TOPOLOGICAL COLOURING BOUNDS AND GRAPH STRUCTURE

by

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Abstract

Lovász invoked topological colouring bounds in proving Kneser's Conjecture. Subsequently, numerous applications of topological techniques to graph colouring problems have arisen. However, even today, little is known about how to construct a graph with a particular topological colouring bound, or about the structure of such graphs. The aim of this thesis is to remedy this deficit.

First, we will perform a review of topological techniques used in bounding the chromatic number and discuss constructing graphs with particular topological colouring bounds. Then we will derive necessary conditions for a graph to attain a particular topological colouring bound, and use these conditions to analyze the structure of such graphs. In particular, we provide support for some open problems in graph theory by verifying the problems under additional assumptions on topological colouring bounds. We also discuss the relationship between colour critical graphs with tight topological colouring bounds and quadrangulations of surfaces.

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Chapter 1

Topological Lower Bounds

1.1 Introduction

This thesis focuses its attention on the study of topological lower bounds on the chromatic number of a graph (which we will sometimes refer to as topological colouring bounds). Such bounds have played significant roles in solving a variety of difficult problems in graph theory over the years beginning, of course, with Lovász's seminal work on proving Kneser's Conjecture by establishing a sharp topological lower bound on the chromatic number of the Kneser graphs [24]. For readers unfamiliar with Kneser's Conjecture, we shall recall that it asserts that the collection of all n-element subsets of a $2n+k$ element set cannot be partitioned into $k+1$ classes, so that every pair of n-sets within the same class has non-empty intersection. Inspired by this problem, the Kneser graphs were introduced as graphs whose vertices were the n-element subsets of a $(2n+k)$-element set and whose edges joined vertices representing disjoint sets. Consequently, colourings of Kneser graphs correspond with partitions of the n-sets into classes such that pairs of vertices in the same class have non-empty intersection. Thus, if we could prove that the Kneser graph corresponding with the n-element subsets of a $2n+k$ element set was not k-colourable, then Kneser's Conjecture would be proven. This was the approach which Lovász took to establishing this conjecture, and it will be our approach as well when we tackle this problem later in this chapter.

Of course, there are now extremely short proofs of Kneser's Conjecture using only elementary methods, such as Greene's proof [13]. However, our interest in this chapter will be in generalizing methods for finding topological colouring bounds from the Kneser graphs to arbitrary graphs. The second chapter of this thesis will then study how one goes about constructing interesting examples of graphs for which these topological lower bounds are sharp, while the third chapter will discuss the structure of such graphs.

Unfortunately, in order to cover so much material, we have had to assume the reader is familiar with a rather wide variety of content. To the best of our ability, we have strove to keep these prerequisites fairly elementary. However, the reader wholly unfamiliar with one of these topics should like consult one of the references provided.

Firstly, we will generally assume that the reader is familiar with the basic material on simplicial complexes and posets contained in Chapter 1 of Matoušek's book on using the Borsuk-Ulam [27]. *Abstract simplicial complexes* are, of course, collections of finite subsets closed under inclusion, and they have associated with them geometric realizations which are topological spaces which may be embedded in $\mathbb{R}^n$ for a sufficiently high n. We will generally ignore the distinction between an abstract simplicial complex and its geometric realization unless there is some reason to distinguish between the two notions. We will also discuss *simplicial maps*, which are maps between simplicial complexes which map simplices to simplices. These maps induce continuous maps between the geometric realizations of the simplicial complexes between which they map. The *face poset* of a simplicial complex K is the poset $P(K)$ which is the set of all non-empty simplices of K ordered by inclusion, while the *order complex* of a poset P is the simplicial complex $\Delta(P)$ whose vertices are the elements of P and whose simplices are all the chains of P. The *barycentric subdivision* of a simplicial complex K is defined by $sd(K) = \Delta(P(K))$. We will also assume that readers are familiar with basic poset terminology such as the notions of chains, anti-chains, linear orders and dual posets.

Readers familiar with basic topology will, of course, recognize barycentric subdivision as a particularly simple example of a homotopy equivalence. More generally, given topological spaces X and Y, a continuous map $f : X \to Y$ is termed a *homotopy equivalence* of X and Y if for some map $g : Y \to X$, $g \circ f = id_X$ and $f \circ g = id_Y$. In this case, we term the map g the homotopy inverse of f. We say that X and Y are homotopy equivalent if there is a homotopy equivalence between them. For any two continuous functions $f, g : X \to Y$, we will say that f and g are *homotopy equivalent* if there exists a continuous function $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Such an H is termed a *homotopy* between f and g. As is standard in the study of maps between topological spaces, we will assume in what follows that, unless otherwise stated, all maps between topological spaces are continuous.

A particularly useful kind of homotopy equivalence between topological spaces is deformation retraction. Given a topological space X, a map $F : X \times [0,1] \to X$ is termed a *deformation retraction* from X onto a subspace $A \subset X$ provided that for each $x \in X$ and $a \in A$, $F(x,0) = x$, $F(x,1) \in A$ and $F(a,1) = a$.

In addition to understanding the notion of homotopy, readers should be familiar with the notion of a Hausdorff space (all topological spaces treated in this thesis will be Hausdorff spaces), as well as quotient spaces. Formally, the *quotient space* of a topological space X together with some equivalence relation $\sim$ on X is a topological space formed from X by identifying all the points identified by $\sim$. The topology of the quotient space, which is usually denoted by $X/\sim$ is given by letting the open sets of $X/\sim$ be those sets whose preimage is open in X under the projection mapping from X onto $X/\sim$. Readers unfamiliar with quotient spaces should consult [33] and Chapter 0 of [16]. In particular, we will freely use the result from Chapter 0 of [16] that taking the quotient of any topological space X by a contractible space is a homotopy equivalence. The only other topological notions we will assume are the basic definitions of the boundary operator and homology groups for simplicial complexes. We will not restate these definitions here, as they are rather lengthy, but these notions are quite elementary, and may be found in either Massey's textbook [26] or Chapter 1 of Munkres' book [32].

We will also freely use basic ideas from graph theory, such as those found in Bondy and Murty's classic textbook [4]. Moreover, we will assume throughout the rest of this thesis, that unless otherwise explicitly stated, all graphs are simple and have no isolated vertices. We will define most of the non-standard notation we use. However, we still feel that it is prudent to inform the reader that in what follows $[n] = \{1, 2, ..., n\}$, 2^A denotes the power set of the set A, and we will write $N_G(v)$ for the neighbourhood of a vertex $v \in V(G)$ in order to clearly distinguish between the neighbourhood of a vertex and the neighbourhood complex (a simplicial complex which we will introduce later in this chapter).

1.2 The Box Complex

In attempting to place lower bounds on the chromatic number of a graph, Lovász began by studying the neighbourhood and Lovász complexes of a graph. However, formulating the bounds we can derive from studying these complexes is rather difficult and technical if one wishes to be precise. Consequently, we will begin our study with a simplicial complex which only became of significant interest in more recent years, but which allows for a clearer and more concise presentation: the box complex.

Informally, the box complex of a graph G may be thought of as the simplicial complex which has as its simplices all complete bipartite subgraphs of G. More precisely, following Matoušek [27], we define the *box complex* of a graph G, denoted by $B(G)$, as the simplicial complex defined on the ground set $V(G) \times [2]$ with the following simplices:

$B(G) := \{A_1 \uplus A_2 : A_1, A_2 \subseteq V(G), A_1 \cap A_2 = \emptyset, G[A_1, A_2] \text{ is complete bipartite}, CN(A_1) \neq \emptyset \neq CN(A_2)\}$.

In this definition, by the notation $A \uplus B$ we denote the set $A \times \{1\} \cup B \times \{2\}$. $G[A_1, A_2]$ is the graph with vertex set $A_1 \cup A_2$ such that $uv \in E(G)$ is an edge of $G[A_1, A_2]$ if and only if $u \in A_1$ and $v \in A_2$. The function $CN : 2^{V(G)} \to 2^{V(G)}$ is the function which maps any set of vertices $A \in 2^{V(G)}$ to $CN(A) := \{v \in V(G) : v \sim a \ \ \forall a \in A\} \subseteq V(G) \setminus A$.

We naturally think of colourings of a graph G as graph homomorphisms from G to K_n. Moreover, it has long been known that graphs together with graph homomorphisms form a category, as do topological spaces together with continuous maps. Thus, in order to place a lower bound on the chromatic number of G, our strategy will be to find a functor from the category of graphs to the category of topological spaces, and then study the obstructions to the existence of graph homomorphisms by studying topological obstructions to the existence of continuous maps between topological spaces. Of course, this is not quite right, as the constant map is always a continuous map between any two non-empty topological spaces, so we will need to place some restrictions on our topological spaces and maps. One such restriction which is particularly well-studied in topology is to require our topological spaces to admit fixed point free involutions, while our maps must not only be continuous, but must commute with the involutions defined (we call such maps equivariant maps). In summary, we would like to have the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow[\text{hom.}]{\text{graph}} & K_n \\ \downarrow & & \downarrow \\ B(G) & \xrightarrow[\text{map}]{\text{equiv.}} & B(K_n) \end{array}$$

In what follows, we will refer to topological spaces with fixed point free involutions (such as the box complex) as $\mathbb{Z}_2$-spaces and the equivariant maps between such spaces as $\mathbb{Z}_2$-maps.

Having introduced all the terminology and ideas necessary, we can now proceed to establish a topological lower bound on the chromatic number in terms of the topology of the box complex. In order to do so, we first establish the following lemma.

Lemma 1. *Let G, H and L be graphs, and let $B(G)$, $B(H)$ and $B(L)$ be their box complexes. Then the following statements hold:*
(a) the function $\nu : B(G) \to B(G)$ which is induced on the simplices of $B(G)$ by the vertex map $\nu(v, i) = (v, 3 - i)$ for all $(v, i) \in V(B(G))$ is a simplicial fixed point free involution on $B(G)$;
(b) given a graph homomorphism $f : G \to H$, the map $B(f) : B(G) \to B(H)$ which is induced on the simplices of $B(G)$ by the vertex map $B(f)(v, i) = (f(v), i)$ for all $(v, i) \in V(B(G))$ is a simplicial $\mathbb{Z}_2$-map;
(c) B is a functor from the category of graphs to the category of $\mathbb{Z}_2$-spaces;
(d) $B(K_n) \cong S^{n-2}$.

Proof. (a) For all $A, B \subseteq V(G)$, $\nu(A \uplus B) = B \uplus A$, so ν is simplicial, as if $G[A, B]$ is complete bipartite, then so is $G[B, A]$. Additionally, $\nu^2(A \uplus B) = A \uplus B$, so ν is an involution. Finally, ν is fixed point free, as, since $A \cap B = \emptyset$, $(A \uplus B) \cap (B \uplus A) = \emptyset$.

(b) As f is a graph homomorphism, it maps complete bipartite subgraphs to complete bipartite subgraphs. Consequently, $B(f)$ is simplicial. Additionally, if we let $\tilde{f} : 2^{V(G)} \to 2^{V(G)}$ be the map induced on the subsets of $V(G)$ by f, then for all $A, B \subseteq V(G)$, $B(f)(\nu(A \uplus B)) = B(f)(B \uplus A) = \tilde{f}(B) \uplus \tilde{f}(A) = \nu(\tilde{f}(A) \uplus \tilde{f}(B)) = \nu(B(f)(A \uplus B))$.

(c) Given (a) and (b), to show that B is a functor from the category of graphs to the category $\mathbb{Z}_2$-spaces, all that remains is to note that that the image under B of the identity homomorphism is the identity $\mathbb{Z}_2$-map, and to observe that for all $f : G \to H$, for all

$g : H \to L$, and for all $A, B \subseteq V(G)$, $B(g \circ f)(A \uplus B) = (\tilde{g} \circ \tilde{f})(A) \uplus (\tilde{g} \circ \tilde{f})(B) = (B(g) \circ B(f))(A \uplus B)$.

(d) The boundary of the n-dimensional cross-polytope $\Diamond^n$ is a triangulation of S^{n-1} which may be naturally represented as an abstract simplicial complex by the n-fold join of two points which we shall call a and b. Here we recall that the *join* of two simplicial complexes X and Y is $\{A \uplus B : A \in X,\ B \in Y\}$. The n-fold join is simply the operation which successively joins n simplicial complexes. If we abbreviate the n-fold join of the simplicial complex $\{\{a\}, \{b\}\}$ by a sequence $x_1x_2x_3...x_n$, where each $x_i = a$ or b, then we observe that the facets of the abstract simplicial complex representing $\Diamond^n$ are all such sequences. Moreover, the antipodal action on $\Diamond^n$ is induced on sequences by the vertex map which takes a to b and b to a.

Similarly, if we fix an ordering of the vertices of K_n and take any bipartition of $V(K_n)$ except for $V(K_n) \uplus \emptyset$ or $\emptyset \uplus V(K_n)$, then we see that (identifying as with vertices on the first side of the bipartition and bs with vertices on the second side), we have an obvious bijection between the facets of $B(K_n)$ and the facets of the abstract simplicial complex representing $\Diamond^n$, excluding the two antipodal facets $aaa...a$ and $bbb...b$. Thus, $B(K_n) \cong S^{n-2}$. □

We now have proved all the results we need in order to establish a topological lower bound on the chromatic number except for one fact. Unfortunately, this fact, the Borsuk-Ulam Theorem, will require a lengthy and technical proof if we restrict ourselves to the ideas developed thus far in this thesis. Consequently, while we state the Borsuk-Ulam Theorem here, we will postpone its proof until later in the chapter, at which point we will derive the Borsuk-Ulam Theorem as a corollary of Ky Fan's Theorem (which we will explicitly prove). Now, Let S^n_a be the n-sphere together with the usual antipodal mapping $x \to -x$. This is certainly a $\mathbb{Z}_2$-space, so we can state the Borsuk-Ulam Theorem as follows.

Borsuk-Ulam Theorem. *If $f : S^n_a \to S^m_a$ is a $\mathbb{Z}_2$-map, then $n \leq m$.*

Now, for any $\mathbb{Z}_2$-space X, let us define the index of X, denoted by $Ind(X)$, as the smallest integer n such that there exists a $\mathbb{Z}_2$-map from X to S^n_a. Similarly, we define the coindex of X, denoted by $Coind(X)$, as the largest integer n such that there exists a $\mathbb{Z}_2$-map from S^n_a to X.

Observation 1. *Let X and Y be $\mathbb{Z}_2$-spaces, and let $f : X \to Y$ be a $\mathbb{Z}_2$-map. Then $Coind(X) \leq Ind(Y)$. In particular, $Coind(X) \leq Ind(X)$.*

Proof. Suppose that there exist $\mathbb{Z}_2$-maps $g : S_a^n \to X$ and $h : Y \to S_a^m$. Then we have the following composition of maps:

$$S_a^n \to X \to Y \to S_a^m$$

Thus, by the Borsuk-Ulam Theorem, $n \leq m$. Consequently, $Coind(X) \leq Ind(Y)$, and, taking $X = Y$ and $f = id_X$, we see that $Coind(X) \leq Ind(X)$. □

Now, arguing similarly, we will establish that the index and, consequently, the coindex of the box complex yield topological lower bounds on the chromatic number.

Theorem 1. *Let G be a graph and $B(G)$ be its box complex. Then $Coind(B(G)) + 2 \leq Ind(B(G)) + 2 \leq \chi(G)$.*

Proof. Suppose that G can be coloured with $\chi(G)$ colours. Then there exists a graph homomorphism $f : G \to K_m$. Thus, by Lemma 1, there exists a $\mathbb{Z}_2$-map $B(f) : B(G) \to S^{\chi(G)-2}$, whence, $Ind(B(G)) \leq \chi(G) - 2$. □

1.3 Topological Connectivity

While we were able to establish topological lower bounds on the chromatic number in the last section using the index and coindex of the box complex, the careful reader will have noticed that we did not provide any straightforward method of computing either of these invariants for an arbitrary $\mathbb{Z}_2$-space. This is because attempting to directly compute these invariants using elementary methods is difficult. Consequently, in applications of topological methods, a more readily computable parameter, the topological connectivity of a $\mathbb{Z}_2$-space (which we will simply refer to as connectivity from this point on), frequently plays an important role in bounding the chromatic number of a graph. The main focus of this section will be on defining this new parameter for an arbitrary $\mathbb{Z}_2$-space and explaining how it can be used to bound the chromatic number of a graph.

To begin with, we will need to define the homotopy groups of a topological space. So, let us fix a base point $a \in S^n$, a topological space X and a base point b in X. Then the

k^{th} homotopy group $\pi_k(X, b)$ of X is the set of homotopy classes of maps $f : S^n \to X$ which map the base point a to the base point b. Now, let us call a topological space X *n-connected* if $\pi_k(X) = 0$ for all $0 \leq k \leq n$. From this definition, we can easily derive a few useful equivalent characterizations.

Proposition 1. *The following are equivalent for all $i \in \mathbb{Z}^+$:*
(a) $\pi_i(X, x_0) = 0$ for all $x_0 \in X$;
(b) every map $f : S^i \to X$ is homotopic to a constant map;
(c) every map $f : S^i \to X$ extends to a map from $D^{i+1} \to X$, where D^{i+1} is $(i+1)$-ball (i.e. the space enclosed by S^i).

Another equivalent formulation of n-connectedness, which we will not prove (for a proof, we refer the reader to [16]), is given by the well-known Hurewicz Theorem. This theorem allows us to compute the connectedness of a simply connected space (i.e. a space X for which $\pi_1(X, x_0) = 0$ for all $x_0 \in X$) by examining its homology groups. This simplifies our study of connectivity considerably for simply connected spaces.

Hurewicz Theorem. *Let X be a nonempty topological space and let $n \in \mathbb{Z}^+$. Then if X is $(n-1)$-connected, then for $2 \leq i \leq n$, and for all $x_0 \in X$, $\pi_i(X, x_0) \cong H_i(X)$.*

Now, let us prove a proposition from which a topological lower bound on the chromatic number based upon the connectivity of the box complex will follow as a corollary.

Proposition 2. *Let X and Y be regular CW-complexes with fixed point free involutions γ and ν. Additionally, suppose that for some $k \geq 0$, $dim X \leq k$, Y is $(k-1)$-connected, and there exists a $\mathbb{Z}_2$-map $f : X^{(d)} \to Y$ for some $d \geq -1$. Then there exists a $\mathbb{Z}_2$-map $g : X \to Y$ which extends f.*

Proof. Firstly, let us note that if $d = -1$, then this means that there is no map f. In this case, let us extend g to the 0-skeleton of X as follows: for each orbit $\{a, b\}$ of the action of $\mathbb{Z}_2$ on X (which consists of two vertices of X), map a to an arbitrary point $y \in Y$, and then map b to $\nu(y)$. This completes the base case in our inductive argument, so we may now proceed to the inductive step.

Suppose that g is defined on the $(i-1)$-skeleton of X for some $i \geq 1$. Then we may extend it to the i-skeleton as follows. Let (σ, τ) be a pair of i-dimensional cells of X

such that $\gamma(\sigma) = \tau$. Note that the boundary of σ ($\partial\sigma$) is an $(i-1)$-sphere. Then, as $i-1 \leq dimX - 1 \leq k-1$, by Proposition 1, the restriction of g to $\partial\sigma$ extends to σ. In order to extend g to τ, we simply compose with the involution γ: $g|_\tau := (g|_\sigma) \circ \gamma$. Performing this extension on each pair of i-dimensional cells of X such that $\gamma(\sigma) = \tau$ completes the proof. □

In what follows, let us denote the connectivity of a topological space X by $conn(X)$.

Corollary 1. *Let X be a $\mathbb{Z}_2$-space. Then $conn(X) + 1 \leq Coind(X) \leq Ind(X)$.*

Proof. Let X be $(k-1)$-connected. Then, since S_a^k is k-dimensional, by Proposition 2, there exists a $\mathbb{Z}_2$-map $g : S_a^k \to X$, whence $Coind(X) \geq k$. Consequently, by Observation 1, $conn(X) + 1 \leq Coind(X) \leq Ind(X)$. □

Corollary 2. *For any graph G, $\chi(G) \geq conn(B(G)) + 3$.*

Proof. This is a trivial consequence of Theorem 1 and Corollary 1. □

1.4 Ky Fan's Theorem and the Borsuk-Ulam Theorem

In the previous sections, we have made repeated use of the Borsuk-Ulam Theorem. This section will be devoted to deriving this theorem as a corollary of one of two equivalent variants of Ky Fan's Theorem, a result which we will make use of in the third section of this thesis. We will begin by proving the Borsuk-Ulam Theorem from the first of our variants of Ky Fan's Theorem, and then we will derive both of the versions of Ky Fan's Theorem we will require from a combinatorial lemma. Ky fan's Theorem is, of course, due to Ky Fan [23], but the proof of the combinatorial lemma we will use is a more recent, constructive result of Prescott and Su [34].

Ky Fan's Theorem.
(1) Let $\mathcal{A}$ be a finite collection of subsets covering S^n, which are either all open or all closed. Furthermore, assume that there is a linear order on $\mathcal{A}$, and that for all $A \in \mathcal{A}$, $A \cap -A = \emptyset$. Then there exist sets $A_1 < A_2 < ... < A_{n+2}$ in $\mathcal{A}$ and a point $x \in S^n$ such that $(-1)^i x \in A_i$ for all $i \in [n+2]$.
(2) Let $\mathcal{A}$ be a finite collection of subsets of S^n such that $\cup_{A\in\mathcal{A}}(A \cup -A) = S^n$, which are

either all open or all closed. Furthermore, assume that there is a linear order on $\mathcal{A}$, and that for all $A \in \mathcal{A}$, $A \cap -A = \emptyset$. Then there exist sets $A_1 < A_2 < ... < A_{n+1}$ in $\mathcal{A}$ and a point $x \in S^n$ such that $(-1)^i x \in A_i$ for all $i \in [n+1]$.

With Ky Fan's Theorem in hand, we will now proceed to swiftly derive the Borsuk-Ulam Theorem.

Borsuk-Ulam Theorem. *If $f : S_a^n \to S_a^m$ is a $\mathbb{Z}_2$-map, then $n \leq m$.*

Proof. Firstly, note that inclusion is always a $\mathbb{Z}_2$-map $i : S_a^m \to S_a^{n-1}$ if $n > m$. Consequently, it suffices to show that there is no $\mathbb{Z}_2$-map $f : S_a^n \to S_a^{n-1}$. To see this fact, let us cover S^{n-1} by closed sets $A_1, ..., A_{n+1}$ such that $A_i \cap -A_i = \emptyset$ as follows. Consider an n-simplex in $\mathbb{R}^n$ containing 0 in its interior, and project its $n+1$ facets centrally from 0 onto S^{n-1}. The images of the faces in S^{n-1} are $n+1$ closed sets $A_1, ..., A_{n+1}$ covering S^{n-1} such that $A_i \cap -A_i = \emptyset$. Therefore, if $f : S_a^n \to S_a^{n-1}$ were a $\mathbb{Z}_2$-map, then $f^{-1}(A_1), ..., f^{-1}(A_{n+1})$ would be a collection of $n+1$ closed sets covering S^n such that for all $i \in [n+1]$, $f^{-1}(A_i) \cap -f^{-1}(A_i) = \emptyset$. This contradicts Ky Fan's Theorem (1). □

We will now proceed to prove the two variants of Ky Fan's Theorem we have introduced. However, in order to do so, we will first need to introduce some definitions.

A *labeling* of a triangulation K of S^n is an assignment of an integer to each vertex of the triangulation. A simplex in a labeled triangulation will be termed an *alternating simplex* if its vertex labels have distinct absolute values and alternate in sign when arranged in order of increasing absolute value. we will also call the sign of the smallest label in absolute value of an alternating simplex the *sign* of that simplex. Similarly, we will say that a simplex is *almost-alternating* if it is not alternating, but it can be made alternating by deleting one of its vertices. The *sign* of an almost-alternating simplex is the sign of any one of its facets. We should note here that the sign of an almost-alternating simplex is easily seen to be well-defined, as if deleting the the vertex with the smallest label in absolute value makes an almost-alternating simplex σ alternating, then the smallest two labels of σ must have the same sign.

Now, Ky Fan's combinatorial lemma does not apply to any triangulation of S^n. In fact, the original proof called for K to be a barycentric subdivision of the octahedral subdivision of S^n. Recall, that the *octahedral subdivision* of S^n is the subdivision of S^n into 2^n n-simplices using the coordinate hyperplanes of $\mathbb{R}^{n+1}$. Such triangulations have an important property which we should recall. We will say that a triangulation K of S^n is *symmetric* if $\sigma \in K$ implies that $-\sigma \in K$. Furthermore, we will say that a symmetric triangulation is *aligned with hemispheres* if there exists a sequence of subcomplexes of K $H_0 \subset ... \subset H_n$ such that each H_d is both homeomophic to a d-ball and contained in the d-skeleton of K, while for $1 \leq d \leq n$, $\partial H_d = \partial(-H_d) = H_d \cap -H_d = H_{d-1} \cup -H_{d-1} \cong S^{d-1}$. It is easy to see that we can use the coordinate hyperplanes in $\mathbb{R}^{n+1}$ to construct a sequence $H_0 \subset ... \subset H_n$ with the aforementioned properties for the octahedral subdivision K of S^n, so we may observe that K is a symmetric triangulation aligned with hemispheres. In a symmetric triangulation K aligned with hemispheres, for any $\sigma \in K$, we will term the minimal H_d or $-H_d$ which contains σ the *carrier hemisphere* of σ. We now have enough terminology available to state and prove Ky Fan's combinatorial lemma in the mildly generalized form due to Prescott and Su [34].

Lemma 2. *Let K be a symmetric triangulation of S^n aligned with hemispheres. Furthermore, suppose that K has a labeling using the labels*
$\{\pm 1, \pm 2, ..., \pm m\}$ such that (i) the labels of antipodal vertices sum to 0 (the labeling is anti-symmetric*) and (ii) the labels at adjacent vertices never sum to 0 (the labeling is* complementary*). Then the number of positive alternating n-simplices is odd and equal to the number of negative alternating n-simplices. In particular, $m \geq n+1$.*

Proof. To prove our lemma, we will construct an auxiliary graph associated with K whose paths have as endpoints either alternating n-simplices or 0-simplices. Consequently, we will derive our theorem by simply counting the endpoints of the paths obtained.

In order to define our auxiliary graph, we will need a few definitions. Let us call an alternating or almost-alternating simplex *agreeable* if the sign of the simplex matches the sign of its carrier hemisphere. For example, if it is agreeable, a simplex with its smallest label negative must be carried by some hemisphere $-H_d$.

Now, let us define the auxiliary graph G. A simplex $\sigma \in K$ carried by the hemisphere $\pm H_d$ is a vertex of G if it is one of the following:

(1) an agreeable alternating $(d-1)$-simplex;

(2) an agreeable almost-alternating d-simplex; or

(3) an alternating d-simplex.

A pair of vertices representing the simplices σ and τ are adjacent if they satisfy the following conditions:

(a) one is a facet of the other;

(b) $\sigma \cap \tau$ is alternating; and

(c) the sign of $\sigma \cap \tau$ and the sign of the carrier hemisphere of $\sigma \cup \tau$ are the same.

Now, let us study the neighbourhood of each of G's vertices. For the moment, we will not discuss simplices carried by either $\pm H_0$ or $\pm H_n$. Naturally, with this restriction, there are three cases: one for each of the kinds of vertices we defined above.

Case 1: If σ is an agreeable alternating $(d-1)$-simplex with carrier $\pm H_d$, then it is a facet of precisely two d-simplices, each of which (as only one vertex is added), must be either agreeable or almost-agreeable in in the same carrier as σ. Thus, the adjacency conditions (a)-(c) are satisfied between these two simplices and σ, so the vertex representing σ in G has degree two.

Case 2: If σ is an agreeable almost-alternating d-simplex with carrier $\pm H_d$, then it has two facets which are agreeable alternating $(d-1)$-simplices (which are obtained by deleting one of the exactly two vertices which appear consecutively and with the same sign in the almost-alternating d-simplex). Adjacency condition (b) is then trivially satisfied, and (c) is satisfied because an almost-alternating simplex must have the same sign as its two alternating facets. None of σ's other facets satisfy (b), and the same is true for any $(d+1)$-simplices of which σ is a facet, so, once again, the vertex representing σ in G has degree two.

Case 3: If σ is an alternating d-simplex with carrier $\pm H_d$, then the only way to obtain an alternating facet of σ whose sign agrees with the sign of the carrier hemisphere of σ is to delete the smallest label in absolute value (if σ is not agreeable) or to delete the largest label in absolute value (if σ is agreeable). Thus, the vertex representing σ has degree at least 1. Additionally, σ is the facet of two simplices: one in H_{d+1} and one in $-H_{d+1}$, but, of course, it may only be adjacent to one of these. Thus, just as before, the vertex representing σ in G has degree two.

So, all those vertices which represent simplices not carried by either$\pm H_0$ or $\pm H_n$ have degree two. If σ is carried by $\pm H_0$, then it is the point $\pm H_0$, which is an alternating simplex with no facets. Thus, the vertex representing it has degree 1. If it is carried by $\pm H_n$, then it is not the facet of any other simplex, so, if it is agreeable almost-alternating, then the vertex representing it has degree two, and, if it is alternating, then the vertex representing it has degree 1. Thus, all vertices of G have degree two except for the vertices representing $\pm H_0$ and those representing alternating n-simplices. Thus, G is a collection of disjoint cycles and paths in which the endpoints of the paths represent either alternating n-simplices or $\pm H_0$.

Now, note that if we start with a path in G $v_1 v_2 ... v_k$ formed from the simplices representing $\sigma_1, \sigma_2, ..., \sigma_k$, and consider the vertices representing the simplices $-\sigma_1, -\sigma_2, ..., -\sigma_k$, then we observe that the vertices representing these simplices form a path in G as well. Let's call this new path the *antipodal path* of our original path. Observe that no path in G may have antipodal endpoints, as, in this case either the central vertex or central pair of vertices would need to represent antipodal simplices (which cannot occur). Thus, all paths of G must come in pairs, which means that the number of endpoints of these paths is a multiple of 4. Consequently, there are twice an odd number of alternating n-simplices in K. Thus, as each positive alternating n-simplex has a negative alternating n-simplex as its antipode, there are an odd number of positive alternating n-simplices and an equal number of negative alternating n-simplices, as required. □

With Prescott and Su's lemma in hand, we can now proceed to establish the precise statement of Ky Fan's theorem which we shall use in a manner quite similar to the remainder of Ky Fan's original proof [23]. To accomplish this goal, we will prove a theorem from which version (1) and version (2) of Ky Fan's theorem follow as corollaries (we will only prove the first version explicitly, as the proof of the second version is nearly identical).

Our proof will require a few basic definitions and facts from measure theory which we will state here. The general topology textbook [33] provides more complete coverage for the interested reader. The *Lebesgue number* of a collection of closed subsets $\mathcal{C}$ of a metric space X is any number $\epsilon > 0$ such that if any subset $A \subseteq X$ of diameter $\leq \epsilon$ intersects all of the elements of some subcollection $\mathcal{C}'$ of $\mathcal{C}$, then the intersection of all the elements of $\mathcal{C}'$ is non-empty. Similarly, the *Lebesgue number* of an open cover $\mathcal{O}$ of a metric space X is any number $\epsilon > 0$ such that if a subset $A \subseteq X$ has diameter $< \epsilon$, then A is contained in at least one element of $\mathcal{O}$. It is a basic result of topology (often called the Lebesgue number lemma) that any open cover of compact metric space (such as the sphere) has at least one Lebesgue number and that any finite collection of closed sets has at least one Lebesgue number. Also, let us define the mesh of a simplicial complex K as the maximum diameter of any simplex in K. Note that taking the barycentric subdivision of K reduces the mesh, so, by taking barycentric subdivisions repeatedly, we can ensure that the mesh of the triangulation of a space is as small as we wish.

Theorem 2. *Let n and m be two positive integers, and let $\{A_1, -A_1, A_2, -A_2, ..., A_m, -A_m\}$ be $2m$ closed subsets of S^nsatisfying the following two conditions:*
(i) for any two antipodal points x and $-x$ in S^n, there exists an index $i \in [m]$ such that $x \in A_i$ and $-x \in -A_i$;
(ii) $A_i \cap -Ai = \emptyset$ for all $i \in [m]$.
Then there exist $n+1$ indices $k_1, k_2, ..., k_{n+1}$ such that $1 \leq k_1 < k_2 < ... < k_{n+1} \leq m$ such that $A_{k_1} \cap -A_{k_2} \cap A_{k_3} \cap -A_{k_4} \cap ... \cap (-1)^n A_{k_{n+1}} \neq \emptyset$.

Proof. Let $\{A_1, A_2, ..., A_m\}$ be a finite collection of closed sets satisfying the hypotheses of our theorem. For each $i \in [m]$, let d_i be the distance between A_i and $-A_i$, and let d_0 be the Lebesgue number of the collection $\{A_1, -A_1, A_2, -A_2, ..., A_m, -A_m\}$ (which exists because S^n is a compact metric space). Then take K to be a simplicial complex formed by taking the barycentric subdivision of the octahedral subdivision of S^n until the mesh of K is strictly less than $min\{d_0, d_1, ..., d_m\}$.

Now, to apply Lemma 2, we need an anti-symmetric labeling of K with no complementary edges by the the labels $\{\pm 1, \pm 2, ..., \pm m\}$. One natural way of specifying such a labeling is given by hypothesis (i) in our theorem. Condition (i) guarantees that for any pair of antipodal points x and $-x$ in S^n, there is at least one index $i \in [m]$ such that $x \in A_i$ and $-x \in -A_i$. So, for each pair of antipodal points, we can simply pick one of these indices i and assign x the label i, while assigning $-x$ the label $-i$. This takes care of anti-symmetry. Also, complementarity is satisfied, as we assumed that the mesh of K was less than each d_i. Thus, by Lemma 2, there exists a positive alternating n-simplex with labels $k_1, k_2, ..., k_{n+1}$ such that $1 \leq k_1 < k_2 < ... < k_{n+1} \leq m$ in K. Due to the way we constructed our labeling, this n-simplex has one vertex in each of the sets $A_{k_1}, -A_{k_2}, A_{k_3}, -A_{k_4}, ..., (-1)^n A_{k_{n+1}}$. Thus, the intersection of these $n+1$ sets in non-empty, as, since we chose the mesh of K to be less than d_0, the n-simplex we have described is a subset of S^n satisfying the conditions of the Lebesgue number lemma. □

The following corollary follows immediately from Theorem 2.

Corollary 3. *Let n and m be two positive integers, and let $\{A_1, -A_1, A_2, -A_2, ..., A_m, -A_m\}$ be $2m$ closed subsets covering the n-ball D^n and satisfying the following two conditions:*
(i) for any two antipodal points x and $-x$ on the boundary S^{n-1} of D^n, there exists an index $i \in [m]$ such that $x \in A_i$ and $-x \in -A_i$;
(ii) $A_i \cap -Ai = \emptyset$ for all $i \in [m]$.
Then there exist $n+1$ indices $k_1, k_2, ..., k_{n+1}$ such that $1 \leq k_1 < k_2 < ... < k_{n+1} \leq m$ such that either $A_{k_1} \cap -A_{k_2} \cap A_{k_3} \cap -A_{k_4} \cap ... \cap (-1)^n A_{k_{n+1}} \neq \emptyset$ or $-A_{k_1} \cap A_{k_2} \cap -A_{k_3} \cap A_{k_4} \cap ... \cap (-1)^{n+1} A_{k_{n+1}} \neq \emptyset$.

Now, finally, let us quickly establish the version of Ky Fan's Theorem which we shall need.

Corollary 4. *Let $\mathcal{A}$ be a finite collection of subsets covering S^n, all of which are either open or closed. Furthermore, assume that there is a linear order on $\mathcal{A}$, and that for all $A \in \mathcal{A}$, $A \cap -A = \emptyset$. Then there exist sets $A_1 < A_2 < ... < A_{n+2}$ in $\mathcal{A}$ and a point $x \in S^n$ such that $(-1)^i x \in A_i$ for all $i \in [n+2]$.*

Proof. Following Ky Fan, let's first prove this result just for closed sets by considering the $(n+1)$-ball D^{n+1} of which S^n is the boundary. For each set A_i in $\mathcal{A}$, let C_i be the closed set which consists of the union of all segments joining the origin to a point of A_i, and let $C_{-i} = -A_i$. Then, applying Corollary 3 to the $(n+1)$-ball together with the $2m$ sets $\{\pm C_i\}$, we see that there must exist $n+2$ indices $1 \leq k_1 < k_2 < ... < k_{n+2} \leq m$ such that either $C_{k_1} \cap -C_{k_2} \cap C_{k_3} \cap -C_{k_4} \cap ... \cap (-1)^{n+1} C_{k_{n+2}} \neq \emptyset$ or $-C_{k_1} \cap C_{k_2} \cap -C_{k_3} \cap -C_{k_4} \cap ... \cap (-1)^{n+2} C_{k_{n+2}} \neq \emptyset$. Since for any index i, $C_{-i} = -A_i \subseteq S^n$, in either case, we are done.

To see that the same result holds for open sets as well, it suffices to recall an easy argument of Greene [13]. Suppose that S^n is covered by a collection of open sets $\mathcal{A} = \{A_1, ...A_m\}$. Now, select a Lebesgue number d_0 for this open cover (which exists, as S^n is compact). This is a number such that for any $x_i \in S^n$, the closed ball given by the closure $\bar{B}(x_i, d_0)$ of the open ball $B(x_i, d_0)$. Note that, by compactness, there exists a finite collection $\mathcal{O}$ of open balls $B(x_i, d_0)$ which cover S^n. Now, for each index j, let C_j be the union of all the closed balls $B(\bar{x_i}, d_0)$ such that $B(x_i, d_0)$ is in $\mathcal{O}$ which are contained in A_j. Then, each of the sets C_j is closed, and the union of all such C_j covers S^n. These sets also satisfy the other hypotheses of our corollary. Consequently, our corollary must hold for this particular collection of closed sets, which is easily seen to imply that it holds for the collection of open sets $\mathcal{A} = \{A_1, ...A_m\}$. □

1.5 Kneser Graphs

In the introduction to this section, we stated that we would prove Kneser's conjecture by examining simplicial complexes associated to any given graph whose topological properties are closely tied the chromatic number of the graph. We then proceeded to show that the box complex was a simplicial complex associated to any graph whose connectivity placed a lower bound on the chromatic number of a graph. However, while the box complex is extremely convenient for theoretical purposes, as it admits an explicit and intuitive fixed point free involution, for actually proving Kneser's conjecture, it is more convenient to work with a different simplicial complex known as the neighbourhood complex of a graph.

For any graph G, the *neighbourhood complex* $N(G)$ is defined to be the simplicial complex on the vertex set $V(G)$ whose maximal simplices with respect to inclusion are the neighbourhoods of the vertices of G. Equivalently, we can define the neighbourhood complex as $N(G) := \{A \subseteq V(G) : CN(A) \neq \emptyset\}$. In the next chapter, a subcomplex of the first barycentric subdivision of the neighbourhood complex known as the Lovász complex will also be of interest to us, so we will define it here as well. The *Lovász complex* $L(G)$ is the simplicial complex whose vertex set is $\{A \subseteq V(G) : CN^2(A) = A\}$ and which has as its simplices all the chains of elements of its vertex set under the ordering given by inclusion. We briefly observe that, as with the box complex, $L(G)$ admits an intuitive fixed point free involution, in this case induced by the action of CN on the vertex set of $L(G)$. To see this fact, observe that for all $A_i, A_j \in V(L(G))$, $A_i \subset A_j \Rightarrow CN(A_i) \neq A_j$ and $CN(A_j) \neq A_i$. Consequently, for any simplex $\mathcal{A}$ corresponding to a chain $A_1 \subset ... \subset A_n$ of vertices of $L(G)$, $\mathcal{A} \cap CN(\mathcal{A}) = \emptyset$. In particular, $\mathcal{A} \neq CN(\mathcal{A})$.

Rather than tediously reproving the results we have already established for the box complex, it is far more convenient to simply prove that $B(G)$, $N(G)$ and $L(G)$ are homotopy equivalent. One easy way to prove this fact is by introducing the notion of a collapse.

Let K be a simplicial complex, and let $\sigma, \tau \in K$. Moreover, suppose that the following statements hold:

(1) $\tau \subset \sigma$ and $dim \sigma = dim \tau + 1$;

(2) σ is a maximal simplex; and

(3) σ is the only maximal simplex of K which contains τ.

Then we say that the simplicial complex $K \setminus \{\tau, \sigma\}$ is an *elementary collapse* of K. We call a sequence of elementary collapses, the first of which is an elementary collapse of K, a collapse of K. It is easy to see that an elementary collapse (and, consequently, a collapse) is a homotopy equivalence, as it is a retraction, and the inclusion map from $K \setminus \{\tau, \sigma\}$ to K is its homotopy inverse. When it causes no ambiguity, we will often denote an elementary collapse from K to $K \setminus \{\tau, \sigma\}$ by (τ, σ).

Finding a collapsing sequence from a simplicial complex to a subcomplex of said complex can be rather complicated, so various tools exist to simplify the problem of finding collapsing sequences. We will examine two such tools: closure operators and discrete Morse functions.

We will call an order-preserving map f from a poset P to itself a descending closure operator if $f^2 = f$ and $f(x) \leq x$ for all $x \in P$. Analogously, we term an order-preserving map f from a poset P to itself a ascending closure operator if $f^2 = f$ and $f(x) \geq x$ for all $x \in P$. That ascending and descending closure operators induce collapses is a necessary result for many of the proofs we will give in this thesis. However, in order to establish this fact, we will need to recall the definition of the *link* of a simplex $\sigma \in K$:

$$lk_K(\sigma) := \{\tau \in K : \sigma \cap \tau = \emptyset \text{ and } \sigma \cup \tau \in K\}.$$

Using the link, Kozlov has given a particularly elegant proof the fact that ascending and descending closure operators induce collapses [21], which we will reproduce here.

Theorem 3. *Let P be a poset, and let f be a descending closure operator. Then the order complex $\Delta(P)$ of P collapses onto $\Delta(f(P))$. Consequently, by considering the dual poset, we observe that the same is true for an ascending closure operator.*

Proof. The proof proceeds by induction on $|P| - |f(P)|$.

If $|P| = |f(P)|$, then f must be the identity map if it is a descending closure operator. Thus, $\Delta(f(P))$ is obtained by $\Delta(P)$ by a trivial collapse (one in which no elementary collapses occur).

Now, let us suppose that $P \setminus f(P) \neq \emptyset$, and let x be one of the minimal elements of $P \setminus f(P)$. Also, note that for any $x \in P$, by the definition of the link and order complex, we have the following equality:

$$lk_{\Delta(P)}(x) = \Delta(P_{>x}) * \Delta(P_{<x}).$$

Now, note that, as x is one of the minimal elements of $P \setminus f(P)$, f fixes every element of $P_{<x}$. Consequently, as f is an order-preserving map, $f(x)$ must be a maximal element of $P_{<x}$. Therefore, we have the following equation:

$$lk_{\Delta(P)}(x) = \Delta(P_{>x}) * \Delta(P_{<f(x)}) * \{f(x)\}.$$

Consequently, $lk_{\Delta(P)}(x)$ is a cone with apex $f(x)$, so if we let $A_1, ..., A_n$ be the simplices of $\Delta(P_{>x}) * \Delta(P_{<f(x)})$ ordered so that the dimension is weakly decreasing, then $(A_1 \cup \{x\}, A_1 \cup \{x, f(x)\}), ..., (A_n \cup \{x\}, A_n \cup \{x, f(x)\})$ is a sequence of elementary collapses leading from $\Delta(P)$ to $\Delta(P \setminus \{x\})$.

Moreover, as f restricted to $P \setminus \{x\}$ remains a descending closure operator, by our induction hypothesis, $\Delta(P \setminus \{x\})$ collapses onto $\Delta(f(P \setminus \{x\})) = \Delta(f(P))$. Thus, composing the two collapses which we have obtained, we see that $\Delta(P)$ collapses onto $\Delta(f(P))$, as required. □

Now, we can easily establish that $B(G)$, $N(G)$ and $L(G)$ are homotopy equivalent by exhibiting explicit descending closure operators linking $B(G)$ to $N(G)$, and then exhibiting an explicit deformation retraction from $N(G)$ to $L(G)$. The second of these proofs is a standard argument which may be found in either Lovász's original paper on Kneser's Conjecture [24] or Matoušek's book on using the Borsuk-Ulam Theorem [27].

Proposition 3. *$B(G)$ is homotopy equivalent to $N(G)$.*

Proof. Firstly, let us note that $N(G)$ is isomorphic to a subcomplex of $B(G)$. In particular, the map $g : N(G) \to B(G)$ defined by $g(A) = A \uplus \emptyset$ for all $A \in N(G)$ is a bijection between the set of simplices of $N(G)$ and the set of simplices $A \uplus \emptyset$ such that $A \uplus \emptyset \in B(G)$.

Now, let us take the face poset of $B(G)$ $P(B(G))$ and define the poset map $f : P(B(G)) \to P(B(G))$ by $f(A \uplus B) \to A \uplus \emptyset$ for all $A \uplus B \in P(B(G))$. Then we note that, for any $A \uplus B, C \uplus D \in B(G)$, if $A \uplus B \subseteq C \uplus D$, then $A \uplus \emptyset \subseteq C \uplus \emptyset$, so f is order-preserving. Moreover, $f^2 = f$ and $f(A \uplus B) = A \uplus \emptyset < A \uplus B$ for any $A \uplus B \in P(B(G))$. Thus, f is a descending closure operator, whence, by Proposition 3, $B(G)$ collapses onto the order complex of $f(P(B(G)))$, which is isomorphic to the first barycentric subdivision of $N(G)$. This isomorphism is induced by the isomorphism g between the vertex sets of $P(N(G))$ and $f(P(B(G)))$ defined in the first paragraph of this proof. □

Proposition 4. *$N(G)$ is homotopy equivalent to $L(G)$.*

Proof. It suffices to construct a deformation retract from the barycentric subdivision of $N(G)$ onto $L(G)$. That is to say, we wish to construct a homotopy from $\Delta(P(N(G))) \times [0, 1]$ to $\Delta(P(N(G)))$ which fixes $L(G)$.

In what follows, let us represent the barycentric subdivision of $N(G)$ by $N_1 := \Delta(P(N(G)))$. The simplicial complex $L(G)$ is a subcomplex of N_1, and it is not difficult to see that $CN^2 : V(N_1) \to V(L(G))$ is a simplicial map of N_1 into $L(G)$, as for any $A, B \in V(N_1)$, $A \subseteq B$ implies that $CN^2(A) \subseteq CN^2(B)$. Now, thinking of our simplicial complexes as topological spaces, we define a function which will be useful in constructing our desired homotopy. Define a map $f : N_1 \to L(G)$ as the canonical affine extension of CN^2. We will now show that f is a homotopy of N_1 with itself which fixes $L(G)$.

Let $x \in N_1$. Furthermore, suppose that the simplex $A_1 \subset A_2 \subset ... \subset A_n$ of N_1 contains x in its interior (this must be the case for some simplex in N_1). Then $f(x)$ lies in the simplex of N_1 spanned by $CN(A_1), CN(A_2), ..., CN(A_n)$. Now, all of $A_1, A_2, ..., A_n, CN(A_1)$, $CN(A_2), ..., CN(A_n)$ are vertices in the first barycentric subdivision of the simplex $CN^2(A_n)$ of $N(G)$, and so both x and $f(x)$ lie in a common simplex. Thus, for any $t \in [0, 1]$, the point $(1 - t)x + tf(x) \in N_1$ is well-defined, and, consequently, $F : N_1 \times [0, 1] \to N_1$ given by $F(x, t) = (1 - t)x + tf(x)$ is the required homotopy fixing $L(G)$. □

At this point, we turn to the proof of Kneser's conjecture. We will establish this result in two steps. Firstly, we will use the theory of closure operators which we have developed to show that the neighbourhood complex of a Kneser graph is homotopy equivalent to the order complex of a particular subposet of the boolean lattice. Then we will determine the connectivity of this order complex using discrete Morse theory. Of course, if we wish to apply discrete Morse theory, we had best introduce the techniques we shall require, so we will now proceed to establish two of the area's most fundamental results by elementary arguments.

Firstly, let us show that the existence of a sequence of elementary collapses from a simplicial complex K to a subcomplex K' of K is equivalent to the existence of a discrete Morse function on K (the primary tool used in discrete Morse theory) which has K' as its collection of critical simplices. More precisely, we will display equivalence with the existence of acyclic matchings, a notion in Chari's [6] reformulation of discrete Morse theory for posets which precisely corresponds with Forman's [11] notion of a discrete Morse function in the original theory.

For any poset P, we define a partial matching on P to be a pair (A, f), where $A \subseteq P$ is a set, and $f : A \to P \setminus A$ is an injective map such that $f(x) > x$ for all $x \in A$. Then, for any partial matching of P, we will term the elements of $P \setminus (A \cup f(A))$ critical elements. Such a partial matching on P is called acyclic if there exists no sequence of distinct elements $a_1, ..., a_n \in A$ (for $n \geq 2$) such that $f(a_1) > a_2, f(a_2) > a_3, ..., f(a_n) > a_1$.

The motivation behind the terminology 'partial acyclic matching' originates in the connection between posets and directed graphs. As we will freely make use of this connection, let us recall it here. Given a poset P, we may always associate with it a digraph $D(P)$ as follows: take $V(D(P)) = V(P)$, and let the arcs of $D(P)$ be the edges of the Hasse diagram of P oriented downward. Then, given a partial matching (A,f) on a poset P, we may construct the digraph $D(A,f)$ associated with the partial matching (A,f) by first taking the digraph $D(P)$ and then reorienting all the edges $\{a, f(a)\}$ such that $a \in A$ upward. Now, note that the collection of edges $\{a, f(a)\}$ such that $x \in A$ form a matching in the undirected graph obtained by forgetting the orientation on the arcs of $D(P)$. Moreover, if (A,f) is acyclic, then the digraph $D(A,f)$ has no directed cycles, as any such directed cycle expressed as a cyclically ordered sequence of elements of P would need to contain a subsequence (for some $n \geq 2$) $(a_1, f(a_1), a_2, f(a_2), ..., a_n, f(a_n))$ such that $a_1, a_2, ..., a_n \in A$ and $f(a_1) > a_2, f(a_2) > a_3, ..., f(a_n) > a_1$, contradicting the fact that (A,f) is a partial acyclic matching. The key point to note here is that given a partial acyclic matching (A,f) on a poset P, we may construct the acyclic digraph $D(A,f)$ whose upward oriented edges form a matching in the undirected graph obtained from $D(A,f)$ by forgetting the orientation of its arcs.

Proposition 5. *Let K be a simplicial complex, and let K' be a subcomplex of K. Then the following are equivalent:*
(a) there is a sequence of elementary collapses leading from K to K'.
(b) there is a partial acyclic matching on the poset $P(K)$ whose critical simplices are precisely those in $P(K')$.

Proof. Firstly, suppose that $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ is a sequence of elementary collapses from K to K'. The let us define a partial matching $f : A := \{x_i\}_{i=1}^{n} \to P(K) \setminus A$ by $f(x_i) = y_i$ for all $i \in [n]$. Now, suppose that (for some $s \in [n]$) there is a sequence of distinct elements $a_1, ..., a_s \in A$ such that $f(a_1) > a_2, f(a_2) > a_3, ..., f(a_s) > a_1$. Then, by the definition of an elementary collapse, we know that the elementary collapse $(a_1, f(a_1))$ precedes $(a_2, f(a_2))$, which precedes $(a_3, f(a_3))$,..., which precedes $(a_s, f(a_s))$, and that, consequently, it is not the case that $f(a_s) > a_1$, which is a contradiction. Thus, (A,f) is a a partial acyclic matching on the poset $P(K)$ whose critical cells are precisely those in $P(K')$, as required.

Now, let us proceed to establish the converse by induction on $|K \setminus K'|$. If $K = K'$, then there is a trivial (empty) sequence of elementary collapses from K to K'. Now, suppose that $K \setminus K' \neq \emptyset$ and that we have a partial acyclic matching (A,f) on $P(K)$ whose critical simplices are precisely those in $P(K')$. Then observe that at least one vertex$a_i \in A$ must be a source of the acyclic digraph $D(A,f)$, as, since K' is a subcomplex of K, it cannot be the case that $a_i < b$ for any $a_i \in A$ and $b \in P(K')$. Thus, vertices $a_i \in A$ may only be directly below vertices $a_i \in A$ or vertices $f(a_i) \in f(A)$. However, if every vertex $a_i \in A$ is either directly below some vertex $a_j \neq a_i$ such that $a_j \in A$ or directly below some vertex $f(a_j)$ such that $a_j \neq a_i$ is in A, then we can construct a directed cycle in $D(A,f)$ as follows. Begin at any $a_i \in A$. Either there is an arc from $a_j \in A$ for some $a_j \neq a_i$ to a_i, or there is arc from $f(a_j) \in f(A)$ for some $a_j \neq a_i$ to a_i. In either case, trace the arc from its endpoint a_i back to the initial point of the arc. This initial point is either a vertex $a_j \in A$, in which case we repeat the previous argument to discover a new vertex, or is a vertex $f(a_j) \in f(A)$, in which case we trace backwards along the arc from a_j to $f(a_j)$ to discover the vertex a_j.

Repeatedly tracing backwards along arcs as described above until we repeat a vertex, we observe that we must eventually repeat a vertex, as there are only finitely many vertices in $A \cup f(A)$. This constructs a directed cycle in $D(A,f)$, which cannot exist, so we see that there must be some vertex $a_i \in A$ which is only directly below the vertex $f(a_i) \in f(A)$. Thus, we can collapse K to $K \setminus \{a_i, f(a_i)\}$, so, as $K \setminus \{a_i, f(a_i)\}$ collapses to K' by our inductive hypothesis, we see that there is a sequence of elementary collapses from K to K', as required. □

However, while the argument above will be extremely useful to us in later sections of this thesis, in order to prove Kneser's conjecture, we shall require a more topological generalization of this result. However, in order for this generalization to make sense, we will need to introduce the idea of a CW-complex.

Following Massey [26], we define a *CW-complex* to be a topological space X together with an ascending sequence of closed subspaces of X

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq ... \subseteq X^n$$

for some $n \in \mathbb{N}$ which satisfy the following conditions:

(i) X^0 has the discrete topology;
(ii) for $n > 0$, X^n is obtained from X^{n-1} by adjoining a collection of n-cells D^n_α to X^{n-1} via maps $f_\alpha : S^{n-1} \to X^{n-1}$ (i.e. X^n is the quotient space of $X^{n-1} \bigsqcup_\alpha D^n_\alpha$ under the identification $x \sim f_\alpha(x)$ for all $x \in \partial D^n_\alpha$, where $\bigsqcup$ denotes disjoint union);
(iii) X is the union of the subspaces X^i for $i \geq 0$.

For any CW-complex X, we will say that a *subspace* $A \subseteq X$ is a subcomplex of X if it is a union of cells of X such that the closure of each cell in A is contained in A. A itself is obviously a CW-complex, as the fact that the closure of each cell $f_\alpha(D^n_\alpha \setminus \partial D^n_\alpha)$ in A is in A guarantees that the image of each attaching map $f_\alpha : S^{n-1} \to X^{n-1}$ is in A. Additionally, we will term a CW-complex X *regular* if all its attaching maps f_α are homeomorphisms.

Before proceeding further, let us note two useful facts about CW-complexes. Firstly, any simplicial complex K is easily seen to be a regular CW-complex by simply taking its n-simplices as its n-cells for each $n \in \mathbb{N}$. Secondly, we may easily define the homology groups of a CW-complex in a manner analogous to our definition of the homology groups of a simplicial complex. It is then possible to prove with some effort [26] that if X is a CW-complex with no n-dimensional cells, then $H_n(X) = 0$. Additionally, it may be shown that that if X is a 0-connected CW-complex with no 1-dimensional cells, then X is 1-connected. These results may seem trivial at first glance, but they do require some technical machinery such as the cellular approximation theorem in order to prove formally. We are not interested in this machinery, so we will omit the proofs of these assertions, referring the reader to [16] for a full development of the theory of CW-complexes.

The particularly insightful reader will likely have noticed by this point that we could have proven Proposition 5 for any regular CW-complex and its subcomplexes using a very similar argument. We will not need this generality, but we will need an extension of this result (which was originally proved by Forman [10], although we prefer to mimic Kozlov's subsequent proof [21]). The proof of this extension will once again require a basic fact about CW-complexes which we will state without proof. This result is proved in nearly all texts which introduce the notion of a CW-complex, as it guarantees that CW-complexes with the same cells and homotopy equivalent attachment maps have the same homotopy type (without which the definition of a CW-complex is very hard to work with). One example of such a text is [21].

Proposition 6. *Let X^n and $\tilde{X}^n$ be two homotopy equivalent topological spaces, and let $h : X^n \to \tilde{X}^n$ be a homotopy equivalence. Moreover, suppose that D_α^{n+1} is a cell with attachment maps $f_\alpha : \partial D_\alpha^n \to X^n$ and $\tilde{f}_\alpha : \partial D_\alpha^{n+1} \to \tilde{X}^n$ such that $h \circ f_\alpha = \tilde{f}_\alpha$. Then the topological space $X^n \sqcup_{f_\alpha} D_\alpha^{n+1}$ is homotopy equivalent to $\tilde{X}^n \sqcup_{\tilde{f}_\alpha} D_\alpha^{n+1}$, where by $X^n \sqcup_{f_\alpha} D_\alpha^{n+1}$, we denote the disjoint union of X^n and D_α^{n+1} together with the identification given by f (and similarly for $\tilde{X}^n \sqcup_{\tilde{f}_\alpha} D_\alpha^{n+1}$).*

In addition to the proposition above, we will also need to know a little bit about the connection between partial acyclic matchings and linear extensions. Recall that a *linear extension* of a poset P is a total order L on the same ground set as P such that if $x \leq y$ in P, then $x \leq y$ in L.

Proposition 7. *A partial matching (A, f) on a poset P is acyclic if and only if there exists a linear extension L of P such that for all $a \in A$, a and $f(a)$ follow consecutively in L.*

Proof. Firstly, let us suppose that L is a linear extension of P which admits a partial matching (A, f) such that for all $a \in A$, a and $f(a)$ follow consecutively in L. Then, for a contradiction, let us also suppose that we have a sequence of distinct elements $a_1, a_2, ..., a_n \in A$ (for some $n \geq 2$) such that $f(a_1) > a_2$, $f(a_2) > a_3$, ..., $f(a_n) > a_1$. Since each pair $(a_i, f(a_i))$ appears consecutively in L, we must also have $f(a_1) > a_1 > f(a_2) > a_2 > ... > f(a_n)$, which contradicts the inequality $f(a_n) > a_1$. Consequently, if L is a linear extension of P which admits a partial matching (A, f) such that for all $a \in A$, a and $f(a)$ follow consecutively in L, then (A, f) is acyclic.

Conversely, if we are given an acyclic matching (A, f) on P, then we can inductively define a linear extension L of P. At each inductive step, we let T be the set of elements of P which are already ordered according to L, and let W be the set of minimal elements of $P \setminus T$. We begin with $T = \emptyset$. Then, at each step, either there exists a critical element $c \in W$ of (A, f), or all the elements of W are in the matching given by (A, f).

If there exists a critical element $c \in W$ of (A, f), then we may simply add c to L as the largest element and proceed with $T \cup \{c\}$. Otherwise, all the elements of W are in the matching given by (A, f). In this case, consider the induced subgraph of $D(A, f)$ induced by $W \cup f(W)$. If there exists an $a \in W$ such that the only element of P smaller than $f(a)$ is a itself, then we may add the elements a and $f(a)$ on top of L and proceed with $T \cup \{a\} \cup \{f(a)\}$. If not, then the outdegree of each element $f(a)$ is at least one. Thus, as each element of the induced subgraph of $D(A, f)$ induced by $W \cup f(W)$ has outdegree at least one, G must have a directed cycle, contradicting the acyclicity of (A, f).

Hence, as P is finite, the inductive process given above always concludes with the required linear extension L of P. □

We can now state the result regarding partial acyclic matchings which we have been pursuing. Here, as we are proving a result about CW-complexes, we will need a more topological notion of an elementary collapse (which is easily seen to generalize the combinatorial notion we introduced earlier). Let X be a topological space, and let Y be a subspace of X. Then we say that Y is obtained from X by an *elementary collapse* if X can be represented as a result of attaching a ball D^n_α to Y along one of the hemispheres of D^n_α. More precisely, Y is obtained from X by an elementary collapse if there exists a map $\tilde{f}_\alpha : D^n_\alpha \to Y$ such that $X = Y \sqcup_{\tilde{f}_\alpha} D^{n-1}_\alpha$ is one of the closed hemispheres bounded by ∂D^n_α.

Proposition 8. *Let X be a regular CW-complex, and let $P(X)$ be its face poset. Moreover, let (A, f) be a partial acyclic matching on $P(X)$, and, for each $i \in \mathbb{N}$, let c_i denote the* *number of critical i-dimensional cells of $P(X)$ with respect to (A, f). Then X is homotopy equivalent to some CW-complex X_c which has precisely c_i i-dimensional cells for each $i \in \mathbb{N}$.*

Proof. Following Kozlov [21], we will proceed by induction on $|P(X)|$. Note that, if $|P(X)| = 1$, then there can be no matched cells, so the proposition is vacuously true. For the induction step, let us take the linear extension L of $P(X)$ such that for all $a\ inA$ a and $f(a)$ follow consecutively in L. That such a linear extension exists is guaranteed by Proposition 7. Additionally, let us suppose that σ is the largest cell in L. Now, either σ is critical or not. Thus, our proof divides into two cases.

Case 1: Suppose that σ is critical. Then let $\tilde{X} = X \setminus Int(\sigma)$, and let $g : \partial\sigma \to X$ be the attaching map of σ in X. The partial acyclic matching on $P(X)$ restricted to $\tilde{X}$ remains acyclic, as all we have done is deleted a cell, and the critical cells of $P(\tilde{X})$ are the same as those of $P(X)$ save that the cell σ is missing. Thus, by induction, there exists a CW-complex $\tilde{X}_c$ with the correct number of i dimensional cells for each i (except that we are off by one in the dimension of σ) and a homotopy equivalence $h : \tilde{X} \to \tilde{X}_c$. Hence, by Proposition 6, $X = \tilde{X} \sqcup_g \sigma \cong \tilde{X}_c \sqcup_{h \circ g} \sigma$. Therefore, if we set $X_c = \tilde{X}_c \sqcup_{h \circ g} \sigma$, then we see that we have $X \cong X_c$, as required.

Case 2: Now, suppose that σ is not critical. In this case, $f^{-1}(\sigma)$ and σ are matched, and, by the proof of the Proposition 7, $f^{-1}(\sigma)$ is maximal in $P(X) \setminus \{\sigma\}$. So, let $\tilde{X} = X \setminus (int(\sigma) \cup int(f^{-1}(\sigma)))$. Now, note that removing $(f^{-1}(\sigma), \sigma)$ is an elementary collapse, so there exists a homotopy equivalence $h : X \to \tilde{(}X)$. At the same time, by induction, there exists a CW-complex $\tilde{X}_c$ with c_i i-dimensional cells for each i and a homotopy equivalence $\tilde{h} : \tilde{x} \to \tilde{X}_c$. Consequently, setting $X_c = \tilde{X}_c$, we see that $\tilde{h} \circ h : X \to X_c$ is the desired homotopy equivalence, completing the proof. □

We are now ready to attack Kneser's Conjecture directly. As we promised at the beginning of this section, we will first show that the neighbourhood complex of any Kneser graph is homotopy equivalent to a particular subposet of the boolean lattice, and then will explicitly find the connectivity of the order complex of this subposet using discrete Morse theory. Before proceeding, however, we should recall the definition of the Kneser graphs and of the definition of the boolean lattice. Given a pair of positive integers n and k, the

Kneser graph $KG_{n,k}$ is the graph whose vertices are all k-element subsets of $[n]$, and in which two vertices are connected if the sets to which they correspond are disjoint. Perhaps the best known example of a Kneser graph is the ubiquitous Petersen graph: $KG_{5,2}$. Given some set X, the *boolean lattice* of X is the poset on the ground set 2^X whose elements are ordered by inclusion.

Lemma 3. *For any positive integers n and k for which the Kneser graph $KG_{n,k}$ is defined, the neighbourhood complex of $KG_{n,k}$ is homotopy equivalent to the order complex of the subposet of the Boolean lattice induced on $\{B \subseteq [n] : k \leq |B| \leq n-k\}$.*

Proof. Firstly, let us define a map $PS_k : 2^{[n]} \to 2^{2^{[n]}}$ which maps any $A \subseteq [n]$ to the collection of all subsets of A which have cardinality k. Then, using this map, let us define an order-preserving map $f : P(N(KG_{n,k})) \to P(N(KG_{n,k}))$ by $f(A) = PS_k(\cup A)$, where A is a non-empty collection of k-subsets of $[n]$, and $\cup A$ is the union of all subsets in the collection A. So, for example, $f(\{\{1,2\},\{2,3\}\}) = \{\{1,2\},\{1,3\},\{2,3\}\}$. Observe that $f^2 = f$ and $A \subseteq f(A)$ for any $A \in P(N(KG_{n,k}))$. Thus, f is an ascending closure operator, so, by Theorem 3, $\Delta(P(N(KG_{n,k})))$ collapses onto $\Delta(im(f))$, which is isomorphic as a poset to $\{B \subseteq [n] : k \leq |B| \leq n-k\}$. □

Wachs [39] and Kozlov [21] have separately shown that order complex of the poset above is homotopy equivalent to a wedge of spheres of dimension $n-2k$. However, their proofs use methods beyond the scope of this thesis, so we will instead establish a simpler result by a much easier proof.

Lemma 4. *The order complex of $\{B \subseteq [n] : k \leq |B| \leq n-k\}$ is $(n-2k-1)$-connected provided that $n-2k$ is at least two.*

Proof. Let's define a partial matching (A,f) on $\{B \subseteq [n] : k \leq |B| \leq n-k\}$ as follows. A consists of all sets in $\{B \subseteq [n] : k \leq |B| \leq n-k\}$ which both do not contain 1 and have cardinality at most $n-k-1$. For any $B \in A$, $f(B) = B \cup \{1\}$. This partial matching is acyclic because one can only go up in the poset by adding a 1 to a set, while it is impossible to go down by deleting 1 from a set. The only simplices which remain unmatched are the sets of cardinality k which contain a 1 and the sets of cardinality $n-k$ which do not contain a 1. Consequently, by Proposition 8, the order complex of $\{B \subseteq [n] : k \leq |B| \leq n-k\}$ is homotopy equivalent to a CW-complex with no cells besides 0-cells and $(n-2k)$-cells.

We know that this complex is 0-connected, as we can easily see that the order complex of $\{B \subseteq [n] : k \leq |B| \leq n-k\}$ is 0-connected provided that$n-2k$ is at least two. Consequently, as the order complex of $\{B \subseteq [n] : k \leq |B| \leq n-k\}$ is homotopy equivalent to a CW-complex which is 0-connected and has no cells of dimension $1 \leq i \leq n-2k-1$, it must be $(n-2k-1)$-connected. □

From the two lemmas we have just proven together with the fact that the connectivity of $N(G)$ provides a lower bound on the chromatic number, we can now effortlessly derive the lower bound on the chromatic number of the Kneser graphs needed in order to establish Kneser's Conjecture.

Theorem 4. *For any positive integers n and k for which the Kneser graph $KG_{n,k}$ is defined, $\chi(KG_{n,k}) \geq n-2k+2$.*

Chapter 2

Constructions

2.1 Introduction

Having introduced a method for finding topological lower bounds on the chromatic number, one naturally becomes interested in when besides for Kneser graphs such bounds are sharp. In particular, we will say that a graph is *topologically k-chromatic* if its chromatic number is k, and one of the topological lower bounds noted in the previous section is tight.

As it turns out, finding interesting classes of topologically k-chromatic graphs is fairly difficult. Cliques, bipartite graphs and odd cycles are obvious examples, but, besides these graphs and the Kneser graphs, it was decades after Lovász's original paper's publication before any significant progress was made in understanding either how to construct topologically k-chromatic graphs or what sort of properties are common to graphs in this class. However, there are a few easy structural observations one can make which are useful in guiding our intuition. We collect these results in the following proposition.

Proposition 9. *Let G be a graph and $N(G)$ be its neighbourhood complex. Then the following hold:*
(a) $N(G)$ is not contractible;
(b) G is both connected and non-bipartite if and only if $N(G)$ is connected (0-connected);
(c) if G has girth ≥ 5, then G is not topologically k-chromatic for any $k \geq 4$.

Proof. (a) The dimension of $N(G)$ is equal to $\Delta(G) - 1$. If $N(G)$ is contractible, then all its homology groups vanish. Consequently, we have that $\chi(G) \geq \Delta(G) + 2$. However, this cannot be, as $\chi(G) \leq \Delta(G) + 1$ for any graph G.

(b) Let us consider the 1-skeleton of $N(G)$ as a graph on $V(G)$. Note that in this new graph, which we will denote by $N(G)$, any two vertices u and v are adjacent in $N(G)$ if and only if u and v are the endpoints of a path of length two in G. Thus, if G is disconnected, then $N(G)$ is disconnected. Similarly, if G is connected and bipartite, then the bipartite classes of G are in different connected components of $N(G)$, so $N(G)$ is disconnected.

Finally, suppose that G is connected and non-bipartite. Then G contains an odd cycle $C = c_1c_2...c_{2r+1}$ for some $r \in \mathbb{N}$. For each vertex $v \in V(G)$, there is a path in $N(G)$ to some vertex c_i in C. If i is odd, then $c_ic_{i-2}...c_1$ is a path from c_i to c_1 in $N(G)$. If i is even, then $c_ic_{i-2}...c_{2r}c_1$ is a path from c_i to c_1 in $N(G)$. In either case, we observe that all vertices of $N(G)$ are connected to the vertex $c_1 \in V(N(G))$. Thus, $N(G)$ is connected.

(c) We will prove that graphs of girth greater than or equal to 5 cannot be topologically k-chromatic for any $k \geq 4$ by showing that the Lovász complex $L(G)$ of such a graph is 1-dimensional. To see this, suppose for a contradiction that $A \subset B \subset C$ is a 2-simplex of $L(G)$. This means that $CN(C) \subseteq CN(B) \subseteq CN(A)$. Thus, as $CN^2(A) = A$, $CN^2(B) = B$, $CN^2(C) = C$, $A \neq B$ and $B \neq C$, there must exist a vertex $u \in (CN(A) \cap CN(B)) \backslash CN(C)$, as well as a vertex $v \in CN(A) \cap CN(B) \cap CN(C)$. Hence, as these two sets are disjoint, and all elements of both A and B (of which there are at least two) are adjacent to both u and v, G contains a 4-cycle, contradicting our assumption that it has girth greater than or equal to 5. □

While the proposition above is not very exciting on its own, its implications for the problem of constructing interesting classes of topologically k-chromatic graphs are significant. First of all, part (a) tells us that $N(G)$ is never contractible. In some sense, this fact indicates why straightforward constructions are elusive. In topology, most constructions involve either gluing together contractible spaces or gluing together spaces along a contractible intersection. However, since graphs and subgraphs never correspond with contractible spaces, most sensible ways of combining graphs to form new graphs such as unions, clique sums and Hajos sums do not have straightforward translations into the realm of topology. We also cannot easily create graph operations mimicking topological construction techniques. Consequently, we will have to be somewhat craftier if we want to create new topologically k-chromatic graphs from known examples.

Part (b) of Proposition 9 tells us, in essence, that connected, topologically 2- and 3-chromatic graphs are precisely the usual bipartite and 3-chromatic graphs, so we may safely restrict our attention to the study of topologically k-chromatic graphs for $k \geq 4$.

Part (c) places a significant restriction on the classes of graphs which may be topologically k-chromatic for $k \geq 4$, and also suggests a place to start in considering topologically k-chromatic graphs. Namely, we might ask what can be said about topologically k-chromatic graphs of girth 4. Are there many such graphs? As it happens, topologically k-chromatic graphs of girth 4 abound for any given k, and we can easily construct a wide variety of such graphs using a graph operation which corresponds with suspending the neighbourhood complex and generalizes the Mycielski construction originally used by Mycielski in 1955 to construct triangle-free graphs of arbitrarily high chromatic number. The details of this construction, will be the focus of the next section, while many of its implications will play prominent roles in the next chapter.

2.2 The Generalized Mycielskian and Suspensions

One early use of topological lower bounds on the chromatic number arising from the study of simplicial complexes associated with graphs is in constructing graphs of arbitrarily large chromatic number. To do so, what is necessary is to pick a construction which has a well-understood effect on the topology of one's complex. In particular, we would like to pick a construction which, given a graph G, produces a new graph which always (or all but finitely often if we can apply the construction an arbitrary number of times) increases the topological lower bound on the chromatic number. One of the most powerful and well-studied of such constructions is the generalized Mycielskian of a graph.

Formally, following Gyárfás, Jensen and Stiebitz [15], we define the *generalized Mycielskian* of a graph G for any fixed $r \geq 1$. $M_r(G) = M_r(G, p_1, p_2, ..., p_r)$, where, for $1 \leq i \leq r$, p_i denotes a bijection $p_i : V(G) \to X_i$. Here, the map p_1 is the identity map on $V(G) = X_1$, and the sets $X_2, ..., X_r$ are any sets in bijective correspondence with $V(G)$ such that all the X_i such that $i \in [r]$ are pairwise disjoint. Additionally, for $1 \leq i \leq r-1$, let $E_i = \{p_i(x)p_{i+1}(y) : xy \in E(G)\}$, and let $E_r = \{p_r(x)z : x \in V(G)\}$, where z is a vertex not contained in any of $X_1, ..., X_r$. Then the vertex and edge sets of the generalized Mycielskian of G are as follows.

$$V(M_r(G)) := (\bigcup_{i=1}^{r} X_i) \cup \{z\} \tag{2.1}$$

$$E(M_r(G)) := E(G) \cup \bigcup_{i=1}^{r} E_i \tag{2.2}$$

While it is actually quite straightforward, the preceding definition may look a tad confusing at first glance, so we shall ease into its use with an easy example and a historically motivated proposition.

Firstly, for our example, we will draw the graph $M_2(C_5)$ to clarify the definition above.

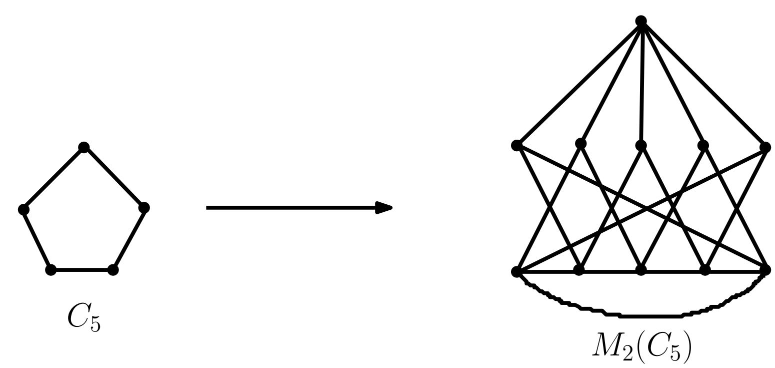


Figure 2.1: An Example of the Generalized Mycielskian Construction

In this example (and in general, as well), the base of $M_2(C_5)$ corresponds with a copy of $G = (V(G), E(G)) = (X_1, E(G))$. Above this first level, we have $r - 1 = 2 - 1 = 1$ successive levels, the i^{th} of which has vertex set X_i. Vertices within a given level are not joined by any edges, but vertices in the i^{th} level are joined to those in the $(i+1)^{th}$ level by the edges in $E_i = \{p_i(x)p_{i+1}(y) : xy \in E(G)\}$. This continues until we reach the r^{th}, all of whose vertices are adjacent to the vertex z.

Readers familiar with the usual Mycielskian construction introduced by Jan Mycielski in 1955 to construct triangle-free graphs of arbitrarily high chromatic number will observe that this usual Mycielskian corresponds with the $r = 2$ case of the generalized Mycielskian construction we have introduced. That $M_2(G)$ is triangle-free provided that G is triangle-free is obvious, so we need make only one observation to conclude that $\{M_2^l(C_5) : l \in \mathbb{N}\}$

is a collection of triangle-free graphs with members of arbitrarily high chromatic number. Moreover, by identical reasoning, for any triangle-free graph G, $\{M_2^l(G) : l \in \mathbb{N}\}$ is a collection of triangle-free graphs with members of arbitrarily high chromatic number.

Proposition 10. $\chi(M_2(G)) = \chi(G) + 1$.

Proof. Let G be a graph with chromatic number k, and let c be a colouring of G which partitions the vertices of G into colour classes $C_1, C_2, ..., C_k$. Firstly, let us note that if we let $c(p_2(v)) = c(p_1(v))$ for all $v \in V(G)$, and let z be the sole element of some new colour class C_{k+1}, then we obtain a $(k+1)$-colouring for $M_2(G)$. Thus, $\chi(M_2(G)) \leq \chi(G) + 1$.

Now, suppose that we have a k-colouring for $M_2(G)$ c which partitions $V(M_2(G))$ into colour classes $C_1, C_2, ..., C_k$. Then note that for each $i \in [k]$, there is some vertex $v_i \in C_i|_{V(G)}$ such that the restriction of the neighbourhood of v_i to $V(G)$ has at least one representative from each colour class C_j satisfying $i \neq j$ (as otherwise we could recolour each vertex in $C_i|_{V(G)}$ with a different colour, resulting in a $(k-1)$-colouring of G). Moreover, $N_G(v_i) \subseteq N_{M_2(G)}(p_2(v_i))$, so $p_2(v_i) \notin C_j$ for any $j \neq i$. Thus, for each $i \in [k]$, the class C_i has some representative $p_2(v_i)$. However, as $z \sim p_2(v)$ for all $v \in V(G)$, $z \notin C_j$ for any $j \in [k]$. Consequently, $\chi(M_2(G)) \geq \chi(G) + 1$, as required. □

At this time, we should also note a mild generalization of the first half of the proof of Proposition 10.

Proposition 11. *For all* $r \in \mathbb{N}$, $\chi(M_r(G)) \leq \chi(G) + 1$.

Proof. Given a k-colouring c of any graph G with chromatic number k, let us colour $M_r(G)$ as follows: for all $i \in [r]$ and for all $v \in V(G)$, $c(p_i(v)) = c(p_1(v)) = c(v)$, and let z be the sole element of some new colour class C_{k+1}. Then this $(k+1)$-colouring is proper precisely because the colouring c of G is proper, so we observe that $\chi(M_r(G)) \leq \chi(G) + 1$, as required. □

However, while, for $r = 1, 2$, $\chi(M_r(G)) = \chi(G) + 1$, it is not the case that this equality holds for general r. In particular, the graph G shown in Figure 2.2 is a counterexample to this assertion, as it satisfies $\chi(M_3(G)) = \chi(G) = 4$.

As we suggested at the beginning of this section, the key to using the generalized Mycielskian construction as a tool for increasing the chromatic number of a graph is understanding

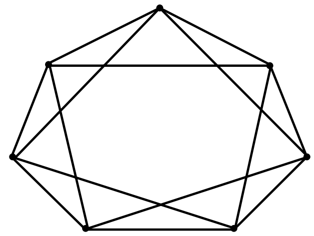


Figure 2.2: A Graph for which Applying the Generalized Mycielskian Construction does not Always Increase the Chromatic Number

its topological properties. Now, we will describe a theorem of Gyárfás, Jensen and Stiebitz [15] which displays precisely how the generalized Mycielskian construction effects the topology of the neighbourhood complex. While not essential, we can make our lives easier by proving this result in two steps. Firstly, we will prove that the result of this construction applied to the neighbourhood complex of a graph G is isomorphic to the neighbourhood complex to the generalized Mycielskian of G. This is essentially just a reorganization of the simplices of the neighbourhood complex of the generalized Mycielskian of a graph into a more convenient form. We will then show that the result construction we have defined on any simplicial complex K is homotopy equivalent to $S(K)$ from which it follows that the neighbourhood complex of the generalized Mycielskian of a graph G is homotopy equivalent to $S(N(G)))$.

Let K be a simplicial complex. Then for $r \geq 1$, we construct a new simplicial complex $M_r(K) = M_r(K, q_1, q_2, ..., q_r)$ as follows. For $1 \leq i \leq r$ let $q_i : V(K) \to Y_i$ be a bijection, where $Y_1, ..., Y_r$ are any pairwise disjoint sets in bijection with $V(K)$ according to the bijection q_i. Additionally, let $\Delta_0 = \{A : \emptyset \neq A \subseteq Y_1\}$, for $1 \leq i \leq r-1$ let $\Delta_i := \{A : \emptyset \neq A \subseteq q_i(B) \cup q_{i+1}(B)$ for some $B \in K\}$, and let $\Delta_r = \{A : \emptyset \neq A \subseteq q_r(B) \cup \{z\}$ for some $B \in K\}$, where z is some additional vertex in none of the Y_i. Then the vertices and simplices of $M_r(K)$ are as follows.

$$V(M_r(K)) := (\bigcup_{i=1}^{r} Y_i) \cup \{z\} \tag{2.3}$$

$$\Delta(M_r(K)) := \bigcup_{i=1}^{r} \Delta_i \tag{2.4}$$

Now, before we dive right into proving Gyárfás, Jensen and Stiebitz's theorem, let us build some intuition about the construction we have just defined. Consider the following pictorial representation of the generalized Mycielskian construction applied to some graph G for some fixed $r \in \mathbb{N}$.

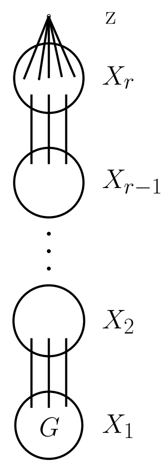


Figure 2.3: A Pictorial Representation of the Generalized Mycielskian Construction

Note that the collection of simplices of $M_r(N(G))$ which we call Δ_0 looks just like the neighbourhood of z, so if it were to happen that $M_r(N(G))$ and $N(M_r(G))$ were precisely the same simplicial complex, then most likely $Y_1 = q_1(V(N(G))) = X_r$. Now, if we make this assumption and want our new construction to correspond with the neighbourhood complex of the generalized Mycielskian, then, as $\Delta_1 := \{A : \emptyset \neq A \subseteq q_1(B) \cup q_2(B)$ for some $B \in N(G)\}$, we must have that $Y_2 = q_2(V(N(G))) = X_{r-2}$. Similar reasoning then continues to determine that $Y_i = q_i(V(N(G))) = X_{r-2i+2}$ until we reach either X_2 or X_1 (which one we arrive at first, of course, depends on the parity of r). If we reach X_2 first (r is even), then, by identical reasoning, the next Y_i must be X_1, after which we are forced to proceed in defining the remaining Y_i by successively increasing odd X_j starting from X_3. For the odd r case, we reason identically, save that we proceed upwards along even indices in defining the remaining Y_i. Eventually, we will then find that we will have defined all sets Y_i in terms of the sets X_j and identified all the simplices of $N(M_r(G))$, except for those corresponding to the neighbourhoods of vertices in X_r, with the simplices in $\bigcup_{i=1}^{r-1} \Delta_i$. However, this is perfect, as we now simply observe that Δ_r consists of precisely those simplices corresponding to the neighbourhoods of vertices in X_r. In summary, we have proved the lemma below.

Lemma 5. *For any graph G and for all $r \geq 1$, $M_r(N(G)) \cong N(M_r(G))$.*

With the lemma above at our disposal, we can now establish Gyárfás, Jensen and Stiebitz's result with relative ease. We will, however, need to introduce a new definition. The *suspension* $S(K)$ of a simplicial complex K is the simplicial complex $K * S^0$. Using this definition, one can easily establish that the geometric realization of $S(K)$ is homotopy equivalent to the quotient space $(K \times [0,1]) \setminus (K \times \{0\}, K \times \{1\})$. We will need both of these characterizations.

Theorem 5. *For any graph G and for all $r \geq 1$, $N(M_r(G))$ is homotopy equivalent to $S(N(G))$.*

Proof. Firstly, let us use the identification in Lemma 5, to note that $M_r(N(G)) = N(M_r(G))$. Thus, it is more than sufficient to show that $M_r(K)$ is homotopy equivalent to $S(K)$ for any simplicial complex K. To do so, let us note that that for any $1 \leq i \leq r$, the simplicial complex defined by $K_i := (Y_i, \{q_i(A) : A \in K\})$ is an isomorphic copy of K. Moreover, using this notation, we observe that $(Y_r \cup \{z\}, \Delta_r)$ is the cone of K_r with apex z, and (Y_1, Δ_0) is a simplex. Consequently, both of these subcomplexes of $M_r(K)$ are collapsible (as simplices are collapsible, and, for any cone $K * \{x\}$, provided that K is non-empty, for any point x, we can always collapse $K * \{x\}$ to $(K * \{x\}) \setminus \{A, A \cup \{x\}\}$, where A is any maximal simplex of K, resulting in the cone $(K \setminus \{A\}) * \{x\}$). So, let's make note of the fact that these subcomplexes are collapsible, and then consider the other subcomplexes of $M_r(K)$.

For each $1 \leq i \leq r-1$, $(Y_i \cup Y_{i+1}, \Delta_i)$ is equal to $K_i * K_{i+1}$. Now, taking advantage of the equivalence between the simplicial and geometric definitions of the join (a proof of this equivalence is sketched in [27]), we observe that as all of the K_i are isomorphic ot K, each of the $K_i * K_{i+1}$ is homotopy equivalent to $K \times I$. Moreover, for each $1 \leq i \leq r-1$, the intersection of $K_{i-1} * K_i$ and $K_i * K_{i+1}$ is K_i. Thus,

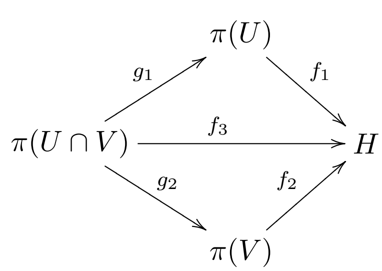
$$(\bigcup_{i=1}^{r} Y_i, \bigcup_{i=1}^{r-1} \Delta_i)$$

is homotopy equivalent to $K \times [0, r-1]$.

Consequently, as $M_r(K)$ consists of the union above together with its collapsible subcomplexes $(Y_r \cup \{z\}, \Delta_r)$ and (Y_1, Δ_0), we see that, upon collapsing these collapsible subcomplexes to single vertices in Y_r and Y_1, respectively, $M_r(K)$ is homotopy equivalent to the quotient space $K \times [0, r-1]/\sim$, where $(x, 0) \sim (y, 0)$ and $(x, r-1) \sim (y, r-1)$ for all $x, y \in K$, which is homotopy equivalent to $K \times I/\sim$, where $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ $x, y \in K$. However, this is precisely the definition of the suspension $S(K)$ of the geometric realization of a simplicial complex K. Therefore, $M_r(K)$ is homotopy equivalent to $S(K)$ for any simplicial complex K, which completes our proof. □

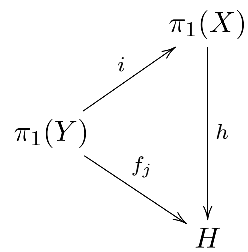
With the theorem we have just proven in hand, all that remains is to determine the connectivity of the suspension of a simplicial complex in terms of the connectivity of the original complex. This relation is determined by a straightforward corollary of the well-known Mayer-Vietoris and Seifert and Van Kampen theorems from algebraic topology. These theorems may be found in nearly any text on the subject, but for proofs of the versions we prefer, we suggest that the interested reader consult [32] and [26], respectively.

Seifert and Van Kampen Theorem. *Let X be a topological space, and suppose that U and V are path-connected open sets of X such that $X = U \cup V$, $U \cap V \neq \emptyset$ and $U \cap V$ is path-connected. Select a base point $x_0 \in U \cap V$ for all fundamental groups under consideration. Then let H be any group, and let f_1, f_2, f_3 be any three group homomorphisms such that the following diagram commutes:*



where the group homomorphisms g_1 and g_2 are induced by the usual inclusion maps.

Then there exists a unique group homomorphism $h : \pi_1(H) \to H$ such that the following diagram is commutative for each of $Y = U, V, U \cap V$ and $j = 1, 2, 3$, respectively:



where i is the group homomorphism induced by the usual inclusion map for each of $Y = U, V, U \cap V$.

Mayer-Vietoris Theorem. *Let K be a simplicial complex, and let K_0 and K_1 be subcomplexes of K such that $K = K_0 \cup K_1$. Additionally, let $A := K_0 \cap K_1$. Then there is an exact sequence*

$$... \to H_p(A) \to H_p(K_0) \oplus H_p(K_1) \to H_p(K) \to H_{p-1}(A) \to ...$$

called the Mayer-Vietoris sequence of (K_0, K_1). Moreover, the same exact sequence exists in reduced homology provided that $A \neq \emptyset$.

Corollary 5. *Let K be an n-connected simplicial complex. Then $S(K)$ is $(n+1)$-connected.*

Proof. Our proof proceeds in three steps. Firstly, we observe that the suspension of a disconnected simplicial complex is 0-connected, as each of the the two new vertices added to K are joined to all of K's vertices by 1-simplices (edges).

Secondly, using the Seifert-Van Kampen theorem, we can establish that the suspension of a 0-connected simplicial complex is 1-connected. To do so, recall that, for some arbitrary points a and b, $S(K) := K * \{a, b\}$, so the complements of a and b, respectively, are $K * \{b\}$ and $K * \{a\}$. Both of these are cones of K, and we showed in the proof of Theorem 5 that cones are collapsible. Thus, $S(K)$ is the union of two contractible simplicial complexes whose intersection is the 0-connected (path-connected) simplicial complex K. Consequently, by the Seifert and Van Kampen theorem (here, we do not concern ourselves with the choice of base point, as, for path-connected spaces, all choices of base points lead to isomorphic fundamental groups [26]), $\pi_1(S(K)) = \pi_1(K * \{a\}) \cup \pi_1(K * \{b\})$ is equal to the free product of $\pi_1(K * \{a\})$ and $\pi_1(K * \{b\})$, both of which are 0. Thus, $\pi_1(S(K)) = \{0\}$, whence $S(K)$ is 1-connected.

Finally, we will use a Mayer-Vietoris sequence in reduced homology to establish that for any simplicial complex K, and for all $p \in \mathbb{N}$, $\tilde{H}_p(S(K)) \cong \tilde{H}_{p-1}(K)$. Taken together with the Hurewicz theorem, this result implies that for all $n \geq 1$, if K is n-connected, then K is $(n+1)$-connected. So, using once more the fact that $S(K)$ is the union of $K * \{a\}$ and $K * \{b\}$, a pair of contractible simplicial complexes with intersection K, we have the following Mayer-Vietoris sequence (in which we let $X := K * \{a\}$ and $Y := K * \{b\}$):

$$... \rightarrow \tilde{H}_p(X) \oplus \tilde{H}_p(Y) \rightarrow \tilde{H}_p(S(K)) \rightarrow \tilde{H}_{p-1}(K) \rightarrow \tilde{H}_{p-1}(X) \oplus \tilde{H}_{p-1}(Y) \rightarrow ...$$

Hence, as $\tilde{H}_p(X)$ and $\tilde{H}_p(Y)$ are both trivial, we observe that $\tilde{H}_p(S(K)) \cong \tilde{H}_{p-1}(K)$, as required. □

In essence, this corollary tells us that if $B(G)$, $N(G)$ or $L(G)$ is n-connected and $\chi(G) = n+3$, then for any $r \in \mathbb{N}$, $M_r(G)$ is topologically $(n+4)$-chromatic. For example, by precisely this reasoning, as we established in Proposition 9 that any connected 3-chromatic graph G has a 0-connected neighbourhood complex, it follows that applying any generalized Mycielskian transformation to such a G yields a topologically 4-chromatic graph. However, we should caution the reader here, as it is not the case that applying any generalized Mycielskian transformation to any topologically k-chromatic graph yields a topologically $(k+1)$-chromatic graph. In particular, in his Ph.D. thesis, Peter Csorba studied non-tidy $\mathbb{Z}_2$-spaces (spaces whose coindex and index are not equal). For such spaces, it may happen that $conn(X) < ind(X) = ind(S(X))$. That there exist graphs whose box complexes have this property is a result of Csorba [7] which will not be discussed in this thesis. Fortunately, in most of our applications, we will obtain our topological lower bounds through determining connectivity, in which case Corollary 5 tells us that applying any generalized Mycielskian construction to a topologically n-chromatic graph yields a topologically $(n+1)$-chromatic graph.

2.3 Categorical Products and Unions of Graphs

In this section, we will study two constructions which have a well-understood effect upon the topology of the box complex of a graph. Understanding how taking categorical products effects the topology of the box complex will allow us to establish a topological version of

Hedeteniemi's conjecture, and will also provide us with insights into some properties of topologically k-chromatic graphs in the next chapter. Applications of the result we shall prove on unions of graphs will not be quite so immediately apparent, but it can frequently be used to simplify theoretical computations involving the box complex when G is a union of induced subgraphs.

Now, let us recall the definition of the categorical product of a pair of graphs. Given a pair of graphs G and H, we say that the *categorical product* $G \times H$ of G and H is the graph with $V(G \times H) := V(G) \times V(H)$ and $E(G \times H) := \{\{g_1h_1, g_2h_2\} : g_1, g_2 \in G, h_1, h_2 \in H, g_1 \sim g_2, h_1 \sim h_2\}$. An important and long studied conjecture in the study of the categorical products is Hedetniemi's conjecture, which we will now state.

Hedetniemi's Conjecture. *Let G and H be graphs. Then $\chi(G \times H) = min\{\chi(G), \chi(H)\}$.*

Of course, proving that $\chi(G \times H) \leq min\{\chi(G), \chi(H)\}$ is easy. In fact, if, without loss of generality, we suppose that $\chi(G) \leq \chi(H)$, then it suffices, to simply colour every vertex $g_ih_i \in G \times H$ with the colour of g_i, since this is a colouring with $\chi(G)$ colours, and for any $g_1h_1 \sim g_2h_2$ in $G \times H$, we must have $g_1 \sim g_2$.

Unfortunately, establishing the lower bound on the chromatic number is far more difficult. However, this makes Hedetniemi's conjecture a natural result to approach using methods for placing topological lower bounds on the chromatic number of a graph. Of particular use to us is the following theorem (which is a special case of a result of Kozlov [22]).

Theorem 6. *For any graphs G and H, $B(G \times H) \cong B(G) \times B(H)$.*

Proof. Our strategy is to define an ascending closure operator on $P(B(G \times H))$ whose image is isomorphic to $P(B(G)) \times P(B(H))$. Note, that here we are using a new product: the product of posets. For a pair of posets P and Q, this product is defined on the ground set $V(P) \times V(Q)$ by $p_1q_1 \leq p_2q_2$ if $p_1 \leq p_2$ and $q_1 \leq q_2$. It is also important to think of the product of CW-complexes carefully. We have not offered any proof thus far that the product space of a pair of simplicial complexes is a simplicial complex, and we will not do so. Instead, we think of $B(G)$ and $B(H)$ as regular CW-complexes, and make note of the definition of the product of two regular CW-complexes offered by Massey [26]. He establishes that this definition actually gives a regular CW-complex which corresponds precisely with the definition of the product of topological spaces. We will use this result, but omit its proof.

Let K and L be a pair of regular CW-complexes. Then we define their product $K \times L$ as the CW-complex which has as its n-skeleton

$$X^n := \bigcup_{p+q=n} K^p \times L^q$$

where K^p is the p-skeleton of K and L^q is the q-skeleton of L. In this case, we see that the product of a p-cell and a q-cell is a single $(p+q)$-cell, and the attaching map of such a product cell is the product of the attaching maps.

With this definition in hand, we easily observe that a $(p+q)$-cell $A_1 \times B_1 \in K \times L$ is contained in a cell $A_2 \times B_2 \in K \times L$ if and only if $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$. Consequently, the order complex operator commutes with the product of posets, so we see that $\Delta(P(B(G)) \times P(B(H))) = \Delta(P(B(G))) \times \Delta(P(B(H))) \cong B(G) \times B(H)$. Thus, if we can find the ascending closure operator we desire, then our proof will be complete.

Firstly, let's denote by $p_G : V(G) \times V(H) \to V(G)$ and $p_H : V(G) \times V(H) \to V(H)$ the usual projection mappings. Then $2^{p_G} : 2^{V(G)\times V(H)} \to 2^{V(G)}$ and $2^{p_H} : 2^{V(G)\times V(H)} \to 2^{V(H)}$ are the corresponding mappings induced on the power set of $V(G) \times V(H)$.

Now, define $f : P(B(G \times H)) \to P(B(G \times H))$ by

$$f(A \uplus B) := (2^{p_G}(A) \times 2^{p_H}(A)) \uplus (2^{p_G}(B) \times 2^{p_H}(B))$$

for all $A \uplus B \in P(B(G \times H))$.

Before proceeding further, we should show that this map is well-defined. To do so, we must demonstrate that for any $A \uplus B \in P(B(G \times H))$, $f(A \uplus B) = (2^{p_G}(A) \times 2^{p_H}(A)) \uplus (2^{p_G}(B) \times 2^{p_H}(B)) \in P(B(G \times H))$. So suppose that $a_G a_H \in 2^{p_G}(A) \times 2^{p_H}(A)$ and that $b_G b_H \in 2^{p_G}(B) \times 2^{p_H}(B)$. Then for each pair of vertices $v_G \in a_G$ and $u_G \in b_G$, $v_G \sim u_G$. Similarly, for each pair of vertices $v_H \in a_H$ and $u_H \in b_H$, $v_H \sim u_H$. Thus, $u_G u_H \sim v_G v_H$, whence all the vertices of $a_G a_H$ are joined to all the vertices of $b_G b_H$ in the graph $G \times H$. Consequently, as we desired, $im(f) \subseteq P(B(G \times H))$.

Now, since the map f which we have defined is order-preserving, while $f^2 = f$ and $A \uplus B \subseteq f(A \uplus B)$ are obvious from the definition of f, consequently, f is an ascending closure operator. Therefore, by Theorem 3, $\Delta(P(B(G \times H)))$ collapses onto $\Delta(im(f))$.

Hence, all that remains is to show that $im(f)$ is isomorphic to $P(B(G)) \times P(B(H))$. So define $g : P(B(G)) \times P(B(H)) \to im(f)$ by

$$g((A_G \uplus B_G) \times (A_H \uplus B_H) := (A_G \times A_H) \uplus (B_G \times B_H)$$

for all $A_G \uplus B_G \in P(B(G))$ and for all $A_H \uplus B_G \in P(B(H))$. The well-definedness of g follows from a similar argument to the one we gave above for the well-definedness of f, and injectivity is trivial. Additionally, g is surjective, as if $(A_G \times A_H) \uplus (B_G \times B_H) \in im(f)$, then there exist complete bipartite subgraphs A and B of $G \times H$ such that $A_G = 2^{p_G}(A)$, $A_H = 2^{p_H}(A)$, $B_G = 2^{p_G}(B)$ and $B_H = 2^{p_H}(B)$. Consequently, by the definition of the categorical product, $A_G \uplus B_G$ is a complete bipartite subgraph of G, and $A_H \uplus B_H$ is a complete bipartite subgraph of H, as required. □

Knowing that the box product of the categorical products of a pair of graphs is homotopy equivalent to the topological product of the box complexes of the graphs is extremely useful, as the behaviour of the fundamental group and homology groups under the topological product is extremely well-understood. In particular, we have the following two well-known propositions (proofs of which may be found in [26] and [32], respectively).

Proposition 12. *If X and Y are path-connected topological spaces, then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.*

Proof. Let X and Y be path-connected topological spaces, and let Z be a topological space. Then, in the product topology, a map $f : Z \to X \times Y$ is continuous if and only if the maps $g : Z \to X$ and $h : Z \to Y$ defined by $f(z) = (g(z), h(z))$ for all $z \in Z$ are both continuous.

Thus, the existence of a map $f : S^1 \to X \times Y$ is equivalent to the existence of a pair of maps $g : S^1 \to X$ and $h : S^1 \to Y$. Similarly, the existence of a homotopy $f_t : S^1 \to X \times Y$ is equivalent to the existence of a pair of homotopies $g_t : S^1 \to X$ and $h_t : S^1 \to Y$. Consequently, the map $r : \pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$ defined by $r([f]) = ([g], [h])$ is a bijection and group homomorphism. Thus, $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$, as required. □

Kunneth Theorem. *Let X and Y be topological spaces, and let F be a field. Then for all $k \in \mathbb{Z}^+$*

$$\bigoplus_{i+j=k} H_i(X, F) \otimes H_j(Y, F) \cong H_k(X \times Y, F).$$

Thus, we easily see that, by Hurewicz's theorem, if $B(G)$ and $B(H)$ are n-connected, for some $n \in \mathbb{N}$, then $B(G) \times B(H)$ is n-connected as well. Indeed, we have now established the following theorem (sometimes called a topological version of Hedetniemi's conjecture).

Theorem 7. *If $B(G)$ and $B(H)$ are $(k-3)$-connected, then $G \times H$ is topologically k-chromatic.*

The proposition we will now state on unions of graphs is a result of Csorba which he used (in conjunction with other tools) in order to determine that the box complex of any chordal graph is homotopy equivalent to a wedge of spheres [8]. We will not reproduce all of Csorba's work on this problem here, but will prove the result on unions of graphs which we discussed at the beginning of this section using discrete Morse theory.

Proposition 13. *Let G be the union of two distinct induced subgraphs G_1 and G_2. Then $B(G)$ is homotopy equivalent to $B(G_1) \cup B(G_2)$.*

Proof. Firstly, let us note that $B(G_1) \cup B(G_2) \subseteq B(G)$, so we may let $S = B(G) \setminus (B(G_1) \cup B(G_2))$. We will also let $N = G_1 \cap G_2$ be the intersection graph of G_1 and G_2 (the induced subgraph of G on the vertex set $V(G_1) \cap V(G_2)$).

Now, let $A_1 \uplus A_2$ be a simplex of S. By the definition of S and the definition of G, $A_1 \cup A_2$ must contain a vertex $u \in V(G_1) \setminus V(N)$ and a vertex $v \in V(G_2) \setminus V(N)$. Additionally, note that we cannot have $u \in A_i$ and $v \in A_{i+1(mod2)}$ for either $i = 1$ or $i = 2$, as $uv \notin E(G)$. Moreover, if for either $i = 1$ or $i = 2$ we have $x, y \in A_i$, then $A_{i+1(mod2)} \subseteq N$ since $G[A_1, A_2]$ is complete bipartite. Consequently, we obtain that if $A_1 \uplus A_2 \in S$, then either
(1) $A_1 \subset N$, $A_2 \cap (V(G_1) \setminus V(N)) \neq \emptyset$ and $A_2 \cap (V(G_2) \setminus V(N)) \neq \emptyset$; or
(2) $A_2 \subset N$, $A_1 \cap (V(G_1) \setminus V(N)) \neq \emptyset$ and $A_1 \cap (V(G_2) \setminus V(N)) \neq \emptyset$.

Now, suppose that we let $V(N) = [n]$ for some positive integer n and let $A \subseteq V(G)$ be a set satisfying $N_G(A) \cap V(N) \neq \emptyset$. For such an A, we then define a vertex $g(A) \in V(N)$ by $g(A) = min\{CN_G(A) \cap V(N)\}$. With this definition in hand, we are ready to define a useful partial matching on $P(B(G))$. Note that for any $A_1 \uplus A_2 \in P(B(G))$, either (1) or (2) holds.

For any $A_1 \uplus A_2 \in S$ satisfying (1), we define $f(A_1 \uplus A_2) = (A_1 \cup g(A_2)) \uplus A_2$ whenever $g(A_2) \in A_1$. Similarly, for any $A_1 \uplus A_2 \in S$ satisfying (2), we define $f(A_1 \uplus A_2) = A_1 \uplus (A_2 \cup g(A_1))$ whenever $g(A_1) \in A_2$.

We want to show that, in either case, f is a partial acyclic matching. To establish this fact, we must first show that f is well-defined. By symmetry, it is sufficient to demonstrate this fact for any $A_1 \uplus A_2 \in S$ which satisfies (1). If $g(A_2) \notin A_1$, then, since $g(A_2) \in CN_G(A_2)$, $(A_1 \cup g(A_2)) \uplus A_2 \in P(B(G))$, as required.

We will show acyclicity by contradiction. Once again, we will only prove the result we need for $A_1 \uplus A_2 \in S$ satisfying (1), but, by symmetry, it holds in both cases. If we want to form a directed cycle in $D(A, f)$, then we may assume that the cycle begins with the simplex $A_1 \uplus A_2$ satisfying (1), and then immediately goes up to the simplex $f(A_1 \uplus A_2) = (A_1 \cup g(A_2)) \uplus A_2$. Note that when we go up, we can only add either a vertex of N and that $A_1 \subseteq N$. Thus, by going upwards, we can only reach simplices which satisfy (1). In going down, we must delete something. If that something is in A_2, then we cannot have a cycle, as we cannot add by a matching to the second set. If we delete from A_1, then the resulting simplex is already matched to a vertex below it by deleting $g(A_2)$. Consequently, once more, we cannot have a cycle.

Thus, as the critical simplices of the matching are the simplices of $B(G_1) \cup B(G_2)$, we are done by Proposition 5. □

2.4 Folds

In the last section, we discussed global constructions one can perform on a graph which have a well-understood effect on the homotopy type of the box complex associated to a graph. However, there is also a place for local operations. Unfortunately, most of our usual operations from graph theory do not seem to have effects on the topology of $B(G)$ which are both useful and easily understood. In order to obtain easily an understood and useful effect (such as homotopy equivalence), we generally have to place fairly stringent demands upon the local structure of G. Perhaps, the best known local operation we can perform on a graph which preserves its homotopy type is folding a graph at a vertex v, and, as we will see, folding requires quite strong conditions on the local structure of G near v.

Let G be a graph, and let $v \in V(G)$. Then we call the graph $G - v$ a *fold* of G if there exists a vertex $u \in V(G)$ such that $u \neq v$ and $N(v) \subseteq N(u)$. We can easily prove folding induces a homotopy equivalence between $B(G)$ and $B(G - v)$ using ascending and descending closure operators.

Theorem 8. *Let $G - v$ be a fold of G. Then $B(G - v) \cong B(G)$.*

Proof. Firstly, note that as $G - v$ is a fold of G, there exists some $u \in V(G)$ such that $u \neq v$ and $N(v) \subseteq N(u)$. Additionally, it will be useful to define X, a subposet of $P(B(G))$, as the the collection of all $A \uplus B \in P(B(G))$ such that $A \cap \{u, v\} \neq v$ and $B \cap \{u, v\} \neq v$. Note that $P(B(G - v)) \subseteq X \subseteq P(B(G))$.

The strategy of our proof will be to construct an ascending closure operator $f : P(B(G)) \to X$, and then we will construct a descending closure operator $g : X \to P(B(G - v))$. So let us begin be defining f on an arbitrary $A \uplus B \in P(B(G))$.

$$f(A \uplus B) = \begin{cases} (A \cup \{u\}) \uplus B & \text{if } v \in A \\ A \uplus (B \cup \{u\}) & \text{if } v \in B \\ A \uplus B & \text{otherwise} \end{cases}$$

Note here, that, by definition of $B(G)$, it cannot be the case that $v \in A$ and $v \in B$. Thus, as $N(v) \subseteq N(u)$, $(A \cup \{u\}) \uplus B$ is a complete bipartite subgraph of X provided that $v \in A$, and, similarly, $A \uplus (B \cup \{u\})$ is a complete bipartite subgraph of X provided that $v \in B$, f is well-defined. Hence, as f is also easily seen to be order-preserving, $f^2 = f$ and $f(A \uplus B) \geq A \uplus B$ for any $A \uplus B \in P(B(G))$, f is an ascending closure operator. Moreover, $im(f) = X$, so $\Delta(P(B(G)))$ collapses onto $\Delta(X)$.

Now, let's show that $\Delta(X)$ collapses onto $\Delta(P(B(G - v)))$. To do so, define $g : X \to P(B(G - v))$ by

$$g(A \uplus B) = \begin{cases} (A \setminus \{v\}) \uplus B & \text{if } v \in A \\ A \uplus (B \setminus \{v\}) & \text{if } v \in B \\ A \uplus B & \text{otherwise} \end{cases}$$

We observe that g is obviously order-preserving, $g^2 = g$ and $g(A \uplus B) \leq A \uplus B$ for any $A \uplus B \in X$, so all that remains in order to show that g is a descending closure operator is to note that g is well-defined. Well-definedness follows from the fact that the image of g corresponds with the complete bipartite subgraphs of G which do not contain v, all of which are represented by elements of X. Moreover, as the complete bipartite subgraphs of G which do not contain v are precisely those represented by simplices in $B(G - v)$, we see that $im(g) = P(B(G - v))$, whence $\Delta(X)$ collapses onto $\Delta(P(B(G - v)))$.

Consequently, as homotopy equivalence is transitive, $B(G - v) \cong B(G)$. $\square$

Here, we should note that folding obviously does not increase the chromatic number of G, and, so, consequently, it is an example of a local operation which not only preserves the homotopy type of the box complex, but also preserves the property of being topologically k-chromatic.

Chapter 3

Structure

3.1 The Zig-Zag Theorem

In the previous chapter, we introduced a number of methods for constructing graphs with topological colouring bounds. However, besides a brief proposition in the introduction of the chapter providing some simple necessary conditions for a graph to have a particular topological colouring bound, we refrained from saying anything about the structure and properties of these graphs. In this chapter, our focus will be upon fleshing out the the structure of graphs which admit topological colouring bounds. Perhaps the most important result in this direction is the Zig-Zag Theorem of Simonyi and Tardos [37]. The result itself is not difficult given Ky Fan's Theorem (which we established in Chapter 1). However, its proof will require studying a slightly different box complex than the one we have used up to this point.

$B_0(G) := \{A_1 \uplus A_2 : A_1, A_2 \subseteq V(G), A_1 \cap A_2 = \emptyset, G[A_1, A_2] \text{ is complete bipartite } \}.$

This complex differs from our previous box complex in that we have dropped the condition that $CN(A_1) \neq \emptyset \neq CN(A_2)$. Consequently, we now have the simplices $V(G) \uplus \emptyset$ and $\emptyset \uplus V(G)$, which were absent from $B(G)$. For the moment, we will not address the relation between the topologies of our two box complexes in favour of swiftly establishing the Zig-Zag Theorem. However, once this is done, we will show that our new box complex is actually homotopy equivalent to the suspension of our old box complex.

Zig-Zag Theorem. *Let G be a graph such that $Coind(B_0(G)) \geq t - 1$, and let c be any* *proper colouring of G with any number of colours. Moreover, suppose that the colours of c are linearly ordered. Then G contains a complete bipartite subgraph $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$ such that all t vertices of this subgraph are assigned distinct colours by c and such that these colours appear alternating on the two sides of the complete bipartite subgraph with respect to their order.*

Proof. As $Coind(B_0(G)) \geq t-1$, there exists a $\mathbb{Z}_2$-map $f : S^{t-1} \to B_0(G)$. We want to apply Ky Fan's theorem to open sets representing the colour classes of the proper colouring given to us by our theorem, so we will now define open sets covering S^{t-1} based upon the colouring c which meet the hypotheses of Ky Fan's theorem.

For any colour i, let's define a set $A_i \subseteq S^{t-1}$ by letting $x \in A_i$ if and only if for the minimal simplex of $B_0(G)$ containing $f(x)$, which we shall denote by $U_x \uplus V_x$, there exists a vertex $z \in U_x$ such that $c(z) = i$. Each of these sets A_i is then the preimage of a union of open sets, so it is open. Additionally, while the collection of sets $\{A_i\}$ may not cover the sphere, we can easily show that $\cup_i (A_i \cup -A_i) = S^{t-1}$. To see this fact, note that if $x \in -A_i$, then $-x \in A_i$, which happens precisely when there exists a vertex $y \in U_{-x}$ such that $c(y) = i$. Now, note that $U_{-x} = V_(x)$, so, for each $x \in S^{t-1}$, either U_x or V_x is non-empty. Consequently, $\cup_i (A_i \cup -A_i) = S^{t-1}$, as required. Now, seeking a contradiction, let us suppose that for some colour i we have $A_i \cap -A_i \neq \emptyset$, and let x be a point in this intersection. Then there exists a vertex $y \in U_x$ and a vertex $z \in V_x$ such that $c(y) = c(z) = i$. However, the definition of $B_0(G)$ tells us that y and z are connected in G, contradicting the fact that c is a proper colouring. Thus, for any colour i, $A_i \cap -A_i \neq \emptyset$, so the collection of open sets $\{A_i\}$ satisfies the hypotheses of version (2) of Ky Fan's Theorem.

Applying version (2) of Ky Fan's Theorem to our collection of sets $\{A_i\}$, we see that for any linear ordering of the colours $i_1 < i_2 < ... < i_t$ of c, there exists a point $x \in S^{t-1}$ such that $(-1)^j x \in A_{i_j}$ for each $j \in [t]$. Consequently, for each such j, there exists a vertex $z_j \in U_{(-1)^j x}$ such that $c(z_j) = i_j$. Now, noting that $U_{(-1)^j x} = U_x$ for even j and $U_{(-1)^j x} = V_x$ for odd j, we see that the complete bipartite subgraph with sides $\{z_j : j \text{ is even}\}$ and $\{z_j : j \text{ is odd}\}$ is a subgraph of G with the properties we desire. □

Consequently, as we may linearly order the colours of any given colouring of a graph G however we like, we actually obtain many totally multicoloured copies of $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$. If we restrict ourselves to examining topologically t-chromatic graphs, then we actually obtain all possible colourings.

The relationship between $B_0(G)$ and $B(G)$ is not difficult to establish, and the proof is quite similar to a number of arguments we made in Chapter, so we state this relationship without proof. In its full generality, this result was originally proved by Csorba in [7].

Proposition 14. *$B_0(G)$ is homotopy equivalent to the suspension of $B(G)$.*

Consequently, we can easily discern the relationship between the coindex of the $B_0(G)$ and the coindex of $B(G)$.

Observation 2. *For any $\mathbb{Z}_2$-space X, $Coind(susp(X)) \geq Coind(X) + 1$.*

Proof. Recall the topological definition of suspension. For any topological space X, the suspension of X is the topological space obtained from $X \times [-1, 1]$ by identifying all the points in $X \times \{-1\}$ and all the points in $X \times \{1\}$. Consequently, if ν is a fixed point-free involution on X, then the map $\tau : X \times [-1, 1] \to X \times [-1, 1]$ defined by $\tau(x, t) = (\nu(x), -t)$ is a fixed point free involution on $X \times [-1, 1]$. Thus, given any $\mathbb{Z}_2$-map $f : S^n_\alpha \to X$, the map $g : susp(S^n_\alpha) \cong S^{n+1}_\alpha \to susp(X)$ defined by $g(x, t) = (f(x), t)$ is a $\mathbb{Z}_2$-map. Consequently, $Coind(susp(X)) \geq Coind(X) + 1$, as required. □

An obvious consequence of this result is that if $Coind(B(G)) \geq t-2$, then $Coind(B_0(G)) \geq t - 1$, so we may apply the Zig-Zag Theorem to graphs with topological colouring bounds of the kinds we considered in the first two chapters. Applications of this kind will play prominent roles in the next two sections of this chapter.

3.2 Circular Chromatic Number

For positive integers p and q, a colouring $c : V(G) \to [p]$ of a graph G is called a *(p,q)-colouring* if, for any pair of adjacent vertices u and v in G, $q \leq |c(u) - c(v)| \leq p - q$. Given this definition, we then define the circular chromatic number of G as $\chi_c(G) := \{\frac{p}{q} : G \text{ has a } (p, q) - \text{colouring}\}$.

Interest in the circular chromatic number has grown in recent years as an interesting refinement of the usual chromatic number. Two useful references on this subject are Bondy and Hell's paper [3], which introduces the circular chromatic number (although they call it the star chromatic number) in an intuitive combinatorial setting and Zhu's wide-ranging survey of the area [43]. In the first of these (along with many other places), one will find the easy fact that for any graph G, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. More interesting for us are two conjectures Zhu proposed near the end of his survey. First, he wondered whether there might be a version of Hedetniemi's conjecture for the circular chromatic number, and then he asked whether the circular chromatic number always equals the chromatic number for Kneser graphs. As far as we know, both of these problems are still open. However, the second of these questions may be answered for Kneser graphs with even chromatic number by the proposition below (originally proved by Simonyi and Tardos [37]). Determining whether or not $\chi_c(G \times H) = min\{\chi_c(G), \chi_c(H)\}$ is, naturally, far harder. However, we will also provide some support for this conjecture by proving this statement for graphs with topological colouring bounds.

Proposition 15. *Let t be an even positive integer, and let G be a topologically t-chromatic graph. Then $\chi_c(G) = t$.*

Proof. This fact follows easily from the Zig-Zag Theorem, as, by said theorem, given any t-colouring c of any topologically t-chromatic graph (for even t) G, G contains a copy of $K_{\frac{t}{2},\frac{t}{2}}$ as a subgraph which is multicoloured by the colours of c. Moreover, given any linear ordering of the colours of c, the colours of c appear in an alternating manner on the two sides of $K_{\frac{t}{2},\frac{t}{2}}$. Thus, if we let these colours be $c_1 < c_2 < ... < c_t$, then, as, for each $i \in [t-1]$, the vertex coloured c_i is adjacent to the vertex coloured c_{i+1}, we have $c_{i+1} \geq c_i + q$, whence $c_t \geq c_1 + (t-1)q$. Additionally, as t is even, the vertices coloured c_1 and c_t are adjacent, so we also have $c_t - c_1 \leq p - q$. Combining our two inequalities, we then have $p - q \geq c_t - c_1 \geq (t-1)q$, whence $\frac{p}{q} - 1 \geq t - 1$, which implies that $\frac{p}{q} \geq t$. Therefore, $t = \chi(G) \geq \chi_c(G) \geq t$, from which it follows that $\chi_c(G) = t$. □

Corollary 6. *Let k and l be positive integers such that k is even and $k \leq l$. Then if G and H are topologically k- and l-chromatic graphs, then $\chi_c(G \times H) = \chi_c(G)$.*

Proof. This corollary follows easily by combining Theorem 7 with Proposition 15. □

3.3 Quadrangulations and Colour-Critical Graphs

In chapter 2, we devoted much effort to examining the topological properties of the generalized Mycielskian construction. However, this construction is also related rather intimately to quadrangulations of the projective plane and edge-critical graphs. For the remainder of this section, when we refer to critical graphs, we will always mean edge-critical graphs. Following Mohar, Simonyi and Tardos [30], we define the *quadrangulation* of a surface S to be a loopless graph embedded on S in such a way that all of its faces are quadrilaterals. The focus of this section will be on connections between the 4-critical topologically 4-chromatic graphs and quadrangulations of the projective plane. However, this focus will not stop us from dealing with more general results, such as the following proposition due to Gyárfás, Jensen and Stiebitz [15].

Proposition 16. *Let G be a k-critical graph ($k \geq 2$). Then if $\chi(M_r(G)) = k+1$ (for some $r \in \mathbb{Z}^+$), then $M_r(G)$ is $(k+1)$-critical.*

Proof. It suffices to show that $M_r(G) - e$ has a k-colouring for every edge $e \in E(M_r(G))$. Following Gyárfás, Jensen and Stiebitz, we will distinguish three cases.

Case 1. If $e \in E(G)$, then there exists a $(k-1)$-colouring of $G - e$, whence, as $M_r(G) - G$ is bipartite, we can extend this colouring to a k-colouring of $M_r(G) - e$.

Case 2. If $e = p_i(a)p_{i+1}(b)$ for some $1 \leq i \leq r-1$, then $ab \in E(G)$. There is a $(k-1)$-colouring c of $G - ab$, as G is k-critical, so, since G is not $(k-1)$-colourable, we must have that $c(a) = c(b)$.

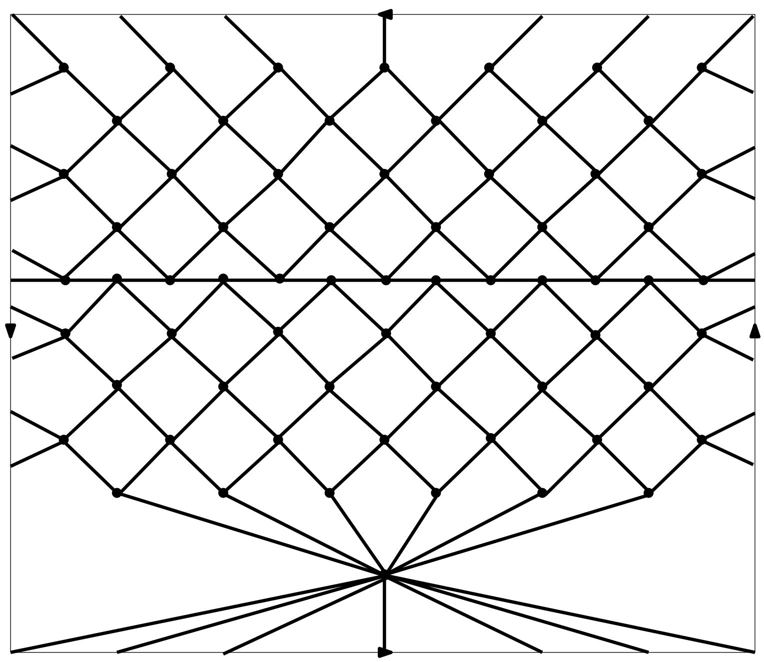
Now, define a map g as follows for any $1 \leq j \leq i$ and any $x \in V(G)$:

$$g(p_j(x)) = \begin{cases} c(x) & \text{if } x \neq b \\ k & \text{if } x = b. \end{cases}$$

Furthermore, let $g(p_{i+1}) = c(x)$ for all $x \in V(G)$. Then g is a k-colouring of the subgraph$M_r(G)[X_1 \cup X_2 \cup ... \cup X_{i+1}]$ which we may extend to a k-colouring of $M_r(G) - e$ by letting $g(p_l(x)) = c(x)$ for all $l \geq i+1$.

Case 3. If $e = p_r(x)z$ for some vertex $x \in V(G)$, then, as G is k-critical, there is a k-colouring c of G such that $c(y) = k$ only for $y = x$. So, if we color each vertex $p_i(x)$ with $c(x)$ and colour the new vertex z with the colour k, then this colouring is a k-colouring of $M_r(G) - e$. □

To connect the 4-critical quadrangulations of the projective plane with the 4-critical generalized Mycielskians, we naturally observe that any generalized Mycielskian of an odd cycle is a 4-critical graph (as, by the results of chapter 2, it is 4-chromatic). It is also a 4-critical quadrangulation of the projective plane, as we can see by considering adding rows (for higher r) and lengthening the central cycle in Figure 3.1.



$M_5(C_{11})$

Figure 3.1: The Generalized Mycielskian Construction Applied to an Odd Cycle Drawn as a Quadrangulation of the Projective Plane

These 4-critical quadragulations of the projective plane have an additional property originally established by Youngs [42]. Youngs proved that a non-bipartite quadrangulation of the projective plane is 4-critical if and only if all of its 4-cycles are facial. Having established that any generalized Mycielskian of an odd cycle is 4-critical, it is not hard to establish a converse.

Corollary 7. *Let G be 3-chromatic. Then $M_r(G)$ is 4-critical if and only if G is an odd cycle.*

Proof. That $M_r(G)$ is 4-critical if G is an odd cycle is established in Proposition 16. Conversely, if $M_r(G)$ is 4-critical, then deleting any one of its edges must result in a 3-chromatic graph. In particular, this holds for any edge $e \in E(G)$. However, observe that if G is 3-chromatic, then if we delete all of its edges except for an odd cycle, the generalized Mycielskian of the resulting graph is still 4-chromatic. This generalized Mycielskian is a subgraph of $M_r(G)$, so, unless G has no edges besides those in some odd cycle, it may not be 4-critical. □

Thus far, we have said nothing about topological coluring bounds in this section. That will now change, as we will first prove and then establish a converse to a result of Mohar, Simonyi and Tardos which shows that all the 4-critical quadrangulations of the projective plane are homotopy equivalent to the 2-sphere.

In order to prove Mohar, Simonyi and Tardos' result, as well as its converse, we must first review some notation and basic results from the theory of covering spaces. Many references exist which cover this subject, but we would recommend chapter 8 of Seifert and Threfall's textbook [36] which proves the standard results on covering spaces which we will state here without proof.

Let S be a topological space and T be a covering space for S. Then the fundamental group $\pi(T)$ of T projects onto a subgroup of $\pi(S)$ which depends on the choice of the initial point of the closed paths of T. All these subgroups are conjugate in $\pi(S)$. If all these subgroups are the same, then the subgroup H to which $\pi(T)$ projects in $\pi(S)$ is normal, and we call T a *regular covering* of S.

Observation 3. *The number of sheets of the covering T of S is equal to the index of the subgroup H in $\pi(S)$. Consequently, if T is a 2-sheeted covering of S, then H is normal, whence T is a regular covering.*

Mohar, Simonyi and Tardos's result relies heavily on the concept of the medial graph. For any graph G embedded on a surface S, the *medial graph* of G, denoted by $M(G)$, and also embedded on S, is defined as follows. The vertices of $M(G)$ corresponding with the edges of G, while we join the vertices of $M(G)$ by edges according to the following rule. For each vertex $v \in V(G)$, let $e_1, ..., e_k$ be the edges incident with v in the cyclic order these edges leave v. Then, for $1 \le i \le k$, we join the vertex representing e_i in $M(G)$ and the vertex representing e_{i+1} in $M(G)$ by an edge. In the same way, we join the vertex representing e_k in $M(G)$ to the vertex representing e_1 in $M(G)$. Consequently, we obtain a 4-regular graph embedded in S which has 2 types of faces. We call a face of $M(G)$ a *star face* if it contains a vertex of G, and term $M(G)$'s other faces *cycle faces*. Below, in Figure 3.2, we have drawn $M(K_4)$ embedded on the projective plane as an example. In this case, the four star faces are the 3-cycles, while the three 4-cycles are cycle faces. Observe that the vertices in a cycle face of $M(G)$ correspond to the edges of a facial walk in G.

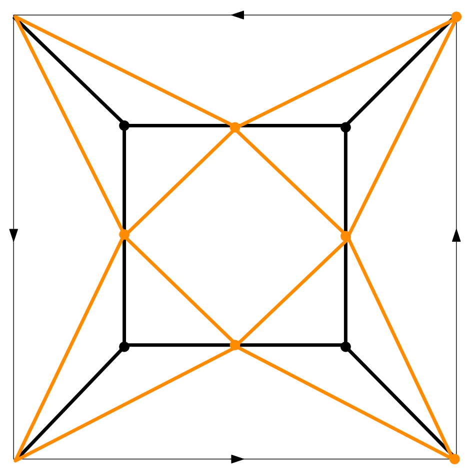


Figure 3.2: The Complete Graph on Four Vertices Drawn as a Quadrangulation of the Projective Plane and its Medial Graph

Usign the notation we have introduced for medial graphs, let f be a map which maps each vertex $v_e \in M(G)$ to the class$\{a\} \uplus \{b\}$ in $B(G)/\nu$, where ν is the usual $\mathbb{Z}_2$-action on $B(G)$ and $e = xy$. Then f extends to the edges of $M(G)$ (mapping them these edges to 1-cells of $B(G)$). f also extends to the faces of $M(G)$ as follows: the image of a cycle face corresponding to the face $xyzt$ in G will be the simplex $\{x, z\} \uplus \{y, t\}$, and the image of the star face containing any vertex $v \in V(G)$ is contained within the simplex $\{v\} \uplus N_G(v)$. This makes f a simplicial (and, consequently, continuous) map $f : S \to B(G)/\nu$.

Now, note that the map f lifts to a map $g : T \to B(G)$, where T is a double cover of S. The involution of T which exchanges points which have the same image in S is an obvious fixed point free involution on T, so we observe that T is a $\mathbb{Z}_2$-space and g is a $\mathbb{Z}_2$-map. Additionally, it is easy to see that T is the union of two disjoint copies of S if and only if G is bipartite. Taking the argument above together with the correspondence we established previously between box complexes and neighbourhood complexes, we have the following proposition.

Proposition 17. *Let G be a non-bipartite quadrangulation of a surface S in which all 4-cycles of G are facial. Then $N(G)$ is homotopy equivalent to some 2-sheeted covering T of S.*

Proof. Consider the maximal simplices of $B(G)$. These correspond with maximal complete bipartite subgraphs of G, which are the face cycles and stars of vertices. That G may not have any vertices of degree strictly less than 3 is clear, and, if G is 3-regular, then we can easily see that T and $B(G)$ are homeomorphic, as we can choose g to be a homeomorphism. Moreover, if G is not 3-regular, then, as the only edges of a d-cycle on the boundary of these cells appear in other cells, we can collapse all higher dimensional cells of $B(G)$ until we obtain a simplicial complex homeomorphic to T. □

Proposition 17 has three important implications. The first two of these are due to Mohar, Simonyi and Tardos, while the last one is new.

Theorem 9. *Let G be a non-bipartite quadrangulation of the projective plane. Then G is topologically 4-chromatic.*

Proof. The upper bound on the chromatic number is proved easily by Youngs [42]. Our interest is in the lower bound.

Consider the map $f : T \to B(G)$ which we constructed above. As T is a double cover of the projective plane which is not a union of two disjoint copies of the projective plane, we observe that T must be the 2-sphere. Thus, there exists a $\mathbb{Z}_2$-map form the 2-sphere to $B(G)$, whence $Coind(B(G)) \geq 2$, which implies that $\chi(G) \geq 4$, as required. □

Theorem 10. *Let G be a non-bipartite quadrangulation of the projective plane in which all of G's 4-cycles are facial. Then $N(G)$ is homotopy equivalent to the 2-sphere.*

Proof. By Proposition 17, T and $N(G)$ are homotopy equivalent. Thus, by the proof of Theorem 9, $N(G)$ is homotopy equivalent to the 2-sphere. □

We can also derive a converse to this result.

Theorem 11. *Let G be a quadrangulation of a surface S in which all the 4-cycles of G are facial and such that N(G) is 1-connected. Then G is a quadrangulation of the projective plane.*

Proof. The neighbourhood complex of G is 1-connected, so the fundamental group of $N(G)$ is trivial. Proposition 17 tells us that $N(G)$ and T have the same fundamental group, so the fundamental group of T is trivial, whence H is the trivial group. Thus, by Observation 3, the fundamental group of S must have order two (as the index of H in $\pi(S)$ must be two). Consequently, $\pi(S) \cong \mathcal{Z}_2$. Therefore, S must be the projective plane, as this is the only surface with fundamental group $\mathcal{Z}_2$. □

Consequently, we see that bipartite quadrangulations of surfaces in which all 4-cycles are facial are disconnected, while the non-bipartite quadrangulations of surfaces in which all 4-cycles are facial separate into those which quadrangulate the projective plane (which are 1-connected), and all other non-bipartite quadrangulations of surfaces in which all 4-cycles are facial (which are 0-connected).

Additionally, let us note that the Zig-Zag Theorem suggests that the 4-critical topologically 4-chromatic graphs all share important properties with the 4-critical quadrangulations of the projective plane. In particular, the following result is a straightforward consequence of the Zig-Ziag Theorem and the definition of a 4-critical graph.

Proposition 18. *Let G be a 4-critical, topologically 4-chromatic graph. Then, for any vertex v of G, there exists a colouring c of G in which v is the only vertex of v coloured with the colour 4. Consequently, in this colouring, v must be incident with 3 distinct multicoloured 4-cycles. By similar reasoning, every edge of G must be an edge of at least two 4-cycles.*

This result generalizes Mohar's result that the number of multicoloured faces in a non-bipartite quadrangulation of the projective plane is odd [31].

3.4 Odd Girth and Counting Colourings

In the second chapter of this thesis, we established that if the girth of a graph G is greater than or equal to 5, then that G is not topologically k-chromatic for any $k \geq 4$. However, using a construction of Lovász and Greenwell [25] together with the results we established on the categorical product of graphs and generalized Mycielskian construction in chapter 2, it is not hard to construct infinitely many topologically k-chromatic graphs for any given k which not only have arbitrarily high odd girth, but are also uniquely colourable. Additionally, using these same results, it is also not hard to see how one can construct topologically k-chromatic graphs, for any positive integer $k \geq 4$, which have any given number of colourings.

To begin with, let us prove Lovász and Greenwell's result.

Theorem 12. *Suppose that G is a connected graph such that any two n-colourings of G colour at least two vertices differently. Then each n-colouring of $K_n \times G$ is induced by either an n-colouring of K_n or an n-colouring of G.*

Proof. Following Lovász and Greenwell, we separate our proof into two cases.

Case 1. Firstly, suppose that there is an $x \in V(G)$ such that an n-colouring c of $K_n \times G$ assigns the vertices of $K_n \times G$ $(1,x),(2,x),...,(n,x)$ different colours. Now, let some vertex y of G be adjacent to x. Then, for any two distinct vertices $r,s \in V(K_n)$, $c(r,y) \neq c(s,x)$, whence $c(s,y) = c(s,x)$. Thus, as G is connected, $c(i,x)$ is independent of x, whence it is induced by the colouring of K_n

Case 2. Conversely, suppose that for all $v \in V(G)$, there exist distinct vertices i,j of K_n such that $c(i,x) = c(j,x)$. For each such x, let us denote this colour by $c'(x)$. Now, fix a point $x \in V(G)$ and define

$$c_k(y) = \begin{cases} c'(y) & \text{if } y \neq x \\ c(k,x) & \text{if } y = x \end{cases}$$

If $u,v \in V(G)$ (we may assume that $u \neq x$) and $u \sim v$, then $c_k(u) = c'(u) = c(i,u) = c(j,u)$ for some $i \neq j$. Additionally, whether $v = x$ or not, $c_k(v) = c(m,v)$ for some m. We may additionally assume that $i \neq m$ (as otherwise we may argue inductively from $j \neq m$). Consequently, $(i,u) \sim (m,v)$, whence $c_k(u) = c(i,u) \neq c(m,v) = c_k(v)$. Thus, c_k is an n-colouring of G.

Now, as the colourings $c_1,...,c_n$ differ only in x, by our assumption on G, they are identical. Thus, for any vertices $i, j \in V(K_n)$, $c(i, x) = c(j, x)$, which implies that the colouring c is induced by c_k of G.

□

From this theorem, a number of corollaries are immediately apparent. These corollaries have consequences for our work in light of the results we derived in Chapter 2 on the topological properties of the categorical product of graphs.

Corollary 8. *If $\chi(G) > n$ and G is connected, then $K_n \times G$ is uniquely n-colourable.*

Corollary 9. *$(K_n)^k$ has exactly k n-colourings.*

Corollary 10. *For any fixed positive integers k and $n \geq 2$, K_n^k and $M_1(K_{n-1}^k)$ are topologically n-chromatic graphs with precisely k n-colourings.*

Proposition 19. *For any positive integers $n, l \geq 3$ there exist infinitely many uniquely n-colourable, topologically n-chromatic graphs with odd girth greater than or equal to l.*

Proof. By Theorem 12, for any graph G, if the chromatic number $\chi(G) > n$ and G is connected, then $K_n \times G$ is uniquely n-colourable. By Theorem 6, is the graph G is at least n-connected, then so is $K_n \times G$. Morevoer, the odd girth of G is equal to the maximum of the odd girth of K_n and the odd girth of G. So, consider letting $G = M_l^n(Q)$ for some non-bipartite quadrangulation of the projective plane all of whose 4-cycles are facial and which has odd girth at least l. That such a quadrangulation exists for any choice of l is easy to see. We construct a few small cases below in Figure 3.3 from which it is simple to construct the rest. Then $G = M_l^n(Q)$ is a topologically $(n+4)$-chromatic which has odd girth l. Consequently, by Theorem 7, the graph $K_n \times G$ is a uniquely n-colourable, topologically n-chromatic graph with odd girth greater than or equal to l, as are all graphs $K_n \times G$ with $G = M_l^r(Q)$ for some $r \geq n$. □

3.5 Concluding Remarks

When we began working on this thesis, our aim was to come to a better understanding of the methods used for constructing graphs with large topological colouring bounds, as well as to examine tools used for efficiently computing topological colouring bounds in a theoretical

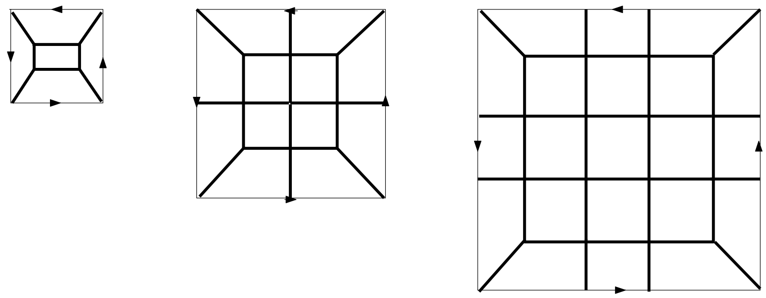


Figure 3.3: Quadrangulations of the Projective Plane all of whose 4-cycles are Facial and which have Odd Girth 3, 5 and 7

setting. As we worked on these problems, we repeatedly found that the graphs we studied seemed to all have a high degree of structural similarity. This observation then motivated the main objective of this thesis, which was to bring together results spread throughout the literature, and, if necessary, prove new results, in order to forge links between structural graph theory and topological colouring bounds.

Of course, this is an extremely hard and broad problem. While we attempted to be as exhaustive as possible, there is no doubt that we have done little more than scratch the surface of the links which may be forged between structural graph theory and topological colouring bounds. An area we feel might be particularly fruitful for future work is the study of k-critical topologically k-chromatic graphs. In the $k = 4$ case, these graphs seem to be closely connected to the 4-critical quadrangulations of the projective plane, while for higher k, while many such graphs exist, the Zig-Zag Theorem places quite stringent demands upon their local structure.

This thesis is also far from a complete account of the constructions which may be used in order to construct graphs with large topological colouring bounds. In particular, in the interest of brevity, we chose to exclude both the Schrijver graphs (which are also sometimes called stable Kneser graph) and Csorba's technique for constructing graph complexes with any possible given topology. Moreover, by placing suitably strict conditions on the graphs at hand, it is frequently possible to show that constructions which do not normally have easily understood topological behaviour have tamer behaviour under these conditions. A number of recent papers in the literature have, in fact, proved interesting results using precisely these sorts of methods. No doubt there is far more to discover here, as well.

In summary, we have reworked, synthesized and frequently generalized a variety of results on the structure of graphs with notable topological colouring bounds, as well as their construction, placing a particular focus upon proving results using elementary combinatorial methods. As we have stated, this is far from the last word on this subject, however, we would hope that it at least provides a starting point from which further exploration into the connections between structural graph theory and topological colouring bounds may begin.

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