A generating tree approach to k-nonnesting arc diagrams

by

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ABSTRACT

This thesis describes a strategy for exhaustively generating series information and enumerating combinatorial classes that can be represented using arc diagrams. We focus on k-nonnesting set partitions, permutations, matchings and tangled diagrams. Results are new functional equations, counting sequences, bijections and asymptotic results for these classes. Our key innovation is a generalized arc diagram in which arcs may have left endpoints, but not right endpoints, and our main tool is generating trees.

Keywords: arc diagram, generating tree, generating function, nestings, crossings, set partition, permutation, matching, tangled diagram, bijection, asymptotics

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Introduction

1.1 Invitation

Sometimes *how* we represent a combinatorial class gives key insight into its structure, enumeration, and asymptotic behaviour. This is the motivation of this thesis. A *combinatorial class* is a finite set, together with a non-negative, integer-valued size function such that the number of elements of any given size is finite. This thesis is centred around the *arc diagram* representation of combinatorial classes. This is an embedded graph encoding of a combinatorial class. Also referred to as a *standard representation* or an *arc annotated sequence*, an arc diagram is a labelled, graphical representation of a combinatorial class. In particular, it is a row of increasing vertices labelled from 1 to n with some arcs between them under restrictions given by the class. Our two central classes of interest are *set partitions* and *permutations*.

A set partition is a partition of a finite domain into non-empty sets, also called blocks. We are be interested in set partitions of $\{1, \dots, n\}$. In the *arc diagram* of a set partitions, a partition block $\{a_1, a_2, \dots, a_j\}$, where $a_1 < a_2 < \dots < a_j$, is represented by the arcs (a_1, a_2), (a_2, a_3), $\dots$, (a_{j-1}, a_j) which, in this case, are always drawn above the vertices. A permutation is a bijective mapping on the set $\{1, \dots, n\}$, and its arc diagram consists of arcs $(a, \sigma(a))$, where the arc is drawn above the row of vertices if $a \le \sigma(a)$, otherwise it is drawn below.

Example 1. *Consider the set partition of* $\{1, 2, \dots, 8\}$*,* $\pi_8 = \{1, 3, 7\}, \{2, 8\}, \{4\}, \{5, 8\}$ *which we denote* $\pi_8 = 137 - 28 - 4 - 56$ *and the permutation* $\sigma_8 = (173)(28)(4)(56) \in \mathbb{S}_8$*. We represent* π_8 *and* σ_8 *using arc diagrams in Figure 1.1.*

There are a variety of statistics that arise in arc diagrams, some are described in Part III, but two particular patterns are natural in this representation: nestings and crossings. They are visualized in Figure 1.2.

We focus on the patterns of k-crossings and k-nestings: a k-crossing (k-nesting) is a set of k arcs in which each of the $\binom{k}{2}$ arcs mutually cross (resp. nest). A 4-crossing and a 4-nesting are depicted in Figure 1.3. Arc diagrams without a k-crossing (k-nesting) are called *k-noncrossing*

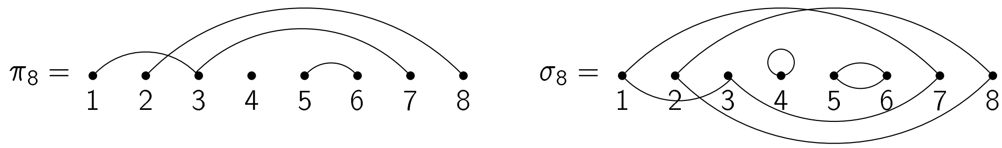


Figure 1.1: Two arc diagrams: π_8 and σ_8

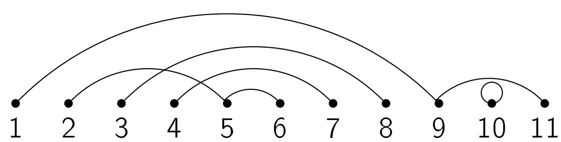
Figure 1.2: A crossing (left); a nesting (right).

Figure 1.3: A 4-crossing (left); a 4-nesting (right).

(resp. *k-nonnesting*).

As we explain in Section 1.2, this is an important pattern in the study of permutation patterns, and RNA secondary structure. In this document we focus mainly on k-nonnesting arc diagrams.

Example 2. *The set partition* $\pi_{11} = 1\,9\,11 - 2\,5\,6 - 3\,8 - 4\,7 - \ 10$ *shown here,*



is 3*-noncrossing, and* 5*-nonnesting.*

1.2 Arc diagrams and crossings

Arc diagrams are useful for illustrating the presence of certain kinds of patterns. We defined a crossing in an arc diagram in Section 1.1, and traditionally this has been the pattern where most interest lay. This is due to a connection to the field of RNA folding. Since the days of Watson and Crick, mathematical models for RNA have continued to emerge, and in 1979 Waterman [55] described the concept of an RNA secondary structure, in which nucleotides were depicted using vertices and the arcs connected nucleotides that formed hydrogen bonds. Initially, the resulting arc diagrams were highly constrained: no arcs were allowed to cross.

It has now become well established that there are cross-serial interactions in RNA: the bonds, or arcs are allowed to cross [48]. Arc diagrams which depicted RNA secondary structures with crossing arcs are referred to as *pseudoknots*, and a variety of differing restrictions on the pseudoknot motif have been examined. These included restricting to even more particular patterns of crossing arcs [52, 1, 46], classifying by genus [50], or according to the maximum number of crossing arcs [22].

Meanwhile, crossings (or lack thereof) in arc diagrams has developed into an active area of pure combinatorial research. As in the world of RNA folding, originally arc diagram studies were dominated by the study of noncrossing diagrams [30, 53, 43]. Then, starting with Touchard [54] and made more explicit by Riordan [51], the total number of matchings with exactly k crossings[1] was determined to be counted by the coefficient of q^k in

$$M_{2n} = \frac{1}{(1-q)^n} \sum_{i \geq 0} (-1)^i \left(\binom{2n}{n-1} - \binom{2n}{n-i-1} \right) q^{\binom{i+1}{2}}.$$

In 1983, M. de Sainte-Catherine [29] proved that the number of matchings with r 2-crossings is equal to the number of matchings with r 2-nestings. This notion of equidistribution between crossings and nestings in arc diagrams was extended dramatically with the innovative and robust bijection of Chen, Deng, Du, Stanley and Yan [20] between matchings with maximum nesting size k and matchings with maximum crossing size k. The bijection used was adapted further in that same article to give that crossings and nestings are equidistributed in set partitions. This gave way to a series of results which showed this equidistribution existed in more combinatorial classes, including permutations [18], graphs [27] and tangled diagrams [23].

These bijections lead to our own interest in the enumeration of arc diagrams with a maximum nesting size: if we can enumerate k-nonnesting arc diagrams, we can also enumerate k-noncrossing diagrams, thereby getting enumerative results which have not been forthcoming.

Indeed, the interest in depicting combinatorial classes using arc diagrams has progressed dramatically since they were first used as a model for RNA folding. Combinatorialists have studied them, both from a structural and enumerative point of view, with crossings and nestings forming the main focus. Furthermore, these restricted structures do not exist in a vacuum, but have counting sequences which relate them to other classic combinatorial classes: as mentioned in Section 1.1, crossings and nestings are connected to pattern avoidance in permutations [26], in matchings they are connected to pairs of nonintersecting Dyck paths [20], and we show and conjecture in Part III that many other relations exists as well.

The heart of the argument that was pioneered by Chen, Deng, Du, Stanley and Yan in 2007 [20] to show crossings and nestings are equidistributed uses a map from the object to a sequence of Young tableaux. Our questions are enumerative, and so our approach is quite different: we study a new class of structures that are essentially arc diagrams "under construction", which we call

[1] not to be confused with k-crossings

open arc diagrams. These are arc diagrams in which an arc may have a left endpoint, but not necessarily a right endpoint, as in Figure 1.4.

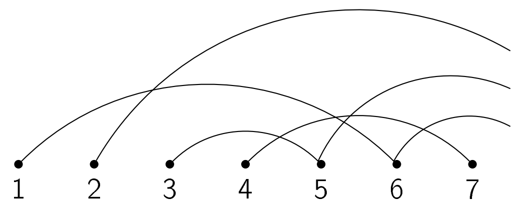


Figure 1.4: Open arc diagram for a set partition.

In Part II, we precisely define open arc diagrams for each combinatorial class we study. These together with generating trees, give access to our main enumerative results. With this representation, generating trees are a natural combinatorial tool to use.

1.3 Generating trees

1.3.1 Philosophy

Many combinatorial classes can be defined recursively. For example, consider a set partition of $\{1, 2, \ldots, n\}$. Element $n+1$ can be inserted into any pre-existing block, or create a block on its own.

Example 3. *Consider the set partition $\pi = 126-34-5 \in \Pi^6$. We can insert 7 in four different locations, resulting in set partitions* $1267-34-5$, $126-347-5$, $126-34-57$ *and* $126-34-5-7$.

This process can be best represented using a tree, where new elements are depicted as children. The tree corresponding to **Example 3** is depicted below.

$126-34-5$

$1267-34-5$ $\quad$ $126-347-5$ $\quad$ $126-34-57$ $\quad$ $126-34-5-7$

Notice: tracking the entire set partition is not necessary. Knowing a *single parameter*, the number of blocks, is enough to determine how many new partitions would be created. In **Example 3**, π had 3 blocks: we can deduce that there will be 4 new partitions constructed. In fact, from our description, we can also determine how many blocks each of the new set partitions will have: three will each have 3 blocks, while one will have 4. The tree above can be redrawn reflecting only this parameter.

3

3 $\quad$ 3 $\quad$ 3 $\quad$ 4

Processes with this property are *generating trees*. When only a single parameter is needed to determine the children, nodes are typically given a label which describes how many children they have. For set partitions, a node labelled $[m]$ will have $m-1$ children labelled $[m]$, and 1 child labelled $[m+1]$. The notation used to describe this particular rewriting rule is:

$$[m] \rightsquigarrow \underbrace{[m][m]\dots[m]}_{m-1 \text{ times}}[m+1]$$

Definition 1. *A* single parameter generating tree *is a rooted labelled tree with the following properties:*

1. *Given the label of a node, the labels of all of its children are determined;*

2. *A node with label $[k]$ has k children labelled $[e_{1,k}], [e_{2,k}], \dots, [e_{k,k}]$.*

Generating trees are specified by first identifying the label of the root, and the defining a set of succession rules (sometimes called rewriting rules).

Generating trees with multiple parameters are similarly defined, but lack property *2*.

Definition 2. *A* multiple parameter generating tree *is a rooted labelled tree with the following property:*

1. *Given the label of a node, the labels of all of its children are determined;*

Generating trees are specified by first identifying the label of the root, and the defining a set of succession rules.

Succession rules were introduced initially by [25] to study *Baxter permutations*, a pattern avoiding permutation we will surprisingly visit again later in this thesis. Formal generating trees were first described by West [56] to study pattern avoiding permutations in general. They were further exploited to enumerate other closely related problems in [6, 7, 31, 32] and [57]. Then in [4] and [5], Barcucci, Del Lungo, Pergola and Pinzani showed that further classical combinatorial structures can be described using generating trees. Banderier, Bousquet-Mélou, Denise, Flajolet, Gardy and Gouyou-Beauchamps in [2] further investigated generating trees, and in particular their connection to their corresponding generating functions, when the label of the tree has one parameter. In particular, if the children are predictable we gain further information about the corresponding generating function, as seen in Section 1.5.

1.3.2 One parameter

In the one parameter case, the label of each node in a generating tree is a single integer. Knowing this value is enough to determine its rewriting rule. In such a case, we say that the generating tree has one parameter. More precisely, a *parameter* is a characteristic or measurable factor that helps to define a system, in this case the generating tree of a combinatorial class. The level or height of the tree encodes the size parameter. The majority of generating trees we study in this thesis have more than one parameter, and as such have vectors as labels. Generating trees specified by a single parameter are increasingly well understood, in particular with regards to the rationality, algebraicity or D-finiteness of the corresponding generating function.

In [2], the authors studied the relationship between a generating tree with one parameter and its generating function. The notation they used to describe the children of a node with label $[k]$ (we use square brackets where they use parentheses) is:

$$[k] \rightsquigarrow [e_{1,k}][e_{2,k}] \dots [e_{k,k}].$$

This describes the succession rule where a node with label $[k]$ (one parameter) has k children with labels $[e_{1,k}]$, $[e_{2,k}]$, $\dots$, $[e_{k,k}]$; Figure 1.5 depicts the start of a generating tree with this succession rule.

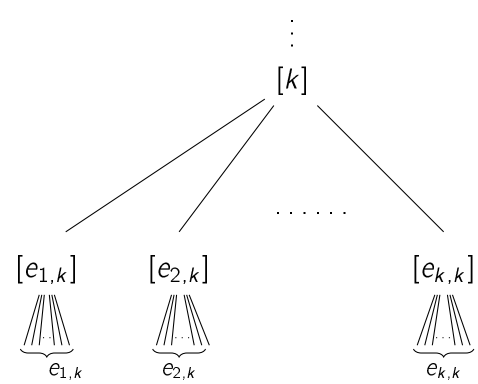


Figure 1.5: Portion of a generating tree with succession rule $[k] \rightsquigarrow [e_{1,k}][e_{2,k}] \dots [e_{k,k}]$.

This formalism allows us to get the functional equation succession:

$$u^k z^n \to z^{n+1}(u^{e_{1,k}} + \dots + u^{e_{k,k}})$$

A label $e_{i,k}$ is exactly the degree of the node with that label. To incorporate the label of the root, s_0, the following notation is used as a shorthand for the entire class:

$$[[s_0], \{[k] \rightsquigarrow [e_{1,k}][e_{2,k}] \dots [e_{k,k}]\}]$$

Remark that the number of nodes at any level is finite, and thus describes a combinatorial class. Set partitions and permutations can each be described and generated using generating trees.

Example 4. *Consider set partitions using block notation. A set partition of* $\{1, 2, \dots, n\}$ *with* m

blocks generates $m+1$ *children which are partitions of* $\{1, 2, \ldots, n+1\}$ *by inserting element* $n+1$ *into each of the* m *blocks, or by adding* $n+1$ *as its own block. Thus, the label of a node is* m*, the number of blocks in the set partition, and the rewriting rule is* $[m] \rightsquigarrow [m+1][m]^{m-1}$ *where* $[m]^{m-1} := \underbrace{[m][m]\ldots[m]}_{m-1 \text{ times}}$.

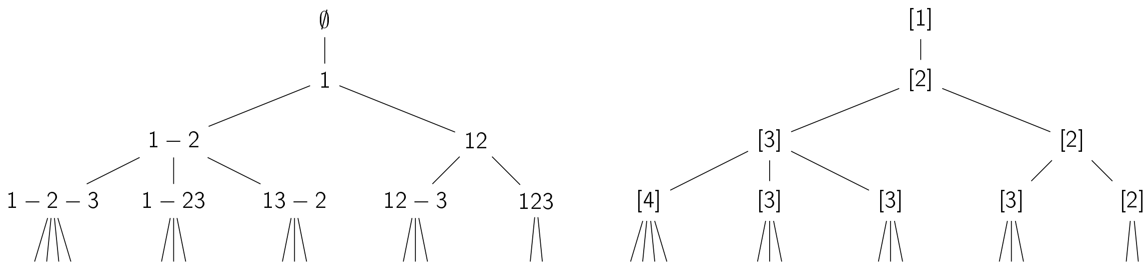


Figure 1.6: The generating tree for set partitions (left) and its labels (right).

The label of the empty set (the root) is [1], *in the notation of Banderier et al., [2] this one parameter generating tree is completely described as:*

$$[[1], [m] \rightsquigarrow [m+1][m]^{m-1}].$$

Example 5. *Consider permutations in one-line notation. A permutation of size* n *generates* $n+1$ *children of size* $n+1$ *by inserting the element* $n+1$ *into the* $n+1$ *positions of the original permutation. The start of the permutation generating tree is given below in Figure 1.7 (left). The number of children of a permutation is simply its size. The corresponding labels are given in Figure 1.7 (right).*

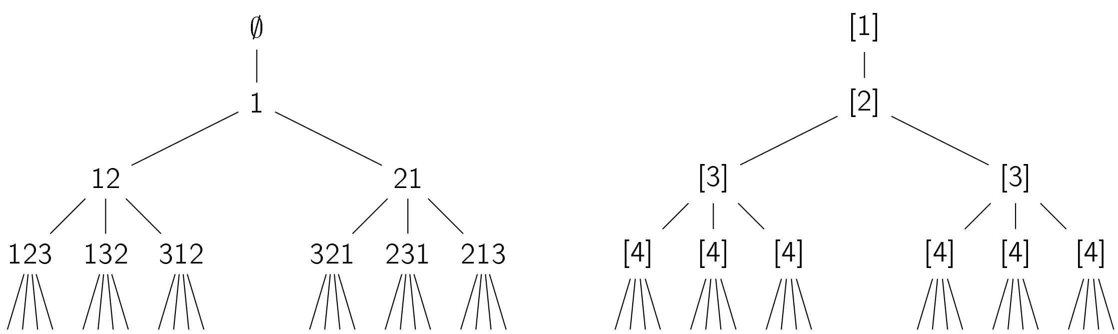


Figure 1.7: The generating tree for permutations (left) and its labels (right).

A permutation which has label n *produces* n *children which each have label* $n+1$*. In the notation established by Banderier et al. [2], the rewriting rule for permutations given in this example is:*

$$[[1], \{[n] \rightsquigarrow [n+1]^n\}]$$

where the first [1] *indicates the number of children of the empty permutation.*

The key property of a generating tree is that exactly one succession rule applies to each possible label of a parent. This is the case in both Example 4 and 5 above. One benefit to describing a generating tree is that in the case of one label, the rewriting rule automatically translates into a functional equation with properties that are well understood, see Section 1.5.

While some classical combinatorial objects may be described using generating trees with only one parameter, more are needed in order to further restrict our classes of combinatorial objects.

1.3.3 Multiple parameters

The conditions of a generating tree can also be satisfied using more than one parameter. As long as the succession rule clearly explains how to derive the children of a node and their corresponding labels, each rule is unique, and there is a label defined for the root of the tree, a generating tree is defined. One example of this is *Baxter permutations*. A permutation $\sigma = \sigma_1 \cdots \sigma_n$ is called a Baxter permutation if there are no indices $i < j < k$ such that $\sigma(j+1) < \sigma(i) < \sigma(k) < \sigma(j)$ or $\sigma(j) < \sigma(k) < \sigma(i) < \sigma(j+1)$.

Example 6. *The permutation $\sigma = 2341$ is a Baxter permutation. The permutation $\sigma = 2413$ is not a Baxter permutation since it violates the first condition when $i = 1$, $j = 2$ and $k = 4$.*

Equivalently, a permutation σ is called a Baxter permutation if for any $i \in \{1, 2, \dots, n-1\}$, σ is either $\sigma = \pi i \pi_- \pi_+ (i+1) \pi'$ or $\sigma = \pi (i+1) \pi_+ \pi_- i \pi'$ where all elements in π_+(resp. π_-) are larger (resp. smaller) than i.

Example 7. *The permutation 3142 is not a Baxter permutation since when $i = 2$, neither condition is satisfied.*

In [38], Gire describes a generating tree for Baxter permutations using a label with *two* parameters: the number of left-to-right maxima and the number of right-to-left maxima of a permutation σ. This is because a Baxter permutation of length $n+1$ can be constructed from a Baxter permutation σ of length n by inserting $n+1$ into σ either just before a left-to-right maximum, or just after a right-to-left maximum. We direct the reader there for more details. The rewriting rule is formally described below.

Lemma 1.3.1 ([38], Gire 1993)**.** *Let σ be a Baxter permutation of length $n \geq 1$ with parameters $[p, q]$. Exactly $p + q$ Baxter permutations can be obtained by inserting $n + 1$ in σ, and their parameters are respectively:*

$$[p, q] \rightsquigarrow [1, q+1], [2, q+1], \dots, [p, q+1], [p+1, q], [p+1, q-1], \dots, [p+1, 1].$$

While we can see that each Baxter permutation with label $[p, q]$ produces $p + q$ children, this property is not standard in the literature. For the generating trees we describe in Chapters 2, 3, 4 and 5, this is not the case, although under some transformations of variables such a property could be recovered. Instead, the labels we describe for the generating trees in these Chapters give strong intuition regarding the objects growth which prefer to keep. Notice though, in the Baxter case we cannot determine the labels of their children based solely on the number of children: it is not a single parameter generating tree.

Example 8. *Consider two nodes which each have* 6 *children, one with label* $[2,4]$ *and the other* $[1,5]$. *The children they produce are:*

$$[2,4] \rightsquigarrow [1,5],[2,5],[3,4],[3,3],[3,2],[3,1]$$

$$[1,5] \rightsquigarrow [1,6],[2,5],[2,4],[2,3],[2,2],[2,1]$$

The node with label $[2,4]$ *produces children which each have* $6,7,7,6,5,4$ *children, while the node with label* $[1,5]$ *gives children which each have* $7,7,6,5,4,3$ *children.*

We return to this class in Section 6.3 where we conjecture a surprising connection between Baxter numbers and a nesting restricted class of open partitions.

1.3.4 Random and Exhaustive Generation

A generating tree construction leads naturally to a random generation algorithm. Given the tree up to depth n, one can uniformly generate objects, knowing the various probabilities. For example, in the case of generating trees with one label [**?**] describe how to generate an object of size n in $O(n \log n)$, given complete information of the number of walks of length n, starting from a state labelled k. This information is computed in $O(n^3)$ time, but only needs to be computed once. We can generalize this very naturally to generating trees with multiple labels.

We provide generating trees of open diagrams, and consequently, this generation strategy must be followed by a rejection stage, if our intended objects are the closed diagrams. If Conjecture 7.3.2 is true, and there is strong evidence that there is, then this rejection stage is not exponential.

On the other hand, exhaustive generation schemes are very simple to implement, as they constitute a traversal of the generating tree to a given level: the tree is constructed to a given level, and the leaves are output. To generate only closed diagrams we do rejection. We can slightly optimize this process in the closed diagram case: at level $n-k$, any node whose diagram has more than k children can be pruned. We can do either a depth first, or breadth first traversal of the generating tree, and these might give very different results. Notice that for our enumerative purposes, the set of children's labels is exactly that: a set; order does not matter. However, if we were interested *generation*, imposing an order on the children's labels would help accomplish this goal. As described in [3], in such a case a random path in the generating tree would uniquely define a combinatorial object.

1.4 Overview: main enumeration

The central enumerative questions that we address are:

Question 1: How many set partitions of $\{1, \ldots, n\}$ are k-nonnesting?

Question 2: How many permutations of $\{1, \ldots, n\}$ are k-nonnesting?

Question 3: Which classes are amenable to the same techniques?

We can answer each of these questions to some extent. For Questions 1 and 2, we determine functional equations which are iterated generate counting sequences. We also find generating trees which construct these classes. While we do not determine a general closed formula for their enumeration, we gain access to new series results, see Chapter 7. Similarly, for Question 3, we determine functional equations for generating functions which enumerate k-nonnesting matchings, and 3-nonnesting tangled diagrams. The series results lead us to find, and conjecture a variety of nontrivial bijective results, which we discuss in Chapter 6.The main contributions of this thesis are summarized in Section 1.7 after a few more introductory remarks.

1.5 Generating functions

A *generating function* is a formal power series whose coefficients encode information about a sequence of numbers A_n. The *ordinary generating function* (OGF) of a sequence is the formal power series

$$A(z) = \sum_{n=0}^{\infty} A_n z^n.$$

Example 9. *Consider the function* $W(z) = \frac{1}{1-2z} = 1 + 2z + 4z^2 + 8z^3 + 16z^4 + 32z^5 + \ldots$. *This* $W(z)$ *is the ordinary generating function of the series* $W_n = 2^n$ *which begins* $1, 2, 4, 8, 16, \ldots$ *The* n^{th} *element of the sequence is captured by the coefficient of* z^n*, which is denoted* $[z^n]$.

The *exponential generating function* (EGF) of a sequence of A_n is the formal power series

$$A(z) = \sum_{n\geq 0} A_n \frac{z^n}{n!}.$$

Example 10. *Consider the function* $P(z) = \frac{1}{1-z} = \sum_{n\geq 0} z^n = \sum_{n\geq 0} n! \frac{z^n}{n!}$. *This is the EGF for permutations, and the start of the series is* $1, 2, 6, 24, 120, \ldots$ *The* n^{th} *element of the sequence is given by the coefficient of* $z^n n!$*, also denoted* $[z^n n!]$.

Generating trees translate directly to *functional equations* satisfied by generating functions.

1.5.1 One parameter

When only one parameter is present in a generating tree, the succession rule used by [2] was given as follows:

$$[[s_0], \{[k] \rightsquigarrow [e_{1,k}][e_{2,k}] \dots [e_{k,k})\}]$$

Recall that a node with label $[k]$ has $[k]$ children, and each of those k children has a label which describes how many children it has: child $i \in \{1, \dots, k\}$, has $e_{i,k}$ children. The following notation also captures this information:

$$[[s_0], \{[k] \rightsquigarrow [e_1(k)][e_2(k)] \dots [e_k(k)]\}].$$

Such notation makes it clear that e_i is a function of k. For such a generating tree, the translation to functional equation proceeds as follows. Let f_n be the number of nodes at level n and s_n be the sum of the labels of those nodes. The ordinary generating function is then $F(z) = \sum_{n \geq 0} f_n z^n$, where $s_n = f_{n+1}$. If $f_{n,k}$ is the number of nodes at level n with label k, then

$$F(u, z) = \sum_{n,k \geq 0} f_{n,k} u^k z^n \text{ and } F_k(z) = \sum_{n \geq 0} f_{n,k} z^n$$

which allows one to recursively determine the functional equation. In [2], the authors studied the links between the structural properties of the succession rule and the corresponding generating functions. For example:

Proposition 1.5.1 ([2], Banderier, Bousquet-Mélou, Denise, Flajolet, Gardy, Gouyou-Beauchamps 2004)**.** *If finitely many labels appear in the tree, then the corresponding generating function is rational.*

Proposition 1.5.2 ([2], Banderier, Bousquet-Mélou, Denise, Flajolet, Gardy, Gouyou-Beauchamps 2004)**.** *Consider the following system:*

$$[[s_0], \{[k] \rightsquigarrow [c_1(k)][c_2(k)] \dots [c_{k-m}(k)][k + a_1][k + a_2] \dots [k + a_m]\}]$$

where $1 \leq a_2 \leq a_2 \leq \dots \leq a_m$ *and the functions* c_i *are uniformly bounded. Let* $C = \max_{i,k}\{s_0, c_i(k)\}$ *and* $\pi_{j,k} = |\{i \leq j : e_i(j) = k\}|$*. If all the series for* $k \leq C$ *are rational, then so is the series* $F(z)$*.*

Proposition 1.5.3 ([2], Banderier, Bousquet-Mélou, Denise, Flajolet, Gardny, Gouyou-Beauchamps 2004)**.** *Let b be a nonnegative integer. For* $k \geq 1$*, let* $m(k) = |\{i : e_i(k) \geq k - b\}|$*. Assume that:*

1. *for all* k*, there exists a forward jump from* k *(i.e.* $(e_i(k) > k$ *for some* i*),*

2. *the sequence* $[m(k)]_k$ *is nondecreasing and tends to infinity.*

Then the (ordinary) generating function of the system has radius of convergence 0*.*

Example 11. *The succession rule* $[[1], [m] \rightsquigarrow [m+1][m]^{m-1}]$ *for set partitions satisfies Prop. 1.5.3. Thus the radius of convergence is* 0 *for the ordinary generating function; we consider the exponential generating function. We get:*

$$\begin{aligned}\tilde{F}(u,z) &= \sum_{n,m\geq 0} f_{n,m}\frac{u^m z^n}{n!}\\ &= u + \sum_{n,m\geq 0} f_{n,m}\frac{z^{n+1}}{(n+1)!}(u^{m+1} + (m-1)u^m)\\ &= u + \int (u\tilde{F}(u,z) + u\tilde{F}_u(u,z) - \tilde{F}(u,z))dz\end{aligned}$$

This is the functional equation. Solving gives the exponential generating function $\tilde{F}(u,z) = u\exp(u(\exp z - 1))$*, which at* $u = 1$ *gives* $\tilde{F}(1,z) = \exp(\exp(z) - 1)$*.*

1.5.2 Multiple parameters

In [12], Bousquet-Mélou describes an approach for solving certain functional equations that arise from generating functions with 2 parameters. The strategy employed makes use of the kernel method, and a certain symmetry of the objects is exploited. The combinatorial objects we will be studying in Part II lack such a symmetry, and so we were unable to adapt her strategy. Indeed we have no strategies for solving the resulting functional equations of generating trees with more than two parameters. Despite this, we iterate the functional equations and get insightful enumerative results.

1.6 Strategy

In answering each of the questions listed in Section 1.1, we follow a procedure which exhaustively generates and enumerates a parameterized combinatorial class: k-nonnesting arc diagrams. Each iteration of the procedure involves the strategy listed in Table 1.1.

In Sections 1.3 and 1.5 we saw how labels are defined in generating trees, and how they can be translated into a functional equation. We did not use an open diagram, we leave that strategy for Part II, but we completed steps **(2)** and **(3)** for set partitions. We can also complete **(4)**.

Example 12. *In Example 4 we showed that the succession rule for set partitions was* $[(1), (m) \rightsquigarrow (m+1)(m)^{m-1}]$*. Then in Example 11 we determined that the corresponding functional equation*

(1) Generalize the arc diagram of the combinatorial class to its corresponding open diagram.
(2) Find a generating tree label and succession rule which tracks nesting statistics.
(3) Translate the generating tree to a functional equation for faster enumeration.
(4) Iterate functional equation to get series data.

Table 1.1: Our strategy for generating and enumerating k-nonnesting arc diagrams.

for set partitions was

$$\tilde{F}(u,z) = u + \int (u\tilde{F}(u,z) + u\tilde{F}_u(u,z) - \tilde{F}(u,z))dz$$

We can extract series information from this functional equation by iterating. We view the equation $F = 1 + z\Phi(F)$ as the system $F^{[n]} = 1 + z\Phi(F^{[n-1]})$. The succession rule gives that the label of the root is (1), thus we input $\tilde{F}(u,z)^{[0]} = u$ into the above equation.

$$\begin{aligned}\tilde{F}(u,z)^{[1]} &= u + \int (u \cdot u + u \cdot 1 - u)dz \\ &= u + u^2 z.\end{aligned}$$

We repeat this process, substituting $\tilde{F}(u,z)^{[1]}$ into our functional equation.

$$\begin{aligned}\tilde{F}(u,z)^{[2]} &= u + \int (u \cdot (u + u^2 z) + u \cdot (1 + 2uz) - (u + u^2 z))dz \\ &= u + \int ((u^3 + u^2)z + u^2)dz \\ &= u + u^2 z + \frac{z^2}{2}(u^3 + u^2)\end{aligned}$$

As this is an EGF, multiplying through by $n!$ recovers the enumerative information encoded in the generating tree:

- *u: the empty set has label (1);*
- *$u^2 z$: there is one set partition of size 1, it has label (2);*
- *$\frac{z^2}{2}(u^3 + u^2)$: multiplying through by $2!$ gives that there are two set partitions of size 2, one has label (3), the other label (2).*

Note: setting $u = 1$ recovers the total number of set partitions.

Following this strategy accounts for the bulk of Part II. Exhaustive generation and enumeration are not the only results that arise from this process however. We also gain access to data which points us toward both asymptotic and bijective questions, each of which is addressed in Part III.

1.7 Summary of contributions

The main contributions of this thesis are found in Part II, where the framework from Section 1.6 is described in detail uses our new tool of open arc diagrams for k-nonnesting set partitions, permutations, matchings and tangled diagrams. It draws on two works: an extended abstract and an article submitted for publication.

[16] Sophie Burrill, Sergi Elizalde, Marni Mishna, and Lily Yen. A generating tree approach to k-nonnesting partitions and permutations. *arXiv:1108.5615 [math.CO]*, 2014+ *(under review, 35 pages)*.

[17] Sophie Burrill, Sergi Elizalde, Marni Mishna, and Lily Yen. A generating tree approach to k-nonnesting partitions and permutations. In *24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012)*, Discrete Math. Theor. Comput. Sci. Proc. AR, pages 409-420. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2012.

Our framework directly constructs our k-nonnesting arc diagrams, returning the actual trees. The succession rules for each of our trees is quite 'nice;' they are explicit and finitely specified. The geometric sums that arise in the corresponding functional equations are relatively compact and efficient. We are above to rewrite in terms of evaluations of the functional equations. The resulting series data, found in Appendix A, largely motivates Part III. In some cases, series results have previously appeared in the literature. As such, we give a variety of bijections between k-nonnesting arc diagrams and other combinatorial classes, and conjecture a surprising connection between nesting restricted arc diagrams and Baxter permutations; see Chapter 6. Furthermore, the data allowed us to conjecture on the asymptotic form of k-nonnesting arc diagrams. We prove upper bounds on their exponential growth factor in Chapter 7, results which are appearing for the first time in this thesis.

In Part III we consider distribution of crossing and nesting statistics in arc diagrams. The equidistribution of these parameters in permutations was first given in:

[18] Sophie Burrill, Marni Mishna, and Jacob Post.On k-crossings and k-nestings of permutations. In *22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010)*, Discrete Math. Theor. Comput. Sci. Proc., AN, pages 593-600. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2010.

Equidistribution between crossing and nesting statistics in open arc diagrams is appearing for the first time in this thesis.

Enumerative results on k-nonnesting arc diagrams exist in the literature. To clarify where our contribution has been made, Table 1.2 summarizes our results in the context of the literature. For each of set partitions, permutations, matchings and tangled diagrams, we determine a functional equation, series results for up to n terms, and an upper bound on the exponential growth factor, g. Some explicit generating functions are previously known; such results are depicted in black. New results are printed in **blue**.

As can be seen from Table 1.2, enumerative results are completely known in the case of matchings. The generating function for tangled diagrams, to be defined in Chapter 5, relies heavily on their connection to matchings. The only other known nontrivial enumerative result for k-nonnesting arc diagrams is in the case of 3-nonnesting set partitions. In [13] the authors also rely on known results in matchings, and remark that their method is unwieldy to extend to larger k. It is here that our strategy shines: we give a method for exhaustively generating and enumerating k-nonnesting arc diagrams which is entirely independent of any already known enumerative matching results. All that is required is that the combinatorial object can be depicted using arc diagrams.

We now begin Part II by focusing on set partitions, and our key innovation in the study of k-nonnesting arc diagrams: the *open arc diagram*.

Comb. class	k	G.F.	**In this thesis**					References
			Func. eq. Corollary	page	**Series** n	**Asymptotics** $r \leq$	page	
Set partitions	2	✓				4		[Folklore]
	3	✓	2.4.2	31	420	9		[13]
	4	✗	**2.5.2**	**36**	**276**	**16**	**92**	
	5	✗	**2.5.2**	**36**	**129**	**25**	**92**	
	6	✗	**2.5.2**	**36**	**32**	**36**	**92**	
	7	✗	**2.5.2**	**36**	**21**	**49**	**92**	
	$k > 7$	✗	**2.5.2**	**36**		$\mathbf{k^2}$	**92**	
Set partitions (enhanced)	2	✓				3		[Folklore]
	3	✓	2.6.4	39	484	8		[13]
	4	✗	**2.6.4**	**39**	**129**	**15**	**93**	
	5	✗	**2.6.4**	**39**	**121**	**24**	**93**	
	6	✗	**2.6.4**	**39**	**37**	**35**	**93**	
	7	✗	**2.6.4**	**39**	**30**	**48**	**93**	
	$k > 7$	✗	**2.6.4**	**39**		$\mathbf{k^2 - 1}$	**93**	
Permutations	2	✓						[18]
	3	✗	**3.3.2**	**49**	**223**	**16**	**93**	
	4	✗	**3.3.4**	**52**	**20**	**36**	**93**	
	5	✗	**3.3.4**	**52**	**16**	**64**	**93**	
	6	✗	**3.3.4**	**52**	**15**	**100**	**93**	
	$k > 8$	✗	**3.3.4**	**52**		$\mathbf{4(k-1)^2}$	**93**	
Matchings	2	✓						[Folklore]
	3	✓	4.2		n	4		[20], [42]
	k	✓	4.3.2		n	$2(k-1)$		[20], [42]
Tangled diagrams	3	✓	5.3	68	60	21		[23]
	k	✓				$4(k-1)^2 + 2(k-1) + 1$		[23]

Table 1.2: Summary of results. Entries in **blue** are new. Table indicates for each k-nonnesting combinatorial class whether the explicit generating function (**G.F.**) is known, the corollary that states the functional equation (**Func. eq**), the number of terms n we compute for each series, and an upper bound on the exponential growth factor r. When an exact exponential growth factor is known, results are depicted in regular font. Functional equations and asymptotic results have corresponding page numbers listed, and relevant references are indicated.

Part II

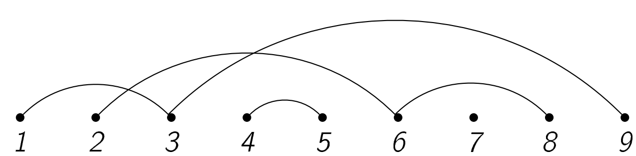
Generating trees for k-nonnesting arc diagrams

Chapter 2

Set partitions

Set partitions are are a classic combinatorial object. Recall that a set partition of $\{1, \ldots, n\}$ is a union of non-overlapping, non-empty subsets, called *blocks*, of $\{1, \ldots, n\}$. We take the convention of labelling the vertices of our arc diagrams from left to right, so that we can refer to left and right endpoints of an arc. The arc diagram representation of a set partition, which we call a *partition diagram*, always has the arcs drawn above the vertices. The partition block $\{a_1, a_2, \ldots, a_j\}$, where $a_1 < a_2 < \cdots < a_j$, is represented by the set of arcs $(a_1, a_2), (a_2, a_3), \ldots, (a_{j-1}, a_j)$.

Example 13. *The set partition* $\pi = 139 - 268 - 45 - 7$ *depicted as an arc diagram:*



Notice that this partition diagram has 2*-nestings, but not* 3*-nestings; it is* 3*-nonnesting.*

Our aim is to determine the number of k-nonnesting arc diagrams. We begin with some history,

2.1 History

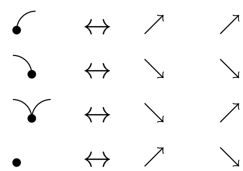
Because set partitions are a classic, fundamental combinatorial object, they have been well studied. It is known that set partitions are enumerated by the *Bell numbers*., B_n which satisfy the following recurrence:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k, \qquad B_0 = B_1 = 1.$$

The Bell numbers also have the EGF

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = e^{e^z - 1}.$$

The enumeration of noncrossing partitions has now become classical: they are counted by the Catalan numbers: $C_n = \frac{1}{n+1}\binom{2n}{n}$. One classical combinatorial object which is enumerated by C_n is *Dyck paths* of length $2n$. Recall that a Dyck path is a path on $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ that never steps below the x-axis, and has step set $\{(1,1),(1,-1)\}$. There is a bijection between set partitions of size n with no crossings (i.e. 2-nonncrossing set partitions) and Dyck paths of length $2n$, where each vertex translates to two steps in the Dyck path according to the following dictionary:



Arcs are connected in the unique way such that no crossing is formed. This dictionary also gives a bijection between 2-nonnesting set partitions and Dyck paths as there is a unique way to connect the arcs such that no nesting is formed. This bijection, and similar ones, are further explored in Chapter 6.

The fact that the number of 2-noncrossing partition diagrams is equivalent to the number of 2-nonnesting partition diagrams was generalized dramatically in [20]:

Theorem 2.1.1 ([20] Chen, Deng, Du, Stanley, Yan 2007)**.** *The number of k-noncrossing set partitions of $\{1,\ldots,n\}$ is equal to the number of k-nonnesting set partitions of $\{1,\ldots,n\}$.*

Thus, if we enumerate the number of k-nonnesting set partitions, we also enumerate the number of k-noncrossing set partitions.

Further enumerative results have been less forthcoming, with one notable exception. In [13], Bousquet-Mélou and Xin enumerated the number of 3-nonnesting set partitions.

Theorem 2.1.2 ([13] Bousquet-Mélou, Xin 2007)**.** *The number $C_3(n)$ of 3-noncrossing set partitions is given by $C_3(0) = C_3(1) = 1$, and for $n \geq 0$,*

$$9n(n+3)C_3(n) - 2(5n^2 + 32n + 42)C_3(n+1) + (n+7)(n+6)C_3(n+2) = 0.$$

Equivalently, the associated generating function $\mathcal{C}(t) = \sum_{n\geq 0} C_3(n)t^n$ satisfies

$$t^2(1-9t)(1-t)\frac{d^2}{dt^2}\mathcal{C}(t) + 2t(5-27t+18t^2)\frac{d}{dt}\mathcal{C}(t) + 10(2-3t)\mathcal{C}(t) - 20 = 0.$$

and has asymptotic form as $n \to \infty$,

$$C_3(n) \sim \kappa \frac{9^n}{n^7},$$

for some positive constant κ.

Theorem 2.1.3 ([13] Bousquet-Mélou, Xin 2007)**.** *For $n \geq 1$, the number of 3-noncrossing set partitions of $\{1, \ldots, n\}$ is*

$$C_3(n) = \sum_{j=1}^{n} \frac{4(n-1)!n!(2j)!}{(j-1)!j!(j+1)!(j+4)!(n-j)!(n-j+1)!} P(j, n)$$

with

$$P(j, n) = 24 + 18n + (5 - 13n)j + (11n + 20)j^2 + (10n - 2)j^3 + (4n - 11)j^4 - 6j^5.$$

Here, the authors translated a family of restricted lattice paths to walks in the non-negative domain, which then, using a recursive construction of the walks, determined a functional equation which was solved using the kernel method. They further conjectured:

Conjecture 2.1.4 ([13] Bousquet-Mélou, Xin 2005)**.** *For every $k > 3$, the generating function of k-nonnesting set partitions is not D-finite.*

Note that a formal power series $f(z)$ is called *D-finite* if there exists polynomials $a_r(z)$, $a_{r-1}(z), \ldots, a_0(z)$ which lie in the field $\mathbb{C}(z)$ of rational functions that satisfy the linear differential equation,

$$a_0(z)\frac{d^r}{dz^r}f(z) + a_1(z)\frac{d^{r-1}}{dz^{r-1}}f(z) + \ldots + a_r(z)f(z) = 0.$$

Mishna and Yen [49] determined functional equations for k-nonnesting set partitions, and described a process for isolating coefficients, giving additional evidence to this conjecture.

With our procedure, we recover known series results and also extend to a different type of nesting pattern, called the *enhanced nesting* in Section 2.6.

The backbone of this thesis is the strategy outlined in Section 1.6 as a series of four steps which can be applied to many combinatorial class restricted to k-nonnesting. To warm up, we first treat set partitions with no nesting restrictions. We then follow through the strategy for those with nesting restrictions.

2.2 Open arc diagrams

To begin, in any arc diagram for partitions, a vertex is one of four types:

1. **fixed point** no incident edges;
2. **opener** the left endpoint of an arc;
3. **transitory** the right endpoint of one arc and the left endpoint of another;
4. **closer** the right endpoint of an arc.

Example 14. *Figure 2.1 shows the diagram of a partition of* $\{1, \ldots, 9\}$*, and illustrates the different types of vertices. The vertex labelled* 2 *is a* fixed point*; the vertices labelled* 1, 4*, and* 7 *are* openers*; vertices* 3 *and* 8 *are* transitories*; and* 5, 6*, and* 9 *are* closers.

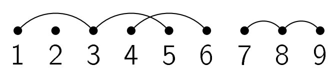


Figure 2.1: Arc diagram representation of $\{1, 3, 5\}\{2\}\{4, 6\}\{7, 8, 9\}$.

An important innovation to this study of k-nonnesting set partitions is a generalization of the arc diagram. This new class of structures are essentially arc diagrams "under construction," which we call *open diagrams*. These are arc diagrams which may have semi-arcs: vertices with only a single left endpoint and no right endpoint. In the language above, we allow opener and transitory vertices which are not 'closed.' We sometimes call such arcs open semi-arcs, or simply semi-arcs. We draw the semi-arcs to a vertical line to the right of vertex n, and retain their order, not allowing the semi-arcs to intersect. A semi-arc with left endpoint i will be denoted by $(i, *)$. Such generalized diagrams are called *open partition diagrams*.

Example 15. *The open partition diagram* π *depicted in Figure 2.7 has arcs* $(1, 3)$*,* $(3, *)$*,* $(4, 6)$*,* $(5, *)$*,* $(7, *)$ *and* $(8, 9)$.

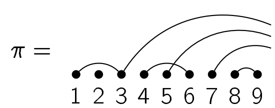


Figure 2.2: Open partition diagram of $\{1, 3, *\}, \{2\}, \{4, 6\}, \{7, *\}, \{8, 9\}$

Notice that a diagram with no semi-arcs represents a usual set partition. Such a diagram is called a *complete* (partition) diagram. Complete diagrams form the subclass of open diagrams with no semi-arcs. This will be important for enumeration purposes.

2.2.1 The label and succession rule

An open partition diagram can be viewed as a future set partition, or as a set partition in progress. This process incrementally adds vertices in numerical order: the added vertices may close semi-arcs, and/or they may open new ones. We first study the generation of open partition diagrams without nesting conditions.

Example 16. *The open partition diagram* π *from Example 15 has infinitely many descendants. Figure 2.3 has two set partitions which have* π *as an ancestor.*

It is worth noticing that open partition diagrams can also be viewed as bi-coloured set partitions: those in which each block is coloured one of two colours. Blocks which are complete, i.e. those

Figure 2.3: Two set partitions with π as an ancestor.

that end in a closer or are fixed points are given one colour, while those that end with a semi-arc have another colour.

Example 17. *The open partition diagram π in Example 15 represents the bi-coloured partition* $\{\mathbf{1},\mathbf{3}\}$, $\{2\}$, $\{4,6\}$, $\{\mathbf{5}\}$, $\{\mathbf{7}\}$, $\{8,9\}$ *where blocks written in bold face are those that end in a semi-arc, and normal fonts indicate proper blocks.*

We begin by describing the generating tree for these open partition diagrams. Given an open partition diagram with n vertices, the added vertex $n+1$ can be any of the four kinds: fixed point, opener, transitory, closer, provided an existing semi-arc is available to be closed by a transitory or closer. Thus the parameter we must track is semi-arcs which are available to be closed.

Example 18. *The open partition diagram π from Example 15 generates 8 diagrams which are seen in Figure 2.4*

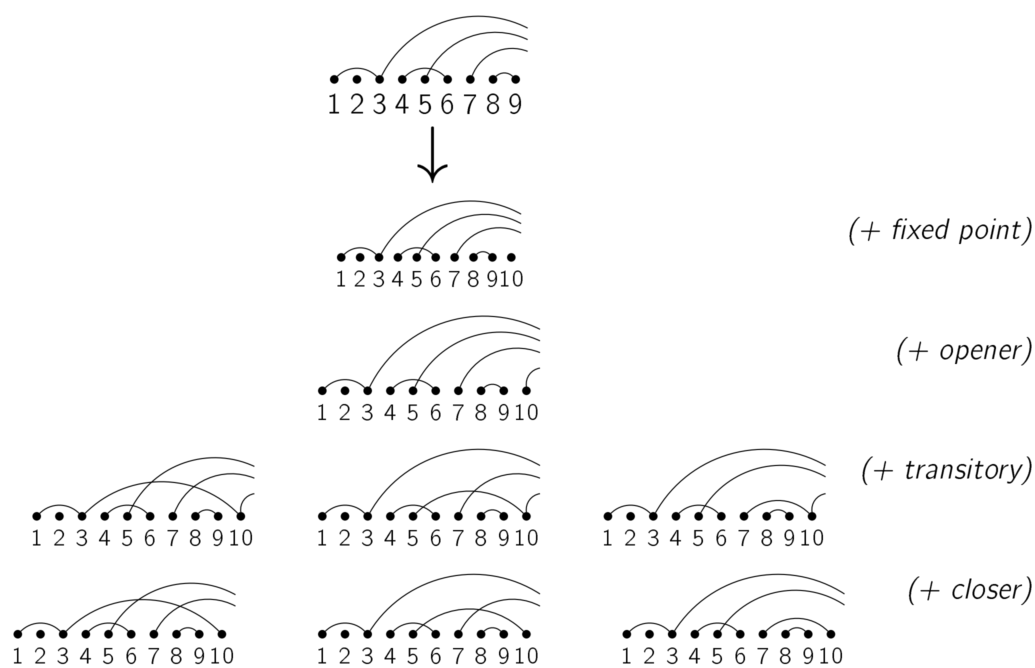


Figure 2.4: Open partition diagram for π and its eight children.

Our label consists of the number of semi-arcs. This is sufficient to describe the number of children and all of their labels. To see this, suppose a diagram π with n vertices has $\ell(\pi) = m$

semi-arcs. Its number of children and their labels (i.e. the number of semi-arcs) are as follows, depending only on the type of added vertex $n+1$:

1. **fixed point** one child with m semi-arcs;
2. **opener** one child with $m+1$ semi-arcs;
3. **transitory** m children, each with m semi-arcs;
4. **closer** m children, each with $m-1$ semi-arcs.

In condensed form:

$$[m] \rightsquigarrow \begin{cases} [m], & & \text{(fixed point)} \\ [m+1], & & \text{(opener)} \\ \underbrace{[m],[m],\dots,[m]}_{m \text{ copies}}, & \text{if } m>0, & \text{(transitory)} \\ \underbrace{[m-1],[m-1],\dots,[m-1]}_{m \text{ copies}}, & \text{if } m>0. & \text{(closer)} \end{cases} \tag{2.1}$$

We denote the set of labels in the succession rule by $\mathrm{succ}([m])$. The root of the tree has label [0] because the empty set partition has no semi-arcs

Notice that if there are no semi-arcs in the parent open diagram, i.e. when $m=0$, the last two rules are trivially empty. The total number of children can be found by summing these cases up: any diagram with m semi-arcs has $2m+2$ children. The number of children of π and their labels are completely determined by $\ell(\pi)$. This means these open partition diagrams are fit for generating tree techniques.

In Section 1.3 we saw the rewriting system of Banderier *et al.* [2] which used the number of children as the parameter in their label. For an open partition diagram with m semi-arcs, this equals $2m+2$. Using their notation, the rewriting rule would be:

$$[[2] : [2m+2] \rightsquigarrow [2m+2][2m+4][2m+2]^m[2m]^m]. \tag{2.2}$$

The start of the generating tree using our succession rule is given in Figure 2.5.

2.2.2 Functional equation

Now we translate to functional equations. We perform this translation using succession rule 2.1.

Let $P(u,z)$ be the bivariate generating function for open set partitions where the exponent of u is the label of the node, which in this case is the number of semi-arcs. Let π be an open partition diagram with $|\pi|=n$ vertices and $\ell(\pi)=m$ semi-arcs. Let $\mathcal{C}(\pi)$ denote the set of children of π. The children $\pi' \in \mathcal{C}(\pi)$ in the generating tree can be of four types: fixed point, opener, transitory

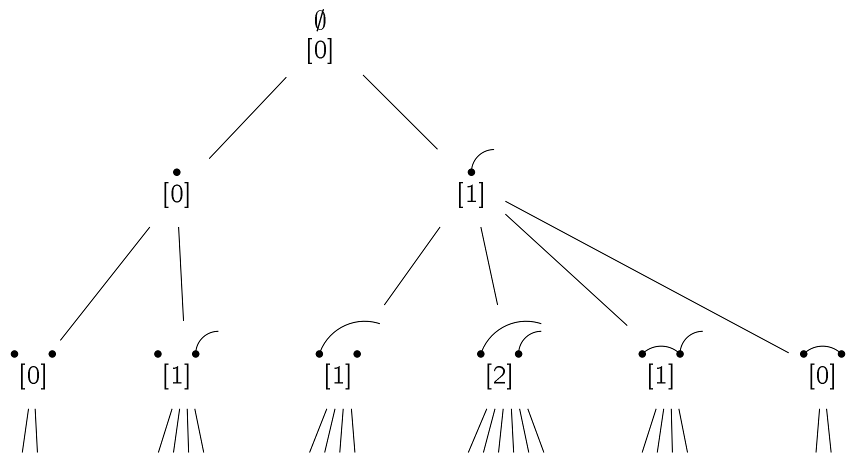


Figure 2.5: Start of generating tree for open set partitions using labelling 2.1

and closer. Its type depends on the type of the added vertex $n+1$, and the labels are described above in 5.1. From this succession rule, it follows that:

$$\sum_{\pi' \in \mathcal{C}(\pi)} u^{\ell(\pi')} = \underbrace{u^m}_{\text{fixed point}} + \underbrace{u^{m+1}}_{\text{opener}} + \underbrace{m\,u^m}_{\text{transitory}} + \underbrace{m\,u^{m-1}}_{\text{closer}},$$

which gives the following generating function recurrence:

$$\begin{aligned} P(u,z) &= \sum_{n,m\geq 0} p(m,n)\frac{u^m z^n}{n!} \\ &= 1 + \sum_{n,m\geq 0} p(m,n)\frac{z^{n+1}}{(n+1)!}\left(u^m + u^{m+1} + mu^m + mu^{m-1}\right) \\ &= 1 + \int \left(P(u,z) + uP(u,z) + uP_u(u,z) + P_u(u,z)\right)\,dz, \end{aligned}$$

where we use the fact that $\int P(z,u)\,dz = \sum p(m,n)\frac{1}{n+1}\frac{z^{n+1}u^m}{n!}$. Differentiating with respect to z we get

$$P_z(u,z) = (1+u)P_u(u,z) + (1+u)P(u,z) = (1+u)\left(P_u(u,z) + P(u,z)\right),$$

In this case, we are able to actually solve this differential equation and get that $P(u,z) = e^{(1+u)(e^z-1)}$. Setting $u=0$ recovers all (regular) set partitions, i.e. those without semi-arcs, and we see the familiar EGF: e^{e^z-1}. Ideally we would always be able to solve the functional equation; however, future chapters show that incorporating further parameters adds enough to the complexity of the equations that it becomes quite a formidable task.

2.2.3 Series data

We can iterate the functional equation and recover the total number of set partitions with no nesting restrictions, but in this case we simply use the explicit generating function $P(z) = e^{e^z-1}$ to recover sequence A000110 in The On-Line Encyclopedia of Integer Sequences [41] when $u = 0$. When $u = 1$, the generating function $P^* = e^{2(e^z-1)}$ gives A001861: the number of bi-coloured set partitions. See Chapter 6 for explicit bijections, and Appendix A for data.

2.3 Future k-nestings

Now we incorporate the nesting restrictions into our open diagrams.

Recall that a k-nesting in a partition diagram is a set of k mutually nesting arcs, that is, arcs $(i_1, j_1), \ldots, (i_k, j_k)$ such that

$$i_1 < i_2 < \cdots < i_k < j_k < j_{k-1} < \cdots < j_1.$$

To generalize the notion of k-nestings to open partition diagrams we define future k-nestings.

Definition 3. *A* **future k-nesting** *is a set of $k-1$ mutually nesting arcs and one semi-arc, such that the left end-point of the semi-arc is to the left of the $k-1$-nesting.*

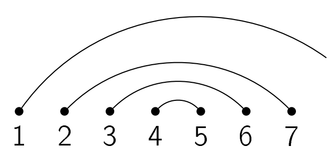


Figure 2.6: An example of a future 4-nesting.

An example is drawn in Figure 2.6. Recall that since semi-arcs do not intersect, a semi-arc that is above another one also has its left endpoint further to the left. Thus we can make the following claim:

Claim 1. *If a semi-arc belongs to a future k-nesting, then any semi-arc above it also belongs to a future k-nesting.*

Notice that it is not true, however, that having multiple semi-arcs above a $k-1$-nesting means that there is an ℓ-nesting in its complete descendants for $\ell > k$.

Example 19. *The open partition diagram π has 2 semi-arcs above a 2-nesting (and thus a future 3-nesting). The partition π' is a descendant of π, and has a 3-nesting, but not a 4 or 5-nesting.*

We can now define k-nonnesting open partition diagrams.

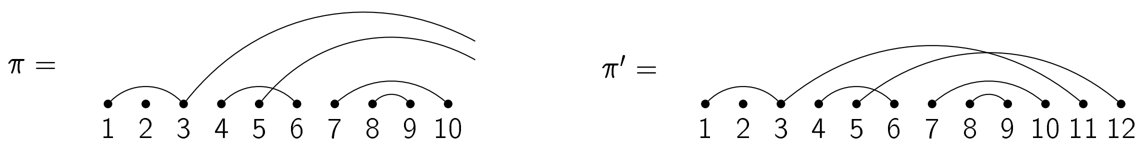


Figure 2.7: An open partition diagram π (left); one of π's complete descendants, π'(right)

Definition 4. *An open partition diagram is* k-nonnesting *if it contains neither regular nor future* k*-nestings.*

A closed/completed diagrams is of course also k-nonnesting in this definition. Our strategy is to pare the generating tree for open partition diagrams by removing all vertices that create future k-nestings, plus all of their children; this is sufficient.

Proposition 2.3.1 ([16], Burrill, Elizalde, Mishna, Yen, 2014+)**.** *Consider the subclass of open partition diagrams generated from the empty diagram left after pruning all subtrees of diagrams with a future* k*-nesting. The elements in this class are precisely the* k*-nonnesting open partition diagrams. In particular, the complete diagrams in this class are precisely the* k*-nonnesting partitions.*

Proof. Any open partition diagram that is a descendant of a diagram with a future k-nesting has either a k-nesting or a future k-nesting. Indeed, the only way to remove a future k-nesting is by closing its top semi-arc, which creates a k-nesting.

On the other hand, starting from a diagram with a k-nesting, and deleting vertices starting from the right, one can find at least one ancestor diagram that contains a future k-nesting. Thus, in order to obtain all k-nonnesting open partition diagrams (and thus all k-nonnesting complete partition diagrams), it is sufficient to generate all open diagrams that avoid future k-nestings. ☐

When we generate all k-nonnesting open partition diagrams, we are generating a superset of the k-nonnesting partitions. After the translation from generating tree to functional equation, we are able to recover the generating function for k-nonnesting partitions by variable specialization, that is setting the variable which marks semi-arcs to 0. In [49] Mishna and Yen gave a construction which generates only k-nonnesting partitions, but the construction we describe here will both handle *enhanced* set partitions (see Section 2.6) and k-nonnesting permutations (see Chapter 3). Furthermore, any concerns that we are over-generating by constructing open diagrams instead of closed ones are eased in Part III where an upper bound on the exponential growth rate for the number of k-nonnesting open diagrams is found.

2.4 3-nonnesting partitions

2.4.1 The label and succession rule

In order to capture the greater complexity of a 3-nonnesting set partition, we consider a label which is a *two* component vector. To each 3-nonnesting open partition diagram π, associate the label $\ell(\pi) = [m, s]$, where m is the total number of semi-arcs and s is the number of semi-arcs in a future 2-nesting. This s is the number of semi-arcs that are above, or to the left of, at least one closed arc. The label of the empty partition diagram ϵ is $\ell(\epsilon) = [0, 0]$.

Example 20. *Consider the open partition diagram σ in Figure 2.8. The two arrows indicate the semi-arcs in future* 2*-nestings. Thus,* $\ell(\sigma) = [4, 2]$. *If vertex 12 is a closer or a transitory vertex closing the semi-arc started at vertex 7, then* $(7, 12)$, $(8, 9)$ *and* $(3, *)$ *form a future* 3*-nesting, which is forbidden in the construction. On the other hand, if vertex* 12 *closes the semi-arc started at vertex 3, then only a* 2*-nesting is created, but no future* 3*-nesting. Vertex* 12 *can also close any of the other two semi-arcs and remain in the class. Consequently, this arc diagram has eight children in our generating tree: the two obtained from adding a fixed point or opener, plus the six diagrams obtained by making vertex* 12 *a closer or transitory vertex.*

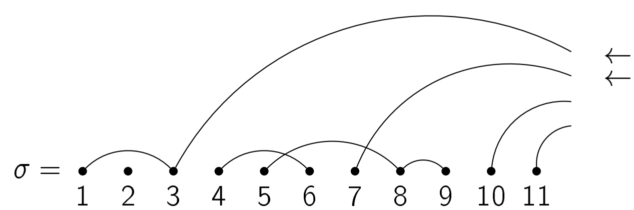


Figure 2.8: An open partition diagram with label [4, 2].

In order to avoid constructing future 3-nestings, we must not close any semi-arc which is in a future 2-nesting, except for the very top, left-most arc, which can always be closed. Any semi-arc which does not belong to a future 2-nesting can also be closed. Furthermore, we are always able to add openers and fixed points, because their addition can not create a future 3-nesting. These observations help describe the succession rule.

We now must show that the label we defined provides sufficient information to determine the number children and their labels.

Suppose a diagram π of size n has $\ell(\pi) = [m, s]$. Its children are as follows, depending on the type of the added vertex $n + 1$:

1. fixed point one child with label $[m, s]$;

2. opener one child with label $[m + 1, s]$;

3. **transitory** (if $s > 0$) $m - s + 1$ or (if $s = 0$) m children, since we can close any of the $m - s$ semi-arcs not in future 2-nestings, plus the top semi-arc in the case $s > 0$;

4. **closer** (if $s > 0$) $m - s + 1$ or (if $s = 0$) m children.

In the next theorem, we address the labels of children. To build intuition, we first return to the open partition diagram from Example 20.

Example 21. *Consider the different diagrams generated by adding a vertex to the diagram in Figure 2.8; $\ell(\sigma) = [4, 2]$ and the application of the succession rule,* $\mathrm{succ}([4, 2])$*, yields the following, sorted by the type of vertex added:*

1. **fixed point** *label* $[4, 2]$*;*

2. **opener** *label* $[5, 2]$*;*

3. **transitory** *in each one of the three children, the number of semi-arcs is preserved, while the number of semi-arcs belonging to future* 2*-nestings depends on the semi-arc that is closed, giving labels* $[4, 3]$, $[4, 2]$ *and* $[4, 1]$ *when closing* $(11, *)$, $(10, *)$ *and* $(3, *)$*, respectively;*

4. **closer** *in each one of the three children, the number of semi-arcs is reduced by one, but otherwise it is analogous to the transitory case, giving labels* $[3, 3]$, $[3, 2]$, $[3, 1]$ *when closing* $(11, *)$, $(10, *)$ *and* $(3, *)$*, respectively.*

Theorem 2.4.1 ([16], Burrill, Elizalde, Mishna, Yen, 2014+)**.** *Let $\Pi^{(2)}$ be the set of* 3*-nonnesting open partition diagrams. To each diagram, associate the label $\ell(\pi) = [m, s]$ if π has m semi-arcs, s of which belong to some future* 2*-nesting. Then the number of diagrams in $\Pi^{(2)}$ of size n is the number of nodes at level n in the generating tree with root label* $[0, 0]$ *for* $n = 0$*, and succession rule given by*

$$[m,s] \rightsquigarrow \begin{cases} [m,s], & & \text{(fixed point)} \\ [m+1,s], & & \text{(opener)} \\ [m,s],[m,s+1],\ldots,[m,m-1], & \text{if } m>0, & \text{(transitory)} \\ [m-1,s],[m-1,s+1],\ldots,[m-1,m-1], & \text{if } m>0, & \text{(closer)} \\ [m,s-1],[m-1,s-1]. & \text{if } m>0 \ \& \ s>0. & \text{(transitory and closer respectively, when } s>0\text{)} \end{cases}$$

The number of 3*-nonnesting set partitions of* $\{1, \ldots, n\}$ *is equal to the number of nodes with label* $[0, 0]$ *at level* n*.*

Proof. We focus on closer vertices, since transitory vertices behave identically except for the first component, and fixed points and openers have already been discussed.

Consider a 3-nonnesting open partition diagram π with $\ell(\pi) = [m, s]$. It must be that $m > 0$, or else there is nothing to close. Closing any semi-arc decreases the total number of semi-arcs by one. Closing the bottom semi-arc turns all the semi-arcs above it into future 2-nestings, producing a diagram with label $[m-1, m-1]$. Closing the second lowest semi-arc converts all remaining semi-arcs except the bottom one into future 2-nestings, and the resulting diagram has label $[m-1, m-2]$. More generally, each of the $m-s$ semi-arcs not belonging to future 2-nestings can be closed in this manner, yielding $m-s$ diagrams with labels $[m-1, m-i]$ for $1 \leq i \leq m-s$.

Finally, if $s > 0$, then π contains a future 2-nesting, and the top semi-arc can be closed without creating a future 3-nesting. This operation removes a future 2-nesting, resulting in label $[m-1, s-1]$.

$\square$

The first few levels of this generating tree are shown in Figure 2.9, and the first 9 levels are given in Appendix B.

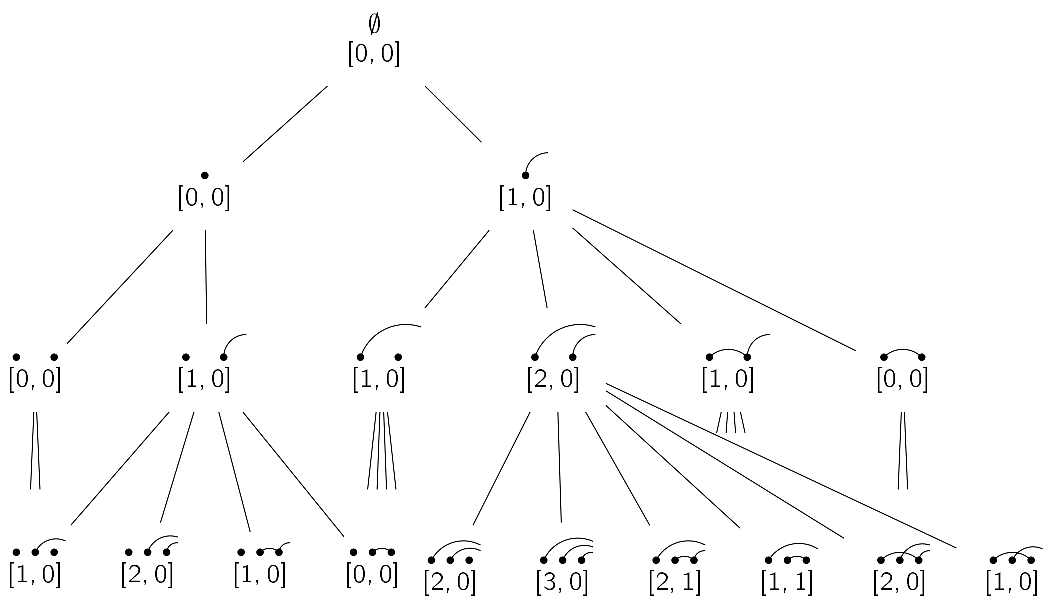


Figure 2.9: Generating tree for 3-nonnesting open partition diagrams with labels.

2.4.2 Functional equation

Given the generating tree for 3-nonnesting open partition diagrams, we can translate its succession rule, given in Theorem 2.4.1, into generating function equations. An evaluation of the resulting generating function will give the function for the class we are primarily interested in: 3-nonnesting set partitions.

Define the multivariate generating function

$$A(u, v; z) = \sum_{\pi \in \Pi^{(2)}} u^m v^s z^{|\pi|} = \sum_{m,s,n} a_{m,s}(n) u^m v^s z^n,$$

where $\Pi^{(2)}$ is the set of 3-nonnesting open partition diagrams, $[m, s]$ are the components of $\ell(\pi)$ in the first sum, and $a_{m,s}(n)$ is the number of 3-nonnesting set partitions π at level n of the generating tree with label $\ell(\pi) = [m, s]$. For the sake of simplicity, we use $A(u, v)$ interchangeably with $A(u, v; z)$.

Corollary 2.4.2 ([16] B., Elizalde, Mishna, Yen 2014+)**.** *The generating function $A(u, v)$ for 3-nonnesting open partition diagrams, with variables u and v marking values m and s in the label, respectively, and z marking the number of vertices, satisfies the functional equation*

$$A(u, v) = 1 + z\left((1 + u)A(u, v) + \left(1 + \frac{1}{u}\right)\left(\frac{A(u, v)}{v(1 - v)} - \frac{A(uv, 1)}{1 - v} - \frac{A(u, 0)}{v}\right)\right). \tag{2.3}$$

Proof. The root π_0 has label $[0, 0]$ and size $n = 0$. Let $\mathrm{succ}([m, s])$ be the set of labels resulting from the application of the succession rule in Theorem 2.4.1 to the label $[m, s]$. The generating tree gives

$$A(u, v) = 1 + \sum_{m,s,n} a_{m,s}(n) z^{n+1} \sum_{[m',s'] \in \mathrm{succ}([m,s])} u^{m'} v^{s'}. \tag{2.4}$$

The terms in the interior sum originating from a fixed point and an opener are straightforward. Let us now compute the terms coming from a transitory vertex. These terms only appear when $m > 0$, which also implies $n > 0$. For $s > 0$, we get

$$\sum_{[m',s'] \in \{[m,s-1],[m,s],[m,s+1],\ldots,[m,m-1]\}} u^{m'} v^{s'} = u^m (v^{s-1} + v^s + v^{s+1} + \cdots + v^{m-1}) = u^m \frac{v^{s-1} - v^m}{1 - v}.$$

For $s = 0$, we get

$$\sum_{[m',s'] \in \{[m,0],[m,1],\ldots,[m,m-1]\}} u^{m'} v^{s'} = u^m (v^0 + v^1 + \cdots + v^{m-1}) = u^m \frac{1 - v^m}{1 - v}.$$

Thus, the contribution in Equation (2.4) from the children obtained by adding a transitory vertex is

$$\sum_{n,m,s>0} a_{m,s}(n) z^{n+1} u^m \frac{v^{s-1} - v^m}{1 - v} + \sum_{n,m>0} a_{m,0}(n) z^{n+1} u^m \frac{1 - v^m}{1 - v}$$
$$= \frac{z}{1 - v} \left(\sum_{n,m,s>0} a_{m,s}(n) z^n u^m v^{s-1} + \sum_{n,m>0} a_{m,0}(n) z^n u^m - \sum_{n,m>0,s\geq 0} a_{m,s}(n) z^n u^m v^m \right).$$

Writing these summations in terms of evaluations of the generating function $A(u, v)$, this expression becomes

$$\frac{z}{1 - v} \left(\frac{A(u, v) - A(u, 0)}{v} + A(u, 0) - A(0, 0) - (A(uv, 1) - A(0, 0)) \right)$$
$$= z \left(\frac{A(u, v)}{v(1 - v)} - \frac{A(uv, 1)}{1 - v} - \frac{A(u, 0)}{v} \right).$$

The computations for the terms coming from a closer vertex are very similar, the only difference being a factor of $1/u$.

Combining the contributions for the four types of vertices, we obtain the desired functional equation. □

2.4.3 Series data

In order to get this series information, we iterate the functional equation. Specifically, we view each equation $F = 1 + z\Phi(F)$ as the system $F^{[n]} = 1 + z\Phi(F^{[n-1]})$. Upon setting $F^{[0]} = 1$, we iterate to get successive terms in the series expansion. After n iterations, we obtain the correct coefficients for z^i for $0 \leq i \leq n$, since the functional equation is of the form $F = 1 + z\Phi(F)$, where Φ is linear in F and its evaluations. Setting the catalytic variables (i.e., those other than z) to 0 results in the univariate generating series for complete diagrams. See Appendix C for code.

We describe the first two iterations of the functional equation given in Corollary 2.4.2 in detail. Here, $F = A(u, v)$, and $\Phi(A(u, v)) = \left((1 + u)A(u, v) + \left(1 + \frac{1}{u}\right) \left(\frac{A(u,v)}{v(1-v)} - \frac{A(uv,1)}{1-v} - \frac{A(u,0)}{v}\right)\right)$. We start with $A(u, v)^{[0]} = 1$. Next,

$$
\begin{aligned}
A(u,v)^{[1]} &= \quad 1 + z\left((1+u)\cdot 1 + \left(1+\frac{1}{u}\right)\left(\frac{1}{v(1-v)} - \frac{1}{1-v} - \frac{1}{v}\right)\right) \\
&= \quad 1 + (1+u)z \\
A(u,v)^{[2]} &= \quad 1 + z\Bigg[(1+u)\cdot(1+(1+u)z) \\
&\qquad + \left(1+\frac{1}{u}\right)\left(\frac{1+(1+u)z}{v(1-v)} - \frac{1+(1+uv)z}{1-v} - \frac{1+(1+u)z}{v}\right)\Bigg] \\
&= \quad 1 + (u+1)z + (u^2+3u+2)z^2.
\end{aligned}
$$

This gives us the very beginning of the counting series for open set partitions with no future 3-nestings. The coefficient of z^2 tells us there is one open diagram with 2 semi-arcs, three with 1 and two with no semi-arcs, which can be verified using the third row of Figure 2.9. The number of completed set partitions are recovered by setting $u = 0$: $A(0,0)^{[2]} = 1 + z + z^2$.

Continuing in this manner gives the counting series for open partition diagrams without 3-nestings when $u = 1$ (not yet in OEIS), and for 3-nonnesting partition diagrams when $u = 0$. See Appendix A for data.

2.5 k-nonnesting set partitions

We now generate and enumerate k-nonnesting set partitions for general k. We follow the same procedure as outlined in Section 1.6 and followed explicitly for 3-nonnesting set partitions in Section 2.4. For convenience, we shift the index and consider $k+1$-nonnesting partitions for the rest of this Section.

Example 22. *Suppose we are generating* 6*-nonnesting open set partitions from the diagram below. This diagram has a regular* 4*-nesting and a future* 5*-nesting. Closing semi-arcs* 4 *or* 5 *with vertex* 15 *creates a future* 6*-nesting, not permissible.*

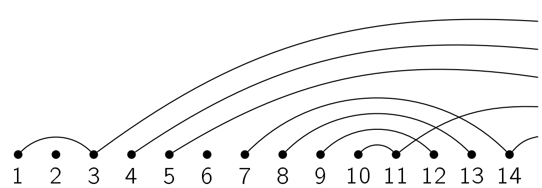


Figure 2.10: A 6-nonnesting open partition diagram.

Thus, in order to construct $k+1$-nonnesting open partition diagrams, we must be able to control the number of future k-nestings. To do this, we need to keep track of j-nestings for $j < k$.

2.5.1 The label and succession rule

Definition 5. *The* nesting index *of a semi-arc is the maximum j such that there is a j-nesting beneath it. Equivalently, the nesting index of a semi-arc is the largest j such that the semi-arc is in a future $j+1$-nesting.*

Example 23. *The open partition diagram in Figure 2.11 has the nesting index of each semi-arc labelled in italics.*

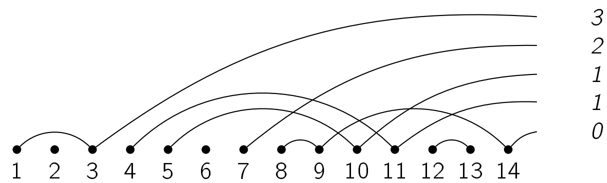


Figure 2.11: An open partition diagram with nesting index of semi-arc written in italics.

We track the distribution of the nesting indices on the semi-arcs. We update their distribution every time we add a vertex, making sure we avoid the appearance of a future $k+1$-nesting. To each $k+1$-nonnesting open partition diagram π, we associate a label with k components $\ell(\pi) = [s_0, \ldots, s_{k-1}]$, where s_i is defined to be the number of semi-arcs with nesting index greater than or equal to i.

Example 24. *The open partition diagram in Figure 2.11 has label* $[5, 4, 2, 1]$.

In our notation, s_0 is the total number of semi-arcs, and s_{k-1} is the number of semi-arcs in a future k-nesting. For $k = 2$, this labelling is consistent with Subsection 2.4.1. Furthermore, note that by definition, $s_0 \geq s_1 \geq \cdots \geq s_{k-1} \geq 0$. The label of the empty partition is $[0, 0, \ldots, 0]$, since it contains no semi-arcs. Next we describe the succession rule.

Each time a semi-arc is closed, the nesting index of those semi-arcs above it, which also have the same nesting index, is increased by one, and the label must reflect change. This is all that needs to be tracked, and thus we can now describe a succession rule for the generating tree.

Theorem 2.5.1 ([16] Burrill, Elizalde, Mishna, Yen, 2014+)**.** *Let $\Pi^{(k)}$ be the set of $k+1$-nonnesting open partition diagrams. To each diagram, associate the label $\ell(\pi) = [s_0, \ldots, s_{k-1}]$, where s_i is the number of semi-arcs with nesting index $\geq i$. Then, the number of diagrams in $\Pi^{(k)}$ of size n is the number of nodes at level n in the generating tree with root label* $[0, 0, \ldots, 0]$*, and* *succession rules given by*

$$
\begin{aligned}
&[s_0, s_1, \dots, s_{k-1}] \rightsquigarrow \\
&[s_0, s_1, \dots, s_{k-1}], && (1)\\
&[s_0+1, s_1, \dots, s_{k-1}], && (2)\\
&[s_0, s_1-1, \dots, s_{j-1}-1, i, s_{j+1}, \dots, s_{k-1}], \quad \text{for } 1 \le j \le k-1 \text{ and } s_j \le i \le s_{j-1}-1, && (3)\\
&[s_0-1, s_1-1, \dots, s_{j-1}-1, i, s_{j+1}, \dots, s_{k-1}], \quad \text{for } 1 \le j \le k-1 \text{ and } s_j \le i \le s_{j-1}-1, && (4)\\
&[s_0, s_1-1, \dots, s_{k-1}-1], [s_0-1, s_1-1, \dots, s_{k-1}-1], \quad \text{if } s_{k-1} > 0. && (5)
\end{aligned}
$$

Proof. The labels arise from adding the following kinds of vertices:

1) a fixed point; 2) an opener 3) a transitory 4) a closer

5) a transitory or a closer that closes the top semi-arc, if the parent diagram has a future k-nesting.

□

2.5.2 Functional equation

As in Section 2.4, the generating tree in Theorem 2.5.1 can be translated to a functional equation. Consider the generating function $Q(v_0, v_1, \dots, v_{k-1}; z) = \sum Q_{s_0,s_1,\dots,s_{k-1}}(n) v_0^{s_0} v_1^{s_1} \dots v_{k-1}^{s_{k-1}} z^n$, where $Q_{s_0,s_1,\dots,s_{k-1}}(n)$ is the number of $k+1$-nonnesting open partition diagrams at level n of the generating tree with label $[s_0, s_1, \dots, s_{k-1}]$. For simplicity, we will use the notation $Q = Q(v_0, \dots, v_{k-1}) = Q(v_0, \dots, v_{k-1}; z)$ and $Q_{\mathrm{s}}(n) = Q_{s_0,\dots,s_{k-1}}(n)$.

The equations for the addition of an opener or a fixed point are analogous to the ones described in Section 2.4.2. Now consider the addition of a transitory vertex that closes a semi-arc with nesting index greater than or equal to j. This corresponds to (3) in Theorem 2.5.1, giving

$$
\begin{aligned}
z \sum_{s_0,\dots,s_{k-1}} & Q_{\mathrm{s}}(n) v_0^{s_0} v_1^{s_1-1} v_2^{s_2-1} \dots v_{j-1}^{s_{j-1}-1} (v_j^{s_j} + v_j^{s_j+1} + \dots v_j^{s_{j-1}-1}) v_{j+1}^{s_{j+1}} \dots v_{k-1}^{s_{k-1}} z^n \\
&= \frac{z}{v_1 \dots v_{j-1}} \sum_{s_0,\dots,s_{k-1}} Q_{\mathrm{s}}(n) v_0^{s_0} \dots v_{k-1}^{s_{k-1}} \left(\frac{1 - v_j^{s_{j-1}-s_j}}{1 - v_j} \right) z^n \\
&= \frac{z}{v_1 \dots v_{j-1}(1 - v_j)} \left(Q - Q(v_0, \dots, v_{j-2}, v_{j-1} v_j, 1, v_{j+1}, \dots, v_{k-1}) \right).
\end{aligned}
$$

Summing over all nesting indices gives

$$
\sum_{j=1}^{k-1} \frac{z}{v_1 \dots v_{j-1}(1 - v_j)} \left(Q - Q(v_0, \dots, v_{j-2}, v_{j-1} v_j, 1, v_{j+1}, \dots, v_{k-2}, v_{k-1}) \right).
$$

If $s_k > 0$, the addition of a transitory corresponds to (5) in Theorem 2.5.1, giving

$$z \sum_{s_0, s_1, \ldots, s_{k-2}, s_{k-1} \geq 1} Q_s(n) v_0^{s_0} v_1^{s_1-1} \ldots v_{k-1}^{s_{k-1}-1} z^n = \frac{z}{v_1 v_2 \ldots v_{k-1}} (Q - Q(v_0, v_1, \ldots, v_{k-2}, 0)).$$

The addition of closers proceeds similarly, and combining the expressions for all four types of vertices, we get the following functional equation.

Corollary 2.5.2 ([16] Burrill, Elizalde, Mishna, Yen 2014+)**.** *The generating function for $k+1$-nonnesting open partition diagrams, with variable v_i marking value s_i in the label and variable z marking number of vertices, denoted $Q = Q(v_0, v_1, \ldots, v_{k-1}) = Q(v_0, v_1, \ldots, v_{k-1}; z)$, satisfies the functional equation*

$$\begin{aligned} Q = 1 + z(1+v_0)\Bigg(Q &+ \frac{1}{v_0 v_1 \ldots v_{k-1}}(Q - Q(v_0, v_1, \ldots, v_{k-2}, 0)) \\ &+ \sum_{j=1}^{k-1} \frac{1}{v_0 v_1 \ldots v_{j-1}(1-v_j)}(Q - Q(v_0, \ldots, v_{j-2}, v_{j-1}v_j, 1, v_{j+1}, \ldots, v_{k-2}, v_{k-1}))\Bigg). \end{aligned}$$

Note that, as k increases, the number of catalytic variables increases as well.

2.5.3 Series data

The iteration is as in Section 2.4.3, to get series information we iterate the functional equations. Setting the catalytic variables (i.e., those other than z) to 0 results in the univariate generating series for complete diagrams. See Appendix C for code.

Table A.1 is found in Appendix A. It presents the initial counting sequences for $k+1$-nonnesting set partitions and relevant references to the On-line Encyclopedia of Integer Sequences [41] for completeness. Note that the first few terms (presented in grey) coincide with the Bell numbers, the nesting condition is first apparent at $n = 2k+1$, or $n = 2k$ in the case of enhanced nestings. We are able to generate many more terms than listed in Table A.1, and indicate the highest n in Table A.6 as well. For for 4 and 5-nonnesting set partitions, we used this data and the *gfun* package of Maple (version 3.53) to try to fit the counting sequence into a differential equation with no success. In the 4-nonnesting set partition case, 276 terms were used, so the order of the equation times the degree of the maximum polynomial is less than 276, and no equation was found. This supports the conjecture of Bousquet-Mélou and Xin.

2.6 Set partitions without enhanced k-nestings

We can also use our procedure to generate and enumerate set partitions according to a slightly different nesting pattern, called an *enhanced nesting*, where fixed points are drawn as loops in their arc diagrams, and an arc over a loop is an *enhanced nesting*.

Definition 6. *An* enhanced k-nesting *is either a k-nesting, or a set of $k-1$ arcs (i_1, j_1), ..., (i_{k-1}, j_{k-1}) and a fixed point i_k (a singleton block in the corresponding partition) such that*

$$i_1 < i_2 < \cdots < i_{k-1} < i_k < j_{k-1} < \cdots < j_1,$$

that is, a $k-1$-nesting with a fixed point inside the innermost arc.

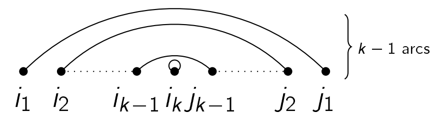


Figure 2.12: Enhanced k-nesting

A comparable definition exists for enhanced k-crossings: an enhanced crossing is a transitory vertex. Furthermore, analogous results to Theorems 2.1.1 and 2.1.2 exist:

Theorem 2.6.1. [[20] Chen, Deng, Du, Stanley, Yan 2007] *The number of set partitions of $\{1, \ldots, n\}$ without an enhanced k-nesting is equal to the number of set partitions of $\{1, \ldots, n\}$ without an enhanced k-crossing.*

Theorem 2.6.2. [[13] Bousquet-Mélou, Xin 2007] *The number $E_3(n)$ of partitions of $\{1, \ldots, n\}$ having no enhanced 3-crossing is given by $E_3(0) = E_3(1) = 1$, and for $n \geq 0$,*

$$8(n+3)(n+1)E_3(n) + (7n^2 + 53n + 88)E_3(n+1) - (n+8)(n+7)E_3(n+2) = 0.$$

Our framework can naturally generate and enumerate set partitions with no enhanced k-nesting.

2.6.1 The label and succession rule

Definition 7. *A* future enhanced k-nesting *is an enhanced $k-1$ nesting together with a semi-arc beginning to its left.*

We define the *enhanced nesting index* of a semi-arc as the largest j such that the semi-arc forms a future enhanced $j+1$-nesting. Like in the (regular) set partition case, the label tracks semi-arcs according to their future enhanced index: the label $\ell(\pi) = [s_0, \ldots, s_{k-1}]$, where s_i is the number of semi-arcs with enhanced nesting index $\geq i$. The generating tree that results will be very similar to the (regular) k-nonnesting set partition case except for the following:

- the addition of a fixed point to an open partition diagram can create a future enhanced 2-nesting.

In other words, after a fixed point has been added, a semi-arc that had nesting index 0 in an open partition diagram has enhanced nesting index of 1.

From this, we get the following variation of Theorem 2.5.1.

Theorem 2.6.3 ([16] Burrill, Elizalde, Mishna, Yen, 2014+)**.** *Let $\widetilde{\Pi}^{(k)}$ be the set of open partition diagrams with neither enhanced $k+1$-nestings nor future enhanced $k+1$-nestings. To each diagram, associate the label $\ell(\pi) = [s_0, \ldots, s_{k-1}]$, where s_i is the number of semi-arcs with enhanced nesting index $\geq i$. Then the number of diagrams in $\widetilde{\Pi}^{(k)}$ of size n is the number of nodes at level n in the generating tree with root label* $[0, 0, \ldots, 0]$*, and succession rule given by*

$$
\begin{aligned}
&[s_0, s_1, \ldots, s_{k-1}] \rightsquigarrow \\
&[s_0, s_0, s_2, \ldots, s_{k-1}], && (1)\\
&[s_0+1, s_1, \ldots, s_{k-1}], && (2)\\
&[s_0, s_1-1, \ldots, s_{j-1}-1, i, s_{j+1}, \ldots, s_{k-1}], \quad \text{for } 1 \leq j \leq k-1 \text{ and } s_j \leq i \leq s_{j-1}-1, && (3)\\
&[s_0-1, s_1-1, \ldots, s_{j-1}-1, i, s_{j+1}, \ldots, s_{k-1}], \quad \text{for } 1 \leq j \leq k-1 \text{ and } s_j \leq i \leq s_{j-1}-1, && (4)\\
&[s_0, s_1-1, \ldots, s_{k-1}-1], [s_0-1, s_1-1, \ldots, s_{k-1}-1], \quad \text{if } s_{k-1} > 0. && (5)
\end{aligned}
$$

Proof. The label for the addition of a fixed point, which is line (1), follows directly from the paragraph before this theorem. The labels for adding other types of vertices are obtained using the same arguments as in Theorem 2.5.1. □

2.6.2 Functional equation

Next we again translate the succession rule into a functional equation. Consider the generating function $P(v_0, \ldots, v_{k-1}; z) = \sum P_{s_0,\ldots,s_{k-1}}(n) v_0^{s_0} \ldots v_{k-1}^{s_{k-1}}$, where $P_{\bar{s}}(n) = P_{s_0,\ldots,s_{k-1}}(n)$ is the number of open partition diagrams that avoid enhanced $k+1$-nestings at level n of the generating tree with label $[s_0, \ldots, s_{k-1}]$. Notice that a fixed point can be considered to be a transitory vertex that connects to itself. The addition of a fixed point or a transitory when $j = 1$ contributes:

$$
\begin{aligned}
&P_{\bar{s}}(n) v_0^{s_0}(v_1^{s_1} + v_1^{s_1+1} + \ldots + v_1^{s_0-1} + v_1^{s_0}) v_2^{s_2} \ldots v_{k-1}^{s_{k-1}} \\
&\qquad + P_{\bar{s}}(n) v_0^{s_0-1}(v_1^{s_1} + v_1^{s_1+1} + \ldots + v_1^{s_0-1}) v_2^{s_2} \ldots v_{k-1}^{s_{k-1}} \\
&= P_{\bar{s}}(n) v_0^{s_0} v_1^{s_1} \ldots v_{k-1}^{s_{k-1}} \left(\frac{1 - v_1^{s_0-s_1+1}}{1 - v_1} \right) + P_{\bar{s}}(n) v_0^{s_0-1} v_1^{s_1} \ldots v_{k-1}^{s_{k-1}} \left(\frac{1 - v_1^{s_0-s_1}}{1 - v_1} \right) \\
&= + \frac{(1+v_0)P - (1+v_0 v_1) P(v_0 v_1, 1, v_2, \ldots, v_{k-1})}{v_0(1-v_1)} \Bigg).
\end{aligned}
$$

Combining with the translation from the addition of all other vertex types, we get the following corollary:

Corollary 2.6.4 ([16] Burrill, Elizalde, Mishna, Yen 2014+)**.** *The generating function for open partition diagrams with neither regular nor future enhanced $k+1$-nestings, with variable v_i marking value s_i in the label and variable z marking the number of vertices, denoted by $P = P(v_0, v_1, \ldots, v_{k-1}) = P(v_0, v_1, \ldots, v_{k-1}; z)$, satisfies the functional equation*

$$\begin{aligned} P = 1 + z \Bigg(& v_0 P + \frac{1+v_0}{v_0 v_1 \ldots v_{k-1}} \left(P - P(v_0, v_1, \ldots, v_{k-2}, 0)\right) \\ & + \sum_{j=2}^{k-1} \frac{1+v_0}{v_0 v_1 \ldots v_{j-1}(1-v_j)} \left(P - P(v_0, \ldots, v_{j-2}, v_{j-1}v_j, 1, v_{j+1}, \ldots, v_{k-1})\right) \\ & \qquad + \frac{(1+v_0)P - (1+v_0v_1)P(v_0v_1, 1, v_2, \ldots, v_{k-1})}{v_0(1-v_1)} \Bigg). \end{aligned}$$

Again, $P(0, v_1, \ldots, v_{k-1}; z)$, which is a function of z only, is the generating function for partitions avoiding enhanced $k+1$-nestings.

2.6.3 Series data

Again, we extract series information by iterating our functional equation, as was done in Sections 2.4.3 and 2.5.3. The resulting initial counting sequences for set partitions without $k+1$ enhanced nestings is found in Table A.2 in Appendix C. The first few terms are presented in grey to indicate they coincide with the Bell numbers. Notice that the counting sequence for open partition diagrams without enhanced future 3-nestings agrees with *Baxter numbers*, a correspondence we further explore in Section 6.3.

We now turn to a different combinatorial class and consider open permutations.

Chapter 3

Permutations

Permutations can be represented using arc diagrams, as seen in Section 1.1, by drawing the cycle structure of the permutation. Given $\sigma \in \mathfrak{S}_n$, the arc diagram of σ has an arc between i and $\sigma(i)$ for each i from 1 to n, and the arc is drawn above the vertices (an *upper arc*) if $i \leq \sigma(i)$, and below the vertices (a *lower arc*) if $i > \sigma(i)$. Such a representation is called a *permutation diagram* of size n.

Example 25. *The permutation* $\sigma = (1\,11\,3)(2\,6\,4\,5)(7\,9)(8)(10)$ *is depicted as a permutation diagram in Figure 3.1.*

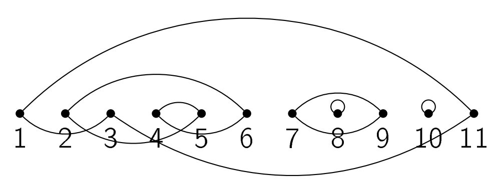


Figure 3.1: The permutation diagram of $\sigma = (1\,11\,3)(2\,6\,4\,5)(7\,9)(8)(10)$.

Notice vertices 8 *and* 10 *are both fixed points, and the loop is drawn in the diagram.*

In a permutation diagram nesting structures are also defined. A subset of k arcs is a *k-nesting* if either

1. all k arcs are upper arcs and form an enhanced k-nesting with the definition from Subsection 2.6 (considering arcs of the form (i, i) to be fixed points), or

2. all k arcs are lower arcs and form a k-nesting with the definition from Section 1.1.

We will refer to the first possibility as an *upper enhanced k-nesting* and the second as a *lower k-nesting*. We are following the literature in this regard: in the paper that addressed crossings and nesting in permutations, Corteel [26] defined nesting with this slight dissymmetry in order to obtain bijections between certain classes of permutations. Later B., Mishna and Post [18] maintained the dissymmetry. Remark that our strategy could also be used to treat a symmetric definition of nestings in permutations.

3.1 History

Permutations of $\{1, \ldots, n\}$ are a classical combinatorial object and as such are well known to be enumerated by $n!$. Enumerating restricted permutations has become an increasingly well studied area [6, 8, 21, 33, 44, 56]. In particular, it is well known that permutations avoiding any 3-length pattern are enumerated by the Catalan numbers. In Section 2.1, we saw that 2-nonnesting set partitions are enumerated by the Catalan numbers. In [18], we gave a bijection between 2-noncrossing partitions and 2-noncrossing permutations by flipping a non-crossing permutation arc diagram upside down, converting the loops to fixed points, and removing the lower arcs. This leads to the following proposition.

Proposition 3.1.1 ([18], Burrill, Mishna, Post 2010)**.** *The set of 2-noncrossing permutations of $\{1, \ldots, n\}$ is enumerated by the n^{th} Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$.*

Equidistribution between crossing and nesting statistics in permutations was also proven in [18].

Proposition 3.1.2 ([18], Burrill, Mishna, Post 2010)**.** *The number of k-nonnesting permutations of $\{1, \ldots, n\}$ is equal to the number of k-noncrossing permutations of $\{1, \ldots, n\}$.*

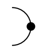
This gives that the set of 2-nonnesting permutations is also enumerated by the Catalan numbers. Finally, as mentioned in Section 1.2, Corteel connected crossings and nestings to permutation patterns as well.

Theorem 3.1.3 ([26] Corteel 2007)**.** *The number of permutations of $\{1, \ldots, n\}$ with k weak exceedances, ℓ crossings and m-nestings is equal to the number of permutations of $\{1, \ldots, n\}$ with $n-k$ descents, ℓ occurrences of the pattern $31-2$ and m occurrences of the pattern $2-31$.*

This was shown by mapping each object to a weighted bicolored Motzkin path. Short of some brute force computation done in [18], enumerative data on k-nonnesting permutations has not been forthcoming for $k > 2$. It is with this in mind that we introduce *open permutation diagrams* so that our strategy can be applied to gain new access to series.

3.2 Open permutation diagrams

As in the set partition case, one of the keys to generating all k-nonnesting permutations is to define a label and identify how the addition of each type of vertex affects that label. In the case of permutations there are five types of vertices:

1. **fixed point**  loop connected to itself;
2. **opener**  left endpoint of an upper and lower arc;
3. **upper transitory**  right endpoint of an upper arc and left endpoint of another;
4. **lower transitory**  right endpoint of a lower arc and left endpoint of another;
5. **closer**  right endpoint of an upper and lower arc;

We treat *open permutation diagrams*, those which also have *open semi-arcs*, *upper semi-transitories*, and *lower semi-transitories*.

Example 26. *Figure 3.4 depicts an open permutation. Vertices* 1, 2, 3, 7 *and* 8 *are openers,* 4 *is a fixed point,* 5 *is a lower transitory, while* 6 *and* 9 *are closers.*

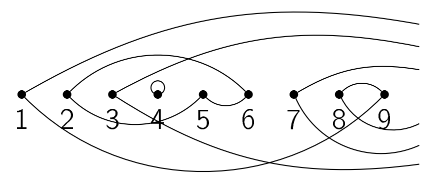


Figure 3.2: An open permutation diagram.

Proposition 3.2.1. *In an open permutation diagram, the number of upper semi-arcs equals the number of lower semi-arcs.*

Proof. With each type of vertex is added, the total number for both upper and lower semi-arcs remains the same:

- adding a fixed point does not introduce semi-arcs;
- adding an opener adds a semi-arc to both the upper and lower portion;
- adding an upper transitory removes and adds a semi-arc to the upper portion, lower semi-arcs remain the same;
- adding a lower transitory removes and adds a semi-arc to the lower portion, upper semi-arcs remain the same;
- adding a closer removes a semi-arc from both the upper and lower portion.

□

Open permutation diagrams can be interpreted as permutations where each cycle of length i can be one of $i+1$ colours.

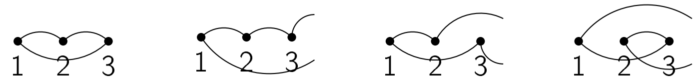


Figure 3.3: A permutation $\sigma = (123)$ with a cycle of length 3 may be coloured in 4 different ways.

Example 27. *The permutation* $\sigma = (123)$ *is a cycle of length* 3*. When we allow semi-arcs, there are* 3 *more diagrams constructed:* $(123*)$*,* $(12*3)$*, and* $(1*23)$*. The four total diagrams are depicted in Figure 3.3.*

From this, it follows that the exponential generating function for open permutations is

$$\frac{1}{1-z}\exp\left(\frac{uz}{1-z}\right)$$

Setting $u = 1$ gives the total number of permutations where each cycle of length i can be $i+1$ different colors, and is sequence A002720 in the OEIS [41].

3.2.1 The label and succession rule

To construct exhaustively all open permutation diagrams, we proceed as in the set partition case and build a diagram of size n from one of size $n-1$. The vertex n can be a fixed point, opener, or an upper transitory, lower transitory or closer, provided their are available semi-arcs to close.

The addition of each type of vertex is translated in a very straightforward manner to a succession rule for open permutations diagrams, where the statistic h, number of upper semi-arcs (and equivalently number of lower semi-arcs) forms the label. The root has label [0] and the succession rule is:

$$[h] \rightsquigarrow \begin{array}{lll} [h], & & \text{(fixed point)} \\ [h+1], & & \text{(opener)} \\ \underbrace{[h],[h],\ldots,[h]}_{h \text{ copies}}, & \text{if } h > 0, & \text{(upper transitory)} \\ \underbrace{[h],[h],\ldots,[h]}_{h \text{ copies}}, & \text{if } h > 0, & \text{(lower transitory)} \\ \underbrace{[h-1],[h-1],\ldots,[h-1]}_{h^2 \text{ copies}}, & \text{if } h > 0. & \text{(closer)} \end{array}$$

We have determined our label and its succession rule.

3.2.2 Functional equation

We translate the succession rule into a functional equation for open permutation diagrams . Let $A(u,z)$ be the bivariate generating function for open permutation diagrams, where the exponent of u is the label of the node: the number of upper semi-arcs. Let σ be an open permutation diagram with $|\sigma| = n$ vertices and $\ell(\sigma) = h$ upper semi-arcs. If $\mathrm{succ}(\sigma)$ is the set of children of σ, then we get the following from our succession rule:

$$\sum_{\sigma' \in \mathcal{C}(\sigma)} u^{\ell(\sigma')} = u^h + u^{h+1} + hu^h + hu^h + h^2 u^{h-1}$$

which gives the following generating function recurrence:

$$\begin{aligned} A(u,z) &= \sum_{n,h} a(h,n) \frac{u^h z^n}{n!} \\ &= 1 + \sum_{n,h} a(h,n) \frac{z^{n+1}}{(n+1)!} (u^h + u^{h+1} + 2hu^h + h^2 u^{h-1}) \\ &= 1 + \int ((1+u)A(u,z) + (1+2u)A_u(u,z) + A_{uu}A(u,z))dz. \end{aligned}$$

In this case, we are able to solve and recover that the EGF for these open permutation diagrams is $\frac{1}{1-z} \exp\left(\frac{zu}{1-z}\right)$. When $u = 1$, we get the enumeration of all open permutation diagrams; remark that $u = 0$ recovers the classic EGF for (regular) permutations, $\frac{1}{1-z}$.

3.2.3 Counting sequence

While we can iterate the functional equation given above, since we have an explicit generating function for open permutations, $A(z) = \frac{1}{1-z} \exp\left(\frac{uz}{1-z}\right)$, we will use it to recover our counting sequences. When $u = 0$, we get $n!$ (A000142), and setting $u = 1$ returns A002720[41]: the number of partial permutations of $\{1, \ldots, n\}$. In Chapter 6 we explicitly demonstrate the bijection, and Appendix A gives data for the counting sequences.

3.3 k-nonnesting permutations

Definition 8. *A* future enhanced upper k-nesting *is an upper enhanced* $k-1$ *nesting together with an upper semi-arc above it beginning to its left. A* future lower k-nesting *is a lower* $k-1$ *nesting together with a lower semi-arc below it beginning to its left.*

Definition 9. *The* enhanced nesting index of an upper semi-arc *is the largest* j *such that the semi-arc is in a future enhanced upper* $j+1$*-nesting. The* nesting index of a lower semi-arc *is the largest* j *such that the semi-arc is in a future lower* $j+1$*-nesting.*

Example 28. *Figure 3.4 depicts an open permutation diagram* σ*. The upper arcs* $(7,12)$*,* $(8,9)$ *and the upper semi-arc* $(3,*)$ *form a future enhanced upper* 3*-nesting. The lower arc* $(7,10)$ *and* *the lower semi-arc* $(3, *)$ *form a future lower* 2*-nesting. The nesting index of each semi-arc is labelled in italics.*

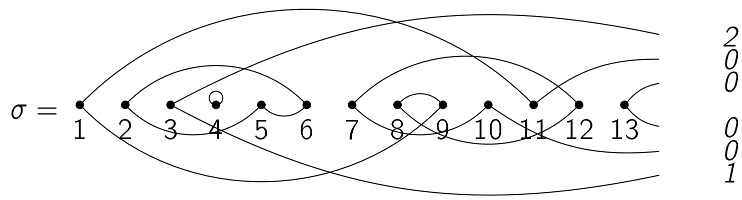


Figure 3.4: An open permutation diagram with nesting index given in italics.

It is still preferable to change the index and generate $k+1$-nonnesting permutations. To each $k+1$-nonnesting permutation diagram, associate a label $[h; \mathbf{r}; \mathbf{s}]$, where h is the number of upper semi-arcs (also the number of lower semi-arcs), $\mathbf{r} = [r_1, \ldots, r_{k-1}]$ is a vector such that r_i is the number of upper semi-arcs with enhanced nesting index greater than or equal to i, and $\mathbf{s} = [s_1, \ldots, s_{k-1}]$ is a vector such that s_i is the number of lower semi-arcs with nesting index greater than or equal to i.

Example 29. *The diagram* σ *in Figure 3.4 has* $\ell(\sigma) = [3; 1, 1; 1, 0]$.

Notice that we can view the upper arcs of an open permutation diagram σ of size n as forming an open partition diagram (enhanced) σ^+, and its lower arcs as forming an open partition diagram σ^- on the vertices $\{1, \ldots, n\}$. If the label of σ is $[h; \mathbf{r}; \mathbf{s}]$, then the label of σ^+ is as in Theorem 2.6.3 and is $[h, \mathbf{r}]$ and the label of σ^- is as described in Theorem 2.5.1 is $[h, \mathbf{s}]$. From this, we also get that $h \geq r_1 \geq \cdots \geq r_{k-1} \geq 0$ and $h \geq s_1 \geq \cdots \geq s_{k-1} \geq 0$.

3.3.1 The case $k = 3$

We first describe the generating tree for the 3-nonnesting case, as it provides (the majority of) insight into how the succession rule works for permutations. Then we describe it for general k.

The label of a $k+1$-nonnesting open permutation is $[h; r_1, \ldots, r_{k-1}; s_1, \ldots, s_{k-1}]$. For $k+1 = 3$, $\mathbf{r}$ and $\mathbf{s}$ vectors only have one element, so for simplicity we use commas instead of semi-colons. The label of a 3-nonnesting open permutation diagram is $[h, r, s]$. Here, $2h$ is the total number of semi-arcs, r is the number of semi-arcs that are in a future enhanced upper 2-nesting, and s is the number of semi-arcs that are in a future lower 2-nesting. The empty diagram has label $[0, 0, 0]$.

Example 30. *The open permutation diagram* σ *given in Figure 3.5 has arrows drawn to indicate the semi-arcs which are part of future* 2*-nestings. There are two such arcs in the upper portion and one lower, with eight semi-arcs total, thus* $\ell(\sigma) = [4, 2, 1]$.

As in the set partition case, we predict the labels of the children of a 3-nonnesting open permutation diagram by tracking total number of semi-arcs and future 2-nestings.

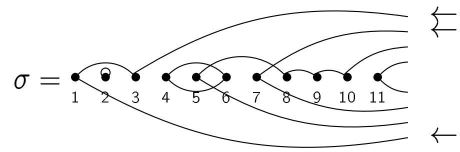


Figure 3.5: An arc diagram with label [4, 2, 1].

Example 31. *Consider the open permutation given above in Figure 3.5. The labels of its children are found by adding vertices of each type as described below:*

1. *Fixed point: one child with label* $[4, 4, 1]$*, since the upper semi-arcs belong now to future enhanced upper* 2*-nestings.*

2. *Opener: one child with label* $[5, 2, 1]$*.*

3. *Upper transitory: closing the upper semi-arcs that are not in future enhanced* 2*-nestings gives the labels* $[4, 2, 1]$ *and* $[4, 3, 1]$*; closing the top semi-arc (the only one in a* 2*-nesting that we are allowed to close) removes one future upper* 2*-nesting, giving the label* $[4, 1, 1]$*.*

4. *Lower transitory: all lower semi-arcs can be closed, and the four resulting labels are* $[4, 2, 0]$*,* $[4, 2, 1]$*,* $[4, 2, 2]$ *and* $[4, 2, 3]$*.*

5. *Closer: we simultaneously and independently close an upper and a lower semi-arc, among those that we are allowed to close. There are three choices for the former and four for the latter, giving twelve children with labels* $[3, 1, 0]$*,* $[3, 1, 1]$*,* $[3, 1, 2]$*,* $[3, 1, 3]$*,* $[3, 2, 0]$*,* $[3, 2, 1]$*,* $[3, 2, 2]$*,* $[3, 2, 3]$*,* $[3, 3, 0]$*,* $[3, 3, 1]$*,* $[3, 3, 2]$*,* $[3, 3, 3]$*, that is,* $\{3\} \times \{1, 2, 3\} \times \{0, 1, 2, 3\}$*.*

Example 32. *We illustrate exhaustively all children and their labels of a particular* 3*-nonnesting open permutation diagram with label* $[2, 0, 0]$ *in Figure 3.6.*

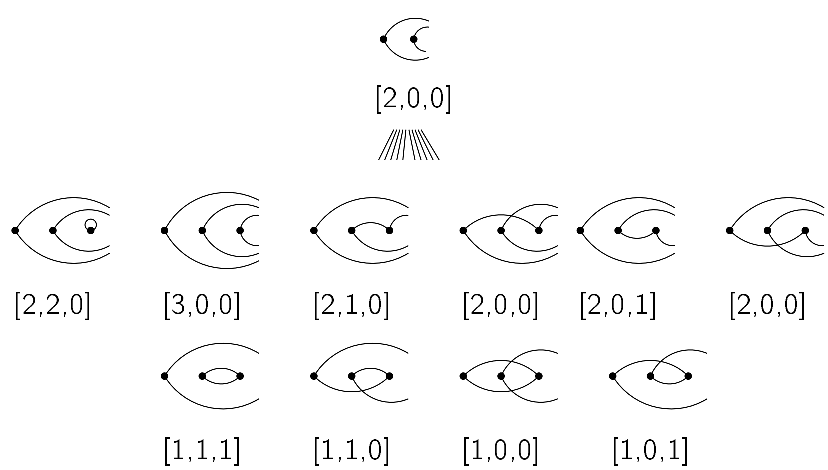


Figure 3.6: A 3-nonnesting open permutation diagram and its children, and labels.

The succession rule is given in Theorem 3.3.1.

Theorem 3.3.1 ([16] Burrill, Elizalde, Mishna, Yen 2014+). *Let $\Sigma^{(2)}$ be the set of 3-nonnesting open permutation diagrams. To each diagram σ, associate the label $\ell(\sigma) = [h, r, s]$, where $2h$ is the total number of semi-arcs, r is the number of semi-arcs in a future enhanced upper 2-nesting and s is the number of semi-arcs in a future lower 2-nesting. Then, the number of diagrams in $\Sigma^{(2)}$ of size n is the number of nodes at level n in the generating tree with root label $[0, 0, 0]$, and succession rule given by*

$$[h,r,s] \rightsquigarrow \begin{cases} [h,h,s], & (1)\\ [h+1,r,s], & (2)\\ [h,i,s], \quad \text{for } \max\{0,r-1\} \le i \le h-1, & (3)\\ [h,r,j], \quad \text{for } \max\{0,s-1\} \le j \le h-1, & (4)\\ [h-1,i,j], \quad \text{for } \max\{0,r-1\} \le i \le h-1 \text{ and } \max\{0,s-1\} \le j \le h-1. & (5) \end{cases}$$

The number of 3-nonnesting permutations of size n is equal to the number of nodes with label $[0, 0, 0]$ at the n-th level of this generating tree.

Proof. The labels correspond to the addition of the following vertices to a diagram of σ:

1. A fixed point, which results in all the upper semi-arcs becoming part of future enhanced 2-nestings;
2. an opener, which produces a new upper semi-arc and a new lower one, neither of which is in a future 2-nesting;
3. an upper transitory closing a semi-arc not belonging to a future enhanced upper 2-nesting or, if $r > 0$, possibly closing the top semi-arc;
4. a lower transitory closing a semi-arc not belonging to a future lower 2-nesting or, if $s > 0$, possibly closing the bottom semi-arc;
5. a closer, which can close any combination of an upper and a lower semi-arc among those allowed to close in parts (3) and (4).

□

We give the start of the generating tree for 3-nonnesting permutations with their labels in Figure 3.7.

With the succession rule defined, we can now translate the generating tree from Theorem 3.3.1 into a functional equation. Let $F(u, v, w) = \sum F_{h,r,s}(n) u^h v^r w^s z^n$ where $F_{h,r,s}(n)$ is the

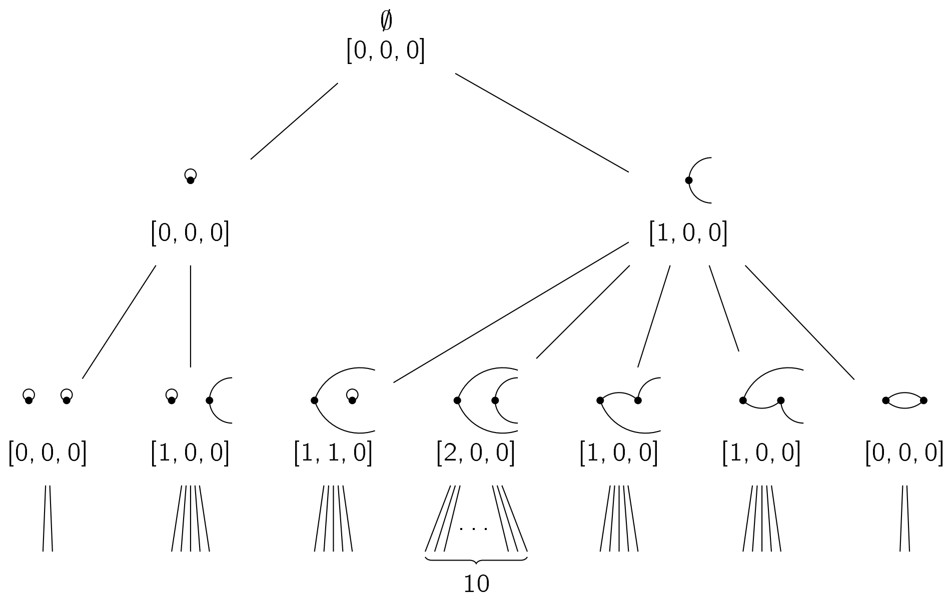


Figure 3.7: Generating tree for 3-nonnesting open permutation diagrams with labels.

number of open permutation arc diagrams at level n with label $[h, r, s]$. The coefficient $F_{0,0,0}(n)$ is the number of 3-nonnesting permutations of $\{1, 2, \ldots, n\}$.

We follow the same process as in the open partition diagram case: we consider each type of vertex and determine its contribution to the functional equation. Its form will be:

$$F(u, v, w) = 1 + z(\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5),$$

where Ψ_i is the contribution for adding a vertex of type (i) for $1 \leq i \leq 5$, which we compute next.

1. Fixed point. Note that case (1) in the succession rule can alternatively be included by extending the range of i in case (3) to include h. Thus, it is simpler to compute $\Psi_1 + \Psi_3$ in item (3) below.

2. Opener. $\Psi_2 = uF(u, v, w)$.

3. Upper transitory and fixed point. $\Psi_1 + \Psi_3 = \frac{F(u,v,w) - vF(uv,1,w)}{1-v} + \frac{F(u,v,w) - F(u,0,w)}{v}$, found using the formula for a finite geometric sum in the expressions below:

$$\sum_{h,s,n} F_{h,0,n}(n) u^h (1 + v + v^2 + \cdots + v^h) w^s z^n \qquad \text{if } r = 0,$$
$$\sum_{h,s,n} F_{h,r,s}(n) u^h (v^{r-1} + v^r + \cdots + v^h) w^s z^n \qquad \text{if } 0 < r \leq i.$$

4. Lower transitory. $\Psi_4 = \frac{F(u,v,w) - F(uw,v,1)}{1-w} + \frac{F(u,v,w) - F(u,v,0)}{w}$.

5. Closer. The addition of a closer to a diagram with label $[h, r, s]$ contributes

$$\Psi_5 = \sum_{h,n} F_{h,r,s}(n) u^{h-1} (v^{\max\{r-1,0\}} + \cdots + v^{h-1})(w^{\max\{s-1,0\}} + \cdots + w^{h-1}) z^n,$$

which can be simplified using finite geometric sum formulas, and separating the case when $r = 0$ or $s = 0$:

$$\begin{aligned}\Psi_5 = &\frac{F(u,v,w) - F(uv,1,w) - F(uw,v,1) + F(uvw,1,1)}{u(1-v)(1-w)} \\ &+ \frac{F(u,v,w) - F(u,0,w) - F(uw,v,1) + F(uw,0,1)}{uv(1-w)} \\ &+ \frac{F(u,v,w) - F(u,v,0) - F(uv,1,w) + F(uv,1,0)}{uw(1-v)} \\ &+ \frac{F(u,v,w) - F(u,0,w) - F(u,v,0) + F(u,0,0)}{uvw}.\end{aligned}$$

Adding all five contributions, we get the following corollary.

Corollary 3.3.2 ([16] Burrill, Elizalde, Mishna, Yen 2014+)**.** *The generating function for 3-nonnesting open permutation diagrams, denoted*

$$F(u,v,w) = F(u,v,w;z) = \sum_{h,r,s,n} F_{h,r,s}(n) u^h v^r w^s z^n,$$

where $F_{h,r,s}(n)$ is the number of diagrams of size n with label $[h, r, s]$, satisfies the functional equation

$$\begin{aligned}F(u,v,w) = 1 + z\Bigg(&uF(u,v,w) \\ &+ \frac{F(u,v,w) - vF(uv,1,w)}{1-v} + \frac{F(u,v,w) - F(u,0,w)}{v} + \frac{F(u,v,w) - F(uw,v,1)}{1-w} \\ &+ \frac{F(u,v,w) - F(u,v,0)}{w} + \frac{F(u,v,w) - F(uv,1,w) - F(uw,v,1) + F(uvw,1,1)}{u(1-v)(1-w)} \\ &+ \frac{F(u,v,w) - F(u,0,w) - F(uw,v,1) + F(uw,0,1)}{uv(1-w)} \\ &+ \frac{F(u,v,w) - F(u,v,0) - F(uv,1,w) + F(uv,1,0)}{uw(1-v)} \\ &+ \frac{F(u,v,w) - F(u,0,w) - F(u,v,0) + F(u,0,0)}{uvw}\Bigg).\end{aligned}$$

This functional equation is useful for generating the series, but we have so far not been able to solve it or find an explicit form for $F_{0,0,0}(n)$, the number of 3-nonnesting permutations.

With the functional equation we iterate to get the counting sequence for 3-nonnesting permutations. For $u = v = 0$, we recover that there are 2 permutations of size 2. The reader can also verify that there are 7 open permutations of length 2: three have 1 semi-arc and no future 2-nesting, one has 2 semi-arcs, one with 1 semi-arc that is also a future 2-nesting, plus the two permutations with no semi-arcs.

3.3.2 The case $k \geq 4$

As in the set partition case, we can generalize our construction to open permutation diagrams avoiding $k+1$-nesting. Recall that each $k+1$-nonnesting open permutation diagram has label $[h; \mathbf{r}; \mathbf{s}] = [h; r_1, r_2, \dots, r_{k-1}; s_1, s_2, \dots, s_{k-1}]$. As already mentioned, we can describe the succession rule of the corresponding generating tree by viewing $[h, \mathbf{r}]$ as the label of the upper set partition, where we consider enhanced nestings (refer to Theorem 2.6.3), and $[h, \mathbf{s}]$ as the label of the lower set partition, where we consider usual nestings (see Theorem 2.5.1). We use $\mathbf{r} - \mathbf{1}$ as a shorthand for $r_1 - 1, r_2 - 1, \dots, r_{k-1} - 1$, and similarly for $\mathbf{s} - \mathbf{1}$. When the parameters r_0 and s_0 are used below in $(3b), (4b)$, etc., they are defined to be equal to h.

Theorem 3.3.3 ([16] Burrill, Elizalde, Mishna, Yen 2014+)**.** *Let $\Sigma^{(k)}$ be the set of $k+1$-nonnesting open permutation diagrams. To each diagram σ, associate the label $\ell(\sigma) = [h; \mathbf{r}; \mathbf{s}] = [h; r_1, r_2, \dots, r_{k-1}; s_1, s_2, \dots, s_{k-1}]$, where $2h$ is the number of semi-arcs, and r_i (resp. s_i) is the number of open upper (resp. lower) semi-arcs of enhanced nesting index (resp. nesting index) greater than or equal to i. Then the number of diagrams in $\Sigma^{(k)}$ of size n is the number of nodes* *at level n in the generating tree with root label* $[0;\mathbf{0};\mathbf{0}]$*, and succession rule given by*

$$
\begin{aligned}
&[h;\mathbf{r};\mathbf{s}] \longrightarrow \\
&\quad [h;h,r_2,\dots,r_{k-1};\mathbf{s}], && (1)\\
&\quad [h+1;\mathbf{r};\mathbf{s}], && (2)\\
&\quad [h;\mathbf{r}-\mathbf{1};\mathbf{s}], \quad \text{if } r_{k-1}\ge 1, && (3a)\\
&\quad [h;r_1-\mathbf{1},\dots,r_{j-1}-1,i,r_{j+1},\dots,r_{k-1};\mathbf{s}], \quad \text{for } 1\le j\le k-1 \text{ and } r_j\le i\le r_{j-1}-1, && (3b)\\
&\quad [h;\mathbf{r};\mathbf{s}-\mathbf{1}], \quad \text{if } s_{k-1}\ge 1, && (4a)\\
&\quad [h;\mathbf{r};s_1-\mathbf{1},\dots,s_{\jmath-1}-1,\imath,s_{\jmath+1},\dots,s_{k-1}], \quad \text{for } 1\le \jmath\le k-1 \text{ and } s_\jmath\le \imath\le s_{\jmath-1}-1, && (4b)\\
&\quad [h-1;\mathbf{r}-\mathbf{1};\mathbf{s}-\mathbf{1}], \quad \text{if } r_{k-1}\ge 1 \text{ and } s_{k-1}\ge 1, && (5a)\\
&\quad [h-1;\mathbf{r}-\mathbf{1};s_1-1,\dots,s_{\jmath-1}-1,\imath,s_{\jmath+1},\dots,s_{k-1}],\\
&\qquad\qquad \text{if } r_{k-1}\ge 1, \text{ for } 1\le \jmath\le k-1 \text{ and } s_\jmath\le \imath\le s_{\jmath-1}-1, && (5b)\\
&\quad [h-1;r_1-1,\dots,r_{j-1}-1,i,r_{j+1},\dots,r_{k-1};\mathbf{s}-\mathbf{1}],\\
&\qquad\qquad \text{if } s_{k-1}\ge 1, \text{ for } 1\le j\le k-1 \text{ and } r_j\le i\le r_{j-1}-1, && (5c)\\
&\quad [h-1;r_1-1,\dots,r_{j-1}-1,i,r_{j+1},\dots,r_{k-1};s_1-1,\dots,s_{\jmath-1}-1,\imath,s_{\jmath+1},\dots,s_{k-1}],\\
&\quad \text{for } 1\le \jmath\le k-1 \text{ and } s_\jmath\le \imath\le s_{\jmath-1}-1, \text{ and for } 1\le j\le k-1 \text{ and } r_j\le i\le r_{j-1}-1. && (5d)
\end{aligned}
$$

Note that for $k=2$, the generating tree in Theorem 3.3.3 agrees with the generating tree defined in Theorem 3.3.1 for $\Sigma^{(2)}$.

Proof. The labels correspond to the addition of the following vertices to a diagram σ:

(1) a fixed point (as in (1) of Theorem 2.6.3);

(2) an opener (as in (2) of Theorem 2.6.3 or (2) of Theorem 2.5.1);

(3*a*) an upper transitory closing the top semi-arc, if σ has a future enhanced upper $k-1$-nesting (as in (5) of Theorem 2.6.3);

(3*b*) an upper transitory (as in (3) of Theorem 2.6.3);

(4*a*) a lower transitory closing the bottom semi-arc, if σ has a future lower $k-1$-nesting ((5) in Theorem 2.5.1);

(4*b*) a lower transitory (as in (3) of Theorem 2.5.1);

(5*a*) a closer that closes both the top and the bottom semi-arcs, if σ has both a future enhanced upper $k-1$-nesting and a future lower $k-1$-nesting;

(5b) a closer that closes the top semi-arc and a lower semi-arc that is not the bottom one, if σ has a future enhanced upper $k-1$-nesting;

(5c) a closer that closes the bottom semi-arc and an upper semi-arc that is not the top one, if σ has a future lower $k-1$-nesting;

(5d) a closer that closes an upper and a lower semi-arc, neither of which is an outermost one.

□

The rules described in Theorem 3.3.3 are now translated to a functional equation. The generating function for $k+1$-nonnesting open permutation diagrams, denoted by

$$F(u; v_1, v_2, \ldots, v_{k-1}; w_1, w_2, \ldots, w_{k-1}; z) = F(u, \mathbf{v}, \mathbf{w}) = \sum_{h,\mathrm{r},\mathrm{s},n} F_{h,\mathrm{r},\mathrm{s}}(n) u^h \mathbf{v}^{\mathrm{r}} \mathbf{w}^{\mathrm{s}} z^n,$$

where $F_{h,\mathrm{r},\mathrm{s}}(n)$ is the number of diagrams of size n with label $[h; \mathbf{r}; \mathbf{s}]$ satisfying the functional equation:

$$F(u, \mathbf{v}, \mathbf{w}) = 1 + z(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5)$$

such that each Φ_i is the contribution for adding a vertex of type (i) in Theorem 3.3.3.

Corollary 3.3.4 ([16] Burrill, Elizalde, Mishna, Yen 2014+)**.** *The generating function for k-nonnesting open permutation diagrams, denoted*

$$F(u; v_1, v_2, \ldots, v_{k-1}; w_1, w_2, \ldots, w_{k-1}; z) = F(u, \mathbf{v}, \mathbf{w}) = \sum_{h,\mathrm{r},\mathrm{s},n} F_{h,\mathrm{r},\mathrm{s}}(n) u^h \mathbf{v}^{\mathrm{r}} \mathbf{w}^{\mathrm{s}} z^n,$$

where $F(u; v_1, v_2, \ldots, v_{k-1}; w_1, w_2, \ldots, w_{k-1}; z)$ is the number of diagrams of size n with label $[h; \mathbf{r}; \mathbf{s}]$ satisfies the functional equation

$$F(u, \mathbf{v}, \mathbf{w}) = 1 + z(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5)$$

where Φ_i, $i \in \{1, 2, 3, 4, 5\}$ are as described above.

Proof. We compute each Φ_i, following the development of the functional equation for 3-nonnesting open permutation diagrams in Section 3.3.1.

1. A fixed point. By extending the range of i in case (3) for upper transitories, the case (1) in the succession rule can alternatively be included in (3b); it is simpler to compute $\Phi_1 + \Phi_3$ in item (3) below.

2. An opener. $\Phi_2 = uF$.

3. An upper transitory and fixed point:

$$\Phi_1 + \Phi_3 = \frac{1}{v_1 \ldots v_{k-1}} \left(F - F|_{v_{k-1}=0}\right) + \sum_{j=1}^{k-1} \frac{1}{v_1 \ldots v_{j-1}(1-v_j)} \left(F - F|_{v_j=1, v_{j-1}=v_{j-1}v_j}\right) + F|_{v_1=1, u=uv_1}$$

4. A lower transitory:

$$\Phi_4 = \frac{1}{w_1 \ldots w_{k-1}} \left(F - F|_{w_{k-1}=0}\right) + \sum_{j=1}^{k-1} \frac{1}{w_1 \ldots w_{j-1}(1-w_j)} \left(F - F|_{w_j=1, w_{j-1}=w_{j-1}w_j}\right)$$

5. A closer:

$$\begin{aligned}
\Phi_5 = {} & \frac{1}{uv_1 \ldots v_{k-1} w_1 \ldots w_{k-1}} \left(F - F|_{v_{k-1}=0} - F|_{w_{k-1}=0} + F|_{v_{k-1}=w_{k-1}=0}\right) \\
& + \frac{1}{uv_1 \ldots v_{k-1}} \sum_{j=1}^{k-1} \frac{1}{w_1 \ldots w_{j-1}(1-w_j)} \left(F - F|_{v_{k-1}=0} - F|_{w_j=1, w_{j-1}=w_{j-1}w_j} + F|_{v_{k-1}=0, w_j=1, w_{j-1}=w_{j-1}w_j}\right) \\
& + \frac{1}{uw_1 \ldots w_{k-1}} \sum_{j=1}^{k-1} \frac{1}{v_1 \ldots v_{j-1}(1-v_j)} \left(F - F|_{w_{k-1}=0} - F|_{v_j=1, v_{j-1}=v_{j-1}v_j} + F|_{w_{k-1}=0, v_j=1, v_{j-1}=v_{j-1}v_j}\right) \\
& + \frac{1}{u} \sum_{j=1}^{k-1} \sum_{j=1}^{k-1} \Bigg(\frac{1}{v_1 \ldots v_{j-1}(1-v_j) w_1 \ldots w_{j-1}(1-w_j)} \\
& \times \left(F - F|_{v_j=1, v_{j-1}=v_{j-1}v_j} - F|_{w_j=1, w_{j-1}=w_{j-1}w_j} + F|_{v_j=1, v_{j-1}=v_{j-1}v_j, w_j=1, w_{j-1}=w_{j-1}w_j}\right) \Bigg)
\end{aligned}$$

Adding these contributions gives the result. □

Remark: when F is evaluated at $u = 0$, i.e. when there are no semi-arcs in an open permutation diagram, the function is only in z. Thus $F|_{u=0}$ is the generating function for (regular) permutations with no $k+1$-nestings.

Table A.3 in Appendix A contains the counting sequence, along with relevant references to the OEIS for completeness. We present the terms which coincide with $n!$ in grey. We used the *gfun* package of Maple (version 3.53) to try fitting the counting sequence for k-nonnesting permutations (for $3 \leq k \leq 6$) into a differential equation. Using 80 terms was not successful, and so we make the following conjecture:

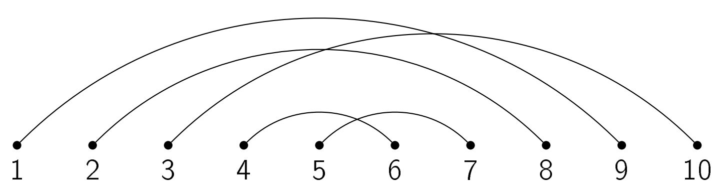
Conjecture 3.3.5. *The ordinary generating function for k-nonnesting permutations is not D-finite for any $k > 2$.*

Chapter 4

Matchings

Matchings are yet another classic, well understood combinatorial object: they are central to understand bijective connections, and essential for the tangled diagram construction in Chapter 5. A (complete) matching on the set $\{1, 2, \ldots, 2n\}$ is a partition of the set into blocks of size 2. There is an edge between two points in the same block.

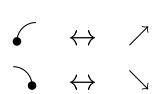
Example 33. *The matching* $\mu = 1\,9 - 2\,8 - 3\,10 - 4\,6 - 5\,7$ *is depicted below.*



Matchings are the most well understood combinatorial class when parameterized according to nestings: full enumerative results are known. That said, we include a treatment of them here to first illustrate that we can recover all known counting results using our strategy, and also because interesting bijections arise when we generate k-nonnesting matchings (see Chapter 6).

4.1 History

Complete, or perfect, matchings of $\{1, 2, \ldots, 2n\}$ are well known to be counted by $(2n-1)!!$. Matchings were the first combinatorial object encoded as an arc diagram in which crossings and nestings were considered. The classic bijection between matchings and Dyck paths is even simpler than that between set partitions and Dyck paths.



This gives that noncrossing matchings of $\{1, 2, \ldots, 2n\}$ are also enumerated by the Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$.

As in the set partition case, this notion of equidistribution between crossing and nestings was greatly generalized by Chen, Deng, Du, Stanley and Yan in 2007 [20].

Theorem 4.1.1 ([20] Chen, Deng, Du, Stanley, Yan 2007)**.** *The number of k-noncrossing matchings of $\{1, 2, \ldots, 2n\}$ is equal to the number of k-nonnesting matchings of $\{1, 2, \ldots, 2n\}$.*

The authors of [20] went even further, and completely enumerated k-nonnesting matchings. They did this by first showing a connection to lattice walks:

Corollary 4.1.2 ([20] Chen, Deng, Du, Stanley, Yan 2007)**.** *The number of k-nonnesting matchings of $\{1, 2, \ldots, 2n\}$ is equal to the number of closed lattice walks of length $2n$ in the set*

$$V_k = \{(a_1, a_2, \ldots, a_{k-1}) : a_1 \geq a_2 \geq \ldots \geq a_{k-1} \geq 0, a_i \in \mathbb{Z}\}$$

from the origin to itself with unit steps in any coordinate direction or its negative.

(Notice: when $k = 2$ this gives the correspondence between nonnesting matchings and Dyck paths). The enumeration of such paths had already been solved by Grabiner and Magyar [40] in 1993. Thus, k-nesting matchings have been enumerated.

Theorem 4.1.3 ([40] Grabiner, Magyar 1993, [20] Chen, Deng, Du, Stanley, Yan 2007)**.** *Let $f_k(n)$ be the number of k-nonnesting matchings of $\{1, 2, \ldots, 2n\}$. Then,*

$$F_k(z) = \sum_n f_k(n)\frac{z^{2n}}{(2n)!} = \det[I_{i-j}(2z) - I_{i+j}(2z)]_{i,j=1}^{k-1},$$

where

$$I_n(2z) = \sum_{j\geq 0} \frac{z^{n+2j}}{j!(n+j)!}$$

is the hyperbolic Bessel function of the first kind of order n [58].

4.2 3-nonnesting matchings

We can recover the known enumerative results for k-nonnesting matchings by noticing that a matching is simply a set partition in which no fixed points or transitory vertices exist: we only need to consider the contribution of openers and closers. The following is a corollary to Theorem 2.4.1.

Corollary 4.2.1. *Let $M^{(2)}$ be the set of 3-nonnesting open matching diagrams. To each diagram, associate the label $\ell(\mu) = [i, j]$ if μ has i semi-arcs, j of which belong to some future 2-nesting.*

Then the number of diagrams in $M^{(2)}$ of size n is the number of nodes at level n in the generating tree with root label $[0,0]$ for $n=0$, and succession rule given by

$$[i,j] \rightsquigarrow \begin{array}{lll} [i+1,j], & & \textit{(semi-opener)} \\ [i-1,j],[i-1,j+1],\ldots,[i-1,i-1], & \textit{if } i \geq 1 & \textit{(closer)} \\ [i-1,j-1]. & \textit{if } i \geq 1 \textit{ and } j > 0 & \textit{(closer when } j>0\textit{)} \end{array}$$

Proof. The labels arise from adding an opener, a closer, and a closer that closes the top semi-arc, if the parent diagram has a future k-nesting. □

The start of the generating tree for open matching diagrams without 3-nestings is given in Appendix B.

This succession rule translates easily to a functional equation. Let $M_{i,j}(n)$ be the number of 3-nonnesting open matching diagrams at level n with label $[i,j]$. Consider $M(u,v;z) := \sum M_{i,j}(n)u^i v^j z^n$, where u marks total number of open arcs and v the number that are part of a future 2-nesting. For simplicity, we write $M(u,v)$ instead of $M(u,v;z)$.

Corollary 4.2.2. *The generating function $M(u,v)$ for 3-nonnesting open matching diagrams, with variables u and v marking variables i and j in the label, respectively, and z marking the number of vertices satisfies the functional equation*

$$M(u,v) = 1 + zuM(u,v) + \frac{z}{u}\left(\frac{M(u,v)-M(uv,1)}{1-v} + \frac{M(u,v)-M(u,0)}{v}\right).$$

4.3 k-nonnesting matchings

The case of k-nonnesting matchings can also be described explicitly using the nesting index. As in Section 2.5.1, let s_i be the number of semi-arcs with nesting index greater than or equal to i. We get the following generating tree:

Theorem 4.3.1. *Let $M^{(k)}$ be the set of $k+1$-nonnesting open partition diagrams. To each diagram, associate the label $\ell(\mu) = [s_0,\ldots,s_{k-1}]$, where s_i is the number of semi-arcs with nesting index $\geq i$. Then, the number of diagrams in $M^{(k)}$ of size n is the number of nodes at level* *n in the generating tree with root label* $[0, 0, \ldots, 0]$, *and succession rule given by*

$$
\begin{aligned}
&[s_0, s_1, \ldots, s_{k-1}] \rightsquigarrow \\
&[s_0 + 1, s_1, \ldots, s_{k-1}], && (1)\\
&[s_0 - 1, s_1 - 1, \ldots, s_{j-1} - 1, i, s_{j+1}, \ldots, s_{k-1}], \quad \text{for } 1 \le j \le k-1 \text{ and } s_j \le i \le s_{j-1} - 1, && (2)\\
&[s_0 - 1, s_1 - 1, \ldots, s_{k-1} - 1], \quad \text{if } s_{k-1} > 0. && (3)
\end{aligned}
$$

Proof. The labels arise from adding the following kinds of vertices: (1) an opener, (2) a closer, and (3) a closer that closes the top semi-arc, if the parent diagram has a k-nesting. □

Since matchings are just set partitions without transitory or fixed point vertices, we easily get the following functional equation result.

Corollary 4.3.2. *The generating function for* $k+1$*-nonnesting open matching diagrams, with variable* v_i *marking value* s_i *in the label and variable* z *marking the number of vertices, denoted* $M = M(v_0, v_1, \ldots, v_{k-1}) = M(v_0, v_1, \ldots, v_{k-1}; z)$, *satisfies the functional equation*

$$
\begin{aligned}
M = 1 + zv_0M + z\Bigg(&\frac{1}{v_0v_1\ldots v_{k-1}}(M - M(v_0, v_1, \ldots, v_{k-2}, 0)) \\
&+ \sum_{j=1}^{k-1} \frac{1}{v_0v_1\ldots v_{j-1}(1-v_j)}(M - M(v_0, \ldots, v_{j-2}, v_{j-1}v_j, 1, v_{j+1}, \ldots, v_{k-2}, v_{k-1}))
\end{aligned}
$$

4.4 Counting sequences

Indeed, k-nonnesting matchings have been enumerated; we included their generating tree and functional equation here for both completeness, and to illustrate the applicability of our procedure. The iterative method described in detail in Chapters 2 and 3 also applies here for extracting counting sequences; data can be found in Appendix A. That said, the main benefit to the treatment of matchings using our strategy becomes most apparent when all catalytic variables are set to 1 giving the number of open matching diagrams without future k-nestings. The counting sequences that arise for $2 \le k \le 7$ are each present in the OEIS, hinting at a rich area of bijective combinatorics. We investigate this further in Chapter 6

We now use our framework to treat a rather different combinatorial class: tangled diagrams.

Chapter 5

Tangled diagrams

A *tangled diagram* [23] is a generalization of matchings, set partitions, and in a sense permutations. A tangled diagram on $\{1,\ldots,n\}$ is a labelled graph on vertices $1,\ldots,n$ drawn on a horizontal line with arcs drawn above the line connecting vertices. As opposed to matchings, set partitions and permutations, *any* vertex may have degree 0, 1 *or* 2. A tangled diagram may have isolated points, plus the types of arcs listed in Figure 5.1.

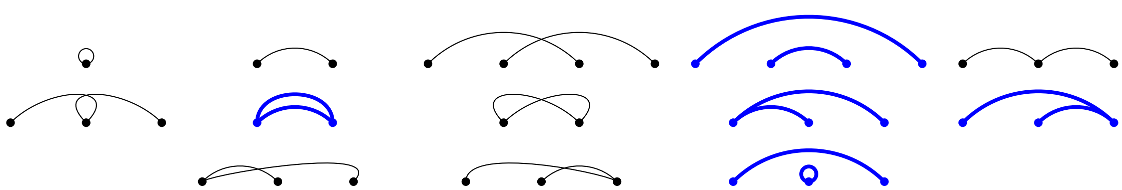


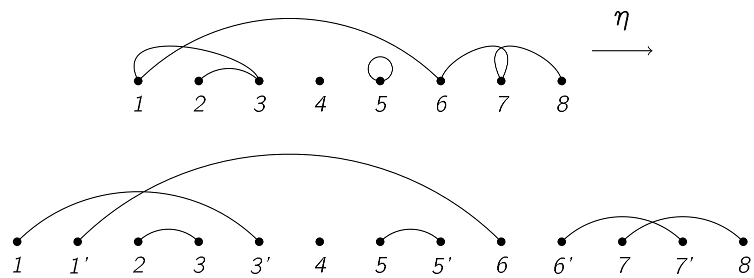
Figure 5.1: All possible interactions of 1 and 2 arcs in a tangled diagram; those in **blue** form nestings

.

Because there is more than one way that two arcs can be drawn, defining a nesting in a tangled diagram requires some care. As in all other cases, we say two arcs (i_1, j_1) and (i_2, j_2) are nesting if $i_1 < i_2 < j_2 < j_1$. If two arcs (i, j_1) and (i, j_2) have a common left endpoint, they can be drawn in two ways: either (i, j_1) is drawn strictly below (i, j_2), which gives a nesting, or not (a crossing). Similarly if two arcs have a common right endpoint. Two arcs with two common endpoints that are drawn one completely below the other is a nesting. Finally, a fixed point (i, i), drawn with a loop at vertex i is considered a nesting if it is below any arc. Each of these possible ways of forming a nesting is depicted blue in Figure 5.1. Similar crossing statistics can be identified. A tangled diagram is k-nonnesting if it does not contain k mutually nesting arcs.

It should be noted that tangled diagrams are closely related to matchings through a map η called **inflation**. Under η, any vertex i that has degree 2 it is inflated to become i and i', with i being connected only to the left most arc in the tangled diagram, and i' being only connected to its right most arc. This map allows us to identify easily k-nestings in tangled diagrams: if the inflated tangled diagram is k-nonnesting under the matching definition of nestings, then the original tangled diagram is also k-nonnesting.

Example 34. *Consider the tangled diagram T_8 below on 8 vertices with 5 of degree two, and its inflation $\eta(T_8)$ on $8 + 5$ vertices each with degree one or less.*



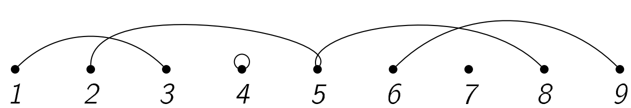
We see that the inflated tangled diagram is 3-nonnesting, and thus the original tangled diagram is 3-nonnesting as well.

5.1 History

Tangled diagrams were introduced by Chen, Qin and Reidys in 2008 [23] and arise in [24], [35] from a biological motivation: the study of RNA folding. Tangled diagrams are able to express all intramolecular interactions of RNA molecules; while nucleotides are known to form classic **A**-**U**, **G**-**C** and **U**-**G** base pairs, they can also form hydrogen bonds which stabilize its structure, thus they need any nucleotide (or vertex) to form two bonds (or arcs). We quickly revisit arc diagrams and RNA folding in Section 6.4.

From a combinatorial point of view, tangled diagrams are a generalization of other classes. A matching is a 1-regular tangled diagram. Set partitions are also tangled diagrams. A *braid* is a tangled diagrams where all vertices of degree two are either loops or crossing arcs.

Example 35. *The following tangled diagram is also a braid.*



Remark that this braid is also 3-noncrossing.

In [23], nestings and crossings were shown to be equidistributed in tangled diagrams.

Theorem 5.1.1. [[23] Chen, Qin, Reidys 2008] *There is a bijection between the set of k-noncrossing and k-nonnesting tangled diagrams.*

Furthermore, due to their relationship to matchings via the inflation map, exact enumeration formulas were accessible.

Theorem 5.1.2. [[23] Chen, Qin, Reidys 2008] *Let $f_k(2n-\ell)$ be the number of k-noncrossing matchings on $2n-\ell$ vertices. The number of k-noncrossing tangled diagrams on $\{1, \ldots, n\}$ without isolated points is given by*

$$\tilde{t}_k(n) = \sum_{\ell=0}^{n} \binom{n}{\ell} f_k(2n-\ell),$$

We recover these known enumerative results in the case of 3-nonnesting tangled diagrams, and our approach is radically different from [23]: we deal directly with the tangled diagram, and have no need to pass through matchings, tableaux, or lattice paths in order to extract enumerative results. All that is required is representation as an arc diagram.

5.2 Open tangled diagrams

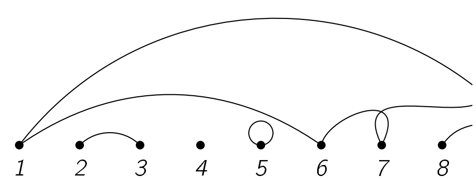
Indeed, tangled diagrams are a good candidate for our strategy. There are more types of vertices, but they are still limited in number:

1. **singleton**
2. **opener**
3. **noncrossing transitory**
4. **closer**
5. **fixed point**
6. **double opener**
7. **crossing transitory**
8. **double closer**

Thus, the procedure is still appropriate. We begin by using our strategy to generate and enumerate tangled diagrams without nesting restrictions.

First, remove the restriction that all semi-arcs must be closed to get *open tangled diagrams.*.

Example 36. *The following is an open tangled diagram:*



In order to define a succession rule, we first identify the parameter we should track: semi-arcs that are available to be closed. If t is an open tangled diagram, let $\ell(t) = [m]$ be the number of semi-arcs in that diagram. The succession rule for open tangled diagrams is

$$[m] \rightsquigarrow \begin{cases} [m], & & \text{(singleton)} \\ [m+1], & & \text{(opener)} \\ \underbrace{[m],[m],\dots,[m]}_{m \text{ copies}}, & \text{if } m>0, & \text{(noncrossing transitory)} \\ \underbrace{[m-1],[m-1],\dots,[m-1]}_{m \text{ copies}}, & \text{if } m>0. & \text{(closer)} \\ [m], & & \text{(fixed point)} \\ [m+2], & & \text{(double opener)} \\ \underbrace{[m],[m],\dots,[m]}_{m \text{ copies}}, & \text{if } m>0, & \text{(crossing transitory)} \\ \underbrace{[m-2],[m-2],\dots,[m-2]}_{m(m-1) \text{ copies}}, & \text{if } m>1. & \text{(double closer)} \end{cases} \tag{5.1}$$

Let $T(u,z) = \sum_{m,n} t_m(n) u^m z^n$ be the bivariate generating function for open set partitions where the exponent of u is the label of the node: the number of semi-arcs. We consider the contribution of each type of vertex:

- singleton: $+\sum_{m,n} t_m(n) u^m z^n = T(u,z)$;
- opener: $+\sum_{m,n} t_m(n) u^{m+1} z^n = uT(u,z)$;
- noncrossing transitory: $+\sum_{m,n} t_m(n) m u^m z^n = uT_u(u,z)$;
- closer: $+\sum_{m,n} t_m(n) m u^{m-1} z^n = T_u(u,z)$;
- fixed point: $+\sum_{m,n} t_m(n) u^m z^n = T(u,z)$;
- double opener: $+\sum_{m,n} t_m(n) u^{m+2} z^n = u^2 T(u,z)$;
- crossing transitory: $+\sum_{m,n} t_m(n) m u^m z^n = uT_u(u,z)$;
- double closer: $+\sum_{m>1,n} t_m(n) m(m-1) u^{m-2} z^n = T_{uu}(u,z)$;

Thus our functional equation for open tangled diagrams is:

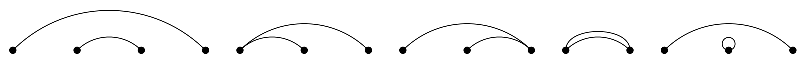
$$T(u,z) = 1 + z\left((2+u+u^2)T(u,z) + (2u+1)T_u(u,z) + T_{uu}(u,z)\right)$$

This functional equation can be iterated to get the counting sequence for open tangled diagrams. Setting $u=1$ gives the total number of open tangled diagrams, and evaluating at $u=0$ returns the total number of tangled diagrams, with no nesting restrictions. The data is presented in Table A.5 in Appendix A. Notice that sequence A125660 in the OIES [41] claims to be the number of tangled diagrams, yet their numbers do not match ours. This is due to a simple mislabelling of that sequence: A125660 is the number of 3-noncrossing tangled diagrams, data which we recover in Section 5.3.

We are now ready to include nonnesting conditions in tangled diagrams.

5.3 3-nonnesting tangled diagrams

Recall that there are five ways to form a nesting in a tangled diagram:



A tangled diagram without a k-nesting is a k-nonnesting tangled diagram.

We next define an appropriate label which tracks nestings in tangled diagrams. We define a *future k-nesting of a tangled diagram* to be a $k-1$-nesting plus a semi-arc above it. Like in the case of set partitions, permutations and matchings, the label for 3-nonnesting tangled diagrams will consist of two parts: the number of semi-arcs, m, and the number of future 2-nestings, s.

Our first observation is that the addition of any vertex of degree 2 can be thought of as adding either an opener or closer (without increasing size), and then applying the succession rule for matchings.

Theorem 5.3.1. *Let $\mathbf{T}^{(2)}$ be the set of 3-nonnesting open tangled diagrams. To each diagram τ, associate the label $\ell(\tau) = [m, s]$ if τ has m semi-arcs, s of which belong to some future 2-nesting. Let $\mathbf{M}([a, b])$ indicate the succession for perfect matchings applied to a diagram with label $[a, b]$. Then the number of diagrams in $\mathbf{T}^{(2)}$ of size n is the number of nodes at level n in the generating tree with root label $[0, 0]$, and succession rule given by*

$$[m,s] \rightarrow \begin{array}{lll} [m,s] & & (1) \\ [m+1,s] & & (2) \\ [m-1,i] & \text{for } \max\{0,s-1\} \le i \le m-1 & (3) \\ \mathbf{M}([m+1,s]) & & (4) \\ \mathbf{M}([m-1,i]) & \text{for } \max\{0,s-1\} \le i \le s-1. & (5) \end{array}$$

The number of 3-nonnesting tangled diagrams of size n is equal to the number of nodes with label $[0, 0]$ at the n-th level of this generating tree.

Proof. The labels arise from adding the following kinds of vertices:

(1) a singleton;

(2) an opener;

(3) a closer;

(4) an application of the matching succession rule to a diagram with one extra semi-arc corresponds to a fixed point, crossing transitory, and double opener;

(5) an application of the matching succession rule to diagrams in which one arc has already been closed corresponds to a noncrossing transitory and closer.

□

An application of this succession rule is found in Appendix C. While effective in generating data, it is not as conducive to functional equation translation. For that reason, we also determine the generating tree for 3-nonnesting tangled diagrams directly.

As before suppose an open tangled diagram τ on n vertices has label $\ell(\tau) = [m, s]$. Its children will have the following labels, depending on the type of added vertex $n + 1$:

1. **singleton** one child with label $[m, s]$;

2. **opener** one child with label $[m + 1, s]$;

3. **noncrossing transitory** $m - s + 1$ children if $s > 0$, or m children if $s = 0$, since we can close any of the $m - s$ semi-arcs not in future 2-nestings, plus the top semi-arc in the case $s > 0$; the children's labels are:

$$[m, s], [m, s + 1], [m, s + 2], \ldots, [m, m - 1], \text{ and } [m, s - 1] \text{ if } s > 0$$

4. **closer** $m - s + 1$ children if $s > 0$, or m children if $s = 0$, since we can close any of the $m - s$ semi-arcs not in future 2-nestings, plus the top semi-arc in the case $s > 0$; the children's labels are:

$$[m - 1, s], [m - 1, s + 1], [m - 1, s + 2], \ldots, [m - 1, m - 1], \text{ and } [m - 1, s - 1]\text{if } s > 0$$

5. **fixed point** one child with label $[m, m]$;

6. **double opener** one child with label $[m + 2, s]$;

7. **crossing transitory** $m - s + 1$ children if $s > 0$, or m children if $s = 0$, since we can close any of the $m - s$ semi-arcs not in future 2-nestings, plus the top semi-arc in the case $s > 0$; the children's labels are:

$$[m, s], [m, s + 1], [m, s + 2], \ldots, [m, m - 1], \text{ and } [m, s - 1] \text{ if } s > 0$$

8. **double closer** closing one arc gives $m-s+1$ children if $s>0$, and m children if $s=0$. Each of those children then closes another arc. The labels of the children are as follows:

$$(\star)\qquad [m,s] \rightarrow \begin{array}{lcl} \textbf{[m-1, s-1]} & \rightarrow & {\color{red}[m-2, s-2]}, \textbf{[m-2, s-1], [m-2, s], [m-2, s+1], [m-2, s+2], \dots, [m-2, m-2]} \\ [m-1,s] & \rightarrow & \textbf{[m-2, s-1]}, [m-2,s], [m-2,s+1], [m-2,s+2], \dots, [m-2,m-2] \\ [m-1,s+1] & \rightarrow & \textit{[m-2, s]}, [m-2,s+1], [m-2,s+2], [m-2,s+3], \dots, [m-2,m-2] \\ [m-1,s+2] & \rightarrow & \textit{[m-2, s+1]}, [m-2,s+2], [m-2,s+3], [m-2,s+4], \dots, [m-2,m-2] \\ \vdots & & \vdots \\ [m-1,s+3] & \rightarrow & \textit{[m-2, s+2]}, [m-2,s+3], [m-2,s+4], [m-2,s+5], \dots, [m-2,m-2] \\ [m-1,m-3] & \rightarrow & \textit{[m-2, m-4]}, [m-2,m-3], [m-2,m-2] \\ [m-1,m-2] & \rightarrow & \textit{[m-2, m-3]}, [m-2,m-2] \\ [m-1,m-1] & \rightarrow & \textit{[m-2, m-2]} \end{array}$$

Note: labels given in **bold** only appear if $s>0$, and the label in red only if $s>1$.

This describes the succession rule for 3-nonnesting open tangled diagrams. We summarize it in the following theorem:

Theorem 5.3.2. *Let $\mathbf{T}^{(2)}$ be the set of 3-nonnesting open tangled diagrams. To each diagram τ, associate the label $\ell(\tau)=[m,s]$, where m is the number of semi-arcs and s is the number of semi-arcs in a future 2-nesting. Then, the number of diagrams in $\mathbf{T}^{(2)}$ of size n is the number of node at level n in the generating tree with root label $[0,0]$, and succession rule given by:*

$$[m,s] \rightarrow \begin{cases} [m,s], & (1) \\ [m+1,s], & (2) \\ [m,i], \quad \text{for } \max\{0,s-1\} \le i \le m-1, & (3) \\ [m-1,i], \quad \text{for } \max\{0,s-1\} \le i \le m-1, & (4) \\ [m,m] & (5) \\ [m+2,s] & (6) \\ [m,i], \quad \text{for } \max\{0,s-1\} \le i \le m-1, & (7) \\ [m-2,a], \quad \text{for } a \le k \le m-2 \text{ and } \max\{0,s-2\} \le a \le m-2, & (8) \end{cases}$$

The number of 3-nonnesting tangled diagrams of size n is equal to the number of nodes with label $[0,0]$ at the n^{th} level of this generating tree.

Proof. This follows from description of the label before theorem. $\square$

We next translate the succession rule from Theorem 5.3.2 to a functional equation. Let $T(u,v)=\sum T_{m,s}(n)u^m v^s z^n$, where $T_{m,s}(n)$ is the number of open tangled diagrams at level n with label $[m,s]$. The coefficient $T_{0,0}(n)$ is the number of 3-nonnesting tangled diagrams of $\{1,\dots,n\}$.

We follow the same process as we have throughout Part II: we consider each type of vertex and determine its contribution to the functional equation. Its form will be:

$$T(u,v) = 1 + z(\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 + \Phi_6 + \Phi_7 + \Phi_8).$$

Each Φ_i is the contribution to the functional equation from the addition of a vertex of type (i) for $1 \leq i \leq 8$. We compute the Φ_i's:

1. Singleton. $\Phi_1 = T(u,v)$.

2. Opener. $\Phi_2 = uT(u,v)$.

3. Noncrossing transitory and fixed point (the case of (5) in the succession rule can alternately be included by extending the range of i in case (3) to include m).
   $\Phi_3 + \Phi_5 = \frac{T(u,v) - vT(uv,1)}{1-v} + \frac{T(u,v)-T(u,0)}{v}$, found using the formula for finite geometric sum on the expressions below:

$$\sum_{m>0} T_{m,0}(n)u^m(1 + v + v^2 + \ldots + v^m)z^n \qquad \text{if } s = 0,$$

$$\sum_{m>0,s} T_{m,s}(n)u^m(v^{s-1} + v^s + \ldots + v^m)z^n \qquad \text{if } s > 0,$$

4. Closer. $\Phi_4 = \frac{1}{u}\left(\frac{T(u,v)-T(uv,1)}{1-v} + \frac{T(u,v)-T(u,0)}{v}\right)$. Found using the formula for finite geometric sum on expressions similar to those in 3.

5. Fixed point. Contribution is included in 3. with noncrossing transitory.

6. Double opener. $\Phi_6 = u^2T(u,v)$.

7. Crossing transitory. $\Phi_7 = u\left(\frac{1}{u}\left(\frac{T(u,v)-T(uv,1)}{1-v} + \frac{T(u,v)-T(u,0)}{v}\right)\right)$.

8. Double closer. Φ_8. See below.

Double closer contribution We need to determine the contribution of a double closer, as indicated by ($\star$) in the (partial) succession rule above. Transcribing carefully, a closer contributes the following to the functional equation:

$$\begin{aligned}\Phi_8 = \sum_{m,s} T_{m,s}(n)u^{m-2}\big[&\big(v^s + v^{s+1} + \ldots + v^{m-2}\big) + \big(v^s + v^{s+1} + \ldots + v^{m-2}\big)\\ &+ \big(v^{s+1} + v^{s+2} + \ldots + v^{m-2}\big) + \ldots + \big(v^{m-3} + v^{m-2}\big) + \big(v^{m-2}\big)\big]\\ &+ \sum_{m,s\geq 1} T_{m,s}(n)u^{m-2}(v^{s-1} + v^{s-1} + v^s + v^{s+1} + \ldots + v^{m-2})\\ &+ \sum_{m,s\geq 2} T_{m,s}(n)u^{m-2}v^{s-2}\end{aligned}$$

$$\begin{aligned}\Phi_8 = \sum_{m,s} T_{m,s}(n)u^{m-2}\big[&(v^{s-1} + v^s + v^{s+1} + \ldots + v^{m-2}) + (v^s + v^{s+1} + \ldots + v^{m-2})\\ &+(v^{s+1} + v^{s+2} + \ldots + v^{m-2}) + \ldots + v^{m-2}\big] - \sum_{m,s=0} T_{m,s}u^{m-2}v^{s-1}\\ &+ \sum_{m,s\geq 1} T_{m,s}(n)u^{m-2}(v^{s-1} + v^s + \ldots + v^{m-2})\\ &+ \sum_{m,s\geq 2} T_{m,s}(n)u^{m-2}v^{s-2}\end{aligned}$$

$$\begin{aligned}\Phi_8 = \sum_{m,s} T_{m,s}(n)u^{m-2}v^{s-1}&(1 + 2v + 3v^2 + \ldots + (m-s)v^{m-s-1}) - \sum_{m,s=0} T_{m,s}(n)u^{m-2}v^{s-1}\\ &+ \sum_{m,s\geq 1} T_{m,s}(n)u^{m-2}v^{s-1}(1 + v + v^2 + \ldots + v^{m-s-1})\\ &+ \sum_{m,s\geq 2} T_{m,s}(n)u^{m-2}v^{s-2}\end{aligned}$$

Recall the formula for finite geometric series and its derivative:

$$1 + v + v^2 + \ldots + v^k = \frac{1 - v^{k+1}}{1 - v}; \qquad 1 + 2v + 3v^2 + \ldots + kv^{k-1} = \frac{kv^{k+1} - (k+1)v^k + 1}{(v-1)^2}.$$

These are used to compute Φ_8:

$$\Phi_8 = \sum_{m,s} T_{m,s}(n)u^{m-2}v^{s-1}\left(\frac{(m-s)v^{m-s+1}-(m-s+1)v^{m-s}+1}{(v-1)^2}\right)$$
$$- \sum_{m,s=0} T_{m,s}(n)u^{m-2}v^{s-1}$$
$$+ \sum_{m,s\geq 1} T_{m,s}(n)u^{m-s}v^{s-1}\left(\frac{1-v^{m-s}}{1-v}\right)$$
$$+ \sum_{m,s\geq 1} T_{m,s}(n)u^{m-s}v^{s-2}$$

$$\Phi_8 = \frac{1}{(v-1)^2u^2}\left[\sum_{m,s} T_{m,s}(n)mu^m v^{m-1}v - \sum_{m,s} T_{m,s}(n)su^m v^m\right.$$
$$-\sum_{m,s} T_{m,s}(n)mu^m v^{m-1}$$
$$\left.+\frac{1}{v}\left(\sum_{m,s} T_{m,s}(n)su^m v^m + \sum_{m,s} T_{m,s}(n)u^m v^m + \sum_{m,s} T_{m,s}(n)u^m v^s\right)\right]$$
$$- \frac{1}{u^2v}\sum_{m,s=0} T_{m,s}(n)u^m v^s$$
$$+ \frac{1}{(1-v)u^2v}\left[\sum_{m,s\geq 1} T_{m,s}(n)u^m v^s - \sum_{m,s\geq 1} T_{m,s}(n)u^m v^m\right]$$
$$+ \frac{1}{u^2v^2}\left[\sum_{m,s\geq 2} T_{m,s}(n)u^m v^s\right]$$

We use the following identities:

$$T(uv,1) = \sum_{m,s} T_{m,s}(n)u^m v^m \qquad \frac{\partial T(uv,1)}{\partial v} = \sum_{m,s} T_{m,s}(n)mu^m v^{m-1}$$

$$\frac{\partial T(uv,w)}{\partial w} = \sum_{m,s} T_{m,s}(n)su^m v^m w^{s-1} \qquad \left.\frac{\partial T(uv,w)}{\partial w}\right|_{w=1} = \sum_{m,s} T_{m,s}(n)su^m v^m.$$

Substituting, we get Φ_8:

$$\Phi_8 = \frac{1}{(v-1)^2u^2}\left[\frac{\partial T(uv,1)}{\partial v}\cdot v - \left.\frac{\partial T(uv,w)}{\partial w}\right|_{w=1} - \frac{\partial T(uv,1)}{\partial v}\right.$$
$$\left. + \frac{1}{v}\left(\left.\frac{\partial T(uv,w)}{\partial w}\right|_{w=1} + T(uv,1) + T(u,v)\right)\right] - \frac{1}{u^2v}\sum_{m,s=0} T_{m,s}(n)u^{m-2}v^{s-1}$$
$$+ \frac{1}{u^2v(1-v)}\left[T(u,v) - T(uv,1) - \sum_{m,s=0} T_{m,s}(n)u^mv^s + \sum_{m,s=0} T_{m,s}(n)u^mv^m\right]$$
$$+ \frac{1}{u^2v^2}\left[T(u,v) - \sum_{m,s=0,1} T_{m,s}(n)u^mv^s\right]$$

$$\Phi_8 = \frac{1}{(v-1)^2u^2}\left[\frac{\partial T(uv,1)}{\partial v}(v-1) - \left(1-\frac{1}{v}\right)\left.\frac{\partial T(uv,w)}{\partial w}\right|_{w=1} - \left(\frac{T(uv,1)-T(u,v)}{v}\right)\right]$$
$$- \frac{1}{u^2v}\left(T(u,0)+T(0,0)\right) + \frac{1}{u^2v(1-v)}\left[T(u,v) - T(uv,1) - T(u,0) + T(uv,0)\right]$$
$$+ \frac{1}{u^2v^2}\left[T(u,v) - T(u,0) - v\cdot\left.\frac{\partial T(u,v)}{\partial v}\right|_{v=0}\right]$$

Adding each of the eight contributions gives the following corollary to Theorem 5.3.2

Corollary 5.3.3. *The generating function for* 3*-nonnesting open tangled diagrams, denoted*

$$T(u,v) = T(u,v;z) = \sum_{m,s,n} T_{m,s}(n)u^mv^sz^n,$$

where $T_{m,s}(n)$ is the number of diagrams of size n with label $[m,s]$, satisfying the functional equation

$$T(u,v) = 1 + z\left\{T(u,v)(1+u+u^2) + \frac{T(u,v)-vT(uv,1)}{1-v} + \frac{T(u,v)-T(u,0)}{v}\right.$$
$$+ \left(\frac{1}{u}+1\right)\left(\frac{T(u,v)-T(uv,1)}{1-v} + \frac{T(u,v)-T(u,0)}{v}\right)$$
$$\frac{1}{(v-1)^2u^2}\left[\frac{\partial T(uv,1)}{\partial v}(v-1) - \left(1-\frac{1}{v}\right)\left.\frac{\partial T(uv,w)}{\partial w}\right|_{w=1} - \left(\frac{T(uv,1)-T(u,v)}{v}\right)\right]$$
$$- \frac{1}{u^2v}\left(T(u,0)+T(0,0)\right) + \frac{1}{u^2v(1-v)}\left[T(u,v)-T(uv,1)-T(u,0)+T(uv,0)\right]$$
$$\left. + \frac{1}{u^2v^2}\left[T(u,v) - T(u,0) - v\cdot\left.\frac{\partial T(u,v)}{\partial v}\right|_{v=0}\right]\right\}$$

With this functional equation, we are able to recover the counting sequence for 3-nonnesting tangled diagrams. When we iterate as in previous chapters, and get sequence A125660 [41] (which, recall, is labelled as the number of tangled diagrams, but is in fact the number of 3-noncrossing tangled diagrams). The sequence listed on the OEIS has only 8 terms listed; we are able to recover significantly more than this, getting to $n = 60$ in under four minutes on a desktop machine. We list the start of the counting sequence in Appendix A. The start of the generating tree is given in Appendix B for $n = 7$.

Generating k-nonnesting tangled diagrams for $k > 3$ is certainly possible using our strategy, however 3-nonnesting was sufficiently technical that we let it rest here.

Part III

Other applications

Chapter 6

An enumeration of bijections

An interesting fact about k-nonnesting arc diagrams is that they are in bijection with k-noncrossing arc diagrams. One advantage of counting data is that the series that arise sometimes enumerate other combinatorial classes. When this happens, we search for bijections between the two classes that are counted by the same sequence. In this chapter, we give and conjecture a variety of bijections between k-nonnesting arc diagrams and lattice paths, Young tableaux and Baxter permutations. In doing so, we illustrate that k-nonnesting arc diagrams are central to a rich area of bijective combinatorics.

Note that we have examined the counting sequences for all k-nonnesting arc diagrams (complete and open) for set partitions, permutations, matchings and tangled diagrams, and up to $k = 8$. If the series appeared in the literature, we either give or conjecture a bijection.

6.1 Bijections to lattice paths

There are a variety of bijections between restricted arc diagrams and lattice paths. Most of them are quite straightforward and can be listed using a dictionary which translates directly between the objects.

6.1.1 Easy

Arc diagrams were initially studied in the context of noncrossing diagrams. There are a series of bijections between (2−)noncrossing arc diagrams and lattice paths. In each case, for every opener-closer sequence there is a *unique way* of completing the arcs such that no crossing is formed. We give the dictionary for each of the following bijections:

- Φ_1: noncrossing matchings of $\{1, 2, \ldots, 2n\} \leftrightarrow$ Dyck paths of length $2n$;
- Φ_2: noncrossing set partiitons of $\{1, \ldots, n\} \leftrightarrow$ Dyck paths of length $2n$;

- Φ_3: noncrossing permutations of $\{1, \ldots, n\} \leftrightarrow$ Dyck paths of length $2n$;
- Φ_4: noncrossing set partitions (enhanced) of $\{1, \ldots, n\} \leftrightarrow$ Motzkin paths of length n.
- Φ_5: open matching diagrams of $\{1, \ldots, n\}$ without future 2-crossings $\leftrightarrow$ left factors of Dyck paths (those that may end above the x-axis) of length n;
- Φ_6: open partition diagrams of $\{1, \ldots, n\}$ without future 2-crossings $\leftrightarrow$ left factors of Dyck paths of length $2n$;

The dictionaries are found in Table 6.1.

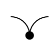
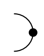
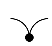
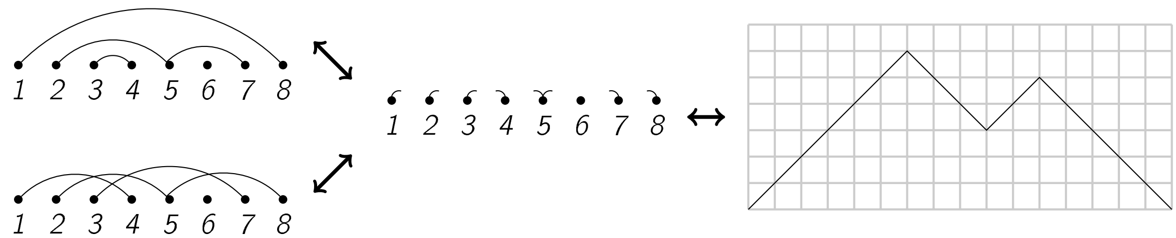
Vertex	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6
	$\nearrow$	$\nearrow\nearrow$		$\nearrow$	$\nearrow$	$\nearrow\nearrow$
	$\searrow$	$\searrow\searrow$		$\searrow$	$\searrow$	$\searrow\searrow$
		$\searrow\nearrow$		$\rightarrow$		$\searrow\nearrow$
		$\nearrow\searrow$				$\nearrow\searrow$
			$\nearrow\nearrow$			
			$\searrow\searrow$			
			$\nearrow\searrow$			
			$\nearrow\searrow$			
			$\searrow\nearrow$			
Sequence:	A000108 $\frac{1}{n+1}\binom{2n}{n}$			A001006 $\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{k+1}\binom{n}{2k}\binom{2k}{k}$	A001405 $\binom{n}{\lfloor n/2 \rfloor}$	A000984 $\binom{2n}{n}$

Table 6.1: Dictionary for bijections Φ_1, Φ_2, Φ_3, Φ_4, Φ_5 and Φ_6.

Note that though straightforward, Φ_3 was first described in [18] by B., Mishna and Post.

Remark that as there is a unique way of connecting an opener-closer sequence such that no crossing is formed, there is *also* a unique way of closing an opener closer-sequence such that no nesting is formed. Similarly for Φ_5 and Φ_6, there is a unique way of closing an opener-closer sequence such that no future 2-crossing (future 2-nestings) is formed. Thus the dictionaries describing bijections Φ_1, Φ_2, Φ_3, Φ_4, Φ_5 and Φ_6 in Table 6.1 also give bijections between these lattice paths and the nonnesting arc diagrams.

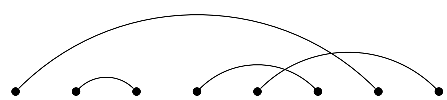
Example 37. *A* 2*-noncrossing (top) and a* 2*-nonnesting (bottom) set partitions are both in bijection with the same Dyck path.*



6.1.2 Harder

Bijections involving k-nonnesting arc diagrams when $k > 2$ are significantly less trivial. In 1989, Gouyou-Beauchamps [39] studied involutions which avoided decreasing subsequences of length 6; those fixed-point free involutions are 3-nonnesting matchings.

Example 38. *Inverting the elements that are joined by an arc in a* 3*-nonnesting matching constructs a fixed point free involution which avoids a decreasing subsequence of length* 6*. The matching* $\mu = 17 - 23 - 46 - 58$*, seen in Figure 6.1 is* 3*-nonnesting, and its corresponding involution avoids decreasing subsequences of length* 6*.*



	1	2	3	4	5	6	7	8
Involution:	7	3	2	6	8	4	1	5

Figure 6.1: 3-nonnesting matchings and its corresponding involution.

In [39], Gouyou-Beauchamps gave a bijection between these involutions and pairs of noncrossing Dyck paths via a recursive construction. Then, in [20], Chen, Deng, Du, Stanley and Yan observed that 3-noncrossing matchings were also in bijection with pairs of noncrossing Dyck paths, giving that 3-noncrossing and 3-nonnesting matchings are in one-to-one correspondence. We describe our own highly visual bijection Φ_7 between pairs of noncrossing Dyck paths and 3-nonnesting matchings. The weighting step in our bijection was inspired by to the weight assigned by Corteel in [26] to weighted bicolored Motzkin paths for a bijection with permutations.

Φ_7: 3-nonnesting matchings $\to$ pairs of noncrossing Dyck paths. To go from a 3-nonnesting matching of $\{1, 2, \ldots, 2n\}$ to a pair of noncrossing Dyck paths of length $2n$:

1. Translate the opener-closer sequence of the μ using the dictionary from Φ_1. This is the upper Dyck path.

2. Consider the vertices in μ from left to right. When a closer is reached, count the number of semi-arcs (those that are *not* completed) to its left, starting with 0.

3. Let x be the number of the semi-arc to which that closer is connected.

4. Label the corresponding step in the Dyck path with x.

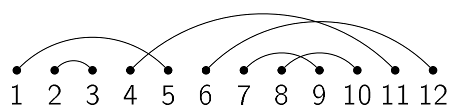
5. Continue labelling the remainder of the Dyck path in this manner, so that all down steps have a label.

6. For each edge labelled x, drop a 'ball' of diameter 1 along the south-east diagonal.

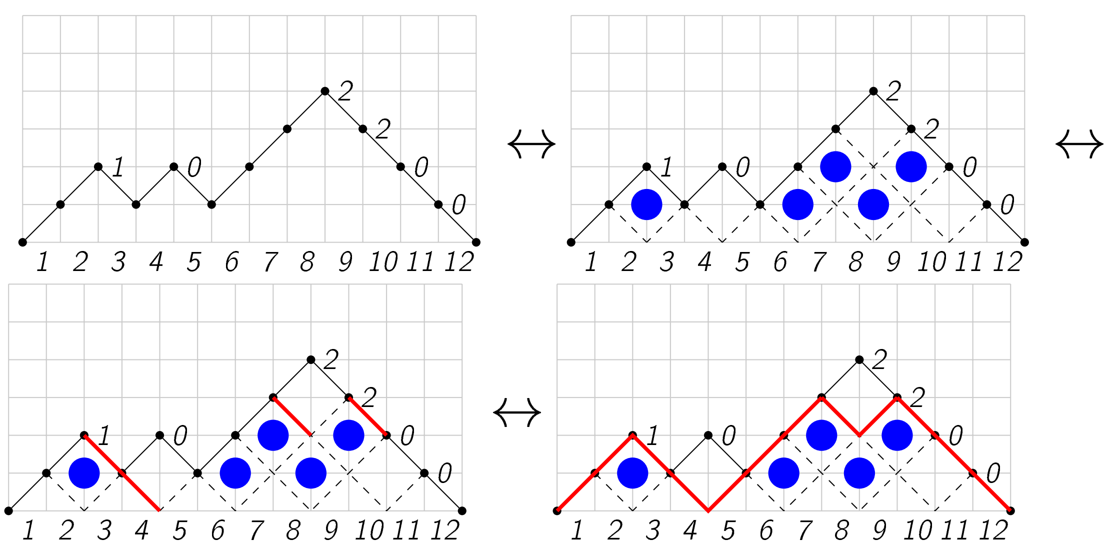
7. Draw edges on top of the upper most ball of each diagonal.

8. Connect these edges in the unique way that forms a Dyck path. This is the lower Dyck path.

Example 39. *Consider the* 3*-nonnesting matchings* $\mu = 15 - 23 - 4, 11 - 6, 12 - 79 - 8, 10$*:*



This corresponds to the following weighted Dyck path:



Φ_7^{-1}: pairs of noncrossing Dyck paths $\rightarrow$ 3-nonnesting matchings. Proceed from left to right.

1. Translate the upper Dyck path to an opener-closer sequence for a matching using the dictionary from Φ_1.

2. Under each peak in the lower Dyck path, draw balls along its south-east diagonal.
3. For each down step in the upper Dyck path, count the number of balls x along its south-east diagonal; label the down step with x.
4. Proceed from left to right in μ. For each closer vertex, consider its corresponding down step in the upper Dyck path, with label x.
5. Starting at vertex 1, count semi-arcs that have not been closed until $x-1$ is reached; close that semi-arc.
6. Repeat until all closers have been considered and no semi-arcs remain. This is the 3-nonnesting matchings μ.

6.2 Bijections using Young diagrams

There are two different instances of Young diagrams arising in bijections with k-nonnesting arc diagrams. The first is in the highly nontrivial bijections between k-nonnesting and k-noncrossing arc diagrams. Pioneered by Chen, Deng, Du, Stanley and Yan in [20], each instance of the bijection goes through a Young tableau. The second instance is an apparent connection between standard Young tableaux of restricted height and open matching diagrams without k-nestings.

6.2.1 Bijections through Young tableaux

All results on equidistribution between k-nonnesting and k-noncrossing arc diagrams were moved dramatically forward with Chen, Deng, Du, Stanley and Yan's 2007 paper, [20] in which they gave a bijection Ψ_1 between k-nonnesting matchings and k-noncrossing matchings using tableaux. They also determined a bijection between k-nonnesting set partitions and k-noncrossing set partitions, both in the regular, Ψ_2, and enhanced, Ψ_3 case. Following their lead, in [18], B., Mishna and Post described a bijection Ψ_4 between k-noncrossing and k-nonnesting permutations. In [23], Chen, Qin and Reidys gave a bijection Ψ_5 between k-nonnesting and k-noncrossing tangled diagrams. Each bijection $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ and Ψ_5 went through an intermediary object: a sequence of *Young diagrams*.

We do not reproduce these bijections here, but point the reader to Chapter 7 for more details where the intermediary object is critical to asymptotic results. In each instance of proving the equidistribution between crossing and nesting statistics with this bijection to a tableau, the authors showed:

k-nonnesting arc diagrams corresponded to tableaux with at most $k-1$ *rows*; k-noncrossing arc diagrams to those with at most $k-1$ *columns*.

This correspondence between crossings and nestings and columns and rows in the tableaux was key to the bijection between k-noncrossing and k-nonnesting arc diagrams. It is also very important to our asymptotic upper bound results in Chapter 7. The bijection between k-nonnesting and k-noncrossing arc diagrams is not limited to completed diagrams. In fact, we can extend bijections $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ and Ψ_5 to incorporate semi-arcs and future k-nestings in each of matchings, set partitions, permutations and tangled diagrams.

We can further make use of sequence of tableaux. Define an *open* tableaux to be λ^0, λ^1, $\ldots$, λ^{2n}, where it is not necessarily the case that $\lambda^{2n} = \emptyset$. First we determine a bijection between k-nonnesting arc diagrams and open tableaux with at most k rows:

1. If vertex i is a semi-arc, consider $\infty + i$ to be its corresponding closer.
2. Proceed as in bijection Ψ_1, Ψ_2, Ψ_3, Ψ_4 or Ψ_5 as appropriate.
3. An arc diagram with m semi-arcs will have $\lambda^{2n} = m$, a row of m cells as its final shape. This determines a bijection between k-nonnesting arc diagrams and open tableaux with at most k rows.

Next, determine a bijection between k-noncrossing arc diagrams and open tableaux with at most k columns.

1. If vertex i is a semi-arc, consider $\infty - i$ to be its corresponding closer.
2. Proceed as in bijection Ψ_1, Ψ_2, Ψ_3, Ψ_4 or Ψ_5 as appropriate.
3. An arc diagram with m semi-arcs will have $\lambda^{2n} = 1^m$, a column of m cells as its final shape. This determines a bijection between k-noncrossing arc diagrams and open tableaux with at most k columns.

Conjugating diagrams so that rows become and columns and columns become rows gives a bijection, Ψ_6, between k-nonnesting and k-noncrossing openarc diagrams.

6.2.2 Standard Young Tableaux

Open matchings with future nesting restrictions have counting sequences that appear in the literature; we saw an example of one such resulting bijection in Φ_5. While other objects also appear, the one that arises repeatedly is the standard Young tableaux (SYT) with height restrictions.

Recall that a Young diagram, or Ferrers diagram, is a finite collection of cells arranged in left-justified rows, with row lengths weakly decreasing. A Young tableau is obtained by filling the cells with elements, and is called a *standard* Young tableau if the entries in each row and column are increasing. The number of rows is referred to as the *height* of the SYT. Table 6.2 gives the start of counting sequences for SYTs with maximum height h, which appears to coincide (up to at least n terms) with open matchings that avoid future k-nestings.

We begin by proving an easy case, the first line of Table 6.2.

n	OEIS	k	h	Start of series
n	A001405	2	2	1, 1, 2, 3, 6, 10, 20, 35, 70, 126, 252, 462, 924, 1716, 3432, 6435, 12870
n	A005817	3	4	1, 1, 2, 4, 10, 25, 70, 196, 588, 1764, 5544, 17424, 56628, 184041, 613470
50	A007579	4	6	1, 2, 4, 10, 26, 76, 231, 756, 2556, 9096, 33231, 126060, 488488, 1948232,
50	A007580	5	8	1, 2, 4, 10, 26, 76, 232, 764, 2619, 9486, 35596, 139392, 562848, 2352064
40	A212916	6	10	1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, 35695, 140140, 568360, 2389192
27	A229068	7	12	1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, 35696, 140152, 568503, 2390466

Table 6.2: Open matchings without future k-nestings appear to have the same counting sequence as SYTs of maximum height h up to n.

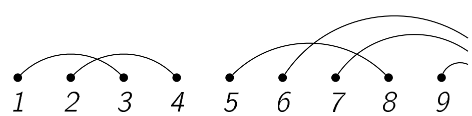
Ψ_7: open matchings without future 2-nestings $\rightarrow$ SYTs with maximum height 2 Proceed from left to right.

1. List the index of each vertex as it is encountered; if it is a closer, place it under its corresponding opener.

2. Since we never encounter a future 2-nesting, drawing cells around the numbers constructs a SYT of maximum height 2.

This mapping is easy to reverse, giving the bijection Ψ_7 between open matchings of $\{1, \ldots, n\}$ and SYTs of n entries with maximum height 2.

Example 40. *Consider the following open matching without future* 2*-nestings:*



We list our vertices and their closer sequence, drawing cells around them to get our SYT of maximum height 2*:*

1	2	5	6	7	9
3	4	8			

In [39], Gouyou-Beauchamps gave a bijection between SYTs with maximum height 4 and pairs of noncrossing left Dyck factors. Combining this result with bijection Φ_5 gives line 2 of Table 6.2, Ψ_8: the set of open matching diagrams on $\{1, \ldots, n\}$ with no future 3-nestings is in bijection with SYTs on n entries with maximum height 4.

For general $k > 3$, we make the following conjecture based on our numerical evidence:

Conjecture 6.2.1. *The set of open matching diagrams on $\{1, \ldots, n\}$ without future k-nestings is in bijection with standard Young tableaux with n entries and maximum height $2k - 2$.*

Observations

- An SYT with 1 row corresponds to the diagram with no completed arcs.
- The number of SYTs with exactly 2 rows corresponds to the number of diagrams which have completed arcs and no future 2-nestings.
- The number of SYTs with exactly 3 rows corresponds to the number of diagrams which have a future 2-nesting.
- The number of SYTs with exactly 4 rows corresponds to the number of diagrams which have a (regular) 2-nesting.

Given that SYTs with maximum height have enumerative formulas where determinants of hyperbolic Bessel functions appear [10], as do the number of 3-nonnesting matchings (regular case) [20], we are confident a generating function proof can be found.

We now discuss a surprising counting sequence which arises when we enumerate open partition diagrams which avoid enhanced future 3-nestings: Baxter numbers.

6.3 A conjecture on Baxter permutations

Baxter numbers appear in surprisingly diverse combinatorial contexts. Define B_n to be the number of Baxter permutations of $\{1, \ldots, n\}$, those permutation $\sigma \in \mathcal{S}_n$, such that there are no indices $i < j < k$ such that $\sigma(j+1) < \sigma(i) < \sigma(k) < \sigma(j)$ or $\sigma(j) < \sigma(k) < \sigma(i) < \sigma(j+1)$

Example 41. *The permutation* 3**5**2**84**1**7**69 *is* **not** *a Baxter permutation. The permutation* 643578912 *is a Baxter permutation.*

The first few terms in the counting sequence for Baxter numbers are:

$$0, 1, 2, 6, 22, 92, 422, 2074, 10754, 58202, 326240, 1882960, 11140560, 67329992, 414499438\ldots$$

(sequence A001181 in the OEIS [41]) and satisfy [25]:

$$B_n = \frac{2}{n(n+1)^2} \sum_{r=0}^{n} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2}.$$

Our study of open diagrams led to the following conjecture:

Conjecture 6.3.1. *The number of open partition diagrams on n vertices with neither regular nor future enhanced* 3*-nestings is B_{n+1}, the number of Baxter permutations of size $n+1$.*

From evaluations of the functional equation for open partitions which avoid future enhanced 3-nestings at $u = v = 1$, we know this is true up to $n = 300$. Baxter numbers are named for Glen Baxter, who in 1964 [9] introduced the class of permutations now known as Baxter permutations. However, B_n also counts many other combinatorial objects. In [37], Fusy give a nice summary of the known combinatorial classes that are counted by B_n; these include monotone 2-line meanders [15], plane bipolar orientations [11], triples of nonintersecting paths [36].

Example 42. *A monotone* 2*-line meander of size* n *is a pair of (self-avoiding) monotone lines which intersect each* n *times. A plane bipolar orientation of size* n *is an acyclic orientation of a planar map (a connected graph embedded in the plane with no edges crossing) with a unique source* s *and a unique sink* t *and* n *edges. The triples of nonintersecting lattice paths of length* n *have step set* $\{(1,0),(0,1)\}$*, are on the grid* $\mathbb{Z}^2$*, have origins* $(-1,1)$*,* $(0,0)$ *and* $(1,-1)$ *and respective endpoints* $(i-1,j+1)$*,* (i,j) *and* $(i+1,j-1)$ *where* $i+j=n$*.*

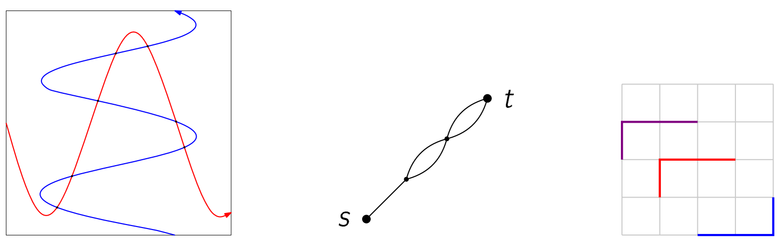


Figure 6.2: 2-line meander ($n=7$), bipolar orientation ($n=5$) and triple of paths ($n=3$).

However, as Fusy notes, all of these classes have antipodal symmetry, which our class does not. This complicates the search for a bijection and suggests a fundamentally different class. There are at least two known generating trees for Baxter objects that appear in the literature, in each case the labels have two components.

Theorem 6.3.2 ([11], Bonichon, Bousquet-Mélou, Fusy 2010)**.** *The generating tree for Baxter permutations with* i *left-right maxima, and* j *right-left maxima has root label* $(1,1)$ *and succession rule:*

$$[i,j] \rightsquigarrow \begin{cases} [1,j+1],[2,j+1],\ldots,[i,j+1), \\ [i+1,j],\ldots,[i+1,2],[i+1,1]. \end{cases}$$

Theorem 6.3.3 ([14], Bouvel, Guibert 2014+)**.** *The following succession rule, with root label* $(0,1)$ *describes a generating tree for Baxter permutations:*

$$[i,j] \rightsquigarrow \begin{cases} [0,j],[1,j],\ldots,[i-1,j], \\ [i,j+1],[i+1,j],[i+2,j-1],\ldots,[i+j-1,2]. \end{cases}$$

Compare with our tree for $\widetilde{\Pi}^{(2)}$,

$$
[i,j] \rightsquigarrow
\begin{array}{llll}
[i,i], & & (1) & (6.1)\\
[i+1,j], & & (2) & (6.2)\\
[i,j],[i,j+1],\ldots,[i,i-1] & \text{if } i>0 & (3) & (6.3)\\
[i-1,j],[i-1,j+1],\ldots,[i-1,i-1] & \text{if } i>0 & (4) & (6.4)\\
[i,j-1],[i-1,j-1] & \text{if } i>0 \;\&\; j>0 & (5) & (6.5)\\
 & & & (6.6)
\end{array}
$$

differs from each of these trees already at the third level which contains 6 elements. Appendix B illustrates the different generating trees.

The following bijection, due to Elizalde and Rubey both extends Φ_7, and offers hope that a correspondence may yet be found.

Theorem 6.3.4 ([34] Elizalde, Rubey, 2014)**.** *There exists a bijection χ_1 between open partition diagrams of size n with no enhanced future 3-nestings and decorated pairs of noncrossing lattice paths with step set $\{(1,0),(0,1)\}$ that start at $(0,0)$ and end at the same position, stay above the main staircase diagonal, where n is the length of the upper path minus the number of decorated corners.*

Proof. We show that the objects are generated by the same generating tree as equation 6.3. To each pair of paths p, associate the label $\ell(p) = [i,j]$ if the endpoint of the paths is i above the main staircase diagonal, and the last east $(1,0)$ step in the lower path is j above the main staircase diagonal. See Figure 6.3 for an example. The number of pairs of lattice paths of size n is the

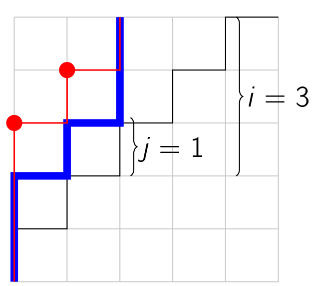
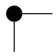
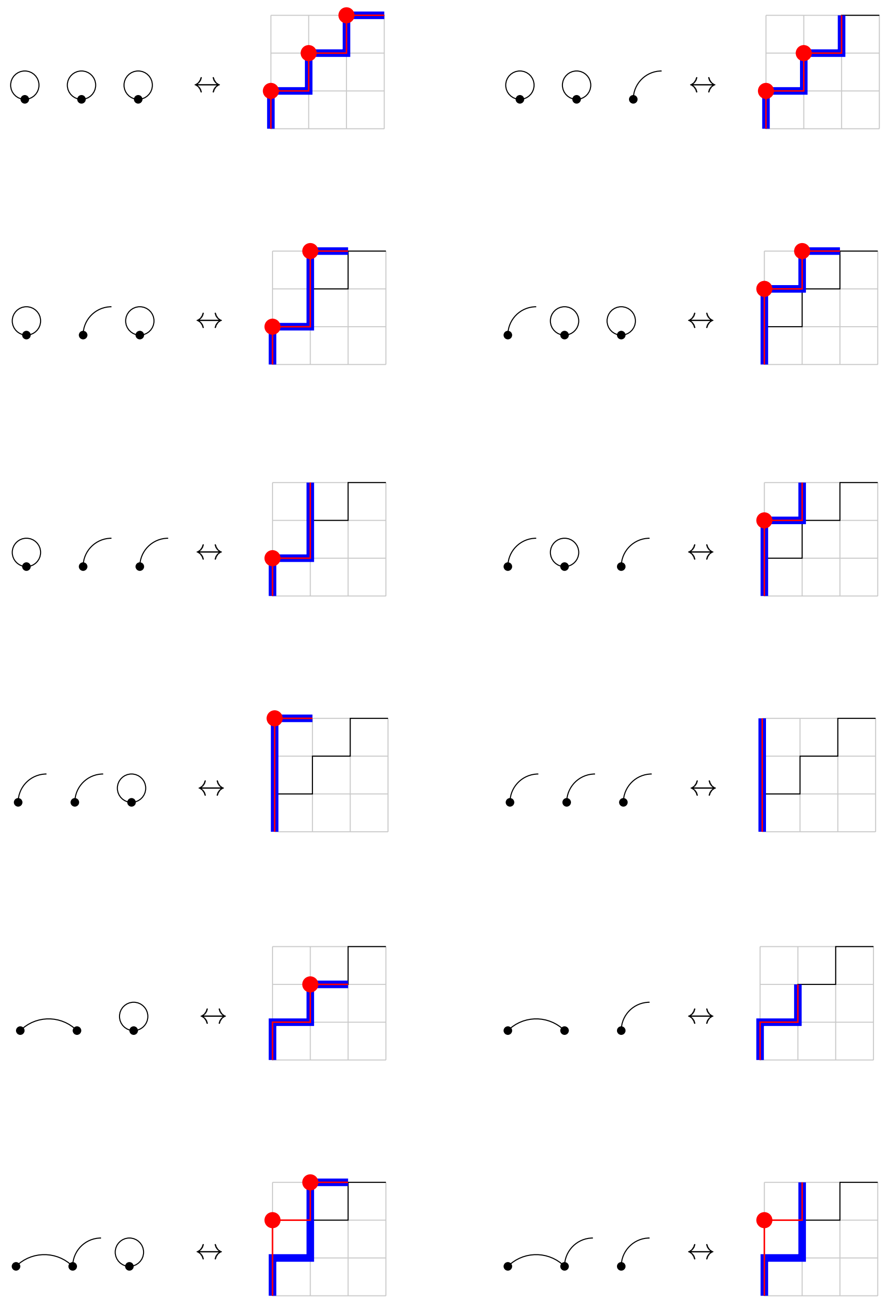
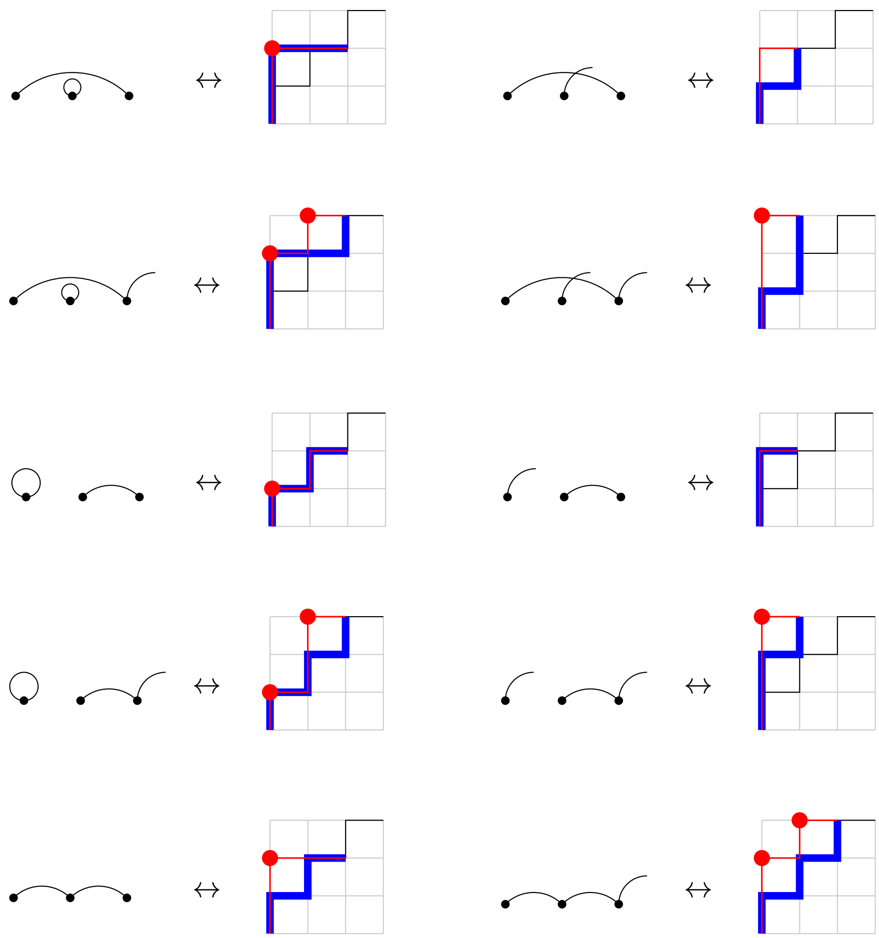


Figure 6.3: A pair of noncrossing lattice paths with label $\ell(p) = [3,1]$.

number of nodes at level n in the generating tree with root label $[0,0]$ for $n = 0$ and succession rule: Then, (1) corresponds the the addition of , which can always be added to a legal pair of paths and does not change either paths distance from the main staircase diagonal. Similarly, (2) corresponds to the addition of $\|$, which increases the distance between the endpoint of the path and the main diagonal staircase. The addition of (3) and $[i,j-1]$ from (5) corresponds to the addition of  as the upper path, and possible heights of the lower path: $j-1$ if it simply remains the same height (staircase increments), $j, j+1, \ldots i$ if it gains height first. Finally, the addition of (4) and $[i-1, j-1]$ from (5) corresponds to the additions of ___ for the upper path, and allowable heights of the lower path. This generating tree is equivalent to the generating tree for open set partitions with no enhanced 3-nestings. $\square$

We illustrate the bijection χ_1 for $n = 3$; there are 22 elements in each case.



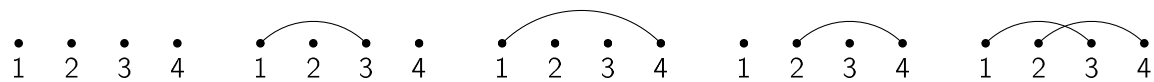


If our conjecture is true, it would be interesting to know which subsets of each of the above Baxter objects correspond to (complete) set partitions with no enhanced 3-nesting. Furthermore, perhaps there is a generalization of Baxter objects which correspond to open partition diagrams without enhanced k-nestings, for $k > 3$.

6.4 A broader picture: growth diagrams

For some, interest in arc diagrams is due to a connection to biology: the quest to understand ribonucleic acid (RNA) and how it folds has lead scientists, and specifically bioinformaticians, to its study. This is because arc diagrams can be used to model the single stranded macromolecule: vertices represent nucleotides, and arcs are used to model the hydrogen bonds between them, and such bonds are unlikely to cross. This area of study is beyond the scope of this work, but we mention it for completeness and to highlight another representation of combinatorial classes which may be fruitful for future work: *growth diagrams*.

In RNA, a hydrogen bond is unlikely to form between nucleotides that are too close to each other. In an arc diagram representation of a combinatorial class, this translates to a minimum arc length requirement. One observation is immediately apparent: equidistribution in crossing and nesting statistics is destroyed. Even with matchings on just 4 vertices this is apparent when we draw all diagrams with no 1-arcs:



;

Notice that all of the above diagrams are nonnesting, but they are not all noncrossing. Because of this, enumerating k-non**crossing** objects with minimum arc length m has the added benefit of biophysical application. When $m = 2$, the answer is straightforward.

Theorem 6.4.1 (Folklore)**.** *2-noncrossing set partitions on* $\{1, \ldots, n\}$ *with minimum arc length 2 are equinumerous with Motzkin paths of length* $n - 1$.

Beyond this, much remains open. Strategies for considering this problem for larger k include a sieve and relating our k-noncrossing objects with other enumerable structures. On the other hand, significantly more can be said for the combinatorial case of k-non**nesting** arc diagrams with a minimum arc length.

We recall a bijection between k-noncrossing diagrams, and fillings of Ferrers diagrams, described in detail by Krattenthaler in [45].

Growth diagrams We can represent an integer partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with a *Ferrers diagram*, which is a left-justified arrangement of squares with λ_i squares in the i^{th} row, $i = 1, 2, \ldots$. Here we follow the conventions of [45] and use the French notation for growth diagrams, aligning in the bottom left corner. We fill the squares of a Ferrers diagram F with non-negative integers, and the corners of the cells are labelled with partitions that satisfy the following:

1. A partition is either equal to its right neighbour or smaller by exactly one square, this is also true for its top neighbour.

2. A partition and its right neighbour are equal if and only if in the column cells of F below them there appears no 1, and if their bottom neighbours are also equal to each other. Similarly, a partition and its top neighbour are equal if and only if in the row of cells of F to the left of them there appears no 1 and if their neighbours are also equal to each other.

A diagram that obeys these conditions is called a *growth diagram*.

Example 43. *We only consider 0-1 fillings of Ferrers diagrams, and suppress the 0s and represent the 1s with X's for clarity.*

We restrict ourselves to growth diagrams that obey the following local rules when consider squares with and without crosses:

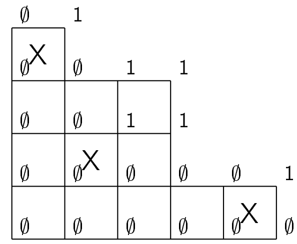
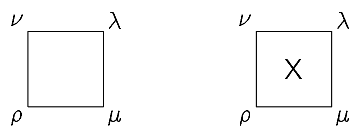


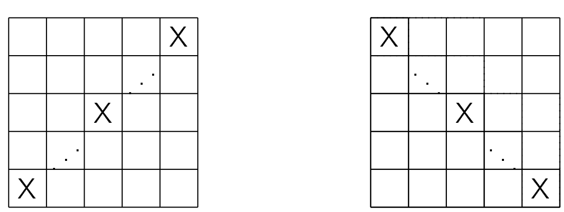
Figure 6.4: A Ferrers diagrams.



1. If $\rho = \mu = \nu$, and if there is no cross in the square then $\lambda = \rho$.
2. If $\rho = \mu \neq \nu$, then $\lambda = \nu$.
3. If $\rho = \nu \neq \mu$, then $\lambda = \mu$.
4. If ρ, μ, ν are pairwise different, then $\lambda = \mu \cup \nu$.
5. If $\rho \neq \mu = \nu$, then λ is formed by adding a square to the $(k+1)$st row of $\mu = \nu$, given that $\mu = \nu$ and ρ differ in the kth row.
6. If $\rho = \mu = \nu$, and there is cross in the square, then λ is formed by adding a square to the first row of $\rho = \mu = \nu$.

In [45], Krattenthaler used these growth diagrams to represent set partitions, and proved equidistribution between crossing and nesting statistics. This representation helps us to enumerate k-nonnesting set partitions with a minimum arc length restriction.

A *NE-chain* of a $0-1$-filling is a sequence of 1's in the filling such that any 1 in the sequence is above and to the right of the preceding 1 in the sequence. A *SE-chain* of a $0-1$-filling is a set of 1's in the filling such that any 1 in the sequence is below and to the right of the preceding 1 in the sequence.



A NE-chain A SE-chain

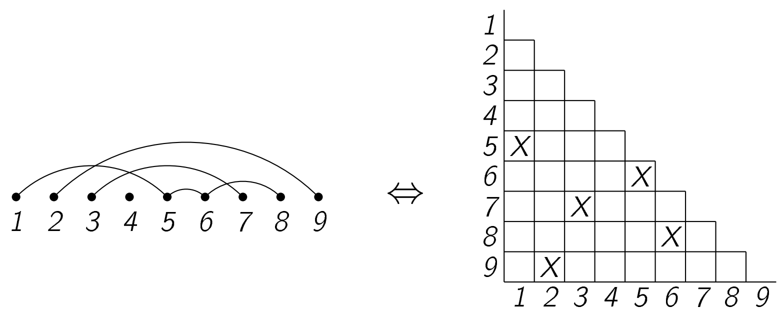
Theorem 6.4.2 ([45], Krattenthaler, 2006)**.** *Let $N(F; n; NE = s, SE = t)$ denote the number of 0-1-fillings of the Ferrers diagrams F with exactly n 1's, such that there is at most one 1 in each column and in each row, and such that the longest NE-chain has length s and the longest*

SE-chain (the smallest rectangle containing the chain being contained in F, has length t. Then, for any Ferrers shape F and positive integers s and t,

$$N(F; n; NE = s, SE = t) = N(F; n; NE = t, SE = s).$$

This result on growth diagrams automatically gives equidistribution between crossing and nesting statistics when we represent set partitions with growth diagrams. Let $\{(i_1, j_1), \ldots, (i_m, j_m)\}$ be the set of arcs in the arcs in the arc diagram representation of a set partitions where (i_r, j_r) indicates an arc is drawn between i_r and j_r. Then in the triangular shaped diagram Δ_n with $n-1$ cells in the bottom row place an X in the cell in the i_r^{th} column and j_r^{th} row from above.

Example 44. *Consider the set partition* $1568 - 29 - 37$ *as depicted in both its arc and growth diagrams:*



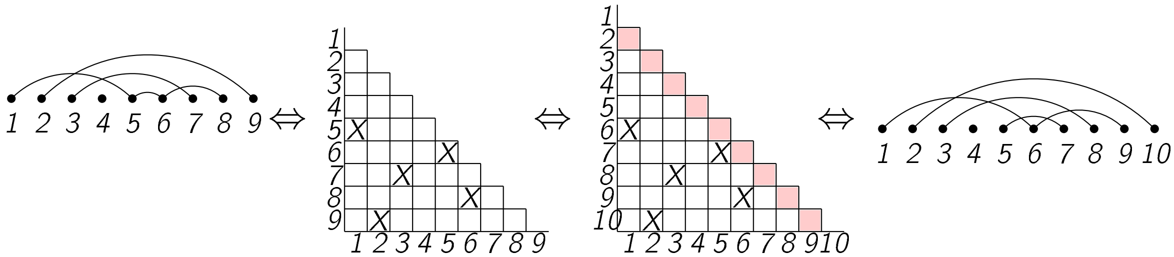
The key is: *a k-crossing corresponds to a SE-chain; a k-nesting corresponds to a NE chain.* Thus, the equidistribution between crossing and nesting statistics in set partitions is simply a specialization of the above theorem on NE-chains and SE-chains in growth diagrams.

We now return to the problem of considering arc diagrams with minimum arc length restrictions. Consider a 1-arc, i.e. that of the form $(i, i+1)$. An X is placed in the i^{th} column and the $i+1^{th}$ row from above; this cell must occur in the diagonal edge of the triangular shape. Because a k-nesting corresponds to a NE-chain of length k, k-nonnesting set partitions with no 1-arcs amounts to considering growth diagrams with NE-chains of length k with no X's in the diagonal cells.

Theorem 6.4.3. *The set of k-nonnesting set-partitions on $n+1$-vertices with no 1-arcs are in bijection with the set of k-nonnesting set partitions on n vertices, with no minimum arc length requirements.*

Proof. Consider an arbitrary k-nonnesting set partition on n-vertices with no minimum arc-length requirements and its triangular growth diagram Δ_k. Add a new blank strip of cells to the diagonal side of Δ_k. This new strip cannot change nesting number, as k-nestings correspond to NE-chains, and a blank box to the north and east may not increase such a chain. The addition of this diagonal strip produces a growth diagram with n cells in the bottom row, $n-1$ in the row above, etc., and thus corresponds to a set partitions on $n+1$ vertices. For reasons stated above, since no cells along the diagonal have an X, the corresponding set partition does not have any 1-arcs. $\square$

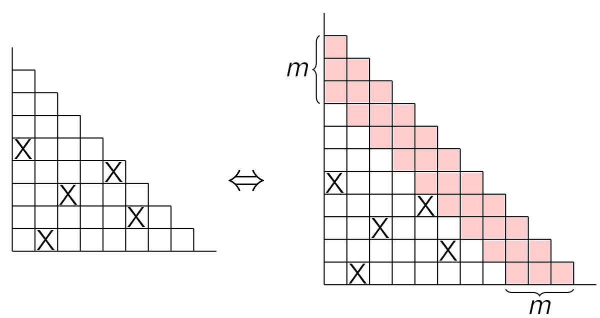
Example 45. *Consider the following* 4*-nonnesting set partition on* $\{1, 2, \ldots, 9\}$ *with 1-arcs and the corresponding* 4*-nonnesting set partition on* $\{1, 2, \ldots, 10\}$ *with no 1-arcs:*



We are not restricted to the case of k-nonnesting set partitions avoiding 1-arcs:

Theorem 6.4.4. *The set of k-nonnesting set partitions on $n+m$ vertices with minimum arc length $m+1$ (no 1-arcs, no 2-arcs, ..., no m-arcs) are in bijection with the set of k-nonnesting set partitions on n vertices with no minimum arc length requirements.*

Proof. Consider an arbitrary k-nonnesting set partitions on n-vertices with no minimum arc length requirements and its triangular growth diagram Δ_k. We show that it is in bijection with a k-nonnesting set partition on m vertices with minimum arc length m. To Δ_k, add m blank strips of cells along the diagonal. Since the addition of blank cells along the diagonal may not increase the number of NE-chains, the set-partition corresponding to this new growth diagram is still k-nonnesting. It also has $n-1+m$ cells in the bottom row, $n-2+m$ cells in the row above, etc. and 1 cell in the top most row. Thus, this new growth diagram corresponds to a set partition on $n+m$ vertices that is k-nonnesting. Any arc of length $\leq m$ corresponds to an X in one of the newly added m blank strips along the diagonal. Since these diagonal strips are blank, the set partition has minimum arc-length $m+1$.



$\square$

Thus, since we can enumerate k-nonnesting set-partitions using the functional equations gained from our generating tree scheme, we can also enumerate k-nonnesting set-partitions *with minimum arc-length restrictions* using this bijection.

Chapter 7

Asymptotic enumeration of k-nonnesting arc diagrams

Analysis of the enumeration data generated in Part II led us to a conjecture about the exponential growth factor of k-nonnesting arc diagram families. Specifically, we derive an upper bound for the exponential growth factor of the number of k-nonnesting arc diagrams, providing a unifying combinatorial description which was largely missing in the literature. One of our main motivations was to estimate the extent of the over-generating inherent in using open diagrams. We conjecture that the exponential growth factor for k-nonnesting diagrams is the same as that for open version, suggesting only a polynomial over-generation. At the heart of these results are the bijections between k-nonnesting arc diagrams and certain types of tableaux.

7.1 Asymptotics in the literature

Asymptotic results exist for some combinatorial classes that are represented using arc diagrams with nesting restrictions. Such results have come from analysis of the generating functions, and have been largely dependent on the known enumerative formulas for k-nonnesting matchings. We summarize the known exponential and subexponential growth factors in Table 7.1. The number of such k-nonnesting arc diagrams have asymptotic form $cn^{\alpha}r^{n}$, with r the exponential growth factor, α the subexponential growth factor and c a constant.

While asymptotic results are indeed known, a combinatorial interpretation is absent in the literature. An exception to this is found in [47] (Prop. 5.8), where Marberg states:

Proposition 7.1.1 ([47] Marberg 2013)**.** *The exponential generating function of set partitions without enhanced k-nestings is the* derivative *of the exponential generating function of set k-nonnesting set partitions.*

This was proved via a bijection between k-nonnesting set partitions on $\{1, \ldots, n+1\}$ and set partitions with no enhanced k-nestings on $\{1, \ldots, n\}$.

Combinatorial class			r	α	Reference	Tableaux type
		k				
1. $c_2(n)$	set partitions	2	4	3/2	Folklore	oscillating
2. $c_3(n)$	set partitions	3	9	7	[13] Prop.1	vacillating
3. $e_3(n)$	set partitions (e)	3	8	7	[13] Prop. 1.	hesitating
4. $f_k(n)$	matchings	k	$2(k-1)$	$(k-1)^2+\frac{k-1}{2}$	[42] Thm. 2	oscillating
5. $t_k(n)$	tangled diagrams	k	$4(k-1)^2+2(k-1)+1$	$(k-1)^2+\frac{k-1}{2}$	[24] Thm. 3.2	vacillating, oscillating

Table 7.1: Summary of known exponential (r) and subexponential (α) growth factors for k-nonnesting arc diagrams, (e) refers to enhanced nestings.

Now we give a unifying combinatorial view of the exponential growth factor for all arc diagrams restricted by crossings and/or nestings. We prove an upper bound on the exponential growth factor for k-nonnesting arc diagrams. This bound is achieved for all k-nonnesting arc diagram families which have a known asymptotic form, and is predicted by experimental data for k-nonnesting arc diagrams without solved enumerative forms. It also holds in the case of *doubly restricted* arc diagrams: those that are both k-nonnesting and ℓ-noncrossing.

In [20], the authors showed that crossing and nesting statistics in set partitions are equidistributed using a bijection that went through an intermediary object: the *vacillating tableaux*. This bijection was brought up in Chapter 6 and in Section 7.2 we describe it. Similar bijections with other combinatorial classes also use tableaux to prove crossing and nesting statistics. Properties of these tableaux allow us to determine our upper bound.

7.2 Bijections with Young tableaux

Proving equidistribution between crossing and nesting statistics in arc diagram representations of combinatorial classes requires highly nontrivial bijections which go through a sequence of tableaux as an intermediary object. Pioneered by Chen, Deng, Du, Stanley and Yan [20] for set partitions and matchings, this strategy was extended to permutations by Burrill, Mishna and Post [18], and tangled diagrams by Chen, Qin and Reidys [23]. In each instance, a sequence of *Young diagrams* was used.

An *integer partition* of $n \in \mathbb{N}$ is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{N}^k$ such that $|\lambda| := \sum_{i=1}^{k} \lambda_i = n$. We can represent λ with a left-justified array of boxes or cells with λ_i boxes in row i, called its *Young diagram*. "Adding a box" to a partition λ means getting a partition μ whose Young diagram is obtained by adding a single box to λ: $|\lambda|+1=\mu$ ("deleting a box" is defined similarly). This inclusion gives a partial order on the set of nonnegative integers, called the Young lattice, seen in Figure 7.1.

The *tableau* $T = (\lambda^0 = \emptyset, \lambda^1, \lambda^2, \dots, \lambda^n)$ is built from walks on the Young lattice that satisfy

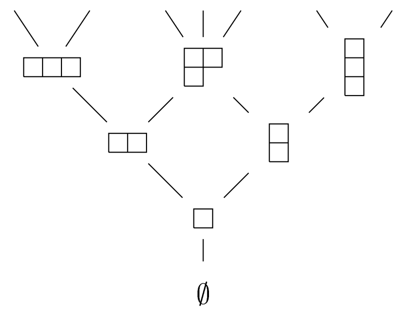


Figure 7.1: The start of the Young lattice.

further restrictions corresponding to the combinatorial class. It is always the case that λ^i is obtained from λ^{i-1} (for $i \in [n]$) by either adding a box, deleting a box, or doing nothing, and $\lambda^0 = \lambda^n = \emptyset$. The further restrictions imposed depend on the type of tableaux in question. In particular:

Matchings	$\leftrightarrow$	*oscillating tableaux*:	those that have distinct neighbouring λ^i's	[20]
Set partitions	$\leftrightarrow$	*vacillating tableaux*:	steps in tableaux come in pairs: $(\emptyset, \emptyset)$, $(\emptyset, +\square)$, $(-\square, \emptyset)$, or $(-\square, +\square)$.	[20]
Set partitions (e)	$\leftrightarrow$	*hesitating tableaux*:	steps in tableaux come in pairs: $(\emptyset, +\square)$, $(-\square, \emptyset)$ or $(+\square, -\square)$.	[20]
Permutations	$\leftrightarrow$	pairs of tableaux	one vacillating and one hesitating	[18]
Tangled diagrams	$\leftrightarrow$	vacillating tableaux	(of length $2n$)	[23]

Example 46. *The following sequence is a vacillating tableau of length* $2n = 16$*:*

$$(\emptyset, \emptyset, \square, \square, \square, \square, \substack{\square\\\square}, \square, \square\square, \square, \square\square, \square, \substack{\square\\\square}, \square, \square, \emptyset, \emptyset).$$

This is an example of an oscillating tableau of length $2n = 8$*:*

$$(\emptyset, \square, \substack{\square\\\square}, \square, \substack{\square\\\square}, \substack{\square\square\\\square}, \square\square, \square, \emptyset)$$

The bijections in [20] are both nontrivial and quite involved. Our aim is not to reproduce them, but instead give a sense of the general procedure and reminders for a familiar reader. Note that in [20] the authors work from right to left; we will follow Marberg's description [47] instead and proceed *left to right*.

In all instance of bijections between our arc diagram representations of combinatorial classes and their corresponding tableaux, the following rules are adhered to:

1. Read the vertices from left to right. A vertex corresponds to a fixed number of steps in a tableau.

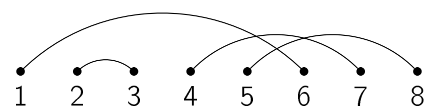
2. If a vertex is an opener, insert its corresponding closer into the tableau.

3. If a vertex is a closer, delete it from the tableau.

4. Other types of vertices correspond to a combination of the above steps, combined with a 'do nothing' step.

In the case of matchings, 1, 2 and 3 are followed exactly.

Example 47. *Consider the matching* $\mu = \{\{1,6\},\{2,3\}\{4,7\},\{5,8\}\}$

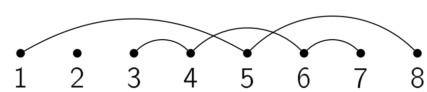


The sequence of closers, inserted when the opener is read, and deleted when the closer is read, from left to right is $(\emptyset, 6, 63, 6, 67, 678, 78, 8, \emptyset)$. *This corresponds to the following oscillating tableaux*

$$(\emptyset, \square, \begin{smallmatrix}\square\\\square\end{smallmatrix}, \square, \square\square, \square\square\square, \square\square, \square, \emptyset).$$

For set partitions (regular crossings and nestings), insertion and deletion of cells depends on the parity of the step.

Example 48. *Consider the set partition* $\pi = 1\,5\,8 - 2 - 3\,4\,6\,7$



Here, each vertex corresponds to two steps in the vacillating tableaux. The closer sequence is $(\emptyset, \emptyset, 5, 5, 5, 5, 54, 5, 56, 6, 68, 8, 87, 8, 8, \emptyset, \emptyset)$. *Its corresponding vacillating tableaux is*

$$(\emptyset, \emptyset, \square, \square, \square, \square, \begin{smallmatrix}\square\\\square\end{smallmatrix}, \square, \square\square, \square, \square\square, \square, \begin{smallmatrix}\square\\\square\end{smallmatrix}, \square, \square, \emptyset, \emptyset).$$

In each bijection between tableaux and arc diagrams, the following principle is key to showing equidistribution.

Principle 1. *A k-nonnesting arc diagram representation of a combinatorial class corresponds to a tableaux with at most $k-1$* rows*; a k-noncrossing corresponds to those with at most $k-1$ columns.*

Indeed, conjugating the tableaux of a k-nonnesting combinatorial object returns the tableaux of a k-noncrossing object.

Example 49. *The matching μ in Example 47 is* 3*-nonnesting and* 4*-noncrossing. The oscillating tableau in bijection with μ has at most* 2 *rows and* 3 *columns. Similarly, the set partition π in Example 48 is* 3*-nonnesting and* 3*-noncrossing; the vacillating tableau in bijection with π has at most* 2 *rows and* 2 *columns.*

In considering the case of 3-nonnesting matchings and their corresponding oscillating tableaux, which, by Principle 1, have at most 2 rows, we made the following observation:

Observation 1. *In a* 3*-nonnesting matching, the transition between Young diagrams* λ^i *in the oscillating tableaux may be one of only four options:*

1. *a cell is inserted into row 1;*
2. *a cell is inserted into row 2;*
3. *a cell is deleted from row 1; or*
4. *a cell is deleted from row 2.*

No cell may be inserted into or deleted from any other row.

This observation only addresses allowable steps in the oscillating tableaux. It omits finer conditions, e.g. all neighbouring λ^i are distinct, and $\lambda^{2n} = \emptyset$; however, it is enough to determine an upper bound for the exponential growth factor.

Proposition 7.2.1. *Let* $f_3(n)$ *denote the number of* 3*-nonnesting matchings, then*

$$\lim_{n\to\infty} f_3(n)^{\frac{1}{n}} \leq 4.$$

Proof. There are exactly 4 transitions that the oscillating tableaux corresponding to a 3-nonnesting matching may take, listed in Observation 1. Thus, $f_3(n) \leq 4^n \Rightarrow \lim_{n\to\infty} f_3(n)^{\frac{1}{n}} \leq 4$. □

In fact, this limit is exists: $f_3(n)f_3(m) \leq f_3(n+m)$. We can consider all 3-nonnesting matchings of length $n+m$ which have two 'factors' (a vertical line drawn through the arc diagram does not intersect an arc) of length n and m. There are fewer of these than the set of all matchings of length $n+m$.

Notice line **4** of Table 7.1 gives the exponential growth factor for k-nonnesting matchings as $g = 2(k-1)$ [42]. Substituting $k = 3$ returns 4 as the exponential growth factor for 3-nonnesting arc diagrams; in this case the upper bound is achieved.

7.2.1 An upper bound on exponential growth factors

We generalize Proposition 7.2.1: tracking the allowable transitions in a k-nonnesting arc diagram's corresponding tableaux gives an upper bound on the exponential growth constant.

Theorem 7.2.2. *Let* $a_k(n)$ *denote the number of* k*-nonnesting arc diagrams of size* n *of a combinatorial class. If a corresponding tableau family exists and admits* m *different transitions,*

$$\lim_{n\to\infty} a_k(n)^{\frac{1}{n}} \leq m.$$

That is, the exponential growth factor of $a_k(n)$ *is at most* m*.*

Proof. If there are m possible transitions, thus $a_k(n) \leq m^n \Rightarrow \lim_{n\to\infty} a_k(n)^{\frac{1}{n}} \leq m$. This limit exists giving an exponential growth factor of at most m since $a_k(m)a_k(n) \leq a_k(m+n)$: the number of k-nonnesting arc diagrams of length $m+n$ with two factors, one of length m and one of length n is less than the total number of k-nonnesting diagrams of size $m+n$. □

We next show that this upper bound is achieved for each result in Table 7.1. In Table 7.2 we give all of the allowable transitions for the corresponding tableaux; summing them returns the upper bounds on the exponential growth factors. In each instance of known asymptotic results, the bound is achieved.

Combinatorial class		**Vertex type**	**Transitions**	**# Rows**	**Total**
1. $c_2(n)$:	2-nonnesting set partitions	fixed point	$\emptyset, \emptyset$	1	
		opener	$\emptyset, +\square$	1	
		closer	$-\square, \emptyset$	1	
		transitory	$-\square, +\square$	1	**4**
2. $c_3(n)$:	3-nonnesting set partitions	fixed point	$\emptyset, \emptyset$	1	
		opener	$\emptyset, +\square$	2	
		closer	$-\square, \emptyset$	2	
		transitory	$-\square, +\square$	4	**9**
3. $e_3(n)$:	3-nonnesting set partitions (enhanced)	fixed point	$+\square, -\square$	1	
		opener	$\emptyset, +\square$	2	
		closer	$-\square, \emptyset$	2	
		transitory	$+\square, -\square$	3	**8**
4. $f_k(n)$:	k-nonnesting matchings	opener	$+\square$	$k-1$	
		closer	$-\square$	$k-1$	$\mathbf{2(k-1)}$
5. $t_k(n)$:	k-nonnesting tangled diagrams	fixed point	$\emptyset, \emptyset$	1	
		opener	$\emptyset, +\square$	$k-1$	
		double opener	$+\square, +\square$	$(k-1)^2$	
		closer	$-\square, \emptyset$	$k-1$	
		double closer	$-\square, -\square$	$(k-1)^2$	
		crossing transitory	$+\square, -\square$	$(k-1)^2$	$\mathbf{4(k-1)^2+}$
		transitory	$-\square, +\square$	$(k-1)^2$	$\mathbf{2(k-1)+1}$

Table 7.2: Allowable tableau transitions for k-nonnesting arc diagrams.

We use Theorem 7.2.2 to determine upper bounds on the exponential growth factor for the combinatorial classes which have been our main focus.

Corollary 7.2.3. *Let $c_k(n)$ denote the number of k-nonnesting set partitions of $\{1, \ldots, n\}$. Then*

$$\lim_{n\to\infty} c_k(n)^{\frac{1}{n}} \leq k^2$$

That is, the exponential growth factor of $c_k(n)$ is at most k^2.

Proof. The corresponding vacillating tableaux of a k-nonnesting set partition has at most $k-1$

rows. Each step in a vacillating tableaux is either **1)** doing nothing twice (1 possible way), **2)** doing nothing, and then inserting a cell ($k-1$ possible ways), **3)** deleting a cell and then doing nothing ($k-1$ possible ways) or **4)** deleting a cell and then inserting a cell ($(k-1)^2$ possible ways). Summing up, we get $1+(k-1)+(k-1)+(k-1)^2 = k^2$ possible steps in the vacillating tableaux. □

Corollary 7.2.4. *Let $e_k(n)$ denote the number of set partitions of $\{1,\ldots,n\}$ without enhanced k-nestings. Then*

$$\lim_{n\to\infty} e_k(n)^{\frac{1}{n}} \leq k^2-1$$

That is, the exponential growth factor of $e_k(n)$ is at most k^2-1.

Proof. The corresponding hesitating tableaux of a set partition with no enhanced k-nestings has at most $k-1$ rows. Each steps is either **1)** inserting a cell, and then deleting a cell ($(k-1)^2$ possibilities), **2)** doing nothing and then inserting a cell ($k-1$ possible ways) or **3)** delete a cell and then do nothing ($k-1$ possibilities). Summing we get k^2-1 possible steps in the hesitating tableaux. □

Corollary 7.2.5. *Let $p_k(n)$ denote the number of k-nonnesting permutations of $\{1,\ldots,n\}$. Then*

$$\lim_{n\to\infty} p_k(n)^{\frac{1}{n}} \leq 4(k-1)^2$$

That is, the exponential growth factor of $p_k(n)$ is at most $4(k-1)^2$.

Proof. A permutation corresponds to a pair of tableaux running in parallel: one hesitating (upper arcs) and one vacillating (lower arcs). Each step in the pair of tableaux is either **1)** doing nothing and then inserting a cell in both tableaux ($(k-1)^2$ possible ways) (opener), **2)** inserting a cell and deleting a cell in the hesitating tableaux, doing nothing twice in the vacillating tableaux ($(k-1)^2$ possibilities) (upper transitory), **3)** doing nothing twice in the hesitating tableaux, and deleting a cell and then doing nothing in the vacillating tableaux ($(k-1)^2$ possibilities) (lower transitory) or **4)** deleting a cell and then doing nothing in both tableaux ($(k-1)^2$ possible ways) (closer). Summing, we get $4(k-1)^2$ possible steps. □

Indeed, the upper bound for the exponential growth factor that we showed in Theorem 7.2.2 is achieved in the cases of all previously known asymptotic results for restricted arc diagrams. Because of this, we make the following conjecture:

Conjecture 7.2.6. *Let $a_k(n)$ denote the number of k-nonnesting arc diagrams of size n of a combinatorial class. If a corresponding tableau family exists and admits m different transitions, then*

$$\lim_{n\to\infty} a_k(n)^{\frac{1}{n}} = m.$$

In order for a combinatorial class represented by an arc diagram to have a corresponding tableau family, there must be an upper bound on the number of arcs adjacent to each vertex. For example, general graphs, while representable with arc diagrams, are not amenable to tableau techniques.

We now use this method to show that over-generating open k-nonnesting arc diagrams and then restricting to those without semi-arcs has the same upper bound. We also conjecture that based on experimental data that this bound is achieved, a conjecture which, if true, means that we are certainly not over-generating by 'too-much'. To do this, we first need to represent open arc diagrams using tableaux.

7.3 'Open' tableaux

In Chen, Deng, Du, Stanley and Yan's 2007 paper [20], the first bijection Ψ given is from a vacillating tableaux $V = (\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \lambda)$ to a pair (P, T) where P is a set partition depicted using an arc diagram, and T is an SYT of shape λ^i. There was no stipulation that $\lambda^i = \emptyset$: this was required in the bijection Φ from partitions to vacillating tableaux. In fact, the authors note that T is an SYT whose content is made up of the maximal elements of some blocks in the set partition, i.e. some blocks are decorated.

Thus, we can exploit this bijection almost exactly to handle open partition diagrams. In the bijection Ψ, the tableaux T is filled with the vertices that are semi-arcs. Future k-nestings are captured in the vacillating tableau with k columns (note the difference) and future k-crossings are seen as k-rows.

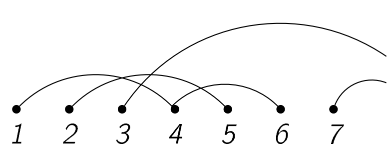
Example 50. *Consider the following vacillating tableaux of length* $2n = 14$ *and shape* $\lambda^{2n} = \ydiagram{2}$.

$$\emptyset, \emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{2}, \ydiagram{2}, \ydiagram{2,1}, \ydiagram{1,1}, \ydiagram{1,1,1}, \ydiagram{1,1}, \ydiagram{1,1}, \ydiagram{1}, \ydiagram{1}, \ydiagram{1}, \ydiagram{2}.$$

The filled SYT are:

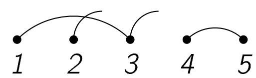
i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
T_i	$\emptyset$	$\emptyset$	$\ytableaushort{1}$	$\ytableaushort{1}$	$\ytableaushort{12}$	$\ytableaushort{12}$	$\ytableaushort{12,3}$	$\ytableaushort{2,3}$	$\ytableaushort{2,3,4}$	$\ytableaushort{3,4}$	$\ytableaushort{3,4}$	$\ytableaushort{3}$	$\ytableaushort{3}$	$\ytableaushort{3}$	$\ytableaushort{37}$

and the corresponding partition is:



We can also determine an analogue to the map Φ found in [20], which is a bijection between set partitions and vacillating tableaux of empty shape. Our bijection Φ' is between open partition diagrams, and vacillating tableaux of shape $\lambda^{2n} = x$, where $x \in \{0, 1, \ldots, n\}$, the number of semi-arcs in the open partition diagram. We treat an incomplete semi-arc as being closed by 'vertex ∞_{+i}'. With this, bijection Φ' becomes identical to Φ in [20].

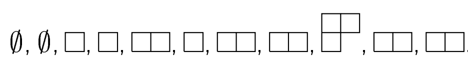
Example 51. *Consider the open set partition $\pi = 13 * -2 * -45$.*



This open partition has the following closer sequence:

$$\emptyset, \emptyset, 3, 3, 3\infty_{+2}, \infty_{+2}, \infty_{+2}\infty_{+3}, \infty_{+2}\infty_{+3}, \infty_{+2}\infty_{+3}5, \infty_{+2}\infty_{+3}, \infty_{+2}\infty_{+3},$$

which gives the vacillating tableau:



Other bijections between open arc diagrams and open tableaux can be defined similarly to bijection Φ'. From this, we get the following corollary to Theorem 7.2.2:

Corollary 7.3.1. *Let $b_k(n)$ denote the number of open arc diagrams of size n of a combinatorial class without future k-nestings. If a corresponding tableau family exists and admits m different transitions, then*

$$\lim_{n\to\infty} b_k(n)^{\frac{1}{n}} \leq m.$$

That is, the exponential growth factor of $b_k(n)$ is at most m.

Note that this limit exists, as the set of open arc diagrams is bounded from below by the set of (regular) arc diagrams.

The following is a corollary to Conjecture 7.2.6 and Corollary 7.3.1:

Conjecture 7.3.2. *Let $b_k(n)$ denote the number of open arc diagrams of size n of a combinatorial class without future k-nestings. If a corresponding tableau family exists and admits m different transitions, then*

$$\lim_{n\to\infty} b_k(n)^{\frac{1}{n}} = m.$$

That is, the exponential growth factor of $b_k(n)$ is m.

7.4 Discussion

Using the functional equations and series data we established in Part II, the number of open partition, permutation and matching diagrams restricted according to future nestings was computed and the number of terms computed in each case is found in Appendix A, and all support the exponential growth factor in Conjecture 7.3.2.

With this, we have shown that the exponential growth factors of k-nonnesting arc diagrams and k-nonnesting open arc diagrams have the same upper bound, which we conjecture is achieved. If the conjecture is true, than we can conclude that we do not over-generate by 'too much'. Figure B.7 in Appendix B depicts the density of 3-nonnesting set partitions that are complete with blue nodes, and those that are open with black, for up to size $n = 9$.

The 'transitions' referenced in Theorem 7.2.2 can also include other restrictions on arc diagrams. For example, *coloured* arc diagrams have recently been studied by [19] for matchings, [47] for set partitions and [59] in the case of permutations. An r-coloured arc diagram has arcs which are one of r different colours. A k-nesting or k-crossing in such a diagram must be monochromatic. In [47], Marberg gave the exponential growth factor of 2-nonnesting 2-coloured set partitions as $r = 9$. We can recover this result by considering the corresponding coloured tableaux. A fixed point corresponds to doing nothing twice, an opener to doing nothing, and then inserting a cell in row 1 in one of two different colours (contribution 2), a closer to removing a cell from row 1 of either colour 1 or colour 2, and a transitory to removing and then inserting a cell (two different colours, one row: contribution 4).

Chapter 8

Conclusions and Open Problems

8.1 Summary

This thesis is centred around the arc diagram representation of combinatorial classes and in particular facilitates enumeration of set partitions, permutations, matchings and tangled diagrams. Motivated by the RNA secondary structure literature and connections to pattern avoidance in permutations, we largely focused on the enumeration and generation of k-nonnesting arc diagrams. Our primary tools were generating trees with multiple parameters, and a more general arc diagram in which not all arcs are complete: open arc diagrams. Through the steps listed first in Table 1.1 and again in Table 8.1 for completeness, we described a construction which enumerates and generates k-nonnesting arc diagrams.

(1) Generalize the arc diagram of the combinatorial class to it corresponding open diagram.
(2) Find a generating tree label and succession rule which tracks nesting statistics.
(3) Translate the generating tree to a functional equation for faster enumeration.
(4) Iterate functional equation to get series data.

Table 8.1: Strategy for generating and enumerating k-nonnesting arc diagrams.

We follow this procedure for each of set partitions (Chapter 2), permutations (Chapter 3), matchings (Chapter 4), and tangled diagrams (Chapter 5). Some counting sequences that arose already had appeared in the literature. In Chapter 6, we described and conjectured a series of bijections from our k-noncrossing arc diagrams to lattice paths and Young diagrams. Lastly, one of the bijections went through sequences of Young tableaux and led us to a result on the upper bound of the exponential growth factor for k-nonnesting arc diagrams, found in Chapter 7. Many open problems have arisen in the course of this thesis.

8.2 Open problems

Some of the questions that arise are broad:

Question 1: What other combinatorial classes can be represented using arc diagrams and treated using the procedure listed in Table 8.1?

Tangled diagrams are arc diagrams in which each vertex may have degree 0, 1 or 2; can we treat diagrams which also allow vertices of degree 3? 4? In [28], de Mier showed equidistribution between crossings and nestings in *labelled graphs*. Is enumeration of such a general class, restricted according to nesting constraints, feasible? In Chapter 5, we noted *braids* are a subset of tangled diagrams; are they amenable to our procedure?

Others are more focused on the enumeration of various classes:

Question 2: Can we solve any of the functional equations that enumerate k-nonnesting arc diagrams for $k > 2$?

In [20], Chen, Deng, Du, Stanley and Yan gave the explicit generating function for k-nonnesting matchings. It involved determinant formulas of the hyperbolic Bessel function of the first kind; can we recover this result using our functional equation? Similarly, in [13], Bousquet-Mélou gave generating functions for the number of 3-nonnesting set partitions. For larger k, they conjectured the following, which we also believe to be true:

Conjecture 8.2.1. [[13] Bousquet-Mélou, Xin 2007] *The number of k-nonnesting set partitions for $k > 3$ is not D-finite.*

We give a similar conjecture on the generating function which counts k-nonnesting permutations:

Conjecture 8.2.2. *The number of k-nonnesting permutations for $k > 2$ is not D-finite.*

We also have questions related to the asymptotic form of k-nonnesting arc diagrams:

Question 3: What is the asymptotic behaviour of k-nonnesting arc diagrams?

The answer is known in the case of matchings, tangled diagrams and 3-nonnesting set partitions, but k-nonnesting set partitions for $k > 3$ and permutations for $k > 2$ are wide open. We have the following conjectures on the exponential growth factor for general k-nonnesting arc diagrams and open arc diagrams:

Conjecture 8.2.3. *Let $a_k(n)$ denote the number of k-nonnesting arc diagrams of size n of a combinatorial class which is represented using an arc diagram. If the corresponding tableau admits m different transitions,*

$$\lim_{n\to\infty} a_k(n)^{\frac{1}{n}} = m.$$

That is, the exponential growth factor of $a_k(n)$ is m.

Conjecture 8.2.4. *The exponential growth factor of open arc diagrams is equal to the exponential growth factor for (complete) arc diagrams.*

Known results agree with these conjectures.

The counting sequences that arose for k-nonnesting arc diagrams sometimes appeared in the literature. While we gave a variety of bijections in Chapter 6, some correspondences were less forthcoming. The following conjectures are based on enumerative data:

Conjecture 8.2.5. *The set of open matching diagrams without future k-nestings is in bijection with standard Young tableaux of maximimum height $2k-2$.*

We showed that this conjecture is true when $k = 2, 3$.

Conjecture 8.2.6. *The number of open partition diagrams on n vertices with neither regular nor enhanced future 3-nesting is B_{n+1}, the number of Baxter permutations of length $n+1$.*

This conjecture is particularly compelling; our combinatorial class is fundamentally different from other objects that are enumerated by Baxter numbers. Open partition diagrams without enhanced future 3-nestings do not have antipodal symmetry, a characteristic of all other Baxter objects. The trees for both our object and two different Baxter objects are given in Appendix B. That said, we know it is true up to $n = 300$, and given the increasing number of combinatorial classes that are known to be in bijection with Baxter permutations, we are extremely interested in solving this conjecture.

Additionally, our open arc diagram innovation can be applied to other areas of mathematics, such as problems motivated by RNA folding. Our technique is potentially more robust than some other methods, and we would love to determine a generating tree that constructed k-non**crossing** arc diagrams **directly**. Ideally it would be able to take minimum arc length restrictions into account. Nestings have a global structure which crossings lack, so such a tree would likely be significantly more technical.

Beyond this, in [50] it was shown that the set of 1-structures, are contained in the set of 4-noncrossing diagrams. It would be interesting to determine if there are other relationships such as this between k-noncrossing diagrams and their genus. One approach to solving this might be to consider all 3472 shadows of genus 2 [50] and determining if they are all k-noncrossing for some k. Furthermore, it might be interesting to explore the gap between 1-structures and 4-noncrossing diagrams: what type of arc diagrams occur?

Lastly, we consider some natural combinatorial problems that arise in the study of k-nonnesting diagrams:

Question 4: Can we uniformly generate random k-nonnesting arc diagrams for set partitions? Permutations?

Question 5: Are there other patterns that we can identify which arc diagrams avoid? Can we enumerate such arc diagrams?

Question 6: In [29], the number of matchings with m 2-crossings was determined. Can we similarly enumerate the number of arc diagrams according to the number of k-nestings?

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Appendix A

Counting sequences

Data is recovered from evaluations of the functional equations. First, when no semi-arcs are present: $u = 0$:

A.1 (Complete) arc diagrams

A.1.1 Set Partitions

$k+1$	OEIS	Initial terms
2	A000108	1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650
3	A108304	1, 2, 5, 15, 52, 202, 859, 3930, 19095, 97566, 520257, 2877834, 16434105, 96505490, 580864901, 3573876308, 22426075431, 143242527870, 929759705415, 6123822269373, 40877248201308
4	A108305	1, 2, 5, 15, 52, 203, 877, 4139, 21119, 115495, 671969, 4132936, 26723063, 180775027, 1274056792, 9320514343, 70548979894, 550945607475, 4427978077331, 36544023687590, 309088822019071
5	A192126	1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115974, 678530, 4212654, 27627153, 190624976, 1378972826, 10425400681, 82139435907, 672674215928, 5712423473216, 50193986895328, 455436027242590
6	A192127	1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213596, 27644383, 190897649, 1382919174, 10479355676, 82850735298, 681840170501, 5828967784989, 51665915664913, 473990899143781
7	A192128	1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644437, 190899321, 1382958475, 10480139391, 82864788832, 682074818390, 5832698911490, 51723290618772, 474853429890994

Table A.1: Counting sequences for $k+1$-nonnesting set partitions.

$k+1$	OEIS	Initial terms
2	A001006	1, 2, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, 310572, 853467, 2356779, 6536382, 18199284, 50852019, 142547559, 400763223, 1129760415, 3192727797, 9043402501, 25669818476
3	A108307	1, 2, 5, 15, 51, 191, 772, 3320, 15032, 71084, 348889, 1768483, 9220655, 49286863, 269346822, 1501400222, 8519796094, 49133373040, 287544553912, 1705548000296, 10241669069576
4	A192855	1, 2, 5, 15, 52, 203, 876, 4120, 20883, 113034, 648410, 3917021, 24785452, 163525976, 1120523114, 7947399981, 58172358642, 438300848329, 3391585460591, 26898763482122, 218263920521938
5	A192865	1, 2, 5, 15, 52, 203, 877, 4140, 21146, 115945, 678012, 4205209, 27531954, 189486817, 1365888674, 10278272450, 80503198320, 654544093035, 5511256984436, 47950929125540, 430240226306346
6	A192866	1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678569, 4213555, 27643388, 190878823, 1382610179, 10474709625, 82784673008, 680933897225, 5816811952612, 51505026270176, 471875801114626
7	A192867	1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644436, 190899266, 1382956734, 10480097431, 82863928963, 682058946982, 5832425824171, 51718812364549, 474782378367618

Table A.2: Counting sequences for set partitions avoiding enhanced $k+1$-nestings.

Terms presented in gray coincide with the Bell numbers.

A.1.2 Permutations

$k+1$	OEIS	Initial terms
2	A000108	1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650
3	A193938	1, 2, 6, 24, 118, 675, 4333, 30464, 230615, 1856336, 15738672, 139509303, 1285276242, 12248071935, 120255584181, 1212503440774, 12519867688928, 132079067871313
4	A193935	1, 2, 6, 24, 120, 720, 5034, 40087, 356942, 3500551, 37343168, 428886219, 5257753614, 68306562647, 934747457369, 13404687958473, 200554264435218, 3118638648191005
5	A193936	1, 2, 6, 24, 120, 720, 5040, 40320, 362856, 3627385, 39864333, 477407104, 6183182389, 86033729930, 1278515941177, 20185987771091
6	A193937	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, 39916680, 478991641, 6226516930, 87157924751, 1306945300264

Table A.3: Counting sequences for $k+1$-nonnesting permutations.

Terms presented in gray coincide with $n!$.

A.1.3 Matchings

$k+1$	OEIS	Initial terms
2	A000108	1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650
3	A005700	1, 3, 14, 84, 594, 4719, 40898, 379236, 3711916, 37975756, 403127256, 4415203280, 49671036900, 571947380775, 6721316278650, 80419959684900, 977737404590100, 12058761323277900
4	A136092	1, 3, 15, 104, 909, 9449, 112398, 1489410, 21562086, 336086022, 5577242292, 97671172836, 1792348213025, 34268124834495, 679376016769260, 13911118850603610, 293220749128031010
5	—	1, 3, 15, 105, 944, 10340, 133055, 1958060, 32279090, 586453658, 11589971918, 246518371679, 5594169454700, 134456679614850, 3402014360391645, 90146180439817440, 2490533922180210720
6	—	1, 3, 15, 105, 945, 10394, 135057, 2023020, 34284920, 647659574, 13471248273, 305364739239, 7477837120725, 196339572677700, 5491374604824840, 162677781287102790, 5079143986721679870

Table A.4: Counting sequences for $k+1$-nonnesting matchings.

Terms presented in gray coincide with $(2n-1)!!$.

A.1.4 Tangled diagrams

$k+1$	OEIS	Initial terms
3	A125660	2,7,39,292,2635,27019,304162,3677313,47036624,629772754,8756958083,125704001433, 1854192548122,28000866597844, 431627186229001,6775008031753481,108068014309278846

Table A.5: Counting sequence for 3-nonnesting tangled diagrams.

Note: data displayed does not represent maximum completed values. In Table A.6 we list the number of terms computed for each k-nonnesting arc diagram representation of the combinatorial classes.

Class	k	n
Set partitions	3	420
	4	276
	5	129
	6	32
	7	21
Set partitions (enhanced)	3	484
	4	129
	5	121
	6	37
	7	30
Permutations	3	223
	4	20
	5	16
	6	15
Tangled diagrams	3	60

Table A.6: Data computed for k-nonnesting arc diagrams.

Appendix B

Generating trees

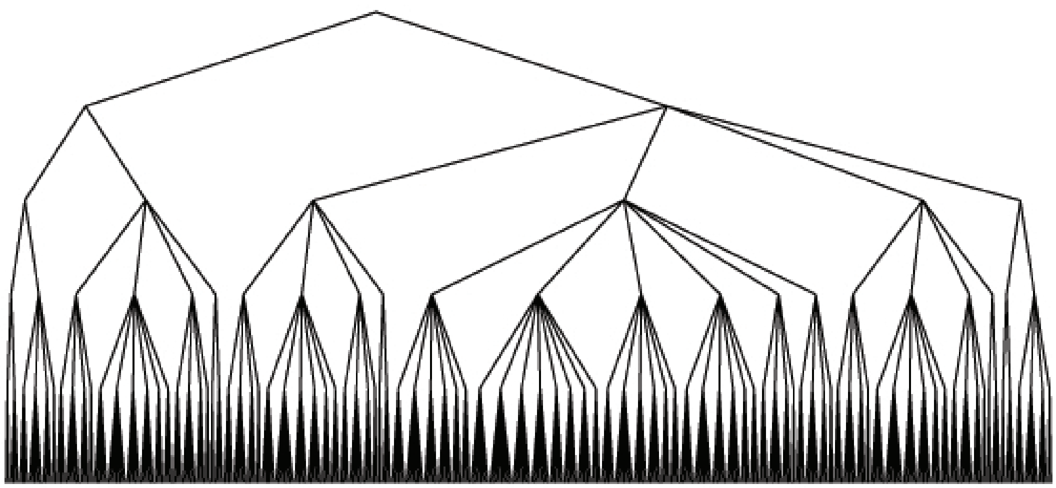

Figure B.1: Generating tree for open set partitions with no future 3-nestings.

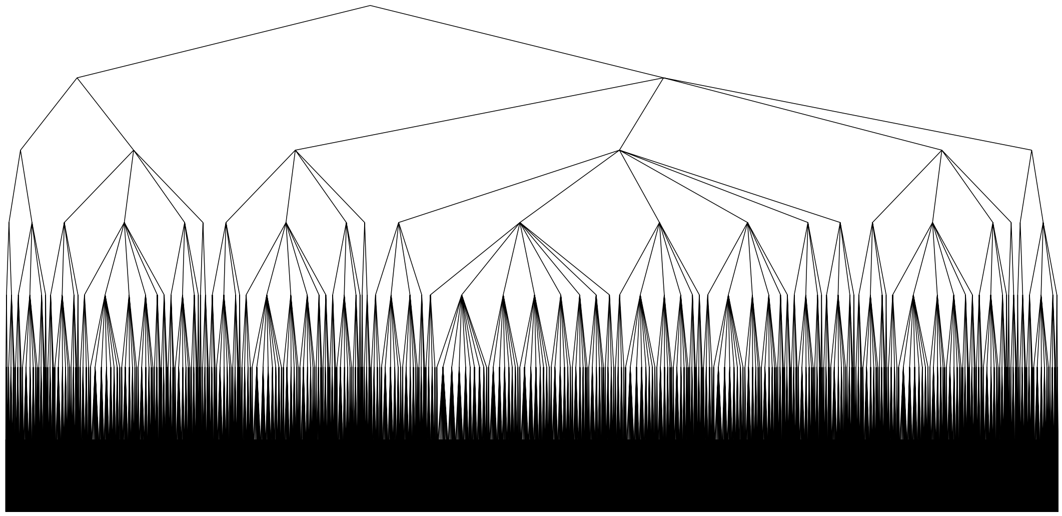

Figure B.2: Generating tree for open set partitions with no enhanced future 3-nestings.

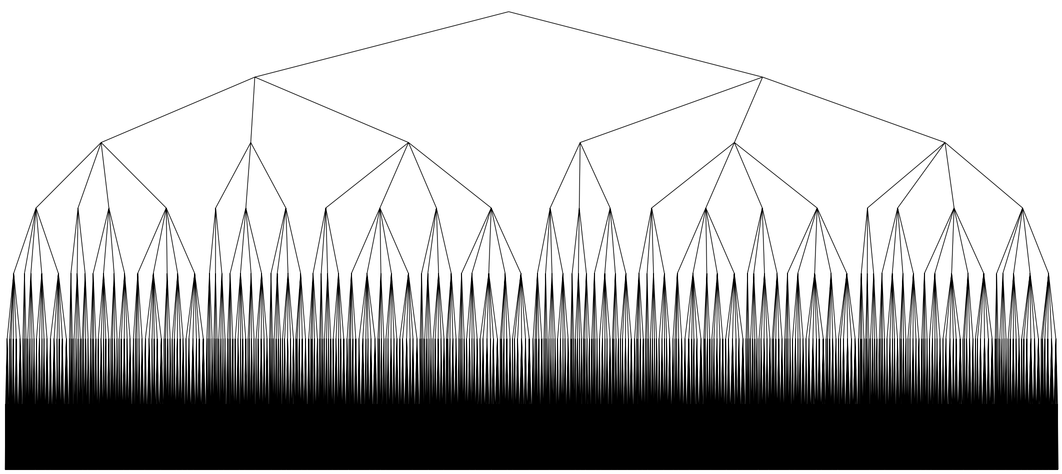

Figure B.3: Generating tree for Baxter permutations.

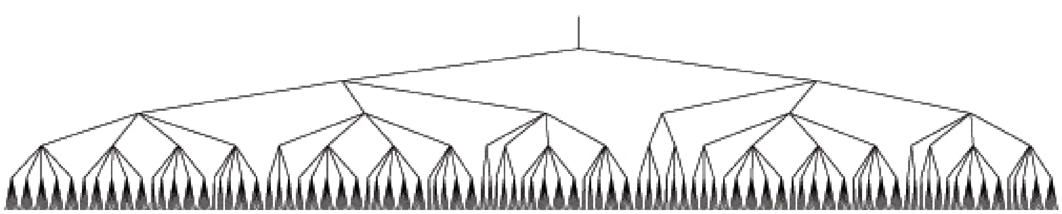

Figure B.4: Alternate generating tree for Baxter permutations.

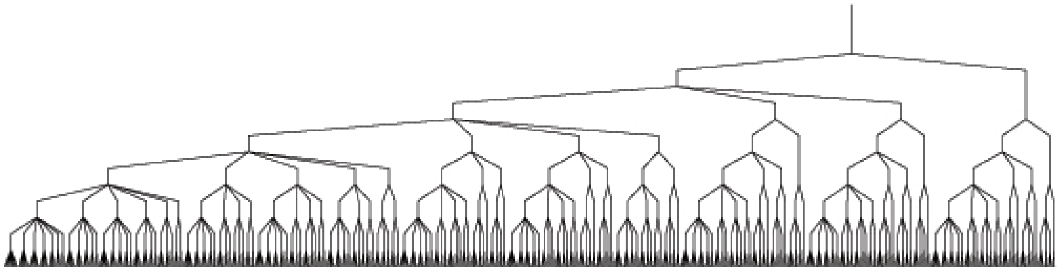

Figure B.5: Generating tree for open matchings with no future 3-nestings.

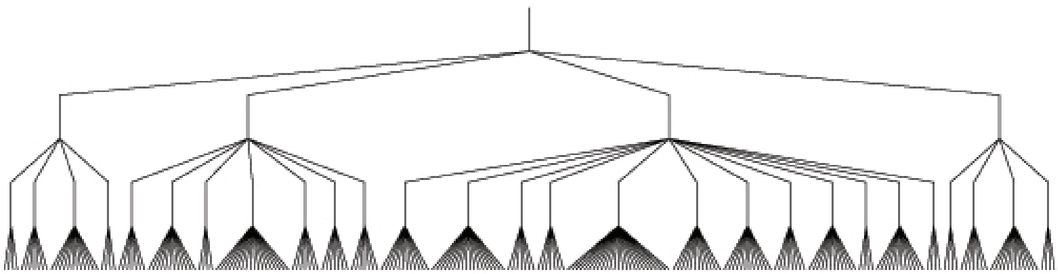

Figure B.6: Generating tree for open tangled diagrams with no future 3-nestings.

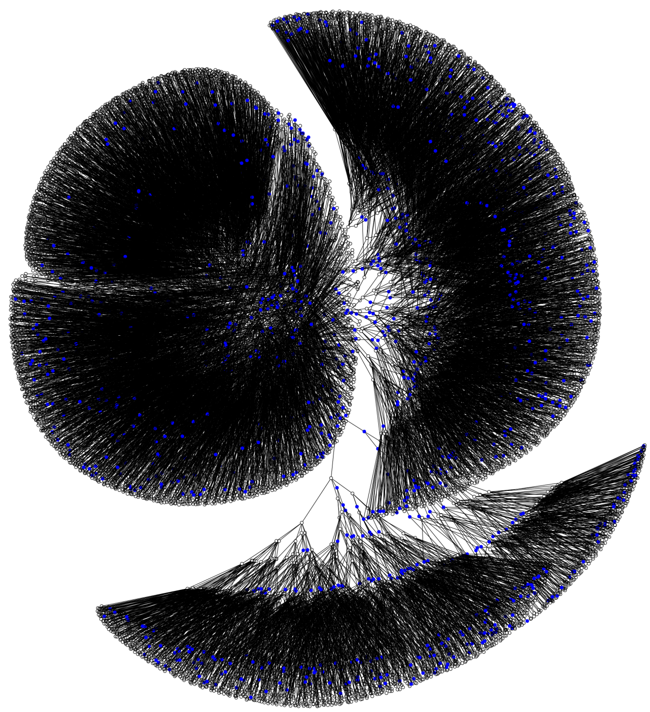

Figure B.7: Density of (complete) 3-nonnesting set partitions.

Appendix C

Maple code

C.1 Succession Rules

C.1.1 Set partitions

The succession rule for generating $k+1$-nonnesting set partitions:

```
RULE1:=proc(label) option remember; #nops(label)=2(k-2)+1
local out, s, ss,i,j,k;
k:= nops(label);
s:= label; ss:= s - [1$k];

out:=
      #1. fixed point
      [s[1], s[2], op(s[3..k])],

      #2. opener
      [s[1]+1, op(s[2..k])];

      #3. transitory - top arc
      if s[k]>0 then
       out:= out, [s[1], op(ss[2..k])];
      fi;

      #4. transitory - other arcs
      for j from 1 to k-1 do
         for i from s[j+1] to s[j]-1 do
           out:= out, [s[1], op(ss[2..j]), i, op(s[j+2..k])];
         od;
      od;

      #5. closer - top arc
      if s[k]>0 then
       out:= out, [s[1]-1, op(ss[2..k])];
```

```
      fi;

      #6. closer - other arcs
      for j from 1 to k-1 do
         for i from s[j+1] to s[j]-1 do
           out:= out, [s[1]-1, op(ss[2..j]), i, op(s[j+2..k])];
         od;
      od;


return [out];
end proc:
```

For example `RULE1([1,0,0]);` **yields**

$$[[1, 0, 0], [2, 0, 0], [1, 0, 0], [0, 0, 0]].$$

The succession rule for generating set partitions without enhanced $k + 1$-nestings:

```
RULE2:=proc(label) option remember; #nops(label)=2(k-2)+1
local out, s, ss,i,j,k;
k:= nops(label);
s:= label; ss:= s - [1$k];

out:=
      #1. fixed point
      [s[1], s[1], op(s[3..k])],

      #2. opener
      [s[1]+1, op(s[2..k])];

      #3. transitory - top arc
      if s[k]>0 then
       out:= out, [s[1], op(ss[2..k])];
      fi;

      #4. transitory - other arcs
      for j from 1 to k-1 do
         for i from s[j+1] to s[j]-1 do
           out:= out, [s[1], op(ss[2..j]), i, op(s[j+2..k])];
         od;
      od;

      #5. closer - top arc
      if s[k]>0 then
       out:= out, [s[1]-1, op(ss[2..k])];
      fi;
```

```
      #6. closer - other arcs
      for j from 1 to k-1 do
         for i from s[j+1] to s[j]-1 do
           out:= out, [s[1]-1, op(ss[2..j]), i, op(s[j+2..k])];
         od;
      od;


return [out];
end proc:
```

For example `RULE2([1,0,0]);` **yields**

$$[[1,1,0],[2,0,0],[1,0,0],[0,0,0]].$$

C.1.2 Permutations

The succession rule for generating $k+1$-nonnesting permutations:

```
RULE3:=proc(label) option remember; #nops(label)=2(k-2)+1
local out, h,r, s, rr,ss,i,j, ii, jj,k;
#r is for top, s is for bottom

k:= (nops(label)-1)/2+1;
h:= label[1];
r:= label[2..k];       rr:= r - [1$k-1];
s:= label[k+1..nops(label)]; ss:= s - [1$k-1];

out:=
      #1. fixed point
      [h, h, op(r[2..k-1]), op(s)],

      #2. opener
      [h+1, op(r), op(s)];

      #3. upper transitory - top arc
      if r[k-1]>0 then
       out:= out, [h, op(rr), op(s)];
      fi;

      #4. upper transitory - other arcs
      for i from r[1] to h-1 do
           out:= out, [h, i, op(r[2..k-1]), op(s)];
      od;
      for j from 2 to k-1 do
         for i from r[j] to r[j-1]-1 do
```

```
     out:= out, [h, op(rr[1..j-1]), i, op(r[j+1..k-1]), op(s)];
   od;
od;
#5. upper transitory - top arc
if s[k-1]>0 then
 out:= out, [h, op(r), op(ss)];

fi;


#6. lower transitory - other arcs
for i from s[1] to h-1 do
     out:= out, [h, op(r), i, op(s[2..k-1])];
od;
for j from 2 to k-1 do
   for i from s[j] to s[j-1]-1 do
     out:= out, [h, op(r), op(ss[1..j-1]), i, op(s[j+1..k-1])];
   od;
od;

#7. closer - top and bottom arcs
if r[k-1]>0 and s[k-1]>0 then
  out:= out, [h-1, op(rr), op(ss)];
fi;


#8 closer top arc + bottom others
if r[k-1]>0 then
  for i from s[1] to h-1 do
     out:= out, [h-1, op(rr), i, op(s[2..k-1])];
   od;

  for j from 2 to k-1 do
   for i from s[j] to s[j-1]-1 do
     out:= out, [h-1, op(rr), op(ss[1..j-1]), i, op(s[j+1..k-1])];
   od;
  od;
fi;

#9 closer top others + bottom arc
if s[k-1]>0 then
  #j=1
  for i from r[1] to h-1 do
     out:= out, [h-1, i, op(r[2..k-1]), op(ss)];
   od;
  #j=2
```

```
  for j from 2 to k-1 do
   for i from r[j] to r[j-1]-1 do
     out:= out, [h-1, op(rr[1..j-1]), i, op(r[j+1..k-1]), op(ss)];
   od;
  od;
fi;


#10 closer: others top + bottom
#j=1
for i from s[1] to h-1 do
  #jj=1
  for ii from r[1] to h-1 do
     out:= out, [h-1, ii,  op(r[2..k-1]),
                     i,  op(s[2..k-1])];
  od;

  #jj>1
  for jj from 2 to k-1 do
    for ii from r[jj] to r[jj-1]-1 do
     out:= out, [h-1, op(rr[1..jj-1]), ii, op(r[jj+1..k-1]),
                    i,  op(s[2..k-1])];

    od;
  od;
od;

#j>1
for j from 2 to k-1 do
   for i from s[j] to s[j-1]-1 do
     #jj=1
     for ii from r[1] to h-1 do
     out:= out, [h-1,  ii, op(r[2..k-1]),
                    op(ss[1..j-1]),  i,  op(s[j+1..k-1])];

     od;

  #jj>1
  for jj from 2 to k-1 do
        for ii from r[jj] to r[jj-1]-1 do
        out:= out, [h-1, op(rr[1..jj-1]), ii, op(r[jj+1..k-1]),
                    op(ss[1..j-1]),  i,  op(s[j+1..k-1])];

   od;od;
od;od;
```

```
return [out];
end proc:
```

For example, `RULE3([2,0,0]);` **yeilds**

$$[[2,2,0],[3,0,0],[2,0,0],[2,1,0],[2,0,0],[2,0,1],[1,0,0],[1,1,0],[1,0,1],[1,1,1]].$$

C.1.3 Matchings

```
RULE4:=proc(label) option remember; #nops(label)=2(k-2)+1
local out, s, ss,i,j,k;
k:= nops(label);
s:= label; ss:= s - [1$k];

out:=
      #1. opener
      [s[1]+1, op(s[2..k])];

      #2. closer - top arc
      if s[k]>0 then
       out:= out, [s[1]-1, op(ss[2..k])];
      fi;

      #3. closer - other arcs
      for j from 1 to k-1 do
         for i from s[j+1] to s[j]-1 do
           out:= out, [s[1]-1, op(ss[2..j]), i, op(s[j+2..k])];
         od;
      od;

return [out];
end proc:
```

For example, `RULE4([3,2,0])` **yeilds**

$$[[4,2,0],[2,2,0],[2,1,0],[2,1,1]].$$

C.1.4 3-nonnesting tangled diagrams

```
RULE5 := proc (label) option remember
 local m, s, out, j;
 m := label[1]; s := label[2];

 out :=
     #1. singleton
      [m, s];
```

```
    #2. opener
     out := out, [m+1, s];

     #3. closer
     out := out, seq([m-1, i], i = max(0, s-1) .. m-1);

     #4. opener + application of matching succession rule
      out := out, op(RULE4([m+1, s]));

     #5. closer + application of matching succession rule
     for j from max(0, s-1) to m-1
          do out := out, op(RULE4([m-1, j]));
     od;

 return [out];
 end proc:
```

For example, `RULE5([1,1])` **returns**

$$[[2,1],[3,1],[1,0],[1,1],[4,1],[2,0],[2,1],[2,2],[2,0],[0,0],[2,1],[0,0]].$$

C.2 Counting sequences

To generate the counting sequences for each type of $k+1$-nonnesting arc diagram:

```
termtolabeltoterm:= proc (term, N,RULEtype )
 option remember;local out;
   out:=RULEtype([seq(degree(term, x[i]), i=1..N)]);
subs(seq(x[i]=1, i=1..N), term)*add(mul(x[i]^out[j][i], i=1..N),
j=1..nops(out))
end proc:

nextlevel:=proc(l, N,RULEtype) option remember;
local i,out, L;
  out:=0;
  L:= convert(l, list);
  out:= add(termtolabeltoterm(L[i], N,RULEtype), i=1..nops(L));
  return out;
end:

level:=proc(n, K, RULEtype) option remember;
    if n=0 then return 1
    else return nextlevel(level(n-1, K, RULEtype), K, RULEtype); fi;
end:
```

For examples, A108305 [41], the number of 4-nonnesting set partitions is returned from:

```
seq(subs(seq(x[i] = 0, i = 1 .. 5), level(n, 3, RULE1)), n = 0 .. 16);
```

A108307, the number of set partitions that avoid enhanced 3-nestings:

```
seq(subs(seq(x[i] = 0, i = 1 .. 5), level(n, 2, RULE2)), n = 0 .. 16);
```

Sequence A193938, 3-nonnesting permutations:

```
seq(subs(seq(x[i] = 0, i = 1 .. 5), level(n, 2, RULE3)), n = 0 .. 16);
```

We get the number of 3-nonnesting matchings from:

To generate the counting sequence of tangled diagrams, K must be 2. The following gives A125660, the number of (3-nonnesting) tangled diagrams:

```
seq(subs(seq(x[i] = 0, i = 1 .. 5), level(n, 2, RULE5)), n = 0 .. 16);
```