Knowledge Growth through Lesson Study: A Case of Secondary Mathematics Teachers' Collaborative Learning

by

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Abstract

Teachers learn their profession throughout their careers but how this happens is elusive. This research attempts to find out how mathematics for teaching (MfT) evolves in individual teachers and whether it can be promoted within a lesson study setting for in-service teacher development. The present small-scale, school-based "ethnographic" qualitative study uses participant observer methodology. A detailed analysis of one lesson study cycle is presented, focusing on a team of secondary mathematics teachers' pre and post-lesson discussions.

Analysis and interpretation of findings are structured initially in four components of knowledge for teaching: mathematical, psychological, didactical, and pedagogical. A fifth component – the philosophical – is identified in the pre-lesson discussion from the data, prompting the extension of the theoretical framework used for this study. The philosophical component assumes normative principles and value decisions, which are interwoven into the subject matter knowledge about mathematics. While it is seldom discussed it seems to be always present and can even be identified as implicit content of teaching. Shifts in teachers' cognition and practice through the lesson study process are noted across all five components of MfT which moreover are discerned to function cohesively and to resist separate analysis. The findings further show certain conditions needed to promote teachers' learning in lesson study settings, including the influential roles of mentorship and of observing one another's practice in the classroom.

The results of this study are consistent with the aim for a supportive collegial network built over time and acting as a source of continued learning and ongoing improvement of teaching practice. They also suggest that the incremental changes observed in teachers' MfT hold a promise for building confident, effective and inspired teaching through sustained professional development activity over time. However, they do not support the view that mathematics, being just distilled common sense, can be taught without intense prior endeavor in the field.

Keywords: In-service mathematics teacher development, Knowledge base for teaching, Lesson study, Mathematics for teaching, Secondary mathematics teachers, Professional learning communities

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List of Acronyms

MfT	Mathematics for Teaching
PCK	Pedagogical Content Knowledge
PUFM	Profound Understanding of Fundamental Mathematics
SMK	Subject Matter Knowledge
TIMSS	Third International Mathematics and Science Study

Chapter 1.

Introduction

1.1. Where and how teachers learn how to teach mathematics

"Where and how do teachers learn to teach mathematics?" This question has been asked many times over in the educational literature, with each answer completing the picture a little bit more. The prevailing views are based on three complementary positions. The first is the view of teaching practice as a collective phenomenon, the second view accounts for it as an individual experiential history, and the third view positions the development of teaching practice through practice itself.

The first position says that teaching is culturally based. In their book, *The Teaching Gap*, Stigler and Hiebert (1999) portrayed teaching practice in three different cultures: American, German, and Japanese. They analysed 50 – 80 samples of videos of mathematics lessons from each of these countries that were collected as part of a larger international study on student achievement, Third International Mathematics and Science Study, conducted in the 1990s (TIMSS) in Grade 8 classrooms. The differences were astonishing and clear, especially between Japan and the U.S.. In Japan, "teaching mathematics for understanding" through structured problem solving was portrayed as being the norm and very distinct from how it occurred in the U.S.. Some of its aspects were described in detail. Concepts are being developed in Japan and seldom just stated, which is mostly the case in the U.S.. Lessons are focused on important mathematics in Japan, while in the U.S. students encounter less challenging mathematics presented in a less coherent way. In Japan, students spend most of their class time learning for understanding by inventing new solutions and by applying concepts to new situations, whereas in the U.S. they spend most of their class time learning terms and practicing procedures. German teaching was portrayed as giving much attention to more challenging content knowledge and developing advanced procedures. In Japan, the students and the teacher work together to develop mathematical concepts compared to the other two countries where students are expected to follow the teacher's lead. Cultural environment has a significant influence on the development of teaching practice, which connects to the second position.

The second position has been described as the "apprenticeship of observation" (Lortie, 1975), that attributes learning about teaching to the personal educational experiences of teachers, which have been shaped over a long period of time of their own schooling when one's personality is most susceptible to forces of social influences in the classroom. For the most part, teachers replicate ways in which they themselves were taught when they were students, in any cultural environment. Both of these first two positions suggest that teaching practice is culturally engrained and self-perpetuating; therefore, it is very difficult to change.

The third positon argues that teachers learn how to teach also from their own practice. This is different from simply accumulating teaching experience. It hints that teaching is actually a learning profession. Leikin and Zazkis (2010) contributed scholarly literature on teachers' opportunities to learn their professional knowledge through teaching.

> Teachers learn both mathematics and pedagogy when teaching. In many situations, teachers' pedagogical knowledge develops when they become aware of unforeseen student difficulties. By analyzing the sources of the students' difficulties and misconceptions, teachers gain further awareness of the concepts and greater appreciation of the structure of mathematical thought. (p. 17)

A question could be raised about whether what teachers are learning through practice on their own is sufficient to transform the educational system at large in a desirable direction as envisioned by educational reforms. According to the three positions, it seems that to reform teaching practice would require overcoming the culturally and personally engrained blueprint along with placing inquiry and self-reflection at the center of the role of teacher. One way to improve both teaching practice and teacher learning could be to create opportunities for professional growth within practice itself.

1.2. Roots of my interest for this research project

Personally, I found the subject of mathematics very interesting throughout my schooling. I have no doubt that it was inspired by the kind of teaching I experienced. However, I had not considered mathematics teaching as my career when I set out on my academic path in the electrical engineering program. At a certain point in my life, after an 8-year career as an engineer, I turned to teaching, following my passion for the subject of mathematics and for assisting young people in their learning of mathematics, which I rediscovered when my own children began their schooling.

I began my teaching career well equipped with mathematical content knowledge and a certification to teach secondary mathematics. However, I did not feel that this was sufficient for my teaching practice. Particularly lacking for me was the integration of my knowledge of mathematics and knowing how to teach it using efficient methods and techniques, which of course is expected to develop through practice and, consequently, implies different levels of professional maturation. So I began to explore various possibilities to incorporate and connect the various forms of knowledge into my own teaching practice. The most influential in this respect were "Certificate Program for Mathematics Teachers," a joint program from the Department of Mathematics and Department of Curriculum Studies at University of British Columbia (UBC) and the Master's Program in Mathematics Education at Simon Fraser University (SFU). Through this work, I became interested in the role of teachers' knowledge for teaching mathematics on the teaching practice and in how this knowledge could be acquired.

It was during the time of my graduate studies in the Master's program in Mathematics Education that I read "The Teaching Gap" by Stigler and Hiebert (1999). I chose this book for my 'book review assignment' in one of my courses in which we were asked to read a book from our field and then present it to the class of participants in our cohort. I also watched the accompanying video that portrayed the differences in the mathematics teaching practices across three cultures: German, American and

Japanese. For each of the three countries, two lessons were presented (from the videos of the TIMSS study that took place in Grade 8 classrooms) as representative of mathematics teaching practice in these countries, one in algebra and one in geometry..

The descriptions of how mathematics is being taught in the U.S. resonated fairly closely with what I became aware of soon after I embarked on my teaching career in Canada, and also with what I have been witnessing since that time. However, what really caught my attention was the description of *lesson study* by the authors as a process of systemic and ongoing, in-service teacher professional development practice, which is used in Japan and that was supposedly a key to developing "good mathematics teaching." Given the observation that "much of what our society expects children to learn, they learn in school, and teaching is most clearly responsible for learning" (Stigler & Hiebert, 1999, p.3), this pointed to an opportunity to restructure schools as places in which teachers can learn too. Naturally, we can meet "good mathematics teaching" in expert teachers in any cultural setting. This includes our own. However, in Japan it was portrayed to be present nationwide, across the board. There, it is deemed to be "cultural." Here, it is more of an exception than a norm.

What really seemed, more than anything else, to be "cultural" about the Japanese approach to improving their education system was the effort, perseverance and vision to employ a systemic method for continuous improvement of the national education system. Their method is not dictated by curriculum changes and swings in philosophical orientations toward education by the powers in charge, but by empirical evidence and constant research into what works in the classroom.

This is how and why I became interested in lesson study. At that time I was working at an independent school in British Columbia, which I call West Coast Academy (WCA), where I was a faculty member and an acting head of the mathematics department responsible for leading the school's mathematics program for grade levels K-12. I sensed that lesson study could offer a perfect setting in which teachers could learn from one another, as well as from their own practice. They could reinvent themselves as learners once again, as they collaborate on various aspects of their mathematics teaching practice.

1.3. Importing lesson study into a local school setting

Simply reading about lesson study and becoming knowledgeable about it did not seem sufficient for me to be able to transfer this practice into a context that is culturally so different and that is not set up to sustain its requirements for successful implementation on a school-wide scale. However, it may have been that my enthusiasm convinced the school administrators. This is particularly true with regard to the Head of School at the time who decided to support my proposal to participate in the Lesson Study Immersion Program (LSIP), which took place in Japan in the summer of 2007 and again in 2010, and was offered by Global Education Resources (GER) as a way for mathematics educators across North America to experience first-hand the process of lesson study, as it is conducted in Japan.

The Lesson Study Immersion Program was organized by Akihiko Takahashi, Makoto Yoshida, and Tad Watanabe, three mathematics education researchers of Japanese origin who have been actively promoting lesson study at the global level. During the two-week study tour, participants of the Lesson Study Immersion Program visited a number of classrooms in which they observed "research lessons" and "post-lesson discussions," which were conducted in an "open-house" format. There were visiting pre-service teachers from a local university who were always accompanied by one or more of their mathematics methods university professors, as well as in-service teachers, both from the host school and from neighbouring schools of the same prefecture in Japan. Live simultaneous translations of all lessons and teacher discussions were provided by our guides, who broadcasted them over a radio channel that we tuned into with our MP3 players and radio receivers, and then listened to through our headphones. Classrooms of 40 to 50 students were observed by as many as 80 teachers and education researchers.

We also learned about the process of textbook development, curriculum development and pre-service teacher education. Things that I saw and learned there convinced me even more that lesson study could very well be a vehicle to raise the quality of instruction and student achievement in mathematics, at least at the level of a single school, such as West Coast Academy, if not on a broader scale. The Head of

School further supported my initiatives and asked me to present to our faculty what I had learned through participating in the Lesson Study Immersion Program that summer of 2007.

This is the story of how lesson study was imported into West Coast Academy. Such a phenomenon of importing of a practice is not unusual. It has been described by researchers who studied the use of resources in schools. Speaking of imported practice, they note that both teachers and students whose principals urge ambitious work will be more likely to exert themselves, while equally able colleagues in schools whose principals prefer less ambitious performance will be less likely to do so (Cohen, Raudenbush, and Ball, 2003).

1.4. Challenges and rationale for implementing lesson study

In our society, we have an unusually high proportion of secondary school teachers who teach mathematics but do not have a formal degree in the subject of mathematics, which can be considered as out-of-field teaching. Ingersoll (1999) undertook a study to investigate how much out-of-field teaching goes on in secondary schools in the U.S. and why. The data show that one-third of all secondary school teachers of mathematics have neither a major nor a minor in mathematics. According to Ingersoll's research, many teachers are assigned by their principals to teach classes that do not match the field of their degree or certification or both.

In British Columbia, teacher candidates who are preparing to enter secondary school teaching, enter schools of education with a four-year degree which is already completed in an area related to their intended teaching subject. It is assumed that they already possess the subject matter knowledge that they will be expected to teach and that the professional program will equip them with specialized pedagogical knowledge. However, upon completing their teacher education program, it frequently occurs that, as in the U.S., teachers are assigned to teach subjects in which they were not formally educated. Reasons for this are complex and not directly relevant to this thesis.

Mathematics teacher training and deployment vary from country to country significantly. Some societies have tightly regulated mathematics teacher education programs, both in terms of university level coursework requirements and in strictly controlled protocols on the qualifications that are required in order to teach the subject in schools, but in British Columbia it is not so.

Therefore, are we counting on teachers to develop their knowledge for teaching mathematics through their teaching practice? If so, there is a strong case for looking into practice-based, ongoing, professional development models in which teachers continue to develop their knowledge through learning from teaching itself, which is supported by their colleagues. There have been a number of such models described in recent years, with slight differences in their emphases. To name just a few, as this might aid in the understanding of the research question of this study, there are the "community of inquiry" (Jaworski, 1998), the "community of practice" (Wenger, 1998) and the "professional learning community" (DuFour, 2004; Burney, 2004). These can all be thought of as having in common the practice of professionals, such as teachers, working together in collaborative groups on developing their practice. These differences are not of much significance for this study as it is not my intent to provide an analysis of what the benefits or shortcomings of each might be.

One of the better known of such models is lesson study in which teachers participate in considerable shared planning, observation and discussion of lessons. This thesis is not about how to structure such environments and teachers' activity within them, nor is it an attempt to compare, evaluate or make a claim that one model is better than another. However, in Chapter 3 of this thesis I will relate why lesson study, and not perhaps *concept study* (Davis & Renert, 2014) or *video club* (Jacobs, Lamb, & Philipp, 2010; Hannah, 2012), was employed as a mode of engagement in this particular empirical research on the activity of one such community of secondary mathematics teachers who worked together to develop their practice.

This thesis will look at the activities of one group of teachers in the context of lesson study and provide a case study analysis aimed at trying to understand the growth in their professional knowledge. Furthermore, the research aims to find out what and how the teachers can learn in such a collaborative setting in order to develop professional knowledge that they need for their practice, through practice itself.

1.5. Focus of the research

I realized that teaching as a profession requires ongoing learning, beyond mere accumulation of experience, in order to enhance and develop the practice as a way of being with mathematics knowledge as portrayed by the *mathematics for teaching* construct (MfT). This construct has gained much attention in the recent educational research literature because "there is now widespread agreement that the quality of primary and secondary mathematics teaching depends crucially on the subject-related knowledge that teachers are able to bring to bear on their work" (Rowland & Ruthven, 2011, p. 1). However, there is no widely-accepted framework for describing teachers' mathematics for teaching. Much more will be said about this construct in the next chapter.

My motivation for this research comes from an inquiry into how mathematics for teaching evolves in individual teachers and whether it can be promoted within a lesson study setting of in-service teacher education. My research is engrained in the teaching practice, exploring the opportunities to improve it and to understand what professional knowledge is needed for effective teaching and what ways of being promote its development. This led to the main research question of this thesis: "What and how can in-service secondary mathematics teachers learn about mathematics for teaching through participating in a practice-based, professional learning community of lesson study?"

Since most research on lesson study is conducted in elementary school settings, it is my hope that answering this question from within a secondary school context will offer a useful contribution to the field of professional development of secondary mathematics teachers.

1.6. Overview of the organization of the thesis

Chapter 1 offers an introduction that explains where my research interest originated and what it hopes to contribute to the field of mathematics education. Chapter 2 provides an overview of the construct of *mathematics for teaching* (MfT) based on scholarly literature and presents a theoretical framework that is used in this thesis. Chapter 3 provides a literature review on *lesson study* as a theoretical research domain. It gives a description of the generic features of lesson study as a model for in-service teacher development as well as surveys empirical studies on in-service professional learning in the context of lesson study. Chapter 4 describes the methodology that is used in the study and situates this study as ethnographic research, providing the rationale for the choices that were made. Chapter 5 looks at the details of the professional learning of the teachers during the process of collaboratively designing for instruction in the context of one lesson study cycle that was conducted by the participants. This particular lesson study cycle acts as the main case study for this research. Chapters 6 and 7 describe the two lesson enactments, which were implemented by two different teachers in their own classrooms and then discussed by the team of teachers during a post-lesson conference. Chapter 8 presents a cross case comparison of the two lesson enactments. Chapter 9 contains results, contributions, conclusions, implications and recommendations for in-service teacher education.

Chapter 2. Professional Knowledge of Mathematics Teachers

Teachers and teacher educators have always been interested in determining the kind of mathematical knowledge needed by teachers to teach effectively. This concept has become known as *Mathematics-for-Teaching* (Davis & Simmt, 2006) or *Mathematical Knowledge for Teaching* (Ball & Bass, 2000; Ball, Thames, & Phelps, 2008). Though, in my opinion, these terms have the same meaning and are interchangeable, and for the purposes of consistency this concept will hereafter be referred to as *Mathematics for Teaching*, or *MfT*. MfT has been a focus for research studies during the last four decades.

This kind of research seeks to gain insight into how the quality of teacher development can be improved. MfT is relevant to this thesis because it has significant implications for teacher training and professional development. This chapter takes a brief look at the history and development of MfT since its early framing in the 1980s. It also reviews important models and theories of mathematics for teaching to date and discusses these models and their implications for research and practice, as well as for this thesis. Finally, it presents the four components of teachers' professional knowledge described by Selter (2001), my preferred way of parsing MfT.

2.1. Mathematics for Teaching

It is widely acknowledged that, while content knowledge is necessary for effective teaching, there is more than just that required (e.g. Cohen, Raudenbush, & Ball, 2003; Monk, 1994). Further explorations have revealed that it is important to consider not only what mathematics teachers need to know, but also "how" they need to know it. Seminal research in this area was conducted by Shulman (1986), who proposed a new construct called *Pedagogical Content Knowledge* (PCK). The PCK construct revealed the additional, specialized knowledge that teachers need to possess in order to teach well.

Recent research in the area of MfT comes from Davis and Renert (2014), who propose the reframing of the basic question about what mathematics teachers need to know in order to teach mathematics as "How must teachers know mathematics for it to be activated in the moment and in the service of teaching?" (p. 34). Their answer to this question is:

> "Mathematics for Teaching is a way of being with mathematics knowledge that enables a teacher to structure learning situations, interpret student actions mindfully and respond flexibly, in ways that enable learners to extend understandings and expand the range of their interpretive possibilities through access to powerful connections and appropriate practice." (p.34)

They affirm the prior research that has taken place in relation to where and how this knowledge is learned. This is accomplished through mathematics courses that are taken in post-secondary settings (both through formal education and teacher training programs), as well as through the practice of teaching. They also argue for the "critical importance of the third site – namely, the community of teachers working together to understand their mathematics for the purposes of teaching it" (p. 12).

This definition implies that there are a number of factors that come into play during the process of teaching. It implies that there is some sort of interplay between the mathematical knowledge that is possessed by a teacher and the pedagogy behind how they convey it to students in a way that enables them to learn. These factors will be examined in more detail in the following sections of this chapter.

In the literature, the idea of specialised knowledge needed for teaching mathematics is referred to using somewhat varying terminology. Some researchers have called it *Mathematics for Teaching* (Davis & Renert, 2014) while others call it *Mathematics in Teaching* (Watson, 2008; Petrou & Goulding 2011; Ruthven, 2011). Still others (Ball, Thames & Phelps, 2008; Turner & Rowland, 2011) have titled it *Mathematical Knowledge for Teaching* (MKT). Rowland and Ruthven (2011) express a distinction between the terms *Mathematical Knowledge 'for' Teaching* and *Mathematical Knowledge 'in' Teaching*. In their description, *Mathematical Knowledge for Teaching* is held individually by the teacher. *Mathematical Knowledge in Teaching* differs from *Mathematical Knowledge for Teaching* in that it takes into account the fact that teaching does not occur independently from the classroom context. By looking at the contributions from various sources, Rowland and Ruthven conclude that current research supports the following view: Trying to assess and develop mathematical knowledge for teaching will not succeed without taking into consideration the context in which teachers work. Looking at mathematical knowledge for teaching from this perspective is what they have termed *Mathematical Knowledge in Teaching.* For the purposes of this thesis, I will continue to use the term *Mathematics for Teaching,* or MfT to refer to all nuances of this construct because it is broad enough to unify the various aspects of teachers' activity related to classroom context.

2.1.1. History of MfT and milestones in its development

The interest in the concept of MfT developed in the 1980s as a reaction to the low standard of mathematical achievement in the U.S. in relation to that in European countries (Petrou & Goulding, 2011). Studies in this field began with a key question that researchers have been asking for a long time. "What kind of knowledge do teachers need to have in order to teach effectively?"

Early ideas on this topic revolved around a simple view of the disciplinary knowledge of mathematics that teachers hold. The study conducted by Begle (1972, 1979), looked at the correlation between two factors: teacher knowledge of mathematics and student understanding of mathematics. Begle found a low correlation between the number of mathematics courses that were taken by teachers at the university level and the performance of their students in the classroom. Other accounts of research on what mathematics teachers need to know in order to teach the subject exist in other literature (e.g., Davis & Renert, 2014; Roland & Ruthven, 2011), and will not be summarized here. According to these accounts, similarly low correlations were found even when the scope of such studies was expanded to include factors such as the content of the courses that were taken, as well as the teachers' grades and their instructional approaches. However, these studies point to the fact that in order to teach effectively, one needs to know the subject matter at a level that surpasses that of the students. The conclusion of these studies was that the content knowledge of the teachers (i.e., their understanding of the subject matter) is necessary, but that as an isolated factor, it is an insufficient precondition for effective instruction.

At this point in the evolution of research into teachers' disciplinary knowledge, the new concept of Pedagogical Content Knowledge (PCK) was introduced by Shulman (1986). He created a categorization of MfT that, over time, has influenced many frameworks of MfT. Shulman's categorization of MfT includes the following seven components of teacher knowledge: general pedagogical knowledge, knowledge of learners and their characteristics, knowledge of educational contexts, knowledge of educational purposes and values, curriculum knowledge, Subject Matter Knowledge (SMK), (i.e., the knowledge of the mathematical content) and PCK, (i.e., the teacher's awareness of how content is established). For Shulman, the latter three categories, SMK, PCK, and curriculum knowledge together comprise a broader category, which he referred to as Content Knowledge. Further details about Shulman's conceptualization of MfT are presented in the following section on models or MfT.

The concept of PCK was the most influential of Shulman's categories, as it fueled a significant amount of further research in the field of mathematics education. Shulman described PCK as:

> " …the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject to make it comprehensible to others" (p. 9).

Shulman's concept of PCK was closely aligned with the similar concept of didactics, which already existed in Europe (Freudenthal, 1983). Therefore, we can loosely say that PCK was the North American version of didactics.

In the 1990's, Shulman's conceptualization of PCK became popular among researchers that studied MfT (Davis & Renert, 2014). An important cross-cultural study that was conducted by Ma (1999) compared the PCK of Chinese and American teachers. The study looked at teachers' knowledge of elementary mathematics and found that the Chinese teachers had a deep knowledge of basic concepts of mathematics and would revisit them often during the course of instruction. This feature was not present in the American teachers who were studied. This provided an important insight into both the content that teachers need to know, as well as how they need to know it in order to convey it effectively to students.

In consideration of the specialized mathematics needed to teach mathematics (i.e., PCK), there are categories of knowledge in addition to the knowledge that is gained through advanced mathematics courses. The articulation of PCK as a component of teachers' knowledge created a distinction between the mathematical knowledge that is to be used and that which is structured to be taught. One cannot teach mathematics simply because one knows how to do mathematics. Teachers need to convey their knowledge in a way that makes it learnable by students.

Following Ma (1999), Ball and Bass (2000, 2003) named this process, which teachers frequently engage in as they teach, *unpacking*. It means that teachers take mathematical ideas apart to make the concepts accessible for learners to learn. This is the reverse of what mathematicians do when they condense ideas in order to make mathematics more efficient. It also suggests an explanation as to why taking mathematical content courses at an advanced level does not necessarily translate to effective teaching, namely because these courses are not intended to break apart mathematical concepts as both teachable and learnable objects. Since the mathematical content that teachers learn in university is often in this condensed form, university level mathematics courses do not help a teacher in unpacking mathematical knowledge. This could explain the low correlation between the number of advanced mathematics courses in a teacher's educational background and the achievement of their students (Adler & Davis, 2006). On the other hand, more recent studies have shown that teachers' mathematical knowledge is related significantly to student achievement gains when this knowledge is conceptualized, not by the proxies as university courses taken, but by the specialized mathematical knowledge and skills that are used in teaching mathematics (Hill, Rowan, & Ball, 2005).

2.1.2. Models and theoretical frameworks of MfT

As mentioned in the previous section, Shulman's work on PCK was pivotal in starting the movement for studies in MfT, as well as in researching what makes a teacher able to convey content in a way that makes it possible for others to understand and learn it. Shulman (1986) identified an area in the research about content knowledge which had been neglected. By observing and describing teacher's actions during lessons, Shulman (1987) identified seven categories of teacher knowledge. He focused on the last three, which he called the *Missing Paradigm in Research on Teaching*.

The first of these three categories of teacher knowledge is *subject matter content knowledge*. Content knowledge is the knowledge about the subject matter that is held by the teacher. It includes facts and awareness of mathematical structure. Ball (1988) described this construct as knowledge of mathematics, which is not specific to teachers, but that is used by anyone who uses mathematics as part of their job, (e.g., accountants, engineers, etc.).

The second category from Shulman's missing paradigm is *curriculum knowledge*. Shulman further divided this category *into lateral curriculum knowledge* - knowledge of the available materials, and *vertical curriculum knowledge*, (i.e., the knowledge of topics, how those topics are addressed before and after, as well as how they are connected.) The third and certainly the most influential category is that of *pedagogical content knowledge* (PCK), which includes content specific representations and examples that are used by teachers in order to help students understand the subject matter. It also includes the strategies that teachers use in order to help students develop their conceptual understanding, including dealing with misconceptions and difficulties.

Although it was influential, Shulman's categorization was not without criticism. Meredith (1995) criticized this categorization for being too narrow in assuming a teacher centered model of teaching and not being applicable to different teaching methodologies. Furthermore, she argued that each teacher develops his or her own form of PCK, depending on individual background and beliefs. This is something that Shulman's model does not account for. Ball, Thames and Phelps (2008) offered a further criticism that Shulman's model does not give a clear distinction between SMK and PCK, which is something that they attempted to rectify in their own refinement of this model. This will be discussed shortly.

A subsequent model that attempted to address some of the criticisms of Shulman's categorization was proposed by Fennema and Franke (1992). It modified Shulman's model to account for the dynamic and interactive nature of knowledge for teaching by including four components: knowledge of content, knowledge of pedagogy, knowledge of student cognition and the beliefs of the teachers. They reasoned that it is the interaction of these components that creates a teacher's knowledge set and determines his or her actions in the classroom. This theory also left room for the possibility that existing knowledge can be changed and new knowledge can be created. Therefore, it moved from a static view of teacher knowledge to a more dynamic, albeit still individualistic, view.

Another model, created by Ball, Hill and Bass (2005) and Ball et al. (2008), attempted to refine Shulman's model by making it applicable to practice. This practice-based theory of MfT created a clearer distinction between SMK and PCK. It divided SMK into three subdivisions: *common content knowledge*, *specialized content knowledge* and *horizon content knowledge*. Common content knowledge is mathematical knowledge for any use. Specialized content knowledge is that which is used in the classroom and is needed to teach well. Horizon content knowledge is parallel to Shulman's vertical content knowledge, or the teacher's awareness of topics that were previously covered and how they relate to subsequent topics in the curriculum.

PCK was also divided into three sub-categories: *knowledge of content and students* (KCS), *knowledge of content and teaching* (KCT) and *knowledge of content and curriculum* (KCC). Knowledge of Content and Students includes awareness and anticipation of student cognition and misconceptions, and selecting appropriate examples and explanations. Knowledge of Content and Teaching involves choosing and setting up activities. Knowledge of Content and Curriculum is parallel to Shulman's curriculum knowledge. This model was significant in that it created a clear distinction between SMK and PCK. It offered a way to study the relationship between particular aspects of teacher knowledge and student achievement. This was a key factor and was important for practice because it implied that a teacher with weak knowledge would pass it on to the students. However, like Shulman's model, it did not give importance to the beliefs of the teacher (Petrou & Goulding, 2011).

Although all of the previously discussed models were important in understanding the nature of the knowledge that is needed by teachers in order to teach effectively, none of the models that were mentioned thus far offered a practical tool to be used in teacher training and professional development. A model that attempted to do this has become known as the *Knowledge Quartet* (Rowland, Huckstep, & Thwaites, 2003).

The Knowledge Quartet was developed in England and Wales to assess the mathematical content knowledge of pre-service elementary mathematics teachers. This model is based on Shulman's framework and reorganizes it to categorize situations in which mathematical knowledge appears in teaching. It also looks at the relationship between SMK and PCK, which is something that Shulman's model does not do (Hashweh, 2005). It is important to note that, although this model does parallel Shulman's, it is not a refinement, as the previously mentioned models are (Ruthven, 2011). This model was created for the purpose of analyzing knowledge in teaching in a classroom context. It is meant to be used as a tool for teacher training and professional development to help teachers understand what categories to look at in order to analyze and reflect on their teaching. It provides a structure, or framework, for teachers to use during their instruction, to reflect on and constructively critique their mathematical knowledge during their own teaching. It can also be used by teacher educators as a means of assessing pre-service teachers' teaching.

The Knowledge Quartet is composed of four categories, each with its own contributory codes that define the category (Turner & Rowland, 2008). These categories are: *foundation*, *transformation, connection* and *contingency*. The foundation category includes codes such as the teachers' content knowledge, beliefs and academic understanding of mathematics, as well as their knowledge and understanding of mathematics pedagogy. The transformation category refers to the practical aspect of knowledge-in-action, both in planning and teaching, or the process whereby the teacher converts knowledge of mathematics into an unpacked form that students can learn. This category also includes the teachers' choice of representations, examples and explanations. The third category, which is connection, involves connecting choices for discrete parts of mathematical content, such as creating links between lessons and parts of lessons, ideas, activity sequences and anticipating the difficulties that students might have with the content. Contingency, the final category, includes reactions to unpredictable classroom events, the ability to deviate from the lesson plan and adapt it according to in-the-moment classroom circumstances and dealing with unanticipated student misconceptions.

A study that was conducted by Turner and Rowland (2008) examined the use of the Knowledge Quartet by five pre-service teachers throughout the course of their four-year teacher education program. It also examined how the Knowledge Quartet contributed to the teachers' development over time. The study found that all of the teachers used the Knowledge Quartet to the extent that it became ingrained in their teaching and self-assessment habits. All five of the teachers reported that they found it to be essential in framing their teaching. They found it to be indispensable as a guide to the aspects of their teaching and the classroom interactions on which they needed to focus, both before and during instruction. It also provided them with a useful tool for self-reflection after the lesson, to aid them in determining what aspects of the lesson went well and which did not, and why. Thus, it helped them identify the areas in which they needed to improve. This has important implications for practice and professional development, because once teachers are trained in understanding and using this tool, they can use it on their own in order to guide their practice throughout their teaching career.

The Knowledge Quartet takes into account many of the criticisms of Shulman's categorization. It addresses Hashweh's (2005) criticism that interaction between SMK and PCK was being ignored and it shows how these two constructs appear in practice. It also addresses Meredith's (1995) criticism that Shulman's framework does not consider the beliefs of the teacher, which have an effect on the kind of PCK they form.

A somewhat divergent concept of MfT was proposed by Watson (2008) in her paper called *Developing and Deepening Mathematical Knowledge in Teaching: Being* *and Knowing.* In her view, models that define MfT, PCK, etc., are of limited value in teacher development because it does not make sense to create a general list of rules and guidelines, which are then applied to specific teaching situations. She claimed that teachers learn to teach simply by teaching, as well as from the students themselves and that this is what deepens their mathematical knowledge and understanding. To illustrate her point, she used approaches toward understanding and dealing with student misconceptions. It is unnecessary to learn categorizations of student errors and to predict them from these categorizations. Errors should not be looked at individually as student misunderstandings, but rather as the thought processes they are having in interpreting mathematical concepts as they learn. In her view, obstacles to understanding are actually part of the process of learning. They relate to how students apply their existing knowledge in new situations. Watson argued that MfT is developed by learning to reason about and understand students' thinking. In the same way, by observing and analyzing one's own teaching and by engaging in self-reflection, one can gain a deeper understanding of mathematical concepts and pedagogy.

Watson also argued that MfT is not individual. To develop MfT, it is necessary to collaborate with others, including other teachers and even students. This moves away from a perspective of teachers as conveyers of knowledge and views knowledge as being constructed collaboratively in a classroom situation. In turn, this would imply that two very similar lessons, which are taught by the same teacher in different classrooms, would turn out differently due to the different knowledge and backgrounds of the students. By viewing teaching and learning from Watson's perspective, it is the mathematical representations that are seen as problematic, not the students. Student errors can be seen as stemming from representations that are inappropriate or inadequate, or from trying to apply existing knowledge to new concepts, which is something that is natural in the course of learning to do mathematics.

A study that was conducted by Watson and her colleagues and researchers at the University of Auckland, (Watson, 2008) looked at teacher educators and teachers in training who were studying mathematics. They structured opportunities for teachers to experience what it is like to learn mathematics as a student. Teachers gained an insight into what it is like to be a student, rather than simply focusing on pedagogical aspects of their teaching. This kind of analysis and self-reflection led to teachers having a greater appreciation and understanding of learning mathematics, which had a positive impact on their practice. For example, it reshaped their sensitivities to teaching conceptually, as opposed to procedurally. An important implication for practice that was generated by this study is that, in a mentor-teacher interaction (and, by extension, a teacher-student interaction), both sides can develop mathematical knowledge. Another practical implication is that MfT is developed by gaining greater and deeper experience in learning mathematics. This idea can be used in training teachers, both pre-service and in-service.

Watson (2008) developed a list of questions that revolve around seven categories, which are meant to guide how a teacher educator can develop MfT in the teachers he or she is mentoring. These questions include, "What mathematical ideas, operations, meanings, representations, etc., are being taught?", "How are they taught?" and, "What are the learner's perceptions likely to be?" (Watson, p. 7). These categories are loosely structured, but not fundamentally different from those in the Knowledge Quartet framework, where they are used to focus activities in practice and self-reflection of one's own teaching.

Further considerations regarding models for MfT

The majority of the models that are discussed in this section stem from Shulman's (1986) framework. They seek to refine it and to address criticisms of it. The Knowledge Quartet was developed specifically as a tool that can be used in a practical context for teacher training and professional development. It is based on Shulman's work, but is restructured to help teachers reflect on their practice.

Watson takes a different stance from Shulman and models of this type, saying that self-reflection and analysis is most important for improving one's teaching practice. When a teacher learns to reflect and critically assess his or her own teaching, this ability can be applied to any teaching situation. She argues against a deficiency view of students, saying that student errors and misconceptions are cognitive processes that are involved in learning and that if a teacher can learn to understand students' thinking based on their errors, this can help the teacher address it in the moment of teaching.

Essentially, Watson's arguments and the Knowledge Quartet both claim that self-reflection is critical in teaching effectively and in developing teacher's professional knowledge. However, the Knowledge Quartet provides more of a structure to guide teachers' and teacher educators' analysis of teaching practice.

Despite all of the research that has been conducted on MfT, there is still disagreement about the forms and functions of subject-related knowledge and PCK, as well as how teachers develop this kind of knowledge. However, there is a growing consensus in the field that MfT not only resides in the teacher, but that collective factors are involved in the development and implementation of MfT, such as curriculum materials, other teachers and even students themselves. An ongoing debate still exists as to the kind of knowledge that is necessary for teachers to possess and whether MfT is an individual construct that describes knowledge held independently by each teacher, or whether it can be viewed as a social construct that is measurable only in the context in which a teacher works. Nonetheless, a view, held by many researchers, is that trying to assess and develop MfT will not succeed without taking into consideration the classroom context in which teachers work (Rowland & Ruthven, 2011).

More coherent approaches need to be developed for characterizing, assessing and developing MfT in a way that can be used by teachers and teacher educators as an ongoing self-assessment and professional development tool, respectively. The next step for research in MfT could be in developing a way for any school and its teachers to use its outcomes in a practical way in order to improve their own practice independently within their schools, without researchers having to come to the classrooms and conduct a study. The question is, "How can we make these models applicable and useful, in a professional development context, and not just for the purposes of research and teacher evaluations?" This would require creating a coherent and evolving model of professional development that teachers can use on a regular basis, rather than just with single professional development seminars.

2.2. Towards another framing of teachers' MfT

In my efforts to try to understand teachers' MfT and its growth in collaborative settings as emerging from their professional conversations during planning for instruction and reflecting on instruction, none of the previously described frameworks provided a suitable fit for the analysis of the data flowing from the research setting of the present study. This is understandable given that these frameworks were developed with different purposes in mind, using different methods, theories, and concepts. For example, the framework of Ball et al. (2008) was developed for the study of teachers' knowledge in practice for the purpose of assessment of their work in the classroom. Therefore, within the collaborative setting of this research, the framework of Ball at al. did not seem suitable as it is both overly specific to individual teacher's MfT and compartmentalized into overly nuanced subcategories of SMK and PCK to be detectable in the small scale qualitative research such as this one. Shulman-inspired models focus on the study of MfT as a theoretical construct which is not the aim of this research. In contrast with the Knowledge Quartet, which is the closest in terms of the purpose of using it in a practical context for the development of teaching practice and for structuring teachers' reflection on it, this study seeks to understand the teachers' knowledge growth in collaborative settings within the context of Practice-Based Professional Development, while the Knowledge Quartet is meant to be learned by the teachers as a special tool to be used to guide their pre-service professional development. As such it is meant to help teachers understand what categories to look at, and how to structure their personal reflection on their own teaching practice for an ongoing use in their professional career. In contrast, the present study could not use this framework because the setting is not one of pre-service teacher training but rather that of experienced teachers' practice in real school setting where non-intrusiveness is an important consideration.

This section presents another framework for the study of teachers' professional knowledge which was used as an analytic framework for this study. It is borrowed from Selter (2001) who proposed the parsing of teachers' background knowledge and awareness into four components: mathematical, psychological, didactical and practical (or pedagogical). This framework was used for all stages in the analysis of data in the study because of the type of data collected and instruments used for this practice-based research.

Next, the four components are described in greater detail, followed by a justification for choosing this particular framework for the study of teachers' MfT.

2.2.1. Mathematical component

Content knowledge of mathematics as a discipline involves the disciplinary knowledge of the teachers of objectified mathematics or, following Shulman (1986), the subject matter knowledge (SMK). According to Schwab (1978), this knowledge consists of substantive and syntactic knowledge.

Substantive knowledge is presented as the knowledge of the subject. It incorporates the knowledge of concepts, key facts, definitions, theories, models and also the way in which mathematical knowledge is structured and organized. Syntactic knowledge is presented as the knowledge about the subject. It incorporates the knowledge of the ways in which mathematical knowledge is generated and validated. It is the knowledge of mathematical processes, such as conjecturing, generalizing, validating, proving and constructing models.

2.2.2. Psychological component

This component consists of teachers' understanding of how students think and learn. Learning is a transformational process of evolving different cognitive schema that is related to a given concept, structured by previous experience in different representation systems. Acting upon new evidence, we are attempting to reconstruct and coordinate those mental structures, often through inner negotiation of meanings, to once again achieve equilibrium. The idea of the human tendency for equilibrium (seeking order and harmony) in the field of consciousness first appeared in Gestalt psychology. It was later adopted in Piaget's theory of equilibration of cognitive structures (Sierpinska, 1994).

According to Piaget (1976), assimilation and accommodation are two operations of the mind that make the equilibration of cognitive structures possible. In his scientific experiments, Piaget proved that children have their own way of thinking and constructing knowledge. In his view, learning and understanding are active and dynamic cognitive processes, which consist of assimilation, accommodation and adaptation. This component also involves the understanding by teachers of the affective domain, such as motivating students to learn and dealing with their emotional responses during the learning process.

Selter agrees with Piaget that children often think differently from how we think or what we assume how they think; furthermore, he expects teachers to learn to understand children's thinking by reflecting on their productions to sensitize their own perspectives on children's ways of thinking.

2.2.3. Didactical component

Teachers often consider different methods of presenting the subject matter in order to convey the new concepts that need to be learned. Usually they base these methods on their own learning experiences, or they use ready-made methods that are presented in the textbook or teacher manual. Freudenthal (1983) used the term "*didactical phenomenology"* to capture the idea of describing a mathematical concept in its relation to the phenomena for which it was created and as it concerns the learning process. In other words, it is a content specific analysis that deals with the question of what there is to know or understand about a certain mathematical notion in the context of school mathematics and prescribed curricula.

The didactical component involves the teacher's transformation of a mathematical concept in order to make it learnable and understandable for the student (didactic transformation). Therefore, it is specific to subject matter. It includes techniques for teaching and knowledge about teaching. It is about the structuring of the lesson, the sequencing of the learning tasks and activities and the organization of the intended outcomes of the learning process. The didactics of mathematics can be thought of as the art of conceiving and conducting conditions that can determine and shape the learning of a piece of mathematical knowledge on the part of a subject. The subject can be an individual or it can be a system, such as a class of students (D'Amore, 2008).

2.2.4. Pedagogical component (concerning the practice of teaching)

Teachers have various conceptions of teaching, which manifest in the pedagogical component of their practice. These conceptions might fall along a continuum of two extremes, such as traditional or progressive, student-centred or teacher-centred. They are connected with the beliefs, experiences and personalities of the teachers. This component relates to the practice of teaching itself and is always present in a classroom teaching situation. Consequently, it is used in the discussion of teaching practice observed in the enacted lessons and in the analysis of the collective reflections by the teachers on their teaching practice.

2.3. Rationale for MfT framework used in this study

The four components from Selter's (2001) conceptualization of teachers' background knowledge and awareness serve in this thesis as a framework to guide the research on the development of MfT in a practice-based environment. It is a simple, clear, and yet comprehensive framework for the analysis of the data across the various teachers' collaborative activities such as planning for instruction as well as reflecting on and evaluating their own teaching practice. It materialized during the research process as the best fit for the evidence, the context, and the implementation of teachers' professional development activity that was studied. The four components that are described above provide the most suitable way for framing the investigation on teachers' knowledge growth in this particular setting in which the study took place. This framing, with the presence of the pedagogical component, takes into consideration the context in which teachers work. Much like Fennema and Franke's (1992) model, this framework accounts for the dynamic and interactive nature of knowledge for teaching, but it has the added benefit in that the codependent character of SMK and PCK can be accounted for through the didactical component of Selter's characterization of teachers' knowledge for teaching. Furthermore, this particular choice of framework allowed for sufficiently broad and yet clearly defined accounting of salient contents of teachers' professional conversations that were held during their professional development activity.

Given the broad scope of MfT, it could be assumed that the main goal of all professional development activity, especially the kind that is ongoing and practice-based, is the development of teachers' MfT. One of the possible ways to achieve the development of MfT is through the professional development practice which originated in Japan, known as *lesson study*. In the next chapter we examine lesson study both as a theoretical field of research and as a practical tool for the professional development of teachers.

Chapter 3. Lesson Study in Literature

This chapter is composed of two parts. The first part relates to the reader my motivation for situating the research on in-service teacher development in the context of lesson study. The second part, which comprises the bulk of this chapter, is dedicated to the literature review on lesson study as a theoretical field of research and as a practical practice for teacher professional development. It begins with a presentation of the model of lesson study as it is practiced in Japan, where it originated. It describes its many varieties, goals and capacities, as well as ways in which it brings about a planned educational change. Next, it presents the process of lesson study, along with its features. Further, this chapter addresses the literature that is related to the implementations of lesson study in the United States, together with the challenges and recommendations that have been recorded in the efforts to transfer this practice outside of Japan. The last part of the chapter examines the literature that is related to lesson study in the context of research on instructional improvement and teacher development, as well as on developments in the research of lesson study itself.

3.1. Lesson study as a context for teacher learning and for research

Lesson study is a method for the development of effective teaching. As a goal, effective teaching is, at the same time, both clear and elusive. Teaching itself is complex and context dependent. Different people will likely have different ideas about what effective teaching might look like in the classroom. It is also true that effective teaching is most easily recognized by the results in terms of the "students' mathematical well-being," by which I mean, not only that the students have conceptual understanding of the topics that they learned, but also procedural fluency, interest for mathematics and confidence that they can do it.

To put this discussion into some perspective, it is clear that educational research has greatly advanced our current understanding of what results in effective teaching. As consumers of such research, teachers and administrators can now consult the research and theory to inform their practice. For example, they can learn about the relative importance of a myriad of factors, such as giving feedback to students, setting high expectations for students, having clear instructional goals, reinforcing effort, class size, cooperative learning, etc. They can even read about the effect sizes and percentile gains that are associated with these conditions, (e.g., Hattie, 2003). However, simply consulting literature that has been written about effective teaching will not, by itself, provide for a teacher's own effective teaching in day-to-day practice. It is my contention that a job-embedded process in which teachers have a chance to closely examine the phenomena at the level of personal practice, seems very promising indeed. There is a growing body of research on embedding the learning in classroom practice as a way for teacher development. While lesson study falls into this category, its distinct feature is that it is a comprehensive and integrative approach as well.

I am looking at lesson study as a context in which teachers conduct their "action research projects" on the learning of mathematical ideas by students. In doing so, I also re-examine the understanding of the content by the teacher and how it could be transformed for learning. It is a kind of fertile ground for advancing the knowledge that teachers hold in regard to teaching specific mathematical topics within school mathematics. It is also a window from which that process can be researched. As a result, lesson study has a dual purpose here.

Freudenthal (1973) stated that, "A science of teaching must start with a science of teaching something." Therefore, building effective teaching amongst other things will most assuredly rest on the building of subject matter expertise by teachers. The term "subject matter expertise" is best captured by the notion of Profound Understanding of Fundamental Mathematics or PUFM (Ma, 1999). Obviously, no "magic pill" exists with which to deliver PUFM to all teachers overnight. However, we could examine how practicing teachers can begin to build some aspects of it through the use of lesson study. Ma's contribution has been to exemplify and describe these important aspects of teachers' knowledge, which seem to matter so much. I am interested in looking at ways in which lesson study can help teachers to access and build up this kind of understanding.

Some researchers have drawn attention to the need to explicate the mechanism by which lesson study works (Lewis, Perry & Murata, 2006). This calls for rendering lesson study as a model for professional development beyond a set of prescriptions on how to conduct it. Such a model would need to specify the connections between the observable features of lesson study and the resulting instructional improvement. This is particularly important for dissemination of the practice of lesson study, in the sense that essential features of lesson study are attended to at the level of implementation, and that rote implementations of surface features, in a recipe-like fashion, can be avoided.

According to Lewis et al., given that there is a growing interest and spread of lesson study in North America, there exists a need for design-based research cycles, which would not only hone the method of lesson study for its applicability to local contexts, but would also build theory about how it works. Such research could also contribute to a practical purpose of providing "usable, actionable and adoptable artefacts that leverage learning in other sites" (Lewis, Perry & Murata, 2006, p.5).

3.2. Lesson study in Japan

It is repeatedly claimed that efforts to improve the teaching of mathematics in North America have produced little change, despite years of reform (Hill & Ball, 2004). In contrast, Japan seems to have been much more successful in bringing about a planned educational change. For example, it has been noted that elementary mathematics and science education in Japan has changed dramatically over the last fifty years, having moved away from lecture style "teaching as telling," and toward "teaching for understanding" (Lewis & Tsuchida, 1997). It has been speculated that one of the main routes by which such systemic change is accomplished is the well-defined and established practice of lesson study (Chokshi & Fernandez, 2005; Lewis, 2000; Stigler & Hiebert, 1999; Watanabe, 2002; Fernandez & Yoshida, 2004), which is a ubiquitous feature of Japanese elementary education. The publication of *The Teaching Gap* in 1999 portrays lesson study as an effective method for systematic improvement of classroom instruction. It helped the practice to sweep the United States as a grass roots movement, springing up in 335 schools, across 32 states. The practice of lesson study became the focus of dozens of conferences, reports and published articles (Lewis, 2006).

I do not aim to provide a comprehensive review of educational literature on lesson study to date, but to discuss some of the principles by which lesson study works, as documented in literature. In addition, I relate this to the North American experience of lesson study. I also address the question of "alternatives."

Until the year 1999, very little educational literature was written in English about the process of lesson study. Stigler and Hiebert based their account of lesson study on research that was conducted by four people (Yoshida, Lewis, Tsuchida, and Shimahara), as well as on informal discussions with teachers and teacher educators throughout Japan. Now we see many more authors associated with the theme of lesson study. Therefore, I review some advances that have been made in this area.

Ecology of lesson study Practice

In what follows, organizational aspects, such as the "ecology" (Lewis, 2006) and the process of lesson study, are summarized to provide a background on how the Japanese education system succeeded in transforming something as complex and "culturally embedded" as teaching (Stigler & Hiebert, 1999).

There is a diverse ecology of lesson study in Japan. Lesson study, with its associated research lesson, comes in several distinct forms, the three most notable being the "in-school research lesson," the "public research lesson" and the "district-based research lesson" (Lewis, 2006). The first kind is practiced throughout Japan. It involves teachers in the school, regardless of how small the school might be. The last kind is a special feature of national schools that are attached to universities. In these national schools, the lesson study work culminates in open house events in which research lessons are conducted throughout the school and are open to teachers from other schools. These events, which are held several times each year, attract up to 3000 teachers a day. Public research lessons obtain public research funds and are charged with a mandate to focus their lesson study on some innovation. The innovations can be of many kinds. For example, they can be in regard to a new approach to teaching and learning, such as problem solving, mathematics journal writing or classroom discussion; or they can be related to the content of some newly developed curricular area; or they might be related to general issues of education, with an attempt to treat them in novel ways, an example of which is the general question, "How can we help students become independent learners?" Schools then study these questions in the context of lesson study and report their findings. The purpose of these specialized lesson studies is to shape and disseminate new curricula, emphases and approaches (Lewis, 1997).

Finally, in the district-based lesson study, teachers work in cross-school groups, with each group working on a specific instructional area of interest to them. They often draw on the work that is conducted by specialized groups in the public lesson studies, with an aim to bringing the new approaches to life in lesson studies with local students. These district-based research lessons are then presented twice a year, during district wide professional development days. In this way, district-based groups provide a pivotal translation point at which local teachers make sense of outside knowledge (Lewis, 2006). Such a structure of lesson study activity acts as a viable system for linking educational policy to practice.

Common, focused and frugal curriculum

It should be noted that the Japanese have a shared national curriculum. This has been deemed to provide teachers with a common set of expectations in regard to the background knowledge that is held by students, as well as with widely shared points of reference. For example, it was reported that the Japanese participants in the Study of Teaching Practice as a Medium for Professional Development workshop "reflected an almost unanimous awareness and acceptance of content and its placement in the national curriculum" (p. 5) such that, when a topic was identified, they responded by knowing the grade level of its presentation and the background knowledge that students would have gained before being exposed to the new concept (Bass, Burrill & Usiskin, 2002). Such shared perspective, which was in sharp contrast with the U.S. participants in this workshop, allegedly contributed to a much more focused discussion by the Japanese. It seems plausible that the national curriculum contributes to a more conducive environment for professional knowledge to flow between the groups that are engaged in various forms of lesson study throughout the country. In addition to the common curriculum across the country, the curriculum is focused on fewer concepts which students are expected to learn at depth.

3.3. Lesson study cycle

In its broadest sense, lesson study is a long-term, professional development process that is centered in the classroom and focused on the students' learning. The centerpiece of lesson study is the so-called *research lesson,* which is developed collaboratively by a group of teachers. The research lesson is an unrehearsed, but well prepared, lesson with real classroom students. It should be noted that, in this context, the term "research" means *teacher-initiated, practice-based inquiry.*

Regardless of the form, the typical lesson study process unfolds as a cycle with the following activities: 1) research and preparation, 2) implementation, 3) reflection and improvement, 4) implementation, 5) reflection and filing of records (see Appendix A for further details). Based on his survey of 35 schools in the Hiroshima area, Yoshida presents a detailed description of how the lesson study cycle unfolds and how it is organized within the school (Yoshida, 1999). Teachers typically engage in two to eight lesson study cycles in each year, with each cycle taking approximately four weeks to complete. In this section, the process of lesson study is summarized, based on the descriptions from a number of sources (Curcio, 2002; Lewis, 2000; Stigler & Hiebert, 1999; Yoshida, 1999).

What characterizes lesson study as a professional development practice by which instructional change can be achieved on a larger scale (beyond that of an individual teacher), is its deliberate and constant attention to change, which is framed and supported by the articulation of an "overarching goal." This goal then becomes a driving force and a focus for what teachers do as they engage in the process of lesson study. Lesson study begins with a group of teachers identifying a particular common goal or vision of education, which will motivate and direct the work of a lesson study group. This goal can be rather general, (e.g., to awaken students' interest in mathematics), or it can be more specific, (e.g., to improve students' understanding of how to add fractions with unlike denominators). The goal can be derived from the mandated curriculum policies at the administration or even the national level, or it can be based on teachers' own classroom experiences in order to address something that has posed particular challenges to their students. Once the goal is chosen, teachers decide on a specific lesson that will transpire this goal, and then they begin planning.

The research and preparation phase is characterized by teachers' collaboration in gathering information on the topic they would teach. This planning stage may include examining the available textbooks and curricular materials, as well as articles that were produced by other teachers who have studied a similar problem. In fact, many teams now begin new lesson study cycles by reviewing student data and by following up on problems in student learning, which surfaced in prior lesson study work (Lewis et al., 2006). They also consider how the lesson fits in the unit of study, how it builds a bridge between prior learning and future lessons, what the anticipated student responses are to the pivotal questions of the lesson, as well as the misconceptions that might arise.

In the case of a mathematics lesson, at all times during this phase, there is an overriding concern for the mathematics, its structure and the attempts to motivate students to learn it (Bass et al., 2002). One teacher, who is usually the teacher who will deliver the lesson, takes the initiative and writes up the lesson plan, incorporating all the information and ideas that arose. The lesson plan is then presented at the all staff meeting for feedback from outside the group. Next, the lesson-study group revises the lesson plan and develops instructional materials. The final version of the lesson plan is distributed in advance, to everyone who will observe the implementation of the research lesson. The lesson is then implemented in a real classroom setting. It is observed by other members of the lesson-study group and often by other faculty in the school as well.

According to Lewis and Tsuchida, it is not uncommon for outside observers, such as educational researchers, well-known classroom teachers as well as administrators, to attend and serve as commentators for the research lesson, thus providing another bridge between policy and practice. In this way, research lessons serve as a nationwide formative research "in which thinkers at the center of a reform get the chance to see how it is being interpreted by the teachers, implemented in the classroom and received by students" (Lewis & Tsuchida, 1997, p. 322). During the observation, teachers gather evidence on student learning. This is usually according to a planned observation protocol. This may include recording what particular students said, did, or wrote down, which reveals how and what they are learning as the lesson progresses. The particular data that is gathered depends upon the issues of interest to teachers in the lesson-study team.

A colloquium (post-lesson discussion) follows the implementation. During this time, the participants evaluate the lesson and reflect on its effect. Typically, the gathering begins with presentations by the teacher who taught the lesson, followed by the people that were involved in its design. This is then followed by a discussion, which could be free or structured. During this process, teachers criticize the parts of the lesson that they considered to be problematic. The focus is on the lesson, not the teacher who taught it. This shifts the focus from a personal evaluation to a self-improvement activity. After all, the research lesson is a product of collaborative effort that is used by teachers as a place to examine and improve their teaching practices.

There are reports on the observation criteria and the professional jargon that is used during these colloquiums. The observed features of the lesson to which the criteria is attached include: 1) the opening problem setting with its motivational focus, 2) the questioning sequence important to mathematical development and connections, 3) purposeful scanning and interaction while observing student work, 4) anticipation of student thinking, 5) blackboard writing as a way of recording and organizing student work, and key mathematical statements and results, 6) raising the level of whole class discussion in the orchestration and probing of student solutions, 7) the teacher's mathematical summary of the lesson with attention to student contributions. According to Bass et al., training individuals to be effective observers of the content development is not an easy task (2002).

In the next stage of the cycle, the lesson-study team revises the lesson, basing their changes on the student misunderstandings that were evidenced. The revisions may include changing the instructional materials, activities, questioning or problems that were posed to the students. The revised lesson is then taught to a different class, usually by another teacher from the team. This class is frequently observed by a larger audience.

This stage is followed by another round of reflecting and evaluating, with a larger number of participants who have observed the research lesson. This discussion includes broader issues, such as what guided the design of the lesson, what was accomplished, what implications can be drawn from its implementation and what parts of the lesson still need rethinking.

Finally, the lesson-study group produces a report that tells the story of their work. These reports are filed for accessibility to a wider audience, either for the school or the district. At times, the report may be published by a commercial publisher, especially in the case of a specialized research lesson with external collaborators.

3.4. Lesson study as a model for teacher development

Lesson study is deemed to be the key factor in providing sustained professional development to teachers in Japan (Stigler & Hiebert, 1999). Many experts agree that what teachers do in the classroom has an effect on what students learn. According to the Third International Mathematics and Science Study (TIMSS), Japanese students outperform most other nations. In particular, results indicate that students in Japan learn more and better mathematics than those in the U.S. One of the reasons for this might be that, in the U.S., concepts are developed only 21.9% of the time. The rest of the time, they are just stated to students. In contrast, in Japan, concepts are developed 83% of the time (Stigler & Hiebert, 1999, p. 61). What teachers do in the classroom in Japan is shaped by their highly developed teaching culture in which the acquisition of knowledge of teaching revolves around the careful design of lessons. Teachers learn both content and pedagogy in the process of developing and reflecting upon a common lesson that is developed by a community of professionals.

A similar approach also serves as a way to mentor those people who are new to the profession. Peterson gives a detailed account of his observations of the teaching of mathematics to students in three Japanese junior high schools that are affiliated with universities (Peterson, 2005). The author found that the process is collaborative, with one sponsor teacher working with two to six student teachers at a time. The focus of the student teaching experience was on planning, teaching, observing and critiquing lessons. Student teachers engage in a collaborative "materials research" and "content analysis" to determine the ways in which the topic might be approached. An example of teaching the systems of linear equations is given, where the main challenge for the student teacher is to come up with the problem for which the use of equations would arise naturally.

Peterson (2005) notes that a problem-solving approach was evident in the lessons that were prepared by the student teachers. The structure of the lesson follows what might be seen as a "cultural script" (Stigler & Hiebert, 1999), in that it starts with presenting a practical problem in mathematics. The problem is usually complex and intriguing. Yet it is accessible because student teachers sequence the lesson so that it builds on previous knowledge. Students usually start by working on the problem individually. This is followed by eliciting different solutions from students, inviting other students to evaluate the effectiveness of the solutions, and then bringing the lesson to a close by summarizing the rules that govern a solution to the problem. At no time did the student teachers specify substitution or elimination as a method for solving the system of equations. Instead, the focus was on finding an answer that satisfied both conditions as they arose from the problem itself.

Several of the articles provide a glimpse of what teachers talk about as they plan the lesson (Peterson, 2005; Yoshida, 1999). Not only are the elements of the lesson, (e.g., the opening problem or the design of instructional materials), considered at substantial detail, but there is also teacher intent that is made explicit during this process.

Peterson's report recounts a discussion that took place between a sponsor teacher and his group of student teachers, at one of the schools that he observed during the collaborative planning session. During the discussion, the sponsor teacher guided the student teacher to be very clear on the intent of the opening question for the lesson so that he would be “better able to handle responses that were not anticipated” (p. 14). It must be noted that, in the Japanese lessons, where concepts are developed through coherent lessons using a problem-solving approach, teachers frequently rely on students as sources of information. In the exchange that is reported by Peterson, the sponsor teacher helped the student teacher to see that the use of variables needs to arise naturally when solutions to systems of linear equations are being taught. He felt strongly that one should not ask students to write equations, but rather to solve problems, which will naturally generate the equations during the problem solving process. Further, he emphasized the importance of helping pupils see the connection between “problems having answers” and “equations having solutions” in the process of converting problems to equations, finding solutions to those equations, and then interpreting these solutions as answers to the original problem. These teachings were not prescriptive. They were provided mostly through discussion and by asking clarifying questions, such as how the pupils would interpret the wording of the problem. In essence, in Japan, the approach to teacher preparation is similar to the approach to teacher development, in the sense that it is collaborative and focused on the lesson, with special attention to lesson design, and then later, to reflection and critique.

Whether in the pre-service or in-service setting, the professional development culture of lesson study provides a real-time observation of teaching and learning, which is followed by a reflective discussion that, together, have a huge impact on everyone involved. As for the frequency of teaching a research lesson, some teachers may serve as the lesson study teacher only a few times in their entire career, while others may serve this role twice in a school year. The impact of lesson study is primarily on the team that created the lesson. There is also a ripple effect on the observers as it creates an opportunity to establish a professional communication and allows participants to take away ideas for implementation and refinement. The live observation of a teacher and class in action, which is continually referred to by the Japanese teachers as “seeing with the eyes,” is considered by teachers to be the essence of their professional growth (Bass, Usiskin & Burrill 2002, p. 22).

Lesson study can be described as a form of action research that is carefully structured and in which teachers engage to improve instruction. In addition, lesson study lays out a model for teacher learning and a clear set of principles, or hypotheses, that is to be tested in practice. Yoshida's study (1999) was the first case study to offer a detailed description of how the process of lesson study unfolds and what it ultimately provides to teachers in Japan. He identified several key principles that lead to improved instruction. These principles were later expanded upon and refined by several other authors, most notably Lewis (2000), and Fernandez & Chokshi (2002).

The overall vehicle in which these principles are embedded is related to the teachers' engagement in research about their teaching. As they plan, refine, teach and re-teach the lesson, they continuously test their hypotheses of what works best in the classroom. They test and refine their theories, and they collect data and interpret their findings. They do that with a clear focus on the thinking and learning processes of the students. Although the process may lack the rigor of scientific research, it is nonetheless thorough and methodical (Yoshida, 1999).

In this process, teachers engage in a form of empirical research of classroom teaching and its effects, essentially attending to both of the two main goals of mathematics education research that are mentioned by Schoenfeld (2000). One of the two main goals of mathematics education research is to examine how mathematical understanding develops. This "pure research" is concerned with questions about the nature of mathematical thinking, teaching and learning. The other main goal is to use such understanding to improve mathematics instruction, thus belonging to the area of "applied research." In lesson study, the first of the goals is put in the service of the second one.

As we examine the most important principles that are identified in the literature about how teachers learn in the context of lesson study, we can notice the links to the two goals that are mentioned above. While Yoshida speaks of these principles as the driving forces that lead to improved instruction, Lewis et al. describe them using the language of "key pathways to instructional improvement," essentially focusing on the benefits of lesson study as a tool for instructional improvement (Lewis, Perry, & Hurd, 2009, p.19). It should be noted that the identified sets of principles overlap significantly across various authors, upon which I elaborate next.

The first one of the principles relates to the “learning about students’ thinking and learning processes.” This is done in the planning and observation phase of the lesson study. In planning, it is a regular practice to anticipate how students will respond to an action of the teacher. Also, a significant amount of time is devoted to designing instructional tasks that would support the desired learning outcome in the most robust way. Generally, the observation records include detailed narrative records of the learning of several students, (i.e., what students said and wrote, how they used instructional materials, what specific instructional supports encouraged understanding and what presented obstacles to learning). As one Japanese teacher put it, “You develop the vision to see children,” (Lewis, 2000, p.14). According to Yoshida (1999), there is a prevalent philosophy among Japanese teachers that one of the criteria for becoming a good teacher is to be able to understand what the students are really thinking.

The second principle relates to the “increased knowledge of the subject matter” and the “increased knowledge of instruction.” This principle is accomplished as the teachers examine existing textbooks and standards, and discuss the essential concepts and skills that need to be in place in order for the students to learn the content of the lesson successfully. They also compare the treatment of the concept in other curricula. They generate many questions about the subject matter and how it should be approached in teaching. They often draw upon the expertise of mathematicians and mathematics educators, deepening both the subject matter and the pedagogical content knowledge of the topic. This is exemplified a number of times across various readings, most notably in the case of a lesson on pattern growth (Lewis et al., 2009). In their report, they provide an account of six elementary teachers who uncovered an interesting paradox. Fourth grade students were able to correctly fill out a table relating two variables. However, they were unable to express the relationship, either in words or by using an equation. The data suggested that the table “spoon-fed” the students. So the teachers redesigned the lesson so that it required the students to organize the data themselves. This led to many more students grasping the mechanism by which the two variables were related. From their experience, teachers derived a broad instructional implication that it is the students that must do the work, not the teachers.

In the effort to offer an explication, or theory, of the mechanism by which lesson study improves instruction, as it relates to the second principle that is mentioned above, two of its aspects, in particular, have been considered (Yoshida, 1999). First, teachers can easily engage in the task of planning instruction because it is concrete and very familiar. This task affords the opportunity to share experiences, knowledge and ideas for instruction. The combination of a collaborative activity and a concrete sharable task produces a common language in which teachers can talk about teaching. In this way, it furthers their understanding of subject matter and instruction. On a side note, there are several edited books in Japan on teachers' terminology with specialized terms whose meanings cannot be found in ordinary dictionaries. The second aspect has to do with the shaping of a shared vision of what good teaching entails. The teachers seem to share very clear ideas of the type of lessons they strive to produce. They envision lessons that can be termed as "student-centered, whole class instruction," the characteristics of which have been touched upon in the discussion of criteria for observation of research lessons.

The third principle is related to "building strong collegial networks." Lesson study builds a community of practice and a supportive environment in which teachers routinely share resources and ideas. According to one Japanese teacher, lesson study is not about a single lesson; rather, it gives her a chance to continue consulting with other teachers. Lewis et al. explain, "Ideally, the interpersonal bridges built during lesson study enable collaboration well beyond the research lesson, increasing the coherence and consistency of the learning environment" and "teachers can greatly improve students' lives by working together as a whole faculty" (2004, p.21). This is consistent with Yoshida's account of teachers at Tsuta elementary school. The teachers believed that the impact of a single teacher upon student learning, during elementary education, is limited. Their work was motivated by a belief in the maximum impact on the overall development of the students when all of the teachers at the school continually try to improve upon their teaching.

3.5. Lesson study in North America

While Stigler and Hiebert (1999) present lesson study as a model for teacher development in positive terms, they also warn that it will likely need to be modified before it can work in the U.S. Although many authors acknowledge that, in the U.S., teaching has been very resistant to change, they argue that lesson study may be the strategy to improve teaching (Fernandez & Chokshi, 2002; Hiebert, Gallimore, & Stigler, 2002; Kelly, 2002).

There are two main reasons for the resistance to change that is described in the literature. One is that teaching in the United States is conducted in an individualistic, isolated fashion (Stevenson & Nerison-Low, 2002). The other is that teachers in the U.S. have a limited knowledge of the concepts they are supposed to teach. The "closed door practice" allows for fewer opportunities to exchange teaching strategies and other professional knowledge. For example, when a phenomenal American teacher retires, all lesson plans and practices that were developed by that teacher also retire. In contrast, when a phenomenal Japanese teacher retires, he/she leaves a legacy that is expanded upon by future teachers (Chenoweth, 2000).

According to Chokshi and Fernandez, (2005), U.S. teachers do not have a way to harvest their collective experiences, share common concerns, and systematically integrate or further refine their knowledge. Research has shown that teachers in North America tend to fall into the same patterns of teaching as they were subjected to themselves, through many years of their schooling (Lortie, 1975), and that the effect of teacher training programs has been limited.

According to Schifter (1998), most teachers in the U.S. have been educated in mathematics that is restricted to the memorization of procedures. They have not had opportunities to learn mathematics from a conceptual perspective. If both isolated practice and limited knowledge continue to survive, there is very little hope for any kind of reform in the U.S., especially as it concerns elementary school mathematics education. Lesson study seems to counter both of these limitations by effectively transforming the personal knowledge of the teacher into a collectively built, widely shared and cohesive, professional knowledge base.

The practice of lesson study in the United States is gaining momentum. In the efforts to provide a more coordinated and deliberate approach to improving instruction, a number of schools and school districts in the U.S. have looked into the Japanese model of lesson study. The Highlands School of the San Mateo-Foster City School District in California is one of the most significant examples of a successful transfer of the Japanese lesson study to North America (Lewis, Perry, Hurd & O Connel, 2006). Lesson study in Highlands began with two individuals who had a vision. At the time of the Lewis et al. report, it was in its sixth year of operation, with all of its teachers participating in two lesson studies each year. There are clear signs that it has become an institutionalized practice. Examples include the decision of administrators to give teachers dedicated time for their work on lesson studies and to replace the evaluations of teachers with observations from lesson studies, as well as to use it as a vehicle for mentoring.

The comments of the teachers reveal that, through lesson study, they achieve greater instructional coherence, and a sense of collective efficacy and mutual responsibility for the learning of the students. According to Lewis et al. (2006), student achievement in mathematics at Highlands surpassed the district and state levels over a three-year period. Moreover, an additional analysis revealed that the net increase in mathematics achievement during the same time for students at Highlands was triple that of students that received instruction in comparable other schools elsewhere.

In the same report, the authors discuss four changes that have taken place during the evolution of lesson study at Highlands, which are seen to have moved the focus of the teachers from the surface features of lesson study to its underlying principles. Many reform efforts have failed when only their surface features were implemented in recipe-like fashion without attending to the underlying rationale. For example, there were serious problems with the way many teachers interpreted the recommendation for the use of manipulatives. Instead of using them to reveal mathematical relationships and to promote reasoning, often teachers simply showed children yet another set of steps to remember. Another example of a surface feature being taken for the actual substance is the case of problem solving, as it is often used in school mathematics. Although many of the problems that are used require some hard thinking, they are not necessarily connected to the content that is to be taught. The point of developing a coherent view of mathematics for the students is then missed.

Schifter (2002) reports that, "In one very disturbing interpretation of the Standards, some teachers agreed on the importance of eliciting ideas from their students, but did not understand that they had a further responsibility to critically analyze those ideas for mathematical soundness" (p. 26). The four changes in the conceptions of the teachers who participated in lesson study, which have accompanied the movement from focusing on surface features to focusing on the underlying principles, are:

- Lesson study is about teacher learning, not just about lessons.

This point refers to the initial conception that is held by teachers. That is that lesson study is about designing perfect lessons, and then later disseminating them. These perfect lessons were seen as the end-goal of all of the effort that was put into the lesson study work. However, the teachers soon began to view it as an opportunity to work on their practice by acting as researchers, testing their knowledge of how students think, and understanding the content and why it is important.

- Effective lesson study hinges on skillful observation and subsequent discussion.

Initially, teachers focused on rather obvious features of student behaviour, such as whether students are engaged, are working on a teacher assigned task or are treating their peers with respect. Gradually, the focus moved toward observing student thinking and how they responded to the development of mathematics during the lesson. Consequently, data collection became more intentional and focused. The subsequent discussion moved from considering the surface features of the lesson to considering the particular aspects of student thinking, such as what solution strategies students came up with, how they organized information and what types of errors they made.

- Lesson study is enhanced by turning to outside sources of knowledge.

In the beginning of the lesson studies at Highlands, teachers worked in groups without enlisting any support from outside. Later on, various content specialists and educators were invited to participate as commentators, observers and team members for the entire duration of a lesson study cycle. In addition to broadening the base of human resources into which they could tap, Highlands' teachers now draw upon an ever increasing range of print materials, including research articles.

- The phases for the lesson study cycle are balanced and integrated.

In the beginning, teachers spent a greater amount of time creating the lesson and less time drawing implications for further practice. They felt that the research lesson was the final performance. As the practice matured, it became viewed as a catalyst for further study and for the improvement of practice.

The study of Lewis et al. (2006) addressed the question of scaling up the Highlands example. They concluded that, in order to repeat the success of Highlands on a wider scale, the following four key changes in the education policy are needed: cross-site learning about lesson study, diverse ecology of lesson study, pathways linking lesson study to textbooks and provision for inside-out reform.

Cross-site learning means that groups of practitioners can share their experiences and learn from each other. As described above, the diverse ecology of lesson study refers to the three aspects of lessons study: a) the focus of the lesson study, (e.g., a particular discipline, vision or a goal), b) the level at which it is conducted (school-based, district-based or public research lesson), and c) the extent of external cooperation involved.

Pathways linking lesson studies to textbooks refer to the Japanese practice of regular assessments of theories and innovations from outside of Japan. After being thoroughly scrutinized through research lessons, these ideas are then incorporated into the Japanese curriculum. The last of the recommended changes considers providing "inside-out" reform. Inside-out reforms are initiated and supported within a school, rather than being brought from outside. Allegedly, in North America, we do not suffer from a lack of good programs, but rather from a lack of the demand for them by teachers.

According to the experience at Highlands, lesson study increases the demand for good programs because it reveals to teachers what is not working. For example, it has been noted that some North American textbooks are flawed in content and include untested ideas.

Another report that was found in the literature concerns the implementation of lesson study in 25 schools in the school district in Jefferson County, Kentucky (Byrum, Jarel & Munoz, 2002). The report focuses on the impact of lesson study on the professional development of teachers. It uses a participant-oriented evaluation model to determine the perceptions of teachers on the effectiveness of lesson study as a model for teacher development. Lesson study seems to embody many of the principles of effective job-embedded professional development. For example, sustained and intensive programs with local teachers in charge are more likely to make an impact than shorter professional development programs and activities that are imposed from outside. Similarly, teacher networks or study groups contribute to greater gains in teacher development, as opposed to traditional classes and workshops.

One of the major findings of this study is that teachers and administrators are enthusiastic about the opportunities to direct their own professional development as it relates to their personal growth. The results of this study indicated that the model was popular among teachers and administrators as it stimulated their desire to become more effective. The two most significant benefits that were mentioned by teachers were the opportunity to collaborate with others and that the knowledge that is gained is comprehensive enough to be applicable to lesson planning, instructional practice and assessment of student learning. Participants agreed that the biggest challenges are the time and cost that are needed to implement the lesson study process properly.

Another lesson study pioneering school is Paterson (NJ) School Number 2. Several reports emerge from the lesson study work at this school (Byrum et al., 2002; Kelly, 2002). According to the principal Lynn Liptak, lesson study has become part of the school culture. Provisions have been made for ensuring that it is not an activity that is engaged in by a few enthusiastic volunteers during after school hours. Rather, it has become a school policy, with the goal of gradual improvement and with time allocated during school hours. At this school, five mathematics lesson study groups meet for 80 to 100 minutes each week, while students move to specialized classes, such as art and physical education. Teachers at the school claim that one of the program's greatest benefits is the improvement of their content knowledge.

Teachers at Peterson also had the benefit of learning from Japanese mentors. For one year, teachers from the Greenwich Japanese School coached the Peterson teachers once a week for two hours. The project was funded by a grant and was introduced through Columbia's Lesson Study Research Group. In addition to adopting the lesson study framework, teachers at Peterson also adopted Asian textbooks, which include fewer topics, systematic development of concepts and a problem solving approach.

Based on the report by Kelly (2002), there seems to be a difference in how lesson study is perceived by teachers in schools where lesson study is mandated, as opposed to where it is voluntary. At Peterson, where participation is voluntary and teachers have Japanese mentors, it is viewed as a "cornerstone of their practice." In contrast, in Community District 2 of New York, where participation is mandatory and there are no external mentors, lesson study is viewed as "just another tool in their professional development kit."

Further on the difficulties of implementing reform practices, Stewart & Brendefur (2005) question the effectiveness of many of the major reform efforts during the past two decades. In particular, they argue that strategic planning and whole-school reform efforts have been mostly ineffective because planning cycles are too long, and implementation is too complex and cumbersome. This leads to fragmentation and overload. To solve these problems and to make schools work better for students, they recommended that small groups of teachers work collaboratively on relatively short-term goals. They wrote, "We must replace complex, long-term plans with simple plans that focus on actual teaching of lessons and units created in true 'learning communities' that promote team-based, short-term thought and action" (p.681).

After several years of working with school districts on various systemic reform efforts, the authors conclude that the best way to bring about positive change at the classroom level is to adopt a model in which small groups of teachers work in collaborative learning communities that are focused on improving day-to-day instruction. The model they propose is the model of lesson study.

3.6. Research on effects of lesson study

Since lesson study is a fairly young practice in North America, it is a matter of time before we can determine under which conditions the practice is sustainable, or even scalable. Its various adaptations to local contexts and their effects have already become a subject of study in its own right. However, research on lesson study almost never involves the hallmarks of experimental research: comparison groups, random assignment and controlled conditions.

According to Lewis et al. (2006), three types of research are needed on lesson study in order to avoid the fate of so many promising reforms that were discarded before being fully understood or well implemented. These are: a) development of a descriptive knowledge base, b) explication of the innovation's mechanism, and c) iterative cycles of improvement research. Furthermore, they propose certain changes in the norms and structure of educational research in order to enhance the field's capacity to study innovations more thoroughly. There is a concern that too many innovations end up buried in the graveyard of educational reforms before they are given a fair chance to take root in wider practice. One change that they propose is rethinking the routes from educational research to educational improvement and recognizing a "local proof route." Another change is building research methods and norms that will enable learning from innovation practitioners. Yet another proposal is increasing the capacity to learn across cultural boundaries.

The idea of recognizing the "local proof route" grew out of the concern that U.S. researchers are already proposing randomized controlled trials, horse-race style comparisons and other summative research, which is designed to determine whether lesson study works. This is in dramatic contrast to the situation in Japan in which lesson study has been used for a century without summative evaluation. Authors speculate that instructional knowledge in Japan progresses according to the "local proof route," where knowledge accumulates through progressive advances in research lessons that are taught in various local contexts throughout Japan, rather than through large-scale or centralized studies.

Lewis at al. (2006) warn against the danger of lesson study becoming just another fad if summative trials of lesson study are conducted. Although it is widely believed that the root cause of educational faddism is the adoption of educational practices that have not been tested through controlled trials, the authors are concerned that, given how little is currently known about the nature and mechanisms of lesson study, this kind of research might actually contribute to falling out of favour. Running summative trials on lesson study while it is still in its immature state in the U.S. might lead to a conclusion that it does not work.

3.7. Lesson study as a vehicle for reforming practice

In North American classrooms, reforms have been elusive because they require teachers to change very basic assumptions about how students learn. According to Lewis & Tsuchida (1997), important features of student-centered learning, (e.g., peer discussion), are often lost when the approach is 'domesticated' to fit the beliefs of U.S. practitioners about instruction. Even in educational systems such as Japan's, where a centralized system allows for policy and textbooks to be mobilized to support new directions, the burden on the teachers to implement changes and to understand new visions of learning is likely to be great.

Lewis & Tsuchida (2007) note, "Before teachers can successfully implement a new curriculum or approach to learning, they need to figure out what it means, see it as important, and figure out how it can be done in their own setting" (p. 324). According to these authors, lesson study is a structure that allows teachers to reinvent policy in the classroom by giving teachers a chance to discuss and test out collaboratively new curricular content and approaches, and by providing an arena for exploring the big ideas and the concrete techniques that bring policy to life in the classroom. As such, lesson study is not simply individual professional development. Rather, it is a research and development system for ongoing improvement of the teaching profession as a whole.

Furthermore, conflicts often arise as shared research lessons are discussed among teachers with differing views, beliefs, values and opinions in regard to education. These conflicts result in lively discussions that enable a kind of dialectic process of refinement until the innovations become the norm. Teachers define collectively, not only the meaning of innovations that were once translated into practice in the classroom, but also how the education's many goals will be balanced across its many facets, such as children's ethical, intellectual, social and emotional development.

While the many swiftly changing demands on North American teachers leave little room for sustained focus on transforming education, the Japanese system emphasizes slow change and expects policy to be reinvented in the classroom. This creates a much more supportive context for reform. Liptak (2002) argues:

> "For too long, professional development time has been allocated to outside experts to "train" teachers rather than given to teachers to reflect collaboratively on their practice. We need to tap outside expertise; we need to improve our content and pedagogical knowledge. But the professional development process needs to occur in the context of our classrooms and be driven as an on-going activity by professional practitioners." (p. 7)

As it is in Japan, schools could apply to become designated research sites. The primary final product of such grants would not be a report that gathers dust, but a day of public research lessons during which teachers demonstrate how they have chosen to invent their subject. Such settings would provide opportunities for educators, researchers and policy makers to discuss the lessons, ask questions, find out what challenges teachers have encountered along the way and share their own views about how the instruction captures or misses the vision behind the policy. It would also remind us that visions of good teaching cannot just be talk. They must be brought to life in actual classrooms, by both teachers and students.

3.8. Research Questions

Flowing from what was presented in Chapters 2 and 3 on MfT and lesson study, and driven by my interest in the in-service mathematics teacher professional development, the main research question of this thesis is: “What and how can in-service, secondary mathematics teachers learn about mathematics for teaching through participating in a practice based, professional learning community of lesson study?” More specifically, this overarching question breaks into the following set of questions:

(a) What can be gained by the teachers who participate in the school-based lesson study initiative?

(b) What are the factors influencing the development of teaching practice in the lesson study setting? This might include visible and invisible features, such as beliefs and attitudes.

(c) What is the nature of the mathematics for teaching that has emerged?

Chapter 4. Methodology

Ethnographic fieldwork through *participant observation* is the principal methodology that is used in this study. The research scope defines this study as a *micro-ethnography,* with a single social unit being studied across multiple social situations within a single social institution (Spradley, 1980, p. 30).

The fieldwork involves a disciplined study of the professional learning of a group of secondary mathematics teachers who undertook the work of learning from one another and with one another, in an ongoing and sustained practice-based setting. The team of secondary mathematics teachers that are the participants of this research were engaged in lesson study practice for three years prior to the onset of the research. Therefore, during the time that this research took place, my role in this team did not include acting as a facilitator. For this team, doing lesson studies together was already an established practice that did not require facilitation. Given this context, lesson studies that were conducted by the team of teachers act as a window through which teaching practice and teacher learning is examined across multiple social situations in order to address the main research question of this thesis, “What and how do in-service secondary mathematics teachers learn about mathematics for teaching through participating in a practice-based, professional learning community of lesson study?”

4.1. Rationale for choosing lesson study

Collaborative inquiry as a model for professional learning has been employed by teachers in many different forms and for a variety of purposes. However, what they all have in common is their aim to increase learning and the effectiveness of teaching. Lesson study is only one model, among many. However, lesson study is unique in that it integrates all of the important aspects of teaching practice – planning for instruction, actual teaching in the class and, finally, evaluating and reflecting upon one’s teaching in light of its impact on the learning of the students. Lesson study works on many levels of learning because, by design, it attends to all of the important phases of teaching practice and is built around a single instructional unit – that is the lesson. For teachers, the lesson is a concrete context for learning.

Lesson study is not meant to be a one-shot experience. Rather, it needs to be ongoing and cyclical. Many of the other types of collaborative models for professional learning are, in a sense, only partial models. By "partial models," I mean either those models of collaboration during which teachers gather together to plan lessons or units of study, without following through on how that worked in practice; or those models that look at instruction by viewing videos of other teachers in the process of teaching lessons, and then discuss such lessons, either with a purpose to observe student thinking, or teacher-student interactions, or other elements that are intended to simulate the practice of reflecting on instruction. Other such partial models involve a collaborative analysis of the mathematical work of the students, usually with the purpose of coming to an understanding of student thinking that produced various kinds of errors. Such models are appropriate in the pre-service teacher programs in which pre-service teachers do not have other options because they do not have their own classrooms, and are still learning bits and pieces of the craft. Therefore, it makes a lot of sense to learn such skills as separate entities.

However, for in-service teachers lesson study as an integrated model is feasible and conducive to professional learning. It could be characterized as being embedded in the place of work, continuous, collaborative, cumulative, professional development practice of the team of teachers. This type of context will be hereafter referred to as *practice-based professional development* (PBPD) (Loucks-Horsley, Stiles, Mundry, Love, & Hewson, 2009).

4.2. Features of this empirical research

This ethnographic design involves four features. Firstly, this qualitative research study is concerned with exploring the nature of teaching practice and of its development within a community of secondary mathematics teachers in their natural work setting. It is a study of the meaning of the professional lives of participants, under real-world conditions. As such, it seeks to represent the views and perspectives of the participants of the study. Particularly, this research aims to provide the view of events, actions, norms and values from the perspective of the participants and, in doing so, to capture the meaning of real-life events as important sets of meanings that are held by the participants in the events. Therefore, it is more concerned with learning from participants than with studying people.

Secondly, there is employment of unstructured data as these appear in natural contexts of teacher collaboration and classroom teaching practice. No interviews or surveys of teachers were used during the time of data collection. As a researcher, I am conscious that a school is a place of work and that it must be respected as such. In order for the normal operations of lesson study to unfold and not be affected by the fact that there is research going on, I made an effort to conduct research as non-intrusive. Obstructions to normal activities of school life were kept to a minimum. The lessons that were chosen as research lessons by the teachers were part of the normal program of study that the teachers had to teach according to the curriculum.

Thirdly, in terms of incorporating concepts and theories into this study, I used an inductive approach primarily, as a way of shifting between data and concepts. For the analysis of data I used the Four-Component Framework detailed in Chapter 2.3, which I extended through thematic data analysis (Creswell, 2002). Investigation of the activity of the secondary mathematics teacher team was carried out through a single lesson study cycle, at a fine grain analysis of teacher interactions, although the research involved three cycles of lesson studies that the team of secondary mathematics teachers implemented in the school year when the research was conducted (Appendix B). "How and what do teachers learn about teaching mathematics when they collaborate on designing to teach a single lesson, and then test those designs in the classroom (as in the tradition of lesson study)?" "What are the qualitative changes in their conceptions and actions?" To answer such questions, the chosen research methodology aims to enable the generation of a theory of practice-based, professional education via a systematic, qualitative procedure. This broad theory about the phenomenon is grounded in the data that was drawn from one cycle of the lesson study process which was analyzed at depth, as mentioned before.

4.3. Setting

West Coast Academy is a suburban, K-12, coeducational, nondenominational, university preparatory, independent school in British Columbia with no selection process or entrance exam for incoming students.

This medium-sized school is a fairly young school that, until recently, struggled to fill all of the spots that were available. Therefore, almost everyone that applied was accepted. For this reason, its student population very closely represents the general student population in public and other non-selective independent schools, in terms of the talent and scholastic aptitude of its students. As is the case with most other independent schools, West Coast Academy receives a partial grant from the government in order to cover the costs of the education of its students. It is obliged to follow the provincial curriculum and hire provincially-accredited teachers. The rest of the cost is covered by the tuition fees, which are set at about two-thirds of that of comparable schools. Therefore, West Coast Academy is considered to be one of the more affordable options for those parents who opt to have their children educated in an independent school.

Despite the fact that the students are diverse as they enter, just as they are at the public schools, there is a promise that all students will be prepared for entry into university when they graduate from West Coast Academy. Naturally, it is expected that the teaching in all subjects will be of high quality to achieve this ambitious goal. To that end, teachers are carefully selected. Their employment contracts are renewed based on their performance and their value to the school. More importantly, teachers are expected and encouraged to continually develop their professional expertise. They are well supported in these pursuits. Specifically, the administration accommodates the timetables of the teachers in order to allow for collaboration in the areas of program design and instructional unit planning.

Three years prior to the start of this research study, the school embarked on adopting the International Baccalaureate (IB) program in Grades K-5. One of the features of this program is the delivery of curriculum through transdisciplinary themes that focus on an inquiry model of learning. In such an environment, teachers were used to engaging in substantial collaborative planning around the topics that they wanted their students to learn, although their activity was not specific to educational goals in the topics of mathematics. However, this setting provided the organizational structure in which teacher teams of neighbouring grade levels could work together if they chose to participate in the in-service professional development initiative in which this study was embedded.

4.4. Context for the study: Collaborative communities of inquiry

Teams of teachers at West Coast Academy acted as inquiry communities. Jaworski speaks of "inquiry communities" in which inquiry is used as a fundamental principle and position for engaging critically with key questions and issues of practice, such as the use of mathematical tasks in classrooms (teachers' practice and perspective), and finding ways of working with teachers in order to promote teaching development (educators' practice and perspective) (Jaworski, 2006). She proposed the use of "inquiry as a tool," which can be used as a strategy that can lead to developing "inquiry as a way of being." An inquiry community is similar to a "professional learning community" (DuFour, 2004). However, it places emphasis on the investigative nature of teaching as a means for teachers to generate knowledge about their practice. In the context of this research, it was used to generate knowledge of the teaching and learning of mathematics.

Lesson study can be thought of as a special form of inquiry community that integrates "teaching as inquiry" with "teaching as design," where each lesson study cycle acts as a teaching experiment, as well as a site for teacher and teaching development. It is not about how teaching is now, or what it should look like if it were to be effective. Rather, it is a close-up look inside the process as it is changing, developing and growing in the awareness and action of the participants in this research.

4.4.1. The beginning of lesson study at West Coast Academy

For three years prior to the start of this research study a pilot project of implementing lesson study took place with only one group of secondary mathematics teachers (including myself) who had become interested in lesson study. This began in September 2007, following my first round of participating in the Lesson Study Immersion Program and my presentation of what I had learned during my study tour in Japan. We began to experiment with lesson study in our own practice, at West Coast Academy. My colleagues were willing to try out the model of lesson study as a way of working together on developing lessons and of testing them in practice. These same teachers are the participants of the research study that is reported in this thesis.

Over the next several years, the four of us developed a number of mathematics lessons together. We observed one another teaching these lessons, and then recorded evidence in terms of the student responses and interactions that took place in the classroom. Afterward, we discussed what we saw, shared with each other what we noticed and what we thought the impact of the teaching practices had been on student learning. During the three years that followed, we designed, tested and evaluated a variety of lessons from Grade 8-11 curricula.

Throughout this process, we were learning and doing mathematics together, and sharing our own personal experiences of how we came to understand certain concepts and mathematical topics. Each time we implemented a “research lesson,” we invited other colleagues from the school to observe and participate in the post-lesson discussion. We supplied them with the lesson plan that we had collaboratively developed, along with the class seating chart. We asked them to focus on one or more aspects of the class, and then record their observations for later discussion.

On two occasions, we timed these lessons to coincide with the province-wide, professional development day, which takes place once a year and at which time all of the schools in the province free up their teachers from their teaching duties in order to allow them to pursue various professional development opportunities of their choice. In two consecutive years on this day, we opened up the school to other teachers and held a “lesson study open house,” in an attempt to replicate what I had seen in Japan during my participation in the Lesson Study Immersion Program.

In the fall of 2009, two years after our team had become actively involved in the practice of lesson study, our school hosted 35 mathematics teachers from various schools who came to observe and discuss two lessons that had been prepared by our team of teachers. Along with their principal, one group of six teachers drove over 400 km to come to West Coast Academy in order to learn about lesson study, in particular, and to also learn about improving mathematics instruction and achievement at their school in general. By that time, West Coast Academy had attained excellent outcomes in its mathematics education program, and joined the rank of the top schools in the province. In British Columbia school achievement data on several standardized mathematics tests, administered at Grades 4, 7, and 10 are publicly available. This was one of the reasons that this group of educators from so far away decided to participate in the open-house, professional development event at West Coast Academy.

By this time, our team of teachers knew that we were creating something very special within the school, not only because of the results on the standardized assessment, but because, for the most part, the students actually enjoyed mathematics. They believed that it is an interesting and important school subject to learn. We could all sense that this was the case. However, it was confirmed by a visiting mathematics teacher from a sister school in China, who asked our Head of School if she could perform a survey on all of the Grade 6-12 students, in order to find out about the attitudes and beliefs that the students at West Coast Academy held toward mathematics. She wanted to compare this with her own school in China.

One of the highlights that I took from her observations was that 95% of all of the students in these grades responded that they see mathematics as an interesting and worthwhile subject, and that they see themselves as successful in it. Moreover, about the same percentage expressed their commitment to pursue it to the highest levels that are offered by the school. This has been validated by the fact that, each year, between 50% and 67% of the students in the graduating class take AP Calculus AB course. They all write their AP exam and achieve an average score of well over 4.0. In short, this attests to a healthy attitude toward mathematics in the student population of West Coast Academy, across grade levels. As a team of mathematics teachers, we were aware of how we were creating this, and we actively pursued ways in which to assist one another to develop as professionals that can, in turn, impact the achievement of the students we teach.

During these three years prior to the commencement of this research study, data were not being collected, although artefacts of lesson study practice were being generated. At that time, the purpose was only to engage in the practice of continual professional development, and to explore the methods and potential benefits of lesson study. Together, we sustained the practice of lesson study with four "research lessons" being conducted each year. Each lesson study cycle spanned over a period of time of approximately one month in duration.

A number of other teachers from the school participated as observers during the enactment of these lessons and they repeatedly expressed an interest in doing this work themselves. They had already gained some familiarity with the process of lesson study from these occasional encounters with our team's work and from informal conversations that took place in the staff room in regards to what we were doing. In this way, the stage was set for other teachers to become involved as well. Our small team of secondary mathematics teachers had laid the groundwork and, in a real sense, it had led the way. In addition, during this time, with the help of my team, I gained a level of understanding of the process, which was sufficient for me to feel confident to lead its implementation across the school and to facilitate the work of other teacher teams.

4.4.2. Implementing lesson study at West Coast Academy and initiation of this research

I explained the nature of the research I wished to do to the school's administration and sought permission for my study from them. The Head of School granted his permission for the research. At the beginning of the 2010/11 school year, he announced to all faculty his approval of the study and expressed his hope that everyone who teaches mathematics, at any grade level in the school, would participate in this in-service professional development initiative.

A meeting was set in which I explained the following to the prospective participants: the purpose of this Practice-Based Professional Development activity, what the teachers would be expected to do and what time commitment was involved. I ensured that the teachers understood that they were not being evaluated and that they were free to withdraw from the study at any time, and for any reason, without consequences. Participation was voluntary. Nonetheless, 16 out of 17 teachers who taught mathematics at the school in the year when the data was being collected participated in the study. All of the names used in this work are pseudonyms (Appendix B shows the teams and the lessons).

Following this initial stage, which provided the groundwork for the implementation of a school-wide, in-service professional development work in the 2010/11 school year, the research study that is reported in this document began. With the exception of one teacher who taught only one class of Grade 7 Mathematics, all of the teachers who taught mathematics at any level from K-12, took on this professional development opportunity. The teachers at West Coast Academy worked together as a "professional learning community" (Wenger, 1998), engaging in shared planning, observation and discussion of research lessons.

To sum up, the actual field study and data collection took place over the period of one school year in a school-wide implementation whereby 16 teachers who taught mathematics in the K-12 grades voluntarily participated in this structured practice-based, collaborative professional development process of lesson study. This was the context in which this research study took place. For the purposes of this thesis, I focus my analysis on one team of teachers, the teachers with whom I had originally begun to implement the lesson study process at West Coast Academy.

4.4.3. The overarching goal for the lesson study implementation at West Coast Academy

The research lesson is like a window through which one can observe many things and learn from them and, consequently, align one's practice accordingly. As lesson study implies situated learning, the teachers act as agents of their own learning guided by their intentions. The intentions of the teachers are explicitly articulated as a set of goals, which are nested inside one another. By design, lesson study as an ongoing, professional development practice involves an articulation and adoption of a long-term goal to which all particular lesson implementations are intended to relate. The role of an overarching goal in lesson study was discussed in Chapter 3.3. For the implementation of lesson study at West Coast Academy, the overarching goal that was chosen was "to build a culture of mathematical thinking in the classroom." This means to promote both the individual student's capacity to think mathematically, as well as that of the classroom community as a whole. Then, the more particular goals for student learning and development were set, with thought given to how the smaller goals would support the larger ones.

Finally, the specific instructional goals of the research lesson were decided upon. In this way, there is socially constructed cohesiveness of goals for student learning and development, which members of the community of practice strived to realize through their practice. A structure of shared goals for student learning provided guidance for teacher activity within their lesson study activity.

4.4.4. Organizational and structural aspects of the implementation of lesson study at West Coast Academy

With respect to the organizational and structural aspects of working together, teacher teams were set according to the proximity of the grade levels that they taught. A scheduled weekly time of one hour was set for each of the teams to work together on their research lessons. Once the teams were decided upon by the teachers, one school administrator went the extra mile in supporting our work by tweaking some of the timetables of the teachers, in order to create these common blocks of time in which the teams could meet.

In working with the teams of teachers, my role varied, depending upon the type of my engagement: teacher of mathematics and a practitioner of lesson study (much like other participants), a mathematics education researcher and a facilitator of lesson study process. Table 1 depicts the structure of teacher teams and my role.

Teacher Teams (# of participating teachers)	My Role
Elementary School Teacher Teams (10) Teachers of grades K-5 2-4 teachers per team Generalists (teach several subjects, typically 4 or 5)	Facilitator and Researcher
Middle School Teacher Team (3) Teachers of grades 6-8 3 teachers in the team (excluding myself) 2 specialists, 1 generalist	Participant, Facilitator and Researcher
Senior School Teacher Team (3) Teachers of grades 9-12 3 teachers in the team (excluding myself) All specialists (2 also teach science courses, 1 teaches mathematics courses only)	Participant and Researcher

Table 1: Organization of the collaborative teacher teams

In the teacher teams where I acted as a facilitator, as shown in Table 1, my role was to guide the participants through the process of lesson study, while keeping the process nonintrusive. What really mattered was the mathematical content that the teachers selected for teaching, as well as what they brought to the table through their participation. Lesson study can only work if teachers see themselves as creating everything that happens. One of my objectives was also to fade out my role as a facilitator over time, relegate the leadership and let the events unfold according to the needs of the group. Usually one teacher would take charge of keeping the record of the meetings. One team of teachers decided to use a wiki. They did so quite successfully to keep the records, share resources and communicate more efficiently. All such decisions came from the participants as a group and were always supported by me.

4.4.5. Considerations in implementation of lesson study at West Coast Academy

Choice of topic for the research lessons

Lesson study provides a tangible and concrete focus on a single lesson, which is easy for teachers to understand and adopt. However, good candidate lessons are not chosen randomly. Three considerations guided the decisions about the mathematical content that would be taught in these research lessons. One was made for the practical reasons of minimizing any obtrusiveness of lesson study on normal day-to-day practice. I wanted to ensure that teachers would not have to go out of their way to teach something fancy, which is not in the curriculum, simply to showcase how wonderful and interesting mathematics lessons can be, while they put their regular teaching program on hold. Secondly, it was also important to retain genuine aspects of teaching practice, without contaminating it with the noise that would likely be introduced by some out-of-context novelty in terms of the content that was being taught. There was an agreement that the lessons would remain curricular in the sense that teachers would teach the content, which they would normally teach and when they would normally teach it, respecting the intended sequence of lessons (either driven by their yearly plan for the program of study for students, or by the textbook that they used in their class). Third, teachers were encouraged to choose lessons that involved concepts that, in their experience, seemed difficult for students to learn. This seemed to be an obvious space for teacher learning.

Challenges in the implementation of lesson study at West Coast Academy

One of the misconceptions about lesson study is that it is primarily about developing perfect lesson plans and exemplary lessons when, in fact, it is predominantly about teachers learning about mathematics for teaching. As a facilitator of lesson study in elementary and middle school teacher teams, I tried to steer teachers away from this interpretation and to help them see that lesson study has much more to do with developing teacher and teaching practice. The way the terms “teacher practice” and “teaching practice” are being used here is meant to capture, not only the instructional aspect of teaching practice, but also the planning for instruction and the reflecting on one’s own instruction.

Within the context of lesson study, teaching practice becomes public and collaborative, in the sense that it involves the collective planning and design of lessons, enactment by one teacher, which is observed by the team of teachers, followed by a collective reflection and evaluation of the team's work in the form of a post-lesson conference. In contrast to some other cultures, our culture is not one in which the public teaching of lessons is commonly practiced. To have an observer in the class usually means that the teacher is being evaluated. To have multiple observers is even more uncommon.

As a facilitator, my role was to help teachers understand the difference between critiquing the lesson, which is the product of the team's effort, and criticizing the teacher who taught the lesson. My role was to help teachers focus on the former, rather than the latter. Nevertheless, there was some apprehension, and even anxiety, associated with being the teacher who teaches the lesson.

At some point during the implementation of lesson study at West Coast Academy, all of the teachers were expected to assume the role of the teacher who enacts the lesson. A number of teachers enacted lessons in several of their team's lesson study cycles. Only one teacher did not participate in the role of enacting the lesson but instead asked another teacher on her team to teach the lesson on her scheduled occasion. While she really seemed to enjoy the whole process, she could not put herself in the position of teaching the lesson publicly.

4.5. Participants of the study

4.5.1. The team of teachers selected for the study

The Senior School Teacher Team (see Table 1), composed of a group of three teachers, whom I shall call Andrew, Gabrielle and Steve, had been my fellow colleagues for five, three and four years at the school respectively, at the onset of this research. As such, the practice of lesson study as an ongoing professional development practice had been well established among us, long before the start of the research. Therefore, my newly adopted identity as a researcher did not interfere with the attitude of the participants toward the research project. They were engaging in the lesson study process in much the same manner as before.

Our principal professional relationship is one of collegiality and trust, based on our common commitment to student success. Stepping into this study, we were aligned in our view of this work being a contribution, both to each of us personally and to our profession as a whole. During the time of the field research, they were the faculty whose teaching assignments involved teaching mathematics to all Grade 9-12 students at the school.

There were several considerations that guided the narrowing of the research to one teacher team for the purpose of this thesis. Firstly, as a researcher I wanted to pursue the study of secondary mathematics teachers' practice and its development because this is my primary area of interest. Secondly, working with this team my influence as the researcher on the data was minimized. This group of teachers had been doing lesson studies together for three years prior to the start of the school-wide implementation. Therefore, this removed the need for me as the researcher to also facilitate the process of lesson study. The team had been introduced to lesson study as a mode of professional development previously and was comfortable with the process. In other words, the noise, or disturbance, which is created by introducing a new practice into the system, was minimized that way. Thirdly, this choice made for a *natural environment* in which to study the research questions. In a way, it simulates how other such teacher teams might be operating in similar contexts of Practice-Based Professional Development, without adding an external influence of "facilitator" or "researcher." Finally, this teacher team completed three full cycles of lesson study during the time of this research and within each cycle, the lesson that the team developed was implemented twice in the classroom, each time with a different teacher teaching the lesson to his or her own class (Appendix B). This feature of the design provided for the possibility of conducting comparative analysis of the two lesson enactments, from which the main topic of this research can be attended to in a more controlled way.

4.5.2. Individual participants: Background and experience

As already noted, Andrew, Gabrielle, Steve and I had been working together as colleagues at the school for a number of years. Our shared commitment to student success and the desire to keep developing as teachers brought us together even more closely because we informally instituted our small professional learning community in order to study how the students best learn mathematics and what we can do as teachers to facilitate this process.

All of the teachers in the team were highly experienced. However, their educational and cultural backgrounds varied. At the time this study took place, Andrew was in his late 50s and in his twenty-fifth year of teaching. He came from an Eastern European educational background where he received a B.Sc. degree in secondary mathematics teaching, from a university that prepares mathematics teachers through a five-year program that is offered within the mathematics department. He obtained a B.Ed. degree and became certified to teach secondary mathematics in British Columbia, after having undergone a provincially accredited university program for foreign-trained teachers. He was a nationally renowned teacher in his country of origin, where he taught for the first 15 years of his career.

Andrew had authored two secondary school mathematics textbooks, which are still in use in that part of the world today. He acted as a mathematics department head, and later on also as a principal of the school, at a magnet school that offered intensive academic preparation in mathematics. For a number of years during that time, he was also involved in mathematics teacher education at the local university, where he taught courses in methods and techniques for teaching secondary school mathematics. At West Coast Academy, he had quickly established himself as a master teacher. He introduced the AP Calculus course into the school's course offerings and, within a span of two years, he developed a very successful program that saw a large proportion of Grade 12 students enrolling in the course.

Gabrielle, who was in her mid-50s and in her thirtieth year of teaching at the time of this research, was educated in the British school system. She held a B.Sc. with a major in Physics and had extensive international teaching experience. For several years, she taught first in South Africa, her country of origin. Then, she moved to North America, where she continued her teaching career for about 20 years, first in the U.S. and then in British Columbia.

During most of her teaching experience, Gabrielle has been teaching both sciences and mathematics, although she regarded herself more as a science teacher. At the time of this study, she was in her fourth year of teaching at West Coast Academy where, in that school year, her teaching assignment included two classes in mathematics (Math 10 and 11) and three classes in sciences (Physics 11 and 12).

During the research period, Steve was in his late 30s and in his twelfth year of teaching. He held a B.Sc. in Chemistry from a university in Australia, his home county, where he taught for the first few years of his career. Then he settled in British Columbia and began to work at West Coast Academy. Steve also held a M.Ed. in School Administration from a university in Australia, which he obtained during a one-year study leave from his teaching position during his fifth year at the school. Steve identified himself as both a science and a mathematics teacher, and he aspired to enter an administrative position in the future. His teaching assignment during the year of observation included three classes in mathematics (Math 9, 10 and 11) and two classes in sciences (Chemistry 12 and AP Chemistry).

Table 2 below summarizes the details of the educational backgrounds of the participants and their teaching experience:

	Educational Background	**Teaching Experience and Teaching Assignment during the Observation Period**
Andrew	B. Sc. in Mathematics B.Ed. with certification in Secondary Mathematics	25 years In his 5th year at West Coast Academy Math 9, Math 11, Math 12, AP Calculus
Gabrielle	B. Sc. in Physics B.Ed. with certification in Secondary Science	30 years In her 4th year at West Coast Academy Math 10, Math 11, Physics 11, Physics 12

Steve	B. Sc. in Chemistry B.Ed. with certification in Secondary Science M.Ed. in School Administration	12 years In his 7th year at West Coast Academy Math 9, Math 10, Math 11, Chem 12, AP Chem

Table 2: Participants' educational background and experience.

4.6. Participants' activity within the lesson study cycle

Collaborative Preparing for Instruction

Andrew, Gabrielle and Steve selected the topic for the research lesson in their first meeting. They also decided among themselves who would be teaching the lesson first and in which class period they would teach it. That teacher would have the responsibility of writing out the lesson plan, based on the instructional decisions that were made by the entire team. This depended on the prior knowledge of the students and on the goals, which the teachers set for learning in the new lesson.

The first meeting was dedicated to planning the lesson. In this meeting, teachers discussed the ways in which the topic might be approached. They also consulted the curriculum document, textbooks and other resources, and shared personal experiences in regards to how they taught the topic in the past and what they might change in order to promote deeper understanding and create situations for students to engage in mathematical thinking.

The second planning meeting was dedicated to revisions of the lesson image that was captured in the first draft of the lesson plan. Details of the lesson were worked out, learning tasks were tried out, instructional materials were decided upon, and exact questions that the teacher would pursue with the class were formulated. In addition, teachers continued to exchange ideas and consult resources between the two meetings.

Production of the lesson plan

Producing the lesson plan is a necessary part of the lesson study process. With it the teacher team communicates their goals and objectives for students learning, as well as how these will be achieved. The teacher that would be enacting the lesson first took the responsibility for writing up the lesson plan based on the input during the pre-lesson discussions.

Lesson enactment

On the designated day for the lesson implementation, all of the teachers from the team would be freed up to participate in the lesson observation. One teacher taught the lesson to their own class, while other teachers recorded their observations using a protocol for lesson observation that was agreed upon by the teachers prior to the implementation.

Post-lesson discussion(s)

After school on the same day, the teachers met again to reflect upon the implemented lesson and assess how the lesson met the intended goals. They examined which instructional supports enabled learning and which ones did not. Based on what teachers noticed and their reasoning about it, they made either minor or major changes for the second lesson enactment, which was taught by another teacher, in a different classroom, within the few days that followed, usually to his or her own class. Another post-lesson discussion followed in which teachers had the chance to discuss how the changes of the revised lesson or the differences in the classroom affected student learning.

Teachers' formal report

I encouraged and hoped that teachers would also produce a formal report on their research lesson in order to capture what they learned and what they felt would be significant enough to be shared with their professional community at large. Producing a formal report on the lesson study experience is a standard practice in Japan, where teachers publish more than educational researchers and where research-lesson reports are available at large bookstores (Takahashi & Yoshida, 2004). However, only one such report was produced by this team of teachers (presented in Appendix E). Teachers publishing reports on their practice is also culturally based and could not be fit in this context where there is no such community with which to share this work.

4.7. Research design

The single social unit that was chosen for this study is a team of secondary mathematics teachers who taught mathematics to Grade 9-12 students at West Coast Academy. This group conducted three cycles of lesson study during the research period. Each lesson study cycle spanned over three to four weeks and is used as the *data collection unit* for the study (Yin, 2010). Lesson study cycle as the data collection unit is an organizational entity that, at the narrower level, consists of the data collection units that capture the practices of planning for instruction, teaching mathematics lessons and reflecting on instruction within the context of the engagement of teachers in lesson studies. This thesis draws upon one particular cycle of the lesson-study activity of the group, which is analyzed in depth in order to capture the teachers' knowledge, routines, norms, habits and details that surface in this kind of collaborative structure.

The role that I assumed within this team for research purposes is that of the *participant observer*. Essentially, each time I visited the setting of the research, my role involved simultaneously serving as both the researcher and as a teacher participant. Ainley (1999) speaks of complementary and conflicting roles that are faced by researchers that conduct school-based research. This dual role raises the need for the articulation of a *psychological frame* based on professional affiliation (Silverman, 2006, p. 83) and for consideration of the differences between the ordinary participant and the participant observer. In the section that follows, I attend to this question.

4.7.1. The role of participant observer in the research

In order to explain my positioning as a researcher within the context of the research, three major differences are considered between the ordinary participant and the participant observer. The first is the consideration of the *dual purpose* (Spradley, 1980, p. 53) with which I, as the researcher, came to the setting of the research. As a participant observer in the lesson study process, I came to the social situation with two purposes - to engage in the activities that were appropriate to the situation and to observe the activities within these social situations (planning for instruction, enacting the lesson in the classroom, reflecting on the lesson in the post-lesson discussion). With this dual purpose comes also the need for record-keeping of observations, which is something that ordinary participants do not do. All lesson-planning sessions and post-lesson discussions were audiotaped. This enabled me to be fully immersed in being a participant during these phases, and then upon reviewing the recordings to fully switch into the researcher's role. The teaching of the research lessons in the classroom was videotaped and field notes of the observations were recorded (most often on the spot). At times, notes were recorded later, after I had left the social situation.

Secondly, a participant observer experiences the social situation in which the research is being conducted both as an insider and as an outsider simultaneously. As an *insider* who participated in the process of lesson study with her colleagues, I engaged in much the same activities as they did. At the same time, the experience of the researcher is one of an *outsider*, in which all that is observed, including myself as the participant, can be taken as objects of study. This positioning required making oneself explicitly aware of things that others take for granted and looking beyond the immediate focus of the activity. Therefore, the role of a researcher inquiring into the practices of mathematics teaching and its development, and the role of a teacher practitioner who is a member of the group in the study, are both intertwined and separate. Processes and teachers' interactions that were recorded as part of the collected data were thus based on an authentic situation, much like one in which no research is taking place.

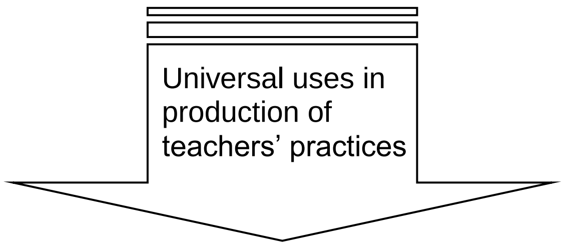
Thirdly, in regards to the degree of involvement, the study reported here can be seen as having a moderate participation involvement by the researcher. During the time of the field study and data collection, I did not teach in the senior school. Therefore, the classes in which the research took place were not my own, but were those of my colleagues who are participants of this study. This also means that I did not act as the teacher who enacted the research lessons when I participated in lesson studies with this team of teachers. Nor did I participate in the production of artefacts that were generated by the teachers. The teacher that was designated to teach the research lesson first, in the extended lesson study cycle, took on the responsibility for writing up the lesson plan. If there were modifications for the second implementation of the research lesson, the teacher enacting it would revise the lesson plan. Likewise, the report that the team produced for one of the research lessons, came out of the collaborative effort of the group with practically no involvement on my part. *Moderate participation* occurs when the researcher seeks to maintain balance between being an insider and an outsider, and between participation and observation (Spradley, 1980, p.60).

4.7.2. Conceptual framework of the study

Behaviour and artefacts, and speech messages that are created by a community of people (such as a team of teachers that are engaged in a Practice-Based Professional Development), are easily accessible to our senses. However, they represent only the surface of the vast reservoir of *cultural knowledge* that is hidden from view. Yet, this knowledge is of fundamental importance because we all use it constantly to generate behaviour and interpret our experience. The cultural knowledge of interest here is mathematics for teaching (MfT) held by the participants of this study which is taken to mean, following Davis & Renert (2014) the "complex network of understanding, disposition, and competencies that are not easily named or measured. The embodied complexity of [MfT] must be experienced – seen, heard, and felt" (p. 3).

Following the developments in research about the construct of MfT, I take this construct to include the formal content knowledge, specialized pedagogical content knowledge, content knowledge that is entailed in the work of teaching, as well as the understanding by the teachers of "emergent mathematics", which is characterized by an "open disposition toward mathematics" (Davis, & Renert, 2014, p. 47). MfT exists at two levels of consciousness: explicit knowledge that people can communicate with relative ease, as well as tacit knowledge, which is outside of our awareness and is highly personal. Tacit knowledge includes one's insights, values and convictions. It is embodied, enacted and taken for granted.

Teachers' MfT	
Explicit	Tacit



Interpreting experience	Generating Behaviour
Artefacts: textbooks, curriculum, student work, … Behaviour and events: student reasoning, problem solving strategies, … Physical environment	Behaviour that is specific to the work of mathematics teachers: explaining a concept, marking tests, selecting/modifying/designing tasks for instruction ... Artefacts: teachers' lesson plans

Figure 1: Universal uses of teachers' MfT for generating behaviour and interpreting experience

In my data analyses I was guided by the ideas presented in Figure 1, which is an adaptation of Spradley's depiction of cultural knowledge, applied to the professional knowledge of secondary mathematics teachers. I use it as a model to think about the methodological framework for researching the main topic of the study (Spradley, 1980, p. 8). In doing field work, I observed what teachers do (behaviour) and what they make use of or create (lesson plans, instructional tasks). I listened to what they say (speech messages) in order to infer their MfT, based on reasoning from evidence. Both explicit and tacit knowledge are revealed through speech, acts and artefacts. Participant observer methodology relies on making inferences about the cultural knowledge of the group of people, based on the meanings that they hold about things that are related to their teaching practice.

4.7.2.1. Integrated model of the research design

In researching the learning of teachers through the practice-based, collaborative, professional development process, we realize how complex phenomena teaching and learning are. Teaching practice is where mathematics for teaching manifests itself and shows how multidimensional this construct is. In my analysis, I focus on pre-lesson discussion and post-lesson reflection, which are intimately connected to the enacted lesson in the classroom. I use the lesson study process as an arena in which to observe what and how teachers learn about mathematics for teaching, and what kind of opportunities for professional growth are afforded through this practice-based collaborative process.

Figure 2 below depicts the overall conceptual design of this study. It gives a comprehensive view of my idea as the researcher on how the research problem will have to be explored. It serves to illustrate the tools and frameworks that were used in this research, for a customized analysis of the process of lesson study through which the research question of this thesis is being pursued: "What and how do in-service secondary mathematics teachers learn about mathematics for teaching through participating in a practice-based, professional learning community of lesson study?" At the center is the lesson study process as a Practice-Based Professional Development method, which consists of three phases that have a dual purpose, practical and theoretic, for this research.

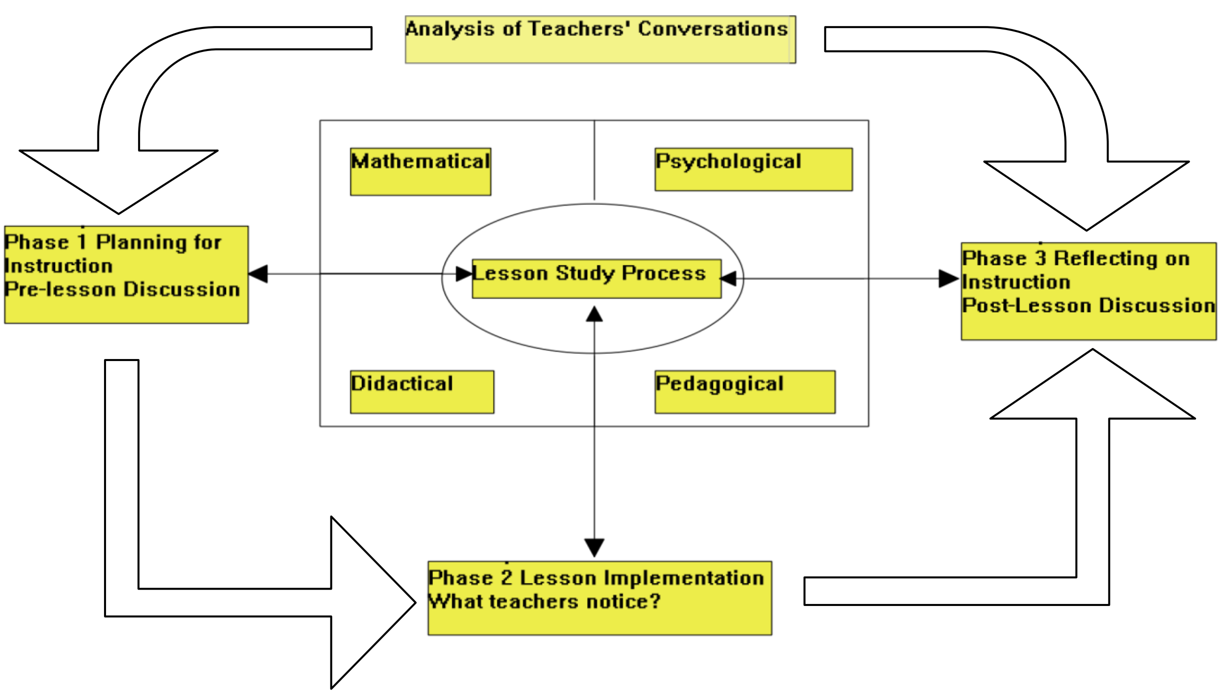


Figure 2: Integrated model of the research design

The practical purpose assumes that the lesson study process offers opportunities for teachers to learn through their activity in all three stages, while from the theoretic perspective this context serves to identify and research the nature of teachers' knowledge growth. As a researcher who is also a participant observer I performed a micro-analysis of one lesson study cycle conducted by the team of teachers who are the participants of this study. In doing so, I analysed the participants' professional conversations from two of the stages of the lesson study process, the pre-lesson and post-lesson discussions. In the latter, I drew on what the participants noticed about the enacted research lessons that they observed, as documented in their lesson observation logs and discussed during the post-lesson discussions.

Planning for instruction and reflecting on instruction are the two collaborative processes, to which I applied my analysis of the teachers' conversations as the primary methodology to extract themes that are related to the MfT construct. Lesson implementation is an individual teacher's performance, as observed by other members of the team. It connects the other two phases into a cycle through which teacher learning is meant to materialize. In this research, the lesson implementation is used only to validate and/or further illuminate the learning of the teachers in the collaborative setting. A detailed analysis of the lesson implementations is beyond the scope of this thesis.

The conceptual framework is founded on the theoretical Four-Component Framework of teachers' MfT (described in Section 2.2 and named in Section 2.3) which lies on a much broader scale, and which was derived from various considerations of the MfT construct, as a suitable theoretical framework for the investigation of how phenomena occur in this particular context. Furthermore, it provides a general representation of the relationships between things related to the phenomena of mathematics teaching practice. On the other hand, the conceptual framework provides a specific direction for this research and enables a description about the relationships between specific variables identified in the study (such as pre-lesson and post-lesson discussion, and lesson enactment). It also outlines the input (teachers' conversations), process (lesson study) and output (claims regarding the participants' learning) of the whole investigation as envisioned by my design of this research.

4.8. Data

As a researcher, I ensured that the techniques of recording observations and interactions did not interfere with the flow of the daily events in the lives of the participants. The fact that the research is using *naturally occurring data* is a deliberate choice and is important. This is also why there are no interviews or surveys in this study during the data collection phase. That would constitute *researcher-provoked data* (Silverman, 2006, p. 201). Naturally occurring data derive from situations that exist independently of the researcher's intervention. Given that the genre of the research methodology is participatory, where the researcher is integrated with the setting, the presence of the researcher is not intrusive.

4.8.1. Sources for data collection

All data were collected during the social situations that were entailed by the lesson study process in which the team of secondary mathematics teachers were engaged. This occurred during the span of one school year in which the research took place over three lesson study cycles. Spradley calls these types of social situations "a network of social situations defined as the same group of people sharing in a variety of activities" (Spradley, 1980 p. 44). The activities of planning for instruction, teaching and observing one another in teaching situations, as well as post-lesson conferencing, were observed in order to discover the patterns of culture as acquired knowledge that are common to this group of teachers.

Data consist of:

- Audio recordings of lesson planning meetings
- Audio recordings of post-lesson discussions
- Video recordings of implemented lessons
- Field notes
- Artefacts that were created in the process: lessons plans and one teacher-produced lesson study report.

With the understanding that all data are affected to some extent by the researcher, (e.g., where the camera is placed or how data is transcribed), the activities that I recorded would exist independently of the researcher, as the intervention did not actively contribute to creating new data.

4.8.2. Data collection procedures

Field notes were taken during all phases of the activity of the teachers, mostly to supplement electronic recordings (audio and video files) with observational data. As a participant observer, I employed the stance that is described by Mason as the *discipline of noticing* (Mason, 2002). It is contended that, "noticing can be sharpened, can be developed and refined as part of personal professional development, even disciplined to form the basis for recognized research" (p. 38). Within this stance, there are a number of well-articulated and clear guidelines to increase impartiality of the researcher and to ensure that descriptions of accounts are fair and accurate.

Mason's notions of "account-of" and "accounting-for" are helpful in separating what happened as could be objectively confirmed by others participating in the social situation from a personal interpretation of what happened (as judgment laden). "An account-of is a description of what was seen, heard, experienced, described in terms which others can recognize, without elaboration, justification, or explanation" (Mason, 2002, p.53). It is a technique to help increase objectivity, by preventing infiltration of evaluation, judgments and explanations into data. Field notes were taken as accounts-of things that happened and were captured as "brief-but-vivid" descriptions.

Artefacts that teachers developed in the process, such as lesson plans, instructional materials, records of observations of research lessons, and one report that was generated by the teacher team, are all forms of reifications that were also taken into account. While the details of the lesson were worked out collaboratively, there was still the task of writing this up as a rather detailed lesson plan. This is one of the distinct characteristics of lesson study – the level of thought that is put into the preparation of the lesson is reflected in its concrete physical form as a detailed lesson plan, encompassing processes of teaching and learning (such as anticipating student reactions/responses, checking for student understanding), as well as the static objects (such as objectives, the task, teacher key questions, specific representations of the content, teacher made instructional resources). The lesson progression, in terms of content exposure (also referred to as critical-input experiences), student processing and classroom interactions, can be quite well envisioned from these lesson plans.

Live data for each lesson study cycle consisted of two to three pre-lesson meetings, two in-class implementations and two post-lesson discussions. This made up for a total of about seven hours of streamed data, in the form of audio or video recordings. However, some interaction among the team members also occurred off actual contact time during the three to four-week duration of the cycle, in particular between the two pre-lesson meetings when the team members worked on shaping their lesson plan. The evolution of the lesson plan document is manifest from its multiple versions, and was kept on a wiki, which was created for the purpose of collaboration, document sharing and commenting.

The teacher that taught the lesson assumed the responsibility for writing up the lesson plan and, if there were modifications to that plan for the second implementation, the teacher that was teaching the revised lesson would modify that lesson plan. Of the three lesson study cycles that took place during the observation period, only the first lesson did not undergo significant revisions. The main body of this research report is based on that particular lesson study cycle as the case study for the investigation of teaching practice and its development, occasioned by the activities in which this team of teachers engaged.

4.8.3. Case study based on one lesson study cycle: Solving of radical equations

The following is a list of data sources that were collected for this particular lesson study cycle on the topic of solving radical equations for Grade 11 students, conducted with the core teacher team, Gabrielle, Andrew, Steve and myself.

i) Pre-Lesson Discussion #1: Selecting the topic to be taught, setting the dates for implementations, brainstorming the possible teaching approaches and settling on one of them, consulting the curriculum and the textbook used for the course, generating tasks to teach the topic.

ii) Pre-Lesson Discussion #2: Engaging with the mathematics that is entailed in this content, refining the lesson plan.

iii) First enactment of the lesson: Andrew's Class

iv) Post-lesson Discussion #1: Discussing Andrew's Class, and using what was learned to prepare for Gabrielle's class

v) Second enactment of the lesson: Gabrielle's Class

vi) Post-lesson Discussion #2: Discussing Gabrielle's Class

vii) Artefact #1: Lesson Plan (Appendix D)

viii) Artefact #2: Teachers' Report on the Mathematics Pedagogy for the teaching of solving of radical equations (Appendix E)

Implementations of research lessons have been videotaped, and the lesson-planning meetings and post-lesson discussions of the teachers were audiotaped to allow for the close analyses of the interactive dynamics of professional discussions among teachers. The recordings of the lessons were shared with the teacher who was teaching the lesson. The teachers received the recordings of their lessons positively. Two of the teachers mentioned that they had never seen themselves in the act of teaching before.

Lastly, it should be pointed out that my primary interest was not to find out whether or not lesson study works, or even how it works exactly. Neither the structure of lesson study, nor its social features alone, are a guarantee for productive professional learning. However, it is of interest for this thesis to find out what can produce critical conditions that may lead to professional learning and consequently to the improvement of instruction.

4.8.4. Methods for data analyses

The data were analysed using qualitative methods for analytic induction, which were performed through the extraction of themes that emerged in the data (Yin, 2010). Methods of content analysis, constant comparison, structural and relational analysis, and thematic categorizations were deployed. The process for category generation stems from the patterns that were expressed by the participants and that were evident in the setting, through analysing the data and by doing conceptual mapping. This was done until patterns emerged, which are internally consistent, but distinct from one another.

As a researcher, I chose the emergent format as a way of approaching the process of doing research, as well as in writing it up (Schostak, 2002). Essentially, the work consisted of inducing themes from texts, which include transcripts of audio and video recordings, using qualitative analysis. According to the emergent format, the researcher "suspends judgment as to the core set of aims, the key research questions, and the nature of the data to be collected [...]. Through a process of critical reflection during the process of engaging in the research the design emerges, shaped by the researcher's engagement with the broad scene of research under study" (Schostak, 2002, p. 83). I chose this format because of the possibilities of providing findings and results that more closely correspond to lived experience and to the realities of the social situations of school life in the context of teachers' learning in the workplace, through the process of lesson study.

The above described format is appropriate for research that is undertaken to study the teaching practice and its development in the social setting of professionals who critically reflect upon their own practice and that of others in their workplace, in order to improve their understanding, decision-making and action. As a researcher, I tried to infer the various aspects of participants' MfT through how they interpreted experience and generated behavior within the context of Practice-Based Professional Development, as illustrated in Figure 1 (p. 68). I applied deconstructive strategies to interrogate the theoretical underpinnings of particular perspectives that the participants of this study hold. This is used to broaden my view as a researcher by examining multiple perspectives in the sense of "inclusion of multiple viewpoints," in order to provide greater confidence in the research findings and in the particular kinds of knowledge claims that the research is making. I sought to employ the kinds of methods and strategies that most directly, adequately and sincerely address the emerging phenomena that relate to the main topic of the research.

Besides identifying themes that characterize the experience, practice and meanings that are expressed by the participants, I also employed scrutiny-based techniques. In particular, I employed the method of searching for missing information, because, sometimes, silences indicate areas in which people are unwilling or afraid to discuss. At other times, absences might indicate primal assumptions that are made by the participants (Ryan & Bernard, 2003). Most importantly, I was entirely aware all along that I am not attempting to prove something (such as how wonderful lesson study is), but rather, to engage in an intellectual inquiry about developing mathematics teaching practice and expertise, by harnessing the potential of the community and workplace.

All of the pre-lesson and post-lesson discussions were transcribed using the NVivo software. Within the inductive research methodology that is described above, I analysed mathematics teachers' talk using methods of *conversation analysis* (Silverman, 2006), which is different from how we might intuitively analyse talk. Conversation analysis is a careful inductive method, which is depicted by the following stages:

- Explore your data without any initial hypotheses.
- Identify some phenomenon worthy of further study.
- Establish how this phenomenon occurs in varying ways in your data.
- Try to account for this variation.

Technically, conversation analysis involves identifying sequences of related talk, examining how speakers take on certain roles or identities through their talk, (i.e., questioner – answerer; leader – follower), and looking for particular outcomes in the talk, (i.e., a request for clarification, a repair, a challenge). It works backwards to "trace the trajectory through which a particular outcome was produced" (Silverman, 2006, p. 222). Doing conversation analysis involves identifying informationally salient speech messages, looking for how meaning is being made, and how actions and purposes are being accomplished through the language being used.

Although the pre-lesson discussions are of a different nature than post-lesson reflections, given the entirely different purposes for which the teachers come together in these two social situations, I use Four-Component Framework described in Section 2.2 to look at both types of situations. In addition, within the analyses of the post-lesson discussions, the method for data identification and filtering to capture what teachers noticed in each of the two enacted research lessons is informed by the van Es and Sherin (2008) noticing framework that is described in the section that follows.

Professional noticing in the collective post-lesson reflection

In Chapters 6 and 7, I examine, describe and analyze the post-lesson discussions after both implementations of the lesson on the topic of solving radical equations. This lesson was taught by two different teachers, in two different classrooms. Each teacher taught the lesson to their own class of Grade 11 students. Determining what the participants noticed as they observed the enacted research lessons is informed by the three-part noticing framework from van Es and Sherin (2008) as a way for data identification and selection from the post lesson conversations. Van Es and Sherin used their noticing framework with pre-service teachers in the context of video club (a practice in which videos of classroom interactions are being collectively watched, and then discussed), in order to train them in the expertise of noticing and to track how learning to notice develops. I used this framework to identify and filter the topics that were brought up in the post-lesson discussions in order to understand what the teachers saw as noteworthy aspects of classroom events and situations that they witnessed during the two lesson enactments.

The three parts of this framework are: attending to noteworthy aspects of classroom events, using knowledge to reason about these events, and making informed connections between the specific classroom events and broader principles of teaching and learning, for the purpose of responding to classroom events during in-the-moment decision making by the teachers. Other researchers view the last stage as separate because it might involve action, whereas the first two do not. That is, in their conceptualization of noticing, they recognize the issues that are related to responding based on what is being noticed. However, they see the decision-making about how to respond as a separate event, which happens only after one notices something of importance about the situation that requires action (Santagata, 2011). However, I take the perspective of van Es and Sherin, who find that these cannot be separated from each other because they often occur at the same time for one event, manifested as a single integrated teaching move. Therefore, the three components are inextricably intertwined, although not all three parts necessarily happen. The framework integrates the teacher's reasoning about how to respond into the construct of professional noticing to elucidate how teachers make effective instructional moves.

Attending to noteworthy aspects of classroom events

I start by looking at what the three teachers in this team brought up in their post-lesson discussions. That is, what they attended to as noteworthy events of the classroom interactions. An event was identified as having been noteworthy in the following ways: a teacher brought it up in the post-lesson discussion, a teacher recorded it in the lesson observation log, a significant length of time was used to discuss it, it included specific details and depth, and/or the teacher said it was important or noticeable (teacher language). The content that could be noticed includes, but is not limited to, the following: noticing students, the teacher teaching the lesson, student-teacher interactions and the mathematics itself.

Using knowledge to reason about classroom events

The second part of this framework describes a teacher's knowledge as consisting of the following: mathematical content knowledge, knowledge of how to present and represent that knowledge for the purpose of understanding, knowledge of the students and how the students think about the content, and knowledge of their specific, local context. When analyzing a classroom event, a teacher uses these kinds of knowledge in order to reason about specific events that they notice in the classroom. As teachers gain more experience in their field and, therefore, more knowledge about these factors, they become better at analyzing situations that occur in this context. This is considered significant because it implies that teachers will be more accurate when reasoning about a classroom situation that pertains to their subject matter, their area of expertise and their own students' thinking (Schoenfeld, 1998, as cited by van Es and Sherin, 2008).

Making connections between specific classroom events and broader principles of teaching and learning

The third stage of the framework for noticing involves connecting a specific event that occurs in a classroom to the broader teaching or learning principle that it represents. Connecting a specific classroom event to a broader principle essentially categorizes that specific event into a more general concept. This can also be expressed as making generalizations from specific events, or labeling a specific event 'as a case of' a more general principle. By doing this, teachers build a more general collection of knowledge, which they can then use in future situations, to reason about similar events and respond during classroom instruction.

4.9. Finding an identity in the context of the research

One of the reasons for which it is important that the researcher be explicit about the role that is adopted and to stay faithful to that decision is to guard against the possibility of "going native" (Silverman, 2006, p. 82). This refers to identifying too much with the participants and losing sight of what the research is about. I was aware of moving between my identity in the field as a teacher and my new identity in this social situation as a researcher. While there was a potential issue that may arise by "being native" (I am a teacher too) in the community where this research was undertaken, there were also advantages to doing research in the place of one's own work and with one's own colleagues as the research participants.

One such advantage is that of having an already established relationship of collegiality and trust, as well as having a shared understanding of the ways of working together when conducting lesson studies. This allowed me to observe the professional development practices of one group of secondary mathematics teachers and to study their disciplinary knowledge of mathematics for teaching within the context of *naturally occurring data*. As we know, observers may change the situation simply by their presence, as can settings that require a facilitator when a novel practice is being introduced into a workspace.

I believe that both “being native” and being seen as “being native” by the participants of the study allowed for a good level of comfort and authenticity in the interactions of participants during the observation. Moreover, given that rapport among the group members was already established long before the start of the research (as was the affiliation around the importance of professional growth), this study did not suffer from what might be the case in short-term studies, which involve an external researcher. In such contexts, the researcher might not obtain authentic data because participants could change their behaviour in order to portray themselves in a way that they believe is more favourable (Merriam, 1998).

4.10. Validating the accuracy of research findings

To establish the credibility of my interpretations and findings, four standard strategies were employed: extended engagement with the participants in the practices of a professional learning community, thick description, triangulation from multiple sources of data and peer debriefing.

Firstly, the teachers who are participants of this study practiced lesson study over a prolonged period of time, before the onset of this study, in its unaltered team composition and ways of working together. This allowed me, as a researcher, to access authentic group practices. Secondly, I chose to employ a thick description of the collaborative work of the participants and their teaching practices, in the final product of this study, giving grounds for the reader to make his or her own interpretation (Hatch, 2002). Thirdly, through triangulation of data that transpired in multiple sources, I was able to corroborate and confirm the findings, particularly those that relate to teachers’ subject matter knowledge and their knowledge for teaching mathematics related to the content areas that were chosen for the lesson studies by the participants. Through comparative analysis of themes within the lesson study cycle and across the three lesson study cycles that were conducted by this group of teachers, I was able to extract the findings to a high degree of accuracy.

Finally, throughout the process of data analysis, I generated interpretations of my findings in the form of short research reports, which were peer reviewed by fellow doctoral students, and then presented at the local conference each year. This process was invaluable in allowing the space for conferring regularly during the initial stages of my analysis and interpretation. The input and feedback from this community prompted me to perform negative case analysis, to search for contradictory evidence and actively re-examine data in light of contextual factors, and to consider alternative interpretations in order to refine my findings and increase the trustworthiness of this work (Glesne, 1998). In addition, the raw data is preserved and is available for scrutiny. Some of it has been reviewed and discussed, in part, with a peer researcher who is the co-investigator in this study, in order to scrutinize and get her reactions to coding, the case summaries and the analytic memos written during data analysis (Miles & Huberman, 1994).

The three chapters that follow are empirical, beginning with the participants' collaborative work related to preparing for instruction.

Chapter 5. Preparing for Instruction

5.1. Overview

This chapter captures what mathematics teachers do and what they attend to as they plan for instruction. As a researcher I aim to understand the goals and purposes of teachers in the concrete context of preparing for the instruction of a particular lesson. Furthermore, I relate to the reader what teachers can learn from engaging in the collaborative process of preparing for instruction.

Planning for instruction is one of the key components of teaching practice and at the same time it is the bulk of the work in the process of lesson study. If we could understand deeply what mathematics teachers do and what drives their decisions when they plan for instruction, we could find out a great deal about *mathematics for teaching* (MfT). When teachers plan for instruction they must bring to bear their knowledge of students and mathematics, of curriculum and materials and of ways to teach and learn the topic most productively, to name just a few.

In this chapter, I describe and analyze how Andrew, Gabrielle and Steve went about planning the lesson on solving radical equations. The lesson plan went through several modifications during the phase of planning for instruction. This phase stretched over a period of three weeks in which the teachers held two planning sessions. After the first planning session, Andrew produced the first version of the lesson plan (Appendix D) which then became the basis for the second planning session. This is not what a standard lesson plan looks like. It is more akin to the workings of the teacher as he/she prepares for instruction. The lesson plan in Appendix D is “idealized” in the sense that it only presents the responses that the teacher desires from the students. It also captures the main tasks that will be presented to students, along with teachers’ questions and desired responses toward which the teacher wants to lead the students.

Themes that surfaced in teachers' discussions during both planning sessions are analysed in detail in what follows in the rest of this chapter, using the Four-Component Framework, described in Section 2.2 and named in Section 2.3. Clearly, a specific mathematical content goal, be it a concept or a skill (most often, it is a combination of both), plays a major role in planning for instruction. The learning goals for instruction are set by the provincial curriculum. For this lesson, the learning outcome is stated as "formulate and apply strategies to solve absolute value equations, radical equations, rational equations, and inequalities" (BC Ministry of Education, 2006, p. 50). Of course, several lessons are usually needed in order to achieve this outcome. Therefore, this lesson was designed to target a small part of the stated curricular outcome, namely the part that relates to the solving of radical equations.

This specific learning outcome is part of a group of learning outcomes that is labeled as "Relations and Functions." In turn, this is a subset of a large set of outcomes that span across all grade levels of schooling and is known as a strand of "Patterns and Relations." This is only one of the goals that teachers have – teaching students the specific mathematical content. Given the nature of the subject area, every specific learning goal or outcome is necessarily interdependent, interconnected and related to a number of other concepts. This provides fodder to inform teaching decisions that are related to the structuring and coordinating of the content in ways that make it suitable for learning.

The topic of solving radical equations is often regarded as a dry topic. However, for the development of algebraic reasoning, it is an important one. It also relates to further work in mathematics, such as to the study of functions. While the teachers could have chosen a "more attractive" topic which would lend itself more easily to "innovative" ways of teaching, such as teaching "through problem solving" or teaching by "situating the learning in the real world context," they did not do so. They chose this particular topic for their lesson study. In fact, any curricular topic could be chosen for this kind of work among teachers and made relevant. However, it is recommended that teachers choose a topic that presents difficulties for the students. Furthermore, any lesson is a good candidate and, perhaps, what is perceived as a mundane topic can be an especially productive space for teacher learning, given that "all task-types are available for deep analysis of mathematical affordances and that such analysis can help teachers develop sensitivity to variations of presentation, layout, symbol use, language, diagram and hence to variations in perception, recognition, interpretation on the part of learners" (Watson, 2008, p. 4). However, such a choice presents an additional challenge for teachers. That is, "How can they motivate students to learn it?" and "Can the incentive come from mathematics itself?"

The main issue of contention was about the teaching approach. Andrew proposed a non-standard teaching approach, which he believed would promote mathematical thinking and logical reasoning. In what follows, I refer to this teaching approach as "Setting Restrictions" (Approach B). This approach was in contrast to the standard way in which the textbook presented the topic and which Steve and Gabrielle had used in their teaching practice previously. I refer to this approach as "Rejecting Extraneous Roots" (Approach A). Steve and Gabrielle had never considered Approach B before and did not initially recognize it as helpful for the students; at the same time, they saw it as more demanding to teach and for students to understand. Naturally, they questioned the merits of Approach B but decided to adopt it and incorporate it into the implementation of this research lesson because they realized that this approach is mathematically sounder, promotes mathematical reasoning and sense-making by the students and is also more elegant.

From deciding on the teaching approach, concerns about students' potential misconceptions arose. To prevent students' possible errors and to set the stage for the development of the process of solving radical equations, which requires certain conceptual understandings, the lesson plan began with evoking some elements of prior knowledge, specifically, that square root of any real number is defined as a non-negative value. As the teachers negotiated the teaching approach through reasoning and justifying their choices, a gap in a teacher's content knowledge was sensed.

One cannot underestimate the critical role of examples in the learning and teaching of mathematics. For a significant portion of time, during the phase of planning for instruction, the team of teachers was engaged in creating examples that were to be used in teaching, with the purpose of illuminating the topic of solving radical equations.

These examples would then be used in the classroom to reveal and help students to discover various aspects of the mathematics of the solving of radical equations in particular, as well as of other equations more generally. The activity of the team can be described as purposeful generation and use of examples to structure the learning of a mathematical idea, which is seen to be "a major feature of being mathematical and also one that characterizes good planning and teaching" (Watson & Barton, 2011). Interestingly, this activity morphed into an exploration of a special class of examples for the sake of the teachers' own understanding.

The discussion revolved around how the topic would be taught to students, why it should be taught in a certain way, what the students would be expected to do and how they should understand the topic. The implicit understanding of the team was that teachers' knowledge can accommodate all these needs for students' conceptual understanding. At the end of the phase of planning for instruction, the teachers spontaneously created and solved different radical equations to explore the example space and the nature of solutions that could occur.

5.2. Analysis of the themes that emerged from the phase of planning for instruction

This section presents an analysis of specific themes that surfaced during the planning stage of the lesson, grouped under the frame for analysis according to the following components: psychological, didactical, mathematical and pedagogical (or practical), as proposed by Selter (2001). I begin with the psychological component first because it seemed to permeate much of the teachers' discussion, although not necessarily in an explicit way. It is often intertwined with other components and is at times difficult to separate from them. This component includes teachers' considerations of student cognition and affect during the learning process and seems to be the main drive behind their work.

5.2.1. Psychological component in the pre-lesson discussion

There was an undercurrent throughout the teachers' discussion that involved consideration of student cognition. The teachers tried to predict where the students would likely experience obstacles in the process of learning this topic.

5.2.1.1. What do we want the student to understand?

When preparing for instruction, teachers often work out the problems that they plan to give their students. This helps them to re-enter the world of the novice and sharpen their instructional goals. Most of the time, teachers prepare their lessons in solitude. It is not clear what sorts of things are going on in the mind of the teacher during this phase. However, it seems that this takes a form of an internal dialog, which is shaped through questioning and responding to reveal the mathematical meaning of the knowledge that is to be taught.

In the team setting, the teachers worked out the main task of the lesson $\sqrt{x+11}-\sqrt{9-x}=2$. While doing so, they engaged in the practice of "anticipating student responses." Teachers' discussion revolved around how the student might fall into a <u>trap</u> of solving the equation by "motoring through." That is, squaring both sides of the equation, without setting the restrictions on the radicals to state that both sides of the equation are positive, before the squaring can be applied. A naïve process of this kind then leads to arriving at "two solutions" in this case. The student would still need to check if either one of them is valid and would not necessarily know the reason for the occurrence of these "faulty solutions." Setting a restriction on the range (the radical needs to be non-negative) would prevent this kind of error entirely. In the words of Andrew, who said, "Actually, a trap wasn't set; people set traps for themselves," one can see his general stance toward mathematical knowledge as being valuable and empowering.

In Andrew's view, students need to consider first where the equation is defined, based on the values of the expressions under the root (the radicand), and then again before squaring the equation, they need to know that they are applying the operation to equal sides (the need to look at the range). This way, they will prevent the setting of a trap for themselves.

> Natasa (N): Then I'm going to do it like grade 8s, square it on both sides.
>
> Andrew (A): So CAN we square it?
>
> N: Well ok.
>
> A: You need to know, is left side positive and is right side positive.
>
> N: So left side is positive under these conditions [pointing to the restrictions set on the domain that had been calculated before] and the right side is positive under the condition that x – 1 is positive.
>
> A: So we know.
>
> N: Ok so a student would probably just go on to square both sides like here, without thinking that this has to be positive, like this has to be positive. [Pointing to the left side of $x-1=2\sqrt{9-x}$ on the board]
>
> A: No it IS positive.
>
> N: It is positive, any square root is always positive.
>
> Gabrielle (G): How do you justify what you just said? Every square root is positive…
>
> N: Because it is how it is defined.
>
> G: According to that or?
>
> A: No, it is defined in mathematics.
>
> N: Like square root of 15 means positive square root of 15.
>
> A: Any number, but positive.
>
> N: Like it means this, whatever that is, but positive not the negative. Because it's defined as such.
>
> A: Or 0 if it is really 0.
>
> G: Ok.
>
> N: But so, if this is positive, 2 times this is still positive [referring to the right side of the equation], so you have to ensure that this is positive [referring to the left side of the equation], which will be if x is greater than 1. So now all of a sudden our interval is actually this only

[pointing to the interval between 1 and 9 on the number line], and the rest has to be cut out.

This is the first occasion where it showed up in the transcripts that Gabrielle was unclear about the definition of square root being taken as positive. She asked about how to justify this. Neither Andrew nor Natasa really heard what she meant.

The above excerpt presented a discussion about how to help the student approach this topic with understanding and not be <u>tricked</u> into thinking that there might be two solutions, or not knowing which one to reject, without having to test it out by substituting into the original equation. However, this requires that the teacher use the knowledge about the square root being positive. In fact, the didactical situation, as it has been set up by the decision to teach the topic using the idea of setting up restrictions on the domain and range of the associated functions, is dependent on this piece of mathematical content knowledge. More will be said about the choice of the teaching approach in the next section (Figure 4).

In this part of the discussion the teachers were still settling on the teaching approach and becoming familiar with the affordances of the task. They proceeded to find the common interval for the possible solutions, based on their prior analysis. At this point in their lesson planning session, they were still in the phase of performing the task themselves and through this personal experience, they were considering and discussing how students might think it through. Here is an example of this kind of discussion among the team, just as they worked out the two candidates for solutions, $x = 5$ and $x = -7$, and then considered both the mathematical implications and implications for teaching.

G: So the kids will say, so that whole section between positive 1 and negative 11 is out of the question. That was foolishness. We thought it might be included, obviously it was not.

A: Say it again?

G: We thought initially that that could be part of the solution or part of the restriction, and now obviously it's not because we see [that *x* must be greater than 1].

A: We encountered additional limitations.

N: Yes, which entered the picture here. But we didn't know it from the start.

A: Well, actually it existed earlier but we didn't see it, we couldn't see.

G: Yes, we thought it was all the way down to -11 but lo and behold, it turns out that it's not.

N: Yeah, now only between 1 and 9. Ok, and then the rest is just usual [..inaudible, calculations]

G: So a student's mistake could be not paying attention to that, the new limitation.

N: Yeah that's right. So I just went on and solved this, and it turns out to be $x_1 = -7$ and $x_2 = 5$. And you know, these both fit fine in the initial interval of permissible values, so if we didn't set the additional limitation we would not know to exclude –7. Just looking at this initial interval that we had [..] –11 to 9, and yeah, both these values are within and both are solutions and all is handy dandy [if we don't pay attention to the range].

From the preceding discussions, it seems that the teachers were set on assisting students to develop a particular view of the process of equation solving, one that would allow them to use their own reasoning and mathematical judgment and productively resolve the conflict of "falling into a trap," "being tricked," or having to consider answers that can be identified earlier on as "foolishness." Clearly, the teachers of this team were concerned with the development of mathematical thinking in their students. When this condition would arise in the process of equation solving, they wanted their students to pay attention, not only to the domain (by setting the radicand expressions to be non-negative), but also to the range (by setting the radical to be non-negative).

5.2.2. Didactical component in the pre-lesson discussion

As the team of teachers was preparing to teach the lesson on radical equations, they were, in fact, solving a problem of how to teach this topic. The problem is one that relates to the organization of a *didactic situation* (Brousseau & Balacheff, 1997), in which the activity of the teacher and the students evolves through the interplay of the interaction with mathematics that the students are supposed to learn. The problem requires a setting up of the conditions (by the teacher) for the appropriation of the target knowledge and its meaning (by the students). The situation is designed as a model of the knowledge, to be taught by the teacher and experienced by the student. In other words, it is specific to the knowledge that is to be taught. Therefore, if the teachers want their students to learn mathematics, the situation cannot be arbitrary in the kind of action and thinking processes that it invites the student to engage in. This is precisely the topic of the next section – looking at how the team of teachers tackled this problem and what they learned along the way, as they deliberated around the question of "How should we teach it?"

5.2.2.1. Connecting to prior knowledge

Teachers often begin their lessons by situating the learning in the realm of what the students already know. In planning a mathematical lesson, it is always important to connect the new concept to prior knowledge in multiple layers, and then to scaffold the learning of the new topic in relation to the backdrop of existing ideas. In their preparation for teaching of this topic, through the work of collaborative planning, the teachers informally made an inventory of essential ideas that will be needed, or that will have to be brought to the fore in the lesson.

The lesson depended upon bringing the students to attend to the values of the variable for which the equation is or is not defined. Prior knowledge of these matters was going to be brought forth through asking students to generate their own examples of the different kinds of equations that they had met before, and then to consider the set of values for which the equations are defined. Rational equations would have been "met before" examples of equations that have restrictions – students would know that the denominator cannot be zero. Also, they had studied the quadratic formula before and had applied it in problem situations where they would have encountered negative value for the discriminant, which they would recognize as "real solution does not exist." The teachers were clear that this was not new knowledge, as the students would know that the radicand cannot be negative when working in the system of real numbers. This part of the discussion by the teachers led to starting the lesson by asking students to generate examples of equations for which they were already familiar and by considering the domains of the corresponding functions. Figure 3 below shows the excerpt from the lesson plan (included in its entirety in Appendix D) that resulted from this part of the team's pre-lesson discussion.

Let's start with linear equations: Give me one example of a linear equation.

Example: $2x - 3 = 7$

What is the set of numbers for which this equation is defined?

Or, does this type of equation have any limitations in its domain?

Quadratic equations: Example: $x^2 - 3x + 2 = 0$

What is the set of numbers for which this equation is defined?

Or, does this type of equation have any limitations in its domain?

Polynomial equations: Example: $3x^3 + 5x^2 - 7x + 1 = 0$

What is the set of numbers for which this equation is defined?

Or, does this type of equation have any limitations in its domain?

Rational equations: Example: $\frac{x}{x+1} - 2x = \frac{2}{x}$

What is the set of numbers for which this equation is defined?

Or, does this type of equation have any limitations in its domain?

$x+1 \neq 0$, so $x \neq -1$ and $x \neq 0$

Therefore x cannot equal -1 or 0.

Figure 3: Excerpt from the lesson plan: Connecting to prior knowledge - Teachers want the students to consider the domain of definition

According to Ma (1999), a profound understanding of subject matter knowledge involves a cross-topic picture and longitudinal coherence in learning, both of which the teachers were attending to.

5.2.2.2. To reject the extraneous roots or to set restrictions?

The choice of the method that is used to teach any topic is fundamentally related to a teacher's understanding of the concept that is to be taught. In their discussion, the teachers consider two didactical approaches to teaching of the topic. Andrew said that he would teach the topic of solving radical equations by "setting restrictions" (Approach B, presented in Figure 4 below). (His reasons for taking this uncommon approach will be discussed later.) This was in contrast to the way the textbook presented the topic, which was by "Rejecting Extraneous Roots," and which was how Gabrielle and Steve had previously been teaching it (Approach A, presented in Figure 4 below). Therefore, the question of how to approach the teaching of the topic made for a considerable amount of discussion.

Andrew's approach brought with it the requirement to consider the domains, which is how the lesson, as planned and later enacted, begins. The lesson was therefore planned to begin with "teacher making or eliciting informational/factual statements," (Watson & De Geest, 2012, p 229) for the purpose of reminding students about domain and range (as presented in Figure 3). This is done in preparation for what comes next in the process of learning the topic and to bring students to attend to what will be needed to understand this topic.

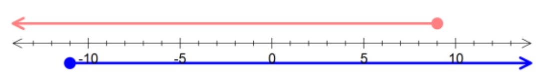
The introductory learning example that was set out in Andrew's first draft of the lesson plan was to solve the following radical equation: $\sqrt{x-3}+x=5$. It was intended for students to run into a situation wherein they come up with two values for *x* as possible candidates for the solution. This was to serve as a springboard into an inquiry about what happened, how this is possible and what the mathematics is telling us about how this could have occurred. This "surprise" was then meant to motivate the students to use the approach of setting the restrictions on the domain (in this case the radicand) and the range (in this case the radical) of the corresponding functions, as they proceeded through the solving of such equations, so as not to "fall into a trap" of thinking that they had made a computational error or that there was some mysterious process taking place, but to know exactly why and how the so-called extraneous roots occurred. To be able to determine in advance what can and cannot be a solution to the given equation requires an analysis of the intervals for the permissible values of the unknown (setting the restriction), by considering the domains and the ranges of corresponding functions, where each side of the equation can be seen as a function and the equation itself as a condition under which the graphs of two functions would intersect, if at all (this is discussed later, in Section 5.2.3).

The planning session began by the teachers' working out the main learning task that would be set for the students in the lesson, which was to solve the equation $\sqrt{x+11}-\sqrt{9-x}=2$, and then considering what this involved mathematically and what a student might do when trying to solve such equations. Below is a brief outline of the process, as was proposed by Andrew in his lesson plan. His method for solving such equations was different from the one that was presented in the student textbook. As we know, the approach that was presented in the textbook often serves as the "implemented curriculum" in practice (Johansson, 2003). However, Andrew was showing an approach that he thought was more mathematical and that was akin to the way that students should learn mathematics in general. Later, when we look at the teachers' conversations, we will understand what he meant by that.

<table>
<tr><td colspan="2">How to solve it?
$\sqrt{x+11}-\sqrt{9-x}=2$
(main learning task)</td></tr>
<tr><td>Approach A: The standard textbook way: "Rejecting Extraneous Roots"</td><td>Approach B: Taken by our team of teachers: "Setting Restrictions"</td></tr>
<tr><td>$\sqrt{x+11}-\sqrt{9-x}=2$
$\sqrt{x+11}=2+\sqrt{9-x}$
$x+11=4+4\sqrt{9-x}+9-x$
$2x-2=4\sqrt{9-x}$
$x-1=2\sqrt{9-x}$
$x^2-2x+1=4(9-x)$
$x^2-2x+1=36-4x$
$x^2+2x-35=0$
$(x+7)(x-5)=0$
$x_1=-7, x_2=5$

Now, we must not forget to check if these values are actually solutions. Substitute them into the original equation and see if they work.</td><td>$\sqrt{x+11}-\sqrt{9-x}=2$

Set the restrictions on the domain, i.e. on the radicand. Values of the expressions under the square root signs must be non-negative for solutions to exist in the set of real numbers.

From $x+11\geq 0$ and $9-x\geq 0$ it follows that $-11\leq x\leq 9$ or visually

We proceed the same way as in Process</td></tr>
</table>

Let’s start by checking out if -7 is a solution.

$$\sqrt{x+11}-\sqrt{9-x}=2$$
$$\sqrt{-7+11}-\sqrt{9-(-7)}=\sqrt{4}-\sqrt{16}=2-4=-2$$

But $-2\neq 2$, so -7 is not the solution. This type of “faulty solution” has a special name in school mathematics – it is called “the extraneous root.”

Now let’s check if 5 is a solution. Again, substitute into the original equation.

$$\sqrt{x+11}-\sqrt{9-x}=2$$
$$\sqrt{5+11}-\sqrt{9-5}=\sqrt{16}-\sqrt{4}=4-2=2$$

This makes the equation true, so 5 is the solution.

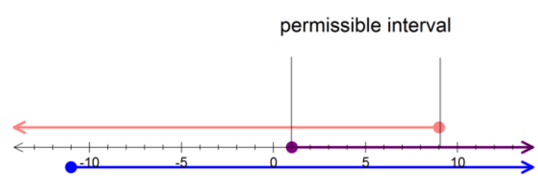
A, being mindful that squaring the equation the first time requires no special restrictions on the variable, since both sides of the equation $\sqrt{x+11}=2+\sqrt{9-x}$ are necessarily positive. To undo the radical, we need to square the equation again at this stage:

$$x-1=2\sqrt{9-x}$$

But before squaring the equation we must set the restrictions (on the range this time): the value of the radical on the right is positive, so the left side is positive too.

$x-1\geq 0$ or $x\geq 1$

Combining this with what we know from before about the permissible values for the solution set, we see that the interval decreases to $1\leq x\leq 9$, or visually



Under this condition we may square both sides of the equation, and upon obtaining 5 and -7, we reject -7 because it is not in the permissible interval. Therefore, 5 is the solution.

Figure 4: Comparison of the two didactical approaches considered by the teachers in the pre-lesson discussion

5.2.2.3. Analysis of the two teaching approaches

While both of these teaching approaches have the potential to satisfy the curricular learning outcome and they both lead to a correct result, we cannot help but notice the procedural emphasis of the textbook approach in comparison to a much more conceptual emphasis of Approach B. In terms of the disciplinary knowledge that is required of a teacher, Approach B is much more demanding. However, in return, the learner would have access to the underlying meaning and reasoning that support learning for understanding. Under Approach A, the only goal seems to be "answer getting" and the process could be carried out purely mechanically. It does not attend to any reasoning as to why these mysterious faulty solutions appear, which one must then remember to test and then reject or accept. On the other hand, Approach B requires a careful consideration of the domains and ranges and a coordination of what that might reveal. It is connected to the mathematics of functions and inverses. It requires reasoning, analysis and synthesis of information.

At its core, the reasoning is based on the knowledge of definitions. In particular, it is based on knowing how square root is defined. While it also requires the use of mathematical skills, or procedural knowledge such as techniques for solving equations, squaring of a binomial, solving of inequalities, examining intervals of values for their intersection, Approach B is fundamentally different from the procedure-dominated Approach A. In terms of mathematical affordances (what is available for the student to learn and the "habits of mind" that are being promoted), these two approaches differ significantly. And yet, as noted before, they both lead to a correct answer and satisfy the curriculum requirements. For the student, however, they provide an entirely different experience of learning mathematics.

Instructional decisions of this type are ultimately in the hands of the teacher, assuming that the teacher has the knowledge and the disposition to make a rich and mathematically connected learning experience available to the students. This is true of any topic, concept and lesson being taught. In regards to the content and the presentation of the topic of study, mathematics teachers often rely on the student textbook. In practice, the textbook becomes a kind of reference, which dictates the *implemented curriculum* (Cuban, 1993).

Additionally, research shows that there is a mismatch between the intended curriculum as prescribed by policy makers, the implemented curriculum as carried out by teachers in their classrooms, and the attained curriculum as the one that is learned by the students (Handal & Herrington, 2003). In their efforts to account for this mismatch, the authors argue that this is due to the fact that “teachers and students work on more limited goals than those proposed by curriculum developers, teacher educators, writers of syllabuses, and textbook authors.” Furthermore, these authors claim that “Mathematics teachers are concerned only with students acquiring facts and performing skills prescribed by the syllabus rather than being concerned about broader educational goals” (p.2). Such arguments paint a view that mathematics teachers deplete the well-designed curriculum and misrepresent the good intentions of textbook authors.

In my experience with this team of teachers, entirely the opposite was true. In every one of the lesson studies, both those that were held during the year of this study and those that were held in the preceding years, when the same team of teachers worked together in a similar fashion, the curriculum, as it was represented by the textbook, was qualitatively surpassed by the teachers every time. This constitutes an empirical proof that such a thing is possible and that it was brought about through the collaboration of teachers in the context of lesson study. The team consistently worked to enrich the instruction, often well beyond that which was envisioned by the textbook writers. Moreover, their work demonstrates great care for “broader educational goals,” perhaps even exceeding the goals that were envisioned by the curriculum writers (see Appendix E).

Following the above discussion, it is true that most teachers of mathematics would likely present the topic of solving radical equations using some variant of Approach A, without having necessarily developed the underlying concept beforehand (Stigler & Hiebert, 1999). This is dictated both by the textbook, which becomes the resource for the implemented curriculum, as well as by the culturally engrained methods of teaching, which are described by the same authors as “practicing procedures”.

While Approach B is didactically more promising in that it sets up the conditions for the learner to understand the principles behind solving such equations, it is not my goal to argue for this approach. The questions I would like to raise instead are, “How do mathematics teachers come to a place where this kind of instructional choice is available to them?” and “How can participating in lesson study aid this development of teachers?”

As the team of teachers planned for instruction, they engaged in a number of practices that are described by Watson (2008) as “taken-as-shared.” These practices include the following: teachers working on mathematical tasks, which they later use in teaching situations, reflecting cooperatively on multiple approaches (by doing so, becoming aware of the obstacles perceived by learners), analyzing task structure, drawing on their own experience as learners and sharing their personal knowledge. These practices constitute a mathematical analysis of the topic that is to be taught. This analysis is then used by the teachers to shape and inform what will be available for students to learn through the lesson.

5.2.2.4. To “factor” or to use the quadratic formula?

The following excerpt illustrates an inquiry into the mathematical practices of a teacher, specifically one that relates to the technical aspect of teaching, when there is a need to solve a quadratic equation.

> G: Now that equation, Andrew, $x^2 - 11x + 28$ will factor. It's a trinomial that could be factored. Again, is there a compelling reason why you used a quadratic formula and didn't just factor it?
>
> A: Yes it is. Because, just of billions of quadratic equations, just a few [will “factor”], compared to let’s say millions of billions maybe of trillions, just a small portion are those that could be solved without using quadratic formula. It means, 99.9% of quadratic equations [will not “factor”]. If we are always choosing nice ones, then it seems like, why bother using the quadratic formula? Because in most situations you can't come to the solution because it's an irrational number. Sometimes, some are rational but many are not. If we choose a nice quadratic equation that we can factor without [using the quadratic formula], it is ok. But I don't insist because if it is a slightly different constant or coefficient, let’s say if it is here 32 instead of 28, you cannot. But it is not visible that you cannot. Then students try and try and lose time. So, I say the best way is to do it directly, you don’t waste your time.
>
> G: It always works, whether it's right or not.

In the lesson plan that Andrew wrote, which is included in Appendix D, we can also see that he always uses the quadratic formula, in spite of the fact that it is possible to factor most of the quadratic equations there into two binomials, which contain whole number constants and where the coefficient of the variable is 1. Many of the examples in the textbook are fabricated so that they can "factor" and it seems that Andrew took his first two learning examples directly from the textbook. However, in his teaching, he does not take the advantage of these prefabricated shortcuts. He prefers to work out the solutions using the quadratic formula because he feels that this is more authentic to how one might work with quadratic equations in real life outside of school. One might say that it is a question of opinion, but there is a significant amount of time involved in school mathematics in learning to factor these prefabricated trinomials, which one really only encounters in school mathematics.

5.2.3. Mathematical component in the pre-lesson discussion

5.2.3.1. Possible gap in teacher's content knowledge

On several occasions during the two pre-lesson discussions, Gabrielle indicated that for her, there is a very fine line of distinction between the two distinct solutions coming from the quadratic equation $x^2 - 9 = 0$ and the $\sqrt{9}$ as a number.

> G: Somewhere along the line maybe your kids will be trained better than mine, but at some point to say, x squared equals 9, and x equals plus or minus square root of 9, would you just write that here, or…?
>
> A: That is on that [..]
>
> G: Oh yes, x squared minus 9 =0, x squared equals 9, x squared, so 3 squared equals 9, and negative 3 squared equals 9.
>
> A: [..] just checking, what could be the solution? Just 3 or …
>
> G: well the difference between saying that as an equation and not saying square root 9 = plus or minus 3
>
> A: When we are solving any equation, we are looking for possible numbers which satisfy the equation. So what would be a possible number, 3 or negative 3. when we finally find, when we see that x = plus or minus 3, or we can say is x is equal to plus minus root of 9. We can say that. But when we know, that is the procedure. We don't know this procedure of let's say [..] we don't know that until we discover it.

And when we discover, we know that it has got to go that way. Because they don't have, of course if you give them just formula, they will say that root of 4 is equal to plus minus 2 thinking that it's the same. Thinking, but if you say, we are looking for numbers, which are the numbers? So we also have that quadratic formula. We have that quadratic formula. But it is not that you take it that way from the start. We developed it, we found it, we went through completing in the square, then square rooting it, then putting the root in the other side then solving finally, again etc. Of course when you give them final formula, they apply, and don't think really much about the way how we did it and why it is plus minus, but it is because of that, and not because the root of 9 is plus minus 3. Root of 9 is just 3.

N: That plus/minus comes because it could be positive 3 or negative 3, when it's a solution of a quadratic equation...

G: Yes and for me that's a **very fine point of distinction**, that I will have to make very clear to my students that we're not just saying what is the square root of 9. Square root of 9 is 3. But if you say x squared equals 9, now you have two possible values for that.

A: Yes, we see x could be negative also. Square, and it gives again 9, so. We are actually looking for the numbers which satisfy this equation. What could x be?

N: You see, if you were to allow, actually once you wrap your head around it, it's necessary. Because if you were to allow let's say the square root of 9 is plus or minus 3, you get a huge problem. Then in every expression where you have a square root you would have to account for both values. This is, we define it when we teach it in grade 8, a square root being a side length of a square whose area is known and whose side we want to find, and so the area is under the root sign, but this thing, the result is a distance, and as a length, it's always positive.

G: Yes.

A: You can define...

G: What I'm saying here is that that is very compelling and **I have not made that distinction as clearly as I should**, you know.

Interestingly, none of the colleagues pointed out to Gabrielle that, if one applies the quadratic formula to the above example, the two solutions are obtained, albeit in a procedural way. However, from the discussion presented above, it can be seen that perhaps the difficulty that she experienced was, in fact, due to the notation. There were two points of inconsistency in her understanding of the square root. Gabrielle saw the

$\sqrt{9}$ primarily as an instruction to "take the square root", so as a verb. For her this process should yield both the positive and the negative value of the number which when squared would equal the radicand. She did not view it as an object in its own right, simply as a number and nothing more. Taking the square root for her meant finding a number which when multiplied by itself produced the radicand, so in this case both positive and negative 3. Secondly, putting the $\pm$ in front of the square root sign did not appear to make much sense to her. She saw it as an operation and not as the sign of the positive number $\sqrt{9}$. For those with a mathematical background, this is well known; however, it is not such a distinct point in the teaching of science, which is Gabrielle's main subject of expertise.

At this point, the colleagues in the team did not identify the challenges that Gabrielle faced and the consequences that this could have on her instruction, despite the fact that she voiced out several times during the pre-lesson discussions her confusion regarding the distinction between square root as a number and the two distinct solutions in the case of certain quadratic equations. They merely pointed out that this is how the square root is defined, but did not go into any explanation as to why.

As we follow the development of this lesson study cycle across both lesson implementations, this particular flaw in Gabrielle's mathematical disciplinary knowledge reappears again and again in different forms. It turned out to have an adverse effect on the instruction itself. In fact, the symbolic and the linguistic representations of square root seem to be at odds with each other in a number of resources that I have checked out. Taking the square root to stand for positive square root is a matter of convention. It is done for the purpose of consistency and convenience of mathematical notation, not because it would be a mathematical necessity *per se*. I discuss this in greater detail in the next section.

5.2.3.2. The mathematics that was absent from the discussion

I was puzzled by Gabrielle's inconsistent usage of the square root notation and also by her persistent inquiry about the concept of square root. This is what prompted me to consult various sources on this topic, particularly how square root is defined in school mathematics.

Most mathematical conventions and definitions are not arbitrary; however, it is important to know which ones are and which ones are not. They are such for a good reason, and it is my premise that this is the knowledge and understanding that a secondary mathematics teacher should have. In what follows I elaborate on the aspects of the mathematics of the topic that I believe would be helpful to have in the mathematics teachers' education: definition of square root, inverse of the quadratic function and quadratic function as a member of the family of power functions, and when the process of equation solving is mathematically valid.

Definition of square root

By definition, we have the following:

$$\sqrt{x^2} = |x|$$

I have examined nine different school textbooks from several countries in order to find out how square root is defined. Only two of the texts include the above definition (Mickelson, 2009, p. 8; Dakic & Elezovic, 2006, p. 120). The rest of them avoid the technical notation and, in some way or another, present either an incomplete or muddled definition of square root, or they remain silent about "what happened to the negative part."

Returning to the issue that Gabrielle was grappling with, if one uses the above definition, there is a clear explanation for what is going on. If the equation to be solved is $x^2 = 9$, then it follows from the above definition:

$x^2 = 9$	Take the square root of both sides. Since both sides are positive we can do this.
$\sqrt{x^2} = \sqrt{9}$	Now use the definition above.
$\lvert x \rvert = \sqrt{9}$	Solve for the unknown.

$x_1 = +\sqrt{9} = +3$ and $x_2 = -\sqrt{9} = -3$ There are two solutions.

${}_1x_2 = \pm\sqrt{9} = \pm 3$ Written in compact form.

The first part of the difficulty is that, for practical reasons, we usually skip from the first step to the last, without ever explicitly referring to the definition. This is fine once the idea is understood. However, it can cause trouble otherwise. From the textbooks that I examined, it is my suspicion that most students never see this definition for square root and also that many teachers do not know about it. The common concept image (Tall & Vinner, 1981) that people carry around is something akin to this: "Square root of a given number is a number which multiplied by itself is equal to the number under the root sign." This is problematic because it leads to the belief that square root of 9 could be negative 3, since $(-3)\cdot(-3) = 9$. Of course, this would be in conflict with the above definition.

The second part of the difficulty is the perceived conflict between the linguistic representation (the way we talk about the square root) and the symbolic representation. This is exemplified by the treatment of this subject in an encyclopedia of mathematical terms. Looking at the entry about square root, this is what this credible resource says (Tanton, 2005, p. 475):

> Any equation of the form $x^2 = a$ with $a \neq 0$ has two distinct solutions. Thus every number different from zero has two distinct square roots. For instance, 3 and –3 are both square roots of 9. By convention, if a is a positive quantity, then $\sqrt{a}$ is used to denote the positive root and $-\sqrt{a}$ the negative square root. For instance, we write $\sqrt{9} = 3$ even though –3 is also a valid square root of 9.
>
> A study of exponents shows that it is appropriate to define a number raised to the half power to mean the square root of that number. Whether that root is positive or negative is left undefined. Thus, for instance, $9^{\frac{1}{2}} = \pm 3$.

As we can see, this text says that square root of 9 is both positive and negative 3 (incorrect). However, it is also saying that $-\sqrt{9}$ by convention denotes the negative root of 9 (correct). To make matters on the meaning of notation even more confusing, we learn from this text that, if the square root is written using exponential notation, it is undefined, whether its value is positive or negative.

This type of conflicting information is not uncommon, even in regular student textbooks. The following is an example from a Singapore Mathematics student textbook that is used in the first year of secondary school (Meng, 2004, p. 104). It appears beneath the heading "Square and Square Roots":

> We know that $4 \times 4 = 16$. This can be written as $4^2 = 16$. We say that 16 is the **square** of 4 and 4 is the **square root** of 16.
>
> 16 is also the square of –4 since $(-4)^2 = -4 \times (-4) = 16$. Thus, –4 is also the square root of 16. This means that 16 has a positive square root 4 and a negative square root –4. We indicate the positive square root of 16 as $\sqrt{16}$, and the negative square root of 16 as $-\sqrt{16}$.

There seems to be a linguistic inconsistency between the process and the object that is being produced through the process. The process of "taking a square root of a number a" could be thought of as producing the two results, $\sqrt{a}$ and $-\sqrt{a}$. However, that does not mean that square root of 16 could be –4.

I mentioned that I reviewed nine different textbooks to determine how the topic of square root is presented in them. Only one was exemplary in its treatment. Here I present one excerpt from its treatment of the topic (Dakić & Elezović, 2006, p. 121):

> For any positive number a we label with $\sqrt{a}$ that *positive* number for which $(\sqrt{a})^2 = a$. We call this number **square root**, **second root**, or simply **root** of number a.
>
> [then four examples are given, with a brief discussion for each of them:
>
> $\sqrt{9}$, $\sqrt{16}$, $\sqrt{-16}$, and $\sqrt{10}$]

The square root of a positive number a is a positive number $\sqrt{a}$ for which the following holds true:

$$(\sqrt{a})^2 = a$$

Furthermore, $\sqrt{0} = 0$. For any real number a it holds

$$\sqrt{a^2} = |a|$$

This is followed by the second example for this topic, which is presented in Figure 5 below.

Let's solve the equation $x^2 = a$

1. If $a < 0$, this equation has no real solutions.

2. If $a = 0$, the solution of the equation is $x = 0$

3. If $a > 0$, we solve the equation this way:

$$x^2 = a$$
$$x^2 - a = 0$$
$$x^2 - (\sqrt{a})^2 = 0$$
$$(x - \sqrt{a})(x + \sqrt{a}) = 0$$

A product is equal to zero if and only if one of its factors is equal to zero:

$$x - \sqrt{a} = 0 \Rightarrow x_1 = \sqrt{a}$$
$$x + \sqrt{a} = 0 \Rightarrow x_2 = -\sqrt{a}$$

Therefore, the solutions of the equation are

$x_1 = \sqrt{a}$ and $x_2 = -\sqrt{a}$

Figure 5: Excerpt from a Croatian first year of secondary school textbook

Inverse of the quadratic function

Once we understand what necessitates such convention, the perceived inconsistency is completely resolved. It has to do with inverses – a perennial theme in mathematics that has to do with the "doing and undoing" (Mason, 2000). When we are looking at a quadratic function, for example $f(x) = x^2$, such functions are symmetric in the y axis.

This means that two different values from the domain map into the same value in the range (codomain). We could say that, apart from the (0, 0) pair which lies at the vertex, that for a single value in the range, there are two values in the domain - the two opposite numbers. For all other points, we have mirror pairs at $(-x, y)$ and (x, y). This means that it is impossible to deduce the sign of its input from the output of this function. In turn, this means that, if we insist that the domain must remain to be all real numbers, the inverse cannot exist.

In order for a function to have an inverse that is a function, each element in the range must correspond to no more than one element in the domain, (i.e., the function has to be an injection), which would only occur if the function is an information preserving one-to-one type of relation. Luckily, this one is not too far off, as it has this regularity that each of its half wings has this property. By convention, we make the quadratic function invertible, simply by reducing the domain, so that only one of the branches of the parabola is used (by convention we take the positive branch).

Now we are really only concerned with the inverse of $f(x) = x^2$ for which we take the domain to be only the values $x \in [0, \infty)$ as now every element of the domain can be paired with exactly one element in the range and there are no unpaired elements left. The entire set of non-negative real numbers is now effectively mapped onto the entire set of non-negative real numbers. In mathematics, we say that it is a bijective function. This means that it is a one-to-one and onto mapping of the entire set of the elements in the domain onto the entire set of the elements in the range. With this small adjustment of taking only the non-negative real numbers, we have a bijection and the inverse now exists.

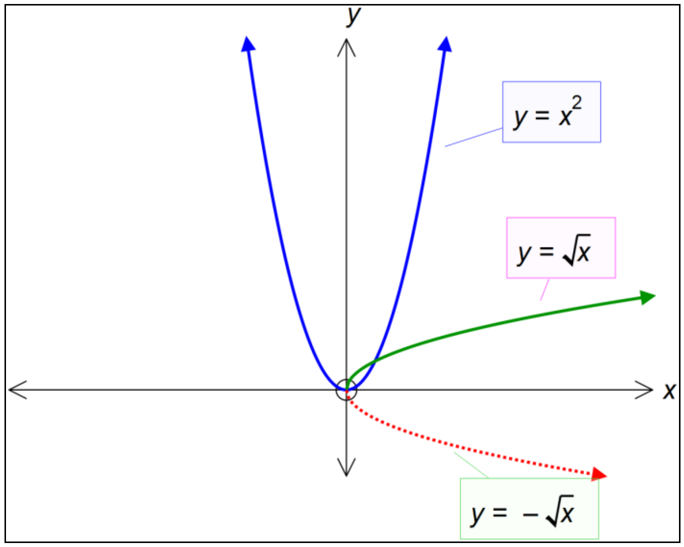


Figure 6: The inverse function of the quadratic function

This inverse of the quadratic function is no other than $y = \sqrt{x}$, represented in the graph above (Figure 6), known also as the principal square root. Clearly, the range of this function is all non-negative real numbers. The same is true of its domain. The opposite of the principal square root is then the negative square root which is denoted as $y = -\sqrt{x}$ and is represented by the other wing, which is also a function.

Furthermore, it would be utterly inconvenient if the symbolic representation of the square root of a number could be taken as both its positive and negative root. In such a case, even simple expressions as $2\sqrt{9} + 3\sqrt{4} - \sqrt{16}$ would have as many as 8 different values if we had to account for two versions for each of the square roots.

In general, most polynomial functions do not have inverses (they only have inverses in cases where the sum of the terms is constantly increasing or constantly decreasing, with respect to change in the independent variable). However, a polynomial that has only one term is essentially a power function. Power functions can be viewed as a family of functions in their own right. By restricting their domain to only non-negative real numbers, each of them now has its inverse in the set of real numbers and is defined in the set of real numbers. Taking the exponent as a parameter, we can view this family of functions and their inverses, which also belong to the same family (Figure 7). Therefore, another good reason for having this convention set as a definition is to achieve this unifying perspective.

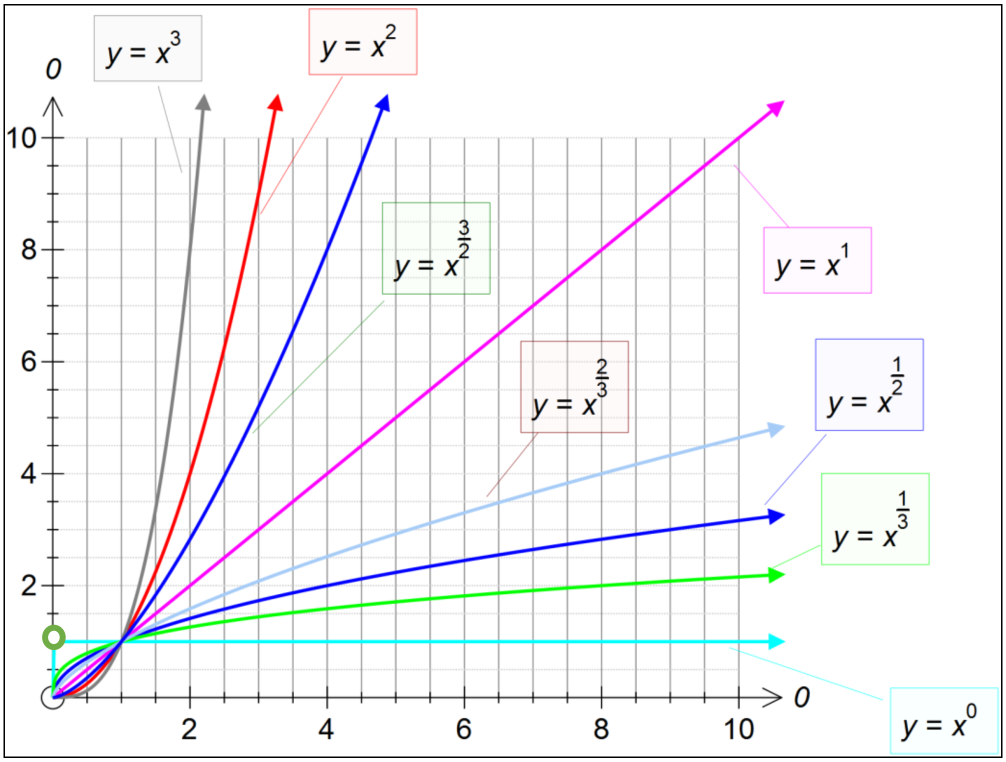


Figure 7: Classifying functions by focusing on the effect of their parameters: Square root function as a member of the family of functions with exponent as a parameter

When is the equation solving process mathematically valid?

In the equation solving process, the basic principle is that we can perform the same operation to both sides of the equation and that this will not change the solution set. However, this is not always true. For example, we cannot multiply the two sides of an equation by 0 because this would produce 0 = 0, which would change the solution set and make it appear as if any value for the unknown is a solution. After such operation is performed, one cannot undo it any more.

In general, the operations that we perform on the equations must be invertible, which is the case only if they are associated with one-to-one functions. This frequently goes unsaid in school mathematics. Performing operations that cannot be uniquely "undone," for some or all values of the variables, will disturb the chain of logical implications in the process of equation solving and we will no longer have an equivalent equation, (i.e., an equation that has an equivalent solution set).

In the case of solving radical equations, the operation of squaring is not invertible. There is a bi-unique value that could have yielded the same result. This disturbs the logical chain of proof. However, by setting the restrictions on the unknown and by being explicit about the conditions under which the equation is being squared (both sides must be positive, for example), the problem of breaking the logical chain is side-stepped because the operation now becomes invertible.

In summary, the confusion arises from the incorrect, but commonly written statement $\sqrt{x^2} = x$, which is not true if *x* is negative and is further reinforced by a related statement $(x^2)^{\frac{1}{2}} = x$ that also can seem plausible to a novice, but is an example of misuse of the exponent law $(x^m)^n = x^{mn}$ which might not be true if *x* is negative and *m* or *n* are not integers. Various paradoxes may arise if we accept calculations that have to do with squaring or taking the square root of both sides of an equation uncritically (Movshovitz-Hadar & Webb, 1998).

5.2.3.3. The mathematics that was present in the pre-lesson discussion

The following four themes from the analysis teachers' conversations during their pre-lesson discussions were identified as belonging primarily to the mathematical component of their MfT.

- **Anticipating student difficulties**

What arose spontaneously during the planning sessions was anticipating student thinking, including their difficulties. Although it is also a prominent feature in the lesson study practice, the teachers in this team were not prompted to engage in it. They did so naturally. This teacher activity is seen as a very important one because it equips the teacher with the knowledge for dealing with students' potential errors and for correcting their understanding while it is being shaped -- a way to act proactively (rather than reactively when the students might have already reinforced their erroneous thinking through premature practice).

The teachers did not anticipate that students would have difficulty with understanding and being able to state for which values of the variable the equation was defined. The question about the domain was seen as non-problematic. However, the question about the range was anticipated to be something that the students might not have thought through carefully, especially not in this context. Therefore, because it represented a critical piece in organizing the didactical situation in this particular way, it was going to be explicitly addressed in the lesson. In other words, the mathematical knowledge that was needed for the success of the lesson as it was being planned was a proper understanding of the square root as a number, which is always positive, and to distinguish that notion from the common experience of the solving of a quadratic equation where generally both the positive and the negative root come into play. Here we take a look at the discourse among the teachers in order to learn how they planned to accomplish this task:

> A: Now we are coming to this point. Take a look at this quadratic equation. $x^2 - 9 = 0$. How do we solve it? So it is $x^2 = 9$. Which numbers satisfy this?
>
> G: 3
>
> A: Let's say 3 squared = 9. Correct. Negative 3 squared = 9
>
> G: Oh yes, alright.
>
> A: So now what are the solutions? x_1 or x_2 = plus or minus 3 [writes ${}_1x_2 = \pm 3$ on the board]. So if we want to go from here, let's say that we have x squared = 5, how much is x? x one and two …
>
> G: Square root 5.
>
> N: Plus or minus.
>
> G: Aaaah, riiiiight!
>
> A: […] so it would be plus minus 5. So, take a note. Root of 5 is that a positive or negative number? It is obviously positive. How do we know it's positive? So what kind of number is 5, is it a positive

> number? So we square it or root it, it's a positive number, so it's greater than 0 so it cannot be negative. But we write that, it looks like this [${}_1x_2 = \pm\sqrt{5}$], but the root itself is not negative. The solution [of a quadratic equation] could be positive or negative, but root is always positive. It's the reason why we put here positive or negative [in front of the root symbol]. If it was not the case, we would simply say root of 5 if that would be both plus/minus, but it is not the case. So we have to put here plus/minus.

The lesson plan, as well as this discussion, captures Andrew's anticipation of this being problematic for the students. However, it also appears that Gabrielle is having considerable difficulties with this idea. Andrew is explicitly trying to distinguish the radical as a number from the case when a positive and a negative version of the same radical (or 0) is generated as a result of solving a quadratic equation.

In my analysis of the above discussion and with the knowledge of what transpired after this, with respect to the inconsistency in Gabrielle's content knowledge regarding the square root, I have found this to be a surprisingly persistent issue. As was noted earlier, this flaw in her mathematical content knowledge did not have an adverse effect on her teaching of physics. However, in the teaching of mathematics, it is a different matter. For example, when solving quadratic equations, as one regularly does, (e.g., whenever an application of the Pythagorean Theorem is being called for), most often only the positive solution of the associated quadratic equation is of any interest. In secondary school physics, we often deal with physical quantities, such as distance, volume, mass, where only the positive values make sense. Therefore, it is a regular practice to completely ignore the negative solution in the solving of quadratic equations.

From the preceding conversation, we see that Gabrielle offers only the positive solution to the two simple quadratic equations that are being brought up by Andrew. At the same time, Gabrielle's competence with the quadratic formula is unproblematic. Obviously she knows that a quadratic equation normally produces two solutions. However, the +/- symbol in front of the radical sign in the quadratic formula is an operation, which acts as an instruction to once add and the other time subtract. Therefore, in a compact way, it embeds the two solutions. This means that Gabrielle is aware of the two solutions when applying the formula. However, she seems not to be aware of this when the simpler version of the quadratic equation is involved, namely the kind that does not involve the linear term. Therefore, for quadratic equations of the form $ax^2 + c = 0$, where the process of solving no longer requires the use of the quadratic formula, the idea that a quadratic equation actually yields two solutions breaks down in Gabrielle's concept image. It could be that the experience of repeatedly ignoring the negative solution in science (in a mathematics class, this is often the case with the applications of Pythagorean Theorem) over many years of a teaching career, caused her to forget to account for it elsewhere, where it might be applicable. It is a pattern in Gabrielle's treatment of simple quadratic equations that result in the form $x^2 = k$ that she would only see $x = \sqrt{k}$ as a solution.

- **What is "extraneous root"?**

The term "extraneous root" came up twice in the conversations among the teachers while planning for instruction. Each time, it is apparent that Andrew is completely unfamiliar with the term. This is very interesting because he is an exceptionally knowledgeable teacher with extensive educational background in mathematics. Throughout this lesson study cycle and also during the other lesson study cycles, he acted as a mentor to both Gabrielle and Steve. Gabrielle and Steve mainly asked questions of Andrew during these lesson planning meetings. Then, the discussion would often unfold with everyone contributing something. However, in relation to the term "extraneous root," it was much different.

> A: So we see that −7 is not the solution, and because we put additional limitation, we will not include that into the solution because it's not the solution [...] just to develop mathematical way of doing things.
>
> G: So do you introduce the terminology of extraneous roots here or do you wait until grade 12?
>
> A: What do you mean?
>
> G: Extraneous roots, as I understand them are roots that the mathematics gives you until you look at your original premise and do a check. Or in this case [...]
>
> A: Even in some textbooks it is, just solve it. And when you come to the solution, check it. But that is a big problem. Because, it could

happen that you have a really long way and you get a very complicated number. To check it, it's not easy! And not to mention that you don't, sometimes, even need to start, because you immediately, you can see there's no solution. Put the starting limitations and there's no match, so you don't even need to start solving. So that is one of the reasons. But also if you want to develop some kind of mathematical reasoning, then maybe we should do that this way.

As we can see from the preceding conversation, Andrew's mathematical practices are at odds with the standard practices that are presented in the textbook. He considers that simply solving the equation, without paying attention to the conditions under which it may be solved, is somehow non-mathematical. He supports his view using two reasons. The first reason is that if the "solutions" turn out to be irrational, checking for which one is the correct one can be arduous. The second reason is that it could be that "there is no solution at all." His argument is one of economizing effort, in this case, computational effort. According to Krutetskii, curtailing mental processes is one of the recognized common features of "acting mathematically" (as cited in Watson & Barton, 2011). The mathematical practice of requiring the consideration of the conditions under which a certain procedure may be performed is an example of being mathematical. Indeed, there is more than economizing effort in Andrew's reasoning for why the topic should be taught this way and not the standard way.

G: So my question down there was that word, extraneous. You don't, you've never used that?

A: No, I would have to admit that I don't know about that.

S: I guess that's what I would call them, you probably don't get to use that word very often, like for the grade 11, but …

G: Because it comes up when you're working with logs, you can't do a log with a negative number either, so if your solution turns up a negative root, then you call it an extraneous root and you reject it. And the thing about extraneous, as I understand the meaning of the word, is that the mathematics without the restriction up at the top, the mathematics allows that solution, that solution is mathematically correct, up to the point where the answer turns out to be some nonsense and you must reject it.

It appears that the term is primarily in use in school mathematics, while in "mathematics proper" its use is limited to specific subfields of mathematics, such as optimization. This is interesting because it points to a phenomenon of overemphasis on the language in the peculiar subculture of school mathematics where, at times, learning the concept itself might get substituted or curtailed by learning a procedure, as well as a term to go along with that procedure. It is easier to train behaviour and to learn language than it is to learn concepts. Learning concepts requires a much greater mental effort, as well as expert teaching.

From a student perspective, the term "extraneous root" can be taken to stand for a syndrome in the equation solving process, when something strange is happening with the mathematics. One need not bother with what is going on. It can be treated as a "black box." Just perform all the work as usual and then check if all of the solutions are good. If they are not, simply call them "extraneous roots," and then discard them as a possible solution.

"What happens mathematically?" "How can this occur in the first place?" "What is the underlying "mechanics" of the process?" The answers to three questions can remain hidden from the student. Of course, the only thing that matters is that the student obtains a correct answer! Whether he or she understood the mathematics is not so important. This could be one interpretation of what the introduction of this term was supposed to accomplish for school mathematics. Another interpretation is that it is a pedagogical tool which is intended to help students accomplish the task of solving equations successfully, even if they do not understand the underlying processes. That understanding can come later, when the student already has a certain facility with procedural knowledge in this topic.

- **Pivotal explanation**

The route along which the teachers were going to travel in the instruction excluded using the term "extraneous root." Instead, the idea was to invoke a *cognitive conflict* regarding the appearance of a faulty solution, and then ask the students, "Could we have known it beforehand?" (See the lesson plan in Appendix D). Andrew wanted to present a mathematically complete explanation for the process of solving radical equations, which would make it clear for the students how these faulty solutions arise in the first place.

Zazkis and Chernoff (2008) introduce the notion of 'pivotal example' to illustrate a means toward a resolution of cognitively conflicting ideas about a mathematical procedure or concept. In their prior experience with respect to solving equations, students could perform basic operations on both sides of the equation without running into inconsistencies, apart from the well-known case of division or multiplication by zero. Their everyday experiences with solving equations would lead them to believe that the process itself is trustworthy because it always gives correct results (as long as it is correctly done). However, when solving radical equations, the process requires the elimination of radicals. Therefore, equations need to be squared. This is an operation that could easily present an inconsistency because it is irreversible in the sense that it is impossible to work out whether a number or its negative had been squared. For the purposes of this discussion, the idea of pivotal example is extended to the idea of a *pivotal explanation* as an explanation, rather than as an example that a teacher uses in an instructional setting to help students resolve the conflict regarding this "sudden apparent misbehaviour" of these types of equations. The following excerpt illustrates the pivotal explanation that Andrew would use in the enactment of his lesson:

> But what if I have let's say, root of 4 is negative 2? Is that correct? Obviously not. Because this is positive [and] this is negative number. We square it, we get 4=4. Correct! So something that wasn't correct, squaring can make it correct. Or, we actually changed this. So we have to be really careful, if that is not possible, we cannot make of something that is not possible, something that is ok. In mathematics, that logic is wrong. So we need to see, before we can square it. Is this side positive, and is the other side positive? Because if that is negative as it was here, we could from something that probably didn't have a solution, get something that has a solution. Or whatever, we have a wrong solution. […] We need to all the time whatever we do, see, is that move or procedure ok because we cannot square something that is negative on one side and positive on the other side, or we cannot consider negative root, or a square root of a negative number we have to be sure it's positive, so this is the way how.

This is an example where several components of MfT overlap in a teacher's discussion (mathematical, psychological, and pedagogical). This pivotal explanation was planned to be used to support student learning during the progression of the lesson at this conceptually critical stage. Andrew felt that, in order to develop a mathematical way of thinking in students, they should be taught to take care and see (in a sense, maintain an awareness) that all of the steps in the process of solving an equation are mathematically valid. In particular, he viewed that squaring of both sides of the equation, without "setting restrictions" first, would be somewhat of a violation of a mathematical principle. Nonetheless, this did not appear to be a behaviour-training teaching move. Rather, it was a case of a teacher teaching a general principle. It is a case of teaching in terms of generalities, rather than techniques.

- **Teachers explore special cases of radical equations for their own understanding**

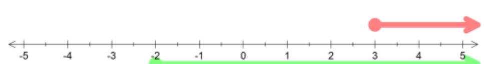
This section focuses on the considerations that guided the mathematical inquiry of the teachers into what it is that they want their examples to exemplify, (i.e., the purpose for their use in instruction). In this part of the teachers' pre-lesson discussion, some examples got rejected, while others got improved upon. Interestingly, the teachers spent the largest amount of time dealing with a special class of radical equations, those whose solution is the empty set. This was in fact one of the major arguments for the team's decision to choose Approach B over Approach A as their teaching approach for this content (see Figure 4).

The cases of radical equations whose solution is the empty set appeared to propel a fruitful mathematical exploration for the teachers. This was spearheaded by the following comment that Andrew made.

> And don't mention that it's sometimes, you don't need to solve. You immediately find at the beginning that there is no solution. And also we say that the solution is that there is no solution. Actually, solution is empty set. So it is solution. You solved it, you said that there is no solution. You actually solved the problem. You didn't find the exact number for the equation, but you solved the problem.

As noted earlier, Andrew mentioned that the chosen process for approaching the topic this way has the advantage of economizing effort, because there are cases when one does not even need to start solving if the process of the analysis of the domains and ranges leads to the conclusion that the solution is an empty set. This comment prompted an inquiry (initiated by Steve) into exactly when, as well as how this advantage can be harnessed. He was intrigued by this, as he clearly had not considered it before in this context. He wanted to understand the variation of such cases and also how this could be presented to the students. He saw the chosen teaching approach as being useful in the case where there is no solution. However, he wondered how much of an advantage there would really be if “there is one solution that works and one that does not.” The discussion began with Steve’s creation of an example in which it could be immediately concluded that the solution is an empty set, based on the reasoning about the radical being non-negative, (i.e., the reasoning about the range of the associated radical functions, in this case, a sum of two radical functions, being all non-negative real numbers).

> S: Can you do something like ... [writing $\sqrt{x+2}+\sqrt{x-3}=-2$ on the board]. I've been trying to say that that's got no solution. Can you use your number line to show that?
>
> A: So it is root of (x+2) + root of (x – 3) = -2. Of course, immediately this is positive, whatever x is, the root is always positive. And this is always positive, so the sum cannot be negative.
>
> S: I'd say, given that both of these have to be positive, it's impossible, but is there some sort of number line you can do to show that?
>
> A: So it is x greater than or equal to -2, and x is greater than 3 [draws the graph of the domain]. So on the number line, it is ok, there is a common area. But the whole equation is not possible. Because adding any two positive numbers, you cannot get a negative number.



This example, created by Steve, is a skillfully chosen example because the solution is an empty set for a particular reason, namely the one that is associated with range. However, Steve wanted to find a common explanation and a representation using a number line (the same way as that which was used in the analysis of the domain to establish for which values of the variable the equation is defined). If we are restricted to the number line alone, such uniform representation is not possible in this case because the matter involves different sets of values; one for the domain and the other for the range. This is somehow seen by him as less than satisfactory. His considerations are pedagogical in their nature, which is another case of overlapping components within the teachers' discussion. This is how the enquiry continues:

> S: So is this the only way you can explain it though is with an [..]
>
> A: That is the limitation for the roots. But you have to look at the whole equation. It is not root, but still has to be positive. Not just root. Always look on your equation. For example, you have like this, root of x over x -1 + root of x +2 =3, let's say.
>
> [writes $\frac{\sqrt{x}}{x-1}+\sqrt{2+x}=3$ on the board]
>
> So this has to be positive, this has to be positive, but this has to be different than 1. Doesn't matter only what is the value under the root, if we are looking at the whole equation. And I'll give one example later like that in class too. To show that we are solving radical equations but we pay attention to everything that comes with mathematics.

In Chapter 6, I present an analysis of the tasks that were actually used in instruction, with attention to their variation. There I present them with graphs in the Cartesian coordinate plane to give a visual access to both the domain and range. Interestingly, this way of representing the equations did not arise in the pre-lesson discussion.

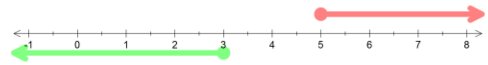
The examples chosen by the teachers are guided by some underlying principles and considerations, which interact with their example spaces (Zodik & Zaslavsky, 2008). Andrew's expansion of the example space in this interaction was prompted by reasoning that one needs to "look at the whole equation," with all its elements. His stance exhibits a mathematical habit of mind, one that prompts a requirement for a careful analysis of the situation and not a mechanical application of a limited skill that is currently being learned. This theme actually appeared frequently in his articulation of the reasoning behind his decision making. In the next segment, the teachers collectively decided to admit Steve's example into the lesson because it served to exemplify the subclass of radical equations whose solution is the empty set. They do so with a slight modification to it, in order to conceal the immediate access to the discovery that the equation "has no solution." These are pedagogical considerations in nature. Teachers want the students to discover something mathematical. This theme surfaces repeatedly as one of the considerations in the generating of examples for instruction (overlap with the didactical component).

> A: So that is a good example, your example is great, because just to show [...]. I have to write it down!
>
> G: I think you should include it in the lesson. This should be part ...
>
> A: Yes of course, that is good. [root of x + 2, calculations] but it could be, could be done even better, +2 =0. So it is not visible. Then it is not visible immediately, but you need to discover it. But how do we discover? [...] So it will be $\sqrt{x+2}+\sqrt{x-3}+2=0$. Again, you have to [..] but not but on the first step, if you don't see it immediately, you'll see it in the next step.

Steve's enquiry continued further and drove the discussion to another expansion of the example space. The next example that was generated is also a representative from the class of radical equations whose solution is the empty set. In contrast with the previous example, this one is meant to highlight that there could be "no solution" for a different reason, this time one that is not based on the analysis of the range (value of the radical must be non-negative), but rather, on the analysis of the domain (values of the radicand expressions must be non-negative). Again, the teaching is meant to emphasize that one needs to "pay attention to everything that comes with mathematics."

> S: Would you have an example though where you can look on your number line and then show basically that you don't have to do anything?
>
> G: So what Steve was asking is, if you can just, from graphing on a number line already also discover that there is no solution? It should be, if the 2 intervals exclude each other.
>
> S: Yeah, something like that.
>
> A: So root of x - 5 and root of 3 - x = 10.
>
> [writes $\sqrt{x-5}+\sqrt{3-x}=10$ on the board]
>
> So it looks ok, positive positive positive. But here is x greater or equal to 5, x less than or equal to 3, solve, no solution. Greater than 5, less

than 3, no solution. And it is also a solution, to discover that the solution is an empty set, it's also a solution. You solved it.



What the team of teachers is doing here can be described as "*structured variation"* (Watson & Mason, 2006). Prompted by Steve's enquiry, Andrew again drew on his personal example space and almost immediately came up with an example to exemplify another situation in which the process of solving does not even need to be initiated because, upon the examination of radicands, it can be concluded that the solution is the empty set. There is no doubt that Gabrielle and Steve could have produced such an example as well and, in fact, Gabrielle articulated the idea behind such examples. However, it seems that at this point, Steve's enquiry was guided by pedagogical considerations and a desire to find some common rule that could be applied to all such cases. He is repeatedly requesting a way to use the number line representation to show that the example he had generated earlier had "no solution."

S: You can't really do that for this though, can you [referring to the previous equation, $\sqrt{x+2}+\sqrt{x-3}=-2$, which had subsequently been modified to $\sqrt{x+2}+\sqrt{x-3}+2=0$]?

A: No, because the set where roots are is ok, but here root gives an empty set, but [...] so, as we said, look at the whole process, not just pay attention to one part.

S: I guess I'd like to be able to do this for that question, but can you?

A: Which one?

S: Well can you do this kind of thing and show that it's impossible for that?

G: No, because they overlap.

S: But I think that's what the kid would say wouldn't they? They'd say, alright, like I can see this here [referring to $\sqrt{x-5}+\sqrt{3-x}=10$], but why can't you justify that one the same way?

A: So then they have to know that this [radical] is always a positive number, or 0. Rather say, non-negative. So non-negative or negative, and here is clearly negative. So addition of two non-

> negative numbers cannot be negative. It cannot. But this is one example and this one is different. Sometimes you can combine it. If you say, here, let's say $\sqrt{3-x}+\sqrt{x-5}=-2$, this itself is not possible [for two reasons]. Addition of square roots, [that is] of 2 positive numbers cannot give negative, but don't do too much in one example. Could happen, you could give whatever you want, so then they have to discover one or the other, but if you want to practice, then you'd rather separate one idea in one example, and another in second example.

The last example that was generated by Andrew, in response to Steve's enquiry, collapses the two possible reasons for which the solution could be the empty set. However, Andrew immediately finds it unsatisfactory for use in instruction because there is "too much in one example." In structuring sense-making with respect to tasks, the notion of variation across different dimensions of variability of a mathematical object can be helpful (Watson & Mason, 2006). The variability that the teachers explored here was across the special class of radical equations, those whose solution is the empty set. Each of the examples showed a different aspect of how or when an equation can be "discovered to have no solution." Table 3 below summarizes the examples that were generated, used for discussion, or adopted for the lesson:

Example	**What the example intends to exemplify**
$\sqrt{x+2}+\sqrt{x-3}+2=0$	Solution is the empty set because of the restriction on the range (i.e. radicals are non-negative).
$\sqrt{x-5}+\sqrt{3-x}=10$	Solution is the empty set because of the restriction on the domain (i.e. the radicand expressions have no overlapping set of values)
$\sqrt{x+9}-\sqrt{11-x}=5$	Solution is the empty set because of the restriction on the range which shows up later in the process of solving.

Table 3: Example variation for radical equations with "no solutions"

One might ask why the teachers placed so much attention on these rather uncommon radical equations. I suspect the answer is that they were finding this to be fertile space for their own learning because it touched upon something that they had not considered before. Although they were all versed in the teaching of this topic, they had not all done it using this particular approach. Therefore, it allowed them to "see things in a new way." It was the novel approach that drove this delving into the inquiry about the uncommon cases.

5.2.4. Pedagogical component in the pre-lesson discussion

In teachers' pre-lesson discussions the pedagogical themes did not dominate in any of the topics that were discussed but the pedagogical component was present implicitly in several instances, as indicated in the analysis of the excerpts. Surprisingly however, teachers' philosophical stance surfaced a few times during their discussion, and was intriguing enough that this prompted me to extend the Four-Component Framework to account for these themes.

5.2.5. Philosophical component in the pre-lesson discussion

It is the professional duty of mathematics teachers to represent mathematics with integrity to the subject and to teach students the "mathematical way of thinking". This aspect of teaching mathematics came up repeatedly throughout the pre-lesson sessions and became a major theme that did not fit into the four components identified by Selter (2001). As such, I identified the philosophical component of teachers' professional knowledge in the empirical data of this study.

As Thom (1973) pointed out, "whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (p. 204). Incidentally, in his studies of the professional knowledge of mathematics teachers, Bromme (1994) developed a topology which included a component which he called *philosophy of school mathematics* and which seems to closely resonate with my findings. Bromme defines this category as "ideas about epistemological foundations of mathematics and mathematics learning and about the relationship between mathematics and other fields of human life and knowledge" (p .74). He purports that the philosophy of a school subject is an implicit content of teaching as well. According to Bromme, the concept of "philosophy" in the teachers' professional knowledge is intended to capture an evaluative perspective on the content of their teaching. It is not a matter of subjective preferences that teachers have regarding the mathematical content that they teach but is part of metaknowledge permeated with implicit epistemology and ontology. As such, this component assumes normative principles and value decisions, which are interwoven into the subject matter knowledge about mathematics.

For these reasons I could not ignore the philosophical component and chose to incorporate it into the framework for analyzing teachers' MfT. From this point onwards I will refer to this extended framework as the 5-Component MfT Framework.

In the rest of this section I present the themes from the pre-lesson discussions that are part of the philosophical component of teachers' professional knowledge.

To teach students the "mathematical way of thinking" and that "math class is meant to prepare for real life"

As a team, the teachers considered it important to nurture the students' *habits of mind*. This can be done, not only through the teaching of the particular concepts that are associated with radical equations, or the procedural knowledge of how to solve radical equations, but in the way that any topic in school mathematics is approached and dealt with. The following excerpt captures the aims and intentions behind this team's instructional design:

> While conceptual understanding is an aspect of mathematical knowledge and pedagogic design based on these understandings is important, it is only part of what constitutes mathematical knowledge. To become a fluent mathematician, a student must learn to act in certain ways in mathematical and other situations, to develop mathematical habits of mind, to enact mathematical modes of enquiry, and to think in terms of these underpinning key understandings. (Watson & Barton, 2011, p. 70)

Therefore, when considering MfT, it is important to look beyond simply how teachers choose to represent and present the content. At least as important is that teachers themselves enact mathematics, which means that they exhibit and embody a "mathematical way of thinking" themselves.

> Steve (S): I wonder what to say if a kid says, "Alright, why can't I just do it and then check my 2 numbers and see which one works?"
>
> A: I said at the end, you probably noticed, that in some textbooks it is done. And it can be done. And I mentioned that problem with the

> Iron Worker's Memorial Bridge. They didn't calculate properly and they thought they will check. And how they check the bridges? When it is finished, put all the trucks full of sand over, one by one. And if it survives, it is ok. So do we need to wait until we finish and check and realize, oh it is not ok.
>
> S: What if they say, but this is not a bridge, it's just a math problem.
>
> A: So this is not real life also, it is school, so we are preparing ourselves for real life. So we need to think and work as we are in the real situation. So we are preparing ourselves for real life situations. Not pretending, "who cares, we'll check at the end." We try, and mathematics logic also requires all the steps – it is mathematics logic - all the steps to be correct, and then for example, Andrew Wiles, that mathematician who was proving some theorem, the Last Fermat's Theorem, for 7 years, and got the result. And after a year of checking, his professor at Princeton found a mistake. And he spent one or 2 years again, because in the meantime, some Japanese mathematicians discovered new part of the theory, that helped him to really solve his problem. So you need, and also he did, point by point check, but didn't see at some point, that there is a mistake. So you cannot wait until the end. Spend 7 years working and then go, oh there was one mistake. You need to check always, go step by step, that is mathematics logic.

In the above excerpt, the two teachers display different philosophical stances towards school mathematics. Andrew shared about the importance of applying mathematical logic throughout the process of equation solving and linked it to engineering as well as to the ways of how a mathematician would think. In his practice, he intends to develop mathematical thinking as an essential competency for life.

To teach students the "elegant way of thinking"

As was indicated in the previous discussion, Steve had some resistance around planning to teach the topic in this non-standard way. He challenged Andrew with his questions about how to justify it from a pedagogical perspective. In the following excerpt, he resolves this dilemma by himself:

> S: But then, you know what my students are going to say, "Alright, if I do it the other way on my test are you going to take marks off?"
>
> A: I would say, in MY test, you will not get full marks. You will get some mark because I usually give even if they don't finish. If the problem is out of 15, so instead of 15 I'll give 5 or 7, whatever.

> S: I wonder if you could say something [to students] about you know, in mathematics we like to have elegant solutions and if you check this it's much more elegant way. Checking in the end is a bit less elegant.
>
> A: Maybe that would be a good reason [...], not just that "we like." We are proposing here mathematics logic. And mathematics logic requires that every step has to be justified.
>
> S: You know at the end it's kind of like guess and check, it's not really pretty. If you do the other way.
>
> A: People do that. But it is just, I would say, short term solution. If you really want to work on long term problems, or actually to learn procedures for long, not just a short period, then you need to do the complete logic, to outline the logic completely.

On the other hand, Andrew holds a strict stance that it is the requirement that is imposed by the mathematical logic and that just "liking it" is an insufficient reason in this context. He also draws attention to the structuring of the content and, in this case, the procedures, in ways that will be remembered by the learner. If a student understands, then the student will also remember and be able to use the knowledge. This is a case where we can see the presence of multiple components (pedagogical and didactical, in addition to philosophical which is dominant).

5.3. Discussion of emergent themes in the stage of planning for instruction

In the process of collaborative planning for instruction the team of teachers engaged the following questions: "What do we want the students to learn?", "How will this be connected to what they already know?", "How will it support their future learning?", "How will the content be made available in the lesson in order to promote student mathematical thinking?", "What kind of tasks and ways of working with them in the classroom will get to the heart of the matter with respect to the integrity and coherence of the subject?", "Why should we use this approach and not another?". The questions prompted an inquiry into the problematics of the concept that was to be taught and into the student learning processes. Resources, especially the time that is available as well as the students' background knowledge, can substantially influence the teachers' decisions about when a certain method of teaching can be applied.

While the commonly used approach of grinding through the equation-solving process and then seeing what that produces in the end might seem more practical, it obscures what the mathematics is doing. It sidesteps the need for reasoning and for performing an analysis of what could be the possible solution set. It may even leave some students mystified or thinking that mathematics does not make sense. If a teacher is mainly concerned with the ability of the students to produce correct results, the approach that is presented in the textbook may seem more practical (Approach A). It certainly requires less cognitive effort on the part of the students. However, it also would not lead to a deep and connected understanding of mathematics, which can be achieved through learning that focuses on mathematics as a sense-making activity.

Steve's questioning about the merits of teaching this topic, using Approach B (setting restrictions) as opposed to using Approach A (testing for correct solution at the end), had to do with this practicality. If the quadratic equation produces two solutions, of which one is correct and the other one is not, would there really be much of a benefit in using Approach B? Even if we accept that Approaches A and B both yield correct answers and it is debatable which one requires more effort, the question still remains: "Is there value in students learning exactly how and why the error gets introduced into the equation?" In other words, do students need to understand the mechanics of the "extraneous roots"?

After some initial hesitation, the team of teachers eventually settled in agreement to use Approach B in the teaching of the research lesson because they saw this approach as better supporting the goal that they had for their students and which they had articulated as the long term goal for student learning - developing a culture of mathematical thinking in the classroom. Part of this culture includes the learners having to develop a habit of reflecting on their experience of doing mathematics and "developing the internal monitor" (Mason, Burton & Stacey, 1982). Curriculum writers also seem to agree with this approach because the need for students to understand how "extraneous roots" arise during the process of equation solving is mentioned in the achievement indicators in the curriculum document.

In Table 4 below, the specific learning outcome of this lesson's topic is presented as it appears in the latest curriculum document for this course (now called Pre-Calculus 11). It is important that there is congruence between the intentions of the curriculum and the way in which teachers interpret it and implement it. How well this can be achieved is closely linked to teacher's MfT (as communicated in Figure 1). The Grade 11 curriculum in British Columbia has been revised and the learning outcomes that are embedded in this lesson are now articulated with more guidance in not only what students need to learn, but also using the language of *achievement indicators* to signal what it means to say that a student understands the topic.

Algebra and Number (continued)	**General Outcome**: Develop algebraic reasoning and number sense.
Specific Outcomes *It is expected that students will:*	**Achievement Indicators** *The following set of indicators* ***may*** *be used to determine whether students have met the specific outcome.*
A3. Solve problems that involve radical equations (limited to square roots) [C, PS, R]	(It is intended that the equations will have no more than two radicals.) 3.1 Determine any restrictions on values for the variable in a radical equation. 3.2 Determine the roots of a radical equation algebraically, and explain the process used to solve the equation. 3.3 Verify, by substitution, that the values determined in solving a radical equation algebraically are roots of the equation. 3.4 Explain why some roots determined in solving a radical equation algebraically are extraneous. 3.5 Solve problems by modeling a situation using a radical equation.

Table 4: Learning outcomes related to solving of radical equations from Pre-Calculus Grade 11 curriculum (BC Ministry of Education, 2008)

We can see in Table 4 that two of these indicators relate very closely to what Andrew, Gabrielle and Steve were aiming to achieve in this lesson in terms of what they hoped their students would learn and understand. These indicators are, "determine any restrictions on values for the variable in a radical equation" and "explain why some roots determined in solving a radical equation algebraically are extraneous." However, on the basis of what was uncovered in this lesson study cycle, with this team of experienced and committed teachers, it is a question and a real concern to what extent this can be supported by the content knowledge of mathematics by most teachers and by the textbooks that they use. The problem of mathematical content knowledge has been briefly touched upon in this chapter, but will be further discussed in the following chapters.

Our group of mathematics teachers constantly strived to show students a very different, more truthful image of mathematics and the processes that are used so that they would get to know it in more meaningful ways. As teachers planned, they considered many things that guided them in making their professional decisions, all of which reflect their MfT. Among the things on which they based their decisions in planning for instruction were:

- How is mathematics being used in the world? These teachers wanted to represent mathematics "honestly."
- What might motivate the students? They considered the affective aspects in relation to the content they needed to teach.
- How will the futures of their students be best served? These are university bound students, many of whom are going into sciences, engineering and business, and so the teachers know that they will need technical skills and proficiency with mathematics subject matter and ability to think critically. To serve them in a right way they make these adjustments and may depart from the ways a standard textbook treats the topic in order to place higher cognitive demands on their students in their instructional approach and also in how they evaluate them (for example, Andrew will not give full marks if the student does not make use of "restrictions").
- How will the topic be developed? What tasks will be used and to what ends – how will that open up the main ideas of the topic for the student?
- How the textbook will be used, if at all. A judgment call is made about critical/cautious consumption of standard textbook/resource materials and adjustments are made where it is felt that this will serve the students better.

In this chapter, I described teachers' practice of preparing for instruction: This group of teachers worked together and used one another as a resource, to create and fine-tune a selection of tasks that would be offered to students in order for them to learn about solving radical equations. They set the learning tasks carefully, with the intent to exemplify something new with each one of them. They discussed in depth what is involved in each task and how that is different from what the other tasks revealed. The work of teaching is being seen as opening up the mathematical content for understanding. The work of creating tasks and/or examples *strategically* involves a great deal of knowledge of both mathematics and of how it can be presented in instruction for the purpose of learning. Usually, this is the work of didacticians (as they are called in Europe, Quebec, and elsewhere – but this specific kind of profession is not so commonly known in the rest of Canada and the Unites States). This group of teachers tackled this without hesitation and grappled with how to effectively target all of the nuances that solving radical equations can offer.

To sum up this part, we saw how Andrew, Gabrielle and Steve strived to empower students mathematically. They wanted their students to ask themselves, "Does an answer to this equation even exist in the field of real numbers?" Next, they want them to inquire about the domain in which the given equation is defined, to perform an analysis of that and to make conclusions about the range of values from which the candidate solution might come. In other words, students need to know much more than the fact that an answer is *right*. They also need to be able to determine *when* it is not right and *why*. Andrew thought that this is important in mathematics. Gabrielle and Steve had not considered this approach to teaching the topic before in their practice. However, as they worked through the tasks together and considered what sorts of thinking students need to do to solve the various radical equations, they too accepted this plan (presented as Approach B in Figure 4).

5.4. Addressing the research questions

What can be gleaned about the professional work of mathematics teachers as they plan for instruction and how might this further the inquiry into the nature of their MfT?

Through the collaborative process of planning for instruction, the teachers faced an opportunity for qualitative change in their understanding of the concept that they chose to teach in the research lesson. Together they decided to use a different teaching approach, proposed by Andrew but novel for Gabrielle and Steve, to achieve the goal of developing mathematical thinking in their students with respect to this topic. They were self-motivated to explore the special class of examples that had the effect of deepening their knowledge and understanding of the topic. Not all examples that were created and solved were later used in instruction. The teachers explored the example space out of their own drive to learn about the affordances of this topic.

Here I offer some partial answers to the research questions of this thesis.

(a) What can be gained by the teachers who participate in the school-based lesson study initiative?

The collaborative work in the stage of preparing for instruction offered the teachers an opportunity to discuss the subject matter of the lesson at depth. It allowed them to analyse alternative teaching approaches to the topic and gain an extended experience with the teaching approaches. The choice of teaching approach is rooted in the teacher's philosophical orientations, which encompass the teacher's values, attitudes and beliefs towards mathematics. These teachers were able and willing to come out of their comfort zones and implement a new teaching approach. As a result of their reasoning about the benefits of modifying the teaching approach to an old topic, and keeping in mind the goal for developing student mathematical thinking, a shift in the stance towards school mathematics occurred as they became aware that they teach mathematics to students for life. In addition, the teachers' discussion around the different teaching approaches prompted an increased attention to student cognition.

(b) What were the factors influencing the development of teaching practice in this lesson study setting? This might include visible and invisible features, such as beliefs and attitudes.

It seems that the most influential factor in this phase of lesson study was the role of the master teacher, Andrew, acting as a mentor.

(c) What is the nature of the mathematics for teaching that has emerged?

Three distinct features of the nature of MfT emerged from this stage of teacher engagement. First, there was an increased awareness to mathematical connections and development of the concept in instruction, including taking into account students' prior knowledge. Second, there was an increased interest to present the topic in a way that will foster conceptual understanding by the students. Third, there was an increased sensitivity to critically evaluate teaching resources and materials.

The above is a presentation of the results flowing out of the preparing for instruction teacher collaborative sessions that partially answer the research questions of this thesis. However, besides these results there were other gains in teacher learning which were not directly pursued by the research questions but were nevertheless identified as important because they show shared understandings that emerged amongst this group of teachers, which could impact their practice. This is linked to the teachers having a common goal to create a culture of mathematical thinking in their classrooms but through this process came to a shared plan of how this could be achieved in relation to the topic. The major gain from the teachers' work on planning for instruction seems to be a shift in teacher's mind to teach the topic relationally and reject teaching it instrumentally. As a team they came to an awareness of the value behind understanding a concept relationally.

> The kind of learning which leads to instrumental mathematics consists of the learning of an increasing number of fixed plans, by which pupils can find their way from particular starting points (the data) to required finishing points (the answers to the questions). […] There is no awareness of the overall relationship between successive stages, and the final goal [...] and the learner is dependent on outside guidance for learning each new 'way to get there'. In contrast, learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point. ~ (Skemp, 1976, p 22)

Students often demand a "recipe" from their teachers, or they exhibit a kind of attitude that implies, more or less overtly, that they prefer to be "told how to do it," and that they would rather not understand *why* it is so. The shared understanding that emerged was that the teacher needs to nurture the curiosity and the desire to know the *why* behind every mathematical result, concept or procedure.

As a final point, the two teachers who will be enacting the lesson accepted the risk that comes with performing in front of the colleague-observers in delivering the lesson that was collectively designed, taking the responsibility for representing it authentically. They took the risk, they were ready to test out their lesson in the classroom and learn from the experience ahead, without being fully aware of the challenges that await them in face of concerns and unanswered questions. In addition, an unanticipated need that surfaced during the analysis of the data was to extend the Four-Component Framework to encompass a fifth component that was identified as the philosophical component.

Chapter 6. First Round: Post-Lesson Reflection Following Andrew's Enactment

6.1. Overview

Lesson study process assumes that teacher learning does not stop with the stage of planning for instruction, but continues to develop through the processes of implementation of the research lesson and the collective post-lesson reflection, which is meant to consolidate the teachers' learning. During the lesson enactment, the teachers in the team are in the role of observers of students' learning processes as resulting from the teaching performance and lesson design. There is an underlying assumption that teachers learn from watching other teachers teach, especially in a situation where the team of teachers share the goals and the instructional design of the lesson. This lesson study cycle featured two implementations of the research lesson on solving radical equations. Andrew taught the lesson first in his own class, which was then followed by the team's discussion. The focus of this chapter is to follow the MfT that emerged in this phase of the lesson study cycle through the team's response to the lesson enactment. Chapter 7 will be a parallel analysis of the second enactment of the lesson taught by Gabrielle to her own students and collectively discussed, as part of the final phase of this lesson study cycle.

In this multidimensional and comprehensive learning situation, teacher observers had the opportunity to test out in practice the lesson that they designed together, which features a different teaching approach than the one that Steve and Gabrielle were used to in their own past practice. The situation was also comprehensive because all the important elements of teaching practice were involved through the stages of planning, implementing, reflecting and evaluating the research lesson and its effects in the classroom.

The analysis of the themes as well as the organization of this section accords with the 5-Component Framework, my contribution to the field, a new framework that I made by extending Selter's four-components of teachers' background knowledge to encompass the fifth component, the philosophical, which was identified as important in the thematic analysis of both pre-lesson and post-lesson discussions. Within the components of this extended framework, which I named in the previous section (in 5.2.5), I use Van Es and Sherin's noticing framework as a tool for detecting the themes that surfaced during the post lesson discussions, as described in the methodology. As mentioned in Section 4.8.4, this three-part noticing framework consists of a) attending to noteworthy classroom events, b) using knowledge to reason about classroom events and c) making connections between specific classroom events and broader principles of teaching and learning. It can be assumed that the events under discussion were noticed and considered noteworthy by the teachers since they surfaced in the post-lesson discussion and were discussed at some length. The three parts of the framework might not occur for a single classroom event and, for many events it might be hard to separate the three parts as described in the interpretation of Jacobs, Lamb, Philipp and Schappelle (2010) of the framework. Their interpretation states that noticing, reasoning and connecting are interrelated and can occur simultaneously within a classroom event. Stated more simply, I use the above mentioned framework as a filtering device. The application of the noticing framework ensures that a classroom event is identified as important if the teachers not only mentioned it, but also reasoned about it and connected it to their practice or philosophy of education.

6.2. Analysis of the themes that emerged from the phase of reflecting on instruction

Within each of the components I first offer what I noticed in the enacted lesson, and then I describe and analyse what the participants noticed and discussed in the post lesson reflection about the classroom events. I had the time and the opportunity to review notes, reflect, and watch video - this was an advantage over the other teachers in noticing. However, what really matters, is what the teachers noticed and brought to the discussion. These were the events that were most noticeable from their point of view after the experience of observing the lesson. Finally, I offer my remarks about the teachers' post-lesson discussion in relation to each of the five components.

In separating the themes from the teachers' post lesson discussions according to the 5-Component MfT Framework, often I was faced with a dilemma about which component a theme best fits into, due to the frequent incidence of multiple components within a single theme. This was particularly difficult in separating the themes into mathematical, pedagogical and didactical components because in practice these are very much intertwined. In such cases, I classified the theme based on the component I determined to be most salient in that theme.

The results are organized into five sections following the 5-Component Framework, each of which has three parts: discussion about what I noticed in the lesson, presentation and analysis of the themes that emerged in the teachers' post-lesson discussion, and my remarks about the teachers' post-lesson discussion.

I start with the mathematical component because the teachers referred to the mathematical tasks that were used in instruction when they discussed various events that were then classified under the other components. This is followed by the presentation of themes that emerged in the post-lesson discussion in the categories of didactical, psychological, pedagogical and philosophical, in this order.

6.2.1. Mathematical component

What I noticed about the enacted lesson

With respect to how Andrew enacted mathematics in the classroom, he used rigour and precision with the mathematical idea of how extraneous roots can seemingly appear out of nowhere when, in fact, there is logic and reasoning behind this. He required the students to use mathematical notation with precision and he communicated the meaning of notation when this was needed. For example, after the student at the board analyzed the possible interval for the solution space, Andrew prompted her to describe her result as $x \in [-11, 9]$ briefly noting that the square bracket means that the endpoints are included (closed interval).

Andrew's professional knowledge was prominent in his instructional decision-making, which appeared to be based on his ongoing analysis of student learning and in his way of provoking critical thinking in students. Drawing on his vast knowledge of mathematics for teaching, Andrew used his simple, yet powerful, pivotal explanation to convey the reasoning behind the manifestation of extraneous roots. With the intention to prevent possible erroneous thinking, he used his knowledge to create skillfully chosen examples on the spot (such as the fact that $-x-4$ is not necessarily negative).

Andrew used four examples of radical equations in his lesson enactment. The teacher discussion was mainly concerned with the two spontaneous tasks and the reasoning behind the teacher's decision to use them. These two spontaneous examples, together with the two planned tasks make up an example space which is the subject of my analysis in what follows next. The choice of examples used in instruction defines to a great extent what is made available for the students to learn, and therefore speaks a great deal about the teacher's MfT.

Analysis of tasks used in instruction: Considering the domains and ranges of associated functions

Zodik and Zaslavsky (2008) studied in-service teachers' knowledge and use of examples in teaching mathematics. They pointed out that, although there are only a small number of studies that focus on teachers' choice and treatment of examples, a better understanding of this craft could lead to improvements in teacher education programs. They examined the underlying considerations that are employed by teachers in making their choices about the use of examples in instruction. They found that, to a large extent, the choice and generation of examples relied on sound mathematical knowledge of the relevant topics.

During the implementation of the research lesson, four tasks were used in the instruction, two of which were preplanned and two were spontaneous examples that emerged during Andrew's instruction as a preventative measure to circumvent possible erroneous thinking by students. Together, these four examples span a rich example space. They are not just simple exercises of the same type, but rather, each of the tasks reveals something different and new about radical equations and about what it means to solve such equations. The analysis of the tasks that were used in instruction is presented here.

Two of the examples were *pre-planned* examples (part of the lesson plan) and two were *spontaneous* (invented in the act of teaching). Spontaneous examples are done in real time, in response to unexpected classroom situations. They require *knowing to act in the moment* (Mason & Spence, 1999). Andrew responded to a common student difficulty when he created the spontaneous examples. Although it is a well-known challenge for students it was not considered in the planning stage.

Next I analyze all four examples (Figures 8 – 11) that were used in Andrew's lesson enactment, in order to understand what mathematics is being made available for the students by this choice. Solving of equations involves a powerful mathematical idea – that of the equivalence of two expressions. Solving the kinds of equations that arise in school mathematics can be seen as finding intersections of functions with each other. In the case of radical equations, we have expressions in *x* on both sides of the equation, with at least one involving a radical, or it may be the case that one side is simply a constant. Essentially, we could say that the equation can be modelled by equating the two functions and looking for the point(s) of intersection(s), if these exist. Let's take a look at the four tasks that are used in instruction from this perspective, using $f(x)=g(x)$ or $f(x)=k$, whatever may be the case. Looking at the solutions graphically reveals the nature of the roots and sheds light on the example space spanned by these tasks.

Task 1 (preplanned): Solve $\sqrt{x-3}=5-x$.

Using $f(x)=\sqrt{x-3}$ and $g(x)=5-x$ we obtain the following graph.

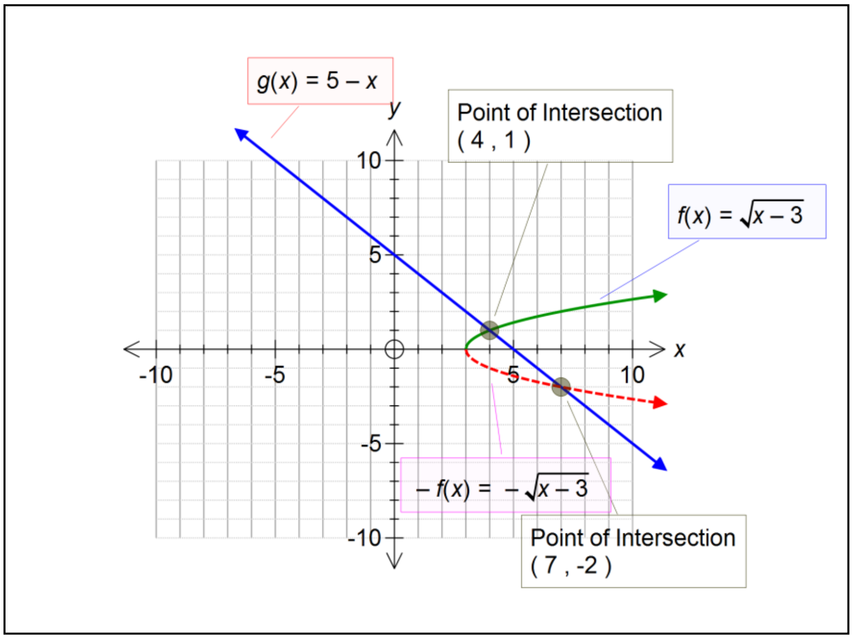


Figure 8: Task 1 interpreted as intersection of associated functions: *x* = 4 is the solution, *x* = 7 is an "extraneous root."

Task 1 was used as the introductory task. In the process of setting the intervals algebraically, the students were meant to discover that the solution had to be in the interval $x \in [3,5]$ where both functions are non-negative. That would lead to the rejecting of x=7 as a possible solution, without the need to test it in the original equation.

Task 2 (spontaneous): Solve $\sqrt{x-3} = -x-5$.

Using $f(x) = \sqrt{x-3}$ and $g(x) = -x-5$ we obtain the following graph.

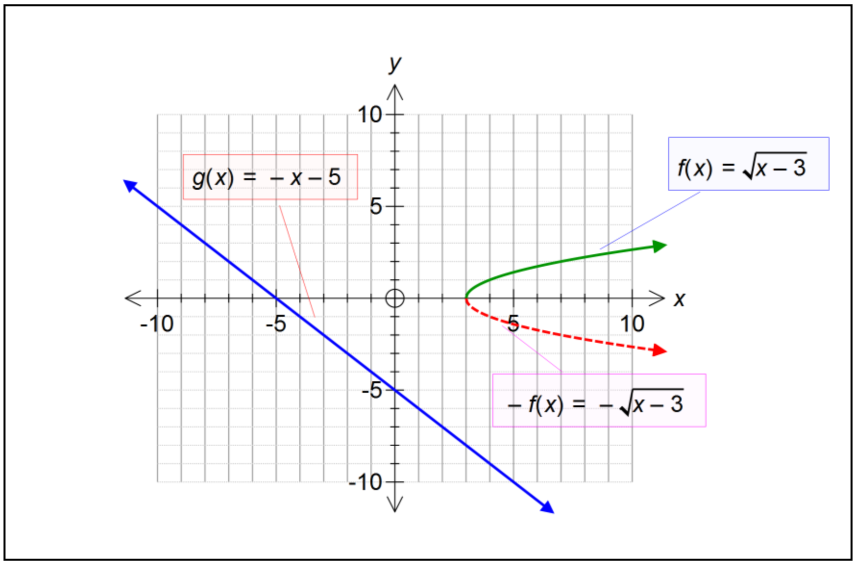


Figure 9: Task 2 interpreted as intersection of associated functions: the two functions do not intersect so the solution is an empty set.

Andrew generated this example on the spot during the lesson. Incidentally, the example revealed two points that the teacher wanted to make, but probably not through the use of a single example. Therefore, another spontaneous example was used right after, in order to distinguish one idea from the other. Firstly, it exemplified that the solution could be an empty set, in which case one does not even need to begin solving the equation. Indeed, in the analysis of permissible values that the students were expected to perform algebraically, it turned out that the potential solutions would lay in mutually exclusive intervals $x \leq -5$ and $x \geq 3$. Secondly, it was meant to exemplify that, even though the right-hand side of the equation seemed to be negative that depends entirely on the value of the variable. It was used to prevent the possibility that students would assume that "adding two negatives gives a negative," which can result from uncritically transferring a property from an arithmetic to algebraic situation. This latter feature was then separately distinguished in the next example that was generated by Andrew in his enactment of the research lesson.

Task 3 (spontaneous): Solve $\sqrt{3-x} = -x-4$.

Using $f(x) = \sqrt{3-x}$ and $g(x) = -x-4$ we obtain the following graph.

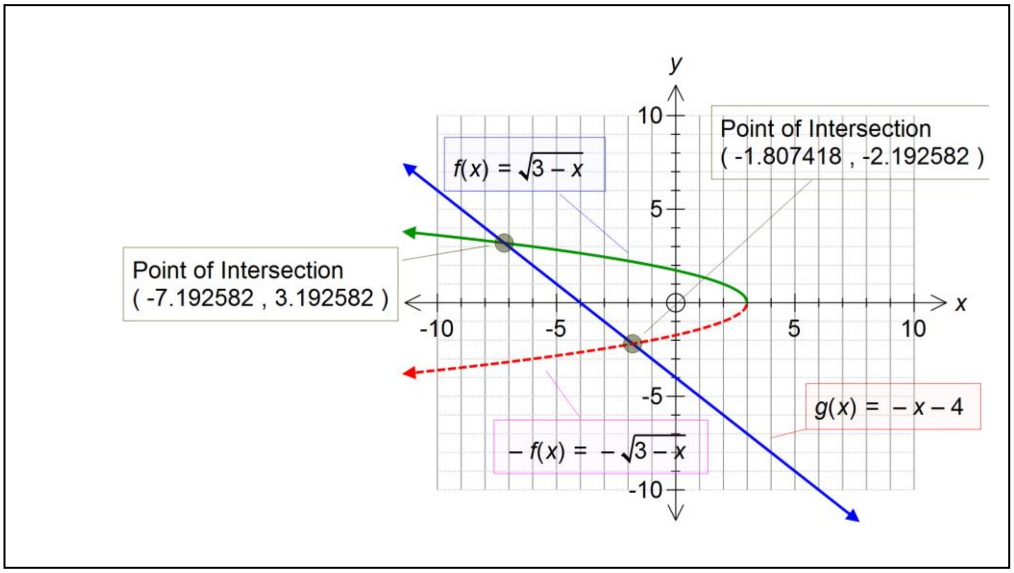


Figure 10: Task 3 interpreted as intersection of associated functions: $x \approx -7.193$ is the solution, $x \approx -1.807$ is the "extraneous root."

In addition to exemplifying that the right-hand side of the equation is not negative everywhere, this equation turned out to have an irrational root. It exemplified the cost-saving benefits of using estimation to determine which of the roots is valid, based on the prior analysis of permissible values of the variable. During the instruction, the teacher did not even use a calculator. Instead, Andrew performed an estimation directly on the quadratic formula, in order to figure out which one of the solutions belonged in the interval of the expected values for the variable $x \in (-\infty, -4]$, thus enacting mathematics in the class and modeling the predictive power of mathematics.

One of the points that was made in the pre-lesson discussion was that the solution is often irrational. Therefore, it is wise to eliminate the need for having to test the "solutions" that turn up when solving the equation "uncritically," whereby one is required to substitute each of these "solutions" into the original equation in order to eliminate any extraneous roots. The argument was that such testing is arduous. Therefore, knowing in advance under which conditions the equation is being solved promoted a mathematical way of thinking.

The last example used in instruction was preplanned. It involved a greater number of steps in the process of equation solving. Depending on the solution process that was taken with this example (squaring the equation as is, or squaring the equivalent equation $\sqrt{x+11} = 2 + \sqrt{9-x}$ as envisioned in the lesson plan, Appendix D), the permissible interval of values turns out to be either $x \in [-1, 9]$ or $x \in [1, 9]$ respectively, requiring that we accept the correct root $x = 5$ and reject the extraneous root $x = -7$.

Task 4 (preplanned): Solve $\sqrt{x+11} - \sqrt{9-x} = 2$.

Using $f(x) = \sqrt{x+11} - \sqrt{9-x}$ and $g(x) = 2$ we obtain the following graph.

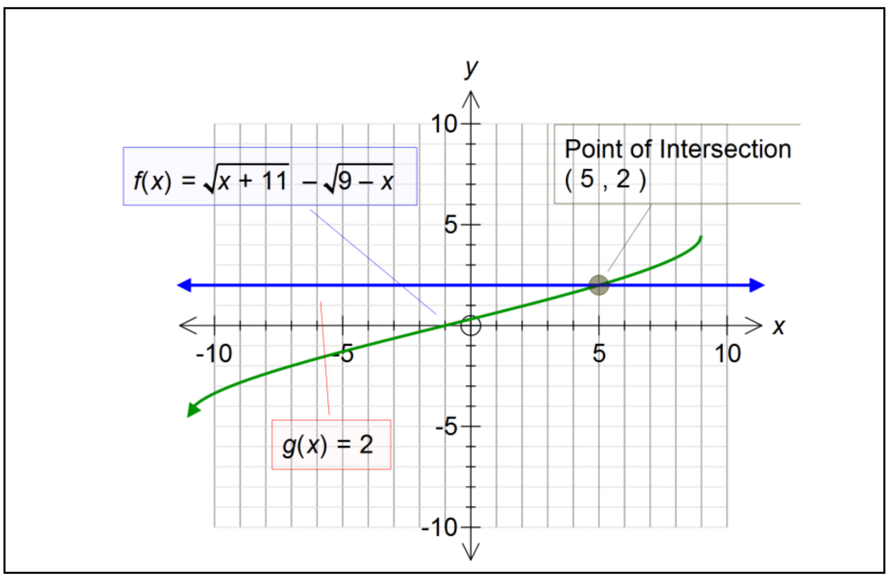


Figure 11: Task 4 interpreted as the intersection of associated functions: *x* = 5 is the solution.

Interestingly, the perspective of looking at the instructional tasks as intersections of graphs of functions in a coordinate plane was not brought up during the stage of planning for instruction because the teachers worked primarily with the algebraic representations of the expressions involved in these equations. It is likely that with a greater variety of experience that the teachers have with this topic, this graphical representation would have come up too as part of considering the mathematical affordances of these the tasks. The domains and ranges of the associated functions, when the process of solving equations is considered from a graphical point of view, were translated into the idea of “limitations” or “restrictions” on the variable x and was therefore considered from an algebraic point of view. This algebraic view uses up more processing power than the graphic view. However, it does not rely on having to think of functions as objects, which is developmentally appropriate given that the students have not yet been exposed to the study of transformations of functions and that the objective of this lesson was of an algebraic nature, (i.e., solving of a certain type of equations). In addition, it is worth noticing that the two preplanned examples involve quadratic equations whose factors are whole numbers. This indicates that they were probably taken from a textbook, whereas the spontaneous examples are of a different kind, which is not commonly represented in standard student textbooks.

Teachers’ discussion

The main theme within the mathematical component of the teachers’ discussion revolved around dealing with students’ misconceptions both from the perspective of predicting and preventing them during the instruction as well as remediating them when these actually occur.

Dealing with students’ misconceptions

The theme of student’s misconceptions appeared throughout the post-lesson discussion. Based on an analysis of the post-lesson transcript, it can be divided into two sub-sections. The first sub-section deals with predicting a potential student misconception based on a teacher’s experience, and consequently adapting instruction in order to prevent it. What was brought up in the discussion was that Andrew departed from the lesson plan to show the students that the value of –x is not always a negative number, but that it depends on the value of x, which plays a part in setting the restrictions. The second sub-section looks at several events in which teachers noticed that a student made an error or demonstrated a misconception. There were two events that had to do with squaring a binomial, where in one case the student omitted the middle term and in another case wrote it last. There was also one event in which the student made a correct conclusion that a number she got as a result of solving an equation was not a solution (it was an extraneous root), but her confusion stemmed from not knowing why this occurred as she could not find any restrictions that would exclude this result from the possible solution set.

- **Predicting and preventing potential errors: Is –x – 5 negative?**

During his teaching, Andrew included/added/changed examples on the spot, making an in-the-moment teaching decision that was not pre-planned and was not even considered during the pre-lesson discussion. Although Andrew followed his lesson plan (Appendix D), including all of the examples that were set out in it, he had to make in-the-moment changes to his lesson. He did this to prevent a potential student error, which he believed students might appropriate or assume on the basis of the types of preplanned examples. Based on his previous experience, he knew that students were likely to make this type of error.

> A: I had to include something I didn't mention in the lesson plan, that equation that was at the beginning […] After the first example, I had to include, I can't remember exactly what it was...
>
> N: But it was a rational equation, a simple rational one, a student came up with.
>
> A: It was a radical equation, but it doesn't matter. I said just that I included something because it seemed it was important to highlight.
>
> N: Oh yes, after that you went, you posed this one, x minus 3…
>
> A: Yes that, with the negative.
>
> N: Yes and then you changed it slightly to actually make it work. So this one was impossible and this one was possible.
>
> A: Yes so these two I put although they weren’t in the lesson plan, because, seeing, I actually later realized I should, that when they see 2 minuses, they think it is negative all the time.

Andrew was referring to the use of examples $\sqrt{x-3} = -x-5$ and $\sqrt{3-x} = -x-4$, the first of which has no solution, because there is no common interval for the value of *x* (the domain of one has to overlap with the interval over which the second has a positive range) and the second one has a solution. The point he was making is that the right-hand side of these equations could be positive and that this depends on the value of the variable. In his experience, students sometimes falsely conclude that $-x$ is negative. This could lead them to think that these equations would have no solution because square root, (i.e., left-hand side of the equation) is, by definition, non-negative. This is an example of how the quality of teacher's understanding of the subject matter bears upon what is available for the students to learn.

- **Dealing with errors that occur: Forgetting to set a limitation on the range**

A student became confused about a problem she was working on at the board, and afterwards the teacher team reasoned about what happened during that event. Initially, the teachers did not completely agree about what had happened. This shows that the reasoning about one classroom event might not be the same for each teacher.

> A: That same girl came later, but for that last question, and said that she found no limitations. She said there wasn't a limitation.
>
> N: Yeah she was sure that negative 7 is also a solution.
>
> A: She knew that it is not a solution, but didn't find any limitation for that, so she had no bases for excluding it based on limitations.
>
> N: Because it was included in the initial interval.
>
> A: No...
>
> N: Yeah the initial interval was between negative 11 and 9 but then it shrunk.
>
> A: The reason why she said, "I actually didn't find limitations," that was this page [looking at the lesson plan]. She said, she squared it, and had 2 roots multiplying, and said, "No limitations." I said, "can that be negative?" She said, "Ah that is the problem, because that could also be negative," and then she put the interval on the board [...] it came out that x has to be greater than negative 1, so negative 7 cannot be a solution. She said, "I forgot."

The teachers noticed that one student was confused about the problem on the board, and wondered how Andrew resolved her confusion. The student knew that -7 cannot be a solution. However, she had not found any limitations. Andrew described what the student's thinking was and how she eventually understood why her process of setting limitations did not yield a correct conclusion that – 7 cannot be the solution of the given equation. Andrew was reasoning about the student's thinking and the way that led her to misapply the newly learned process of setting restrictions on the range of the associated function, and he explained this event to other teachers. Andrew was committed to assist an individual student in order to find the error that she herself was concerned about, and this is how the learning for this student occurred.

- **Dealing with errors that occur: Expanding a binomial**

The teachers also noticed another instance of a misconception by a student which occurred during the lesson. They spent some time discussing and debating about what had happened and why. During the lesson, a student that was solving an equation at the board was expanding $(a-b)^2$ and forgot to include the middle term $-2ab$. The team of teachers reasoned about why this error had occurred, because the students had already been taught this concept. Finally, it was concluded that the student must have 'forgotten' and that this is a common occurrence with this particular concept. Here I include an abbreviated excerpt from the post-lesson discussion transcript, in order to demonstrate this point.

> A: […] I just wanted them to use what they should already know. But when I [..] I didn't even think that they don't know A plus B squared. Again, they can multiply. But they have done it so many times, this table of multiplication. […]
>
> N: It must be something wrong with how we're teaching it, because I was really surprised that these really bright kids didn't know this.
>
> [….]
>
> A: So I know that they forget, they will forget.
>
> N: But that's a problem even at the university level, I've heard a math prof say how in first year calculus, they have a problem with that, about 15 percent of university students who take math in University have this exact same problem.
>
> [….]
>
> A: They will forget something that I am teaching now also. Forgetting is a natural process.

The teachers were reasoning as to the cause of this error. They were using their experience with this common student misconception. Natasa thought that the reason for this problem is of a didactical nature because it relates to the way in which the concept is introduced while Andrew attributed it the natural process of forgetting when knowledge is not in use for a longer period of time.

On the other hand, before and during the lesson, Andrew seems to have assumed that the students knew how to expand this algebraic identity. He attributed this misconception to the students 'forgetting' the steps that are involved in this expansion. This situation shows that Andrew and Natasa have a different opinion about the reason behind this widespread error. Natasa said that, if a student doesn't know something or if they make an error in something they should know, they simply haven't been taught properly. However, Andrew's view is that students can forget certain things after they had learned them, even if those things were previously familiar to them. This event is discussed later under the theme of "Condensing Mathematical Processes," because this was another theme that arose within the same event that was discussed by the team.

The teachers discussed another event that occurred during the lesson. Gabrielle brought up that a student who was solving an equation at the board was going to square both sides of the equation, but stopped herself before she made the error and then solved it correctly. Even though in the end the student did not make a mistake, it was interesting that the teachers noticed this event and discussed it for such a significant length of time. The teachers were reasoning extensively about the student's thinking.

> G: The other one that was interesting was when they were, you know when they moved that two, they had $\sqrt{x+3}$, I can't remember what it was, $-\sqrt{4-x}=2$. And they moved that to the side and they got $\sqrt{x+3}=2+\sqrt{4-x}$ and then she got to the point when she was going to square both sides. She was inclined to do that. But then she stopped, and pondered.
>
> A: I didn't intervene. Maybe she stopped, but she did, at the moment when she wrote that, I thought she forgot the middle term. But she didn't even stop, I didn't even say a word, she continued and filled everything. Did you notice that?
>
> G: Yes.

N: You did not have to intervene right?

A: No.

N: No I didn't see any problem with that.

A: I was thinking that she would forget the middle term.

N: But she put it last, right?

G: Yes, that's right. I was anticipating that she would do that. Because that's the way it looked when she started. But no, she, it became a non-issue because she figured it out. Interesting.

Both Andrew and Gabrielle interpreted the student's thinking in the same way. They thought that the student would forget the middle term, which she did not (she just wrote it last, after writing the two square terms first). The fact that both teachers had the same impression and had been expecting the same error suggests that they were making a similar connection between that classroom event and a broader principle of learning, which is to let the student hesitate and correct her own error, without the interference of the teacher.

Remarks about the teacher discussion

In the post lesson discussion, Andrew shared about his use of the two unplanned examples of radical equations and the reasons behind why he thought they were needed, in order to prevent the common erroneous thinking by the students. It is interesting that this point was not further discussed by the teachers. Although the teachers did not seem to be surprised by Andrew's move to use the spontaneous examples, they still thought it was a good idea and decided to use them in the second enactment too. The mathematical component in the discussion was rather modestly represented. It is my contention that this was because of two reasons. Firstly, the mathematics within the lesson was presented to students without flaws and the student learning progressed according to the plan. Secondly, the teachers were confident with their own knowledge about this topic.

6.2.2. Didactical component in the post-lesson discussion

What I noticed about the enacted lesson

Andrew prepared the terrain for the new lesson by placing an emphasis on the tools that are needed for thinking about the new concept, therefore reminding the students about the prior knowledge that was needed for the progress of their understanding of the new concept. He highlighted the important idea that a square root is always positive. This was a mathematical key point that was required for the development of student understanding in the intended lesson.

In this way, Andrew situated the new knowledge to be learned in the context of what students already knew. After having students generate the examples of equations that they already studied, he asked them what the term “radical” meant (both outside and inside of mathematics). He did this to introduce the new topic for the lesson. He asked the students to consider the values for which the equations they generated would be defined, thus directing their attention to the idea of restrictions on the domains of the corresponding functions. By situating the learning in this way, Andrew communicated that mathematics is a coherent and connected body of knowledge.

Often, the work of mathematics teachers is described as involving *unpacking* of mathematical concepts and procedures (Davis & Renert, 2014). However, in Andrew’s teaching practice, there is also a case in which he insists that the students use packed knowledge when expanding the square of a binomial. He considers that students should be able to utilize the already established knowledge automatically and fluently, in order to free up their mental capacity for acquiring the new concept. Davis and Renert point out: “An important component of research mathematician’s work is to collect their thinking into compact formulations – that is, packing – it is the teachers’ task to perform the reverse operation”(p.24). As is shown in Andrew’s practice, both of these processes are represented and deliberately fostered. When concepts are being *developed,* it is the “unpacking” that takes precedence and, when already developed concepts are being *applied,* it is the “packing” that is encouraged.

Teachers' discussion

Condensing mathematical processes: Squaring a binomial

At a couple of points in the post-lesson discussion, teachers noticed that Andrew wanted his students to use a mathematical shortcut in the process of squaring a binomial, rather than allowing the student to write out the long form. One of the events that was discussed was about a student who tried expanding $(a-b)^2$ and Andrew insisted that the student should not write out two separate brackets. The student forgot to write the middle term and came up with an incorrect solution. Steve's reasoning was that, if the student had been allowed to write out two separate brackets, he would not have made the mistake of omitting the middle term. Andrew provided the reasoning behind his choice to encourage the student not to write out the long form of multiplying two binomials and carrying out the multiplication according to the distributive property.

> G: Well the embarrassing moment that came out with A minus B squared, uh, the A squared minus B squared, the typical moment right?
>
> S: It's pretty classic, how come you wouldn't let him write out two separate brackets?
>
> A: I wanted them to know like, for example multiplication, 3 times 4. 4, 4, and 4 is 12. It is the same as multiplying. We know that 3 times 4, so we have to know. Sometimes you have really long expressions, so you simply need to know a faster way than multiplying them out the long way. I just wanted them to use what they should already know.

Andrew explained that there was a slower way and a more condensed, economical way, which was true for many other mathematical concepts. He used multiplication in an analogy, saying that we could add all of the numbers separately in order to solve a multiplication problem, but this would be too time-consuming. Therefore, we need to know the shortcut of multiplication. Andrew teaches students to curtail mental processes, which is one of the features of "acting mathematically," as identified by Krutetskii (as quoted in Watson, 2011). Using the same reasoning, in Andrew's opinion, students should also be able to expand $(a-b)^2$ without writing out both brackets, in order to be more efficient. In this instance, teachers were reasoning about what happened in this situation and how the student error could have been prevented. In his own classroom, Steve would have encouraged the students to write out both brackets so that they would not make the error of missing the middle term.

> S: Because that's why I did this, so they don't screw it up like I say, write it out like this and then you won't miss that middle bit. Which is probably why he was doing it I think, so I told him to [...] because I reckon other times, otherwise you just get a lot of that don't you, and they miss a little bit.

Andrew justified his choice about how he handled the situation. Once mathematical concepts have been developed through lessons and students were familiar with them, they needed to condense the mathematical processes. Andrew was tying this to his knowledge and style of pedagogy, and to principles of teaching and learning of mathematics. Andrew expected his students to use these previously developed understandings and skills in order to promote mathematical efficiency and thinking.

This led to the second instance that arose during this post-lesson discussion, which was about learning new knowledge versus applying already learned skills, in the context of learning mathematics. This was related to the previous event and to using the skills that were developed in the study of the concept more fluently. Andrew explained that for practical reasons, students need to become skilled with a concept once it has been developed in the class and students have understood it. This is important both for the purpose of efficiency and to allow time for students to focus on the learning of the new concept. In other words, his argument is about freeing up the mental capacity to do more intellectual work.

> A: We are doing trigonometry in grade 12 now. And I developed with them all possible formulas. Not just those that are in the textbook, because you need that for Calculus. And they are saying, "Give us the formula sheet for the test." No sheet, you have to know that (the formulae)! We developed every formula. I didn't just give them the formula.

> A: If they don't know, develop it. Then it forces them to understand. If you have to every time go to the formula sheet, and choose which one to use, you don't promote mathematical thinking at all. Students need to know exactly which one to use when.

> A: But as I said, for trigonometry, it is just because of practical reasons. You are faster. And in many situations on the test, if you have to think about every part, extra, and think, is that this way, you have to be skilled. Do things you already know fast, and think about new things.

Gabrielle was in agreement with Andrew around this point and further supported it with her own experience of teaching the solving of linear equations in Grade 9. In the following excerpt she refers to the shortcut of transposing the terms in process of equation solving.

> G: There are a few things that you do in grade 9, that when you get to grade 10, like you've got to do that step in your head. I don't want to see you write down minus 2 minus 2 when you're solving an equation. That step has got to be in your head.

Scaffolding student learning: "Could we have known it before?"

In the discussion, Gabrielle brought up how Andrew skillfully guided the students towards what they should be paying attention to. By doing so he scaffolded their understanding of the main idea of the lesson, which was the reason behind the appearance of the extraneous roots.

> G: This is what you've encouraged right from the get go. Could we anticipate that this was not the right root? Could we have known that there was a question right before we started doing any of the mathematics? And you can do that with absolute value too because the right hand side cannot be negative. It cannot. So, they're already looking at that before you even start solving it, and then going back to check that they're anticipating, that one of the roots or both of the roots may be false, may be extraneous. So that's good.

Remarks about the teacher discussion

In relation to the mathematical and didactical components, the most prominent example was Andrew's use of his pivotal explanation of how, why and when the extraneous roots arise in the process of equation solving. With the pivotal explanation, he revealed what the mathematics was doing and how the student can use this knowledge to set the restrictions on the variable and, therefore, recognize the correct solution, without having to test it.

After this piece of knowledge was developed and consolidated, Andrew expected the students to use it with rigour. When they submitted their solutions to the first example, $\sqrt{x-3}=5-x$, only one student used the idea of setting the restrictions, while others used the "testing" method. Consistent with his view, which was expressed in the pre-lesson discussion, Andrew rewarded the student who used the new knowledge with a bonus mark. He encouraged the others to do so as well. This was fruitful because, in the second and more difficult example, $\sqrt{x+11}-\sqrt{9-x}=2$ more students who submitted their notebooks demonstrated their use of this new knowledge. It is interesting that none of the teachers in the team discussed this didactical approach that Andrew uses for his ongoing assessment of students learning of the concepts and which allows him to know exactly where each student stands with respect to the new knowledge to be learned as it is being learned during the lesson progression.

6.2.3. Psychological component in the post-lesson discussion

What I noticed about the enacted lesson

Of interest in Andrew's lesson implementation was how he made the mathematics available to the students. He engaged the students with the content and used the element of surprise to drive the investigation into what the mathematics was doing. That is how he held their attention throughout the lesson. The highlight of the lesson in my opinion was the moment when Andrew scaffolded the students' understanding from the seed of the idea to look at the range of the associated function, to a fully-fledged concept that he wanted to teach the students. The following discussion was about how one can know that 7 is not the solution of $\sqrt{x-3}=5-x$ without having to test.

A: So, we got the solution. Are you satisfied?

Ss: No.

A: Why not?

Ss: Because you can't have 7 as a solution.

A: Why can't we have 7 as a solution?

S: Because if you plug in 7 for x, then it doesn't, 7 doesn’t work, it won't equal the same on the right side.

A: Ok, that is obviously correct. But could we avoid that before coming to the solution? [pause] Yes?

S: You can set restrictions that whatever is under the square root has to be greater than 0.

A: Please come to the board and to show me the restrictions. What kind of restrictions do you mean?

Student comes to the board and sets the restriction on the domain writing that $x-3\geq 0$ *and concluding that x has to be greater than or equal to 3.*

A: Ok, so your restriction is x greater than 3. So what do you do? It is greater than 3. Both are greater than 3!

Students’ laughter.

A: So does it mean 7 <u>is</u> the solution? [pause 10 sec] But we don't know it is the solution. We checked, it is not, but maybe we are wrong, maybe it is but we just don't know how to substitute into the first equation. [pause] Could we know before we come to this point, could we know that it is really not possible for 7 to be the solution? [pause 7 sec] We could! And there was the place where we should know that, but we just went over it, like who cares. But you see, we have to care because otherwise… Yes?

S: Cuz when you're, you can’t have x greater than x or smaller than 5.

A: Come here, because I''m not sure what you are thinking.

[student comes to board]

A: What did you say? 5 minus x [...] could be equal. Or? Ok so, or x what?

Teacher is scaffolding the student’s work at the board.

S: [inaudible]

A: x, so we have

S: [inaudible]

A: You see, we have 5 minus x

S: Yeah when we started this one,

A: So that is not the case ok? why did you put that. Why did you put less than 0?

S: Yeah well if [...], then these will be, i don't know,..

A: So that is not a good thing [...] Ok we can solve it but I'm interested in the reasoning, why we cannot put, why do we have to put that?

S: [inaudible]

The student provides the correct reasoning about the requirement to set the restriction on the range of values that the radical can take ($5-x\geq 0$ *) which reveals that x cannot equal to 7.*

A: Bravo! Now that is good, that is two bonus marks, bravo! You are the only student in the class who understood why we, because, if that you see, this is root. And root, under the root we said has to be positive. So what is the result of this root? Positive. And this, what does this then have to be? Something that is positive. Something that is positive cannot be equal negative. If we say, does 3 equal negative 3? No. But square it, 9 equals 9, it's equal. You see? That something which wasn't possible, if we squared both sides, we make it possible. So of course if you then put into the equation something that cannot be. So before we square, we have to be sure that it is positive because that is positive. We cannot say that is positive and this is negative and if you involve 7, what would happen? 7 [...] 3 is [..] root of 4 is 2, [...] minus 7 is [...] 2 equals negative 2. That is not possible. But when we square it, [..] we didn't care. Who cares! But as you see we have to care.

Andrew praised a student who realized an important point about setting $5-x\geq 0$. He did so in front of the whole class. He also rewarded the student with a bonus mark. Acknowledging students for the mathematical thinking they display in a classroom is one of the consistent pedagogical practices that Andrew has been using throughout his teaching career. After that he consolidated the main point of the lesson and used his pivotal explanation to include the other students in what the student at the board had discovered. The Japanese teachers have a special term for this phenomenon; they call it "children's twitters — mumbled nuggets of inchoate thoughts that teachers can mold into the fully formed concept they are trying to teach" (Green, 2014).

Teachers' discussion

Praising students

Steve commented on Andrew's praising of a student for her effort in coming to the conclusion that Andrew was trying to emphasize in the lesson. Steve connected what he noticed to a broader principle, which is that praising a student for thinking something through and getting it right is very important. It helps to motivate students and encourages them to think mathematically.

> And then the key idea right, the girl at the front, I don't know her name, she was the first one who said the thing about 5 - x being greater than or equal to 0 and you said, "yes, two bonus marks" and really emphasized that that's the part we want to emphasize today."

Raising the bar

Steve noticed that during the lesson, a couple of students used more knowledge than the others. During the post-lesson discussion, he said that these students were particularly smart. Andrew's response to Steve's comment highlighted a different stance about how mathematical knowledge is developed. He brought up the difference between being smart and being trained. He pointed out that the two students had just come from China, where they had already learned radical equations. Therefore, according to Andrew, one cannot infer how smart someone is from this kind of situation; one can only infer how trained or experienced someone is.

> S: Both of those Chinese girls are smart.
>
> A: I wouldn't say so, because they already finished Math 12 back home, in China. So they know.
>
> S: Ooh right.
>
> A: So they know that. […] And there was a guy […] he finished math 12 during the summer. And I remember I gave these problems for bonus marks, and he was the first to solve them. They said "It is not fair. He is always first, we cannot (compete)." I said, "Wait for 2 weeks and you will see." And after 2 weeks, because they had a challenge, they rose to the challenge, and they also could be sometimes even before him. So it's always good to have someone like these 2 girls. Although I know they finished Math 12, it is good to have them to challenge others in the class. And it's interesting, they

> are not better than student S and they are not better than student E, although they finished. So when you say smart, I never really say, even if they look smart. It is a skill."

Andrew's reasoning about this event was that, having students in the class that are more knowledgeable than the others was a positive thing, even though initially, other students might feel that it was unfair or that they were disadvantaged because they had yet to learn the concept. He believed that in the long run, having more advanced students in the class helped to set high standards and was therefore motivational to students. He used such conditions deliberately to raise the bar.

Next, the teachers discussed another instance of how Andrew used the fact that some students were more knowledgeable in mathematics than others in order to motivate the other students. He accomplished this by praising the students who already knew more than the others, even though he was already aware that they were more knowledgeable.

> G: Yeah they have advantage right?
>
> A: But it is good to have them, they are challenging the class, and I pretend like I don't know that. Whenever they know, I give a bonus mark. And they like it also. Even though they actually know that it is something easy for them."

In his practice, Andrew praises and rewards the students for their effort and for demonstrating their mathematical thinking. In his view, praising a more knowledgeable peer challenges other students too because they must work harder in order to bridge the gap between their performance and that of the more advanced students. This practice that Andrew shared with the rest of the teachers showed how he manages to engage at the same time two groups of students, one that already knows the concept and the other that is in the process of learning it.

Remarks about the teacher discussion

The teachers' post lesson discussion revealed the difference in the way the teachers attribute the role of disposition versus effort to the learning of mathematics. Steve sees the students who know mathematics as being smart whereas Andrew sees the students who know mathematics as having had more training and experience. This stance in teachers' perception of how humans learn has been described in the literature as *growth mindset* versus *fixed mindset* where the former attributes the learning gains to effort and struggle and the later to innate ability (Dweck, 2006).

6.2.4. Pedagogical component in the post-lesson discussion

What I noticed about the lesson

As a result of Andrew's application of a pedagogical principle, the idea of the radical always being positive was affirmed and objectified as a taken-as-shared piece of knowledge. He exposed the naïve thinking of a high achieving and confident student to teach the class the mathematical convention, which the students should have already known. This helped to resolve the students' ignorance regarding this definition in a humorous way.

Next, an introductory problem was given to be solved, first by the students, independently. Andrew let the students discover on their own "that one of the answers is faulty," and then he raised the question of how this can be possible and whether they could have known it before. He fostered independent mathematical thinking in his students by engaging their mathematical curiosity in this way.

Andrew called on the weaker students from the beginning, in order to help them get invested in the inquiry about the appearance of the "faulty solutions" and to ensure that no student was left behind. In addition, he demanded demonstration of knowledge from the students in two ways. Firstly, various students were called up to contribute on the board. Andrew was scaffolding their learning and guided their work when necessary, providing space for the students to make mathematical decisions and to see their implications and deal with the consequences.

In addition, he used his routine in which he expected the rest of the students to submit their notebooks with solutions to the same problem as the one that was being worked on at the board. For submitting a correct solution, but only if it featured the desired process of setting the restrictions beforehand, they were rewarded with a bonus mark. He used this method on a regular basis, as a way to give formative feedback to the students. Moreover, he gave short commentaries regarding these solutions aloud, in order to teach the entire class. For example, he did not accept the solutions that did not present an analysis of the restricted intervals. This is also a type of formative assessment that he uses regularly in his practice. Essentially, in his class, most of the intellectual work and mathematical thinking was done by the students.

Teachers' discussion

Teaching moves intended to engage the students in the learning process

Andrew shared about his commonly used pedagogical practice in which he purposely invites to the board students that he knows might not be able to solve the mathematical problem. He does this in order to keep them alert and attentive throughout the class and also to give them a chance to learn as they try to solve a problem. He uses this pedagogical move throughout the lessons, but he uses it with the weaker students at the beginning of the lesson in order to not leave any students behind. In the following excerpt he shared about this pedagogical principle.

> You probably notice that at the beginning I didn't call students who are good at mathematics. And I usually do that, maybe equally, students who don't know math well. And those who don't know, they learn when they come to the board. And also not allow them to sleep during the class. So I knew that at the beginning some of these students may not be brilliant at math.

During the lesson, Andrew used humor at a particular point as he explained a concept. This event was brought up in the post-lesson discussion. The teachers used this instance to discuss a broader principle, namely that using humor during instruction can make a concept more memorable.

> This place, where you wanted to get out the fact that square root of a number, [main] square root is defined to be positive, that was very nicely done. It was, you know, the pause and there was some humor there, and it took some time, and what do you think and what do you think, and before it, I think it's going to be memorable, it's going to stay with them.

Gabrielle noticed that during the lesson, one student was repeatedly helping another student. She wondered whether it was a regular occurrence in the class and if it was permitted. It seemed that she was comparing her own pedagogy to that of Andrew and perhaps thinking about whether she should use peer-tutoring during her own lesson. This could be an example of one teacher reasoning about another teacher's pedagogy.

> G: […] I noticed student E out of his seat helping student A quite a bit. Is it, do they do that? Are they allowed to work with each other?
>
> A: They are allowed when it is not a test.
>
> G: Yes.
>
> A: So it is good, peer tutoring.
>
> G: No, I agree.

Although Gabrielle accepted the peer-tutoring as a good idea, she wondered whether Andrew encouraged it in his class and purposely included it as part of his pedagogy, or if it just happened spontaneously in his class.

Teacher uses the student's mistake to teach other students an important mathematical convention

The teachers brought up the point that Andrew strategically called upon a particular student in order to expose a common student misconception. He chose a student who was confident and who Andrew knew could handle making a mistake in front of the whole class. According to the lesson plan, the teachers anticipated that the students will not know that the square root is defined as non-negative, and had built the preventative measure to deal with this convention when they planned the lesson. Andrew carefully selected a student who could handle making the error in front of the whole class.

> N: This place, where you wanted to get out the fact that square root of a number, [main] square root is defined to be positive, that was very nicely done […]
>
> G: Yes and you had the right person at the front there, it's good you had student X up there because she can take it.
>
> N: Yes she is so good and yet she got it wrong.

> G: Yeah she was fumbling around there but I mean, when everyone, you're right, when everyone other than Student X got it after most said positive 3 you know then, she's sure enough of herself that she could say, "I screwed up" you know, whereas some of the other ones, that may have been a little too touchy. If they make a mistake in front of everybody, if they're not sure of themselves to start with, it can be devastating. She showed it for the rest of the lesson. So that's great, you had the right person up there.

The pedagogical principle of having the whole class learn from the mistake that was made by a very good student was used purposefully by the teacher in this specific event, which the teachers discussed.

Remarks about the teacher discussion

At the opening of the post-lesson discussion Andrew shared an important point with regard to his decision-making and the pedagogical reasons behind his decisions, which were not further discussed by the group. This point was about ensuring that the weaker students participate in the learning process from the beginning of the lesson.

On the other hand, the teachers discussed at some length the student learning they observed. They noticed how freely students became engaged in peer support. Throughout the class, students had been discussing what was happening on the board and sharing their thinking about the task at hand. Therefore, the teachers were wondering if such conditions fostered the learning. The discussion did not involve any reference as to how Andrew managed to establish this kind of climate in his classroom where students were all focused on the task during their peer to peer interactions instead of wandering off into private conversations.

Andrew's teaching practice exhibited a specific way of student learning, which could be captured as a pedagogical principle. That is, "it is easier for the student to learn through the doing than through listening to the teacher."

6.2.5. Philosophical component in the post-lesson discussion

What I noticed about the enacted lesson

While on several occasions, the teachers expressed their concern about how a student would cope with making a mistake in front of the class, what was not discussed was how Andrew eased the discomfort of the student in such situations. He extended the responsibility for the student not knowing the mathematics to the broader social context. When several students at the board failed to correctly expand $(a-b)^2$, Andrew said:

> Now I would ask you, who was your teacher in Grade 10? […] You see, you are not embarrassing yourself, not knowing that. If you come to university and you don't know what I was teaching you (I would be embarrassed too). So it is not just you. Other people are involved also."

In this excerpt, Andrew made the point that knowledge is gained collaboratively, and he communicated to his students his view that knowledge is built with a shared responsibility between the student and the teacher.

Another example of a teacher's philosophical stance surfacing in the middle of instruction and essentially becoming an implicit content of teaching, is related to the philosophy about mathematics and the learning of mathematics. It can be seen as a normative element in that it projects the teacher's values to students.

> So that is not a good thing […] Ok, we can solve it but **I'm interested in the reasoning**, why we cannot (proceed blindly), why do we have to put (the restrictions on the domain of definition)?

Andrew's care for the students was expressed through his caring for their mathematical thinking and the ways in which mathematics can empower them in all other areas of life.

> We cannot say we don't care for mathematics knowledge. If you don't follow mathematics logic you will make mistakes even in social studies and some other subjects. Because mathematics logic is logic. It may be the clearest logic of all. For example in the real world you can make some conclusions about opinions, but in mathematics opinions (don't count).

Teachers' discussion

Teachers did not discuss the philosophical aspect of the teachers' MfT as part of their observation of Andrew's enactment of the lesson.

Remarks about the teacher discussion

Although there were a variety of opportunities to discuss Andrew's philosophical stance regarding school mathematics, conversations of this component of the MfT did not surface in the team's discussion. This does not mean that the philosophical component was not present in the enactment – it is always implicit in every teaching because it reflects the teacher's attitudes towards teaching, learning, students and towards the subject matter knowledge.

6.3. Addressing the research questions

The post-lesson discussion following the first round of teaching disclosed that the teachers were more focused on the pedagogical and psychological components of the MfT than they were in the pre-lesson discussion. I argue that this shift is a natural consequence given the context of teachers observing another teacher in the act of teaching, when the practical side of teaching becomes exposed and invites the observers to compare the various features of classroom practice with their own. This can be interpreted as the teachers' MfT being extended along these two components the most during this phase.

Here I offer some partial answers to the research questions of this thesis as flowing from the first post-lesson discussion.

(a) What can be gained by the teachers who participate in the school-based lesson study initiative?

Observing another teacher in their practice gives the opportunity to learn about the following: how the teacher relates to the students across various aspects (i.e., how the teacher handles students' mistakes, how the teacher engages the weaker and less motivated students in the learning, how the teacher fosters peer learning), how the teacher establishes the expectations and the classroom climate (i.e., the system of rewards, use of mathematical notation). This opens the possibility for teachers to consider modifying their own practice (i.e., Gabrielle contemplated allowing her students to talk to each other about the learning task during her lesson).

There were both shared understandings and differences that emerged which could also be considered as a fertile ground for teachers' knowledge growth. For example, the teachers agreed that student mistakes are a necessary part of learning, but they did not seem to have a common view on how to handle them. This dialectic contrast itself promotes teachers' learning through the discussion and reasoning about it. They also agreed that allowing peer to peer interactions in class is a good thing and can promote student learning but this pedagogical principle may not be applied in the same way by the teachers. Further differences were identified in the teachers' views about how mathematical learning occurs (fixed and growth mindsets) and about the need for condensing mathematical processes once the concept had already been learned.

(b) What are the factors influencing the development of teaching practices? This might include visible and invisible features, such as beliefs and attitudes.

Observing other teachers in their practice seems to be the most important factor influencing the development of teaching practices. Teaching and learning happen concurrently and this in itself provides the observing teachers with plenty of opportunity for growing their own professional knowledge in all of its dimensions. This means that the participants gain different insights from the same situated learning conditions, and thus learn different things which may not be measurable. The performance piece complements what teachers often only verbally exchange with one another but rarely see with their own eyes, because of the isolated conditions in which they work.

(c) What is the nature of the mathematics for teaching that has emerged?

In this stage of the lesson study cycle an important aspect of MfT was discerned, and that is how all these five components function together in a cohesive way in the practice of teaching. The diagram in Figure 12, which I shall call 5-Component MfT

Venn Diagram, illustrates the notion that these components are not isolated, but in fact interact and overlap. The diagram contains 32 regions capturing the idea that any subset of the 5 components of teacher's knowledge can be present in a given instructional episode. The diagram indicates that each episode discussed by the teachers could be mapped into a region depending on which subset of the five components of MfT it revealed, and some regions in the diagram would be more populated while others could be empty. As mentioned before, it was often the case that the themes emerging from the teachers' discussion of a single classroom event reflected several components at once. The discussion about how the teacher handled remediation of the student error that occurred during the instruction in order to highlight that the square root is defined to be non-negative brought together didactical, pedagogical, psychological and mathematical components to be exhibited all at once. On the other hand, the philosophical component was never discussed even though it was clearly expressed by the teacher in his teaching, and has caught my attention and prompted the extension of the original Four-Component Framework. This is a case of phenomena that occurred but were not touched upon in the discussion.

The diagram prompts a consideration of how much each of the components was represented in the teachers' professional conversations. The density of representation of each component in the teachers' talk could be measured in NVivo through the coding of transcribed text, however this is beyond the scope of this thesis which is more concerned with qualitative than quantitative analysis.

Further on the connection between the diagram presented in Figure 12 and the results of this study, it is important to note that teaching is complex and as such it would be rare to see that any single classroom event involved only one component of teacher's MfT. It could very well be that all components are present all the time, with each being accentuated to a varying degree in any given classroom episode. This implies that it would be possible to analyse each classroom episode in a way that would identify the presence and possibly the degree of each component, and by doing so to pinpoint the exact region of the diagram into which the event could be mapped. As such, the diagram is only meant to communicate about the nature of MfT that has emerged.

However, the nature of working with a theoretical framework is that it allows the researcher to parse and isolate what is being studied in order to better understand the phenomena, which in this case is the teachers' knowledge growth along the five components.

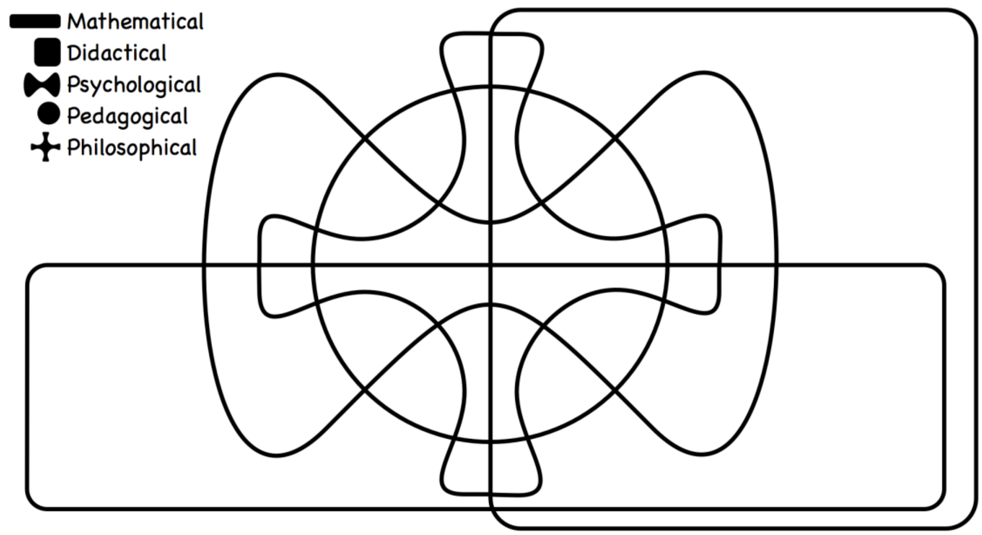


Figure 12: Five-Component MfT Framework as Manifesting in the Secondary Mathematics Teachers' Team at West Coast Academy

In the 5-Component MfT Venn Diagram, the philosophical and pedagogical components are situated in the center of the structure of MfT, signaling that in essence teachers are responsible for both growing their own knowledge and sharing it with others. On the other hand, mathematics as subject matter knowledge is the base of MfT.

Chapter 7. Second Round: Post-Lesson Reflection Following Gabrielle's Enactment

This chapter is about the post-lesson discussion that followed the second teaching of the lesson. It parallels the previous chapter, as described in the overview section (Section 6.1), in the methods of analysis of data and in the presentation of the themes that emerged in the teachers' post-lesson reflection. The themes are again organized according to the 5-Component MfT framework and presented in the same order as in Chapter 6, with one exception - within each component the teachers' discussion precedes my noticing, which in some cases acts only as an extension of what the teachers noticed, and in other cases I noticed events that did not appear in the teachers' discussion.

7.1. Synopsis

Prior to her lesson enactment, Gabrielle expressed several concerns which foreshadowed what transpired in her teaching. This is the reason why I include the synopsis of the relevant events that happened prior to the post-lesson discussion. Afterwards, I follow with the presentation of the themes from the post-lesson discussion that followed Gabrielle's implementation.

7.1.1. Prior to Gabrielle's enactment

The final part of the post-lesson discussion that followed Andrew's enactment of the research lesson was dedicated to looking at how to incorporate what was observed and learned into the second teaching of the lesson, which was to be done by Gabrielle in her own class on the following day. In a sense, this segment was in fact a pre-lesson discussion to the second round of teaching, where the teachers' goals and expectations are usually to refine the lesson based on the evidence from the first implementation.

However, there were no suggestions for improvement, although it was acknowledged that the two spontaneous examples that Andrew used in his instruction would be incorporated into the revised lesson plan.

Gabrielle talked about how prepared she thought her students were for this lesson and what their background knowledge of related concepts was. The students had been learning about solving absolute value equations to that point. She expressed a belief that they had been prepared to start learning about radical equations because of the similarities between the two topics. She expected that her students would understand some points of the upcoming research lesson quickly because they had been "set up to a point" to be ready for radicals, given that previously they had been learning absolute value equations.

> G: Students may surprise me, you know, they may be able to get it. So I've been setting them up to a point without talking about radicals because we've been doing absolute value. Making them look at descriptions before they start.

Gabrielle drew a comparison between absolute value and square roots by saying that neither can be negative. She believed that this would help her students to understand square roots and be able to quickly learn how to set restrictions in the process of finding the solutions in the case of radical equations.

Gabrielle complimented Andrew's lesson a lot. At the same time, she raised concerns a number of times about how her own lesson will go in the next implementation. She specifically worried about whether she would do as good a job and whether she would remember to include everything she had observed and written in her notes (i.e., the in-the-moment invented examples that Andrew used).

> G: I hope I remember to cover all those little points that you managed to bring in in my lesson tomorrow. There's a lot, there was a lot of learning going on I think.

Gabrielle was specifically concerned about whether she would remember to include all of the examples that Andrew did and if she would be able to get through all of the material.

> G: I express some of the things a little differently, it's just, it's really just a formality of how you draw a number line, I put a circle there and the line, and you do that, you know, it's minimal. It's really not of any consequence. As I say, and I really mean this, I hope I can remember to include all of those, look at my notes! If I can teach all of them [..], It'll be really, really good!

Gabrielle also anticipated that there would be some things in her lesson that she may gloss over because they would be too time-consuming and not as important.

> G: There were some things that I probably will just gloss over because it will be too, I could take 10 or 15 minutes to get it across to the kids, and it's not worth it. It's not the main point of the lesson. So I anticipate that.

Gabrielle acted unsure but the team was unaware what her challenges were and it seems that she too was unable to identify and articulate the problem that she had. It seemed as if her concerns were about whether she would be able to live up to her own expectations as she reflected upon the differences between her own teaching practice and that of Andrew. Perhaps she sensed some issues regarding her content knowledge for teaching as well. There was a disturbance in Gabrielle's subjective understanding of the content knowledge of objectified mathematics that was embedded in the curriculum and that she was about to enact in her lesson.

7.1.2. Gabrielle's lesson enactment

Both the team's and Gabrielle's expectations were high given that it was the second implementation and that the lesson was revised to include a better example set than the one initially planned. Because Gabrielle's lesson was scheduled just the day after the first implementation, impressions from Andrew's lesson were still very fresh and the teachers' learning from having experienced a different teaching approach to the old topic may have not yet fully consolidated. Gabrielle's concerns that she expressed the day before during the final segment of the post-lesson discussion that followed Andrew's implementation may have been a sign that the learning process was still underway.

Right at the beginning of the lesson, the teaching plan required the establishment of the fact about the square root being defined as non-negative. Gabrielle did not manage to convey to her students that it is by definition that it is so; instead, she was asking them for a proof of this fact. This was the root of the confusion which lingered in the classroom interactions for a good portion of the class time and was not resolved. The quality of instruction suffered, as did students' learning. Instead of the four examples, only two were completed, and the third one was only started. On two occasions Gabrielle accidentally misled the student who was trying to make sense of the mathematics that was supposed to be learned. None of the team members anticipated such an outcome, especially given that the issue of square root was clarified in the lesson planning conversation on several occasions and Gabrielle had observed this lesson being implemented by Andrew just the day before.

The post-lesson discussion was somewhat stifled by what happened during the lesson enactment. The teachers were more concerned with acknowledgement of the facets of the lesson that went well than with discussing the mathematics that went wrong.

7.2. Themes from the post-lesson discussion

7.2.1. Mathematical component in the post-lesson discussion

Teachers' discussion

Gabrielle and the square root of 9

This section deals with Gabrielle's handling of the students' misconception that the square root of 9 is both positive and negative 3, which was revealed when the students responded to Gabrielle's question about this. This episode was noticed during the lesson and discussed in the post-lesson discussion by the teachers. At the beginning of the discussion, Andrew pointed out that he noticed there was a problem with $\sqrt{9}$. The problem to which Andrew was referring dealt with the mathematical convention that, while both 3^2 and $(-3)^2$ result in 9, the reverse is not true, namely that $\sqrt{9}$ does not equal both 3 and -3; it is only equal to 3, by the use of the definition that states that square root is always positive.

Next, the teachers discussed how this misconception was handled. They turned to a discussion in relation to how the explanation was unclear to students. At this point, Andrew instructed Gabrielle on two mathematical ideas that she missed in her instruction to students. Firstly, once a number is squared, it is impossible to reverse the process and determine whether it was the negative or the positive number that was squared. In her instruction, Gabrielle told the students that the process of squaring cannot be simply reversed. However, she did not explain why and how this related to the current lesson. The following conversation captures Andrew's reasoning about how student confusion could have been avoided.

> A: [...] at some point you didn't know what to say 'cause students were asking questions, and you said, "you can't simply reverse," and that was true--you can't simply reverse because if you reverse, you have $-3=3$, so you cannot simply reverse.
>
> G: That was what I needed, that was the key. That whole little thing there...

Gabrielle seemed to have only in this moment realized the power of the simple explanation that had been referred to previously as a pivotal explanation.

Secondly, a student in the class asked the teacher about the example $\sqrt{(-3)^2}$. The student claimed that, since the square root cancels the squaring, the result of this should be -3. Gabrielle had agreed with him, saying that, in this case, it is so because "it is a function inside a function." Referring to this classroom event, Andrew explained to Gabrielle that square root and square cancel out only when there is a positive number under the root. Otherwise, one needs to follow the order of the operations. He used his knowledge of mathematics, one of the subcomponents of MfT, which in this case encompasses both knowledge of objectified disciplinary mathematics as well as knowledge of curricular mathematics, to reason about the misconception that surfaced in the class but that Gabrielle did not resolve.

The discussion that was related to Gabrielle's mathematical knowledge gap concluded with Gabrielle's confession about her struggle and her promise that she would go back to the unresolved issue the following day in class.

> G: So that's a really, I think I lay the groundwork of that by saying "we'll come back to that and we'll talk more about it" but we had to move on and we had to. Cause otherwise, I was stalling. I had no idea where to go with this, let's move on.

This move allowed the discussion to proceed to other matters.

Students' mathematical practices

An event related to a student's mathematical practice of verifying the solution of an equation was discussed by the teachers. Steve noticed that one student had an unusual way of verifying that one side of an equation was equal to the other side. It can be assumed by the language he used (specifically the phrase 'I noticed') that this event was considered to be noteworthy. Gabrielle also stated that she noticed the same event using the same language.

> S: One thing I noticed, Student J had a funny way of verifying the--that the left side is equal to the right side, basically like solving an equation, rather than seeing that the left side is equal to the right side, by evaluating each side separately. So she, when she arrived to square root of 1 plus 4 equals 5, then she ported the 4 over there, so she was mixing solving the equation with verifying that the left side equals to the right side so I don't know but I've seen in most books they recommend the left side to be separately evaluated and the right side separately, and then…
>
> A: I wouldn't pay attention to that.
>
> S: No I wouldn't either but it's just interesting that students had this thing.
>
> G: She may be the only one that does that. You pointed that out now and I noticed it when she did it and I thought, hmm, because I don't…I will end up with all of this equals all of that, is that true? I push that word, "is it true?." Are we still truthful here? Does this still equal that?

The teachers noticed that the student used an informal way of checking the answer to an equation, one that was different from what the textbooks recommend and how these teachers teach it. For this reason, they were attributing this way of solving to the student's own unique thought process. Gabrielle justified the student's method as something that the student had developed herself, since they didn't teach the process of verifying the correctness of the solution to an equation this way and the student probably did not see this in a textbook either. Andrew added that he thought that the fact that the student had a different way of verifying the correctness of her solution was irrelevant and that he wouldn't pay attention to that. Presumably, this was because the method still led the student to the correct conclusion and, therefore, there was no problem. This could be speaking to a greater pedagogical belief that if something is not causing a problem and is not incorrect, it does not need to be dealt with. Students should be able to have different ways of reasoning about mathematics.

Steve is of the same opinion, that the student should solve things in their own way, as long as they know when that way works and when it does not. This is another way of saying that, as long as it is not causing a problem, a student can use his or her own methods. This reasoning by Andrew and Steve shows the connection between this specific classroom event and the broader principle of teaching, which is that there are different ways of thinking about and solving mathematical problems. All of these ways are valid, as long as they are not leading to misconceptions.

What I noticed about the lesson

Within the mathematical component of MfT, two points attracted my attention: (Non)conveying of a mathematical definition and imprecise use of mathematical notation.

The following episode was initiated by a number of students, asserting that $\sqrt{9}$ was both positive and negative 3. Gabrielle's response was to try and convince them otherwise, by the use of reasoning. She did not bring this out as a matter of mathematical definition.

93	T	So is it true to say that? [Teacher writes $\sqrt{9} = \pm 3$ on the board] . Think think think. You cannot do the square root of a negative number. We agree on that. So are both of those solutions legal? Can we do that?
94	Ss	Yes.
95	T	Really? **How do you prove that?** Yes S13?
96	S13	You could do the opposite and you square it [meaning both sides], like you do negative (3) squared and you get 9.

97	T	Ok, ok so you are looking at a technique, ok [teacher looks at her notes for an extended amount of time]. I'm having an issue with this [points to the minus symbol in front of the 3]. The square root of 9 has to be a positive number. So is it possible to have those two roots? [pause] I think what you may be confusing this with, is if we had, let me think how you can, how you express this [looks at the notes for a few seconds]. If we said, yes, 3 squared equals 9, and negative 3 squared also equals 9 [writes both statements on the board $3^2 = 9$ and $(-3)^2 = 9$]. That's undoubtedly true. But think about going in reverse. If you squared this [pointing at – 3], yes it is 9. But if you do the square root of 9, it cannot be a negative number.
98	Ss	Uh, … [grudging and sounding confused]...
99	T	Ok we're really going down on this one. Do you have any suggestions? Yes S12.

Gabrielle asked the students to prove something that could only be known through a definition, and she referred to the number as a solution when there was no equation being considered here. This is an example of a teacher unintentionally misguiding students' mathematical thinking.

The following episode resulted from the statement that Gabrielle wrote on the board: $\sqrt{9} = \pm 3$. She wrote it because students said so. Then she tried to find a way to convince the students that it is incorrect. The following episode shows a student who offered a correct interpretation. However, Gabrielle misunderstood what he meant. In the excerpt, Gabrielle tries to draw a distinction between $\sqrt{9}$ as a number and when the result $\pm\sqrt{9}$ arises as a solution to a quadratic equation.

100	S12	I realize that positive or negative root 9 equals positive negative 3. Before the root sign, put positive negative [..]
101	T	So you're suggesting that we do this? [T writes a plus/minus sign in front of square root of 9, so now it says $\pm\sqrt{9} = \pm 3$ on the board] .
102	S12	Yeah.
103	T	Ugh. [pause] Ok, I'm … I'm not ... [briefly looks at her notes]. What if we, what if we had this? [T writes $x^2 - 9 = 0$ on the board] So x squared equals? S8?
104	S8	9.

105 T Ok, so x squared equals 9. [T writes $x^2 = 9$ on board under the previous statement] ... So, x equals, you're doing the square root of both, right?

At this point, Gabrielle wrote $\sqrt{x^2} = \sqrt{9}$, without rewriting the statement, and without putting the $\pm$ symbol in front of $\sqrt{9}$. She simply put the root symbol over each of the expressions, on the original statement. This was confusing for the student because it appeared that in one instance the symbol $\sqrt{9}$ is interpreted as just positive 3 and in another instance it is interpreted as both positive and negative 3 (in case it comes from solving the quadratic equation). This dependence of the "history" as to where the object came from was mathematically unsettling for S12 and so he proposed the notation as $\sqrt{x^2} = \pm\sqrt{9}$, feeling that it would afford a meaningful way to get out of this dilemma. Because of her imprecise use of notation, the same symbol $\sqrt{9}$ meant different things, depending on the context from which it came. What Gabrielle wrote was mathematically incomplete and proved to be confusing for the students. Since students had previously studied absolute value equations, it would have been productive and correct to instruct the class that $\sqrt{x^2} = |x|$ and therefore the correct line after what she wrote would be |x| = 3, and since she has just taught absolute values, the solution would then be $x = \pm 3$. The root of the confusion was both imprecision of notation and omission of definition. As well, Gabrielle did not explain to S12 why his proposed notation was incorrect (or is it?). Since the root must be positive, the left-hand side as written cannot be equal to negative root 9.

105 (con t'd) T So in this case, x could be plus 3 or minus 3, you're right.

[*T applies the operation of square rooting, and writes under the last statement* $x = \pm 3$.]

Now that's solving an equation for a variable, for an unknown.

[*T points to that which was just written on the board*].

This is not. This is just thinking a number.

[*T points to sqrt 9 from previous discussion, which is still on the board from before*].

So I want you to try and see the difference between what we said here and what we said here. This is just a number. Can a square root of [...] I'm going to take this off S12, because this is really not the issue here.

[*T erases the +/- symbol in front of the sqrt 9 that S12 had previously suggested, changing the statement to sqrt 9 = +/- 3, and pointing to sqrt 9*].

Square root of 9 has to be positive. So I'm not [...] Um. A square root of 9 is just one number. If you punch that into a calculator you come up with just one number. Don't bother. I don't want you to do that. Put your calculators away. [..] Is that number positive or negative?

106 Ss Positive.

[*Some students still tried it on the calculator and are now more convinced*].

107 T Positive. It didn't say negative, did it? [meaning the calculator]

108 T So the square root of anything has to be positive. Now do you remember what we've been talking about saying that, the absolute value of x cannot be a negative number?

[*T writes* $|x| = -2$ *; pause 5 sec*].

The same principle is coming today here.

[*T erases* $|x| = -2$ *that she just wrote on the board*].

The square root of any number cannot be a negative number. So this is not acceptable.

[*T erases the minus sign in front of the 3; the board now shows* $\sqrt{9} = +3$].

That is [*pointing to + sign*]. And I'm trying to find something in here that I can convince you of that.

[*T looks at her notes; 6 sec pause*].

This is true. If you square both sides, if you square both numbers, whether the number is positive or negative, you will get a positive answer.

[*T points to* $3^2 = 9$ *and* $(-3)^2 = 9$, *the two statments that are still there from previous discussion*].

So this always has to be positive.

[*T points to* $\sqrt{9} = +3$].

109 S5 Is that one only positive or negative 3 because we have the square root of the (perfect) square?

110 T No, it could be any number. I'm using a 9 because it has a perfect square answer. But it could be any number.

In this long excerpt, I noticed that the teacher stimulated divergent thinking in the students. For instance, she used the example with absolute value, to draw a parallel that was not mathematically appropriate here. This can actually mislead students, given that it is in relation to the range of the absolute value function and not about how a radical number is defined. As evident from this interaction, students were confused and in the end, Gabrielle had no other choice but to ask them to check it on the calculator. It seems that in this case, it would be more productive to stir the students to convergent thinking by establishing the rule from a mathematical standpoint, stating simply that that is how it is defined in mathematics. This was an example where, on several occasions, the students were misled in their understanding of mathematics.

This is an example where my noticing both overlaps and extends what was brought up in the post-lesson discussion. The participants and I all identified the gap in the teacher's knowledge; however, how this gap impacted instruction and student learning was not discussed.

Remarks about the teacher discussion

Softening the teacher's misconception

During the first part of the post-lesson discussion Andrew pointed out that Gabrielle's explanation about how the square root of a number cannot be negative was unclear. He used 'softened' language, such as "many things come up at the same time," "can't explain everything," "you're under a lot of pressure to finish" and "you didn't make any mistakes, just a bit unclear."

> A: …So, but you, many things came at the same time. So you just can't simply explain everything. And they have different ideas, it takes time. It is the reason we have just one class, clear next time. And also, you are under the pressure to finish, you are in front. Do you remember that anecdote you said yesterday?
>
> G: I'm always a little nervous to begin with, and then I…
>
> A: You cannot show in front of the students that you are with doubt because you have to be right. And you actually could be right if you had a little bit more time but you're under pressure but you cannot show students, let's say, that I'm afraid of anybody.
>
> G: So that's a really, I think I lay the groundwork of that by saying, "We'll come back to that and we'll talk more about it." but we had to move on and we had to…Because otherwise, I was stalling. I had no idea where to go with this, let's move on. […]
>
> A: You didn't make any mistakes so far. So just probably, that was a little bit unclear.

Andrew was softening the event during which the students were confused by Gabrielle's explanation, by using language that was intended to help Gabrielle feel better about what happened. He noticed that she was not being clear about the mathematical concept and did not explain it properly to the students. However, he justified that this event could be attributed to the fact that there was not enough time in the lesson to deal with every misconception thoroughly and that the teacher needed to act confidently about the subject matter. In a way, he was protecting her professional confidence.

7.2.2. Didactical component in the post-lesson discussion

Teachers' discussion

Successful aspects of the lesson

After the short discussion about the compromised part of the lesson implementation, the teachers' discussion then turned to the events of the lesson with which Gabrielle said she was pleased.

> G: […] But what I thought they got well was the overlap of the two, the two restricted areas, what couldn't happen and what could. That was good, I thought that was really good.
>
> S: Yes, you gave it a lot of attention, pointing out that nice little area there. It landed well with the students.

Gabrielle herself pointed out a part of the process of solving radical equations, which she thought she had explained well during the class. She stated that, even though her definition and explanation of a square root was unclear to students, she was very successful at getting across the idea of how to look for restrictions when solving a radical equation.

The second event that the teacher team noticed as being a successful part of the Gabrielle's instruction was how she led students to make mathematical conclusions, rather than pushing them. This event can be seen as noteworthy because all of the teachers had noticed it, had discussed it and were in agreement that this was an effective way of teaching.

> S: But you have a nice way of gently doing it, like you let them come up with it themselves.
>
> A: But that requires time.
>
> S: You don't push it like you don't push your conclusion.
>
> A: And you cannot, if you have to go slow, go slow. If you push, then you make mistakes.
>
> N: And you lose them.

> A: So I rather go slowly, and don't finish. Then we have next class, then we can jump ahead, when they did get it.
>
> G: And I'm not conscious of doing that. I don't do that deliberately so that's just years and years and years of teaching and knowing what works already.

The teachers were discussing about how students learn mathematics and how lessons should be paced. Students need time to process the information and work on their understanding of the concept. The teachers agreed that it was better to go slowly and not finish, than to finish, but lose the students in the process. Knowing how much time to give to students in order for them to consolidate their learning, as well as how much to push them, had become automatic for Gabrielle because of the many years of teaching experience that she had. This event is being connected to the broader principle of teaching and learning, which is that a teacher cannot push students to learn something; they must learn at their own pace when they are ready.

What I noticed about the lesson

Gabrielle's method of teaching this content was based on developing procedural knowledge of setting the restrictions which was divorced from the reasoning of why one should be doing this in the first place. In the absence of a resolution about the definition of square root, the procedure that she believed that she had successfully taught had limited bases for the development of conceptual understanding of the topic. The part of her lead up that reinforced that the domains of the radical expressions had to overlap would have been fine. Where it would break down is for discussions of considerations of the range. Essentially, her teaching method was incompatible with the teaching approach chosen by the team for this lesson study.

In addition to what was already mentioned, I noticed her implicit way of presenting the processes of mathematical thinking. For example, in setting the restrictions on the variable, Gabrielle used an implicit and verbalized technique in order to arrive at the possible values of the variable in the process of setting limitations. During Gabrielle's entire lesson, there was no solving of inequalities that was presented on the board as an explicit process by which to arrive at the interval of permissible values for the solution of the given radical equation. Rather than setting the restrictions as an inequality, and then solving it, Gabrielle used a verbalized guess-and-check method to arrive at the restrictions on the variable. This may have left some students wondering where the restrictions came from, or what they meant, or even what the purpose was behind stating them. I infer that Gabrielle's teaching relied upon prior knowledge at a level at which the students were expected to understand it intuitively. I wonder whether this was an intentional part of Gabrielle's teaching methodology, or if it was simply an oversight. It is interesting that this was not discussed by the teachers.

Remarks about the teacher discussion

Interestingly, the teachers did not discuss how Gabrielle's teaching missed the goal that was previously set in the planning of the lesson. Despite having a clearly laid out lesson plan which she co-created and to which she frequently referred during the implementation of the lesson, she steered off into a direction where the main activity was for students to experience a procedure of setting restrictions on a variable instead of to understand the reasoning behind it.

7.2.3. Psychological component in the post-lesson discussion

Teachers' discussion

Ways to motivate students

In this part of the discussion Andrew and Gabrielle were comparing their students across classes and their motivational techniques for engaging them in the learning process. This included how they get students involved and motivated, and how they question students. They were connecting the phenomenon that the students in each class were different, to the principle that a class of students needed to become accustomed to a particular method of instruction in order for that method to work. It takes a while to get students used to a particular teaching style. For example, Andrew's students are used to being questioned, and to coming to the board and moving around. However, if Gabrielle tried to implement this style in her own class, it might not work as well right away. Even in Andrew's class, this method of questioning took a significant amount of time to become successful, implying that it is not simply an issue of teaching style, but also how accustomed students are to that method.

> A: It is many years struggle, I remember first 3 years I was struggling to energize the students, to actually make them act, it is not easy. When you come to some new class, it takes time to start doing that because they just aren't used to doing that.

In the above excerpt Andrew revealed that a teacher's consistency and perseverance in his requirements builds up an active, energized class. Andrew described his method of questioning students, whether they knew the answer or not, in order to keep them active and alert. During Gabrielle's enactment of the lesson, she was also questioning students and having them come to the board. However, she only called up students who volunteered to participate.

Furthermore, Andrew reasoned that the difference between the levels of engagement in the two classes cannot be due to the different classes' knowledge level because many of his students did not know the mathematics either. Therefore, it is evident that he was attributing the difference in the engagement between the two classes to differences in pedagogy. The pedagogical suggestion that Andrew was implicitly making to Gabrielle was that, in order to increase students' attention and participation, she should question students constantly and have them come to the board, even if they do not volunteer to contribute.

Another related point of comparison that was made by the teachers dealt with the amount of student involvement that occurred in the course of the lesson. Gabrielle had noticed that Andrew's students were doing more of the mathematical work than Andrew was doing in his own class during the implementation of this lesson. However, in her class, it was she who was doing more of the work.

> G: […] But I think that's a big difference here. Your kids are excited about getting it. My kids are surprised if they get it. And I think that's probably a big difference in the dynamic of the class, they would much rather sit there and let me do the work. So to get them to come up and do the work is more of an effort than I think in your class.

Although both research-lesson implementations used the same lesson plan and essentially employed the same structure for the learning path, it is clear from this quotation and from the analysis of the transcripts of the two lessons, that in Gabrielle's class, she is doing the majority of the mathematical work, whereas in Andrew's class, it is the students that are at the board solving the problems. Gabrielle had noticed this and was reasoning that, because the students in her class were not as motivated, it takes a lot more effort to engage them in learning. This again ties into the broader principle of teaching that it is the class experiences, which students have had, as well as the style of teaching and learning that the teacher promotes, which determines how engaged the students will be during the class.

What I noticed about the lesson

In the classroom students were asking a lot of questions because they were confused by Gabrielle's unclear presentation of the topic. On the other hand, the questions that she posed to students were mostly intended to guide the students through the procedure and not to foster conceptual understanding. The nature of her questions was such that they required short declarative answers. Some examples of such questions from Gabrielle's lesson enactment are, "What do you think is the first thing we have to do?", "Ok, Student K, what would be our next step?". Other than questioning students, in this lesson Gabrielle did not use any other motivational techniques to engage the students in learning.

Remarks about the teacher discussion

The teachers' discussion about who does most of the work in the classroom and why, reveals that it is easier for Gabrielle to do the mathematical work herself than to get the students to do it. She ascribes this to the characteristic of her class more than to the style of her teaching. I find it unusual that a teacher expressed that her students have low expectations of themselves. Could that be a reflection of her expectations of them?

7.2.4. Pedagogical component in the post-lesson discussion

Teachers' discussion

Students correct one another spontaneously

The teachers discussed an event during which a student showed a misconception related to solving of an inequality. The student was working on the task which at some point in the process involved multiplying both sides of an inequality by -1. The student had forgotten to reverse the direction of the inequality sign. Before the teacher could intervene and correct him, the other students jumped in and corrected him instead.

> S: So there was a student who suggested x must be less than or equal to -9, I think he said, but then other students corrected him. So I'm just wondering whether all the students understand the idea of, you know, the sign flips if you have to multiply through by -1 ?
>
> G: I think most of them do, we looked at that yesterday and we actually had a discussion, what if you had -x is less than -5. You're changing two signs, do you change that twice? That was a good question, and I said, "Let's figure it out." So yeah. I think they're interested in it, they may not know deeply what the actual implication is, but they have a working knowledge of why we come up with an inequality when we are looking at what makes x equal.

The teachers wondered whether all of the students knew that, when both sides of an inequality are multiplied by -1, the direction of the inequality sign must be changed. Gabrielle's assumption was that, since so many of the students participated in the correction, most of them must be aware of this process in solving inequalities. Her pedagogical move was to let the students collaboratively and independently resolve the issue.

Gabrielle shared that the students had been working on this idea the previous day so they must have had a 'working knowledge' of what it was. Instead of just telling students that the direction of the sign changes when the inequality is multiplied through by -1, Gabrielle said that, when this topic came up in her previous class, she had the students figure it out by themselves, rather than simply giving them the rule. Even though this specific event did not occur in this particular lesson, it is related to it, and shows that Gabrielle used the broader principle of teaching and learning, which is that students will learn better if they work something out for themselves.

Another event within this theme involved teachers' discussion that, if students are in a learning environment where it is normal and acceptable to make mistakes or to not know something, learning becomes more effective.

> G: And if it [students making mistakes in front of the class] happens often enough which I'm sure it does, then there's no stigma attached, because everyone does that sometimes.

> N: Student Y, even she made a mistake the other day.
>
> G: Yeah and even Student X is going to make a mistake, Student Z is going to make a mistake so it doesn't [...]
>
> S: The point is to have them risk taking. Because then they're learning.
>
> A: When you call them, if they know or they don't know, it doesn't matter. At least 10 minutes after that they are here.

In a class, such as Andrew's, where students are constantly working on the board and being questioned (particularly those who do not know much), there are many opportunities for students to make errors and to be corrected in front of the class. Because this is such a common and acceptable occurrence, there is no stigma attached to "being wrong" in front of other students. Thus, the class becomes a safe environment that is conducive to learning because students are free to take risks, which they would not be willing to take in a more threatening classroom situation.

Gabrielle noticed that many students had made mistakes, but were comfortable doing so. She reasoned that, if a teacher, such as Andrew, had established an environment in which students are constantly being questioned and made to work on the board, they will make many errors, as will the teacher, from time to time. This, in turn, will make the students more willing to take risks because they know there will be no negative consequences for "being wrong." Students are not embarrassed when they are incorrect. Therefore, they have more opportunities for learning. This connects to the broader principle of teaching and learning that, in order to learn, you must practice something and learn from the mistakes you make along the way.

Pedagogical styles

One theme that surfaced several times was about the pedagogical styles that different teachers employ in their class and how this affects student learning. Gabrielle and Andrew discussed the characteristics of their specific group of students and of their own teaching styles to account for the differences in how the students in their classes learn mathematics. Here they were also making connections to broader principles of teaching and learning, by saying that two groups of students can be very different because of the teachers that they have, and the teaching style of the teacher as well as because of the background knowledge that the students possess. Here is an example from the post-lesson discussion that illustrates this point. It is induced by Andrew's observation that, in his own class, students were more engaged in learning than they were in Gabrielle's class.

> A: We all know that all classes are not equal; courses are not equal. And that it's easier to work with some classes than others. Also, I don't know, do you question them when you have them without us being here, all the time?
>
> G: Pretty much.
>
> A: But what I'm doing, I always question. And they are used to getting questioned. They are jumping they are moving around, not really, this class. Is it because they are a little less knowledgeable than other classes? Not so [...] some of them are struggling so no wonder but, still if you are insisting on asking, then they will learn. Doesn't matter, you remember yesterday, I called students, I knew that they would not know.
>
> N: And quite quickly you said no and then you sent them back to their seat and called up another student and like that, all the time, right? So they strived to actually know.

What I noticed about the lesson

I noticed that there was no attempt made by the teacher to prevent errors. Instead, they were dealt with when they occurred. Within the realm of student - teacher interactions, I noticed that the students in Gabrielle's class asked many questions. However, on several occasions, they either received incorrect answers, or were misled. At times, their questions were not answered at all, or they were not heard. The most explicit example of a question that was answered incorrectly came from the following interaction between Gabrielle and S12, in regard to $\sqrt{(-3)^2}$.

111	S12	But if you have root negative 3 squared?
112	T	Negative 3 squared is most definitely 9.
113	S12	No, no, I mean if this equation is the root of negative 3 squared, then [...]
114	T	Could you just come up and write it, I can't visualize it.
115		[S12 comes and writes the following on the board: $\sqrt{(-3)^2}$]

116	S12	Because I think [..] because [in a new line, writing $((-3)^2)^{\frac{1}{2}}$ on the board]. So it's equal to −3, right? [Student completes the original statement above, and writes $\sqrt{(-3)^2} = -3$, essentially "proving" that $\sqrt{9} = -3$. Then looks at the teacher for an extended time, waiting for her response. T looks at S12's writing, pondering for 10 sec].
117	T	Ok so you're doing a function inside a function though, you are doing two things that cancel each other out. So yes, this cancels that out. This neutralizes that so whatever you have, it will be that answer, yes. But that's two things going on at once, I wanna try and avoid doing that at the moment. Alright?

Remarks about the teacher discussion

Andrew and Gabrielle compared their classes and reasoned about where the differences in the different levels of engagement and learning came from. Andrew got the sense that Gabrelle's students were not used to being questioned as much as his students were. This is very interesting because in her lesson enactment she posed nearly twice as many questions as he did (62 versus 35). More about the comparison between the two lesson enactments follows in the next chapter (see Table 5 in Chapter 8). The perception that Andrew had could be due to the nature of Gabrielle's questioning, which was more procedural than conceptual or to the fact that Gabrielle calls only upon students who volunteer their answers while others can remain passive throughout the lesson. In addition, her language, both spoken and written (i.e., mathematical notation), was informal.

7.2.5. Philosophical component in the post-lesson discussion

Teachers' discussion

The philosophical component did not emerge in the post-lesson discussion following Gabrielle's enactment.

What I noticed about the lesson

Gabrielle holds the authority over the knowledge

The philosophical component showed up in Gabrielle's enactment in the way she placed herself as the authority that holds the knowledge. Regarding the critical piece of knowledge that was needed for the development of the lesson, namely the fact that the square root is being taken as non-negative, she referred to this as "having laid down the law" and that this fact needed to be used as such because she said so, placing the authority on herself, rather than on the mathematical convention or definition. The following episode comes from the beginning to solve the first task, $\sqrt{x-3} = 5 - x$.

180	T	Can we just gently walk down to this without running into any walls? What about this [pointing to 5 – x on the right side of the equation]? Bear in mind that **I layed down the law and told you this**. [points to older equation on the other side of the board, the one that says $\sqrt{9} = +3$]. What about this? [points back to the radical equation the class is currently solving] What we've got is a radical term equals something here. And what about the something here?
181	T	Do you see? It has to be positive. Yes, because Mrs. G [*meaning herself*] said so. So what do we need, do we need to do anything about our right hand side?
182	Ss	Restrict it?
183	T	Do you think so, S6? Do you think a restriction here would be good?
184	S6	Yes, because it can't be negative.
185	T	We can't allow it to be negative, yes. So what would be our restriction?

"How useful is it to know ahead of time that something may not work? In the real world we have to do that all the time."

In her enactment, Gabrielle not only used a new approach to teaching an old topic, but also used what was discussed amongst the team in the pre-lesson discussion as a rationale to convey to students why doing the mathematical reasoning ahead of time pays off later in solving difficult problems in the real world.

> Ok, so plugging in the value that we have suspicions about actually proved to be, our suspicions were proved to be correct. Absolutely, it

> cannot be [..], that's outside the combined zone. So when we say [when we] made this groundwork, we could see straight away when we got to here, that one of these answers is extraneous. It looks fine, the algebra is beautiful, it's all there. But, it had to break a rule. And the rule is that the answer has to come from this interval here. It has to be a value of x between those 2 [values]. Ok, so what we're taking away from all of this, as far as writing equations are concerned, if you look very carefully at your original statement, and figure out what is legitimate and what is not, you already know where you're going. You know your answer has to be somewhere in here. So when you get an answer, look at that, look at that and say, oh no, or, "we're fine", alright? So it's a really smart idea to assess the possible red flag areas before you start. Now, in a lot of cases, we just plunge through all of this, we get the answers and go back and check them and say, "Oh! It didn't work!" How useful is it to know ahead of time that something may not work? In the real world we have to do that all the time. Cuz if we're gonna build a bridge and say, "Oh yeah, I think it will be fine," and then the first truck that drives over the bridge falls into the river. "Oh, I'm sorry." Wouldn't it have been good to know that we'd done something wrong right at the beginning? Ok so, there is a good reason to go through that process first, and see what's up. Ok, now have a look at, what about an equation like this? [writes a new equation on board, $\sqrt{x-3} = -x-5$] What do you think? Have a look at it and see if there's anything you should be paying attention to here. Is it possible to solve that problem? Who thinks it is?

Remarks about the teacher discussion

As remarked before, in the corresponding section following Andrew's lesson enactment, here too the teachers' discussion did not touch upon the philosophical component; however, this component of MfT was still present as shown in the excerpt above. Moreover, it is an implicit content of teaching. It is obvious from Gabrielle's enactment that her philosophical stance was influenced by the collaborative work in the phase of collaborative preparing for instruction of this topic.

7.3. Addressing the research questions

The post-lesson discussion following Gabrielle's enactment exposed the bonding that this teacher team developed through the years of working and learning together. Over time, they created a safe and secure environment for their professional development. They knew each other's practice quite well and have seen one another in the act of teaching multiple times before; however, interestingly, they did not identify Gabrielle's challenges that had surfaced after Andrew's lesson implementation, and which Gabrielle expressed through her concerns about how her lesson enactment would go the day after. Neither could Gabrielle articulate her problem. Her mathematical content knowledge gap which surfaced at the beginning of the lesson enactment surprised everyone.

In the discussion, Andrew brought up the mathematics that went wrong but none of the teachers elaborated upon it. Instead, they excused it and trusted Gabrielle that this would be fixed on the following day. The discussion that followed veered off mathematics into pedagogical, didactical and psychological issues. Gabrielle was complimented for and was aware that her presentation of the technical and procedural aspects of the topic went well.

As in the previous chapter, I address the research questions again, offering further answers to them, as flowing from the second lesson enactment.

(a) What can be gained by the teachers who participate in the school-based lesson study initiative?

With respect to what the teachers gained from this experience was a sharpened understanding about the function of the teacher's questioning during the class and its effect on students' engagement and motivation for class work. Two other related but minor themes were the role of students' mistakes in their learning and the share of mathematical work done by the teacher versus the students during the class. These conversations in complement with the observed teachers' enactments and their respective classroom learning cultures provide the opportunity for the teachers to reflect upon and consequently transfer this experience into their future practice.

The most prominent outcome that came to light in this stage of the lesson study cycle seems to be the supportive collegial network that gets built over time and acts as a source of continuous learning, offering the opportunity for ongoing improvement of teaching practice.

(b) What are the factors influencing the development of teaching practice in the lesson study setting? This might include visible and invisible features, such as beliefs and attitudes.

The performance piece seems to be of critical importance here for three reasons. First, it gives the team members the opportunity to observe, compare and change their own practice, as well as challenge that of a colleague's. This is the nature of learning. Second, it gives the performing teacher the chance to put into practice something new and different (i.e., teaching the topic using Approach B instead of A) and get the feedback from the colleagues. Third, it may be as it was in this instance that a teacher eventually learns mathematics for teaching out of her own experience, practice, and even failure.

(c) What is the nature of the mathematics for teaching that has emerged?

This post-lesson discussion further confirmed the complexity of the nature of MfT. Specialized mathematical content knowledge for teaching is balanced with psychological, didactical, pedagogical and philosophical components. Although the philosophical component did not emerge amongst this group of teachers in the discussion explicitly, it is there implicitly pinned to the goals that teachers agreed upon: to develop mathematical thinking in the students that they teach. Besides the collectively set goal, which is shaped by the philosophical component of the individuals, each teacher holds their own personal value system that is embedded in their philosophical component of the MfT and which is reflected in their individual teaching practice.

7.4. Epilogue

> *By three methods we may learn wisdom: first, by reflection, which is noblest; second, by imitation, which is easiest; and third by experience, which is the bitterest.*
>
> *Confucius*

Here I present Gabrielle's personal reflection on the lesson study experience, which she shared with me in an interview that was conducted three years after the data was collected for the main study. This interview was prompted by my aspiration to investigate the effects of this professional development work on her practice. Did the experience from the collaborative work of this group of teachers, through lesson study, impact the subjective level of MfT and transfer into the teaching practice? The interview questions are presented in Appendix F.

In response to the question that asked what attracted her to lesson study, Gabrielle said that the school in which she worked before she joined West Coast Academy asked the teachers to enroll in some form of professional development, outside of school. She chose to join a lesson study group that was organized by the Sigma Institute (pseudonym), at a local university where a group of teachers participated in the lesson study community of practice, and that was led by guest mathematicians and mathematics educators. She was attracted to the idea of learning with, as well as from, other in-service teachers. She found that the experience resonated with her professional needs. Thus, she chose to participate in it as a sustained practice over a number of years.

> You know, we were required to do something outside of school for our professional development, and this looked like something much more interesting than the other stuff that was out there. I had no idea what it was about, and just went to see what it was about and then we went on to carry it for what, 4 or 5 years.

The second question asked how this experience compared to other professional development opportunities in which she engaged over the years of her career. She offered the following:

> I think it is so useful, I feel that right from the word go from my first job in 1978 , learning from other teachers is really, to me, much more useful than learning from an expert. And I think part of that is the cult of personality, you know that Stand and Deliver guy, Jaime Escalate, who taught that Brooklyn class, the kids to do Calculus... And so many people tried to follow up on that, and they really couldn't, some of them had relative success, others not. Even programs like that, still do that, they ebb and flow, and I'm sure it is the personality of the person who is in control – so, take that away, you don't really have much of a program. And I think that when <u>we</u> get together as colleagues, like at Sigma, or here at the school, everyone is a classroom teacher - nobody is this deity, like Escalante, and a few others – who are wonderful, and we should learn from them, but I think for actual teachers in the trenches - we are the experts on classroom warfare – not the ones who have done this wonderful stuff that … we can't copy, 'cause the personality. That's my little theory. And that's why I think the work at Sigma, and then to do it here at the school, was so terrific.

> We are all teachers, the whole range, and we have all something to learn from each other, and everyone who was there is invested in their job, and wanted to find out more and do better, and so on. What more fertile ground can you find? … I truly believe that, with all my heart, we learn more from each other than we do from the deities. It shouldn't be exclusive, but for every-day teaching … we have all this accumulated experience, and have learned from our own mistakes… so how do these rules of teaching work and under what circumstances, and when do you need to be creative and do something completely different.

In response to the third question regarding which lesson study cycle sticks out for Gabrielle the most and for what reasons, she shared a number of important points. Incidentally, she mentioned all three lessons that her team worked on during the research period.

> Umm… the first one that I did was fascinating, because [I saw] how much preparation is really necessary to give a really good lesson, and that collaboration that went into it. And then when I implemented that in the classroom and just went, it was so exciting to see how those kids were learning because of this huge amount of effort that we have done. [It was] much more than, "Oh, ok, what are we doing today, kids?," which is very often the default, with the amount of time that we have.

> It is great if you can prepare your lesson ahead of time, and think of the points and what you are going to do. But also, there is no time to go back to it [after teaching the lesson], and say, "Oh, what could I

> have done better?," because you are already up there for the next one. I think that makes our job really difficult.

First, Gabrielle pointed out the challenges that teachers face when they are rushed from class to class and do not have the time to prepare their lessons and reflect upon their practice. Teachers commonly work in isolation. On average, they have an 80% teaching load, meaning that of their full time employment as a teacher, 80% involves contact time teaching students in the class and the rest is for other activities such as planning and marking student work. This is in sharp contrast with the conditions that secondary mathematics teachers have in China. Perhaps the PUFM that Ma (1999) talks about is attained, in a large part, by the teachers in China because of their favourable professional working conditions. These include common office space, collaborative time to plan for instruction and study the teaching materials, observing each other's practice, marking the work of the students and reflecting on instruction together.

> And when I repeated that lesson the next year, and the year after, you know, and so forth, it started becoming just easy, because I knew it. And I knew all the things that the kids would stick on, I could, I knew ahead of time that was going to happen, so it was, I still feel that it was a fabulous lesson.
>
> Q: Do you still remember which lesson that was?
>
> A: It was the second lesson in quadratic graphing. The first one was just what the parabola looks like and what the characteristics are – not much shifting or expanding – just the very $y=x^2$. And then the one that I gave as a lesson study was, "Now apply it to a word problem!." You know, how do you flip it, and expand it, what are your roots and intercepts … So the second, pretty much the second lesson on quadratics ... I can do it in my sleep now.

In the above excerpt, Gabrielle is referring to the third research lesson that was conducted by the team (see Appendix C for the schedule of the team's lesson study activity during the data collection period), for which she was responsible to write the lesson plan. Here she communicates how she incorporated new knowledge of MfT into her practice and strengthened her confidence level.

> And the other one, was the one that Andrew did, the one on radical equations, that I have his notes [meaning the Report, presented in Appendix E], I have given his lesson several times, and shared it with

> other people. It covered aspects that I would never think of on my own. So you know, again, a much, much better lesson than what I would have done on my own or without that input. So that, I think is invaluable.

Interestingly, Gabrielle mentioned the lesson in the main study of this research and conveyed her enthusiasm to share her newly found MfT with other colleagues.

> I don't think it's feasible to adopt this without making some major changes in the way our time is scheduled. But I still think it is very valuable to keep going, even if it is just isolated for such nuggets, like to develop the first lesson in each unit. Over time, it could become a syllabus bank of these really first-rate lessons.

Here Gabrielle suggests that collaborative work should continue, despite the challenges of limited time in the current scheduling practices of teachers, with the purpose of creating a shared knowledge bank of first-rate lessons. However, as some other researchers have already pointed out, it is not about creating perfect lessons. Rather, it is about the learning opportunities of the teachers (Lewis et al., 2009).

> And that one, on Circle Geometry, that was so much fun! It was different than saying, "Here, pick up your protractor, and measure some angles…." And particularly - I find my joy is geometry, because I love the pictures, and I draw the pictures all the time, to see it - that visual part. It is more exciting to me than the algebraic part. And you know, do you remember when we did the "art and math," when we proved some obscure theorem – and the kids were drawing that, and proving the sides belonged to an equilateral triangle? [Discussion ensued about Napoleon's Theorem, which came up in one of the meetings when planning for the lesson, and which later Gabrielle used with the students in a special event held at the school as part of the academic enrichment week]. Yeah, but it was so cool, because yeah, they were learning, how to … sure, it was manipulating compass and all that, but still, they got a lot out of that, it was fun.

With the above thoughts, Gabrielle expressed joy and inspiration that she experienced through the collaborative work of the lesson study on circle geometry.

Next, Gabrielle was responding to the fourth interview question, which was set to reveal the nature of mathematical knowledge for teaching that had emerged, for her personally, from the lesson study on radical equations. For this purpose, I asked

Gabrielle, “Going to the lesson that we did on solving radical equations, can you remember, and perhaps describe, what it was that you learned from that experience?”

> Absolutely! I guess that was hovering around here, that I was aware of, but watching Andrew do that, made it so clear that, not only was it necessary to introduce this particular step early on, but it helped the kids from then on. And that was limits, or you know, the restrictions. Restrict before you start doing anything. Look at what you’ve got, and restrict. Now go ahead and solve it, and da-da-da, and restrict your answer and -oh, and then go back to your original restrictions and you can find that overlap [where could the solution be]. And this didn’t matter because it was not in the solution space, […] but look at that little bit there in the middle, in that double enhanced line. Again, visual, for me is so clear… It was, I have never ever thought of looking at that, say root (x plus 5) equals x, ok, we have to limit that value. And you’ve got to restrict the right side too, because the radical cannot equal a negative value. Restrict right now, and say, somewhere in there is your solution, but it is not the whole thing. And then under these conditions we can continue solving. Yeah, that was a real revelation for me! It helped so much.

What Gabrielle expressed here reveals that she deepened her understanding of the subject matter knowledge and transferred the new teaching approach (Approach B) into her future practice. This could indicate a transformation in her perception of what is her role as a teacher in promoting mathematical thinking in her students. When she became clear about the problematic of the mathematics that were to be taught, she was empowered to be able to convey it to her students. Her knowledge of how to teach mathematics was strengthened.

The following excerpt indicates that Gabrielle’s conceptual understanding of mathematics deepened as she resolved the challenges that she had in her enactment of this lesson during the study. Prompted by her experience of the impact on the learning of the students, the need was created for clarity in her own understanding and, subsequently, for clarity in instruction. The conversation was in regard to how one could explain the distinction between solutions of a quadratic equation having possibly both positive and negative solutions (opposites, in the case of $ax^2 + c = 0$, for example $x^2 - 9 = 0$) but that the radical itself is defined as positive, (i.e., $\sqrt{9} = 3$).

> Well, half of the students will say it is 3, and the other half will remember that it is plus/minus. If too many students are just giving

> you the answer as x=3, I would teach or confirm it by asking, "well, factor this side, and then see what are your solutions." [i.e. referring to the use of "difference of squares"]… That was the sticking point for some students, and also to say that square root of 9 is plus or minus 3. That is different than to say that the solutions to this equation are plus or minus 3. That was something that I learned too: "Ah, this has to be made so clear to the students!." Because some kids got it right away, and some were like, "Really?." So it became a 10-minute thing with half the class. It can be confusing, because it is quite philosophical.
>
> … and incidentally, to go back to this, as looking, setting, evaluating your possible solution set – well, just looking at it, we need to make sure that … this concept is also useful for other things, like absolute value, logarithms, … To say, well, what are we allowed to have here? There are many possible things here, but not all of them are allowed, they are not legal. And you start to narrow your focus, it is not an unlimited set, and that really helps kids to see.

Gabrielle's alluding to the square root issue as "philosophical" seems to be in relation to the fact that it is a matter of a mathematical convention and not a mathematical necessity per se.

Responding to the final question of the interview, which was in regard to what the other team members contributed to her growth as a teacher, and to her MfT specifically, Gabrielle made three points.

Her first point expressed the "bifocal view" (Ball & Bass, 2002; Sleep, Ball, Boerst & Bass, 2009; Ball, Thames & Phelps, 2008), or the ability to see the mathematics through the eyes of the student, as well as through her own perspective simultaneously. This is a much desired quality of MfT (captured by all three categories of Ball and colleagues' framework for analyzing Mathematical Knowledge for Teaching – Knowledge of Content and Curriculum, Knowledge of Content and Students, and Knowledge of Content and Teaching).

> Yes, when I was working on that lesson study, and I was very new to the whole concept and how it could work, and you looked at my 1-page document, which then grew into a 4-page document… there were so many things that I haven't thought about. That breadth that you have, and also Andrew's capacity to see where the kids are, so yeah, it helped me realize that they are not all going to see it the way I see it. You remember that student X, he always had a different way of seeing things.

From this excerpt, it is clear that Gabrielle's capacity for the bifocal view had evolved, at least in part, through Andrew's influence.

Gabrielle's second point was in regard to the opportunities for learning that were created for her through the collaborative process. The goal-oriented and purposeful activity that teaching is by its very nature was enhanced for her by the shared experience of other colleagues, which they brought to bear on the process.

> I taught bits of math here and there, but most of my career I taught physics, and so I don't have … and I'm aware of that, and to be able to get, to be able to steal your experience and Andrew's experience, and be able to share - that collegiality is wonderful. I had a lot of fun teaching math at this school, and it was probably because of that.

The last point that Gabrielle made in the interview was about the collegial network that was created through this work. She spoke of Andrew as an ally to whom she can turn for professional advice when she is faced with a challenging situation in her teaching practice.

> And another thing, you know, the first year that Andrew was not here any more, and then I taught the Math 11 Honours kids, those that were Andrew's babies before, and you know, he so kindly gave me the unit tests that he gave for this enriched class, and his exams. And whenever I had a problem, because these were super-smart kids, I would not know the answer – it was usually on something fairly obscure, like … you know, something that most kids would go like, "Huh?," but there would be those few, and I would immediately go on the e-mail: "Andrew, help! There are kids who are too smart for me. And he would immediately fire back to me a really nice response, about how you would do that and explain things, and we carried on like that for the whole year, and it was really nice of him. I miss him a lot."

I can join Gabrielle with the same sentiment in regard to Andrew's unique contribution, which he espoused on our teaching practice, through his caring personality and expertise in MfT. These are reflected in both Andrew's deep commitment to the learning of mathematics by the students and to the raising of the level of instruction in the classroom. In this professional development initiative, we were fortunate to have had a master teacher in our midst, from whom we could all learn and grow, and who continues to impact our practice. Furthermore, this exceptional composition of the team greatly determined the outcome of the Practice-Based Professional Development endeavour, as well as the results of this study.

Chapter 8. Cross case comparison

Innovations in the curriculum and teaching methods are successful only when what the teacher does with these innovations is taken into account (Steiner, 1987). This chapter presents the findings stemming from the case study of the lesson study cycle which featured two enactments of the lesson on solving radical equations.

8.1. Comparison of the two lesson enactments

Kilpatrick argued that:

> Two classrooms in which the same curriculum is supposedly being "implemented" may look very different; the activities of teacher and students in each room may be quite dissimilar, with different learning opportunities available, different mathematical ideas under consideration, and different outcome achieved (as quoted in Tarr, Chávez, Reys & Reys, 2006, p.191).

For the purpose of research, I argue that we can tighten the similarities even more by looking, not only at the same curriculum, but also at the same lesson. The lesson can be collaboratively planned and discussed, taught by two experienced teachers who work in the same school setting, with a similar population of students, and still be found to be dissimilar. This is indeed intriguing. It raises these questions, "What exactly might account for these differences?", "Are they really that significant?" and "What can we learn from all of this about the nature of MfT that has emerged?"

In order to address these questions, I portray a comparison of the two lesson enactments across the five components of the 5-Component MfT Framework, used in this thesis; the first enactment of the lesson by Andrew and the second by Gabrielle. Microanalysis of the two lessons or their evaluation is out of the scope of this research; rather, in this comparison the focus is on bringing out the features that address the research questions.

Both teachers taught their lesson based on the same lesson plan. They followed the same sequence of microtasks (Watson & De Geest, 2012), but orchestrated the development of mathematical ideas differently. Andrew's teaching used learner response to develop interaction and meanings, through scaffolding student thinking, repertoire of skills, and structural understanding of the complexity and coherence of the mathematical ideas that were central to the lesson. For example, Andrew repeatedly directed the class to the central mathematical inquiry about the phenomenon of extraneous root by asking, "Could we have known it before?" This was the pivotal question that led to a *cognitive dissonance* (Festinger, 1962), which was resolved through the pivotal explanation. It offered a strong, explicit and mathematical rationale to support student understanding.

Lesson episodes can be viewed through the lens of the mathematical purposes that are present in these episodes. The episodes from the two lessons throw up differences in "what is being made available for the learners to experience and exactly how the teacher expects learners to engage with mathematical objects and ideas" (Watson & De Geest, 2012). In Andrew's lesson, there was attention to the mathematical purposes of analysis, discussion of implications, deductive reasoning, explanation and justification, use of generalization, synthesis and connection, as well as affirmation and objectification of the newly learned knowledge.

Gabrielle taught the lesson next. She was inspired by Andrew's performance and intended to follow the same lesson plan, involving the same sequence of microtasks. She even adopted the spontaneous examples that he created during his lesson, in order to enhance the learning outcomes for her students. This shows how open-minded Gabrielle was about the new learning adventure, even though she had never used this didactic approach before.

Her lesson enactment focused much more heavily on personal and public orientation toward concepts, methods, properties and relationships, and less on the analysis, synthesis and objectification of the new knowledge. In the process, she ran up against a situation in which she was unaware of what she did not know. Thus, she misled the students by requiring a proof, rather than by offering an explanation that is based on the definition. This resulted in a significant loss of instructional time. Consequently, only two of the four intended learning tasks were completed by the class.

The findings in this study are consistent with the findings of Watson and De Geest (2012), whose research results indicated that good non-specialist teachers can entice students into mathematically authentic ways of thinking, generate mathematical ideas, and effectively engage students with mathematical concepts. However, when compared to mathematics specialists, they were less able to communicate how the ideas, which the students are working on, relate to the subject in its fullest sense. The non-specialist teachers offered fewer opportunities to reason deductively, and discuss implications and connections within mathematics.

This finding is related to Brousseau's discussion about the reproduction of didactical situations as producing the same effects from the point of view of meaning. This has implications for the learning of the teacher. Brousseau argued that good reproduction of a lesson is one that, under the same circumstances, gives an identical development and the same meaning to items of knowledge that are acquired by the students. In contrast, a bad reproduction of a lesson gives an identical "development," but a different meaning, to acquired items of knowledge (Brousseau & Balacheff, 1997). To conclude this part, this study shows that, what really matters, is the way in which the mathematical content is made available for the learners in the sense of what meanings the students ascribe to what they are learning.

8.2. Technical features of the two lessons

Table 5 below presents a quantification of three technical aspects of teacher activity for each of the lesson implementations: number of turns of "teacher talk," number of words uttered by the teacher, and the number of questions the teacher asked. Regarding the student engagement, also three features are quantified: number of times a student comes to work on the board, the amount of time individual students work on the board, and the number of students who submitted their notebook to receive teacher's feedback. Both lessons were of the same duration (60 minutes) and were following the same lesson plan.

	Andrew's Lesson	Gabrielle's Lesson
# of turns the teacher takes	84	159
# of words the teacher says	2889	4456
# of questions the teacher asks	35	62
# of times a student comes to work on the board	6	4
Total time students work on the board	15.5 minutes	6.2 minutes
# of times students submit notebooks	14	0

Table 5: Comparison of Selected Lesson Features

The salient features of the two lessons show that Andrew required that his students demonstrate their knowledge (students submit their notebooks for examination for each task that is worked on; students come to the board to work on the problem more frequently and for a longer time; students are being called up to the board). In contrast, Gabrielle expects only a declaration of knowledge (students mostly answer from their seat; students come up to the board less frequently; students come up to the board only when they volunteer to do so). It is a unique feature and an established practice in Andrew's classroom that students voluntarily bring their notebook to the teacher's desk for feedback on the task that is being worked on. Andrew checks individual students' work on the spot in the classroom and provides feedback on their work to the entire class, which acts as a kind of formative assessment not only for an individual student but for the entire class.

The complexity and coherence in the lesson development is different. This is indicated by both the amount and the complexity of the questions that are asked by each of the teachers. Could it be that the sheer amount of teacher talking can interfere with how much mathematical activity can be pursued by the student? In his meta-analytic study of effect sizes on learning, Hattie found that, what matters the most is not what the teacher does specifically, but rather, what the teacher gets the students to do (cited in Watson & De Geest, 2012). Gabrielle herself noticed and reflected on her teaching practice, pointing out that she does most of the work. In contrast, in Andrew's class, students do most of the mathematical work.

8.3. Teacher's mathematical knowledge matters

There is a consensus in the educational research literature that content knowledge of mathematics is a core component of teacher competence. In particular, what is required is the conceptual understanding of the topic that is to be taught. Teachers need to know from a more advanced standpoint the content that they are responsible for teaching. They need to understand how it relates to both prior and subsequent development along the learning trajectories (Baumert, Kunter, Blum, Brunner, Voss, Jordan, Klusmann, Krauss, Neubrand, & Tsai, 2010). Although the presence of the subject matter knowledge of mathematics is not, in itself, a guarantee for effective teaching practice and can remain inert, the findings of this thesis show that knowing how to teach mathematics (PCK) is inconceivable without knowing mathematics (SMK).

The two teachers in the main study differed in the breadth and depth of their conceptual understanding of the subject, which reflected acutely on the quality of their instruction in this particular lesson. This argument can be supported by the differences in the time management of the two lesson enactments, the way the teachers used the symbolic language, precision of notation, and by how the content was made available for learning.

Andrew's class effectively resolved the distinction between the square root being a positive number and the case of solutions of a quadratic equation having both positive and negative roots in a matter of three minutes of instructional time. In the second enactment in the other class, 14 minutes were used and the content was inaccurate and incomplete. In addition, Gabrielle's facilitation of productive classroom discourse was compromised in several instances, as was her ability to interpret student responses, and analyze student errors and difficulties. Her use of notation was unclear. It conflated the ideas that were central to the lesson.

On the theoretical level, the two enactments bring out the implications of mathematical content knowledge on instruction. However, on the level of personal understanding of the concepts that were taught, the collaborative activity of the teachers flashed out the awareness of the impact that a flaw in mathematical content knowledge can have on the quality of instruction. At the end of the previous chapter in Section 7.4., the Epilogue, I presented my interview with Gabrielle, which was conducted at the conclusion of this study, and in which she self-reported how she resolved the issue for her future practice.

8.4. On teachers' learning of mathematics (SMK)

Gabrielle became aware of the flaw in her subject matter knowledge through the process of lesson study. However, she could not articulate it. The whole team experienced the persistent nature of this particular flaw in her knowledge because it kept reoccurring in different forms throughout the lesson study cycle. It seems that we, as a team, failed to remediate Gabrielle's content knowledge flaw before her teaching of the lesson. She asked for clarification and pointed out repeatedly that this was "a very subtle thing" for her and that she was unsure if she would be able to convey the topic adequately to her students. However, either she was not heard by her colleagues or it was not well explained to her. Although it was pointed out by other members of the team that that is how it is defined in mathematics, this was not an adequate explanation. What should have been discussed is the role of definition in mathematics and, in particular, why it is that we define square root to be non-negative. This was discussed in section 5.2.3 under subtitle "Mathematics that was Absent from Discussion."

In addition, the teachers uncovered mathematically inadequate treatment of the topic of solving radical equations in the commonly used textbook for school mathematics. This is a direct result from their lesson study activity. Examples of errors and flawed treatment of mathematical concepts in student textbooks have been noted (Hyndman & McLoughlin, 2012). For teachers to be able to develop a critical eye for evaluating instructional resources is a desirable outcome of professional development effort.

8.5. On teachers' learning of how to teach mathematics (PCK)

To be able to enter into the world of the learner, again and again, is a critical component of the ability to teach because this is being called for in the moments of teaching once one has already become an expert in the content that is being taught. Davis & Renert (2014, p. 24) point out:

> Teachers need to be able to do more than to unpack a concept and to pry apart various instantiations. They also have to be able to provide learners with the expert assistance needed to blend instantiations into a coherent concept. In other words, the expertise associated with using mathematics is quite different from the expertise associated with teaching mathematics. In the vocabulary of expert vs novice, teachers are experts who are able to think like novices.

However, it is not enough for a teacher to be knowledgeable about the multiple interpretations and representations of basic operations and various concepts, such as those provided in teacher manuals. In addition to what the above quote communicates, we should also keep in mind that teachers need the expertise with *using* mathematics too, and making judgments about when to expect automaticity with already learned processes, because they need to be able to lead the students out of thinking like novices, to being able to apply that knowledge. In this study, an example of this was encountered in teachers' discussion about expanding a binomial with automaticity in order to free up the capacity for development of a new concept. This is part of a teacher's philosophical stance regarding the role of school mathematics as a way to prepare students for life and careers.

Chapter 9. Conclusion and implications

My motivation for this work came from my core belief that, if it is taught and learned correctly, mathematics is a subject that empowers humans and liberates the mind. Like music, it is for everyone. I care about how students experience mathematics and what amount of it is available for them to learn in a classroom.

As a developed country, we value mathematics and commit a great deal of school time in the life of a young person, from a very early age, all the way through the entire span of compulsory education to learning of the subject. It is of great importance to pay attention to the results of such a huge investment of human time. Are we really graduating students who have the confidence, interest, skills and knowledge to pursue career paths of their choice? Do we, as a society/country/economy, get the required/optimal number of well-prepared students entering and succeeding in the studies of sciences, engineering, architecture and so on, all of which require a great deal of mathematical competence?

Clearly, what is made available for the students to learn, and even what kind of mindsets about mathematics they will acquire depends, to a great extent, on the teachers by whom they are taught during their school life. Teachers enact mathematics in the classroom and this is directly related to their MfT. Teachers acquire their MfT in layers, over time: from a blueprint set through their own schooling experience, followed by formal university education in specific subjects, to the teacher training programs, and then finally through their own teaching practice, which could last for thirty years or more.

Therefore, it would stand to reason that, if we wish to improve the student achievement in mathematics and their prospects in life, we need to use teachers at large as a lever to do so. Consequently, in-service teacher professional development can hold some promise. We all know that the accumulation of years of experience does not, in itself, guarantee good teaching. Rather, it is a matter of deliberate cultivation, which a teacher can do in a variety of ways that suit her/his needs and personality.

There is an increased attention to professional development initiatives that rely on building communities of teachers. While researchers may not yet fully agree on what mathematical knowledge, skills and "habits of mind" teachers need to have in order to teach mathematics effectively, there seems to be a growing consensus that embedding teachers' learning into their everyday work, through a careful examination of their practice and classroom artifacts, increases the likelihood that this learning will be meaningful (Smith, Stein & Silver, 1999; Lampert & Ball, 1998; Lieberman, 1995).

In this research, I used lesson study as a window through which to study teachers' learning in a setting that is characterized as collaborative, comprehensive, cumulative and ongoing. What and how do in-service secondary mathematics teachers learn about mathematics for/in teaching, through participating in a practice-based, professional learning community of lesson study? We do not know much about successful local implementations and adaptations of lesson study, especially in the context of secondary school mathematics, in terms of what the teachers gain from engaging in this form of professional development, what makes it sustainable and what of it transfers into practice.

In the remainder of this chapter, I recap the main research findings, identify the contribution that this study makes to the field of mathematics education, state the implications for practice and for policy, indicate areas for further research, and discuss the limitations of this study.

9.1. Answering the research questions of the study

When setting out to do this research, I was driven by wanting to find out how mathematics for teaching evolves in individual teachers and whether it can be promoted within a lesson study setting of in-service teacher development. In each empirical chapter (Chapters 5, 6, and 7), the answers to the sub-questions flowing out of the particular phases of the lesson study cycle have been addressed. I now offer my answer to the overarching question, “What and how can in-service secondary mathematics teachers learn about mathematics for teaching through participating in a practice-based, professional learning community of lesson study?”, which encapsulates the results of this thesis. In what follows I offer the answers to all three specific research questions of this study and present what this group of teachers gained through participating in this professional development opportunity.

9.1.1. Shifts in teachers’ cognition and practice through the stages of lesson study process across the five components

In response to, “What is the nature of change in the teachers who participate in the school based lesson study initiative?”, I look across the stages of the entire cycle of this lesson study process to follow the nature of teachers’ knowledge growth across the five components of their mathematics for teaching. Naturally, these shifts are intertwined and co-dependent. Their occurrence was mediated by the collective interpretation of the curricular outcome and other teacher activities in the stage of planning for instruction, the enactment of the two lessons while being observed by colleagues, and the post-lesson reflection. I now flesh out what crystalized from beginning to the end of this lesson study cycle and recast it through the lens of the five component framework for the study of the teachers’ MfT.

9.1.1.1. Mathematical

In the beginning of this lesson study cycle in their pre-lesson discussion the teachers chose the solving of radical equations as the topic for their research lesson. With this they immediately faced the decision about how they will teach the topic (the didactical and mathematical components intertwine here). They reasoned about and negotiated the choice of the teaching approach to be used in the classroom (Process A versus Process B). The arguments that tipped the gauge were the teachers’ realization that Process B would better foster the mathematical, elegant way of thinking (philosophical component enters here). It was recognized by the teachers to be a more demanding approach, and by implementing it they would raise their level of instruction. This inspired the teachers to explore the example space of a special class of radical equations with which they had little experience before. The teachers designed tasks whose solution is the empty set, but they did not really use them all in instruction. However, these tasks seemed to deepen the pedagogical content knowledge of the teachers. It also provided a rationale for teaching the topic in this more demanding way (pedagogical component enters here) which would promote students' mathematical understanding through this sound cognitive approach (mathematics as a sense-making activity; psychological component enters here). This is a special contribution of lesson study in that it gave the chance for teachers to play with mathematics and deepen their conceptual understanding of the topic.

During the first implementation three prominent mathematical events emerged: use of pivotal explanation, teacher's insertion of two spontaneous examples, and teacher's enactment of mathematics in the solving of one spontaneous example. During the second implementation the mathematical component was compromised in the lesson and left the students confused. The teacher came up to a challenge that she could not resolve on the spot and she promised that she would come back to it. This is an example where lesson study tendered the teacher and the entire team the space to experience the challenge of unexplored and unpracticed terrain (that teaching with Approach B instead of Approach A provoked), which in the end enabled teachers' learning.

Instances of teacher learning are also evident from the artefact that is presented in Appendix E. This is the teachers' report, which represents a reification of the team's consolidated collective learning and what emerged from the post-lesson discussion. A thorough analysis of this artefact shows how the entire cycle of this lesson study gelled into a summary of outcomes, as an amalgam of a lesson plan and the lived experience of its enactment, with commentaries that were based on what was observed. Observations included such things as what to pay attention to, turning points in the lesson progression, mathematical implications, what to expect from the students and what kinds of errors may occur (i.e., the students' mistake of saying):

"$\sqrt{(-3)^2} = -3$, therefore $\sqrt{9} = -3$ "

The above example of erroneous student thinking has entered this document on the basis of what actually happened in the class.

9.1.1.2. Didactical

When the teacher team decided in the pre-lesson discussion to go with the different approach for teaching the solving of radical equations from the one that they were used to before, this became a learning opportunity for them as they could broaden their repertoire of the methods and techniques for teaching this particular topic. This approach required a different kind of background knowledge to be activated in the classroom prior to teaching; in addition, wider connections within mathematics needed to be established in the teaching. Under this new approach, there was the requirement to consider the domains of definition and the range of possible values of the corresponding functions. The didactical approach also generated the possibility to visualize what the mathematics is doing, which was then represented as the limitations of possible solution intervals graphically on the number line (but not in the coordinate plane). This opened up the space for the teachers to enrich their expertise in representing of the mathematical ideas, explaining of the concepts involved, and facilitating of student learning.

In the first implementation the teacher met the goal set in the planning stage. He raised the level of complexity of working on the tasks by encouraging the students to employ analysis, reasoning and mathematical rationale, with the aim to help students reach a mathematically meaningful interpretation for the occurrence of extraneous roots. In the second implementation, the teacher relapsed into the emphasis of the procedural aspect of the topic that was being taught. As a consequence, the student learning was impoverished.

The teachers' report on their learning experience that was produced at the conclusion of this lesson study cycle demonstrates that this team took a critical position about how this topic is treated in the textbooks (see Appendix E). They came out of the experience convinced that their approach better supports students' mathematical thinking than the one offered in the textbook.

9.1.1.3. Psychological

In the stage of planning for instruction, the teachers were motivated to design a lesson which would foster students' understanding of the topic as well as their level of engagement in the learning process. Teachers anticipated where potential student errors could occur and they built into their instruction ways to prevent misconceptions that could form in the students' interpretation of the concepts involved.

During the enactment of the first lesson, the teacher stimulated student thinking with questioning and scaffolding their learning up at the board. He also motivated them with constant formative feedback. He made use of a student's error by turning it into a teachable moment for the benefit of the entire class, and he used humour to disperse any discomfort that the students may feel. This was one of the best students in the class that everybody looked up to, and so it can be interpreted as a teacher's motivational technique. In the second implementation, the teacher guided the students through the process of solving radical equations, doing most of the mathematical work herself. She also frequently questioned the students; however, the nature of her questions was such that they required declaration and not demonstration of knowledge.

In their concluding document (Appendix E) the teachers agreed that the students should be exposed to different kinds of radical equations in order to learn to pay attention to the different nuances that can be embedded in these equations. They felt that was important for the development of flexible thinking and preventing of rigid and expected solution paths.

9.1.1.4. Pedagogical

In the pre-lesson discussion, pedagogical topics did not emerge, which could be explained by the fact that the practice of teaching was not the focus of the conversation. In the first lesson enactment, the pedagogical component of MfT manifested in the teacher's intentions and actions to engage the weaker students in the learning process right from the beginning of the lesson and hold their attention throughout by questioning them and scaffolding their learning. The teacher made sure that all students were participating doing their mathematical work. He encouraged peer collaboration and tutoring amongst the students while they were working on the mathematical tasks. After equipping the students with the necessary requisite tools (i.e., prior knowledge that is needed for the development of the main ideas of the lesson) he let the students independently discover the process by which they could set up the restrictions and proceed with solving of the equation without violating the logical process, and which led to knowing how to recognize without testing the faulty solution(s). In the second lesson enactment, the teacher showed a more permissive pedagogical style which was marked by its more informal language. She engaged only the students who volunteered to share their answers. In addition, the teacher let the students in the class handle independently the mistake made by the student working on the board. The stage of lesson enactment offered the teachers the opportunity to observe and compare their own pedagogical style and motivational techniques (psychological component) with that of the performing teacher's and see first-hand the influence of that on student learning and engagement. Common understandings emerged among the team members regarding the influence of teachers' questioning and students' mistakes on the learning process.

9.1.1.5. Philosophical

This component of MfT emerged in the pre-lesson discussion. It was evident that it shaped the teachers' decisions about the goals for instruction and the didactical approach that would be taken for this research lesson. The philosophical component is implicit in every teacher's practice, and in this study it showed up in both teachers' enactments as well. In the second enactment, a shift in the teacher's philosophical stance was observed. It seems that choosing to participate voluntarily in this kind of collaborative professional development also reflects a teacher's philosophical stance. In particular, it promotes their readiness to try out and develop new approaches. Nonetheless, the philosophical component is very rarely expressed in the teachers' discussions explicitly, although it is an important component of MfT because the teacher's viewpoint on why we should learn mathematics influences the students' perception and attitude towards mathematics.

The findings of this thesis are consistent with Thomson's, who came in her empirical study of three teachers to the following conclusion, (as cited in Steiner, 1987, p. 8):

> Teachers' beliefs, views, and preferences about mathematics and its teaching, regardless of whether they are consciously or unconsciously held, play a significant, albeit subtle, role in the teachers' characteristic pattern of instructional behavior. In particular, the observed consistency between the teachers' professed conceptions of mathematics and the manner in which they present the content strongly suggests that the teachers' views, beliefs, and preferences about mathematics do influence their instructional practice.

On the other hand, the study of this thesis indicates that in a collaborative professional learning setting through the process of lesson study this component can be developed. This is significant given that conceptions and beliefs that teachers hold about mathematics and mathematics teaching are difficult to change and yet they are intimately related to teachers' work and instructional practice.

9.1.1.6 Conclusions with respect to the first research question

The professional learning of the teachers is a very complex phenomenon and we have not developed reliable tools with which to capture and measure it. The data of this study show changes at a descriptive level. This means that they were noticed. In each of the five components of teachers' MfT, the learning occurred. In the mathematical component, teachers explored the example space of a special class of radical equations, those whose solution is the empty set, and consequently deepened their knowledge of mathematics. In the didactical component, the teachers experienced a new approach to teaching an old topic, by which they developed a new technique for presenting and situating the development of concepts to be taught. In the psychological component, the teachers' understanding of student learning, thinking, and prevention of possible errors expanded. In the realm of the pedagogical component, the teachers furthered their knowledge about how to keep their students engaged in the learning. Finally, in the philosophical component, which is arguably most difficult to change, a teacher presented a shift in her philosophical stance in her lesson enactment.

Further research would be needed in order to confirm the evidence of teacher learning and, especially, its transfer into practice. It could be that the only reliable measure of the professional learning of the teacher would be gains in student achievement, as a result of their increased understanding. Nonetheless, the claim here is that professional growth is implicit in the process of lesson study because it guarantees at least content related conversations among teachers and, at best, genuine mathematical activity, which can open up a new way of seeing the mathematics that they are supposed to teach.

I argue that individual schools could go a long way in transforming mathematics teaching practice and student achievement by harnessing the potential of learning in practice, for practice by a professional learning community of in-service teachers. Incremental changes that were observed in teachers' MfT hold a promise for building confident, effective and inspired teaching over time of sustained professional development activity. As mentioned earlier, two other lesson study cycles were conducted by the same team of teachers during the research period, as shown in Appendix C. These further corroborate and support the major findings about teachers' knowledge growth in this particular collaborative setting.

9.1.2. Factors influencing the development of teaching practice in the lesson study setting

I now address the second question, "What are the factors influencing the development of teaching practice in the lesson study setting? This might include visible and invisible features, such as beliefs and attitudes." The following have been found to be essential factors that influence the teachers' knowledge growth:

1) The entire team of teachers teaching mathematics is involved, but participation is voluntary. It is assumed that, if teachers want to raise the level of their teaching practice, they are driven and open to learn. Certainly, the premise is that the teachers have the conditions, time and energy to engage in such activity.

2) There is at least one member in the team that can act as a catalyst, based on his/her profound understanding of mathematics, extensive experience in teaching and aptitude for seeing the students' points of view, all of which are recognized by other members of the team.

3) Teachers verifiably learn from watching one another teach in this setting. It seems that the performance piece, or the enactment of the lesson, is an essential part of this learning.

4) There is a detailed lesson plan developed by the team and used to communicate the goals of the lesson and the lesson progression as envisioned by the team (and made visible for the external observers too). It is assumed that the teacher's enactment is a response to both the lesson plan and students' learning process.

9.1.3. Nature of MfT that has emerged from the study

Finally, I address the third question of the study, "What is the nature of the mathematics for teaching that has emerged?". Through all the stages within the lesson study cycle it was observed how multifaceted MfT is and how the components that were identified interact and overlap in teaching practice. Furthermore, all the five components were strengthened both on the collective and individual levels. The collaborative work stimulated harmonization of the components of MfT.

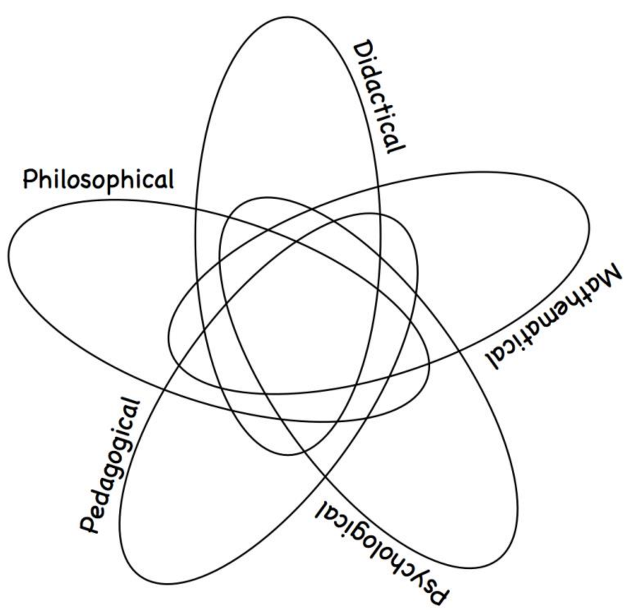


Figure 13: The Five Components in Harmony Comprise MfT

Figure 13 above shows how the five components of MfT are ideally in balance in a teaching practice. I argue that a teacher should strive to equally develop all components of MfT in order to develop responsive and responsible teaching practice. Teachers develop their MfT through their teaching practice, as well as through their formal education in mathematics at university, teacher training program, and in-service professional development. A practice based professional development activity such as lesson study is a comprehensive way to support MfT development throughout the professional career of a teacher and to help harmonize the components. The nature of good teaching practice seems to be in taking care that not one of the components is underdeveloped or neglected. This is my vision for teaching practice and my theoretical contribution stemming from this research.

9.1.4. Other research findings

This study reaffirms that acquiring mathematics for teaching is a complex and dynamic process in which teachers learn incrementally about mathematics and how to teach it both on the individual and collective level. In their unique expression, given by the specific topics of the research lessons and specific setting, the findings exposed the following features of the professional knowledge and/or learning of the teachers, which was afforded through this Practice-Based Professional Development. The following findings of this research are further articulations of what has already been identified in some prior studies:

- Teachers' conceptual understanding of mathematics and the quality of their subject matter knowledge is a critical part of their MfT and influences what is available for the students to learn and experience in the classroom.

- Teachers' deliberations over teaching a concept differently, with specific and overt aims to promote understanding and mathematical thinking in their students, resulted in deepening their own understanding of the associated mathematical content and promoted more meaningful interpretation of the curriculum and how to implement it in practice.

- Teachers departed from the habitual ways of presenting the topic by critically evaluating other approaches brought up by other members of the team. Through this process, they developed a shared understanding about the teaching approach that they decided to test out through implementing the research lesson.

- When teachers created their learning tasks, new mathematical insights into the topic emerged.

- Teachers exceeded the (standard) textbook presentation of the topic in different ways for each of the three specific research lessons (see Appendix C) that were designed by the team during the research period by:

    - teaching a concept through problem solving, and moving away from purely empirical learning to learning that encourages abstractive reflection (lesson study on parabola),

    - teaching a geometric concept through proof and deductive thinking, moving away from learning a concept by purely inductive reasoning (lesson study on circle geometry), and

    - shifting from procedure-oriented teaching (in a recipe-like fashion) to a sense-making logic-oriented teaching (lesson study on radical equations), moving away from instrumental learning (via a black-box mechanism) toward relational, sense-making learning (via understanding why, how, and when a certain procedure works).

- Teachers created challenging mathematical tasks to nurture mathematical thinking in their students and they scaffolded their learning. There is indication that shared understanding about how to engage all students in the learning process emerged from their collaborative work. Teachers derived a broad instructional implication that "it is the students that must do the work, not the teachers" (which is a replicated result from reviewed literature on lesson study).

- Teachers tried to predict student misconceptions and how to circumvent them (in the pre-lesson discussions), and they reasoned about student thinking processes that they observed during lesson enactment. Through this process, they strengthened their own mathematical understanding and gained insight about how students think.

- Deficiencies in a teacher's mathematical content knowledge impeded the teacher's ability to analyze and make sense of student mathematical thinking.

- Teachers presented the new topic with attention to coherence within mathematics. The new concept was introduced in a framework of existing related knowledge and there was an explicit consideration of the use of the new knowledge for further learning.

- Teachers formed a collegial network that extended past this study and they continued to use one another for professional learning. This phenomenon indicates a need for mentorships and peer coaching.

- Teachers engaged in productive mathematical conversations, which influenced their practice in the classroom, beyond that which was included explicitly in the lesson plans that were produced (some transfer of the "philosophical component" had occurred between the teachers).

The study confirms all of the above characteristics of the knowing and learning by teachers, in spite of it being a small-scale research setting. Lesson study work contributed to tangible transformations of teachers' MfT. The processes of collaborative planning, observed enactment and post-lesson reflection seem to provoke myriad of opportunities for teacher professional growth. The study of teaching itself becomes a science. Teachers can hypothesize about various problems of teaching and learning, and then test them in practice. They can vary some conditions and run through an experiment in the classroom again, refining their personal knowledge of mathematics and their craft knowledge of teaching, through an iterative process. This continuously moves them in the right direction.

As in any scientific research, there is a peer-review process in the end. That is the post lesson colloquium in which the teacher who enacted the lesson has a chance to gain the most, while the rest of the team members critique their own teaching design and the enacting teacher's performance. This enables the professional learning of the teachers to occur.

9.2. Contributions of this study

This small scale research study provides an existence proof that desirable changes in MfT are achievable in local settings. Moreover, the changes that were detected span across a wide spectrum of facets of MfT. This proves that lesson study as a vehicle for in-service professional development practice for teachers can be extremely effective and viable.

All of the teachers involved in the study learned through their engagement in lesson study, but the extent to which each teacher learned as well as the nature of the content of their learning depended on the MfT developed in each of the participants. They each learned different things. Understandably, in such small group of teacher participants, the master teacher has fewer opportunities to learn from other teachers, but can still learn something from observing other teachers enactments.

The unexpected and surprising outcome of this study is a realization of how slow the learning of mathematics can be, especially if one has to undo, or relearn differently, an important aspect of the subject after years of (mal)practice. If a teacher has a deficiency in the subject matter knowledge, we cannot expect that it will be rectified through the practice of teaching (i.e., teacher learning from her/his students) or even from some initial conversations and explanations from colleagues.

Another contribution is the creation of the 5-Component MfT Framework with its inclusion of the philosophical component that surfaced in this study as an important part of teachers' mathematics for teaching, affecting their planning as well as their performance and decision making in the classroom. Furthermore, none of the known MfT frameworks spell out the philosophical component of teacher's MfT as the present research does. Although this component is different in kind than other components that are modelled in these frameworks, it is nevertheless important and needs to be accounted for and attended to.

9.3. Implications for practice

Schools and school districts, which are looking at ways in which to improve teacher effectiveness and student achievement, might already be looking into possibilities for implementing some sort of systemic, ongoing, collaborative professional development practice. It is not a trivial matter to develop a culture of continuous, cumulative, job-embedded, in-service teacher education, such as lesson study. As such, it will likely require a great deal of wisdom and support from those persons who are charged with the responsibility of the setting of teaching loads and assignments, as well as the calendar of school events and general priorities, in the effort to uphold the educational vision of their institution. In this section, I discuss the benefits of implementing a Practice-Based Professional Development model, such as the one in the West Coast Academy's experience, as well as the challenges that need to be overcome if such practices are to be sustained.

It is a fact that professional development costs resources, most notably time and energy, which could be put toward other things that are important for education. If not planned wisely, lesson study may disrupt normal operations of the school.

A common concern is how to arrange for coverage during the implementation of the research lesson when all of the members of the team are to be present because their scheduled classes would be left unattended. At West Coast Academy, there were difficulties in ensuring that the post-lesson discussion was timely and not rushed. Rarely was it possible for teachers to meet right after the lesson implementation and, often, there was not sufficient time to meet between classes. Alternatively, if the post-lesson discussions were left for after-school hours, teachers had other duties, such as coaching or running extracurricular clubs. Ideally, the post-lesson discussion would be held right after the lesson implementation, when the evidence that the observers have gathered is fresh and current. In contrast, in Japan the lesson study events are set in the school calendar from the start of the school year. The research lessons are set for different subject areas, on the same day, two or three times a year. Furthermore, all classes are dismissed for teachers' professional development, apart from the classes in which the research lessons are being implemented. This arrangement frees up all of the teachers to observe these research lessons in their subject area and to participate in the post-lesson discussions.

At West Coast Academy, the initiative for implementing a school-wide Practice-Based Professional Development model has come from within and, in our case, the administrators nourished and assisted the process. They appreciated that someone had taken the time to think things through and was willing to provide leadership in the area of professional development for the school faculty. The fact that the initiative came from within, rather than being imposed from the top or brought from outside, was favourable because there was freedom to dovetail the work to match what seemed to be most relevant to the teachers, given their current set of circumstances. Regardless of how a systemic Practice-Based Professional Development model is introduced, it is important to determine the needs and weaknesses of the teachers to create an educational program for in-service teachers.

This kind of Practice-Based Professional Development model has the potential to get the entire faculty involved. Therefore, it can provide a comprehensive and low-cost professional development strategy for the entire school. No one would be left out and the school could begin to move toward a planned continual improvement. This is important, particularly for independent schools, not only because there is competition for students between them, but also because this could be one of the defining features that sets an independent school apart. All private schools do this in some sense. They define themselves upon some set of values (religious or otherwise), and then serve the specific population that shares that set of values.

Ideally, the faculty also upholds the same set of shared values. This means that, to a large extent, a "university preparatory school" values high academic standards for all students and is committed to providing high quality education that will enable their students, to not only enter the university of their choice, but also to succeed in their chosen program of study. This kind of environment naturally calls for faculty that embodies the principles that are associated with these values, one of which is the continual cultivation of professional knowledge.

Teachers can benefit in many ways. Professionally, there is an improved knowledge of the subject matter and instruction, the capacity to observe students and understand their needs, and an improved ability to connect the daily practice to long-term goals for student development and to short-term goals for the learning of a particular unit of study or lesson within that unit. In addition to teachers developing their individual practice, there are teams of teachers working together to expand their collective repertoire of teaching techniques and resources. More experienced and knowledgeable teachers could act as mentors and, collectively, teachers could choose to work on manifesting the long term goals that they set for student development.

In their teams, teachers interpret the curriculum and study the resources. They share their personal knowledge of mathematics (or of another subject) and of how they understand or have come to understand a certain concept. Through this process, they make the "what is there to teach" a live experience that is shared by practicing teachers within a school community. When "what is there to teach" is re-examined from multitudes of perspectives on "how to teach it," the curriculum can come alive. Teachers can build a deeper understanding of the mathematical content through the variety of ways in which it can be opened up for learning (didactical transformation). This expands their repertoire of teaching methods and techniques. However, in order for that to happen, there needs to be a culture of teachers that is studying the curriculum and reinventing themselves as learners in the process. Through such work, teachers become more knowledgeable and confident. They become deeply invested in the content that they are about to teach. They build up a level of enthusiasm and an urgency to communicate it to their students, through their carefully designed learning activities and tasks. Observed by other team members, the process of classroom instruction and its occasioning of student learning in this way becomes like a scientific experiment.

As we know, the prescribed curriculum is a fixed document, which is consulted once in a while, usually to see if everything that needed to "be covered" was actually "covered," or to see if a textbook that is in use is "well aligned with the curriculum". We witness curriculum changes every 6-10 years and, yet, there is little change to practice. It is not so much about what the curriculum sets as it is about how the teachers interpret that curriculum, which will affect change in practice. Therefore, it would be wise to accompany any implementation of a new curriculum with some kind of Practice-Based Professional Development, as is proposed here, given that the way in which teachers interpret the curriculum bears so much on the way they implement it.

Where the Practice-Based Professional Development model includes observing classrooms in action (with a visible lesson plan present), as is the case with lesson study, there is a palpable sense of the blending of teaching and learning. Teachers are clearly putting themselves in the position of a learner, a collective learner as well, and they are using the classroom environment, their students, and other colleagues to improve their teaching practice. Students benefit from seeing their teachers stretch in their efforts to figure out how to best instruct them. In my experience, students feel honoured and are grateful for receiving this kind of special treatment. On some level, they probably perceive themselves as contributing to the better learning of other students, perhaps many generations after their own. Indeed, this is so. Most likely this is not at a conscious level. However, it does stand to consider that classrooms are complex environments and what we learn from our students has far reaching effects for our future practice.

Further, the textbook that is used in the class shapes the instruction to a large extent. There were a number of issues pointed out about the way in which the topics, which were the object of learning in the three lesson studies that were conducted during the research period by this team of teachers, were presented, developed and treated in the standard textbooks. A few examples include the difficulties concerning the definition of square root, reliance on inductive approaches in teaching of geometric theorems, and reliance on perception-based thinking in presenting the effects of the leading coefficient in the quadratic function. Additionally, there is an overemphasis on the quadratics that can be solved by factoring, when, in reality, such quadratics are rarely encountered.

In solving radical equations, not much perspective is being given on the reasons for the appearance of "extraneous roots." The point that is being made is that these were the textbook deficiencies, which the teachers discovered through the process of lesson study. Schools need textbooks that meet the following criteria: (a) the way the content is presented has to support the development of mathematical thinking, (b) mathematics needs to be treated with a sense of integrity to the subject, (c) topics need to be complete in their treatment and application, (d) they have to be precise in their use of notation, (e) it would be desirable if they involved an explanation of the phylogeny of the underlying historical development of important ideas that shaped mathematics, and reflect a way of working mathematically that will serve students in the long run (not just follow a sort of "answer getting" paradigm). As is shown in our study, the lesson study process can open the eyes of the teachers to be able to discern such shortcomings and become more critical in evaluating the adequacy of a resource for use in the classroom.

In addition, this study showed how the "teaching-out-of-field" phenomenon can have a serious negative impact on student learning and that pedagogical content knowledge (how to teach a topic) is inconceivable without the subject matter knowledge. We have teachers with diverse educational backgrounds who are in the practice of teaching mathematics and who have not had a prior assessment of their MfT. This situation creates the need for mentorships. With their deep and vast knowledge of MfT and experience, they can contribute to the professional development of other teachers. Mentorship programs would also be beneficial for the beginner mathematics teachers. Lesson study is a means to guarantee content related communication among teachers and could, therefore, be used as a vehicle for providing such mentorships for the enhancement of knowledge of mathematics as a discipline and of how to teach it to others in a practical sense.

Lastly, inclusion of "knowledgeable others" from university teacher education programs could also greatly improve the quality and the level of MfT that is afforded through the practice of lesson study, especially where lesson study open houses can be held, featuring larger forums for the study of teaching and learning.

It would be safe to conclude that every lesson study team draws on the strengths of its participants, which vary from one team to another. Every lesson study cycle is unique because it is embedded in the live experience of the particular content that is being developed by particular teachers, and then taught to particular students. Therefore, the knowledge pool might become limited over time, for a group of teachers, if no fresh sources are engaged.

Mathematics educators and mathematicians can contribute to the quality of the learning experiences of the teachers, with their knowledge of how mathematical concepts had developed historically and how they connect within the wider field of mathematics. For example, in our main study, the role of definition in mathematics could have been expounded and would help the teachers to understand that mathematical definitions are not arbitrary, but are there for a good reason, such as to provide coherence within mathematics.

9.4. Changing the culture: Ongoing learning for in-service teachers

What is required is changing the culture from practicing in isolation with no feedback from colleagues, to a collaborative approach that would allow for a shared experience and a space in which teachers can enrich their disciplinary knowledge of mathematics for teaching. Lesson study is intended to do just that, not as a prescription for how to teach mathematics, but to develop the sensitivities and sensibilities of the teachers, through the process of inquiry into what constitutes effective practice.

Some teachers take on the hard path of learning only from personal experience, extensive study on their own and practicing in isolation. Perhaps it is possible to develop professionally through relentless striving for clarity, a sort of deep and honest self-reflection, and constant examination of what worked in practice and what did not. In any case, the professional development of practicing teachers is very important. However, it is insufficiently attended to. If we agree that “improving the quality of teaching must be front and center in efforts to improve students’ learning … much of what our society expects children to learn, they learn at school, and teaching is the activity most clearly responsible for learning" (Stigler & Hiebert, p. 4), then the idea of a community of practice as a vehicle for improving the quality of teaching is worth exploring further.

It is certainly more effective if these kinds of efforts become the enterprise of a larger society so that greater numbers of teachers have access to a network of people with shared goals, and a range of already developed expertise and accumulated experience in various aspects of the practice of mathematics teaching, as well as who can provide feedback or act as sounding boards in this process of communal developing of teachers.

9.5. Implications for further research

This research shows that the feedback that was given in the post-lesson discussions was very mild, toned down in its critique, focused more on student engagement and teacher performance, and not touching on the subject matter knowledge and on fostering critical thinking in the students, which were the real issues in the lesson enactment described in this study. This raises the questions, "Why did the constructive critique not happen as a part of the learning process for the teachers?" "Is giving and accepting constructive feedback and critique a learned skill that requires special training to develop?" This thesis did not expound on this issue. However, it is obviously very important for the developing of MfT in collaborative settings in which a teacher's performance in the classroom is being observed for the purpose of professional development.

While performing, a teacher reveals his/her values, beliefs and attitudes (philosophy of school mathematics). This has a direct influence on student learning and it shapes the class ecology through the history of the relationship between the teacher and the students. Bromme (1994) points out that a teacher's philosophy of school mathematics, which captures both the teacher's philosophy about mathematics and the learning of mathematics, is an implicit content of teaching as well because it includes normative elements; moreover, "the effect of teachers' philosophy of school mathematics on their teaching is much more strongly verified empirically than the amount of subject-matter knowledge" (p. 79).

The data from the post lesson-discussion showed very little constructive feedback on the teaching performance, even within a group of teachers that had developed trust and had a commitment to professional growth and was supporting it in one another through the collective learning process of lesson study. In our culture, professional and personal identity are so closely bonded that a critique of a colleague's professional performance is perceived as criticism. It seems that we have to learn to delineate constructive critique from criticism in order to harness the experience of collective reflection for the learning process.

In contrast, in Japan, where the collective culture is highly developed in the teaching profession, such reflective practice and professional critique of the performance of the colleagues is accepted and valued as a part of the professional development of the teachers. Moreover, teachers in Japan go through a lengthy mentorship process before being launched into practice, through which they seem to have acquired observation skills, and have become equipped to give and accept constructive critique and feedback. In addition, they have become more open to public scrutiny of their practice versus the isolated, closed practice that teachers in our culture experience. We would advance much further if we made the shift from perceiving a constructive feedback as a way of learning and improving our professional practice, instead of as a personal criticism.

Does the cumulative collective reflection on enacted teaching practice engender the development of self-reflection? This would be important to find out, based on what we already know about the influence of reflective practice on effective teaching.

9.6. Limitations of the study

I acknowledge the limitations of the ethnographic design in that it can only go as far as showing how people respond to particular settings, but is unable to answer how these settings were constructed. In the introduction it was mentioned that lesson study was adopted at West Coast Academy as a result of an import of a method for developing of teaching practice, but how this setting was actually constructed over time is not part of this study, nor can it be answered in more generalizable way. Furthermore, transplanting a culturally-based practice into a new cultural environment is challenging for the researcher because it requires numerous adaptations to fit the local context.

Regarding the use of deductive approach to incorporating concepts and theories in the research design, I acknowledge the risk of the premature loss of any fresh insights into the real-world events that are being studied. However, the field of mathematics education is already rich with the knowledge of the concepts regarding mathematics for teaching. The primary purpose was to study how mathematics for teaching affects both the practice of teaching and developing of teaching practice.

According to Jaworski (1998) we do not have a theory for the development of teaching. It could be that such theory may end up representing all of the complexities that are actually found in the process. Because it is grounded in the data, it adequately addresses the particularities of the situation (setting and individuals), thus producing a better explanation than we have now.

9.7. Personal reflections

I started the journey of this research enthused about how lesson study can change the learning culture of teachers at our school, hoping that what would be learned could lead to a model for in-service professional development at large. However, through this research I also experienced important personal transformation and learning in several areas that I wish to share with the reader.

First, concerning the broad area of theory and practice, I have read and studied an abundance of educational literature on MfT. My knowledge and awareness about the complexity of teachers' professional knowledge and its role in teaching practice deepened. I connected lesson study as a practical way of teachers' professional development with the theoretical construct of MfT. I sensed that lesson study offers the kind of environment in which teachers could grow their collective and individual MfT.

The prospect of teachers' knowledge growth continued to inspire me to engage the entire school community of teachers to participate in this form of professional development as well as to personally persevere in the journey of this research. It undoubtedly had an effect on my daily teaching practice as well. It is the first time through this research that I became aware of my philosophical stance regarding mathematics in general and school mathematics in particular, and of how vitally it influences my professional practice. Moreover, through the collaborative interaction with my colleagues from the team that participated in this research, especially through observations of their classroom practice and collective reflections, I recognized my personal professional growth along all five components of MfT. Another benefit for all participants was the collegial network that we formed through this work.

Second, as a researcher I was challenged by having to employ a viable methodology that would be reliable enough for interpreting the data and answering the research questions of this thesis. I learned how important the choice of a suitable framework is, and I experienced the freedom to define and create the one that fits the data that was generated in the process of this ethnographic research, which aims at all times to preserve the authenticity of the data and of their interpretation by the researcher. Often times it seemed that I was forging a new path, never walked upon. Situating myself as a participant observer in a community of teachers whose knowledge growth I study presented a unique and certainly novel situation, for which a clearly formulated methodological path was not being laid out ahead for me.

Third, I realized that lesson study as a way for in-service teachers' professional learning is not trivial to set up, and is even harder to sustain. As described earlier, there are certain conditions that are necessary for nurturing this kind of highly committed professional development. From my experience in West Coast Academy, our team of teachers sustained the practice over 5 years altogether, after which for different reasons the team disintegrated, while all other teacher teams did not continue the practice beyond the year in which the research took place. This experience could lead to a conclusion that lesson study needs to be culturally engrained before it can be sustained in a school environment, as we see in the case of Japan. Admittedly, it is difficult to transplant the practice from one cultural environment to another. Regardless, through my experience I am confident that lesson study is an effective and valuable way of in-service teacher professional development, in any cultural environment. Through this work I gained the knowledge and understanding of the process of lesson study to the point that I would be able to create in the future sustainable Practice-Based Professional Development environments for development of teaching practice.

Lastly, I realized that the outcomes of lesson study as a model for professional development are difficult to identify and measure as MfT itself is too. This study has shown that the teachers that participated in this research gained MfT through the collaborative practice of lesson study, but each of the participants learned different things and contributed in different ways. Besides the teachers' openness and willingness to learn from each other, the composition of the team and the knowledge that they bring into the collaborative situation seems to play an important role. In Japan, the learning environment of lesson study is enhanced with greater involvement by "knowledgeable others", typically university professors from schools of education, while in our case we were lucky enough to have a master teacher in the team. However, the question remains, "What did the master teacher learn?" If there were several of them in the team, the learning would likely be more explicit and enhanced for everyone.

9.8. Concluding thoughts

Just as when teachers teach their students through problem solving and they present their students with a new problem to solve, their purpose is not just for the students to be able to solve this particular problem, but also to use what they learned from that process in order to become better problem solvers. The intention is to use that experience to be able to solve a host of problems of a similar type. The problem is intended to act as an example from which students are meant to generalize and apply what they have learned to other similar problem situations later on in their life. That is, if we assume that the problem was a novel one for the students and not just an exercise.

In much the same way, the lesson that the teachers prepare, implement and discuss, in the context of lesson study, is not just about that one lesson. It would not make much sense to spend several weeks developing just one lesson. It would make even less sense to try to create a "perfect lesson" that would serve as a model, or even worse, a prescription on how to teach a certain topic.

The purpose of a lesson that is developed through the process of lesson study is to serve as an example from which teachers can learn many things about their practice, and then use that knowledge across many other lessons that teachers will face throughout their practice. Such research lessons entail a close examination of the impact of teaching on learning. They act as a stepping stone for developing instructional practice in general. By generalizing from a single lesson study cycle, teachers can shape their practice, which is at the heart of any professional development activity.

By looking at a full cycle of a single lesson that is developed, implemented and discussed in two enactments, by two different teachers within one lesson study team, we can begin to separate the properties of the lesson itself (which are related to the specific learning goals and the content that needs to be taught) from the teaching practice of the teacher who is teaching the lesson. We can also begin to understand the impact of the practice-based collaboration among the teachers on their individual development in the realm of "mathematics for teaching."

Critical reflection on the teaching practice in post-lesson discussion seems to be essential in enhancing the mathematics for teaching at the individual and collective level. In the case of Japanese culture of lesson study, "knowledgeable others" offer the impartial critical eye in the reflective discussion of the enacted lessons.

While there is a consensus that teachers' mathematics-for-teaching (Davis & Simmt, 2006) is a complex, dynamic and tacit body of knowledge, which is very difficult to assess reliably, there seems to be little agreement on exactly what this knowledge is.

W*hat* should be known to teach well is elusive. However, *how* such knowledge should be held has been shown quite explicitly on several specific domains of mathematics for teaching. For example, Liping Ma's research revealed that mathematics for teaching rests firmly on what she called "profound understanding of fundamental mathematics," and which she explicated quite extensively for several topics of elementary mathematics, such as multi-digit subtraction with regrouping, multi-digit multiplication, and division of fractions (Ma, 1999). With such understanding teachers are seen to be able to move in their subject easily and naturally, and in a way that allows them to effectively plan for instruction in order to avoid the typical student misconceptions, and to respond efficiently to a great variety of possible student errors. It is less clear how mathematics teachers are to acquire this kind of profound and connected knowledge, how such knowledge is to be held and used in the classroom, how it could be recognized, and exactly what constitutes such knowledge.

Lesson study holds promise as a context in which mathematics teaching could be developed systematically and in which such knowledge could be deepened, both at the level of individual teacher, as well as within a community of teachers. It can also act as a window for educational research to examine and explicate teachers' mathematics-for-teaching.

References

Adler, J., & Davis, Z. (2006). Opening another black box: Researching mathematics for teaching in mathematics teacher education. Journal for Research in Mathematics Education, 270-296.

Ainley, J. (1999). Who Are You Today? Complementary and Conflicting Roles in School-Based Research. For the Learning of Mathematics, 19(1), 39-47.

Ball, D. L. (1988). Research on Teaching Mathematics: Making Subject Matter Knowledge Part of the Equation.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. Multiple perspectives on the teaching and learning of mathematics, 83-104.

Ball, D. L., & Bass, H. (2002, May). Toward a practice-based theory of mathematical knowledge for teaching. In Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group (pp. 3-14).

Ball, D., & Cohen, D. (1999). Toward a Practice-Based Theory of Professional Education. Teaching as the learning profession. San Francisco: Jossey-Bass.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide?.American Educator, 29(1), 14-46.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching what makes it special?. Journal of teacher education, 59(5), 389-407.

Bass, H., Usiskin, Z. P., & Burrill, G. (2002). Studying Classroom Teaching as a Medium for Professional Development: Proceedings of a US-Japan Workshop (9th, Tokyo/Makuhari, Japan, July 31-August 6, 2000). Washington, DC: National Academy Press.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., ... & Tsai, Y. M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. American Educational Research Journal, 47(1), 133-180.

BC Ministry of Education. (2006). Principles of Mathematics 10 to 12: Integrated Resource Package 2006. Taken from http://www.bced.gov.bc.ca.

BC Ministry of Education. (2008). WNCP Common Curriculum Framework for Grades 10-12 Mathematics: Pre-Calculus Grade 11. January 2008. Taken from http://www.bced.gov.bc.ca.

Begle, E. G. (1972). Teacher knowledge and student achievement in algebra (Vol. 9). Palo Alto, California: School Mathematics Study Group.

Begle, E. G. (1979). . Critical Variables in Mathematics Education: Findings from a Survey of the Empirical Literature.

Bromme, R. (1994). Beyond subject matter: A psychological topology of teachers' professional knowledge. In Biehler R. (Ed.). Didactics of mathematics as a scientific discipline. Dordrecht: Kluwer Academic Publishers, 73-88.

Brousseau, G., & Balacheff, N. (1997). Theory of didactical situations in mathematics: Didactique des mathématiques, 1970-1990 (p. 336). M. Cooper, R. Sutherland, & V. Warfield (Eds.). Dordrecht: Kluwer academic publishers.

Burney, D. (2004). Craft Knowledge: The Road to Transforming Schools. Phi Delta Kappan. v85, n7, p526.

Byrum, J. L., Jarrell, R., & Munoz, M. (2002). The perceptions of teachers and administrators on the impact of the lesson study initiative.

Chenoweth, K. (2000, December 21). Inadequate teacher training puts U.S. students behind in math, science. The Washington Post, pT9.

Chokshi, S., & Fernandez, C. (2005). Reaping the systemic benefits of lesson study. Phi Delta Kappan, v86(n9) .

Cohen, D. K., Raudenbush, S. W., Ball, D. L. (2003). Resources, Instruction, and Research. Educational Evaluation and Policy Analysis, 25(2), pp. 119-142.

Creswell, J. W. (2002). Educational research: Planning, conducting and evaluating, quantitative and qualitative research. New Jersey: Pearson.

Cuban, L. (1993). How Teachers Taught: Constancy and Change in American Classrooms, 1890-1990. Research on Teaching Series. New York: Teachers' College Press.

Curcio, F. R. (2002). A User's Guide to Japanese Lesson Study: Ideas for Improving Mathematics Teaching. National Council of Teachers of Mathematics. Reston, VA.

D’Amore, B. (2008). Epistemology, didactics of mathematics and teaching practice. Mediterranean Journal for Research in Mathematics Education, Vol. 7, 1, 1-22.

Dakic, B., & Elezovic, N. (2006). Matematika 1. Zagreb: Element.

Davis, B., & Renert, M. (2014). The math teachers know: Profound understanding of emergent mathematics. New York, NY: Routledge.

Davis, B., & Simmt, E. (2006) Mathematics-for-teaching: an ongoing investigation of the mathematics that teachers (need to) know. Educational Studies in Mathematics, 61(3), 293-319.

DuFour, R. (2004). Schools as learning communities. Educational Leadership. V61(n8), p(6-11).

Dweck, C. (2006). Mindset: The new psychology of success. Random House LLC.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact.

Fernandez, C. (2005). Lesson study: A means for elementary teachers to develop the knowledge of mathematics needed for reform-minded teaching? Mathematical Thinking & Learning: An International Journal, v7(n4), p265.

Fernandez, C., & Yoshida, M. (2004). Lesson study: A Japanese approach to improving mathematics teaching and learning. Routledge.

Fernandez, C., & Chokshi, S. (2002). A practical guide to translating lesson study for a U.S. setting. Phi Delta Kappan, v84(n2), p128. .

Festinger, L. (1962). A theory of cognitive dissonance (Vol. 2). Stanford university press.

Freudenthal, H. (1973). Mathematics as an educational task. Springer.

Freudenthal, H. (1983). Didactical phenomenology of mathematical structures. Dordrecht, NL: D. Reidel.

Glesne, C. (1998). Becoming qualitative researchers: An introduction. 2nd edition. White. Plains, NY: Longman.

Green, E. (2014, July 23). Why do Americans stink at math? New York Times. Retrieved from http://www.nytimes.com

Handal, B., & Herrington, A. (2003). Mathematics teachers’ beliefs and curriculum reform. Mathematics Education Research Journal, 15(1), 59-69.

Hannah, W.G., (2012). Examining what in-service elementary teachers notice when presented with instances of students' mathematical thinking. In Van Zoest, L. R., Lo, J.J., & Kratky, J.L. (Eds.). Proceedings of the 34th annual meeting of the North American.

Hashweh, M. Z. (2005). Teacher pedagogical constructions: a reconfiguration of pedagogical content knowledge. Teachers and Teaching, 11(3), 273-292.

Hatch, J. A. (2002). Doing qualitative research in education settings. Albany: State University of New York Press.

Hattie, J. (2003). Teachers make a difference: what is the research evidence?(p. 4). Camberwell: Australian Council for Educational Research.

Hiebert, J., Gallimore, R., & Stigler, J. W. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one?. Educational researcher, 31(5), 3-15.

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. Journal for research in mathematics education, 330-351.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. American educational research journal, 42(2), 371-406.

Hyndman, J., McLoughlin, J. (2012). PIMS and mathematics education in BC. *CMS Notes*, 44(5). Retrieved from https://cms.math.ca

Ingersoll, R. M. (1999). The problem of underqualified teachers in American secondary schools. Educational Researcher, 28(2), 26-37.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for Research in Mathematics Education, 169-202.

Jaworski, B. (1998). Mathematics teacher research: Process, practice and the development of teaching. Journal of Mathematics Teacher Education, 1(1), 3-31.

Jaworski, B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. Journal of Mathematics Teacher Education, 187-211.

Johansson, M. (2003). Textbooks in mathematics education: a study of textbooks as the potentially implemented curriculum.

Kelly, K (2002). Lesson study: Can Japanese methods translate to U.S. schools? Harvard education letter, 18(3), 4-7. .

Lampert, M., & Ball, D. L. (1998). Teaching, Multimedia, and Mathematics: Investigations of Real Practice. The Practitioner Inquiry Series. Teachers College Press, 1234 Amsterdam Avenue, New York, NY 10027.

Leikin, R., & Zazkis, R. (Eds.). (2010). Learning through teaching mathematics: development of teachers' knowledge and expertise in practice (Vol. 5). Springer.

Lewis, C. (2000). Lesson study: The core of Japanese professional development.

Lewis C. C., Perry, R. R. & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and North American case. Journal of Mathematics Teacher Education, 12(4), 285-304.

Lewis, C., Perry, R., Hurd, J., & O Connell, M. P. (2006). Lesson study comes of age in North America. Phi delta kappan, 88(4), 273.

Lewis, C., Perry, R., & Murata, A. (2006). How should research contribute to instructional improvement? The case of lesson study. Educational Researcher, v35(n3), p3.

Lewis, C., & Tsuchida, I. (1997). Planned educational change in Japan: The case of elementary science instruction. Journal of Education Policy, 12(5), 313-331.

Lieberman, A. (1995). Practices that support teacher development: Transforming conceptions of professional learning. Innovating and Evaluating Science Education: NSF Evaluation Forums, 1992-94, 67.

Liptak, L. (2002). It’s a matter of time: Scheduling lesson study at Peterson, NJ School 2. Research for Better Schools Currents, V (2), 6-7.

Lortie, D. C., & Clement, D. (1975). Schoolteacher: A sociological study (pp. 1-54). Chicago: University of Chicago Press.

Loucks-Horsley, S., Stiles, K. E., Mundry, S., Love, N., & Hewson, P. W. (2009). Designing professional development for teachers of science and mathematics. Corwin Press.

Ma, L. (1999). Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States. Mahwah, NJ: Lawrence Erlbaum Associates.

Mason, J. (2000). Asking mathematical questions mathematically. International Journal of Mathematical Education in Science and Technology, 31(1), 97-111.

Mason, J. (2002). Researching Your Own Practice: The Discipline of Noticing. Routledge-Falmer, London.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. Educational Studies in Mathematics, 38(1-3), 135-161.

Mason, J., Burton, L., & Stacey, K. (1982). Thinking mathematically. AMC, 10, 12.

Meredith, A. (1995). Terry's learning: some limitations of Shulman's pedagogical content knowledge. Cambridge Journal of Education, 25(2), 175-187.

Meng, S.K. (2004). New Elementary Mathematics Syllabus 1. Singapore: Marshall Cavendish Education.

Merriam, S. B. (1998). Qualitative research and case study applications in education. San Francisco: Jossey-Bass.

Mickelson, R.J., & Mickelson, P. (2012). Theory and problems for Pre-Calculus 11. Crescent Beach Publishing.

Miles, M. B., & Huberman, A. M. (1994). Qualitative data analysis: An expanded sourcebook. Sage.

Monk, D. H. (1994). Subject area preparation of secondary mathematics and science teachers and student achievement. Economics of Education Review, 13(2),125-145.

Movshovitz-Hadar, N., & Webb, J. (1998). One equals zero and other mathematical surprises. Emeryville, CA: Key Curriculum Press.

Peterson, B. E. (2005). Student teaching in Japan: The lesson. Journal of Mathematics Teacher Education, v8(n1).

Petrou, M., & Goulding, M. (2011). Conceptualising teachers' mathematical knowledge in teaching. In Mathematical knowledge in teaching (pp. 9-25). Springer Netherlands.

Piaget, J. (1976). Piaget's theory (pp. 11-23). Springer Berlin Heidelberg.

Rowland, T., Huckstep, P., & Thwaites, A. (2003). The knowledge quartet. Proceedings of the British Society for Research into Learning Mathematics,23(3), 97-102.

Rowland, T., & Ruthven, K. (2011). Introduction: Mathematical knowledge in teaching. In Mathematical knowledge in teaching (pp. 1-5). Springer Netherlands.

Ruthven, K. (2011). Conceptualising mathematical knowledge in teaching. InMathematical knowledge in teaching (pp. 83-96). Springer Netherlands.

Ryan, G. W., & Bernard, H. R. (2003). Techniques to identify themes. Field methods, 15(1), 85-109.

Santagata, R. (2011). From teacher noticing to a framework for analyzing and improving classroom lessons. Mathematics teacher noticing: Seeing through teachers' eyes, 152-168.

Schifter, D. (1998). Learning mathematics for teaching: From a teachers' seminar to the classroom. Journal of mathematics teacher education, 1(1), 55-87.

Schifter, D. (2002). Elementary Mathematics Education in the United States. In Bass, H., Usiskin, Z. P., Burrill (Eds). Studying classroom teaching as a medium for professional development: Proceedings of a U.S.- Japan workshop (9th, Tokyo/Makuhari, japan, july 31-august 6, 2000), 21-28.

Schoenfeld, A. (2000). Purposes and methods of research in mathematics education. Notices of the AMS, v46(n6), p 461.

Schostak, J.F. (2002). Understanding, Designing and Conducting Qualitative Research in Education. Framing the Project. Open University Press.

Selter, C. (2001). Understanding - the underlying goal of teacher education. Marja van den Heuvel (Ed.): Proceedings of the 25th Conference of the Group for the Psychology of Mathematics Education. Volume 1. Utrecht: Freudenthal instituut 2001, pp. 198-202 (long version in: Erich Wittmann (Ed.): mathe 2000 - selected papers. Dortmund: Universität 2001, pp. 213-231).

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational researcher, 15(2) 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. Harvard educational review, 57(1), 1-23.

Sierpinska, A. (1994). Understanding in Mathematics. London, Washington D.C.: Falmer Press.

Silverman, D. (2006). Interpreting qualitative data. Sage.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. Mathematics Teaching (77), 20-26.

Sleep, L., Ball, D. L., Boerst, T. A., & Bass, H. (2009). Combining the development of practice and the practice of development in teacher education. The Elementary School Journal, 109(5), 458-474.

Smith, M. S., Stein, M. K., & Silver, E. A. (1999). The development of professional developers: Learning to assist teachers in new settings in new ways. Harvard educational review, 69(3), 237-270.

Spradley, J. P. (1980). Participant observation. London: Harcourt Brace College Publishers.

Steiner, H. G. (1987). Philosophical and epistemological aspects of mathematics and their interaction with theory and practice in mathematics education. For the Learning of Mathematics, 7-13.

Stevenson, H.W. & Nerison-Low, R. (2002). To sum it up: Case studies of education in Germany, Japan and the United States. Washington D.C.: Office of Educational Research and Improvement, Department of Education. .

Stewart, R. A., & Brendefur, J. L. (2005). Fusing lesson study and authentic achievement. Phi Delta Kappan, v86(n9) .

Stigler, J.W., & Hiebert, J. (1999). The teaching gap: Best ideas from the world's teachers for improving education in the classroom. New York: Summit Books.

Takahashi, A., & Yoshida, M. (2004). Lesson-Study communities. Teaching children mathematics, 10(9), 436-443.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. Educational studies in mathematics, 12(2), 151-169.

Tanton, J. (2005). Encyclopedia of Mathematics. New York: Infobase Publishing.

Tarr, J. E., Chávez, Ó., Reys, R. E., & Reys, B. J. (2006). From the written to the enacted curricula: The intermediary role of middle school mathematics teachers in shaping students' opportunity to learn. School Science and Mathematics, 106(4), 191-201.

Thom, R. (1973). Modern mathematics: Does it exist. Developments in mathematics education, 194-209. University Press.

Turner, F., & Rowland, T. (2008). The knowledge quartet: A means of developing and deepening mathematical knowledge in teaching. In Mathematical Knowledge in Teaching Seminar Series: Developing and Deepening Mathematical Knowledge in Teaching, Loughborough University, Loughborough. Retrieved on January (Vol. 30, p. 2013).

Turner, F., & Rowland, T. (2011). The Knowledge Quartet as an organising framework for developing and deepening teachers' mathematics knowledge. In Mathematical knowledge in teaching (pp. 195-212). Springer Netherlands.

Van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. Teaching and Teacher Education, 24(2), 244-276.

Watanabe, T. (2002). Learning from Japanese lesson study. Educational Leadership, v59(n6), p36. .

Watson, A. (2008). Developing and deepening mathematical knowledge in teaching: being and knowing. In MKiT 6, Nuffield Seminar Series, 18th March, at University of Loughborough.

Watson, A. (2008). School mathematics as a special kind of mathematics. For the Learning of Mathematics, 3-7.

Watson, A., & Barton, B. (2011). Teaching mathematics as the contextual application of mathematical modes of enquiry. In Mathematical Knowledge in Teaching (pp. 65-82). Springer Netherlands.

Watson, A., & De Geest, E. (2012). Learning coherent mathematics through sequences of microtasks: Making a difference for secondary learners. International Journal of Science and Mathematics Education, 10(1), 213-235.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. Mathematical Thinking and Learning, 8(2), 91-111.

Wenger, E. (1998). Communities of Practice: Learning, Meaning and Identity. Cambridge University Press.

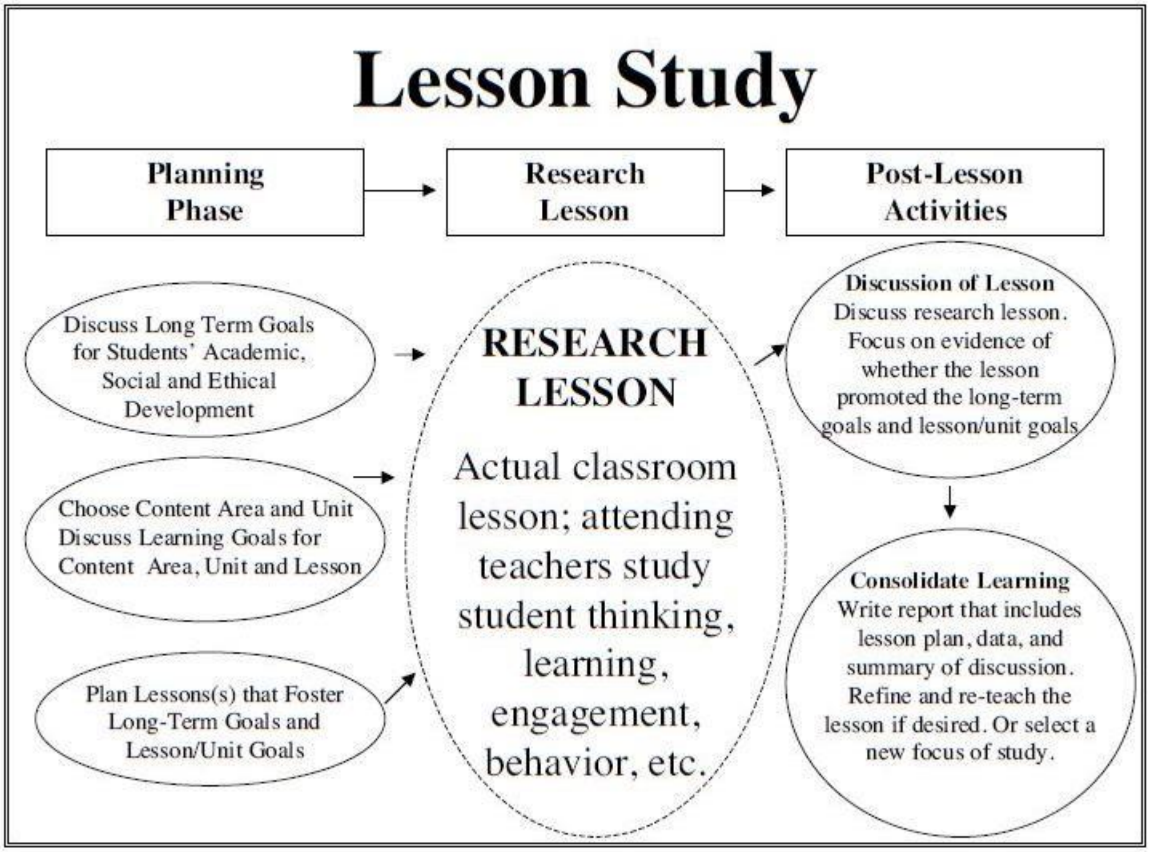
Yin, R. K. (2010). Qualitative research from start to finish. Guilford Press.

Yoshida, M. (1999). Lesson study [Jugyokenkyu] in elementary school mathemtics in Japan: A case study. A paper presented at the American Educational Research Association 1999 Annual Meeting, Montreal, Canada.

Zazkis, R., & Chernoff, E. J. (2008). What makes a counterexample exemplary?. Educational Studies in Mathematics, 68(3), 195-208.

Zodik, I., & Zaslavsky, O. (2008). Characteristics of teachers' choice of examples in and for the mathematics classroom. Educational Studies in Mathematics, 69(2), 165-182.

Appendix A. Lesson Study Process/Cycle



Excerpted from Catherine Lewis, "Lesson Study: A Handbook for Teacher-Led Improvement of Instruction," Philadelphia: Research for Better Schools, 2002.

All three lesson study cycles conducted at West Coast Academy by the secondary mathematics teachers' team were extended cycles as they included the second teaching of the revised lesson; that is, each research lesson involved an extended post-lesson conferencing where modifications to the first lesson plan based on teachers' reasoning and feedback were made. The revised lesson was then implemented by a different team member with a different class of students. The process then concluded with another post-lesson discussion. In one of the three cycles conducted by the secondary mathematics teachers' team this process was followed by a written report.

Appendix B. The Context in which the Study Took Place: Teacher Teams and Lessons Implemented

This Appendix provides an overview of the organization of the “Practice Based Professional Development" (PBPD) as implemented in the study “Collaborative inquiry: How to Build a Culture of Mathematical Thinking in a Classroom?” at the West Coast Academy over the course of one school year. The lessons were curriculum based, chosen by the participants, collaboratively planned, implemented (taught by a single teacher, observed by the entire team), discussed and reflected upon.

Teacher Team	Participants	“Research Lessons” Implemented by the Team
A: Kindergarten, Grade 1	Jennifer Sophia	Lesson 1 (Gr 1): Measuring with a non-standard unit Lesson 2 (K): Number bonds (making up a given sum) Lesson 3 (Gr 1): Subtract from 10 (subtraction with regrouping with numbers up to 40)
B: Grade 2, 3	Paula Colette Fiona	Lesson 1 (Gr 3): Comparing capacities of containers Lesson 2 (Gr 2): A pattern to colour (fraction problem) Lesson 3 (Gr 3): Mass Lesson 4 (Gr 3): Creating a pictograph Lesson 5 (Gr 3): Properties of 3-D shapes Lesson 6 (Gr 2): Curious subtraction
C: Grade 4, 5	Erica Meaghan Laura Ursula Bianca	Lesson 1 (Gr 5): Solving word problems involving decimals Lesson 2 (Gr 4): Problem solving involving capacity Lesson 3 (Gr 4): Adding and subtracting unlike fractions Lesson 4 (Gr 5): Categorizing numbers based on the remainder[1]
D: Grade 6, 7, 8	Denisse Shelly Wilma	Lesson 1 (Gr 6): Decimal multiplication Lesson 2 (Gr 8): Pythagorean theorem
E: Grade 9, 10, 11, 12	Andrew Gabrielle Steve	Lesson 1 (Gr 11):Solving radical equations Lesson 2 (Gr 9): Circle geometry: Relationship between the central and the inscribed angle Lesson 3 (Gr 11): Graphing a parabola (the effect of the leading coefficient)

Teacher Team A, B, C (Kindergarten, Grade 1-5): Elementary grades
Teacher Team D (Grade 6-8): Middle grades
Teacher Team E (Grade 9-12): High school grades

[1] Lesson plan for this lesson, called “Do I have a Window Seat or an Isle Seat?”, was borrowed from http://hrd.apec.org/images/c/c6/G5WindowSeat_or_AisleSeat_LessonPlan.pdf

Appendix C. Teacher Team E as the Focus of the Study: Professional Development Activity at the Lesson Level

According to the protocol for all teams at the onset of the study, the teams would develop the lessons collaboratively. Each lesson would be then implemented by two different teachers in two different classes – preferably each teacher would teach the lesson to his/her own class of students. There were two exceptions, when a teacher would teach the lesson to a class other than her own (second implementations of Lessons 2 and 3).

The teacher that was going to teach the lesson first would be the expected to take the initiative to write up the lesson plan. This was agreed upon to aid the process of communication during the planning stage and also to serve as documentation and a tool for observation.

After the first implementation of the lesson, which would be followed by a post-lesson discussion, the lesson would be revised if teachers felt this was necessary. These revisions would usually be minor in content. The second implementation took place within several days of the first implementation, and often already the next day.

In this team of teachers, to roles in terms of 1^{st} and 2^{nd} teaching of the lessons were set as follows:

Lesson (Grade): Title	**Teacher who taught the lesson first and wrote the lesson plan**	**Teacher who taught the lesson second and revised the lesson plan (if needed)**
Lesson 1 (Gr 11):Solving radical equations	Andrew	Gabrielle
Lesson 2 (Gr 9): Circle geometry: Relationship between the central and the inscribed angle	Steve	Andrew
Lesson 3 (Gr 11): Graphing a Parabola: Engineer's Arch	Gabrielle	Steve

The main part of the study focuses on the cycle of professional development activities of planning, implementation, observation, and post-lesson discussions for Lesson 1 in Grade 11, Solving radical equations.

Appendix D. Artefact of a Teacher-Written Document Lesson Plan: Solving Radical Equations (as produced by Teacher Team E and written up by Andrew)

SOLVING RADICAL EQUATIONS
(Grade 11 Math Class)

First – What kind of equations have we learned so far?

Here we will review some types of equations:

-linear equations,

-quadratic equations

-polynomial equations

-rational equations

-radical equations

We have many other equations, like exponential equations, logarithmic equations, trigonometric equations, etc. However, this time, we will talk just about first group of equations.

Let’s start with linear equations: Give me one example of a linear equation.

$2x - 3 = 7$

What is the set of numbers for which this equation is defined?

Or, does this type of equation have any limitations in its domain?

Quadratic equations: Example: $x^2 - 3x + 2 = 0$

What is the set of numbers for which this equation is defined?

Or, does this type of equation have any limitations in its domain?

Polynomial equations: Example: $3x^3 + 5x^2 - 7x + 1 = 0$

What is the set of numbers for which this equation is defined?

Or, does this type of equation have any limitations in its domain?

Rational equations: Example: $\frac{x}{x+1} - 2x = \frac{2}{x}$

What is the set of numbers for which this equation is defined?

Or, does this type of equation has any limitations in its domain?

$x+1 \neq 0$, so x $\neq$ -1 and x $\neq$ 0

Therefore x cannot equal -1 or 0.

Before we move to the radical equations, the question is, what does the term **radical** in mathematics mean?

It means the result of any root. Here we will explore even roots, which means – square root or any other even root.

Example:

$$\sqrt{15}$$

What is the value of this root? Is it positive or negative?

What would be the solution of this quadratic equation?

$$x^2 - 9 = 0$$

The solution is:

$x^2 = 9 \Rightarrow x = \pm\sqrt{9} \Rightarrow x = \pm 3$ or $x_1 = 3$ and $x_2 = -3$

Do you notice the difference? The solutions of this quadratic equation are two numbers, one positive and one negative. The value of $\sqrt{15}$ is just one number, and that number has to be positive. That means, the result of the square root is always positive.

What could be the value of the root: $\sqrt{x+4}$ Is it positive or negative?

The value of the root, if exists, will be a positive number or zero.

When does this number exist?

It exists when the value under the root is positive or zero, because there is no real value of the root when the negative number is under the root. That means, the value x + 4 has to be positive or zero.

x + 4 $\geq$ 0 or x $\geq$ - 4.

What do you think would be a radical equation then? What would it look like?

Radical equations: Example: $\sqrt{x-3} + x = 5$

What is the set of numbers for which this equation is defined?

What are the limitations? What have we said about the values of a square root?

So, we have to look at the expression under the root first. It has to be positive or zero, in order for the equation to have a solution.

x – 3 $\geq$ 0 or x $\geq$ 3.

Then, under this condition we can start solving this equation. In order to solve the equation we will leave the root on the left side of the equation, and move x to the right side.

$\sqrt{x-3}$ = 5 – x

To solve this equation we have to eliminate the root. If we square both sides of the equation, the root will be eliminated.

May we do that? What do you think? Before we answer that question, let eliminate the root without any limitation.

$\sqrt{x-3}$ = 5 – x $/^2$ (square both sides of the equation)

x – 3 = (5 – x)² (then simplify the equation)

x – 3 = 25 – 10x + x² (now, solve this quadratic equation)

x² - 11x + 28 = 0

$$x_{1,2} = \frac{11 \pm \sqrt{11^2 - 4 \cdot 28}}{2} = \frac{11 \pm \sqrt{121-112}}{2} = \frac{11 \pm \sqrt{9}}{2} = \frac{11 \pm 3}{2} \Rightarrow x_1 = 4 \text{ and } x_2 = 7$$

Seems that the solutions of this radical equation are: $x_1 = 4$ and $x_2 = 7$.

Are they really the solutions of the given radical equation?

Check it! How can we check if these are the solutions of this equation? Substitute the solutions into the given equation:

$\sqrt{x-3} = 5 - x$

For $x_1 = 4 \Rightarrow \sqrt{4-3} = 5-4 \Rightarrow \sqrt{1} = 1$

The equation is satisfied, because 1 = 1.

On the other hand, for $x_2 = 7 \Rightarrow \sqrt{7-3} = \sqrt{4} = 2$ and right side 5 – 7 = - 2 $\Rightarrow$ 2 = -2,

But this is not correct. What does this mean?

It means that $x_2 = 7$ is not the solution of the given equation.

Could we know it beforehand?

Yes, we could know that if we did our work properly. Namely, we made a mistake squaring both sides of the given equation without checking the right side, because:

Left side of the equation is a square root, and we know that it has to be positive. If so, then the right side of the equation has to be positive, too, because it is an equation, and left side has to be equal to the right side.

If the right side of the equation was negative, squaring it we made it positive, and we cannot just change the sign of one side in the equation without changing the other side, too.

So, we had to make limitation that the right side is positive too. That means:

$5 - x \geq 0 \Rightarrow 5 \geq x$ or $x \leq 5$

Now, the limitation under which we may square both sides of the equation is $x \leq 5$, and the solution $x_2 = 7$ does not satisfy that limitation. That means that $x_2 = 7$ is not the solution of the given equation.

This way, putting limitations through the process of solving the equation, **we know in advance what can and what cannot be the solution**, and will not fall into the trap of taking the result which is not the solution.

That was the discussion of the solution of one of the radical equations. The same principle we have to apply whenever we need to solve a radical equation. That means that we have to take all the precautions whenever we do the next step in the process of solving such equations. It applies not only to the roots and the sign of the expressions, but also to the denominators of the fractions in they exist in the equation.

Now, we will solve another equation:

$\sqrt{x+11} - \sqrt{9-x} = 2$

In the process of solving, we need to determine for which values of the variable the equation could have solutions. That means the values under the roots have to be non-negative.

$x + 11 \geq 0$ and $9 - x \geq 0$ or

$x \geq -11$ and $x \leq 9$

Combining these two limitations we get:

$-11 \leq x \leq 9$

And this is the limitation under which we will continue to solve this radical equation.

In the process of solving we will follow the same procedure as in the above solved equation.

So, if we want to gradually get rid of the roots, we have to square both sides of the equation. But first we have to make sure that the sides are positive. We should notice that it would be easier if we move one of the roots to the right side of the equation. If we move the second one, we will get:

$\sqrt{x+11} = 2 + \sqrt{9-x}$

We notice that the left side is always positive because there is just a square root there, and the result of any square root has to be nonnegative value. On the right side we have the sum of one positive number and a square root which is nonnegative, so the entire sum is positive. Then, we can square both sides of this new equation.

x + 11 = 4 + 4$\sqrt{9-x}$ + 9 – x $\Rightarrow$

2 x – 2 = 4$\sqrt{9-x}$ Dividing the equation by 2, we get:

x – 1 = 2$\sqrt{9-x}$

Because we still have one root we have to eliminate it. This means we need to square both sides. But before we can do so, we have to make sure that both sides are nonnegative, as we did in the process of solving the first equation. Let's do it:

x – 1 $\geq$ 0 or x $\geq$ 1

Now we have to combine this limitation with the first one: - 11 $\leq$ x $\leq$9

Combined, they will give final limitation:

1 $\leq$ x $\leq$9

We may now square both sides:

x^2 - 2x + 1 = 4 (9 – x) $\Rightarrow$ x^2 - 2x + 1 = 36 – 4x $\Rightarrow$ x^2 + 2x – 35 = 0

$$x_{1,2} = \frac{-2 \pm \sqrt{4+140}}{2} = \frac{-2 \pm 12}{2} \Rightarrow x_1 = -7 \text{ and } x_2 = 5$$

If we compare the results with the previous limitations we see that x_1 = - 7 cannot be the solution, because it doesn't satisfy the limitations.

So, the only solution is $x = 5$

Appendix E. Artefact of a Teacher-Written Document: Teachers' Report on Teaching of Radical Equations after the Team concluded the Lesson Study Cycle (as produced by Teacher Team E and written up by Andrew)

Questions the Teacher Does:	Expected Answers:	Points of Evaluation
[Part 1: Connecting to Prior Knowledge]		
Here we will review some types of equations: In which grade we began to learn equations?	Grade 7 and 8.	
What kind of equations we learned so far?	Answer: -linear equations, -quadratic equations -polynomial equations -rational equations	Students learned equations in the earlier grades, but a little bit of review would be helpful.
We have many other equations, like exponential equations, logarithmic equations, trigonometric equations, etc., but this time we will talk just about first group of equations.		
Let start with linear equations: Give me one example of linear equation.	$2x - 3 = 7$	It is expected that students will know to solve linear equations.
Do we know how to solve this equation. Or, what is the set of numbers for which this equation is defined? Does this type of equation have any limitations in its domain?	This equation is defined on the set of real numbers, or $x \in R$ (Symbol $\in$ means – belongs) and our equation does not have any limitations.	
Quadratic equations: Give us an example of a quadratic equation.	$x^2 - 3x + 2 = 0$	
What is the set of numbers for which this equation is defined? Or, does this type of equation have any limitations in its domain?	Again, $x \in R$, and the given equation doesn't have any limitation in its domain.	Students learned to factor quadratic trinomials in grade 10, and the solutions of a quadratic equations in grade 11, (what means –recently)
How will you solve this equation? Do you think it is the best way for	To factor it!	

solving a quadratic equation? What if the solutions are rational or irrational numbers and the factoring is difficult to do? Do we know that the equation has real number as a solution? How can you express that statement? Polynomial equations: Do you know any example of a polynomial equation? What is the set of numbers for which this equation is defined? Or, does this type of equation have any limitations in its domain? Rational equations: Would you be able to write an example of a rational equation? What is the set of numbers for which this equation is defined? Or, does this type of equation have any limitations in its domain? Or, do you see any 'red flag' in this equation? What will happen if x = 0 ? --------------------- **[Part 2: Introducing the Topic]** Before we move forward, the question is – what the term radical in mathematics means? Give me an example!	Yes, I think it is good way of solving it. Then, it is better to solve using the formula for the solutions of a quadratic equation. Yes, if the discriminant of a quadratic equation is positive or zero, then the solution is a real number. $D = b^2 - 4ac \geq 0$ $3x^3 + 5x^2 - 7x + 1 = 0$ Domain: $x \in R$ and it means that this equation does not have any limitations in its domain. $\frac{x}{x+1} - 2x = \frac{2}{x}$ We know that division by zero is not possible, so we need to exclude that possibility. It means that the denominators in this equation have to be different than zero: $x + 1 \neq 0 \Rightarrow x \neq -1$ and $x \neq 0$ It is not possible to divide by zero, so for x = 0 the equation is not defined! ------------------------------ That means the result of any root. Example: $\sqrt{15}$, $\sqrt[3]{29}$, $\sqrt[8]{12}$ $\sqrt[11]{6}$, and etc.	but could expect that they need some kind of review. We could expect that they don't remember the condition for the quadratic equations to have real solutions. Domain is always some kind of a problem for students. Limitations and the reasons to look for them in any equation, especially in a rational function is important, and it is expected that students will have some doubt. ------------------------ Students had learned radical numbers, but we don't expect for everybody to know them. It is known that students

What is the value of the root $\sqrt{15}$? Is it positive or negative?	Square root is always positive number or zero (or nonnegative), because the value under the root has to be nonnegative in order for square root to exist in the real area (the set of real numbers), so the result has to be nonnegative, too.	have difficulties when square root is in question. The most common mistake students make is to say that the root of a number is $\pm$ (positive or negative).
Here we will explore roots of perfect "numbers," what means – square root or any other root that turns out to be rational. But first, we have to know the sign of the square root of a positive number. What is the result of $\sqrt{9}$?	Students had a few seconds to think about the possible result, and first who answered, said: The result of $\sqrt{9} = \pm 3$ The answer is not correct, but the teacher asked other students in the classroom, and every other student said that the result is just +3. The teacher was questioning again and again, and finally the student who first said that the result is ± 3 changed her mind and said that she also thinks that the result is just +3.	So we expect today the students to say that $\sqrt{9}$ is ± 3
Now take a look at this quadratic equation: $x^2 - 9 = 0$ What would be the solution of this quadratic equation?	We will start with: $x^2 = 9$ Consider first these two equalities: $(+3)^2 = 9$ and $(-3)^2 = 9$, Looking at them, we can conclude that the values of x could be: x = 3 and x = - 3 or $x = \pm\sqrt{9} \Rightarrow x = \pm 3$	We also expect that most students will know how to solve quadratic equation, and everybody should know how to solve the equation $x^2 = 9$.
Compare this result to $\sqrt{9}$. Do you notice the difference? And why is it so?	As we said, $\sqrt{9}$ is always equal 3, (not $\pm$ 3). The given quadratic equation has two solutions, one positive and one negative, while $\sqrt{9}$ is just positive number.	
What do you think is the value of $\sqrt{(-3)^2}$	First student: We will cancel the square and the root, so the result is - 3.	This is also students' usual mistake – to cancel out square and square root, so the result is -3.

But we said that the result of any square root with the positive value under the square root is positive number! Then, how the result could be negative?	Order of operations! First, we have to square (- 3), which is positive number 9. Then, we will find the square root of 9, which is 3.	We need to pay attention to this problem and explain it better, so that students don't make that mistake in the future.
Is there any other way to express this result?	Yes, we could say that: $\sqrt{(-3)^2} = \lvert -3 \rvert = 3$	
What could be the value of the root: $\sqrt{x+4}$? This is called radical expression. Is this root positive, zero, or negative?	Again, the value of a square root is always nonnegative, so the value of $\sqrt{x+4}$, if it exists, has to be nonnegative.	At the beginning, students usually forget to make the statement about the value under the root – to be sure that it is nonnegative. To reinforce that, we need to question them again and again about the possible values for the value under the root.
When that number exists?	That number exists when the value under the root is nonnegative, (there is no real value of the root if the value under the square root is negative number). That means: $x + 4 \geq 0 \Rightarrow x \geq -4$	
I hope that we now know what a radical is.		
[Part 3: Solving Radical Equations] What do you think that would be a radical equation? Can you give an example?	Example: $\sqrt{x-3} + x = 5$	
What is the set of numbers for which this equation is defined? What are the limitations? What we have said for the values of a square root?	We have to look at the expression under the root first. We know that it has to be positive or zero in order for the root to exist: $x - 3 \geq 0$ or $x \geq 3$.	
Then, under this condition we can start solving the given equation. In order to solve the equation we will keep the root on the left side of the equation, and move x to the right side.	$\sqrt{x-3} = 5 - x$	We predict that students will try to square both sides of this equation first, without making any limitations for the right side. To show them how serious this mistake is, we let them square it first, to see what the consequences are if we don't pay enough
To solve this equation we have to eliminate the root. How to do that?		
May we do that? What do you think? Before we answer that question, let's	If we square both sides of the equation, the root will be eliminated.	

eliminate the root without any limitation.		attention.
	$\sqrt{x-3} = 5 - x \quad /^2$ (square both sides of the equation) $x - 3 = (5 - x)^2$ then simplify the equation: $x - 3 = 25 - 10x + x^2$ $x^2 - 11x + 28 = 0$ Now, solve this quadratic equation: $x_{1,2} = \frac{11 \pm \sqrt{11^2 - 4 \cdot 28}}{2} = =$ $\frac{11 \pm \sqrt{121 - 112}}{2} = \frac{11 \pm \sqrt{9}}{2} =$ $= \frac{11 \pm 3}{2} \Rightarrow x_1 = 4$ and $x_2 = 7$	Here, when students had to square the right side, the student tried to multiply (5 – x)(5 – x) but the teacher insisted for student to know the formula: $(a - b)^2 =$ $a^2 - 2ab + b^2$ and to apply it here.
Are we satisfied? Are $x_1 = 4$ and $x_2 = 7$ really the solutions of the given radical equation? Check it!	Seems that the solutions of this radical equation are: $x_1 = 4$ and $x_2 = 7$	
How to know if the zeros $x_1 = 4$ and $x_2 = 7$ really are the solutions of the given equation?	If we substitute these solutions into the given equation, we will know whether these are the solutions.	
Check $x_1 = 4$ first.	Substitute the solutions into the given equation: $\sqrt{x-3} = 5 - x$ For $x_1 = 4 \Rightarrow$ $\sqrt{4-3} = 5 - 4 \Rightarrow \sqrt{1} = 1$ and the equation is satisfied, (because 1 = 1).	Students really didn't realize that they made a mistake by not making statements about the right side of the equation.
Now, check for $x_2 = 7$.	On the other hand, for $x_2 = 7 \Rightarrow$ $\sqrt{7-3} = \sqrt{4} = 2 \Rightarrow 2 = -2$, which is not correct.	Now we underline the problem, making sure that students really see the need for making statements (limitations) before we square both sides of the radical equation.
What that means?	That means that $x_2 = 7$ is not the solution of the given equation.	
Could we know it beforehand?		
OK, we could, but seems that we did not care. How come? Where is the	Yes, we could know that if we did	

mistake?	our work properly. We made the mistake when squaring both sides of the given equation without checking the right side, because: -Left side of the equation is a square root, and we know that it has to be nonnegative. If so, then the right side of the equation has to be nonnegative too, because it is an equation, and left side has to be equal to the right side, or: $5 - x \geq 0$ -If the right side of the equation was negative, by squaring we made it positive. We cannot just change the sign of one side in the equation without changing the other side too.	Knowing that students will have difficulties in deciding what to do with equations like this, we offered one very simple example, end asked questions first, before explaining how to approach the solution of this equation.
Now, the limitation under which we may square both sides of the equation is x $\leq$ 5, and the solution $x_2 = 7$ does not satisfy that limitation. That means that we could know much earlier that $x_2 = 7$ is not the solution of the given equation.	-So, we had to make limitation that the right side is nonnegative too. That means: $5 - x \geq 0 \Rightarrow 5 \geq x$ or $x \leq 5$	
Take a look at this radical equation: $\sqrt{x-1} = -2$ Find the limitations, if they exist, for this equation:	First, value under the root has to be positive or zero: $x - 1 \geq 0$, or: $x \geq 1$	
What about right side of the equation? Could it ever be negative?	No, on the left side is the root, and the result of any root could be just positive or zero. So, the right side too has to be nonnegative.	
This way, putting limitations through the process of solving the equation, we know in advance what could and what could not be the solution, and we will not fall into the trap of taking the result that is not the solution. Even more, sometimes, like in this example, we see the solution immediately, namely – the		We think that students now have elementary knowledge about the

solution is the empty set ($x \in \varnothing$), or, the equation does not have the solution. That means that we solved this radical equation even before we started the process of solving it. That way, putting the limitations first, we shorten the solving process. That was the discussion of the solution of one of the radical equations. The same principle we have to apply whenever we need to solve a radical equation. It means that we have to take all the precautions whenever we do the next step in the process of solving such equations. It applies not only to the roots and the signs of the expressions in the equation, but also to the denominators of the fractions, if they exist, in the equation.		procedure for solving radical equations, but that is just a beginning. In order to truly understand the process, more examples have to be done. This was one of them.
----------------------- Now, we will solve another equation: $\sqrt{x+11} - \sqrt{9-x} = 2$ What we need to determine in the process of solving this equation?	----------------------- First, for this equation we need to determine the values of the variable for which the equation is defined. It means that we need to find the set of numbers for which the values under the roots are nonnegative: $x + 11 \geq 0$ and $9 - x \geq 0$ or $x \geq -11$ and $x \leq 9$ and combining these two limitations we get: $-11 \leq x \leq 9$ This is the limitation under which we will continue to solve our radical equation.	--------------- It is not always the same way of solving a radical equation, so we need to show the best way. To do so, we need to show the difficulties when we don't choose the most efficient way. Here, on purpose, we choose a little bit more complicated way. Then, we return to the easier way, moving negative root to the right side of the equation.
So, if we want to gradually get rid of the roots, what we have to do? But before we square both sides, first we have to make sure that these sides are positive. How we can do that?	In the process of solving this equation we will follow the same procedure as in the above solved equation. To square both sides! We could start with:	In order to understand the procedure better, we need to repeat again and again that procedure with different equations. Here is one, different than previous ones.

Could we do that in a different way and to avoid this part of the solution?	$\sqrt{x+11}-\sqrt{9-x} \geq 0 \Rightarrow$ $\sqrt{x+11} \geq \sqrt{9-x} \quad /^2 \Rightarrow$ $x + 11 \geq 9 - x \Rightarrow$ $2x \geq -2 \Rightarrow$ $x \geq -1$ Yes we could, because we should notice that moving the negative root to the right side of the equation: $\sqrt{x+11} = 2 + \sqrt{9-x}$ we will get both sides always positive. We notice that the left side is always nonnegative because there is just square root, and the result of any square root has to be nonnegative value. On the right side we have the sum of one positive number and a square root, and then the sum has to be nonnegative too. Now we can square both sides of this new equation. $x + 11 = 4 + 4\sqrt{9-x} + 9 - x$ $\Rightarrow \quad 2x - 2 = 4\sqrt{9-x}$ Dividing the equation by 2, we get: $x - 1 = 2\sqrt{9-x}$	Doing the steps in the procedure, we always learn something new. This time it was avoiding two roots on the same side of the equation, what makes the process a little bit shorter.
We still have one root to eliminate, and that means – to square both sides. But does the rule of checking for possible limitations apply here, too? How to do so? Let's do it! Notice that this limitation already includes the limitation $x \geq -1$ How to continue this process?	Yes, it applies! We always have to make sure that both sides in any equation are nonnegative before squaring them. The same way as we did in the beginning of solving this equation. We only need to check the expression on the left (because we already know that the right side is always nonnegative): $x - 1 \geq 0$ or $x \geq 1$. Now we have to combine this limitation with the first one: $-11 \leq x \leq 9$ Combined, they will give final	Every time, after finding the solutions, we reminded the students to check limitations made earlier, in order to see are they really the solutions of the given radical equation.

	limitation: $1 \le x \le 9$ We may now square both sides: $x^2 - 2x + 1 = 4(9 - x) \Rightarrow$ $x^2 - 2x + 1 = 36 - 4x \Rightarrow$ $x_{1,2} = \frac{-2 \pm \sqrt{4+140}}{2} = = \frac{-2 \pm 12}{2} \Rightarrow x_1 = -7,\ x_2 = 5$ If we compare the results with the previous limitations we see that $x_1 = -7$ cannot be the solution, because it doesn't satisfy the limitations, ($x \ge 0$). So, the only solution is: $x = 5$	Many different situations may occur in different radical equations, and it is good to try to solve as many different types of radical equations as possible.
-------------------- Some interesting examples of radical equations: Take a look at this radical equation: $\sqrt{x+2} + \sqrt{x-3} + 2 = 0$ Can you say, without solving the equation, does it have a solution? What are the limitations?	------------------------	------------------ This time, an equations is given where we could, if we have enough experience in solving such equations, recognize that the solution is empty set, or – the equation does not have a solution.
	The limitations regarding the square roots are: $x \ge -2$ and $x \ge 3$, and the common solution is: $x \ge 3$	We need to underline that in this case we don't need to spend any time in solving the equation.
Looking just at the limitation we could conclude that there is a possible solution of the equation. But take another look: Can it be that the sum of two square roots, which are nonnegative numbers, added to 2 gives zero as the result?	No, it is not possible.	We always challenge the students questioning them about the possible solutions, making them aware of possible shortcuts in the process of solving radical equations.
Instead of judging the left side, we can rearrange the equation and then look for limitations: $\sqrt{x+2} + \sqrt{x-3} = -2$ Question again: Could the sum of two		This was such example,

<table>
<tr>
<td>nonnegative numbers be a negative number?
Or, it is better to say – the solution of this equation is the empty set: $x \in \varnothing$.
Conclusion: This means that the solution of this equation is the set with no elements in, or as we already said, the equation doesn't have the solution in the set of real numbers, or $x \in \varnothing$.
Sometimes we simply say – the equation doesn't have the solution, meaning that there is no solution in the set of real numbers.</td>
<td>Answer is clearly – no, so there is no value of x that satisfies the equation.</td>
<td>and we can again remind the students about the possibilities we need to explore.</td>
</tr>
<tr>
<td>-----------------
Now take a look at another radical equation:
$$\sqrt{x-5}+\sqrt{3-x}=1$$
Do you think that this equation have the solution (in the set of real numbers)?

I hope that now you know what the symbol $x \in \varnothing$ means? How do we read it?

Then, what is the conclusion?

Looking at the last two equations we can notice that sometimes we can came to the conclusion about the solutions faster if we analyze the equation before we start solving it.
That way we will have solution at the very beginning, and not waste our time going through the process of solving, because in this case it is not necessary.</td>
<td>----------------------
If we write the limitations, it would be:
$x \geq 5$ and $x \leq 3$,
or combined:
$x \in \varnothing$

We already said that symbol $\varnothing$ means – empty set, or set with no elements. For the symbol $\in$ we also said that we read it as – belongs.

So, the expression $x \in \varnothing$ we would read: x belongs to empty set. In other words, there is no such number *x* in the area of real numbers which satisfies given equation. Or, as we used to say – the equation doesn't have the solution (in the set of real numbers).

The equation does not have the solution in the area of real numbers.</td>
<td>------------------

Students will eventually find the examples in the textbooks, -
for example Mathpower 11
- where they give the only way of solving radical equations by checking the solutions et the end by substituting the solution into the given equation.
That is not the way of developing mathematical thinking in students' math education.</td>
</tr>
<tr>
<td colspan="3">You may notice that in some of the textbooks they don't pay attention to the limitations. They simply solve the equation and then substitute the solution into the given equation. If that solution satisfies the equation, they accept it as a solution, and if not, they reject it.
But what if the solution was an irrational number and we had approximated it? It could happen that although it is the (approximate) solution, we may come to the conclusion that it is not, because of rounding numbers. On the other hand, that way makes the calculations unnecessary longer, not to mention some other problems. So, we will stick to the method explained at the beginning of the lesson, putting the limitations first whenever it is necessary.</td>
</tr>
</table>

Appendix F. Interview Questions

1. What attracted you to lesson study? You began to participate in it before you started working at West Coast Academy. We actually met when you joined the community of teachers at Sigma Institute, where some teachers were engaged in this work. What brought you there?
2. As a form of professional education, how does this compare to other professional development opportunities that you participated in during your career as a teacher?
3. Which lesson study cycle that we did together sticks out in your memory the most? For what reason(s)?
4. Going to the lesson that we did on Solving Radical Equations, can you remember and perhaps describe what was it that you learned through that experience?
5. In what ways did other group members of the team contribute to this work? Do you recall some of the things that you learned from your colleagues that had an effect on your practice, or on how you understand some parts of mathematics differently now, or on your knowledge of ways of teaching certain topics?