MINIMUM COST HOMOMORPHISMS TO DIGRAPHS

by

Mohammad Mahdi Karimi

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APPROVAL

Name: Mohammad Mahdi Karimi

Degree: Doctor of Philosophy

Title of thesis: Minimum Cost Homomorphisms to Digraphs

Examining Committee: Dr. Ramesh Krishnamurti
Chair

Dr. Arvind Gupta, Senior Supervisor

Dr. Pavol Hell, Senior Supervisor

Dr. Andrei A.Bulatov, SFU Examiner

Dr. Derek G. Corneil, External Examiner,
Professor of Computer Science,
University of Toronto

Date Approved: February 3, 2009

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Abstract

For digraphs D and H, a homomorphism of D to H is a mapping $f: V(D) \rightarrow V(H)$ such that $uv \in A(D)$ implies $f(u)f(v) \in A(H)$. Suppose D and H are two digraphs, and $c_i(u)$, $u \in V(D)$, $i \in V(H)$, are nonnegative integer costs. The cost of the homomorphism f of D to H is $\sum_{u \in V(D)} c_{f(u)}(u)$. The minimum cost homomorphism for a fixed digraph H, denoted by MinHOM(H), asks whether or not an input digraph D, with nonnegative integer costs $c_i(u)$, $u \in V(D)$, $i \in V(H)$, admits a homomorphism f to H and if it admits one, find a homomorphism of minimum cost. Our interest is in proving a dichotomy for minimum cost homomorphism problem: we would like to prove that for each digraph H, MinHOM(H) is polynomial-time solvable, or NP-hard. Gutin, Rafiey, and Yeo conjectured that such a classification exists: MinHOM(H) is polynomial time solvable if H admits a k-Min-Max ordering for some $k \geq 1$, and it is NP-hard otherwise.

For undirected graphs, the complexity of the problem is well understood; for digraphs, the situation appears to be more complex, and only partial results are known. In this thesis, we seek to verify this conjecture for "large" classes of digraphs including reflexive digraphs, locally in-semicomplete digraphs, as well as some classes of particular interest such as quasi-transitive digraphs. For all classes, we exhibit a forbidden induced subgraph characterization of digraphs with k-Min-Max ordering; our characterizations imply a polynomial time test for the existence of a k-Min-Max ordering. Given these characterizations, we show that for a digraph H which does not admit a k-Min-Max ordering, the minimum cost homomorphism problem is NP-hard. This leads us to a full dichotomy classification of the complexity of minimum cost homomorphism problems for the aforementioned classes of digraphs.

Keywords: homomorphism; minimum cost homomorphism; polynomial time algorithm; NP-hardness; dichotomy

Subject Terms: Graph Theory; Graph Homomorphism; Digraphs; Graph Algorithms

To my family, my teachers, and my friends

One who quested, found.

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Chapter 1

Introduction

The minimum cost homomorphism problem was first introduced, in the context of undirected graphs, in [48, 85]. There, it was motivated by a real-world problem, called Level of Repair Analysis (LORA). For a complex engineering system containing perhaps thousands of assemblies, sub-assemblies, components etc. organized into $\ell \geq 2$ levels of *indenture* and with $r \geq 2$ possible repair decisions, LORA seeks to determine an optimal provision of repair and maintenance facilities to minimize overall life-cycle costs. Barros [11] and Riley [12] provide a generic integer programming formulation of the LORA optimization problem for systems with ℓ levels of indenture and r possible repair decisions. A special case with $\ell = 2$ and $r = 3$, which is called LORA-BR, is of particular importance because it corresponds to several interesting real world problems, see Barros and Riley [12].

Let us refer to the first level of indenture in LORA-BR as *subsystems* $s \in S$ and the second level of indenture as *modules* $m \in M$. The distribution of modules in subsystems can be given by a bipartite graph $G = (V_1, V_2; E)$ with partite sets $V_1 = S$ and $V_2 = M$. For arbitrary $s \in V_1$ and $m \in V_2$, $sm \in E$ if and only if module m is in subsystem s.

There are $r = 3$ available repair decisions for each level of indenture: discard, local repair and central repair, labelled respectively D, L, C (subsystems) and d, l, c (modules). Assume we also know additive nonnegative integer costs (over a system life-cycle) $c_z(v)$ of prescribing repair decision z for a subsystem or module v. We wish to minimize the total cost of available repair options to the subsystems and modules subject to the following constraints.

If a module m occurs in subsystem s (i.e., $sm \in E$) we impose the following logical restrictions on the repair decisions for the pair (s, m) motivated through practical considerations:

$R_1 : D_s \Rightarrow d_m$,
$R_2 : l_m \Rightarrow L_s$,

where D_s, d_m denote the decisions to discard subsystem s, module m, respectively, etc. Note that even though module m may be common to several subsystems, we are required to prescribe a unique repair decision for that module. R_1 has the interpretation that a decision to discard subsystem s necessarily entails discarding all enclosed modules. R_2 is a consequence of R_1 and a policy of "no backshipment" which rules out the local repair option for any module enclosed in a subsystem which is sent for central repair [12].

For a pair of graphs $H = (V(H), E(H))$ and $B = (V(B), E(B))$, a mapping $k : V(B) \to V(H)$ such that if $xy \in E(B)$ then $k(x)k(y) \in E(H)$ is called a *homomorphism* of B to H. Let $F_{BR} = (Z_1, Z_2; T)$ be a bipartite graph with partite sets $Z_1 = \{D, C, L\}$ (subsystem repair options) and $Z_2 = \{d, c, l\}$ (module repair options) and with edges $T = \{Dd, Cd, Cc, Ld, Lc, Ll\}$. Observe that any homomorphism k of G to F_{BR} such that $k(V_1) \subseteq Z_1$ and $k(V_2) \subseteq Z_2$ satisfies the rules R_1 and R_2. Indeed, let $u \in V_1$, $v \in V_2$, $uv \in E$. If $k(u) = D$ then $k(v) = d$, and if $k(v) = l$ then $k(u) = L$.

Now LORA-BR can be formulated as the following graph-theoretical problem: for a fixed bipartite graph $F_{BR} = (Z_1, Z_2; T)$, we are given a bipartite graph $G = (V_1, V_2; E)$, with nonnegative integer costs $c_z(v)$, $z \in Z_i, v \in V_i$ as input, and we verify whether G admits a homomorphism k to F_{BR} such that $k(V_1) \subseteq Z_1$ and $k(V_2) \subseteq Z_2$ (If no homomorphisms of G to F_{BR} exists, then the problem has no feasible solution), and if it admits one, we find a homomorphism k of G to F_{BR} that minimizes the following aggregation:

$$\sum_{v \in V_1 \cup V_2} c_{k(v)}(v) \tag{1.1}$$

where $k(V_i) \subseteq Z_i$.

We call the expression in (1.1) the *cost* of k.

The graph-theoretical formulation of LORA-BR can be naturally extended as follows: The above problem with F_{BR} replaced by an arbitrary *fixed* bipartite graph $F = (Z_1, Z_2; T)$ is called the *general LORA problem with* $\ell = 2$.

The formulation of the LORA problem in terms of particular homomorphisms led the authors of [48] to introduce the minimum cost homomorphism problems for general 'undirected graphs': for a fixed undirected graph H, we are given an input graph G with costs $c_z(u)$ of mapping each vertex $u \in V(G)$ to each vertex $z \in V(H)$, and the problem is to verify whether G admits a homomorphism to H, and if it admits one, find a homomorphism k that minimizes $\sum_{u \in V(G)} c_{k(u)}(u)$.

For undirected graphs, the complexity of the problem is well understood; for digraphs, the situation appears to be more complex, and only partial results are known. In this thesis, we study the complexity of the minimum cost homomorphism problem for directed graphs (digraphs).

The thesis is structured as follows. In the remaining sections of this chapter, we first introduce several classes of digraphs which have been studied for different derivatives of the digraph homomorphism problem. After that, we will discuss current results concerning the minimum cost homomorphism problem.

Chapter 2 is devoted to the different versions of the constraint satisfaction problem. In Chapter 3, we introduce tools useful for the study of the complexity of the minimum cost homomorphism problems.

Chapters 4 and 5 cover the main results of this thesis, concerning digraphs with some loops (meaning at least one loop). In Chapter 4, we give a full dichotomy classification of the minimum cost homomorphism problem for reflexive digraphs. Chapter 5 is devoted to oriented cycles with some loops, and we study the minimum cost homomorphism problem for this subclass of digraphs as a first step toward a dichotomy for oriented graphs with some loops.

In Chapters 6 and 7, we study minimum cost homomorphism problem for quasi-transitive digraphs and locally in-semicomplete digraphs, respectively. Specifically, the class of locally in-semicomplete digraphs is the largest class of irreflexive digraphs for which such dichotomy classification is proved.

1.1 Definitions

A *relational structure* D consists of a finite set of vertices, denoted by $V(D)$, and a finite number of relations $R_1, R_2, \ldots, R_t$ on $V(D)$, of arities $r_1, r_2, \ldots, r_t$ respectively. The vector $(r_1, r_2, \ldots, r_t)$ is called the *type* of D. A relational structure D is *complete* if each $R_i = (V(D))^{r_i}$.

A *digraph* D is a relational structure with only one binary relation $A = A(D)$. An element (u,v) of A is called an *arc* of D, and denoted by $uv \in A(D)$. For a digraph D, if $uv \in A(D)$ we say that u *dominates* v or v is *dominated* by u, and denote by $u \rightarrow v$. For sets $X, Y \subset V(D)$, $X \rightarrow Y$ means that $x \rightarrow y$ for each $x \in X, y \in Y$. If uv is an arc of D, we say that u is an *in-neighbor* of v and v is an *out-neighbor* of u . The number of in-neighbors (out-neighbors) of v is called the *in-degree* (*out-degree*) of v. We call $V(D)$ the *vertex set* and $A(D)$ the *arc set* of D. A digraph D is *symmetric*, or *reflexive*, or *irreflexive*, etc., if the relation A is symmetric, or reflexive, or irreflexive, etc., respectively. Note that a reflexive digraph is a digraph such that every vertex has a loop and an irreflexive digraph is a digraph without loops. We stress that a digraph D is not assumed irreflexive by saying that D is a *digraph with possible loops*. If a digraph D has at least one loop, then we say that D is a *digraph with some loops*. From now on, whenever we do not stress that a digraph is a digraph with some loops or a digraph with possible loops or a reflexive digraph, we assume that it is irreflexive.

Symmetric digraphs are more conveniently viewed as (undirected) graphs. In fact, each pair of symmetric arcs uv, vu in the arc set of a digraph D can be replaced by an edge uv in its corresponding undirected graph G. Formally, a *graph* G is a set $V = V(G)$ of vertices together with a set $E = E(G)$ of edges, each of which is a two-element set of vertices. We say that u and v are *adjacent* if $uv \in E(G)$. If we allow loops to a graph G, i.e., edges that only consist of one vertex, we have a *graph with possible loops*. If every vertex has a loop, we have a *reflexive graph*. In this thesis, directed (undirected) graphs have no parallel arcs (edges) and parallel loops. We always denote the edge set of an undirected graph G by $E(G)$ and the arc set of a digraph D by $A(D)$.

We say that D' is a *subgraph* of a digraph D, if $V(D') \subseteq V(D)$ and $A(D') \subseteq A(D)$. Also, D' is an *induced subgraph* of D if it is a subgraph of D and contains all the arcs of D amongst the vertices of D'. For a digraph D, we denote by $D[X]$ the subgraph of D induced by $X \subseteq V(D)$.

Let D be a digraph with possible loops. An arc $xy \in A(D)$ is *symmetric* if $yx \in A(D)$. We denote by $S(D)$ the *symmetric subgraph* of D, i.e., the undirected graph with $V(S(D)) = V(D)$ and $E(S(D)) = \{uv : uv \in A(D) \text{ and } vu \in A(D)\}$. We also denote by $U(D)$ the *underlying graph* of D, i.e., the undirected graph with $V(U(D)) = V(D)$ and $E(U(D)) = \{uv : uv \in A(D) \text{ or } vu \in A(D)\}$. A digraph D is *connected* if $U(D)$ is connected. We denote by $B(D)$ the bipartite graph obtained from D as follows. Each vertex v of D

gives rise to two vertices of $B(D)$ - a *white* vertex v' and a *black* vertex v''; each arc vw of D gives rise to an edge $v'w''$ of $B(D)$. Note that if D is a reflexive digraph, then all edges $v'v''$ are present in $B(D)$. A digraph H is an *extension* of D if H can be obtained from D by replacing every vertex x of D with a set S_x of independent vertices such that $xy \in A(D)$ if and only if $uv \in A(H)$ for each $u \in S_x, v \in S_y$. The *converse* of D is the digraph obtained from D by reversing the directions of all arcs. Finally, we denote by $I(D)$ the irreflexive digraph D' obtained from a digraph D with possible loops by removing all existing loops.

To construct 'bigger' digraphs from 'smaller' ones, we will often use the following operation called *composition*. Let D be a digraph with vertex set $\{v_1, v_2, \ldots, v_n\}$, and let $G_1, G_2, \ldots, G_n$ be digraphs which are pairwise vertex-disjoint. The composition $D[G_1, G_2, \ldots, G_n]$ is the digraph H with vertex set $V(G_1) \cup V(G_2) \cup \ldots \cup V(G_n)$ and arc set $(\bigcup_{i=1}^{n} A(G_i)) \cup \{g_i g_j : g_i \in V(G_i), g_j \in V(G_j), v_i v_j \in A(D)\}$.

An *oriented path* P is a sequence of distinct vertices $[b_0, b_1, \ldots, b_p]$ such that for each $i \in \{0, 1, \ldots, p-1\}$, either $b_i b_{i+1} \in A(P)$ (a *forward* arc of P) or $b_{i+1} b_i \in A(P)$ (a *backward* arc of P), and P has no other arcs. The direction in which P is traversed is emphasized by saying that b_0 is the *initial vertex* of P, and b_p is the *terminal vertex* of P, respectively.

An *oriented cycle* C is a digraph obtained from an oriented path P by identifying its initial and terminal vertices. Thus an oriented cycle C can be given by a circular sequence of vertices $[b_0, b_1, \ldots, b_p, b_0]$, such that, for each $i \in \{0, 1, \ldots, p\}$, either $b_i b_{i+l} \in A(C)$ (a *forward* arc of C) or $b_{i+1} b_i \in A(C)$ (a *backward* arc of C), and C has no other arcs. (Subscript addition is taken modulo $p+1$.) Since we do not distinguish an initial vertex of an oriented cycle, we usually choose the most convenient vertex to start listing C. In this thesis, we will always consider the direction $b_0 b_1 \ldots b_p b_0$ in which the number of forward arcs is not smaller than the number of backward arcs. This way, the *net length* of C is the difference between the number of forward arcs and the number of backward arcs and hence is always nonnegative. An oriented cycle C is *balanced* if its net length is zero; otherwise C is *unbalanced*. A digraph D is *balanced* if all its oriented cycles are balanced; otherwise D is *unbalanced*. Let C be an oriented cycle with possible loops. The net length of C , denoted $\lambda(C)$, is equal to the net length of $I(C)$.

A *directed cycle* (respectively, a *directed path*) is an oriented cycle (respectively, oriented path) in which all edges are in the same direction. We denote a directed cycle (respectively, path) with k vertices by $\vec{C}_k$ (respectively, $\vec{P}_k$). A digraph D is *acyclic*, if it does not contain any directed cycle $\vec{C}_k$. A digraph D is *strongly connected* (or, just, *strong*) if, for every pair

x, y of distinct vertices in D, there exists a directed path from x to y, denoted by (x, y)-path. A *strong component* of a digraph D is a maximal induced subgraph of D which is strong. A *strong component digraph* of a digraph D, abbreviated by $SCD(D)$, is obtained by contracting each strong component D_i of D into a single vertex v_i and placing an arc from v_i to v_j, $i \neq j$ if and only if there is an arc from D_i to D_j [3]. ($SCD(D)$ is also known as the *condensation* of D, cf. [92].) Observe that $SCD(D)$ is acyclic. We call a strong component an *initial strong component* if its corresponding vertex in $SCD(D)$ is of in-degree zero. A vertex u of digraph D is a source (sink) if it has in-degree (out-degree) zero. A digraph D is *smooth* if it has no sources and no sinks. An *oriented graph* D is a digraph which does not contain $\vec{C}_2$.

A digraph D is *bipartite* if $U(D)$ is bipartite. For a bipartite digraph $H = (V, U; A)$, where V and U are its partite sets, $H^{\rightarrow}$ is the subgraph induced by all arcs directed from V to U, $H^{\leftarrow}$ is the subgraph induced by all arcs directed from U to V, and $H^{\leftrightarrow}$ is the subgraph induced by all 2-cycles of H, i.e., by the set $\{xy : xy \in A, yx \in A\}$.

A digraph D is *semicomplete* if $U(D)$ is a complete graph. We say that a digraph D is *semicomplete k-partite digraph* (or, semicomplete multipartite digraph when k is immaterial) if $U(D)$ is a complete k-partite graph. A digraph D is *locally in-semicomplete* if for every vertex x of D, the in-neighbors of x induce a semicomplete digraph. A *tournament* is a semicomplete digraph which does not have any symmetric arc. An acyclic tournament on p vertices is denoted by TT_p and called a *transitive tournament.* The vertices of a transitive tournament TT_p can be labeled $1, 2 \ldots, p$ such that $ij \in A(TT_p)$ if and only if $1 \leq i < j \leq p$. For $p \geq 2$, we denote by TT_p^- the digraph obtained from TT_p by deleting the arc $1p$. A digraph D is *quasi-transitive* if, for every triple x, y, z of distinct vertices of D such that xy and yz are arcs of D, there is at least one arc between x and z.

Let H be a digraph with possible loops and $<$ a linear ordering of $V(H)$. Two arcs $ab, cd \in A(H)$ are called a *crossing pair* if $a < c$, and $d < b$.

Definition 1.1.1 *A linear ordering $<$ of $V(H)$ is a* Min-Max ordering *if, for each crossing pair $ab, cd \in A(H)$ we have $ad, cb \in A(H)$.*

Clearly, if H has no crossing pair then $<$ is a Min-Max ordering.

Definition 1.1.2 *Let $k \geq 2$ be an integer. A digraph H admits a k-Min-Max ordering if the following conditions hold:*

- *H admits a homomorphism f to a directed k-cycle $0, 1, \ldots, k-1, 0$, i.e., every arc of H is an arc from $V_i = f^{-1}(i)$ to $V_{i+1} = f^{-1}(i+1)$ for some $i \in \{0, 1, \ldots, k-1\}$, and*
- *there is a linear ordering $<$ of vertices of each $V_i = f^{-1}(i)$, so that for each crossing pair $ab, cd \in A(H)$ ($a < c$, and $d < b$) where $a, c \in V_i$, and $b, d \in V_{i+1}$ we have $ad, cb \in A(H)$,*

where all indices $i+1$ are taken modulo k.

A graph G is *chordal* if it does not contain an induced subgraph isomorphic to an undirected cycle C_k for $k \geq 4$. An *asteroidal triple* of a graph G is a triple of mutually non-adjacent vertices such that for any two vertices of the triple there exists a path in G between them that avoids the neighborhood of the third vertex in the triple. We say that a graph G is *AT-free* if G does not contain any asteroidal triple.

The *intersection graph* of a family $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$ of sets is an undirected graph G with $V(G) = \mathcal{F}$ in which S_i and S_j are adjacent just if $S_i \cap S_j \neq \emptyset$. Note that by this definition, each intersection graph is reflexive. An undirected graph isomorphic to the intersection graph of a family of intervals on the real line is called an *interval graph.* If the intervals can be chosen to be inclusion-free, the graph is called *proper interval graph.* Thus both interval graphs and proper interval graphs are reflexive. By a result of Lekkerkerker and Boland [79], we have the following characterization of interval graphs.

Theorem 1.1.3 *[79] A graph G is interval if and only if G is chordal and AT-free.*

We refer to *claw, net*, and *tent* as the digraphs, shown in Figure 1.1. There is a nice induced subgraph characterization of proper interval graphs due to Wegner [91].

Theorem 1.1.4 *[91] Let G be a reflexive graph. G is a proper interval graph if and only if it does not contain an induced undirected cycle C_k, with $k \geq 4$, or an induced claw, net, or tent.*

The *intersection bigraph* of two families $\mathcal{F}_1 = \{S_1, S_2, \ldots, S_n\}$ and $\mathcal{F}_2 = \{T_1, T_2, \ldots, T_n\}$ of sets is the bipartite graph with $V(G) = \mathcal{F}_1 \cap \mathcal{F}_2$ in which S_i and T_j are adjacent just if $S_i \cap T_j \neq \emptyset$. Note that by this definition an intersection bigraph is irreflexive (since it is a bipartite graph). A bipartite graph isomorphic to the intersection bigraph of two families of intervals on the real line is called an *interval bigraph.* If the intervals in each family

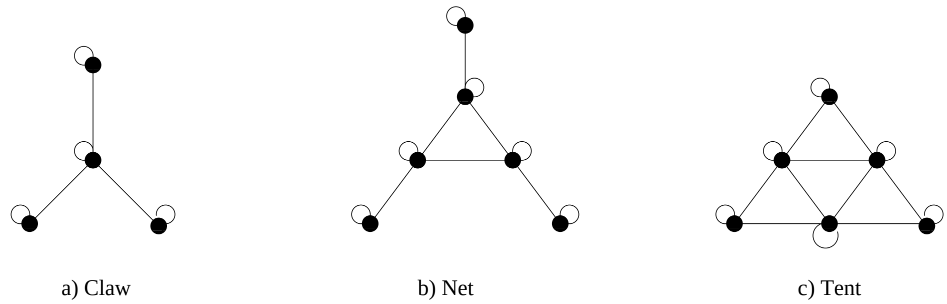

Figure 1.1: The claw, the net, and the tent.

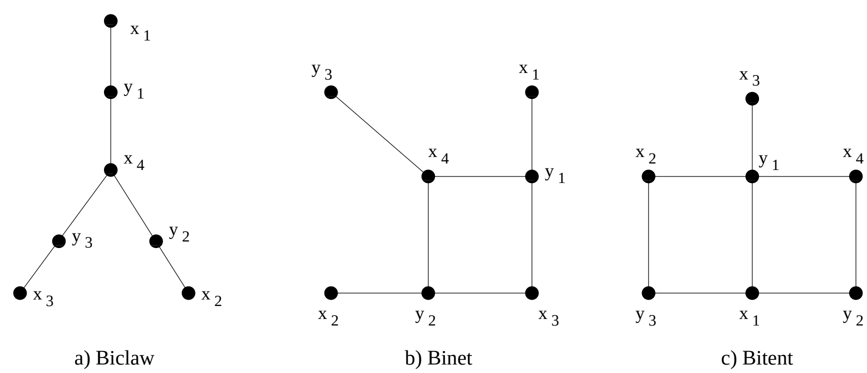

Figure 1.2: The biclaw, the binet, and the bitent.

$\mathcal{F}_i$ can be chosen to be inclusion-free, the graph is called a *proper interval bigraph*. Thus both interval and proper interval bigraphs are irreflexive. Let *biclaw, binet*, and *bitent* be the digraphs, shown in Figure 1.2. A Wegner-like characterization (in terms of forbidden induced subgraphs) of proper interval bigraphs is given in [62].

Theorem 1.1.5 *[62] Let G be a bipartite graph. G is a proper interval bigraph if and only if it does not contain an induced undirected cycle C_{2k}, with $k \geq 3$, or an induced biclaw, binet, or bitent.*

1.2 Minimum Cost Homomorphisms

For digraphs D and H, a mapping $f : V(D) \rightarrow V(H)$ is a *homomorphism of D to H* if $uv \in A(D)$ implies that $f(u)f(v) \in A(H)$. Suppose D and H are two digraphs, and $c_i(u)$, $u \in V(D)$, $i \in V(H)$, are nonnegative integer costs or $+\infty$. (We treat $+\infty$ as a special value, with the property that $+\infty + x = +\infty$ for any x.) The cost of the homomorphism f of D to H is $\sum_{u \in V(D)} c_{f(u)}(u)$. The *minimum cost homomorphism problem for a fixed digraph H*, denoted by MinHOM(H), asks whether or not an input structure D, with nonnegative integer costs $c_i(u)$, $u \in V(D)$, $i \in V(H)$, admits a homomorphism f to H, and if it admits one, asks to find a homomorphism of minimum cost. Equivalently, we can define a decision version of this problem as follows: Given an input digraph D, together with costs $c_i(u)$, $u \in V(D)$, $i \in V(H)$, and an integer k, decide if D admits a homomorphism to H of cost not exceeding k. We refer to the former version of this problem in this thesis unless mentioned otherwise. Note that MinHOM(H) is NP-hard in the former version if and only if it is NP-complete in the later version. Due to this fact, we will use the term NP-hard even when we deal with the decision version of MinHOM(H).

The minimum cost homomorphism problem seems to offer a natural and practical way to model many optimization problems. Special cases include for instance the list homomorphism problem [60, 61] and the optimum cost chromatic partition problem [57, 66, 69] (which itself has a number of well-studied special cases and applications [74, 88]).

There is an extensive literature on the minimum cost homomorphism problem, e.g., see [48, 49, 50, 51, 52, 53, 54]. These and other papers study the dichotomy of MinHOM(H) for various families of directed and undirected graphs. In particular, the authors of [49] proved a dichotomy classification for all undirected graphs with possible loops.

Theorem 1.2.1 *[49] Let H be a connected graph with possible loops. If H is a proper interval graph or a proper interval bigraph, then the problem MinHOM(H) is polynomial time solvable. Otherwise, MinHOM(H) is NP-hard.*

In contrast to undirected graphs, it is still an open problem whether there is a dichotomy classification for the complexity of MinHOM(H) when H is a digraph with possible loops. Gutin, Rafiey, and Yeo [50] conjectured such a classification. We will study this dichotomy in Chapter 2.

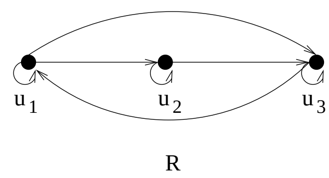

Figure 1.3: The digraph R.

Motivated by the paper of Bang-Jensen, Hell, MacGillivray [8] that classifies the complexity of homomorphism problem for the class of semicomplete digraphs, Gutin et al. began studying MinHOM(H) for this class. The following theorem is the main result of [51].

Theorem 1.2.2 *[51] For a semicomplete digraph H, MinHOM(H) is polynomial time solvable if H is acyclic or $H = \vec{C}_2$. Otherwise, MinHOM(H) is NP-hard.*

The dichotomy for semicomplete digraphs has been generalized by Gutin and Kim for semicomplete digraphs with possible loops in [54]. Let R be the digraph, shown in Figure 1.3. The following theorem is the main result of [54].

Theorem 1.2.3 *[54] Let H be a semicomplete digraph with possible loops. If H is one of the following digraphs, then MinHOM(H) is polynomial time solvable. Otherwise, it is NP-hard.*

- *The digraph $H = \vec{C}_k$ for $k = 2$ or 3.*
- *The digraph $H = TT_k[D_1, D_2, \dots, D_k]$ where D_i for each $i = 1, \dots, k$ is either a single vertex without loop, or a reflexive semicomplete digraph which does not contain R as an induced subgraph and each $U(S(D_i))$ is a connected proper interval graph.*

As a generalization of semicomplete digraphs, semicomplete multipartite digraphs is the second class that has been examined for dichotomy. However, showing a dichotomy for this class is not as straightforward as for semicomplete digraphs. To overcome this difficulty, Gutin et al. studied semicomplete k-partite digraphs for $k \geq 3$, and semicomplete bipartite digraphs separately. The following dichotomy has been shown for the former case in [53].

Theorem 1.2.4 *[53] Let H be a semicomplete k-partite digraph, $k \geq 3$. If H is an extension of $TT_k, \vec{C}_3$ or TT_p^- ($p \geq 4$), then MinHOM(H) is polynomial time solvable. Otherwise, MinHOM(H) is NP-hard.*

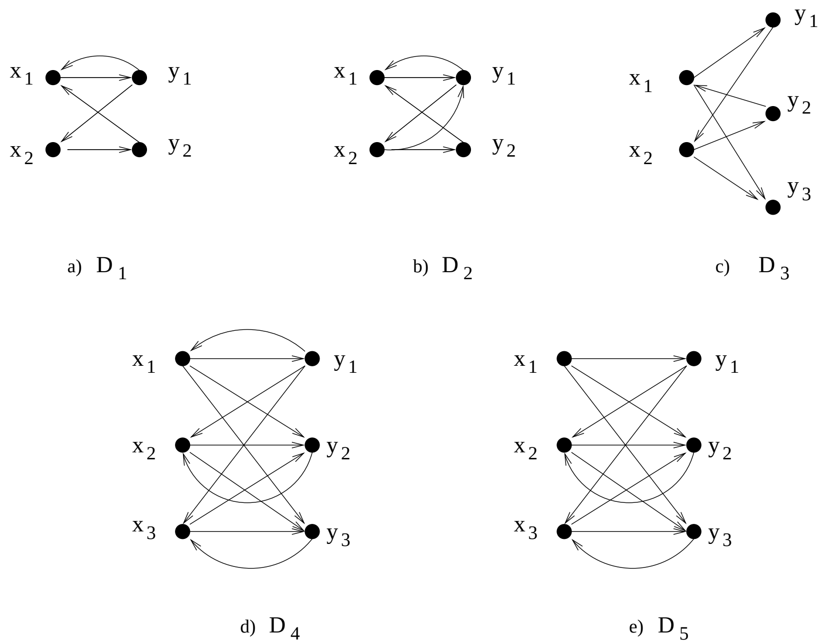

Figure 1.4: Forbidden digraphs $D_1, \ldots, D_5$.

Although Theorem 1.2.4 gives us a nice characterization of polynomial cases of k-partite semicomplete digraphs, there is not such a characterization for semicomplete bipartite digraphs. In fact, instead of having a few induced subgraphs to characterize polynomial cases for semicomplete bipartite digraphs, we have a family of forbidden digraphs $\mathcal{F}$. A digraph H belongs to the forbidden family $\mathcal{F}$ if H or its converse is isomorphic to one of the five digraphs, shown in Figure 1.4 or $U(H^s)$, where $s \in \{\rightarrow, \leftarrow, \leftrightarrow\}$, is isomorphic to the bipartite claw, bipartite net, bipartite tent (See Figure 1.2), or an even cycle with at least six vertices. The following theorem shows a dichotomy for semicomplete bipartite digraphs [52].

Theorem 1.2.5 *[52] Let H be a semicomplete bipartite digraph. If H does not contain any digraph of $\mathcal{F}$ as an induced subgraph, then MinHOM(H) is polynomial time solvable. Otherwise, MinHOM(H) is NP-hard.*

The class of oriented cycles is another interesting class of digraphs for which a MinHOM dichotomy is known. We will see this dichotomy in Chapter 5 when we study minimum cost homomorphism problem for oriented cycles with some loops.

Chapter 2

Constraint Satisfaction Problems

In this chapter, we review different variants of the problem of existence of homomorphism between two relational structures and their restrictions to digraphs (relational structures with only one binary relation). The reader can skip this chapter with no loss of continuity.

2.1 HOM and CSP

Let D and H be two relational structures of the same type with relations $R_1, R_2, \ldots, R_t$ and $S_1, S_2, \ldots, S_t$ respectively. A *homomorphism* of D to H, written as $f : D \to H$ is a mapping $f : V(D) \to V(H)$ such that $(v_1, v_2, \ldots, v_{r_i}) \in R_i$ implies that $(f(v_1), f(v_2), \ldots, f(v_{r_i})) \in S_i$, for all $i = 1, 2, \ldots, t$. If $D \to H$ we shall say that D is *homomorphic* to H. Note that if H is a complete relational structure, then any relational structure D with the same type as H, is homomorphic to H. Two structures such that each is homomorphic to the other are called *homomorphically equivalent*. A homomorphism f of D to H is an *isomorphism*, if f is bijective and the inverse of f is also a homomorphism. An isomorphism of D to D is called an *automorphism* of D.

Given two relational structures D and H of the same type, a *constraint satisfaction problem* asks whether there exists a homomorphism of D to H. This formulation of constraint satisfaction problem was first introduced by Feder and Vardi [39]. Let H be a fixed relational structure. The *constraint satisfaction problem* CSP(H) asks whether or not an input relational structure D, of the same type as H, admits a homomorphism to H. Specifically, if H is a digraph, we call this problem digraph H-*colouring* or *homomorphism problem* for digraph H, and denote it by HOM(H). We say that a relational structure D is a *core* if it has no proper substructure D' to which D admits a homomorphism. It is easy to see that each relational structure including digraph is homomorphically equivalent to a unique core [60].

The study of constraint satisfaction problems has largely been undertaken within the artificial intelligence (AI) community. The pioneering work was undertaken in the early 1970 by Montanari in a slightly different formulation [83]. Since then, these problems have been used to model many problems in different areas such as graph theory, machine vision, and data bases [39, 82, 80]. However, our focus in this section is on the theoretical aspects of constraint satisfaction problems. Specifically, we are interested in a *dichotomy* (polynomial or NP-complete) of CSP and HOM for different relational structures or digraphs H, respectively.

It is worth noting that it is likely that there exist problems in NP which are neither polynomial time solvable nor NP-complete. Indeed, Ladner [77] has shown that if $P \neq NP$, there are NP problems that are neither polynomial nor NP-complete, also called *intermediate problems* - in fact there must be an infinite hierarchy of such (non-polynomially equivalent) problems. Feder and Vardi [39] investigated which subclasses of NP have the same computational power as NP, and which do not (and hence might not contain intermediate problems). They define the class MMSNP, monotone monadic strict NP without inequality, and show that for this class Ladner's argument does not immediately apply, however, removing either of 'monotone', 'monadic', or 'without inequality' restrictions gives the full computational power of NP. Furthermore, they show that MMSNP is polynomial time equivalent to the class CSP [39, 76]. This observation motivated Feder and Vardi to raise the following conjecture [39].

Conjecture 2.1.1 *[39] For any relational structure H, the problem CSP(H) is NP-complete or polynomial time solvable.*

There are several other forms of this conjecture for CSP(H) in the literature, see, e.g., [19, 78, 84]. The primary motivation to examine the dichotomy for CSP(H) goes back to Hell and Nešetřil [59], and Schaeffer [87]. The first form examines undirected graphs and gives us the following dichotomy [59].

Theorem 2.1.2 *[59] Let H be an undirected graph with possible loops. If H is bipartite or has a loop, then HOM(H) is polynomial time solvable. Otherwise, HOM(H) is NP-complete.*

The second form classifies which CSP(H) are NP-complete and which are polynomial time solvable when H is a Boolean relational structure, i.e., $V(H) = \{0, 1\}$. This result has been later generalized for structures H with up to three vertices [20]. For the case of Boolean structures H, Schaeffer [87] has established a dichotomy in terms of four well known operations on tuples (AND, OR, MAJORITY, and XOR). The *OR* operation on two tuples $(a_1, a_2, \ldots, a_s)$ and $(b_1, b_2, \ldots, b_s)$ is the tuple $(z_1, z_2, \ldots, z_s)$ where each $z_i = a_i \vee b_i$ ($z_i = 1$ unless both $a_i = b_i = 0$, in which case $z_i = 0$). The *AND* operation on two tuples $((a_1, a_2, \ldots, a_s)$ and $(b_1, b_2, \ldots, b_s)$ is the tuple $(z_1, z_2, \ldots, z_s)$ where each $z_i = a_i \wedge b_i$ ($z_i = 0$ unless both $a_i = b_i = 1$, in which case $z_i = 1$). The *MAJORITY* operation on three tuples $(a_1, a_2, \ldots, a_s)$, $(b_1, b_2, \ldots, b_s)$, and $(c_1, c_2, \ldots, c_s)$ is the tuple $(z_1, z_2, \ldots, z_s)$ where each z_i is the majority value (0 or 1) of a_i, b_i, c_i. The *XOR* (exclusive OR, also known as *MINORITY*) operation on three tuples $(a_1, a_2, \ldots, a_s)$, $(b_1, b_2, \ldots, b_s)$, and $(c_1, c_2, \ldots, c_s)$ is the tuple $(z_1, z_2, \ldots, z_s)$ where each z_i is the exclusive-or value of a_i, b_i, c_i (equal to 1 if the number of 1's amongst a_i, b_i, c_i is odd, and 0 otherwise). Schaeffer proved the following fact [87].

Theorem 2.1.3 *[87] Suppose H is a relational structure with $V(H) = \{0, 1\}$ and relations $S_1, S_2, \ldots, S_p$. then CSP(H) is NP-complete, except in the following, polynomial time solvable, cases:*

1. *each S_i contains the s_i-tuple $(0, 0, \ldots, 0)$; or*
2. *each S_i contains the s_i-tuple $(1, 1, \ldots, 1)$; or*
3. *each S_i is closed under the OR operation; or*
4. *each S_i is closed under the AND operation; or*
5. *each S_i is closed under the MAJORITY operation; or*
6. *each S_i is closed under the XOR operation.*

It is worth noting that Conjecture 2.1.1 has not been proved for digraphs H. It has been shown by the authors of [39] that a dichotomy for digraph H-colouring problems would imply the entire dichotomy of Conjecture 2.1.1. The following two theorems clearly show this fact [39].

Theorem 2.1.4 *[39] Every constraint-satisfaction problem is polynomially equivalent to a digraph H-colouring problem, where H is an unbalanced digraph.*

Theorem 2.1.5 *[39] Every constraint-satisfaction problem is polynomially equivalent to a digraph H-colouring problem, where H is a balanced digraph.*

Many results have been proved for digraph H-colouring when H is restricted to special families of digraphs H [7, 8, 9, 10, 31, 43, 55]. For instance, dichotomy is known to hold for the case when $U(H)$ is a cycle [31], or path [55], or complete graph [8] (H is a semicomplete digraph). The most general result is due to Barto et al. [13] for smooth digraphs, conjectured earlier by Hell and Bang-Jensen [7].

Theorem 2.1.6 *[13] Let H be a smooth digraph. If each component of the core of H is a directed cycle, HOM(H) is polynomial time solvable. Otherwise HOM(H) is NP-complete.*

This theorem is a generalization of Theorem 2.1.2, since a connected graph H (subclass of the class of smooth digraphs) has a core which is a directed cycle if and only if the graph is bipartite or has a loop.

2.2 ListHOM and ListCSP

A *list constraint satisfaction problem* for a fixed relational structure H, denoted ListCSP(H), asks whether or not an input structure D, with lists $L_u \subseteq V(H), u \in V(D)$, admits a homomorphism f to H in which all $f(u) \in L_u, u \in V(D)$. In particular, if H is a digraph, we call this problem *list homomorphism problem* for digraph H, and denote it ListHOM(H).

ListCSP(H) tend to be more manageable than CSP(H). Many natural applications of homomorphisms, such as frequency assignment, scheduling, and so on, tend to have additional constraints expressible by lists. Finally, it turns out that many algorithms for graph homomorphisms adapt very naturally to lists [59].

In the literature [15, 63], ListCSP's are often investigated as *conservative* CSP's or CSP's for *conservative structures*. A structure H is *conservative* if H contains all unary (arity one) relations $S \subset V(H)$. Indeed, any instance (G, l) of ListCSP(H) can be transformed to an instance G' of CSP(H') where H' is the structure H augmented with all unary relations S, and G' is constructed from G by setting unary relations $R = \{v \in V(G) | l(v) = S\}$. In this transformation, it is easy to see that G is a positive instance of ListCSP(H) if and only if G'

is a positive instance of CSP(H'). By a similar argument, any instance G' of CSP(H') for conservative H' can be transformed to an instance (G, l) for ListCSP(H). To show this, let $S_1, S_2, \ldots, S_n$ be the unary relations added to H, and $R_1, R_2, \ldots, R_n$ be their counterparts in G', respectively. One can easily see that if $R_i \cap R_j \neq \emptyset$ and $S_i \cap S_j = \emptyset$, there is no homomorphism from G' to H'. So, we suppose that $S_i \cap S_j \neq \emptyset$ when $R_i \cap R_j \neq \emptyset$. Let us now construct the pair (G, l) as follows: G is obtained from G' by removing all unary relations $R_1, R_2, \ldots, R_n$ from G' and $l(u) = \bigcap_{u \in R_i} S_i$. It remains again to see that G is a positive instance of ListCSP(H) if and only if G' is a positive instance of CSP(H'). Thus, ListCSP(H) is polynomial time equivalent to CSP(H') where H' is a conservative structure. Bulatov [15] has shown a dichotomy for all conservative CSP problems, and hence for all ListCSP problems. (We remark this result does not immediately imply a dichotomy for any of HOM, or CSP).

Theorem 2.2.1 *[15] For any relational structure H, ListCSP(H) is NP-complete or polynomial time solvable.*

It is worth noting the earlier results of Feder, Hell, and Huang [34, 35, 36] for undirected graphs, which motivated this theorem. To study these results, let us first define an interesting subclass of graphs.

A graph G is a *circular-arc graph* if G is the intersection graph of a family of arcs of a circle. A graph G is a *bi-arc graph* if there exists a family of arcs of a circle with two distinguished points p and q where each vertex of G is associated with two arcs (N_x, S_x) such that N_x contains p but not q and S_x contains q but not p, and for any two vertices x, y the following holds: (i) if $xy \in E(G)$ then $N_x \cap S_y = \emptyset$ and $N_y \cap S_x = \emptyset$, and (ii) if $xy \notin E(G)$ then $N_x \cap S_y \neq \emptyset$ and $N_y \cap S_x \neq \emptyset$. Note that a bi-arc representation can not contain bi-arcs (N_x, S_x) and (N_y, S_y) where $N_x \cap S_y = \emptyset$ and $N_y \cap S_x \neq \emptyset$. Finally, a graph G is a *weak interval bigraph* if it is a bipartite graph whose complement is a circular arc graph.

Theorem 2.2.2 *[34] Let H be a reflexive graph. If H is an interval graph, then ListHOM(H) is polynomial time solvable. Otherwise ListHOM(H) is NP-complete.*

Theorem 2.2.3 *[35] Let H be an irreflexive graph. If H is a weak interval bigraph, then ListHOM(H) is polynomial time solvable. Otherwise ListHOM(H) is NP-complete.*

Theorem 2.2.4 *[36] Let H be an undirected graph with possible loops. If H is a bi-arc graph, then ListHOM(H) is polynomial time solvable. Otherwise ListHOM(H) is NP-complete.*

2.3 MinHOM and MinCSP

Suppose D and H are two relational structures, and $c_i(u)$, $u \in V(D)$, $i \in V(H)$, are nonnegative integer *costs* $+\infty$. (We treat $+\infty$ as a special value, with the property that $+\infty + x = +\infty$ for any x.) The *cost of the homomorphism* f of D to H is $\sum_{u \in V(D)} c_{f(u)}(u)$. The *minimum cost constraint satisfaction problem* for a fixed relational structure H, denoted by MinCSP(H), asks whether or not an input structure D, with nonnegative integer costs $c_i(u)$, $u \in V(D)$, $i \in V(H)$, admits a homomorphism f to H and if it admits one, find a homomorphism of minimum cost. If H is a digraph, we call this problem the *minimum cost homomorphism problem*, and denote it MinHOM(H).

The minimum cost constraint satisfaction problem was introduced in [30] as a generalization of the minimum cost homomorphism problem earlier appeared in several other papers [48, 49, 50, 51]. The authors of [30] considered both D and H and the nonnegative integer costs $c_i(u)$, $u \in V(D)$, $i \in V(H)$ as inputs to the problem. In this framework, it has been shown in [30] that the hard instances of the minimum cost homomorphism problem are inapproximable as well. To our knowledge, this is the first inapproximability result for this problem (For further details regarding approximability we refer to [2].)

2.4 sHOM and sCSP

In the standard framework of constraint satisfaction problem, defined in Section 2.1, a constraint is usually taken to be a relation R_i of an input structure D, specifying the allowed combinations of values S_i of a fixed structure H. In a certain sense, these constraints are "exact", or "crisp".

The constraint satisfaction framework can be enhanced by extending the definition of a constraint to include also a soft constraint. Let us define a *soft* constraint as follows: Let H be a fixed complete relational structures with relations $S_1, S_2, \ldots, S_k$ and let each $\mathcal{F}_i$ for $1 \leq i \leq k$, be a finite set of functions $f : S_i \rightarrow \mathcal{R}^+$, where $\mathcal{R}^+$ is the set of the nonnegative real numbers. For an arbitrary relational structure D of the same type as H with relations

$R_1, R_2, \ldots, R_k$, a *soft* constraint is a function $\mathcal{C}_i : R_i \rightarrow \mathcal{F}_i$. Let us call i the *arity* of the soft constraint $\mathcal{C}_i$.

Let H be a fixed complete relational structure and $\mathcal{F} = \cup_{i=1}^{k} \mathcal{F}_i$. An instance of a *soft constraint satisfaction problem* sCSP$(H, \mathcal{F})$ contains an input relational structure D (of the same type as H with relations $R_1, R_2, \ldots, R_k$) with a set of soft constraints $\mathcal{C}_i : R_i \rightarrow \mathcal{F}_i, 1 \leq i \leq k$, and the problem is to find a homomorphism h of D to H which minimizes the following aggregation:

$$\sum_{i=1}^{k} \sum_{r \in R_i} \mathcal{C}_i(r)(h(r))$$

(Note that D is always homomorphic to H as H is a complete relational structure.) In particular, if H is a complete digraph, we call this problem *soft homomorphism problem*, and denote it sHOM$(H, \mathcal{F})$.

The soft constraint satisfaction problem is sufficiently flexible to allow us to express a wide range of problems such as MinHOM(H), where all $c_i(u)$ are bounded by a constant integer c. (Since they are assumed integers, we have each $c_i(u) \in \{0, 1, \ldots, c\}$.) For any instance of MinHOM(H), where all $c_i(u)$ are bounded by a constant integer c, we can define a corresponding instance of soft constraint satisfaction problem $sCSP(H', \mathcal{F})$.

Let H' be a complete relational structure with the same vertex set as H, i.e., $V(H') = V(H)$, and a set of complete relations S_1', S_2', where the arity of S_i' is i. Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{F}_1$ contains all unary functions from $S_1' = V(H)$ to $\{0, 1, \ldots, c\}$ and $\mathcal{F}_2$ contains only one binary function f from S_2' to $\{0, +\infty\}$ defined as follows:

$$f(uv) = \begin{cases} 0 & \text{if } uv \in A(H) \\ +\infty & \text{otherwise} \end{cases}$$

An instance of MinHOM(H) contains an input digraph D and nonnegative integer costs $c_i(u), i \in V(H), u \in V(D)$ bounded from above by a constant c. Now, we choose an instance of sCSP$(H', \mathcal{F})$ as follows. The input relational structure is D' with the same vertex set as D, i.e., $V(D') = V(D)$, and a set of relations R_1', R_2', where R_1' is a complete unary relation and $R_2' = A(D)$. Now, it remains to choose the set of soft constraints. We choose for the binary constraint $\mathcal{C}_2$ the mapping from R_2' to $\mathcal{F}_2$, which takes all elements of R_2' to f. We choose for the unary constraint $\mathcal{C}_1$ the mapping which takes each $u \in V(D)$ to the mapping f_u which is equal to $c_i(u), i \in V(H)$. It is easy to observe that, for a mapping h of $V(D)$ to $V(H)$ which is a homomorphism, the following aggregation

$$\sum_{i=1}^{2} \sum_{r' \in R'_i} \mathcal{C}_i(r')(h(r'))$$

is the cost of this homomorphism in MinHOM(H), and if h is not a homomorphism, then the above aggregation $+\infty$. Hence, by solving sCSP($H', \mathcal{F}$), we will determine whether D admits a homomorphism h to H and if it admits one, find a homomorphism of minimum cost.

As for CSP(H), there have been some efforts to find a dichotomy for sCSP($H, \mathcal{F}$). The main contribution to this problem goes back to Cohen et al. [22, 23, 24]. The authors of [24] took the first step towards a systematic analysis of the complexity of sCSP($H, \mathcal{F}$) of arbitrary arity over arbitrary finite domain $V(H)$. This leads first to a dichotomy for sHOM($H, \mathcal{F}$) when H has only unary and binary relations [22], and second to a dichotomy for sCSP($H, \mathcal{F}$) when H has a boolean domain $V(H) = \{0, 1\}$ [24]. The later case is beyond the scope of this thesis. So, we only consider the former case in the rest of this section.

A binary function $\phi : W^2 \rightarrow \mathcal{R}^+$ is called *submodular* with respect to an ordering $<$ of W, if for all $x, y, u, v \in W$, we have

$$\phi(\min\{x, u\}, \min\{y, v\}) + \phi(\max\{x, u\}, \max\{y, v\}) \leq \phi(x, y) + \phi(u, v).$$

The following fact is the main result of [22].

Theorem 2.4.1 *[22] Let H be a digraph with possible loops. The sHOM($H, \mathcal{F}$) is polynomial time solvable if there exists an ordering $<$ such that all functions of $\mathcal{F}$ are submodular with respect to $<$. Otherwise, sHOM($H, \mathcal{F}$) is NP-hard.*

The theorem in [22] is in fact more general, covering all relational structures, having only unary and binary relations rather than digraphs.

Theorem 2.4.2 *[22] Let H be a relational structure having only unary and binary relations. sCSP($H, \mathcal{F}$) is polynomial time solvable if there exists an ordering $<$ such that all functions of $\mathcal{F}$ are submodular with respect to $<$. Otherwise, sCSP($H, \mathcal{F}$) is NP-hard.*

2.5 Algebraic Approaches to CSP

There has been a lot of interest [7, 8, 9, 10, 19, 31, 39, 43, 55, 78, 84] for more than a decade to verify the conjecture of Feder and Vardi (see Conjecture 2.1.1) for CSP problems.

However, this conjecture is still open, although good progress has been made. Among various techniques used, algebraic techniques have been most successful. In this section, we define two algebraic notions, polymorphism and multimorphism, and review the results related to these notions.

2.5.1 Polymorphisms

Let A be a finite set. An *operation* on A is a mapping $f : A^n \to A$ for some nonnegative integer n. A *polymorphism* (of order k) of H is an operation $f : (V(H))^k \to V(H)$ on $V(H)$, such that $(v_1^j, v_2^j, \ldots, v_{r_i}^j) \in S_i$ for $j = 1, 2, \ldots, k$ implies that $(f(v_1^1, v_1^2, \ldots, v_1^k), f(v_2^1, v_2^2, \ldots, v_2^k), \ldots, f(v_{r_i}^1, v_{r_i}^2, \ldots, v_{r_i}^k)) \in S_i$, for all relations S_i of H. For the purpose of this discussion, we shall focus on polymorphisms f that are *idempotent*, i.e., satisfy $f(x, x, \ldots, x) = x$ for all vertices $x \in V(H)$. Note that every structure H admits some polymorphisms of order k. For each $i \leq k$, we have a polymorphism called the *i-th projection*, defined by $\pi_i(v_1, v_2, \ldots, v_k) = v_i$. A structure H is *projective* if the only polymorphisms of H are $f \circ \pi_i$ where f is an automorphism of H. The following theorem has been proved in [67, 68].

Theorem 2.5.1 *[67, 68] Let H be a projective relational structure. Then CSP(H) is NP-complete.*

It is easy to see that each HOM(H) is a special case of ListHOM(H), obtained by setting all lists to $V(H)$. Similarly, MinHOM(H) generalizes ListHOM(H) by setting $c_i(u) = 0$ if $i \in L_u$ and $c_i(u) = 1$ otherwise. Thus we have the following corollary from Theorem 2.5.1.

Corollary 2.5.2 *Let H be a projective digraph with possible loops. Then MinHOM(H) is NP-hard.*

It was shown by Luczak and Nešetřil [81] that almost all structures are projective, and hence have NP-complete CSP problems. The class of projective structures does not include all NP-complete cases of CSP. In fact, there exists some non-projective relational structures with NP-complete CSP [19]. However, admitting any polymorphism other than projection by H is good evidence to candidate CSP(H) as a polynomial time solvable problem. Three well known such polymorphisms are majority operation, Mal'tsev operation, and semilattice operation [19, 21, 67, 68].

A *majority* operation is a ternary operation f on A satisfying $f(u, u, v) = f(v, u, u) = f(u, v, u) = u$ for all $u, v \in A$. A *Mal'tsev* operation is a ternary operation f on A satisfying

$f(u,u,v) = f(v,u,u) = v$ for all $u, v \in A$. A *semilattice* operation is a binary operation f on A satisfying $f(u,u) = u, f(u,v) = f(v,u)$ and $f(u, f(v,w)) = f(f(u,v), w)$ for all $u, v, w \in A$.

Theorem 2.5.3 *[19, 21, 67, 68] Let H be a relational structure. If H admits a polymorphism which is a majority operation, or a Mal'tsev operation, or a semilattice operation, then CSP(H) is polynomial time solvable.*

All three of these operations are special cases of a more general operation called *Taylor operation*. We say that an operation f is *inclusive in position i*, if it satisfies an identity involving two variables, with different entries in position i. More precisely, there exist choices $u_j, v_j \in \{u, v\}, j = 1, 2, \ldots, k$, with $u_i \neq v_i$, such that the identity $f(u_1, u_2, \ldots, u_k) = f(v_1, v_2, \ldots, v_k)$ holds for all $u, v \in A$. A k-ary operation f is a *Taylor operation* if f is inclusive in each position $1 \leq i \leq k$. The following fact has been proved by the authors of [78].

Theorem 2.5.4 *[78] Let H be a relational structure which does not admit any Taylor operation as a polymorphism. Then CSP(H) is NP-complete.*

Larose and Zádori [78] have also conjectured that all structures H which admit a Taylor operation as a polymorphism, have polynomial time solvable CSP(H).

Recall that a structure H is conservative if H contains all unary (arity one) relations $S \subset V(H)$. The class of conservative relational structures is a large subclass of relational structures for which a dichotomy is known [15]. It is easy to see that every polymorphism of order k of H satisfies the condition: $f(u_1, \ldots, u_k) \in \{u_1, \ldots, u_k\}$, for all $u_1, \ldots, u_k \in V(H)$. Such an operation is said to be *conservative*. Let $f|_B$ denote the restriction of an operation f onto a set B. The following result gives us a full dichotomy for conservative CSP problems [15].

Theorem 2.5.5 *[15] The problem CSP(H) for a conservative relational structure H is polynomial time solvable if, for any 2-element subset $B \subset V(H)$, there exits a polymorphism f^B of H such that $f^B|_B$ is either the semilattice operation, or the majority operation, or the Mal'tsev operation. Otherwise, CSP(H) is NP-complete.*

Recall that each ListCSP(H) is polynomially equivalent to a CSP(H') where H' is a conservative structure obtained from H by augmenting H with all unary relations. Furthermore, we have shown in Chapter 1 that MinCSP(H) generalizes LisCSP(H). These facts lead us to the following corollary.

Corollary 2.5.6 *Let H be a digraph with possible loops and let H' be a conservative structure obtained from H by augmenting H with all unary relations. If there is a 2-element subset $B \subset V(H')$ with no polymorphism f^B such that $f^B|_B$ is a semilattice operation, or a majority operation, or a Mal'tsev operation, then MinHOM(H) is NP-hard.*

2.5.2 Multimorphisms

For crisp constraint satisfaction problems, we defined polymorphisms and discussed that if a relational structure H has a polymorphism other than projections, CSP(H) has a good chance to be polynomial time solvable. Recall that sCSP generalizes CSP by admitting soft constraints rather than crisp constraints. To recognize tractable sCSP problems, the authors of [24] introduced a new algebraic notion, called *multimorphism*. Multimorphism is a natural generalization of polymorphism.

Throughout the rest of this section, the ith component of a tuple t will be denoted $t[i]$. Let $f : A^m \rightarrow \mathcal{R}^+$ be a function, where A is a fixed set, and $\mathcal{R}^+$ is the set of the nonnegative real numbers. We say that $g : A^k \rightarrow A^k$ is a *multimorphism* of f if, for any list of k-tuples $t_1, t_2, \ldots, t_m$ over A we have

$$\sum_{i=1}^{k} f(g(t_1)[i], g(t_2)[i], \ldots, g(t_m)[i]) \leq \sum_{i=1}^{k} f(t_1[i], t_2[i], \ldots, t_m[i])$$

k is called the *arity* of multimorphism g.

Let H be a fixed complete relational structure, and let $\mathcal{F}$ be the set of functions, defined in Section 2.4. We say that $g : V(H)^k \rightarrow V(H)^k$ is a multimorphism of $\mathcal{F}$, if g is a multimorphism of each function $f \in F$. It has been shown in [24] that if $\mathcal{F}$ admits particular multimorphisms, then sCSP($H, \mathcal{F}$) is polynomial time solvable. Among such polymorphisms, we only study $\langle$Min, Max$\rangle$ polymorphism.

A *$\langle$Min, Max$\rangle$ multimorphism* is a binary multimorphism $g : A^2 \rightarrow A^2$, where $g(x, y) = \langle \text{Min}(x, y), \ \text{Max}(x, y) \rangle$. Let A be a totally ordered set. Recall that a binary function $f : \ A^2 \rightarrow \mathcal{R}^+$ is called submodular, if there exists an ordering of elements of A such that for all $x, y, u, v \in A$, we have

$$\phi(\min\{x,u\}, \min\{y,v\}) + \phi(\max\{x,u\}, \max\{y,v\}) \leq \phi(x,y) + \phi(u,v).$$

It is easy to see that a binary function f is submodular if and only if it admits a $\langle$Min, Max$\rangle$ multimorphism. We close this chapter by noting that Theorem 2.4.2 can be equivalently restated in terms of admitting a $\langle$Min, Max$\rangle$ multimorphism by $\mathcal{F}$ rather than each binary function f be submodular.

Chapter 3

Tools

In this chapter, we introduce several tools required to study the minimum cost homomorphism complexity. In particular, we introduce new combinatorial techniques to prove NP-hard cases of minimum cost homomorphism problems.

3.1 MinHOM Dichotomy

Recall that a linear ordering $<$ of $V(H)$ is a Min-Max ordering if $a < c, d < b$, and $ab, cd \in A(H)$ imply that $ad \in A(H)$ and $cb \in A(H)$. The following theorem is folklore [51].

Theorem 3.1.1 *[51] Let H be a digraph with possible loops. MinHOM(H) is polynomial time solvable if H admits a Min-Max ordering.*

A directed cycle $\vec{C}_k, k \geq 2$ is a well known example which does not admit a Min-Max ordering, but MinHOM($\vec{C}_k$) is polynomial time solvable [51]. The authors of [53] have shown this fact is also true for extensions of $\vec{C}_k$. (Extensions are defined on page 4.) More precisely, the following Proposition has been proved in [53].

Proposition 3.1.2 *[53] Let H be an irreflexive digraph. If MinHOM(H) is polynomial time solvable then, for each extension H' of H, MinHOM(H') is polynomial time solvable.*

The same authors also proposed k-Min-Max ordering (see Definition 1.1.2) as a property which covers all digraphs H for which MinHOM(H) is polynomial time solvable, but H does not admit a Min-Max ordering. To show that MinHOM(H) is polynomial time solvable for digraphs H with k-Min-Max ordering, Gutin et al. have polynomially reduced MinHOM(H) to the minimum weight cut problem in a network, which is solvable in polynomial time [3].

Theorem 3.1.3 *[52] Let H be a digraph with possible loops. If H admits a k-Min-Max ordering for some $k \geq 2$, then MinHOM(H) is polynomial time solvable.*

It is easy to interpret the usual Min-Max ordering as conforming to the same definition of k-Min-Max ordering with $k = 1$, via the trivial homomorphism f to a vertex with a loop. Thus we shall understand k-Min-Max orderings to include Min-Max orderings. For simplicity of description, we will also say that a 1-Min-Max ordering of H is the usual Min-Max ordering. Very recently, Gutin, Rafiey, and Yeo conjectured that all digraphs H for which MinHOM(H) is polynomial time solvable, should have a k-Min-Max ordering for some $k \geq 1$ [50].

Conjecture 3.1.4 *[50] Let H be a digraph with possible loops. Then MinHOM(H) is polynomial time solvable if H admits a k-Min-Max ordering for some $k \geq 1$. Otherwise, MinHOM(H) is NP-hard.*

Clearly, it is the NP-hardness part of this conjecture which is the open part of it. We remark that the NP-hardness part of this conjecture can be shown, if one gives a nice characterization of digraphs with k-Min-Max orderings. We note that, in particular, if these digraphs can be characterized by a few forbidden induced subgraphs, then the NP-hardness part easily follows. Indeed, it is sufficient to prove that minimum cost homomorphism is NP-hard for all these induced subgraphs. However, it is not easy to characterize digraphs with these orderings for general digraphs with possible loops. So, we do not consider the minimum cost homomorphism problem in its full generality, but rather focus on restricted classes of digraphs for which we can characterize the digraphs with k-Min-Max orderings with a few forbidden induced subgraphs (or a few forbidden families of induced subgraphs).

It follows from the definition of k-Min-Max ordering that a digraph H with some loops (meaning at least one loop), can not admit a k-Min-Max ordering for some $k \geq 2$. This leads us to a simple form of the MinHOM conjecture for digraphs with some loops.

Conjecture 3.1.5 *Let H be a digraph with some loops. Then MinHOM(H) is polynomial time solvable if H admits a Min-Max ordering. Otherwise, MinHOM(H) is NP-hard.*

3.2 New Methods

It was mentioned before that it is the NP-hardness part of Conjecture 3.1.4 which is the open part of it. In this chapter, we discuss several techniques which will be used later to prove the NP-hardness part of Conjecture 3.1.4 for special classes of digraphs. We begin with a few simple observations.

Let D be a digraph with possible loops. Recall that $B(D)$ is the bipartite graph obtained from D as follows. Each vertex v of D gives rise to two vertices of $B(D)$ - a *white* vertex v' and a *black* vertex v''; each arc vw of D gives rise to an edge $v'w''$ of $B(D)$. Note that if D is a reflexive digraph, then all edges $v'v''$ are present in $B(D)$. The following observation is easily proved by setting up a natural polynomial time reduction from MinHOM($B(H)$) to MinHOM(H) [49].

Proposition 3.2.1 *[49] Let H be a digraph with possible loops. If $B(H)$ is not a proper interval bigraph, then MinHOM(H) is NP-hard.*

The next observation is folklore, and proved by obvious reduction, cf. [54]. Recall that $S(H)$ is the symmetric subgraph of H.

Proposition 3.2.2 *[54] Let H be a digraph with possible loops. If $S(H)$ is neither a proper interval graph nor a proper interval bigraph, then MinHOM(H) is NP-hard.*

Since proper interval graphs are reflexive and proper interval bigraphs are irreflexive, we obviously have the following corollary.

Corollary 3.2.3 *Let H be a reflexive digraph. If $S(H)$ is not a proper interval graph, then MinHOM(H) is NP-hard.*

The following proposition allows us to prove that MinHOM(H) is NP-hard when MinHOM-(H') is NP-hard for an induced subgraph H' of H.

Proposition 3.2.4 *[51] Let H be a digraph with possible loops and H' be an induced subgraph of H. If MinHOM(H') is NP-hard, then MinHOM(H) is NP-hard.*

Given this fact, we are able to determine the complexity of MinHOM(H) by checking the complexity of MinHOM for some basic structures which are induced subgraphs of H.

So, to verify Conjecture 3.1.4, first of all, we have to study some basic classes of digraphs such as directed tree, oriented cycles, and semicomplete digraphs.

The other interesting tools which are used in this thesis, are described in the following propositions. The general idea is to seek a special digraph D having a set of special homomorphisms to H.

Proposition 3.2.5 *Let D and H be two digraphs. Suppose D and H have two pairs of vertices u, v and x, y, respectively such that:*

(a) there is a homomorphism of D to H which maps both u and v to y;

(b) there is no homomorphism of D to H which maps both u and v to x;

(c) there is a homomorphism of D to H which maps u to y and v to x;

(d) there is a homomorphism of D to H which maps v to y and u to x.

Then MinHOM(H) is NP-hard.

Proof: We will construct a polynomial time reduction from the maximum independent set problem to MinHOM(H). Let G be an arbitrary undirected graph. We replace every edge $u'v' \in E(G)$ by the digraph D such that $u' = u$, and $v' = v$. We will denote this digraph by D'. Let all costs $c_i(t) = 0$ for $t \in V(D') - V(G)$, $i \in V(H)$, and $c_y(t) = 1$, $c_x(t) = 0$ for $t \in V(G)$, and $c_i(t) = +\infty$ for $i \in V(H) - \{x, y\}$, $t \in V(G)$. There is always a homomorphism of finite cost from D' to H. (We can map all vertices of G to y). Let f be a homomorphism of D' to H with finite cost and let $S = \{u \in V(G) : f(u) = x\}$. Then, S is an independent set in G since we cannot assign color x to both u and v in $V(G)$ whenever there is an edge between them. Observe that the minimum cost homomorphism will assign as many vertices of $V(G)$ as possible with color x.

Conversely, suppose we have an independent set I of G. Then we can build a homomorphism f of D' to H such that $f(u) = x$ for all $u \in I$ and $f(u) = y$ for all $u \in V(G) \setminus I$.

Hence, a minimum cost homomorphism f of D' to H yields a maximum independent set of G and vice versa, which completes the proof. ⋄

Proposition 3.2.6 *Let D and H be two digraphs with costs $c'_i(u), i \in V(H), u \in V(D)$, where there is at least one $c'_i(u)$ which is $+\infty$. Suppose D and H have two pairs of vertices u, v and x, y, respectively such that:*

(a) $c'_x(u) = c'_x(v) = 0$, $c'_y(u) = c'_y(v) = 1$, *and all other costs are either* $+\infty$ *or zero;*

(b) there is a homomorphism with cost two of D *to* H *which maps both* u *and* v *to* y*;*

(c) there is no homomorphism with finite cost of D *to* H *which maps both* u *and* v *to* x*;*

(d) there is a homomorphism with cost one of D *to* H *which maps* u *to* y *and* v *to* x*;*

(e) there is a homomorphism with cost one of D *to* H *which maps* v *to* y *and* u *to* x.

Then MinHOM(H) is NP-hard.

Proof: We will construct a polynomial time reduction from the maximum independent set problem to MinHOM(H). Let G be an arbitrary undirected graph. We replace every edge $u'v' \in E(G)$ by the digraph D such that $u' = u$, and $v' = v$. We will denote this new digraph obtaining from G by D'. Let $c_y(t) = 1$, $c_x(t) = 0$ for $t \in V(G)$, and $c_i(t) = +\infty$ for $i \in V(H) - \{x, y\}$, $t \in V(G)$, and $c_i(t) = c'_i(t)$ for $t \in V(D) - \{u, v\}$, $i \in V(H)$.

There is always a homomorphism of finite cost from D' to H. (We can map all vertices of G to y). Let f be a homomorphism of D' to H with finite cost and let $S = \{u \in V(G) : f(u) = x\}$. Then, S is an independent set in G since we cannot have a homomorphism of finite cost of D to H which maps both u and v to x. Observe that the minimum cost homomorphism will assign as many vertices of $V(G)$ as possible with color x.

Conversely, suppose we have an independent set I of G. Then we can build a homomorphism f of D' to H such that $f(u) = x$ for all $u \in I$ and $f(u) = y$ for all $u \in V(G) \setminus I$.

Hence, a minimum cost homomorphism f of D' to H yields a maximum independent set of G and vice versa, which completes the proof. $\diamond$

Now, we can develop a technique to prove the NP-hardness of MinHOM(H) when H does not admit a k-Min-Max ordering. Indeed, we have to look for a digraph D which fulfills the conditions of Proposition 3.2.5 or 3.2.6. This technique has been partially used in [48, 49, 50, 51, 52, 53, 54], but it has not been expressed in this form so far. We remark that it is not always easy to find such a digraph D. So, we should seek new tools. One idea is to restrict H to a special class of digraphs and find a new technique which is specially designed for that class of digraphs. With this perspective, we customize Proposition 3.2.5 for oriented cycles with some loops C leading to the following proposition.

Proposition 3.2.7 *Let C be an oriented cycle with some loops having a loop on an arbitrary vertex z and let D be a digraph. Suppose D has a pair of distinct vertices u, v, and C has a pair of not necessarily distinct vertices x, y, distinct from z such that:*

(a) there is a homomorphism f_1 of D to C which maps both u and v to z;

(b) there is no homomorphism of D to C which maps u to x and v to y;

(c) there is a homomorphism f_2 of D to C which maps u to x and v to z;

(d) there is a homomorphism f_3 of D to C which maps u to z and v to y,

Then MinHOM(C) is NP-hard.

Proof: If x and y are not distinct then by Proposition 3.2.5, MinHOM(C) is NP-hard. Thus, we may assume that x and y are distinct vertices. This way, there should be a vertex w in C such that there are two internally disjoint oriented paths $P = ww_1 \dots w_n z$, and $Q = ww'_1 \dots w'_m z$ from w to z in C, and at least one of x and y is only in one of these oriented paths. Without loss of generality assume that this vertex is x, which is in P, and we have $x = w_i, 1 \leq i \leq n$. Note that w may be equal to y if x and y are adjacent. If w and y are distinct vertices then we may assume that $y = w'_j, 1 \leq j \leq m$ in Q.

Now, we will construct a polynomial time reduction from the maximum independent set problem to MinHOM(C). Let G be an arbitrary undirected graph. We replace every edge $u'v' \in E(G)$ by the digraph D' consisting of a set of special vertices $\{u', v', r, s, t\}$, and a set of digraphs between some pairs of these vertices as follows:

- there is an oriented path $u'u_1 \dots u_{i-1}s$ from u' to s, which is exactly isomorphic to the oriented path from w to x in P.

- there is an oriented path $su_{i+1} \dots u_n t$ from s to t, which is exactly isomorphic to the oriented path from x to z in P.

- if w and y are distinct vertices then there is an oriented path $v'v_1 \dots v_{j-1}r$ from v' to r, which is exactly isomorphic to the oriented path from w to y in Q. If $w = y$ then $v' = r$ and there is no oriented path between them.

- if w and y are distinct vertices then there is an oriented path $rv_{j+1} \dots v_m t$ from r to t, which is exactly isomorphic to the oriented path from y to z in Q. If $w = y$ then $v' = r$, and there is an oriented path $rv_1 \dots v_m t$ from r to t, which is isomorphic to Q.

- there is a digraph D_1 between s and r, which is isomorphic to D, and this isomorphism maps s to u and r to v.

For simplicity, let us rename s to u_i and r to v_j. We will denote this digraph obtained from G by D''.

We assign the costs as follows:

$c_z(a) = 1$, $c_w(a) = 0$ for $a \in V(G)$, and $c_b(a) = +\infty$ for $b \in V(C) - \{z, w\}$, $a \in V(G)$;

$c_{w_{i'}}(u_{i'}) = c_z(u_{i'}) = 0$ apart from $c_b(u_{i'}) = +\infty$ for $b \in V(C) - \{z, w_{i'}\}$;

$c_{w'_{j'}}(v_{j'}) = c_z(v_{j'}) = 0$ apart from $c_b(v_{j'}) = +\infty$ for $b \in V(C) - \{z, w'_{j'}\}$;

$c_b(a) = 0$ for $b \in V(C)$, $a \in V(D_1) - \{u_i, v_j\}$;

There is always a homomorphism of finite cost from D'' to C. (We can map all vertices of D'' to z). Let f be a homomorphism of D'' to C with finite cost and let $S = \{u' \in V(G) : f(u') = w\}$. Then, S is an independent set in G since we cannot assign color w to both u' and v' in $V(G)$ whenever there is an edge between them. In fact, if $f(u') = f(v') = w$ then $f(u_i) = w_i = x$ and $f(v_j) = w'_j = y$ as the homomorphism has finite cost. Hence, f is a homomorphism of D_1 (which is isomorphic to D) to C such that it maps u_i (correspondingly u in D) to x and v_j (correspondingly v in D) to y, contrary to part (b). Observe that the minimum cost homomorphism will assign as many vertices of $V(G)$ as possible with color w.

Conversely, suppose we have an independent set I of G. Then we can build a homomorphism of finite cost f of D'' to C such that $f(u') = w$ for all $u' \in I$ and $f(u') = z$ for all $u' \in V(G) \setminus I$. To do so, it is enough to show that if there is an edge between $u'v'$ in G, there are homomorphisms g_1, g_2, and g_3 of the gadget D' to C such that:

- $g_1(u') = z$ and $g_1(v') = z$;
- $g_2(u') = w$ and $g_2(v') = z$;
- $g_3(u') = z$ and $g_3(v') = w$;

We will build these homomorphisms respectively as follows:

- $g_1(u') = z$, $g_1(a) = z$ for $a \in V(D')$, $g_1(v') = z$;
- $g_2(u') = w$, $g_2(u_{i'}) = w_{i'}$, $g_2(t) = z$, $g_2(v_{j'}) = z$, $g_2(v') = z$, $g_2(a) = f_2(a)$ for $a \in V(D_1)$;

- $g_3(v') = w$, $g_3(v_{j'}) = w'_{j'}$, $g_3(t) = z$, $g_3(u_{i'}) = z$, $g_3(u') = z$, $g_3(a) = f_3(a)$ for $a \in V(D_1)$;

Hence, a minimum cost homomorphism f of D' to C yields a maximum independent set of G and vice versa, which completes the proof. ◇

The next proposition is a slightly different form of Proposition 3.2.7 which will be used later in proving dichotomy for oriented cycles with some loops.

Proposition 3.2.8 *Let C be an oriented cycle with some loops having a loop on an arbitrary vertex z and let D be a digraph with cost $c'_i(u), i \in V(C), u \in V(D)$, where $c'_i(u)$ is either $+\infty$ or zero, and there is at least one $c'_i(u)$ which is $+\infty$. If D has a pair of distinct vertices u, v, and C has a pair of not necessarily distinct vertices x, y, distinct from z, and the following conditions hold:*

(a) $c'_x(u) = 0, c'_y(v) = 0$;

(b) there is a homomorphism with cost zero f_1 of D to C which maps both u and v to z;

(c) there is no homomorphism with finite costs of D to C which maps u to x and v to y;

(d) there is a homomorphism with cost zero f_2 of D to C which maps u to x and v to z;

(e) there is a homomorphism with cost zero f_3 of D to C which maps u to z and v to y,

then MinHOM(C) is NP-hard.

Proof: We will construct a polynomial time reduction from the maximum independent set problem to MinHOM(C). Let G be an arbitrary undirected graph. We replace every edge $u'v' \in E(G)$ by the digraph D', introduced in the proof of Proposition 3.2.7. We will denote this digraph obtained from G by D''. The costs are exactly like the costs in the proof of Proposition 3.2.7, apart from:

$c_b(a) = c'_b(a)$ for $b \in V(C)$, $a \in V(D_1) - \{u_i, v_j\}$;

where D_1 is isomorphic to D, and there is a one to one correspondence between vertices of D and D_1. Let f be a homomorphism of D'' to C with finite cost and let $S = \{u' \in V(G) : f(u') = w\}$. Since, there is no homomorphism of finite costs of D_1 (isomorphic to D) to C which maps u to x and v to y, then, S is an independent set in G.

Conversely, suppose we have an independent set I of G. Then we can build a homomorphism of finite cost f (similar to the proof of Proposition 3.2.7) of D'' to C such that $f(u') = w$ for all $u' \in I$ and $f(u') = z$ for all $u' \in V(G) \setminus I$.

Hence, a minimum cost homomorphism f of D' to C yields a maximum independent set of G and vice versa, which completes the proof. $\diamond$

We close this Section by introducing another tool. Here, we reduce the maximum independent set problem for three-partite graphs to MinHOM(H) when H does not admit a k-Min-Max ordering. Let us denote by $\mathcal{I}_3$ the independent set problem for 3-partite graphs: given a 3-partite graph G and a positive integer k, $\mathcal{I}_3$ asks whether G has an independent set of cardinality at least k. The optimization version of $\mathcal{I}_3$, the maximum independent set problem for 3-partite graphs, attempts to find the largest independent set in a 3-partite graph G. This problem has been useful for proving NP-hardness of minimum cost homomorphism problems for undirected graphs [49], and we use it here for digraphs.

Proposition 3.2.9 *[49] The problem of finding a maximum independent set in a 3-partite graph G (even given the three partite sets) is NP-hard.* $\diamond$

The following proposition will be extensively used in this thesis for proving dichotomies for different classes of digraphs.

Proposition 3.2.10 *Let D_0, D_1, D_2, and H be four digraphs. Suppose each $D_i, 0 \le i \le 2$ has a pair of distinct vertices u_i, v_i and H has three pairs of vertices $x_i, y_i, 0 \le i \le 2$ such that:*

- *(a) there is no homomorphism of D_i to H which maps u_i to x_i and v_i to x_{i+1};*
- *(b) there is a homomorphism of D_i to H which maps u_i to x_i and v_i to y_{i+1};*
- *(c) there is a homomorphism of D_i to H which maps u_i to y_i and v_i to x_{i+1};*
- *(d) there is a homomorphism of D_i to H which maps u_i to y_i and v_i to y_{i+1},*

where all indices are taken modulo 3. Then MinHOM(H) is NP-hard.

Proof: We construct a polynomial time reduction from the maximum independent set problem for 3-partite graphs to MinHOM(H). Let G be a graph whose vertices are partitioned into independent sets U, V, W. We construct an instance of MinHOM(H) as

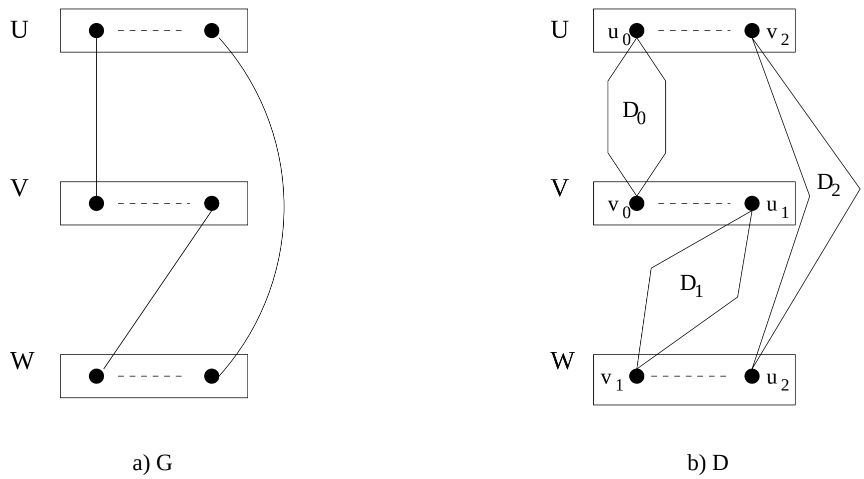

Figure 3.1: (a) A three-partite graph G. (b) The digraph D obtained from G.

follows: the digraph D is obtained from G as shown in Figure 3.1 by replacing each edge $uv, u \in U, v \in V$ of G with the digraph D_0 where $u = u_0$, and $v = v_0$, and replacing each edge $vw, v \in V, w \in W$ of G with the digraph D_1 where $v = u_1$, and $w = v_1$, and replacing each edge $uw, u \in U, w \in W$ of G with the digraph D_2 where $w = u_2$, and $u = v_2$. Let all costs $c_i(t) = 0$ for $t \in V(D) - V(G)$ and $i \in V(H)$, and let all costs $c_i(t) = +\infty$ for $i \in V(H)$, $t \in V(G)$, apart from $c_{y_0}(t) = 1$, $c_{x_0}(t) = 0$, for $t \in U$, $c_{y_1}(t) = 1$, $c_{x_1}(t) = 0$, for $t \in V$, and $c_{y_2}(t) = 1$, $c_{x_2}(t) = 0$, for $t \in W$.

There is always a homomorphism with finite cost of D to H. (We can map all vertices of U to y_0, all vertices of V to y_1, and all vertices of W to y_2). Let f be a homomorphism of D to H with finite cost and let $S = \{u \in V(G) : f(u) = x_i, \text{for some } i, 0 \leq i \leq 2\}$. Then, S is an independent set in G: for instance suppose the contrary that $f(u) = x_0$, $f(v) = x_1$, and $uv \in E(G)$. Then f is a homomorphism of D_0 to H with $f(u_0) = x_0$ and $f(v_0) = x_1$ (note that $u = u_0$, and $v = v_0$), contrary to condition (a). (The other possibilities are similar.)

Conversely, suppose we have an independent set I of G, and $I_U = I \cap U$, $I_V = I \cap V$, and $I_W = I \cap W$. Then we can build a homomorphism f of D to H such that $f(u) = x_0, u \in I_U$, $f(v) = x_1, v \in I_V$, $f(w) = x_2, w \in I_W$, $f(u) = y_0, u \in U - I_U$, $f(v) = y_1, v \in V - I_V$, $f(w) = y_2, w \in W - I_W$. Conditions (b), (c), and (d) guarantee that such a homomorphism exists. Hence, a minimum cost homomorphism f of D to H yields a maximum independent set of G and vice versa, which completes the proof. $\diamond$

Chapter 4

Reflexive Digraphs

In this chapter, we give a full dichotomy classification of the complexity of MinHOM(H) for reflexive digraphs; this is the first dichotomy result for a general class of digraphs - our only restriction is that the digraphs are reflexive. We shall give a combinatorial description of reflexive digraphs with a Min-Max ordering, in terms of forbidden induced subgraphs. Our characterization yields a polynomial time algorithm for the existence of a Min-Max ordering in a reflexive digraph. It also allows us to complete a dichotomy classification of MinHOM(H) for reflexive digraphs H, by showing that all problems MinHOM(H) where H does not admit a Min-Max ordering are NP-hard. This verifies a conjecture of Gutin and Kim from [54]. This chapter is mostly based on [46].

4.1 Structure and Forbidden Subgraphs

For a reflexive digraph H, it is easy to see that $<$ is a Min-Max ordering if and only if for any j between i and k, we have $ik \in A(H)$ imply $ij, jk \in A(H)$. (Clearly, a Min-Max ordering has the property, by the definition applied to ik and jj. Conversely, the property implies that $is \in A(H)$ and $jr \in A(H)$ if j and s are between i and r or conversely - by considering the arcs ir respectively js; in the remaining cases $i < s < r < j$ or $s < i < j < r$ we apply the property to the two arcs ir and js.) For a bipartite graph H (with a fixed bipartition into white and black vertices), a *bipartite Min-Max ordering* is an ordering $<$ such that $<$ restricted to the white vertices, and $<$ also restricted to the black vertices satisfy the condition of Min-Max orderings, i.e., $i < j$ for white vertices, and $s < r$ for black vertices, and $ir, js \in E(H)$, imply that $is \in E(H)$ and $jr \in E(H)$. Recall the definitions of $S(H)$

and $B(H)$ from Chapter 1. The following theorems have been shown in [49].

Theorem 4.1.1 *[49] Let H be a bipartite graph. H admits a bipartite Min-Max ordering if and only if H is a proper interval bigraph.*

Theorem 4.1.2 *[49] Let H be a reflexive graph. H admits a Min-Max ordering if and only if H is a proper interval graph.*

Since both reflexive and bipartite graphs admit a characterization of existence of Min-Max orderings by forbidden induced subgraphs, our goal will be accomplished by proving the following theorem. It also implies a polynomial time algorithm to test if a reflexive digraph admits a Min-Max ordering.

Theorem 4.1.3 *A reflexive digraph H admits a Min-Max ordering if and only if*

- *$S(H)$ is a proper interval graph, and*
- *$B(H)$ is a proper interval bigraph, and*
- *H does not contain an induced subgraph isomorphic to H_i with $i = 1, 2, 3, 4, 5, 6$.*

The digraphs H_i are depicted in Figure 4.1. Recall that proper interval graphs and proper interval bigraphs are characterized by a set of forbidden induced subgraphs introduced by Theorem 1.1.4 and 1.1.5, respectively. The resulting forbidden subgraph characterization is summarized in the following corollary. Note that forbidden subgraphs in $S(H)$ directly describe forbidden subgraphs in H, and it is easy to see that each forbidden induced subgraph in $B(H)$ can also be translated to a small family of forbidden induced subgraphs in H.

Corollary 4.1.4 *A reflexive digraph H admits a Min-Max ordering if and only if $S(H)$ does not contain an induced undirected cycle $C_k, k \geq 4$, or claw, net, or tent, $B(H)$ does not contain an induced undirected cycle $C_{2k}, k \geq 3$, or biclaw, binet, or bitent, and H does not contain an induced H_i with $i = 1, 2, 3, 4, 5, 6$.*

We proceed to prove the Theorem.

Proof: Suppose first that $<$ is a Min-Max ordering $<$ of H. It is easily seen that $<$ is also a Min-Max ordering of $S(H)$, and hence $S(H)$ is a proper interval graph by

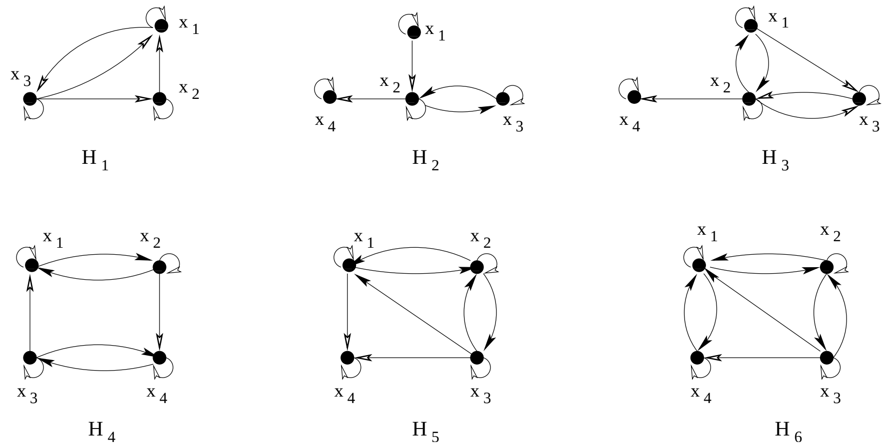

Figure 4.1: The obstructions H_i with $i = 1, 2, 3, 4, 5, 6$.

Theorem 4.1.2. It is also easy to see that $<$ applied separately to the corresponding white and black vertices of $B(H)$ is a bipartite Min-Max ordering of $B(H)$, and thus $B(H)$ is a proper interval bigraph by Theorem 4.1.2. To complete the proof of necessity, we now claim that none of the digraphs $H_i, i = 1, 2, 3, 4, 5, 6$ admits a Min-Max ordering. We only show this for H_3, the proofs of the other cases being similar. Suppose that $<$ is a Min-Max ordering of H_3. For the triple x_1, x_2, x_3, we note that x_2 must be between x_1 and x_3 in the ordering $<$, as otherwise the arcs between x_2 and x_1, x_3 would imply that $x_1x_3 \in E(S(H))$. Without loss of generality assume that $x_1 < x_2 < x_3$. Since x_1 and x_4 are independent and $x_1x_2 \in E(S(H))$, we must have $x_4 > x_1$. A similar argument yields $x_4 < x_3$; however, $x_1 < x_4 < x_3$ is impossible, as $x_1x_3 \in A(H)$ but $x_1x_4 \notin A(H)$.

To prove the sufficiency of the three conditions, we shall prove the following claim.

Lemma 4.1.5 *If $S(H)$ admits a Min-Max ordering and $B(H)$ admits a bipartite Min-Max ordering, then either H admits a Min-Max ordering, or H contains an induced H_i (or its converse) for some $i = 1, 2, 3, 4, 5, 6$.*

Proof: Suppose $<$ is a bipartite Min-Max ordering of $B(H)$. A pair u, v of vertices of H is *proper for* $<$ if $u' < v'$ if and only if $u'' < v''$ in $B(H)$. We say a bipartite Min-Max ordering $<$ is *proper* if all pairs u, v of H are proper for $<$. If $<$ is a proper bipartite Min-Max ordering, then we can define a corresponding ordering $\prec$ on the vertices of H, where

$u \prec v$ if and only if $u' < v'$ (which happens if and only if $u'' < v''$). It is easy to check that $\prec$ is now a Min-Max ordering of H.

Suppose, on the other hand, that the bipartite Min-Max ordering $<$ on $B(H)$ is not proper. Thus there are vertices v', u' such that $v' < u'$ and $u'' < v''$. Suppose there is no vertex s' such that $s'v'' \in E(B(H))$, $s'u'' \notin E(B(H))$: then we can exchange the position of v'' and u'' in $<$ and still admits a bipartite Min-Max ordering. Furthermore, this exchange strictly increases the number of proper pairs in H: any w with $u'' < w'' < v''$ and $u' < w'$ creates a new improper pair u, w but also creates a new proper pair v, w (and the pair u, v is also a new proper pair). Analogously, if there is no vertex t'' such that $u't'' \in E(B(H))$, $v't'' \notin E(B(H))$, we can exchange u', v' and increase the number of proper pairs in H. Suppose we have performed all exchanges until we reached a bipartite Min-Max ordering $<$ which admits no more exchanges. Then there are two possibilities: either $<$ is now proper, and H admits a Min-Max ordering as above, or $<$ is still not proper, and one of the following two cases must occur (up to symmetry):

Case 1: $s'v'', v't'' \in E(B(H))$ and $s'u'', u't'' \notin E(B(H))$.

It is easy to see that since $<$ is a bipartite Min-Max ordering, we must have $u' < s'$ and $t'' < u''$. (Note that means that $s'' \neq t''$.) Since $u'u'', v'v'' \in E(B(H))$, by the same argument we must have $u'v'', v'u'' \in E(B(H))$; and similarly we obtain $s't'' \notin E(B(H))$. If both $v's''$ and $t'v''$ are edges of $B(H)$ then u, v, s, t induce a claw in $S(H)$: indeed in $B(H)$, we have the edges $v't'', t'v'', v'u'', u'v'', v's'', s'v''$ and the non-edges $u't'', s'u'', s't''$. This is a contradiction, as $S(H)$ is assumed to admit a Min-Max ordering, i.e., be a proper interval graph.

If neither $v's''$ nor $t'v''$ is an edge of $B(H)$, then if $u's''$ is an edge of $B(H)$, then s, v, u induce a copy of H_1 in H, and if $, t'u''$ is an edge of $B(H)$, then t, v, u induce a copy of H_1. Thus consider the case when $u's'', t'u'' \notin E(B(H))$. If $t's'' \in E(B(H))$, then $s', s'', t', t'', v', v''$ would induce a copy of C_6 in $B(H)$, contrary to our assumption that $B(H)$ admits a bipartite Min-Max ordering, i.e., is a proper interval bigraph. Thus $t's'' \notin E(B(H))$ and t, s, v, u induce a copy of H_2 in H.

If only one of $v's''$ or $t'v''$ is an edge of $B(H)$, assume first that $v's'' \in E(B(H))$ and $t'v'' \notin E(B(H))$. If $t'u''$ is an edge of $B(H)$, then t, v, u induce a copy of H_1 in H, and if $t's''$ is an edge of $B(H)$, then t, v, s similarly induce a copy of H_1; thus asume that $t'u'', t's'' \notin E(B(H))$. Note that $u's'' \in E(B(H))$, else the vertices $u', u'', v', t'', t', s'', s'$ would induce a biclaw in $B(H)$, contrary to $B(H)$ being a proper interval bigraph. It now follows that s, t, u, v induce a copy of H_3 in H. If $v's'' \notin E(B(H))$ and $t'v'' \in E(B(H))$, the proof is similar, except we obtain copies of H_1 and the converse of H_3.

Case 2: $s'v'', u't'' \in E(B(H))$ and $s'u'', v't'' \notin E(B(H))$.

We again easily observe that we must have $u' < s'', v'' < t''$, and $u'v'', v'u'' \in E(B(H))$. If $s'' = t''$ we obtain a copy of H_1 induced by u, v, s in H; hence we assume that $s'' \neq t''$. Suppose first that $u's'', t'v'' \notin E(B(H))$. We have $s' < t'$ and $t'' < s''$, and so $t's'', s't'' \in A(H)$, implying that u, v, s, t induce a copy of H_4 in H. Suppose next that both $t'v'', u's'' \in E(B(H))$. If $v's''$ is not an edge of $B(H)$, vertices u, v, s induce a copy of H_1 in H, and if $t'u''$ is not an edge of $B(H)$, vertices u, v, t induce a copy of H_1 in H. Thus we have $v's'', t'u'' \in E(B(H))$. Now we have $t' < s'$ and $s'' < t''$, and hence $t's'', s't'' \in E(B(H))$. This is impossible, since u, v, s, t would induce a copy of C_4 in $S(H)$. Finally, if only one of $t'v'', u's''$ is an edge of $B(H)$, say $u's'' \in E(B(H))$ and $t'v'' \notin E(B(H))$ (the other case is symmetric), then with the same argument as above, $v's'' \in E(B(H))$, $s't'' \in E(B(H))$, and s, t, u, v induce (depending on which of the pairs $t'u'', t's''$ are edges of $B(H)$) one of H_1, H_5 (or its converse), or H_6 (or its converse). $\diamond$

4.2 Complexity

If H admits a Min-Max ordering, then MinHOM(H) is polynomial time solvable, see Theorem 3.1.1. Now using our forbidden induced subgraph characterization we can prove that reflexive digraphs H without a Min-Max ordering yield NP-hard MinHOM(H) problems. Note that we already know that MinHOM(H) is NP-hard if $S(H)$ is not a proper interval graph, and MinHOM(H) is NP-hard if $B(H)$ is not a proper interval bigraph. (See Propositions 3.2.2 and 3.2.1.) Recall now Proposition 3.2.4. It states that MinHOM(H) is NP-hard when MinHOM(H') is NP-hard for an induced subgraph H' of H. Therefore, to show a dichotomy for reflexive digraphs H, it is sufficient to prove that MinHOM(H) is NP-hard for digraphs $H = H_1, \ldots, H_6$. Among these digraphs, H_1 is a reflexive semicomplete digraph. Thus the complexity of MinHOM(H_1) is known by Theorem 1.2.3: indeed MinHOM(H_1) is NP-hard. So it remains to prove that MinHOM(H) is NP-hard, when $H = H_2, \ldots, H_6$. The following lemmas show this fact.

Lemma 4.2.1 *The problem MinHOM(H_2) is NP-hard.*

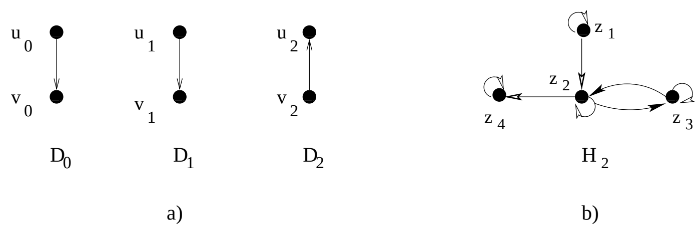

Figure 4.2: (a) The digraphs D_0, D_1, and D_2. (b) H_2.

Proof: The NP-hardness of MinHOM(H_2) easily follows from Proposition 3.2.10. Let D_0, D_1, and D_2 be the digraphs depicted in Figure 4.2.(a) and let $x_0 = z_1, y_0 = z_2, x_1 = z_4, y_1 = z_2, x_2 = z_3, y_2 = z_2$, where z_1, z_2, z_3, z_4 are vertices of H_2 in Figure 4.2.(b). It is easy to check the digraphs $D_i, 0 \leq i \leq 2$ with pairs u_i, v_i, respectively, and H_2 with three pairs $x_i, y_i, 0 \leq i \leq 2$, fulfill the conditions of Proposition 3.2.10, and thus MinHOM(H_2) is NP-hard.

Lemma 4.2.2 *MinHOM(H_3) is NP-hard.*

Proof: The NP-hardness of MinHOM(H_3) easily follows from Proposition 3.2.10. Let D_0, D_1, and D_2 be the digraphs depicted in Figure 4.3.(a) and let $x_0 = z_1, y_0 = z_2, x_1 = z_4, y_1 = z_2, x_2 = z_3, y_2 = z_2$, where z_1, z_2, z_3, z_4 are vertices of H_3 in Figure 4.3.(b). It is easy to check the digraphs $D_i, 0 \leq i \leq 2$ with pairs u_i, v_i, respectively, and H_3 with three pairs $x_i, y_i, 0 \leq i \leq 2$, fulfill the conditions of Proposition 3.2.10, and thus MinHOM(H_3) is NP-hard. $\diamond$

Recall that the decision version of MinHom(H) is the following problem: Given an input digraph D, together with nonnegative costs $c_i(u)$, $u \in V(D)$, $i \in V(H)$, and an integer k, decide if D admits a homomorphism to H of cost not exceeding k. In the rest of this section we consider the decision version of MinHOM(H) to prove the NP-hardness of MinHOM(H) when $H = H_4, H_5, H_6$.

Lemma 4.2.3 *MinHOM(H_4) is NP-hard.*

Proof: Recall that $\mathcal{I}_3$ is the independent set problem for three-partite graphs. We construct a polynomial time reduction from $\mathcal{I}_3$ to MinHOM(H_4). Let X be a graph whose vertices are partitioned into independent sets U, V, W, and let k be a given integer. An

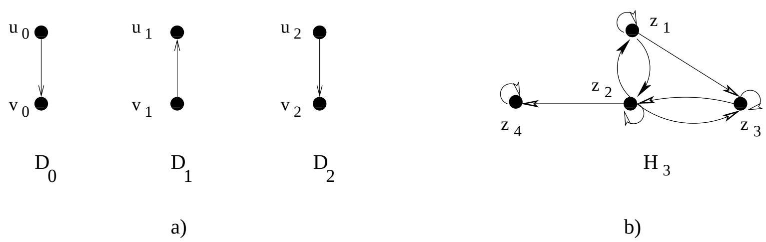

Figure 4.3: (a) The digraphs D_0, D_1, and D_2. (b) H_3.

instance of MinHOM(H_4) is formed as follows: the digraph D is obtained from X by replacing each edge uv of X with $u \in U, v \in V$ by an arc vu, replacing each edge uw of X with $u \in U, w \in W$ by a directed path $um_{uw}w$, and replacing each edge vw of X with $v \in V, w \in W$ by a directed path $vm_{vw}w$. The costs are defined by (writing for simplicity $c_i(y)$ for $c_{x_i}(y)$ where $x_i, 1 \leq i \leq 4$ is a vertex of H_4 in Figure 4.1) $c_1(u) = 1, c_3(u) = 0$ for $u \in U$; $c_2(v) = 0, c_3(v) = 1$ for $v \in V$; $c_4(w) = 0, c_1(w) = 1$ for $w \in W$; $c_3(m_{uw}) = c_4(m_{uw}) = |V(X)|$ for each edge uw of X with $u \in U, w \in W$; $c_2(m_{vw}) = c_4(m_{vw}) = |V(X)|$ for each edge vw of X with $v \in V, w \in W$; and $c_i(m) = 0$ for any other vertex $m \in V(D) - V(X)$, and $c_i(y) = |V(X)|$ for any other vertex $y \in V(X)$.

We now claim that X has an independent set of size k if and only if D admits a homomorphism to H_4 of cost $|V(X)| - k$. Let I be an independent set in X. We can define a mapping $f : V(D) \to V(H_2)$ as follows:

- $f(u) = x_3$ for $u \in U \cap I$ and $f(u) = x_1$ for $u \in U - I$
- $f(v) = x_2$ for $v \in V \cap I$ and $f(v) = x_3$ for $v \in V - I$
- $f(w) = x_4$ for $w \in W \cap I$ and $f(w) = x_1$ for $w \in W - I$
- $f(m_{uw}) = x_2$ when $f(u) = x_1$, and $f(m_{uw}) = x_1$ when $f(u) = x_3$ for each edge uw of X with $u \in U, w \in W$
- $f(m_{vw}) = x_3$ when $f(w) = x_4$ and $f(m_{vw}) = x_1$ when $f(w) = x_1$ for each edge vw of X with $v \in V, w \in W$

This is a homomorphism of D to H_4 of cost $|V(X)| - k$.

Let f be a homomorphism of D to H_4 of cost $|V(X)| - k$. Then, all $c_{f(u)}(u), u \in V(D)$ are either zero or one. Let $I = \{y \in V(X) \mid c_{f(y)}(y) = 0\}$ and note that $|I| \geq k$. It can be again seen that I is an independent set in X, as if $uw \in E(X)$, where $u \in I \cap U$ and $w \in I \cap V$ then $f(u) = x_3$ and $f(w) = x_4$, thus, $f(m_{uw}) = x_3$ or $f(m_{uw}) = x_4$. However, the cost of homomorphism is greater than $|V(X)|$, a contradiction. The other cases can also be treated similarly. $\diamond$

Lemma 4.2.4 *MinHOM(H_5) is NP-hard.*

Proof: We similarly construct a polynomial time reduction from $\mathcal{I}_3$ to MinHOM(H_5): this time the digraph D is obtained from X by replacing each edge uv of X with $u \in U, v \in V$ by an arc uv; replacing each edge uw of X with $u \in U, w \in W$ by arcs um_{uw}, wm_{uw}; and replacing each edge wv of X with $w \in W, v \in V$ by a directed path $wm_{wv}v$. The costs are defined by (writing for simplicity $c_i(y)$ for $c_{x_i}(y)$ where $x_i, 1 \leq i \leq 4$ is a vertex of H_5 in Figure 4.1) $c_1(u) = 1, c_2(u) = 0$ for $u \in U$; $c_2(v) = 1, c_4(v) = 0$ for $v \in V$; $c_3(w) = 1, c_1(w) = 0$ for $w \in W$; $c_1(m_{uw}) = c_2(m_{uw}) = |V(X)|$ for each edge uw of X with $u \in U, w \in W$; $c_1(m_{wv}) = c_4(m_{wv}) = |V(X)|$ for each edge wv of X with $w \in W, v \in V$; $c_i(m) = 0$ for any other vertex $m \in V(D) - V(X)$, and $c_i(y) = |V(X)|$ for any other vertex $y \in V(X)$.

We again claim that X has an independent set of size k if and only if D admits a homomorphism to H_5 of cost $|V(X)| - k$. Let I be an independent set in D. We can define a mapping $f : V(D) \to V(H_2)$ by $f(u) = x_2$ for $u \in U \cap I$ and $f(u) = x_1$ for $u \in U - I$; $f(v) = x_4$ for $v \in V \cap I$ and $f(v) = x_2$ for $v \in V - I$; $f(w) = x_1$ for $w \in W \cap I$ and $f(w) = x_3$ for $w \in W - I$; $f(m_{uw}) = x_3$ when $f(u) = x_2$, and $f(m_{uw}) = x_4$ when $f(u) = x_1$, for each edge uw of X with $u \in U, w \in W$; $f(m_{wv}) = x_3$ when $f(w) = x_3$ and $f(m_{wv}) = x_2$ when $f(w) = x_1$, for each edge wv of X with $w \in W, v \in V$. This is a homomorphism of D to H_5 of cost $|V(X)| - k$.

Let f be a homomorphism of D to H_5 of cost $|V(X)| - k$. Then, all $c_{f(u)}(u), u \in V(D)$ are either zero or one. Let $I = \{y \in V(X) \mid c_{f(y)}(y) = 0\}$ and note that $|I| \geq k$. It can be seen that I is an independent set in X, as if $uw \in E(X)$, where $u \in I \cap U$ and $w \in I \cap V$ then $f(u) = x_2$ and $f(w) = x_1$, thus, $f(m_{uw}) = x_1$ or $f(m_{uw}) = x_2$. However, the cost of homomorphism is greater than $|V(X)|$, a contradiction. The other cases can also be treated similarly. $\diamond$

Lemma 4.2.5 *MinHOM(H_6) is NP-hard.*

Proof: The proof is again similar, letting the digraph D be obtained from X by replacing each edge uv of X with $u \in U, v \in V$ by an arc uv; replacing each edge uw of X with $u \in U, w \in W$ by a directed path $um_{uw}w$; and replacing each edge vw of X with $v \in V, w \in W$ by an arc wv. The costs are defined by (writing for simplicity $c_i(y)$ for $c_{x_i}(y)$ where $x_i, 1 \leq i \leq 4$ is a vertex of H_6 in Figure 4.1) $c_1(u) = 0, c_2(u) = 1$ for $u \in U$; $c_3(v) = 0, c_1(v) = 1$ for $v \in V$; $c_4(w) = 0, c_3(w) = 1$; $c_1(m_{uw}) = c_4(m_{uw}) = |V(X)|$ for each edge uw of X with $u \in U, w \in W$; and letting $c_i(m) = 0$ for any other vertex $m \in V(D) - V(X)$, and $c_i(y) = |V(X)|$ for any other vertex $y \in V(X)$.

It can again be seen that X has an independent set of size k if and only if D admits a homomorphism to H_6 of cost $|V(X)| - k$: letting I be an independent set in D, we define a mapping $f : V(D) \to V(H_2)$ by $f(u) = x_1$ for $u \in U \cap I$ and $f(u) = x_2$ for $u \in U - I$; $f(v) = x_3$ for $v \in V \cap I$ and $f(v) = x_1$ for $v \in V - I$; $f(w) = x_4$ for $w \in W \cap I$ and $f(w) = x_3$ for $w \in W - I$; $f(m_{uw}) = x_3$ when $f(u) = x_2$ and $f(m_{uw}) = x_2$ when $f(u) = x_1$ for each edge uw, $u \in U, w \in W$. This is a homomorphism of D to H_6 of cost $|V(X)| - k$.

Let f be a homomorphism of D to H_6 of cost $|V(X)| - k$. Then, all $c_{f(u)}(u), u \in V(D)$ are either zero or one. Let $I = \{y \in V(X) \mid c_{f(y)}(y) = 0\}$ and note that $|I| \geq k$. It can again be seen that I is an independent set in X. ◇

We have proved the following result, conjectured in [54].

Theorem 4.2.6 *Let H be a reflexive digraph. If H admits a Min-Max ordering, then MinHOM(H) is polynomial time solvable. Otherwise, it is NP-hard.*

Chapter 5

Oriented Cycles with Some Loops

Homomorphisms to oriented cycles have been investigated in a number of papers [31, 50, 56, 64]. In particular, Feder has provided a dichotomy for HOM(H) [31] and Gutin et al. have provided a dichotomy for MinHOM(H) [50] when H is an irreflexive oriented cycle. In this chapter, we obtain a full dichotomy for MinHOM(H) when H is an oriented cycle with some loops. In fact, we verify Conjecture 3.1.5 for oriented cycles with some loops (meaning at least one loop). Furthermore, we shall argue that this constitutes an important step toward a dichotomy for all oriented graphs with some loops. This chapter is mostly based on [71].

5.1 Preliminaries

Let $P = b_0 b_1 \ldots b_p$ be an oriented path. We assign *levels* to the vertices of P as follows: we set $l(b_0) = 0$, and $l(b_{t+1}) = l(b_t) + 1$, if $b_t b_{t+1}$ is forward and $l(b_{t+1}) = l(b_t) - 1$, if $b_t b_{t+1}$ is backward. Let $[p]$ be the set $\{0, 1, \ldots, p\}$. We say that P is *of type* r if $r = \max\{l(b_i) : i \in [p]\} = l(b_p)$ and $0 \leq l(b_t) \leq r$ for each $t \in [p]$.

The following Proposition was first proved in [58]; see also [31, 93] and Lemma 2.36 in [60].

Proposition 5.1.1 *[58] Let P_1 and P_2 be two oriented paths of type r. Then there is an oriented path P of type r that maps homomorphically to P_1 and P_2 such that the initial vertex of P maps to the initial vertices of P_1 and P_2 and the terminal vertex of P maps to the terminal vertices of P_1 and P_2. The length of P is polynomial in the lengths of P_1 and*

P_2.

We will use the following notation in this chapter: $L(P) = \min\{l(b_j) : j \in [p]\}$, $H(P) = \max\{l(b_j) : j \in [p]\}$, $V_L(P) = \{b_t : l(b_t) = L(P), t \in [p]\}$, and $V_H(P) = \{b_t : l(b_t) = H(P), t \in [p]\}$.

Let $C = b_0b_1 \ldots b_pb_0$ be an oriented cycle. Recall from Chapter 1 (page 5) that we will always consider $b_0b_1 \ldots b_pb_0$ as the direction in which the number of forward arcs is not smaller than the number of backward arcs. We can assign *levels* to the vertices of C as follows: $l(b_0) = k$, where k is a non-negative integer, and $l(b_{t+1}) = l(b_t) + 1$, if b_tb_{t+1} is forward and and $l(b_{t+1}) = l(b_t) - 1$, if b_tb_{t+1} is backward. Clearly, the value of each $l(b_i)$, $i \in [p]$, depends on both k and the choice of the initial vertex b_0. We refer to $P^C_{b_0}$ as the oriented path $b_0b_1 \ldots b_pb_0$ obtained from the oriented cycle $C = b_0b_1 \ldots b_pb_0$ such that the first b_0 and the last b_0 are distinct vertices. (We "open C at b_0".) This way, each vertex of $P^C_{b_0}$ has a unique counterpart in C. So, when we refer to a vertex in $P^C_{b_0}$, one can imagine its corresponding vertex in C.

The following notation is extensively used in the rest of this chapter: $L(C) = \min\{l(b_j) : j \in [p]\}$, $H(C) = \max\{l(b_j) : j \in [p]\}$, $V_L(C) = \{b_t : l(b_t) = L(C), t \in [p]\}$, and $V_H(C) = \{b_t : l(b_t) = H(C), t \in [p]\}$.

Recall that an oriented cycle C is balanced if its net length is zero. Note that if C is balanced, the vertices of C that belong to $V_L(C)$ and $V_H(C)$, do not change by changing the initial vertex b_0 and k. We say that a balanced oriented cycle $C = b_0b_1 \ldots b_pb_0$ is *of the form* $(l^+h^+)^q$ with $q \geq 1$, if there is an initial vertex $b_0 \in V_L(C)$ such that $P = C - b_pb_0$ can be written as $P = x_1P_1y_1R_1x_2P_2y_2R_2 \ldots x_qP_qy_qR_q$, where $x_i \in V_L(C),\ y_i \in V_H(C)$ for each $i \in [q]$, and P_i, R_i are oriented paths such that P_i (respectively, R_i) contains no vertex of $V_L(C)$ (respectively, $V_H(C)$). We write l^+h^+ instead of $(l^+h^+)^1$. (see Figure 5.1 for an example of balanced oriented cycles of the form l^+h^+.)

In a balanced oriented cycle C of the form l^+h^+, a vertex $b_0 \in V_L(C)$ is the *absolute base*, if there is no vertex $u \in V_L(C)$ between b_0 and the first vertex of $V_H(C)$ in the direction $b_0b_1 \ldots b_pb_0$. Correspondingly, the path $P^C_{b_0} = b_0b_1 \ldots b_pb_0$ is called the *absolute predecessor path.*

Recall that for each oriented cycle with possible loops C, the net length of C , denoted $\lambda(C)$, is equal to the net length of $I(C)$, where $I(C)$ is the oriented cycle obtained from C by removing all existing loops. Consider the oriented cycle with the vertex set $C^0_4 = \{1, 2, 3, 4\}$,

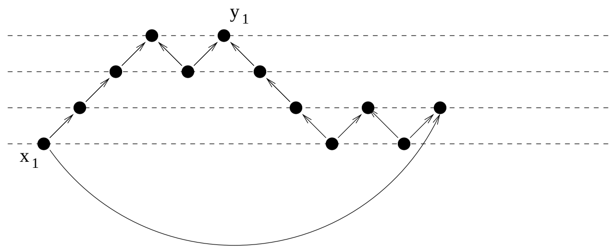

Figure 5.1: A balanced oriented cycle of the form l^+h^+. The higher dashed lines, the higher levels.

and the arc set $\{12, 32, 14, 34\}$. The next theorem follows from the main result of [50] for irreflexive oriented cycles.

Theorem 5.1.2 *[50] Let C be an irreflexive oriented cycle.*

- *If C has net length $k \geq 2$, then it has a k-Min-Max ordering and MinHOM(C) is polynomial time solvable.*
- *If C has net length $k = 1$, then it has a Min-Max ordering and MinHOM(C) is polynomial time solvable.*
- *If C is balanced of the form l^+h^+ or $C = C_4^0$, then C has a Min-Max ordering and MinHOM(C) is polynomial-time solvable. For all other balanced oriented cycles C, MinHOM(C) is NP-hard.*

5.2 Dichotomy

In this section, we provide a full dichotomy for MinHOM(C) when C is an oriented cycle with some loops. To do that, first of all, we partition the class of oriented cycles with some loops to three subclasses: the first contains all oriented cycles with some loops C such that the net length $\lambda(C)$ is more than one; the second contains all C such that $\lambda(C) = 1$; the third contains all C such that $\lambda(C) = 0$. We will verify Conjecture 3.1.5 for each of these subclasses separately.

Consider the oriented cycle C_3^1 with the vertex set $\{1, 2, 3\}$, and the arc set $\{12, 23, 13\}$, and the reflexive directed cycle C_2^2 with the vertex set $\{1, 2\}$ and the arc set $\{11, 22, 12, 21\}$.

Lemma 5.2.1 *Let C be an oriented cycle with possible loops and and let C' be an oriented cycle with some loops obtained from C by adding a loop to a vertex z of C, which is neither a source nor a sink. If $I(C') \neq C_3^1$ and $C' \neq C_2^2$, then MinHOM(C') is NP-hard.*

Proof: Since z is neither a source nor a sink, then there is a vertex x, dominating z, and a vertex y, dominated by z in C'. Note that $x = y$ if $I(C')$ is a directed cycle of length 2. Consider the digraph D with the vertex set $\{u, v\}$, and the arc set $\{uv\}$. Then there is no homomorphism from D to C', which maps u to x and v to y unless $I(C') = C_3^1$ or $C' = C_2^2$, meeting condition (b) of Proposition 3.2.7. The following homomorphisms meet conditions (a), (c), and (d) of Proposition 3.2.7, respectively:

- $f_1(u) = z$, and $f_1(v) = z$;
- $f_2(u) = x$, and $f_2(v) = z$;
- $f_3(u) = z$, and $f_3(v) = y$;

Hence, MinHOM(C') is NP-hard. ⋄

Lemma 5.2.2 *Let C be an oriented cycle with possible loops and and let $C' \neq C_2^2$ be an oriented cycle with some loops obtained from C by adding a loop to a vertex z of C. If MinHOM(C) is NP-hard, then MinHOM(C') is also NP-hard.*

Proof: Let C'' be the directed cycle with the vertex set $\{1, 2\}$ and the arc set $\{11, 12, 21\}$, where MinHOM(C'') is NP-hard by Lemma 5.2.1. C' has a symmetric arc (u dominates v and v dominates u) if and only if $C' = C_2^2$ or $C' = C''$. Since MinHOM(C'') is NP-hard, the current lemma is true for $C' = C''$. On the other hand, it is trivial to check that C has a Min-Max ordering and MinHOM(C) is polynomial time solvable, when $I(C') = C_3^1$; hence the current lemma is also true for oriented cycles C' for which $I(C') = C_3^1$.

Now, let us assume that C' is not C_2^2, C'', and all oriented cycles C' for which we have $I(C') = C_3^1$. If z is neither a source nor a sink, then MinHOM(C') is NP-hard by Lemma 5.2.1. Thus, we assume that z is either a source or a sink; without loss of generality assume that it is a sink. Moreover, as we exclude C_2^2 and C'', $I(C')$ (equivalently, $I(C)$) will not have any symmetric arc.

Now, we will construct a polynomial time reduction from MinHOM(C) to MinHOM(C'). An instance of MinHOM(C) contains an input digraph D with n vertices and the costs $c_i(u)$,

$u \in V(D), i \in V(C)$. Let all costs $c_i(u)$ be bounded from above by a constant m. We can partition the vertices of D to four sets as follows:

- U_1, where each vertex $u \in U_1$ has a loop;
- U_2, where no vertex $u \in U_2$ has a loop, and no vertex of U_2 is a source or sink in D;
- U_3, where no vertex $u \in U_3$ has a loop, and every vertex of U_2 is a source in D;
- U_4, where no vertex $u \in U_4$ has a loop, and every vertex of U_2 is a sink in D.

It is easy to check that there is no homomorphism of D to C which maps $u \in U_i, i = 1, 2, 3$ to z in C. To make an instance of MinHOM(C'), let us keep D as the input digraph and change the costs as follows: $c'_b(a) = c_b(a)$ for $a \in V(D), b \in V(C') - \{z\}$, and $c'_z(u) = nm + 1, u \in U_i, i = 1, 2, 3$ apart from $c'_z(u) = c_z(u), u \in U_4$. Observe that if MinHOM(C') returns a minimum cost homomorphism f of D to C' with a cost more than nm, then there is no homomorphism from D to C. Moreover, if MinHOM(C') returns a minimum cost homomorphism f of D to C' with a cost less than $nm + 1$, then f is a minimum cost homomorphism of D to C as well. Finally, if there is no homomorphism from D to C', then there is no homomorphism of D to C. ⋄

5.2.1 Oriented Cycles C with $\lambda(C) \geq 2$

Theorem 5.2.3 *Let C' be an oriented cycle with some loops such that $\lambda(C') \geq 2$. If $C' = C_2^2$, then MinHOM(C') is polynomial time solvable. Otherwise, MinHOM(C') is NP-hard.*

Proof: It is trivial to see that C_2^2 has a Min-Max ordering. Thus, we will assume that $C' \neq C_2^2$. To prove this theorem, it is sufficient (by Lemma 5.2.2) to show that MinHOM(C) is NP-hard, where C is an oriented cycle obtained from C' be removing all loops but the loop of z. Since the net length of $I(C')$ is more than one, $I(C')$ is not equal to C_3^1. Thus, if z is neither a source nor a sink in $I(C)$, MinHOM(C) is NP-hard by Lemma 5.2.1. In what follows, we prove that when z is either a source or a sink in $I(C)$, then MinHOM(C) is NP-hard. Without loss of generality, we assume that z is a sink in $I(C)$.

To show that MinHOM(C) is NP-hard, we will construct a digraph D, which meets the conditions of Proposition 3.2.7. First of all, consider the oriented path $P_z^C = za_1a_2 \ldots a_nz$.

For simplicity, let us denote the last z by z', i.e., $P_z^C = za_1a_2\ldots a_nz'$. It follows from the definition of P_z^C and the net length of $I(C)$ that $l(z') - l(z) \geq 2$. Hence, we will always have a vertex $x \neq z$ in P_z^C such that $l(x) - l(z) = 1$. Among such vertices, we will choose x as the first vertex with $l(x) - l(z) = 1$, met in the direction $za_1a_2\ldots a_nz'$ of P_z^C. On the other hand, if $z' \notin V_H(P_z^C)$, there must be a vertex x' such that $l(x') - l(z') = 1$ and x' is the first vertex with $l(x') - l(z') = 1$, met in the direction $z'a_n\ldots a_2a_1z$ of P_z^C. Let us now focus on two paths $P_{zx} = za_1a_2\ldots a_ix, i \geq 2$, and $P_{x'z'} = x'a_j\ldots a_nz', j \geq 2$. Let s be an arbitrary vertex of $V_L(P_{zx})$. If $z \notin V_H(P_z^C)$, we will also consider an arbitrary vertex of $V_L(P_{x'z'})$, denoted by s'. Now, we construct the digraph D, which meets the conditions of Proposition 3.2.7, as follows:

Case 1: Suppose that x' does not exist

Let w be the first vertex, met in the direction $z'a_n\ldots a_2a_1z$ of P_z^C, such that $l(z')-l(w) = l(x) - l(s)$, and Let y be the first vertex, met in the direction of $wa_{i'}\ldots a_nz'$, such that $l(y) - l(w) = l(z) - l(s)$. Note that $y \neq z'$. It is easy to see that $P_{wz'} = wa_{i'}\ldots a_nz'$, and $P_{sx} = sa_{j'}\ldots a_ix$ are of type $r = l(z') - l(w)$, and $P_{wy} = wa_{i'}\ldots y$, and $P_{sz} = s\ldots a_1z$ are of type $r' = l(z) - l(s)$. Applying Proposition 5.1.1, we can construct two oriented paths P_1 of type r and P_2 of type $r' = r - 1$, with terminal vertices u_1, v_1, and u_2, v_2, which map homomorphically to $P_{wz'}, P_{sx}$, and P_{wy}, P_{sz}, respectively, such that u_1 (respectively, u_2) maps to x, and z' (s, and w), and v_1 (respectively, v_2) maps to s and w (z, and y). To construct D, we will join these two oriented paths at vertex v_1 of P_1, and u_2 of P_2. Let $u = u_1$ and $v = v_2$. One can easily check that u, v of D and x, y, z, z' of P_z^C, which have unique counterparts in C, meet the conditions (a), (c), and (d) of Proposition 3.2.7. To show that condition (b) of this lemma also holds, it is enough to see that the net length of D is one, i.e., $l(u) - l(v) = 1$; however, $l(x) - l(y) \leq 0$ in P_z^C.

Case 2: Suppose that x' exists and $l(z) - l(s) \neq l(z') - l(s')$

Let $r = l(z) - l(s)$, and $r' = l(z') - l(s')$ in P_z^C. First, we assume that $r > r'$; hence $r \geq (l(x') - l(s'))$, as $l(x') - l(s') = r' + 1$. Let w be the first vertex, met in the direction of $P_{zs} = za_1\ldots s$, such that $l(z) - l(w) = l(x') - l(s')$, and let y be the first vertex, met in the direction of $P_{wz} = wa_{i'}\ldots a_1z$, such that $l(y) - l(w) = l(z') - l(s')$. Note that $y \neq z$. It is easy to check that $P_{s'x'} = s'\ldots a_jx'$, and $P_{wz} = wa_{i'}\ldots a_1z$ are of type $r' + 1$, and $P_{wy} = wa_{i'}\ldots y$, and $P_{s'z'} = s'\ldots a_nz'$ are of type r'. As for Case 1 , we can apply Proposition 5.1.1 to find P_1 and P_2 with terminal vertices u_1, v_1, and u_2, v_2 for $P_{s'x'}, P_{wz}$, and $P_{s'z'}, P_{wy}$, respectively, such that u_1 (respectively, u_2) maps to x', and z (s' and w)

and v_1 (respectively, v_2) maps to s' and w (z' and y). To construct D, we will join these two oriented paths at vertex v_1 of P_1, and u_2 of P_2. Let $u = u_1$ and $v = v_2$. One can easily check that u, v of D and x', y, z, z' of P_z^C, which have unique counterparts in C, meet conditions (a), (c), and (d) of Proposition 3.2.7. To show that condition (b) of this lemma also holds, it is enough to see that the net length of D is one, i.e., $l(u) - l(v) = 1$; however, $l(x') - l(y) \geq 4$ in P_z^C.

Second, we assume that $r < r'$. Then, the only difference is that w and y are in $P_{s'z'}$ rather than P_{sz}, and x, y are the representative pair of P_z^C rather than x', y in Proposition 3.2.7. Then $l(x) - l(y) \leq 0$, as $l(z') - l(z) \geq 2$; hence condition (b) of Proposition 3.2.7 holds as $l(u) - l(v) = 1$.

Case 3: Suppose that x' exists and $l(z) - l(s) = l(z') - l(s')$

Let $w \neq z$ (respectively, $w' \neq z'$) be a vertex of $P_{zx} = za_1 \dots a_i x$ (respectively, $P_{x'z'} = x'a_j \dots a_n z'$) with $l(z) = l(w)$ ($l(w') = l(z')$), and let $r = l(z) - l(s) = l(z') - l(s')$. One can easily check that $P_{s'z'}, P_{s'w'}, P_{sz}, P_{sw}$ are of type r. Applying Proposition 5.1.1, we can construct two oriented paths P_1 and P_2 of type r, with terminal vertices u_1, v_1, and u_2, v_2, which map homomorphically to $P_{s'z'}, P_{sw}$, and $P_{s'w'}, P_{sz}$, respectively, such that u_1 (respectively, u_2) maps to w, and z' (s, and s'), and v_1 (respectively, v_2) maps to s and s' (z, and w'). To construct D, we will join these two oriented paths at vertex v_1 of P_1, and u_2 of P_2. Let $u = u_1$ and $v = v_2$. One can easily check that u, v of D and w, w', z, z' of P_z^C, which have unique counterparts in C, meet conditions (a), (c), and (d) of Proposition 3.2.7. To show that condition (b) of this lemma also holds, it is enough to see that the net length of D is zero, i.e., $l(u) - l(v) = 0$; however, $l(w) - l(w') \leq -2$ in P_z^C. ⋄

5.2.2 Oriented Cycles C with $\lambda(C) = 1$

Before we give a dichotomy for this subclass of oriented cycles with some loops, we distinguish two special vertices s and t of oriented cycles with net length one. These two vertices play an important role in our study of this subclass of oriented cycles with some loops.

Lemma 5.2.4 *Let $C = b_0 b_1 \dots b_p b_0$ be an oriented cycle of net length one and b_0 be an arbitrary vertex of C. Let t be the first vertex of $V_H(P_{b_0}^C)$, met in the direction $b_0 b_1 \dots b_p b_0$, and s be the last vertex of $V_L(P_{b_0}^C)$, met in the direction $b_0 b_1 \dots b_p b_0$. Then the pair s, t in C is independent of the choice of b_0.*

Proof: Recall that each vertex of $P_{b_0}^C$ has a unique counterpart in C. So, when we refer to a vertex in $P_{b_0}^C$, one can imagine its corresponding vertex in C. Let $P_{b_0}^C = b_0b_1\dots b_pb_0$ and $P_{a_0}^C = a_0a_1\dots a_pa_0$ be two arbitrary oriented paths starting from b_0, and a_0, respectively. For simplicity, we replace the last b_0 and a_0 in $P_{b_0}^C$ and $P_{a_0}^C$ with b_{p+1} and a_{p+1}, respectively, i.e., $P_{b_0}^C = b_0b_1\dots b_pb_{p+1}$ and $P_{a_0}^C = a_0a_1\dots a_pa_{p+1}$. We will show that t is the first vertex of $V_H(P_{a_0}^C)$, met in the direction $a_0a_1\dots a_pa_{p+1}$. For s the proof is similar.

Note that both $P_{b_0}^C$ and $P_{a_0}^C$ traverses $I(C)$ in the same direction by the assumption of traversing oriented cycles in the direction of positive net length. Now, the following cases may happen:

Case 1: Suppose a_0 occurs on the oriented path from b_0 to t (inclusively)

It is easy to see that no vertex on the oriented path from a_0 to t has a level equal or greater than t in $P_{a_0}^C$. Recall that $\lambda(C) = 1$, i.e., $l(b_{p+1}) - l(b_0) = 1$. Now, we show that no vertex of $P_{a_0}^C$ on the oriented path Q from t to a_{p+1} has a level more than t. In fact, the portion of Q which is from t to b_{p+1} does not have such a vertex. Suppose, on the other hand, that this vertex occurs in the portion of Q from b_{p+1} to a_{p+1}. This is impossible since $l(b_{p+1}) - l(b_0) = 1$, i.e., the vertices of this portion in $P_{a_0}^C$ are exactly in one level more than the same vertices in the same portion in $P_{b_0}^C$ reaching at most to the level of t in $P_{a_0}^C$. (Recall that the levels of vertices of this portion in $P_{b_0}^C$ are strictly less than the level of t.) Hence, t is also the first vertex of $V_H(P_{a_0}^C)$, met in the direction $a_0a_1\dots a_pa_{p+1}$.

Case 2: Suppose a_0 occurs on the oriented path from t to b_{p+1} (inclusively)

Let Q be the portion of $P_{a_0}^C$ from b_{p+1} to a_{p+1}. Since t is the first vertex of $V_H(P_{b_0}^C)$, met in the direction $b_0b_1\dots b_pb_0$, it is easy to see that t is also the first vertex of $V_H(Q)$, met in the direction $b_{p+1}\dots a_pa_{p+1}$. Now, we show that all vertices of $P_{a_0}^C$ on the oriented path Q' from a_0 to b_{p+1} have levels less than t. In fact, there might be some vertices of the same level as t on Q' when we see Q' as a portion of $P_{b_0}^C$. However, since $l(b_{p+1}) - l(b_0) = 1$, these vertices of this portion Q' in $P_{a_0}^C$ are exactly in one level less than t in $P_{a_0}^C$, as t occurs on Q. Hence, t is also the first vertex of $V_H(P_{a_0}^C)$, met in the direction $a_0a_1\dots a_pa_{p+1}$. ⋄

Let C be an oriented cycle with some loops such that $\lambda(C) = 1$. In this subsection, we assume that s and t are fixed vertices of $I(C)$ introduced in Lemma 5.2.4. Recall that C_3^1 is an oriented cycle with the vertex set $\{1, 2, 3\}$, and the arc set $\{12, 23, 13\}$.

Theorem 5.2.5 *Let C' be an oriented cycle with some loops such that $\lambda(C') = 1$. If C' is one of the following digraphs, then MinHOM(C') is polynomial time solvable. Otherwise,*

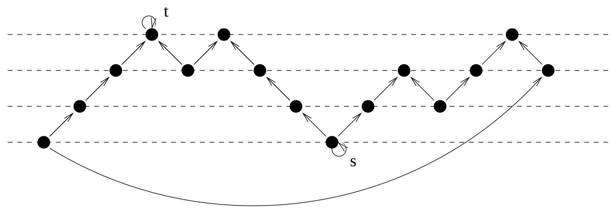

Figure 5.2: Dashed lines represent levels.

MinHOM(C') is NP-hard.

(a) Any oriented cycle C' such that $I(C') = C_3^1$.

(b) Any oriented cycle C' such that $I(C') \neq C_3^1$, and C' has at most two loops, which are the loops of s and t. (as defined earlier.)

Proof: It is trivial to check that C' has a Min-Max ordering when $I(C') = C_3^1$. Thus, we assume that $I(C') \neq C_3^1$. To prove part (b), suppose at least one of s and t has a loop, and no other vertex of C' has a loop. We wish to obtain a Min-Max ordering $\ll$ for C'. Let b_0 be an arbitrary vertex of C'. In what follows $l(u)$ represents the level of u in $P_{b_0}^{C'}$. Once we have $P_{b_0}^{C'}$, we can order the vertices of C' with the following rules (note that we do not order b_{p+1}, since it is a copy of b_0):

1. If $l(u) < l(v)$ then $u \ll v$;

2. If $l(u) = l(v)$, and u has been met earlier than v in the direction $b_0b_1 \dots b_pb_{p+1}$, then $v \ll u$.

Consider that t has the highest, and s has the lowest order in $\ll$. Thus, there is no crossing pair including arcs ss or tt. It is also easy to check that there is no crossing pair between the other arcs. Hence, $\ll$ is a Min-Max ordering. (see Figure 5.2.)

Now, it remains to prove that if a vertex z of C' other than s and t has a loop then MinHOM(C') is NP-hard. To do so, it is sufficient by Lemma 5.2.2 to show that MinHOM(C) is NP-hard, where C is an oriented cycle obtained from C' by removing all loops but the loop of z. Now, if z is neither a source nor a sink in $I(C)$, then MinHOM(C) is NP-hard by Lemma 5.2.1. So, we assume that z is either a source or a sink in $I(C)$. Without loss of generality we assume that z is a sink in $I(C)$.

Consider the oriented path $P_z^C = za_1a_2\dots a_nz$. For simplicity, let us denote the last z by z', i.e., $P_z^C = za_1a_2\dots a_nz'$. It follows from the definition of P_z^C and the net length of $I(C)$ that $l(z') - l(z) = 1$. Observe that since $z \neq t$, we will always have vertices $x, q \neq z, z'$ in P_z^C such that $l(x) - l(z) = 1$, and $l(q) - l(z) = 0$. Among such vertices, we will choose x and q as the first vertices with $l(x) - l(z) = 1$ and $l(q) - l(z) = 0$, met in the direction $za_1a_2\dots a_nz'$ of P_z^C. On the other hand, if $z' \notin V_H(P_z^C)$, there must be a vertex x' such that $l(x') - l(z') = 1$, otherwise x' does not exist. Among such vertices, we will choose x' as the first vertex with $l(x') - l(z') = 1$, met in the direction $z'a_n\dots a_2a_1z$ of P_z^C. Let us now focus on two paths $P_{zx} = za_1a_2\dots a_ix$ and $P_{x'z'} = x'a_j\dots a_nz'$. Let s be an arbitrary vertex of $V_L(P_{zx})$. If $z \notin V_H(P_z^C)$, we will also consider an arbitrary vertex of $V_L(P_{x'z'})$, denoted by s'. We now construct a digraph D, which meets the conditions of Proposition 3.2.7:

Case 1: Suppose that x' does not exist

Since x' does not exist, we have $z' \in V_H(P_z^C)$. Let w be the first vertex, met in the direction $z'a_n\dots a_2a_1z$ of P_z^C, such that $l(z') - l(w) = l(q) - l(s)$, and let y be the first vertex, met in the direction of $wa_{i'}\dots a_nz'$, such that $l(y) - l(w) = l(z) - l(s)$. Note that $y \neq z'$, as $z \neq t$. It is easy to see that $P_{wz'} = wa_{i'}\dots a_nz'$, $P_{sq} = sa_{j'}\dots a_iq$, $P_{wy} = wa_{i'}\dots y$, and $P_{sz} = s\dots a_1z$ are all of type $r = l(z') - l(w) = l(z) - l(s)$. Applying Proposition 5.1.1, we can construct two oriented paths P_1 and P_2 of type r, with terminal vertices u_1, v_1, and u_2, v_2, which map homomorphically to $P_{wz'}, P_{sq}$, and P_{wy}, P_{sz}, respectively, such that u_1 (respectively, u_2) maps to q, and z' (s, and w), and v_1 (respectively, v_2) maps to s and w (z, and y). To construct D, we will join these two oriented paths at vertex v_1 of P_1, and u_2 of P_2. Let $u = u_1$ and $v = v_2$. One can easily check that u, v of D and q, y, z, z' of P_z^C, which have unique counterparts in C, meet conditions (a), (c), and (d) of Proposition 3.2.7. To show that the condition (b) of this lemma also holds, it is enough to see that the net length of D is zero, i.e., $l(u) - l(v) = 0$; however, $l(q) - l(y) = -1$ in P_z^C.

Case 2: Suppose that x' exists and $l(z) - l(s) \neq l(z') - l(s')$

D is constructed exactly like Case 2 in the proof of Theorem 5.2.3. Note that $l(u) - l(v) = 1$ in D. If $r' < r$, $l(x') - l(y) = 3$ as $l(z') - l(z) = 1$; hence condition (b) of Proposition 3.2.7 holds as $l(u) - l(v) = 1$. One the other hand, if $r < r'$, then $l(x) - l(y) = 1$, which does not necessarily guarantee that condition (b) of Proposition 3.2.7 holds, as $l(u) - l(v) = 1$. However, since $l(x') - l(x) = 1$, there is no homomorphism, mapping u to x and v to y due to the existence of x'. In fact, such a homomorphism f maps D to the oriented cycle, existing between x and y in P_z^C, such that $f(u) = x, f(v) = y$, and there is a vertex u' in D, for which we have $f(u') = x'$, since x' is in the oriented cycle between x, and y. We denote by $l_D(u)$, the level of vertex u in D. We can easily see that $l_D(u') > l_D(u)$, as $l(x') > l(x)$. This is a contradiction, since both P_1 and P_2 are of type $r+1$ and r and no vertex of D has a level more than u.

Case 3: Suppose that x' exists and $l(z) - l(s) = l(z') - l(s')$

D is constructed exactly like Case 3 in the proof of Theorem 5.2.3. To show that condition (b) of Proposition 3.2.7 also holds, it is enough to see that $l(u) - l(v) = 0$; however, $l(w) - l(w') = -1$ in P_z^C. ◇

5.2.3 Oriented Cycles C with $\lambda(C) = 0$

We begin this subsection by introducing two special pairs of vertices s_1, s_2 and t_1, t_2 for oriented cycles C of the form l^+h^+. Let C be a balanced oriented cycle of the form l^+h^+, and let b_0 and $P_{b_0}^C$ be the absolute base and absolute base path of C, respectively. (Note that for each balanced oriented cycle C of the form l^+h^+, the absolute base is unique.) Let t_1 be the first vertex and t_2 be the last vertex of $V_H(P_{b_0}^C)$, met in the direction $b_0b_1 \ldots b_pb_0$, and s_1 be the first vertex and s_2 be the last vertex of $V_L(P_{b_0}^C)$, met in the direction $b_0b_p \ldots b_1$. It is easy to see that all four of these vertices are fixed in C and $b_0 = s_1$, as C is balanced of the form l^+h^+. Note that t_1, t_2 (respectively, s_1, s_2) are not necessarily distinct; we can assume a case where $|V_H(P_{b_0}^C)| = 1$. (respectively, $|V_L(P_{b_0}^C)| = 1$.) For an oriented cycle with some loops C, s_1, s_2, t_1, t_2 are defined as s_1, s_2, t_1, t_2 in $I(C)$, respectively. Recall that C_4^0 is an oriented cycle with the vertex set $C_4^0 = \{1, 2, 3, 4\}$, and the arc set $\{12, 32, 14, 34\}$.

Theorem 5.2.6 *Let C' be an oriented cycle with some loops such that $I(C')$ is balanced. If C' is one of the following digraphs, then MinHOM(C') is polynomial time solvable. Otherwise, MinHOM(C') is NP-hard.*

(a) Any oriented cycle C' such that $I(C') = C_4^0$, and C' has at most two loops.

(b) Any oriented cycle C' such that $I(C')$ is of the form l^+h^+, and C' has at most two loops, which are the loops of either s_1 and t_2 or s_2 and t_1 (as defined earlier).

Proof: If $I(C')$ is not of the form l^+h^+ and $I(C') \neq C_4^0$, then MinHOM($I(C')$) is NP-hard; hence by Lemma 5.2.2, MinHOM(C') is NP-hard. It is trivial to check that C' has a Min-Max ordering, when $I(C') = C_4^0$, and C' has at most two loops. If $I(C') = C_4^0$, and C'

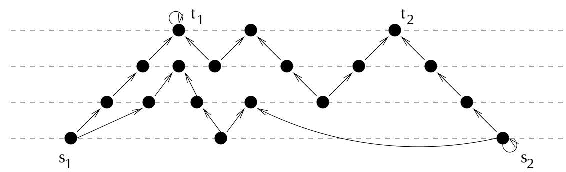

Figure 5.3: Dashed lines represent levels. The higher the dashed lines, the higher levels. The further right the vertex, the lower the order.

has three or four loops, then $B(C')$ has a binet as an induced subgraph and MinHOM(C') is NP-hard by Proposition 3.2.1.

Now, suppose that C' is an oriented cycle with some loops such that $I(C') \neq C_4^0$ is of the form l^+h^+, and C' has at most two loops, which belong to either s_1 and t_2 or s_2 and t_1. Without loss of generality, assume that at least one of s_2 and t_1 has a loop and no other vertex of C' has a loop. (see Figure 5.3.) We split the oriented cycle C' into two oriented paths P_1, and P_2 from s_1 to s_2. In what follows $l_{P_1}(u)$ (respectively, $l_{P_2}(u)$) represents the level of u in P_1 (respectively, P_2), where $l_{P_1}(s_1) = 0$, and $l_{P_2}(s_1) = 0$. Since $I(C')$ is of the form l^+h^+ and $I(C') \neq C_4^0$, one of P_1 or P_2, say P_1, contains all the vertices of $V_H(P_{s_1}^{C'}) = V_H(C')$, and P_2 contains all the vertices of $V_L(P_{s_1}^{C'}) = V_L(C')$. Hence, $l_{P_2}(u) < l_{P_1}(t_1)$ for all $u \in V(P_2)$. We wish to obtain a Min-Max ordering $\ll$ for C'. We can order the vertices of C' with the following rules:

1. If $u \in P_i, i = 1,2$,$v \in P_j, j = 1,2$, and $l_{P_i}(u) < l_{P_j}(v)$ then $u \ll v$;

2. If $u, v \in P_i, i = 1,2$, and $l_{P_i}(u) = l_{P_i}(v)$, and u has been met earlier than v in the direction $s_1 \ldots s_2$ in P_i, then $v \ll u$.

3. If $u \in P_1$, $v \in P_2$, and $l_{P_1}(u) = l_{P_2}(v)$, then

   3.1. if u is in the oriented path between s_1 and t_1 in P_1, then $v \ll u$;

   3.2. otherwise, $u \ll v$.

Note that t_1 has the highest and s_2 has the lowest order in $\ll$. Thus, there is no crossing pair including arcs s_2s_2 or t_1t_1. It is also easy to check that there is no crossing pair between the other arcs since $I(C') \neq C_4^0$ is of the form l^+h^+, and $l_{P_2}(u) < l_{P_1}(t_1)$ for all $u \in V(P_2)$; hence, $\ll$ is a Min-Max ordering. (see Figure 5.3.)

It remains to prove that MinHOM(C') is NP-hard for all oriented cycles C' with some loops, where $I(C')$ is of the form l^+h^+, and C' does not fulfill the conditions of part (b). Let b_0 be the absolute base of C'. The following lemmas cover this fact.

Lemma 5.2.7 *Let C' be an oriented cycle with some loops such that $I(C')$ is balanced and of the form l^+h^+. If a vertex z of C', which is neither in $V_H(P_{b_0}^{C'})$ nor in $V_L(P_{b_0}^{C'})$, has a loop, then MinHOM(C') is NP-hard.*

Proof: It is sufficient by Lemma 5.2.2 to show that MinHOM(C) is NP-hard, where C is an oriented cycle obtained from C' by removing all loops except for the loop of z. Now, if z is neither a source nor a sink in $I(C)$, then MinHOM(C) is NP-hard by Lemma 5.2.1. So, we assume that z is either a source or a sink in $I(C)$. Without loss of generality, we assume that z is a sink in $I(C)$.

Consider the oriented path $P_z^C = za_1a_2\dots a_nz$. For simplicity, let us denote the last z by z', i.e., $P_z^C = za_1a_2\dots a_nz'$. Since z is neither in $V_H(P_{b_0}^C)$ nor in $V_L(P_{b_0}^C)$, we will always have a unique vertex $x \neq z, z'$ in P_z^C such that $l(x) - l(z) = 1$, and x is the first vertex with $l(x) - l(z) = 1$, met in the direction $za_1a_2\dots a_nz'$ of P_z^C. On the other hand, there is also a unique vertex x' such that $l(x') - l(z') = 1$, and x' is the first vertex with $l(x') - l(z') = 1$, met in the direction $z'a_n\dots a_2a_1z$ of P_z^C. Let us now focus on two paths $P_{zx} = za_1a_2\dots a_ix$ and $P_{x'z'} = x'a_j\dots a_nz'$. Let s (respectively, s') be an arbitrary vertex of $V_L(P_{zx})$ (respectively, $V_L(P_{x'z'})$). The following cases may happen:

Case 1: Suppose that $l(z) - l(s) \neq l(z') - l(s')$

D is constructed exactly like Case 2 in the proof of Theorem 5.2.3. Note that $l(u)-l(v) = 1$ in D. If $r' < r$ (respectively, $r < r'$), $l(x') - l(y) = 2$ (respectively, $l(x) - l(y) = 2$) as $l(z') - l(z) = 0$; hence condition (b) of Lemma 5.2.1 holds since $l(u) - l(v) = 1$.

Case 2: Suppose that $l(z) - l(s) = l(z') - l(s')$

D is constructed exactly like Case 3 in the proof of Theorem 5.2.3. Since $l(u) - l(v) = 0$ and $l(w) - l(w') = 0$, condition (b) of Proposition 3.2.7 is not easily implied. However, due to the existence of x, this condition also holds. In fact, if a homomorphism f of D to C exists such that $f(u) = w, f(v) = w'$, it must map the vertices of D to the oriented cycle between w and w' in P_z^C. Thus, there is a vertex u' in D, for which we have $f(u') = x$, as x is in the oriented cycle between w and w'. We denote by $l_D(u)$, the level of vertex u in D. It is easy to see that $l_D(u') > l_D(u)$, as $l(x) > l(w)$. This is a contradiction since both P_1 and P_2 are of type r and no vertex of D has a level more than u. ◇

Lemma 5.2.8 *Let C' be an oriented cycle with some loops such that $I(C')$ is balanced and of the form l^+h^+. If a vertex z of $V_H(P_{b_0}^{C'})$ (respectively, $V_L(P_{b_0}^{C'})$), which is neither t_1, nor t_2 (respectively, neither s_1 nor s_2) has a loop, then MinHOM(C') is NP-hard.*

Proof: Without loss of generality, assume that $z \in V_H(P_{b_0}^{C'})$, and clearly it is a sink. Similar to the proof of Lemma 5.2.7, we consider C, which is an oriented cycle obtained from C' by removing all loops but the loop of z.

Consider the oriented path $P_z^C = za_1a_2 \dots a_nz$. For simplicity, let us denote the last z by z', i.e., $P_z^C = za_1a_2 \dots a_nz'$. Since $|V_H(P_{b_0}^C)| \geq 3$, we will always have a vertex $x \neq z, z'$ in P_z^C such that $l(x) - l(z) = 0$, and x is the first vertex with $l(x) - l(z) = 0$, met in the direction $za_1a_2 \dots a_nz'$ of P_z^C. On the other hand, there is another vertex $y \neq z, z'$ such that $l(y) - l(z') = 0$, and y is the first vertex with $l(y) - l(z') = 0$, met in the direction $z'a_n \dots a_2a_1z$ of P_z^C. Note that x and y are distinct vertices, as $|V_H(P_{b_0}^C)| \geq 3$, and both of them are in $V_H(P_{b_0}^C)$, as $z \in V_H(P_{b_0}^C)$. Let us now focus on two paths $P_{zx} = za_1a_2 \dots a_ix$ and $P_{yz'} = ya_j \dots a_nz'$. Let s (respectively, s') be an arbitrary vertex of $V_L(P_{zx})$ (respectively, $V_L(P_{yz'})$). Without loss of generality, assume that $l(z') - l(s') \leq l(s) - l(z)$. Observe that neither s nor s' is in $V_L(P_z^C)$, as $I(C)$ is of the form l^+h^+, and $z \neq t_1, t_2$. In other words, there exists a vertex $s'' \in V_L(P_z^C)$, which is in the oriented path $P_{xy} = x \dots y$ of P_z^C.

Let w be the first vertex, met in the direction $z'a_n \dots a_2a_1z$ of P_z^C, such that $l(z') - l(w) = l(x) - l(s)$. It is easy to see that $P_{wz'} = wa_{i'} \dots a_nz'$, and $P_{sx} = sa_{j'} \dots a_ix$ are of type $r = l(z') - l(w)$, and $P_{wy} = wa_{i'} \dots y$, and $P_{sz} = s \dots a_1z$ are of type $r' = l(z) - l(s) = r$. Applying Proposition 5.1.1, we can construct two oriented paths P_1, and P_2 of type r, with terminal vertices u_1, v_1, and u_2, v_2, which map homomorphically to $P_{wz'}, P_{sx}$, and P_{wy}, P_{sz}, respectively, such that u_1 (respectively, u_2) maps to x, and z' (s, and w), and v_1 (respectively, v_2) maps to s and w (z, and y). To construct D, we will join these two oriented paths at vertex v_1 of P_1, and u_2 of P_2. Let $u = u_1$ and $v = v_2$. One can easily check that u, v of D and x, y, z, z' of P_z^C, which have unique counterparts in C, meet conditions (a), (c), and (d) of Proposition 3.2.7.

Note that $l(u) - l(v) = 0$ in D. Since $l(x) - l(y) = 0$, the condition (b) of Proposition 3.2.7 is not easily implied, as $l(u) - l(v) = 0$. However, there is no homomorphism, mapping u to x and v to y due to existence of s''. In fact, such a homomorphism f maps D to the oriented path, existing between x and y in P_z^C, such that $f(u) = x, f(v) = y$, and there is a vertex u' in D, for which we have $f(u') = s''$ as s'' is in the oriented cycle between x, and y. This way, we must have: $l_D(u) - l_D(u') = l(x) - l(s'') = l(z') - l(s'')$, which is a contradiction since D does not contain u' with such a level. $\diamond$

Lemma 5.2.9 *Let C' be an oriented cycle with some loops such that $I(C')$ is balanced and of the form l^+h^+. If distinct vertices t_1 and t_2 (respectively, s_1 and s_2) have loops, then MinHOM(C') is NP-hard.*

Proof: Without loss of generality, we prove this lemma for t_1 and t_2, and we consider C, which is an oriented cycle obtained from C' by removing all loops but the loops of t_1, and t_2.

Let $z = t_1$ and $x = t_2$. Consider the oriented path $P_z^C = za_1a_2 \dots a_nz'$. Let s be an arbitrary vertex of $V_L(P_{zx})$, where $P_{zx} = za_1a_2 \dots a_ix$. Observe that $s \notin V_L(P_z^C)$ since $I(C)$ is of the form l^+h^+. Hence, there exists a vertex s', which is in the oriented path $P_{xz'} = xa_{i+1} \dots a_nz'$ of P_z^C, where $l(s) - l(s') = 1$. Among such vertices, we will choose s' as the first vertex with $l(s) - l(s') = 1$, met in the direction $z'a_n \dots a_2a_1z$ of P_z^C. Let y be the first vertex, met in the direction of $P_{s'z'} = s'a_{i'} \dots a_nz'$, such that $l(y) - l(s') = l(z) - l(s)$. Note that $y \neq z'$. We will virtually assume a vertex x' such that x dominates this vertex. (Note that $x = t_2$ has a loop in C.) It is easy to see that $P_{s'z'} = s'a_{i'} \dots a_nz'$, and $P_{sx'} = sa_{j'} \dots a_ixx'$ are of type $r = l(z') - l(s')$, and $P_{s'y} = s'a_{i'} \dots y$, and $P_{sz} = s \dots a_1z$ are of type $r' = r - 1$.

Applying Proposition 5.1.1, we can construct two oriented paths P_1 of type r and P_2 of type r', with terminal vertices u_1, v_1, and u_2, v_2, which map homomorphically to $P_{s'z'}, P_{sx'}$, and $P_{s'y}, P_{sz}$, respectively, such that u_1 (respectively, u_2) maps to x', and z' (s, and s'), and v_1 (respectively, v_2) maps to s and s' (z, and y). One can easily see that all vertices that these homomorphisms map to x', can also be mapped to x, since x has a loop in C. Now, we construct a digraph D, which fulfills the conditions of Proposition 3.2.8. To construct D, we will join these two oriented paths at the vertex v_1 of P_1, and the vertex u_2 of P_2. Let $u = u_1$ and $v = v_2$. Let all $c_i(u) = 0$ apart from $c_i(u) = +\infty, i \in V(P_{xs'}) - \{x, s'\}, u \in V(D)$, where $P_{xs'} = xa_{i+1} \dots s'$.

One can easily check that u, v of D and x, y, z, z' of P_z^C, which have unique counterparts in C, meet the conditions (a), (b), (d), and (e) of Proposition 3.2.8. To show that condition (c) of this lemma also holds, it is enough to see that there is no homomorphism of D to C, which maps u to x, and v to y, unless one of the vertices of D maps to a vertex $i \in V(P_{xs'}) - \{x, s'\}$, i.e., the cost of homomorphism is infinity, meeting condition (c) of

Proposition 3.2.8. Thus, MinHOM(C) is NP-hard. $\diamond$

Lemma 5.2.10 *Let C' be an oriented cycle with some loops such that $I(C')$ is balanced and of the form l^+h^+. If s_1 and t_1 (or s_2 and t_2) have loops, and $|V_H(P_{b_0}^{C'})| \geq 2$, $|V_L(P_{b_0}^{C'})| \geq 2$, then MinHOM($C'$) is NP-hard.*

Proof: Without loss of generality, we prove this lemma for s_1, t_1, and we consider C, which is an oriented cycle obtained from C' by removing all loops but the loops of s_1, and t_1.

Let $z = t_1$. Consider the oriented path $P_z^C = za_1a_2 \ldots a_nz'$. Since $|V_H(P_{b_0}^C)| \geq 2$, we will always have a vertex $x \neq z, z'$ in P_z^C such that $l(x) - l(z) = 0$. Among such vertices, we will choose x as the last vertex with $l(x) - l(z) = 0$, met in the direction $za_1a_2 \ldots a_nz'$ of P_z^C. Let s' be an arbitrary vertex of $V_L(P_{zx})$, where $P_{zx} = za_1a_2 \ldots a_ix$. Observe that $s' \notin V_L(P_z^C)$, as $I(C)$ is of the form l^+h^+. Thus, all vertices of $V_L(P_z^C)$ are in the oriented path $P_{xz'} = xa_{i+1} \ldots a_nz'$. One can easily see that s_1 is the last vertex of $V_L(P_z^C)$, met in the direction $za_1a_2 \ldots a_pz'$, since s_1 is the first vertex of $V_L(P_{b_0}^C)$, met in the direction $b_0b_p \ldots b_1$ of $P_{b_0}^C$. Let s'' be the first vertex of $V_L(P_z^C)$, met in the direction $za_1a_2 \ldots a_pz$. It is easy to see that s'' and s_1 are distinct as $|V_L(P_z^C)| = |V_L(P_{b_1}^{C'})| \geq 2$.

It is easy to see that $P_{s''x} = s'' \ldots a_{i+1}x$, and $P_{s_1z'} = s_1a_{j'} \ldots a_pz'$ are of type $r = l(z') - l(s_1)$.

Applying Proposition 5.1.1, we can construct an oriented path P_1 of type r with terminal vertices u_1, v_1, which maps homomorphically to $P_{s''x}, P_{s_1z'}$, such that u_1 maps to x, and z', and v_1 maps to s'' and s_1. Let P_2 be an oriented path with terminal vertices u_2 and v_2, isomorphic to $P_{zs''} = za_1 \ldots s''$, where u_2 maps to s'', and v_2 maps to z with this isomorphism. To construct D, we will join these two oriented paths at vertex v_1 of P_1, and u_2 of P_2. Let $u = u_1$ and $v = v_2$. One can easily check that u, v of D and x, s_1, z, z' of P_z^C, which have unique counterparts in C, meet the conditions (a), (c), and (d) of Proposition 3.2.7. Note that when $f(v_1) = s_1$, then all vertices of P_2 can map to s_1.

Observe that $l(u) = l(v)$ in D, and there is a vertex $u' \neq u, v$ in D, for which $l(u') = l(u) = l(v)$. However, the oriented path between x and s_1 does not contain any vertex w with the same level as x, since x was the last vertex with the highest level, met in the direction $za_1a_2 \ldots a_nz'$ of P_z^C. Thus, there is no homomorphism of D to C, mapping u to x, and s_1 to v, meeting condition (b) of Proposition 3.2.7. $\diamond$

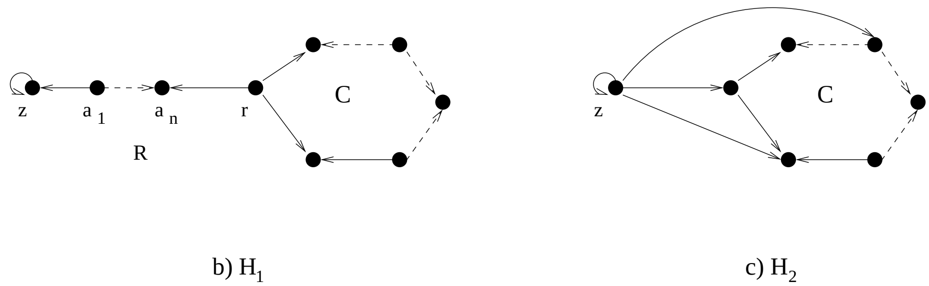

Figure 5.4: H_i, $i = 1, 2$.

Thus, if we have any loop (or a set of loops) which does not satisfy the condition of part (b) for a balanced oriented cycle with some loops of the form l^+h^+, then MinHOM(C') is NP-hard by the previous four lemmas. This completes the proof of this theorem. $\diamond$

5.3 Oriented Graphs

Our new dichotomy for oriented cycles with some loops is an important step towards a MinHOM dichotomy for oriented graphs with some loops. Recall that oriented graphs do not have $\vec{C}_2$ as an induced subgraph. Thus, oriented graphs with some loops do not have C_2^2 as an induced subgraph. On the other hand, MinHOM(C) is NP-hard for all oriented cycles C with some loops when $\lambda(C) \geq 2$, except for $C = C_2^2$ by Theorem 5.2.3. Hence, if an oriented graph H contains an induced oriented cycle C with some loops and $\lambda(C) \geq 2$, then MinHOM(H) is NP-hard by Proposition 3.2.4 and Theorem 5.2.3. We conjecture that this fact will also hold when an oriented graph with some loops H contains an irreflexive oriented cycle C with $\lambda(C) \geq 2$ as an induced subgraph.

Conjecture 5.3.1 *Let H be an oriented graph with some loops. If H contains an irreflexive oriented cycle C with $\lambda(C) \geq 2$ as an induced subgraph, then MinHOM(H) is NP-hard.*

Let H be an oriented graph with some loops such that H contains an irreflexive oriented cycle C, with $\lambda(C) \geq 2$. It is easy to show that H must contain at least one of the following cases as an induced subgraph: (a) the digraph H_1, consisting of C, a vertex z with a loop, and an oriented path $R = za_1 \ldots a_n r$ between z and a vertex r of C; (b) the digraph H_2, consisting of C, a vertex z with a loop, and at least two arcs between z and some vertices of C. (See Figure 5.4.) Thus, to show that MinHOM(H) is NP-hard, it is sufficient by Proposition 3.2.4 to show that MinHOM(H_i), $i = 1, 2$ is NP-hard. The authors of [65] have shown that if an irreflexive digraph H has an induced directed cycle of length k and oriented cycle C of net length n not divisible by k, then MinHOM(H) is NP-hard. Since loops are special cases of directed cycles (directed cycles with only one vertex), we think that the same approach used in [65], can be applied to show that MinHOM(H_i), $i = 1, 2$ is NP-hard. We conclude that if Conjecture 5.3.1 holds, one should only seek a dichotomy for oriented graphs having no oriented cycles or having oriented cycles with possible loops C, with $\lambda(C) \leq 1$.

Chapter 6

Quasi-Transitive Digraphs

Along with semicomplete digraphs and semicomplete multipartite digraphs, quasi-transitive digraphs are the most studied families of generalizations of tournaments [3]. Thus, it is a natural problem to seek a dichotomy for quasi-transitive digraphs. As with semicomplete digraphs and semicomplete multipartite digraphs, structural properties of quasi-transitive digraphs play a key role in proving this dichotomy. We hope that the study of well-known classes of digraphs will eventually lead to a proof of full dichotomy for irreflexive digraphs.

A digraph H is quasi-transitive if, for every triple x, y, z of distinct vertices of H such that xy and yz are arcs of H, there is at least one arc between x and z. In this chapter, we always assume that H is connected, as otherwise we can study the problem for each component of H. The following two sections study the tractable and intractable MinHOM(H) for different quasi-transitive digraphs H.

6.1 Polynomial Cases

Let H be a quasi-transitive digraph. Recall the definitions of extension of H and $B(H)$ from Section 1.1. The following theorem gives us a sufficient condition for tractability of MinHOM(H) when H is a quasi-transitive digraph.

Theorem 6.1.1 *Let H be a quasi-transitive digraph. Then MinHOM(H) is polynomial time solvable if H is one of the following digraphs.*

- *H is $\overrightarrow{C_2}$*
- *H is an extension of $\overrightarrow{C_3}$*

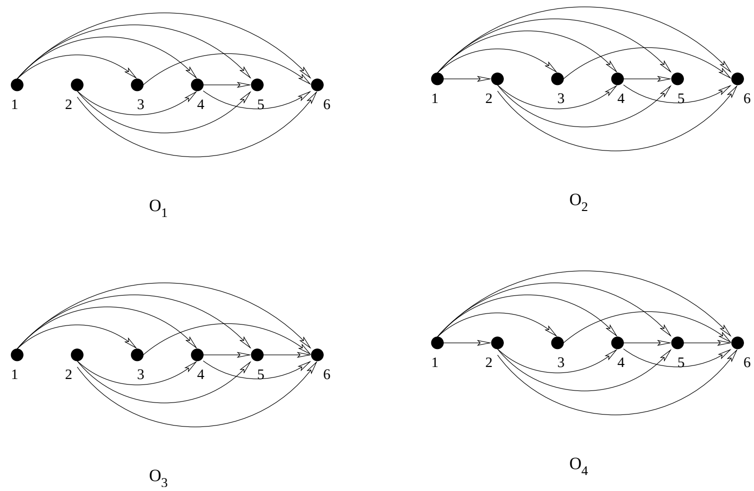

Figure 6.1: The obstructions O_i with $i = 1, 2, 3, 4$

- *H is acyclic, $B(H)$ is a proper interval bigraph and H does not contain O_i with $i = 1, 2, 3, 4$ as an induced subdigraph. (See Figure 6.1.)*

Proof: If H is $\overrightarrow{C_2}$ or an extension of $\overrightarrow{C_3}$, then it has a 2-Min-Max ordering or a 3-Min-Max ordering, and thus MinHOM(H) is polynomial time solvable.

Now assume that H is acyclic. Then, for every triple x, y, z of distinct vertices of H such that $xy, yz \in A(H)$, we must have $xz \in A(H)$. Let us call this property the *transitivity* of H. We remind the reader of the definitions of bipartite Min-Max ordering, proper bipartite Min-Max ordering, and proper pairs for $B(H)$ from Chapter 4. For the bipartite graph $B(H)$ (with a fixed bipartition into white and black vertices), a bipartite Min-Max ordering is an ordering $<$ such that $<$ is restricted to the white vertices, and $<$ is also restricted to the black vertices satisfy the condition of Min-Max orderings, i.e., $i < j$ for white vertices, and $s < r$ for black vertices, and $ir, js \in E(H)$, imply that $is \in E(H)$ and $jr \in E(H)$). A pair u, v of vertices of H is proper for $<$ if $u' < v'$ if and only if $u'' < v''$ in $B(H)$. We say a bipartite Min-Max ordering $<$ is proper if all pairs u, v of H are proper for $<$.

Recall Theorem 4.1.1 that $B(H)$ has a bipartite Min-Max ordering if and only if it is a proper interval bigraph. We will show that a bipartite Min-Max ordering of $B(H)$ can be transformed to produce a proper bipartite Min-Max ordering of $B(H)$, and thus a Min-Max ordering of H. Suppose $<$ is a bipartite Min-Max ordering of $B(H)$ which is not proper.

That is, there are vertices x', y' such that $x' < y'$ and $y'' < x''$. In the remaining part of this proof, we will show that we can always exchange the positions of x' and y' or the positions of x'' and y'' in $<$ whenever we have an improper pair x, y and $<$ is a bipartite Min-Max ordering of $B(H)$.

Suppose that for every pair of vertices c'' and d'' such that $d'' < c''$ and $x'd'', y'c'' \in E(B(H))$, we have both $x'c''$ and $y'd''$ in $E(B(H))$. Then we can exchange the positions of x' and y' in $<$ while perserving the Min-Max property. Furthermore, it can be checked that this exchange strictly increases the number of proper pairs in H: if a proper pair turns into an improper pair or vice versa by this exchange, then one of the two vertices of this pair must be x or y. Clearly the improper pair consisting of x and y is turned into a new proper pair. Suppose that vertex w constitues a pair with x or y which is possibly affected by the exchange. Observe that we have $x' < w' < y'$ or $y'' < w'' < x''$. When w lies between x and y in both partite sets in $B(H)$, the improper pairs (w, x), (w, y) are transformed to proper pairs by the exchange of x' and y'. When $x' < w' < y'$ and w'' is not between x'' and y'', there is a newly created proper pair and improper pair respectively, which compensate the effect of each other in the number of proper pairs in H. Similarly, there is no change in the number of proper pairs of the form (w, x) or (w, y) when $y'' < w'' < x''$ and w' is not between x' and y'. Hence, the exchange increases the number of proper pairs at least by one.

Analogously, we can exchange the positions of x'' and y'' in $<$ if for every pair of vertices a' and b' such that $b' < a'$ and $a'x'', b'y'' \in E(B(H))$, we have both $a'y''$ and $b'x''$ in $E(B(H))$. This exchange does not affect the Min-Max ordering of $B(H)$ and strictly increases the number of proper pairs as well.

Suppose, to the contrary, that we performed the above exchange for every improper pair as far as possible and still the Min-Max ordering is not proper. Then, there must be an improper pair x and y with $x' < y'$, $y'' < x''$ in $<$ which satisfies the following conditions: 1) there exist vertices c'' and d'', $d'' < c''$ such that $x'd'', y'c'' \in E(B(H))$ and at least one of $y'd''$ and $x'c''$ is missing in $B(H)$. 2) there exist vertices a' and b', $b' < a'$ such that $b'y'', a'x'' \in E(B(H))$ and at least one of $b'x''$ and $a'y''$ is missing in $B(H)$.

Note that a, d and x are distinct vertices in H since otherwise, the edges $a'x''$ and $x'd''$ induce $\overrightarrow{C_2}$ or a loop in H. With the same argument b,c and y are distict vertices in H. On the other hand, by transitivity of H, the edges $a'x''$ and $x'd''$ imply the existence of edge $a'd''$ in $E(B(H)$. Similarly, there is an edge $b'c''$ in $E(B(H))$. Note that we do not have

$x'x''$ and $y'y''$ in $E(B(H))$ as H is irreflexive.

We will consider cases according to the positions of a', b', c'', d'' in the ordering $<$. We remark the two edges $b'y''$ and $y'c''$ cannot cross each other. That is, they either satisfy $b' < y'$ and $y'' < c''$, or $y' < b'$ and $c'' < y''$, since otherwise there must be an edge $y'y''$ by the Min-Max property, which is a contradiction. Similarly, the two edges $a'x''$ and $x'd''$ cannot cross each other, since otherwise there must be an edge $x'x''$ by the Min-Max property, which is a contradiction. Hence we have either $x' < a'$ and $d'' < x''$, or $a' < x'$ and $x'' < d''$.

If $y' < b'$ and $c'' < y''$, then the positions of all vertices are determined immediately so that we have $x' < y' < b' < a'$ and $d'' < c'' < y'' < x''$. On the other hand, when $b' < y'$ and $y'' < c''$ we can place the edges $a'x''$ and $x'd''$ in two ways, namely to satisfy either $x' < a'$ and $d'' < x''$ or $a' < x'$ and $x'' < d''$ due to the argument in the above paragraph. In the latter case, however, the positions of all vertices are determined as well and this is just a converse of the case when $y' < b'$ and $c'' < y''$. Therefore we may assume that $x' < a'$ and $d'' < x''$ whenever $b' < y'$ and $y'' < c''$.

CASE 1 $b' < y'$ and $y'' < c''$ ($x' < a'$ and $d'' < x''$)

We say that $u \le v$ for $u, v \in V(B(H))$ if and only if $u < v$ or u is v. There are the following cases to consider. We show that in every case we have a contradiction.

Case 1-1 $y' < a'$ and $d'' < y''$

The two edges $a'd'', y'c'' \in E(B(H))$ imply the existence $y'd'' \in E(B(H))$ by the Min-Max property. The edge $y'd''$, however, together with $b'y'' \in E(B(H))$ enforce the edge $y'y'' \in E(B(H))$, which is a contradiction.

Case 1-2 $y' \le a'$ and $y'' \le d''(< x'')$

Case 1-2-1: $b' < x'$. We know that $a'd'' \in E(B(H))$. We can easily see $y'd'' \in E(B(H))$ since $<$ is a Min-Max ordering. (Note that $y'c'', a'd'' \in E(B(H))$.) Now consider the two vertices c'', d''. The existence of $y'd'' \in E(B(H))$ enforces $x'c'' \notin E(B(H))$. On the other hand, however, we must have the edge $x'c'' \in E(B(H))$ due to edges $b'c'', x'd'' \in E(B(H))$ and the Min-Max property, a contradiction.

Case 1-2-2: $x' \le b' < y'$. If $x' = b'$ or $y'' = d''$ then $x'y'' \in E(B(H))$ since $b'y'' \in E(B(H))$ and $x'd'' \in E(B(H))$. If $x' < b'$ and $y'' < d''$, it is easy to see that we have $x'y'' \in E(B(H))$ by the Min-Max property. (Note that $b'y'', x'd'' \in E(B(H))$.) With $a'x'', x'y'' \in E(B(H))$, the transitivity of H implies $a'y'' \in E(B(H))$. However, the two edges $y'c'', a'y'' \in E(B(H))$ and the Min-Max property enforce that $y'y'' \in E(B(H))$, a contradiction.

Case 1-3 $(x' <)a' \leq y'$ and $y'' \leq d''(< x'')$

Case 1-3-1: $x'' < c''$. We will show that we cannot avoid having the edge $x'c'' \in E(B(H))$. Once this is the case, the two edges $x'c''$ and $a'x''$ imply the existence of edge $x'x'' \in E(B(H))$, which is a contradiction.

If $x' \leq b'$, we again easily observe that $x'y'' \in E(B(H))$ and thus, $x'c'' \in E(B(H))$ by transitivity of H and $x'y'', y'c'' \in E(B(H))$. On the other hand, when $b' < x'$ we have $x'c'' \in E(B(H))$ again by the Min-Max property and the two edges $b'c'', x'd'' \in E(B(H))$.

Case 1-3-2: $c'' \leq x''$. We again easily observe that $y'x'' \in E(B(H))$ by the Min-Max property and the two edges $y'c'', a'x'' \in E(B(H))$.

If $x' \leq b'$, the Min-Max property implies $x'y'' \in E(B(H))$. Since H does not contain $\overrightarrow{C_2}$ as an induced subgraph, this is a contradiction.

If $b' < x'$, it is again implied that $b'x'' \in E(B(H))$ by transitivity of H and $b'y'', y'x'' \in E(B(H))$. The two edges $b'x'', x'd'' \in E(B(H))$ enforce $x'x'' \in E(B(H))$ by the Min-Max peoperty, which is a contradiction.

Case 1-4 $(x' <)a' \leq y'$ and $d'' < y''$

We will show that we cannot avoid having the edge $b'x'' \in E(B(H))$. Once this is the case, by focusing on two vertices a', b', and the arc $b'x'' \in E(B(H))$, we can easily see that $a'y'' \notin E(B(H))$. On the other hand, however, we must have the edge $a'y'' \in E(B(H))$ due to edges $a'd'', b'y'' \in E(B(H))$ and the Min-Max property, a contradiction.

If $x'' = c''$, we trivially have $b'x'' \in E(B(H))$. If $x'' < c''$, the Min-Max property and the two edges $b'c'', a'x'' \in E(B(H))$ imply $b'x'' \in E(B(H))$. If $x'' > c''$, the Min-Max property and the two edges $a'x'', y'c'' \in E(B(H))$ imply $y'x'' \in E(B(H))$. For $b'y'', y'x'' \in E(B(H))$, we again have $b'x'' \in E(B(H))$ by the transitiviety of H. This completes the argument.

CASE 2 $y' < b'$ and $c'' < y''$

In this case, we claim that H has one of O_i with $i = 1, 2, 3, 4$ as an induced subgraph. Remember that $x' < y' < b' < a'$ and $d'' < c'' < y'' < x''$. On the other hand, by transitivity of H, we have $a'd'', b'c'' \in E(B(H))$. Since $<$ is a bipartite Min-Max ordering, $\{a'x'', a'y'', a'c'', a'd'', b'y'', b'c'', b'd'', y'c'', y'd'', x'd''\} \subset E(B(H))$. Now by the primary assumptions on the pairs a', b' and c'', d'', we have $b'x'', x'c'' \notin E(B(H))$; hence $y'x'', x'y'' \notin E(B(H))$ as $<$ is a bipartite Min-Max ordering. It is easy to see from the set of edges existing in $B(H)$ that a, b, x, y, c, d are distinct vertices in H. Let us define $H' = H[\{a, b, x, y, c, d\}]$. As H' is acyclic we do not have symmetric arcs in H'.

From $E(B(H'))$, we have $\{ax, ay, ac, ad, by, bc, bd, xd, yc, yd\} \subset A(H')$ and $xy, yx, bx, xc \notin A(H')$. We can easily see that $xb \notin A(H')$, since otherwise from $xb, by \in A(H')$ and the transitivity of H' we must have $xy \in A(H')$, a contradiction. With the same argument we see that $ba, cx, dc \notin A(H')$. Therefore we can only add a subset of $S = \{ab, cd\}$ to the previous arc subset of H' mentioned above, and each such subset of S makes H' isomorphic to one of O_i with $i = 1, 2, 3, 4$, via the isomorphism g where $g(a) = 1, g(b) = 2, g(x) = 3, g(y) = 4, g(c) = 5, g(d) = 6$. ◇

6.2 Complexity

We begin this section with the following lemma showing that MinHOM(H) is NP-hard when $H = Q_i, i \in \{1, 2, 3, 4\}$, as depicted in Figure 6.1. Recall that the decision version of MinHOM(H) is the following problem: Given an input digraph D, together with nonnegative costs $c_i(u)$, $u \in V(D)$, $i \in V(H)$, and an integer k, decide if D admits a homomorphism to H of cost not exceeding k. An extended decision version of MinHOM(H) allows the costs to be negative, with a lower bound C, i.e., $c_i(u) \geq C, u \in V(D), i \in V(H)$. It is easy to see that the regular and the extended decision versions of MinHOM(H) are polynomially equivalent. In the following lemma, we show that the extended decision version of MinHOM(H) is NP-hard when $H = Q_i, i \in \{1, 2, 3, 4\}$, and hence so is the regular decision version.

Lemma 6.2.1 *Let H' be an arbitrary digraph over vertex set $\{1, 2, 3, 4, 5, 6\}$ such that*

$$\{13, 14, 15, 16, 24, 25, 26, 36, 45, 46\} \subseteq A(H'),$$

$$A(H') \subseteq \{12, 13, 14, 15, 16, 24, 25, 26, 36, 45, 46, 56\}.$$

Let H be H' or its converse. Then MinHOM(H) is NP-hard.

Proof: Recall that $\mathcal{I}_3$ is the independent set problem for three-partite graphs. We construct a polynomial time reduction from $\mathcal{I}_3$ to MinHOM(H). Let X be a graph whose vertices are partitioned into independent sets U, V, W, and let k be a given integer. We construct an instance of MinHOM(H) as follows: the digraph D is obtained from X by replacing each edge uv of X with $u \in U, v \in V$ by an arc uv, replacing each edge vw of X with $v \in V, w \in W$ by an arc vw, and replace each edge uw of X with $u \in U, w \in W$ by an arc $um_{uw}, n_{uw}m_{uw}, n_{uw}w$, where m_{uw}, n_{uw} are new vertices. Let us assign the costs as follows:

$c_2(u) = 0$, $c_1(u) = 1$, $c_3(v) = 0$, $c_4(v) = 1$, $c_5(w) = 0$, $c_6(w) = 1$, $c_3(m_{uw}) = c_3(n_{uw}) = -|V(X)|$, $c_i(m_{uw}) = c_i(n_{uw}) = |V(X)|$ for $i \neq 3$; apart from these, set all costs to $|V(X)|$.

We now claim that X has an independent set of size k if and only if D admits a homomorphism to H of cost $|V(X)| - k$. Let I be an independent set in D. We can define a mapping $f : V(D) \to V(H)$ as follows:

- $f(u) = 2$ for $u \in U \cap I$, $f(u) = 1$ for $u \in U - I$
- $f(v) = 3$ for $v \in V \cap I$, $f(v) = 4$ for $v \in V - I$
- $f(w) = 5$ for $w \in W \cap I$, $f(w) = 6$ for $w \in W - I$.
  When $uw \in E(X)$:
- If $f(u) = 2, f(w) = 6$ then set $f(m_{uw}) = 6, f(n_{uw}) = 3$.
- If $f(u) = 1$ and $f(w) \in \{5, 6\}$ then set $f(m_{uw}) = 3, f(n_{uw}) = 1$,

One can verify that f is a homomorphism from D to H, with cost $|V(X)| - k$.

Let f be a homomorphism of D to H of cost $|V(X)| - k$. Note that we cannot assign color 3 to both n_{uw} and m_{uw} simultaneously due to the arc $n_{uw}m_{uw}$. Hence, all the costs $c_{f(u)}(u)$, for the vertices $u \in V(X)$ are either zero or one, and for each edge $uw \in E(X)$, the costs $c_{f(m_{uw})}(m_{uw})$ and $c_{f(n_{uw})}(n_{uw})$ sum up to zero.

Let $I = \{u \in V(X) \mid c_{f(u)}(u) = 0\}$ and note that $|I| = k$. It can be seen that I is an independent set in D, as if for example $uw \in E(D)$, where $u \in I \cap U$ and $w \in I \cap W$ then $f(u) = 2$ and $f(w) = 5$, which implies that $f(m_{uw}) \neq 3$ and $f(n_{uw}) \neq 3$ contrary to f being a homomorphism of cost $|V(X)| - k$. $\diamond$

Let us now partition the class of quasi-transitive digraphs into two subclasses: the first is the class of acyclic quasi-transitive digraphs; the second is the class of quasi-transitive digraphs having at least one cycle. The following two lemmas cover the NP-hard cases of for these two subclasses of quasi-transitive digraphs H.

Lemma 6.2.2 *Let H be an acyclic quasi-transitive digraph. If $B(H)$ is not a proper interval bigraph or H contains at least one of O_i with $i = 1, \ldots, 4$ as an induced subgraph, then MinHOM(H) is NP-hard.*

Proof: If $B(H)$ is not a proper interval bigraph then MinHOM(H) is NP-hard by Proposition 3.2.1. If H contains at least one of O_i with $i = 1, \ldots, 4$ as an induced subgraph, then MinHOM(H) is NP-hard by Lemma 6.2.1 and Proposition 3.2.4.

Lemma 6.2.3 *Let H be a quasi-transitive digraph which is neither acyclic nor $\overrightarrow{C_2}$ nor an extension of $\overrightarrow{C_3}$. Then MinHOM(H) is NP-hard.*

Proof: We can easily observe that H has a directed cycle $\overrightarrow{C_k} = 0, 1, \ldots k-1, 0$ for $k \geq 2$. If this cycle is $\overrightarrow{C_2}$, then there is a vertex $k+1$ outside this cycle which is adjacent with one of the vertices in $\overrightarrow{C_2}$, as H is connected and is not $\overrightarrow{C_2}$. Furthermore, the quasi-transitivity of H enforces $k+1$ to be adjacent with both vertices in this cycle, and the cycle $\overrightarrow{C_2}$ together with $k+1$ induce a semicomplete digraph. By Theorem 1.2.2 and Proposition 3.2.4, MinHOM(H) is NP-hard in this case. Therefore, we assume that H does not have any symmetric arc hereafter.

Note that H cannot have an induced cycle $\overrightarrow{C_k} = 0, 1, \ldots, k-1, 0$ of length greater than 3. Otherwise, by quasi-transitivity of H a chord appears in the cycle, a contradiction. Hence we may consider only $\overrightarrow{C_3}$ as an induced cycle of H. Choose a maximal induced subdigraph H' of H which is an extension of $\overrightarrow{C_3}$ with partite sets X_1, X_2 and X_3. Clearly such a subdigraph H' exists.

By assumption, we have $H' \neq H$. Hence, there exists a vertex x in $H \backslash H'$ which is adjacent with at least one vertex of H'. Without loss of generality, suppose that $x \rightarrow 1$, for some $1 \in X_1$. As H is quasi-transitive, the vertex x must be adjacent with every vertex of X_2. There are two possibilities.

Case 1. $x \rightarrow 2$ for some $2 \in X_2$. Then x is adjacent with every vertex $3 \in X_3$ due to quasi-transitivity. Consider the subdigraph induced by x, $1, 2$ and a vertex of X_3. MinHOM(H) is NP-hard by Theorem 1.2.2 and Proposition 3.2.4.

Case 2. $X_2 \rightarrow x$. Then there is an arc between x and each vertex of X_1 by quasi-transitivity. If $1' \rightarrow x$ for some $1' \in X_1$, x is adjacent with every vertex of X_3 and MinHOM(H) is NP-hard by Theorem 1.2.2 and Proposition 3.2.4. Else if $x \rightarrow X_1$, there is a vertex $3 \in X_3$ which is adjacent with x since otherwise, $H' \cup \{x\}$ is an extension of $\overrightarrow{C_3}$, in contradiction to the maximality assumption. Again MinHOM(H) is NP-hard by Theorem 1.2.2 and Proposition 3.2.4. $\diamond$

The following theorem is the main result for quasi-transitive digraphs which easily follows from Theorem 6.1.1, Lemma 6.2.2, and Lemma 6.2.3.

Theorem 6.2.4 *Let H be a quasi-transitive digraph. Then MinHOM(H) is polynomial time solvable if H is one of the following digraphs.*

- *H is $\overrightarrow{C_2}$*

- *H is an extension of* $\overrightarrow{C_3}$
- *H is acyclic, $B(H)$ is a proper interval bigraph and H does not contain O_i with $i = 1, 2, 3, 4$ as an induced subdigraph.*

Otherwise, MinHOM(H) is NP-hard.

Chapter 7

Locally In-Semicomplete Digraphs

The class of locally in-semicomplete digraphs was first introduced in [6] as a generalization of tournaments. This class contains a wide variety of digraphs ranging from very sparse digraphs such as a directed path to very dense ones such as semicomplete digraphs. It has been shown in [3, 4, 5, 6] that the locally in-semicomplete digraphs have very nice properties leading to a nice reconstruction of them (See Theorem 7.2.4). In this chapter, we verify the minimum cost homomorphism conjecture for locally in-semicomplete digraphs, the largest class of irreflexive digraphs for which a dichotomy has been proved. It is worth noting that the reconstruction of locally in-semicomplete digraphs introduced in [3] plays a key role towards this dichotomy. This chapter is mostly based on [47].

Recall that a digraph H is locally in-semicomplete if for every vertex x of H, the in-neighbors of x induce a semicomplete digraph. Throughout this chapter, we always assume that the fixed digraph H is locally in-semicomplete unless stated otherwise. To show a MinHOM dichotomy, first of all, we partition the class of locally in-semicomplete digraphs into three subclasses: the first consists of all strongly connected locally in-semicomplete digraphs; the second consists of all non-strong locally in-semicomplete digraphs having at least one directed cycle; the third consists of all acyclic locally in-semicomple digraphs. We will verify Conjecture 3.1.4 for each of these subclasses separately.

7.1 Strong Locally In-Semicomplete Digraphs

We start to investigate the complexity of MinHOM(H) by considering the strongly connected case. Due to Proposition 3.2.4, in many cases it suffices to focus on small subgraphs and

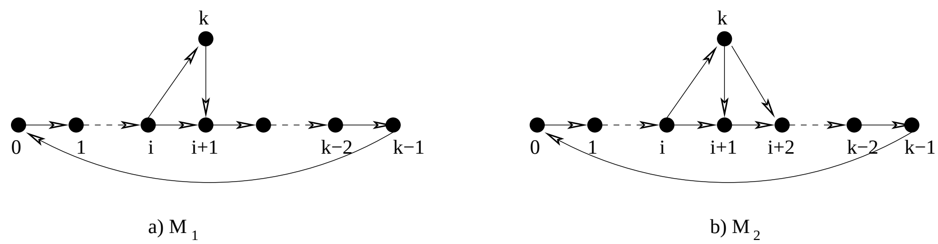

Figure 7.1: M_1 and M_2.

prove that they are NP-hard instead of looking at the whole digraph. In the arguments which will follow, we shall sometimes omit to mention Proposition 3.2.4 when it is obvious from the context. The following Lemma has been proved in [54].

Lemma 7.1.1 *[54] Let H be a digraph obtained from $\overrightarrow{C_k} = 0, 1, \ldots, k-1, 0, k \geq 2$, and an additional vertex k. MinHOM(H) is NP-hard if k is dominated by at least two vertices of the cycle and no other arc exists.*

Let M_1 and M_2 be the digraphs shown in Figure 7.1, obtained from a directed cycle $\overrightarrow{C_k} = 0, 1, \ldots, k-1, 0, k \geq 2$ and an extra vertex k. The following two lemmas are important tools for characterizing the strong locally in-semicomplete digraphs H with tractable MinHOM(H).

Lemma 7.1.2 *Let M_1 be a digraph obtained from $\overrightarrow{C_k} = 0, 1, \ldots k-1, 0, k \geq 2$, and an additional vertex k. MinHOM(M_1) is NP-hard if there are two consecutive vertices $i, i+1$ in $\overrightarrow{C_k}$ such that $i \rightarrow k$ and $k \rightarrow i+1$, and no other arc exists.*

Proof: Without loss of generality, assume that the vertex k is dominated by $k-1$ and dominates 0. To show that MinHOM(M_1) is NP-hard, we construct the digraph D which fulfills the conditions of Proposition 3.2.5. Let D be the following digraph.

$V(D) = \{u_0, u_1, \ldots, u_{k(k+1)-1}\} \cup \{u'', v'', u', v', u, v\}$

$A(D) = \{u_i u_{i+1} : 0 \leq i \leq k(k+1)-1\} \cup \{u_{2k-1}u', u'u, u_{k(k+1)-2}v', v'v\} \cup \{u''v'', u''u_0, v''u_0\}$

where the addition is taken modulo $k(k+1)$.

Observe that in any homomorphism f of D to M_1, we must have $f(u_0) = 0$. Once we assign the first k vertices $u_0, \ldots, u_{k-1}$ color $0, \ldots, k-1$, the vertex u_k is assigned with either color 0 or color k. If we opt for color 0, then through the whole remaining vertices

$u_k, \ldots, u_{k(k+1)-1}$ we must assign these vertices with colors along the $k-$cycle $0, 1 \ldots, k-1$ in M_1. Else if we opt for color k, then we must assign the whole remaining vertices with colors along the $(k+1)-$cycle $0, 1 \ldots, k$ in M_1. To see this, suppose to the contrary that we assign the vertices $u_0, \ldots, u_{k(k+1)-1}$ in M_1 with colors along the $k-$cycle s times and with colors along the $(k+1)-$cycle t times, where $0 < t < k$. Then, we have the following equation.

$k \cdot (k+1) = s \cdot k + t \cdot (k+1)$

which again implies

$(k+1)(k-t) = s \cdot k$

Knowing that the least common multiple of k and $k+1$ is $k(k+1)$, this leads to a contradiction. Hence, $(f(u_0), \ldots, f(u_{k(k+1)-1}))$ coincides with one of the following sequences:

$(0, 1, \ldots, k-1, \ldots, 0, \ldots, k-1)$: the sequence $0, 1, \ldots, k-1$ appears $k+1$ times; or

$(0, 1, \ldots, k, \ldots, 0, \ldots, k)$: the sequence $0, 1, \ldots, k$ appears k times.

If the first sequence is the actual one, then we have $f(u_{2k-1}) = k-1$, $f(u') \in \{0, k\}$, $f(u) \in \{0, 1\}$, $f(u_{k(k+1)-2}) = k-2$, $f(v') = k-1$ and $f(v) \in \{0, k\}$. If the second one is the actual one, then we have $f(u_{2k-1}) = k-2$, $f(u') = k-1$, $f(u) \in \{0, k\}$, $f(u_{k(k+1)-2}) = k-1$, $f(v') \in \{0, k\}$ and $f(v) \in \{0, 1\}$. In both cases, we can assign both of u and v color 0. Furthermore, by choosing the right sequence, we can color one of u and v with color 1 and the other with color 0. However, we cannot assign color 1 to both u and v in a homomorphism. Let $y = 0, x = 1$. Then x, y, u, v and the digraph D fulfill the conditions of Proposition 3.2.5. $\diamond$

Lemma 7.1.3 *Let M_2 be a digraph obtained from $\overrightarrow{C_k} = 0, 1, \ldots k-1, 0, k \geq 3$, and an additional vertex k. MinHOM(M_2) is NP-hard if there are three consecutive vertices $i, i+1, i+2$ such that $i \rightarrow k$ and $k \rightarrow \{i+1, i+2\}$, and no other arc exists.*

Proof: Without loss of generality, assume that vertex k is dominated by $k-1$ and dominates 0 and 1. To show that MinHOM(M_2) is NP-hard, we construct a digraph D which fulfills the conditions of Proposition 3.2.5. Let D be defined as in the proof of Lemma 7.1.2.

Observe that in any homomorphism f of D to M_2, we must have $f(u_0) = 0$. And also by the same argument discussed in the proof of Lemma 7.1.2, the vertices of the $k(k+1)-$cycle in D must be assigned with vertices either along the $k-$cycles, $0, 1, \ldots, k-1$ and $k, 1, \ldots, k-1$, or the $(k+1)-$cycle $0, 1, \ldots, k$ in M_2. If the vertices of $k(k+1)-$cycle in D are assigned with $k-$cycles in M_2, then we have $f(u_{2k-1}) = k-1$, $f(u') \in \{0,k\}$, $f(u) \in \{0,1\}$, $f(u_{k(k+1)-2}) = k-2$, $f(v') = k-1$ and $f(v) \in \{0,k\}$. If the vertices of $k(k+1)-$cycle in D are assigned with $(k+1)-$cycles in M_2, then we have $f(u_{2k-1}) = k-2$, $f(u') = k-1$, $f(u) \in \{0,k\}$, $f(u_{k(k+1)-2}) = k-1$, $f(v') \in \{0,k\}$ and $f(v) \in \{0,1\}$. In both cases, we can assign both of u and v color 0. Furthermore, by choosing the right sequence, we can color one of u and v with color 1 and the other with color 0. However we cannot assign color 1 to both u and v in a homomorphism. Let $y = 0, x = 1$. Then x, y, u, v and the digraph D fulfill the conditions of Proposition 3.2.5. $\diamond$

The next theorem characterizes the tractable cases of strongly connected locally in-semicomplete digraph.

Theorem 7.1.4 *Let H be a strongly connected locally in-semicomplete digraph. Then MinHOM(H) is polynomial time solvable if H is a directed cycle. Otherwise, MinHOM(H) is NP-hard.*

Proof: If H is $\overrightarrow{C_k}$, then it has a k-Min-Max ordering, and thus MinHOM(H) is polynomial time solvable. Hence assume that H is nontrivial and is not a directed cycle. Then it contains at least one induced cycle $\overrightarrow{C_k}$, $k \geq 2$, and a vertex $x \notin V(\overrightarrow{C_k})$ which dominates at least one vertex, say z, of $\overrightarrow{C_k}$. If $k = 2$, x is adjacent with both vertices of $\overrightarrow{C_2}$ and MinHOM(H) is NP-hard by Theorem 1.2.2. If $k = 3$ then there should be an arc between x and the vertex z' of $\overrightarrow{C_3}$ that dominates z . For any combination of arcs between x and the third vertex z'' of $\overrightarrow{C_3}$, x, z, z', z'' induced a digraph which is either O_1 or M_1 or M_2. Thus, MinHOM(H) is NP-hard. Else if $k \geq 4$, the following observations can be made on H since it is locally in-semicomplete and $\overrightarrow{C_k}$ is induced.

(a) The vertex x cannot be domintated by more than two vertices of $\overrightarrow{C_k}$. Otherwise $\overrightarrow{C_k}$ has a chord, contrary to the assumption that $\overrightarrow{C_k}$ is an induced directed cycle.

(b) If x is dominated by two vertices of $\overrightarrow{C_k}$, then these vertices appear consecutively on $\overrightarrow{C_k}$. Otherwise $\overrightarrow{C_k}$ has a chord, leading to the same contradiction as Observation (a).

By Observation (a), we have the following three cases according to the number of vertices by which x is dominated by. We show that in each case H inevitably has an induced subgraph for which the problem is NP-hard.

Case 1: No vertex of $\overrightarrow{C_k}$ dominates x.

Recall that x dominates $z \in \overrightarrow{C_k}$. It is straightforward to see that x dominates every vertex of the cycle since H is locally in-semicomplete. Hence, according to Lemma 7.1.1, MinHOM(H) is NP-hard.

Case 2: Only one vertex, say y, of $\overrightarrow{C_k}$ dominates x.

We may assume without loss of generality that z is the only vertex which is dominated by x among the vertices from the (z, y)-path (meaning the directed path from z to y) on $\overrightarrow{C_k}$. Observe that x dominates every vertex of (y, z)-path on $\overrightarrow{C_k}$ except for y. Consider the subgraph induced by the union of (z, y)-path on $\overrightarrow{C_k}$, x and the immediate predecessor w of z on $\overrightarrow{C_k}$. If $y = w$, i.e., $y \rightarrow z$ on $\overrightarrow{C_k}$, MinHOM(H) is NP-hard by Lemma 7.1.2. Else if $y \neq w$ but $y \rightarrow w$, MinHOM(H) is NP-hard by Lemma 7.1.3. (Note that we have the converse of the digraphs introduced in Lemma 7.1.3). Otherwise MinHOM(H) is NP-hard by Lemma 7.1.2.

Case 3: Exactly two vertices, say y_1 and y_2, of $\overrightarrow{C_k}$ dominate x.

By Observation (b), we have $y_1 \rightarrow y_2$. Again we may assume that z is the only vertex which is dominated by x among the vertices from the (z, y_1)-path on $\overrightarrow{C_k}$. Consider the subgraph induced by the union of (z, y_1)-path on $\overrightarrow{C_k}$, x and y_2. If $y_2 \rightarrow z$, MinHOM(H) is NP-hard by Lemma 7.1.3. Otherwise MinHOM(H) is NP-hard by Lemma 7.1.2. ⋄

7.2 Non-Strong Locally In-Semicomplete Digraphs

Before we start to show a dichotomy for non-Strong Locally in-semicomplete digraphs, let us define out-branching and path-mergeability. We say that a digraph D is a *directed tree* if $U(D)$ (meaning underlying graph of D) is a tree. An *oriented tree* is a directed tree without any $\vec{C}_2$. An *out-tree* is an oriented tree T with only one vertex r of in-degree zero (called *the root of* D). A subgraph T of a digraph D is a *spanning oriented tree* of D if $U(T)$ is a spanning tree in $U(D)$ and T is an oriented tree. A subgraph T of digraph D is an *out-branching* if T is a spanning out-tree of D. The following is a basic characterization of digraphs with out-branchings.

Proposition 7.2.1 *[3] A connected digraph D contains an out-branching if and only if D has exactly one initial strong component, or equivalently, $SCD(D)$ has only one vertex of in-degree zero.*

A digraph D is *path-mergeable* if for any choice of vertices x, y of $V(D)$ and any pair of internally disjoint (x, y)-paths P,Q, there exists an (x, y)-path R in D such that $V(R) = V(P) \cup V(Q)$. The following two propositions are due to Bang-Jensen, see [4].

Proposition 7.2.2 *[4] Let D be a digraph which is path-mergeable and let $P = xx_1 \dots x_r y$, $P' = xy_1 \dots y_s y$, $r, s \geq 0$ be internally disjoint (x,y)-paths in D. The paths P and P' can be merged into one (x,y)-path P^* such that vertices from P(respectively, P') remain in the same order as on that path.*

Proposition 7.2.3 *[4] Every locally in-semicomplete digraph is path-mergeable.*

We remind the reader of the definitions of $SCD(H)$ from Chapter 1. $SCD(H)$ is obtained by contracting each strong component H_i of H into a single vertex v_i and placing an arc from v_i to v_j, $i \neq j$ if and only if there is an arc from H_i to H_j. The next theorem was first proved for locally in-tournaments in [5] and later slightly modified into a more general statement in [6].

Theorem 7.2.4 *[6] Let H be a connected non-strong locally in-semicomplete digraph. Then the following holds for H.*

> *(a) Let A and B be distinct strong components of H. If a vertex $a \in A$ dominates a vertex in B, then $a \rightarrow B$.*
> *(b) H has only one initial strong component, or equivalently $SCD(H)$ has an out-branching.*

Corollary 7.2.5 *Let H be a connected non-strong locally in-semicomplete digraph and consider the strong components of it. If H has a non-trivial initial strong component other than a directed cycle or a non-trivial non-initial strong component, then MinHOM(H) is NP-hard.*

Proof: By Theorem 7.1.4, every strong component of H must be a directed cycle, otherwise, MinHOM(H) is NP-hard. Now, suppose a non-initial strong component B is nontrivial, i.e., $|B| \geq 2$. It follows from Theorem 7.2.4 that there exists a vertex a such that $a \rightarrow B$. Choose an induced cycle C from B and let H' be the subgraph induced by $V(C) \cup \{a\}$. Then MinHOM(H') is NP-hard by Lemma 7.1.1. $\diamond$

Theorem 7.2.4 and Corollary 7.2.5 above tell us that if MinHOM(H) is polynomial time solvable for a non-strong locally in-semicomplete digraph H, the structure of H is globally 'acyclic' once we shrink the initial strong component to a vertex.

In the next two subsections we will show that a locally in-semicomplete digraph H for which MinHOM(H) is polynomial time solvable has a special structure.

7.2.1 Locally In-semicomplete Digraphs Having a Cycle

Let $\mathcal{N}$ be the class of connected non-strong locally in-semicomplete digraphs having a non-trivial directed cycle C as an initial strong component where the other strong components are trivial.

Lemma 7.2.6 *Let O_1 be a digraph obtained from a directed cycle $\overrightarrow{C_k} = x_1x_2 \dots x_kx_1$, $k \geq 2$, and the digraph D with vertex set $x_{k+m}, x_{k+m+1}, x_{k+m+2}, m \geq 0$, and arc set $\{x_{k+m}x_{k+m+1}, x_{k+m}x_{k+m+2}, x_{k+m+1}x_{k+m+2}\}$ by joining x_k to x_{k+m} with the directed path $x_kx_{k+1} \dots x_{k+m}$ (see Figure 7.2.). Then MinHOM(O_1) is NP-hard.*

Proof: To show that MinHOM(O_1) is NP-hard, we construct the digraph D which fulfills the conditions of Proposition 3.2.5. Consider the digraph D, shown in Figure 7.2. D consists of a set of special vertices $\{u, v, u_0\} \cup \{u_i, u'_i : 1 \leq i \leq k-4\} \cup \{v_1, v_2\}$, and a set of directed paths existing between them as follows:

- for every u'_i, $1 \leq i \leq k-4$, there is a directed path of length $m+2$ from u_{i-1} and a directed path of length $m+1$ from u_i to u'_i;
- there is a directed path of length $m+2$ from u_{k-4} to u;
- there is a directed path of length $k+m-1$ from u_0 to v_1;
- there is a directed path of length 1 from v_2 to v_1;
- there is a directed path of length 2 from v_2 to v.

Let $x = x_{k+m}$ and $y = x_{k+m+2}$. In what follows, we show that x, y, u, v and the digraph D fulfill the conditions (a)-(d) in the Proposition 3.2.5. Let f be a homomorphism with $f(v) = x_{k+m}$. Then $f(v_2) = x_{k+m-2}$, $f(v_1) = x_{k+m-1}$, $f(u_0) = x_k$. On the other hand let h be a homomorphism with $h(u) = x_{k+m}$. Then $h(u_{k-4}) = x_{k-2}$, $h(u_{k-5}) = x_{k-3}, \dots$

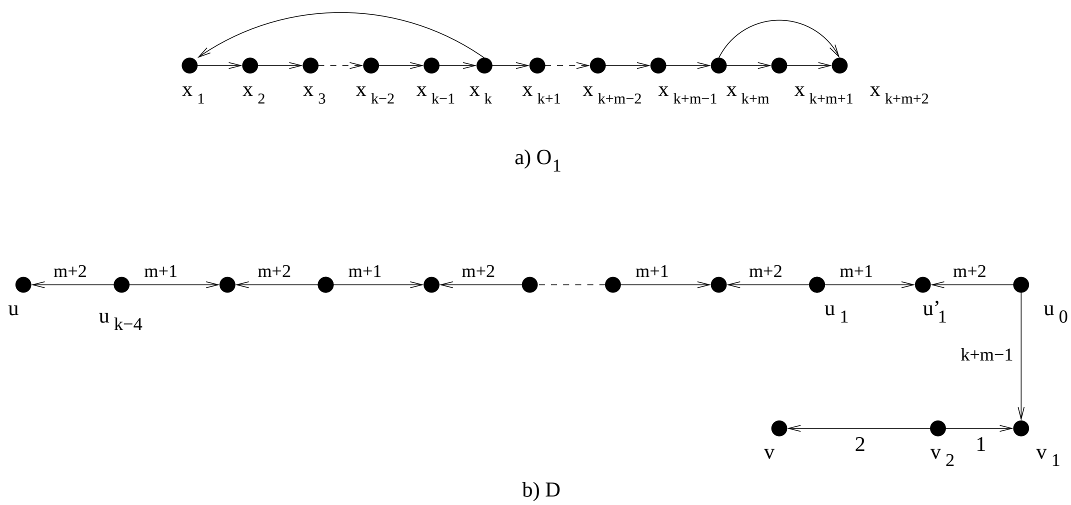

Figure 7.2: (a) O_1. (b) The digraph D. Note that each arc represents a directed path with the length marked beside it.

and $h(u_0) = x_2$. Hence, condition (b) is satisfied. The followings are homomorphisms that satisfy conditions (a),(c) and (d).

(a) $f(u_0) = x_3$,$f(u_1) = x_4, \ldots, f(u_{k-4}) = x_{k-1}$ and $f(u) = x_{k+m+2}$. $f(v_1) = x_{k+m+2}$, $f(v_2) = x_{k+m}$ and $f(v) = x_{k+m+2}$.

(c) $f(u_i) = x_k$ for each i and $f(u) = x_{k+m+2}$. $f(v_1) = x_{k+m-1}$, $f(v_2) = x_{k+m-2}$ and $f(v) = x_{k+m}$.

(d) $f(u_0) = x_2$,$f(u_1) = x_3, \ldots, f(u_{k-4}) = x_{k-2}$ and $f(u) = x_{k+m}$. $f(v_1) = x_{k+m+1}$, $f(v_2) = x_{k+m}$ and $f(v) = x_{k+m+2}$.

$\diamond$

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the family of digraphs, introduced in Lemma 7.2.6 and Lemma 7.1.1, respectively. The following theorem is the main result of this section.

Theorem 7.2.7 *Consider a digraph $H \in \mathcal{N}$. If H does not contain any digraph in $\mathcal{O}_1$ and $\mathcal{O}_2$ as an induced subgraph, then H has a k-Min-Max ordering for some $k \geq 2$ and MinHOM(H) is polynomial time solvable. Otherwise, MinHOM(H) is NP-hard.*

Proof: It is easily derived from Lemma 7.2.6, and Lemma 7.1.1, that if H contains a digraph in $\mathcal{O}_1$ or $\mathcal{O}_2$ as an induced subgraph, then MinHOM(H) is NP-hard.

We say that H has a bypass if there exist two vertices u and v in H such that there are two different directed paths from u to v. First of all, we will show that by excluding the digraphs in $\mathcal{O}_1$ and $\mathcal{O}_2$, H has no bypass, and later we will see if H does not have any bypass, then it has a k-Min-Max ordering, where k is the length of the directed cycle $\overrightarrow{C_k}$ which is the initial strong component. (See Corollary 7.2.5.)

Suppose that there exists a pair u, v, which makes a bypass. Note that since $\overrightarrow{C_k}$ is the initial strong component, no vertex of H outside of $\overrightarrow{C_k}$ can dominate the vertices of $\overrightarrow{C_k}$. It follows that v can not be a vertex of $\overrightarrow{C_k}$, since if v is a vertex of $\overrightarrow{C_k}$, all vertices of two different paths including u must be a vertex of $\overrightarrow{C_k}$, which means there are two different directed paths from u to v on $\overrightarrow{C_k}$, a contradiction.

Let us assume that there are two directed paths P and Q from u to v with sequences $ux_1x_2 \ldots x_pv$ and $uy_1y_2 \ldots y_qv$, respectively. Moreover, we will choose a pair u, v such that $x_1 \neq y_1$. Now, two following cases may happen:

Case 1: Either x_1 or y_1 is a vertex of $\overrightarrow{C_k}$.

If x_1 is a vertex of $\overrightarrow{C_k}$, then we can easily show that u is the predecessor of x_1 on $\overrightarrow{C_k}$, while y_1 is not a vertex of $\overrightarrow{C_k}$. Since a locally in-semicomplete digraph is path-mergeable, using Proposition 7.2.2, we can find a new (u, v)-path R on H such that it includes all vertices of P and Q and the vertices of P and Q remain in the same relative order. In the path R, the vertex x_1 must be immediately after u, as otherwise some vertex not in $\overrightarrow{C_k}$ dominates it. Thus, there exists a directed path of the form $ux_1 \ldots x_iy_1, i \geq 1$. If $i = 1$ then x_1 dominates y_1 and we have a digraph in $\mathcal{O}_2$ as an induced subgraph. Otherwise, since H is locally in-semicomplete, u dominates all $x_j, 1 \leq j \leq i$ including x_2, where again H contains a digraph in $\mathcal{O}_2$.

Case 2: Neither x_1 nor y_1 is a vertex of $\overrightarrow{C_k}$.

There is a directed path of length $m', m' \geq 0$, from a vertex s of $\overrightarrow{C_k}$ to u since $SCD(H)$ has an out-branching. On the other hand, with the same argument as Case 1, there must exist a directed path of the form $ux_1 \ldots x_iy_1, i \geq 1$ or $uy_1 \ldots y_jx_1, j \geq 1$. Hence, it follows that there exists a transitive tournament of length three, a *transitive triple*, starting from u as H is locally in-semicomplete. Let us choose a transitive triple, for which the starting vertex has the minimum distance $m, m \geq 0$ from the cycle $\overrightarrow{C_k}$. We will refer to this transitive triple as minimal transitive triple. It is easy to check that the directed path from $\overrightarrow{C_k}$ to

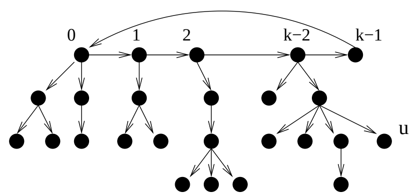

Figure 7.3: H without any bypass.

this transitive triple is an induced path, since otherwise this transitive triple is not minimal. Moreover, no vertex other than the starting vertex of the path can dominate a vertex of this path and the minimal transitive triple, as otherwise we have case 1. Hence, H contains a digraph in $\mathcal{O}_1$ as an induced subgraph, a contradiction.

Since H does not have any bypass, H has out-trees $T_0, \ldots, T_{k-1}$, which are the components of the digraphs remaining after removing the k arcs of the cycle $\overrightarrow{C_k}$ as the initial strong component. Let T_u be the out-tree to which u belongs. (See Figure 7.3.)

Suppose r is a fixed vertex in $\overrightarrow{C_k}$. We will denote the vertex, which dominates u by $P(u)$, and the distance of u from this fixed vertex r of $\overrightarrow{C_k}$ by $l(u)$ or level of u.

It is easy to see that H retracts to $\overrightarrow{C_k}$. In fact, each vertex $u \in V(T_i)$ for which $(i + l(u)) \equiv s \pmod{k}$ is mapped to the vertex $s \in \overrightarrow{C_k}$ in this retraction. Hence we can partition all vertices of H to k independent sets $V_1, V_2, \ldots V_k$ where V_i consists of the vertices of H which are mapped to vertex i of $\overrightarrow{C_k}$.

Let us order the vertices of each independent set V_i by the linear ordering $\ll$ as follows:

- if $l(u) < l(v)$, we have $u \ll v$.
- if $l(u) = l(v)$, $P(u) \neq P(v)$, and $P(u) \ll P(v)$, we have $u \ll v$.
- if $l(u) = l(v)$ and $P(u) = P(v)$, we arbitrarily order u and v.

It remains to see that this ordering is a k-Min-Max ordering. To do this, we will show that we can not have the following situation in the ordering: $m, n \in V_i$ and $r, s \in V_{i+1}$, where $m \ll n$, $s \ll r$, and $mr, ns \in A(H)$.

Suppose such a case occurs. By definition, we know that $P(r) = m$, and $P(s) = n$. Suppose first that $l(s) < l(r)$ then it is trivial to see that $l(n) < l(m)$; hence $n \ll m$, contrary to our assumption that $m \ll n$. Suppose next that $l(r) = l(s)$. We have $P(r) \neq P(s)$ and $P(r) \ll P(s)$, and so $r \ll s$, a contradiction. $\diamond$

7.2.2 Acyclic Locally In-Semicomplete Digraphs

Let $\mathcal{A}$ be the class of connected acyclic locally in-semicomplete digraphs. For any $H \in \mathcal{A}$ we have $SCD(H) = H$. Thus, by Theorem 7.2.4, H, and any induced connected subgraph of H have an out-branching, and only one vertex of in-degree zero. However, any digraph H from this family may have multiple out-branchings. We consider a particular out-branching of H, denoted by $T(H)$, constructed on the same vertex set $V(H)$ recursively as follows: Let $H \in \mathcal{A}$ and r be the unique vertex of H of in-degree zero. Let $C_1, \ldots, C_i$ be the components of $H - r$ and r_i be the unique vertex of C_i of in-degree zero. (Note that each $C_i \in \mathcal{A}$.) We place the arcs $rr_1, rr_2, \ldots, rr_i$ in $T(H)$ and add to $T(H)$ all arcs of $T(C_1), T(C_2), \ldots, T(C_i)$.

We say that a non-trivial H has *one stem* (or we also say H is *one-stem*), if $i = 1$ in the above definition for $T(H)$. Otherwise, H is *multi-stem*. In this subsection, we verify Conjecture 3.1.4 separately for one-stem and multi-stem H.

The *level* of a vertex x, denoted by $l(x)$, is the length of the (r, x)-path in $T(H)$. The *parent* of a vertex u, denoted by $P(u)$, is a unique vertex which dominates u in $T(H)$. A *child* of a vertex u is a vertex v which is dominated by u in $T(H)$. A vertex v is an *ancestor* of vertex u, if there is a (v, u)-path from v to u in $T(H)$. For any $u, v \in V(H)$, the *join* of u and v, denoted by join(u ,v), is the maximum level common ancestor of u and v in $T(H)$ (Note that this vertex is unique in $T(H)$). A *subjoin*$_v$*(u)* is a vertex w which is in the directed path between join(u ,v) and u in $T(H)$. The following fact is easily derived from the definitions.

Observation 7.2.8 *Let H be in $\mathcal{A}$ and $uv \in A(H)$. Then u is an ancestor of v (in $T(H)$).*

Proof: Suppose the contrary that u is not an ancestor of v in $T(H)$. Note that v is definitely not an ancestor of u on $T(H)$, as otherwise H has a cycle. Thus, neither u nor v is the ancestor of the other one in $T(H)$. So, there are two disjoint paths P and Q from join(u,v) to u and v in $T(H)$, respectively. It is easy to see that P and Q are the longest paths from join(u,v) to u and v in H. Since $uv \in A(H)$ and H is path-mergeable, there is a path R in H from join(u,v) to v such that it includes all vertices of P and Q; hence R is the longest path from join(u,v) to v in H, contrary to the assumption that Q is the longest path from join(u,v) to v. $\diamond$

We can easily see by Observation 7.2.8 that if $l(u) \geq l(v)$ then $uv \notin A(H)$. The vertex v is the *minimal dominating ancestor* of u in $T(H)$, denoted by $MDA(u)$, if $v \rightarrow u$, and for all vertices $v' \neq v$ that $v' \rightarrow u$, $l(v') > l(v)$.

The following lemma proved in [53], is extensively used in this subsection.

Lemma 7.2.9 *[53] Let F_1 be given by $V(F_1) = \{x_1, x_2, x_3, x_4\}$, $A(F_1) = \{x_1x_2, x_2x_3, x_3x_4, x_1x_4, x_2x_4\}$. Then MinHOM($F_1$) is NP-hard.*

Since H is acyclic and locally in-semicomplete, the following observation is trivial.

Observation 7.2.10 *Let H be in $\mathcal{A}$, $uv \in A(H)$ and X be the set of all vertices between u and v in $T(H)$ (including u and v). If H does not contain F_1 as an induced subgraph, then X induces a transitive tournament in H.*

CASE 1: One-Stem Digraphs

As H has only one stem then r can not be the join of any pair u and v in H. So, for each pair u, v, the join(u, v) has a parent in H. In the following four Lemmas, we assume that H is in $\mathcal{A}$ and it has only one stem.

Lemma 7.2.11 *Let T be a transitive tournament with at least two vertices and the unique source v_1, and $F_2(k)$ be the digraph obtained from T with k vertices and three other vertices u_1, u_2, and u_3 such that $V(T) \rightarrow \{u_1, u_2\}$, $(V(T) - v_1) \rightarrow u_3$, $u_1 \rightarrow u_2$, and there is no other arc in $A(F_2(K))$. Then MinHOM($F_2(K)$) is NP-hard.*

Proof: See Section 7.3. ◇

Lemma 7.2.12 *Let F_3 be given by $V(F_3) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $A(F_3) = A_1 \cup A_2$, where $A_1 = \{x_1x_2, x_2x_3, x_2x_4, x_3x_4, x_2x_5, x_5x_6, x_2x_6\}$ and A_2 is any subset of $\{x_1x_3, x_1x_4, x_1x_5, x_1x_6\}$. Then MinHOM($F_3$) is NP-hard.*

Proof: See Section 7.3. ◇

Lemma 7.2.13 *Let $\mathcal{F}_4$ denote the family of all digraphs H satisfying all the following conditions for $u, v \in V(H)$:*

- *$l(u) - l(x) = l(v) - l(x) = 2$, where x is the join(u,v);*
- *$xu \in A(H) \setminus A(T(H))$;*
- *$l(MDA(P(v))) < l(MDA(P(u)))$.*

Then for any H in $\mathcal{F}_4$, MinHOM(H) is NP-hard.

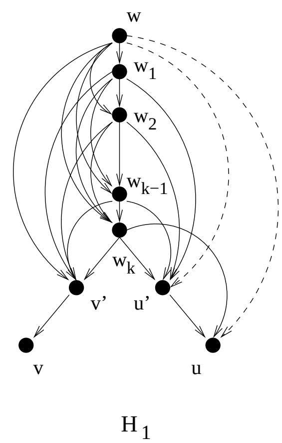

Figure 7.4: H_1. The dashed arcs are missing. There may be additional arcs st, where s is an ancestor of t in $T(H_1)$.

Proof: As $l(MDA(P(v))) < l(MDA(P(u)))$, there is always a common ancestor of $P(u)$ and $P(v)$ in $T(H)$ such that it dominates $P(v)$ in H, but it does not dominate $P(u)$ in H. From now on, we will denote $P(u), P(v)$, and this common ancestor by u', v', and w. We will also assume that w has the maximum level among such ancestors in $T(H)$. Let us enumerate all common ancestors of u', v' in $T(H)$ with level more than $l(w)$ by $w_1, \ldots, w_k$ (Note that $w_k = x$). If H contains F_1 as an induced subgraph then MinHOM(H) is NP-hard by Lemma 7.2.9, otherwise $w, w_1 \ldots, v'$, and $w_1, w_2, \ldots, u'$ induce two transitive tournaments by Observation 7.2.10. Since w does not dominate u', it also does not dominate u by the same observation. This leads us to the structure shown in Figure 7.4.a, denoted as H_1. In this figure, there may be additional arcs st, where s is an ancestor of t in $T(H_1)$. We will prove in Section 7.3 that MinHOM(H_1) is NP-hard.

Lemma 7.2.14 *Let $\mathcal{F}_5$ denote the family of all digraphs H such that H contains vertices u, v where $l(u) = l(v)$, and arc $wu \in A(H) \setminus A(T(H))$, where w is a subjoin$_v$(u). Then for any H in $\mathcal{F}_5$, MinHOM(H) is NP-hard.*

Proof: Among such vertices u and v in H, we choose u and v so that they have the minimum level. Let us enumerate all ancestors of u and v in $T(H)$ from join(u,v) to u and v by $u_1, u_2, \ldots, u_k$ and $v_1, v_2, \ldots, v_k$, $k \geq 2$, respectively. (see Figure 7.5.) Depending on whether $v_{k-1}v$ is in the arc set or not, we will have either H_2 or H_3 in Figure 7.5. For

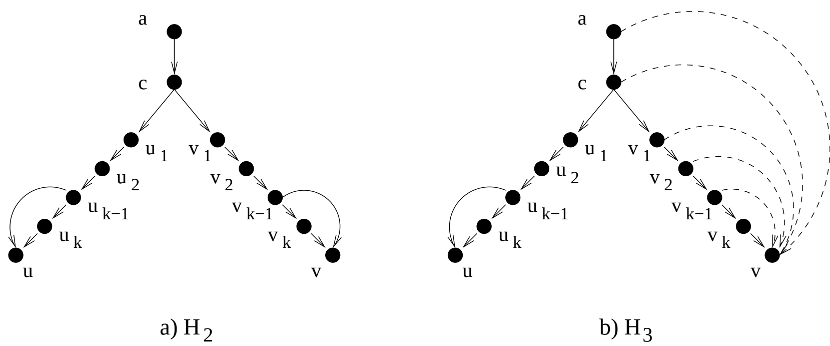

Figure 7.5: (a) H_2 (b) H_3. The dashed arcs are missing. There may be additional arcs st, where s is an ancestor of t in $T(H_2)$ or $T(H_3)$.

the later case H_3, if H contains F_1 as an induced subgraph then MinHOM(H) is NP-hard by Lemma 7.2.9, otherwise since $v_{k-1}v$ is missing, no other vertex than v_k dominates v by Observation 7.2.10. In both cases, there may be additional arcs st, where s is an ancestor of t in $T(H_2)$ or $T(H_3)$. We will prove in Section 7.3 that MinHOM(H_2) and MinHOM(H_3) is NP-hard. $\diamond$

Lemma 7.2.15 *Let $\mathcal{F}_6$ denote the family of all digraphs H satisfying all the following conditions for $u, v \in V(H)$:*

- *$l(u) = l(v)$, and $P(u) = P(v)$;*
- *$l(MDA(v)) < l(MDA(u))$;*
- *u and v lie respectively on the path P and Q of $T(H)$ such that there is an arc $v'v'' \in A(H) \setminus A(T(H))$ on Q, where $l(v) \leq l(v') = l(v'') - 2$, and a vertex u' in P, where $l(v') + 1 = l(u') = l(v'') - 1$.*

Then for any H in $\mathcal{F}_6$, MinHOM(H) is NP-hard.

Proof: As $l(MDA(v)) < l(MDA(u))$, there is always a common ancestor of u and v in $T(H)$ such that it dominates v, but it does not dominate u. From now on, we will denote this common ancestor by w. We will also assume that w has the maximal level among such ancestors in $T(H)$. Let us enumerate all common ancestors of u, v, which have level more that $l(w)$ by $w_1, \ldots, w_k$. If H contains F_1 as an induced subgraph then MinHOM(H)

is NP-hard by Lemma 7.2.9, otherwise $w, w_1 \ldots, v$, and $w_1, w_2, \ldots, u$ induce two transitive tournaments by Observation 7.2.10. Since w does not dominate u, it also does not dominate any vertex of P with level more than $l(u)$ by the same Observation. Note that w_i also does not dominate any vertex of P with level more than $l(u)$, as otherwise we will have one of the forbidden digraphs in Lemma 7.2.13, for which MinHOM(H) is NP-hard. On the other hand, the directed path p' from w_k to u' is also an induced path, since otherwise, one of the digraphs of Lemma 7.2.14 appears. This leads us to a structure like Figure 7.6.a, denoted as H_4. There may be additional arcs st, where s is an ancestor of t in $T(H_4)$. We will prove in Section 7.3 that MinHOM(H_4) is NP-hard. $\diamond$

Lemma 7.2.16 *Let $\mathcal{F}_7$ denote the family of all digraphs H satisfying all the following conditions for $u, v \in V(H)$:*

- *$l(u) = l(v)$, and $P(u) = P(v)$;*
- *$P(v)$ dominates a child of v;*
- *u and v lie respectively on path P and Q of $T(H)$ such that there is an arc $v'v'' \in A(H) \setminus A(T(H))$ on Q, where $l(v) \leq l(v') = l(v'') - 2$, and a vertex u' in P, where $l(v') + 1 = l(u') = l(v'') - 1$.*

Then for any H in $\mathcal{F}_7$, MinHOM(H) is NP-hard.

Proof: As H has only one stem, there is always a vertex w which dominates join(u,v) in $T(H)$. We will refer to join(u,v) by x. The directed path P' from x to u' is an induced directed path, otherwise we will either have F_3 or one of the forbidden digraphs of Lemma 7.2.14, for which we already proved MinHOM(H) is NP-hard. If H contains F_1 as an induced subgraph then MinHOM(H) is NP-hard by Lemma 7.2.9, otherwise since P' is an induced directed path, w can not dominate any vertex of P' other than w and x by Observation 7.2.10. According to that, we will have one of the structures H_5 or H_6, shown in Figure 7.6.b or c. There may be additional arcs st, where s is an ancestor of t in $T(H_5)$. We will prove in Section 7.3 that MinHOM(H) is NP-hard, when H is H_5 or H_6. $\diamond$

We now handle all the forbidden subgraphs from Lemmas 7.2.9, 7.2.11, 7.2.12, 7.2.13, 7.2.14, 7.2.15, 7.2.16. Let $\mathcal{F}_1 = \{F_1\}$, $\mathcal{F}_2 = \{F_2(k) : k = 2, 3, \ldots\}$, $\mathcal{F}_3 = \{F_3\}$ and $\mathcal{F} = \bigcup_{i=1}^{7} \mathcal{F}_i$. Let us call $\mathcal{F}$ the *forbidden family*.

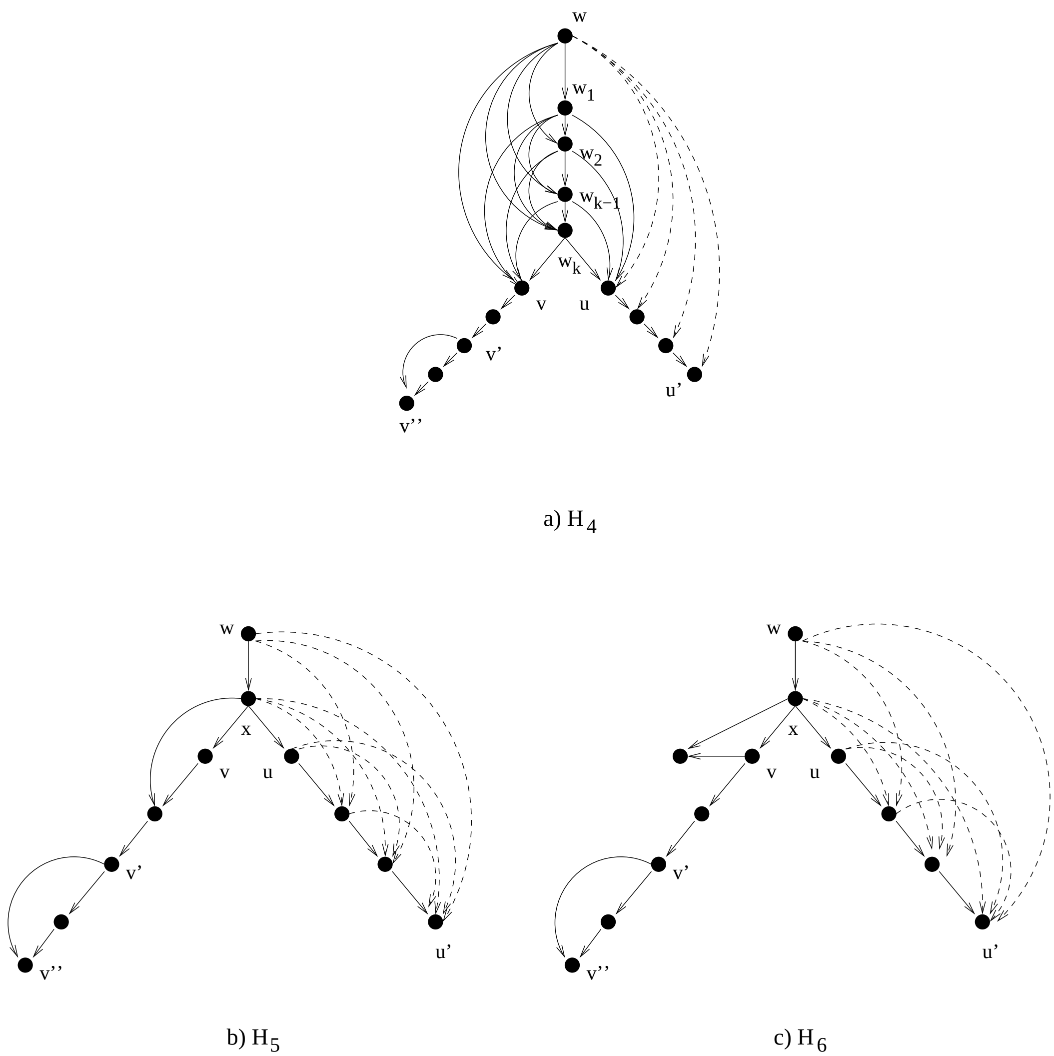

Figure 7.6: (a) H_4. The dashed arcs are missing. There is no arc between w_i and any vertex between u and u'. (b)H_5. (c) H_6. There may be additional arcs st, where s is an ancestor of t in $T(H_4)$.

Theorem 7.2.17 *Let H be in $\mathcal{A}$ and assume it has one stem. If H does not contain any digraph in the forbidden family $\mathcal{F} = \bigcup_{i=1}^{7} \mathcal{F}_i$ as an induced subgraph, then it has a Min-Max ordering and MinHOM(H) is polynomial time solvable. Otherwise, MinHOM(H) is NP-hard.*

Proof: It is easily derived from Lemmas 7.2.9, 7.2.11, 7.2.12, 7.2.13, 7.2.14, 7.2.15, 7.2.16, that if H contains a digraph in the forbidden family $\mathcal{F}$ as an induced subgraph, then MinHOM(H) is NP-hard.

Now, assume that H does not contain any digraph in the forbidden family $\mathcal{F}$ as an induced subgraph. Let us order the vertices of H by the linear ordering $\ll$ as follows. Let $u \ll v$ if

1. $l(u) < l(v)$, or
2. $l(u) = l(v)$ and $P(u) \ll P(v)$, or
3. $l(u) = l(v)$, $P(u) = P(v)$, and $MDA(u) \ll MDA(v)$, or
4. if $l(u) = l(v)$, $P(u) = P(v)$, and $MDA(u) = MDA(v)$
   (a) $P(u)$ dominates a child of u, or
   (b) u and v lie respectively on path P and Q of $T(H)$ such that there is an arc $v'v'' \in A(H) \setminus A(T(H))$ on Q, where $l(v) \leq l(v') = l(v'') - 2$, and a vertex u' in P, where $l(v') + 1 = l(u') = l(v'') - 1$.

Otherwise, order $u \ll v$ or $v \ll u$ arbitrarily.

Because H does not contain any of the digraphs in $\mathcal{F}_3, \mathcal{F}_5$, and $\mathcal{F}_7$ as an induced subgraph, we see easily that if $u \ll v$ by Rules 4.(a) or 4.(b), then we can not have $v \ll u$ by these rules.

Now let us prove that $\ll$ is a Min-Max ordering. Throughout the remainder of the proof, we consider two arcs $uu', vv' \in A(H)$ with $u \ll v$ and $v' \ll u'$. We will try to derive $uv', vu' \in A(H)$.

We claim that at least one of uu', vv' is in $A(H) \setminus A(T(H))$. Indeed, if both of them are in $T(H)$, then $u = P(u')$, $v = P(v')$, Which is easily led to a contradiction by Rules 4.(a) and 4.(b).

Suppose $vv' \in A(T(H))$. Then $uu' \in A(H) \setminus A(T(H))$ by the previous argument and the path R between u and u' on $T(H)$ induces a transitive tournament by Observation 7.2.10.

Note that vertex v cannot be an ancestor of u in $T(H)$ as $l(u) \leq l(v)$. Now, suppose u is not an ancestor of v in $T(H)$. Recall that $l(u) \leq l(v)$, $l(u') \geq l(v')$ and $l(u') \geq l(u)+2$. Thus, it easily follows that $l(u) = l(v)$, since otherwise there exists a forbidden subgraph from $\mathcal{F}_5$ including vertices v', u'' and arc $wu'' \in A(H) \setminus A(T(H))$ where u'' and w are vertices in R and $l(v') = l(u'')$. Since $u \ll v$, we can recursively see that $u'' \ll v''$ by Rule 2. Since $l(u'') = l(v'')$ and $P(u'') = P(v'')$, we must have $MDA(u'') \ll MDA(v'')$, by Rule 3 and 4.(b). This would imply that H contains a digraph from $\mathcal{F}_6$ as an induced subgraph. Therefore, we may conclude that u is an ancestor of v.

There are two cases to consider: (a) v is not in R (b) v is in R. Let us denote join(u',v') by x. In both cases, we claim that $l(v') - l(x) \leq 2$. Otherwise, there are two vertices u_1, and u_2 such that $l(u_2) = l(v')$, and $u_1u_2 \in A(H) \setminus A(T(H))$, where u_1 is a subjoin$_{v'}(u_2)$, leading to a digraph in $\mathcal{F}_5$.

In case (a), since v is not in R, we have $l(v') - l(x) = 2$. In this case, we also have $l(u') = l(v')$, as otherwise we shall encounter a digraph in $\mathcal{F}_7$. Let us denote $P(u')$ by u''. Then $u' \ll v'$ by applying Rule 4.(a) and Rule 2 recursively, unless $MDA(v) \ll MDA(u'')$. If $MDA(v) \ll MDA(u'')$, H has a digraph in $\mathcal{F}_4$ leading to a contradiction.

In case (b), H has the arc vu'. Suppose we do not have the arc uv'. Then we must have $v = x$, as otherwise v' is an ancestor of u', i.e., $uv' \in A(H)$ by Observation 7.2.10. (Note that $l(v') - l(v) = 1$ by assumption.) Since uv' is missing, if $l(u') = l(v')$, then $MDA(u') \ll MDA(v')$, implying that $u' \ll v'$, a contradiction. On the other hand, if $l(u') > l(v')$, we have F_2 as an induced subgraph leading to a contradiction.

Finally let us assume $vv' \in A(H)\setminus A(T(H))$. Then, we must have $uu' \in A(H)\setminus A(T(H))$, since $l(u) \leq l(v) \leq l(v')-2 \leq l(u')-2$. Now we can consider $P(v')v'$ and vv'' instead of vv', where v'' is the child of v on the path from v to v' in $T(H)$. Then the previous argument for $vv' \in A(T(H))$ can be applied, which means that we have the arcs uv' and vu'. $\diamond$

CASE 2: Multi-Stem Digraphs

Let $\mathcal{B}$ be the subclass of $\mathcal{A}$ consisting of all $H \in \mathcal{A}$, such that each stem of H has a Min-Max ordering. For any multi-stem digraph $H \notin \mathcal{B}$, MinHOM(H) is NP-hard by Theorem 7.2.17. So, we should only study the digraphs $H \in \mathcal{B}$. It was mentioned before that any two stems of a multi-stem digraph H only share the root r of $T(H)$ and these stems are different components of H after removing r. In the following six lemmas, we assume that $H \in \mathcal{B}$, $u,v,w \in V(H)$, join(u,v) = join(u,w) = join(v,w) = r, and u,v,w are in different stems of H.

Lemma 7.2.18 *Let $\mathcal{G}_1$ denote the family of all digraphs H satisfying all the following conditions:*

- $l(u) = l(v) = l(w) = l(r) + 2$;
- $ru, rv, rw \in A(H) \setminus A(T(H))$.

Then for any H in $\mathcal{G}_1$, MinHOM(H) is NP-hard.

Proof: The digraph H' induced by r, u, v, w and the vertices between r and each of u, v, and w in $T(H)$, is quasi-transitive and MinHOM(H') is NP-hard by Theorem 6.2.4. $\diamond$

Lemma 7.2.19 *Let $\mathcal{G}_2$ denote the family of all digraphs H satisfying all the following conditions:*

- $l(u) - l(r) > 2$, $l(v) - l(r) > 2$, $l(w) - l(r) > 2$;
- $u'u, v'v, w'w \in A(H) \setminus A(T(H))$, *where* $u' \neq r$, $v' \neq r$, *and* $w' \neq r$.

Then for any H in $\mathcal{G}_2$, MinHOM(H) is NP-hard.

Proof: We can easily see that H contains a structure like E_1 in Figure 7.7 as an induced subgraph. The longest paths from r to u, v, and w in E_1 are some paths in $T(E_1)$. There may be additional arcs st in the structure, where s is an ancestor of t in $T(E_1)$. We will show in Section 7.3 that MinHOM(E_1) is NP-hard. $\diamond$

Lemma 7.2.20 *Let $\mathcal{G}_3$ denote the family of all digraphs H satisfying all the following conditions:*

- $l(u) - l(r) > 2$, $l(v) - l(r) > 2$;
- $l(w) = max(l(u), l(v))$;
- $u'u, v'v \in A(H) \setminus A(T(H))$, *where* $u' \neq r$, $v' \neq r$.

Then for any H in $\mathcal{G}_3$, MinHOM(H) is NP-hard.

Proof: We can easily see that H contains a structure like E_2 in Figure 7.7 as an induced subgraph. The longest path from r to u, v, and w in E_2 are some paths in $T(E_2)$. There may be additional arcs st in the structure, where s is an ancestor of t in $T(E_2)$. We will show in Section 7.3 that MinHOM(E_2) is NP-hard. $\diamond$

Lemma 7.2.21 *Let $\mathcal{G}_4$ denote the family of all digraphs H satisfying all the following conditions:*

- $l(u) - l(r) > 2$, $l(v) - l(r) > 2$;
- $l(w) = max(l(u) - 1, l(v))$;
- $u'u, ru'', v'v \in A(H) \setminus A(T(H))$, *where* $u' \neq r$, $v' \neq r$, *and* u'' *is in the same stem as* u', *where* $l(u'') - l(r) = 2$.

Then for any H in $\mathcal{G}_4$, MinHOM(H) is NP-hard.

Proof: We can easily see that H contains a structure like E_3 in Figure 7.7 as an induced subgraph. The longest path from r to u, v, and w in E_3 are some paths in $T(E_3)$. There may be additional arcs st in the structure, where s is an ancestor of t in $T(E_3)$. Note that u'' is not necessarily in the longest path P between r and u in $T(H)$. In this figure, we have shown both possibilities. However, at least one of them is sufficient for E_3. We will show in Section 7.3 that MinHOM(E_3) is NP-hard. ◇

Lemma 7.2.22 *Let $\mathcal{G}_5$ denote the family of all digraphs H satisfying all the following conditions:*

- $l(u) - l(r) > 2$, $l(v) - l(r) > 2$;
- $l(w) = max(l(u) - 1, l(v) - 1)$;
- $u'u, ru'', v'v, rv'' \in A(H) \setminus A(T(H))$, *where* $u' \neq r$, $v' \neq r$, u'' *is in the same stem as* u' *with* $l(u'') - l(r) = 2$, v'' *is in the same stem as* v' *with* $l(v'') - l(r) = 2$.

Then for any H in $\mathcal{G}_5$, MinHOM(H) is NP-hard.

Proof: We can easily see that H contains a structure like E_4 in Figure 7.7. The longest path from r to u, v, and w in E_4 are some paths in $T(E_4)$. There may be additional arcs st in the structure, where s is an ancestor of t in $T(E_4)$. Similar to the proof of Lemma 7.2.21, there are two possibilities for each of u'' and v'' in this figure. However, at least one of them is sufficient for E_4. We will show in Section 7.3 that MinHOM(E_4) is NP-hard. ◇

Lemma 7.2.23 *Let $\mathcal{G}_6$ denote the family of all digraphs H satisfying all the following conditions:*

- $l(u) = l(v) = l(r) + 2$;
- $\{ru, rv, yw\} \in A(H) \setminus A(T(H))$, *where* $y \neq r$;
- *there exist two vertices* w_1 *and* w_2 , *where* w_1 *is in the same stem as* u, *and* w_2 *is in the same stem as* v *and* $l(w_1) = l(w_2) = l(w)$.

Then for any H *in* $\mathcal{G}_6$, *MinHOM(*H*) is NP-hard.*

Proof: We can easily see that H contains a structure like E_5 in Figure 7.7. The longest path from r to w, w_1, and w_2 in E_5 are some paths in $T(E_5)$. There may be additional arcs st in the structure, where s is an ancestor of t in $T(E_5)$. Note that there are two possibilities for u and v in this figure. However, at least one of them is sufficient for E_5. We will show in Section 7.3 that MinHOM(E_5) is NP-hard. ◇

We now handle all the forbidden subgraphs from Lemmas 7.2.18, 7.2.19, 7.2.20, 7.2.21, 7.2.22, 7.2.23. Let us define the *forbidden family* $\mathcal{G} = \bigcup_{i=1}^{6} \mathcal{G}_i$.

Theorem 7.2.24 *Let* H *be a multi-stem digraph in* $\mathcal{B}$. *If* H *does not contain any digraph in the forbidden family* $\mathcal{G} = \bigcup_{i=1}^{6} \mathcal{G}_i$ *as an induced subgraph, then it has a Min-Max ordering and MinHOM(*H*) is polynomial time solvable. Otherwise, MinHOM(*H*) is NP-hard.*

Proof: One can easily see by Lemmas 7.2.18, 7.2.19, 7.2.20, 7.2.21, 7.2.22, 7.2.23 that if H contains a digraph in $\mathcal{G}$ as an induced subgraph then MinHOM(H) is NP-hard. Thus, we assume that H does not contain any digraph in the forbidden family $\mathcal{G}$ as an induced subgraph.

Let $p_1, p_2, \ldots, p_l$ denote all stems of H. We shall explain how to partition the stems into sets A_1 and A_2 to obtain a Min-Max ordering $\prec$ for each of A_1 and A_2 (preserving ordering $\ll$ for each stem) and combine them to obtain a Min-Max ordering $\lhd$ for H.

The stems of H can be categorized into four subsets as follows:

- S_1 is the set of all stems having only one arc in $A(H) \setminus A(T(H))$, and the arc is rv' with $l(v') - l(r) = 2$.
- S_2 is the set of all stems having at least one arc $uv \in A(H) \setminus A(T(H))$ where $u \neq r$ and not having any arcs of the form rv' in $A(H) \setminus A(T(H))$.
- S_3 is the set of all stems having two arcs $uv, rv' \in A(H) \setminus A(T(H))$ where $u \neq r$, and $l(v') - l(r) = 2$.

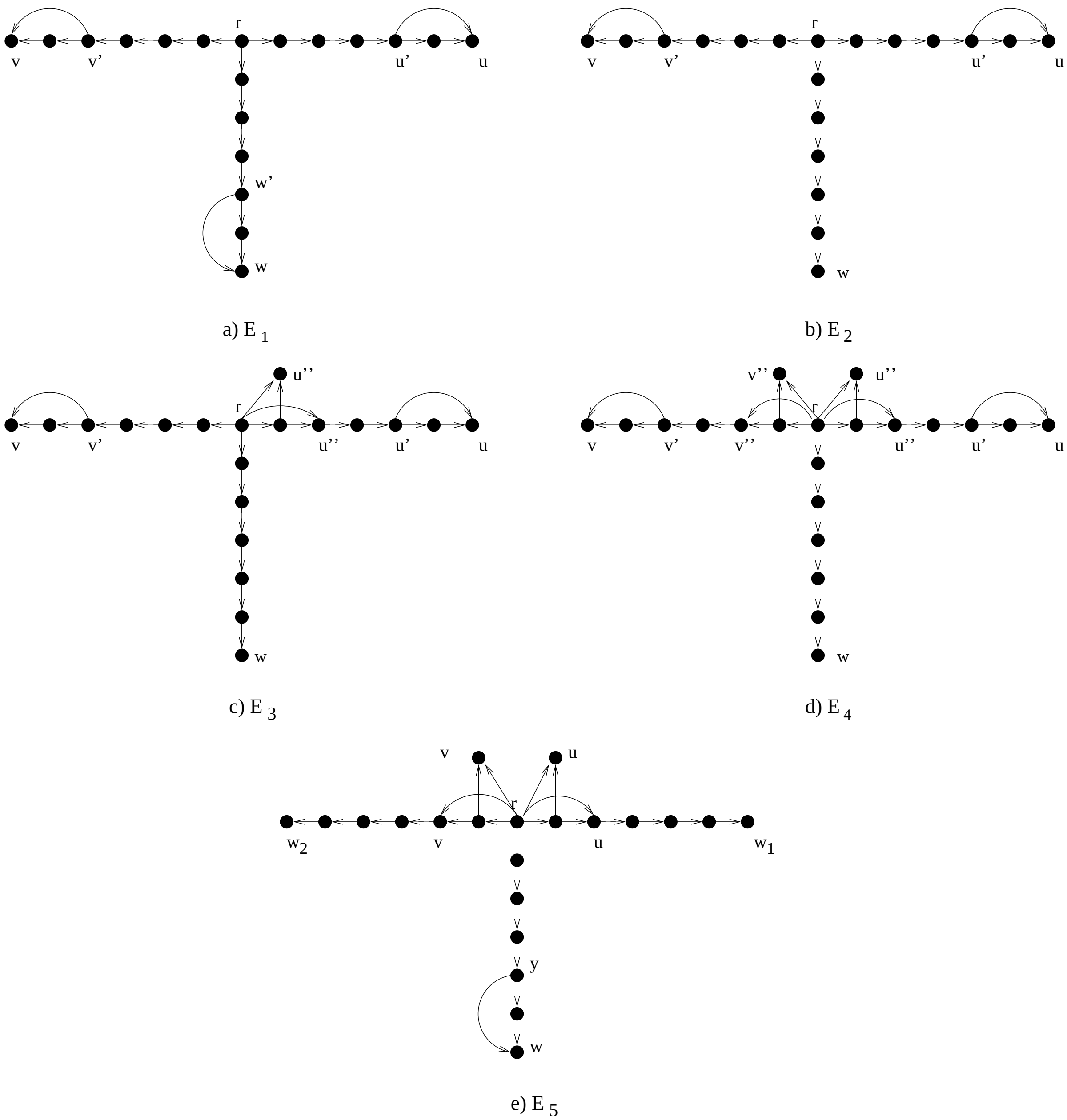

Figure 7.7: (a) E_1. (b) E_2 with $l(w) = max(l(u), l(v))$. (c) E_3 with $l(w) = max(l(u) - 1, l(v))$. (d) E_4 with $l(w) = max(l(u) - 1, l(v) - 1)$ (e) E_5 with $l(w_1) = l(w_2) = l(w)$. There may be additional arcs st, where s is an ancestor of t in $T(E_i), i = 1, \ldots, 5$. There are two possibilities for u'' in E_3, two possibilities for each of u'' and v'' in E_4, and two possibilities for each of u and v in E_5.

- S_4 is the set of all stems without any arc in $A(H) \setminus A(T(H))$.

We now define $\prec$ on a set of stems, preserving $\ll$ for each stem. We only need to order vertices u and v, which are in different stems. In the ordering $\prec$, we will always obey the following rules:

1. if $l(u) < l(v)$, then $u \prec v$.
2. if $l(u) = l(v)$, and $P(u) = P(v) = r$ then
    1. if u is a vertex of a stem in $S_1 \cup S_3$, and v is a vertex of a stem not in $S_1 \cup S_3$ then $u \prec v$.
    2. if u is a vertex of a stem in S_2, and v is a vertex of a stem in S_4, then $v \prec u$.
    3. else, order u and v arbitrarily.
3. If $l(u) = l(v)$ and $P(u) \prec P(v)$, then $u \prec v$.

Now, we are going to find a proper partition of stems into A_1 and A_2 so that $\prec$ is a Min-Max ordering for A_i. To do that, first, we will construct a graph G from H which satisfies the following statement: G is 2-(vertex) colorable if and only if the stems of H can be partitioned into two families A_1 and A_2 such that the vertices in each family have a Min-Max ordering $\prec$. Second, we will show that if H has no induced subgraph in $\mathcal{G}$, then the constructed G will be 2-colorable and so the stems can be so partitioned.

For each stem p of H, we introduce two measures $s_1(p)$ and $s_2(p)$. The first measure $s_1(p)$ denotes the length of a longest path in the stem p. The second measure is defined as follow.

$$s_2(p) = \begin{cases} \infty & \text{if} \quad p \in S_1 \cup S_4; \\ l(u') & \text{if} \quad p \in S_2 \\ l(u') - 1 & \text{if} \quad p \in S_3 \end{cases}$$

where u' is a vertex with minimum level among all vertices which are the ending vertices of some arc $uu' \in A(H) \setminus A(T(H)), u \neq r$ in p.

Suppose $\prec$ is a Min-Max ordering of the sets A_1 and A_2. Given two stems p and q with $s_1(p) \geq s_2(q)$ or $s_1(q) \geq s_2(p)$, we clearly must have p and q in different A_i.

Now we are ready to give the construction of a graph G from H. The vertex set of G is $\{p_1, \ldots, p_l\}$, i.e., there is a one-to-one correspondence between $V(G)$ and the stems of

H. Let us denote by $S_1, \ldots, S_4$ the subset of vertices of G corresponding to the stems in $S_1, \ldots, S_4$, respectively. The edge set of G is created as follows.

1. S_4 is an independent set.

2. $S_1 \cup S_3$ is a clique.

3. $S_2 \cup S_3$ is a clique.

4. For a pair (p, q) which is not covered in 1,2, and 3, we have $(p, q) \in E(G)$ unless $s_1(p) < s_2(q)$ and $s_1(q) < s_2(p)$.

Given the rules for the ordering $\prec$ of the vertices of each A_i, it is not difficult to check that two vertices p and q are not adjacent in G if and only if $\prec$ induces a Min-Max ordering for the vertices of the corresponding stems p, q in H which belong to the same family A_i. Note that there is no edge between S_1 and S_4.

We assert that the length of the largest induced cycle of G is at most 4. Indeed, if there is an induced cycle C whose length is at least 5, at least one vertex u in C must be from S_4. Otherwise, there are three vertices of C, which are either in $S_1 \cup S_3$ or $S_2 \cup S_3$, i.e., they make a clique, a contradiction. Since S_4 is independent, and there is no edge between S_1 and S_4, then the neighbors of u are in $S_2 \cup S_3$, i.e., there is an edge between these neighbors; contrary to the assumption that C is an induced cycle.

In fact, G has no cycle of length 3. Suppose a cycle C consists of three vertices x, y and z. Then without loss of generality there are the following six possibilities: (a) $x, y, z \in S_1 \cup S_3$ (b) $x, y, z \in S_2 \cup S_3$ (c) $x, y \in S_2$ and $z \in S_1 \cup S_4$ (d) $x \in S_2$, $y \in S_3$ and $z \in S_1 \cup S_4$ (e) $x, y \in S_3$ and $z \in S_4$ (f) $x, y \in S_1$ and $z \in S_2$. It is straightforward to see that each of (a)-(f) leads to a digraph in $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_6$, respectively. Therefore G is a bipartite graph, i.e., 2-colorable.

Let A_1 and A_2 be the families of stems of H obtained from a 2-coloring of G. It is clear that for every pair of stems p and q from A_i, $\prec$ is a Min-Max ordering for the digraph induced by the vertices of the union of these two stems. Since no arc exists between stems, the crossing pairs can only have both arcs inside an stem or have one arc in one stem and the other arc in another stem. Since we have shown before that $\prec$ is a Min-Max ordering for each stem and for every pair of stems, then , $\prec$ is a Min-Max ordering for A_1, and for A_2. Note that r comes first in the ordering of both families. It is easy to see that the reverse

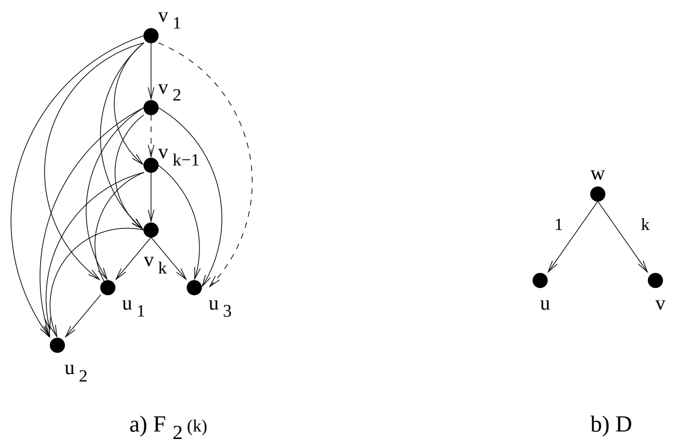

Figure 7.8: (a) F_2. The dashed arc is missing. (b) The digraph D. Note that each arc represents a directed path with the length marked beside it.

of a Min-Max ordering is also a Min-Max ordering. Now, since A_1 and A_2 do not share any vertex and arc except the root r that comes first in the ordering $\prec$ in both families A_1 and A_2, it is easy to see the ordering $\lhd$ obtained from $\prec$ in A_1 and the reverse of $\prec$ in A_2 is well defined for H, i.e., $v \lhd r \lhd u$ for $u \in A_1, v \in A_2$, and it is a Min-Max ordering. $\diamond$

7.3 NP-hardness

Proof of Lemma 7.2.11: Consider the digraph D, shown in Figure 7.8. D consists of a set of special vertices $\{u, v, w\}$, and a set of directed path existing between them as follows.

- there is a directed path of length k from w to v.
- there is a directed path of length 1 from w to u.

Let $x = u_3$ and $y = u_2$. It is easy to see that the conditions of Proposition 3.2.5 are satisfied for F_2 and D with vertices x, y and u, v, respectively. $\diamond$

Recall that $\mathcal{I}_3$ denotes the independent set problem for 3-partite graphs: given a 3-partite graph G and a positive integer k, $\mathcal{I}_3$ asks whether G have an independent set of cardinality at least k.

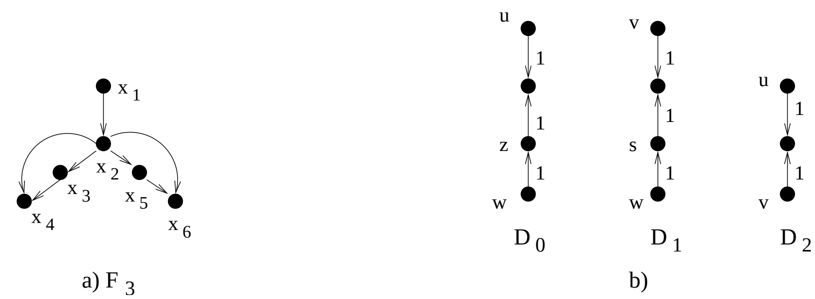

Figure 7.9: (a) F_3. There may be additional arcs st, where s is an ancestor of t in $T(F_3)$. (b) The digraphs D_0, D_1, and D_2. Note that each arc represents a directed path with the length marked beside it.

Proof of Lemma 7.2.12: We now construct a polynomial time reduction from $\mathcal{I}_3$ to the decision version of MinHOM(F_3). Let G be a graph whose vertices are partitioned into independent sets U, V, W, and let k be a given integer. We construct an instance of MinHOM(F_3) as follows: the digraph D is obtained from G by replacing each edge uv of G with the digraph D_2, replacing each edge uw of G with the digraph D_0, and replacing each edge vw of G with the digraph D_1 in Figure 7.9.

The costs are defined by $c_{x_5}(u) = 0, c_{x_2}(u) = 1$ for $u \in U$, $c_{x_3}(v) = 0, c_{x_2}(v) = 1$ for $v \in V$, and $c_{x_2}(w) = 0, c_{x_1}(w) = 1$, for $w \in W$. All other $c_i(y) = +\infty$ for $y \in V(G)$. All $c_i(y) = 0$ for $y \in V(D) - V(G)$ apart from $c_{x_5}(z) = +\infty$, and $c_{x_3}(s) = +\infty$, where z and s are special vertices of D_0, and D_1, shown in Figure 7.9.

We now claim that G has an independent set of size k if and only if D admits a homomorphism to F_3 of cost $|V(G)| - k$. Let I be an independent set in G. We can define a mapping $f : V(G) \to V(F_3)$ as follows:

- $f(u) = x_5$ for $u \in U \cap I$ and $f(u) = x_2$ for $u \in U - I$
- $f(v) = x_3$ for $v \in V \cap I$ and $f(v) = x_2$ for $v \in V - I$
- $f(w) = x_2$ for $w \in W \cap I$ and $f(w) = x_1$ for $w \in W - I$

This is a homomorphism of G to F_3 of cost $|V(G)| - k$.

Let f be a homomorphism of G to F_3 of cost $|V(G)| - k$. Then, all $c_{f(u)}(u), u \in V(D)$ are either zero or one. Let $I = \{y \in V(G) \mid c_{f(y)}(y) = 0\}$ and note that $|I| \geq k$. It can

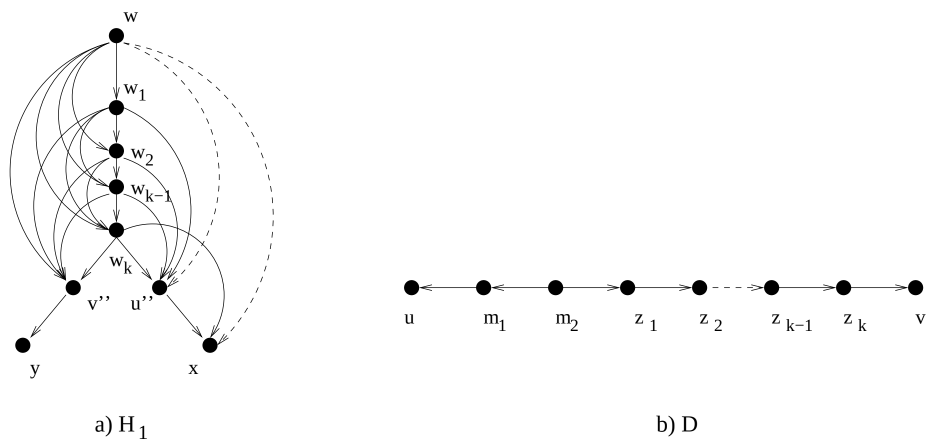

Figure 7.10: (a) H_1. The dashed arcs are missing. There may be additional arcs st, where s is an ancestor of t in $T(H_1)$. (b) The digraph D.

be seen that I is an independent set in G: for instance when $uv \in E(G)$ with $u \in I \cap U$ and $v \in I \cap V$, then $f(u) = x_5$ and $f(v) = x_3$, contrary to f being a homomorphism or a homomorphism of finite cost. (The other possibilities are similar.) $\diamond$

Proof of Lemma 7.2.13: We will construct the digraph D which fulfills the conditions of Proposition 3.2.6. Let D be the digraph shown in Figure 7.10 , whose vertex set and arc set are as follows:

$V(D) = \{u, m_1, m_2, z_1, z_2 \ldots, z_k, v\}$

$A(D) = \{m_1u, m_2m_1, m_2z_1, z_1z_2, z_2z_3, \ldots, z_{k-1}z_k, z_kv\}$

Let all costs $c_i(t) = 0$ for $t \in V(D) - \{u, v\}$, $i \in V(H_1)$ a part from $c_{w_i}(m_1) = +\infty$, $1 \leq i \leq k$, and $c_{u''}(z_k) = +\infty$. We also have $c_x(u) = c_x(v) = 0$, $c_y(u) = c_y(v) = 1$, $c_i(u) = c_i(v) = +\infty$ for $i \in V(H_1) - \{x, y\}$. Then, there are homomorphisms f_1, f_2, f_3 with finite costs from D to H_1 such that:

- $f_1(u) = f_1(v) = y$

  The other vertices of D may be mapped by f_1 as follows: $f_1(m_1) = v''$, $f_1(m_2) = w$, $f_1(z_i) = w_i$, $1 \leq i \leq k-1$, $f_1(z_k) = v''$.

- $f_2(u) = x$, and $f_2(v) = y$

  The other vertices of D may be mapped by f_2 as follows: $f_2(m_1) = u''$, $f_2(m_2) = w_1$, $f_2(z_i) = w_{i+1}$, $1 \leq i \leq k-1$, $f_2(z_k) = v''$.

- $f_3(v) = x$, and $f_3(u) = y$

  The other vertices of D may be mapped by f_3 as follows: $f_3(m_1) = v''$, $f_3(m_2) = w$, $f_3(z_i) = w_i$, $1 \leq i \leq k$,

On the other hand, there is no homomorphism of finite cost, which maps both u, and v to x. Suppose to the contrary, there exists such a homomorphism f. Since $c_{u''}(z_k) = +\infty$ and $f(v) = x$, then $f(m_2) = w$. However, as wu'' is missing and $c_{w_i}(m_1) = +\infty$, $1 \leq i \leq k$, then it is impossible to have $f(u) = x$. $\diamond$

Proof of Lemma 7.2.14: We now construct a polynomial time reduction from $\mathcal{I}_3$ to MinHOM(H_2). Let G be a graph whose vertices are partitioned into independent sets U, V, W, and let k be a given integer. We construct an instance of MinHOM(H_2) as follows: the digraph D is obtained from G by replacing each edge uv of G with the digraph D_2, replacing each edge uw of G with the digraph D_0, and replacing each edge vw of G with the digraph D_1 in Figure 7.11.b.

The costs are defined by $c_{x_1}(u) = 0, c_c(u) = 1$ for $u \in U$, $c_{y_1}(v) = 0, c_c(v) = 1$ for $v \in V$, and $c_c(w) = 0, c_a(w) = 1$, for $w \in W$. All other $c_i(y) = +\infty$ for $y \in V(G)$. All $c_i(y) = 0$ for $y \in V(D) - V(G)$ apart from $c_{x_1}(z) = +\infty$, and $c_{y_1}(s) = +\infty$, where z and s are special vertices of D_0, and D_1, shown in Figure 7.11.b.

We now claim that G has an independent set of size k if and only if D admits a homomorphism to H_2 of cost $|V(G)| - k$. Let I be an independent set in G. We can define a mapping $f : V(G) \rightarrow V(H_2)$ as follows:

- $f(u) = x_1$ for $u \in U \cap I$ and $f(u) = c$ for $u \in U - I$
- $f(v) = y_1$ for $v \in V \cap I$ and $f(v) = c$ for $v \in V - I$
- $f(w) = c$ for $w \in W \cap I$ and $f(w) = a$ for $w \in W - I$

This is a homomorphism of G to H_2 of cost $|V(G)| - k$.

Let f be a homomorphism of G to H_2 of cost $|V(G)| - k$. Then, all $c_{f(u)}(u), u \in V(D)$ are either zero or one. Let $I = \{y \in V(G) \mid c_{f(y)}(y) = 0\}$ and note that $|I| \geq k$. It can be seen that I is an independent set in G: for instance when $uv \in E(G)$ with $u \in I \cap U$ and $v \in I \cap V$, then $f(u) = x_1$ and $f(v) = y_1$, contrary to f being a homomorphism or a homomorphism of finite cost. (The other possibilities are similar)

To prove that MinHOM(H_3) is NP-hard, we construct an instance of MinHOM(H_3) by replacing each edge uv of G with the digraph D_2, replacing each edge uw of G with the

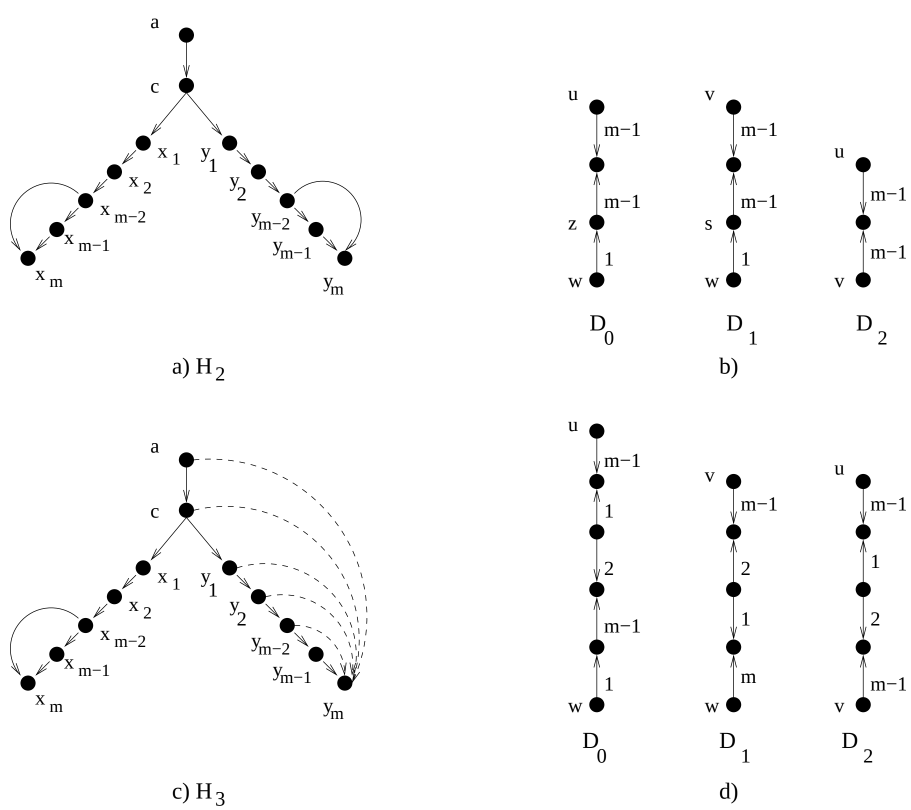

Figure 7.11: (a) H_2. (b) The digraphs D_0, D_1, and D_2. (c) H_3. (d) The digraphs D_0, D_1, and D_2. Note that each arc represents a directed path with the length marked beside it. There may be additional arcs st, where s is an ancestor of t in $T(H_2)$ or $T(H_3)$.

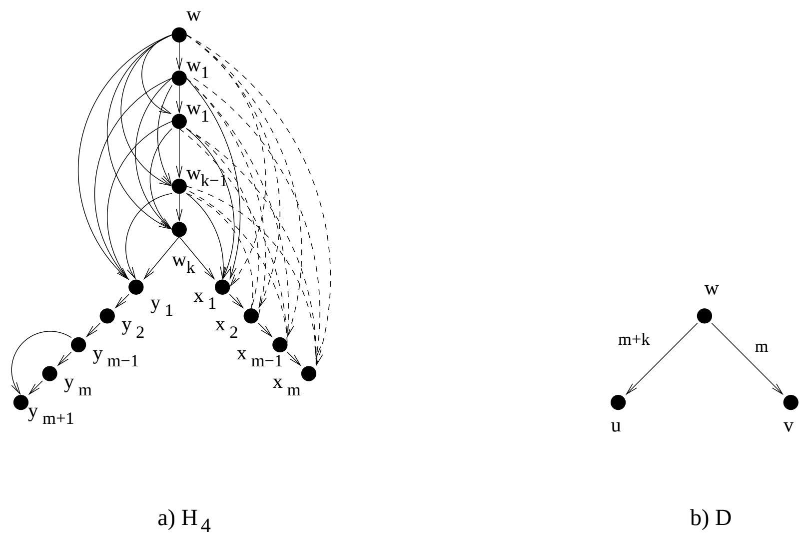

Figure 7.12: (a) H_4. The dashed arcs are missing. There is no arc between w_i and $x_j, j \geq 2$. There may be additional arcs st, where s is an ancestor of t in $T(H_4)$. (b) the Digraph D. Note that each arc represents a directed path with the length marked beside it.

digraph D_0, and replacing each edge vw of G with the digraph D_1 in Figure 7.11.d. The other parts of the proof are similar to MinHOM(H_2). $\diamond$

Proof of Lemma 7.2.15: Consider the digraph D, shown in Figure 7.12. D consists of a set of special vertices $\{u, v, w\}$, and a set of directed paths existing between them as follows.

- There is a directed path of length m from w to v.
- There is a directed path of length $m + k$ from w to u.

Let $x = x_m$ and $y = y_{m+1}$. It is easy to see that the vertices u, v, x, y and the digraph D fulfill the conditions of Proposition 3.2.5. $\diamond$

Proof of Lemma 7.2.16: Consider the digraph D, shown in Figure 7.13. D consists of a set of special vertices $\{u, z_1, z_2, z_3, v\}$, and a set of directed path existing between them as follows:

- There is a directed path of length $m + 1$ from z_1 to u.
- There is a directed path of length 2 from z_1 to z_2.

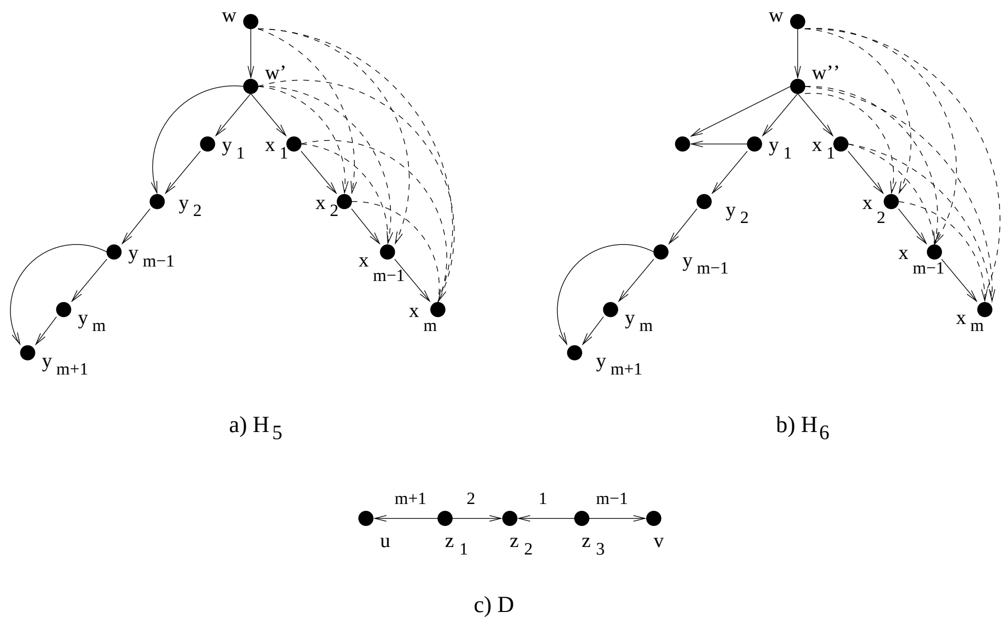

Figure 7.13: (a) H_5. (b) H_6. There may be additional arcs st, where s is an ancestor of t in $T(H_5)$ or $T(H_6)$. (c) The digraph D. Note that each arc represents a directed path with the length marked beside it.

- There is a directed path of length 1 from z_3 to z_2.
- There is a directed path of length $m-1$ from z_3 to v.

Let $x = x_m$ and $y = y_{m+1}$. It is easy to see that the vertices u, v, x, y and the digraph D fulfill the conditions of Proposition 3.2.5. This is also true for H_6. ◇

Proof of Lemma 7.2.19: Without loss of generality, we assume that $i \leq j \leq k$ in Figure 7.14 for E_1. Let D_0, D_1 and D_2 be the digraphs, shown in Figure 7.14.b and let $x_0 = a_1, x_1 = b_1, x_2 = c_1, y_0 = y_1 = y_2 = r$. Then it is easy to see that the digraphs D_0, D_1, D_2 and the pairs u_i, v_i and x_i, y_i for $i = 0, 1, 2$ fulfill the conditions of Proposition 3.2.9. ◇

Proof of Lemma 7.2.20: Without loss of generality, we assume that $i \leq j$ in Figure 7.15. Then the proof of Lemma 7.2.19 can be applied to this case as well without any change. ◇

Proof of Lemma 7.2.21: Let D_0, D_1 and D_2 be the digraphs

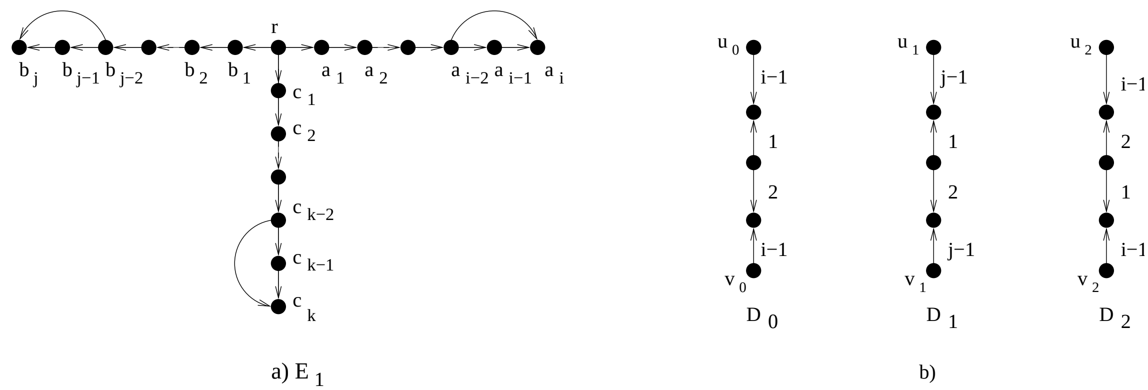

Figure 7.14: (a) E_1. $i \leq j \leq k$. There may be additional arcs st, where s is an ancestor of t in $T(E_1)$. (b) The digraphs D_0, D_1, and D_2. Note that each arc represents a directed path with the length marked beside it.

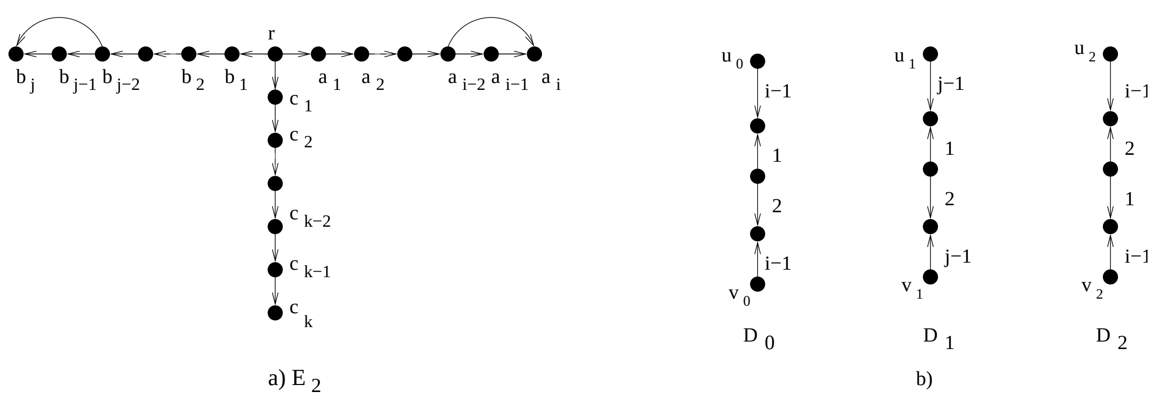

Figure 7.15: (a) E_2. $i \leq j$. There may be additional arcs st, where s is an ancestor of t in $T(E_2)$. (b) The digraphs D_0, D_1, and D_2. Note that each arc represents a directed path with the length marked beside it.

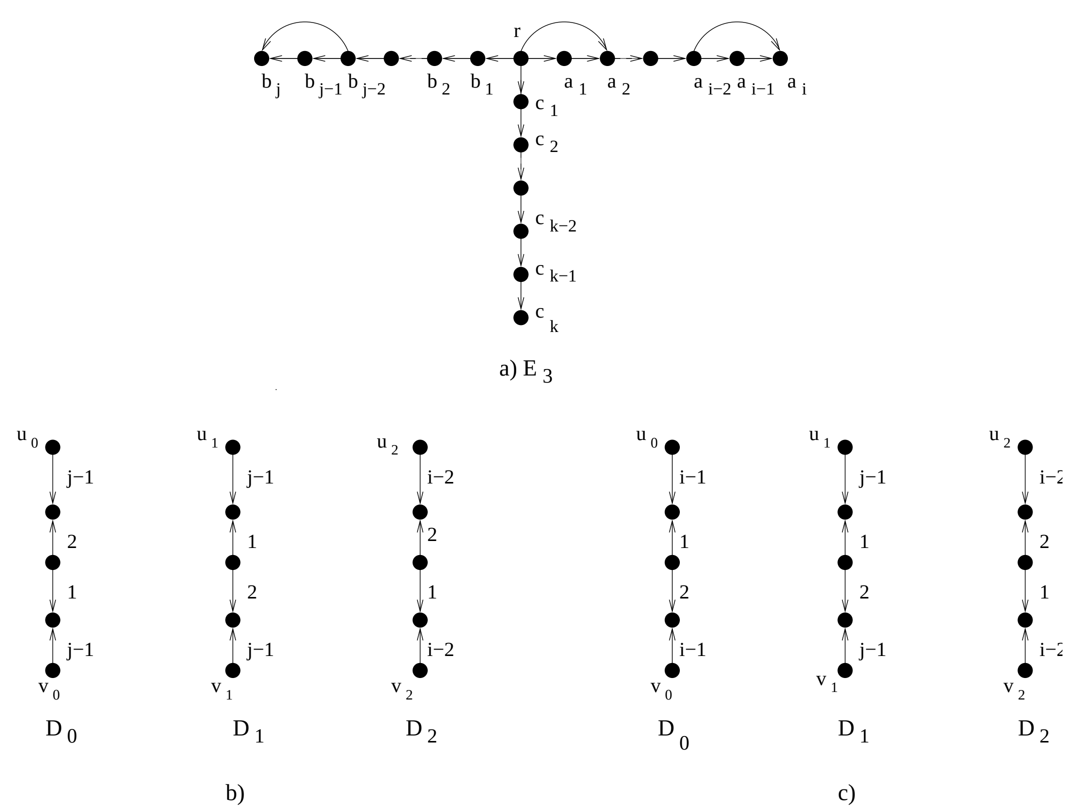

Figure 7.16: (a) E_3. $k = Max(j, i-1)$. There may be additional arcs st, where s is an ancestor of t in $T(E_3)$. (b) The digraphs D_0, D_1, and D_2 when $j \leq i-1$. (c) The digraphs D_0, D_1, and D_2 when $j > i-1$. Note that each arc represents a directed path with the length marked beside it.

- in Figure 7.16.b if, $j \leq i-1$ and E_3 is like Figure 7.16.(a) (ra_2 exists in the arc set of H).
- in Figure 7.16.c if, $j > i-1$ and E_3 is like Figure 7.16.(a) (ra_2 exists in the arc set of H).
- in Figure 7.17.b if, $j \leq i-1$ and E_3 is like Figure 7.17.(a) (ra_2 does not exist in the arc set of H).
- in Figure 7.17.c if, $j > i-1$ and E_3 is like Figure 7.17.(a) (ra_2 does not exist in the arc set of H).

Now let $x_0 = a_1, x_1 = b_1, x_2 = c_1, y_0 = y_1 = y_2 = r$. Then it is easy to see that the digraphs D_0, D_1, D_2 and the pairs u_i, v_i and x_i, y_i for $i = 0, 1, 2$ fulfill the conditions of Proposition 3.2.9. ◇

Proof of Lemma 7.2.22: We know that if rc_2 is in the arc set of E_4 in Figure 7.18, then MinHOM(E_4) is NP-hard by Lemma 7.2.18. Thus suppose that rc_2 is missing. Without loss of generality, assume that $i \leq j$. Let D_0, D_1 and D_2 be the digraphs, shown in Figure 7.18.b and let $x_0 = a_1, x_1 = b_1, x_2 = c_1, y_0 = y_1 = y_2 = r$. Then it is easy to see that the digraphs D_0, D_1, D_2 and the pairs u_i, v_i and x_i, y_i for $i = 0, 1, 2$ fulfill the conditions of Proposition 3.2.9. ◇

Proof of Lemma 7.2.23: We know that if rc_2 is in the arc set of E_5 in Figure 7.19, then MinHOM(E_5) is NP-hard by Lemma 7.2.18. Thus suppose that rc_2 is missing. Let D_0, D_1 and D_2 be the digraphs, shown in Figure 7.19.b and let $x_0 = a_1, x_1 = b_1, x_2 = c_1, y_0 = y_1 = y_2 = r$. Then it is easy to see that the digraphs D_0, D_1, D_2 and the pairs u_i, v_i and x_i, y_i for $i = 0, 1, 2$ fulfill the conditions of Proposition 3.2.9. ◇

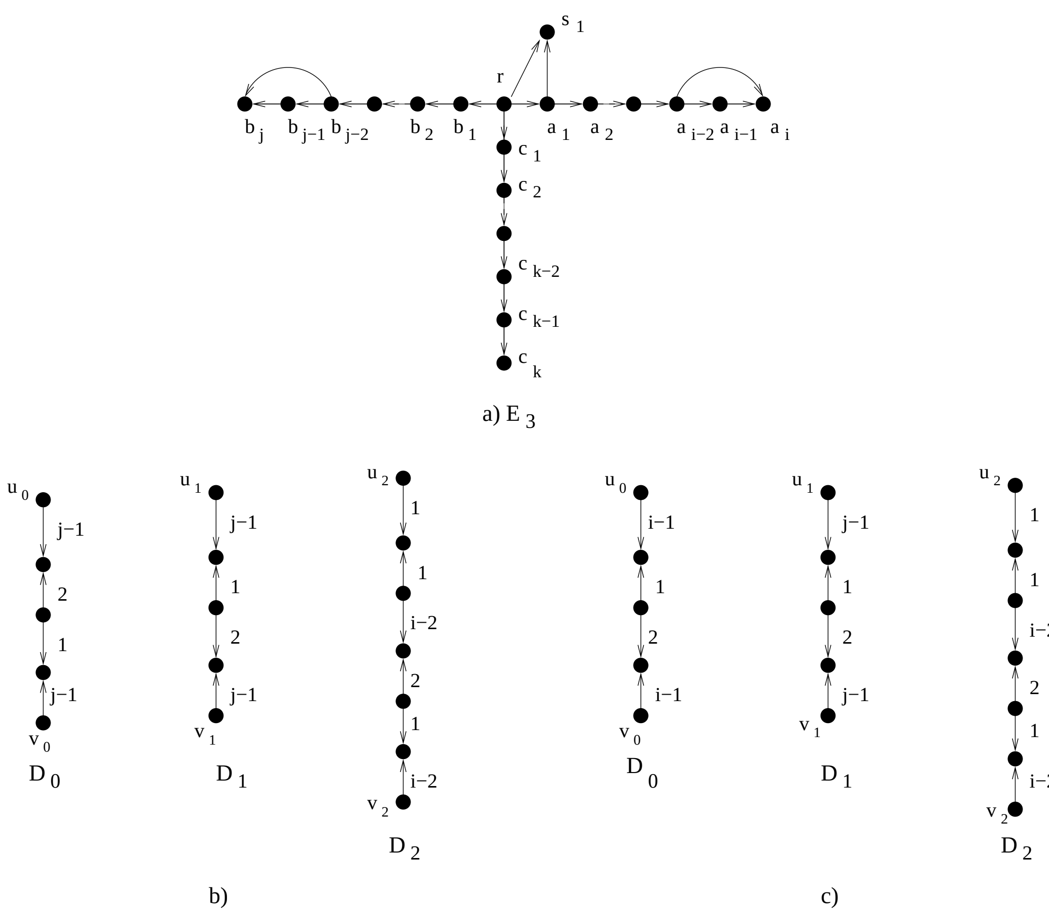

Figure 7.17: (a) E_3. $k = Max(j, i-1)$. There may be additional arcs st, where s is an ancestor of t in $T(E_3)$. (b) The digraphs D_0, D_1, and D_2 when $j \leq i-1$. (c) The digraphs D_0, D_1, and D_2 when $j > i-1$. Note that each arc represents a directed path with the length marked beside it.

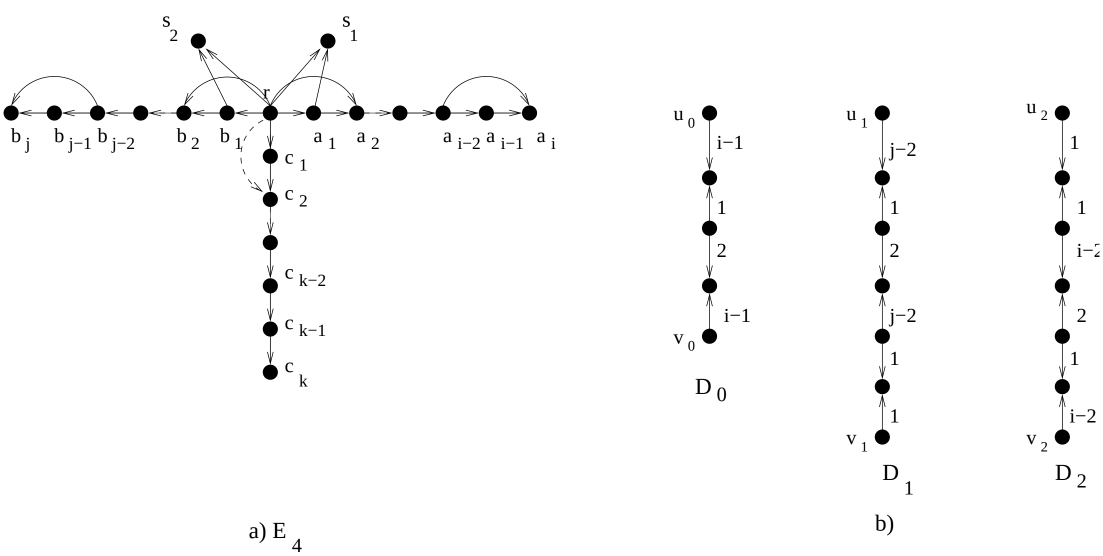

Figure 7.18: (a) E_4. $i \leq j$, $k = Max(j-1, i-1) = j-1$, the dashed arc is missing. There may be additional arcs st, where s is an ancestor of t in $T(E_4)$. The arcs ra_2, and rb_2 may be replaced by the arc sets $\{rs_1, a_1s_1\}$ and $\{rs_2, b_1s_2\}$.(b) The digraphs D_0, D_1, and D_2. Note that each arc represents a directed path with the length marked beside it.

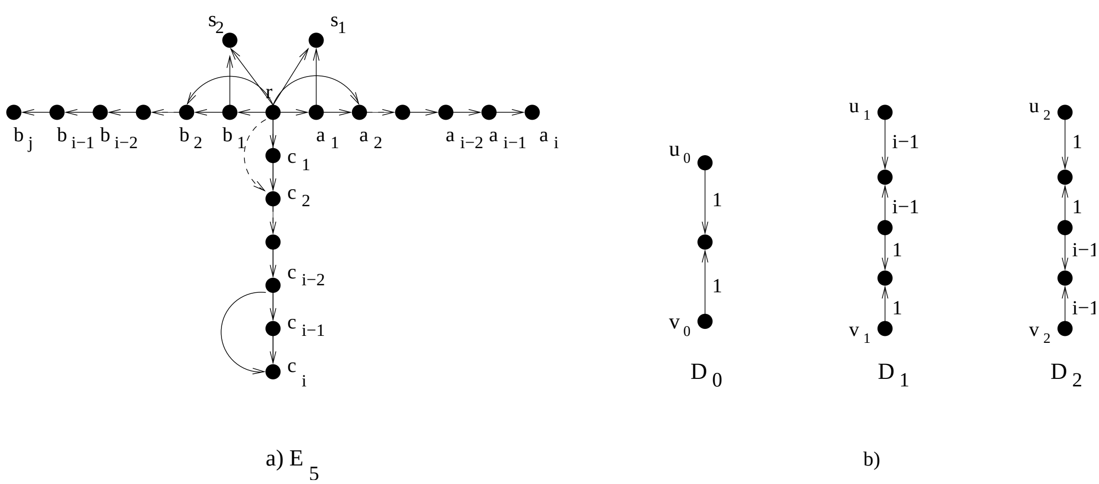

Figure 7.19: (a) E_5. the dashed arc is missing. There may be additional arcs st, where s is an ancestor of t in $T(E_5)$. The arcs ra_2, and rb_2 may be replaced by the arc sets $\{rs_1, a_1s_1\}$ and $\{rs_2, b_1s_2\}$. (b) The digraphs D_0, D_1, and D_2. Note that each arc represents a directed path with the length marked beside it.

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