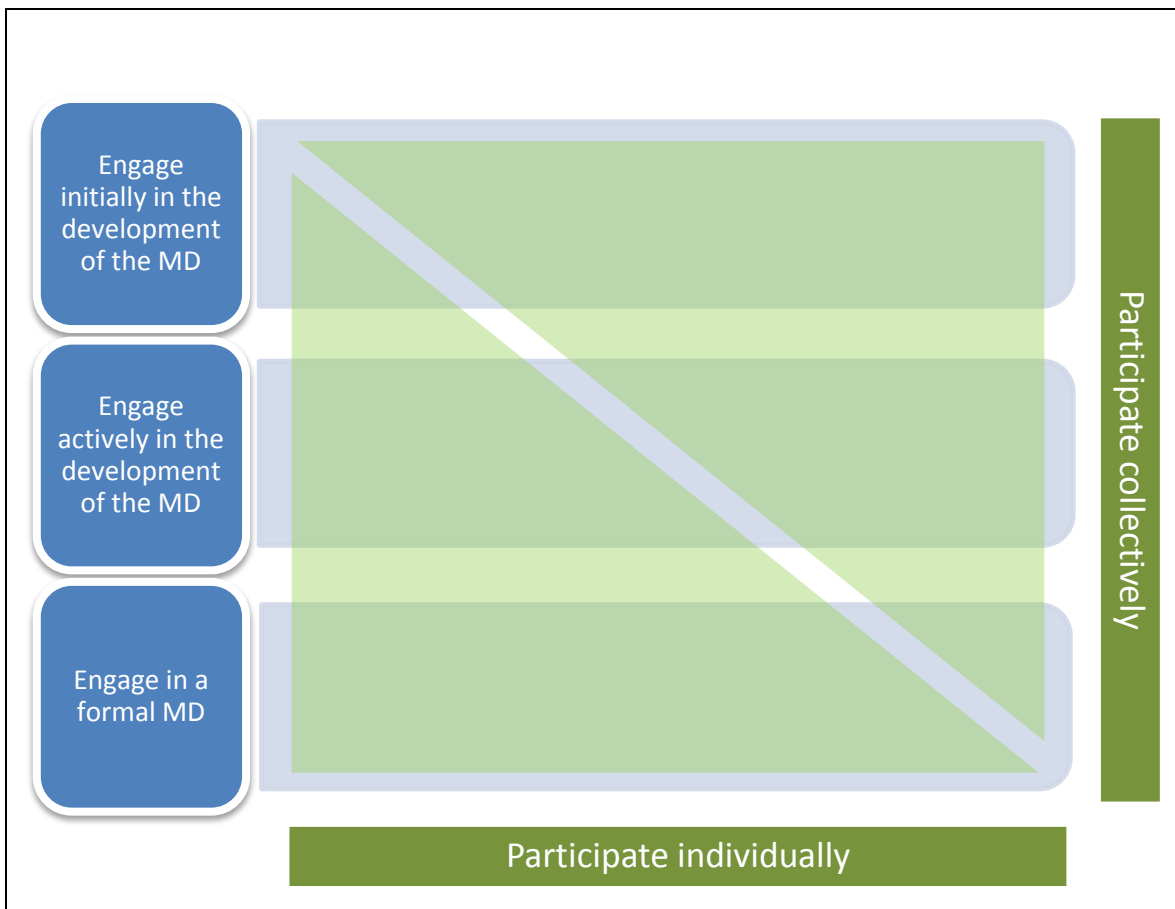


Figure 41. Types of participation and patterns of communication in the development of the mathematical discourse

My study raises important consequences about the way coherent or diverging discourses are engaged in pair-work or group-work mathematical exploratory activities using DGEs. The analysis shows how different attention towards the DGE can serve to cause two students' discourse to diverge. The student pairs were told to discuss what they saw in the sketch containing five page tabs; the different pages designed in the sketch contributed to the student pairs' agreement and disagreement as they had to decide when to move on to a new page or go back to a previous one. For example, in Larry and Ivy's early exploration, Ivy tapped Larry's arm to signal Larry to turn the page, which was agreed by Larry. By contrast, moments after they turned to Page 2, Larry and Ivy were in disagreement when Larry tried to turn to the Page 3 and Ivy's called out, "wait". Using a communicational framework, agreeing on when to turn a page can reveal what the student/pair was attending to or thinking about, especially by observing the

student/pair's discourse after a page was turned. In Larry and Ivy's case, Ivy was likely interested in finding more about what she later called "strange" about the area-accumulating function, yet Larry was not interested in pursuing it, since he was ready to move on to the next page. Indeed, as shown in a later analysis, Larry seemed quite satisfied with being able to state the sign of $A(x)$ and was not interested in exploring the shape of $A(x)$. This suggests that the students were no longer engaged in one commensurable discourse at the end of their exploration like they once were.

In terms of verbal communication, the student pairs used particular words to develop their mathematical discourse individually or collectively. There were ample evidence showing that the student pairs complemented each other's communication such as finishing their partners' utterances verbally, which support that the student pairs were communicating in one commensurable discourse. Besides complementary speech, their gestures and dragging also complemented their partners'. Examples of dragging that complemented their partners' speech include: "the tangent line is increasing" (Ana and Tammy), and "is that area" (Sam and Mario) "is this slope" (Huang and George). These utterances were spoken while the speaker's partner was dragging on the touchscreen-based DGE. The simultaneous and complementary communication by two different interlocutors shows coherent mathematical thinking by both discussants. In addition, the use of pronouns "we" and "let's" as opposed to "I" and "you" was characteristic of two students who were communicating in sync. All student pairs had used the pronoun "we" to communicate as a collective, which indicates that the student pairs were thinking as a collective at that moment. In contrast, at one moment, Huang and George engaged in conflicting discourses while both students were dragging simultaneously. After dragging 'a' and 'x' respectively for five seconds simultaneously, Huang asked George, "what are you doing," implying that he did not understand what George was doing. This suggests the students' dragging routines had given rise to a commognitive conflict between the students.

Hence, my study argues that in order to capture individual or collective participation in the development of mathematical discourse, it is important to attend to student pairs' linguistic and non-linguistic communication. The importance of a multimodal communicational approach was shown in the analysis of Sam and Mario.

Although Mario did not speak much throughout the task, he was actually engaged in the exploratory activity as much as his partner, Sam. His participation can be found in his non-verbal communication, such as gesturing the shape of a quadratic function, as well as dragging rapidly to show the shape of the area-accumulating function. This communication was either a response to Sam or later responded to by Sam, which suggests that the student pairs were developing the mathematical discourse collectively. Conversely, Mario's use of the scribing gestures was never responded to nor initiated a response by Sam. Therefore, it can be said that his scribing gestures offered a case of Mario communicating intra-personally rather than interpersonally.

As touchscreen technology continues to enhance digital experience of learners, its affordances for mathematical learning, especially as it is utilised in the form of touchscreen-based DGEs, is an emerging area of study. There is a need to understand *what* mathematics is communicated and *how* it is communicated in an era of touchscreen DGEs. This sub-section focuses the latter, by addressing how individualistic or collective mathematical thinking are manifested in exploratory activities with touchscreen-based DGEs. I used Sfard's communicational approach to enrich this discussion. In particular, I identified the kinds of words, gestures and touchscreen-dragging that were characteristic of individual and collective discussion. Consistent with a previous discussion, a multimodal view of communication was essential for framing this discussion. Studying student pairs' speech, gestures and dragging synchronically provides information on whether or not they are communicating coherently at the given moment, while a diachronic analysis helps to see whether or not their discussion becomes conflicted during the course of the activity. Conflicting discussion can be detected by a disagreement on when to turn the page, questioning the action of one's partner, non-complementary use of speech, gestures, and dragging, as well as the use of pronouns in a singular form ("I" or "you") as opposed to a plural form ("we").

This developed framework for understanding individual or collective participation within exploratory activities with DGEs is especially relevant for classroom teaching because it informs when students are discussing as a pair or individually. This framework can be used by classroom teachers to decide when intervention is needed to help bring the discussion back in sync. Hence, my study extends Wells' (2014) notion of

“teaching from the sideline” by addressing linguistic and non-linguistic features of student pairs’ discourse for informing the teaching of exploratory activities with touchscreen-based DGEs.

8.2.3. Saming, reification and encapsulation

The third and final dimension proposed in my framework is the type of process used to expand in one’s mathematical discourse. As shown in the study, three Sfardian processes were evident during these discussions: saming, reification, and encapsulation (Figure 42).

Saming	Reification	Encapsulation
<ul style="list-style-type: none"> • The act of calling different things the same name. • Examples: <ul style="list-style-type: none"> • Calling two objects "the same" by speech • Using gestures to realise a function geometrically while referring to the function algebraically by speech • Calling two sets of traces "the same" while dragging 	<ul style="list-style-type: none"> • The act of replacing of talk about processes with talk about objects. • Examples: <ul style="list-style-type: none"> • Using a pronoun to reify a function as a discursive object, the ordered pair $(x, f(x))$ • Using gestures to enact the movement of an ordered pair $(x, f(x))$ • Speaking about the movement of $(x, f(x))$, such as, "it's going up [or down]" while dragging 	<ul style="list-style-type: none"> • The act of assigning a noun or pronoun (signifier) to a specific set of discursive objects, so that some of the stories about the members of this set that have, so far, been told in plural may now be told in singular. • Examples: <ul style="list-style-type: none"> • Calling the set of all $(x, f(x))$ by "this graph" • Using gestures to realise the graph of a function geometrically as an object • Using rapid dragging to obtain the traces faster, thereby realise the function as an object.

Figure 42 Types of processes in the development of the mathematical discourse

The analysis of Katie and Ivy’s communication brings to light the importance of saming, reification and encapsulation for developing one’s mathematical discourse. The students’ discourse revealed that, although they both “found” something, their realisations of the area-accumulating function were significantly different. Ivy’s discourse was a reified one; she talked of the area under ‘f’ as the green point, the *ordered pair* $(x, A(x))$. She used verbs in present-continuous tense to describe the movement of the green point as “going up” and “going down”, which shows that she was thinking of the

green point as a singular, as *an* ordered pair. On the other hand, Katie consistently used the “graph” as a noun to refer to the set of all green traces and the verb “is” to state the relationship between ‘f’ and the “graph”. Accompanied with her two statements, “this graph *is* the derivative of this graph,” she used gestures to realize the shape of the respective “graphs” on the page. These gestures and word use suggest her realizations of both functions as mathematical and geometrical objects. Thus, Katie not only reified the process of area-accumulation as the ordered pair $(x, A(x))$, she also encapsulated the set of all ordered pairs into a new discursive object, a *graph*.

My analysis supports Sfard’s (2008) theory in that encapsulation and reification are two different processes for creating new discursive objects. While some cognitive theories (Dubinsky, 1991; Gray & Tall, 1994) make no distinction between the two, this study shows the merit of distinguishing between them. Analysis of Ivy’s discourse shows that reification does not always lead to encapsulation. Hence, combining the two processes as one, as in Dubinsky (1991), may be problematic. Ivy’s difficulty with making connections between ‘f’ and the “green traces” may be attributed to her reified discourse of area as an ordered pair but not as an encapsulation. This may have inhibited her participation in the subsequent discussion. Conversely, Katie’s ability to see the “green traces” as a single graph may have facilitated her learning.

This study makes a case for studying students’ communication and changes in communication in the learning of calculus. In particular, I argue that attending to students’ use of naming, reification and encapsulation may help address difficulties in learning mathematics, especially in terms of dynamic aspects of calculus. As I mentioned in Chapter 2, a number of studies reported students’ difficulties in creating graphical representation of a function’s derivative and area-accumulating function (Tall, 1986; Ubuz, 2007). The results of my study shed light in this area, suggesting that encapsulation is needed to think about derivative and area-accumulation as functions (graphs) and to work with relations between them at an object-level. Reflecting on my role as a teacher during the task, it seems that my questioning and gestures to signify the set of ordered pairs as an object may have facilitated the process of encapsulation for some students. With regard to the design of the technology, the *Hide/Show* buttons allowed the students to talk about their ideas gradually one button at a time, but more

importantly, also evoked the idea of functions as objects that could be hidden or shown at once. In addition, as the students saw the green traces “grow” on the screen, the *Tracing* tool potentially contributed to the visual mediation of area as a function (Ferrara, Pratt, & Robutti, 2006). Yet, other students did not clue in to thinking about the set of green traces as an object—the graph of the area-accumulating function—even after I had prompted them about its shape. This raises question about what improvements in the sketch design could be made to help facilitate the process of encapsulation for these students. More details about improvement to sketch design is discussed in Section 9.1.

To revisit, I proposed a framework for understanding student pairs’ discourse within exploratory activities using touchscreen-based DGEs. The proposed framework consists of three dimensions: (1) patterns of communication in the development of the mathematical discourse, (2) individual or collective participation and (3) processes used to expand in one’s mathematical discourse. These lenses can be applied to examine the same discursive activity in different ways, as illustrated in Figure 43. For example, a student pair may be participating collectively, in which they use saming to talk about the functionality of the DGE as “the same” (engage initially in the development of the MD). Or, they may use saming to talk about the “sameness” of $y=x$ and $y=x+3$ in terms of slopes (engage actively in the development of the MD). Finally, they may use saming to refer to the green traces as points on the antiderivative function for the chosen $f(x)$ (engage in a ‘formal’ MD). These examples show that student pairs can use saming to engage in the development of the mathematical discourse differently. The mutual inclusiveness of the proposed categories allows for a more robust description of student-pair communication with touchscreen-based DGEs. While the proposed dimensions are not exhaustive, they are meant to highlight the results of the study from a communicational perspective.

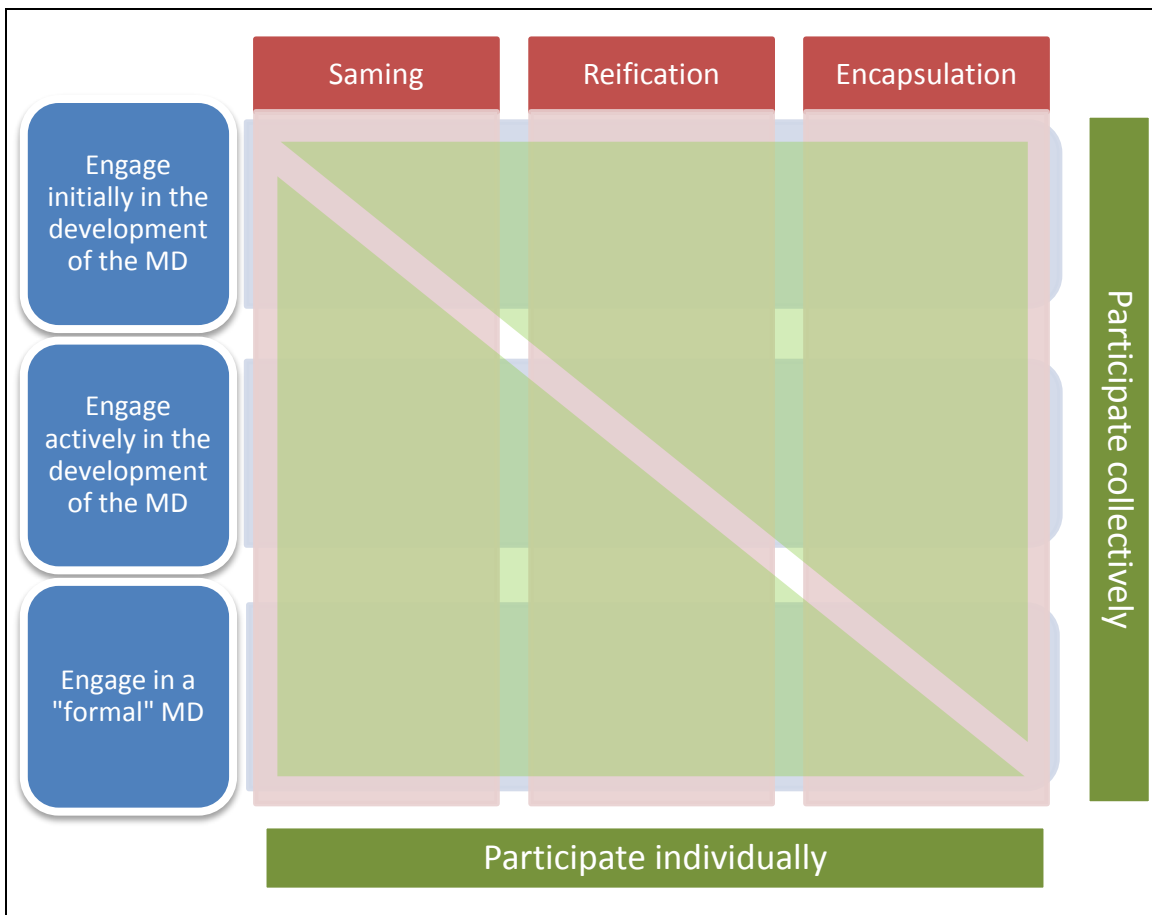


Figure 43. A framework for understanding the student-pairs' mathematical discourse during exploratory activities with touchscreen-based DGEs

8.3. The significance of a multimodal communicational approach for bilingual learners

In this section, I focus my discussion on implications of my study with regards to bilingual learners' competence in mathematical activities with touchscreen-based DGEs. The participants of both Parts I and II of the study were bilingual learners who had been studying in Canada for two to four years at the time of study. When their bilingual background is considered, it can be argued that non-linguistic communication was a significant form of communication for the bilingual learners. As I mentioned in Sub-section 8.2.1, my study provides ample evidence that gestures took on a prevalent role in the students' communication. For example, gestures were used extensively as a

dynamic visual mediator to communicate the movement or shape of the green traces, while deictic gestures or gestures in the form of actual realisations accompanied locative nouns (“here”) and pronouns (“this” or “it”) to communicate what was presented by the sketch. Significantly for bilingual learners, these gestures could reduce the number of words or even replace the words to be spoken in a sentence, simultaneously reducing the language demands on bilingual learners.

Dragsturing emerged in the touchscreen interaction with DGEs and fulfilled the dual function of dragging and gesturing. Dragsturing was repeatedly demonstrated by the student pairs for questioning and exploring calculus ideas, as well as for developing routines of conjecturing and verifying calculus relationships. Initially, the students seemed unsure as to what to make of the sketch; dragging enabled them to formulate their questions about its behaviour. Then, they began to explore and conjecture the relationship of the two functions in both geometrical and algebraic terms through dragging and gesturing. Again, these results suggest that non-linguistic communication, in the form of dragsturing, was an important mode of communication for bilingual learners. Bilingual learners do not have the luxury of a comprehensive English vocabulary; thus, the dragging modality afforded them a non-verbal form of communication. Where monolingual learners have the ease to communicate verbally while dragging or gesturing, bilingual learners may rely on dragging or gesturing to communicate mathematically in the absence of speech. In other words, the language demands on bilingual learners in mathematical communication can be potentially reduced by the use of dragsturing.

As discussed previously, the presence of a dynamic visual mediator gave rise to new conversational structures in the students’ discourse. In particular, gesture-gesture and gesture-utterance sequences were observed repeatedly in the communication. Related to this, *dragsturing* allowed two students to converse simultaneously without interfering with each other. Thus, the results concur with Sfard (2009) in that utterances and gestures (or dragging) serve complementary functions. More importantly, these results were in tune with Grosjean’s contention that bilingual learners blend multiple competencies in mathematical activities. As it has been shown throughout the analysis, the student pairs communicated about significant calculus ideas without speaking in long

sentences. They also demonstrated significant mathematical Discourse practices even though they spoke in “broken” English and their utterances were incomplete at times. For example, Mario and Jay were able to express the encapsulation of the set of all $(x, A(x))$, predicted the shape of $A(x)$ and conjectured that $A(x)$ was the antiderivative of $f(x)$ by using a combination of rapid-dragging and limited speech. These findings support Grosjean’s analogy, in the sense that bilingual learners communicate by blending speaking, dragging and gesturing, like hurdlers who blend jumping and sprinting competence. Personally, I find this analogy extremely applicable and helpful to understand the importance of non-linguistic communication for bilingual learners.

As another personal anecdote, I have always had the impression that bilingual learners are “quiet” and reluctant to participate in classroom discussion. Prior to the study, I identified Mario who fitted this description; to me, he was a student who seldom participated verbally in whole-class discussions. However, the analysis showed that Mario was not as “quiet” as he seemed to be. Rather, he participated actively and coherently with Sam during their exploratory activity with DGE. By participation, I do not mean just “talk”, but listening and watching also matter. This analysis was achieved by adopting a sociocultural view (Moschkovich, 2007a), together with attending to the interplay between linguistic and non-linguistic communication. Methodologically, the organisation of the transcript was helpful to illuminate his participation, by informing the “speaker”, “gesturer” and “dragger” and the utterances that were spoken while a dragging and gesturing was synchronically performed. Through this analysis, it can be shown that Mario, whom I saw as a typical “quiet” and “passive” bilingual learner, was actually participating and learning actively in the mathematical activity.

The student pairs Sam/Mario and Larry/Ivy did not share a common home-language. Despite this, they communicated effectively with each other using a combination of linguistic and non-linguistic communication. Results like these have implications for day-to-day teaching and learning in classrooms where individual bilingualism is manifested (Planas, 2014). In particular, gestures and dragging enabled individual bilinguals, who do not share a common home-language, to engage in communication by means other than the verbal language alone. This implication is different from the one previously discussed concerning the language demands on

bilingual learners. It is argued here that in multilingual classrooms where learners do not share a common home-language, which it is often the case in major cities in Canada, there is a need to widen the view of *language*, defined by Sfard as tools for communication, to include non-linguistic tools. Above all, the study points to the use of DGEs and pair-work activities for facilitating meaningful discussion of mathematical ideas and development in one's mathematical discourse in today's increasingly multilingual classrooms.

More broadly speaking, the results of the study also argue for an expanded view of mathematical communication for monolingual and bilingual learners alike. In other words, focussing on speech alone is not sufficient to capture one's competence fully in mathematical communication. This calls attention to assessment practices that place excessive emphasis on linguistic (written) features of mathematical discourse, which consequently run the risk of dismissing opportunities for learners to demonstrate their mathematical learning by non-linguistic means (gestures and diagrams). Although my study recognises that an expanded view of communication is beneficial for both monolinguals and bilinguals, I have chosen to highlight, in this section, its significance in uncovering bilingual learners' competence in mathematical communication using a sociocultural lens. This line of work is much needed in the field of linguistic diversity in mathematics education to challenge a deficit model that focusses on what bilingual learners cannot do and do not know. I suggest that this discussion can also be extended for cross-disciplinary work towards a "proficiency-based approach" for minority groups of learners, where much research has been already been done to empower mathematics learners with special needs (Peltenburg, van den Heuvel-Panhuizen & Robitzsch, 2012; van den Heuvel-Panhuizen, 2015). The potential for a "proficiency-based approach" in future research is promising, and it is also valuable for critiquing a normative paradigm in mathematics education.

Chapter 9. Conclusion

Life is good for only two things: discovering mathematics and teaching mathematics. (Siméon Poisson)

In this final chapter, I provide responses to my research questions posed in Chapter 4. In doing so, I revisit key results from both Parts I and II of my study and show how they contribute to research in the fields of linguistic diversity, multimodality in mathematical thinking and DGE-based learning. At the end, I reflect on how this study has *changed* me personally as I continue my journey of being a bilingual mathematics teacher and life-long learner.

9.1. Responses to research questions

The first of my research questions explores bilingual learners' use of linguistic and non-linguistic modes of communication in mathematical thinking and learning: *How do bilingual learners utilise linguistic and non-linguistic modes of communication during pair-work on mathematical activities with the use of touchscreen-based DGEs?* My study reveals that linguistic and non-linguistic modes of communication complement one another in bilingual learners' communication. This important interplay between verbal and gestural communication enabled the participants to communicate effectively about significant calculus ideas.

Results from Part I of my study clearly show that word use and gestures evolved in the presence of different types of visual mediator. In general, I observed distinct verb use when student pairs moved from one environment to another. Also observed was a more prevalent use of “scribing gestures” and “measuring gestures” when students were talking about static diagrams. It could be said that the “scribing gestures”, which reflected an algebraic way of thinking about calculus, and the “measuring gestures”,

which were used to convey change of distance, were situated in relation to the type of visual mediator present, one that was static in nature.

In terms of gestures performed in a dynamic environment, my most significant finding concerns the act of touchscreen-dragging, which had a dual meaning of gesturing *and* dragging. Theoretically, my chosen framing of Sfard (2008, 2009) was effective for understanding the kinds of mathematical thinking-communicating that goes on in the act of dragsturing or gesturing. In addition, my distinction between static diagrams as static visual mediators and dynamic sketches as dynamic visual mediators allowed me to explore the three-way relationship among types of visual mediator, mathematical thinking and modes of communication. As I explained in Chapter 8, the results of Part I have important implications for classroom teaching, since they show that mathematical thinking is not located solely in the students' heads, but also in the moving hands, those which interact with the given visual mediators. In order to develop and assess certain aspects of mathematical discourse in the study of functions and calculus, I argue that providing situations for students to communicate these ideas in both static and dynamic environments, as well as adopting a multimodal view of communication, is beneficial.

While the results from Part I of my study address the situatedness of mathematical thinking and modes of communication, Part II of my study explores the development of mathematical discourse about a target concept, the area-accumulating function. In response to the first research question, I developed a framework for understanding student pairs' communication during pair-work on exploratory activities with touchscreen-based DGEs. Three dimensions for studying student pairs' development of mathematical discourse were proposed: the first identified the patterns of communication, the second identified the type of participation, while the third identified the kind of Sfardian processes used in the development of mathematical discourse. Upon studying the interplay between linguistic and non-linguistic modes of communication, I found recurrent use of certain words, gestures and touchscreen-dragging that were characteristic of each sub-category about student pairs' development of mathematical discourse using touchscreen-based DGEs. While I will not restate the

details of each sub-category in this chapter, I want to comment on two important discussions about my proposed framework.

The first is that touchscreen-dragging was a significant mode of communication for the student pairs throughout their exploration with the DGEs: it was used for engaging in different levels of development, types of participation and kinds of mathematical processes during the development of mathematical discourse. For example, a student pair may use touchscreen-dragging to engage in “saming” to show similarities about two functions; they may also use touchscreen-dragging to talk about the movement of a point $(x, f(x))$ which suggests that they have a reified discourse about $f(x)$, or they may use what I term “rapid dragging” as a way to encapsulate the set of points $(x, f(x))$ as one object, the graph of $f(x)$. My findings about the *versatility* and *consistent* use of touchscreen-dragging in mathematical activities informs future research both in the areas of conversation analysis, in what defines a *turn*, and DGE-based learning, in terms of the affordances of the touchscreen for facilitating temporal mathematical thinking in the act of *dragsturing*.

Secondly, with regard to implications for teaching, my proposed framework concurs with my literature review in terms of the how opportunities for student pairs or groups to “talk within” and “talk about” mathematics are beneficial for the development of mathematical discourse. A familiarity with or understanding of the functionalities offered by the DGEs is needed before student pairs can engage actively in their development of the mathematical discourse. On the other hand, my proposed dimension examining individual or collective development of the mathematical discourse can be used by classroom teachers to decide when intervention is needed to help bring the discourse back in sync.

My second research question asks, *what kinds of mathematical Discourse practices do bilingual learners engage in, and what kinds of calculus ideas are communicated during pair-work on mathematical activities with the use of touchscreen-based DGE?* For this question, I focused on my participants’ backgrounds as bilingual learners, and I used a sociocultural view of bilingual learners along with Sfard’s communicational framework to identify their competence in mathematical activities with

touchscreen-based DGEs. In general, they used a combination of their home language, non-verbal communication and the given visual mediators to engage in a variety of mathematical Discourse practices, such as naming, defining, reasoning, comparing, conjecturing, verifying and predicting. My study also revealed that the student pairs engaged in different mathematical Discourse practices when prompted by static diagrams as opposed to dynamic sketches. This analysis was achieved by adopting a *non-deficit* model and a *multimodal* communicational approach for understanding bilingual learners' communication, since bilingual learners utilise much more than verbal language alone to communicate mathematically.

Related to this, my results showed that when student pairs engaged in mathematical Discourse practices, they frequently used non-mathematical terms as well as verbs in present-continuous tense ([verb]+ing), and their talk often did not resemble a theorem-like discourse. For example, the participants shared a mutual understanding of what “this” or “it” meant, even without stating what they meant explicitly. These findings are illuminative because, traditionally, much time and effort in mathematics teaching is devoted to building fluency in the use of mathematical terms and in developing a formal mathematical discourse. Yet, my study shows that students can engage in reasoning, conjecturing, verifying and predicting practices *without* necessarily working with this type of fluency. While I am not suggesting that this fluency is unimportant, I do argue that more time and effort should be devoted to letting student pairs or groups discuss and explore mathematically even without the “right” language. This is because when learners are not evaluated for the language they use, the demands of using “proper” language is reduced, which may help them focus on the mathematical ideas and engage mutually in mathematical Discourse practices. In addition, the nature of the mathematical activities, in which students were asked to engage in an open-ended discussion of what they saw on the touchscreen-based DGEs in pairs, was helpful for fostering active engagement of mathematical Discourse practices.

To respond to the second part of the research question, in terms of the calculus ideas communicated, I found the use of DGEs to be instrumental in facilitating dynamic and temporal—motion-based and action-oriented—thinking in calculus. There is an overlap between my responses to this part of the research question and to the third

research question, and therefore, I will not elaborate further here. However, from the perspective of highlighting bilingual learners' competence, again, I underscore the interplay between linguistic and non-linguistic modes of thinking about dynamic and temporal aspects of calculus. I suggest that in multilingual classrooms where learners do not share a common home language, there is a need to widen the view of *language*, defined by Sfard as tools for communication, to include non-linguistic tools. I call for more research to adopt a multimodal communicational approach to examine mathematical thinking and learning in all classroom contexts, and especially in today's increasingly multilingual mathematics classroom.

The last of my research questions asks, *what is the role of technology for facilitating bilingual learners' communication during pair-work on mathematical activities with the use of touchscreen-based DGEs?* I realised that to respond to this question was not as simple as I had anticipated. This was because the role of the technology could not be isolated and studied on its own, since it was integral to all aspects of students' calculus thinking and modes of communication. As I emphasised previously, there is a three-way interaction among the touchscreen DGEs, modes of communication and mathematical content communicated when student pairs participate in mathematical tasks with touchscreen-based DGEs. Therefore, rather than isolate the role of technology, I comment on three specific functionalities and affordances of the touchscreen DGEs that may have impacted students' calculus communication as my response to this research question.

My design of dynamic sketches was aimed to highlight dynamic aspects of calculus, exploit touchscreen dragging and connect algebraic to geometric representations of calculus. With respect to the first aim, it was clear that the student pairs communicated a motion-based and dynamic sense of calculus when interacting with the touchscreen-based DGEs. This was reflected in their use of verb tense along with particular words and gestures that imply motion, temporality and change. It was interesting to note that the sense of change communicated was *continuous*, as indicated by expressions like "smaller and smaller", "greater and greater" and "closer and closer". Besides, the students seemed to be able to notice what was *variant* and what was *invariant*, as exemplified by, "As the ladder falls, we can see that x ... *is increasing* and y

is decreasing, but z remains constant.” These forms of verbal communication were complemented by students’ gestures or *dragsturing*, which takes me back to the interrelationship among touchscreen DGEs, modes of communication and mathematical content communicated.

Secondly, and to my surprise, the touchscreen-dragging affordance of the DGE impacted students’ calculus thinking more than I had anticipated. The *dragging* affordance helped the students formulate questions about the sketch and later was used to describe the variance and invariance they saw on the sketch. It also helped them generate conjectures about the changes and see them as being functionally dependent on their dragging actions. Further, the *touchscreen* affordance was exploited when the students utilised touchscreen-dragging as a form of gesturing, which I term *dragsturing*, to communicate a dynamic and temporal sense of calculus. Importantly, the design of the sketches facilitated the blending of touchscreen-dragging and gesturing. Had the design of the sketch featured a slider in order to change the numerical values, the gesturing of the shape of the function such as gesturing about the change of the function would not be readily blended in the dragging with one’s finger. This raises question about future sketch design. Specifically, I suggest that future sketch designs should exploit *dragsturing*, rather than introduce sliders, for conveying continuous change.

Lastly, there was strong evidence that the students communicated about covariance and graphs of derivative and area-accumulating functions in a robust manner, rather than just algebraically. This was one of my aims when designing the sketches, since my review of literature pointed out that both high school and college calculus students have significant difficulties dealing with multiple representations and working with simultaneous change of variables—particularly with the graph of area-accumulating functions. Returning to the results of my study, I see encouraging evidence that the student pairs were engaged in meaningful discussion about the change of area (geometrically) and the change of the green trace (numerically and graphically) as both being dependent on the change of ‘ x ’, and this demonstrates their competence to work with covariation and simultaneous change of variables. Moreover, the touchscreen-dragging and tracing affordances of the DGE, along with the dynamic visual mediation and the effect of colour-coding different mathematical objects, facilitated processes of

saming, *reification* and *encapsulation* to take place during exploratory activity. It is worth noting that the student pairs from Part II of the study had experience with working dynamic sketches before the time of the study. Their prior experience interacting with touchscreen DGEs likely helped them in terms of attending to change and exploiting the dragging modality. In other words, I contend that similar results may only be replicated if the student pairs had a degree of experience in exploring and discussing calculus concepts with touchscreen-based DGEs.

Having talked about the positive effect of the touchscreen DGEs on students' calculus thinking, I now reflect on the role of the paper-and-pencil task as well as missed opportunities with regards to the use of technology that could have made an impact in the study. While the dynamic properties of the DGEs supported the students' noticing of *change*, the paper-and-pencil task achieved something different: the students noticed the domain and range of the functions when they had to draw the graphs physically on the whiteboard. Therefore, the two types of task were complementary in facilitating students' thinking about the shape of the graph and its domain and range respectively. In terms of missed opportunities, I suspect that introducing a Hide/Show button to reveal the area-accumulating function, $A(x)$, as one object may help facilitate the process of encapsulation. This may also lead to an exploration of the fact that the constant ' a ' in $A(x) = \int_a^x f(t)dt$ does not affect the shape of the area-accumulating function, but rather, it affects only the vertical translation of the graph. As the participants seemed to find it difficult to make sense of the vertical movement of the green point, perhaps a vertical movement of the green graph as a whole would be more effective for mediating the effect of changing ' a '. More research in this area is needed to test such hypotheses.

9.2. Personal reflection and concluding remarks

Throughout this dissertation, I have referred to calculus as a study of *change*. In this light, as I am at the end of this very doctoral research, I found myself experiencing some "calculus" in me. I say this in the sense that every aspect of conducting this research, from the literature review to the analyses of data and discussion of results, has given me new insight into my dual role as a mathematics educator and mathematics

education researcher. In this last section of my dissertation, I reflect on what has been a valuable and rewarding endeavour for me both from a teaching and from a research perspective, and the lasting changes that I experienced through this journey.

First and foremost, this research experience has changed my understanding of what it means to teach and learn mathematics in multilingual classrooms. As I wrote in the introduction, I entered my doctoral study out of interest to learn more about bilingual mathematics learners and about myself. I began to acquire the “language” that I needed to make sense of the complexities of teaching in multilingual contexts after reading the research literature. For example, I could relate my experience closely to the teacher participants in Adler (2001), who identified the dilemma of code-switching, the dilemma of mediation and the dilemma of transparency as sources of tension that arise generally between explicit language teaching and home- or everyday-language learning in exploratory settings. However, being able to describe the tension was not enough for me; I wanted to explore ways to address these tensions—the many times that I found myself struggling or looking for the “right” word to use while teaching.

Reflecting on this research study, I realise that adopting Moschkovich’s sociocultural view of bilingual learners may offer insight into addressing the teaching tensions identified by Adler and that I also experienced. My study suggests that rather than viewing explicit and implicit language as sources of tension, teachers can foster opportunities for home- or everyday-language learning in exploratory settings—in other words, foster an environment for active participation of valued mathematical Discourse practices *before* they explicitly teach students to develop their formal mathematical discourse. As I have shown in this study, my students understood the expectation of moving towards a “formal” mathematical discourse when they were near the end of the activity. For example, there were observed increases in the number of words spoken, particularly in mathematical terms, and decreases in the number of gesturing and dragging turns in the last ten-minute interval for each student pairs’ communication that I analysed.

This was, in my opinion, a result of my modelling what was considered mathematically acceptable in my classroom community. Moreover, I checked in with the

pairs from time to time to make sure that they could “talk about” mathematics after their “talk within” mathematics and that I could model an “acceptable” mathematical discourse. Given these opportunities, my students engaged in valued mathematical Discourse practices using informal language and gradually developed a more “formal” mathematical discourse during the activity. On the other hand, the tension that I felt between explicit and implicit language teaching was insignificant to me. When I checked in with the pairs early in their exploration, the students talked to me using a combination of informal and formal mathematical language, and I did not feel the need to correct their language because even I was able to understand what they meant by “this” and “that”. Later, the students began to move towards a more formal discourse on their own and sometimes with the help of my prompts.

From this experience, I realised that the tension I felt between using implicit and explicit language teaching can be reduced by adopting a non-deficit model and by modelling a mathematical discourse that is acceptable in my classroom community. I learned not to evaluate students’ talk, even if they were using incorrect vocabulary or their home language, and to encourage them to communicate non-verbally because I now understand that so much about mathematical ideas can be communicated with gestures, with dragging and by referring to static diagrams. Echoing Moschkovich, I felt that making a distinction between every day and mathematical discourse may be problematic because of the mutual inclusiveness of the two. Instead, I am more interested when my students engage in mathematical Discourse practices because, to me, that is what *doing mathematics*, is all about.

Looking forward, I plan to take these ideas into my own mathematics classroom. As a limitation to my study, I did not carry out the research within a real class setting; rather, I simulated a classroom setting by inviting one or two pairs of my students to discuss and explore with the DGEs in my classroom outside of school hours. Although the tension of teaching in a class of up to thirty will be multiplied by the increase number of pairs or groups formed as well as the diversity of languages present, I believe that it is not an impossible task: students in multilingual classrooms can develop their mathematical communicative competence through careful revoicing, modelling, providing tools and opportunities for them to discuss mathematics meaningfully and,

above all, adopt a *sociocultural* view and *multimodal* view of communication. If I could say one thing to mathematics teachers in major cities across Canada, where a large part of the population of students is made up of individual bilingual learners, I would like to highlight the notion that learning mathematics is much more than a cognitive and mental process; it is one closely tied with communicating, participating acceptably socially and culturally, exploring, reasoning, conjecturing, verifying, predicting, etc.—*and*, doing all of these in different *modes*: speaking, gesturing and interacting with visual mediators. Although I would say the context of Canadian multilingual classrooms are less politically-charged than the ones in other parts of the world, I believe that the above ideas can serve to inform mathematics teachers what it means to teach mathematics in multilingual classrooms outside of the Canadian context.

Besides having a keen interest in bilingual mathematics learners, I also wanted to explore, in particular, the teaching and learning of mathematics with the use of DGEs in my classroom context. Before the study, I hypothesised that the linguistic demands on bilingual learners' mathematical communication may be reduced when touchscreen-based DGEs are used. More generally, I also wanted to investigate the effect of using touchscreen-based DGEs on calculus thinking for all learners. Indeed, my study helped answer these questions that I had proposed before the study. Revisiting the transcripts of students' talk about calculus in a touchscreen, dynamic environment, I understood that they could not have talked this way without the technology being present. The student pairs engaged in exploring, reasoning, conjecturing and verifying among other mathematical Discourse practices that I valued in my classroom in part because of the dynamism pertained in the DGEs, while the touchscreen-dragging modality afforded them to communicate mathematically without using long sentences. Because of these findings, I realised that the DGE is not merely a visual representation of calculus, it is part-and-parcel of the students' discourse about calculus at a given moment, which also played a key role in their change of discourse over time, what Sfard calls learning.

Over the course of four years of implementing consistent use of touchscreen-based DGEs in my calculus lessons, I have noticed that my students' understanding of the graphs of derivative and area-accumulating functions has improved. Their discourse about simultaneous and continuous change is much more developed than some years

ago when I taught it without technology. Many students told me how much they appreciated learning with exploratory tasks with DGEs because the work helped them make sense of calculus ideas, and more importantly, made the learning much more interesting. I could see their enthusiasm during their discussions in pairs or trios, where they were actively engaged not only verbally but also with their hands. All of these observations have changed the way I approach my teaching of calculus: I could no longer teach calculus without using DGEs or introduce calculus ideas without facilitating exploratory activities in the beginning of my lessons. My passion for the use of DGEs for teaching mathematics have grown since this research; during the last three years, I have presented about my lessons incorporating DGEs at conferences for teachers' professional development at a provincial level, and I am looking forward to more opportunities to make an impact on high school mathematics teaching in the professional community.

Besides learning about the role of touchscreen-based DGEs on bilingual learners' communication patterns and mathematical thinking, I have also gained a deeper understanding about tool-based learning environments in general. Taking on the role of researcher, I have become interested in tool-based learning environments other than touchscreen DGEs. In particular, I would like to utilise my study's findings as the basis for investigating communication patterns and mathematical thinking in other tool-based learning environments such as 3D printing environments. I understand that the relationship among tools, communication patterns and mathematical thinking is very intricate, and research in this area will enlighten mathematics teaching and learning with tools greatly in the era of digital technology.

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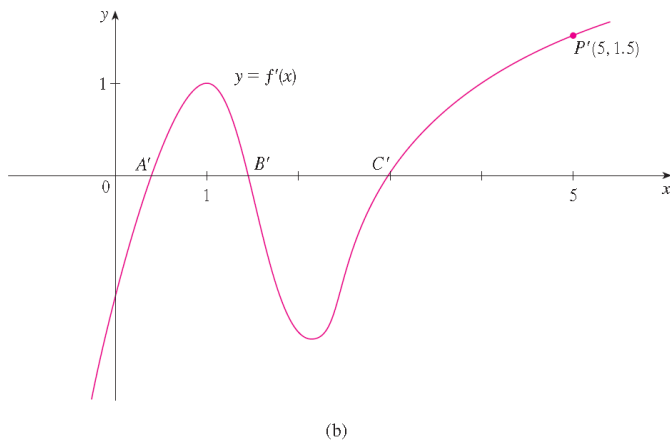
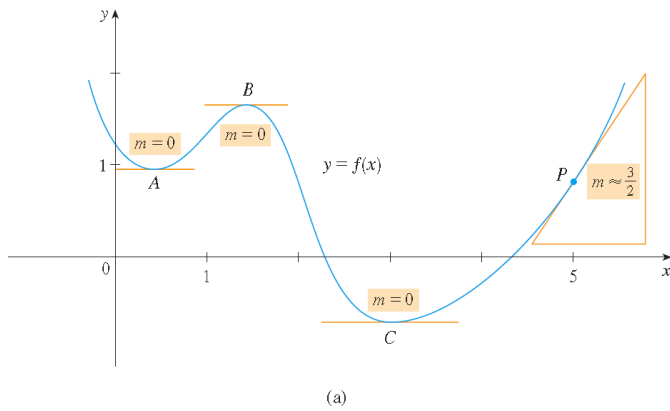
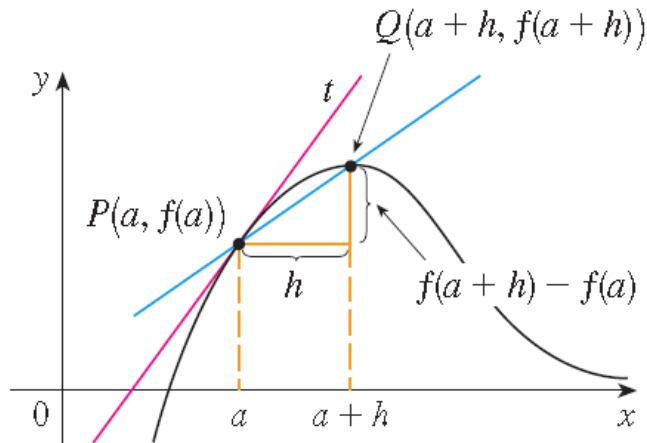
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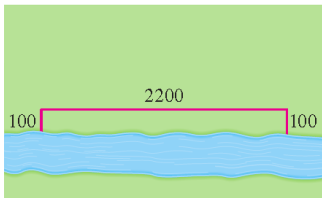
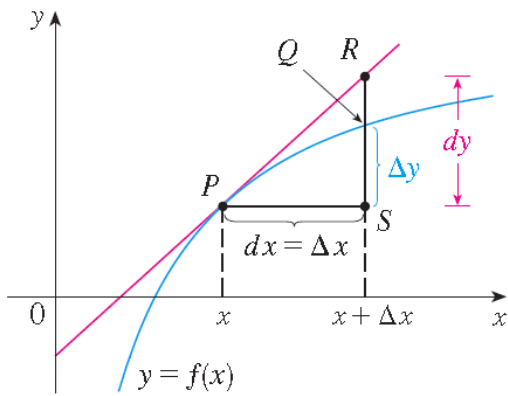
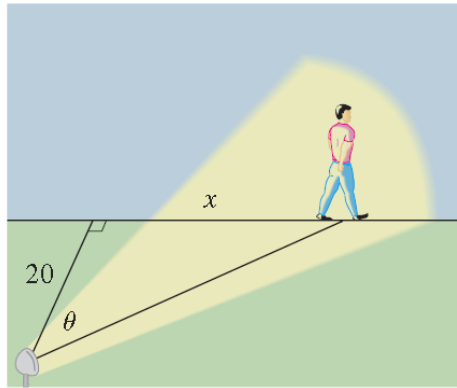
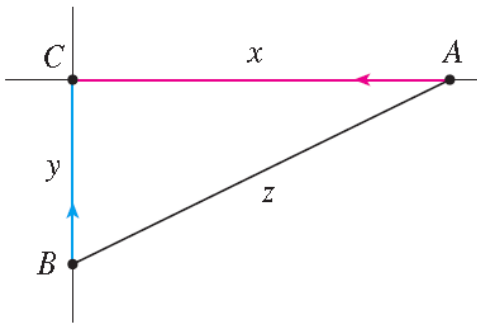
Appendix A.

Static diagrams used for Part I of the study

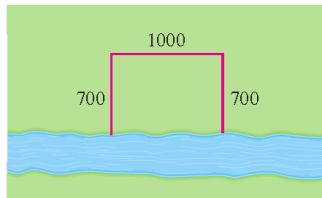
The following are figures taken from Stewart (2008) and used for Part I of the study.



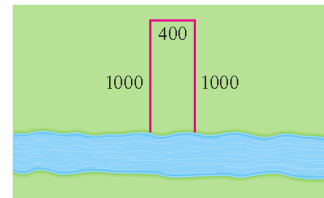
□



$$\text{Area} = 100 \cdot 2200 = 220,000 \text{ ft}^2$$



$$\text{Area} = 700 \cdot 1000 = 700,000 \text{ ft}^2$$



$$\text{Area} = 1000 \cdot 400 = 400,000 \text{ ft}^2$$

FIGURE 1

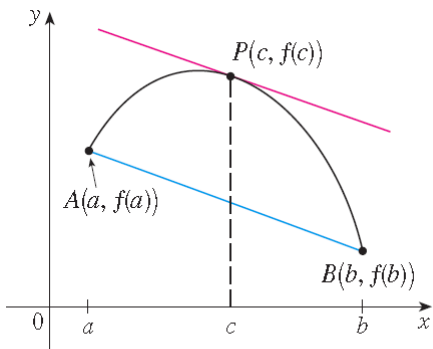


FIGURE 3

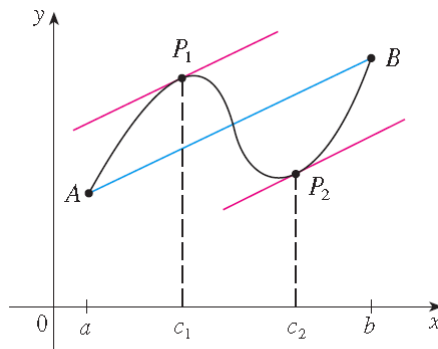


FIGURE 4

Appendix B.

Transcript of introduction and instructions to the participants during Part II of the study

The following transcript is taken during the first 1:50, in the beginning of Larry and Ivy's engagement of the task. It illustrates the introduction and instructions that I had given to the participants during Part II of the study

Table B1

	Timespan	Content	Sp -er	Dr -er	Ge -er
1	0:00.0 - 1:32.6	I am, I am. Um...so, this is...I'm going to show you this applet. In case anything happen, you know how to find this applet. This is the, um, Sketch Explore app. It's ready to go. So, there are four pages. At the end of the four pages, I'd like you to try. There's something to try. You might want to use one of these to draw on, or to write on. So physically, you're just gonna explore the buttons. Explore the sketch, we call this a sketch. And try to figure out something. You have not seen this before, so there's no worry about right or wrong answer. Anything that you can come up with is good, right? And so you talk about what you see and what, see if you can learn something or understand something. At the end you can try this. I-I think, I suggest that you do these four pages first, like, in the order, and then at the end you can try and also you can come back to it, any time. So, I'm hoping this will take around fifteen, thirty minutes. It doesn't matter how long you take. But, I might, uh, come in and just to, check in and see. If you guys have a question, I can actually answer it. So, uh, are you guys ok seeing this? So I've adjusted the lighting so that it's like, not too light, 'cause then my camera can see the...if it's too light it will reflect and it cannot see. Can you guys see it well? So, you guys can... are ok right? So I'll, the angle there that's so I can see that as well. So, try to just talk about it. K, any questions?	O		
2	1:32.6 - 1:33.5	No.	L		

3	1:33.4 - 1:50.4	Just as it is, kind of like in a class, right? I would have given you an iPad, I would ask you some questions, but right now, you're actually trying to learn it yourself. So we can...you guys can start. At the end, if you want, here are, here's the, something to write on, if you need it, okay?	O		
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