ENQUIRIES INTO UNDERGRADUATE STUDENTS' UNDERSTANDING OF COMBINATORIAL STRUCTURES

by

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Abstract

Philosophers and educational researchers have pondered and studied understanding for many years. From an educational perspective, illuminating how understanding is formed and improved can provide valuable theoretical and pedagogical insights.

Combinatorics is an important field of research, with vast applications in mathematics and other fields of science. It is also included in many undergraduate curricula. In this thesis, I examine students' understanding of combinatorics based on two specific research questions: How do students understand a new definition? How do students improve their understanding of a concept when it is challenged?

In one study, I examined students' initial understanding of a new definition and the corresponding concept image. I presented students with a new definition, *trization*, and a set of tasks carefully chosen to reveal different aspects of their concept image of the new definition. For this purpose I considered example generation, use of formulae, use of representations, and connections the students make to their existing knowledge and concept images.

The study revealed that most students did not generate examples; however, they expected examples to be presented to them. Many students could find the appropriate formula for counting the number of possible trizations, after they were exposed to a few related tasks. Finding the formula helped them make a connection with their previous knowledge. Although a few students made this connection, they did not make any other anticipated connections. Many students' first attempt to understand trization was to create a pictorial representation, which, they did not consider significant in their understanding. However, algebraic representation was seen as a necessary and often sufficient form of understanding.

In the other study, I developed a methodology, *mediated successive refinement*, to help students change an inappropriate concept image. This methodology was based on learner-generated examples, and it encouraged students not only to reflect on their own examples, but also to reflect and modify their peers'.

This study identified the different scenarios that can occur when a student's concept image is changing. It established mediated successive refinement as a methodological tool for providing valuable research data, and a pedagogical tool for helping students improve their understanding.

Keywords: Mathematical understanding, combinatorics, mediated successive refinement, concept image, post-secondary education, learner-generated example

To the loves of my life,

Babak and Pedrom,

for the wonderful life they have given me...

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Chapter 1

Introduction

Discrete mathematics explores the properties and relations among discrete structures. It includes but is not limited to graph theory, coding theory, discrete optimization, and counting techniques. Discrete mathematics is a relatively new branch of mathematics that has continually gained importance, not only within mathematics but also in other fields of science such as physics, computer science and chemistry. Combinatorics is one of the major components of discrete mathematics. Combinatorial problems generally deal with *enumeration* (counting certain discrete structures), *existence* (examining the existence of certain discrete structures), *construction* (constructing certain discrete structures), and *optimization* (finding the maximum, minimum, or optimal discrete structure under certain conditions).

Combinatorics is a part of the curriculum in schools and universities in North America as well as in many other parts of the world. Its connection to other branches of mathematics and other fields of science makes combinatorics an important subject in the mathematics curriculum. In addition to its different applications, combinatorics offers a unique educational opportunity in mathematics: without much prior knowledge of mathematics, one can solve many creative, interesting, and challenging combinatorial problems.

There are many popular puzzles and games that encourage and even require combinatorial thinking. One example of a combinatorial puzzle gaining widespread popularity is a number placement puzzle called *sudoku.* A completed sudoku is a 9×9 *Latin square* with an additional condition on each of the nine smaller 3×3 squares. Sudoku puzzles are very popular, and they can be found in many magazines and newspapers. Additionally, numerous books and electronic games are written for sudoku enthusiasts. Many people with little or no interest in traditional mathematics solve sudoku puzzles regularly, yet still claim that they do not like mathematics and that "sudoku is not really mathematics".

The popularity of sudoku is so widespread that USA TODAY published an article on the "hardest sudoku puzzle" in 2006[1]. The puzzle was designed by a Finnish mathematician, Arto Inkala, who claimed he had designed the hardest sudoku puzzle in the world. The puzzle, named AI Escargot, is depicted in Figure 1.1[2] for sudoku lovers.

I believe that, given the appropriate pedagogical attention, combinatorics has the potential to become one of the most popular and interesting parts of mathematics in schools.

In this chapter, I first present my personal history and why I am interested in mathematics, particularly combinatorics, and in education. Then I present the motivation for this study and its purpose. Finally, I outline the organization of this

[1]Retrieved from `http://www.usatoday.com/news/offbeat/2006-11-06-sudoku_x.htm` on October 3, 2008.

[2]Retrieved from `http://www.keskiespoo.net/~arinkala/aisudoku/en/AIsudoku_Top10s1_en.pdf` on October 3, 2008.

1					7		9	
	3			2				8
		9	6			5		
		5	3			9		
	1			8				2
6					4			
3							1	
	4							7
		7				3		

Figure 1.1: The world's hardest sudoku "AI Escargot"

thesis.

1.1 Personal history

In the Fall of 2007, I was interviewed for CMESG's newsletter[3] by Gladys Sterenberg. I include parts of that interview here to provide a context for my background and interest in mathematics and education.

Shabnam: ... I am a PhD student in mathematics education at Simon Fraser University. My supervisor is Dr. Rina Zazkis. I am a full-time instructor at Langara College, which is a two-year college in Vancouver. My background is in mathematics. I came from Iran to Canada in 1996, after I finished my bachelor degree in applied mathematics at Sharif University of Technology. I always loved mathematics, since childhood, and for me studying mathematics was a very natural choice. I was the top student in all my math classes in high school. Even in middle school I remember my teacher asked me to teach our geometry class a couple of times. Besides math, I didn't like any other subject. So I didn't know (or even try to learn) much of anything else, just enough to get the grades I needed and get out of high school. To get into university in Iran, we had to write a national test that included all the subjects.

[3]The Canadian Mathematics Education Study Group, November 2007 issue, retrieved from `http://publish.edu.uwo.ca/cmesg/` on September 28th, 2008.

That was a very scary thought for my parents, seeing how much I loved math and that I was not spending any time studying anything else. So they decided to make me study other things... Funnily enough I would hide math books in the middle of chemistry or literature to read math and let them think I was studying other things.

Gladys: Why do you think you loved mathematics? What helped spark this childhood interest?

Shabnam: Well, my interest in mathematics was sparked a few times during my childhood and youth. When I was in fifth grade, I had an accident. A motorcycle hit me and I had to undergo surgery and was hospitalized for a month. Then I had to stay home for another month. All of this happened during the fall of 1983 or 1984. Anyway, I was going to miss school for a while. My grade five teacher, who never saw me (I had the accident right before the first day of school), was a very generous and kind woman. She heard about me from the principal of the school and decided to come to our home a few days a week to teach me what she was teaching in school that day. Her kindness is not describable and I will never forget her. She used to bring me math books as gifts as well. I loved those books.

Gladys: What an amazing story about such a caring teacher.

Shabnam: The second time that my love for math increased (and I guess it stayed with me up to this day) was when I was in grade 9. There was a war going on between Iran and Iraq. That year was the last year of war and Iraq attacked Tehran with missiles. So the schools were closed for six of the nine months. However, the final exams were still written and we had to study on our own and write the finals to pass the grade we were on. Going through was hard with bombs and missiles; one really didn't care much about studying. So I went into all my exams unprepared. For a student who was used to being 100% prepared all the time, it seemed like I had no clue what was going on. However, when the results came back, I got 95-100% on all my math courses, and I barely passed my biology. That was the point that I knew I am a math person and that it comes naturally to me. That caused my attraction to mathematics as a career. So I studied mathematics as my specialty in high school.

[...]

The reason I liked mathematics (besides the things I mentioned) is that when I do mathematics it takes me to a world that is calm, beautiful and certain. There are no maybe's or it depends. I trust what I do, and I trust the results. It is my hiding place I guess...

Gladys: I am interested in knowing more about how math can be a hiding place for you.

Shabnam: Mathematics is a hiding place for being creative. I think when you do math, you find something new, something that is yours, and it is a big achievement. I am sure people find this pleasure in different things, my husband finds it in physics for example. Also it is not just a hiding place, it is more like a sanctuary. Somewhere where I feel comfortable and relax.

Gladys: Could you tell me more about why you were interested in teaching?

Shabnam: I became interested in teaching when my middle school teacher asked me to teach a few geometry classes. My teaching inspired some students and I got a very positive feedback from the teacher and students. Later when I reflected on my academic life, I realized how much I am indebted to many of my teachers for being who I am and where I am. That was another reason why I wanted to be a teacher. I wanted to be one of those good teachers that people remember fondly.

Gladys: I certainly can understand your desire to be a teacher since you have shared so poignantly their impact on you.

Shabnam: After less than one year of teaching at Langara College, I felt something is missing from my teaching: Knowledge about teaching and learning. So I decided to apply for my next degree in Mathematics education with Rina, whom I knew from the time I was studying mathematics at SFU. I got accepted to study mathematics education at SFU in 2002 and I have been there ever since. My research interest is in exploring students' understanding and misunderstanding in general, however, I have concentrated on students' understanding of combinatorics. Combinatorics was a natural choice for me, since it was my research interest in my masters degree as well. I have studied two groups of students for this purpose, one group is social science and arts students in the college where I teach, and the other group is math and science students at SFU. The goal is not a comparison of the two groups though; I just look at the understanding and misunderstandings of each group in a different way.

I studied discrete mathematics, and in particular combinatorics, when I was in high school in Iran. It was a very interesting subject and different from the other branches of mathematics we had studied before. It was called Modern Mathematics in school, a well-deserved name. It was a different kind of mathematics, and learning it was much like playing a game. The whole concept was very interesting to me, from drawing the attractive graphs to counting different arrangements and sets. It all appealed to me as a person who loved playing games. In university, while I was studying mathematics, I still enjoyed my discrete mathematics courses more than all the other courses. In graduate school my choice was clear: I studied graph theory. I enjoyed carrying my coloured pens and pencils for colouring my graphs, and claiming that I was the only mathematics graduate student who was getting a Master's degree in colouring.

1.2 The study: motivation, context, purpose, and research questions

During the last year of my Master's degree in mathematics I was invited to teach a discrete mathematics course, called MACM 101, in the computing science department of Simon Fraser University. Over 200 students were enrolled in this class. Teaching this course provided me with an opportunity to observe many students and how they learned combinatorics. The students enrolled in MACM 101 were generally mathematics, engineering, or science majors, who are usually quite interested in mathematics. After I started my teaching position at Langara college, I started teaching a course called Finite Mathematics, which includes combinatorics. While teaching Finite Mathematics, I had another opportunity to interact with students, this time more closely in a smaller classroom setting. The students' background in the Finite Mathematics course varied, but most students were more interested in social sciences and humanities. Some students wanted to study commerce, while others were pre-service elementary school teachers. The mathematical background of most of the students in Finite Mathematics was not very extensive. From my experience of teaching these classes os varying sizes, to students with different backgrounds, I realized that combinatorics is a difficult topic for most students to grasp. I found many students had particular difficulty in combinatorics in the following areas:

- recognizing the appropriate combinatorial structure;
- making connections between different concepts;
- interpreting the problem statement correctly;
- using the formula correctly;
- verifying the solution.

Observing these difficulties motivated me to examine students' understanding of combinatorics in more detail. In particular, I had the following questions:

1. How do learners understand a new combinatorial structure? How do they approach a new concept?
2. What are the obstacles in understanding a combinatorial structure? What are students' main difficulties in solving combinatorial problems? Are there obstacles and problems specifically related to combinatorics?
3. How do learners modify their existing knowledge to obtain a better understanding?

There are very few studies in mathematics education related to combinatorics. Hence, it is not difficult to identify and discuss many gaps in the existing research in this area. The previous studies all suggested that students have difficulty in solving and understanding combinatorics, and identified some of these difficulties and obstacles. Hence, studying the previous research helped me answer the second question. However, to the best of my knowledge, there is no study that directly deals with students' understanding of combinatorics and how to make students aware of their own understanding. Hence, the first and last questions remained unanswered by previous studies.

Understanding is one of the most fascinating aspects of education. It is quite difficult, if not impossible, to give a precise definition of understanding. People describe understanding as one of the objectives of education, but often reveal different interpretations when they are asked to describe what they mean by understanding. Hence, in studying understanding, it is very important to describe what it is that we mean by understanding.

In this study, I first introduce some aspects of understanding as they relate to learning combinatorics. Then, I investigate students' understanding occurring in two situations. The first situation is understanding something new for the first time. I examine the methods that students employ to learn a new definition and how they apply it to approach some related tasks. Furthermore, I examine how they make connections between the new concept and the old concepts that they had learnt before. The second situation is when a previous understanding is proven to be inadequate or inappropriate, and there is a need for the previous understanding to be modified or even replaced by a more complete and appropriate understanding. I developed a pedagogical framework to help students become more aware of their own understanding and to recognise when they had not formed an adequate understanding of a concept. Additionally, I used this method to investigate the change in students' concept image through the different tasks posed to support this methodology.

In my point of view, when one discusses understanding it is important to consider both of these aspects of understanding, namely forming understanding of a new concept and modifying an existing understanding. After all, understanding is not something that happens momentarily and is then complete. It is a process that takes time and effort. The process of understanding a concept begins when someone is faced with that concept for the first time, and that understanding continues to be modified and grow every time the existing understanding is challenged or proven to be inadequate.

1.3 Organization of thesis

In Chapter 2, I present a short history of combinatorics, and show how it became a branch of mathematics. I also explain the place of combinatorics in the curriculum and present a brief summary of research supporting the inclusion of combinatorics in the curriculum, and its educational value.

Chapter 3 presents a review of previous research in combinatorics in the context of mathematics education. Then I present a summary of students' common errors and issues that was discussed in the previous studies. Furthermore, I display an example of error analysis.

Chapter 4 starts with a brief introduction to the theory of understanding as described by Sierpinska (1994). She described the components of understanding as understanding subject, object of understanding, basis of understanding, and mental operations. I describe the relation between the notion of concept image, as developed by Tall and Vinner (1981), and the notion of basis of understanding. Furthermore, I present the two frameworks that I developed: a theoretical one for examining students' understanding of a new concept; and a methodological and pedagogical one for examining students' changing concept image when they face an inadequacy in their existing concept image.

In Chapter 5 I contextualize the methodologies chosen to carry out the two studies described in this thesis. In this chapter, I wish to take the reader through the series of events and thoughts that guided my research choices.

In the Chapters 6 and 7, I describe two studies that employ both of the frameworks presented in Chapter 4.

The first study, presented in Chapter 6, examines students' understanding of a new definition in the context of their previous concept image of related or similar combinatorial structures. The participants are presented with a definition that they have not seen before, and with a set of tasks for which they can use either their previous knowledge or the new definition. Each task is specifically designed to reveal how the students think about the new definition in relation to their old knowledge and how, in the process of solving the problem, they explore the connections between their new and old knowledge.

The second study, presented in Chapter 7, is designed to examine how students change their concept image, if and when they are faced with a problematic area in their concept image. For this study I developed a methodology, based on example generation tasks, to encourage students to change their concept image by creating an environment where each student can reflect on their own and their peers' thoughts through their generated examples. Students' original inadequate concept images changed through these sets of tasks, and this change is examined through the framework explained in Chapter 4.

A summary of findings is presented in Chapter 8, along with the contributions of the two studies.

Chapter 2

History of Combinatorics in mathematics and the curriculum

2.1 Introduction

Combinatorics is the study of ways to list and arrange elements of discrete sets according to specified rules (Cameron, 1994). Combinatorics includes many interesting topics such as graph theory, design theory, cryptography, and coding theory (Rosen, 2000). Combinatorics and computer science have a major overlap, which is mostly present in theoretical computer science and complexity theory. In addition to computer science, combinatorial methods are used in various other fields such as operations research (for example scheduling and vehicle routing), electrical engineering (some network problems), molecular biology (maps of DNA), and chemistry (isomer enumeration techniques) (Graham, Grötschel, & Lováz, 1995). Combinatorics has so many applications in physics that there have been several conferences devoted to investigating how combinatorics and combinatorial methods can contribute to research in physics, for example a conference called "Conference on Combinatorics and Physics" was held in Bonn, Germany, in March of 2007[1]. Combinatorics is also used in many different fields within mathematics, such as probability theory, linear algebra, number theory, and topology (Cameron, 1994). There is an intrinsic simplicity about (many) combinatorial problems, which makes the problems easy to understand, but at the same time it is usually challenging to find the appropriate way to solve the problem: "[...] good combinatorial reasoning is largely a matter of knowing exactly when to add, multiply, subtract, or divide" (Zeitz, 2007, p. 188).

In this chapter, I introduce a brief history of combinatorics and how it emerged as a field within mathematics. Furthermore, I describe the place of combinatorics in the mathematics curriculum.

2.2 Combinatorics as a field within mathematics

The art of counting the number of ways to list and arrange elements of discrete sets according to specific rules, known as combinatorics, existed centuries before combinatorics became an integral part of mathematics. It seems that, unlike many other areas of mathematics, the ancient Greeks were not interested or were not familiar with this fascinating part of mathematics. This may be due to the lack of interest or knowledge about this part of mathematics, which some attribute to Greek philosophy and culture (Biggs, 1979), or may be simply a historical coincidence, or due to the historical evidence of Greek work in this area having been lost. As a result, it seems that combinatorics originated from eastern cultures: Hindus had a major role in the for-

[1] http://www.mpim-bonn.mpg.de/Events/2007/Combinatorics+and+Physics/

mation of combinatorics (6^{th} century B.C.) and before them, the Chinese (7^{th} century B.C.) seem to have been the pioneers (Suzuki, 2002). The motivation for developing this kind of mathematics seems to be embedded in eastern culture and philosophy. There is evidence that after the occupation of parts of India by Islamic forces in the 7^{th} century A.D., combinatorics, as well as many other fields of eastern knowledge started to travel towards west. It was not until the 17^{th} century, with Pascal's *Traité* followed by Leibniz's *Dissertatio de Arte Combinatoria*, that the "combinatorial art" became an accepted branch of learning (Biggs, 1979).

The earliest example of combinatorics is from the famous book of "I Ching" (Book of Changes) from China. I Ching is a philosophical book that dates back to ancient Chinese civilization, perhaps as early as the year 3000 B.C. One part of the system of I Ching is based on two symbols, the Yang (—) and the Yin (- -). These symbols were combined into strings of three symbols (trigrams) and strings of six symbols (hexagrams). If these hexagrams had been represented using a binary system, their impact on mathematical development might have been enormous. However, according to Biggs (1979) there is no evidence of systematic listing or use of binary numbers in the oldest edition of I Ching. In fact, the oldest arrangement of these hexagrams appears to be attributed to King Wen, who arranged them while he was imprisoned in 1050 B.C.

In Figure 2.1[2] we see King Wen's arrangement of I Ching hexagrams. The general arrangement consists of 32 pairs, where each pair is related in one of the following two ways:

1. There are eight hexagrams that have the same arrangement if their order is

[2]Retrieved from `http://www.changecycle.org/images/kingWenArrangement.jpg` on June 24, 2008

reversed. Each of these hexagrams are paired with their negation. The negation of a hexagram is another hexagram where each yin is replaced with a yang and vice versa. For example, 1^{st} and 2^{nd} arrangements in Figure 2.1.

2. The rest of the hexagrams are paired with their reverse arrangement. For example, 5^{th} and 6^{th} arrangements in Figure 2.1.

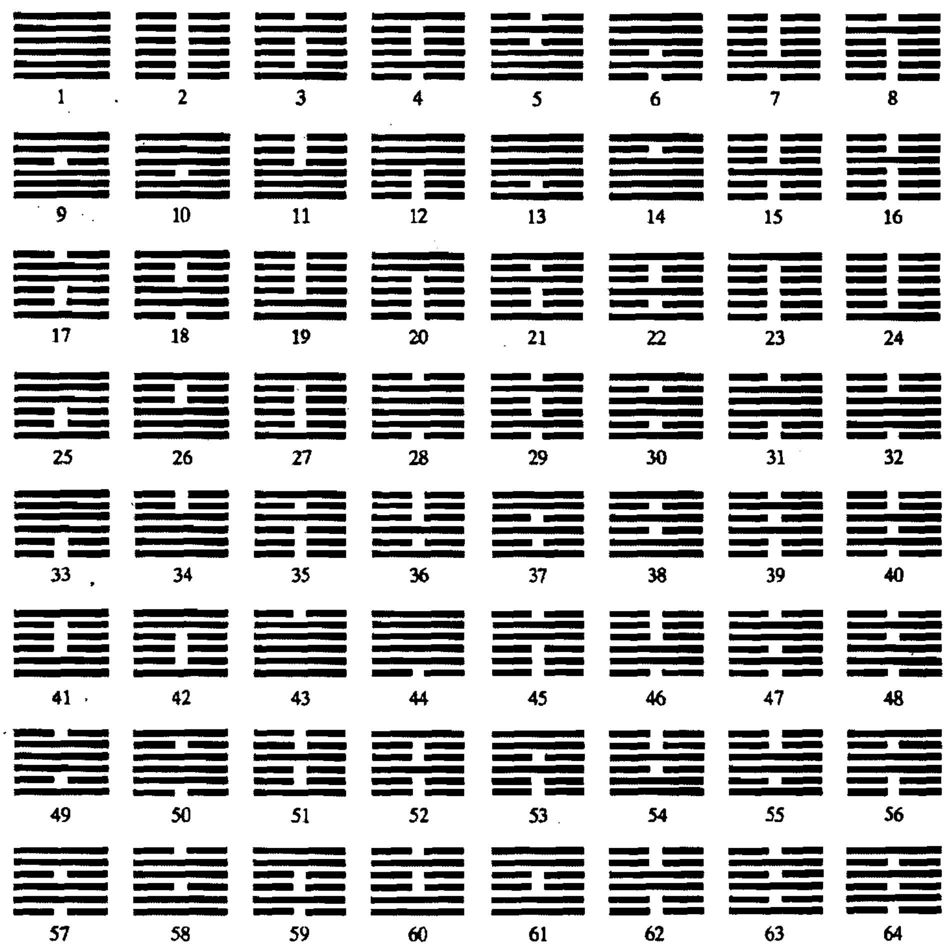

Figure 2.1: King Wen's order of I Ching hexagrams

King Wen's arrangement is symmetric in the sense of pairing the hexagrams, but no other mathematical significance has been attributed to it. According to Aiton and Shimao (1981, p.83), King Wen's arrangement "lacks even a superficial resemblance to a numerical order." However, Leibniz—who was introduced to the book of I Ching—mistakenly concluded that the ancient Chinese had precedence in the discovery of binary arithmetic (Biggs, 1979).

Another major contribution to this field is the creation of magic squares (also known as Lo Shu squares) by the ancient Chinese (Biggs, 1979). There is much debate about when Lo Shu squares were created. Some claim they were known from 2800 B.C., but according to Biggs this claim most likely is not valid and the evidence points to a timeline close to the 1st century A.D. (Biggs, 1979). What is not a matter of debate is the important role of magic squares in combinatorics, because it is the first study of possible arrangements satisfying certain conditions. A magic square of order n is an $n \times n$ grid. In each cell of this grid, there is a number from 1 to n^2 such that no number is repeated and the sum of the numbers in each row, each column, and each main diagonal is equal. Although the ancient Chinese had the first example of a magic square (of order 3), there is no evidence that they had any other square of higher order until the work of Yang Hui in the 13th century. However, by the 13th century other mathematicians had discovered higher order magic squares.

Although the Chinese seem to be the pioneers in this field, the Hindus' contribution to the development of combinatorics is historically more significant. In the Hindu culture, an example of a probable combinatorial calculation is from the text of Susruta who claimed (correctly) that there are 63 combinations of the six tastes (bitter, sour, salty, astringent, sweet and hot). Because the number of combinations is too small, historians are not sure if the problem was solved using non-systematic listing, systematic listing, or other combinatorial methods (Suzuki, 2002). Nonetheless, it seems that the general rules to find numerical solutions to combinatorial problems (without listing all the possibilities) were found very gradually over centuries. There is evidence that Bhaskara (12^{th} century) knew the formulae for both permutations (with or without repetition) and combinations (without repetition) of objects. He discussed the idea of combinatorics in architecture, music, and medicine (Smith, 1958). It appears that he may have known of these formulae from mathematicians such as Mahariva from the 9^{th} century (Biggs, Lloyd, & Wilson, 1995; Suzuki, 2002). There is also reason to believe that earlier mathematicians such as Brhatsamhita of Varahamihira during the 6^{th} century may have known these formulae. The reason is that Brhatsamhita had made some statements about enumeration problems, one of which is that the number of ways of choosing 4 ingredients from 16 is 1820. He mentions this number without any further explanation, and it seems unlikely that the number was obtained by listing of all 1820 cases. He may have used the arithmetic triangle (also known as Pascal's triangle) to arrive at this result. It is also probable that he knew about the combination formula. Another significance of this plain statement is that it suggests these calculations were commonplace knowledge, and that may be why Brhatsamhita did not think it was necessary to explain how he got the number 1820 (Biggs, 1979).

Around the 12^{th} century, formulae for combination and permutation were present in different languages and different contexts. Rabbi Ben Ezra was a 12^{th} century mathematician and astrologer. It is interesting to note that he used the theory of combinations in astrology, rather than mathematics. The reason for this may be that combinatorics was not yet recognized as a field within mathematics. Ben Ezra used combinations to find all the different possibilities for the conjunctions of the seven planets then known. What is interesting is that the method used by Ben Ezra is recursive, and uses the sum of series. According to Ginsburg (1923):

1. The method used by Ben Ezra for calculating what we call n choose k, or $\binom{n}{k}$, can be written in modern notation as $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \ldots + \binom{k-1}{k-1}$.

2. There is evidence that Ben Ezra knew that $\binom{n}{k} = \binom{n}{n-k}$.

There are some claims that the Hindu mathematicians knew about the arithmetic triangle, around 200 B.C. but there is no evidence supporting these claims. In the 12^{th} century, the Persian mathematician and poet Omar Khayyam discussed the use of arithmetic triangle for extracting roots of numbers. Unfortunately, the details of his work in the area of extracting roots did not survive, but there is evidence that he had extensively worked with this triangle. We do not know if the relation between the entries of the triangle (known as the binomial coefficients at that time) and the combination numbers were apparent to the ancient mathematicians (Biggs, 1979). Al-Tusi displayed and discussed arithmetic triangle in the 13^{th} century, and he also explained the method of extracting roots using this triangle, as it was an established method (Biggs, 1979).

The arithmetic triangle also appeared in a Chinese text written around the 14^{th} century by Chu Shih-Chieh. This text indicated that the triangle was known to Chia Hsien around the 12^{th} century. This may have been passed on to the Chinese from the Hindu or Moslem scholars, or it may have been an independent discovery (Biggs, 1979; Biggs et al., 1995). The triangle was used frequently after the 16th century in the European scholars' works. The first evidence that the triangle was used for combination numbers is from the work of Herigonus in the 17^{th} century. Pascal was the first mathematician to provide comprehensive analysis of the properties of the triangle, in the 17^{th} century (Suzuki, 2002). Pascal may have made only a few discoveries about the triangle on his own, but he explained many of the properties clearly, and he made sure that the importance of the triangle was well established. In the western world the arithmetic triangle is called Pascal's triangle, although it was discovered some centuries before Pascal's 17^{th} century writing on the triangle. The historical account given above explains why the triangle is instead called Khayam's triangle in Persian mathematics texts, and in the Chinese texts it is called either Shih-Chieh's triangle or sometimes it is named by Shih-Chieh's predecessor Yang Hui's triangle.

As Biggs puts it: "It seems appropriate to end this account of the roots of combinatorics with Pascal, for his *Traité* contains what is probably the first recognizably modern treatment of the elements of the subject" (Biggs, 1979, p. 132). Pascal's Traité was translated to English as "Treatise on the Arithmetical Triangle", which was written in two parts, one was the treatise on the arithmetic triangle, and the second part was written on the uses of this triangle. Pascal's Traité was followed by Leibniz's work on combinatorics (Cambridge University Library). Subsequently, "combinatorial art was an accepted branch of learning" (Biggs, 1979, p. 132).

2.3 Combinatorics in the curriculum

Although combinatorial methods existed for many centuries, the development of combinatorics as a research field within mathematics is relatively recent. Combinatorics can be seen as modern mathematics, because of its recent evolution during the 17^{th} and the 18^{th} century. With the evolution of computers, combinatorics has found a new importance in the field of mathematics and theoretical computer science, both as a research field and as an educational field. "The interaction of discrete mathematics and computers has made possible powerful new applications, has focused attention on new kinds of problems and forced us to look at traditional mathematics in new ways" (Gardiner, 1991, p. 10). With recent advances in technology and the inclination of many students to study computer science, combinatorics and discrete mathematics in general, have become a necessity for completing a degree in computer science and engineering. Many science and engineering undergraduate programs have at least one discrete mathematics course in their first or second year.

In the early 1980's, there was a call to include discrete mathematics in the undergraduate curriculum as a branch of mathematics that was required for computer science majors (Siegel, 1986). In 1986, the Mathematical Association of America (MAA) assembled a committee on discrete mathematics in the first two years of undergraduate studies. The committee report contains ten recommendations on the inclusion of discrete mathematics in undergraduate programs, among them the following (Siegel, 1986, p. 91):

1. Discrete mathematics should be a part of the first two years of the standard mathematics curriculum at all colleges and universities.
2. Discrete mathematics should be taught at the intellectual level of calculus.

   [...]

8. All students in sciences and engineering should be required to take some discrete mathematics as undergraduates. Mathematics majors should be required to take at least one course in discrete mathematics.

9. Serious attention should be paid to the teaching of the calculus. Integration of discrete methods with the calculus and the use of symbolic manipulators should be considered.

10. Secondary schools should introduce many ideas of discrete mathematics into the curriculum to help students improve their problem-solving skills and prepare them for college mathematics.

Discrete mathematics was created as a course to serve the mathematical needs of computer science majors, and this may have been the reason that many discrete mathematics textbooks include so many different (and sometimes disconnected) parts of mathematics, such as logic, number theory, set theory, inductive reasoning and induction, recursion, relations and functions, generating functions, graph theory, and counting techniques. I have taken this list from the content list of a popular discrete mathematics textbook (Grimaldi, 2004), but many of the textbooks have similar content. Regardless of the textbook, any discrete mathematics course includes combinatorics as an integral part. At the post-secondary level, in addition to a specific undergraduate course in discrete mathematics, topics in introductory combinatorics are explored by undergraduates in a variety of other courses, such as probability, statistics, and even some liberal arts courses.

In 1989, The National Council of Teachers of Mathematics (NCTM) published Curriculum and Evaluation Standards for School Mathematics. This publication, which served as a guideline for schools across the United States, recommended that discrete mathematics become an essential part of the school curriculum. This declaration shifted the attention of mathematics educators to the role of discrete mathematics in the school curriculum. It included a recommended list of topics from discrete mathematics, to be included in high school mathematics. This list included, but was not limited to (NCTM, 1989):

- Represent problem situation using discrete structures such as finite graphs, matrices, sequences, and recurrence relations;
- Develop and analyze algorithms;
- Solve enumeration and finite probability problems.

The NCTM publication also emphasized that the inclusion of discrete mathematics was not solely for its application to computer science, but because these topics represent "useful mathematical ideas that have assumed increasing importance for all students" (NCTM, 1989). The 1991 NCTM yearbook was dedicated to discrete mathematics across the curriculum, from pre-school to grade 12 (Kenney & Hirsch, 1991). In NCTM's more recent document, The Principles and Standards of School Mathematics (NCTM, 2000), we also see an explicit attention to the teaching and learning of discrete mathematics:

> As an active branch of contemporary mathematics that is widely used in business and industry, discrete mathematics should be an integral part of the school mathematics curriculum, and these topics naturally occur throughout the other strands of mathematics.
>
> [...]
>
> Three important areas of discrete mathematics are integrated within these Standards: combinatorics, iteration and recursion, and vertex-edge graphs. These ideas can be systematically developed from prekindergarten through grade 12.

The attention to discrete mathematics from the NCTM created increased attention for this topic in Canada as well. In British Columbia, combinatorics is included in the Math 12 Principles Integrated Resource Package 2006 as a part of learning outcomes for probability and statistics. The topics that are covered are enumeration techniques, Pascal's triangle, and the binomial theorem. In Ontario, combinatorics is included in the grade 11 curriculum, and in the grade 12 curriculum there is a separate course on geometry and discrete mathematics (Jonker & Lidstone, 2005). In many other countries, discrete mathematics is also an important part of school and undergraduate curriculum. For example, in Iran there is a separate course established for discrete mathematics in high school. This course was mandatory for all students studying mathematics-physics stream in high school.

2.4 The educational value of combinatorics

It is important to teach combinatorics, for many reasons. One reason is the importance of combinatorics in teaching and learning probability. Inhelder and Piaget (1975) have stated that their studies led them to expect that "the formation of the ideas of chance and probability depend in a very strict manner on the evolution of combinatorial operations." Batanero, Navarro-Pelayo, and Godino (1997) also have discussed the important and inseparable role of combinatorics in teaching and learning probability. One instance in which the connection between probability and combinatorics is apparent is when all the outcomes are equally likely to happen. If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the probability of E, denoted by $P(E)$, is $\frac{\text{the number of outcomes in } E}{\text{the total number of outcomes}}$.

For example, let us consider the following problem: What should be the length of a password – consisting only of numerical digits – if you want to be sure that the probability that someone figures it out is less than 1 in 1,000,000? This problem can be easily reduced to a combinatorial problem, and can be restated as: If we want to have 1,000,000 different numerical passwords – made with numerical digits – how many digits should they have?

In addition to its role in learning probability, combinatorial reasoning can be used to solve many other problems in a more elegant and intuitive way. As an example of combinatorial reasoning, the equivalence of $\binom{n}{k}$ and $\binom{n-1}{k-1} + \binom{n-1}{k}$ can be shown by symbolic manipulation, or it can be proved using combinatorial reasoning in a shorter and more elegant way. The combinatorial reasoning for this equivalence is: If you want to choose k people from a group of n people (which includes someone called Jim), you can either choose Jim to be in your group, and then choose $k-1$ other people from the rest of the group that has $n-1$ members, or not choose Jim and choose k people from $n-1$ people. The combinatorial method is more interesting and more educationally significant than the algebraic method, because it emphasizes and helps the understanding of the formula.

Combinatorial capacity is a fundamental way of formal thinking. Since combinatorics does not depend on calculus and a high level of algebra, it has suitable problems for many grades and levels. Usually, very challenging and stimulating problems can be discussed with students without extensive prior knowledge. For example: without use of algebra, find how many non-negative integer solutions there are to the equation $x_1 + x_2 + x_3 + x_4 = 7 (x_i \geq 0)$.

Many applications from different fields can be presented to make a connection between mathematics and those fields, or between different parts of mathematics.

In NCTM (2000), one of the standards is connection. The main points of connection are to:

- Recognize and use connections among mathematical ideas;
- Understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
- Recognize and apply mathematics in contexts outside mathematics.

As an example, graph theory is used in organic chemistry, physics, and computer science. Pascal's triangle has many applications in different parts of mathematics, connecting combinatorics with algebra. Fractals and combinatorics are also connected: if we colour the odd numbers in Pascal's triangle, Sierpinski's triangle will be obtained.

The question: "Should we teach discrete mathematics in schools?" has been discussed in parts of the NCTM 1991 yearbook and the NCTM Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989). In the 1989 document, the NCTM recommends that discrete mathematics, and subsequently combinatorics, should be taught to all students. The reason for these recommendations has been elaborated in the 1991 yearbook by Hart (1991):

1. "Mathematics is alive": There are many unsolved problems in discrete mathematics that can be presented to students with very little background in mathematics. An example is the traveling salesman problem.

2. "Problem solving and modeling are important": Discrete mathematics offers unique methods for problem solving, which are accessible to students with little or no algebraic skills. Furthermore, discrete mathematics provides us with many new ways to model interesting problems, such as the scheduling problem.

3. "Discrete mathematics has many applications": Discrete mathematics can have applications outside of mathematics, some of these applications were mentioned before, for example in chemistry, physics, and industry.

4. "Discrete mathematics complements and enriches the traditional curriculum": Some problems can be solved using either a discrete or a continuous approach, some can be solved only using a discrete approach and some can be solved only using a continuous approach. Making these connections has become more important with the introduction of graphing calculators, which operate on discrete basis, and in business calculus where the problems are fundamentally discrete, but are solved using continuous methods for simplicity.

In 2005, I attended the working group that was dedicated to the study of the role and effect of discrete mathematics in the high school and university curriculums at the Canadian Mathematics Education Study Group (CMESG). We discussed many reasons why discrete mathematics is a good addition to the current mathematics curriculum in schools. Some of the reasons are that discrete mathematics (Jonker & Lidstone, 2005, p. 46):

- is accessible for learners;
- is an excellent source of problems that are engaging and allow for multiple solutions;
- requires little technical language and therefore readily promotes communication skills;
- has the potential for drawing learners into other areas of mathematics;

- helps promote the popular image of mathematics as part of human culture (an example is a very popular new puzzle called sudoku);
- is essential to the study of computer science.

In this section, I discussed the educational value of discrete mathematics in the curriculum. However, it is very important to recognize that any topic has the tendency to become dull and boring, and discrete mathematics is no different. The factors that can detract from the educational value of a topic are many, and in the next section, I will mention a few.

2.5 Constraints of including discrete mathematics in the curriculum

"School mathematics all too easily degenerates into a succession of meaningless routines. And discrete mathematics has characteristics that make it vulnerable to such degenerations" (Gardiner, 1991, p. 12). As an example of such a degeneration, Gardiner (1991, p. 12) discusses counting problems: "The educational value of much simple discrete mathematics lies precisely in the fact that it forces students to *think* about very elementary things, such as systematic counting" (italics in the original). He stresses that this can be easily reduced "to a number of manageable and predictable steps, or rules, *requiring an absolute minimum of thinking*" (italics in the original). The danger of an overwhelming and over-stuffed curriculum is that it can cause any topic, especially a mathematical one, to degenrate into a set of rules and procedures which require minimal creativity and thinking. In this way, the educational value of the topic can be lost.

In some universities in Canada, discrete mathematics courses have been removed from the requirements of the engineering program, because of the low performance of students and their negative experience with this course (CMESG 2005 discussions).

Although combinatorics has the potential to be a rich addition to the school and post-secondary curricula, the research in the field of mathematics education about this subject is slim. In the next chapter I present a selection of these studies.

Chapter 3

Mathematics Education Research and Combinatorics

3.1 Introduction

Combinatorics is a growing field of mathematics that has enjoyed increased representation in the curriculum in recent years. Unfortunately, mathematics education research has not yet caught up with this trend, and not much research has been done in this field. However, available findings in the previous studies point to similar problems and areas of difficulty in students' understanding of combinatorics. In this chapter, I review a selection of mathematics education research related to combinatorics. Some of these studies have focused on problem-solving strategies, some on reasoning, and some on problems that students are facing in solving combinatorial problems. The focus of this chapter is on the issues and problems that have been recognized in previous studies in the area of understanding and problem solving in basic combinatorics. The first part of the chapter describes typical student problems and errors, based on the findings of the previous studies. The final part of this chapter illustrates a few of these problematic areas with examples.

3.2 Background

Although mathematics education research in the area of combinatorics is not extensive, there are a few substantial studies and an ongoing project examining different aspects of combinatorics from an educational point of view.

A few of the studies examined combinatorics in an educational context as a necessary prerequisite to probability. For instance, Freudenthal described his first-year university probability course as being systematically and logically connected to combinatorics. Hence, in his treatment of probability he inevitably discussed combinatorics (Freudenthal, 1972, p. 598). As an example, he described his teaching of the binomial distribution using Pascal's triangle and combinations. Batanero, Godino, and Navarro-Pelayo (1997) also considered learning combinatorics as an essential part of learning probability. In fact, Carmen Batanero has a research group at the University of Granada in Spain that concentrates on statistics education, that has published about a half a dozen papers on educational aspects of combinatorics and related matters. In a recent book chapter, Batanero and Sanchez (2005, p. 242) asserted: "Combinatorics is more than a calculus tool for probability, but it does play an important role in probability."

Fischbein, Pampu, and Minzat (1970) discussed the importance of teaching and age in learning and understanding combinatorics, even at the elementary level for children. Maher and Martino (1996, 2000) studied students' understanding of combinatorics in a longitudinal case study. Their main finding was that students truly came to understand mathematics through the reinvention of mathematics they are trying to understand. As part of their long-term study, they also discussed the development of mathematical reasoning in children through young adulthood.

Eizenberg and Zaslavsky (2003, 2004) discussed students' verification strategies of their own solutions. They identified one of students' main difficulties in solving combinatorial problems to be "the lack of well-established and reliable verification strategies" (Eizenberg & Zaslavsky, 2003, p. 390). They explained that combinatorics fosters collaboration among students, who cannot be certain of the correctness of the solution because there are "hardly any useful and readily accessible ways to verify it" (Eizenberg & Zaslavsky, 2003, p. 399). They also believed that creating a collaborative environment can help students overcome some of the difficulties they face in solving combinatorial problems. They identified five strategies for verification of solution by students. The first strategy is verifying by reworking the solution, which was most frequently used. The second strategy is verifying by adding justification to the solution, which they found helpful in finding minor mistakes. The third strategy is verifying by considering the reasonableness of the solution. Eizenberg and Zaslavsky (2003, 2004) found this strategy is not used much, and attributed this to the fact that estimating an answer is not very easy in most problems. The fourth strategy is verifying by modifying some components of the problem. For example by making a problem with smaller numerical components while maintaining the essence and content of the problem. They found that although this strategy is very powerful, it needs to be explicitly taught. They also acknowledged that maintaining the structure of the problem is not an easy task for many students. The fifth and final strategy discussed is verifying by using a different solution method, which was used frequently and was found very helpful. However, to use this strategy effectively, students need to gain experience in using multiple approaches to solve the same problem. In general, the authors asserted that through teaching verification strategies, and creating instances of successful verification, students will become aware of these strategies and may start verifying their solutions on their own initiative.

English (1991, 1996, 2005) studied children's development of combinatorial reasoning and their strategies for solving combinatorial problems. Her findings can be briefly itemized as follows:

- Combinatorics has the potential to be used to create excellent situations for children to "engage in self-directed learning" (English, 1991, p. 472).

- Combinatorial problems are accessible at different levels, and can be solved using different approaches and not just by employing an algebraic formula. Hence, it can provide an opportunity for children to discover alternatives to the rote memorization of formulae for solving mathematical problems.

- Young children can solve problems that are more sophisticated, given the right problems in the right situations: "Children's level of achievement in school mathematics is not a reliable predictor of their ability to solve novel problems" (English, 1996, p. 108).

- Educators and students need to pay attention to the structure of combinatorial problems. "This is especially important across all problem types to enable children to develop conceptual understanding, transfer their learning to related situations, and create new problems for sharing with others" (English, 2005,

p. 137–138).

Fischbein and Gazit (1988), Hadar and Hadass (1981), Batanero, Navarro-Pelayo, and Godino (1997), and Batanero, Godino, and Navarro-Pelayo (1997) have all discussed students' difficulties in solving different combinatorial problems. These difficulties are discussed in the next section in detail as these topics relate to my study directly.

3.2.1 Different combinatorial configurations

One way to organize different types of combinatorial problems is by breaking them into the following three categories: Enumeration, Existence, and Optimization.

- **Enumeration:** Counting the different combinatorial configurations. For example, counting the number of different passwords of length five one can make using ten digits.
- **Existence:** Determining whether or not a specific combinatorial configuration exists. For example, if the edges of the complete graph[1] with six vertices are edge coloured with either blue or red, then there always exists a monochromatic triangle in this graph, namely a triangle all of whose edges have the same colour. This is not true for complete graphs with fewer than six vertices[2].
- **Optimization:** Finding the optimum combinatorial configuration. For example, finding the least number of colours needed to colour a planar map, if no two adjacent countries are coloured the same[3].

[1]Complete graph is a graph where every vertex is connected to every other vertex by exactly one edge.

[2]This is a special case of Ramsey's theorem.

[3]This problem is solved by the famous four colour theorem.

In this study, I concentrate mainly on enumeration problems (although, this does not imply that the other two categories are of less importance to mathematics education). The reason for this choice is that enumeration problems are more commonly used in the early combinatorics curriculum. Hence they have been discussed more frequently in the mathematics education literature as well. Enumeration problems can be generally categorized into three groups:

1. **Arrangement:** Order of the elements within the configuration matters.

    - Limited repetition allowed: 'How many 3-letter words can one make with the letters FINITELY?' (Note that there is no repetition except for the letter 'I', of which we have two.)
    - Unlimited repetition allowed: 'How many 3 digit numbers can you make?' (For each digit we have 10 choices, so there are $10 \times 10 \times 10 = 1000$ numbers, which also makes a connection to basic arithmetic and decimal representation.)
    - No repetition allowed: 'How many ways can 5 people sit around a circular table?'

2. **Combination or selection:** Selection of elements from a set such that the order of the elements within the configuration (selection) does not matter.

    - Limited repetition allowed: 'How many different fruit baskets containing at least one piece of fruit can one make from 5 oranges, 3 peaches, and 10 bananas?'
    - Unlimited repetition allowed: 'How many ways can you choose 3 roses if there are red and white roses available?'

- No repetition allowed: 'How many ways can Kimmy choose 3 of her friends to invite for dinner if she has 10 friends?'

3. **Partition:** Placement of n objects into m cells, with k_i objects in cell i ($i = 1, \ldots, m$), and $k_1 + k_2 + \ldots + k_m = n$. 'In how many ways can Mr. Wong divide 10 different stamps among his 3 children? He can give all ten stamps to one child.'

There are other ways to categorize these basic enumeration problems. For example, Batanero, Navarro-Pelayo, and Godino (1997) presented a different categorization of these problems. However, I chose the above categorization because of its clear distinction among the different categories, and because the language is closer to the current combinatorics textbooks commonly used in North America (e.g. Grimaldi (2004)).

In the next section, I examine and analyze some of the common errors that students make while solving enumeration problems.

3.3 Common errors and issues

Some studies point to a few common errors and misconceptions in solving combinatorial problems (Batanero, Navarro-Pelayo, & Godino, 1997; Fischbein & Gazit, 1988; Hadar & Hadass, 1981). In particular, Batanero, Navarro-Pelayo, and Godino (1997) presented 14 types of error in their research on students' attempts to solve a set of enumeration problems. Hadar and Hadass (1981) also discussed seven obstacles in solving combinatorial problems. I have summarized some of the errors and misconceptions of these two studies, organizing them into two general categories: errors related to combinatorial structure, and issues concerning general problem solving skills.

3.3.1 Errors related to combinatorial structure

These errors are specific to combinatorial problems. They depict the most frequent difficulties that students face in solving combinatorial problems.

Error of order

This error occurs when learners do not recognize if the order in the given combinatorial situation is important or not. Batanero, Navarro-Pelayo, and Godino (1997) displayed this error with an example when the pupil confused combination with permutation or vice versa.

Error of repetition

This error happens when learners do not recognize whether repetition is allowed, and if allowed whether it is limited or unlimited. Batanero, Navarro-Pelayo, and Godino (1997) described that students often considered repetition to be possible when it is not, or vice versa.

Error of indistinguishable/distinguishable elements

This error happens when students do not recognize if some of the elements that are presented in the problem are distinguishable or not. For example the number of ways to give 5 identical cookies to two kids is enumerated differently than the number of ways of giving 5 different cookies to 2 kids. Batanero, Navarro-Pelayo, and

Godino (1997) identified that students often considered indistinguishable elements as distinguishable or vice versa.

Error of over- or under-counting

This is the case where some configurations are counted more than once, or some configurations are not counted. This error is considered by Batanero, Navarro-Pelayo, and Godino (1997) as the error of exclusion, which means they only considered under-counting. However, during my teaching I have witnessed many instances where the students over-count as well. Hence, I have generalized this type of error to include both over- and under-counting.

Non-systematic listing

This error is seen at different ages, with regard to many problems with different degrees of difficulty. The main issue is not being able to follow a systematic method for listing all possibilities. This can be seen in adults who have been recently introduced to combinatorics (Hadar & Hadass, 1981), and in children (English, 1991). It has also been mentioned by Fischbein and Gazit (1988) and Batanero, Navarro-Pelayo, and Godino (1997).

3.3.2 Issues concerning problem solving skills

Some errors are developed because of learners' lack of general problem solving skills. Geroge Polya identified four general steps in mathematical problem solving (Polya, 1957). These four steps do not present a prescription for solving a problem; rather they are descriptive of the process of problem solving. However, as has been illustrated in previous research, many students have difficulty with these basic problem-solving steps as well. I next describe some general problem-solving skills that have been recognized in previous research as necessary for solving problems in general, and combinatorial problems in particular.

Understanding the statement of the problem

Understanding the problem and what it is asking, which is the first step of problem solving described by Polya (1957), is often a major problem for students. This problem can be attributed to many things such as lack of interest on the students' part, or misunderstanding the verbal statement of the problem, or the ambiguous phrasing of the problem. Many students have a poor understanding of the problem, even after reading it a few times. Batanero, Navarro-Pelayo, and Godino (1997) identified three typical misinterpretations of the statement of the problem. The first was not understanding the mathematical model described by the problem. The second was transforming a single problem into a compound problem. The third was failing to correctly interpret some of the "key" verbs in the problem. Hadar and Hadass (1981) also observed students who failed to identify the set of events in the question, and therefore answered a question that was not asked.

Devising a plan

Devising a plan is the second step of problem solving as described by Polya (1957). In fact, devising a plan is often the most challenging part of problem solving. Polya (1957) attributed different factors to one's ability to devise a plan, such as knowledge and experience. He used the metaphor of "gathering the material to build a house" for planning to solve a problem. Although having the material does not necessarily mean that the house is ready, nonetheless, the material is needed for the house. One way to plan to solve a problem is to unpack the problem into smaller problems. Hadar and Hadass (1981) described mathematical language as having a compact nature, and combinatorial problems especially are usually phrased compactly. Hence, the students often need to unpack a problem into a set of smaller problems before they can solve it.

Multiplication and addition

Confusing multiplication and addition is a common mistake for students in general. There is an extensive body of research in mathematics education about students' confusion of multiplication and addition in elementary arithmetic. Additive and multiplicative structures have been examined in detail. For example, to calculate the area of a rectangle, some students add the two sides instead of multiplying them. This problem is particularly notable in combinatorics. Batanero, Navarro-Pelayo, and Godino (1997) describe this error as the error of operation.

Understanding the formula

There are two main formulae used in basic counting problems. One is the permutation formula[4] $P(n,r) = \frac{n!}{(n-r)!}$ and the other is the combination formula[5] $\binom{n}{r} = C(n,r) = \frac{n!}{r!(n-r)!}$. If students do not understand these formulae, they cannot use them properly nor understand the limitations of their use. For many students, rote memorization of formulae is the easy way to solve problems. Therefore, they associate the permutation

[4] $P(n,r)$ means the number of arrangements of r objects from a set of n different objects.
[5] $C(n,r)$ or $\binom{n}{r}$ means the number of selections of r objects from a set of n different objects.

formula with any counting problem where order matters, and use the combination formula for the rest. Batanero, Navarro-Pelayo, and Godino (1997) described a few instances when learners correctly identified the structure of the problem, but did not recall the correct formula. Fischbein and Gazit (1988, p. 197) observed that incorrect use of formulae was "the most frequent type of systematic mistake" among their subjects.

Fixing the parameter or variable

Fixing the parameter or variable is often one of the principal methods of solving a mathematical problem which involves a general parameter. This method helps understand the problem better, find a pattern or reflect on the problem in a more specific and reachable way. Hadar and Hadass (1981) noted that adult students taking combinatorics for the first time did not generally use this strategy when facing such problems.

Generalizing

Hadar and Hadass (1981) observed that even if students were able to fix the parameter or variable, many were still not able to generalize the problem and find a pattern. The fact that students often had difficulty in generalizing, prevented them from thinking about abstract notions in mathematics and eventually from solving many problems.

Verifying the solution

Eizenberg and Zaslavsky (2004) asserted that verifying solutions can help students discover their errors in many cases, and help them to solve the problem correctly after reflecting on their errors. They described the step of verifying the solution as a crucial step, especially in combinatorial problems, since the problems are more complex. Polya (1957) described the last step of problem solving as "looking back", which also meant to reexamine and verify the solution. However, as Eizenberg and Zaslavsky (2004) noted, many students did not automatically verify their solutions.

In the next section I show, using an example, how to apply error analysis to understand students' problem solving strategies in detail. These are possible errors that I encountered in my teaching.

3.4 An example of error analysis

Here I present an example to describe how the error analysis can be used to analyse students' solutions.

Question: How many 3-letter sequences can one make with the letters FINITELY? (Note that there is no repetition except for the letter 'I', of which we have two.)

This problem is an arrangement with limited repetition. One of the possible solutions is to brake the problem into three smaller cases:

- With at most one "I": the only letters that are possible to use are F, I, N, T, E, L, Y (a total of 7 letters), of which we want to use 3 to make a word. There are $P(7,3)$ such sequences.
- With two "I's": In this case we have to choose two places to put the "I's" and we have to put one of the letters F, N, T, E, L, Y in the place of the third letter. There are $C(3,2) \times P(6,1)$ such sequences.

The final step is to add the number of the two parts: $P(7,3) + C(3,2) \times P(6,1)$

A solution generated by one of the students was $P(6,3) + P(7,3) + P(8,3)$. In this case, it seems that the student knew that the order of the 3 letters chosen is important. She or he also broke the problem into 3 cases. But the student did not take into consideration that we do not have 7 or 8 different letters to choose from, since the use of "I" for one or two positions is then mandatory and we have only 6 letters to choose from for the remainder of the positions. This presents an example of over-counting which means that some configurations have been counted more than once. For instance, the sequence FTL has been counted three times, once in each of the three parts of the solution.

Another answer presented by a few students was $\frac{C(8,3)}{2!}$. This answer indicates some understanding and some misunderstanding of the problem. For example, the student seemed to recognize that we need to choose 3 letters from the 8 letters and since "I" may be repeated the answer should be divided by 2!. However, there are two errors associated with this solution:

- **Error of order:** use of combination instead of permutation.
- **Under-counting:** dividing the whole solution by 2!, which mistakenly assumes (if understood properly) that 2 identical "I's" were present in all configurations.

The above example shows how the categorization of errors can help us analyse students' solutions and get a better and deeper insight into their possible misunderstandings and errors involving enumeration problems.

3.5 Summary

Previous accounts of mathematics education research in the area of combinatorics suggest that combinatorics, because of its use in other areas of mathematics, statistics, and in sciences, is an important subject to address in the mathematics curriculum. However, there is a lack of detailed and extensive research into educational aspects of this branch of mathematics.

The existing research is mainly concerned with students' difficulties with combinatorial structure and problem solving. These difficulties are explored as they relate to combinatorics in particular, or as they relate to students' general problem-solving ability. These previous studies answer, although not all, of the initial questions presented in Chapter 1. However, to the best of my knowledge, there are no studies that directly examine students' understanding of combinatorics, and in particular how students understand new concepts and modify old concepts to achieve a better understanding.

In Chapters 6 and 7, I describe two investigations, each examining a different aspect of students' understanding of combinatorics. The data in both studies revealed that the above errors occurred repeatedly and that they appears to represent the real difficulties and obstacles that students face, regardless of their mathematical background.

Chapter 4

Understanding Combinatorics: a Theoretical Perspective

4.1 Introduction

Examining learners' understanding is a challenging task. In fact, it seems to be very difficult to understand and describe someone else's understanding completely. However, we can identify some factors that can be used to paint a reasonable picture of a someone's understanding. Choosing the appropriate factors, and the setting to examine them, constitutes the framework in which one can explore and describe understanding.

In this chapter, I describe the framework that I developed to examine students' understanding of combinatorics. This framework is based on the theory of understanding and its components as described by Sierpinska (1994), the notion of concept image developed by Tall and Vinner (1981), and students' initial understanding of a definition by Dahlberg and Housman (1997).

4.2 What is understanding?

Philosophers have been studying understanding for centuries. They have examined many aspects of understanding, such as what it means, what constitutes it, what types of understanding there are, whether we can understand everything, what are different theories of understanding, etc. Sometimes we can fail to distinguish between knowledge and understanding, as there is a strong connection and dependence between the two. I believe that knowledge is necessary for understanding, but it is not sufficient. However, some knowledge cannot be achieved without proper understanding of its background. Therefore, understanding requires knowledge, and at some level furthering and making knowledge requires understanding. "Descartes placed knowledge at the head of the mainstream philosophic agenda, where it remained for three hundred years" (R. Mason, 2003, p. 2). He describes the aim of philosophers as achieving better understanding, not more knowledge.

Different meanings and metaphors have been associated with understanding. Understanding as seeing, understanding as being able to, understanding as meaning, or understanding as interpretation are just a few of these associations. Even after considering many meanings and metaphors for understanding, it remains very difficult to understanding in general. Sierpinska quoted Ajdukiewicz for a definition of understanding: "a person understands an expression if on hearing it he directs his thoughts to an object other than the word in question" (Sierpinska, 1994, pp. 28-9). This is, of course, a very vague definition of understanding. However, it ties nicely to what Tall and Vinner (1981) called concept image. According to Tall and Vinner

(1981), concept definition is “a form of words used to specify that concept” (p. 2), and concept image is “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes”(p. 2). This cognitive structure includes but is not limited to all forms of representations, formulae, and examples of the given concept. It can also include the connections between that concept and other concepts. In the following section, I describe the different components of understanding and describe the relation between concept image and understanding.

4.3 Components of understanding

There are many factors that affect and contribute to the process of understanding such as various reasonings, examples, previous knowledge and experience, figurative speech, and practical and intellectual activities (Sierpinska, 1994, p. 72). Based on the definition of Ajdukiewicz, Sierpinska (1994) identified the four main components of an act of understanding as: the understanding subject, the object of understanding, the basis of understanding, and the mental operations.

The first component is the ‘understanding subject’, which is the person who understands. In the mathematics education literature, understanding subjects are the participants in the study. Usually some description of understanding subjects is included in each research report. This description can vary greatly in detail depending on the focus of the report and the research methodology.

The second component is the ‘object of understanding’, which is what the understanding subject intends to understand. Objects of understanding can be content specific such as concepts and problems, or language specific such as mathematical formalism or texts. What the object of understanding is, I believe, it is very important to clarify, for both pedagogical and research purposes.

The third component is the 'basis of understanding', which is what the understanding subject's thoughts are directed to when trying to understand something. Some examples that Sierpinska (1994) discussed as the basis of understanding are representations, mental models, and apperception (understanding based on previous experience).

The last component is the 'mental operations', that is the work of the mind that directs or links the object of understanding with its basis. There are a few mental operations discussed by Sierpinska (1994) such as identification, discrimination, generalization, and synthesis. Identification is classifying an object with a set of other objects. This set can be a set of known objects with or without a name, or it can be a set of unknown objects with a common characterization. Discrimination is identification of one or more differences between two objects. Discrimination can be just the recognition that the objects are different, or it can be the identification of differences as well. Generalization is recognition of an object as a special case of another object. Synthesis, which is sometimes also called schema in the literature, is defined by Sierpinska (1994, p. 60) as:

> [...] the search for a common link, the unifying principle, a similitude between several generalizations, and their grasp as a whole (a certain system) on this basis.

While it may be difficult to define understanding as a whole, and what constitutes good understanding, it is often easier to think about understanding and study it in terms of these four components.

The notion of concept image is closely related to the notions of basis of understanding and mental operation. In fact, the basis of understanding is the understanding subject's concept image of the object of understanding. Mental operations are the cognitive activities that support and modify the concept image or the basis of understanding. Examining different aspects of the basis of understanding and the mental operations can reveal some part of students' understanding.

Before discussing understanding any further, I want to acknowledge many other factors that can potentially affect the act of understanding, such as psychological or sociological conditions. However, in this study, I will not discuss these dimensions. Rather, I shall concentrate on a cognitive approach to understanding.

4.4 Theoretical framework

In this thesis I present two studies. In Chapter 6, I present a study in which I examine how learners place a *new* object of understanding within the existing concept image by making connections between the existing concept image and the new object and thereby creating a concept image for a newly introduced concept. In Chapter 7, I examine how learners refine and adjust an *existing* concept image, which may or may not be adequate, to achieve a better understanding of the concept.

In what follows, I describe a framework I developed for understanding the process presented in Chapter 6. This framework describes some of the factors that can illustrate the students' concept image formation and how learners connect a new concept image to their old network of related concept images. I also describe a framework for understanding the process presented in Chapter 7, that describes the change in the learners' concept image when they are confronted with a problematic area in their existing concept image.

4.4.1 A glance into the formation of the concept image of a newly introduced concept

According to Tall (1988), an insight into the variety of concept images can help teaching and learning by suggesting ways of giving students a more coherent concept image. Having a coherent concept image can help students achieve a more complete understanding of the concept, which is what will remain with them. Tall (1988), quoting Vinner, contrasted the importance of concept image with that of concept definition. He noted that concept definitions usually remain inactive or even forgotten, whereas concept images are almost always evoked during thinking.

Based on the distinction between concept image and concept definition, Dahlberg and Housman (1997) developed a theory in order to examine the initial understanding of students who were given a new definition. They identified four basic learning strategies being used by students: example generation, reformulation, decomposition and synthesis, and memorization. They argued that students who used example generation had a more sophisticated initial understanding of the definition.

Having examined the different aspects and components of understanding in the literature, I developed a framework that examines students' understanding of a new concept. This framework is based on the one introduced by Dahlberg and Housman (1997). However, it is tailored to examine students' understanding of combinatorics. This framework enables me to examine students' concept images from a variety of perspectives, based on their responses to the tasks in the interviews.

As mentioned before, concept image is the total cognitive structure that is associated with the concept. It can be very difficult, if not impossible, to access the concept image as a whole. However, by choosing appropriate tasks, related to that concept, and a careful examination of the tools and strategies that students use in approaching these tasks, we can reveal parts of their concept image. These tools can also help the students to expand their existing concept image and as a result understand the concept better.

There are many such tools; however, I decided to focus on four of them because, after a preliminary examination of data, they seemed to be the most appropriate for studying students' understanding of elementary combinatorics. These tools are: students' use of *examples*, use of known *formulae*, and the *representations* that are used to understand the content and communicate one's thoughts. The mental operation of making *connections* between previous knowledge and the task at hand is also an important part of furthering one's understanding and should be also considered. Examining each of these factors can reveal a part of the concept image or mental operation which is evoked by the tasks. These revelations can paint a reasonable picture of learners' concept image, and how it is modified to accommodate a new concept or achieve a better understanding of the old concepts. In the remaining part of this section, I describe each of these factors and how they contribute to examining learners' concept image.

Examples

Examples have a substantial role in mathematical explorations. They contribute to the creation of new ideas and concepts, as well as to the understanding and exploration of existing mathematical structures. Mathematicians often create examples that embody the important characteristics of a concept, to further their understanding of the structures they are exploring. They use examples to study the boundary cases, to show something to their peers, to explore the existence of some structures, to guide and then check a proof, to disprove a conjecture, to refine definitions, to identify patterns and generalization, etc.

Examples also have a fundamental role in teaching and learning mathematics. Teachers often present examples in mathematics classrooms, to help students understand and explore different topics. Students are often used to having examples presented to them by their teachers. We can contrast this typical use of examples in the classroom with their use by experienced mathematicians. Mathematicians deliberately create their own examples for a better understanding of a concept or a structure, whereas students are usually presented with examples and they observe them. In fact, educators are often asked by their students to provide more examples. This dependency on outside sources, such as teachers or textbooks, to provide examples can be potentially problematic.

Hazzan and Zazkis (1999) distinguished the pedagogical effect of teacher-generated examples (learning *from* examples) with that of learner-generated examples (learning *with* examples). Learning from examples can be problematic, because of the possible miscommunication or misunderstanding of the intent of the example. The teacher knows what makes an example exemplary and why he or she has chosen that particular example to exemplify a particular mathematical situation. However, it is not clear if the students can also see what their teacher considers important and exemplary in that example. From experience, often I can see that students concentrate on different aspects of an example, than those that were intended. So even though examples generated by teachers are certainly an important teaching tool, they should be supplemented by learner-generated examples to enable students to understand the purpose of examples in a deeper and more constructive way, also to require more involvement and engagement from the students, to assist their learning.

Watson and Mason (2005) discussed the issues that can arise from students' inability to generate their own examples. These authors offer a method to encourage students to generate their own examples, and be more creative with these examples. They suggest ways of making a shift from textbook- or teacher-generated examples to learner-generated examples. One of the benefits they identified is that when learners are invited by the teacher to construct their own examples, it helps them to think about the topic in a different way and it gives them an opportunity to gain a new understanding of the underlying concepts.

In light of this, I categorize three different types of examples in a learning situation:

1. Passive examples: these examples are presented by the teacher or the textbook to illustrate an instance of a concept, so as to aid students with understanding that concept.

2. Learner-generated examples: these are examples that are generated by students when they are prompted by their teacher.

3. Active examples: these examples are created by students on their own initiative and without any prompt from their teacher, in order to aid their understanding.

In traditional textbooks and many classrooms we often see that examples are used after the introduction of a concept to help students understand that particular concept. Therefore, students are used to "being given examples" to help them understand a concept. In fact, passive examples are the most common source of examples used in a typical classroom. They are generally seen as an efficient way to illustrate a concept and its special cases. However, they usually increase students' dependency on teacher-generated examples and, moreover, they create a limited concept image for students. Students often create a library of passive examples to help them solve problems. This type of problem solving is based on rote memorization, and requires a minimal amount of creativity.

Frequent use of learner-generated example tasks can help students develop the habit of using active examples. Watson and Mason (2005) discussed many ways and settings in which students can be asked to generate their own examples. In addition to their pedagogical benefits, learner generated examples can be an important research tool. Zazkis and Leikin (2007) discussed some of the key benefits of observing the collection of examples that students generate for a particular concept, as a research tool. They believed that studying these example sets can provide a window into the students' understanding of mathematics.

Generating active examples is an effective strategy for learning and discovering. Dahlberg and Housman (1997, p. 293) described the effect of such examples: "the generation of and reflection on examples provided powerful stimuli for eliciting learning events". If learners attempt to generate examples, without an outsider prompting them to do so, they can experience the true advantage and benefit of examples in learning and understanding new concepts.

Formulae

The use of formulae is a prominent method of calculation in mathematics, physics, and engineering. Use of formulae is often easy and efficient. Many students' first and automatic approach to solving problems is to rely on formulae. In fact, many students reduce their approach to mathematics to a set of formulae that they need to memorize and recall when solving problems. The foundation of this method of solving problems is the creation of a library of previously-seen problems and formulae. When faced with a new problem, such students search this library and try to find a problem that closely resembles the problem at hand. If there is no such problem in their library then there is a road block in the way of solving that problem.

In combinatorics, especially in counting, there are very few formulae; furthermore, each formula is presented with the typical situations where it can be used. That makes this part of mathematics an easy target for rote memorization of formulae. Sometimes formulae are used as a substitute for understanding, and students' impression and experience of mathematics is reduced to the use of a set of formulae. As noted in chapter 3, incorrect use of formulae was observed by Fischbein and Gazit (1988, p. 197) to be "the most frequent type of systematic mistake" in solving combinatorial problems. There are so many different problems and different situations, which makes memorization problematic.

I believe it is important to examine the role of formulae in students' concept image, to examine if a formula is any more than an instrumental object for the purpose of faster calculations. Students often trust formulae so much that they do not rely on common sense and intuition, even if the answer they achieve with the formula is irrelevant to the context of the problem. There are different ways that students use formulae:

1. Calculations and solving problems: students can use formula to solve problems more quickly and efficiently.

2. Recognition of a structure: formula can help students identify two structures as isomorphic or recognise them as non-isomorphic by comparing the formula that can be used to count the number of possibilities in each structure.

3. Substitution for understanding: instead of formulae being a part of students' understanding, it becomes their entire understanding of a concept.

Using a formula is an effective way of solving some types of problems, but by no means can it be a substitute for understanding mathematics, nor can mathematics be reduced to a set of formulae that should be memorized. So even though having the formula is a necessary part of concept image for some concepts, it should not be the whole concept image.

Representations

Communicating any idea or thought, and understanding others people's ideas or thoughts, requires some form of representation. The most common representation of thought is in the form of language, oral or written. To understand someone's thoughts or to communicate effectively there is a need for a common representation. Mathematics uses common representations as well as its own unique representation and language. Hence, to understand a mathematical statement or structure, one needs to understand its different representations.

The representation used to communicate one's thoughts exposes some details about that thought. The kinds of representations that students use can reveal how they think about a concept and can be yet another window to the complex concept images of learners.

The use of representations has been widely examined in mathematics education literature. Principles and Standards for School Mathematics (NCTM, 2000) contains a specific standard for students' use of mathematical representations:

> Instructional programs from prekindergarten through grade 12 should enable all students to–
>
> - create and use representations to organize, record, and communicate mathematical ideas;
> - select, apply, and translate among mathematical representations to solve problems;
> - use representations to model and interpret physical, social, and mathematical phenomena.

In Principles and Standards for School Mathematics (NCTM, 2000), authors encouraged students to "represent their mathematical ideas in ways that make sense to them, even if those representations are not conventional". However, they asserted that the students need to be familiar with and understand the conventional representations to be able to communicate their ideas and understand the existing ideas that are expressed in the conventional forms.

I want to limit the use of the term "representation" to include only the basic tools that students use to convey their idea or understand a concept. In this work my focus is on examination of learners' use of visual (pictorial) and symbolic (algebraic) representations.

The Mental operation of making connections

One of the most important mental operations involved in an act of understanding is making connections between different structures to create a coherent image of each structure within the general schema of the concept.

Principles and Standards for School Mathematics (NCTM, 2000) emphasizes the importance of the ability to make connections. The outcome that is expected was discussed in detail as one of the standards for school mathematics across different grades:

> Instructional programs from prekindergarten through grade 12 should enable all students to
>
> - recognize and use connections among mathematical ideas;
> - understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
> - recognize and apply mathematics in contexts outside of mathematics.

As mentioned before, Sierpinska (1994) identified four mental operations involved in the act of understanding: identification, discrimination, generalization, and synthesis. I use the term "making connections" to represent all these operations in the context of understanding combinatorial structures. In particular, a connection can be made between two combinatorial structures by identification or discrimination.

Making connections is ultimately important in mathematical understanding because: "Viewing mathematics as a whole also helps students learn that mathematics is not a set of isolated skills and arbitrary rules" (NCTM, 2000).

4.4.2 Changes in the concept image and the act of understanding

In the framework of concept image, the act of understanding can be described as the creation of a new concept image, or the modification of an existing concept image, so that the concept image embodies the actual concept in a more comprehensive and integral way. Hence, examination of the learners' concept image over time enables the researcher to get a better insight into the learners' understanding. In the previous section, I described some of the factors, namely examples, formulae, representations, and making connections, that can illustrate learners' concept images. In this section, I explain the framework in which various types of change in the learners' concept images can be described.

Vinner (1991) described different scenarios of what might happen when a student is faced with a concept definition of something for which she or he has a previous concept image. For example, a student might have a concept image of a two-dimensional coordinate system as two perpendicular axes. Later, the teacher may define a coordinate system as any two intersecting straight lines. The three scenarios are given as (Vinner, 1991, p. 70):

1. The concept image may be changed to include also coordinate systems whose axes do not form a right angle. (This is the satisfactory reconstruction or accommodation).

2. The concept image may remain as it is. The definition cell will contain the teacher's definition for a while but this definition will be forgotten or distorted after a short time [...]. (In this case the formal definition has not been assimilated.)

3. Both cells will remain as they are. The moment the student is asked to define a coordinate system he will repeat his or her teacher's definition, but in all other situations he or she will think of a coordinate system as a configuration of two perpendicular axes.

The above scenarios are surely a good start for thinking about possible changes to the concept image; however, they can be modified to have more descriptive power by making them more specific, as follows:

1. Persistent concept image: The concept image remains as it is, and the student carries on doing what he or she was doing before the task. No learning event occurred.

2. Inappropriate concept image without improvement: The concept image has changed, or a new concept image has been formed, but it is still not an appropriate concept image. The process of learning led the student to a misunderstanding.

3. Inadequate concept image with improvement: The concept image has changed, or a new concept image has been formed. It is not yet an adequate concept image, but it is an improvement from the previous concept image. The process of learning has started.

4. Fragmented concept image: The concept image may remain as it is, and the student creates a separate cell to store the new way of seeing the concept without any connection to the previous concept image. The process of learning has started.

5. Flexible concept image: The concept image may be changed to include or accommodate this new way of seeing the concept. An appropriate concept image is formed and the appropriate connections are made. A learning event has occurred.

In the first scenario, nothing that can be considered educationally significant has happened. In the second scenario, the student has replaced one inappropriate concept image with another inappropriate concept image. In the third scenario, the learner has replaced or changed an inappropriate concept image with another inadequate concept image. However, the new concept image is an improved image in comparison to the old concept image. In the fourth scenario, the student has added the concept definition to his/her concept image, but in a way that it has not affected the existing concept image, i.e. the students has not made the connection between the definition and the concept. In the fifth scenario, the learner has changed the concept image "successfully" and has achieved a better understanding.

4.5 Summary

Accessing different concept images of a concept is very important for both teachers and students. By accessing different concept images, teachers can see the concept from different perspectives and teach it from a variety of view-points to accommodate different students and different learning styles. Students need to access different concept images to understand the concept better, to be able to identify the concept, and to see its connection to other concepts using the variety of concept images.

In this chapter I discussed some aspects of understanding and connected the basis of understanding to the notion of concept image. As it is difficult, if not impossible, to measure ones' understanding precisely, I decided to focus on two specific aspects of understanding. First I introduced some indicators to examine students' concept image of a newly formed understanding. Second I categorized the types of change that can occur in the concept image when students are faced with incompatibility between their existing concept image and the appropriate concept image.

Chapter 5

My Journey

5.1 Introduction

In this chapter, I describe my journey as an educator and an educational researcher, in order to provide a context for my two studies, described in Chapters 6 and 7. This chapter is written in a narrative format, and described the events in chronological order to allow the reader to follow me through my journey as a teacher/researcher.

As mentioned in Chapter 1, my interest in mathematics was in large influenced by my teachers through my experience as a student. Hence, there is little doubt that my education and educational experience had a substantive impact on my professional and personal interests, and eventually directed me toward my studies in mathematics education.

I started my studies in mathematics education when I was employed as a full-time mathematics instructor at Langara college. My main motivation in pursuing research in pedagogy was to become a better teacher and hopefully influence my students in a positive way. Even though I knew that my thesis needed a firm theoretical grounding, I wanted it to have a strong practical and pedagogical implication as well. Hence, from the very beginning, I chose to perform at least part of my research in my own classroom, assuming the role of teacher/researcher.

Being a teacher provided me with ample opportunities to observe my students' endeavors and struggles to learn. They faced many obstacles and challenges, and yet they thrived for more knowledge. The students' motivation to learn combined with my motivation to become a better and more effective teacher inspired me to pursue a robust research on students' learning that would inform my teaching and that of other interested teachers.

The decision to perform two separate studies was made when I decided to approach two distinct groups of students who were learning combinatorics in post-secondary level. These two groups were distinguished by their interest and background in mathematics; one group, college students, who were less than enthusiasts about mathematics, also had a weaker background in it, and another group, university students, were more enthusiastic about mathematics and had a stronger background in it. The reason I chose to study these two group was not to compare their abilities or learning, but to gather richer data from different groups of students.

In the next section, I describe the study that arose directly from my classroom in the college, and later in the chapter, I will describe the other study that I performed in the university.

5.2 The study of changing a concept image

5.2.1 Motivation and background

During the Spring of 2004, I attended the meeting of Canadian Mathematics Education Study Group in Université Laval. In this meeting, I attended a working group called "Learner-generated examples as space for mathematical learning" led by Anne Watson, Rina Zazkis, and Nathalie Sinclair. The working group explored the role of learner-generated examples as a pedagogical tool. The ways to help students construct and reconstruct their knowledge of mathematics through a set of example-generation tasks were discussed in a series of group meeting spanning over three days.

This workshop not only influenced my teaching, but it also directed my research by providing me with an opportunity to examine the role of examples in mathematics classrooms, how we can encourage students to generate their own examples, and the effects of example-generation on students' learning. I also examined learner generated examples as a data-collection tool. Both studies described in this dissertation have a flavor of learner-generated examples.

In the Fall of 2004, I was teaching a course called finite mathematics. This course was designed for students who did not wish to take calculus, but required a post-secondary mathematics course for their degree. The course was geared toward arts and social sciences students. My exposure to the role of learner-generated examples a few months prior to teaching this course, inspired me to apply this method in my classroom.

The first part of the course was logic. Students showed a particular difficulty with the meaning and purpose of truth tables and their relation to logical operators.

To emphasize the theoretical aspect of logical operators and how the truth tables are constructed, I asked students to define a new logical operator using truth table. Then I asked them to use their logical operator and answer a few questions similar to the ones in the text-book, only replacing some of the existing logical operators with the ones students created. This was the first time I exposed students to a task involving learner-generated example. After this first task proved to be useful in students' understanding and was received well by students, I decided to continue assigning similar tasks once every few classes when the opportunity arose. In doing so, I recognized the potential for using learner-generated examples as a methodological tool for gathering data for my research within the first month of the term.

5.2.2 Ethical approval

Recognizing the potential for research in my classroom led me to the administrative part of my research, that is, applying for and receiving the ethical approval which is required for research on human subjects.

When I approached the dean of Instruction at Langara college to apply for ethical approval he asked me to first get the approval from Simon Fraser University and then from the mathematics department chair at the college and then seek approval from the dean. Consequently, I applied for ethical approval from Simon Fraser University in September of 2004. Since I had planned from the beginning to perform research at both Langara college and Simon Fraser University I applied for ethical approval for research on students at both institutions at once.

Upon receipt of approval from the university and the department chair at the college in October of 2004, I immediately applied for the formal approval from the dean of instruction at Langara, and received the approval in the beginning of November of 2004.

The approval was granted based on the voluntary participation of students and that their grades were not affected by their participation in and results of the research. I, personally, found this requirement fair, and I took it one step further, and decided to study a concept that was not going to be included in any of the exams or assignments. Luckily, I was presented with the perfect opportunity to do so. The students were also informed that their participation was voluntary and it would not affect their course evaluation in any direct or indirect format. Also none of the students' real names is used in this dissertation.

After the receipt of the ethical approval from university and college, which coincided with teaching the section on combinatorics, I started investigating students' responses to example-generation tasks to examine their understanding of combinatorics. This investigation provided an opportunity to observe and examine formation and evolution of different concept images by students. However, in one occasion I was presented with an intriguing opportunity. When I presented the students with a seemingly simple combinatorial problem, most of them offered a similar inappropriate solution. The study that is described in Chapter 7 was created from this seized opportunity.

5.2.3 Experience of being a teacher-researcher

In this study, I was a teacher and a researcher, which had its advantages and disadvantages. There are many studies regarding the role of teacher as a researcher in education in general such as Adelr (1993) and Cochran-Smith and Lytle (1999), and in mathematics education in particular such as Atkinson (1994) and Jaworski (1998). There are also different names associated with this type of research, such as "reflective practice", "action research", and "teacher research". There are, of course, subtle differences between these terms, but they all reference the dual role of a *teacher-researcher.* However, compared to reflective practice, action research and teacher research are the more "systematic enquiry" of certain complexity that are made public (Atkinson, 1994, p.385).

Cochran-Smith and Lytle (1999) describe the trends of teacher enquiry and research along with some critiques of this methodology. The authors claim that although the teacher-research movement needs to address the critiques, it is a movement on the rise and it has been proven to be very useful, not only for the teachers themselves, but for the larger community of researchers and teachers. Adelr (1993, p.166) describes the importance and legitimacy of reflective practice as a form of research for teacher educators and invites educators to share their "wisdom of practice with those engaged in similar work".

Jaworski (1998) discusses the evolution of teacher research and its positive influence on teachers' own practice and teaching of mathematics. Atkinson (1994) outlines and discusses the tensions and struggles involved in the double role of teacher and researcher. She distinguishes teaching and researching as two distinct modes of operating and claims (p.398):

> The type of thinking needed for action research sits somewhere uncomfortably between the quick intuitive judgements of the teacher and the more rational and explicit analysis of the researcher.

In fact, my experience echo's dilemmas and tensions acknowledged by Atkinson (1994) on the struggles of the dual role of teacher/researcher. There is a tension, specially when one has to decide to assume only one of the two roles, because of the conflicts between research and teaching. Atkinson (1994) outlines fourteen conflicts between research and teaching, some of which I experienced personally, for example: the need to act instantaneously and intuitively when teaching and the need to plan well ahead of time and rationally. I also had difficulty balancing teaching and research. One of the main difficulties I faced was that even though I interviewed some of my students, these interviews became tutorial sessions, and thus I was not able to use the data collected from these interviews in my research. In the interview setting, the teacher role took-over the researcher role. Another consideration as a teacher was making sure that students were not at any disadvantage as far as their learning and the flow of their class were concerned, because of my research.

Another difficulty was that for ethical purposes, I did not include the data from my students' examinations and assignments in this study. I assured all participants that anything that is a part of my research is done purely for that purpose, and will not be used for grading or evaluation in any form. Therefore, participation in this research had no effect on students' grades, and it was completely voluntary. This created some uncertainty during data gathering about the number of participants who would complete all the tasks. In spite of these circumstances, the participation rate was very high among students and the tasks were taken seriously. Out of sixteen students registered for the course, fourteen participated in the classes, and from these fourteen students, twelve participated in the research and in all the tasks.

However, this dual role presented me with some advantages as well. The main advantage was my extensive familiarity with my students, their abilities and backgrounds. I was aware not only of their mathematical background, but also what they have been taught in the class. I could use this extensive familiarity to put the data in context, which in turn helped the interpretation and analysis of the data.

I also had the advantage of choosing and employing the method of teaching that I found most beneficial for my students in understanding the concepts. I had ample time to interact with and observe each of the students in class and in my office hours. Another advantage was getting an instant feedback on the teaching and pedagogical methods I employed in this dual role.

In the next section, I discuss the study that was performed in Simon Fraser University.

5.3 The study of forming a concept image

5.3.1 Motivation and background

In this study, I wanted to examine students' formation of a new concept image, how they connect the newly formed concept image to their existing images of related concepts. Hence, I created a new definition to present to students, which had strong connections to their existing knowledge of combinatorics, yet was not discussed before with the students. I designed a set of tasks to exemplify different kinds of questions to give me a more complete picture and assessment of students' concept image. These questions dealt with problem-solving, example-generation, and exploring connections. This study was inspired by an article by Dahlberg and Housman (1997) as will be explained in more detail in Chapter 6.

5.3.2 Ethical approval

As mentioned in the previous section, I had received the ethical approval for my study at Simon Fraser University at the same time as my study at Langara college. The same as in the previous study, the participation was voluntary and the interviews and research outcomes had no part in the course evaluation, and all the names discussed in this dissertation are pseudonyms. In fact, the instructor of the course was unaware of the results of the interviews, and the participants were assured that the interviews were not related to their class evaluations in any way and that the results would not be shared with their instructor.

5.3.3 Experience of being a researcher from outside

The target participants of this study were first-year university students who had a strong background and interest in mathematics, and were pursuing a degree that required them to take more than calculus. The purpose of the study was not to compare this group of students to the students with less background in mathematics who were enrolled in finite mathematics at Langara college. The purpose was instead to obtain richer data by examining understanding of different groups of students attending post-secondary institutions. I knew that the students enrolled in MACM 101[1] would be a good target for my study.

After having experienced the challenges and rewards of my dual role of teacher and researcher in the previous study, I found new struggles and advantages as a researcher

[1]MACM 101 is the first part of Discrete Mathematics course at Simon Fraser University.

from outside in my second study in the university. My first struggle was to find an instructor for the specific course who would agree to accommodate my research in his or her classroom. To find the willing instructor, I only had one chance every semester, as MACM 101 was (at the time of this study) only offered one section per semester.

To make my chances better at finding a willing instructor I decided to consider the following factors:

1. *Time.* Being a teacher myself, I knew the value of time for the instructor. So I tried to minimize the time I took from the class and not to interrupt the flow of the course.

2. *Inconvenience.* I recognized that my presence as a researcher will create some degree of inconvenience for the instructor. I decided to minimize this inconvenience by as much as possible by arranging one short meeting with the instructor to get a general sense of the class and the teaching method.

3. *Schedule.* Since combinatorics was taught toward the end of the semester, I decided to postpone the interviews until the end of the term when the students have studied and had time to reflect on the necessary parts of the course content.

I was lucky to find a willing instructor in the Fall of 2005. I asked the instructor for five minutes at the end of a class toward the end of the term and also asked him/her to spend some time with me to describe her teaching and the content of the course at her convenient time. The instructor was kind enough to grant me all my wishes. She/he met with me once and provided me with a handout that was given to students about the related material, and also described her/his teaching method and was very interested in my research. It was surely a very positive experience to work with an instructor who valued my research and accommodated all my requests. Then it was the time to encourage the students to volunteer for participating in the study in the five minutes I had at the end of the class, when most students were eager to get dismissed. I offered to tutor the students in exchange for their time for the interviews. I thought that this offer would interest some of them at least, specially since it was toward the end of the term and close to final exams. Eight students volunteered to participate, but non of them wanted the tutoring; they just participated because they were interested in research and in mathematics. Out of these eight students, five of them stayed after the interview to ask me questions about my research and the tasks in the interview. This was one of most encouraging moments of my research.

5.4 Summary

In my experience both methods of research (teacher/research or outside-researcher) were rewarding and they involved some struggles and challenges. One of the main struggles of the dual role teacher/researcher was that it was difficult to find the balance between the two and whenever I needed to make a choice between these two roles, I always gave the priority to my role as a teacher. One of the main struggles in being an outside-researcher is that you do not know the students well, and you do not know how they were taught or what they learning experience has been, which makes the analysis of data more challenging.

What stood out the most in both studies was the willingness of the participants to volunteer their time and trust me with their thoughts and knowledge. I hope that my analysis of the data they provided me is respectful of their endeavors, hard work, and achievements.

In this chapter, I explained the methodological context of my two main studies. In the next two chapters, Chapter 6 and 7, I present these two studies in detail, not in the chronological order, but in the order that best describes the process of understanding.

Chapter 6

New Definitions, Old Concepts: Exploring the Connections

A definition is a brief statement that describes the nature of something or the meaning of a word, a phrase, or an expression. In mathematics, definitions are used to name mathematical structures or relations. Mathematical definitions are generally precise, minimal, and elegant. Each definition describes a class of objects exactly, so that all members of that class can be identified and all nonmembers can be excluded using the definition. Ideally definitions are minimal because they "should not contain parts which can be mathematically inferred from the other parts of the definition" (Vinner, 1991, p. 65). Despite the importance of definition to mathematicians, there are many reasons why mathematical definitions are not pedagogically effective, and do not aid students' understanding (Vinner, 1991).

In deed, there is an abundance of research about the role of definitions in teaching and learning mathematics (Edwards, 1999; Vinner, 1991; Leikin & Winicki-Landman, 2000). Vinner (1991) examined the pedagogical and epistemological role of definitions in mathematics. Edwards (1999) concentrated on students' understanding of the importance of definition. Leikin and Winicki-Landman (2000) explored the relations between different definitions in detail. Dahlberg and Housman (1997) examined students' initial understanding of a new definition. Moreover, these authors acknowledged the importance of understanding formal definitions for students in advanced mathematics courses.

Students are expected to understand and use mathematical definitions in most post-secondary mathematics courses. In combinatorics, like other disciplines of mathematics, there are formal definitions for specific structures. One of the textbooks that is used widely in first-year discrete mathematics courses offers the following definition for a combination (Grimaldi, 2004, p. 15):

> If we start with n distinct objects, each selection, or combination, of r of these objects, with no reference to order, corresponds to $r!$ permutations of size r from the n objects. Thus the number of combinations of size r from a collection of size n is
>
> $$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}, \quad 0 \leq r \leq n.$$

Then the author adds:

> A word to the wise! When dealing with any counting problem, we should ask ourselves about the importance of order in the problem. When order is relevant, we think in terms of permutations and arrangements and the rule of product. When the order is not relevant, combinations could

play a key role in solving the problem.

This is the only place where a combination is defined. This definition of combination is intertwined with the formula that is used to count the number of combinations. The author further explains the definition by distinguishing combination from permutation, emphasizing the importance of order. Each definition is followed by a few examples that highlight some properties of the structure for students. But how much do students really understand the definition that is presented to them? How do they connect it to the object that the definition describes? What is needed for students to understand the definition? What does it mean to understand the definition from the students' point of view, the teachers' point of view, and the mathematical point of view? Explicit reformulation of these general questions to my research questions are presented later in this chapter.

To answer some the above questions, in this chapter I examine students' understanding and use of a definition of a combinatorial structure that they have not seen before. I also examine how students connect this new definition to what they have learned before, and how they place this structure within the general schema of the previous familiar combinatorial structures.

I first describe the research objective and the setting in which the study took place. I explain the definition and the tasks that were given to students, and the purpose of each task. Then I present the results, and analyse them using the framework described in Chapter 4. At the end of the chapter I summarize how this study contributes to the research of students' understanding of combinatorics.

6.1 Research objective

The objective of this study was to examine how learners understand a new combinatorial structure and how they approach a new concept. Using the framework that I developed, described in Chapter 4, I looked at some of the factors that contributed to the participants' concept image. Specifically, the following research questions are addressed:

- How do learners understand a new combinatorial structure? How do they approach a new concept? In particular:
  - Do they use examples to get a better understanding of the structure?
  - How do they use known formulae? Do they understand the formulae they use? Can they create new formulae to solve a problem?
  - What kinds of representations do they use to describe their thinking and represent their solutions? Do they use different kinds of representation? Do they use appropriate representations? Do they use a graphical approach or an algebraic approach?
  - What kinds of connections do they make to their previous knowledge? How do they use their previous knowledge to learn this new definition? Do they concentrate on the differences and similarities between the new structure and the other structures they have known previously?
  - How do they approach tasks relating to this new concept?
  - On what aspects of this new structure do they concentrate more?

6.2 Research setting

6.2.1 The course

Discrete Mathematics I, *MACM 101*, is the first part of a two-semester course att Simon Fraser University. This course is required for students with a major or minor in mathematics, computer science, engineering, as well as other programs such as management and systems science, cognitive science, and geographic information science.

In this course, students explore introductory topics in discrete mathematics. The list of topics in MACM 101 includes logic, number theory, set theory, formal reasoning and induction, functions and relations, automata theory and languages, and counting methods. The pre-requisite of this course is the high-school course Math 12. On average the enrollment of this course is about 200 students. Therefore, the instruction was designed to accommodate students in a large classroom setting.

The textbook used in this course is Grimaldi (2004). In the particular semester that I performed this study, the instructor covered the counting methods towards the end of the semester.

6.2.2 Participants

The participants in this study were students enrolled in MACM 101 in the Fall of 2005. I visited the class one day, at the end of the class, and asked for volunteers to participate in clinical interviews on the topic of combinatorics. Eight students volunteered, most of whom were interested in participating because of their interest in the course and in mathematics in general. In fact, in the questionnaire that was given to them prior to the interview, Don, Hanna, Lola, Nadia, and Pam each identified themselves as a person who loves mathematics. Anna, Felix, and Matt identified themselves as people who are taking mathematics because they have to, but they were also okay with it. None of the participants felt uncomfortable with or disliked mathematics.

I do not claim that this is the usual trend in such classes. However, because the participation in this study was on voluntary, I could safely assume that people who did volunteer were more confident students with a stronger background and interest in mathematics than most of their classmates. This assumption was confirmed by students' response to a question about their background and interest in mathematics.

6.2.3 Data collection

A few days before the interviews, I emailed the participants a definition, instructions, and a short questionnaire about their mathematical background. I designed a definition specifically for the purpose of this research[1]. However, participants had seen similar definitions before, such as the definitions of combination, permutation, and multinomial coefficient. The participants were asked initially to read and try to understand the definition that was sent to them before the interview, and write down what they did to understand it. The main part of the data was collected during clinical interviews, where students were presented with a set of tasks, that were designed to examine their understanding of combinatorics in general, and of this new definition in particular. Measuring students' understanding of this new definition, however, was not the main objective of this research. My focus was on the method or methods they

[1]The definition will be described in the next section.

used to try to understand this new definition, how they connected it to their previous knowledge, and finally how they used this definition and their previous knowledge in solving problems.

The interviews were semi-structured, conducted individually, and were audio-recorded. The semi-structure format was chosen to allow for changes to be made to the interview questions, and adding new questions. Each interview lasted between 40 and 60 minutes. The students were encouraged to think out loud, and to write as much as possible. The data comprised the transcription of the audio files, together with the students' writing before and during the interview.

6.2.4 Tasks

A few days before the interview, the participants were given a definition that they had not seen before. During the interview they were given a set of tasks, composed of problems, example-generation tasks, and general questions. The objective of my study was to examine how the students placed this new definition within the schema that existed before the interview and how they used their previous knowledge to understand the new definition and solve problems. To design the tasks in this study, I drew on several ideas from Dahlberg and Housman (1997).

The first task for the participants was to try to understand the definition. The only instruction given to the students was:

> **Instruction.** Please read the following mathematical definition. Try to understand this definition in any way that you normally would. Please try to write down everything you are thinking and take a note of what you are doing, to understand this definition. And bring it to the interview.

DEFINITION: Trization of a set of n distinct elements is a placement of these elements into 3 different cells, with k_i objects in cell $i, i = (1, 2, 3)$, and $k_1 + k_2 + k_3 = n$. The order of objects in each cell does not matter. The number of trizations of a set with n elements with k_i objects in cell $i, i = (1, 2, 3)$ is denoted by $T(n : k_1, k_2, k_3)$.

Not only was this definition one that the students had not previously encountered, it does not even exist in the literature, and so students could not seek help in understanding it from online or library resources. Three of the eight students did try to google the definition and said that it did not exist in the context of combinatorics. I wanted the language of the definition be familiar to the students and to have a direct relation to the concepts that were previously explored in the class. However, because I was not teaching that class, I tried to create the definition based on the language used in the notes provided by the instructor. The students had been introduced to a "*counting formula*" called the multinomial coefficient in two ways. The first one was using arrangements of objects with some repeated symbols:

If there are n objects with n_i objects of type i (for $i = 1, \ldots, k$), where $n = n_1 + n_2 + \ldots + n_k$, then there are $\frac{n!}{n_1!.n_2!....n_k!}$ arrangements of these n objects.

The second version was described as distributing n different objects into k different cells where each cell i has n_i objects in it:

The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \ldots, k$ is: $\frac{n!}{n_1!.n_2!....n_k!}$ where $n = n_1 + n_2 + \ldots + n_k$.

While the students had seen the above definition in class prior to the interview, it was interesting to see if they could make a connection between the these definitions. Specifically, would they see that the number of trizations of n objects is exactly the same as the multinomial coefficient with $k = 3$? How would they try to understand trization? Do they see the combinatorial structure hidden in the definition?

On the day of the interview, participants were presented with a set of nine tasks, which were organized into three categories. The first category involved problem-solving tasks, the second one consisted of example-generation tasks, and the last one comprised connections tasks. I will now specify each of these tasks, stating one possible solution or intended outcomes, and discuss the purpose of presenting them to students.

Category 1. Problem-solving tasks

The first category, problem solving, consisted of four tasks. For all the problem-solving tasks, participants were given total freedom to use their method of choice. They did not need to use trization. In fact, they were informed at the outset that not all of the problems could be solved using trization. In addition, participants were permitted to use their class notes, textbook, and calculator if they chose to do so.

> TASK 1: How many different 6-digit numbers can be made with digits 1, 2, and 3, if 1 can be used only once, 2 can be used only twice, and 3 can only be used three times. □

On the one hand, this task was a routine problem for students. Students had seen very similar problems before and could solve this one easily. On the other hand, there is a non-obvious structure in this task, which actually can be *identified* as a trization. The participants were asked to solve this problem using any method, but were also given a chance to examine the structure in order to see if it fitted the criteria for trization. The solution of the problem could be expressed as the multinomial coefficient $\frac{6!}{1!2!3!}$. Alternatively, the problem can be viewed as a trization if each cell represents a digit and each place value represents an object, and the number of objects in each cell is the number of times that digit can be repeated. In this case, we have 6 objects (the place values) which we want to place in cells 1, 2, and 3. The problem specifies that these should be only one object in cell 1, two objects in cell 2, and three objects in cell 3.

> TASK **2**: How many ways are there to select a president, a vice-president, and a secretary from a group of 50 people? □

This task was another straightforward problem, which was quite easy for the students to solve. The solution could be achieved in a few different ways, all of which simplify to $50 \times 49 \times 48$. The goal of this task was to see if the learners could *discriminate* the structure of this task and that of trization, and realize that we cannot use trization to solve this problem.

> TASK **3**: In how many ways can a father distribute 10 different stamps among his 3 children if he wants to give 5 to the oldest one, 3 to the middle one, and 2 to the youngest one. □

This task was given as a clear example to be identified by participants as a trization. The problem could be solved by counting all the possible trizations or $T(10 : 5, 3, 2)$. To evaluate this, participants could use their previous knowledge by using the product of multiple combinations $\binom{10}{5}\binom{10-5}{3}\binom{10-5-3}{2}$ or else use the multinomial theorem directly to get $\frac{10!}{5!3!2!}$. This problem was placed here to act as the bridge connecting the learners' previous knowledge of combinatorial structures with the new structure of trization.

> TASK 4: How many ways can you put a class of 18 people in 3 different groups if
>
> 1. Each group contains exactly 6 people.
> 2. Each group contains at least 4 people.
> 3. There is no restriction on the number of people in the groups.
>
> □

This task included three parts. Each part was slightly different in its wording and in what the students were asked to do. However, the solution to each part differed greatly.

The first part had a very similar structure to Task 3. It could be easily recognized as trization. The solution could be presented as $\binom{18}{6}\binom{12}{6}\binom{6}{6}$ or $T(18:6,6,6)$.

Solving the second part, however, was considerably more challenging than the first part. In fact, one way to solve this problem was to find all the possible breakdowns of 18 people into three groups of 4 or more people and then find the sum of all the possibilities. In short, the solution is $\sum_{i=0}^{6}\sum_{j=0}^{6-i}\binom{18}{4+i}\binom{18-4-i}{4+j}$. Another way was to list the possible trizations of 18 people in 3 groups of minimum 4 people, and add those possibilities: $3T(18:4,4,10)+6T(18:4,5,9)+6T(18:4,6,8)+3T(18:4,7,7)+3T(18:5,5,8)+6T(18:5,6,7)+T(18:6,6,6)=\sum_{i,j,k\geq 4}T(18:i,j,k)$.

The purpose of this task was to observe how participants approached this challenging problem. I did not expect the participants to find the final solution; however, I wanted to observe how they approached the problem, what representations they used, and whether they recognized the level of difficulty of this problem as being much higher than for the first part. One of the anticipated incorrect solutions was to first assign four people to each group and then distribute the remaining six people without restriction. This method generates a much larger answer than the correct solution because of over-counting.

The third part of this task was considerably easier than the second part. However, the way it was worded could have convinced the participants otherwise. The goal of this part of the problem was to see whether those participants who were unsuccessful in solving the second part of the task would attempt to solve this problem or would simply give up based on the level of the difficulty of the second part of the problem. The anticipated solution to the third part is 3^{18}: each of the 18 people can be placed in any one of the three groups, so there are three independent choices of location for each of the 18 people.

Category 2. Example-generation tasks

Tasks 5, 6, and 7 were example-generation tasks. Participants were asked to generate examples and to find the formula for trization.

> TASK **5**: Give an example of a problem whose solution can be found by $T(12:5,7,0)$. Explain your answer. □

The purpose of this task was to encourage learners to generate their own examples. This task was designed to see their general approach towards the structure of trization.

For example one possible problem could be, "You have 12 different skirts and want to put them in your drawers. You want to put 5 in the first drawer, 7 in the second and as the last drawer is full, you cannot put any skirts in it. How many ways can you do that?"

> TASK 6: Can you give another one that is different from the first example? And another? □

This task was given to see how diverse the students' thinking could be about the structure of trization. For example, because the last cell did not have any elements in it, it could be completely ignored from the structure. It could in fact become a normal combination problem. For example, a possible problem is, "You have 12 friends and you want to invite 5 of them to your party, how many ways can you do that?" Another one could be "How many different strings of letters can you make with five A's and seven B's?"

> TASK 7: What is the formula for finding $T(n : k_1, k_2, k_3)$? □

This task was assigned to see if the participants could find the formula for counting all possible trizations in general. Also, if they did find the formula, would it help them to see its connection to the multinomial coefficient and combinations? The answer could be $\binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}$ or its simplified form $\frac{n!}{k_1!k_2!k_3!}$.

Category 3. Connection tasks

The last two tasks, 8 and 9, were given to explore the connections explicitly, especially if they were not apparent from the previous parts of the interview.

TASK 8: Do you see any relation between combination and trization? If yes, describe how. □

The anticipated answer was to see trization as a more general case of combination, or in other words $\binom{n}{k} = T(n : k, n-k, 0)$.

TASK 9: Have you seen the definition of trization or anything similar to it before? If the answer is yes, can you explain? □

This task was another attempt to examine the connections that students made between their old knowledge and trization. They had seen the multinomial coefficient previously, which is the general case for trization.

6.3 Results and analysis

As mentioned previously, the participants were given a definition. which they had never seen before. Then they were presented with a set of tasks which were directly or indirectly related to that definition. I designed these tasks to reveal students' concept image of simple combinatorial structures.

In what follows, I discuss the results of the study and analyze the responses. The analysis is presented across the tasks and organized by the factors that were discussed above, namely examples, formula, representations, and connections.

6.3.1 Examples

I consider three different types of examples in a learning situation: active examples, passive examples, and learner-generated examples.

Active examples

The interviews revealed that most participants did not actively generate examples. In fact, based on their writing the only instance of a learner taking the initiative in this way was Pam, who used an active example to understand the definition of trization before the interview.

In Figure 6.1, we can see Pam's attempt to understand the definition of trization. In her attempt, she uses an example with $n = 8$ and $k_1 = 4$, $k_2 = 3$, and $k_3 = 1$ and another one with $n = 8$ and $k_1 = 4$, $k_2 = 2$, and $k_3 = 2$. She lists two different possibilities for her first example and one possibility for the second one, and discovers the different permutations within a cell does not create a new trization.

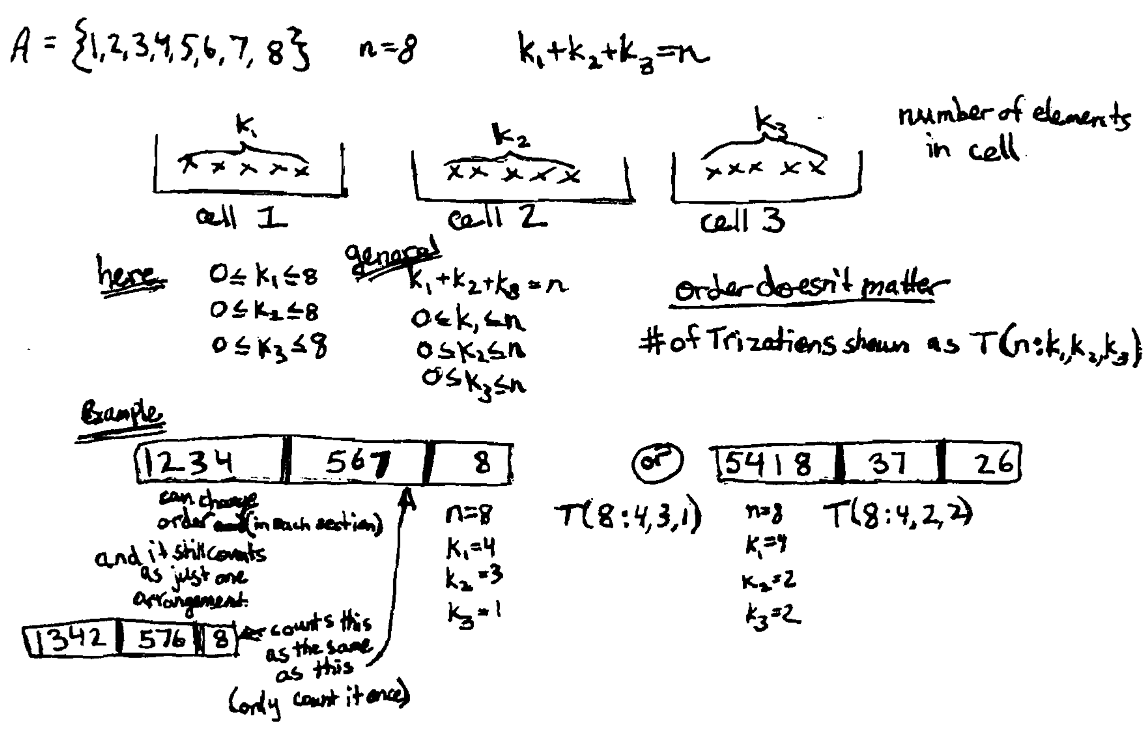

Figure 6.1: Pam's notes for understanding the definition of trization

The lack of use of active examples by all students except Pam shows that example generation is not a part of their routine for understanding or solving problems. Unlike experienced mathematicians, students seem to rely on their teacher to create examples for them, or do not appreciate creating examples.

Passive examples

As described before, passive examples are the examples that are created by the teacher or the textbook for students. The data revealed participants' use of passive examples, similar to the ones that they had seen in class, to solve the tasks in the interview. However, the data also revealed that they sometimes make mistakes because of their reliance on an imperfect memory.

For the second part of task 4, participants were asked to find the number of ways to distribute 18 people in 3 different groups where each group has at least 4 people in it. Matt, in his second attempt, recognized that if he distributed 4 people in each group first, then he could distribute the remaining 6 people in any of the groups freely. He then decided to consider all possible ways of distributing 6 objects in 3 groups and add them. He then recalled the formula $\binom{n+r-1}{r}$ based on a similar problem that he had seen in the class. Matt was not able to justify why the formula works, other than the fact that he had seen it in a similar problem in the class.

EXCERPT 1

Matt: ... Let's start with 4, 4, 10 then 4, 5, 9...
[He keeps writing the different possibilities]

I: And then you have to continue like that. So that's a lot, yes? A lot of cases.

Matt: Yeah.

I: Do you know how many cases are there to do this? If you wanted to write it up how many cases would you have to consider?

Matt: OK. Well there are 6 things to distribute within 3 groups... I can't think of it any more. **I used to know these things.** I know, I know, I know. **I just can't seem to picture it anymore.**

I: No? It's OK.

Matt: Hmmmmm... **I know this but somehow it's just not coming.**

I: That's fine.

Matt: Somehow I... Oh, these are... I think it is... **just my guess. I don't know... I'm just going to guess...** $\binom{3+6-1}{6}=\binom{9}{6}$.

I: But 3+6-1 is...

Matt: Oh, 8. Sorry. It's 8.

[...]

Matt: OK. Would be $\binom{8}{6}$. OK. Would be 28.

I: So 28 [cases]?

Matt: Yeah. **That's my guess. I am not sure though.**

I: OK. Why is it your guess? **Where does this formula come from?**

Matt: **She didn't really explain. But there is the problem she gave us.**

I: What problem? For what purpose?

Matt: **For "cookie jar" problem. Where there are 6 say like 3 different jars and we are going to take like 6 cookies from the 3 jars... yeah... something like that.**

I: Oh, how many ways can you put it in.

Matt: How many ways can you take 6 cookies out of 3 jars.

I: Aaah, OK.

Matt: So here. How many ways can you put 6 cookies into 3 jars.

In the above excerpt we see that Matt used his teacher's "cookie jar" example to help him to solve the problem at hand. The "cookie jar" problem asks how many ways one can put n indistinguishable cookies in k distinguishable jars. This is an instance of a student relying on a passive example to find a solution to a task in the interview. We will see other instances of the use of passive examples by students in the following excerpts as well. In fact, the data revealed that the use of passive examples by participants was one of the most popular methods of solving problems.

It appears that Matt was unsure of his solution, because he did not recall where the formula came from, and if it was applicable here. He used phrases such as "not sure", "don't know", or "guess" a few times in this excerpt. The root of this uncertainty could be that he was not able to rely on a formula whose rationale he did not understand, even though he knew at least one instance of that formula being used in a passive example he had seen before. It could also be possible that he was using these phrases to protect himself against any possible mistakes, even if he trusted his answers.

Learner-generated examples

Tasks 5 and 6 explicitly required participants to come up with examples of their own. These tasks were designed to examine participants' concept image of trization and the possible connection between trization and combination. The outcome of these tasks exposed the weakness of students' ability or willingness to generate non-trivial examples.

I anticipated three general types of valid examples. The first type was the trivial example, where there are twelve distinguishable objects to be distributed in three distinguishable groups of five, seven, and zero objects. The second type was recognizing that there is no need for the existence of the third group since we are not putting anything in it. Finally, the most sophisticated type of example I anticipated to be created was when there is an empty cell in a trization, it becomes in effect a combination. In response to Task 5, out of eight participants, four created a trivial example of trization, three created an inappropriate example and one could not give any example. Only after the second round of example-generation, in Task 6, one participants created the third type of example.

Trivial learner-generated examples

Out of eight participants, four created a trivial example of trization in their first attempt to generate an example. For instance, Felix generated a trivial example involving twelve stamps and 3 kids, where the first kid receives five stamps, the second receives seven, and the last kid receives none.

EXCERPT **2**

Felix: [...] OK, this can – as we have 12 stamps and give them to 3 kids – do I need to write this?

I: Sure, if you can – or you can just say it, that's fine. 12 stamps you want to give it to 3 kids...

Felix: And make sure the first kid gets 5 and the second gets 7 and the other get nothing.

However, after a little time and prompt from me, Felix generated a less trivial example for second example-generation task. He recognized that there is no need to have three kids in his example, if the third kid gets no stamps:

EXCERPT **3**

I: OK. Now if I ask you to give me another problem, which is very, very different, maybe even structured only different, can you come up with another one?

Felix: 12 stamps, give them to 3 kids and at least one kid - every kid - gets at least 1.

I: 12 to 3 kids, at least every kid gets 1?

Felix: Yeah.

I: Would it be $T(12 : 5, 7, 0)$?

Felix: Oh, OK. *[Figuring question for awhile.]* For the definition, is this just a number?

I: Yeah.

Felix: A specific number?

I: Yeah. Because you see everything is a constant, yes?

[...]

Felix: 12 stamps *[figuring question for awhile]*. Oh, no. Two kids and they can pick from 12 stamps and first kid can choose 5 times and the second choose 7 times.

I: So 2 kids will do?

Felix: Yes.

I: You don't need 3?

Felix: No.

Three other people generated similar trivial examples for their first attempt. When these three participants were asked to generate another example, they generated an inappropriate example, namely an example that does not embody all the properties of a trization, or embodies some properties that trization does not have.

Inappropriate learner-generated examples

Some examples revealed that the participants had a limited understanding of trization. For example, five of the eight participants considered indistinguishable objects at least once in their example generation.

Pam was able to correct her mistake while she was generating an example for Task 5. She first considered twelve cookies or candies that were to be divided between three children, where the first received five, the second child received seven, and the last child received none. After I asked her to clarify if the candies were identical or not, she first asserted that they were identical, but later changed her mind.

EXCERPT 4

I: This one is to make an example that the solution can be found with the trization of 12, 5, 7, 0. So, like you are a teacher and you want to design a problem whose solution would be that. What would you say?

Pam: OK. You could have a problem that would say you have 12 candies, cookies or something and there are 3 children and the first child gets 5, the second child gets 7 and the third child gets none, poor little boy or girl, because that way you've got 12 identical candies because it doesn't matter what order you put them in.

I: So 12 identical candies?

Pam: Right. There's 12 candies.

I: So are they identical or not and how should you know if they are identical or not?

Pam: If they were identical then there would only be one way to split it so that this person gets 5 and this person gets 7 and this person gets 0 because they get 5 things that are all the same. It doesn't matter if they get those 5 or 3 of those ones and 2 of those ones. *[She points to some non-existent objects when she refers to candies].* **So I don't know if this** *[she points to the trization formula]* **would solve it if they were identical or if they were not identical.**

I: How would you know? If you wanted to check?

Pam: If they are identical then the answer is going to be 1 and if they are not identical then it is going to be a different answer?

I: But trization is only good for one of these situations. If it's identical or not identical or some are identical, trization is good for which one?

Pam: Trization is good for distinct elements. So they are not identical.

While Pam was thinking about the solution of the problem that she generated, she realized that the answer is 1 if the candies are identical, and so deduced that the candies cannot be identical. When she tried to generate the second example for Task 6, she was sure about the structure she was going to use and found the example quickly.

EXCERPT 5

Pam: OK. If we have 12 people and we need 5 managers and 7 regular employees and 0 people to get fired. So we are going to have our trization of 12 people and we are going to pick 5 of them to be managers and 7 of them to be employees and we are not going to fire any of them.

Learner-generated example that illustrated the connection between combination and trization

In response to Task 6, which asked the participants to generate *another example* of $T(12 : 5, 7, 0)$, Lola figured that if $k_3 = 0$, then we do not need three groups. Lola's response is different than that of Felix in Excerpt 3, because she came up with the example on her own, without me prompting her.

EXCERPT **6**

I: So now, can you give another example that uses the same trization but very different... as different as you can imagine this.

Lola: Just using the same trization?

I: Using the same trization. The same numbers but a very different problem. As different as you can imagine.

Lola: OK...

[She writes: "12 people to put in 2 different groups which has 5 people and 7 people in each group."]

I: Very good. So instead of three, this time you have two groups, yes?

Lola: Yeah.

I: What's the reason that you can go with two instead of three?

Lola: Because the third group doesn't get anything.

Lola recognized that we did not need three groups if there was no element in the third group.

Summary

The data revealed that, with one exception, students did not use active examples to understand trization. Neither were they very skilled with generating examples, even when they were specifically asked to do so.

The data reconfirmed that without explicit teaching students have difficulty in generating their own examples, specially innovative ones. In Chapter 7, I describe a methodology that I developed which uses example generation in the classroom and encourages students to generate their own examples.

6.3.2 Formulae

Finding and using formulae has a special role in mathematics. Formulae are used mainly because of their efficiency for calculations. However, there are some pitfalls along with the use of formulae that one does not understand. The formula can be used in an inappropriate place, or there may be a mistake in using the formula resulting in an undetected wrong answer, and the formula can be forgotten easily. In what follows I describe some of the advantages and pitfalls of using formula by students as suggested by the data.

Identification of the structure, appropriate use of formula

In the first problem-solving task, Task 1, participants were asked to find how many 6-digit numbers can be made with 1, 2, 2, 3, 3, 3. In response to this task in the interview, Don quickly recognized the structure and solved the problem using a formula he recalled from the class. He used the formula for combinations with repetition[2] and got $\frac{6!}{2!3!}$. Don was not able to justify the formula except by reference to a similar problem he had seen in the class.

EXCERPT 7

I: So $\frac{6!}{2!3!}$. OK, very good. How did you get that answer?

Don: From class when we were doing rearranging like if it was UNUSUAL or something and the top would be how many ever you had and the bottom would be repetition.

I: Oh, OK. Do you know where that formula comes from?

Don: No, not exactly.

I: No? Can you like think about where it comes from or do you think it's too hard? Do you think it is easy to figure out or hard to figure out where this formula comes from?

Don: I can't think of it off the top of my head.

The problem that Don refered to, UNUSUAL, was a problem that he had seen in class. The problem was: "How many different arrangements can you make with the letters in the word UNUSUAL". Don was making a connection from a structure that he had seen previously in the class to the structure of the task at hand to solve the problem. He observed that UNUSUAL problem asks for different arrangements of a sequence where some letters are repeated, and that the task in the interview asks the same question. Therefore, he used identification to make the connection to a previously known structure to solve this task. However, he did not know how the formula is obtained and why it works.

Don seemed confident in his solution, based on a passive example "UNUSUAL" he had seen before. In the previous section, we saw Matt's solution to the second

[2]The number of arrangements of n objects, where there are n_i objects of type i for $1 \leq i \leq k$, and $\sum_{i=1}^{k} n_i = n$, is $\frac{n!}{n_1!n_2!\cdots n_k!}$.

part of Task 4 in Excerpt 1. Even though both Matt and Don were not sure where the formula came from, and in contrast to Dan, Matt appeared to be unsure of his solution.

Attempting to recall a formula without recognizing the structure

In the second part of Task 4, participants were asked to find the number of ways to put 18 people in 3 different groups of size at least four. In her attempt to solve this problem, Pam tried to recall a formula from the class:

EXCERPT 8

Pam: I don't remember. I'm not sure how...

I: Do you know something, that would do this, that you are trying to remember?

Pam: I'm trying to remember. Because we've got 4 people here and then we've 4 people here OK, right. Alright, so we have 18 people minus 12 divided by... $\binom{18-12+1}{3}$ so, $\binom{7}{3}$ maybe? $\frac{7!}{3!4!}$.

I: Why did you choose this method? why did you say $18 - 12 + 1$? Where did that come from?

Pam: I am attempting to recall a formula that is n minus - no, OK I think I might have picked the wrong number... $\binom{n-r+1}{r}$. So that should actually be 6, maybe? No. **There's some formula that's $n - r + 1$ choose something.**

I: So when is that formula used? $\binom{n-r+1}{r}$.

Pam: **That formula is used when you have identical things that you are splitting into different groups or identical groups that you are splitting different things into.**

I: So what is identical here?

Pam: The groups.

I: It says different groups.

Pam: Different groups. **Alright then that formula doesn't work.** So the number of ways we would have to arrange this... **I don't remember.**

In the first part of interview, Pam tried to remember something from the class, but she was not able to remember it completely. Her focus was on the recollection of the formula and not on the task at hand. Then she remembered a formula which did not correspond to the structure of the problem. After her mistake was pointed out, she said that she did not remember. Pam was using a library of different problems whose solution she had seen before, and was trying to find a problem in that library similar to the one in the interview. She had difficulty because she could not remember a problem similar to the one at hand.

Not recalling the correct formula

For the third part of Task 4, Anna was also trying to recall a formula. In this task 18 people are divided into three different groups with no restrictions on the number of people in each group.

EXCERPT 9

Anna: No idea. **Eighteen to the third? Eighteen to the number of groups?**

I: Why? Why would you think it would be 18 to the power of groups?

Anna: **Because it is an easy formula and that seems like the easiest case.**

I: What is 18 to the power of three? Where did it come from?

Anna: **Because we have some formula that we use in class that is like n^r or r^n, one of the two.**

I: Oh.

Anna: For permutations.

I: OK.

Anna: So n would be the number of people times the groups, would be three.

Again we can see that Anna had an idea about which formula to use, but did not use it correctly. She also showed signs of uncertainty and admitted that she did not know if the answer was n^r or r^n. Using a formula without justification, or not knowing where it came from, can create many mistakes.

Finding a formula

When the participants were asked to find a formula for trization in Task 7, after attempting the earlier problems, six of the eight participants were able to find the correct formula. Anna was one of the people who found the formula for trization correctly.

EXCERPT **10**

I: So, can you find a formula to find $T(n : k_1, k_2, k_3)$?
Anna: Maybe $\binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}$.

Summary

There are relatively few formulae presented in counting methods of a typical discrete mathematics course. This makes it tempting for students to memorize the formulae that are given and the situations in which they are useful, and make little or no attempt to understand the concept. This could potentially create a problem for learners when they need to use the concepts in more creative ways.

The common thread in the data was that formulae were a big part of students' concept image, and the use of formula for problem-solving was a routine method. Most students did not question the validity of the formula they used, or where it came from, or where else it could be used. They just knew the types of problems which they could solve using a certain formula, and tried to make connections from those problems to the new ones they saw. Their reliance on the use of formulae sometimes helped them find the correct solution quickly. However, it also created uncertainty for the validity of their solutions, or caused mistakes due to imperfect recall or use of an inappropriate formula.

During the interviews, most of the students at some point felt the need to recall a formula, and to remember what the formula they recalled was good for. This reliance on memorization rather than understanding also left them vulnerable to making mistakes while recalling these mysterious formulae. On the other hand, being sufficiently comfortable with using formulae also presented some positive results. For example, students were able to solve easier or more straightforward problems quickly and efficiently. More importantly, they could make some connections between trization and combination, or trization and multinomial coefficient, later in the interview, by just finding the formula for counting the number of trizations.

In general, using formulae is a big part of students' conception of understanding of mathematical structures. On some occasions, this creates difficulties; on others, it helps them to understand the concepts, and gives them confidence in solving problems.

6.3.3 Representation

In this section, I examine participants' use of visual and symbolic representations. By visual representation I mean any kind of drawings, graphs, or pictures. Symbolic representation includes the use of symbols (numbers or letters) and their basic manipulations. This is another attempt to get a more comprehensive picture of the learners' concept image.

Use of visual representation to understand the definition

To understand the definition of trization, five of the participants used drawings that were almost identical. During the interviews, I realized that the mental image of trization, represented by these drawings, did not correspond to what was intended by the definition.

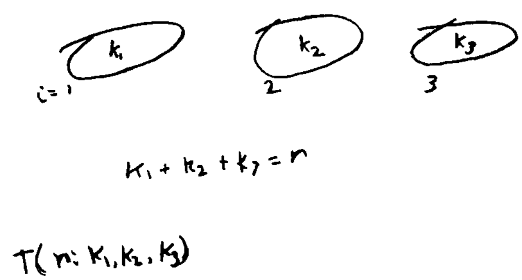

Figure 6.2: Visual representation of trization by Don

The frequent use of drawing shows that a visual representation seemed to help students to achieve an initial and partial understanding of the definition. However, they used visual representations during problem solving less frequently.

Use of visual representation to approach a task

For solving Task 1, Matt started by listing all the possibilities, but he soon figured out that this method will take a long time. Then he started to make a tree diagram.

Even though drawing the tree diagram was going to take a long time as well, Matt explained "it's how I get it".

EXCERPT 11

Matt: ... drawing everything... It's gonna take forever.

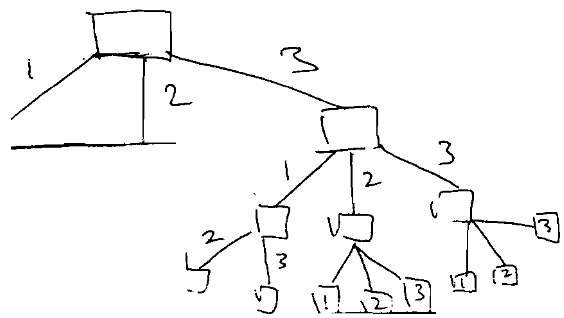

Figure 6.3: Tree representation by Matt

I:	It's going to take a long time.
Matt:	But at least it's how I get it.
I:	So, I believe you can do that tree. I'm sure you can finish this off, but is there any other way, shorter way that you can do this?
Matt:	If I knew how to use this, I think I probably... *[Points to the definition of trization.]*

At this point Matt was thinking that the only way he knew how to solve this problem was by using a tree diagram. Later in the interview, after he was exposed to different tasks, he had another chance to think about this problem. At this point he had successfully found a formula to count the number of trizations:

EXCERPT **12**

Matt:	Oh, I am so stupid. Just 6-digit numbers. That means exactly it's exactly 6 digits. [...] I forgot this. We learned it a long time ago.
I:	Do you think you can use trization for this?
Matt:	Yeah, yeah. This is trization too because... Oh, I can't believe it. *Then he writes* $\frac{6!}{1!2!3!}$ *or* $\frac{6!}{2!3!}$.

Matt was able to shift from one representation (visual) to another (symbolic) to solve the problem. The ability to use multiple representations, and to shift from one to another, is an attribute of successful problem solving.

Hanna also used a tree diagram in her initial attempt to solve Task 1. However, she was not able to improve her initial tree diagram method.

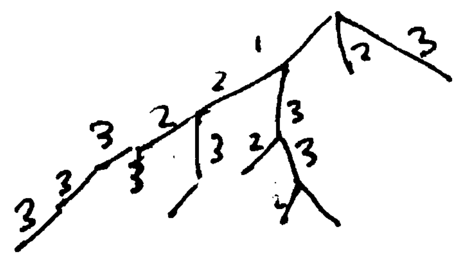

Figure 6.4: Tree representation by Hanna

Hanna did not switch from her initial visual representation to a more appropriate representation to solve this task. Often, the inability to use multiple representations limits one's ability to solve problems. Having a limited or inadequate concept image may be related to the inability to use multiple representations. On the other hand, with a good understanding of a concept comes a more complete concept image and that may contribute to one's ability to use multiple representations.

Use of symbolic representation for calculation

Matt attempted to find the numerical value of $\binom{8}{6}$ for the second part of Task 4 using Pascal's triangle. Parts of his engagement with this task were included in Excerpt 1. To calculate $\binom{8}{6}$, he started by writing down the numbers in the first few rows of Pascal's triangle. He wrote down the rows completely up to the seventh row. For the eighth row he completed only the first three numbers in that row, and for the ninth row he also completed only the first three numbers. He knew that $\binom{8}{6} = \binom{8}{2}$, so to save time he did a partial completion of the triangle.

EXCERPT **13**

Matt: Somehow I... Oh, these are... I think it is... **just my guess. I don't know... I'm just going to guess...** $\binom{3+6-1}{6} = \binom{9}{6}$.

I: But 3+6-1 is...

Matt: Oh, 8. Sorry. It's 8.

[...]

I: So that's how many cases you have of these?

Matt: That would be like...

[He wrote down a few rows of Pascal's triangle to calculate $\binom{8}{6}$ *and located 28.]*

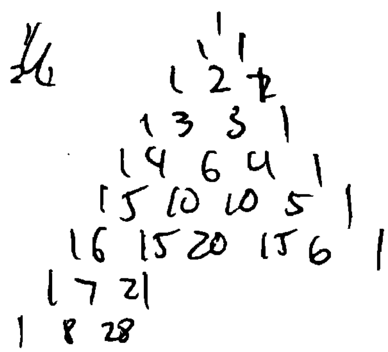

Figure 6.5: Matt's Pascal's triangle

I: You're drawing Pascal's Triangle?

Matt: Yeah, just so I can get the number.

[...]

Matt: OK. Would be $\binom{8}{6}$. OK. Would be 28.

The use of Pascal's triangle to find $\binom{8}{6}$ shows that Matt has strongly connected the concept of using the Pascal's triangle to evaluation of the number of combinations (at least for small numbers). Matt explained that he did not use the formula for the evaluation because he thought it is faster to write down the rows of Pascal's triangle.

Summary

The type of representation that is used to understand or solve a problem varies from one person to another. The use of different representations also influences and is influenced by how one views, understands, thinks about, and solves a problem. The data revealed that students did not use visual representation often, beyond their initial use for understanding the definition. The reason this is significant, is that there may be a lack of appreciation for visual representations by students (which can be rooted in many social and pedagogical experiences), which limits their formation of a more complete concept image; the concept image that includes multiple representations of that concept. The data also showed that there is a substantial appreciation for algebraic representations in students' perception of mathematics. This appreciation was evident in the data presented in the previous section on use of formulae.

6.3.4 Connections

The participants made some connections between combinatorial structures during the interview. What I mean by connection, in this context, is recognizing two or more combinatorial structures as the same or recognizing a structure as a special case of a more general structure (identification), or recognizing them as different with an emphasis on what makes them different (discrimination). I describe some of the connections that were made by participants, and also some connections that I anticipated would be made, but which the participants failed to make.

Identification

As described earlier in this chapter, the data revealed that the participants frequently associated part or all of a problem with another problem that they had seen before. For example, consider the "cookie jar" problem in Excerpt 1, or the "UNUSUAL" problem in Excerpt 7 or the "sum of x's" in Excerpt ??. The learners seemed to use this kind of identification with problems they had seen before, in order to remember and apply some formula and thereby to solve the problem. This exhibited learners' reliance on passive examples to solve problems.

Matt recognized the first part of Task 4 as similar toa problem that he had seen before in his class, and tried to use that problem to solve the task at hand.

EXCERPT **14**

Matt:	Oh, we had this from... in MACM class, I think. Oh yeah, we did have a similar formula, I think. Or maybe not.
I:	Do you remember what it was?
Matt:	No. She was just showing how... a way to do it... to do something with the cards, right. So it's like 52 and say take 13 and say 52 minus 13 and take 13 again and you just keep doing it until the 13 card is left. And then she showed us something similar to that, I guess. I think. If I remember properly. So this would be... Something like that... sort of.

Matt wrote $\binom{52}{13}\binom{52-13}{13}\binom{52-13-13}{13}\binom{13}{13}$ as the solution to the problem he had seen in the class, and explained that these two problems are like each other.

In response to the same task, Nadia wrote "$\frac{18!}{6!6!6!}$ or $\binom{18}{6}\binom{12}{6}\binom{6}{6}$". When I asked her why she used that formula, she explained that it was a formula she had been taught to solve a similar problem,namely in how many ways can we distribute 36 soccer players into 4 different groups.

EXCERPT **15**

I: OK. So then, why is that? Do you know?

Nadia: Because that's the formula I was taught. It reminds me of a problem we did in class where we had 36 like soccer students or soccer girls or whatever and we had to put them on 4 teams. This is the same as if you were going to take the 18 people and choose 6 of them for the first group and then take 9 people and choose 6 of them for the next group not 12 and then take the last 6 and choose them. And if you do this out, you get this.
[Points to $\frac{18!}{6!6!6!}$].

This kind of identification seems to work well for problems that have exactly the same combinatorial structure. However, when the participants were asked to solve a problem that they could not identify with a previously seen problem, they showed great difficulty in knowing how to proceed. Therefore, relying only on the library of known problems weakened learner's problem solving ability when faced with novel problems. However, the data revealed that this method of approaching problems enabled many participants to solve a class of problems quickly and correctly.

Discrimination

Some of the participants recognized connections by differentiating between two combinatorial structures. Don solved Task 2 by first choosing 3 people out of 50 and then assigning each a position: $\binom{50}{3}P(3,3)$. I asked him if this problem could also be solved using trization.

EXCERPT **16**

I: OK, so do you think this problem can be solved with trization or not?

Don: Are you thinking if there was 4 cells and not 3?

I: If there were 4 cells you could?

Don: I think so.

I: So what would you define for each cell to be if there was 4 cells? Let's say we had another definition for 4... quadrization, for example.

Don: OK. If one cell was the president, one was VP, and one was secretary, and then one was just everybody else.

Don recognized the difference between the structure of trization and the structure of Task 2. He identified the main difference as being the number of cells in each of the structures, and described a method for obtaining the correct numerical solution via an appropriate generalization of trization. However, later in the interview Don was not successful in recognizing the connection between combinations and trization, i.e. in recognizing a combination as a trization with two cells, so that $T(n : r, n-r, 0) = \binom{n}{r}$.

Only one student, Lola, identified a structural connection[3] for Task 8. The task was to describe any relation between combination and trization. Lola observed that trization is "like triple-combination".

EXCERPT 17

I: [...] Do you see any relation between what you have here and what you have there? *[Pointing to the formula for combination and the formula for trization that she had found before]* Are they kind of similar? What's their difference, what's their similarity?

Lola: I think it's quite the same because like these three: $k_1 + k_2 + k_3 = n$ and this is the same as when you add these together, you'll get n. This is kind of like triple combinations – I don't know.

I: Exciting. OK. It's like a triple-combination – nice name. Better than trization that I came up with.

Lola: *[Laughs]* – OK.

What Lola meant by "triple-combination" was later clarified as having to choose k_1 different objects from n different objects, then k_2 different objects from $n - k_1$

[3]By structural connection I mean that the connection was discovered based on the combinatorial structure, and not with use of formula.

different objects, and finally k_3 different objects from $n - k_1 - k_2$ (which is equal to k_3) different objects.

Two other participants noted the connection based on their respective formula for each structure. They identified the "formula" for trization was multiplication of three combinations:

$$\binom{n}{k_1} \times \binom{n-k_1}{k_2} \times \binom{n-k_1-k_2}{k_3} = T(n; k_1, k_2, k_3) \tag{6.1}$$

6.4 Conclusion

In Chapter 4, I described the importance of gaining insight into students' concept images in order to examine their understanding of mathematics. Then I narrowed down the general notion of the concept image to three parts, namely students' use of formulae, representations, and examples. I also considered the connections students make among different combinatorial structures as a very important mental operation in the act of understanding.

In this chapter, I have analysed the data through the lens of the framework I developed and described in Chapter 4. Use of learner-generated examples to help students understand a concept is now well established and researchers are now using these examples to get better insight into students' understanding. Since the examples that students generate paints a picture of a part of their concept image, I used example-generation tasks for this purpose. Students were not able to perform well on these tasks, not necessarily because they did not understand trization, but because they were not used to the concept of example generation. Apart from one participant, they also did not actively generate examples to gain a better understanding of a concept or problem.

The data revealed that formulae were a major part of participants' concept image, and that in using the concept they relied strongly on knowing the formula. It also revealed that many of the students did not know how those formulae were obtained and exactly which structures they were designed to count. However, when they were expected to find the formula for trization, most of them were able to do so, after they had been exposed to some related tasks.

In problem solving, most of the students had a tendency to recall a formula from the class or textbook, without paying much attention to how the formula is obtained and how it can be used in different situations. The participants displayed some kind of memorization method to solve the problems. They had difficulty in solving, or sometimes even attempting to solve, problems that were not familiar to them.

Use of multiple representations, and the ability to switch from one representation to another, enhances one's ability to understand mathematical structures. It gives one multiple viewpoints for the same object, each of which may carry very unique insights, without which the understanding may be incomplete. The data, however, revealed that most students did not use multiple representations, and mainly adhered to symbolic representations. Use of a visual representation was common in the initial attempt to understand the definition, but students did not seem to value their visual representation as a part of their understanding of the definition. They claimed that they did not find it useful, since they could not apply it in problem solving. In problem solving, only two students used tree diagrams to solve a problem.

The data revealed that students were generally able to make a connection between the tasks and problems they had seen before. However, most participants showed difficulty in recognizing trization as a more general, but very similar, structure to combination. Most participants also failed to realize that the trization formula was a special case of the multinomial coefficient.

Even though the participants were students with a strong mathematical background who performed generally well in their courses, the data revealed that they had limited understanding of combinatorial structures, and showed difficulty in working independently. To a great extent, they relied on the examples that they had seen in their class to approach the tasks, they experienced great difficulty when faced with a task for which they could not recall a similar example.

Chapter 7

Mediated Successive Refinement

Critical thinking and reflection on one's thoughts can be considered among the main pillars of contemporary educational theory. However, it appears that at this time classrooms are currently geared more towards the transmission of knowledge than the construction of knowledge and critical thinking. In order to achieve a better understanding of "how to think" and "how to be creative", students need a safe environment where they are encouraged to create and are not penalised for making mistakes. To learn effectively, they also need to be asked to reflect on their own, as well as their peer's, mistakes. Inspired by these premises, in this chapter I introduce a new pedagogical methodology called "mediated successive refinement".

Mediated successive refinement emerged from a seized opportunity. The opportunity arose in the Fall of 2004, when all but one of the students in my Finite Mathematics class responded to a problem with substantially the same inappropriate solution. Their reasoning was so persuasive that even the single student with the correct answer was convinced! Rather than simply correct their mistaken reasoning, I decided to invite them to reflect on their inappropriate solution by asking them to generate an example of a problem whose solution is their solution. Later they were given a chance to reflect on their collective responses and solve the problems their peers generated. I later realized that mediated successive refinement could go further than a pedagogical methodology, and can be used as a research tool to gather data as well.

Mediated successive refinement consists of different cycles, in each of which the students interact with their teacher (the mediator), or with each other, to refine and reexamine their thoughts on a particular question or structure.

In this chapter, I describe the original setting in which this approach emerged and how it evolved, and then I describe the methodology and the results of applying it. Finally, I describe the different theories that contributed to the formation of this methodology.

7.1 The setting

Educational ideas often arise in large research projects, with a lot of advanced planning and many rounds of examination performed by educational researchers. However, they can sometimes emerge from a very specific opportunity, seized by the teacher, in a small classroom. Mediated successive refinement was born and employed in the latter setting.

The advantages of this setting are: knowing the students much better, being aware of their mathematical background and what they have been taught in the class, and getting instant feedback on the teaching and learning methods being employed.

7.1.1 The course

The research was performed the Fall of 2004 when I was teaching an elective first year mathematics course called finite mathematics at Langara College. The course covered introduction to logic, elementary counting techniques, statistics and probability.

This course was designed for students in liberal arts, social sciences, or business. The course also accommodated students who did not wish to take calculus, but needed a first year mathematics course. The minimum requirement was grade B in Math 11 or its equivalent. This course was not the direct pre-requisite of any other course. Therefore, although the course topics were predetermined and needed to be introduced, there was flexibility in teaching the course, such as how much time to allot to each topic and the depth of the coverage of each topic. This flexibility permitted more creativity in teaching style.

7.1.2 The participants

There were 16 students registered in the course, 12 of whom took part in this study. I classified all but one of those students into three groups, according to their mathematical background.

The first group (group A) consisted of two students, Anna and Amos, who met minimum or close to minimum requirements for the course. The students in this group did not take any mathematics course after grade 11 in high school.

The second group (group B) consisted of five students: Beth, Barb, Bob, Becky, and Brenda. These students completed some mathematics courses at the college, such as pre-calculus or basic algebra. However, they did not complete any university transfer mathematics course, such as calculus, prior to taking the course.

The students in groups A and B generally were required to take the course to fulfill the mathematics requirements of their program. Three of these students wanted to gain admission to PDP[1]. The other four students in groups A and B enrolled in social science or business programs, requiring some post secondary mathematics course but not necessarily a calculus course.

In the last group, group C, there were four students: Calvin, Cate, Carol, and Carlos. They had taken most of the first and second year mathematics courses at the college (such as calculus and linear algebra), and wanted to take some "easy" mathematics course while they finished their studies in college and prepared to transfer to a university. Three of the students in this group wanted to study commerce, while the fourth wanted to study physics.

One student, Dan, did not provide me with his background information.

7.1.3 The data

The data were gathered during a two week period while I was teaching counting methods. The data were collected from students' written responses to two sets of tasks and classroom interactions. Detailed notes were also taken after each class where the data were collected. I did not collect any data from exams or assignments, to assure the students that their coursework was separated from my research and that their participation in the research was voluntary.

[1]Professional Development Program or PDP is a program designed for preparation and certification of prospective teachers for teaching in schools in British Columbia.

7.2 The method

Mediated successive refinement consists of three cycles of interaction between the teacher and the students. Figure 7.1 depicts these cycles. The first cycle starts with a problem that is presented to the students by the teacher. We call this problem the "initial problem". If many students make substantially same mistakes, then there is an opportunity to use mediated successive refinement. We call this popular incorrect solution the "initial solution". In the second cycle, the teacher explains the initial problem and presents the solution to the initial problem. Then in her role as mediator, the teacher asks the students to reflect on the initial solution and asks them to design a problem whose solution is the initial solution. The second cycle ends with the students' presentation of their "generated problems". In the third cycle, the teacher collects and redistributes the generated problems and asks the students to solve all the generated problems and to reflect on those problems, while keeping in mind the initial problem and solution. The students are also asked to modify the problems that are not be solved by the initial solution. The goal of this modification is to change these problems so that they are solved by the initial solution. Finally, after the third cycle, the teacher summarizes problems and their solutions and structures.

7.3 The results

I developed the method of mediated successive refinement for a pedagogical reason: to encourage and help my students overcome a common misconception. However, the study also revealed how students deal with conceptual change during the tasks and cycles of the methodology.

Teacher	Students
The first cycle Teacher poses a problem (The initial problem)	Students attempt to solve it. A particular incorrect solution appears to occur frequently. (The initial solution)
The Second cycle Teacher explains the initial problem and clarifies the solution. Then asks students to provide examples of a problem whose solution is the initial solution.	Students present their examples (The generated problems).
The Third cycle Teacher collects the generated problems and redistributes them among students. Then asks students to reflect on some of their peers' generated problems, identify those that are not solved by the initial solution and modify them so that the solution now applies.	Students discuss and analyse their peers' examples.
Teacher summarizes problems and their solutions.	

Figure 7.1: Mediated successive refinement

To understand how learners modify their existing concept image to obtain a better understanding in the context of mediated successive refinement, I developed the following research questions:

- What can trigger a change or the need for a change in the learners' current understanding?
- How can learners become aware of their inappropriate understanding? In particular, does mediated successive refinement help students to become more aware of their own understanding?
- What can the method of mediated successive refinement reveal about the process of change in students' understanding? What are the changes that can occur in students' understanding?
- How does this instructional intervention influence students' understanding of the concept?

The results of this study are presented in chronological order, to accommodate the description of the method as the story unfolds.

7.3.1 *The first cycle:* The initial problem and solution

In one of the practice sessions, students were asked: "In how many ways can you choose two groups of 3 people from 8 people to serve on two different committees?" Students were given time to think about this problem. The problem was presented to the class towards the end of the section on counting methods. In fact, it was supposed to be a very simple problem for students to solve at that point in the course. The anticipated solutions were $\binom{8}{3}\binom{5}{3}$ or $\binom{8}{3}\binom{8}{3}$, depending on whether or not a person can be in two groups.

One student did not answer the question at all, and one solved the problem correctly. Eight of the twelve students gave the incorrect initial solution of $2 \times \binom{8}{3}$. When I asked for justification, they said that because we want to choose 3 people out of 8, we count $\binom{8}{3}$, and since we are doing this twice (for two groups), the answer will be $2 \times \binom{8}{3}$. The remaining two students obtained the same numerical answer as thee eight, but expressed it as $\binom{8}{3} + \binom{8}{3}$. One of these two, Dan wrote:

> Firstly, A group needs 3 people. So the solution is $C(8,3)$. Then I need to find the solution for B group. In this question we are not given the information of what kind of groups they are. So we could think same person could be in the different group. We could choose 3 people from 8 also for B group. The ways of choosing people for B group are $C(8,3)$ as A group.
>
> So the answer can be $C(8,3) + C(8,3) = 2 \times C(8,3)$.

Dan paid attention to the fact that the problem was not clear. He asked if a person could be in both groups or in only one group. He redefined the problem with a new assumption that a person could be in both groups. He then solved part of the problem for choosing 3 people for each group. However, in the last step of the solution, he added the two parts, instead of multiplying them.

I did not anticipate this solution at that point in the course; after all, these students had already solved many more difficult counting problems. This problem was supposed to be a very easy warm-up task. The fact that most of the students in the class were convinced by this solution alerted me that this was not a small mistake made by a few students, but perhaps indicative of a more general problem, and that maybe there was an obstacle in the students' understanding of this kind of combinatorial structure. The one student who had an appropriate answer to this problem initially. But she was so convinced by her peers' reasoning that she attempted to find the numerical value of her adequate solution on her calculator to compare with her peers' inappropriate solution to see if there was any difference. She was amazed that the values were not equal and that one solution must therefore have been wrong. Yet, she was not quite sure if it was her solution that was incorrect or her peers'!

Students' responses to this problem motivated me to further investigate their understanding of $\binom{8}{3} \times 2$ as opposed to $\binom{8}{3} \times \binom{8}{3}$, and what each expression meant to them. Therefore, after explaining the appropriate solution, I asked the students to design a combinatorial problem whose solution is $\binom{8}{3} \times 2$ or $\binom{8}{3} + \binom{8}{3}$. My goal for this example generation task was to invite the students to think about a particular combinatorial structure that could be counted using their formula. In doing this, I was hoping that they would understand how $\binom{8}{3} \times 2$ or $\binom{8}{3} + \binom{8}{3}$ and $\binom{8}{3} \times \binom{8}{3}$ count different combinatorial structures.

7.3.2 *The second cycle:* The generated problems

Eleven students completed the task of the second cycle, which was generating a problem whose solution is $\binom{8}{3} \times 2$ or $\binom{8}{3} + \binom{8}{3}$. Their responses can be organized into the four categories described in Table 7.1.

The problems were diverse in their stories and themes; however, there was a clear trend in the structure of the generated problems: the majority of problems had the

Type of response	Frequency
Problems with the solution $\binom{8}{3} \times 2$	2 out of 11
Problems with the solution $\binom{8}{3} + \binom{8}{3}$	0 out of 11
Problems with the solution $\binom{8}{3} \times \binom{8}{3}$	7 out of 11
Problems without a relevant combinatorial structure	2 out of 11

Table 7.1: Summary of students' responses to the problem-generation task

solution $\binom{8}{3} \times \binom{8}{3}$ rather than the requested $\binom{8}{3} \times 2$. This suggested that the previous concept image of the structure of $\binom{8}{3} \times 2$ had persisted and the majority of students were still unclear about how to distinguish the associated combinatorial structures.

Eleven students responded to this task, and I have chosen seven of their problems to discuss here. Each of these problems is presented in the students' original language and format, so there may be some grammatical/spelling errors.

Problems whose solution is $\binom{8}{3} \times 2$

The only problem with the appropriate structure and unambiguous wording was designed by Bob.

> PROBLEM 1: You have 8 coins from 8 different currencies. How many ways can you select 3 coins if they must be either all heads or all tails?

The structure of the problem is based on a two-stage experiment. In the first stage we are choosing 3 out of 8 currencies, which can be done in $\binom{8}{3}$ ways, and in the second stage we have the 2 options of putting them all head or all tails. The general structure of the Problem 1 can be depicted in Figure 7.2. (Note that $\binom{8}{3} = 56$)

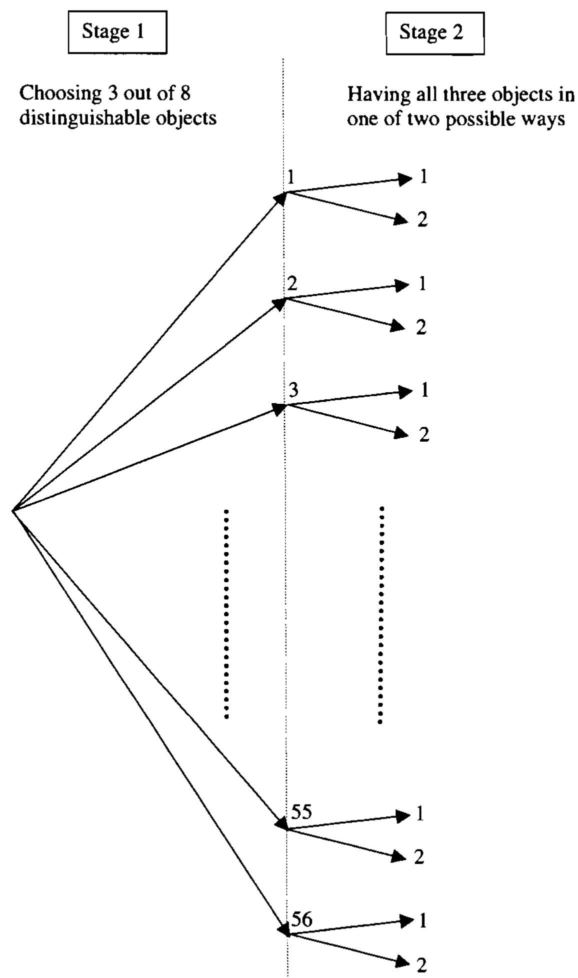

Figure 7.2: General structure for problems with solution $\binom{8}{3} \times 2$

Cate, who was the only student to solve the original problem correctly in the class, generated a problem whose solution could be $\binom{8}{3} \times 2$ depending on how the wording of her problem is interpreted.

> PROBLEM 2: There are 8 people participating in an experimental tests, but only 3 of them to be selected into either control group or experimental group. How many ways to select them?

This problem consists of two stages, where in the first stage we choose 3 out of 8 people in $\binom{8}{3}$ ways. Because of the wording of this problem, one can consider two possibilities for the second stage of the problem. One possibility is that all the three chosen people should be in the same, either experimental or control, group. Hence, by the rule of product the total number of possibilities is $\binom{8}{3} \times 2$. However, the original wording of the problem could also be understood as putting each of the three people in either group, in which case the solution would be $\binom{8}{3} \times 2^3$. The general structure of this problem (if considered in the second setting) is depicted in Figure 7.3.

Problems whose solution is $\binom{8}{3} + \binom{8}{3}$

None of the students generated a problem whose combinatorial structure is counted by $\binom{8}{3} + \binom{8}{3}$. The anticipated problem could have a structure equivalent to having two distinguishable groups and 8 different objects in each of those groups, as in Figure 7.4. In how many ways can we choose either 3 objects from group one or 3 objects from group two?

Problems whose solution is $\binom{8}{3} \times \binom{8}{3}$

Seven students generated problems whose solution is $\binom{8}{3} \times \binom{8}{3}$. This reconfirmed the initial observation that many students did not distinguish structures counted by $\binom{8}{3} \times \binom{8}{3}$ and by $\binom{8}{3} \times 2$. This was despite the original clarification in the class about what each of these two structures entailed, and what they meant. The following four

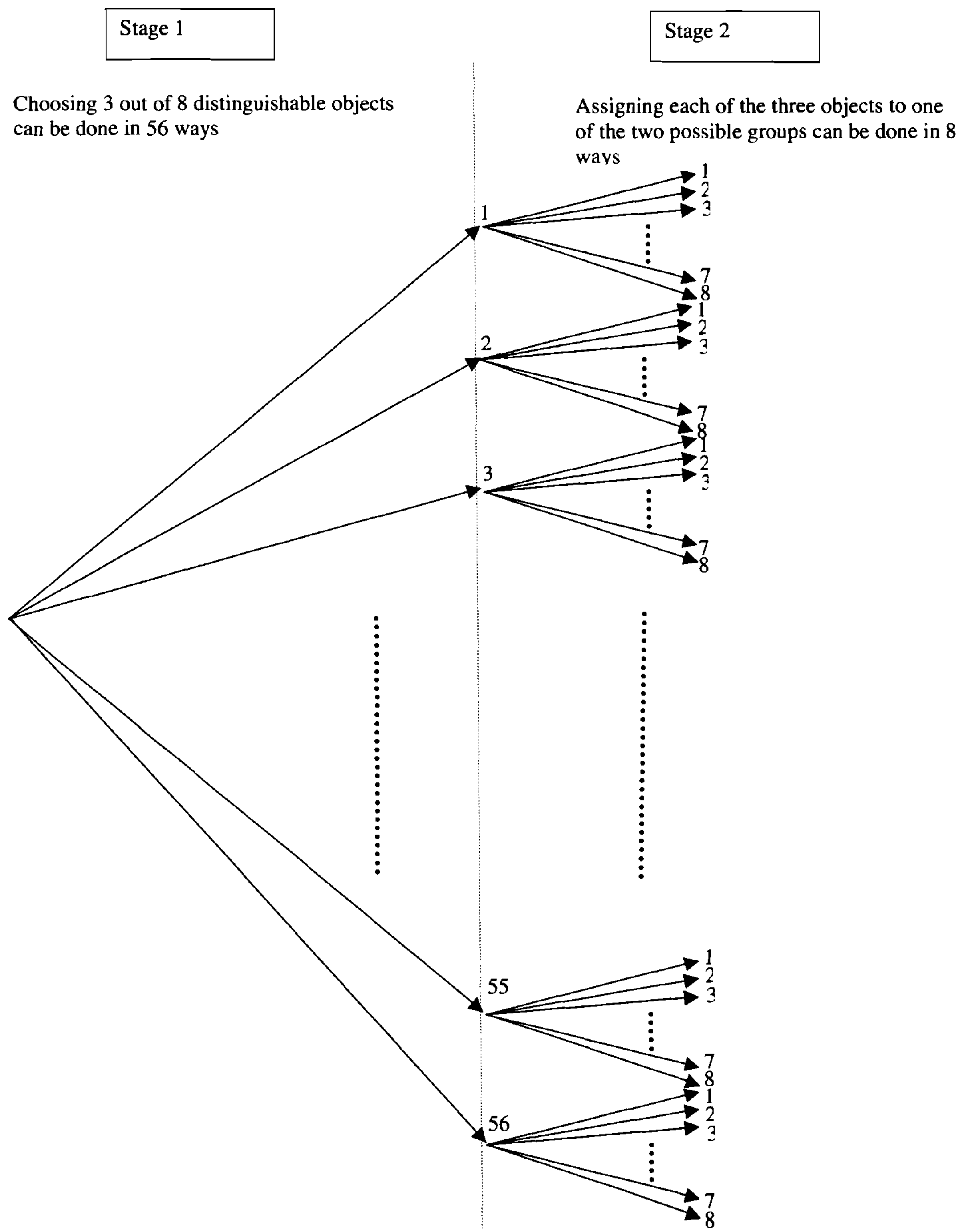

Figure 7.3: General structure for problem with solution $\binom{8}{3} \times 2^3$

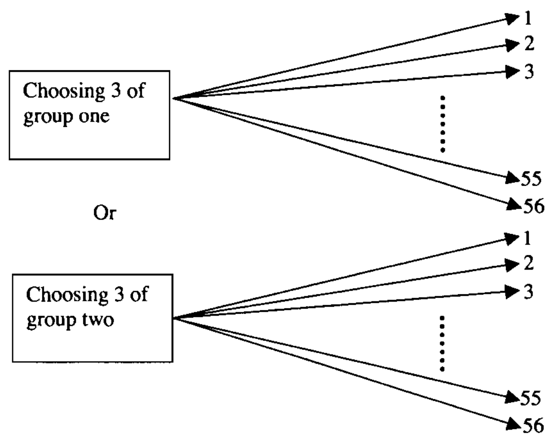

Figure 7.4: General structure for problems with solution $\binom{8}{3} + \binom{8}{3}$

of the seven problems for relative clarity of their wording or for their potential interest to students via their colourful story.

Carol's problem, Problem 3, had a very similar structure to the initial problem, except for the emphasis that a person can be in two groups.

> PROBLEM 3: In a group of 8 people, if we choose 3 people randomly, and then put these people back and choose 3 people randomly out of these 8 people again, how many ways can we get 6 people out of these 8 people in this way?

The solution to Problem 3 is $\binom{8}{3} \times \binom{8}{3}$, but, like many other students, Carol believed that the solution is $\binom{8}{3} \times 2$. Becky created a problem which is structurally very similar to Problem 3. However, Becky was more creative with the story of her problem.

> PROBLEM 4: There are 2 after school activities–ceramics and creative writing with an hour of leeway time between them, making it possible for

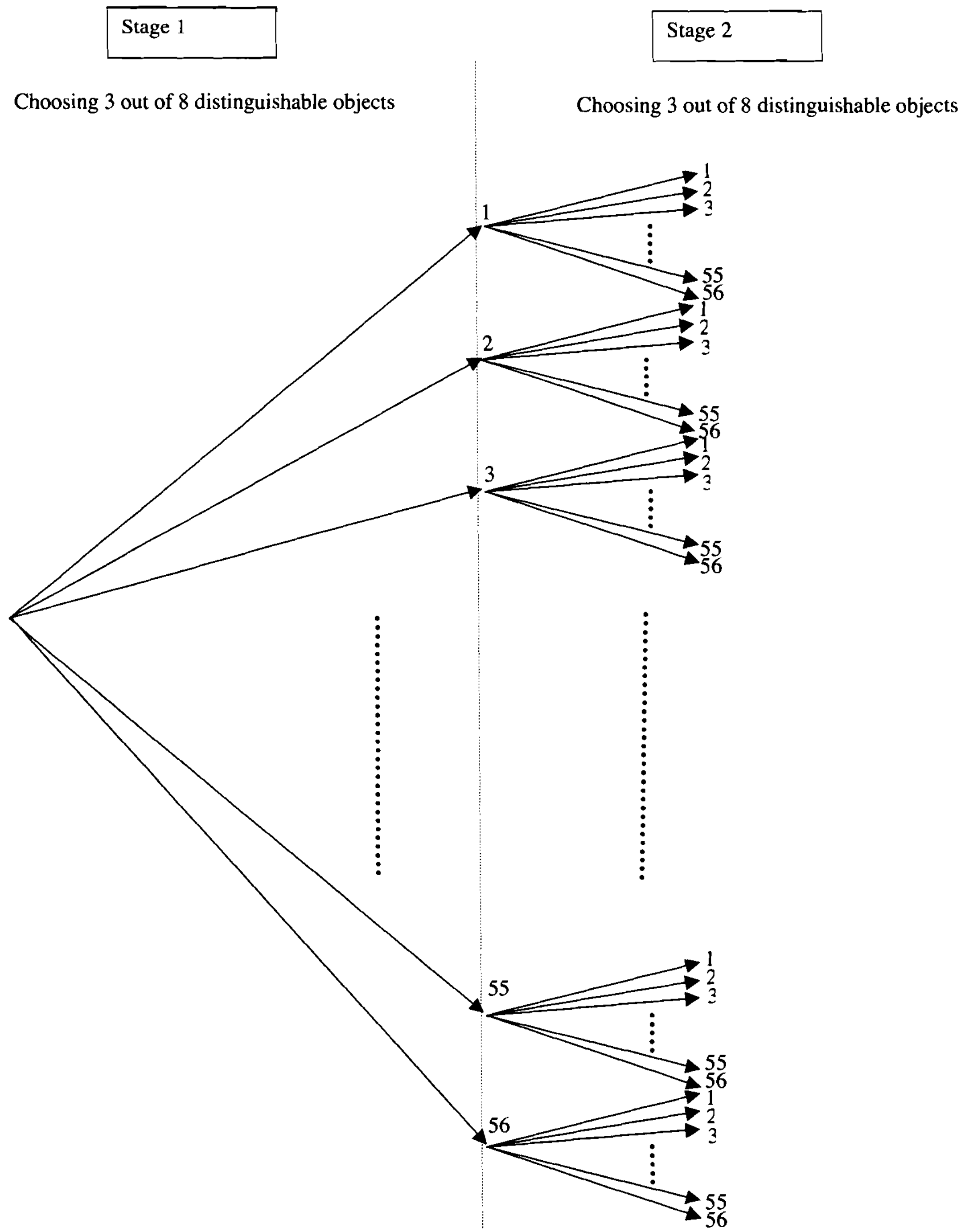

Figure 7.5: General structure for problems with solution $\binom{8}{3} \times \binom{8}{3}$

> 1 person to do both. There are only 3 spots left available in each activity but 8 people that want to participate in both. How many ways can you choose the 2 groups if the same people can be in both?

In Problem 4 we also see the emphasis on the fact that one person can be in both groups. Anna designed a problem with a similar structure to the above problems. Anna's problem has one of the clearest wordings of all the generated problems.

> PROBLEM 5: You have 2 ice-cream cones that you are going to make. There are 8 different toppings and you only put 3 toppings on each. How many possibilities do you have for both ice-cream cones?

Carlos came up with a detailed story as a background to his problem.

> PROBLEM 6: George wants to have vacation to Mexico. He needs to buy 3 pieces of shirts and pants. He decides to go to The Bay[2] to buy these clothes. At The Bay, he go to Guess clothing section. The sales person tells him there are 8 types of each. But George only needs 3 pieces. How many possibilities are there for George to choose 3 out of 8 pieces of shirts and 3 pieces out of 8 pieces of pants?

The general structure of this type of problem can be seen in Figure 7.5. This type of problem generally consisted of two stages, where the first stage has $\binom{8}{3}$ possibilities and each of those possibilities has $\binom{8}{3}$ possibilities at the second stage, which (by the rule of product) results in a total of $\binom{8}{3} \times \binom{8}{3}$.

In all these examples, students generated problems that are structurally identical to the original problem. Possible reasons are that they did not pay attention to the

[2]The Bay is a department store in Canada.

original problem, or that they believed their problem to be different from the original problem.

Problems with irrelevant combinatorial structure

Brenda's problem does not have a combinatorial structure. The wording is unclear and the additive structure is artificially imposed.

> PROBLEM **7**: What is the sum of choosing 3 people out of 8 people, repetition allowed?

I included this problem in the questionnaire in order to observe how students reflect on this type of question, what attracts their attention, and how they attempt to find the solution.

7.3.3 *The third cycle:* Reflection on the generated problems

At this point, in the third cycle, I asked the twelve students to reflect on and solve their collective problems, seven of which were presented in a questionnaire.

To choose the problems for the questionnaire, I first decided to include at least one problem from each type of response. Then I decided to include more problems with the structure $\binom{8}{3} \times \binom{8}{3}$, because it was the most frequent type of response. The chosen problems were the ones with diverse themes, and more clear wording.

Before distributing the questionnaire to the students, I discussed the initial problem and initial solution one more time and I answered their questions. In this cycle, I asked the students to solve and reflect on at least three of the seven problems of their choice in the questionnaire, and to answer the following questions:

1. Can this problem be solved using $\binom{8}{3} \times 2$?

2. If the answer is yes, explain why. If the answer is no, can you modify the problem so it can be solved using $\binom{8}{3} \times 2$?

The goal of this task was to familiarize the students with their peers' responses, and to encourage them to reflect on a set of their collective examples. In doing this, they had a chance to become aware of how others think, to reflect on their own problem again in relation to a large set of problems, and to modify either their own or their peers' problem according to their new understanding of the aforementioned combinatorial structures.

The results are presented in two parts. First, I present the results for three of these seven problems. Later, I present the results based on general trends seen in the students' responses to the questionnaire as a whole.

Analysis of the responses to three of the problems

In this part, I present and analyze the students' responses for three of the seven problems in the questionnaire (problems 2, 5, 7), representing the greatest diversity of problem structure and student responses. I have chosen Problem 2 because of its "almost" appropriate structure. My goal was to examine the students' attention to the language of the problem and what it can entail, and how inappropriate wording can change the problem so much. Problem 5 was chosen, from many problems with the same structure, because all students chose to solve it. Problem 7 was chosen because of its unique perspective including the artificial imposition of "sum".

Results for Problem 2 Ten students responded to Problem 2. The problem was: "There are 8 people participating in an experimental tests, but only 3 of them to be selected into either control group or experimental group. How many ways to select them?"

The results are summarized in Table 7.2. As described before, Cate's problem had two possible interpretations, leading to the solutions $\binom{8}{3} \times 2$ and $\binom{8}{3} \times 2^3$. Since the problem was given to the class without any clarification, I anticipated the students would first clarify what was meant by the question and then attempt to solve it. Without any clarification, there could be no clear solution.

Nonetheless, no one clarified the problem or mentioned that the wording was unclear. In fact, no one presented the solution $\binom{8}{3} \times 2^3$. However, this solution was out of ability of students, and it was not expected.

Five students offered an appropriate modification of Cate's problem. However, all five adequate modifications presented an inappropriate solution to Cate's problem. They solved the problem by $\binom{8}{3} \times \binom{8}{3}$ or $\binom{8}{3} \times \binom{5}{3}$.

Four of these adequate modifications only added the fact that you need to put all three people together in one of the two groups. However, Becky offered a slightly different modification:

MODIFICATION 1

Becky: You are conducting an experiment and you have already selected your control group. You have 8 people to choose from for the experimental group and another 2 people to choose from as the head of the project. How many ways can you do this?

Ways to choose for head of project: $C(2,1)$ Ways to choose experimental group: $C(8,3)$ Total ways to choose from: $C(2,1) \times C(8,3)$

The reason that Becky's modification is different is that she is choosing 4 people in the end, instead of 3.

There were two more students who solved the problem inadequately, using their initial concept image, by $\binom{8}{3} \times \binom{8}{3}$ or $\binom{8}{3} \times \binom{5}{3}$. They presented no modification.

Two students did not solve Cate's problem and presented a modification that was not only inadequate, but also irrelevant to Cate's problem.

Only one student, Carlos, thought the solution was $\binom{8}{3} \times 2$, but the reason he gave for this was not appropriate. Carlos's reasoning for why he thought the answer is $\binom{8}{3} \times 2$ was:

MODIFICATION **2**

Carlos: Yes, $C(8,3) \times 2$ since $C(8,3)$ for selected only control group and another $C(8,3)$ to be selected for experimental group.

Carlos misinterpreted the statement of the problem. He assumed that two groups of 3 are selected, one for each of the experimental and control groups. However, given his misunderstanding, he still thought that the solution would be $\binom{8}{3} \times 2$.

Results for Problem 5 This problem was designed by Anna, which stated: "You have 2 ice-cream cones that you are going to make. There are 8 different toppings and you only put 3 toppings on each. How many possibilities do you have for both ice-cream cones?"

This problem was designed to have the solution $\binom{8}{3} \times 2$, but does not. In fact, seven out of eleven responses had the same combinatorial structure as Problem 5, having the solution $\binom{8}{3} \times \binom{8}{3}$. Most of these problems could be easily modified to have the intended structure. In this part, I discuss how students solved Problem 5

Solving the problem	Modifying the problem	Frequency
Inadequate solution $[\binom{8}{3} \times \binom{8}{3}$ or $\binom{8}{3} \times \binom{5}{3}]$	Adequate modification	5 out of 10
Inadequate solution $[\binom{8}{3} \times \binom{8}{3}$ or $\binom{8}{3} \times \binom{5}{3}]$	No modification	2 out of 10
No solution	Inappropriate modification	2 out of 10
Adequate solution $[\binom{8}{3} \times 2]$	Inadequate reasoning	1 out of 10

Table 7.2: Summary of students' responses to Problem 2

and describe their proposed modification. All twelve students attempted to solve this problem. The results are summarized in Table 7.3.

Solving the problem	Modifying the problem	Frequency
Appropriate solution	Adequate modification	5 out of 12
Appropriate solution	Potentially adequate modification	1 out of 12
Appropriate solution	No modification	1 out of 12
Appropriate solution	Inadequate modification	3 out of 12
Inadequate solution	No modification	2 out of 12

Table 7.3: Summary of students' responses to Problem 5

As we can see in Table 7.3, ten of twelve students solved the problem correctly, by $\binom{8}{3} \times \binom{8}{3}$. Half of these students also proposed an appropriate modification to make the problem have the intended solution of $\binom{8}{3} \times 2$. The usual modification was to change the problem from making two ice cream cones, into making one ice cream cone with a choice of two cones. For example, Bob modified the problem as follows:

MODIFICATION **3**

Bob: You are making an ice cream cone. You have eight different types of ice cream and you are going to use 3 scoops of different flavors. You can put them all either into a normal cone or a waffle cone.

Bob's modification to the problem is minimal, meaning that he kept the story and the context of the problem and only modified enough to get the intended solution. In fact every modification, except Brenda's, was minimal. Brenda's modification was not relevant, and did not have the appropriate structure.

MODIFICATION 4

Brenda: There are 8 customers waiting in a line at the bank. There are two tellers (A and B) who would serve only 3 of the 8 customers. How many ways can customers be served?

In fact, if the eight people were in a line, then the first 3 people would have been served by either teller, which would create the solution 2^3. Even if we ignore the line and say that there are 8 people of whom 3 would be served by either teller (which is not really what can be understood from the wording of the problem), then the solution would be $\binom{8}{3} \times 2^3$.

Results for Problem 7 Problem 7 was designed by Brenda. The problem was: "What is the sum of choosing 3 people out of 8 people, repetition allowed?"

She apparently used the word "sum" to impose the structure $\binom{8}{3} + \binom{8}{3}$. But that is not the only shortcoming of this problem. She does not mention that the "choosing" should be done "twice".

Five people, including Brenda, solved this problem using $8 \times 8 \times 8$, which would have been an appropriate answer if we were choosing 3 people from 8 people considering the order and one person could have been chosen more than once. However, that was not what the problem asked. Five people also explicitly mentioned that the problem

General issues	Frequency
Thought the problem was unclear	5 out of 12
Thought it did not make sense to design a problem like that	2 out of 12
Noticed "not mentioning how many times"	3 out of 12
Thought (mistakenly) the solution was $8 \times 8 \times 8$	5 out of 12

Table 7.4: Summary of students' responses to Problem 7

was unclear or it did not make sense. Even though the problem was not clear and many students expressed this lack of clarity, all twelve students chose it as one of their three questions to solve.

Modification	Frequency
Offered adequate modification	1 out of 12
Offered potentially adequate modification	3 out of 12
Offered inadequate modification	2 out of 12
Offered no modification	6 out of 12

Table 7.5: Summary of students' modification to Problem 7

There were a variety of modifications offered by students for this problem. There were a few students whose modifications were potentially correct. Anna offered one of these modifications:

MODIFICATION **5**

Anna: You have 8 different people, a mix of men and women, and you want to select 3 either all men or all women.

The structure of this problem is very close to being appropriate; however, we need to have 8 men and 8 women. Carol offered a modification which was even closer to being appropriate, which remained problematic by retaining Brenda's word "sum":

MODIFICATION **6**

Carol: What is the sum of choosing 3 out of 8 people and let these people all in group A or group B?

This is another example of artificial imposition of the additive structure to get the structure "adequately".

Analysis of the responses across the whole set of problems

The analysis of the responses was based on the change in the students' concept image after completing the tasks. The final task was for students to solve at least three problems from the questionnaire; if the answer was $2 \times \binom{8}{3}$ they needed to explain the reason, and if not, they needed to modify the problem so that the modified problem would have the desired solution.

Previously, I described in detail the responses to three problems in the questionnaire, concentrating on the students' responses to each problem separately. In this part, I present the students' responses to the questionnaire as a whole, concentrating on the general trends in students' responses. The results are examined through the framework of change in the concept image that was presented in chapter 4. Assuming that every student had an inadequate concept image of the structure of $2 \times \binom{8}{3}$ (by confusing it with $\binom{8}{3} \times \binom{8}{3}$) at the start, the objective is to examine how their concept image had changed after completion of the tasks.

The results are summarized in Table 7.6.

Persistent concept image A persistent concept image is a concept image that has not been affected at all by the tasks, and hence no learning event has occurred. There was no evidence from the data of this study that any of the students' concept images remained exactly the same. In other words, there were no students in the class

The kind of change in the concept image	**Frequency**
Persistent concept image	0 out of 12
Inappropriate concept image without improvement	3 out of 12
Inappropriate concept image with improvement	2 out of 12
Fragmented concept image	1 out of 12
Flexible concept image	6 out of 12

Table 7.6: Summary of students' responses to the whole questionnaire

who responded to these tasks with exactly the same concept image as prior to these tasks.

Inappropriate concept image without improvement In this case, the students presented **contradictory** answers that were consistent with neither their original concept image, nor with the concept itself. They usually modified or solved each problem in the questionnaire in a different inappropriate way. For example, Amos presented the following modification for Problem 5.

MODIFICATION 7

Amos: We have 1 ice cream cone that we want to make. There are 8 different toppings and we only can choose 2 groups of 3 toppings for our ice cream. In how many ways can we make our ice cream?

In the above modification Amos was using one ice cream cone and two groups of 3 toppings, which does not make sense in the context of this problem. However, he solved the Problem 6 and gave the following reasoning:

MODIFICATION 8

Amos: This problem can be solved using $2 \times \binom{8}{3}$ because for only shirts he has 3 out of 8 choices then the number of shirts and pants are the same so it will be $2 \times \binom{8}{3}$.

Which shows his responses are contradictory. As both Problem 5 and Problem 6 have the same combinatorial structure, and Amos is responding to them differently.

Calvin focused on an inappropriate idea to change his concept image. His emphasis was on whether we could choose one person to be in two groups or not. For example his solution and modification of Problem 2 was:

MODIFICATION 9

Calvin: [The solution is] $C(8,3) \cdot C(5,3)$ Because the control group and experimental group are mutually exclusive the first group will have eight people to choose from, whereas the second group will only have five people to choose from. To modify this, so that it can be solved by $2 \cdot C(8,3)$ the two groups would have to be not mutually exclusive. For example, two experimental groups or two control groups.

Inappropriate concept image with improvement When the students' inappropriate concept image is **replaced** by yet another inappropriate concept image, or another inappropriate concept image is **added** to the previous one, it still forms an inappropriate concept image. For example, Anna formed another inappropriate concept image of the structure, which was evident in all her modifications in the questionnaire. For example, she modified her own, Problem 5, in the following way:

MODIFICATION 10

Anna: There are 8 different colours of ice cream cone toppings. You have to choose three that are either all red or all blue.

Anna's modifications had almost the same consistent structure. For example, she modified Problem 7 as follows:

MODIFICATION 11

Anna: You have 8 different people, a mix of men and women, and you want to select three that are either all men or all women.

In this case, Anna's newly formed inappropriate concept image was closer to a concept image of the correct structure. Hence we can consider Anne's new concept image an improvement over her old concept image. This suggests that the process of learning has begun and some degree of better understanding has been achieved.

Fragmented concept image Fragmented concept image, in the context of my study, takes place when there is a previous inappropriate concept image and the appropriate concept image is added to it. In this case, students were answering and modifying questions sometimes appropriately, with adequate reasoning, and sometimes inadequately under the influence of their previous concept image.

Brenda had a mix of adequate and inadequate answers. This shows that she added the appropriate concept image of the structure $2 \times \binom{8}{3}$, but also retained her old inappropriate concept image as well. For example, she modified Problem 2 adequately by adding the clarification that all 3 people need to be either in the control group or or in the experimental group. However, for Problem 4 she replied:

MODIFICATION **12**

Brenda: Yes, [it can be solved by] $2 \times \binom{8}{3}$. [Because] there are two groups and the same people can be in both groups.

To solve Problem 4 Brenda went back to her old concept image. Furthermore, she solved Problem 6 adequately by $\binom{8}{3} \times \binom{8}{3}$, but modified it as follows:

MODIFICATION **13**

Brenda: George wants to buy 3 pieces of shirts out of 8 different types and 3 pieces of pants out of 8 different pants. There are two stores A and B. From A you can buy all 3 shirts and from B you can buy all 3 pants. How many ways can you do that?

Modification 13 is structurally identical to Problem 6, which reveals Brenda may have formed a fragmented concept image.

Flexible concept image When the concept image is flexible, there is a possibility for change. In this case, this flexibility is used as a term to present a positive change in the students' concept image, meaning that they have successfully changed or reconstructed their old concept image to form an appropriate or more complete understanding of the concept.

Becky's responses to the questionnaire revealed that she had corrected her initial inappropriate concept image of the structure $2 \times \binom{8}{3}$ and had formed an appropriate concept image. For example, she had understood the structure in a sense that it needs to have two stages, where one stage has 2 outcomes and the second stage has $\binom{8}{3}$ outcomes. I have presented one of her modifications, Modification 1, previously. She made similar adjustments to the other problems in the questionnaire as well. However, she had only shown one kind of understanding of the structure $2 \times \binom{8}{3}$ which was the case when in the first stage one chooses 3 objects from 8 and in the next stage chooses 1 object from 2 possibilities.

Carol modified all the questions to have the same structure as well. All her modifications consisted of choosing 3 people from a group of 8 people in the first stage and then putting these 3 people in group A or group B.

Bob also modified the problems adequately. For example, to modify Problem 6 he wrote:

MODIFICATION **14**

Bob: George needs 3 outfits for his vacation and he has 8 to choose from. He can either go to Mexico or Saskatchewan for his vacation.

Bob modified four problems and all of them adequately. Likewise, Barb modified two problems adequately. For example she modified Problem 5 in the following way:

MODIFICATION **15**

Barb: There are 8 different toppings and 2 cones to choose from, waffle and sugar. You want to choose 3 to put on either the waffle or sugar cone. How many possibilities are there?

Beth showed some signs of concept image change, and modified one problem adequately. But since she only modified that one problem and did not provide reasoning for the rest of her solutions, I could not decide if she belonged to this group or had fragmented concept image. However, being an optimist, I have considered her in this group.

These students revealed an appropriate concept image, consistent in their responses to the questionnaire. However, it is not very clear if they had multiple appropriate concept images to this structure or a single correct one, and if they faced a different kind of problem with the structure $2 \times \binom{8}{3}$, whether they could recognize it.

Cate, who was the only person with the appropriate initial solution, but was also confused when presented with her peers' solution, cleared up her confusion by the end of the tasks. In addition to the adequate modification to four problems and appropriate answers to the rest, she offered different appropriate concept images for her modifications. Meaning that unlike her peers in this group, she modified problems based on different ways of looking at the structure $2 \times \binom{8}{3}$. For example, she modified Problem 6 as follows:

MODIFICATION **16**

Cate: How many possibilities are there for George to choose 3 out of 8 pieces of shirts or 3 pieces out of 8 pieces of pants.

In Modification 16 we see that Cate used the concept image that involves "**or**" which is a different way of looking at the structure $2 \times \binom{8}{3}$ from her modification of Problem 2 which was originally designed by herself.

MODIFICATION **17**

Cate: There are 8 people participating in an experiment tests. If 3 of them are selected into control group, the rest will be in experimental group. If 3 of them are selected into experimental group, the rest will go to control group. How many ways can we select them?

Cate revealed two different exemplifications of appropriate concept images for the same structure $2 \times \binom{8}{3}$.

7.4 Conclusion

Acquiring knowledge and understanding of a concept involves construction of new knowledge and reconstruction of parts of the old knowledge, which are somehow connected to this concept, to create a coherent unified schema. This reconstruction is necessary, especially when the learner finds that the new knowledge is not compatible with their previous concept image or with parts of their old knowledge. When faced with inconsistencies, some learners ignore the apparent conflict and create a new compartment for their new concept image so it does not interfere with their old concept image. This results in a fragmented understanding of the concept, which can lead to further misconceptions and inconsistencies. Some learners keep using their old concept image and ignore the new knowledge, hence, they resist learning the new concept. Others may decide to abandon the old concept image and adapt the new one. They not only find mathematics to be inconsistent, but also they believe that the rules of mathematics changes according to the course, the teacher, or the situation. They adapt to the new knowledge, but they also get fragmented understanding of the concept. In fact, the only way to deal with these inconsistencies is to reconcile new and old concept image, by reflecting on previous misconceptions, recognizing the cause of those misconceptions, and reconstructing the conceptual structure and concept image. Mediated successive refinement accommodates reflection on and reconstruction of old concept images to allow for constructing a new concept image, and connecting it to the old knowledge. In this study, the concept was the multiplicative structure of multi-stage experiments (to count the number of possibilities we needed to multiply the number of outcomes in each stage). However, the students' approach was an additive structure (they added the number of outcomes in each stage). By using the mediated successive refinement, I invited the learners to reflect on their understanding of additive structures and what they represent to point out why the adequate solution does not posses an additive structure.

This method creates a personal discourse (between the student and her/himself) and a public discourse (among the students and perhaps the teacher or researcher). This method can help us elevate our grasp of students' understanding because the learners themselves will contribute to their questionnaire.

Chapter 8

Conclusion

This study was inspired by my personal interest in discrete mathematics and combinatorics, observing my students' difficulties with these topics, and the insufficient body of research in this area of mathematics education. I posed three general questions in chapter 1. In this chapter, I revisit each of these questions and describe the results addressing each of these question. Furthermore, I describe the contributions of the two studies discussed in chapters 6 and 7. Finally, I present the limitations of this research and some suggestions for further studies.

8.1 Approaching a new concept

In chapter 6, I described a study that investigated students' understanding of a new concept. The study was designed to answer some of the initial questions that were posed in chapter 6: How do learners understand a new combinatorial structure? How do they approach a new concept? In particular I asked the following questions:

- Do they use examples to get a better understanding of the structure?

- How do they utilize known formulae? Do they understand the formulae they use? Can they create new formulae to solve a problem?

- What kinds of representations do they use to describe their thinking and represent their solutions? Do they use different kinds of representation? Do they use appropriate representation? Do they use the graphical approach or the algebraic approach?

- What kinds of connections do they make to their previous knowledge? How do they use their previous knowledge to learn this new definition? Do they concentrate on the differences and similarities between the new structure and the other structures they have known previously?

- How do they approach tasks relating to this new concept?

- On what aspects of this new structure do they concentrate more?

I investigated students' understanding of a new definition that they had not seen before through a set of tasks. The data revealed that learners generally did not use active examples to understand and approach a new combinatorial structure. However, they used the examples of the concepts that they had seen in the class extensively (passive examples). It was also revealed that when the learners were asked to generate examples (LGE), they had some difficulty in going beyond the trivial examples and to be creative.

Most participants used some graphical approach in their initial understanding of the definition of trization. However, they did not accept their graphical understanding as adequate. Their image of understanding mathematical structures required an algebraic representation, i.e. a formula. The data also revealed that participants heavily relied on rote memorization and application of formulae rather than understanding them. Memorization of formulae helped students to solve some problems very efficiently and correctly, but when they could not remember a formula correctly, they could not proceed. Also when they could not indentify a formula, because of the unfamiliar structure, they had difficulty approaching the problem. On the other hand, many participants were successful in finding a formula for trization after they had formed an understanding of the concept.

I found that participants had difficulties making the connection between the new concept and their existing concept image of the related concepts. The only instance when the participants did recognize a connection was through the use of formulae. A few participants recognized trization as "multiple combinations", or its relation to binomial theorem, when they found the formula for counting the possible trizations. However, none of the participants were able to make any connection without the use of formula. Furthermore, another anticipated connection $T(n : k, n-k, 0) = \binom{n}{k}$ was not made by participants.

Considering that the participants were students with generally strong mathematical background, the findings revealed that they rely heavily on the use of formulae to solve the tasks presented to them. They often had difficulty approaching problems that did not provide them with a recipe and a ready-to-use formula for its solution.

8.2 Obstacles and difficulties

When I began this research, I had many questions. Some of these questions were related to the obstacles and difficulties that students face in learning combinatorics.

Some of these questions were answered by the previous studies in the literature.

What are the obstacles in understanding a combinatorial structure? What are the students' main difficulties in solving combinatorial problems? Are there obstacles and problems specifically related to combinatorics?

In the previous studies and existing research in mathematics education students' difficulties and errors in combinatorics were discussed. I presented a summary of these prior findings in Chapter 3. I also distinguished between the errors that are specific to combinatorics and those that are related to general problem solving. This distinction was important for the purpose of my study, since recognizing that there are problems specific to combinatorics reassured me of the need and importance for further research in this area. However, there are some follow up questions:

- How can students deal with these difficulties and obstacles?
- What is the teacher's role in helping students deal with these obstacles?

These questions, along with observing my students' difficulties, led me to the study of change in the learner's concept image. This study inspired mediated successive refinement. I present a the summary of its findings in the next section.

8.3 Modification of an inadequate understanding

Students, especially in post secondary-education, have strongly formed preconceptions. These preconceptions are sometimes in agreement with those of the mathematics community's conceptions and sometimes they are in conflict. Students' preconceptions that are in conflict (misconceptions) with that of the general mathematical community (scientific conception), can create obstacles for their learning. Nussbaum and Novick (1982, p. 184) claim that:

> When a student retains and continues to use his preconception to interpret classroom information, he is likely to give it meaning which differs from or even conflicts with the meaning intended by his teacher. It is possible that the learner is not even aware of this gap and that he is perfectly satisfied with his own interpretation, thinking that such was also his teacher's intention.

Mediated successive refinement helps students deal with their misconception by first recognizing their misconception and reflecting on not only their own concept image, but their peers' concept image as well, through a set of problems that they generated collectively. This process began as an instructional tool; however, I also used it as a research tool to answer the following questions that were initially posed in chapter 7.

How do learners modify their existing concept image to obtain a better understanding? In particular:

- How can the learners become aware of their inappropriate understanding? In particular does mediated successive refinement help students to become more aware of their own understanding?

- What can the method of mediated successive refinement reveal about the process of change in students' understanding? What are the changes that can happen in students' understanding?

- How does this instructional intervention influence students' understanding of the concept?

When a common misconception is revealed, we can use mediated successive refinement to help students achieve a better understanding of their misconception and modify it to reach a better understanding. There may be many factors that trigger a change in students' concept image. One of these triggers is to encourage learners to reflect on their own inadequate concept image by asking them to embody that image example-generation. Furthermore, by reflecting on the examples generated by their peers they get yet another opportunity to become more aware of their own understanding. It provides learners with an opportunity to reexamine their understanding by asking them to modify their generated problems. As mentioned in chapter 7, the data revealed that every student had some kind of change in their concept image through the employment of this methodology. In fact, for most students some kind of learning event did happen and their concept image was refined, hence they achieved a better understanding. However, a few students also replaced an inappropriate concept image with another inappropriate concept image.

Another finding of this study was the high level of participant's interest in the tasks. Even though participation in this study was completely voluntary, many students took part and completed all the tasks. They appreciated the opportunity to interact with their peers through their generated examples and described their overall experience as positive.

8.4 Contributions

The general objective of this study was to examine students' learning and understanding of combinatorial structures in post-secondary education. In the course of this study, I identified different factors that could paint a picture of students' understanding of combinatorial structures and examined the different kinds of change that can happen in students' concept image. Furthermore, I developed the methodology of mediated successive refinement to help students examine their own understanding and refine their concept image by comparing and contrasting their concept image with that of their peers. This methodology was also utilized as a research tool for gathering data for examining the gradual change in students' concept image. This study contributed to two aspects of understanding, the theoretical aspect and the methodological and pedagogical aspect. I briefly recap these contributions.

8.4.1 Theoretical contributions

As mentioned before, in addition to its application in different fields of science and mathematics, combinatorics can be an interesting topic to be explored in the curriculum. However, many students find it challenging not only to solve combinatorial problems, but also to verify their solutions. Sometimes a solution to a combinatorial problem can be very convincing and elegant, but incorrect. Showing that incorrectness could prove very challenging.

There are many valuable books and papers written about problem solving, strategies, pedagogy, etc. In this study, my goal was not to reexamine or reiterate those, but to examine students' development of understanding and concept image through what seemed to be their methods of solving elementary counting problems. To the best of my knowledge, there is no research that specifically pertains to the examination of concept image and understanding of students in combinatorics. I examined the creation of a new concept image and the modification of an existing one. In both studies, the data confirmed that students do face many challenges and difficulties such as the errors that were presented in the previous research. Moreover, I developed a framework to explain and examine different aspects of students' understanding of combinatorial structures.

Revealing concept image

A major contribution of this work is the development of a framework that can paint a picture of students' concept image and their understanding for the educator and researcher. The framework consists of examination of different factors while students are asked to solve some problems. These factors are: students' use of different kinds of examples (active, passive, and LGE), different kinds of representations (graphical and algebraic), their use of and reliance on formulae, and the connections they make to their previous knowledge.

Evolving concept image

Another contribution of this study is examining the change in students' concept image. The investigation of their maturing concept image revealed that there are different levels of change in the concept image. These different levels of change are as follows:

- Persistent concept image
- Inappropriate concept image without improvement

- Inappropriate concept image with improvement
- Fragmented concept image
- Flexible concept image

This allows a more in depth examination and analysis of the change in students' concept image.

8.4.2 Pedagogical and methodological contributions

According to Batanero, Navarro-Pelayo, and Godino (1997), one of the important aspects in research in the domain of teaching and learning combinatorics is understanding students' difficulties in this field and identifying variables that might influence this difficulty. By using mediated successive refinement, I encouraged learners to reflect on different aspects of their own understanding of a particular structure and its relation to other structures that they had encountered previously. In addition, I provided them with an opportunity to observe and reflect on the examples that were generated by their peers. I also utilized their responses to identify some of the difficulties and challenges that learners face in understanding combinatorial structures.

Use of learner-generated examples provides many beneficial opportunities for the learner. However, research shows that generating examples is not an easy task for students. In their interviews, Hazzan and Zazkis (1999) noticed some difficulties for participants in generating their own examples. Dahlberg and Housman (1997) also noted that some students were reluctant or unwilling to generate examples, were unsure of their responses, and needed the confirmation of the interviewer. In the previous chapter, I described students' difficulties with generating example, both in the tasks that required them to do so, and in other tasks in which example generation could have helped their understanding.

I believe that by asking students to generate their own examples, in practice such as it was done in mediated successive refinement, they can overcome their difficulties with the use and generation of examples. The ability to generate examples enables learners to learn mathematics as a constructive activity and it additionally enables them to examine their own understanding of the concepts.

This exercise helped students to practice being critical in relation to the problems they were solving. Because they knew the problems were designed by their peers, they felt that they could think about the structure of the problem with a critical eye. This is a very important pedagogical aspect of these tasks.

This method seems to help students achieve a better understanding. At the very least it helps them to think about, and compare and contrast their peers' and their own inadequate understanding in a critical way.

Mediated successive refinement also served as a tool for gathering data. The dual role of mediated successive refinement as a pedagogical and methodological tool arose from the dual role that I played as the teacher and researcher in the class. Researchers have previously employed learner-generated examples as a methodological tool to gather data and they have proven it to be a successful and effective methodology (Bogomolny, 2006). Mediated successive refinement takes example generation one step further, exploring students' thoughts on their peers' examples as well as their own.

8.5 Limitations and further research

There are a few limitations to each of these two studies. The first limitation in both studies was the sample size. In the first study, there were only eight students who volunteered from a class of over 200 students. These were students who were more confident in their mathematical abilities and had a considerably stronger background in mathematics. Hence, the findings of this study may not be representative of a typical similar classroom. In the second study, the class size was small, which is what was needed to perform the study.

Another limitation was the timing of the interviews. Combinatorics was taught in the last weeks of the course, and by the time that the students had enough information to participate in the study, it was the end of the semester. Hence, there was no time to have a followup interview and reexamine students' understanding of trization at a later time, after their initial interview.

For future research, it would be interesting to examine not only initial understandings, but also how that understanding evolves through time. Also one could examine how mediated successive refinement can be employed to examine the change in students' inappropriate concept image of trization.

Another direction for future research to examine the effect of mediated successive refinement as opposed to another instructional tool, such as a classic problem-solving session, examining how each contributes to students' awareness of their problematic conceptions and to refining their misconceptions.

8.6 Last words

> What are the significant products of research in mathematics education? I propose two simple answers:
>
> 1. The most significant products are the transformations in the being of the researchers.
> 2. The second most significant products are stimuli to other researchers and teachers to test out conjectures for themselves in their own context.
>
> J. Mason (1998, p. 357)

This study had a considerable influence on me, both as a teacher and as a researcher. I found the research very rewarding, while the experience was humbling. Besides the main research findings that were discussed before, I learned a lot as a teacher in the process of research. I learned that given the opportunity many students were open to participating in the research study, performing the tasks even with the knowledge that their participation did not affect their evaluation. They generously dedicated their time and resources to participating in my research without expecting any reward. Most participants demonstrated their appreciation for understanding the concepts by staying long after the interviews were finished to discuss the tasks and topics from the interview. I also learned that taking chances in the classroom to address students' misunderstandings in depth is not only worth its while to help the students, but also it helped me as the teacher to see the root of the students' problems and to address them accordingly.

There are many opinions about education in general and mathematics education in particular. Through careful research, some of these opinions become valuable educational theories or pedagogical tools, and some become myths. I hope that this study has shown that mediated successive refinement belongs to the former.

References

Adelr, S. A. (1993, April). Teacher education: Research as reflective practice. *Teaching and Teacher Education*, *9*(2), 159-167.

Aiton, E. J., & Shimao, E. (1981). Gorai Kinzō's study of Leibniz and the I Ching hexagrams. *Annals of Science*, *38*(1), 71–92.

Atkinson, S. (1994). Rethinking the principles and practice of action research: the tensions for the teacher-researcher. *Educational Action Research*, *2*(3), 383-401.

Batanero, C., Godino, J. D., & Navarro-Pelayo, V. (1997). Combinatorial reasoning and its assessment. In J. Garfeld & I. Gal (Eds.), *The assessment challenge in statistics education.* Amsterdam, The Netherlands: ISI & IOS Press.

Batanero, C., Navarro-Pelayo, V., & Godino, J. D. (1997). Effect of the implicit combinatorial model on combinatorial reasoning in secondary school pupils. *Educational Studies in Mathematics*, *32*(2), 181–199.

Batanero, C., & Sanchez, E. (2005). What is the nature of high school students' conceptions and misconceptions about probability? In G. A. Jones (Ed.), *Exploring probability in school, challenges for teaching and learning.* New York, NY: Springer.

Biggs, N. L. (1979). The roots of combinatorics. *Historia Mathematica*, *6*, 109-136.

Biggs, N. L., Lloyd, E. K., & Wilson, R. J. (1995). The history of combinatorics. In I. Graham, M. R. L. Grötschel, & L. Lovász (Eds.), *Handbook of combinatorics* (Vol. 2). Cambridge, MA: The MIT Press.

Bogomolny, M. (2006). *The role of example-generation tasks in students' understanding of linear algebra.* Unpublished doctoral dissertation, Simon Fraser University.

Cameron, P. J. (1994). *Combinatorics: Topics, techniques, algorithms.* Cambridge: Cambridge University Press.

Cochran-Smith, M., & Lytle, S. L. (1999). The teacher research movement: A decade later. *Educational Researcher, 28*(7), 15-25.

Dahlberg, R. P., & Housman, D. L. (1997). Facilitating learning events through example generation. *Educational Studies in Mathematics, 33*(3), 283–299.

Edwards, B. (1999). Revising the notion of concept image/concept definition. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, p. 205-210). Columbus, OH: The ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Eizenberg, M. M., & Zaslavsky, O. (2003). Cooperative problem solving in combinatorics: the inter-relations between control processes and successful solutions. *Journal of Mathematical Behavior, 22*(4), 389–403.

Eizenberg, M. M., & Zaslavsky, O. (2004). Students' verification strategies for combinatorial problems. *Mathematical thinking and learning, 6*(1), 15–36.

English, L. D. (1991). Young children's combinatoric strategies. *Educational Studies in Mathematics, 22*, 451-474.

English, L. D. (1996). Children's construction of mathematical knowledge in solving novel isomorphic problems in concrete written form. *Journal of Mathematical Behavior*, *15*, 81-112.

English, L. D. (2005). Combinatorics and the development of children's combinatorial reasoning. In G. A. Jones (Ed.), *Exploring probability in school, challenges for teaching and learning.* New York, NY: Springer.

Fischbein, E., & Gazit, A. (1988). The combinatorial solving capacity in children and adolescents. *Zentralblatt für Didaktitk der Mathematik*, *5*, 193-198.

Fischbein, E., Pampu, I., & Minzat, I. (1970). Effects of age and instruction on combinatory ability in children. *The British Journal of Educational Psychology*, *40*(3), 261-270.

Freudenthal, H. (1972). *Mathematics as an educational task.* Dordrecht, The Netherlands: Springer.

Gardiner, A. D. (1991). A cautionary note. In M. J. Kenney & G. R. Hirsch (Eds.), *Discrete mathematics across the curriculum, K-12: 1991 yearbook.* Reston, VA: National Council of Teachers of Mathematics.

Ginsburg, J. (1923). Rabbi Ben Ezra on permutations and combinations. *Mathematics Teacher*, *15*, 347-356.

Graham, R. L., Grötschel, M., & Lováz, L. (1995). *Handbook of combinatorics.* Cambridge: MIT Press.

Grimaldi, R. P. (2004). *Discrete and combinatorial mathematics: an applied introduction* (5th ed.). Boston, MA: Addison-Wesley.

Hadar, N., & Hadass, R. (1981). The road to solve combinatorial problem is strewn with pitfalls. *Educational Studies in Mathematics*, *12*, 435-443.

Hart, E. W. (1991). Discrete mathematics: An exciting and necessary addition to the secondary school curriculum. In M. J. Kenney & G. R. Hirsch (Eds.), *Discrete mathematics across the curriculum, K-12: 1991 yearbook.* Reston, VA: National Council of Teachers of Mathematics.

Hazzan, O., & Zazkis, R. (1999). A perspective on "give an example" tasks as opportunities to construct links among mathematical concepts. *Focus on Learning Problems in Mathematics, 21*(4), 1-14.

Inhelder, B., & Piaget, J. (1975). *The origin of the idea of chance in children.* New York, NY: W. W. Norton & Company Inc.

Jaworski, B. (1998). Mathematics teacher research: Process, practice and the development of teaching. *Journal of Mathematics Teacher Education, 1*(1), 3-31.

Jonker, L., & Lidstone, D. (2005). Discrete mathematics working group report. In P. Liljedahl (Ed.), *Proceedings of the 2005 annual meeting of the canadian mathematics education study group* (pp. 37–48). Ottawa: CMESG/GCEDM.

Kenney, M., & Hirsch, C. (1991). *Discrete mathematics across the curriculum, K-12, 1991 yearbook.* Reston, VA: National Council of Teachers of Mathematics (NCTM).

Leikin, R., & Winicki-Landman, G. (2000). On equivalent and non-equivalent definitions: part 1. *For the Learning of Mathematics, 20*(1), 17-21.

Maher, C. A., & Martino, A. M. (1996). The development of the idea od mathematical proof: A 5-year case stuy. *Journal for Research in Mathematics Education, 27*(2), 194-214.

Maher, C. A., & Martino, A. M. (2000). From patterns to theories: conditions for conceptual change. *The Journal of Mathematical Behavior, 19*(2), 247–271.

Mason, J. (1998). Researching from the inside in mathematics education. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity.* Dordrecht, The Netherlands: Kluwer Academic Publishers.

Mason, R. (2003). *Understanding understanding.* New York: SUNY Press.

NCTM. (1989). *Curriculum and evaluation standards.* Reston, VA: National Council of Teachers of Mathematics.

NCTM. (2000). *Principles and standards of school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Nussbaum, J., & Novick, S. (1982). Alternative frameworks, conceptual conflict and accommodation: Toward a principled teaching strategy. *Instructional Science, 11*(3), 183-200.

Polya, G. (1957). *How to solve it* (2nd ed.). Prlncton, NJ: Princeton University Press.

Rosen, K. (2000). *Handbook of discrete and combinatorial mathematics.* Washington, DC: CRC Press.

Siegel, M. (1986). *Report of the committee on discrete mathematics in the first two years.* Washington, DC: Mathematical Association of America.

Sierpinska, A. (1994). *Understanding in mathematics.* London: The Falmer Press.

Smith, D. E. (1958). *History of mathematics.* New York: Dover.

Suzuki, J. (2002). *A history of mathematics.* New Jersey: Prentice Hall.

Tall, D. (1988). Concept image and concept definition. In J. d. Lange & M. Doorman (Eds.), *Senior secondary mathematics education* (p. 37-41). Utrecht: OW&OC.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (p. 65-80). Dordrecht: Kluwer Academic Press.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity : learners generating examples.* Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Zazkis, R., & Leikin, R. (2007). Generating examples: From pedagogical tool to a research tool. *For the Learning of Mathematics*, *27*(2), 15-21.

Zeitz, P. (2007). *The art and craft of problem solving* (2nd ed.). New York: John Wiley & Sons, Inc.