ALGORITHMS FOR WAVELENGTH ASSIGNMENT AND CALL CONTROL IN OPTICAL NETWORKS

by

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Abstract

Routing and channel assignment is a fundamental problem in computer/communication networks. In wavelength division multiplexing (WDM) optical networks, the problem is called routing and wavelength assignment or routing and path coloring (RPC) problem: given a set of connection requests, find a routing path to connect each request and assign each path a wavelength channel (often called a color) subject to certain constraints. One constraint is the distinct channel assignment: the colors (channels) of the paths in the same optical fiber must be distinct. Another common constraint is the channel continuity: a path is assigned a single color. When a path may be assigned different colors on different fibers, the RPC problem is known as the routing and call control (RCC) problem. When the routing paths are given as part of the problem input, the RPC and RCC problems are called the path coloring and call control problems, respectively. Major optimization goals for the above problems include to minimize the number of colors for realizing a given set of requests and to maximize the number of accommodated requests using a given number of colors. Those optimization problems are NP-hard for most network topologies, even for simple networks like rings and trees of depth one. In this thesis, we make the following contributions. (1) We give better approximation algorithms which use at most $3L$ (L is the maximum number of paths in a fiber) colors for the minimum path coloring problem in trees of rings. The $3L$ upper bound is tight since there are instances requiring $3L$ colors. We also give better approximation algorithms for the maximum RPC problem in rings. (2) We develop better algorithms for the minimum and maximum RPC problems on multi-fiber networks. (3) We develop better algorithms for the call control problem on simple topologies. (4) We develop carving-decomposition based exact algorithms for the maximum edge-disjoint paths problem in general topologies. We develop and implement tools for computing optimal branch/carving decompositions of planar graphs to provide a base for the branch/carving-decomposition based algorithms. These tools are of independent interests.

Keywords: communication networks; path coloring; call control; approximation algorithms; trees of rings; planar branch-decomposition; computational study

To my family

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Chapter 1

Introduction

In recent years, Internet traffic has increased enormously due to the various bandwidth-intensive applications, such as video-conferencing, video-on-demand [106] and peer-to-peer (P2P) applications [116]. Optical networks, which employ optical fibers as the carrier, can provide the huge bandwidth needed. The bandwidth of optical fibers is about 50THz, which is much higher than that of conventional carriers such as copper wires. The huge bandwidth of optical fibers is usually utilized through multiplexing. *Wavelength Division Multiplexing* (WDM) is a multiplexing technology widely used in optical networks. It allows multiple channels to be carried on the same fiber by assigning a different wavelength to each channel. A network is called all-optical (or single-hop) network if optical signals remain in optical form (without conversion to electrical form) from source to destination. A fundamental problem in all-optical WDM networks (and in circuit-switched networks in general) is that given a set of connection requests (source-destination pairs) in a network, find a path for each request (*routing*) and assign each path a channel such that the paths with the same channel do not share any communication link in the network (*channel assignment*). The problem is also known as the off-line routing and channel assignment problem. In this thesis, we study the off-line routing and channel assignment problem and related problems in optical networks.

1.1 Routing and Wavelength Assignment in WDM Optical Networks

In all-optical WDM networks, each channel is supported by a wavelength or a *color* and the routing and channel assignment is known as *routing and wavelength assignment* or *routing and path coloring* (RPC). We will use the two terms interchangeably. The wavelength assignment sub-problem has two basic constraints. The first constraint, called distinctive channel assignment constraint, requires that the paths must be assigned different colors if they are on the same fiber. The second constraint, called the channel continuity constraint, requires that each path must be assigned the same color on every link from source to destination. There are two natural optimization problems. One is to minimize the total number of colors for accepting and coloring all the given routing requests in a network. This problem is called the *minimum routing and path coloring* (Min-RPC) problem. The other is to select a maximum subset of requests, route and color the selected requests with a given number of colors. The problem is called the *maximum routing and path coloring* (Max-RPC) problem. Algorithms for the Min-RPC problem are important to reduce the resource required for realizing a set of requests, and they are widely used in network planning and design stages. Algorithms for the Max-RPC problem are critical for improving the performances of a network when the resources in the network are not enough to support all connection requests, and they are widely used in network design and operating stages. In the Max-RPC problem, each request may be assigned a weight (e.g., the profit obtained if the request is accepted). The goal is to maximize the total weight of accepted requests. This generalized problem is called the maximum weight routing and path coloring problem (weighted Max-RPC). In all the problems defined above, routing paths may be given as part of the input (i.e., a set of pre-specified paths is given instead of a set of requests), and only the path coloring sub-problem needs to be solved (we will use "PC" instead of "RPC" in the abbreviations). All these problems are NP-hard for general networks [92, 105].

The routing and path coloring problems are first studied in single fiber optical networks, i.e., each link has a single fiber (thus paths with the same color must be edge-disjoint). Recently, there are renewed interests in the multifiber optical networks. In these networks, each link e has $\mu(e)$ parallel fibers, and at most $\mu(e)$ paths can use the same color on link e. Thus, in multifiber optical networks, more than one path may use the same color on a link. All the problems defined above can be studied in the multifiber environment. The path

coloring problem in multifiber networks is often referred to as *path multicoloring* (PMC).

The *routing and call control* problem is closely related to the routing and path coloring problem, and can be defined as follows: Given a set R of requests in a graph $G = (V, E)$ with every edge $e \in E(G)$ assigned a non-negative capacity $c(e)$, find a maximum subset of R such that each request in the subset is assigned a path, and the number of paths on any edge e is at most $c(e)$. Each request may be given a positive weight, and the goal is to accept a subset of requests with maximum total weight. When routing paths are given as part of the input, the problem is simply called the *call control* problem. The call control problem can be considered as a variant of the path coloring problem in which a path may use different colors on different links (the channel continuity constraint is relaxed) and the number of available colors on different links may be different. When $c(e) = 1$ for every edge $e \in E(G)$, the routing and call control problem is called the *maximum edge-disjoint paths* (MEDP) problem. The Max-RPC problem with one available color is the same as the MEDP problem. The *maximum edge-disjoint paths with pre-specified paths* (MEDPwPP) problem is the same as the call control problem with $c(e) = 1$ for every edge $e \in E(G)$, or the Max-PC problem with one available color.

The call control problem is also closely related to the maximum path multicoloring (Max-PMC) problem in multifiber optical networks. The Max-PMC problem asks to maximize the number of paths that can be colored by a given number w of colors in a multifiber optical network. When $w = 1$, this is simply a maximum edge-disjoint paths problem in a single fiber network, and is a call control problem (with edge capacity $c(e) = \mu(e)$ on edge e) in a multifiber network. Solving the call control problem is an important step towards solving the Max-PMC problem in the iterative greedy approach.

The routing and path coloring problem and the related problems have been extensively studied in general topologies and in various special topologies, including the tree, a connected graph without cycles; the ring, a cycle with at least three nodes; and the tree of rings, a set of rings connected through a tree structure. The ring and tree of rings are popular topologies in optical networks. For example, a 140-node metropolitan area network of 12 rings with a conceptual tree of rings topology has been deployed in Jacksonville, Florida, based on the Optical Metro 3500 Multiservice Platform of Nortel (see [2, 3] for the conceptual topology). In this thesis, we focus on the path coloring and call control problems. We study the Min-PC problem on trees of rings, the Max-RPC problem on rings, the path multicoloring and call control problems on trees with special properties. We also study the MEDP and Max-PC

problems on more general topologies. We give details of our studies in the following section.

1.2 Contributions of the Thesis

In this section, we give an outline of the problems that we solved in this thesis. For each problem considered, we first give the previously known best result, and then show our contributions. In what follows, a network is a single fiber network (i.e., each link has one fiber) unless it is explicitly referred to as a multifiber network. An undirected network is expressed by an undirected graph, and a directed network is expressed by a directed graph. Optical networks are directed because the optical amplifier, a key element to boost the optical signals in a fiber, is directed. However, the undirected network model is an abstract theoretical model, and has been used in many previous studies. In the following discussion, graphs are undirected unless explicitly stated as directed when we give a specific result on the problems we are interested in.

1.2.1 Contributions on Special Topologies

We first give the results we obtained on special topologies. Given a set of paths in a network, the maximum number L of paths on any link of the network is a lower bound on the number of colors required for coloring the set of paths. For the Min-PC problem, the number of colors required is often evaluated in the unit of L.

The path coloring problem on trees of rings

A tree of rings is an important topology for optical networks, with several subrings connected to a main ring, sub-subrings to subrings, and so on, to form a larger network. An algorithm using at most $4L$ colors is known for the Min-PC problem on the undirected trees of rings with arbitrary node degrees [47]. The algorithm has an approximation ratio of 4. A 2-approximation algorithm is known for a restricted class of trees of rings with maximum node degree four [44]. A $3L$ and 2-approximation (resp. 2.5-approximation) algorithm is given for a restricted class of trees of rings with maximum node degree four (resp. six) in [25]. It is also shown in [25] that at least $3L$ colors are needed for some instances in the trees of rings with degree four. It has been an interesting open problem whether $3L$ is the upper bound for the Min-PC problem on the entire class of trees of rings. We settle this

problem by giving a $3L$ algorithm for trees of rings with arbitrary node degrees. We further give a 2.75-approximation algorithm which improves the previous approximation ratio of 4. For the restricted class of trees of rings with node degree six (resp. eight and ten), we give a $3L$ and 2-approximation (resp. 2.5-approximation) algorithm which improves the previous results. Our algorithms are based on novel applications of edge-coloring of multigraphs and efficient local greedy path coloring schemes.

Our algorithms work for directed trees of rings as well, with an upper bound of $6L$ (this improves the $8L$ upper bound of [47]).

The Min-RPC problem in trees of rings has been studied in [92, 105]. Using the cut-one-link approach, they obtained 3-approximation and 10/3-approximation algorithms for the Min-RPC problem in undirected and directed trees of rings, respectively. For undirected trees of rings with arbitrary degree, our algorithms imply a 3-approximation algorithm without using the cut-one-link approach.

The maximum routing and path coloring problem on rings

The Max-PC, Max-RPC, weighted Max-PC and weighted Max-RPC problems in rings can all be approximated with a ratio of 1.58 using the iterative greedy method. Various algorithms aim to improve this ratio. In particular, the Max-PC, Max-RPC and weighted Max-PC problems in rings can all be approximated with ratios better than 1.5 [33]. We show that the weighted Max-RPC problem in rings can be approximated with a ratio of 1.5, improving the previous 1.58-approximation result. Our approach is to use a combination of the cut-one-link method and the maximum matching method introduced in [98] for approximating the (unweighted) Max-RPC problem in rings.

The path multicoloring problem in generalized stars

A multifiber optical network has more than one fiber per link. If every link has the same number k of fibers, the network is called a *k-fiber network*. Otherwise, the network is called a *non-uniform network*. For every even $k > 1$, the path multicoloring problem is known solvable in polynomial time in a k-fiber undirected star (also known as the depth-1 tree), a tree in which one node has degree greater than one and all other nodes have degree exactly one [87, 88]. This should be contrasted to the single fiber case ($k = 1$), which is NP-hard. The complexity of the problem for arbitrary odd k ($k \geq 3$) was not known. We show that

for every odd $k \geq 3$, the Min-PMC and Max-PMC problems in a k-fiber star are NP-hard. The NP-hardness results also hold for a spider (also known as the generalized star), a tree in which one node has degree more than two and all other nodes have degree one or two. We give efficient algorithms for the Min-PMC problem in non-uniform stars with even number of fibers in every link and k-fiber (k even) spiders. We also give a $(1 + \frac{1}{k-1})$-approximation algorithm for the Min-PMC problem in k-fiber spiders for every odd $k \geq 3$.

We have some algorithmic results for the Max-PMC problem. We give an optimal algorithm for the Max-PMC problem in non-uniform stars with even fibers in every link. We also give an optimal algorithm and a 1.58-approximation algorithm for the Max-PMC problem in k-fiber (k even) spiders and non-uniform spiders, respectively. The algorithms for spiders rely on an optimal algorithm for the call control problem.

The call control problem in trees

The call control problem is solvable in undirected stars (depth-1 trees) with arbitrary capacities, but is NP-hard and Max SNP-hard in depth-3 trees even if the capacities are either one or two [63]. An interesting question is, what is the boundary to separate the class of trees for which the call control problem is in P from those for which the problem is NP-hard.

For this question, we have obtained the following results. We show that the call control problem is NP-hard and MAX SNP-hard even in depth-2 trees with capacities 1 or 2. We give a polynomial time algorithm for the call control problem in double-stars which are special depth-2 trees. These results suggest that the boundary is in depth-2 trees, and the call control problem is in P or NP-hard, depending on the node degrees of the trees. We also give 2-approximation and 3-approximation algorithms for the weighted call control problem on depth-2 and depth-3 trees, respectively. This improves the previous 4-approximation algorithm for the problem on those trees. We show that the call control problem in spiders can be solved optimally. All of our algorithms depend on a subroutine which optimally solves the restricted weighted call control problem on arbitrary trees in which all paths contain a same node of the tree.

1.2.2 Contributions on General Topologies

Results given in Section 1.2.1 are obtained on special topologies. In this section, we first briefly review the maximum edge-disjoint paths problem, then we show an approach for

solving the problem in more general topologies.

The maximum edge disjoint paths problem

Given a set of k source-destination pairs in a graph $G = (V, E)$, the maximum edge-disjoint paths (MEDP) problem is to connect as many of these pairs as possible using edge disjoint paths. The MEDP problem has been known to be NP-hard in directed graphs with only two terminal pairs [58]. In directed graphs, the MEDP problem is NP-hard to approximate within a factor better than $\Omega(|E|^{\frac{1}{2}-\epsilon})$ unless $P = NP$ [67], and can be approximated with ratio $O(\min\{|V|^{2/3}, |E|^{1/2}\})$ [40]. In undirected graphs, the MEDP problem is solvable for any fixed k [111], but is NP-hard for general value k. The MEDP problem is hard to approximate within ratio $\log^{\frac{1}{2}-\epsilon}|V|$ for any fixed $\epsilon > 0$ unless $NP \subseteq ZPTIME(|V|^{poly(\log|V|)})$ [9]. Polylogarithmic approximation algorithms are only known for some special topologies. Although these results are very important theoretically, the hidden constants behind are usually large and the practical performance of them is unknown but is likely far from optimal. We design exact algorithms for the MEDP problem in planar graphs. Our algorithms are based on carving decompositions of the planar graphs.

Branch/Carving decomposition based algorithms

Recently, there has been increasing interest in the tree/branch/carving-decomposition based method for solving optimization problems. A tree/branch/carving-decomposition of a graph is a way to partition the graph into pieces recursively. There are two major steps in a tree/branch/carving-decomposition based algorithm for solving a problem: (1) computing a tree/branch/carving-decomposition with a small width and (2) applying a dynamic programming algorithm based on the decomposition to solve the problem.

To develop a tree/branch/carving decomposition based algorithm, we need a tool to find a tree/branch/carving decomposition. Step (2) usually runs in time exponential in the width of the decomposition computed in Step (1). Thus, it is important to compute a decomposition with small width. Developing such a tool is non-trivial, since deciding the treewidth/branchwidth/carvingwidth of a general graph is NP-hard [113]. There are good news for planar graphs. Given a planar graph G with n vertices and an integer β, Seymour and Thomas give a decision procedure which decides if G has a branchwidth/carvingwidth at least β in $O(n^2)$ time [113].

We propose efficient implementations of the decision procedure of [113]. Our implementations run much faster and use less memory than a straightforward implementation reported in [70], and can compute the branchwidth of some planar graphs with one hundred thousand edges. We further develop divide-and-conquer algorithms for finding the optimal branch/carving decomposition of planar graphs. Our decomposition finding algorithms are much faster than those reported in previous studies [71], and can compute the optimal branch decomposition of some planar graphs with 50,000 edges. This provides the base for using the branch/carving decomposition based method to solve optimization problems.

Carving decomposition based algorithm for the MEDP problem

The MEDP problem in directed trees of bounded degree, and the MEDPwPP problem in bounded degree trees of rings are both optimally solvable. The optimal algorithms are based on dynamic programming as well. Note that bounded degree trees and bounded degree trees of rings both have bounded carvingwidth. Furthermore, trees and trees of rings are both planar. A nature question is, whether the MEDP and MEDPwPP problems in planar graphs with bounded carvingwidth are optimally solvable? We show that the maximum edge-disjoint paths problem (with pre-specified paths) can be solved optimally in planar graphs with bounded carvingwidths. We give a dynamic programming algorithm based on an optimal carving decomposition of the planar graphs. (For the MEDP and MEDPwPP problems, and other problems related to path subsets, it seems that the carving-decomposition is a better choice than the tree/branch-decomposition.) Our experimental results show that the algorithm can compute a set of maximum edge-disjoint paths with reasonable load on graphs of practical size. We also show that the maximum path coloring problem is solvable in graphs with small carvingwidth if the number of available colors is also small, using carving-decomposition based method.

1.3 Thesis Outline

The rest of the thesis is organized as follows. We give the preliminaries of the thesis and review some previous work related to path coloring and call control in Chapter 2.

In Chapter 3, we study the path coloring problem on trees of rings. We give an efficient algorithm which solves the Min-PC problem on a tree of rings with an arbitrary (node) degree using at most $3L$ colors and achieves an approximation ratio of 2.75 asymptotically,

where L is the maximum number of paths on any link in the network. The $3L$ upper bound is tight since there are instances of the Min-PC problem that require $3L$ colors even on a tree of rings with degree four. We also give approximation algorithms for the Min-PC problem on trees of rings with bounded degrees.

In Chapter 4, we first study the call control problem in trees. We show that the call control problem is NP-hard and MAX SNP-hard even in depth-2 trees with capacities 1 or 2. We then give efficient algorithms for the call control problem in several classes of bounded depth trees, and the restricted call control problem on arbitrary trees in which all paths contain a same node of the tree. In the remaining part of Chapter 4, we study the related maximum path coloring problem. We mainly focus on multifiber optical networks. We show that for every odd integer $k \geq 3$, the Min-PMC and Max-PMC problems in k-fiber stars are NP-hard. We give efficient algorithms for the Min-PMC problem in multifiber stars and spiders with even number of fibers in every link. We also give several algorithms for the Max-PMC problems in multifiber stars and spiders. We show that the maximum weight routing and path coloring problem in rings can be approximated with a ratio of 1.5.

In Chapter 5, we develop efficient algorithms for computing optimal branch/carving decompositions of planar graphs. We first give several efficient implementations of Seymour and Thomas decision procedure which, given an integer β, decides whether an input graph has the branchwidth at least β or not. Our implementations are faster and use less memory than previous implementations, and can compute the branchwidth of some instances with one hundred thousand edges. Using the decision procedure as a subroutine, we further give several divide-and-conquer algorithms for computing optimal branch decompositions of planar graphs. Our implementations of the divide-and-conquer algorithms are fast and can compute an optimal branch decomposition of some instances with 50,000 edges.

Exact algorithms for the maximum edge-disjoint paths problem in planar graphs are given in Chapter 6. We first show that the maximum edge-disjoint paths problem can be solved optimally in planar graphs with bounded carvingwidths. We give a dynamic programming algorithm based on an optimal carving decomposition of the planar graphs. Experimental results show that the algorithm can compute a set of maximum edge-disjoint paths with reasonable load on graphs with practical size.

We summarize the thesis and discuss directions for future work in the final chapter.

Chapter 2

Preliminaries and Related Work

In this chapter, we will introduce the preliminaries of the thesis and review some related work. In Section 2.1, we give the graph notation and the definitions for bipartite graph, matching and graph coloring. These concepts can be found in most graph theory books, such as [21] and [121]. We also give some well known results on edge-coloring of multigraphs, and on matching and its generalization. We will define several important parameters related to the routing and path coloring problem. We will also introduce several popular network topologies that are frequently used in WDM optical networks. In Section 2.2, we review the previous results for the routing and path coloring problem and its variants in single-fiber and multifiber optical networks, and for the routing and call control problem. The network topologies considered include chains, rings, trees, trees of rings, and meshes.

2.1 Preliminaries

We use standard graph notation which can be found in a graph theory book such as [121]. An undirected network is expressed by a simple undirected graph G with node set $V(G)$ and edge set $E(G)$, where an edge $\{u, v\} \in E(G)$ expresses a link between u and v. We use n for $|V|$ and m for $|E|$. We use $\delta_G(u)$ for the node degree of $u \in V(G)$, and $\Delta(G) = \max\{\delta_G(u) | u \in V(G)\}$ for the maximum node degree of G. A *vertex coloring* of a graph G is an assignment of colors to the vertices of G such that adjacent vertices are given different colors. The *chromatic number* of G is the minimum number of colors required in a vertex coloring of G.

A *circuit* of a graph G is a sequence of consecutive edges that begins and ends at the

same node with no repeated edge. A component of a graph G is a maximal connected subgraph of G. An *Euler circuit* of a component of G is a circuit which contains all edges of the component. A component of G has an Euler circuit if every node of the component has even degree.

We use a *path* for a simple path in a graph (i.e. a sequence of consecutive edges with no repeated node). An undirected path between node u and node v is denoted by $u - v$. The *length* of a path is the number of edges in the path. We use length-i path to denote a path with i edges.

We define some well used topologies here. A *chain* of n nodes is a path of $n-1$ edges. A *tree* of n nodes is a connected graph with $n-1$ edges. A *star* is a tree with one node (called *center*) having degree greater than one and all other nodes having degree one. A *spider* is a graph obtained by replacing every edge in a star by a chain. These chains are called *legs* of the spider. Spiders are also called *generalized stars.* A *ring* is a cycle with at least three nodes. The ring is the simplest graph in which routing is an issue. A *tree of rings* can be defined recursively as follows: a single ring is a tree of rings; the graph obtained by connecting a new ring and an existing tree of rings at exactly one node is also a tree of rings. See Figure 2.1 for examples of the topologies.

Given a path p in a graph G, we say p is on an edge e (resp. a node v) if p contains e (resp. v). Given a set P of paths in a graph G, for an edge $e \in E(G)$ and a subset $Q \subseteq P$ we use $L_Q(e) = |\{p|p \in Q \text{ and } p \text{ is on } e\}|$ and $L_Q = \max_{e \in E(G)} L_Q(e)$ to denote the *load* of Q on edge e and the load of Q in G, respectively. We will simply use $L(e)$ and L for $L_P(e)$ and L_P, respectively, and call L the load. We use $\omega(P)$ to denote the *clique number* of P, i.e., the maximum number of pairwise intersecting paths in P. For a set P of paths, both L and $\omega(P)$ are lower bounds on the number of colors required to color P. For a set P of paths in a graph G, we say P is *w-colorable* if each path of P can be assigned one of the w colors and paths with the same color are edge-disjoint. Such a coloring is referred to as a *valid w-coloring.* Similarly, a set of routing requests R is said w-colorable if there exists a set P of paths such that each request in R is assigned one of the paths in P and P is w-colorable.

Given a set $W = \{\lambda_1, \lambda_2, ...\}$ of colors and a set P of paths, a color assignment from W to P is called a *valid coloring* if each path in P is assigned a single color from W and the paths with the same color are edge-disjoint. Finding a valid coloring for P is also called *coloring* P.

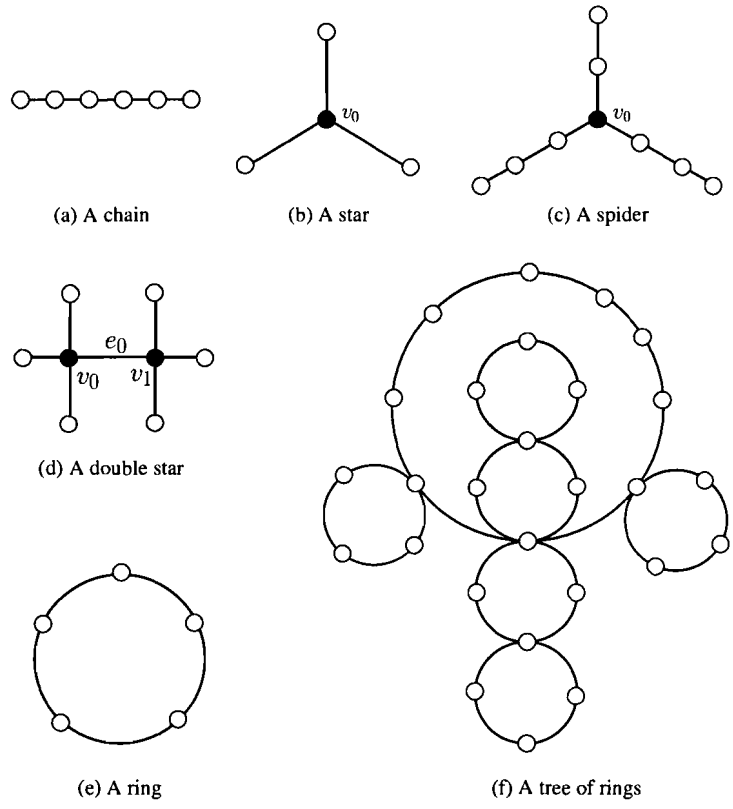


Figure 2.1: Some well-used topologies.

A well used strategy for the Min-PC problem is the *first-fit coloring*: Given a set $W = \{\lambda_1, \lambda_2, ...\}$ of colors and a set P of paths, the paths in P are colored one by one in arbitrary order, and a path $p \in P$ is assigned a color λ_i with the smallest index i such that no path of $P \backslash \{p\}$ already colored by λ_i intersects with p.

Given an undirected graph $G = (V, E)$, we can obtain a symmetric directed graph $D = (V, A)$ by replacing every edge $e \in E(G)$ with two arcs with opposite directions. We use (u, v) to denote an arc with tail u and head v. We use m for $|A|$, the number of arcs in D. We use directed graphs for symmetric directed graphs in this thesis, unless otherwise stated. A directed path is a sequence of consecutive arcs with no repeated node. We use $u \rightarrow v$ to denote a directed path from node u to node v. Given a set P of directed paths in a directed graph D, we can define the load of P on an arc and the load of P in D in a similar way as in the undirected case. In the following discussion, we will state whether the

graph under consideration is undirected or directed when we give a specific result on the problems we are interested in. Other graphs constructed (to help solve these problems) are undirected unless otherwise stated.

Given a set P of paths in an undirected graph G (resp. directed graph D), the *conflict graph* associated with P is the undirected graph $G_c(P, E_c)$ with the node set P such that each node of G_c corresponds to a path in P and two nodes of G_c are adjacent if and only if the corresponding paths in P share an edge of G (resp. an arc of D). The path coloring problem for P in G is equivalent to the vertex coloring problem in the corresponding conflict graph $G_c(P, E_c)$.

An *independent set* in a graph G is a set of pairwise non-adjacent vertices. A graph G is *bipartite* if the node set $V(G)$ is the union of two disjoint independent sets of G. The *clique number* of a graph G is the maximum size of a set of pairwise adjacent vertices (called *clique*) in G. A *matching* in a graph G is a set of edges with no shared endpoints. A *maximal matching* in a graph is a matching that cannot be enlarged by adding an edge. A *maximum matching* is a matching of maximum size among all matchings in the graph. A maximum matching can be computed in polynomial time, both in bipartite graphs [74] and in general graphs [91]. The maximum weight cardinality k matching is a maximum weighted matching of size k. It can also be found in polynomial time [60].

Given a (multi)graph G with vertex set $V(G)$ and edge set $E(G)$, an edge-coloring of G is an assignment of colors to the edges of G such that every pair of edges incident to the same vertex are given different colors. We call such an assignment a *valid edge-coloring* of G. It is NP-hard to find a minimum edge-coloring of a graph [73]. Recall that $\Delta(G)$ is the maximum degree of G. The following results on edge-coloring are well known.

Proposition 2.1.1 *[114] A valid edge-coloring of a multigraph G using at most $\lfloor 3\Delta(G)/2 \rfloor$ colors can be found in $O(|E(G)|(\Delta(G) + |V(G)|))$ time.*

Proposition 2.1.2 *[95] A valid edge-coloring of a multigraph G using at most $\max\{\lfloor (11\Delta(G)+8)/10 \rfloor, l(G)\}$ colors can be found in $O(|E(G)|(\Delta(G) + |V(G)|))$ time, where*

$$l(G) = \max\left\{ L(H) = \left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil \;\middle|\; H \text{ is a subgraph of } G \text{ with } |V(H)| \geq 3 \right\}$$

is a lower bound on the number of colors for the edge-coloring of G.

If the number of nodes of a multigraph G is bounded by a constant, then an optimal

edge-coloring of G can be computed in polynomial time, using a dynamic programming approach [46].

Given a multigraph graph G and an integer t, the maximum edge t-coloring problem is to edge-color a maximum subset of edges of G using at most t colors.

The *degree-constrained subgraph* (DCS) is a generalization of the matching, and can be defined as follows. Given a graph G and functions $b_1, b_2 : V(G) \rightarrow \mathbb{N}$ (set of non-negative integers), a DCS is a subgraph M of G such that for each node $v \in V(G)$, $b_1(v) \leq \delta_M(v) \leq b_2(v)$. We call $b_1(v)$ and $b_2(v)$ the *lower bound capacity* and *upper bound capacity* of v, respectively. We say a node v is *matched* by edge e in a DCS if v is an end-node of the edge e in the DCS. A DCS with the maximum number of edges can be found (or reported non-existing) in $O(\sqrt{\sum_{v \in V(G)} b_2(v)}|E(G)|)$ time [59]. In the weighted DCS problem, each edge has a real-valued weight. The goal is to maximize the total weight of the edges in the subgraph M. A DCS of maximum weight can be found in $O((\sum_{v \in V(G)} b_2(v)) \min\{|E(G)| \log |V(G)|, |V(G)|^2\})$ time [59]. The algorithms of [59] work for multigraphs as well, and a DCS of maximum weight in multigraphs can be found in $O(|E(G)|^2(\log |V(G)|)(\log \mu_{max}))$ time, where μ_{max} is the maximum edge multiplicity. The DCS problem is similar to the well-known b-matching problem. For a given function $b : V(G) \rightarrow \mathbb{N}$, a b-matching is a set M of edges such that the number of edges in M incident to any node v is at most $b(v)$. See pages 257-259 of [65] for details.

We say an algorithm is a *ρ-approximation algorithm* for a minimization (resp. maximization) problem if the algorithm runs in polynomial time and the value SOL of any solution produced by the algorithm satisfies $\frac{SOL}{OPT} \leq \rho$ (resp. $\frac{OPT}{SOL} \leq \rho$), where OPT is the value of an optimal solution. If such an algorithm exists, we say the problem is ρ-approximable. If one can show that such an algorithm does not exist, under the commonly believed $P \neq NP$ or similar conjecture, then the problem is inapproximable up to factor ρ. In this thesis, if an algorithm achieves an approximation ratio of $\alpha + \epsilon$, for some constant $\alpha \geq 1$ and a small additive constant $\epsilon > 0$, the running time is usually exponential in $1/\epsilon$. In particular, a *polynomial time approximation scheme* or PTAS is an algorithm that can achieve an approximation ratio $1 + \epsilon$, for any fixed $\epsilon > 0$, in time that is polynomial in the input size. The notion of MAX SNP was introduced in [100]. An optimization problem does not have PTAS if it is MAX SNP-hard.

2.2 Previous Work

The (off-line) path coloring and call control problems have been extensively studied in both communication and graph theory communities. There have been rich literatures on these problems. In this section, we review previous work related to the path coloring and call control problems. It should be noted that for all problems studied in this thesis, if the graph considered is a chain, then any algorithm for the undirected chain works for the directed chain, by considering each direction separately. Similarly, if the graph considered is a ring and the paths are fixed, then any algorithm for the undirected ring works for directed ring as well, by considering the clockwise and counter-clockwise directions separately.

2.2.1 The Minimum Path Coloring Problem

The path coloring problem is NP-hard in general graphs. In fact, for every graph G, there is a set P of paths in a grid-like graph such that the conflict graph of P is isomorphic to G. Thus, the path coloring problem is in general as hard as the vertex coloring problem. The vertex coloring problem is hard to approximate with ratio $n^{1-\epsilon}$, for any fixed $\epsilon > 0$ [68]. Thus, the path coloring problem is hard to approximate within ratio $|P|^{1-\epsilon}$ as well. It makes sense to develop approximation algorithms for the path coloring problem in special graphs often found in communications networks. In this section, we review results for the path coloring problem in chains, trees, rings, trees of rings, and meshes. For trees (chains are also trees), the PC and RPC problems are the same, since there is a unique path between any two nodes. For graphs containing cycles, the PC and RPC problems are different because there might exist more than one way to route a connection request. We will also review the RPC problem in such topologies.

Min-PC in chains.

Given a set P of paths in an undirected chain, the conflict graph G_c of P is an interval graph (an interval graph is a graph whose nodes correspond to a set of intervals on a chain, and two nodes are adjacent if and only if the corresponding intervals overlap). Since an interval graph is a perfect graph, the chromatic number of G_c is equal to the clique number of G_c [121]. Thus, P has a valid coloring using L colors. There is a simple greedy algorithm that colors P using L colors: process the nodes in the chain one by one from left to right; when processing a node v, consider the paths with left endpoint v in an arbitrary order and

assign a path p a color λ such that no path intersecting p is already colored by λ [38]. It is not hard to see that the above algorithm colors P using exactly L colors, since at the time a path p is colored, no path with left endpoint on the right of v is already colored and at most $L-1$ colored paths intersect p. The Min-PC problem in directed chains can be solved by considering each direction separately. The algorithm for path coloring in chains is often used as a subroutine for path coloring in more complex topologies.

Min-PC in trees.

The Min-PC problem in trees has received much attention in the past decade. The Min-PC problem in undirected stars is known to be equivalent to the edge-coloring in multigraphs [105], a well-studied problem in graph theory community. Since the edge-coloring problem in multigraphs is NP-hard, the Min-PC problem in undirected stars (thus undirected trees) is also NP-hard. The Min-PC problem in undirected stars remains NP-hard even if the given set of paths has load $L = 3$ [48, 82]. Similar result holds for the directed case: the Min-PC problem in directed trees (with depth at least 2) remains NP-hard even if the given set of paths has $L = 3$ [48, 82]. The proofs use reductions from the edge-coloring problem in graphs with maximum degree three [73].

Before we discuss various algorithms for the Min-PC problem in undirected and directed trees, we will give a general framework of these algorithms. All known deterministic algorithms for the Min-PC problem in trees are greedy in the sense that they process the nodes of the tree in a breadth first search (BFS) order, and paths are never re-colored once they are colored. The algorithms work as follows: do a breadth first search of the nodes starting from an arbitrary node of the tree, then process the nodes in this BFS order. When processing a node u, paths on nodes with BFS number strictly smaller than u are already colored, and their colors will not be changed; all uncolored paths on u are colored and colors are re-used as much as possible, according to some scheme. This color reusing scheme is important and differs for different algorithms.

It was proved in [48] and [105] that the Min-PC problem in undirected trees can be reduced to the edge-coloring problem in multigraphs. There is a greedy algorithm that accomplishes such task. Suppose we process the nodes of the tree according to the BFS order defined above, and we are now processing node u. Let v be the parent of u. All paths on v are already colored. We are now coloring the paths on u, and they can be colored by reducing to the edge-coloring problem. Let P' be the set of paths on u that are

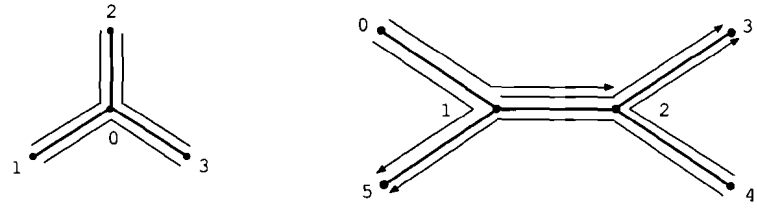


Figure 2.2: (a) A $3L/2$ lower bound on undirected trees; (b) A $5L/4$ lower bound on directed trees.

already colored. Their colors cannot be changed. However, paths in P' are on the edge (u, v) and thus must have been assigned distinct colors. In any valid edge-coloring of the graph constructed from the paths on u, the edges corresponding to paths in P' are assigned different colors. Thus, having pre-colored paths P' does not complicate the edge-coloring.

From the above discussion, one can see that any approximation algorithm for the edge-coloring problem can be used as an algorithm for the Min-PC problem in undirected trees, with the same performance guarantee. In particular, an asymptotic 1.1-approximation algorithm follows from the edge-coloring algorithm of [95]. Note that the Min-PC problem in bounded degree undirected trees can be solved optimally since the edge-coloring problem in multigraphs with constant number of vertices can be solved optimally in polynomial time using a dynamic problem approach (see Theorem 3.1.7 of [46]). For a set P of paths in an undirected tree, there is a valid path coloring of P using at most $\lfloor 3L/2 \rfloor$ colors [105]. This result follows from Shannon's edge-coloring bound and the equivalence of path coloring in undirected trees and edge-coloring in multigraphs. The $\lfloor 3L/2 \rfloor$ bound is tight. Consider for example an undirected 3-star with three paths each of which connects a different pair of leaf nodes (see Figure 2.2(a)). The paths have load two but need three colors. The bound also holds for arbitrarily large load, by adding duplicate paths.

The Min-PC problem in directed trees is somewhat different. The Min-PC problem in directed spiders can be solved optimally in polynomial time. In fact, a stronger result is known: for a set P of directed paths in a directed tree D, P can be colored by L colors if and only if D is a directed spider [64, 122]. The Min-PC problem in directed binary trees is already NP-hard [48]. The reduction is from circular arc graph coloring. The Min-PC problem remains NP-hard in directed trees of depth-2 even if the given set of paths has load

three [48, 82]. The Min-PC problem remains NP-hard in directed trees of depth-3 even if the given set of paths has load two [47]. On the positive side, it was proved in [75] that in a directed tree, any set P of paths can be colored by at most $\lceil 5L/3 \rceil$ colors. The algorithm has asymptotic approximation ratio 5/3. It works in a greedy way, as described before. When processing a node u, it extends the partial coloring to include all the paths on u. In doing so, it reduces the problem to the edge-coloring problem of a bipartite graph, in which some edges are already colored.

No local greedy algorithm can do better than $5L/3$. The following result was given in [52]: Let A be a deterministic greedy Min-PC algorithm in directed trees. There exists an algorithm called ADV which, on any input $\delta > 0$ and integer $L > 0$, outputs a binary directed tree T and a pattern of communication requests P of load L on T, such that A colors P with at least $(5/3 - \delta)L$ colors.

For directed binary trees, a better randomized algorithm is known. It was proved in [16] that any set P of paths in a directed binary tree can be colored by at most $7L/5$ colors with high probability (thus the algorithm has approximation ratio 7/5). Better approximation algorithms are also known for bounded degree directed trees. It was proved in [35] that there exist randomized $1.511 + o(1)$-approximation and $1.336 + o(1)$-approximation algorithms for the Min-PC problem in bounded degree directed trees and binary directed trees, respectively, providing that the load is not small. The $1.511 + o(1)$-approximation algorithm improves a previous $1.61 + o(1)$-approximation algorithm of [34]. The algorithms in [34] and [35] first formulate the Min-PC problem as an integer linear program (ILP), then solve the relaxed linear program (LP) optimally, and get an integral solution through randomized rounding. This approach has also been used by other researchers to obtain good approximation algorithms for many other problems, such as the edge-disjoint paths problem [40, 41]. It works as follows. For a valid coloring of a set of paths, the paths with the same color are edge-disjoint. Thus, the Min-PC problem can be rephrased as covering all the paths with the minimum sets of edge-disjoint paths. It can be formulated as follows:

$$\begin{array}{lll} \text{Minimize} & \sum_{I \in \mathcal{I}} x(I) & \\ \text{Subject to} & \sum_{I \in \mathcal{I}: p \in I} x(I) \geq 1, & p \in P \\ & x(I) \in \{0, 1\}, & I \in \mathcal{I}. \end{array}$$

where $\mathcal{I}$ denotes the collection of all sets of edge-disjoint paths. This integer linear program is NP-hard to solve. One can relax the integral constraint to $0 \leq x(I) \leq 1$. The relaxed

linear program is the formulation for the *fractional path coloring* problem, which is to color a set of paths with fractional colors such that the sum of the fractional colors assigned to any path is at least one. The fractional path coloring problem is NP-hard in general as well [65], but is solvable for some graphs (in particular, undirected rings and bounded degree directed trees). The fractional solution can then be rounded to get an integral solution using randomized rounding techniques introduced in [104].

On the lower bound side, it was proved in [85] that there is a set P of paths with load L in a directed binary tree such that any valid coloring of P requires at least $5L/4$ colors. This lower bound is for all kinds of algorithms, deterministic or randomized, greedy or non-greedy. In Figure 2.2(b), each arrow represents $L/2$ paths. There are 5 arrows which represent a total of $5L/2$ paths. No more than two paths can be assigned the same color, thus at least $5L/4$ colors are needed. Closing the gap (between the $5L/4$ lower bound and the $5L/3$ upper bound) is an interesting and challenging open problem.

The special case of symmetric paths in directed trees was studied in [36]. For any two nodes s and t, the two directed paths $s \rightarrow t$ and $t \rightarrow s$ are called *symmetric.* A set of directed paths is called symmetric if it can be partitioned into pairs of symmetric paths. If each pair of symmetric paths is assigned the same color, a set of symmetric paths can be colored using the $3L/2$ algorithm for undirected trees. In this way, a $3L/2$ upper bound can be obtained for the case of symmetric paths. If the two symmetric paths are not required to use the same color, then a $1.367L + o(L)$ upper bound is known for directed binary trees [37]. This is better than the $7L/5$ bound for directed binary trees in which paths are not symmetric. In general, however, it is not known whether the Min-PC problem is easier if the set of paths is restricted to be symmetric. In particular, no better lower bound is known in the symmetric case. Caragiannis *et al.* showed that for every $\delta > 0$, there exists an integer $l > 0$, a directed binary tree and a set P of symmetric paths on the tree with load $L = 4l$ such that no algorithm can color P using less than $(5/4 - \delta)L$ colors [36].

Min-PC in rings.

The Min-PC problem in undirected rings is NP-hard [61]. The Min-PC problem in undirected rings is also known as circular arc graph coloring, since the conflict graph is a circular arc graph (a circular arc graph is a graph whose nodes correspond to a set of paths on a ring, and two nodes are adjacent if and only if the corresponding paths overlap). It was shown in [118] that any set P of paths with load L in an undirected ring can be colored by

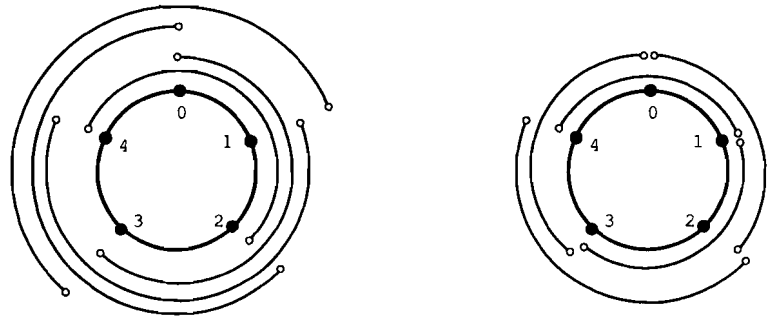


Figure 2.3: (a) A $2L-1$ lower bound on undirected rings; (b) A $\frac{3}{2}\omega(P)$ lower bound on undirected rings.

$2L-1$ colors. The algorithm is very simple: select one node v which is an end-node of some paths, and color the paths with v as an internal node using at most $L-1$ colors, and the remaining paths (which are on an undirected chain) using at most L colors. The algorithm has an approximation ratio of 2 since L is a lower bound on the number of colors required for any optimal solution. The $2L-1$ upper bound is tight, since there are instances that require this many colors. For example, consider five paths which are pairwise intersecting and cover the whole ring (see Figure 2.3(a)). The load is three, but five colors are needed. For any integer $L \geq 2$, it is easy to construct a set P of paths with load L such that $2L-1$ colors are needed to color P.

A generalization of the $2L-1$ upper bound is known. Let l be the minimum number of paths in P that are needed to cover the whole ring. Tucker showed that if $l \geq 4$, then $\lfloor \frac{3}{2}L \rfloor$ colors suffice to color P [118]. This result is further generalized in [119]. It was shown that if $l \geq 5$, then $\lceil (\frac{l-1}{l-2})L \rceil$ colors suffice to color P, and this bound is tight.

A better deterministic approximation algorithm is known. It was proved in [77] that any set P of paths in an undirected ring can be colored by $\frac{3}{2}\omega(P)$ colors. The algorithm has approximation ratio 1.5, which is the current best ratio among all deterministic algorithms. Roughly speaking, the algorithm works as follows. Let L be the load of the given set P of paths. The algorithm first chooses $\lfloor \frac{L}{2} \rfloor$ sets of paths along the clockwise direction of the ring, and then chooses at most L sets of paths along the counter-clockwise direction of the ring. Each of these sets contains a set of edge disjoint paths and thus can be colored by

a single color. If there is no path left during the two phases, then obviously at most $\lfloor \frac{3}{2}L \rfloor$ colors are used. Otherwise, more than $\lfloor \frac{3}{2}L \rfloor$ colors are needed, but it was shown that the clique number $\omega(P)$ of the set P of paths is greater than L and at most $\frac{3}{2}\omega(P)$ colors are needed. The $\frac{3}{2}\omega(P)$ upper bound is also tight. Consider a set of five paths as shown in Figure 2.3(b). The conflict graph is a pentagon which has a maximum clique of size two, but has chromatic number three.

The current best (randomized) approximation algorithm for the Min-PC problem in undirected rings achieves an asymptotic approximation ratio of $1 + 1/e$ (about 1.37) with high probability, if the load is not small [84]. The algorithm first transforms the problem into an integral multi-commodity flow problem, then solves the relaxed linear programming problem, and finally obtains a solution using a randomized rounding technique introduced in [104]. Caragiannis *et al.* obtained the following generalized result: if the load is not small, there exists a polynomial time randomized algorithm which has approximation ratio $1 + \ln(\frac{l-1}{l-2}) + o(1)$ with high probability, where l is the minimum number of paths required to cover the ring [35].

The Min-PC problem in directed rings can be solved similarly by considering each direction separately, since the paths are fixed.

Min-RPC in rings.

As mentioned before, the ring is the simplest graph in which routing is an issue. The routing and path coloring problem seems to be harder than the path coloring sub-problem alone (for many topologies other than tree). In fact, it is not clear how to simultaneously solve the routing problem and the path coloring problem. Most algorithms solve the two sub-problems in two steps: first the requests are routed, then the obtained paths are colored.

The Min-RPC problem is NP-hard for both undirected and directed rings [48]. The Min-RPC problem in undirected rings was studied in [105], and the problem in directed rings was studied in [92]. They proposed the following method: remove an arbitrary link from the ring and obtain an chain, then route (uniquely) all the requests on the chain, and color the paths using the optimal path coloring algorithm for chains. This approach has an approximation ratio of two (for both undirected and directed rings). Suppose the paths routed on the chain have load L and thus are colored by L colors. Consider an edge e on the chain with load L. Any optimal algorithm (for the Min-RPC problem on rings) has to route the requests corresponding to paths on e along one of the two possible routes on the ring.

This results in a load of at least $\lceil L/2 \rceil$ on the ring and thus requires at least $\lceil L/2 \rceil$ colors. Better approximation algorithms are known for undirected rings. In particular, Kumar gave an algorithm which uses at most $OPT \cdot (1.5 + 1/2e + o(1)) + O(\sqrt{OPT \cdot \ln n})$ colors with high probability [83]. The approximation ratio is close to $1.5 + 1/2e$ if OPT is large, but may be greater than 2 if OPT is not large enough. Cheng gave an approximation algorithm which has an approximation ratio of $2 - 1/\Theta(\log n)$ with high probability [42]. This value holds for all values of OPT, although asymptotically it is not better than Kumar's. We are not aware of any deterministic algorithm which achieves a constant approximation ratio strictly better than 2.

Min-PC in trees of rings.

The Min-PC problem in trees of rings is clearly NP-hard, since a ring is the simplest tree of rings, and the Min-PC problem in rings is NP-hard, for both undirected and directed cases. The Min-PC problem in undirected trees of rings was first studied in [44]. They gave a 2-approximation algorithm for undirected trees of rings of maximum degree four (i.e., each node can be contained in at most two rings). Erlebach showed that a set P of paths on an undirected tree of rings TR can be colored by at most $4L$ colors [47]. The $4L$ algorithm can be extended to work for directed trees of rings, with an upper bound of $8L$. His algorithm works as follows.

Algorithm GreedyColoring (TR, P):

1. Initially, all paths are uncolored.
2. Process each node u of TR in a depth first search (DFS) order starting from an arbitrary node $s \in V$ as follows:

   Let P_u be the set of uncolored paths on u. Assign every path p of P_u, in arbitrary order, the color λ with the smallest index (break tie arbitrarily) such that no path intersecting p is already colored by λ.

The coloring strategy used for P_u in the algorithm is the *first-fit strategy*. Although the first-fit approach is simple (and effective sometimes), it is not always the best possible. It was shown in [25] that a set P of paths on an undirected tree of rings with maximum degree eight can be colored by at most $3L$ colors. The $3L$ algorithm is based on a processing order first proposed by Erlebach [47], and a better color reusing strategy when processing each

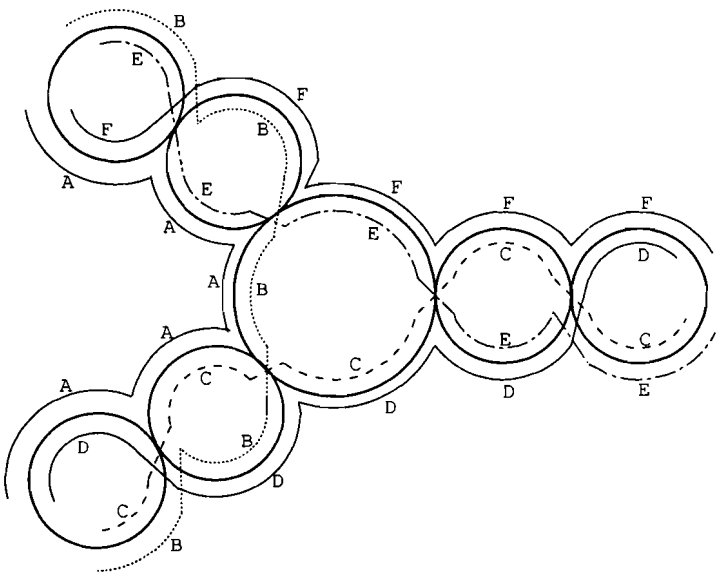


Figure 2.4: A $3L$ lower bound on undirected trees of rings.

node. They further gave a 2-approximation algorithm for undirected trees of rings with degree at most four and a 2.5-approximation algorithm for undirected trees of rings with degree at most six.

There are instances that require $3L$ colors even on undirected trees of rings with maximum degree four [25]. An example of such instances is as follows: Let $P = A \cup B \cup C \cup D \cup E \cup F$ be the set of paths, with each subset having $L/2$ (L is even) paths, as shown in Figure 2.4. The maximum number of paths on any link in the tree of rings is L. Any two paths in P must be assigned different colors since they intersect with each other. There are a total of $3L$ paths in P, thus $3L$ colors are needed. This lower bound shows that in the worst case one cannot do better than $3L$ even for undirected trees of rings with node degree four.

Min-RPC in trees of rings.

The Min-RPC problem in trees of rings is NP-hard, following from the NP-hardness of the problem in rings. The following method is known for the Min-RPC problem on trees of rings: cut one link from each ring in the given tree of rings and obtain a tree, then route (uniquely) the requests and solve the Min-PC problem in the obtained tree. It is known

that through this cut-one-link method, any algorithm for the Min-PC problem in undirected trees using at most αL colors can be used to obtain a 2α-approximation algorithm for the Min-RPC problem in undirected trees of rings [105]. The same result holds for the directed case as well [92]. The best upper bounds (in terms of load) are $3L/2$ and $5L/3$ for the Min-PC problem in undirected and directed trees, respectively. Thus, one can obtain 3-approximation and 10/3-approximation algorithms for the Min-RPC problem in undirected and directed trees of rings, respectively.

For undirected trees of rings with degree at most eight, a 3-approximation algorithm not based on cut-one-link can be obtained as follows: route optimally the requests such that the load L is minimized, then use the $3L$ algorithm of [25] to color the obtained paths. Since L is the optimal load and thus is a lower bound for the Min-RPC problem, the algorithm has approximation ratio 3. It is interesting to design approximation algorithms with ratios strictly better than 3 and 10/3 for the Min-RPC problem in undirected and directed trees of rings, respectively.

Min-RPC in meshes.

The Min-PC problem in meshes is equivalent to the vertex color problem in general graphs. Thus, any hardness or algorithmic result on vertex coloring applies to the Min-PC problem in meshes.

The Min-RPC problem in $N \times N$ undirected square mesh networks is NP-hard, since the special case of edge-disjoint paths is already NP-hard [103]. It was shown in [103] that the number of colors needed can be approximated with a constant factor. The author formalized the problem as a sequence of integer linear programs. It was then shown that the solutions to the relaxations provide a constant factor approximation to the number of colors needed. The ratio was established through a complicated randomized rounding procedure, which proceeds iteratively. In each iteration, the solution gets closer to an integral solution. After a small number of iterations, the solution is near-integral. The author was then able to convert it into an integral solution without much increasing in the number of colors. The above procedure gives a constant factor approximation to the number of colors needed. Unfortunately, the argument is non-constructive and does not give a routing and path coloring. Nevertheless, the author showed that the Min-RPC problem can be approximated within a factor of $poly(\ln \ln N)$. This improves a previously best ratio of $O(\ln N)$ [80] (which is an improvement over an even earlier ratio of $O(\ln^2 N)$).

2.2.2 The Maximum Path Coloring Problem

In this section, we review results for the maximum (routing and) path coloring problem in chains, trees, rings, and trees of rings. The maximum routing and path coloring (Max-RPC) problem can be defined as follows: Given a set R of requests and an integer $w > 0$, find a maximum cardinality subset $R' \subseteq R$ of requests that is w-colorable. The maximum path coloring (Max-PC) problem is similar to Max-RPC, but the paths are given as part of the input. For trees, Max-RPC and Max-PC coincide since the routing is unique. When $w = 1$, the Max-RPC (resp. Max-PC) problem is equivalent to the MEDP (resp. MEDPwPP, the maximum edge-disjoint paths with pre-specified paths). The MEDP problem has attracted much attention in theoretical computer science, and we will review it in more details in Section 2.2.5.

Before we introduce the individual algorithms for Max-PC in various graphs, we describe a generic approach (hereinafter will be called *iterative greedy approach*). Suppose we have a ρ-approximation algorithm for the MEDPwPP problem (equivalently, a ρ-approximation algorithm for the Max-PC problem with only one color). To solve the Max-PC problem, we call the algorithm for the MEDPwPP problem, select a maximum cardinality subset $P' \subseteq P$ of pairwise disjoint paths, and remove the set P' of paths from P (i.e., $P := P \setminus P'$). The procedure is repeated w times. Each selected subset is colored by a distinct color. The union of the w chosen sets is taken as the solution for the Max-PC problem. It is known that if the MEDPwPP algorithm has approximation ratio ρ, then this iterative algorithm for the Max-PC problem has approximation ratio $\frac{1}{1-e^{-1/\rho}}$ (which is at most $\rho+1$) [54]. This simple approach sometimes gives very good approximation for the maximum path coloring problem.

Max-PC in chains.

The Max-PC problem in undirected chains was studied in [38]. It was shown that a simple greedy algorithm can solve the problem optimally. They also gave a clever linear time implementation. The algorithm is very simple and works as follows: process the nodes of the chain one by one from left to right; when processing a node v, consider greedily the paths with right endpoint v, and include a path p if p can be assigned a color λ such that no path intersecting p is already colored by λ. The weighted Max-PC problem in chains can also be solved optimally in polynomial time, by reducing to a minimum cost network flow

problem. The same result holds for the directed chains. This algorithm is often used as a subroutine for solving the maximum path coloring problem in more complex topologies.

Max-PC in trees.

When $w = 1$, the Max-PC problem is simply the MEDP problem, and is known solvable in undirected trees [63]. Erlebach proved that the MEDP problem is NP-hard and MAX SNP-hard in directed trees [49]. The MAX SNP-hardness of the MEDP problem in directed trees implies that it does not have polynomial time approximation scheme (PTAS) unless $P = NP$. Erlebach also gave a $5/3 + \epsilon$-approximation algorithm for the MEDP problem in directed trees, where ϵ can be chosen arbitrarily small [49]. The Max-PC problem in undirected and directed trees with $w > 1$ was studied in [46]. It was showed that the Max-PC problem is solvable if w and the degree of the (undirected or directed) tree are both bounded by a constant, using a dynamic programming approach. If either w or the degree is unbounded, the problem is NP-hard. For arbitrary w, Erlebach used the iterative greedy approach to obtain 1.58-approximation algorithms for undirected trees and bounded degree directed trees, and a 2.22-approximation algorithm for directed trees with arbitrary degree. The iterative greedy approach seems to be one of the best known tools so far for dealing with the Max-PC problem in trees (and many other topologies).

The Max-PC problem in undirected stars is NP-hard [48, 105]. It is equivalent to the maximum edge t-coloring problem in multigraphs. The latter problem was studied by Feige *et al.* [56]. They showed that for every fixed $t \geq 2$ there is some $\epsilon > 0$ such that it is NP-hard to approximate maximum edge t-coloring within a ratio better than $\frac{1}{1-\epsilon}$. They also gave an approximation algorithm whose ratio tends to α as $t \to \infty$, where α is the best approximation ratio for the edge-coloring in multigraphs. The current best approximation algorithm for the edge-coloring problem has asymptotic approximation ratio 1.1. Thus, the maximum edge t-coloring problem can be approximated with an asymptotic ratio 1.1 as $t \to \infty$. Accordingly, the Max-PC problem in undirected stars can be approximated with an asymptotic ratio 1.1 if the number of available colors w approaches infinity.

The Max-PC problem in directed stars and spiders can both be solved optimally in polynomial time as follows: first select a maximum subset of paths with load at most w; then color the selected paths using w colors. The first step is actually a call control problem (see Section 2.2.4). It can be solved optimally in polynomial time for directed stars, since it is equivalent to the b-matching problem in bipartite graphs, which is known to be solvable.

It can also be solved optimally for directed spiders, since Erlebach *et al.* showed that the call control problem in directed spider can be solved optimally in polynomial time by reducing to a network flow problem [55]. The second step can also be done in polynomial time, since in any directed spiders (thus stars), any set of paths with load w can be colored by w colors [64, 122].

Max-PC and Max-RPC in rings.

For general values of w, the Max-PC and Max-RPC problems (and the weighted cases) on rings are NP-hard, for both the undirected and directed cases [98, 99]. This follows easily from the NP-hardness of the corresponding minimization problems. Several methods are well used for solving these problems on rings. Many existing work use either a single method or a combination of several methods. The first method, called the *cut-one-link* method, simply ignores one link of the ring and solves the problem on the obtained chain using an algorithm of [38] or its variants. (We have already seen the use of the cut-one-link approach in Section 2.2.1.) The second method, called the *maximum matching* method, tries to use one color for two requests (or paths) according to a matching found in the auxiliary graph constructed. The third method is the *iterative greedy approach* already discussed, which solves the problem by calling w times an approximation algorithm for the single color case. Intuitively, the iterative greedy method may be better than the maximum matching method, since the iterative greedy method may color more than two paths using a single color, while the maximum matching method always colors at most two paths using a single color.

The Max-PC problem on rings was first studied by Wan and Liu [120], who gave a polynomial time exact algorithm for the MEDPwPP problem in rings, and then used the iterative greedy approach to get a 1.58-approximation algorithm (for both the undirected and the directed cases). The approximation ratio was improved to 1.5 in undirected rings [99] (the algorithm works for directed rings with the same performance ratio). The 1.5-approximation algorithm is based on cut-one-link and maximum matching. Let e be an arbitrary edge on the ring, P_e be the set of paths on e and P_c be the set of paths not on e. The algorithm first cuts the ring at e and then selects a maximum w-colorable subset of paths from P_c. The algorithm then tries to color some paths in P_e, using colors each of which is used to color only one paths in P_c. This latter step is done based on a maximum matching in a bipartite graph with bipartitions P_e and P_c.

Very recently, Caragiannis gave a 4/3-approximation algorithm for the Max-PC problem in undirected rings [33]. His algorithm is based on cut-one-link and a variant of the iterative greedy method. Essentially, he showed that if OPT/w is large, then the cut-one-link approach ensures a good approximation ratio; otherwise (OPT/w is small), the iterative greedy method has a good approximation ratio. The algorithm works for Max-PC in directed rings with the same approximation ratio. In the same paper, he also gave a randomized 1.49015-approximation algorithm for the weighted Max-PC problem in rings.

For the Max-RPC problem in undirected and directed rings, the iterative greedy algorithm also works and has approximation ratio 1.58. Nomikos *et al.* gave a 1.5-approximation algorithm for undirected rings and a 11/7-approximation algorithm for directed rings, both based on two separate algorithms, cut-one-link and maximum matching, and the best of the two is taken as the final solution [98]. Caragiannis improved the ratios to 4/3 and 1.41257 for undirected and directed rings, respectively, by using cut-one-link with iterative greedy algorithm [33].

Max-PC and Max-RPC in trees of rings.

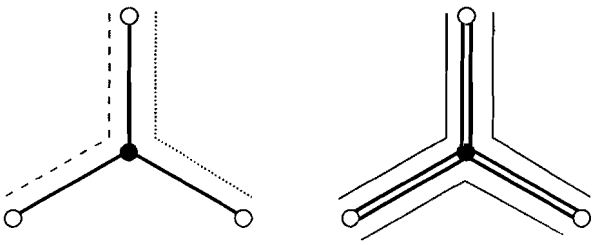
When $w = 1$, the Max-PC and Max-RPC problems reduce to MEDPwPP and MEDP problems, respectively. The MEDPwPP problem is NP-hard and MAX SNP-hard and can be approximated with ratios 4 and 8 in undirected and directed trees of rings, respectively [47]. It was also shown in [47] that the MEDP problem is NP-hard in undirected and directed trees of rings, and any α-approximation algorithm for MEDP in trees gives a 3α-approximation algorithm for MEDP in trees of rings, both in the undirected case and in the directed case. Since the MEDP problem in undirected and directed trees has exact algorithm and $5/3+\epsilon$-approximation algorithm, respectively, the MEDP problem can be approximated with ratio 3 and $5+\epsilon$ in undirected and directed trees of rings, respectively.

The Max-PC and Max-RPC problems with arbitrary w are both NP-hard in trees of rings. Currently, there is no algorithm better than the iterative greedy approach. Simple calculations show that the iterative greedy approach has approximation ratios close to 4.5 (resp. 8.5) for Max-PC in undirected (resp. directed) trees of rings, and has approximation ratios close to 3.5 (resp. 5.5) for Max-RPC in undirected (resp. directed) trees of rings.

2.2.3 The Path Multicoloring Problem

Many optical networks have multiple parallel optical fibers in a link. Such a network is known as a *multifiber optical network*. For a link e with $\mu(e)$ fibers, up to $\mu(e)$ routing paths can use the same color on link e. Note that in directed multifiber networks, both directions of a link should have the same number of fibers. Figure 2.5(a) shows a set of three paths in a single fiber undirected star. The paths need three colors, since they are pairwise intersecting. On the other hand, in Figure 2.5(b), the three paths need only one color in a 2-fiber undirected star. Thus, less colors are needed in multifiber optical networks, at the cost of extra fibers. The path coloring problem in multifiber optical networks is known as the *path multicoloring* (PMC) problem [97]. Recently, there are renewed interests in multifiber optical networks and the PMC problem [53, 54, 86, 87, 88, 89, 96, 97]. Inapproximable results were given in [10, 11, 12]. As in the single fiber case, there are two optimization problems in multifiber optical networks: the minimum path multicoloring problem (denoted as Min-PMC), and the maximum path multicoloring (Max-PMC) problem. Multifiber networks are distinguished into two types according to the number of fibers in every link. One is a uniform network in which every link has the same number of fibers. The other is a non-uniform network in which different links may have different number of fibers. We will use *k-fiber network* to denote a uniform network with k fibers in every link in the undirected case. In a directed k-fiber network, both directions of a link have k fibers. When $k = 1$, the Min-PMC and Max-PMC problems are simply the conventional Min-PC and Max-PC problems, respectively. In the single fiber case, the load L is a lower bound on the number of colors needed. Similarly, in the multifiber case, $w_{lb} = \max_{e \in E} \lceil \frac{L(e)}{\mu(e)} \rceil$ is a lower bound for the number of colors required. The maximization is taken over both directions of a link in the directed case.

The Max-PMC problem asks to maximize the number (or the weight) of paths that can be colored by a given number w of colors in a multifiber optical network. When $w = 1$, this is simply a maximum edge-disjoint paths problem in a single fiber network, and is a call control problem (with edge capacity $c(e) = \mu(e)$ on edge e) in a multifiber network. Thus, a natural way to solve the Max-PMC problem is the iterative greedy approach (see Section 2.2.2). Erlebach *et al.* showed that if the call control algorithm has an approximation ratio of ρ, then the iterative greedy approach has an approximation ratio of $\frac{1}{1-e^{-1/\rho}}$ (which is at most $\rho + 1$) [54]. The proof is similar to the one in [120].



(a) a single fiber star (b) a 2-fiber star

Figure 2.5: A single fiber undirected star and a 2-fiber undirected star.

Min-PMC and Max-PMC in uniform k-fiber networks.

The Min-PMC problem in k-fiber undirected chains can be solved optimally in polynomial time [96]. In fact, the optimal algorithm uses exactly w_{lb} colors. For the Max-PMC problem, suppose we have w colors. One can reduce the Max-PMC problem to the following call control problem: set the capacity of an edge e to be $w \times k$. Then solve the call control problem optimally. The paths in the solution to the call control problem can be colored by w colors, since $w_{lb} = (w \times k)/k = w$.

For every even $k > 1$, the Min-PMC problem is known solvable in polynomial time in k-fiber undirected stars [87, 88]. The Max-PMC problem in even k-fiber undirected stars can be solved optimally by first reducing to a call control problem with every edge e having capacity $k \times w$. The Min-PMC and Max-PMC problems are NP-hard for multifiber undirected and directed binary trees. This should be contrasted to the path coloring problem in bounded degree undirected trees, which is known solvable. If path lengths are restricted to at most three, the Min-PMC and Max-PMC problems in 2-fiber undirected trees are still NP-hard. If path lengths are restricted to four, the Min-PMC and Max-PMC problems in multifiber directed or undirected trees are NP-hard. An upper bound of $3L/(2k)$ is known for the Min-PMC problem in undirected k-fiber trees. This is a generalization of the $3L/2$ upper bound in the single fiber case, and the bound is tight.

The Min-PMC problem in k-fiber undirected rings is NP-hard, for every $k > 1$, as shown in [87]. An upper bound of $\lceil (k+1)/k \cdot L - 1 \rceil$ is also established in the same paper.

Min-PMC and Max-PMC in non-uniform multifiber networks.

The Min-PMC and Max-PMC problems in non-uniform multifiber undirected chains can both be solved optimally in polynomial time, in a similar way as in the uniform case [96]. The Min-PMC problem in undirected and directed rings can be approximated with a ratio of two, no matter whether the paths are fixed or not [96]. The Max-PMC problem can be approximated with a ratio of 1.58 in undirected and directed rings if the paths are fixed, and with a ratio slightly worse than 1.58 in undirected rings if the paths are not fixed, using the iterative greedy approach.

The Min-PMC problem can be solved optimally in directed stars, and can be approximated with ratio 1.5 in undirected stars [96]. The Min-PMC problem in undirected and directed spiders can be approximated with ratios 2.5 and 2, respectively [96]. The Min-PMC problem in undirected and directed trees can be approximated with a ratio of 4, a by-product of [41]. The Max-PMC problem in undirected trees can be approximated with a ratio of 2.542, using the iterative greedy approach.

2.2.4 The Routing and Call Control Problem

The *routing and call control* (RCC) problem can be defined as follows: Given a set $R = \{(s_i, t_i) \mid i = 1, ..., k\}$ of requests in a graph $G = (V, E)$ where each edge $e \in E$ is assigned a non-negative integer capacity $c(e)$, find a maximum subset $R' \subseteq R$ such that the requests in R' can be routed without violating the capacity constraint, i.e., at most $c(e)$ requests use edge e. Such a subset of requests is called a *routable* set. In the directed case, each direction of a link is assigned a capacity, and the capacity constraint is violated only if the number of paths on the same direction of a link exceeds the capacity in that direction. The problem is known as the *call control* problem if the paths are given. Each request (s_i, t_i) may be assigned a positive weight w_i, and the goal is to find a routable set with maximum total weight. In this case, the problem is known as the weighted routing and call control problem. Further, each request (s_i, t_i) may be associated with an integer demand $d_i \geq 1$, and the request can be realized only if the demand d_i is fully satisfied. If the request is required to be on a single path, i.e., unsplittable, then the problem is known as the *unsplittable flow problem* (UFP). In this thesis we assume that the requests are not splittable. Of course, for graphs like trees, there is no splittablity problem since there is a unique path between any two nodes. The unsplittable flow problem is clear NP-hard, since when the graph is a single

edge with all demands going across it, the problem simply becomes the Knapsack problem, which is NP-hard [62]. When every edge has unit capacity, the routing and call control problem is known as the maximum edge-disjoint paths problem, which will be discussed in Section 2.2.5. Any inapproximability result for the MEDP problem holds for the routing and call control problem.

The unsplittable flow problem can be modeled as the following integer linear programming problem when the paths are given:

$$\begin{aligned} \text{Maximize} \quad & \sum_{i=1}^{k} w_i x_i \\ \text{Subject to} \quad & \sum_{i:e \in Path(i)} d_i x_i \leq c(e), \quad \forall e \in E \\ & x_i \in \{0,1\}, \qquad\quad i = 1, ..., k. \end{aligned}$$

When paths are not given, the formulation is similar but more complex. In the formulation, $Path(i)$ is the path along which the request i is routed. The constraint basically says that the total demands of the requests routed on edge e can be at most $c(e)$. This formulation is quite generic. For example, setting d_i to one gives the formulation for the weighted call control problem, and setting both d_i and w_i to one gives the (unweighted) call control problem. Since the unsplittable flow problem is in general NP-hard, the above integer linear program cannot be solved in polynomial time. However, the relaxed linear program (by setting $x_i \in [0,1]$) can be solved in polynomial time. The ratio between the fractional solution and the integral solution is usually called the *integrality gap*. In general, the integrality gap could be large for arbitrary graph. For example, it was shown in [63] that there is a grid-like 3-regular graph in which the integrality gap is $\Omega(\sqrt{n})$ for the MEDP problem.

In the following discussion, we will mainly review the call control problems. We will also review the routing and call control problems in rings. The unsplittable flow problem will be discussed in Section 2.2.4. The discussion of this section is restricted to undirected graphs.

Call control in chains.

The call control problem in undirected chains was studied in [5], and it was shown that the problem can be solved optimally in polynomial time, using a simple greedy algorithm. They also gave a linear time implementation. The weighted call control problem can also be solved in polynomial time, by a similar approach as [38]. If all edge capacities are the same, the problem is equivalent to the Max-PC problem, since any set of paths with load L can be colored by L colors in undirected chains (see Section 2.2.1).

Call control in trees.

The call control problem in undirected trees was studied in [63]. They showed that the call control problem in undirected stars is equivalent to the b-matching problem in multigraphs, and can be solved optimally in polynomial time. They also showed that the call control problem in undirected trees is equivalent to a generalization of b-matching called *cross-free-cut b-matching.* They further showed that the call control problem is NP-hard and MAX SNP-hard even in depth-3 undirected trees with capacities one or two. The reduction is from the maximum three-dimensional matching problem. They formulated the call control problem as an integer linear program and showed that the dual of the relaxed linear program is exactly the minimum multi-cut problem in undirected trees (the minimum multi-cut problem asks for a minimum number of edges whose removal disconnects all the given pairs in a graph). They also gave a primal-dual algorithm that achieves approximation ratio two. The primal-dual method works as follows in approximation algorithms: start with arbitrary solutions to the primal and dual linear programs, and make alternate improvements to each, until good integral solutions to both are found. The improvements are guided by the complementary slackness conditions.

The weighted call control problem in undirected stars is equivalent to the weighted b-matching problem which can also be solved optimally in polynomial time. The weighted call control problem in undirected trees was studied in [41]. They also formulated the problem as an integer linear program, and showed that the relaxed linear program has integrality gap at most 4. This gives a 4-approximation algorithm. Their algorithm implies the following result. Let J be any (multi)set of requests, and k be an integer, such that for any edge e in the tree, at most $k \times c(e)$ of the paths contain e. Then J can be partitioned into $4k$ routable sets. This latter result also implies that the (non-uniform) path multicoloring problem on undirected trees can be approximated with ratio 4 (see Section 2.2.3 and [54]).

Routing and call control in rings.

The call control problem in undirected rings is solvable in polynomial time [5]. This is achieved by calling the optimal chain algorithm iteratively.

The routing and call control problem in undirected rings was studied in [7], and it was shown that there is an algorithm that admits at least $OPT - 3$ requests, where OPT is the maximum number requests that can be admitted in any optimal solution. The algorithm is

based on linear programming. They first formulate the problem as an integer linear program, and then solve the relaxed linear program. They obtain a solution by some sophisticated rounding techniques. The complexity of the problem is still unknown.

The weighted call control and weighted routing and call control problems in undirected rings can both be approximated with ratio two, using a simple cut-one-link approach [6]. However, the complexity status of both problems is still unknown. In particular, it was shown in [6] that the exact matching problem in bipartite graphs can be reduced to the weighted call control problem in undirected rings, thus the weighted call control problem in undirected rings is at least as hard as the exact matching problem. The exact matching problem is known solvable in random polynomial time (RP) [93], but no (deterministic) polynomial time algorithm is known. Finding the complexity of these problems is interesting but probably challenging as well.

Routing and call control in trees of rings.

We are not aware of any algorithmic results on the routing and call control problem in trees of rings (whether paths are given or not), when the edge capacities can be arbitrary. The unit edge capacity case will be discussed in Section 2.2.5. The call control problem in trees of rings with unit edge capacities has also been studied from a different prospective, with the goal of rejecting as few requests as possible [8].

The unsplittable flow problem.

Recall that in the unsplittable flow problem, each request (s_i, t_i) has an integer demand $d_i \geq 1$. Let $d_{max} = \max_{i=1}^{k} d_i$ be the maximum demand, and $c_{min} = \min_{e \in E} c(e)$ be minimum edge capacity. The result for the unsplittable flow problem generally requires $d_{max} \leq c_{min}$, which has been known as the *no-bottleneck* assumption. Without this assumption, the unsplittable flow problem is hard to approximate within a factor of $\Omega(m^{1-\epsilon})$ unless $P = NP$ [17]. The unsplittable flow problem in undirected chains has been studied extensively. In the uniform capacity case (every edge has the same capacity, not necessarily one), the first constant factor approximation algorithm was given in [102], who obtained a 6-approximation algorithm. The ratio was then improved to 3 [19], and then to $2+\epsilon$ [32]. For the unsplittable flow problem in undirected chains with arbitrary capacity, the first constant factor (close to 80) approximation algorithm was given in [39]. This ratio was then improved to $2 +$

ϵ [41]. In [18], it was shown that there is a quasi-polynomial time (i.e., in time $2^{poly\log(n)}$) approximation scheme for the unsplittable flow problem on undirected chains and rings, provided that all capacities and demands are integers bounded by $2^{poly\log(n)}$. Previously, it was not known whether the unsplittable flow problem in chains is MAX SNP-hard or not. This result rules out a MAX SNP-hard result unless $NP \subseteq DTIME(2^{poly\log(n)})$.

The unsplittable flow problem in undirected stars is also known as the demand matching problem, which has also been proved to be MAX SNP-hard [115]. The demand matching problem can be defined as follows: Given a graph $G = (V, E)$ with each node v assigned an integer capacity b_v, each edge e having an integral demand d_e and a weight w_e, find a subset $M \subseteq E$ with maximum total weight such that the sum of the demands of the edges incident to v is at most b_v. Obviously, when $d_e = 1$ for every edge $e \in E$, the demand matching is actually a b-matching. For the demand matching problem in bipartite graphs, the integrality gap is between $2\frac{1}{2}$ and $2\frac{4}{5}$. For general graphs, the integrality gap is between 3 and $3\frac{3}{10}$.

The unsplittable flow problem in undirected trees was studied in [41]. The problem was formulated as an integer linear program. It was shown that the integrality gap for the demand version is at most 11.542 times that for the unit demand case. The latter has an integrality gap of 4, thus the integrality gap for the demand version is at most 48.

2.2.5 The Maximum Edge-disjoint Paths Problem

The maximum edge-disjoint paths (MEDP) problem can be defined as follows: Given a set of k source-destination pairs in a graph G, connect as many of these pairs as possible using edge disjoint paths. It is not hard to see that the maximum edge-disjoint paths problem is a special case of the routing and call control problem in which every edge has unit capacity. The maximum edge-disjoint paths problem is regarded as one of the classic NP-hard problems. In fact, its decision version is one of Karp's original NP-complete problems [78]. It has received much attention during the past several decades. In the maximum edge-disjoint paths with pre-specified paths (MEDPwPP) problem, a set of paths (instead of source-destination pairs) is given. Of course, for graphs like trees, there is no routing problem. In the weighted case, each routing request (or path) may be given a positive weight, and the goal is to maximize the total weight of accepted requests (or paths).

In undirected general graphs, the MEDP problem is solvable for any fixed k [111], but is NP-hard for general value k. The MEDP problem is hard to approximate within ratio

$\log^{\frac{1}{2}-\epsilon} n$ for any fixed $\epsilon > 0$ unless $NP \subseteq ZPTIME(n^{poly(\log n)})$ [9]. However, there is no corresponding poly-logarithmic upper bound on the approximation ratio for all undirected graphs yet, and whether such upper bound exists or not is an outstanding open problem. Until now, poly-logarithmic approximation algorithms are known only for the following undirected graphs: trees with parallel edges, grids and grid-like graphs, expanders, even-degree planar graphs, and planar graphs in which certain congestion on the edges is allowed (see [79] and the references therein). The general approach for approximating MEDP in these graphs is to decompose the graph into node disjoint induced subgraphs, and then find a large routable set within each subgraph. The union of the solutions for the subgraphs is taken as the solution for the problem on the original graph.

The maximum edge-disjoint paths problem has been known to be NP-hard in directed graphs with only two terminal pairs [58]. It can also be approximated with ratio $O(\sqrt{m})$ on directed graphs $D = (V, A)$ using a simple greedy algorithm [81]. The ratio was refined to $O(\min\{n^{2/3}, m^{1/2}\})$ in [40]. On the negative side, the MEDP problem is NP-hard to approximate in directed graphs within a factor better than $\Omega(m^{\frac{1}{2}-\epsilon})$ unless $P = NP$ [67].

The weighted MEDP problem in undirected stars is equivalent to the maximum weight matching problem in general graphs, and can be solved optimally [63]. The weighted MEDP problem in undirected trees can also be solved optimally, using the maximum weight matching algorithm as a subroutine [63]. The MEDP problem is MAX SNP-hard in directed trees, and the reduction is from the maximum three-dimensional matching problem. The problem can be solved in polynomial time for directed stars and spiders. For bounded degree directed trees, the problem can also be solved in polynomial time, using a dynamic programming approach [46]. For directed trees with arbitrary degree, the problem can be approximated with ratio $\frac{5}{3} + \epsilon$, for every fixed $\epsilon > 0$ [49]. The algorithm consists of one bottom-up pass followed by a top-down pass. The reason for using two passes is that some paths have to be in an intermediate state during the first pass. The weighted MEDP problem in directed trees cannot be solved this way, but an algorithm with approximation ratio $\frac{5}{3} + \epsilon$ is given in [51], using a completely different method.

The MEDP problem in undirected rings can be solved in polynomial time as follows [120]. Let e be an edge in the ring, and R be the set of routing requests. In any set of edge disjoint paths, either no path is on e or exactly one path uses e. For the latter case, we can try every request $(s, t) \in R$. We then delete all edges used by (s, t) from the ring, and solve the MEDP problem in the obtained undirected chain in polynomial time. Similarly, the

MEDP in directed rings, and the MEDP problem with pre-specified paths in undirected and directed rings can also be solved in polynomial time. The weighted MEDP problem (and the fixed paths version) can be solved in polynomial time in undirected and directed rings using a similar approach.

The MEDP problem, whether paths are fixed or not, is MAX SNP-hard in directed and undirected trees of rings [47]. When paths are not fixed, there is a 3-approximation algorithm for undirected trees of rings, and a $5+\epsilon$-approximation algorithm for directed trees of rings. When paths are fixed, there are 4-approximation and 8-approximation algorithms for undirected and directed trees of rings, respectively. The weighted MEDP and MEDPwPP problems in trees of rings can be approximated within the same ratio as in the unweighted cases.

Chapter 3

Path Coloring on Trees of Rings

This chapter studies the minimum path coloring (Min-PC) problem on trees of rings. In Section 3.1, we first give a polynomial time algorithm which uses at most $3L$ colors for the problem on trees of rings with arbitrary degrees (recall that L is the maximum number of paths on any link in the network). This improves the previous $4L$ and 4-approximation result of [47]. The $3L$ upper bound is tight since there are instances of the Min-PC problem that require $3L$ colors even on a tree of rings with degree four. Our algorithm is based on a processing order proposed in [47] and novel applications of edge-coloring of multigraphs. In Section 3.2, we give two efficient path coloring schemes for trees of rings and some useful facts on edge-coloring of multigraphs. Based on these techniques, in Section 3.3, we show that the $3L$ algorithm achieves an approximation ratio of 2.75 asymptotically for the Min-PC problem on trees of rings with arbitrary degrees. In Section 3.4, we further give a $3L$ and 2-approximation (resp. 2.5-approximation) algorithm for the Min-PC problem on trees of rings with degree at most six (resp. eight and ten). The algorithms on trees of rings with bounded degrees are of independent interest and improve the previous 2-approximation algorithm for trees of rings with degree four [25, 44], and 2.5-approximation algorithm for trees of rings with degree six [25]. Our $3L$ result also implies a 3-approximation algorithm for the Min-RPC problem on trees of rings. The algorithm does not use the cut-one-link approach, and gives an alternative approach for solving the Min-RPC problem. This approach might provide a better fault-tolerance than the cut-one-link approach. Our $3L$ algorithm implies a $6L$ algorithm for the Min-PC problem on directed trees of rings.

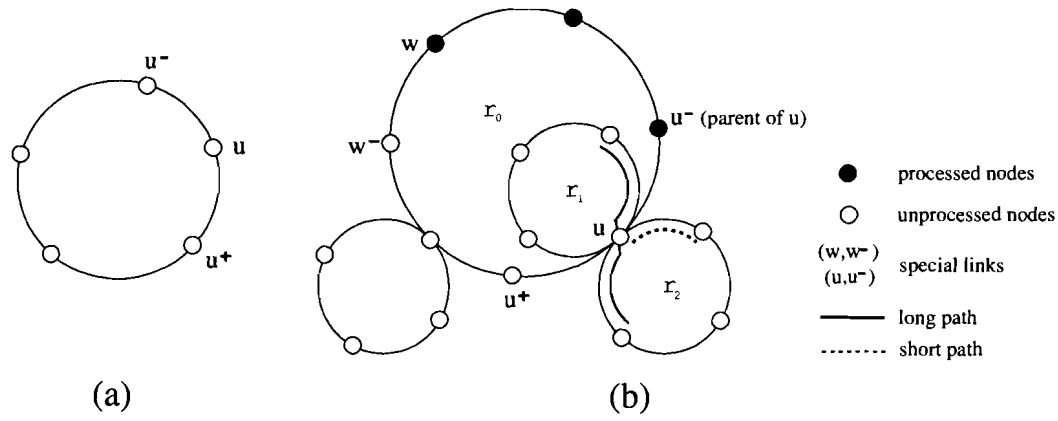


Figure 3.1: Illustration of some terms defined on a tree of rings TR.

3.1 The 3L Upper Bound

In this section, we give an algorithm which solves the Min-PC problem in trees of rings with arbitrary degrees using at most $3L$ colors. We start with some more definitions. A tree of rings network is denoted by a graph TR with node set $V(TR)$ and link set $E(TR)$. For TR, we have the following property.

Proposition 3.1.1 *For any node $u \in V(TR)$, a path on u can be on at most two rings which contain u.*

For a node u in a ring of TR, we denote u^- as the neighbor of u in the counter-clockwise direction and u^+ as the neighbor of u in the clockwise direction in the ring (see (a) of Figure 3.1).

We say a set of elements is assigned distinct colors if for any two different elements in the set, the elements are assigned different colors. Throughout this chapter, we will denote W_P as the set of colors assigned to a set P of paths, and denote W_{uv} as the set of colors assigned to the paths on a link (u, v) of TR.

The 3L algorithm.

We give a simple algorithm, called ALG3.1, which uses at most $3L$ colors for the Min-PC problem on TR with an arbitrary degree and show that the $3L$ upper bound is tight. We

Procedure Framework(TR,P)
Input: A set P of paths in TR.
Output: A valid coloring from $W = \{\lambda_1, \lambda_2, ...\}$ to P.
begin
1. Fix a DFS (depth-first search) order, starting from
   a node (say u_0) of degree two, on the nodes of TR.
2. Process the starting node u_0.
3. Process the other nodes u in the DFS order.
   Let r_0 be the ring which contains u and the parent of u.
   3.1 Color the set P_0 of uncolored paths on u and r_0.
   3.2 Color the set P_1 of other uncolored paths on u.

end.

Figure 3.2: A framework of algorithms for the Min-PC problem on trees of rings.

first give a framework in Figure 3.2 for all algorithms in this chapter. Paths are colored in some order defined later. At any stage of the coloring procedure, a path is called *colored* if it has been assigned a color, otherwise *uncolored*. In the algorithm, processing a node u means coloring the uncolored paths on u. We call a node u *processed* if the coloring process for u has been completed, otherwise *unprocessed*. Notice that before the coloring process for node u, some paths on u may have been colored due to the processing of other nodes. Node u is still called unprocessed if the coloring process for u has not been performed even all paths on u have been colored due to the processing of other nodes. Our algorithm processes the nodes of TR in the DFS (depth-first search) order proposed in [47]. For a node u, its *parent* is the node from which u is reached in the DFS order (see (b) of Figure 3.1). A link is called *special* if it connects a processed node and an unprocessed node (see (b) of Figure 3.1). There are either 0 or 2 special links in a ring in TR. A path on a special link is colored and only such a path has a possibility to intersect with an uncolored path. We assume that in Step 1, the nodes in the same ring are searched in the clockwise direction in the DFS order. Notice that a node of degree two always exists in a finite TR.

The steps of Algorithm ALG3.1 are given in the framework in Figure 3.2. In Step 2, we first assign colors of W to the paths on link (u_0, u_0^-) by the first-fit coloring. Next we assign the uncolored paths on link (u_0, u_0^+) the colors of $W \setminus W_{u_0 u_0^-}$ by the first-fit coloring.

In Step 3, the parent of node u in the DFS order is node u^- in some ring which is called r_0. If u appears in $k+1$ rings, the other k rings are denoted by r_i, $1 \leq i \leq k$ (see (b) of Figure 3.1). Let Q_0 be the set of paths on special links (u, u^-) or (w, w^-). In Step 3.1, we color P_0 using the colors of $W \setminus W_{Q_0}$ by the first-fit coloring.

In Step 3.2, we convert the path coloring problem to the edge-coloring problem of a multigraph G_u with rings r_i $(0 \leq i \leq k)$ as vertices and all paths on u as edges. By Proposition 3.1.1, a path on u is on either one ring or two rings. A path on u is called a *long path* if it is on two rings, otherwise a *short path* (see (b) of Figure 3.1). To eliminate self-loops, we introduce a vertex s_i for every r_i in G_u. More specifically, G_u is defined as: $V(G_u) = \{r_i, s_i | 0 \leq i \leq k\}$, and

$$\begin{aligned} E(G_u) &= \{(r_i, r_j, p) \mid p \text{ is a long path on } r_i \text{ and } r_j, 0 \leq i < j \leq k\} \\ &\cup \{(r_i, s_i, p) \mid p \text{ is a short path on } u \text{ and } r_i, 0 \leq i \leq k\}, \end{aligned}$$

where (x, y, p) denotes an edge between vertices x and y with label p. From Proposition 3.1.1, there is a one-to-one correspondence between the paths on u and the edges in G_u. Assume that a valid edge-coloring for G_u has been found and let $C_{G_u} = \{\mu_1, \mu_2, ...\}$ be the set of virtual colors used for the edge-coloring. We use the mapping $f_1 : C_{G_u} \rightarrow W$ defined below to color the corresponding paths on u. Let Q_1 be the set of colored paths on u before Step 3.2 and C_{Q_1} be the set of virtual colors assigned to the edges (x, y, p) of G_u with $p \in Q_1$. The mapping f_1 is defined as follows:

1. For each $\mu_i \in C_{Q_1}$ assigned to edge (x, y, p) with $p \in Q_1$, $f_1(\mu_i) = \lambda_j$, where $\lambda_j \in W_{Q_1}$ is the color assigned to path p before Step 3.2.

2. For each $\mu_i \in C_{G_u} \setminus C_{Q_1}$, f_1 maps μ_i to a $\lambda_j \in W \setminus W_{Q_1}$ with the smallest available index j such that $C_{G_u} \setminus C_{Q_1}$ is assigned distinct colors.

Since all paths of Q_1 are on ring r_0, edges (x, y, p) with $p \in Q_1$ are given distinct virtual colors in the edge-coloring of G_u. From this and the above definition, f_1 is a function from C_{G_u} to W which implies that for any two edges (in G_u), the corresponding paths are colored by the same real color if and only if these two edges are colored by the same virtual color. Also, f_1 does not change the colors of the paths which were colored before Step 3.2.

To apply the edge-coloring of G_u in Step 3.2 as shown above, it is required that Q_1 has been assigned distinct colors. In other words, no color has been given to more than one

path in Q_1. A set of colored paths is called a *0-set* if the paths are assigned distinct colors. We say the 0-set condition is true on a ring if the set of paths on special links of the ring is a 0-set. As shown later, the 0-set condition is kept true for each ring of TR in Algorithm ALG3.1. This is critical in applying the edge-coloring of G_u in Step 3.2.

Algorithm ALG3.1 colors the paths step by step. In each step, there are a set of colored paths and a set of paths to be colored. The following lemma is useful to get the total number of colors from that used for coloring in each step.

Lemma 3.1.2 *Given a set Q of colored paths and a set R of uncolored paths, assume that at most w colors have been used for Q, and that a subset Q' of Q contains every colored path intersecting with a path of R. If an algorithm colors R such that the coloring for R and the previous coloring for Q' give a valid coloring for $Q' \cup R$ using at most w colors, then $Q \cup R$ can be colored with at most w colors.*

Proof: From the condition of the lemma, $|W_R \setminus W_{Q'}| = |W_R \cup W_{Q'}| - |W_{Q'}| \leq w - |W_{Q'}|$. Since each path of $Q \setminus Q'$ is edge-disjoint with any path of R, all colors of $W_Q \setminus W_{Q'}$ can be used as the colors for R. Therefore, $Q \cup R$ can be colored with at most w colors. □

By the above lemma, if an algorithm colors R such that at most w colors are used for $Q' \cup R$ in every step, it solves the Min-PC problem with at most w colors. In what follows, we only analyze the number of colors used in each step for $Q' \cup R$. Let Q_u be the set of colored paths and P_u be the set of paths to be colored when node u is being processed in Algorithm ALG3.1.

Theorem 3.1.3 *Algorithm ALG3.1 solves the Min-PC problem on TR with n nodes and degree $2(k+1)$ using at most $3L$ colors in $O(nkL(k+L))$ time.*

Proof: In Step 2, since there are at most L paths on any link of TR, there are at most $2L$ paths on u_0. Therefore, the paths on u_0 can be colored with at most $2L$ colors. Since each path is given a distinct color, the 0-set condition is true for every ring of TR after this step. We now show that Algorithm ALG3.1 colors $Q_u \cup P_u$ using at most $3L$ colors for every node u in TR.

In Step 3, assume that Q_u has been colored with at most $3L$ colors and the 0-set condition is true for every ring of TR. In Step 3.1, $|P_0| \leq L$ because each path in P_0 is on link (u, u^+). Therefore, P_0 can be colored with at most L colors. Since Q_0 defined for

Step 3.1 of Algorithm ALG3.1 contains every colored path intersecting with a path of P_0 and $|Q_0| \leq 2L$, $Q_0 \cup P_0$ (thus $Q_u \cup P_0$ by Lemma 3.1.2) can be colored with most $3L$ colors. Since $Q_0 \cup P_0$ is assigned distinct colors, after Step 3.1 the 0-set condition is true for r_0 and Q_1 defined for Step 3.2 of Algorithm ALG3.1 is a 0-set. The latter is critical in Step 3.2.

In Step 3.2, Q_1 contains every colored path intersecting with a path of P_1. By the edge-coloring of G_u, the definition of mapping f_1, and the 0-set condition of Q_1, the set of paths on ring r_i and node u is assigned distinct colors. Also f_1 does not change the color of any path in Q_1. Therefore, f_1 colors P_1 such that the colorings for P_1 and Q_1 give a valid coloring for $Q_1 \cup P_1$. The number of colors used for $Q_1 \cup P_1$ is $|C_{G_u}|$. Since there are at most L paths on any link of TR, there are at most $2L$ paths on node u and any ring r_i. Therefore, $\Delta(G_u) \leq 2L$. By Proposition 2.1.1, a valid edge-coloring of G_u can be found using $|C_{G_u}| \leq 3L$ colors. Thus, at most $3L$ colors are used for $Q_1 \cup P_1$, implying at most $3L$ colors for $Q_u \cup P_u$. Since the set of paths on u and any ring r_i is assigned distinct colors, the 0-set condition holds for every ring.

Summarizing the above, the algorithm solves the Min-PC problem on TR using at most $3L$ colors. The edge-coloring of multigraph G_u is the dominant part in Algorithm ALG3.1 for the time complexity. Since $\Delta(G_u) \leq 2L$, $|V(G_u)| \leq 2(k+1)$, and $|E(G_u)| = O(kL)$, by Proposition 2.1.1, the edge-coloring of G_u takes $O(kL(k+L))$ time. Since Algorithm ALG3.1 executes Steps 3.1 and 3.2 $O(n)$ times, the time complexity of the algorithm is $O(nkL(k+L))$. □

It is known that there are instances which require $3L$ colors for the Min-PC problem on trees of rings [25]. This lower bound implies that in the worst case one cannot do better than $3L$ even for trees of rings with node degree four. Algorithm ALG3.1 achieves the $3L$ tight upper bound for trees of rings with arbitrary degrees. Since L is a lower bound on the number of colors for any optimal solution, Algorithm ALG3.1 achieves an approximation ratio of 3 for the Min-PC problem on TR with an arbitrary degree. The algorithm can be used to obtain a 3-approximation algorithm for the Min-RPC problem on trees of rings as following. First, for a given set of connection requests, a path for each request can be found efficiently such that L is minimized [47]. Then, the set of found paths is colored by Algorithm ALG3.1 using at most $3L$ colors. Since the load L is optimal, it is also a lower bound for the original Min-RPC problem. In this way, the approximation ratio of 3 is achieved without using the cut-one-link approach. Our $3L$ algorithm also implies a

$6L$ algorithm for the Min-PC problem on directed trees of rings with two directed links, one in each direction, between a pair of adjacent nodes. It is interesting to improve the approximation ratio for the Min-PC problem on directed trees of rings.

3.2 Preparation for Improvement

In Algorithm ALG3.1, the 0-set condition is kept for every ring for edge-coloring G_u in Step 3.2 in a straightforward way. One observation is that the 0-set condition may be too strict for solving the Min-PC problem on TR since two paths on special links of a ring can have the same color if they are edge-disjoint. Better approximation ratios may be achieved if the 0-set condition is relaxed. Another observation is that we may use less colors for the edge-coloring of multigraph G_u if a more advanced algorithm like that in [95] is used. In this section, we first give two schemes for coloring paths on trees of rings with the 0-set condition relaxed. The path coloring schemes make more efficient use of colors. Then we show some properties of multigraph G_u related to its edge-coloring. The path coloring schemes and properties of G_u will be used in the following sections to get algorithms with better approximation ratios.

3.2.1 Efficient Path Coloring Schemes

We first introduce the notion of β-set which is an extension of 0-set. A color for a set of colored paths is called a *multi-color* if the color has been assigned to two paths in the set. For an integer $\beta \geq 0$, a set of colored paths is called a *β-set* if each color is assigned to at most two paths, the paths with the same color are edge-disjoint, and the number of multi-colors for the path set is at most β. We say the β-set condition is true on a ring if the set of paths on special links of the ring is a β-set. For any given integer β with $0 \leq \beta \leq L$, the schemes given below use as few colors as possible to keep the β-set condition for every ring.

Recall that P_0 and P_1 are the sets of paths to be colored in Step 3.1 and Step 3.2 of the framework in Figure 3.2, respectively. We give a scheme for coloring P_0 and a scheme for coloring a subset of P_1. The scheme for P_0, called S31, works as follows. Assume that the β-set condition is true for every ring before P_0 is colored. Recall that Q_0 is the set of paths on special links (u, u^-) or (w, w^-). Let $W^m_{Q_0} \subseteq W_{Q_0}$ be the set of multi-colors for Q_0. Then from the β-set condition, $|W^m_{Q_0}| \leq \beta$. Define A_0 (resp. B_0) to be the set of paths on link

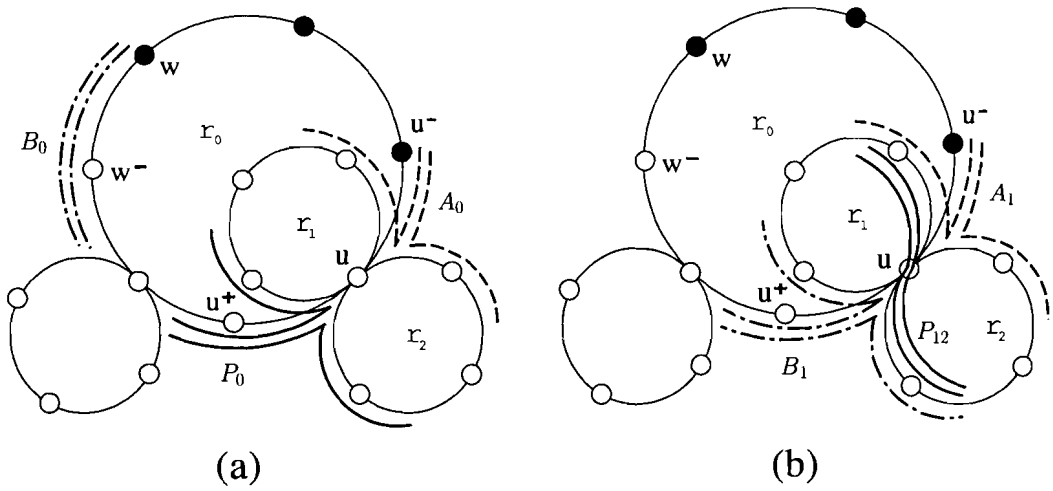


Figure 3.3: The sets of paths related to Schemes S31 and S32.

(u, u^-) (resp. on (w, w^-)), each of which has a color in $W_{Q_0} \setminus W^m_{Q_0}$ (see (a) of Figure 3.3). Then $|A_0| + |W^m_{Q_0}| \leq L$, $|B_0| + |W^m_{Q_0}| \leq L$, and $A_0 \cup B_0$ is assigned distinct colors. We construct a graph G_0 with

$$V(G_0) = P_0 \cup A_0 \text{ and } E(G_0) = \{(p, q) \mid p \text{ and } q \text{ are edge-disjoint}\}.$$

We find a maximum matching M_0 of G_0. Notice that G_0 is bipartite and for each pair $(p, q) \in M_0$, $p \in P_0$ and $q \in A_0$. We select $\min\{|M_0|, \beta - |W^m_{Q_0}|\}$ pairs from M_0. For each selected pair (p, q), assign the color of $q \in A_0$ to p. We assign the remaining paths of P_0 the colors of $W \setminus W_{Q_0}$ by the first-fit coloring. As shown later, the β-set condition is true for every ring after P_0 is colored.

The second scheme, called S32, is used to color a subset of P_1. More specifically, S32 is used to color the long paths on rings r_i and r_j $(i, j \neq 0, i \neq j)$ subject to the condition that every colored path on r_i or r_j is also on r_0 when S32 is called. Without loss of generality, we assume that $r_i = r_1$ and $r_j = r_2$ for simplicity. Let $P_{12} \subseteq P_1$ be the set of long paths on rings r_1 and r_2 (see (b) of Figure 3.3). Recall that Q_1 is the set of colored paths on u before Step 3.2. Then every path of Q_1 is on r_0. Let $Q'_1 \subseteq Q_1$ be the set of colored long paths on links (u, u^-) or (u, u^+) and on rings r_1 or r_2. We define $W^m_{Q'_1} \subseteq W_{Q'_1}$ to be the set of multi-colors for the set Q'_1. From the β-set condition, $|W^m_{Q'_1}| \leq \beta$. Define A_1 (resp. B_1) to be the set of long paths on link (u, u^-) (resp. on (u, u^+)) and on rings r_1 or r_2,

each of which has a color in $W_{Q_1'} \setminus W^m_{Q_1'}$ (see (b) of Figure 3.3). Then $|A_1| + |W^m_{Q_1'}| \leq L$, $|B_1| + |W^m_{Q_1'}| \leq L$, and $A_1 \cup B_1$ is assigned distinct colors. We construct a graph G_1 with

$$V(G_1) = P_{12} \cup A_1 \text{ and } E(G_1) = \{(p,q) \mid p \text{ and } q \text{ are edge-disjoint}\}.$$

We find a maximum matching M_1 of G_1. For each pair $(p,q) \in M_1$, either $p \in P_{12}$ and $q \in A_1$ or $p, q \in P_{12}$. We select $\min\{|M_1|, \beta - |W^m_{Q_1'}|\}$ pairs from M_1. For each selected pair (p,q) with $q \in A_1$, assign the color of q to p. For each selected pair (p,q) with $p, q \in P_{12}$, assign the pair a distinct color from $W \setminus W_{Q_1'}$ by the first-fit coloring (p and q share the same color, but different pairs are given different colors). Let W' be the set of colors assigned to the selected pairs (p,q) with $p, q \in P_{12}$. We assign each of the remaining paths of P_{12} a color from $W \setminus (W_{Q_1'} \cup W')$ by the first-fit coloring such that the set of the remaining paths of P_{12} is assigned distinct colors.

Let OPT_0 (resp. OPT_1) be the number of colors required to color $P_0 \cup A_0$ (resp. $P_{12} \cup A_1$) in an optimal solution.

Lemma 3.2.1 *$OPT_0 = |P_0| + |A_0| - |M_0|$ and $OPT_1 = |P_{12}| + |A_1| - |M_1|$.*

Proof: It is easy to see that $|P_0| + |A_0| - |M_0|$ colors are sufficient for $P_0 \cup A_0$. We prove that $|P_0| + |A_0| - |M_0|$ colors are also necessary. Notice that each color is used by at most two paths in $P_0 \cup A_0$. Assume to the contrary that there is a valid coloring which uses $OPT' = k_1 + k_2 < |P_0| + |A| - |M_0|$ colors, where each of the k_1 colors is used by one path and each of the k_2 colors is used by two paths. Since M_0 is a maximum matching, $k_2 \leq |M_0|$. The total number of paths colored by OPT' is

$$|P_0| + |A_0| = k_1 + 2k_2 \leq k_1 + k_2 + |M_0| < |P_0| + |A_0| - |M_0| + |M_0| = |P_0| + |A_0|,$$

a contradiction.

The proof for $OPT_1 = |P_{12}| + |A_1| - |M_1|$ is similar, noting that a color is used for at most two paths in $P_{12} \cup A_1$. □

Lemma 3.2.2 *Scheme S31 colors P_0 such that the colorings for Q_0 and P_0 give a valid coloring for $Q_0 \cup P_0$ using at least $|Q_0 \cup P_0| - \beta$ and at most $\min\{|Q_0 \cup P_0|, \max\{|Q_0 \cup P_0| - \beta, OPT_0 + L\}\}$ colors. Furthermore, $Q_0 \cup P_0$ is a β-set.*

Proof: Clearly, at most $|Q_0 \cup P_0|$ colors are used, paths with the same color are edge-disjoint, and a color is used for at most two paths in $Q_0 \cup P_0$. If $|M_0| > \beta - |W^m_{Q_0}|$ then there

are exactly β multi-colors for $Q_0 \cup P_0$ and $|Q_0 \cup P_0| - \beta$ colors are used. Otherwise, there are at most β multi-colors for $Q_0 \cup P_0$ and at least $|Q_0 \cup P_0| - \beta$ colors are used. In the latter case, the number of colors used is $|A_0| + |B_0| + |W^m_{Q_0}| + |P_0| - |M_0|$. From Lemma 3.2.1 and $|B_0| + |W^m_{Q_0}| \leq L$, at most $OPT_0 + L$ colors are used. □

Lemma 3.2.3 *Scheme S32 colors P_{12} such that the colorings for Q'_1 and P_{12} give a valid coloring for $Q'_1 \cup P_{12}$ using at least $|Q'_1 \cup P_{12}| - \beta$ and at most $\min\{|Q'_1 \cup P_{12}|, \max\{|Q'_1 \cup P_{12}| - \beta, OPT_1 + L\}\}$ colors. Furthermore, $Q'_1 \cup P_{12}$ is a β-set.*

Proof: Clearly, at most $|Q'_1 \cup P_{12}|$ colors are used, paths with the same color are edge-disjoint, and each color is used for at most two paths in $Q'_1 \cup P_{12}$. If $|M_1| > \beta - |W^m_{Q'_1}|$ then there are exactly β multi-colors for $Q'_1 \cup P_{12}$ and $|Q'_1 \cup P_{12}| - \beta$ colors are used. Otherwise, there are at most β multi-colors for $Q'_1 \cup P_{12}$ and at least $|Q'_1 \cup P_{12}| - \beta$ colors are used. In the latter case, the number of colors used is $|A_1| + |B_1| + |W^m_{Q'_1}| + |P_{12}| - |M_1|$. By Lemma 3.2.1 and $|B_1| + |W^m_{Q'_1}| \leq L$, at most $OPT_1 + L$ colors are used. □

3.2.2 Edge-coloring of Multigraphs

For a (multi)graph G, $l(G)$ defined in Proposition 2.1.2 is a lower bound on the number of colors for the edge-coloring of G. The multigraph G_u constructed in Step 3.2 of Algorithm ALG3.1 has maximum degree $2L$, and $l(G_u)$ can be as large as $3L$. Thus, a direct application of more advanced edge-coloring algorithms (such as that of [95]) in Step 3.2 of Algorithm ALG3.1 cannot improve the approximation ratio. In this subsection, we show some properties of G_u when $l(G_u)$ is large. These properties, Schemes S31 and S32, and the application of a more advanced edge-coloring algorithm in Step 3.2 will be used to improve the approximation ratio of Algorithm ALG3.1.

Lemma 3.2.4 *For any subgraph H of G_u, if $L(H) > \lceil 2.5L \rceil$ then $|V(H)| = 3$.*

Proof: If $L(H) > \lceil 2.5L \rceil$, then clearly $|V(H)| \geq 3$. Therefore, it suffices to show that $L(H) \leq \lceil 2.5L \rceil$ if $|V(H)| > 3$. Consider two cases. If $|V(H)| = 2j$ $(j \geq 2)$,

$$\begin{aligned} L(H) &= \left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil = \left\lceil \frac{|E(H)|}{j} \right\rceil \\ &\leq \left\lceil \frac{\sum_{u \in V(H)} d(u)}{2j} \right\rceil \leq \left\lceil \frac{|V(H)| \times 2L}{2j} \right\rceil = 2L. \end{aligned}$$

If $|V(H)| = 2j + 1$ $(j \geq 2)$,

$$\begin{aligned} L(H) &= \left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil = \left\lceil \frac{|E(H)|}{j} \right\rceil \leq \left\lceil \frac{\sum_{u \in V(H)} d(u)}{2j} \right\rceil \\ &\leq \left\lceil \frac{|V(H)| \times 2L}{2j} \right\rceil = \left\lceil (1 + \frac{1}{2j}) \times 2L \right\rceil \leq \lceil 2.5L \rceil. \end{aligned}$$

□

Lemma 3.2.5 *For subgraphs H_1 and H_2 of G_u with $L(H_1) > \lceil 2.5L \rceil$ and $L(H_2) > \lceil 2.5L \rceil$, $V(H_1) \cap V(H_2) = \emptyset$.*

Proof: To prove the lemma by contradiction, assume that $V(H_1) \cap V(H_2) \neq \emptyset$. By Lemma 3.2.4, both H_1 and H_2 have three vertices. There are two cases to consider: $|V(H_1) \cap V(H_2)| = 1$ and $|V(H_1) \cap V(H_2)| = 2$. For the first case, the total number of edges in $H_1 \cup H_2$ is

$$|E(H_1 \cup H_2)| = |E(H_1)| + |E(H_2)| > \lceil 2.5L \rceil + \lceil 2.5L \rceil \geq 5L.$$

However,

$$|E(H_1 \cup H_2)| \leq \frac{\sum_{u \in (V(H_1) \cup V(H_2))} d(u)}{2} \leq \frac{5 \times 2L}{2} = 5L,$$

a contradiction.

For the second case, assume that $V(H_1) \cap V(H_2) = \{r_a, r_b\}$. Let $m(r_a, r_b)$ be the number of multi-edges between r_a and r_b. The total number of edges in $H_1 \cup H_2$ is

$$|E(H_1 \cup H_2)| = |E(H_1)| + |E(H_2)| - m(r_a, r_b) > 5L - m(r_a, r_b).$$

However,

$$|E(H_1 \cup H_2)| \leq d(r_a) + d(r_b) - m(r_a, r_b) \leq 4L - m(r_a, r_b),$$

a contradiction. □

For multigraph G_u, let F_u be the graph obtained by contracting each subgraph H of G_u with $L(H) > \lceil 2.5L \rceil$ into a single vertex v_H. More precisely, let $V' = \cup_{H:L(H) > \lceil 2.5L \rceil} V(H)$ and $E' = \cup_{H:L(H) > \lceil 2.5L \rceil} E(H)$. Graph F_u is defined by

$$V(F_u) = \{v_H | L(H) > \lceil 2.5L \rceil\} \cup (V(G_u) \setminus V')$$

and

$$E(F_u) = E(G_u) \setminus E',$$

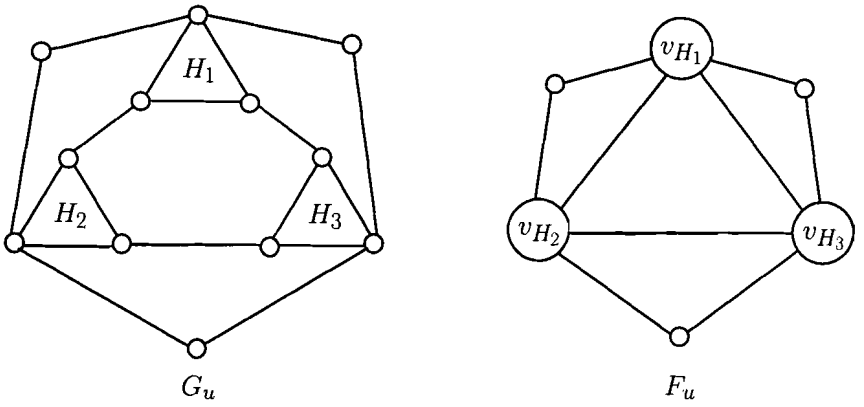


Figure 3.4: A multigraph G_u and its contracted graph F_u.

where for each edge in $E(G_u) \setminus E'$, if an end vertex of the edge is in $V(H)$, the end vertex is replaced by v_H in $E(F_u)$. From Lemma 3.2.5, $v_{H_1} \neq v_{H_2}$ for $H_1 \neq H_2$. We call F_u the *contracted graph* of G_u. Figure 3.4 gives an example of G_u and F_u.

Lemma 3.2.6 *$l(F_u) \leq \lceil 2.5L \rceil$ and $d(v_H) < L$.*

Proof: The degree of v_H in graph F_u is

$$d(v_H) = \sum_{u \in V(H)} d(u) - 2 \times |E(H)| < |V(H)| \times 2L - 2 \times \lceil 2.5L \rceil \leq 3 \times 2L - 5L = L.$$

After every subgraph H with $L(H) > \lceil 2.5L \rceil$ is contracted to a vertex v_H in F_u, from $d(v_H) < L$, any subgraph of F_u with three vertices including v_H has at most $\lfloor \frac{d(v_H)+2L+2L}{2} \rfloor < \lceil 2.5L \rceil$ edges. Therefore, by Lemma 3.2.4, $l(F_u) \leq \lceil 2.5L \rceil$. □

3.3 A 2.75-approximation Algorithm

Applying the results of the previous section, we show a better approximation algorithm for the Min-PC problem on TR with an arbitrary degree. By Proposition 2.1.2, the edge-coloring of G_u can be done with at most $\lceil 2.5L \rceil$ colors if $l(G_u) \leq \lceil 2.5L \rceil$. On the other hand, if $l(G_u) > \lceil 2.5L \rceil$, we can contract G_u into F_u with $l(F_u) \leq \lceil 2.5L \rceil$ (Lemma 3.2.6) and then apply the edge-coloring algorithm of [95] to F_u. Each contracted subgraph H has three vertices, corresponding to three rings containing node u, and the paths corresponding

to the edges in H can be colored by Schemes S31 and S32, with a properly chosen integer β. For simplicity, in what follows, we sometimes refer to the edges in the multigraph G_u and the corresponding paths on node u of TR without distinguishing them, if there is no confusion.

Our algorithm, called ALG3.2, follows the framework in Figure 3.2. Step 2 of ALG3.2 is the same as that in Algorithm ALG3.1. Step 3.1 uses Scheme S31. By Lemma 3.2.2, $|Q_0 \cup P_0| \leq 3L$, and $OPT_0 \leq OPT$, at most $\min\{3L, \max\{3L - \beta, OPT + L\}\}$ colors are used for $Q_0 \cup P_0$. In Step 3.2, we color P_1. Similar to Algorithm ALG3.1, we convert the path coloring problem to the edge-coloring problem of multigraph G_u, but we use the algorithm of [95] to solve the edge-coloring problem. There are two cases.

Case 1: $l(G_u) \leq \lceil 2.5L \rceil$.

We apply the algorithm of [95] to G_u. Since Scheme S31 is used for Step 3.1, Q_1 is a β-set. If Scheme S31 is used with a $\beta > 0$, two paths of Q_1 may have been colored by the same multi-color from W_{Q_1}. To get a valid coloring from W to the paths of G_u, for each pair of paths $p, q \in Q_1$ with the same multi-color from W_{Q_1}, we re-assign a new virtual color $\mu_{pq} \notin C_{G_u}$ to p and q. Let C'_{G_u} (resp. C'_{Q_1}) be the set of virtual colors assigned to the paths of G_u (resp. Q_1) after the re-assignment. We map C'_{G_u} to W by mapping f_1 defined in Section 3.1 to get a valid coloring from W to the paths of G_u. More specifically, we perform the following:

1. For each $\mu_i \in C'_{Q_1}$ assigned to edge (x, y, p) with $p \in Q_1$, $f_1(\mu_i) = \lambda_j$, where $\lambda_j \in W_{Q_1}$ is the color assigned to path p before Step 3.2.

2. For each $\mu_i \in C'_{G_u} \setminus C'_{Q_1}$, f_1 maps μ_i to a $\lambda_j \in W \setminus W_{Q_1}$ with the smallest available index j such that $C'_{G_u} \setminus C'_{Q_1}$ is assigned distinct colors.

Since Q_1 is a β-set, $|C'_{G_u}| \leq |C_{G_u}| + \beta$. Also notice that $\Delta(G_u) \leq 2L$, and $\lfloor (11\Delta(G_u) + 8)/10 \rfloor \leq \lfloor 2.2L + 0.8 \rfloor \leq \lceil 2.5L \rceil$ for any positive integer L. From Proposition 2.1.2 and $l(G_u) \leq \lceil 2.5L \rceil$, the valid coloring uses at most $\lceil 2.5L \rceil + \beta$ colors. This suggests a small β. However, the upper bound $\min\{3L, \max\{3L - \beta, OPT + L\}\}$ in Step 3.1 suggests a large β. To minimize $\max\{\lceil 2.5L \rceil + \beta, 3L - \beta\}$, we choose $\beta = \lfloor 0.25L \rfloor$ for Scheme S31 in Step 3.1. Notice that $\lceil 2.5L \rceil + \lfloor 0.25L \rfloor \leq 3L - \lfloor 0.25L \rfloor = \lceil 2.75L \rceil$.

Case 2: $l(G_u) > \lceil 2.5L \rceil$.

From Lemma 3.2.4, there is at least one subgraph H of G_u with $L(H) > \lceil 2.5L \rceil$ and $|V(H)| = 3$. There are two subcases depending on whether ring r_0 is a vertex of some H or

not.

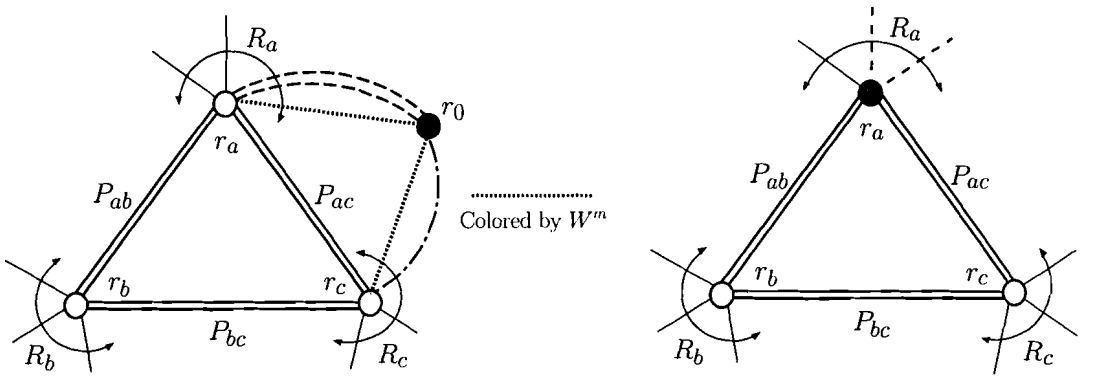
Case 2.1: Ring r_0 is not a vertex of any subgraph H of G_u with $L(H) > \lceil 2.5L \rceil$.

We contract G_u to F_u. From Lemma 3.2.6, $l(F_u) \leq \lceil 2.5L \rceil$. We then apply the algorithm of [95] to F_u and get a valid coloring for the paths corresponding to the edges of F_u by the mapping f_1 as we did in Case 1. After this we color the paths corresponding to the edges in each contracted subgraph H by virtual colors of $C_H = \{\nu_1, \nu_2, ...\}$, using Scheme S32 as a subroutine (the details will be given shortly). Notice that some paths between ring r_0 and a ring of H may have been colored by multi-colors. Because those multi-colors are also multi-colors for the paths on a ring of H, we need to subtract the number of those multi-colors from $\lfloor 0.25L \rfloor$ to get β for Scheme S32 to keep the $\lfloor 0.25L \rfloor$-set condition for each ring. We need some more definitions to formally define the β for Scheme S32.

Assume $V(H) = \{r_a, r_b, r_c\}$. We use P_{ij} $(i, j = a, b, c; i \neq j)$ for the set of long paths in H on r_i and r_j, and use R_i $(i = a, b, c)$ for the set of paths not in H but on r_i (see (a) of Figure 3.5). Notice that $R_a \cup R_b \cup R_c$ has been colored and contains every colored path intersecting with a path of $P_{ab} \cup P_{ac} \cup P_{bc}$. Let $Q' = R_a \cup R_b \cup R_c$ and $W^m_{Q'}$ be the set of multi-colors for Q'. Since paths with a color from $W^m_{Q'}$ are on r_0, from the $\lfloor 0.25L \rfloor$-set condition on r_0, $|W^m_{Q'}| \leq \lfloor 0.25L \rfloor$.

For any two paths p and q with a multi-color $\lambda_m \in W^m_{Q'}$, there are two cases. Case (i), p and q are on r_0 and a single ring of H (say r_a, the dashed edges in (a) of Figure 3.5). Case (ii), p and q are on r_0 and two rings of H (say r_a and r_c, the dotted edges in (a) of Figure 3.5). In Case (i), λ_m is a multi-color for the ring of H (say r_a). In Case (ii), λ_m is not a multi-color for any ring of H. Let $W^m = \{\lambda_m | \lambda_m \in W^m_{Q'}$ is used in Case (ii)$\}$. Then at most $|W^m_{Q'}| - |W^m|$ colors of $W^m_{Q'}$ are multi-colors for each ring of H. From this, we take $\beta = \lfloor 0.25L \rfloor - |W^m_{Q'}| + |W^m|$ for applying Scheme S32 as a subroutine. To color $P_{ab} \cup P_{ac} \cup P_{bc}$ by virtual colors from C_H, we first assign $P_{ab} \cup P_{ac}$ distinct virtual colors from C_H. After this, a path colored with a virtual color from C_H on r_b or r_c must be in $P_{ac} \cup P_{ab}$ and thus is on r_a. Subject to this condition, we color the paths of P_{bc} with virtual colors from C_H using Scheme S32 with $\beta = \lfloor 0.25L \rfloor - |W^m_{Q'}| + |W^m|$, r_a, r_b, r_c corresponding to r_0, r_1, r_2 in the description of S32 in Section 3.2.1, respectively, $P_{ab} \cup P_{ac}$ corresponding to Q'_1, and P_{bc} corresponding to P_{12}.

After $P_{ab} \cup P_{ac} \cup P_{bc}$ is colored by virtual colors of C_H, we map the virtual colors to the colors of W. In the mapping, we try to use the colors of W_{R_a} to paths in P_{bc}. Similarly, we try to use the colors of W_{R_b} (resp. W_{R_c}) to paths in P_{ac} (resp. P_{ab}). Notice that the colors



(a) r_0 is not in H (b) $r_a = r_0$

Figure 3.5: Paths in and incident to subgraphs H with $L(H) > \lceil 2.5L \rceil$.

of W^m are not used in the mapping to keep the $\lfloor 0.25L \rfloor$-set condition on each ring. Let $C_{P_{ij}}$ $(i, j = a, b, c; i \neq j)$ be the set of virtual colors for P_{ij}. We define mapping $f_2 : C_H \rightarrow W$ to color the paths in H as follows.

1. Select $|W_{R_a} \setminus W^m|$ virtual colors from $C_{P_{bc}} \setminus (C_{P_{ab}} \cup C_{P_{ac}})$ arbitrarily, and f_2 maps each selected color ν_i to a $\lambda_j \in W_{R_a} \setminus W^m$ such that the selected virtual colors are assigned distinct real colors.

   Similarly, select $|W_{R_b} \setminus W^m|$ (resp. $|W_{R_c} \setminus W^m|$) virtual colors from $C_{P_{ac}} \setminus (C_{P_{ab}} \cup C_{P_{bc}})$ (resp. $C_{P_{ab}} \setminus (C_{P_{ac}} \cup C_{P_{bc}})$), and f_2 maps each selected virtual color ν_i to a $\lambda_j \in W_{R_b} \setminus W^m$ (resp. $\lambda_j \in W_{R_c} \setminus W^m$) such that the selected virtual colors are assigned distinct real colors.

   Let C_s be the set of selected virtual colors.

2. For each $\nu_i \in C_H \setminus C_s$, f_2 maps ν_i to a $\lambda_j \in W \setminus (W_{R_a} \cup W_{R_b} \cup W_{R_c})$ with the smallest available index j such that $C_H \setminus C_s$ is assigned distinct colors.

The intuition of Step (1) of f_2 is to use as many colors of $W_{R_a} \cup W_{R_b} \cup W_{R_c}$ for $E(H)$ as possible. It is shown later that $|C_{P_{bc}} \setminus (C_{P_{ab}} \cup C_{P_{ac}})| \geq |W_{R_a} \setminus W^m|$, $|C_{P_{ac}} \setminus (C_{P_{ab}} \cup C_{P_{bc}})| \geq |W_{R_b} \setminus W^m|$, and $|C_{P_{ab}} \setminus (C_{P_{ac}} \cup C_{P_{bc}})| \geq |W_{R_c} \setminus W^m|$. This implies that Step (1) of f_2 can be done.

Case 2.2: Ring r_0 is a vertex of some H with $L(H) > \lceil 2.5L \rceil$.

Assume $V(H) = \{r_a = r_0, r_b, r_c\}$ (see (b) of Figure 3.5). Notice that $P_{ab} \cup P_{ac} \cup R_a$ has been colored in Step 3.1 and every colored path on r_b or r_c is also on r_a. We color P_{bc} by Scheme S32 with $\beta = \lfloor 0.25L \rfloor$, r_a, r_b, r_c corresponding to r_0, r_1, r_2 in the description of S32 in Section 3.2.1, respectively, $P_{ab} \cup P_{ac}$ corresponding to Q'_1, and P_{bc} corresponding to P_{12}. Next we color R_b (resp. R_c), trying to use the colors of P_{ac} (resp. P_{ab}). Note that a color used by P_{ac} may have already been assigned to a path in P_{ab} or P_{bc}, and thus cannot be assigned to R_b without violating the $\lfloor 0.25L \rfloor$-set condition for ring r_b (similar situation exists for the set R_c and the ring r_c). To keep the $\lfloor 0.25L \rfloor$-set condition for rings r_b and r_c and for the set $R_a \cup R_b \cup R_c$, we do not use multi-colors for R_b and R_c. More specifically, let $W^m_{Q'}$ be the set of multi-colors for $Q' = P_{ab} \cup P_{ac} \cup R_a$ and $W^m_{Q''}$ be the set of multi-colors for $Q'' = P_{ab} \cup P_{ac} \cup P_{bc}$. Let $W_{P_{ab}}$ (resp. $W_{P_{ac}}$) be the set of colors for the paths of P_{ab} (resp. P_{ac}). For each path $p \in R_b$ (resp. $p \in R_c$), assign p a color from $W_{P_{ac}} \setminus (W^m_{Q'} \cup W^m_{Q''})$ (resp. $W_{P_{ab}} \setminus (W^m_{Q'} \cup W^m_{Q''})$) such that R_b (resp. R_c) is assigned distinct colors. Finally, we contract the H to one vertex, called r'_0, in multigraph G_u to get another multigraph G'_u. In G'_u, $r'_0 \notin V(H)$ for any subgraph H of G'_u with $L(H) > \lceil 2.5L \rceil$ (by Lemma 3.2.6) and the set of paths incident to r'_0 is colored and satisfies the $\lfloor 0.25L \rfloor$-set condition (this will be shown in the proof). We solve the edge-coloring of G'_u as in previous cases to get a valid path coloring.

Theorem 3.3.1 *Algorithm ALG3.2 solves the Min-PC problem on TR with n nodes and degree $2(k+1)$ using at most $\min\{3L, \max\{\lceil 2.75L \rceil, OPT + \lfloor 1.25L \rfloor\}\}$ colors in $O(nkL(k + L^{1.5}))$ time.*

Proof: We show that for every node u, Algorithm ALG3.2 colors $Q_u \cup P_u$ with at most $\min\{3L, \max\{\lceil 2.75L \rceil, OPT + \lfloor 1.25L \rfloor\}\}$ colors. In Step 2, we get a valid coloring for $Q_u \cup P_u$ and the $\lfloor 0.25L \rfloor$-set condition for every ring of TR with $2L$ colors. In Step 3.1, by Lemma 3.2.2, we get a valid coloring for $Q_0 \cup P_0$ and the $\lfloor 0.25L \rfloor$-set condition with at most $\min\{3L, \max\{\lceil 2.75L \rceil, OPT + L\}\}$ colors. In Step 3.2, for Case 1 of $l(G_u) \leq \lceil 2.5L \rceil$, by the $\lfloor 0.25L \rfloor$-set condition, the paths corresponding to edges in G_u can be colored with at most $\lceil 2.75L \rceil$ colors. Thus, we can get a valid coloring for $Q_u \cup P_u$ with at most $\min\{3L, \max\{\lceil 2.75L \rceil, OPT + \lfloor 1.25L \rfloor\}\}$ colors, and the $\lfloor 0.25L \rfloor$-set condition holds after the coloring.

For Case 2 of $l(G_u) > \lceil 2.5L \rceil$, in Case 2.1, we contract G_u into F_u and solve the edge-coloring of F_u. The paths corresponding to the edges of F_u can be colored with at most

$\lceil 2.75L \rceil$ colors, according to Lemma 3.2.6 and the $\lfloor 0.25L \rfloor$-set condition. For each subgraph H of G_u with $L(H) > \lceil 2.5L \rceil$, the paths in H are colored by virtual colors of C_H and f_2 maps the virtual colors to real colors of W. To see Step (1) of f_2 can be done, we show that $|C_{P_{bc}} \setminus (C_{P_{ab}} \cup C_{P_{ac}})| \geq |W_{R_a} \setminus W^m|$, i.e., the number of virtual colors used only by P_{bc} is greater than or equal to the number of real colors used only by R_a. By Lemma 3.2.3, $|C_H| \geq |E(H)| - \beta$, where $\beta = \lfloor 0.25L \rfloor - |W^m_{Q'}| + |W^m|$ (recall that $Q' = R_a \cup R_b \cup R_c$, $W^m_{Q'}$ is the set of multi-colors for Q', and W^m is the subset of $W^m_{Q'}$ such that the two paths with a color from W^m are incident to different vertices of H). Since $|E(H)| > \lceil 2.5L \rceil$,

$$|C_H| \geq |E(H)| - \beta > \lceil 2.25L \rceil + |W^m_{Q'}| - |W^m| \geq \lceil 2.25L \rceil.$$

Therefore,

$$\begin{aligned} |C_{P_{bc}} \setminus (C_{P_{ab}} \cup C_{P_{ac}})| &\geq |C_H| - (|C_{P_{ab}}| + |C_{P_{ac}}|) \\ &\geq |C_H| - (|P_{ab}| + |P_{ac}|) \\ &> \lceil 2.25L \rceil - (d(r_a) - |R_a|) \\ &\geq \lceil 0.25L \rceil + |W_{R_a}| \\ &\geq |W_{R_a} \setminus W^m|. \end{aligned}$$

Similarly, $|C_{P_{ac}} \setminus (C_{P_{ab}} \cup C_{P_{bc}})| \geq |W_{R_b} \setminus W^m|$ and $|C_{P_{ab}} \setminus (C_{P_{ac}} \cup C_{P_{bc}})| \geq |W_{R_c} \setminus W^m|$. Summarizing the above, f_2 gives a valid coloring for $R_a \cup R_b \cup R_c \cup P_{ab} \cup P_{ac} \cup P_{bc}$. In addition, all the colors of $R_a \cup R_b \cup R_c$, except those in W^m, are mapped to the virtual colors in $C(H)$. The following calculations will show that the total number of colors used by $R_a \cup R_b \cup R_c \cup E(H)$ is at most $|C(H)| + |W^m|$ after the mapping f_2.

Since $|P_{ab} \cup P_{ac}| \leq 2L$, $P_{ab} \cup P_{ac}$ can be colored with at most $2L$ distinct colors. Notice that $|E(H)| = |P_{ab} \cup P_{ac} \cup P_{bc}|$. By Lemma 3.2.3, $|C_H| \leq \max\{|P_{ab} \cup P_{ac} \cup P_{bc}| - \beta, OPT + L\} = \max\{|E(H)| - (\lfloor 0.25L \rfloor - |W^m_{Q'}| + |W^m|), OPT + L\}$. The number of real colors used for $R_a \cup R_b \cup R_c \cup P_{ab} \cup P_{ac} \cup P_{bc}$ is at most, noting that $|E(H)| \leq \lfloor \frac{6L - |Q'|}{2} \rfloor \leq 3L - |W^m_{Q'}|$ and

$|W^m| \leq \lfloor 0.25L \rfloor$,

$$\begin{aligned}
& |W_{R_a} \cup W_{R_b} \cup W_{R_c}| + \max\{|E(H)| - (\lfloor 0.25L \rfloor - |W^m_{Q'}| + |W^m|), \\
& \quad OPT + L\} - (|W_{R_a} \setminus W^m| + |W_{R_b} \setminus W^m| + |W_{R_c} \setminus W^m|) \\
\leq\ & \max\{|E(H)| - \lfloor 0.25L \rfloor + |W^m_{Q'}| - |W^m|, OPT + L\} + |W^m| \\
\leq\ & \max\{3L - |W^m_{Q'}| - \lfloor 0.25L \rfloor + |W^m_{Q'}| - |W^m|, OPT + L\} + |W^m| \\
=\ & \max\{\lceil 2.75L \rceil, OPT + L + |W^m|\} \\
\leq\ & \max\{\lceil 2.75L \rceil, OPT + \lfloor 1.25L \rfloor\}.
\end{aligned}$$

Now we show the $\lfloor 0.25L \rfloor$-set condition is true for every ring. Notice that the paths of $R_a \cup R_b \cup R_c$ are given distinct virtual colors of C_{G_u} in the edge-coloring of F_u because all these paths are incident to the same vertex v_H. Therefore, only the paths incident to r_0 may be colored by real multi-colors when a valid coloring from W to the paths of F_u is found. This implies that $W_{R_i} \cap W_{R_j} \subseteq W^m$ $(i, j = a, b, c,\ i \neq j)$. From this, sets $(W_{R_i} \setminus W^m)$ $(i = a, b, c)$ are pairwisely disjoint. Recall that $C_{P_{ab}}$ and $C_{P_{ac}}$ are the sets of virtual colors from C_H assigned to P_{ab} and P_{ac}, respectively. The mapping f_2 selects a subset of $C_{P_{ab}}$ (resp. $C_{P_{ac}}$), assigns the subset distinct real colors from $W_{R_c} \setminus W^m$ (resp. $W_{R_b} \setminus W^m$), and assigns the remaining colors of $C_{P_{ab}} \cup C_{P_{ac}}$ distinct real colors from $W \setminus (W_{R_a} \cup W_{R_b} \cup W_{R_c})$. From $(W_{R_b} \setminus W^m) \cap (W_{R_c} \setminus W^m) = \emptyset$ and the fact that the paths in $P_{ab} \cup P_{ac}$ are given distinct virtual colors of C_H, the mapping f_2 assigns the paths in $P_{ab} \cup P_{ac}$ distinct real colors not in $W_{R_a} \setminus W^m$. Therefore, the $\lfloor 0.25L \rfloor$-set condition is true for r_a. There are at most $\beta = \lfloor 0.25L \rfloor - |W^m_{Q'}| + |W^m|$ virtual multi-colors of C_H for P_{bc} and each of them is mapped to a distinct real color in $W_{R_a} \setminus W^m$ or $W \setminus (W_{R_a} \cup W_{R_b} \cup W_{R_c})$. Therefore, there are at most $|W^m_{Q'}| - |W^m| + \beta = \lfloor 0.25L \rfloor$ real multi-colors for the paths on each of rings r_b and r_c. That is, the $\lfloor 0.25L \rfloor$-set condition is true for every ring.

In Case 2.2, $Q' = R_a \cup P_{ab} \cup P_{ac}$ has been colored with at most $2L$ colors and is a $\lfloor 0.25L \rfloor$-set. In addition, $P_{ab} \cup P_{ac}$ contains every colored path intersecting with a path of P_{bc}. By Lemma 3.2.3 and $|E(H)| \leq 3L$, $Q'' = P_{ab} \cup P_{ac} \cup P_{bc}$ is colored with at most $\min\{3L, \max\{\lceil 2.75L \rceil, OPT + L\}\}$ colors. From this and Lemma 3.1.2, $R_a \cup P_{ab} \cup P_{ac} \cup P_{bc}$ is colored with at most $\min\{3L, \max\{\lceil 2.75L \rceil, OPT + \lfloor 1.25L \rfloor\}\}$ colors. On the other hand, by Lemma 3.2.3, Q'' is colored with at least $|E(H)| - \lfloor 0.25L \rfloor$ colors. Since $|E(H)| > \lceil 2.5L \rceil$, $|W^m_{Q'}| \leq \lfloor 0.25L \rfloor$, $|W^m_{Q''}| \leq \lfloor 0.25L \rfloor$, $W^m_{Q''} \setminus W^m_{Q'} \subseteq W_{P_{bc}}$, and $|W_{P_{ab}} \cup W_{P_{ac}} \cup W_{P_{bc}}| =$

$|E(H)| - |W^m_{Q''}|$, we have

$$\begin{aligned} |W_{P_{ab}} \setminus (W^m_{Q'} \cup W^m_{Q''})| &\geq |W_{P_{ab}} \setminus W_{P_{bc}}| - |W^m_{Q'}| \\ &\geq (|E(H)| - |W^m_{Q''}| - |W_{P_{ac}}| - |W_{P_{bc}}|) - |W^m_{Q'}| \\ &> \lceil 2.25L \rceil - (\delta(r_c) - |R_c|) - \lfloor 0.25L \rfloor \\ &\geq |R_c|. \end{aligned}$$

Similarly, $|W_{P_{ac}} \setminus (W^m_{Q'} \cup W^m_{Q''})| \geq |R_b|$. Therefore, R_c (resp. R_b) is assigned distinct colors of $W_{P_{ab}} \setminus (W^m_{Q'} \cup W^m_{Q''})$ (resp. $W_{P_{ac}} \setminus (W^m_{Q'} \cup W^m_{Q''})$), and $R_a \cup R_b \cup R_c \cup P_{ab} \cup P_{ac} \cup P_{bc}$ is colored with at most $\min\{3L, \max\{\lceil 2.75L \rceil, OPT + \lfloor 1.25L \rfloor\}\}$ colors. The $\lfloor 0.25L \rfloor$-set condition holds for $R_a \cup R_b \cup R_c$ and each of rings r_a, r_b, and r_c, because no multi-color is introduced when coloring R_b and R_c. By solving the edge-coloring of G'_u, we can get a valid coloring for $Q_u \cup P_u$ with at most $\min\{3L, \max\{\lceil 2.75L \rceil, OPT + \lfloor 1.25L \rfloor\}\}$ colors.

The edge-coloring of G_u takes $O(kL(k+L))$ time by Proposition 2.1.2. It takes $O(L^{2.5})$ time to color a subgraph H of G_u with $L(H) > \lceil 2.5L \rceil$ (since H has degree at most $2L$, the graph constructed in Scheme S32 has $O(L)$ vertices, and it takes $O(L^{2.5})$ time to find a maximum matching in such graph [91]). There can be $O(k)$ such subgraphs H in G_u. Therefore, it takes $O(kL(k+L) + kL^{2.5}) = O(kL(k+L^{1.5}))$ time in Steps 3.1 and 3.2. The algorithm executes these steps $O(n)$ times. The time complexity of the algorithm is $O(nkL(k+L^{1.5}))$. □

Since $L \leq OPT$, $\min\{3L, \max\{\lceil 2.75L \rceil, OPT + \lfloor 1.25L \rfloor\}\}/OPT \leq \lceil 2.75L \rceil / L$. Thus Algorithm ALG3.2 achieves an approximation ratio of 2.75 asymptotically.

It seems difficult to extend the approach used in Algorithm ALG3.2 to improve the approximation ratio of 2.75 for a tree of rings with arbitrary degrees. One possible direction is to lower the threshold value to some $T < \lceil 2.5L \rceil$ for subgraph H. However, this will introduce the following problems. First, a subgraph H may have five or more vertices. A new scheme for coloring H is needed. Second, after the contraction of H in G_u, the resulting graph F_u may still have $l(F_u) > T$. To apply the edge-coloring algorithm, we may need to contract F_u as well. A new mapping function for converting the virtual colors of edge-coloring to real colors is needed. It is difficult to solve either of the problems. Nevertheless, in the next section, we show that the approach of Algorithm ALG3.2 can be used to derive algorithms with improved approximation ratios for bounded degree trees of rings.

3.4 Algorithms for Bounded Degrees

The ideas for the 2.75-approximation algorithm can be used to design more efficient algorithms for the Min-PC problem on trees of rings with bounded degrees. Actually, Schemes S31 and S32 shown in Section 3.2.1 imply a $3L$ and 2-approximation algorithm for the Min-PC problem on trees of rings with degree at most six. We first give the algorithm explicitly and then describe algorithms for degrees eight and ten.

3.4.1 Algorithm for Degree Six

The algorithm, called ALG3.3, follows the framework in Figure 3.2. Step 2 of ALG3.3 is the same as that in Algorithm ALG3.1. Step 3.1 uses Scheme S31. In Step 3.2, we first use Scheme S32 to color the long paths in P_{12}. Then we color the short paths on r_1 and those on r_2. Let Q' be the set of all long paths on u and r_1. We assign the short paths on r_1 the colors of $W \setminus W_{Q'}$ by the first-fit coloring such that the set of short paths is assigned distinct colors. Let Q'' be the set of all long paths on u and r_2. We assign the short paths on r_2 the colors of $W \setminus W_{Q''}$ by the first-fit coloring such that the set of short paths is assigned distinct colors.

Theorem 3.4.1 *Algorithm ALG3.3 solves the Min-PC problem on TR with n nodes and degree at most six using at most* $\min\{OPT + L, 3L\}$ *colors in* $O(nL^{2.5})$ *time.*

Proof: To prove the theorem, we take $\beta = L$. We show that Algorithm ALG3.3 colors $Q_u \cup P_u$ using at most $\min\{OPT + L, 3L\}$ colors for every node u in TR.

In Step 2, $Q_u = \emptyset$ and $|P_u| \leq 2L$. At most $2L \leq \min\{OPT + L, 3L\}$ colors are used for $Q_u \cup P_u$. Obviously, the L-set condition is true for every ring after Step 2. In Step 3.1, by Lemma 3.2.2 and $|Q_0 \cup P_0| \leq 3L$, at most $\min\{3L, \max\{3L - L, OPT_0 + L\}\}$ colors are used for $Q_0 \cup P_0$. Since $OPT_0 \leq OPT$ and $L \leq OPT$, $\max\{3L - L, OPT_0 + L\} \leq OPT + L$ and at most $\min\{OPT + L, 3L\}$ colors are used for $Q_u \cup P_0$. By Lemma 3.2.2, the L-set condition is true for every ring after Step 3.1.

In Step 3.2, recall that $Q'_1 \subseteq Q_1$ is the set of colored long paths on links (u, u^-) or (u, u^+). Each path of Q'_1 (resp. P_{12}) is on one (resp. two) of the four links incident to u in r_1 and r_2, implying $|Q'_1| + 2|P_{12}| \leq 4L$. From this and $|Q'_1| \leq 2L$, we have $|Q'_1 \cup P_{12}| \leq 3L$. Notice that Q'_1 contains every colored path intersecting with a path of P_{12}. By Lemma 3.2.3, $OPT_1 \leq OPT$, $L \leq OPT$, and $|Q'_1 \cup P_{12}| \leq 3L$, at most $\min\{OPT + L, 3L\}$ colors are used

for $Q'_1 \cup P_{12}$. By Lemma 3.2.3, the L-set condition is true for every ring after the coloring of P_{12}.

Since there are at most $2L$ paths on the two links of r_1 (resp. r_2) that are incident to node u, all the paths on the two links in r_1 (resp. r_2), including all the short paths, can be colored with $2L$ colors. Obviously, the L-set condition is true for every ring after the short paths are colored. Thus, Algorithm ALG3.3 colors $Q_u \cup P_u$ with at most $\min\{OPT+L, 3L\}$ colors and keeps the L-set condition for every ring.

A tree of rings TR with n nodes has $O(n)$ links. There are at most $O(nL)$ paths in a tree of rings with load L. To reduce the time complexity, we first construct a conflict graph G_c whose vertex set is P and two vertices of G_c are adjacent if the corresponding paths of P intersect with each other in TR. The conflict graph can be constructed in $O(nL^2)$ time, assuming that each path of P is given as a linked list of links of TR. The algorithm executes Steps 3.1 and 3.2 $O(n)$ times. The first-fit coloring takes $O(L^2)$ time to color L paths. It takes $O(L^2)$ time to construct a graph G_u of $O(L)$ vertices, by checking the conflict graph (there is an edge between two vertices of G_u if there is no edge between the two vertices in the conflict graph). It takes $O(L^{2.5})$ time to find a maximum matching of the graph [91]. Therefore, Steps 3.1 and 3.2 take $O(L^{2.5})$ time. The time complexity of the algorithm is $O(nL^{2.5})$. □

3.4.2 Algorithm for Degree Eight

The algorithm for degree eight, called ALG3.4, is similar to Algorithm ALG3.2, but uses a special scheme for the edge-coloring of multigraph G_u. Since the tree of rings considered has degree eight, G_u has at most four vertices r_i. Since the paths with an end vertex of s_i of G_u are short paths which can be easily colored with $2L$ colors after the long paths are colored, in what follows, we assume that G_u has only vertices r_i and edges corresponding to long paths. We first show an optimal edge-coloring algorithm for a multigraph with four vertices. We follow the notation used for Algorithm ALG3.2. Especially, for a subgraph H of multigraph G_u with $V(H) = \{r_a, r_b, r_c\}$, we use P_{ij} $(i, j = a, b, c; i \neq j)$ for the sets of long paths in H on r_i and r_j, and use R_i $(i = a, b, c)$ for the sets of paths not in H but on r_i.

Lemma 3.4.2 *An edge-coloring of multigraph G_u with four vertices can be done using at most* $\max\{\Delta(G_u), l(G_u)\}$ *colors in* $O(|E(G_u)|)$ *time.*

Proof: If $l(G_u) > \Delta(G_u)$, then there exists a subgraph H with three vertices which has $L(H) = l(G_u)$. To see this, assume that for any subgraph H with three vertices, $L(H) < l(G_u)$. The remaining vertex, which is not in H, has degree at most $\Delta(G_u)$. Then

$$l(G_u) \leq \left\lceil \frac{L(H) + \Delta(G_u)}{2} \right\rceil < l(G_u),$$

a contradiction. For the subgraph H with $L(H) = l(G_u)$, assume that $V(H) = \{r_a, r_b, r_c\}$ (see Figure 3.6). We first color the edges $E(H)$ using $l(G_u)$ distinct colors. From $E(H) = l(G_u) > \Delta(G_u)$, we have

$$|P_{bc}| = |E(H)| - (d(r_a) - |R_a|) \geq |R_a|.$$

Thus, all edges of R_a can be colored by the colors used for P_{bc}, since each edge of R_a does not share a common vertex with any edge of P_{bc}. Similarly, all edges of R_b (resp. R_c) can be colored by the colors used for P_{ac} (resp. P_{ab}). Therefore, G_u can be edge-colored with at most $l(G_u)$ colors.

For the case of $l(G_u) \leq \Delta(G_u)$, assume that $d(r_0) = \Delta(G_u)$. We first color the edges incident to r_0 by $\Delta(G_u)$ distinct colors. Assume that the remaining vertices of G_u are r_a, r_b, and r_c (see (a) of Figure 3.6). Let H be the subgraph with $V(H) = \{r_0, r_b, r_c\}$. Then $|E(H)| \leq l(G_u) \leq \Delta(G_u)$, and we have

$$|R_a| = d(r_0) - (|R_b| + |R_c|) = \Delta(G_u) - (|E(H)| - |P_{bc}|) \geq |P_{bc}|.$$

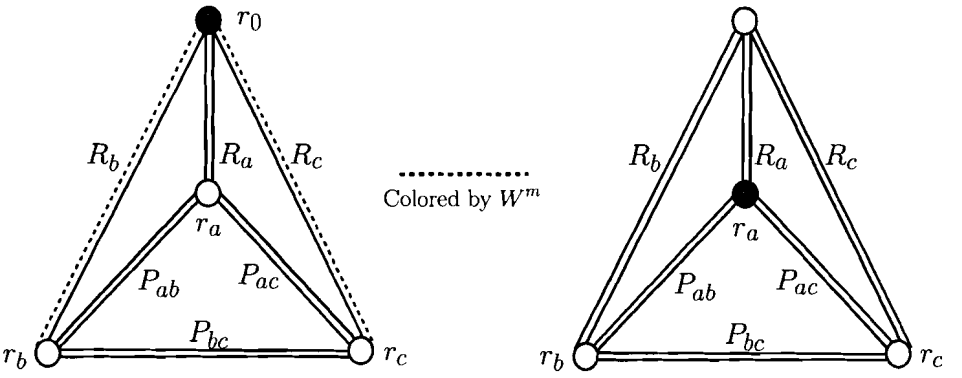
So, all edges of P_{bc} can be colored by the colors used for R_a. Similarly, all edges of P_{ab} (resp. P_{ac}) can be colored by the colors used for R_c (resp. R_b). Thus, G_u can be edge-colored with at most $\Delta(G_u)$ colors.

The algorithm first needs to find the larger number of $l(G_u)$ and $\Delta(G_u)$. This takes $O(|E(G_u)|)$ time. The coloring takes $O(|E(G_u)|)$ time. Thus, the time complexity of the algorithm is $O(|E(G_u)|)$. □

Algorithm ALG3.4 follows the framework of Figure 3.2. Step 2 of ALG3.4 is the same as that in Algorithm ALG3.1. Step 3.1 uses Scheme S31 to color P_0 taking $\beta = \lfloor 0.5L \rfloor$. In Step 3.2 of ALG3.4, to color P_1, we convert the path coloring problem to the edge-coloring problem of multigraph G_u. Similar to Algorithm ALG3.2, there are two cases.

Case 1: $l(G_u) \leq 2L$.

In this case we edge-color G_u by the algorithm given in Lemma 3.4.2 using at most $2L$ virtual colors, re-assign virtual colors to the paths which have been assigned multi-colors of



(a) r_0 is not in H (b) $r_a = r_0$

Figure 3.6: Paths in and incident to the subgraph H with four vertices and $L(H) > 2L$.

W before Step 3.2, and apply the mapping f_1 to color the paths of P_1, as we did in Case 1 of Step 3.2 of Algorithm ALG3.2.

Case 2: $l(G_u) > 2L$.

In this case, there is a subgraph H of G_u with $L(H) = l(G_u)$. There are two subcases.

Case 2.1: Ring r_0 is not a vertex in H.

We color the paths in H by virtual colors of $C_H = \{\nu_1, \nu_2, ...\}$. Assume that $V(H) = \{r_a, r_b, r_c\}$. Notice that $R_a \cup R_b \cup R_c$ has been colored. Let $W^m_{Q'}$ be the set of multi-colors for $Q' = R_a \cup R_b \cup R_c$. From the $\lfloor 0.5L \rfloor$-set condition, $|W^m_{Q'}| \leq \lfloor 0.5L \rfloor$. Let W^m be the subset of $W^m_{Q'}$ such that the two paths with a color from W^m are incident to different vertices of H (see (a) of Figure 3.6). Similar to Case 2.1 of Algorithm ALG3.2, we color $P_{ab} \cup P_{ac}$ by distinct virtual colors and P_{bc} by Scheme S32 with

$$\beta = \min\{\lfloor 0.5L \rfloor - |W^m_{Q'}| + |W^m|, |E(H)| - 2L\},$$

using virtual colors of C_H. Then we apply mapping f_2 in the same way as that in Algorithm ALG3.2, using as many as possible of the colors of W_{R_a}, W_{R_b}, and W_{R_c} to color P_{bc}, P_{ac}, and P_{ab}, respectively.

Case 2.2: Ring r_0 is a vertex of H.

Assume $V(H) = \{r_a = r_0, r_b, r_c\}$ (see (b) of Figure 3.6). Notice that $R_a \cup P_{ab} \cup P_{ac}$ has been colored with at most $2L$ colors. Let $W^m_{Q'}$ be the set of multi-colors for $Q' = P_{ab} \cup P_{ac}$.

Similar to Case 2.2 of Algorithm ALG3.2, we color P_{bc} by Scheme S32 with

$$\beta = \max\{|E(H)| - \lceil 2.5L \rceil, |W^m_{Q'}|\},$$

using colors of $W_{R_a} \setminus (W_{P_{ab}} \cup W_{P_{ac}})$ first and then colors of $W \setminus (W_{R_a} \cup W_{P_{ab}} \cup W_{P_{ac}})$. After this we assign R_b distinct colors from $W'_{P_{ac}} = W_{P_{ac}} \setminus (W_{R_a} \cup W_{P_{ab}} \cup W_{P_{bc}})$ first and then from $W \setminus (W_{R_a} \cup W_{P_{ab}} \cup W_{P_{ac}} \cup W_{P_{bc}})$ by the first-fit coloring such that R_b is assigned distinct colors. Similarly, we assign R_c distinct colors from $W'_{P_{ab}} = W_{P_{ab}} \setminus (W_{R_a} \cup W_{P_{ac}} \cup W_{P_{bc}})$ first and then from $W \setminus (W_{R_a} \cup W_{R_b} \cup W_{P_{ab}} \cup W_{P_{ac}} \cup W_{P_{bc}})$ by the first-fit coloring.

Theorem 3.4.3 *Algorithm ALG3.4 solves the Min-PC problem on TR with n nodes and degree eight using at most* $\min\{3L, OPT + \lceil 1.5L \rceil\}$ *colors in* $O(nL^{2.5})$ *time.*

Proof: We prove that Algorithm ALG3.4 colors $Q_u \cup P_u$ with at most $\min\{3L, OPT + \lceil 1.5L \rceil\}$ colors for every node u. Similar to the proof for Algorithm ALG3.2, if $l(G_u) \leq 2L$, we get a valid coloring for $Q_u \cup P_u$ with at most $\min\{3L, OPT + \lceil 1.5L \rceil\}$ colors and the $\lfloor 0.5L \rfloor$-set condition holds after the coloring.

Assume that $l(G_u) > 2L$ in Step 3.2. In Case 2.1, by Lemma 3.2.3, the paths in H are colored with at least $|E(H)| - \beta \geq 2L$ virtual colors, where $\beta = \min\{\lfloor 0.5L \rfloor - |W^m_{Q'}| + |W^m|, |E(H)| - 2L\}$, $Q' = R_a \cup R_b \cup R_c$, and $W^m_{Q'}$ is the set of multi-colors on Q'. Let $C_{P_{ij}}$ $(i, j = a, b, c; i \neq j)$ be the subset of virtual colors of C_H assigned to P_{ij}. We have

$$|C_{P_{bc}} \setminus (C_{P_{ab}} \cup C_{P_{ac}})| \geq |C_H| - |C_{P_{ab}}| - |C_{P_{ac}}| \geq 2L - (d(r_a) - |R_a|) \geq |W_{R_a}|.$$

Similarly, $|C_{P_{ac}} \setminus (C_{P_{ab}} \cup C_{P_{bc}})| \geq |W_{R_b}|$ and $|C_{P_{ab}} \setminus (C_{P_{ac}} \cup C_{P_{bc}})| \geq |W_{R_c}|$. Thus Step (1) of mapping f_2 can be done and we get a valid coloring for $R_a \cup R_b \cup R_c \cup E(H)$. The number of colors used is, noting $|E(H)| \leq \lfloor \frac{6L - |Q'|}{2} \rfloor \leq 3L - |W^m_{Q'}|$ and $|W^m| \leq \lfloor 0.5L \rfloor$,

$$\begin{aligned}
& \max\{|E(H)| - \beta, OPT + L\} + |W^m| \\
\leq\ & \max\{|E(H)| - (\lfloor 0.5L \rfloor - |W^m_{Q'}| + |W^m|), |E(H)| - (|E(H)| - 2L), \\
& \qquad OPT + L\} + |W^m| \\
\leq\ & \max\{3L - |W^m_{Q'}| - \lfloor 0.5L \rfloor + |W^m_{Q'}| - |W^m|, OPT + L\} + |W^m| \\
=\ & \max\{\lceil 2.5L \rceil, OPT + L + |W^m|\} \\
\leq\ & OPT + \lceil 1.5L \rceil.
\end{aligned}$$

$P_{ab} \cup P_{ac}$ are given distinct real colors. There are at most β virtual multi-colors of C_H for P_{bc} and each of them is mapped to a distinct real color in $W_{R_a} \setminus W^m$ or $W \setminus (W_{R_a} \cup W_{R_b} \cup$

W_{R_c}). Notice that $W_{R_i} \cap W_{R_j} \subseteq W^m$ for $i,j = a,b,c; i \neq j$. Therefore, there are at most $|W^m_{Q'}| - |W^m| + \beta = \lfloor 0.5L \rfloor$ real multi-colors for the paths on each of rings r_a, r_b, and r_c. That is, the $\lfloor 0.5L \rfloor$-set condition is true for every ring.

In Case 2.2, P_{bc} is colored by Scheme $S32$ with $\beta = \max\{|E(H)| - \lceil 2.5L \rceil, |W^m_{Q'}|\}$, where $Q' = P_{ab} \cup P_{ac}$ and $W^m_{Q'}$ is the set of multi-colors on Q'. Since $|E(H)| \leq 3L$, $\beta \leq \lfloor 0.5L \rfloor$. Notice that Q' contains every colored path intersecting with a path of P_{bc}. By Lemma 3.1.2 and Lemma 3.2.3, $R_a \cup E(H)$ can be colored with at most $\min\{3L, \max\{\lceil 2.5L \rceil, OPT+L\}\} \leq \min\{3L, OPT + \lceil 1.5L \rceil\}$ colors. Recall that we choose $\beta = \max\{|E(H)| - \lceil 2.5L \rceil, |W^m_{Q'}|\}$ in the algorithm. Consider which of the two values β takes. Assume that $\beta = |E(H)| - \lceil 2.5L \rceil$. We show that R_b can be assigned distinct colors of $W'_{P_{ac}}$. Notice that

$$|W'_{P_{ac}}| \geq |W_{P_{ac}} \setminus W_{P_{bc}}| - |W_{P_{ac}} \cap (W_{R_a} \cup W_{P_{ab}})|.$$

By Lemma 3.2.3, at least $\lceil 2.5L \rceil$ colors are used for H. Since $W_{P_{ac}} \cap (W_{R_a} \cup W_{P_{ab}})$ is a subset of the multi-colors on $W_{R_a} \cup W_{P_{ab}} \cup W_{P_{ac}}$, from the $\lfloor 0.5L \rfloor$-set condition on ring r_a, $|W_{P_{ac}} \cap (W_{R_a} \cup W_{P_{ab}})| \leq \lfloor 0.5L \rfloor$. Therefore,

$$\begin{aligned} |W'_{P_{ac}}| &\geq (\lceil 2.5L \rceil - |W_{P_{ab}}| - |W_{P_{bc}}|) - \lfloor 0.5L \rfloor \\ &\geq \lceil 2.5L \rceil - (d(r_b) - |R_b|) - \lfloor 0.5L \rfloor \\ &\geq |R_b|. \end{aligned}$$

From this, R_b can be assigned distinct colors of $W'_{P_{ac}}$. Similarly, R_c can be assigned distinct colors of $W'_{P_{ab}}$. Thus $R_a \cup R_b \cup R_c \cup E(H)$ can be colored with at most $\min\{3L, OPT+\lceil 1.5L \rceil\}$ colors, and from $\beta = |E(H)| - \lceil 2.5L \rceil \leq \lfloor 0.5L \rfloor$ the $\lfloor 0.5L \rfloor$-set condition holds for $R_a \cup R_b \cup R_c$ and each of rings r_a, r_b, and r_c.

Assume that $\beta = |W^m_{Q'}|$. By the proof above, we can assume that $|W'_{P_{ac}}| < |R_b|$ or $|W'_{P_{ab}}| < |R_c|$. We further assume, without loss of generality, that $|W'_{P_{ac}}| < |R_b|$ and $|W'_{P_{ab}}| < |R_c|$ (the other two cases can be proved similarly). From $\beta = |W^m_{Q'}|$ and Scheme S32, P_{bc} is assigned distinct colors and $W_{P_{bc}} \cap (W_{P_{ab}} \cup W_{P_{ac}}) = \emptyset$. From $E(H) > 2L$ and $d(r_a) \leq 2L$, $|P_{bc}| > |R_a|$ which implies that all colors of $W_{R_a} \setminus (W_{P_{ab}} \cup W_{P_{ac}})$ are used for P_{bc}. Therefore, $R_a \cup E(H)$ is colored with at most $E(H) - |W^m_{Q'}|$ colors. To color R_b and R_c,

$$(|R_b| - |W'_{P_{ac}}|) + (|R_c| - |W'_{P_{ab}}|)$$

additional colors are needed. Since $W_{P_{bc}} \cap (W_{P_{ab}} \cup W_{P_{ac}}) = \emptyset$,

$$\begin{aligned} & (|R_b| - |W'_{P_{ac}}|) + (|R_c| - |W'_{P_{ab}}|) \\ = \; & (|R_b| + |R_c|) - (|W_{P_{ac}} \setminus (W_{R_a} \cup W_{P_{ab}})| + |W_{P_{ab}} \setminus (W_{R_a} \cup W_{P_{ac}})|) \\ \leq \; & (|R_b| + |R_c|) - (|W_{P_{ac}}| + |W_{P_{ab}}| - |W_{P_{ac}} \cap W_{R_a}| - |W_{P_{ab}} \cap W_{R_a}| \\ & -2|W_{P_{ab}} \cap W_{P_{ac}}|). \end{aligned}$$

Since $(W_{P_{ac}} \cap W_{R_a}) \cup (W_{P_{ab}} \cap W_{R_a}) \cup (W_{P_{ab}} \cap W_{P_{ac}})$ is a subset of the multi-colors on $R_a \cup P_{ab} \cup P_{ac}$, from the $\lfloor 0.5L \rfloor$-set condition on ring r_a, $|W_{P_{ac}} \cap W_{R_a}| + |W_{P_{ac}} \cap W_{P_{ab}}| + |W_{P_{ab}} \cap W_{R_a}| \leq \lfloor 0.5L \rfloor$ (noting that $W_{R_a} \cap W_{P_{ac}} \cap W_{P_{ab}} = \emptyset$). Since $W^m_{Q'}$ is the set of multi-colors on $Q' = P_{ab} \cup P_{ac}$,

$$|W_{P_{ac}}| + |W_{P_{ab}}| - |W_{P_{ab}} \cap W_{P_{ac}}| = |P_{ac}| + |P_{ab}| - |W^m_{Q'}|.$$

Also notice that $|P_{ac}| = |E(H)| - d(r_b) + |R_b|$ and $|P_{ab}| = |E(H)| - d(r_c) + |R_c|$. Summarizing the above,

$$\begin{aligned} & (|R_b| - |W'_{P_{ac}}|) + (|R_c| - |W'_{P_{ab}}|)) \\ \leq \; & |R_b| + |R_c| - (|P_{ac}| + |P_{ab}| - |W^m_{Q'}| - \lfloor 0.5L \rfloor) \\ = \; & d(r_b) + d(r_c) + |W^m_{Q'}| + \lfloor 0.5L \rfloor - 2|E(H)|. \end{aligned}$$

Since at most $|E(H)| - |W^m_{Q'}|$ colors are used for $R_a \cup E(H)$, $d(r_b), d(r_c) \leq 2L$, and $|E(H)| > 2L$, the total number of colors used for $R_a \cup R_b \cup R_c \cup E(H)$ is bounded by

$$d(r_b) + d(r_c) + |W^m_{Q'}| + \lfloor 0.5L \rfloor - 2|E(H)| + |E(H)| - |W^m_{Q'}| \leq \lceil 2.5L \rceil.$$

Obviously, the $\lfloor 0.5L \rfloor$-set condition holds for $R_a \cup R_b \cup R_c$ and each of rings r_a, r_b, and r_c.

The edge-coloring of G_u takes $O(L)$ time by Lemma 3.4.2. It takes $O(L^{2.5})$ time to color the subgraph H of G_u with $L(H) > 2L$. The first-fit coloring takes $O(L^2)$ time to color L paths. Therefore, it takes $O(L^{2.5})$ time in Steps 3.1 and 3.2. The algorithm executes these steps $O(n)$ times. The time complexity of the algorithm is $O(nL^{2.5})$. □

It seems difficult to generalize the approach of Algorithm ALG3.4 for the Min-PC problem on trees of rings with larger constant degrees, although a similar but more complicated analysis shows that the 2.5 approximation ratio is achievable for degree at most ten.

3.4.3 Algorithm for Degree Ten

The algorithm for degree ten, called ALG3.5, is similar to Algorithm ALG3.4, but uses a special scheme for the edge-coloring of multigraph G_u. Since the tree of rings considered has degree ten, G_u has at most five vertices. For a multigraph with five vertices, Lemma 3.4.2 can be extended as follows.

Lemma 3.4.4 *An edge-coloring of multigraph G_u with five vertices can be done using at most* $\max\{\Delta(G_u)+1, l(G_u)\}$ *colors.*

Lemma 3.4.4 follows from the 1.1 edge-coloring algorithm [95]. In the 1.1 edge-coloring algorithm, if a critical path does not contain two vertices with the same missing color, then it has 3, 5 or 7 vertices. $1.1\Delta+0.8$ colors are needed to ensure a valid edge-coloring. This is not needed in a multigraph with at most 5 vertices. The lemma is true for any multigraph with at most eight vertices.

Algorithm ALG3.5 follows the framework of Figure 3.2. We follow the notation used for Algorithm ALG3.4. Especially, for a subgraph H of multigraph G_u with $V(H) = \{r_a, r_b, r_c\}$, we use P_{ij} $(i,j = a,b,c; i \neq j)$ for the sets of long paths in H on r_i and r_j, and use R_i $(i = a,b,c)$ for the sets of paths not in H but on r_i. We use R_d to denote the set of paths not on any ring of H (see Figure 3.7(b)).

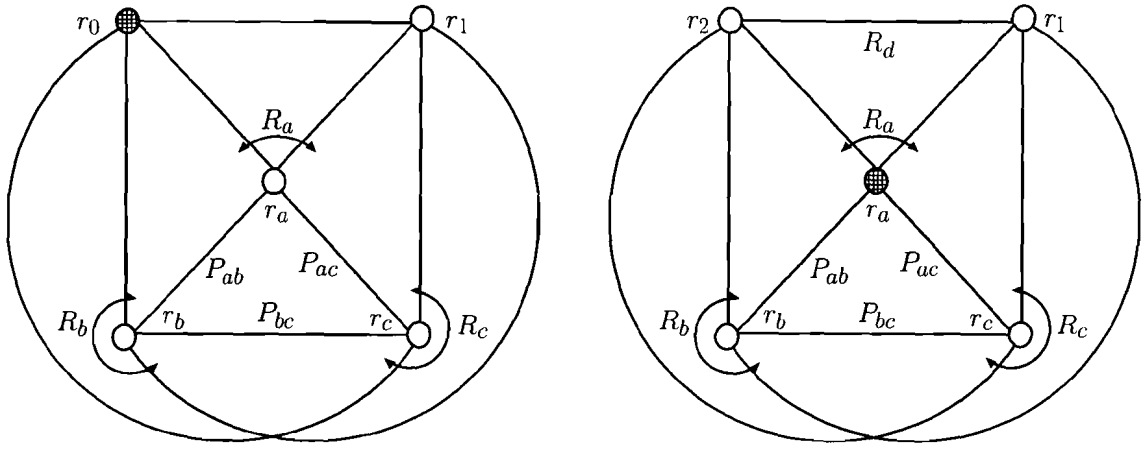
Step 2 of ALG3.5 is the same as that in Algorithm ALG3.1. Step 3.1 uses Scheme S31 to color P_0 taking $\beta = \lfloor 0.5L \rfloor$. In Step 3.2 of ALG3.5, to color P_1, we convert the path coloring problem to the edge-coloring problem of multigraph G_u. Similar to Algorithm ALG3.4, there are two cases.

Case 1: $l(G_u) \leq 2L$.

In this case we edge-color G_u by the algorithm given in Lemma 3.4.4 using at most $2L+1$ virtual colors, re-assign virtual colors to the paths which have been assigned multi-colors of W before Step 3.2, and apply the mapping f_1 to color the paths of P_1, as we did in Case 1 of Step 3.2 of Algorithm ALG3.4. It is easy to see that f_1 colors the paths in G_u by at most $2L+1+\lfloor 0.5L \rfloor = \lfloor 2.5L \rfloor + 1$ colors if the $\lfloor 0.5L \rfloor$-set condition is true before the step.

Case 2: $l(G_u) > 2L$.

We give some more definitions. Edges incident to r_0 in G_u are already colored. Let Q_1 be the set of paths on r_0 (corresponding to edges incident to r_0 in G_u), $W^m_{Q_1}$ be the set of multi-colors in W_{Q_1}, and $Q^m_1 \subseteq Q_1$ be the set of edges each of which is colored by



(a) r_0 is not in H (b) $r_0 = r_a$

Figure 3.7: Paths in and incident to the subgraph H with five vertices and $L(H) > 2L$.

a multi-color ($|Q_1^m| = 2|W_{Q_1}^m|$). For a set of edges E in G, we use $G - E$ to denote the subgraph of G obtained by removing the edges in E.

Notice that by definition, $L(G_u) \leq l(G_u)$. The value $l(G_u)$ could be achieved on a subgraph with three vertices ($L(H) = l(G_u)$ for some $H \subseteq G_u$ with $|V(H)| = 3$), or on graph G_u ($L(G_u) = l(G_u)$). There are two cases.

Case 2A: If there is a subgraph $H \subseteq G_u$ with $|V(H)| = 3$, and $L(H) = l(G_u)$, then we consider the following two cases.

- Case (i): Ring r_0 is not a vertex of the subgraph H.

  We first assign each edge in $G_u - (E(H) \cup Q_1)$ a distinct color not in W_{Q_1} by the first-fit coloring. These edges are incident to r_1 in Figure 3.7(a). Then we contract r_0 and r_1 to a single node r_0' (throw away any loop edges), and obtain a new graph G_u' which has four vertices. In G_u', only edges incident to r_0' are colored, and they form a $\lfloor 0.5L \rfloor$-set. We color the uncolored edges in G_u' as in Case 2.1 of Algorithm ALG3.4.

- Case (ii): Ring r_0 is a vertex of the subgraph H.

  We contract r_1 and r_2 to a single node r_1' (throw away any loop edges), and obtain a new graph G_u' which has four vertices (see Figure 3.7(b)). We color the uncolored

edges in G'_u as in Case 2.2 of Algorithm ALG3.4. We then color the edges in R_d with distinct colors not in $W_{R_a} \cup W_{R_b} \cup W_{R_c}$.

Case 2B: If $L(G_u) = l(G_u)$, and for all subgraph $H \subseteq G_u$ with $|V(H)| = 3$, $L(H) < l(G_u)$, then $l(G_u)$ is achieved on graph G_u itself (obviously $L(G_u) \leq \lceil 2.5L \rceil$, see Lemma 3.2.4). Let $G'_u = G_u - Q_1^m$. If $l(G'_u) \leq 2L$, then we edge-color G'_u with at most $2L+1$ colors (from Lemma 3.4.4), and then apply mapping f_1. Otherwise, $l(G'_u) > 2L$. If $L(G'_u) = l(G'_u)$, then we edge-color G'_u using $l(G'_u)$ colors and apply mapping f_1. Otherwise ($L(G'_u) < l(G'_u)$), there is a subgraph $H' \subseteq G'_u$ such that $L(H') = l(G'_u)$. Let H be the subgraph in G_u induced by the three vertices of H'. We then consider the two cases in the same way as Case 2A.

Theorem 3.4.5 *Algorithm ALG3.5 solves the Min-PC problem on TR with n nodes and degree ten using at most* $\min\{3L, OPT + \lceil 1.5L \rceil + 1\}$ *colors in* $O(nL^{2.5})$ *time.*

Proof We only consider Case 2 of $l(G_u) > 2L$ in Step 3.2 (other cases are similar to Algorithm ALG3.4). The coloring of the edges in H and edges in $R_a \cup R_b \cup R_c$ (the edges not in H but incident to the vertices of H) is essentially the same as in the degree eight case. There are two important facts: (1) in Case (i) of Case 2A, edges incident to r_1 are given distinct colors not in W_{Q_1}, and (2) in Case (ii) of Case 2A, R_b and R_c are given distinct colors (this is true following from the coloring process in Case 2.2 of Algorithm ALG3.4). In both Case (i) and Case (ii), the contracted node has degree at most $2L$, if the subgraph H has at least $2L$ edges (this condition is always true when we do the contraction). In the following proof, our main effort is to show that $G_u - E(H)$ can be edge-colored by at most $\lceil 2.5L \rceil$ colors.

In Case 2A, there is a subgraph $H \subseteq G_u$ with $|V(H)| = 3$, and $L(H) = l(G_u)$. We show that $G_u - E(H)$ has at most $\lceil 2.5L \rceil$ edges. This is true if $L(H) = l(G_u) > \lceil 2.5L \rceil$, since $G_u - E(H)$ has at most $\frac{2L \times 5}{2} - L(H) < 5L - \lceil 2.5L \rceil \leq \lceil 2.5L \rceil$ edges. If $2L < L(H) = l(G_u) \leq \lceil 2.5L \rceil$, then the number of edges in $G_u - E(H)$ is at most $2 \times L(G_u) - L(H) \leq 2l(G_u) - L(H) = l(G_u) \leq \lceil 2.5L \rceil$. Thus $G_u - E(H)$ can be edge-colored by at most $\lceil 2.5L \rceil$ colors. The rest of the proof is the same as the degree eight case.

In Case 2B, $l(G_u)$ is achieved on graph G_u itself (obviously $L(G_u) \leq \lceil \frac{2L \times 5}{4} \rceil = \lceil 2.5L \rceil$). Let $G'_u = G_u - Q_1^m$. If $l(G'_u) \leq 2L$, then G'_u can be edge-colored by at most $2L+1$ colors, and the total number of colors used for G_u is $2L + 1 + |W_{Q_1}^m| \leq \lfloor 2.5L \rfloor + 1$.)

Assume $l(G'_u) > 2L$. If $L(G'_u) = l(G'_u)$, G'_u can be edge-colored by $l(G'_u)$ colors. The total number of colors used for G_u is

$$l(G'_u) + |W^m_{Q_1}| = L(G'_u) + |W^m_{Q_1}| = \left\lceil \frac{2L(G_u) - 2|W^m_{Q_1}|}{2} \right\rceil + |W^m_{Q_1}| = L(G_u) \leq \lceil 2.5L \rceil.$$

On the other hand, if $L(G'_u) < l(G'_u)$, there is a subgraph $H' \subseteq G'_u$ such that $L(G'_u) < L(H') = l(G'_u)$. Consider the following two cases.

In Case (i) (ring r_0 is not a vertex in the subgraph H), $L(G'_u) = \frac{2L(G_u)-2|W^m_{Q_1}|}{2} = L(G_u) - |W^m_{Q_1}|$, and $L(H') = L(H)$. Thus $L(G_u) - |W^m_{Q_1}| < L(H)$. The edges in $G_u - E(H)$ are colored by at most

$$(2L(G_u) - L(H) - 2|W^m_{Q_1}|) + |W^m_{Q_1}| < L(G_u) \leq \lceil 2.5L \rceil$$

colors.

In Case (ii) (ring r_0 is a vertex in the subgraph H), define $W^{m_1}_{Q_1} = W_{R_a} \cap (W_{P_{ab}} \cup W_{P_{ac}})$ (the set of multi-colors each of which is used to color one edge in H and one edge not in H), $W^{m_2}_{Q_1} = W^m_{R_a}$ (the set of multi-colors in W_{R_a}), and $W^{m_3}_{Q_1}$ be the set of multi-colors each of which is used to color two edges in $P_{ab} \cup P_{ac}$. Then $|W^m_{Q_1}| = |W^{m_1}_{Q_1}| + |W^{m_2}_{Q_1}| + |W^{m_3}_{Q_1}|$. We have $L(G'_u) = \lceil \frac{2L(G_u)-2|W^m_{Q_1}|}{2} \rceil = L(G_u) - |W^m_{Q_1}|$, $L(H') = L(H) - |W^{m_1}_{Q_1}| - 2 \times |W^{m_3}_{Q_1}|$. Thus,

$$L(G_u) - |W^m_{Q_1}| < L(H) - |W^{m_1}_{Q_1}| - 2 \times |W^{m_3}_{Q_1}|$$

which implies

$$L(G_u) - L(H) - |W^{m_2}_{Q_1}| < -|W^{m_3}_{Q_1}|.$$

The edges in $G_u - E(H)$ are colored by at most

$$\begin{aligned} (2L(G_u) - L(H) - 2 \times |W^{m_2}_{Q_1}|) + |W^{m_2}_{Q_1}| &= 2L(G_u) - L(H) - |W^{m_2}_{Q_1}| \\ &< L(G_u) - |W^{m_3}_{Q_1}| \leq L(G_u) \leq \lceil 2.5L \rceil \end{aligned}$$

colors.

The time complexity of Algorithm ALG3.5 is the same as Algorithm ALG3.4, which runs in $O(nL^{2.5})$ time. □

It is not clear whether this approach can be used for trees of rings with degree more than ten. For a multigraph with more than five vertices, the five vertices subgraph H may have $L(H) > 2L$. However, we do not have an algorithm that uses at most $OPT + L$ colors for degree more than six.

3.5 Summary

We gave a $3L$ and (asymptotic) 2.75-approximation algorithm for the Min-PC problem on trees of rings with arbitrary degrees. The $3L$ upper bound is tight. We also presented a $3L$ and 2-approximation (resp. 2.5-approximation) algorithm for the Min-PC problem on trees of rings with degree at most six (resp. eight and ten). An interesting problem is to improve the 2.75-approximation ratio. A possible approach is to color the edges of multigraph G_u, allowing two edges with a common vertex in a given subset of edges sharing the same color. Another direction for the future work is to find better algorithms for the Min-PC problem on trees of rings with constant degrees. Our results imply a 3-approximation algorithm for the Min-RPC problem on a tree of rings. To our best knowledge, this is the first 3-approximation algorithm for this problem without using the cut-one-link strategy. We are not aware of any algorithm with performance ratio better than 3 for the Min-RPC problem on trees of rings, even when the tree of rings has bounded degree. It would be challenging to break this barrier. Our $3L$ algorithm also implies a $6L$ algorithm for the Min-PC problem on directed trees of rings with two directed links, one in each direction, between a pair of adjacent nodes. It is interesting to improve the approximation ratio for the Min-PC problem on directed trees of rings.

Chapter 4

Call Control and Maximum Path Coloring

The goal of the call control problem is to accommodate a maximum number of call requests subject to the bandwidth constraint of the links in the networks. The maximum path coloring problem, on the other hand, is to accept a maximum subset of paths that can be colored by a given number of colors. The call control problem with unit link capacity coincides with the maximum path coloring problem with one available color. Multifiber optical networks have multiple parallel fibers per link. For the path coloring problem in multifiber optical networks, each set of paths colored by a same color has maximum load bounded by the number of fibers on each link and is a feasible solution for the call control problem in which the capacity of an edge is equal to the number of fibers on that edge. The call control problem and the path multicoloring problem are closely related. In this chapter, we study these problems in tree and ring networks. We first study the call control problem in trees in Section 4.1. Then we study the path multicoloring problem in trees in Section 4.2. We study the maximum routing and path coloring problem in rings in Section 4.3.

4.1 Call Control in Bounded Depth Trees

In this section, we study the call control problem on bounded depth trees which are well used topologies in communication networks. In Section 4.1.1, we show that the call control problem is NP-hard and MAX SNP-hard even in depth-2 trees with capacities 1 or 2. Our

proof is a straightforward revision on the reduction for proving the hardness results of depth-3 trees in [63]. In Section 4.1.2, we give a polynomial time algorithm for the call control problem in a special class of depth-2 trees called double-stars. These results suggest that depth-2 trees are a boundary topology for which the call control problem is in P or NP-hard, depending on the node degrees of the trees. We also give 2- and 3-approximation algorithms in Section 4.1.3 for the weighted call control problem on depth-2 and depth-3 trees, respectively. This improves the previous 4-approximation algorithm for the problem on those trees. We show that the call control problem in spiders can be solved optimally in Section 4.1.4. All of our algorithms depend on a subroutine which solves the following restricted weighted call control problem on arbitrary trees in polynomial time: Given a set P of paths in a tree with all paths contain a same node of the tree, find an admissible subset $P' \subseteq P$ such that $w(P') = \sum_{i:p_i \in P'} w_i$ is maximized. This subroutine is of independent interest and is given in Section 4.1.5. A key technique used in our algorithms is to convert the call control problem in trees to the problem of finding a maximum degree constrained subgraph in auxiliary graphs. In Section 4.1.6, we show that the weighted call control problem in *any graphs* can be solved optimally if all the paths have length at most 2.

We begin with some definitions. A *rooted tree* is a tree in which a node r is selected as the root. All trees in this section are rooted trees unless otherwise stated, and will be denoted by T. The *level* of a node v in a tree is the length of the path from v to the root r which has level 0. The *depth* of a tree is the maximum level among all nodes of the tree. A depth-i tree is a tree with the maximum level i. An edge with a level-i end-node and a level-$(i+1)$ end-node in a tree is called a level-$(i+1)$ edge. A depth-1 tree is also called a star and the root r of the depth-1 tree is called the center node of the star. A *double-star* is a depth-2 tree in which two nodes have degree greater than one and all other nodes have degree one (see Figure 4.1 (a) for an example).

4.1.1 Hardness of Call Control in Depth-2 Trees

Given three pairwise disjoint sets X, Y, Z, $|X| = |Y| = |Z|$, and a set $S = \{(x_i, y_j, z_k) | x_i \in X, y_j \in Y, z_k \in Z\}$ of triples, the three-dimensional matching problem is to find the maximum number of disjoint triples (two triples are disjoint if they do not have a common element in any dimension). The three-dimensional matching problem is NP-hard and MAX SNP-hard, even if the number of occurrences of any element in X, Y or Z is bounded by a constant [76]. Garg *et al.* prove that the call control problem is NP-hard and MAX

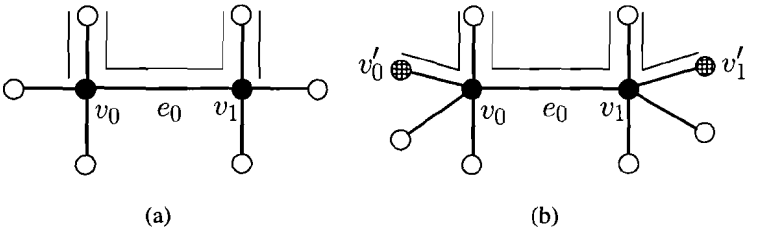


Figure 4.1: (a) A double-star, and (b) the double-star after pre-processing.

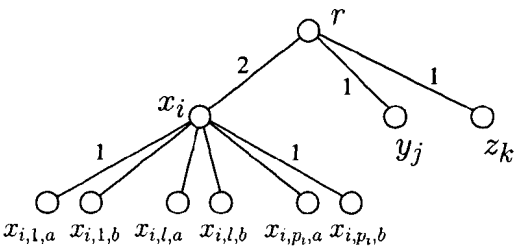


Figure 4.2: The depth-2 tree for the NP-hardness proof of the call control problem.

SNP-hard for trees by reducing the three-dimensional matching problem to the call control problem on depth-3 trees with edge capacities 1 and 2 [63]. This work actually proves a stronger result: the call control problem is NP-hard and MAX SNP-hard for depth-3 trees with edge capacities 1 and 2.

We observe that the three-dimensional matching problem can be reduced to the call control problem on depth-2 trees with edge capacities 1 and 2.

Theorem 4.1.1 *The call control problem is NP-hard and MAX SNP-hard for depth-2 trees with edge capacities 1 and 2.*

Proof Given an instance X, Y, Z, S of the three-dimensional matching problem, we first construct a depth-2 tree T with root r. For every $x_i \in X$, $y_i \in Y$, and $z_i \in Z$, there are level-1 nodes x_i, y_i, and z_i in T, respectively. Assume that each x_i appears in S p_i times. Then in T each x_i has $2p_i$ children $x_{i,l,a}$ and $x_{i,l,b}$ $(1 \leq l \leq p_i)$. There are $3|X|$ level-1 nodes and $2|S|$ level-2 nodes in T. We assign edges $\{r, x_i\}$ $(1 \leq i \leq |X|)$ capacity 2 and all other edges capacity 1. Figure 4.2 shows the construction of T. We number the occurrences of x_i in S arbitrarily from 1 to p_i, and the l^{th} occurrence corresponds to the two level-2 nodes

$x_{i,l,a}$ and $x_{i,l,b}$. Next we construct a call control instance P on T. If $(x_i, y_j, z_k) \in S$ is the l^{th} occurrence of x_i, we include three call requests, $\{x_{i,l,a}, x_{i,l,b}\}$, $\{x_{i,l,a}, y_j\}$, and $\{x_{i,l,b}, z_k\}$ into P. Notice that P has $3|S|$ call requests.

For any disjoint subset $S' \subseteq S$, we construct an admissible subset $P' \subseteq P$ as follows. Let (x_i, y_j, z_k) be the l^{th} occurrence of x_i in S. If $(x_i, y_j, z_k) \in S'$, then P' contains $\{x_{i,l,a}, y_j\}$ and $\{x_{i,l,b}, z_k\}$, otherwise (i.e., $(x_i, y_j, z_k) \notin S'$), P' contains $\{x_{i,l,a}, x_{i,l,b}\}$. Then P' is an admissible set and $|P'| = |S'| + |S|$. Next we show that given an admissible subset $P' \subseteq P$ with $|P'| = t + |S|$, a disjoint subset $S' \subseteq S$ with $|S'| = t$ can be constructed. Let $P_i \subseteq P$ be a maximum subset of paths with an end-node in the subtree rooted at x_i that can be admitted. Then P_i has $p_i + 1$ paths and contains the two paths for $\{x_{i,l,a}, y_j\}$ and $\{x_{i,l,b}, z_k\}$ for some l, and $p_i - 1$ paths for $\{x_{i,m,a}, x_{i,m,b}\}$ for $m \neq l, 1 \leq m \leq p_i$. Since $|P'| = t + |S|$, P' has t such subsets P_i. For every P_i, let (x_i, y_j, z_k) be the corresponding triple. Let S' be the set of triples (x_i, y_j, z_k) corresponding to those P_i's. Then each $x_i \in X$ appears in at most one triple of S'. From the capacities of edges $\{r, y_j\}$ and $\{r, z_k\}$, each $y_j \in Y$ or $z_k \in Z$ appears in at most one triple of S'. Thus, a maximum disjoint subset of S can be computed if and only if a maximum admissible subset of P can be computed. □

Notice that our proof of Theorem 4.1.1 follows a similar argument of [63] where a depth-3 tree is used.

4.1.2 Call Control in Double-stars

From the previous section, we know that the call control problem is NP-hard in depth-2 trees. It is also known that the call control problem can be solved in polynomial time for depth-1 trees. Now we explore the subset of depth-2 trees for which the call control problem is in P. More specifically, we give polynomial time algorithms for the call control problem in double-stars.

Let T be a double-star with two centers v_0 and v_1 (see Figure 4.1 (a)). If edge $e_0 = \{v_0, v_1\}$ has a constant capacity then the call control problem on T can be solved by a rather straightforward enumeration approach: Given a set P of paths on T, the number of paths on e_0 in any admissible subset $P' \subseteq P$ is at most $c(e_0)$. Since $c(e_0)$ is a constant, we can enumerate in polynomial time all possible subsets Q such that Q has at most $c(e_0)$ paths on e_0. For each enumerated subset Q, we find a maximum admissible subset Q_i of $\{p | p \in P \setminus Q, p \text{ is not on } e_0\}$ in the star with center v_i $(i = 0, 1)$. Then the maximum set

$Q_0 \cup Q_1 \cup Q$ over all Q's is a maximum admissible subset P' of P in T. The overall running time is polynomial in the input parameters. Similarly, if $c(e_0)$ is arbitrarily large but there are at most $O(\log n)$ paths of length three in P then we can solve the call control problem optimally by first enumerating all possible subsets of length-3 paths on e_0 in any optimal solution, in $2^{O(\log n)} = O(n^c)$ time for some constant $c > 0$, and then solving two call control problems in the two stars.

The enumeration approach does not give a polynomial time algorithm for the double-stars if $c(e_0)$ is arbitrarily large and P has more than $O(\log n)$ length-3 paths. The difficulty lies in how to choose the length-3 paths in an optimal solution. We now give Algorithm $ALG4.1$ which solves the call control problem in double-stars in polynomial time. For simplicity, we do the following pre-processing. If there is a set Q of paths with v_0 as an end-node, then we create a new leaf node v_0' and a new edge $\{v_0, v_0'\}$ with capacity $|Q|$, and extend the paths in Q to v_0' (see Figure 4.1 (b)). We do a similar pre-processing for node v_1. It is easy to see that after the pre-processing, all end-nodes of every path are leaf nodes in T, there are only length-2 and length-3 paths, and no length-2 path contains both v_0 and v_1. From now on, we use P to denote the set of pre-processed paths. We also do the following pre-processing on the capacities: $c(e) := \min\{c(e), L(e)\}$. This does not affect the solution, but helps to reduce the time complexity.

Given a double-star T with centers v_0 and v_1, let T_i $(i = 0, 1)$ be the star with center v_i obtained by removing edge $e_0 = \{v_0, v_1\}$. Edges of T_i have the same capacities as the corresponding edges in T. We define $E_0 = E(T_0)$ and $E_1 = E(T_1)$. Let $P_i = \{p | p \in P, p \text{ is on edges of } T_i \text{ only}\}$ and OPT_i' be a maximum admissible subset of P_i in T_i $(i = 0, 1)$. OPT_i' can be computed using the algorithm of [63]. Notice that there are only length-2 paths in OPT_i'. Let OPT be a maximum admissible subset of P in T. Then

$$|OPT_0'| + |OPT_1'| \leq |OPT| \leq \min\left\{|OPT_0'| + |OPT_1'| + c(e_0), \left\lfloor \frac{1}{2} \sum_{e \in E_0 \cup E_1} c(e) \right\rfloor\right\}. \quad (4.1)$$

We define $OPT_i \subseteq OPT$ to be the subset of paths in OPT using only edges in T_i $(i = 0, 1)$. Notice that OPT_i is not necessarily a subset of OPT_i'.

In Algorithm $ALG4.1$, we first perform the pre-processing described above, then transform the call control problem in T to a maximum weight DCS problem in an auxiliary graph H, and finally show that an optimal solution of the DCS problem can be converted to an optimal solution for the call control problem.

The auxiliary graph H is constructed as follows. For each edge e in $E_0 \cup E_1$, we create a node u_e in H with $b_1(u_e) = b_2(u_e) = c(e)$. For each length-2 path on $e_i, e_j \in E_0$ or $e_i, e_j \in E_1$, we create an edge $\{u_{e_i}, u_{e_j}\}$ in H (H is a multigraph in general, see Figure 4.3). These edges are shown as solid edges in Figure 4.3. For each length-3 path on a leaf edge $e_i \in E_0$ and a leaf edge $e_j \in E_1$, we create an edge $\{u_{e_i}, u_{e_j}\}$ in H. These edges are shown as dashed edges in Figure 4.3. We create one additional node u in H, and set $b_1(u) = b_2(u) = \sum_{e\in E_0} c(e) + \sum_{e\in E_1} c(e) - 2 \times g$, where g is an integer between $|OPT'_0| + |OPT'_1|$ and $\min\{|OPT'_0| + |OPT'_1| + c(e_0), \lfloor \frac{1}{2} \sum_{e\in E_0\cup E_1} c(e) \rfloor\}$. We create $c(e_i)$ parallel edges $\{u, u_{e_i}\}$, for each $e_i \in E_0 \cup E_1$. These edges are shown as the dash-dotted edges in Figure 4.3. We give each edge $\{u_{e_i}, u_{e_j}\}$ with $e_i \in E_0$ and $e_j \in E_1$ a weight of $1 - \epsilon$, for some small positive ϵ, say $\epsilon = 1/|P|$, and give all other edges a weight of 1. Notice that H is in general a multigraph: if $c(e) > 1$ or there are two paths with the same end-nodes in T, then edges may represent multiple parallel edges in Figure 4.3(b).

Algorithm $ALG4.1$ finds a maximum weight DCS M in H (if there exists one), using the algorithm of [59], for every possible values of g between $|OPT'_0| + |OPT'_1|$ and $\min\{|OPT'_0| + |OPT'_1| + c(e_0), \lfloor \frac{1}{2} \sum_{e\in E_0\cup E_1} c(e) \rfloor\}$. For each found M, Algorithm $ALG4.1$ checks if the set of paths corresponding to the edges of M is admissible. As shown later, a maximum admissible set P' can be obtained from a maximum weight DCS M for $g = |OPT|$.

Theorem 4.1.2 *Algorithm $ALG4.1$ solves the call control problem in double-stars in polynomial time.*

Proof The sum of the capacities of the nodes in H, excluding u, is $\sum_{e\in E_0} c(e) + \sum_{e\in E_1} c(e)$. Since each edge in M is incident to two nodes of H and $b_2(u)$ edges of M are incident to node u, the number of edges $\{u_{e_i}, u_{e_j}\}$ with $e_i, e_j \in E_0 \cup E_1$ that can be included in any DCS M is exactly

$$\frac{\sum_{e\in E_0} c(e) + \sum_{e\in E_1} c(e) - b_2(u)}{2} = \frac{2g}{2} = g.$$

Assume that there is an OPT which has $k \leq c(e_0)$ length-3 paths. Then for $g = |OPT| = |OPT_0| + |OPT_1| + k$, there is a DCS M in H such that M has exactly $|OPT_0| + |OPT_1|$ edges $\{u_{e_i}, u_{e_j}\}$ with $e_i, e_j \in E_0$ or $e_i, e_j \in E_1$, k edges $\{u_{e_i}, u_{e_j}\}$ with $e_i \in E_0$ and $e_j \in E_1$, and $b_2(u)$ edges $\{u, u_{e_i}\}$ with $e_i \in E_0 \cup E_1$. The weight of this M is $w(M) = \sum_{e\in E_0\cup E_1} c(e) - |OPT| - k\epsilon$. For any DCS M' of H with $l > k$ edges $\{u_{e_i}, u_{e_j}\}$ with $e_i \in E_0$ and $e_j \in E_1$, the weight of M' is $w(M') = \sum_{e\in E_0\cup E_1} c(e) - |OPT| - l\epsilon$ which is smaller than $w(M)$.

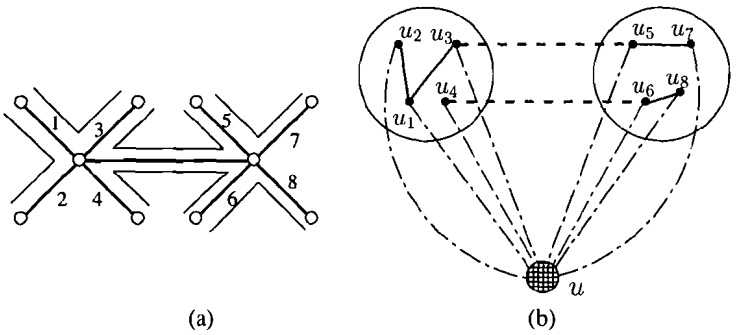


Figure 4.3: (a) An instance of the call control problem in a double-star T with $c(e) = 1$ for every $e \in E(T)$, and (b) an instance of the DCS problem in an auxiliary graph H.

Therefore, any maximum weight DCS must contain at most k edges $\{u_{e_i}, u_{e_j}\}$ with $e_i \in E_0$ and $e_j \in E_1$. Let P' be the set of paths in P corresponding to the edges in a maximum weight DCS in H. Then P' contains at most $k \leq c(e_0)$ length-3 paths and does not violate the capacity constraint of any leaf edge of T either. So P' is an admissible subset of P in T. Since P' contains exactly $|OPT_0| + |OPT_1| + k = |OPT|$ paths, P' is a maximum admissible subset.

From Inequality (4.1) and the fact that $ALG4.1$ computes a maximum weight DCS of H for every possible value of g between $|OPT'_0| + |OPT'_1|$ and $\min\{|OPT'_0| + |OPT'_1| + c(e_0), \lfloor \frac{1}{2} \sum_{e \in E_0 \cup E_1} c(e) \rfloor\}$, and checks the corresponding subset of P, $ALG4.1$ finds a maximum admissible subset $P' \subseteq P$ in T.

The auxiliary graph H has $|E(T)|$ nodes, and less than $(|P| + |E(T)|)$ edges. The maximum edge multiplicity of H is bounded by the load L of P in T. Thus, a maximum weight DCS in H can be found in $O((|P| + |E(T)|)^2 (\log |E(T)|)(\log L))$ time. The total running time of $ALG4.1$ is $O((|P| + |E(T)|)^2 (\log |E(T)|)(L \log L))$, since the maximum weight DCS algorithm is called $c(e_0) \leq L$ times. □

The complexity of the weighted call control problem in double-stars is not known. It seems that our method above cannot be used for the weighted case, since we already assign weights to edges in the graph H constructed in the (unweighted) call control problem and it is not clear how to assign weights to edges in the weighted call control problem. Nevertheless, we show in Section 4.1.3 that the weighted call control problem in depth-2 trees (thus in double-stars) can be approximated with ratio two.

4.1.3 Weighted Call Control in Depth-2 and Depth-3 Trees

The weighted call control problem can be solved optimally in depth-1 trees (stars) [63], and it can be approximated with a ratio of four in arbitrary trees [41]. The algorithm of [41] does not seem to have a performance ratio better than four when applied to bounded depth trees. In this section, we show that the weighted call control problem can be approximated with ratios two and three in depth-2 and depth-3 trees, respectively. Recall that the (unweighted) call control problem is already NP-hard in depth-2 trees, and can be approximated with a ratio of two. Our algorithms use the optimal weighted call control algorithm in stars [63], and an algorithm developed in Section 4.1.5 that solves in polynomial time the weighted call control problem for the instances in which all paths contain the root in arbitrary trees. This algorithm is of independent interest since it optimally solves the weighted call control problem in arbitrary trees for a restricted class of instances.

Given a set P of paths in a depth-2 or depth-3 tree T, we pre-process the paths as follows. For every internal node u of T, if there is a set Q of paths with u of T as an end-node, then we create a new leaf node u' and a new edge $\{u, u'\}$ with capacity $|Q|$, and extend the paths in Q to u. The pre-processing does not change the depth of T or the value of the optimal solution. Now we assume that the end-nodes of every path of P are leaf nodes of T. Let r be the root of T.

We first give Algorithm $ALG4.2$ for the weighted call control problem in a depth-2 tree T. The algorithm works as follows.

1. Let $P_1 \subseteq P$ be the subset of paths such that each path of P_1 contains root r. Find a maximum admissible subset of P_1 in T using the algorithm $ALG4.5$ described later, and denote the solution by SOL_1.

2. Find a maximum admissible subset of $P_2 = P \setminus P_1$ of remaining paths (that does not contain r) in T, and denote the solution by SOL_2.

3. Output the set of SOL_1 or SOL_2 which has the maximum weight as the final solution SOL.

For the weighted call control problem in a depth-3 tree T, we give a 3-approximation algorithm $ALG4.3$ as follows.

1. Let $P_1 \subseteq P$ be the subset of paths such that each path of P_1 contains root r. Find a

maximum admissible subset of P_1 in T using the algorithm $ALG4.5$ described later, and denote the solution by SOL_1.

2. Find a maximum admissible subset of $P_2 = P \setminus P_1$ of remaining paths (that does not contain r) in T using Algorithm $ALG4.2$, and denote the solution by SOL_2.

3. Output the set of SOL_1 or SOL_2 which has the maximum weight as the final solution SOL.

Theorem 4.1.3 *$ALG4.2$ and $ALG4.3$ are 2-approximation and 3-approximation algorithms for depth-2 and depth-3 trees, respectively.*

Proof Let P_1 be the subset of P such that each path of P_1 contains root r of T. We first assume that a maximum admissible subset of P_1 in T can be computed in polynomial time. By this assumption, we get SOL_1 in Step 1 of $ALG4.2$. In Step 2, the set P_2 of paths can be partitioned into several subsets of paths, with each subset of paths on a star obtained by removing r and the edges incident to r in the depth-2 tree. Thus, the maximum admissible subset of P_2 can be found in polynomial time by the algorithm in [63]. Let OPT be a maximum admissible subset of P in T, $OPT_1 = OPT \cap P_1$ and $OPT_2 = OPT \cap P_2$. Then $OPT_1 \cap OPT_2 = \emptyset$, $OPT_1 \cup OPT_2 = OPT$, $w(OPT_1) \leq w(SOL_1)$ and $w(OPT_2) \leq w(SOL_2)$. Thus,

$$\begin{aligned} w(OPT) = w(OPT_1) + w(OPT_2) &\leq w(SOL_1) + w(SOL_2) \\ &\leq 2\max\{w(SOL_1), w(SOL_2)\} \leq 2w(SOL), \end{aligned}$$

and Algorithm $ALG4.2$ has an approximation ratio of 2.

Similarly, let OPT be a maximum admissible subset of P in the depth-3 tree T. Then $w(OPT_1) \leq w(SOL_1)$. The set P_2 of paths can be partitioned into several subsets of paths, with each subset of paths on a depth-2 tree obtained by removing r and the edges incident to r in the depth-3 tree. Thus, $w(OPT_2) \leq 2w(SOL_2)$, since $ALG4.2$ is used to find SOL_2 and it is a 2-approximation algorithm. Therefore,

$$\begin{aligned} w(OPT) = w(OPT_1) + w(OPT_2) &\leq w(SOL_1) + 2w(SOL_2) \\ &\leq 3\max\{w(SOL_1), w(SOL_2)\} \leq 3w(SOL), \end{aligned}$$

and $ALG4.3$ has an approximation ratio of 3.

The assumption that a maximum admissible subset of P_1 in T can be computed in polynomial time is shown by Theorem 4.1.8 in Section 4.1.5. This completes the proof. □

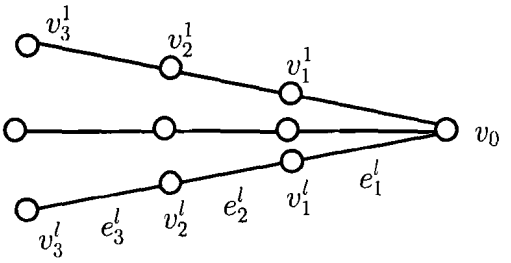


Figure 4.4: A spider network.

4.1.4 Call Control in Spiders

In this section, we study the call control problem in spiders. The result will be used in Section 4.2.2 to develop algorithms for the Max-PMC problems in spiders. We first give some more definitions. Let P be a set of paths in a spider G, and $c : E(G) \rightarrow \mathbb{N}$ be a capacity function. We say a subset $Q \subseteq P$ of paths is feasible if $L_Q(e) \leq c(e)$ for all $e \in E(G)$. Notice that any path in G can have edges in at most two legs of G. A path is called a short (resp. long) path if it is on edges of only one leg (resp. two legs). Let P_S (resp. P_L) be the set of short (resp. long) paths of P. For the convenience of description, we draw G as shown in Figure 4.4. The legs of G are numbered from 1 to Δ. For each leg l with $n_l + 1$ nodes, the node set and edge set are defined as $V^l = \{v_0, v_i^l | 1 \leq i \leq n_l\}$ and $E^l = \{e_i^l | e_1^l = (v_0, v_1^l), e_i^l = (v_{i-1}^l, v_i^l), 2 \leq i \leq n_l\}$. For simplicity, we occasionally use v_0^l for v_0. Node v_{i-1}^l is called the *right-node* of e_i^l and node v_i^l is called the *left-node* of e_i^l. Similarly, we can define the left-node and right-node of short paths. For a long path on legs l and m, we will also use "left-nodes" to denote its *endpoints* on legs l and m. For P in G and $1 \leq l \leq \Delta$, let $P^l \subseteq P$ be the set of paths, each of which is on an edge in leg l. Let $P_S^l = P^l \cap P_S$.

The following algorithm $ALG4.4$ is used to solve the call control problem in spiders.

1. For each leg l of G, process the nodes from $v_{n_l-1}^l$ to v_0. When we process node v_i^l, we check the paths of P_S^l with right-node v_i^l in an arbitrary order, and accept as many of these paths as possible subject to capacities $c(e)$. The processing for leg l is finished when node v_0 is processed. Let SOL_S be the set of paths accepted in processing all legs of G.

2. For each edge e of G, let $c'(e) = c(e) - L_{SOL_S}(e)$ be the new capacity function. Find a maximum cardinality subset SOL_L of P_L subject to new capacities $c'(e)$, using Algorithm $ALG4.5$ described later.

3. Output $SOL_S \cup SOL_L$ as the final solution.

In order to prove that Algorithm $ALG4.4$ gives an optimal solution for the call control problem, we need to prove the following two claims: (1) The two steps can both be done optimally in polynomial time, and (2) the two-step approach gives an optimal solution for the original problem.

By the following proposition, the first step can be done optimally in polynomial time.

Proposition 4.1.4 *[5] The call control problem can be solved optimally in polynomial time for chains.*

This result is a generalization of the algorithm for k-coloring of interval graphs [38]. The greedy algorithm for chains processes the nodes of a chain one by one from left to right. When processing node v, the paths with right-node v can be included into SOL_S in an arbitrary order (subject to the capacity constraint), and the solution is still optimal. This property is critical to the correctness of Algorithm $ALG4.4$.

The second step can be done optimally in polynomial time, i.e., the call control problem can be solved optimally for spiders if all the paths are long paths. This will be shown in Section 4.1.5.

The two-step algorithm

We now show that the two-step algorithm $ALG4.4$ gives an optimal solution for the call control problem in spiders.

Theorem 4.1.5 *The solution $SOL_S \cup SOL_L$ of Algorithm $ALG4.4$ is optimal for the call control problem in spiders.*

Proof For a given set $P = P_S \cup P_L$ of paths in a spider G, where P_S is the set of short paths and P_L is the set of long paths, let OPT be an optimal solution, OPT_S be the set of short paths in OPT, and OPT_L be the set of long paths in OPT. Then $|OPT| = |OPT_S| + |OPT_L|$, and $|OPT_S| \leq |SOL_S|$. Let $Q = SOL_S \cup OPT_L$. Recall that $c(e)$ is the capacity of edge e for the call control problem and $c'(e) = c(e) - L_{SOL_S}(e)$ is the new capacity of e in Algorithm

*ALG*4.4. If $L_Q(e) \leq c(e)$ for every $e \in E(G)$ then $|OPT_L| \leq |SOL_L|$, since OPT_L is a subset of P_L with $L_{OPT_L}(e) \leq c'(e)$ and SOL_L is the maximum cardinality subset of P_L with $L_{SOL_L}(e) \leq c'(e)$ for $e \in E(G)$. In this case,

$$|OPT| = |OPT_S| + |OPT_L| \leq |SOL_S| + |SOL_L| = |SOL|,$$

implying SOL is optimal and we are done.

Assume that $L_Q(e_i^l) > c(e_i^l)$ for some edge e_i^l in leg l. We show that a new OPT' with $|OPT'| \geq |OPT|$ can be obtained from OPT by replacing OPT_S^l with SOL_S^l and replacing OPT_L^l with a subset of OPT_L^l, for every such a leg l, such that $SOL_S \cup OPT_L'$ is a feasible solution. From the argument above, this implies $|OPT'| \leq |SOL|$. For a long path $p \in P_L$ on an edge of leg l, let $p(l)$ be the segment of p from v_0 to the endpoint of p in leg l. Define $Q_l = \{p(l) | p \in OPT_L\}$. Let Q_l' be the maximum cardinality subset of paths in Q_l which can be accepted on leg l subject to the new capacity function c'. Then $|Q_l'| < |Q_l|$. Let OPT_S^l (resp. P_S^l) be the subset of paths in OPT_S (resp. P_S) which is on edges of leg l. Then $SOL_S^l \cup Q_l'$ is the maximum cardinality subset of paths in $P_S^l \cup Q_l$ that can be accepted subject to the capacity function c (see the remarks following Proposition 4.1.4). $OPT_S^l \cup Q_l$ is a feasible subset of paths in $P_S^l \cup Q_l$ subject to the capacity function c. Thus, $|OPT_S^l \cup Q_l| \leq |SOL_S^l \cup Q_l'|$. In OPT, we replace OPT_S^l by SOL_S^l, replace OPT_L^l by $\{p | p \in OPT_L^l, p(l) \in Q_l'\}$, and use OPT' to denote the set of paths obtained. Then $|OPT| \leq |OPT'|$ and $L_{OPT'}(e_j^m) \leq c(e_j^m)$ $(1 \leq m \leq \Delta,\ 1 \leq j \leq n_m)$. However, OPT' has the additional property that for $Q' = SOL_S \cup OPT_L'$, $L_{Q'}(e_j^l) \leq c(e_j^l)$ $(1 \leq j \leq n_l)$.

Rename OPT' as the new OPT, and perform the above procedure as long as $SOL_S \cup OPT_L$ is not feasible. This process terminates after at most $\Delta(G)$ rounds (since G has $\Delta(G)$ legs). At this point, $SOL_S \cup OPT_L$ is feasible, and we have proved that the solution produced by algorithm *ALG*4.4 is optimal. □

4.1.5 Call Control in Trees with Central Paths

We give Algorithm *ALG*4.5 which solves the weighted call control problem optimally in an arbitrary tree T, if all the paths contain a same node of T. Suppose the tree T is rooted at node r, and all the paths in P contain r (we call such paths *central paths*). After proper pre-processing as in previous sections, we assume that no path in P contains r as an end-node. We use T_v to denote the subtree rooted at a node $v \in V(T)$. For a node $v \in V(T) \setminus \{r\}$, the unique neighbor of v whose level is one smaller than that of v is called the *parent* $p(v)$

of v, and we use e_0^v to denote the edge $\{v, p(v)\}$. For a non-leaf node v, all neighbors of v whose level are one larger than that of v are called the *children* of v, and we use $e_1^v, ..., e_d^v$ to denote the edges between v and its children (assuming v has $d = \delta(v) - 1$ children).

An important observation is that for a non-leaf node $v \in V(T) \setminus \{r\}$, a path on any edge e_i^v $(1 \leq i \leq d)$ must be on edge e_0^v as well. Let $Q \subseteq P$ be a set of central paths. Then $\sum_{i=1}^{d} L_Q(e_i^v) \leq L_Q(e_0^v)$. Similarly, we have $\sum_{i=1}^{d} L_{P\setminus Q}(e_i^v) \leq L_{P\setminus Q}(e_0^v)$. If $\sum_{i=1}^{d} L_{P\setminus Q}(e_i^v) \geq L(e_0^v) - c(e_0^v)$, then $L_Q(e_0^v) \leq c(e_0^v)$ for every e_0^v because

$$\begin{aligned} L_Q(e_0^v) = L(e_0^v) - L_{P\setminus Q}(e_0^v) &\leq L(e_0^v) - \textstyle\sum_{i=1}^{d} L_{P\setminus Q}(e_i^v) \\ &\leq L(e_0^v) - (L(e_0^v) - c(e_0^v)) = c(e_0^v). \end{aligned} \tag{4.2}$$

On the other hand, if $\sum_{i=1}^{d} L_{P\setminus Q}(e_i^v) < L(e_0^v) - c(e_0^v)$, then at least $L(e_0^v) - c(e_0^v) - \sum_{i=1}^{d} L_{P\setminus Q}(e_i^v)$ paths on e_0^v (but not on e_i^v, $1 \leq i \leq d$) cannot be included in Q if $L_Q(e_0^v) \leq c(e_0^v)$. These observations are essential to our algorithm.

We give Algorithm $ALG4.5$ which solves optimally the call control problem in trees with only central paths. We reduce the call control problem for the set P of paths containing r in T to the DCS (degree constrained subgraph) problem in an auxiliary graph H constructed below. Let v be the l^{th} child of r. We use E_l to denote $\{e_l^r\} \cup E(T_v)$ $(1 \leq l \leq \delta(r))$. Clearly any path of P is on exactly two children of r. For each path $p \in P$ on the l^{th} and m^{th} children of r, we create two nodes y_p^l and y_p^m, and an edge $e_p = \{y_p^l, y_p^m\}$ in H. These nodes and edges are called *path-nodes* and *path-edges*, respectively. For each path-node y_p^l, we set $b_1(y_p^l) = 0$ and $b_2(y_p^l) = 1$. For every edge $e \in E_l$ $(1 \leq l \leq \delta(r))$, we create in H a node $u(e)$. We create an edge $\{u(e), y_p^l\}$ if path p is on edge $e \in E_l$. Nodes $u(e)$ and edges $\{u(e), y_p^l\}$ are called *aux-nodes* and *aux-edges*, respectively. We set the capacities of the aux-nodes in the following order. The capacity of an aux-node corresponding to a non-leaf edge $e_0^v \in E(T)$ is set only if the capacities of all aux-nodes $\{u(e) | e \in E(T_v)\}$ have been set. For a leaf node v in T, the aux-node corresponding to the leaf edge e_0^v is $u(e_0^v)$, and

$$b_1(u(e_0^v)) = b_2(u(e_0^v)) = \max\{L(e_0^v) - c(e_0^v), 0\},$$

and for any non-leaf node $v \in V(T) \setminus \{r\}$,

$$b_1(u(e_0^v)) = b_2(u(e_0^v)) = \max\{L(e_0^v) - c(e_0^v) - \sum_{e \in E(T_v)} b_2(u(e)), 0\}.$$

We find a maximum cardinality DCS in the constructed graph H. We prove that the set of the paths corresponding to the path-edges in the maximum cardinality DCS is an optimal

solution of the call control problem for P in T. Let OPT be an optimal solution for the call control problem in T.

Lemma 4.1.6 *There exists a DCS of cardinality* $|OPT| + \sum_{e \in E(T)} b_2(u(e))$ *in* H.

Proof Given an optimal solution OPT in T, a DCS M in H can be constructed as follows. We first include into M all the path-edges corresponding to the paths in OPT. Since each path-node is incident to only one path-edge, $b_1(v) \leq \delta_M(v) \leq b_2(v)$ for all nodes of M. Next, we process every aux-node $u(e)$ ($e \in E(T)$) to include into M $b_2(u(e))$ aux-edges incident to aux-node $u(e)$. The processing order of $u(e)$ ($e \in E(T)$) is based on the order of the nodes in $V(T)$. Again, an aux-node corresponding to a non-leaf edge $e_0^v \in E(T)$ is processed only if all aux-nodes $\{u(e) | e \in E(T_v)\}$ have been processed. For a leaf node $v \in V(T)$, since $L_{OPT}(e_0^v) \leq c(e_0^v)$, there are at least $\max\{L(e_0^v) - c(e_0^v), 0\}$ path-nodes adjacent to $u(e_0^v)$ that are not matched by the edges of M prior to the processing of $u(e_0^v)$. So $b_2(u(e_0^v))$ aux-edges incident to $u(e_0^v)$ can be included into M such that $b_1(v) \leq \delta_M(v) \leq b_2(v)$ for all nodes of M after processing $u(e_0^v)$. Similarly, for any non-leaf node $v \in V(T) \setminus \{r\}$, at least

$$b_2(u(e_0^v)) = \max\{L(e_0^v) - c(e_0^v) - \sum_{e \in E(T_v)} b_2(u(e)), 0\}$$

path-nodes adjacent to $u(e_0^v)$ are not matched by the edges of M prior to the processing of $u(e_0^v)$. Therefore, $b_2(u(e_0^v))$ edges incident to $u(e_0^v)$ can be included into M such that $b_1(v) \leq \delta_M(v) \leq b_2(v)$ holds for all nodes of M after processing $u(e_0^v)$. When all aux-nodes are processed, the constructed DCS M satisfies $b_1(v) \leq \delta_M(v) \leq b_2(v)$ for every $v \in V(H)$ and has $|OPT|$ path-edges. The cardinality of M is

$$|OPT| + \sum_{e \in E(T)} b_2(u(e)).$$

□

Lemma 4.1.7 *Let SOL be the set of paths corresponding to path-edges in any DCS of* H. *Then SOL is a feasible solution for the call control problem in* T.

Proof We show that $L_{SOL}(e) \leq c(e)$ holds for every $e \in E(T)$. Let M be the given DCS in H. Then $\delta_M(u(e)) = b_1(u(e)) = b_2(u(e))$ for $e \in E(T)$. For any leaf node v of T, $b_1(u(e_0^v)) = b_2(u(e_0^v)) = \max\{L(e_0^v) - c(e_0^v), 0\}$ and at least $\max\{L(e_0^v) - c(e_0^v), 0\}$ paths on

edge e_0^v are not included in SOL. From this, $L_{SOL}(e_0^v) \leq c(e_0^v)$ holds. For any non-leaf node $v \in V(T) \setminus \{r\}$, if $\sum_{e \in E(T_v)} b_2(u(e)) \geq L(e_0^v) - c(e_0^v)$ ($b_2(u(e_0^v)) = 0$), then at least $L(e_0^v) - c(e_0^v)$ paths on edges in $E(T_v)$ are not included in SOL, and $L_{SOL}(e_0^v) \leq c(e_0^v)$ holds (see Inequality (4.2)). Otherwise, $b_2(u(e_0^v)) = L(e_0^v) - c(e_0^v) - \sum_{e \in E(T_v)} b_2(u(e))$ and there are $b_2(u(e_0^v)) + \sum_{e \in E(T_v)} b_2(u(e)) = L(e_0^v) - c(e_0^v)$ path-nodes of $\{y_p^l | p \text{ on } e_0^v, p \in P\}$ which are matched by aux-edges incident to aux-nodes $\{u(e) | e \in E(T_v) \cup \{e_0^v\}\}$. From this, there can be at most $c(e_0^v)$ path-edges in M whose corresponding paths are on edge e_0^v. This implies $L_{SOL}(e_0^v) \leq c(e_0^v)$. Thus SOL is feasible. □

Theorem 4.1.8 *There is an optimal polynomial time algorithm for the call control problem in an arbitrary tree if the set P of paths contains a same node of the tree.*

Proof Any DCS contains at most $\sum_{e \in E(T)} b_2(u(e))$ aux-edges and at most $|OPT|$ path-edges (by Lemma 4.1.7). Thus, a maximum cardinality DCS contains exactly $\sum_{e \in E(T)} b_2(u(e))$ aux-edges and $|OPT|$ path-edges (Lemma 4.1.6). Let SOL be the set of paths corresponding to the path-edges in the maximum cardinality DCS. Then SOL contains $|OPT|$ paths. According to Lemma 4.1.7, SOL is feasible. Thus, SOL is optimal. Since a maximum cardinality DCS can be found in polynomial time, the theorem holds. □

Algorithm $ALG4.5$ can be extended to work for the weighted call control problem in trees with only central paths. We make the following modifications to the above construction. Each path-edge e_p in H is assigned a weight equal to the weight of the corresponding path p in T. Each aux-edge is assigned a very small positive weight ϵ, where $\epsilon < \min_{p \in P} w(p)/(|P| + \sum_{e \in E(T)} b_2(u(e)))$. We can then show that the path-edges in a maximum weight DCS in H correspond to the paths in an optimal solution for the weighted call control problem in T.

4.1.6 Call Control with Length-2 Paths

From the proof of the NP-hardness and MAX SNP-hardness for the call control problem in depth-2 trees, we can see that the call control problem is NP-hard and MAX SNP-hard even if the path length is restricted to at most 3. On the other hand, we show that the weighted call control problem can be solved optimally in *any graphs*, if the path length is restricted to at most 2, even if the edge capacities are arbitrary. Let $G = (V, E)$ be the input graph for the call control problem and $c(e)$ be the capacity of an edge $e \in E(G)$. Without loss of generality, we may assume that all paths have length exactly 2, after proper pre-processing.

We give an algorithm which optimally solves the weighted call control problem when all the paths have length 2.

1. Construct a multigraph H as follows. For each edge $e \in E(G)$, construct a node u_e in H, and let $b_1(u_e) = 0$ and $b_2(u_e) = c(e)$. For any path p on edges e_1 and e_2 in G, construct an edge $e_p = \{u_{e_1}, u_{e_2}\}$ in H, and give e_p the same weight as p.

2. Find a maximum weight DCS M in H. The paths in G corresponding to the edges in M are taken as the solution for the weighted call control problem in G.

To see the above algorithm gives an optimal solution for the weighted call control problem with paths length 2, we notice that for any optimal solution for the weighted call control problem, there is a DCS with the same weight in H. On the other hand, for any DCS M in H, the corresponding paths in G do not violate any edge capacity constraint, since the node capacity constraint in H translates to edge capacity constraint in G. Thus, a maximum weight DCS in H corresponds to an optimal solution for the weighted call control problem in G.

4.1.7 Remarks

We have shown that the call control problem is NP-hard and MAX SNP-hard even in depth-2 trees. We give polynomial time optimal algorithms for the call control problem in double-stars which are special depth-2 trees, and for the call control problem in spiders. The double-star has depth two but has only two nodes with degree greater than one. The spider may have unbounded depth but only has one node with degree greater than two. These results suggest that whether the call control problem is in P or NP-hard depends largely on the node degree and depth of the trees. We have shown that the weighted call control problem is optimally solvable in arbitrary trees if all the paths contain a same node of the tree, while the weighted call control problem in any graphs can be solved optimally if all the paths have length at most 2. The pattern of the paths may also play an important role in the solvability of the call control problem.

4.2 Path Multicoloring in Multifiber Star and Spider Networks

In this section, we study the path multicoloring problem in WDM optical trees with multiple parallel fibers. We focus on the hardness of the PMC problems in stars and spiders in Section 4.2.1. Recall that the Min-PMC and Max-PMC problems in 1-fiber stars are NP-hard [48, 105]. For every even $k > 1$, the problems are known polynomial time solvable in k-fiber stars [87, 88]. A natural question here is whether there are efficient algorithms for the problems in k-fiber stars for every $k > 1$. We give a negative answer to this question by showing that for every odd integer $k \geq 3$, the Min-PMC and Max-PMC problems in k-fiber stars (and thus spiders) are NP-hard. These results are contrasted to the even k case. We give efficient algorithms for the Min-PMC problem in non-uniform stars with even number of fibers in every link and k-fiber (k even) spiders. The results above suggest that the evenness of the number of fibers plays an important role in the polynomial solvability of the problems. By the result for spiders of even fibers and the delete-one-fiber approach, we have a $(1 + \frac{1}{k-1})$-approximation algorithm for the Min-PMC problem in k-fiber spiders for every odd $k \geq 3$.

We study the Max-PMC problems in Section 4.2.2. By using the algorithm for the Min-PMC problem in stars of even fibers as a subroutine, we get an efficient algorithm for the Max-PMC problem in non-uniform stars with even fibers in every link. We also give an efficient algorithm and a 1.58-approximation algorithm for the Max-PMC problem in k-fiber (k even) spiders and non-uniform spiders, respectively. The algorithms for spiders rely on an optimal algorithm for the call control problem which has been developed in Section 4.1.4.

4.2.1 Hardness of the PMC Problem in Stars

In this section, we prove that the path multicoloring problem in k-fiber stars (and thus spiders) is NP-hard for every odd $k \geq 3$. We also give efficient algorithms for the Min-PMC problem in the non-uniform multifiber stars with even number of fibers in every edge and the k-fiber (k even) spiders.

NP-hardness result

Efficient algorithms for the Min-PMC problem in k-fiber (k even) stars have been known [87, 88]. When each edge of the star has one fiber, the Min-PMC problem becomes conventional path coloring problem and is NP-hard [48, 105]. Note that an efficient algorithm for the path coloring problem in stars with bounded maximum degree is known [48]. We show that the Min-PMC problem in k-fiber stars with unbounded maximum degree is NP-hard for every odd $k \geq 3$. We first give the proof for $k = 3$ and then generalize the proof to arbitrary odd k. We reduce the decision version of the path coloring problem in single fiber stars to the decision version of the path multicoloring problem in 3-fiber stars. The NP-completeness of the decision problem implies that both the Min-PMC and the Max-PMC problems are NP-hard.

The decision version of the path coloring problem in single fiber stars can be stated as follows: Given a set P of paths in a single fiber star and an integer $w > 0$, is P w-colorable? The decision version of the path multicoloring problem in k-fiber stars can be defined similarly: Given a set P of paths in a k-fiber star and an integer $w > 0$, is P w-colorable?

Theorem 4.2.1 *The Min-PMC problem in 3-fiber stars is NP-hard.*

Proof Let G_1 be the 3-fiber star with $V(G_1) = \{v_i | 0 \leq i \leq 3\}$ and $E(G_1) = \{(v_0, v_i) | 1 \leq i \leq 3\}$ (see Figure 4.5(a)). Let Q_1 be the set of $3w - 1$ paths between v_2 and v_3, $Q_1' = \{q_2, q_3\}$, where q_2 is the path between v_1 and v_2 and q_3 is the path between v_1 and v_3, and $P_1 = Q_1 \cup Q_1'$. Obviously there is a valid w-coloring for P_1. On the other hand, in any valid w-coloring for P_1, each color must be used by exactly three paths on (v_0, v_2) and three paths on (v_0, v_3) since the load on each of the two edges is $3w$. By the definition of Q_1, there are $w - 1$ colors each of which is used by three of the $3w - 1$ paths, and exactly one color (say λ) which is used by two of the $3w - 1$ paths of Q_1. The paths q_2 and q_3 of Q_1' can only be colored by λ. Thus, in any valid w-coloring of P_1, paths q_2 and q_3 must be assigned the same color.

Let G_w be the 3-fiber star with $V(G_w) = \{v_i | 0 \leq i \leq 2w+1\}$ and $E(G_w) = \{(v_0, v_i) | 1 \leq i \leq 2w+1\}$ (see Figure 4.5(b)). G_w can be considered as the star obtained from w copies of G_1 by merging the w edges (v_0, v_1) in the copies into one edge. Let Q_j be the set of $3w - 1$ paths between v_{2j} and v_{2j+1} and $Q_j' = \{q_{2j}, q_{2j+1}\}$ for $1 \leq j \leq w$, where q_{2j} is the path

between v_1 and v_{2j} and q_{2j+1} is the path between v_1 and v_{2j+1}. Let $P_w = \cup_{j=1}^{w}(Q_j \cup Q'_j)$ in G_w. Then it is easy to find a valid w-coloring for P_w. By the analysis on G_1 above, in any valid w-coloring of P_w, the paths in each Q'_j must be given the same color. For any pairs Q'_{j_1} and Q'_{j_2} $(1 \leq j_1 \neq j_2 \leq w)$, there are four paths in Q'_{j_1} and Q'_{j_2}, the four paths are on edge (v_0, v_1), and the edge has three fibers. Therefore, in any valid w-coloring of P_w, the two paths in any Q'_j are assigned the same color and any two pairs Q'_{j_1} and Q'_{j_2} $(j_1 \neq j_2)$ are assigned different colors. This implies that each of the w colors is used by exactly one set of Q'_j.

We are now ready to give the reduction. Given a single fiber star G with $V(G) = \{u_l | 0 \leq l \leq \Delta\}$ and $E(G) = \{(u_0, u_l) | 1 \leq l \leq \Delta\}$, and a set P of paths in G, we create a 3-fiber star G' with $V(G') = \{v_0\} \cup \{v_i^l | 1 \leq l \leq \Delta, 1 \leq i \leq 2w+1\}$ and $E(G') = \{(v_0, v_i^l) | 1 \leq l \leq \Delta, 1 \leq i \leq 2w+1\}$. G' can be considered as the star obtained from Δ copies G_w^l $(1 \leq l \leq \Delta)$ of G_w by merging the Δ centers of the copies into one center. For $1 \leq l \leq \Delta$, let P^l be the set of paths in G_w^l as P_w defined for G_w above. Let $\hat{P} = \{\hat{p} | p \in P\}$, where $\hat{p}$ is a path between v_1^l and v_1^m if p is a path between u_l and u_m $(l, m \neq 0)$ and $\hat{p}$ is a path between v_0 and v_1^l if p is a path between u_0 and u_l $(l \neq 0)$. Let $P' = \hat{P} \cup (\cup_{l=1}^{\Delta} P^l)$.

We show that there is a valid w-coloring for P' in G' if and only if P is w-colorable in G. Assume P is w-colorable in G. In G', we color each path $\hat{p} \in \hat{P}$ using the color of the corresponding path $p \in P$ in G. By the definition of $\cup_{l=1}^{\Delta} P^l$, it is easy to find a valid w-coloring for $P' \setminus \hat{P}$. Combining the colorings for $\hat{P}$ and $P' \setminus \hat{P}$ gives a valid w-coloring of P' in G', since on each edge (v_0, v_1^l) in G', each color is used by at most one path in $\hat{P}$ and by exactly two paths in $P' \setminus \hat{P}$. On the other hand, if there is a valid w-coloring for P', then for each edge (v_0, v_1^l) of G', each of the w colors is used by exactly two paths in $P' \setminus \hat{P}$. Thus, on each edge (v_0, v_1^l) of G', each color is used by at most one path in $\hat{P}$. We can obtain a valid coloring for the set P of paths in G by coloring each path $p \in P$ using the color for path $\hat{p}$ in G'. The above reduction clearly runs in polynomial time. □

By a generalization of the reduction in the proof for Theorem 4.2.1, we have the following result.

Theorem 4.2.2 *The Min-PMC problem in k-fiber stars is NP-hard for every odd $k \geq 3$.*

Proof Suppose $k = 2k_0 + 1$ $(k_0 \geq 1)$. Let G_1 be a k-fiber star with $V(G_1) = \{v_i | 0 \leq i \leq 3\}$ and $E(G_1) = \{(v_0, v_i) | 1 \leq i \leq 3\}$. Let Q_1 be the set of $kw - 1$ paths between v_2 and v_3,

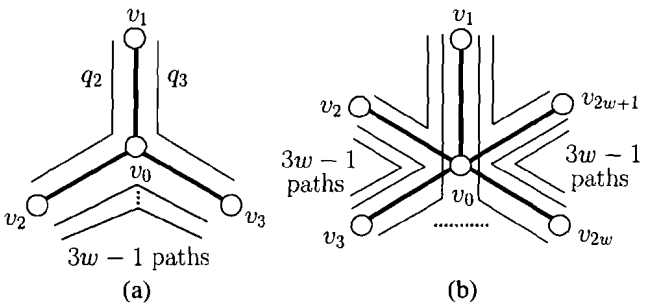


Figure 4.5: Stars G_1 and G_w.

$Q'_1 = \{q_2, q_3\}$, where q_2 is the path between v_1 and v_2 and q_3 is the path between v_1 and v_3, and $P_1 = Q_1 \cup Q'_1$. It is easy to prove that P_1 is w-colorable and in any valid w-coloring of P_1, paths q_2 and q_3 must be assigned the same color. Let G_{k_0w} be a k-fiber star with $V(G_{k_0w}) = \{v_i | 0 \leq i \leq 2k_0w + 1\}$ and $E(G_{k_0w}) = \{(v_0, v_i) | 1 \leq i \leq 2k_0w + 1\}$. G_{k_0w} can be considered as the star obtained from k_0w copies of G_1 by merging the k_0w edges (v_0, v_1) in the copies into one edge. Let Q_j be the set of $kw - 1$ paths between v_{2j} and v_{2j+1} and $Q'_j = \{q_{2j}, q_{2j+1}\}$ for $1 \leq j \leq k_0w$, where q_{2j} is the path between v_1 and v_{2j} and q_{2j+1} is the path between v_1 and v_{2j+1}. Let $P_{k_0w} = \cup_{j=1}^{k_0w}(Q_j \cup Q'_j)$ in G_{k_0w}. Again P_{k_0w} is w-colorable and in any valid w-coloring of P_{k_0w}, the two paths in any Q'_j are assigned the same color and each of the w colors is used by exactly k_0 pairs from $\{Q'_j | j = 1, ..., k_0w\}$. To see this last point is true, suppose some color λ is used by $k'_0 < k_0$ pairs. Each of the remaining $w - 1$ color can be used by at most k_0 pairs, since there are $2k_0 + 1$ fibers per edge. Thus, the total number of pairs colored by the w colors is at most $k_0(w - 1) + k'_0 < k_0w$, a contradiction. Thus, the claim is true.

We are now ready to give the reduction. Given a single fiber star G with $V(G) = \{u_l | 0 \leq l \leq \Delta\}$ and $E(G) = \{(u_0, u_l) | 1 \leq l \leq \Delta\}$, and a set P of paths in G, we create a k-fiber star G' with $V(G') = \{v_0\} \cup \{v_i^l | 1 \leq l \leq \Delta, 1 \leq i \leq 2k_0w + 1\}$ and $E(G') = \{(v_0, v_i^l) | 1 \leq l \leq \Delta, 1 \leq i \leq 2k_0w + 1\}$. G' can be considered as the star obtained from Δ copies $G^l_{k_0w}$ $(1 \leq l \leq \Delta)$ of G_{k_0w} by merging the Δ centers of the copies into one center. For $1 \leq l \leq \Delta$, let P^l be the set of paths in $G^l_{k_0w}$ as P_{k_0w} defined for G_{k_0w} above. Let $\hat{P} = \{\hat{p} | p \in P\}$, where $\hat{p}$ is a path between v_1^l and v_1^m if p is a path between u_l and u_m $(l, m \neq 0)$ and $\hat{p}$ is a path between v_0 and v_1^l if p is a path between u_0 and u_l $(l \neq 0)$. Let $P' = \hat{P} \cup (\cup_{l=1}^{\Delta} P^l)$. It

is easy to prove that P' is w-colorable in G' if and only if P is w-colorable in G. □

Efficient algorithm for Min-PMC problem in stars

Efficient algorithms for the Min-PMC problem in k-fiber (k even) stars have been known [87, 88]. We have just shown that the Min-PMC problem is NP-hard for k-fiber (k odd) stars. A natural question is, does the evenness of the number of fibers play an important role in the polynomial solvability of the Min-PMC problem in stars? In this subsection, we give an efficient algorithm for the Min-PMC problem in the non-uniform multifiber star G with an even number $\mu(e)$ of fibers in every edge e. This suggests that the evenness is a key in the polynomial solvability for the problem. A path in a star is called a *short path* (resp. *long path*) if the path is on one edge (resp. two edges). Let P be the given set of paths in G and $w_{lb} = \max_{e \in E(G)} \lceil \frac{L(e)}{\mu(e)} \rceil$. Then w_{lb} is an obvious lower bound on the number of colors required for coloring P. Algorithm $ALG4.6$ shown below uses exactly w_{lb} colors (thus is optimal), if $\mu(e)$ is even for every $e \in E(G)$.

1. For any edge e with odd $L(e)$, add one short dummy path on e. Let P' be the union of P and the set of dummy paths, and $P_L \subseteq P$ be the set of long paths in P. The number of paths in $P' \setminus P_L$ (short paths) is even, since $\sum_{e \in E(G)} L_{P'}(e)$ is even, each path of P_L is on two edges and $\sum_{e \in E(G)} L_{P_L}(e)$ is even.

   Create a multigraph G' with $V(G') = V(G)$ and there is an edge between v_i and v_j in G' if there is a path between v_i and v_j in star G. Notice that every node in multigraph G' has even degree and there is an Euler circuit in every connected component of G'.

2. Find an Euler circuit for every connected component of multigraph G' and orient each Euler circuit. Then each path in star G is assigned a direction by the oriented Euler circuits. Discard the dummy paths. For any edge e in star G, the load of the paths with the same direction on e is at most $\lceil \frac{L(e)}{2} \rceil$ for each direction.

3. For each leaf node in star G, partition the outgoing paths into sets, each of which has w_{lb} paths (except the last set, which may have less than w_{lb} paths). Similarly, partition the incoming paths of each leaf node into sets, each of which has w_{lb} paths. For the center v_0, partition the incoming (resp. outgoing) short paths into incoming (resp. outgoing) sets each of which has w_{lb} paths. Let V_{in} (resp. V_{out}) be the collection

of sets of incoming (resp. outgoing) paths. Then each path $p \in P$ is in exactly one set of V_{in} and exactly one set of V_{out}.

4. Create a bipartite (multi-)graph with bipartitions V_{in} and V_{out}. For each path $p \in P$ in a set of V_{in} and a set of V_{out}, create an edge e_p between the corresponding nodes in V_{in} and V_{out}.

5. The bipartite graph has node degree w_{lb}, and its edges can be colored optimally by w_{lb} colors [43]. Color each path p using the color of its corresponding edge e_p.

Theorem 4.2.3 *Algorithm ALG4.6 solves the Min-PMC problem in non-uniform stars with even number $\mu(e)$ of fibers in every edge e using $w_{lb} = \max_{e \in E(G)} \lceil \frac{L(e)}{\mu(e)} \rceil$ colors in polynomial time.*

Proof For any node u of the bipartite graph, the edges incident to u are assigned different colors. Thus, paths in any set of V_{in} (or V_{out}) are assigned different colors. The color repetition at any edge e incident to a leaf node v of the star is just the total number of incoming and outgoing sets of v. When $L(e)$ is even, the color repetition at edge e is at most

$$\begin{aligned} 2\left\lceil \frac{L(e)/2}{w_{lb}} \right\rceil &\leq 2\left\lceil \frac{L(e)/2}{\lceil L(e)/\mu(e) \rceil} \right\rceil \\ &\leq 2\left\lceil \frac{L(e)/2}{L(e)/\mu(e)} \right\rceil = 2\left\lceil \frac{\mu(e)}{2} \right\rceil = \mu(e). \end{aligned}$$

The last equality holds since $\mu(e)$ is even. Similarly, when $L(e)$ is odd,

$$\lceil L(e)/\mu(e) \rceil = \lceil (L(e)+1)/\mu(e) \rceil$$

for even $\mu(e)$. The color repetition at edge e is at most

$$\begin{aligned} 2\left\lceil \frac{\lceil L(e)/2 \rceil}{w_{lb}} \right\rceil &\leq 2\left\lceil \frac{\lceil L(e)/2 \rceil}{\lceil L(e)/\mu(e) \rceil} \right\rceil \\ = 2\left\lceil \frac{(L(e)+1)/2}{\lceil (L(e)+1)/\mu(e) \rceil} \right\rceil &\leq 2\left\lceil \frac{(L(e)+1)/2}{(L(e)+1)/\mu(e)} \right\rceil = \mu(e). \end{aligned}$$

Therefore, the color repetition is always bounded from above by $\mu(e)$, the number of available fibers on edge e. Thus, the w_{lb}-coloring is valid. Algorithm $ALG4.6$ runs in polynomial time, since the construction of the multigraph and the bipartite graph can be done in polynomial time, and the Euler circuit and the bipartite graph edge-coloring can be found in polynomial time. □

The f-coloring of multigraphs is an extension of the edge-coloring, and can be defined as follows. Given a multigraph G, a node capacity function $f : V(G) \rightarrow \mathbb{N}$, color the edges in $E(G)$ such that each color is used by at most $f(v)$ edges incident to node v. The goal is to minimize the number of colors used (this number is usually called the *f-chromatic index* of G). The edge-coloring problem is a special case of the f-coloring problem in which $f(v) = 1$ for all $v \in V(G)$. The f-coloring problem is NP-hard and an asymptotic (9/8)-approximation algorithm for the problem is known [94]. It is easy to show that the f-coloring problem in multigraphs is equivalent to the Min-PMC problem in stars (following the reduction in [48]). Thus, we can have an efficient algorithm for the f-coloring problem if $f(v)$ is even for every node v.

Efficient algorithm for Min-PMC problem in k-fiber spiders

We give an efficient algorithm *ALG*4.7 for the Min-PMC problem in k-fiber (k even) spiders. Let G be a k-fiber spider with the center node v_0. Given a set P of paths in G, $\lceil \frac{L}{k} \rceil$ is a lower bound on the number of colors for coloring P. Algorithm *ALG*4.7 works as follows.

1. For every edge e of G, if $L(e) < \lceil \frac{L}{k} \rceil k$ then add unit-length dummy paths on e until $L(e) = \lceil \frac{L}{k} \rceil k$. Let Q be the set of dummy paths on all edges of G.

2. Let $P_0 \subseteq (P \cup Q)$ be the set of paths on the center node v_0 of G. Color P_0 using Algorithm *ALG*4.6 for the Min-PMC problem in stars.

3. For every leg in G, color the paths of $P \setminus P_0$ in the leg by the algorithm for a chain regarding the segments of paths of P_0 in the leg as pre-colored paths.

Theorem 4.2.4 *Algorithm ALG*4.7 *solves the Min-PMC problem in k-fiber (k even) spiders using $\lceil \frac{L}{k} \rceil$ colors in polynomial time.*

Proof By Theorem 4.2.3 the number of colors used in Step 2 is $\lceil \frac{L}{k} \rceil$, since the load of P_0 is $\lceil \frac{L}{k} \rceil k$ (k even) and the number of colors needed is simply the load divided by k. Furthermore, on each edge incident to the central node v_0, each color is used by exactly k paths in P_0. In Step 3, the paths on each leg can be colored by $\lceil \frac{L}{k} \rceil$ colors. This is true since the chain algorithm of [96] works for the uniform case in which a subset of the paths is already colored and all the pre-colored paths are on the same endpoint of the chain. □

Algorithm $ALG4.7$ can be used to derive an approximation algorithm for the Min-PMC problem in k-fiber (k odd) spiders by the delete-one-fiber approach used in [97] for stars. For a k-fiber spider with odd $k \geq 3$, we consider the network as a $(k-1)$-fiber network and apply Algorithm $ALG4.7$ to solve the Min-PMC problem by $\lceil \frac{L}{k-1} \rceil$ colors. Since $\lceil \frac{L}{k} \rceil$ is a lower bound on the number of colors required for any optimal solution, this approach gives a $(1 + \frac{1}{k-1})$-approximation algorithm.

4.2.2 Max-PMC Problems in Stars and Spiders

In this section, we study the Max-PMC problem in multifiber stars and spiders. For a multifiber network with $\mu(e)$ fibers on edge e and w colors, any optimal solution for the Max-PMC problem has load at most $\mu(e) \times w$ on edge e. Consider the call control problem in the same network with edge capacity $c(e) = \mu(e) \times w$ on edge e. The optimal solution for the call control problem is an obvious upper bound on the optimal solution for the Max-PMC problem. The results in this section will use this observation. Our optimal algorithms for the Max-PMC problem achieve this upper bound.

Max-PMC problem in stars

The proofs of Theorems 4.2.1 and 4.2.2 imply that the Max-PMC problem is also NP-hard in k-fiber (k odd) stars. We show that the Max-PMC problem can be solved optimally in stars with even number of fibers. Let G be a star with an even number $\mu(e)$ of fibers in every edge e and P be a set of paths in G. Recall that in any optimal solution for the Max-PMC problem, the load of an edge e is at most $\mu(e) \times w$. Thus, we can reduce the Max-PMC problem to the call control problem as follows: Assign an edge e a capacity of $c(e) = \mu(e) \times w$ and solve the call control problem with P as the input set of paths and c as the capacity function. This call control problem can be solved optimally since it can be formulated as a b-matching problem [63]. Let $P' \subseteq P$ be the optimal solution for this call control problem. Then $|P'|$ is an upper bound on the cardinality of the optimal solution for the Max-PMC problem in G. The chosen set P' of paths can be colored by $\max_{e \in E(G)} \lceil \frac{\mu(e) \times w}{\mu(e)} \rceil = w$ colors using Algorithm $ALG4.6$ in Section 4.2.1, since $\mu(e)$ is even for every $e \in E(G)$. Thus, P' is an optimal solution for the original Max-PMC problem.

The same argument holds for the weighted Max-PMC problem in stars with non-uniform even fibers: one simply selects a maximum weight subset of paths with load at most $\mu(e) \times w$

on edge e (this can also be done in polynomial time by reducing to the weighted b-matching problem), and then color the chosen set of paths using w colors.

Max-PMC in k-fiber (k even) spiders

We show that the Max-PMC problem can be solved optimally in in k-fiber (k even) spiders. Let G be a k-fiber (k even) spider and P be a given set of paths in G for the Max-PMC problem. We first select a maximum cardinality subset $P' \subseteq P$ of paths with load at most kw. This can be done in polynomial time since it is a special case of the call control problem in spiders (in which $c(e) = kw$ for every edge e), and can be efficiently solved using Algorithm $ALG4.4$ in Section 4.1.4. The selected set P' of paths has load at most kw, and can be colored by $\frac{kw}{k} = w$ colors, using Algorithm $ALG4.7$ for k-fiber (k even) spiders (see Section 4.2.1). $|P'|$ is an upper bound on the cardinality of the optimal solution for the Max-PMC problem. Thus, P' is an optimal solution for the Max-PMC problem.

Max-PMC problem in non-uniform spiders

The Max-PMC problem is NP-hard in spiders with non-uniform fibers ($\mu(e)$ can be different for different edges and can be even or odd), since the Max-PMC problem is NP-hard in k-fiber (k odd) stars (which are non-uniform spiders by definition). We solve the problem using the standard approach of calling the call control algorithm as a subroutine. Note that the call control problem is equivalent to the Max-PMC problem with $w = 1$. Suppose we have an approximation algorithm for the call control problem. Let G be a spider with $\mu(e)$ fibers in edge $e \in E(G)$ and P be a set of paths in G. Consider the call control problem with capacity function $c(e) = \mu(e)$ for all $e \in E(G)$. Then we call the algorithm for this call control problem, select a maximum cardinality subset of paths $P' \subseteq P$, and remove the paths P' from P (i.e., $P \leftarrow P \setminus P'$). The procedure is repeated w times. Each selected subset is colored by a distinct color. The union of the w chosen sets is taken as the solution for the Max-PMC problem. It is known that if the call control algorithm has approximation ratio ρ, this iterative greedy algorithm for the Max-PMC problem has approximation ratio $\frac{1}{1-e^{-1/\rho}}$ [54]. The call control problem in spiders is polynomial time solvable (as shown in Section 4.1.4). Thus, $\rho = 1$, and the iterative greedy algorithm for the Max-PMC problem in spiders has an approximation ratio of $\frac{1}{1-e^{-1}}$ (about 1.58). Summarizing the above, we have the following theorem.

Theorem 4.2.5 *There is a 1.58-approximation algorithm for the Max-PMC problem in non-uniform spiders.*

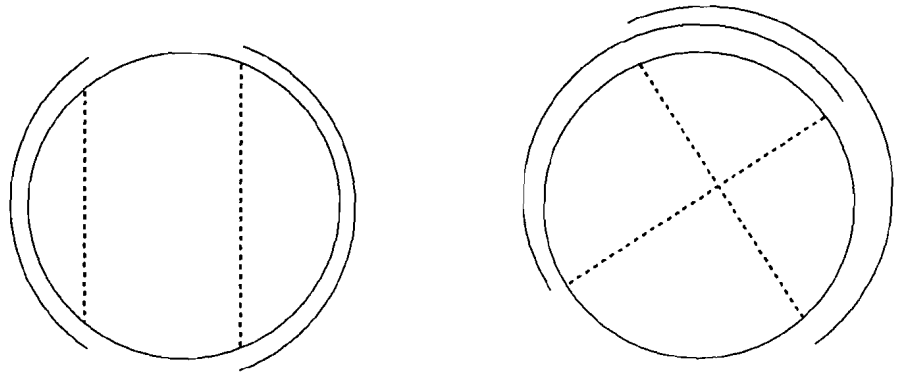
4.3 The Weighted Max-RPC Problem on Rings

In this section, we study the weighted maximum routing and path coloring problem in rings. Recall that for the weighted Max-RPC problem, we are given a set S of routing requests, a weight function $w : S \rightarrow \mathbb{Z}^+$, and a set of k colors. Each request can be routed either clockwise or counterclockwise. A feasible solution is a subset $S' \subseteq S$ such that each request $(s,t) \in S'$ is routed (through one of the two possible routes) and assigned one of the k colors, with no two requests using the same color if they are routed through the same edge. The goal is to find a feasible solution which has the maximum total weight. For a set of routing requests S, we define $w(S) = \sum_{s \in S} w(s)$ as the total weight of all the requests in S. We define $w(p) = w(s)$ if p is the routing path for s. For a set P of paths, we define $w(P) = \sum_{p \in P} w(P)$ as the total weight of all the paths in P. The optimal solution is denoted by OPT (which is used for both the set of requests and the total weight of the requests).

The following proposition will be used several times.

Proposition 4.3.1 *[38] The weighted Max-RPC problem can be solved optimally in polynomial time for chain networks.*

Our algorithm first discards one edge e from the given ring, and solves the weighted Max-RPC problem on the obtained chain, using the optimal algorithm of [38] for chains. The result is refined by considering every edge e on the ring as a candidate for deletion. Note that this second step is not necessary in the unweighted case [98], but is critical in the weighted case. Our algorithm uses a second approach, namely the maximum weight matching method, to see if a better result is possible. A (weighted) *compatible graph* G_c is constructed as follows. Each routing request on the ring corresponds to a node in G_c with the same weight. Two nodes are adjacent in G_c if and only if the corresponding requests are parallel. (Two requests are parallel if and only if their end-points do not interleave on the ring, otherwise are crossing, see Figure 4.6. A pair of parallel requests may be routed without overlapping, while a pair of crossing requests cannot.) In addition, for each node v just created, we add a duplicated node v^d with weight 0 and connect v and v^d by a new



(a) Parallel requests (b) Crossing requests

Figure 4.6: Parallel (compatible) and crossing (incompatible) requests.

edge. (The introducing of the dummy node v^d is necessary, since in an optimal solution, some request may be assigned a color that is not used by any other request.) To use the weighted matching algorithm, we define the weight of an edge in G_c to be the sum of the weights of its two end-points. A maximum weighted matching of cardinality at most k in G_c is found and the set of connection requests is routed and colored according to this matching, where k is the given number of colors. Our algorithm outputs the maximum of the two approaches. The detailed algorithm is shown in Figure 4.7.

Let OPT be an optimal solution (the selected subset of routing requests and the resulting paths). Given an edge e on the ring, the paths in OPT can be divided in two subsets (with respect to e): OPT_e^2 which contains all the paths on edge e (solid lines in Figure 4.8), and OPT_e^1 which contains all the paths not on edge e (dashed lines in Figure 4.8). Then $OPT = OPT_e^1 + OPT_e^2$, and $OPT_e^1 \leq SOL_1$ (since SOL_1 is the maximum of the optimal solutions for the chain obtained by avoiding an edge e, among all edges e on the ring). In the following discussion, we assume there exists an optimal solution OPT (with routes chosen and paths colored), and we will compare the weight of our solution to the weight of the optimal. Note that the assumption of having an optimal solution OPT is for the convenience of the proof. In the algorithm, we do not need the explicit knowledge of OPT.

For any two distinct edges e_1 and e_2 on the ring, let $P_{e_1} \subseteq OPT$ be the set of paths on e_1 (but not on e_2), $P_{e_2} \subseteq OPT$ be the set of paths on e_2 (but not on e_1), and $P_{e_1e_2} \subseteq OPT$

ALG4.8 Output $\max\{SOL_1, SOL_2\}$

SUB$_e$ Let e be an edge of the ring. Route requests so that the resulting set P of paths does not use e. Color a maximum weight subset of paths in P using an optimal algorithm for chains (Proposition 4.3.1). Denote the solution by SOL_e.

ALG$_c$ Call SUB_e for every edge e of the ring. Output the maximum weight SOL_1. ($\mathrm{SOL}_1 = \max_{e \in E} \mathrm{SOL}_e$).

ALG$_m$ Construct a compatible graph G_c. Find a maximum weight matching of cardinality at most k in G_c. Route each pair of matched requests in a compatible manner and assign a distinct color to the pair. The solution is denoted by SOL_2.

Figure 4.7: The algorithm for the weighted Max-RPC

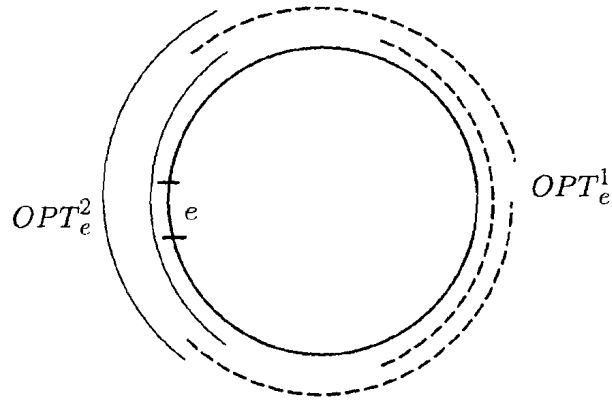


Figure 4.8: Different sets of paths in OPT.

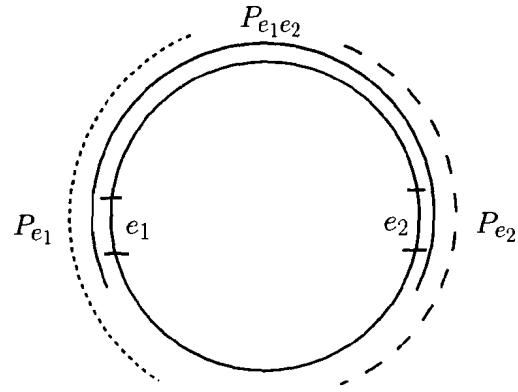


Figure 4.9: Illustration of the idea behind the proof of Lemma 4.3.2.

be the set of paths on both e_1 and on e_2 (see Figure 4.9). P_{e_1}, P_{e_2} and $P_{e_1e_2}$ are pairwise disjoint sets. We have the following lemma:

Lemma 4.3.2 $SOL_2 \geq w(P_{e_1}) + w(P_{e_2}) + w(P_{e_1e_2})$.

Proof Consider the paths in $P_0 = P_{e_1} \cup P_{e_2} \cup P_{e_1e_2}$. At most two paths in P_0 are assigned the same color, since paths in P_0 pass through at least one of the two edges e_1 and e_2, P_0 is a subset of OPT, and paths in OPT have valid colorings. Paths in $P_{e_1e_2}$ are assigned distinct colors, and do not share colors with paths in either P_{e_1} or P_{e_2}. Paths in P_{e_1} may share colors with paths in P_{e_2}. Let $P'_{e_1} \subseteq P_{e_1}$ (resp. $P'_{e_2} \subseteq P_{e_2}$) be the subset of paths each of which shares a color with some path in P_{e_2} (resp. P_{e_1}). Then $|P'_{e_1}| = |P'_{e_2}|$, and $|P_{e_1}| - |P'_{e_1}| + |P_{e_1e_2}| + |P_{e_2}| \leq k$. We can construct a matching M_P in G_c as follows. Let v_p be the node in G_c corresponding to the request assigned path p. For each path $p \in P'_{e_1}$, let $p' \in P'_{e_2}$ be the path which has the same color as p, and we include into M_P the edge between v_p and $v_{p'}$ in G_c. For each of the remaining paths $p \in P_{e_1e_2} \cup (P_{e_1} \setminus P'_{e_1}) \cup (P_{e_2} \setminus P'_{e_2})$, we include into M_P the edge between v_p and its duplicated node v_p^d in G_c. It is not hard to see that we have selected a total of $|P_{e_1}| - |P'_{e_1}| + |P_{e_1e_2}| + |P_{e_2}| \leq k$ edges from G_c, each selected pair can be colored using one color, and requests corresponding to paths in P_0 are all selected. Thus in a matching of cardinality at most k, we can have at least all the requests corresponding to paths in P_0, and the selected requests can be colored by k colors. The matching M_P has weight $w(P_{e_1}) + w(P_{e_2}) + w(P_{e_1e_2})$. The lemma is true since SOL_2 is

a maximum weight cardinality-k matching, and thus has weight at least equal to the weight of the matching M_P just constructed. □

From Lemma 4.3.2, we have the following observations. If $w(P_{e_1e_2}) = 0$ (i.e., no path is on both e_1 and e_2), then $OPT^2_{e_1} = w(P_{e_1})$, $OPT^2_{e_2} = w(P_{e_2})$, and $\min\{OPT^2_{e_1}, OPT^2_{e_2}\} \leq \frac{SOL_2}{2}$. Assume $OPT^2_{e_1} \leq OPT^2_{e_2}$ (the other case is symmetric),

$$\begin{aligned} OPT &= OPT^1_{e_1} + OPT^2_{e_1} \\ &\leq SOL_1 + 0.5SOL_2 \\ &\leq 1.5SOL. \end{aligned}$$

In other words, $SOL \geq OPT/1.5$. This gives a good bound on SOL. For an optimal solution OPT, there may not exist two edges e_1 and e_2 such that $w(P_{e_1e_2}) = 0$ holds. However, we can show that for any optimal solution OPT, there exists two distinct edges e_1 and e_2, and another optimal solution $OPT1$ such that $OPT1 = OPT$ and no path in $OPT1$ passes through both e_1 and e_2. Then, we use Lemma 4.3.2 to show our algorithm has an approximation ratio of 1.5. In what follows, we will try to find an $OPT1$ and two edges e_1 and e_2 that suit for this purpose.

To describe the procedure, we need some more definitions. Each path p on the ring corresponds to an arc from some node x to y along the ring in the clockwise direction. We call x the *left-node* and y the *right-node* of p, and denote them by p^L and p^R, respectively. Given an edge e on the ring, a path p in a set of paths P on e has the left-most left-node (with respect to e) if p^L is closest to e along the countclockwise direction. Similarly, a path p in a set of paths P has the right-most right-node (with respect to e) if p^R is closest to e along the clockwise direction. A node x is on the left of node y if y is on the arc from x towards e in the clockwise direction.

Given an optimal solution OPT, the *identifying* procedure works as follows:

1. Pick up an arbitrary edge e_1 of the ring, and repeat Steps 2-4 until return.
2. If there is no path on e_1, return.
3. If there is only one path p on e_1, then identify any edge not on p as e_2, and return.
4. Otherwise, there are at least two paths on e_1. Let p_1 be the path on e_1 with the left-most left-node, and p_2 be the path on e_1 with the right-most right-node (break tie arbitrarily).

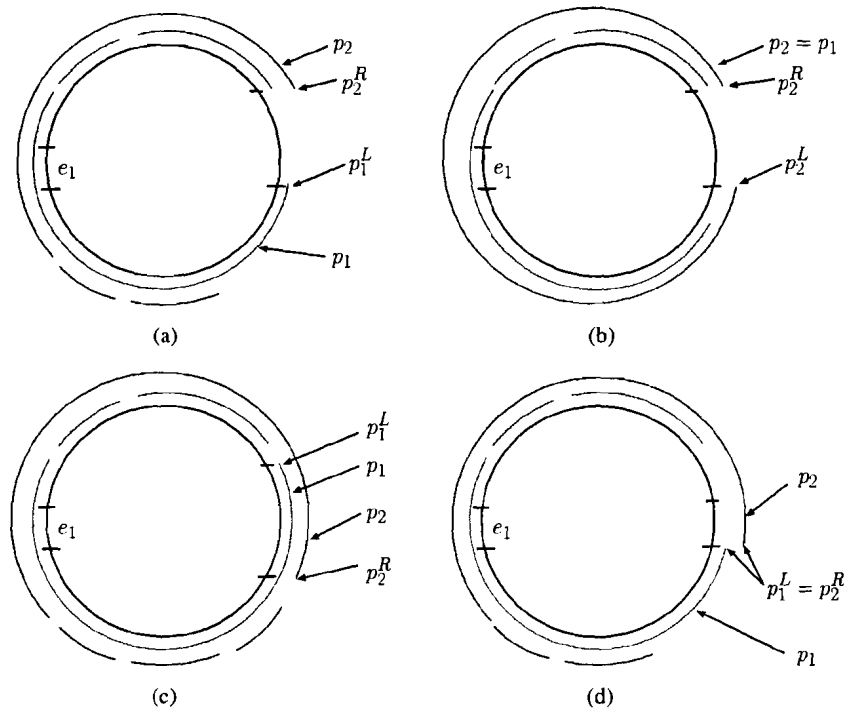


Figure 4.10: The procedure for identifying two edges on the ring.

(a) If p_2^R is on the left of p_1^L (see Figure 4.10(a)), or p_1 is the same path as p_2 (Figure 4.10(b)), then identify any edge on the clockwise segment from p_2^R to p_1^L as e_2, and return.

(b) Otherwise, p_1 and p_2 cover the whole ring (see Figures 4.10(c) and 4.10(d)). Re-route p_1 and p_2 (using the only other possible route), then switch the colors of them. Go back to Step 2.

The identifying procedure will clearly terminate, since initially there are a finite number paths on e_1, and each time Step 4(b) is executed, the number of paths on e_1 is reduced by two. It is easy to see that when the procedure terminates, we have re-routed some of the paths in OPT (but did not change the set of accepted requests) and identified an edge e_2 such that no path passes through both e_1 and e_2. The paths p_1 and p_2 found in Step 4(b) are called a *mutual-support pair*. Note that the requests corresponding to p_1 and p_2 are parallel. However, they were routed in an incompatible manner in OPT. (It may seem

strange that OPT routes requests in such a way. However, we do not make any assumption on how OPT works as long as the result is optimal.) For any such pair, we have re-routed them avoiding edge e_1, exchanged their colors, and kept the paths which share colors with them unchanged, without changing the color of any other path. The resulting paths still have valid colorings. The weight of OPT is not changed, but some of the weight of $OPT^2_{e_1}$ has been shifted to $OPT^1_{e_1}$. Let $OPT1$ be the set of paths obtained after the procedure. Then $OPT = OPT1$. We are now comparing OPT with SOL obtained using $ALG4.8$.

Theorem 4.3.3 *The algorithm ALG4.8 achieves an approximation ratio of* 1.5.

Proof Consider any optimal solution and apply the identifying procedure. If the procedure returns in Step 2, then no path in the resulting $OPT1$ passes through edge e_1, and $OPT1 = OPT1^1_{e_1} \leq SOL_1$. Thus SOL is optimal. If the procedure returns in either Step 3 or Step 4(a), then we have identified two edges e_1 and e_2 such that no path in $OPT1$ passes through both e_1 and e_2. Using Lemma 4.3.2, we can show that $OPT1 \leq 1.5SOL$. Summarizing the above, $OPT \leq 1.5SOL$ in all cases. Thus, our algorithm achieves an approximation ratio of 1.5.

The algorithm clearly runs in polynomial time, since the Max-PC algorithm for chains and the maximum weight cardinality k matching algorithm both run in polynomial time, and we call the Max-PC algorithm for chains only a polynomial number of times. □

The approximation ratio of 1.5 achieved by our algorithm for the weighted Max-RPC is worse than the 4/3 ratio achieved for the (unweighted) Max-RPC problem in [33]. The algorithm of [33] used a combination of the cut-one-link method and an advanced version of the iterative greedy method. As mentioned earlier, the iterative greedy method may be more efficient than the maximum matching method. It is not clear whether the approximation ratio of our algorithm can be improved if we use the iterative greedy method instead of the maximum matching method. It seems that the analysis of [33] cannot be extended to the weighted case.

4.4 Summary

We have shown that the call control problem is NP-hard and MAX SNP-hard even in depth-2 trees with edge capacities one or two. We give polynomial time optimal algorithm

for the call control problem in double-stars which are special depth-2 trees. These results suggest that depth-2 trees are a boundary topology for which the call control problem is in P or NP-hard, depending on the node degrees of the trees. We give polynomial time optimal algorithm for the call control problem in spiders. We also give 2-approximation and 3-approximation algorithms for the weighted call control problem in depth-2 and depth-3 trees, respectively. We show that the weighted call control problem is optimally solvable in arbitrary trees if all the paths contain a same node of the tree. We show that the weighted call control problem in any graphs can be solved optimally if all the paths have length at most 2.

We have shown that the Min-PMC and Max-PMC problems are NP-hard in k-fiber (k odd) stars. This should be contrasted to the even k case which can be solved optimally. We give optimal algorithms for the following problems: the Min-PMC and Max-PMC problems in non-uniform stars with even fibers, the Min-PMC and Max-PMC problems in k-fiber (k even) spiders. We also obtain a 1.58-approximation algorithm for the Max-PMC problem in spiders with non-uniform fibers, using our call control algorithm for spiders as a subroutine.

We have given a 1.5-approximation algorithm for the weighted Max-RPC problem in rings. This improves the previous 1.58-approximation algorithm. Note that the Max-PC, Max-RPC, and the weighted Max-PC problems in rings can all be approximated with ratio better than 1.5. It would be interesting to design approximation algorithm with ratio less than 1.5 for the weighted Max-RPC problem in rings.

Chapter 5

Branch/Carving Decomposition Based Algorithms

In previous chapters, we have given efficient algorithms for the path coloring problem and the call control problem in various networks. Our algorithms only work on the topologies for which they are designed, and the techniques do not seem to be applicable in other topologies, or different problems in the same topology. Algorithms that work for a broader class of problems in a broader class of topologies are highly desirable. Most of our algorithms in previous chapters give only approximate solutions that are constant factors away from optimal solutions. Although approximate solutions can be computed in polynomial time, in some applications, an optimal solution is desired even at the cost of exponential computation time. Recently, there are increased interests in the exact algorithms for optimization problems. Many of the exact algorithms use dynamic programming method based on a tree/branch/carving decomposition of the graph. A graph of small treewidth/branchwidth/carvingwidth admits efficient dynamic programming algorithms for many NP-hard problems on the graph. A key step in these algorithms is to find a tree/branch/carving decomposition of small width for the graph. In this chapter, we propose efficient algorithms for computing optimal branch/carving decomposition of planar graphs. The contents of Sections 5.1 $\sim$ 5.4 in this chapter are joint work with Qian-Ping Gu, Marjan Marzban, Hisao Tamaki, and Yumi Yoshitake, and appeared in the Proc. of the 10th SIAM Workshop on Algorithm Engineering and Experiments (ALENEX'08) [24].

In Section 5.1, we review some related work on optimal branch decompositions of planar

graphs. We give formal definitions of branchwidth and branch decomposition in Section 5.2. All known algorithms for the planar branch decomposition use Seymour and Thomas algorithm (called ST Procedure for short in what follows) which, given an integer β, decides whether G has the branchwidth at least β or not in $O(n^2)$ time. In Section 5.3, we propose efficient implementations of ST Procedure. We first review ST Procedure and give some observations which provide the base of our efficient implementations, then describe our implementations, and finally present the computational results. The computational results of our implementations show that the branchwidth of a planar graph can be computed in a practical time and memory space for some instances of size about one hundred thousand edges. Previous studies report that a straightforward implementation of the algorithm is memory consuming, which could be a bottleneck for solving instances with more than a few thousands edges. Our results suggest that with efficient implementations, the memory space required by the algorithm may not be a bottleneck in practice. Applying our implementations, an optimal branch decomposition of a planar graph of size up to several thousands edges can be computed in a reasonable time, using the edge-contraction method which runs in $O(n^3)$ time [66]. We describe the edge-contraction method and the computational results in Section 5.4.

Although the edge-contraction method can compute an optimal branch decomposition for planar graphs of practical size in a reasonable time, it is still time consuming for graphs with larger size. In Section 5.5, we propose divide-and-conquer based algorithms of using ST Procedure to compute optimal branch decompositions of planar graphs. Our algorithms have time complexity $O(n^3)$. Computational studies show that our algorithms are much faster than the edge-contraction algorithms and can compute an optimal branch decomposition of some planar graphs of size up to 50,000 edges in a practical time.

Branch-decomposition based algorithms have been explored as an approach for solving many NP-hard problems on graphs. Our results suggest that the approach could be practical. We will use the carving-decomposition based method to solve exactly the edge-disjoint paths problem in the next chapter.

5.1 Previous Work

The notions of branchwidth and branch decompositions are introduced by Robertson and Seymour [110] in relation to the more celebrated notions of treewidth and tree decompositions [108, 109]. A graph of small branchwidth (or treewidth) admits efficient dynamic programming algorithms for a vast class of problems on the graph [15, 28]. There are two major steps in a branch/tree-decomposition based algorithm for solving a problem: (1) computing a branch/tree decomposition with a small width and (2) applying a dynamic programming algorithm based on the decomposition to solve the problem. Step (2) usually runs in exponential time in the width of the branch/tree decomposition computed in Step (1). So it is extremely important to decide the branchwidth/treewidth and compute the optimal decompositions. It is NP-complete to decide whether the width of a given general graph is at least an integer β if β is part of the input, both for branchwidth [113] and treewidth [14]. When the branchwidth (treewidth) is bounded by a constant, both the branchwidth and the optimal branch decomposition (treewidth and optimal tree decomposition) can be computed in linear time [29, 31]. However, the huge constants behind the Big-Oh make the linear time algorithms only theoretically interesting.

One hurdle for applying branch/tree-decomposition based algorithms in practice is the difficulty of computing a good branch/tree decomposition because of the NP-hardness and huge hidden constants problems. Recently, the branch-decomposition based algorithms with practical importance for problems in planar graphs have been receiving increased attention [45, 57]. This is motivated by the fact that an optimal branch decomposition of a planar graph can be computed in polynomial time by Seymour and Thomas algorithm [113] and the algorithm is reported efficient in practice [70, 71]. Notice that it is open whether computing the treewidth of a planar graph is NP-hard or not. The result of the branchwidth implies a 1.5-approximation algorithm for the treewidth of planar graphs. Readers may refer to the recent papers by Bodlaender [30] and Hicks *et al.* [72] for extensive literature in the theory and application of branch/tree-decompositions.

Given a planar graph G of n vertices and an integer β, Seymour and Thomas give a decision algorithm which decides if G has a branchwidth at least β in $O(n^2)$ time [113]. Using ST Procedure as a subroutine, they also give an edge-contraction algorithm which constructs an optimal branch decomposition of G. The edge-contraction algorithm calls ST Procedure $O(n^2)$ times and runs in $O(n^4)$ time. Gu and Tamaki [66] give an improved

algorithm which calls ST Procedure $O(n)$ times and runs in $O(n^3)$ time to construct the branch decomposition. Hicks proposes a divide and conquer heuristic algorithm to reduce the number of calls for ST Procedure [71]. Computational studies show that the heuristic is effective in reducing the calls but has the time complexity of $O(n^4)$ [69, 71]. All known algorithms for computing the optimal branch decomposition of a planar graph rely on ST Procedure; thus, an efficient implementation of the procedure plays a key role in computing the branch decompositions. A straightforward implementation of ST Procedure requires $O(n^2)$ bytes of memory which is reported in [70] a bottleneck for solving large instances with more than 5,000 edges. Hicks proposes memory friendly implementations in the cost of performing re-calculations and increasing the running time of ST Procedure to $O(n^3)$ [70]. The time and memory space required by ST Procedure limit the size of planar graphs for which the optimal branch decompositions can be computed in practice. Hicks reports that the edge-contraction algorithm of [113] can solve some instances of about 2,000 edges and the divide and conquer method can solve some instances of about 7,000 edges in a practical time [69, 71].

5.2 Optimal Branch Decomposition of Planar Graphs

We give formal definitions of several terms that are mainly used in this chapter. Again, terms not defined here may be found in a standard textbook on graph theory. Let G be a graph. A *branch decomposition* of G is a tree T_B such that the set of leaves of T_B is $E(G)$ and each internal node of T_B has node degree 3. For each edge e of T_B, removing e separates T_B into two sub-trees. Let E' and E'' be the sets of leaves of the subtrees. The width of e is the number of vertices of G incident to both an edge in E' and an edge in E''. The width of T_B is the maximum width of all edges of T_B. The *branchwidth* of G is the minimum width of all branch-decompositions of G.

The algorithms of Seymour and Thomas [113] for branchwidth and branch decomposition are based on another type of decompositions called *carving decompositions.*

A carving decomposition of G is a tree T_C such that the set of leaves of T_C is $V(G)$ and each internal node of T_C has node degree 3. For each edge e of T_C, removing e separates T_C into two sub-trees and the two sets of the leaves of the sub-trees are denoted by V' and V''. The width of e is the number of edges of G with both an end vertex in V' and an end vertex in V''. The width of T_C is the maximum width of all edges of T_C. The *carvingwidth*

of G is the minimum width of all carving decompositions of G. Notice that the carving decomposition is defined for more general graphs in [113]. The definition allows positive integer lengths on edges of the graphs. The width of e in T_C for the weighted graph is defined as the sum of lengths of edges with an end vertex in V' and an end vertex in V''.

Let G be a planar graph with a fixed embedding. Let $R(G)$ be the set of faces of G. The *medial graph* [113] $M(G)$ of G is a planar graph with an embedding such that $V(M(G)) = \{u_e | e \in E(G)\}$, $R(M(G)) = \{r_s | s \in R(G)\} \cup \{r_v | v \in V(G)\}$, and there is an edge $\{u_e, u_{e'}\}$ in $E(M(G))$ if the edges e and e' of G are incident to a same vertex v of G and they are consecutive in the clockwise (or counter clockwise) order around v. $M(G)$ in general is a multigraph but has $O(|V(G)|)$ edges. Seymour and Thomas [113] show that the carvingwidth of $M(G)$ is exactly twice the branchwidth of G and an optimal carving decomposition of $M(G)$ can be translated into an optimal branch decomposition of G in linear time. To decide whether a planar graph G has the branchwidth at least an integer β, ST Procedure actually decides whether $M(G)$ has the carvingwidth at least 2β.

Proposition 5.2.1 *(Seymour and Thomas [113]) Given a planar graph G of n vertices and an integer β, $bw(G) = cw(M(G))/2$, ST Procedure decides if $bw(G) \geq \beta$ by computing if $cw(M(G)) \geq 2\beta$ in $O(n^2)$ time, and an optimal carving decomposition of $M(G)$ can be translated into an optimal branch decomposition of G in $O(n)$ time.*

A face $r \in R(G)$ and an edge $e \in E(G)$ are incident to each other if e is a boundary of r in the embedding. Notice that an edge e is incident to exactly two faces. For a face $r \in R(G)$, a vertex v is incident to r if v is an end vertex of an edge incident to r. For a face $r \in R(G)$, let $V(r)$ and $E(r)$ be the sets of vertices and edges incident to r, respectively. For a vertex $v \in V(G)$, let $E(v)$ be the set of edges incident to v.

The *planar dual* G^* of G is defined as that for each vertex $v \in V(G)$, there is a unique face $r_v^* \in R(G^*)$; for each face $r \in R(G)$, there is a unique vertex $v_r^* \in V(G^*)$; and for each edge $e \in E(G)$ incident to r and r', there is a unique edge $e^* = \{v_r^*, v_{r'}^*\} \in E(G^*)$ which crosses e.

A *walk* in a graph G is a sequence of edges $e_1, e_2, ..., e_k$ of G, where $e_i = \{v_{i-1}, v_i\}$ for $1 \leq i \leq k$. A walk is *closed* if $v_0 = v_k$. The length of a walk is the number of edges in the walk. For two vertices u and v in a graph G, the distance $d(u, v)$ is the minimum length of all walks between u and v. The walk with distance $d(u, v)$ is a shortest path between u and v.

5.3 Empirical Study on Branchwidth of Planar Graphs

In this section, we propose efficient implementations of ST Procedure. Our implementations can be classified into two groups. Group (1) does not perform re-calculations and runs in $O(n^2)$ time. The most memory efficient implementation in this group can compute the branchwidth of some instance of size up to one hundred thousand edges with 500Mbytes memory and in a couple of hours. Group (2) performs re-calculations and can compute the branchwidth of the instance of one hundred thousand edges with 200Mbytes of memory. The implementations in Group (2) may run in $O(n^3)$ time in the worst case. All of our implementations still use $O(n^2)$ bytes of memory. However, the constants behind the Big-Oh are much smaller than those in a straightforward implementation. In contrast, the results of this thesis and those of [70] show that straightforward implementations can only handle instances of size up to about 5,000 edges within 1Gbytes of memory. Our most time efficient implementation is faster than the straightforward one by a factor of $3 \sim 15$. Compared with the previous memory friendly implementations of [70], our most memory efficient implementations of Group (1) and Group (2) use at most 1/4 memory and 1/8 memory and run faster by a factor of $100 \sim 400$ and a factor of $100 \sim 200$, respectively. Notice that the CPU used in [70] has frequency 194MHz and the CPU used for testing our implementations has frequency 3.06GHz, so we need to keep in mind this difference of speed when we compare the running time.

The results of this section suggest that the memory size required by ST Procedure may not be a bottleneck for computing the branchwidth and optimal branch decomposition of a planar graph in practice. Our implementations also imply more efficient algorithms which call ST Procedure to find the optimal branch decompositions. We will discuss the decomposition algorithms in Sections 5.4 and 5.5.

5.3.1 Seymour and Thomas Procedure

We give a brief review of ST Procedure and readers may refer to [113] for more details of the decision procedure. ST Procedure is often called the *rat-catching* algorithm because it can be intuitively described by a rat catching game introduced in [113]. We first review the game and then give a formal description of ST Procedure.

Rat catching game

In this game, there are two players, a rat and a rat-catcher. The game is on a planar graph G of a fixed embedding, with a face and an edge of G interpreted as a room and a wall of a room, respectively. The rules for the game are as follows.

(R1) The rat-catcher selects a room.

(R2) The rat selects a corner of a room (a vertex of G).

(R3) The rat-catcher selects a room adjacent to the current room and moves to the wall between the two rooms (the edge of G incident to the current face and the selected face). The rat-catcher generates a noise of a fixed level that may make walls noisy. The condition of making a wall noisy will be given later.

(R4) The rat moves to a different corner via walls or stays at the current corner. The rat can not use a noisy wall but can use as many quiet walls as possible in one move.

(R5) The rat-catcher moves to the room it selected and can not change its mind to move back to the previous room. The rat-catcher keeps making noise.

(R6) If the rat is in a corner, all walls incident to the corner are noisy, and the rat-catcher is in a room with this corner, then the rat-catcher catches the rat and wins the game. Otherwise goto (R3).

Now we give the condition on a wall becoming noisy. For the planar dual G^* of G, let v_r^* and e^* be the vertex and edge of G^* corresponding to the face r and edge e of G, respectively. Let k be the noise level produced by the rat-catcher. When the rat-catcher is on edge e, edge f is noisy if and only if there is a closed walk of length smaller than k containing edges e^* and f^* in G^*. Similarly, when the rat-catcher is in face r, edge f is noisy if and only if there is a closed walk of length smaller than k containing vertex v_r^* and edge f^* in G^*. The rat-catcher wins the game if the rat is at a vertex v with node degree smaller than k and the rat-catcher is in a face incident to v. The rat wins the game if there is a scheme by which the rat can escape from the rat-catcher for ever. We use $RC(G,k)$ to denote the rat catching game on G and k. Seymour and Thomas show that the rat wins the game $RC(G,k)$ if and only if G has carvingwidth at least k and give ST Procedure which, given G and k, computes the outcome of the game $RC(G,k)$ [113].

ST Procedure

Now we present ST Procedure using the language of the game $RC(G,k)$. Our presentation is different from the original one which is based on a notion called antipodality [113]. For a graph G with maximum node degree at least k, the rat always wins the game $RC(G,k)$ because the rat will never get caught if it stays at a vertex with node degree at least k. So we assume that G has maximum node degree smaller than k in the following discussion. Given G and k, ST Procedure computes an escaping scheme for the rat or decides no such scheme exists. The escaping scheme is represented by a collection of vertex subsets and subgraphs of G by which the rat can escape from the rat-catcher for ever. The collection contains a non-empty subset of vertices of G (a subset of corners) for every face and a non-empty subgraph of G (a subset of corners and quiet walls) for every edge.

Given G and k, we define G_e to be the subgraph of G obtained by deleting noisy edges from G when the rat-catcher is on edge e. More specifically, $V(G_e) = V(G)$ and

$$E(G_e) = \{f \mid \text{every closed walk of } G^* \text{ containing } e^* \text{ and } f^* \text{ has length at least } k\}.$$

Notice that every edge of G_e is quiet when the rat-catcher is on e. For each face $r \in R(G)$, we define

$$S_r = \{(r,v) | v \in V(G)\} \text{ and } S = \cup_{r \in R(G)} S_r.$$

For each edge $e \in E(G)$, we define

$$T_e = \{(e,C) | C \text{ is a connected component of } G_e\}.$$

Let $T = \cup_{e \in E(G)} T_e$. Then the game $R(G,k)$ can be described by a bipartite graph $H(G,k)$, where the vertex set of $H(G,k)$ is $S \cup T$ and there is an edge between $(r,v) \in S$ and $(e,C) \in T$ if face r is incident to edge e and v is a vertex of C. The vertices of $H(G,k)$ can be interpreted as the states of the game: a $(r,v) \in S$ represents that the rat-catcher is in face r and the rat is at vertex v, and a $(e,C) \in T$ expresses that the rat-catcher is on edge e and the rat is at a vertex of C and can use the edges of C to move. The edge between $(r,v) \in S$ and $(e,C) \in T$ indicates the possible state transitions of the game: when the rat is at v and the rat-catcher moves from face r to edge e, the game state transits from (r,v) to (e,C); or when the rat is at some vertex of C and moves to v, and the rat-catcher moves from edge e to face r, the game state transits from (e,C) to (r,v).

A game state of $R(G,k)$ is called a *losing state* if the rat will lose the game at the state. To compute an escaping scheme for the rat, ST Procedure deletes the losing states from $H(G,k)$. For a face $r \in R(G)$ and a vertex v incident to r ($v \in V(r)$), (r,v) is a losing state because the rat gets caught if the rat is at v and the rat-catcher is in r. For an edge e incident to face r, (e,C) is a losing state if for every vertex v of C, (r,v) is a losing state. To see this, assume that the rat-catcher is on edge e and the rat is at some vertex of C. The rat-catcher may move to r or r', the other face incident to e, in the next step. In either of the moves, the rat can only move to a vertex v of C. If the rat-catcher moves to r then the game transits to (r,v) at which the rat will get caught. If the rat-catcher moves to r' then the rat is at some vertex of C. The rat-catcher can move back to e and then to r, and the rat will get caught. Similarly, if (e,C) is a losing state then for every face r incident to e and every vertex v of C, (r,v) is a losing state.

For every edge $e \in E(G)$, ST Procedure initializes set X_e to include all states of T_e. For every face $r \in R(G)$, ST Procedure initializes set X_r to include all states of S_r and then deletes (r,v) from X_r for every $v \in V(r)$. After this initial deletion step, for each face r and each edge e incident to r, if there is a state (e,C) such that for every vertex v of C state (r,v) has been deleted, then the state (e,C) is deleted from X_e. If this deletion is done then for the other face r' incident to e, state (r',v) is deleted from $X_{r'}$ for every vertex v of C. This deletion may result in further deletions of losing states. The deletion process is repeated until no further deletion is possible. It is shown in [113] that graph G has carvingwidth at least k if and only if after the deletion process finishes, X_r and X_e are not empty for every $r \in R(G)$ and every $e \in E(G)$. The collection of non-empty X_r and X_e for every face r and every edge e is an escaping scheme for the rat. Below is a simplified version of the formal description of ST Procedure [113]. We remark that ST Procedure decides if the carvingwidth is at least k for more general planar graphs. It allows weighted input graphs with positive integer lengths on edges.

ST_Procedure

Input: A non-null connected planar graph G with a fixed embedding, a planar dual G^* of G, an integer $k \geq 0$.

Output: Decides if G has carvingwidth at least k.

1. If the maximum node degree of G is at least k then output G has carvingwidth at least k and terminate.

2. For each face $r \in R(G)$, let $X_r = S_r$.

   For each edge $e \in E(G)$, compute G_e and let $X_e = T_e$. For each $(e, C) \in X_e$ and the faces r and r' incident to e, let $c(r, e, C) = |V(C)|$ and $c(r', e, C) = |V(C)|$, where $V(C)$ is the set of vertices of C.

3. For each face r and each state $(r, v) \in X_r$ with $v \in V(r)$, put (r, v) to a stack L and delete (r, v) from X_r.

4. If L is empty then goto the next step.

   Otherwise, remove a state x from L.

   Assume that $x = (r, v)$ is a state for a face ($x \in S$). For each edge e incident to r, find the state $(e, C) \in X_e$ such that C contains v. Decrease $c(r, e, C)$ by one. If $c(r, e, C)$ becomes 0 and $(e, C) \in X_e$ then put (e, C) to L and delete (e, C) from X_e.

   Assume that $x = (e, C)$ is a state for an edge ($x \in T$). If there is a face r incident to e such that $c(r, e, C) > 0$ then for each vertex v of C and $(r, v) \in X_r$ put (r, v) to L and delete (r, v) from X_r.

   Repeat this step.

5. If X_r is non-empty for every $r \in R(G)$ and X_e is non-empty for every $e \in E(G)$ then output G has carvingwidth at least k, otherwise output G has carvingwidth smaller than k.

Notice that we can stop ST Procedure and conclude that the rat loses the game when some X_r becomes empty. The reason is that if all states of X_r are deleted, all states of X_e for e incident to r will be deleted; then all states for face r' incident to e will be deleted; and finally all states for every face and edge will be deleted. Similarly, the rat loses the game if some X_e becomes empty.

To compute G_e for each e, ST Procedure needs to find the quiet edges when the rat-catcher is on edge e. An edge f is quiet and will be included in G_e if every closed walk in G^* that contains e^* and f^* has length at least k. More specifically, let $e^* = \{u^*, v^*\}$ and $f^* = \{x^*, y^*\}$. Edge f is included in G_e if and only if $d(u^*, x^*) + d(v^*, y^*) + 2 \geq k$ and $d(u^*, y^*) + d(v^*, x^*) + 2 \geq k$. A solution for the all-pairs shortest path problem of G^* will suffice for the distances required in computing G_e for all $e \in E(G)$.

Theorem 5.3.1 *(Seymour and Thomas [113])*
Given a planar graph G of n vertices and integer $k \geq 0$, ST Procedure decides if G has carvingwidth at least k or not using graph $H(G,k)$ in $O(n^2)$ time and $O(n^2)$ bytes of memory.

To decide the branchwidth of G, the input to ST Procedure is the medial graph $M(G)$ and the branchwidth of G is $k/2$ if the carvingwidth of $M(G)$ is k.

Observations for efficient implementations

We give some observations on the game $RC(G,k)$ that can be used for efficient implementations of ST Procedure. By the definition of the game $RC(G,k)$, a state (r,v) is a losing state if $v \in V(r)$ for G with maximum node degree smaller than k. ST Procedure makes use of this sufficient condition to delete the losing states at the initial step of the deletion process for each face r. We observe that if we can find and delete more losing states at the initial step for each face r, then ST Procedure may run faster and use less memory. We prove the following sufficient condition for finding more losing states.

Lemma 5.3.2 *For a face r and a vertex v in graph G with maximum node degree smaller than k, (r,v) is a losing state if there exist two faces s and t incident to v such that there are*
(1) a closed walk W_1 in G^* with length smaller than k that consists of the shortest path from v_r^* to v_s^*, the clockwise walk from v_s^* to v_t^* around r_v^*, and the shortest path from v_t^* to v_r^*; and (2) a closed walk W_2 in G^* with length smaller than k that consists of the shortest path from v_r^* to v_s^*, the counter-clockwise walk from v_s^* to v_t^* around r_v^*, and the shortest path from v_t^* to v_r^*.

Proof Assume that W_1 and W_2 exist. Then for every edge e incident to v in G, e^* is either in W_1 or W_2 and e is noisy when the rat-catcher is in r. Let $e_1^*, ..., e_j^*$ be the edges in the shortest path from v_r^* to v_s^*. Assume that the rat is at v and the rat-catcher is in r. Since all edges incident to v are noisy, the rat can not move away from v. Next, the rat-catcher can move to edge e_1. Since all edges incident to v are noisy when the rat-catcher is on e_1, the rat has to stay at v. Similarly, the rat has to stay at v when the rat-catcher is on edge $e_i, 1 \leq i \leq j$. So the rat-catcher can move to face s using edges $e_1, ..., e_j$ and catch the rat at v. □

Once the shortest paths from v_r^* to all other vertices of G^* have been computed, it is easy to see the time for checking if (r, v) is a losing state by the condition of Lemma 5.3.2 is proportional to the node degree of v. Therefore, it takes $O(n)$ time to check (r, v) for a face r and all $v \in V(G)$. For each face r, let $U(r)$ be the set of vertices that for every $v \in U(r)$, (r, v) is a losing state computed by the sufficient condition of Lemma 5.3.2. From Theorem 5.3.1, we have the following result.

Theorem 5.3.3 *Given a planar graph G of n vertices and $k \geq 0$, ST Procedure decides if G has carvingwidth at least k in $O(n^2)$ time and $O(n^2)$ bytes of memory when the losing states (r, v), $v \in U(r)$, are deleted at the initial step of the deletion process for each face r.*

For each face $r \in R(G)$, we define G_r to be the subgraph of G obtained by deleting the noisy edges from G when the rat-catcher is in face r. That is, $V(G_r) = V(G)$ and

$$E(G_r) = \{f \mid \text{every closed walk of } G^* \text{ containing } v_r^* \text{ and } f^* \text{ has length at least } k\}.$$

Notice that every edge of G_r is quiet when the rat-catcher is in r. Recall that G_e is the quiet subgraph of G when the rat-catcher is on edge e. Our next observation is that for every edge e incident to face r, $E(G_r) \subseteq E(G_e)$, because v_r^* is an end vertex of e^* and therefore the set of closed walks of G^* containing vertex v_r^* and edge f^* includes all closed walks of G^* containing edges e^* and f^*. From this, a component of G_r is a subgraph of some component of G_e. Hence, when the rat-catcher moves from face r to edge e and the rat is at any vertex of some component D of G_r, the component of G_e on which the rat can move is the same one which contains D as a subgraph. Thus, when the rat-catcher is in face r, the states of the game can be expressed by

$$S'_r = \{(r, D) | D \text{ is a connected component of } G_r\}.$$

Let $S' = \cup_{r \in R(G)} S'_r$. The game $RC(G, k)$ can be described by a bipartite graph $H'(G, k)$, where the vertex set of $H'(G, k)$ is $S' \cup T$ and there is an edge between $(r, D) \in S'$ and $(e, C) \in T$ if face r is incident to edge e and D is a subgraph of C. For a face r and a component D of G_r, (r, D) is a losing state if for every vertex v of D, (r, v) is a losing state. For an edge e incident to face r, state (e, C) is a losing state if for every component D of G_r that is a subgraph of C, (r, D) is a losing state. Similarly, if (e, C) is a losing state then for every face r incident to e and every component D of G_r that is a subgraph of C, (r, D) is a losing state. Summarizing the above and from Theorem 5.3.1, the following result holds.

Theorem 5.3.4 *Given a planar graph G of n vertices and $k \geq 0$, ST Procedure decides if G has carvingwidth at least k using graph $H'(G,k)$ in $O(n^2)$ time and $O(n^2)$ bytes of memory.*

When graph $H'(G,k)$ is used for the game $RC(G,k)$, X_r is initialized as S'_r for each face $r \in R(G)$ in ST Procedure. Compared with S_r, S'_r may have less game states and thus require less memory.

During the deletion process of ST Procedure, losing states are deleted from sets X_r and X_e. Our another observation is that the elements of X_e for an edge e incident to faces r and r' at a step of ST Procedure can be computed in $O(n)$ time from the elements of X_r and $X_{r'}$ at that step. This gives an option for implementing ST Procedure that does not keep but dynamically computes X_e from X_r and $X_{r'}$ during the deletion process.

Theorem 5.3.5 *Given a planar graph G of n vertices and $k \geq 0$, ST Procedure can decide if G has carvingwidth at least k or not in $O(n^3)$ time and $O(n^2)$ bytes of memory if for each edge e, X_e is not kept but dynamically computed during the deletion process.*

Proof For an edge e incident to faces r and r', the set X_e is needed when an element of X_r or $X_{r'}$ is deleted during the computation and when ST Procedure terminates. So, we can compute X_e in $O(n)$ time once there is an element deleted from X_r or $X_{r'}$. Since there are $O(n)$ elements in $X_r \cup X_{r'}$, X_e is computed $O(n)$ times. The total time for computing X_e for all $e \in E(G)$ is $O(n^3)$. From Theorem 5.3.1, the theorem holds. □

The re-calculation of edge data used in our implementations is different from the re-calculation in the previous study of [70], where face data are re-calculated for some faces and each re-calculation for a face r involves a computation of T_e for each e incident to r.

Finally, it is easy to see that if all states of S_r (or S'_r) for some face r are losing states then for every face r', all states of $S_{r'}$ ($S'_{r'}$) are losing states and the rat loses the game.

Observation 5.3.6 *If X_r becomes empty for some face r during the deletion process then graph G has carvingwidth smaller than k.*

By this observation we can terminate ST Procedure when some X_r becomes empty. This may save the computation time when the rat loses the game.

5.3.2 Efficient Implementations

Let G be a connected planar graph with a given embedding and $V(G) = \{v_1, ..., v_n\}$. We first describe a straightforward implementation (called *Naive*) of ST Procedure and then propose several improvements on the implementations of ST Procedure. Those improvements try to reduce both the memory space and running time of ST procedure.

Naive implementation

A straightforward implementation of ST Procedure would use graph $H(G, k)$ for deciding the outcome of the game $RC(G, k)$. We use the following data structure for graph $H(G, k)$ in Naive.

- For each face $r \in R(G)$, a Boolean array B_r (of n elements) is assigned such that $B_r[i]$ is used to indicate if $(r, v_i) \in X_r$ or not. A list of $|E(r)|$ elements is used to keep the edges incident to r.
- For each edge $e \in E(G)$, the two faces r and r' incident to e are kept. All components of G_e are kept in a list. Each component of G_e is given an index and component C_j is kept in the jth element of the list. The element of the list for C_j contains the set of vertices of C_j, $c(r, e, C_j)$, $c(r', e, C_j)$, and a Boolean variable indicating if (e, C_j) has been deleted from X_e or not. An integer array I_e (of n elements) is used to indicate which component a vertex is in. If v_i is a vertex of C_j then $I_e[i]$ is set to j.
- In addition to the face and edge data, a stack L is used and a distance matrix is kept for the all pairs shortest distances in the dual graph G^* of G.

It is easy to check that the Naive implementation runs in $O(n^2)$ time. A simple calculation shows that Naive implementation requires about $40n^2$ bytes of memory when G is a medial graph. Since there are many single vertex components in G_e and the operating system may have a minimum memory allocation size of 16 bytes, the memory usage in practice is close to $50n^2$ bytes.

Common improvements

We first describe two common improvements which are used in all of our efficient implementations. When we say *processing* a face r or an edge e, we mean deleting a losing state from X_r or X_e.

The first common improvement is that we define a processing order of the faces in our implementations. We put losing states (r, v) to the stack for only one face at a time. When there are losing states of multiple faces to be included to the stack, we group the losing states according to the faces, and give an order on the groups to be put to the stack. Only the group at the top of the order is put to the stack at a time. A face which has been processed is given a higher priority to be put to the stack. The processing order on the faces is used to define a subset $Q \subseteq R(G)$ and to restrict the rat-catcher moving within the faces of Q. Given a subset Q of $R(G)$, let $S_Q = \cup_{r \in Q} S_r$, $S'_Q = \cup_{r \in Q} S'_r$, and $T_Q = \cup_{e \in E(r), r \in Q} T_e$. We start with a small Q and perform the deletion process for the subgraph of $H(G, k)$ induced by the vertices of $S_Q \cup T_Q$ (or the subgraph of $H'(G, k)$ induced by the vertices of $S'_Q \cup T_Q$) until no deletion is possible. Then we enlarge Q by including a new face and repeat the deletion process. Q is enlarged gradually until $Q = R(G)$. By Observation 5.3.6, ST Procedure may stop at a small Q when the rat-catcher wins the game. Also, for a given subset Q, the losing states are deleted from X_r and X_e ($r \in Q, e \in E(r)$), and after the deletion, the data for X_r and X_e can be compressed before Q is enlarged. This helps in reducing the time and memory of ST Procedure.

The second common improvement is that we use a parsimonious data structure for edge data. We observe that there are many single vertex components in edge data. This makes the list of components for each edge very big. We keep the same face data as those in Naive. For each edge e, a component of G_e is called *non-trivial* if it has at least one edge otherwise called *trivial*. We only assign an index to a non-trivial component and keep a list of non-trivial components. We decide the integer type for I_e based on the number of non-trivial components in G_e. The length of the integer type for I_e is just big enough to encode the indices of non-trivial components of G_e. A trivial component $C = \{v_i\}$ is not kept in the list and $I_e[i]$ is used to indicate if (e, C) has been deleted from X_e or not. Further, if a non-trivial C_j has at least a constant fraction of n (δn) vertices then the set of vertices of C_j is not kept in the list. If there are at most a constant number (c) of non-trivial components then the sets of vertices of the components are not kept in the list. In these cases, when an access to vertices of a non-trivial component is needed, we check I_e to find the vertices of the component. It is easy to see that this does not increase the order of the time complexity of the implementation. A smaller δ saves more memory but may give a larger running time. Similarly, a larger c saves more memory but may increase the running time. We have chosen $\delta = 1/100$ and $c = 100$ in this study. A distance matrix is used to keep the all pairs shortest

distances. We decide the integer type for the distance matrix based on the input integer k to ST Procedure. When G is a medial graph, we can reduce the required memory size to about $4n^2$ bytes if one-byte integer arrays are used for each I_e and the distance matrix, and to about $7n^2$ bytes if two-byte integer arrays are used.

More improvements

Improvement A_1 This improvement is based on Theorem 5.3.3. In A_1, the elements (r, v), $v \in U(r)$, are deleted from X_r and put to the stack at the initial step of the deletion process for face r. From Lemma 5.3.2, $U(r) \supseteq V(r)$ and computational studies show that $|U(r)|$ is usually much larger than $|V(r)|$. Therefore, A_1 gives a room for improving both the running time and memory space.

Improvement D_1 The features of D_1 can be expressed by dynamic data creation and data compression. In D_1 the data for a face (edge) are created only when ST Procedure starts to process the face (edge). When some losing states are deleted, the face/edge data are compressed. More specifically, when ST Procedure is to perform the first deletion for a face r, $U(r)$ is computed and array B_r of n elements is created. After the losing states (r, v_i), $v_i \in U(r)$, are deleted from X_r, vertices of $V(G) \setminus U(r)$ are re-indexed and array B_r is compressed to indicate if (r, v_i) has been deleted for vertices v_i of $V(G) \setminus U(r)$ only. Similarly, when ST Procedure is to perform the first deletion for an edge e, G_e is computed and the edge data are created. Let r and r' be the two faces incident to e. We create two integer arrays I_e and I'_e for e. If the vertices of $V(G) \setminus U(r)$ have been re-indexed and B_r has been compressed then I_e is compressed accordingly. Similarly, array I'_e is compressed for the vertices of $V(G) \setminus U(r')$.
To calculate $U(r)$ and G_e, the shortest distances from vertex v_r^* to all other vertices in the planar dual graph G^* of G are needed for each face r in G. When a distance matrix is used to keep the shortest distances, we need to solve $|R(G)|$ single source shortest path problems. In D_1, the distance matrix is discarded. When we process a face r, we create the data for r and the data for I_e for all e incident to r. Since each edge is incident to two faces r and r', the total number of single source shortest path calculations is bounded by $2|E(G)|$. When G has n vertices and is a medial graph, $|R(G)| = n + 2$ and $|E(G)| = 2n$. From this, if the distance matrix is used, we need to solve $n + 2$ single source shortest path problems while we need to solve at most $4n$

such path problems if D_1 is applied.

Combining D_1 with A_1, the required memory size is now about $5n \times q$ bytes if one-byte integer arrays are used for I_e and I'_e and about $9n \times q$ if two-byte integer arrays are used, where q is the average of $|V(G) \setminus U(r)|$. For the Delaunay triangulation instances tested, q is less than $0.3n$ (instances dependent).

Improvement A_2 This improvement is based on Theorem 5.3.4. For each face r, instead of S_r, A_2 initializes X_r to include all states of S'_r.

Improvement A_3 This improvement is based on Theorem 5.3.5 and performs re-calculation for edge data. A_3 keeps the face data once they are created but keeps the edge data for only a pre-defined maximum number of edges. Once this number is reached A_3 starts to delete the entire X_e for some edge e. If a deleted X_e is needed again, X_e is re-computed from X_r, where r is incident to e.

Improvement D_2 In D_2, we use a bit vector B_r for the data of face r, with one bit for one element of X_r. The memory size for face data is 1/8 of that when a one-byte Boolean array is used. But more complex bit operations have to be used.

It is easy to check that all improvements except A_3 do not change the order of running time of ST Procedure. However, applying A_3, the running time of ST Procedure may become $O(n^3)$.

5.3.3 Computational Results

All of our efficient implementations use common improvements. In our implementations with any of improvements A_2, A_3 and D_2, improvements A_1 and D_1 are always used. We do not mention A_1 and D_1 explicitly in those implementations. We test Naive and Implementations A_1, A_1D_1, A_2, A_2D_2, A_3, A_3D_2, A_2A_3, and $A_2A_3D_2$. To show that our implementations work well for a broad class of planar graphs, three classes of instances are used in the test: one class is the benchmark instances from previous studies, and the other two classes are random planar graphs generated by two well-used software libraries, LEDA and PIGALE, respectively. More specifically, Class (1) of instances includes Delaunay triangulations of point sets taken from TSPLIB [107]. Those instances are used as test instances in the previous studies [70, 71]. The instances in Class (2) are generated by the LEDA

library [1, 90]. LEDA generates two types of planar graphs. One type of the graphs are the randomly generated maximal planar graphs and their subgraphs obtained from deleting some edges. Since the maximal planar graphs generated by LEDA always have branchwidth four, the subgraphs obtained by deleting edges from the maximal graphs have branchwidth at most four. The graphs of this type are not interesting for the study of branchwidth and branch decompositions. The other type of planar graphs are those generated based on some geometric properties, including Delaunay triangulations and triangulations of points uniformly distributed in a two-dimensional plane, and the intersection graphs of segments uniformly distributed in a two-dimensional plane. We will report the results on the intersection graphs. The instances in Class (3) are generated by the PIGALE library [4]. PIGALE randomly generates one of all possible planar graphs with a given number of edges based on the algorithms of [112]. We use Naive and our implementations to compute the carvingwidth of the medial graphs of the instances (i.e., the input graph to ST Procedure is not an instance itself but the medial graph of the instance). Our implementations are tested on a computer with Intel(R) Xeon(TM) 3.06GHz CPU, 2Gbytes physical memory and 8Gbytes swap memory. The operating system is SUSE LINUX 10.0, and the programming language we used is C++.

We compute an upper bound on the carvingwidth as the initial guessed input integer k to call ST Procedure. It is known that the branchwidth of a planar graph of n vertices is at most $\sqrt{4.5n}$ [57]. From this, $2\sqrt{4.5n}$ is an upper bound on the carvingwidth of the medial graph of an instance of n vertices. We follow a similar approach in [70] to compute another upper bound l: Let $M(G)$ be a medial graph of a planar graph G of n vertices. For each face r of $M(G)$ which corresponds to a vertex in G, we compute the eccentricity of v_r^* (the length of the longest shortest paths from v_r^* to all other vertices) in the planar dual $M(G)^*$. We initialize l as twice as the minimum eccentricity among all v_r^*'s. Finally, we take $k = \min\{2\sqrt{4.5n}, l\}$. Either the linear search or the binary search can be used to find the carvingwidth starting from the initial guessed k. In the linear search, when the rat-catcher wins, k is decreased by two and ST Procedure is called again until the rat wins the game. In the binary search, we call ST Procedure to search for the carvingwidth between k (upper bound) and the node degree of $M(G)$ (which is four and a lower bound). For the instances in Classes (1) and (2), the eccentricity-based guess is very close to the carvingwidth and k always takes the value of l. The linear search uses a smaller number of iterations to find the carvingwidth than the binary search. For instances in Class (3), the eccentricity-based

guess could be very large and k may take $2\sqrt{4.5n}$ for large instances. Since $2\sqrt{4.5n}$ is still far away from the carvingwidth, the binary search does a better job. One may run the linear search and binary search in parallel and take the results from the one which finishes earlier.

Computation time and memory

Table 5.1 shows the computation time of Naive and efficient implementations for the carvingwidth of the medial graphs of the instances in Class (1). In the table, *Itr* is the number of iterations in the linear search. Table 5.2 shows the memory size (in megabytes) of those implementations. Only the data for relatively large instances are given in the tables.

For the instances in Class (1), one-byte integer arrays are used for each edge and the distance matrix. The most time efficient implementation is A_1 which is faster than Naive by a factor of at least 10 and uses at most 1/10 memory of Naive. With more improvements, the memory requirement is further reduced but the running time is slightly increased. The effect of data compression in Improvement D_1 is significant. The memory used by A_1D_1 is only about $1/3 \sim 1/4$ of that by A_1. Improvement A_2 is effective in reducing the memory size. In general, the number of non-trivial components is small for both faces and edges, and thus the memory saving is big. The memory used by Improvement D_2 for face data is 1/8 of that when one-byte Boolean arrays are used. When the memory for face data becomes dominating, Improvement D_2 reduces memory requirement significantly. The most memory efficient implementation without re-calculation is A_2D_2 which is faster than Naive by a factor of $8 \sim 9$. For Instance *pla*33810 which has 101,367 edges (corresponding to 101,367 vertices in the input medial graph), A_2D_2 uses about 500Mbytes memory, which is about $\frac{1}{20}n^2$ bytes, where n is the number of vertices in the input medial graph. Compared with Naive, the memory saving is by a factor of about 1000.

Improvement A_3 performs re-calculation for edge data. The performance depends on the maximum number of edges that are kept. This maximum number can be chosen based on the size of available memory. In general, a larger maximum number gives an implementation which uses more memory but runs faster (less re-calculations). The maximum number of kept edges is 500 for the results presented unless otherwise stated explicitly. Among all implementations, $A_2A_3D_2$ is the most memory efficient one. $A_2A_3D_2$ is faster than Naive by a factor of $6 \sim 7$. For Instance *pla*33810, $A_2A_3D_2$ uses less than 200Mbytes memory, which is about $\frac{1}{50}n^2$ bytes. Compared with Naive, the memory saving is by a factor of about 2500. The memory used by Implementation $A_2A_3D_2$ can be further reduced to about

Table 5.1: Computation time (in seconds) of Naive and efficient implementations for Class (1) instances. An X in the table indicates that the implementation runs out of 2Gbyte memory for that instance.

Instances	Number of edges	bw	Itr	Computation time (in seconds)								
				Naive	A_1	A_1D_1	A_2	A_2D_2	A_3	A_3D_2	A_2A_3	$A_2A_3D_2$
pr1002	2,972	21	2	51.2	4.23	5.38	6.07	6.35	5.58	5.83	6.52	6.98
rl1323	3,950	22	2	95.7	6.47	8.0	9.19	9.56	8.65	9.05	12.3	13.1
d1655	4,890	29	2	158	11.3	15.0	17.3	17.6	15.7	16.7	21.6	22.3
rl1889	5,631	22	2	195	13.8	16.6	20.1	20.8	19.2	20.4	29.5	30.5
u2152	6,312	31	3	X	24.5	35.5	41.8	42.4	39.9	42	61.6	62.4
pr2392	7,125	29	2	X	21.1	25.8	32.1	32.8	31.8	31.3	49.1	50.4
pcb3038	9,101	40	2	X	31.6	41.3	50.8	51.9	44.6	49.7	83.2	74.2
fl3795	11,326	25	2	X	63.7	80.2	99	104	86.3	98	155	163
fnl4461	13,359	48	2	X	67.4	92.4	116	119	97.1	110	185	185
rl5934	17,770	41	2	X	151	197	245	249	213	241	385	391
pla7397	21,865	33	2	X	246	296	376	385	393	453	606	629
usa13509	40,503	63	4	X	X	1,061	1,359	1,371	1,241	1,386	2,154	2,165
brd14051	42,128	68	2	X	X	1,061	1,418	1,417	1,226	1,361	2,274	2,282
d15112	45,310	78	3	X	X	2,070	2,810	2,852	2,379	2,598	4,549	4,603
d18512	55,510	88	2	X	X	X	2,315	2,321	2,100	2,241	3,752	3,756
pla33810	101,367	100	5	X	X	X	12,379	12,614	X	14,747	20,482	21,734

Table 5.2: Memory usage (in megabytes) of Naive and efficient implementations for Class (1) instances. An X in the table indicates that the implementation runs out of 2Gbyte memory for that instance.

Instances	Number of edges	Maximum Memory Usage (Mbyte)								
		Naive	A_1	A_1D_1	A_2	A_2D_2	A_3	A_3D_2	A_2A_3	$A_2A_3D_2$
pr1002	2,972	413	39	16	8	8	10	9	8	7
rl1323	3,950	734	66	23	11	10	15	13	11	9
d1655	4,890	1,188	99	30	14	11	17	14	13	10
rl1889	5,631	1,424	130	46	18	14	28	21	16	12
u2152	6,312	X	161	41	17	14	26	20	16	13
pr2392	7,125	X	204	66	21	17	35	26	19	15
pcb3038	9,101	X	328	76	25	21	36	27	22	19
fl3795	11,326	X	504	132	58	42	66	63	40	23
fnl4461	13,359	X	698	158	39	32	66	43	33	26
rl5934	17,770	X	1,226	358	67	51	155	86	50	35
pla7397	21,865	X	1,850	436	123	85	238	144	83	44
usa13509	40,503	X	X	1,534	220	153	498	271	149	79
brd14051	42,128	X	X	1,600	215	149	580	283	149	82
d15112	45,310	X	X	1,795	227	156	508	256	156	86
d18512	55,510	X	X	X	284	198	706	328	194	106
pla33810	101,367	X	X	X	814	508	X	876	507	198

155Mbytes for Instance *pla*33810 with a slightly increase in the running time, if we keep at most 50 edges.

The instances in Class (2) are generated by the LEDA function *random_planar_graph* [1]. We have tested our implementations on instances of Delaunay triangulations and triangulations of points randomly distributed in a two-dimensional plane, and intersection graphs of segments. Our implementations have similar performances for the Delaunay triangulations and triangulations instances as those for the instances in Class (1). Table 5.3 gives the computation time and memory of Naive and Implementations A_1, A_2D_2, and $A_2A_3D_2$ for instances of intersection graphs of segments. In the table, *Itr* is the number of iterations in the linear search. The instances of intersection graphs of segments may have a large number of non-trivial components for edges and faces, and two-byte integer arrays are used to represent the edge data. Therefore, the memory usage is considerably larger than the Delaunay instances of the same size. As shown in the table, our efficient implementations are faster and use much less memory than Naive.

Instances of Class (3) are generated by the PIGALE library [4]. PIGALE provides a number of planar graph generators. Since 2-connected planar graphs are the most interesting class of graphs in the study of branchwidth and branch decompositions, we selected the function for generating 2-connected planar graphs. The function, given the number m of edges, randomly generates one of all possible 2-connected planar graphs of m edges. The output graph is usually a multi-graph with parallel edges. Since parallel edges are not interesting for branchwidth finding, we specify the function to produce simple 2-connected graphs. With a given m, the function outputs a 2-connected random planar graph with m' edges. Normally m' is smaller than m, since parallel edges are not kept and there are performance considerations [4]. Table 5.4 gives the computation time and memory of Naive and Implementations A_1, A_2D_2, and $A_2A_3D_2$. In the table, *Itr* is the number of iterations in the binary search. The instances in Class (3) may have a small number of non-trivial edge components, but we still use two-byte integer arrays for the edge data. For this class of instances, the eccentricity-based guess is usually bad. For example, the medial graph of Instance PI37730 has carvingwidth 12, but the eccentricity-based guess is 8974. For large instances tested, the binary search always finishes earlier than the linear search. The number of iterations in the binary search is about 10 for large instances, and this prohibits us from solving very large instances in a reasonable time. The memory usage for the PIGALE instances is very small, compared to the instances of Classes (1) and (2) even two-byte

integer arrays are used for edge data.

For the instances in Class (3), Implementation A_2D_2 is very memory efficient. This indicates that the numbers of non-trivial components for faces in those instances are small. The gap between the running time of Implementation A_1 and that of Implementation A_2D_2 is a little bigger for instances of this class than the gap for instances in the other two classes. This can be explained by the following reasons. Naive and Implementation A_1 keep the shortest distance matrix, while A_2D_2 discards the matrix. As analyzed in Improvement D_1, Naive and A_1 need to solve $n+2$ single source shortest path problems while A_2D_2 may need to solve $4n$ such problems, where n is the number of vertices of the input medial graph. A_2D_2 also needs to calculate the components of G_r for every face r and there is no such computation in Naive and A_1. Notice that each of the shortest distances calculation, the computation of components of G_r for all r, and the deletion process takes $O(n^2)$ time. For instances of Classes (1) and (2), the time of deletion process is larger than the sum of the other two. However, the deletion process runs faster for the instances of Class (3) than for instances of Classes (1) and (2). In this case, the shortest distances calculation and the computation of components of G_r may become a dominating part of the total running time.

Among all implementations, the most time efficient one is A_1. Compared with Naive, A_1 is faster by a factor of $3 \sim 15$. The memory saving of A_1 is also significant. A_1 can solve an instance of about 20,000 edges in Class (1) and instances of about 15,000 edges in Classes (2) and (3), while Naive can only solve instances of size up to about 5,000 edges for all three classes. The most memory efficient implementation without re-calculation is A_2D_2. It can solve an instance of about 100,000 edges in Class (1) by about 3.5 hours and 500 Mbytes, an instance of about 60,000 edges in Class (2) by about 1.5 hours and 1.5Gbytes memory, and an instance of about 100,000 edges in Class (3) by about 14 hours and 200 Mbytes. Implementation $A_2A_3D_2$ is the most memory efficient one among all implementations. It can solve an instance of about 100,000 edges in Class (1) by about 6 hours and 200 Mbytes, an instance of about 100,000 edges in Class (2) by about 6 hours and 1.4Gbytes, and an instance of about 70,000 edges in Class (3) by by about 14 hours and 160 Mbytes. All implementations without using A_3 have time complexity $O(n^2)$ since no re-calculation for edge data is performed. In the worst case, Implementation $A_2A_3D_2$ may perform re-calculation repeatedly for some edges and has time complexity $O(n^3)$. However, the worst case scenario has not been observed and the running time of Implementation $A_2A_3D_2$ is at most as twice as that of Implementation A_2D_2 for most instances.

Table 5.3: Computation time and memory of intersection graphs of segments generated by LEDA. An X in the table indicates that the implementation runs out of 2Gbyte memory for that instance.

Instances	Number of edges	bw	Itr	Time (seconds)				Memory (MByte)			
				Naive	A_1	A_2D_2	$A_2A_3D_2$	Naive	A_1	A_2D_2	$A_2A_3D_2$
rand1300	2,030	7	6	51.1	10.4	13.1	16.3	181	32	8	8
rand1900	3,029	8	5	102	15.7	18.7	25.8	389	68	13	12
rand3050	5,032	9	4	283	32.5	34.2	61.5	1,179	180	32	22
rand6000	10,261	12	2	X	95.5	101	147	X	724	92	41
rand8700	15,090	14	3	X	292	370	849	X	1,559	160	71
rand11500	20,279	13	2	X	X	557	2,002	X	X	269	120
rand17000	30,433	14	2	X	X	1,334	3,190	X	X	529	216
rand22500	40,622	18	2	X	X	2,156	2,760	X	X	758	326
rand28000	50,901	18	3	X	X	4,002	6,993	X	X	1,112	494
rand33000	60,398	20	2	X	X	5,633	8,540	X	X	1,472	624
rand54000	100,037	22	2	X	X	X	21,417	X	X	X	1,382

Table 5.4: Computation time and memory of random instances generated by PIGALE. An X in the table indicates that the implementation runs out of 2Gbyte memory for that instance.

m	Instances	Number of edges	bw	Itr	Time (seconds)				Memory (MByte)			
					Naive	A_1	A_2D_2	$A_2A_3D_2$	Naive	A_1	A_2D_2	$A_2A_3D_2$
2,400	PI1180	2,022	7	5	23	6.73	9.08	10.4	196	32	8	7
	PI1182	2,016	7	5	22.6	7.44	10.4	12.1	190	32	8	7
	PI1186	2,029	6	5	23.5	6.56	9.24	10	205	32	7	7
	PI1193	2,019	6	5	19.1	7.39	10.5	11.1	156	32	7	7
	PI1207	2,029	9	5	30.6	5.41	6.64	7.04	167	32	7	7
6,000	PI2995	5,043	7	6	156	58.7	93.3	96.5	1,034	178	14	13
	PI2996	5,015	8	7	210	72.1	115	117	1,119	176	16	14
	PI2998	5,049	7	6	163	58.3	92.4	102	1,090	178	17	15
	PI3017	5,063	8	7	197	70.9	111	117	1,112	179	15	14
	PI3018	5,074	7	6	158	57.4	88.2	95.5	986	180	15	14
12,000	PI5940	10,016	7	8	X	289	522	563	X	683	28	26
	PI5992	10,101	7	8	X	286	558	580	X	695	27	25
	PI5998	10,144	7	8	X	304	555	583	X	701	26	24
	PI6043	10,146	8	8	X	299	550	576	X	701	30	27
	PI6067	10,173	7	8	X	312	555	584	X	705	26	25
18,000	PI8950	15,097	10	9	X	907	1,771	2,089	X	1,541	48	39
	PI8977	15,065	9	9	X	913	1,791	1,971	X	1,535	43	38
	PI8986	15,058	8	8	X	765	1,559	1,643	X	1,533	42	39
	PI8995	15,080	9	9	X	885	1,787	1,911	X	1,538	41	38
	PI9020	15,053	8	8	X	807	1,582	1,688	X	1,532	44	37
24,000	PI11974	20,071	9	9	X	X	3,646	3,702	X	X	46	44
35,000	PI17495	30,003	7	9	X	X	8,597	8,610	X	X	70	67
46,000	PI22640	40,074	5	9	X	X	14,163	14,210	X	X	94	89
56,000	PI27671	50,095	8	10	X	X	24,684	24,702	X	X	118	114
66,000	PI32943	60,634	8	10	X	X	36,909	37,076	X	X	156	138
76,000	PI37730	70,022	6	10	X	X	49,136	49,180	X	X	188	160

Computation time of one iteration

To find the carvingwidth of a planar graph, ST Procedure is usually called multiple times. The number of calls (iterations) is instance dependent. In computing the branch decompositions, the computation time of one iteration is an important measure for the time efficiency. Table 5.5 shows the computation time of Naive and Implementations A_1, A_2D_2, and $A_2A_3D_2$ in the iteration when the rat wins the game and the iteration when the rat-catcher wins the game with the noisy level k closest to the carvingwidth for some instances in Classes (1), (2), and (3). From Observation 5.3.6, the deletion process of ST Procedure may terminate earlier when the rat-catcher wins the game. It can be seen from the table that ST Procedure generally uses much less time when the rat-catcher wins for instances of Classes (1) and (2). For instances in Class (3), the computation time when the rat wins is not much different from that when the rat-catcher wins because the time of the deletion process is not a dominating part of the total running time.

Comparison with previous works

Hicks proposes a straightforward implementation *rat* and two memory friendly implementations *comprat* and *memrat* of ST Procedure [70]. The implementations are tested using instances of Class (1) on a SGI Power Challenge with 6×194 MHz processors, 1Gbytes of physical memory, and 1Gbytes of swap space. To compare our results with Hicks', we quote some data of [70] in Table 5.6. An M in the table indicates that the implementation runs out of 2Gbyte memory for that instance. From the table, *rat* runs out of 2Gbyte memory for instances of rl1889 (5,631 edges) and larger. Our Naive implementation can solve rl1889 but runs out of 2Gbyte memory for instances of u2152 (6,312 edges) and larger. This confirms that straightforward implementations of ST Procedure are memory consuming. The memory used by *memrat* for Instance brd14051 (the largest one reported in [70]) is about 600Mbytes. For the same instance, A_2D_2 uses about 1/4 and $A_2A_3D_2$ uses about 1/8 of the memory of *memrat*. Implementation A_1 is faster by a factor of $200 \sim 500$, Implementation A_2D_2 is faster by a factor of $100 \sim 400$, and Implementation $A_2A_3D_2$ is faster by a factor of $100 \sim 200$ than *comprat* and *memrat* for large instances. Notice that the CPU used in [70] has frequency 194MHz and the CPU used our studies has frequency 3.06GHz, so we need to keep in mind this difference of speed when we compare the running time.

Table 5.5: Computation time (in seconds) of several implementations for k close to the carvingwidth $2bw$. An X in the table indicates that the implementation runs out of 2Gbyte memory for that instance.

Instances	Number of edges	Computation time (seconds) for k							
		$k = 2(bw+1)$				$k = 2bw$			
		Naive	A_1	A_2D_2	$A_2A_3D_2$	Naive	A_1	A_2D_2	$A_2A_3D_2$
rl1889	5,631	87.6	0.196	0.526	0.545	79.2	8.55	17.5	27.2
usa13509	40,503	X	X	41.9	61.2	X	X	1,118	1,888
d15112	45,310	X	X	687	1,153	X	X	1,578	2,632
rand3050	5,032	59.4	4.33	5.43	10.7	48.4	20.7	22.7	42.7
rand22500	40,622	X	X	9.2	14.4	X	X	1,714	2,368
rand33000	60,398	X	X	1,203	1,789	X	X	3,825	6,300
PI2995	5,043	12.1	10.2	17.4	18	10.3	9.91	17.5	18.3
PI22640	40,074	X	X	1,670	1,680	X	X	1,673	1,676
PI32943	60,634	X	X	3,725	3,727	X	X	3,471	3,471

Table 5.6: Computation time (in seconds) of *rat*, *comprat*, and *memrat* quoted from Table 1 of [70]. An M in the table indicates that the implementation runs out of 2Gbyte memory for that instance.

Instances	Number of edges	bw	Itr	Computation time (in seconds).		
				rat	*comprat*	*memrat*
pr1002	2,972	21	2	338	448	562
rl1323	3,950	22	3	876	1,519	1,590
d1655	4,890	29	3	1,318	1,608	2,206
rl1889	5,631	22	3	M	3,931	4,012
u2152	6,312	31	4	M	3,207	4,704
pr2392	7,125	29	3	M	3,813	5,167
pcb3038	9,101	40	4	M	13,817	15,865
fl3795	11,326	25	3	M	18,469	17,142
fnl4461	13,359	48	4	M	35,933	51,305
rl5934	17,770	41	3	M	73,468	66,461
pla7397	21,865	33	2	M	65,197	53,564
usa13509	40,503	63	1/2	M	M	413,861
brd14051	42,128	68	3	M	M	594,468

5.4 The Edge-contraction Method for Branch Decompositions

Seymour and Thomas give an algorithm, which is known as *edge contraction method*, for computing an optimal branch decomposition of a planar graph [113]. The contraction of an edge e in a graph G is to remove e from G, identify the two end vertices of e by a new vertex, and make all edges incident to e incident to the new vertex. We denote by G/e the graph obtained by contracting e in G. Given a 2-connected planar graph G, the algorithm of Seymour and Thomas computes an optimal branch decomposition of G by a sequence of edge contractions of the medial graph $M(G)$ of G as follows: First the carvingwidth cw of $M(G)$ is computed by ST Procedure. An edge e of $M(G)$ is *contractible* if the carvingwidth of $M(G)/e$ is at most cw and $M(G)/e$ is 2-connected. Next, a contractible edge e of $M(G)$ is found by ST Procedure and $M(G)$ is contracted to graph $M(G)/e$. The contraction is repeated on $M(G)/e$ until the graph becomes one with three vertices. A optimal carving decomposition of $M(G)$ with width at most cw is constructed based on the sequence of edge contractions.

Proposition 5.4.1 *(Seymour and Thomas [113]) Let $e = \{x, y\}$ be a contractible edge of $M(G)$, x_e be the new vertex identifying $\{x, y\}$ in $M(G)/e$, and T'_C be an optimal carving decomposition of $M(G)/e$. Then the carving decomposition T_C obtained by adding links $\{x_e, x\}$ and $\{x_e, y\}$ to T'_C is an optimal carving decomposition of $M(G)$.*

Finally, the branch decomposition of G is obtained from the carving decomposition of $M(G)$ in linear time (Proposition 5.2.1). It is proved in [113] that for any 2-connected planar graph there is a contractible edge and for a 2-connected planar graph G, $M(G)$ is 2-connected. To check if an edge is contractible, ST Procedure is used to test if $M(G)/e$ has carvingwidth at most cw. In the worst case, all edges may be checked to find a contractible one and for a graph of n vertices, the algorithm of Seymour and Thomas may call ST Procedure $O(n)$ times for one contraction and $O(n^2)$ times in total. So the time complexity of the algorithm is $O(n^4)$.

We call a contractibility test on an edge a *positive one* if the edge is tested contractible, otherwise a *negative one*. Gu and Tamaki give an algorithm which uses a better strategy to find positive tests [66]. When a negative test is obtained on an edge then the edge will not be tested again unless a necessary condition for that edge to be contractible is satisfied. By

this improvement, the algorithm of Gu and Tamaki avoids the repeated negative tests on a same edge, calls ST Procedure $O(n)$ times, and has time complexity $O(n^3)$ for computing an optimal branch decomposition of a planar graph.

We test the $O(n^4)$ time algorithm of Seymour and Thomas and the $O(n^3)$ time algorithm of Gu and Tamaki for instances in three classes using a number of heuristics to select edges for testing the contractibility. Implementation A_1, the most time efficient one, is used as ST Procedure. Both the algorithms have the minimum number of negative calls and running time when the round robin edge selection heuristic is used. Our computational results show that optimal branch decompositions of planar graphs of a few thousands edges can be computed in a practical time. For most instances tested, repeated negative tests are not observed on any edge in the algorithm of Seymour and Thomas. So the advantage of the algorithm of Gu and Tamaki is not shown by those instances when the round robin edge selection heuristic is used. On some other edge selection heuristic, more repeated negative tests are observed in the algorithm of Seymour and Thomas. In this case, the algorithm of Gu and Tamaki has much less negative calls and runs faster than the algorithm of Seymour and Thomas. The details can be found in [24].

5.5 Branch Decomposition of Large Planar Graphs

For large instances, computing optimal branch decompositions is still time consuming by the edge contraction method. Hicks reports that the divide-and-conquer approach is more practical to compute the branch decomposition of planar graphs [69, 71]. In this approach, first the branchwidth β of a graph G is computed. Let S be a set of vertices that separates G into two subgraphs. Roughly speaking, a partition by S is valid if $|S| \leq \beta$, and each subgraph has branchwidth at most β (a formal definition on the valid partition will be given later). Next a valid partition of G is found. In this step, ST Procedure is used to test if each subgraph has branchwidth at most β. If a valid partition is found, then the branch decomposition of each subgraph is computed recursively. The branch decomposition of G is constructed from the decompositions of the subgraphs. How to find a valid partition efficiently is a key for this approach. Hicks proposes the cycle method for computing a valid partition [69, 71]. Notice that there is no guarantee on the existence of a valid partition in a recursive step. The edge-contraction method is used to make a progress in the cycle method when a valid partition can not be found. In the worst case, the cycle method has

time complexity $O(n^4)$. Computational results show that the cycle method is faster than the edge-contraction method by a factor of about $10 \sim 30$ on average for the Delaunay triangulation instances [71].

In this section, we propose divide-and-conquer based algorithms for computing planar branch decompositions. Our algorithms are similar to the cycle method in finding a valid partition but make effort to balance the sizes of subgraphs. Our algorithms also use the edge-contraction method to make a progress when a valid partition can not be found, as is done in previous study [71]. In the worst case, our algorithms run in $O(n^3)$ time. We tested our algorithms and the $O(n^3)$ time edge-contraction algorithm [66] on several classes of planar graphs. Computational results show that our algorithms are faster than the edge-contraction algorithm by a factor of $200 \sim 300$ for Delaunay triangulation instances of more than 5,000 edges. Using the more efficient implementations of ST Procedure of [24], our algorithms can compute optimal decompositions for some instances of size up to 50,000 edges in a practical time. Previous results of the cycle method [69, 71] are obtained by a slower computer and a less efficient implementation of ST Procedure than those in this study. To compare our algorithms with the cycle method on a same platform, we implemented the unaltered cycle method [71] using the more efficient implementation of ST Procedure. Computational results show that our algorithms are faster than the unaltered cycle method by a factor of more than 10 for the Delaunay triangulation instances. Notice that our implementation of the unaltered cycle method is a straightforward one based on the information available in the published literature [69, 71].

Our results suggest that the optimal branch decompositions of large planar graphs can be computed in practice. Our divide-and-conquer algorithms are efficient tools for finding such branch decompositions. This may make the branch-decomposition based algorithms more attractive for many problems in planar graphs.

5.5.1 Divide-and-conquer Based Algorithms

Following the divide-and-conquer approach used in the cycle method [69, 71], we first describe a framework for our algorithms. Given a planar graph H with carvingwidth k, let C be a set of edges (cut set) that partitions H into subgraphs H_1 and H_2. For each H_i $(i = 1, 2)$, define H_i' to be the graph obtained by adding a new vertex v_i' and the edge set $\{\{u, v_i'\} | u \in V(H_i) \cap V(C)\}$ to H_i (see Figure 5.1). Intuitively, H_i' is the graph of H_i and a vertex v_i' representing the part of H other than H_i. The partition by C is valid if $|C| \leq k$,

and each of H_i' has carvingwidth at most k. Below is the framework for our algorithms.

1. Given a planar graph G, compute the medial graph $M(G)$ and the carvingwidth k of $M(G)$ by ST Procedure and let $H = M(G)$.

2. If $|E(H)| > c$ (c is a constant)

   - then try to find a valid partition of H:

     Partition H into subgraphs H_i $(i = 1, 2)$ by a set C of edges with $|C| \leq k$. If every H_i' has carvingwidth at most k for $i = 1, 2$, then a valid partition is found.

   - else compute the carving decomposition of H by enumeration.

3. If a valid partition is found

   - then goto Step 2 to compute the carving decomposition of every subgraph H_i' recursively; and construct the carving decomposition of H from the carving decompositions of the subgraphs.

   - else call an edge-contraction algorithm to contract an edge e of H such that the contracted graph H/e has carvingwidth at most k; goto Step 2 to compute the carving decomposition of H/e; and construct the carving decomposition of H by Proposition 5.4.1.

4. Construct the branch decomposition of G from the carving decomposition of $M(G)$ (Proposition 5.2.1).

Lemma 5.5.1 *An optimal branch decomposition of G can be computed by the framework.*

Proof By Proposition 5.4.1, if an optimal carving decomposition of H/e has been found then an optimal carving decomposition of H can be constructed. Assume that a valid partition of H is found and optimal carving decompositions T_1 and T_2 have been constructed for subgraphs H_1' and H_2' in the valid partition. We assume that T_1 has a leaf node u_1 corresponding to v_1' and T_2 has a leaf node u_2 corresponding to v_2', added in Step 2. Let $e_1 = \{u_1, w_1\}$ be the link of T_1 and $e_2 = \{u_2, w_2\}$ be the link of T_2. We get a carving decomposition T_C of H by first connecting T_1 and T_2 using a new link $\{w_1, w_2\}$ and then discarding links $\{u_1, w_1\}$ and $\{u_2, w_2\}$. Obviously, each internal node of T_C has degree three. Each link of $E(T_C) \setminus \{w_1, w_2\}$ has the same width as that of the corresponding link in T_1

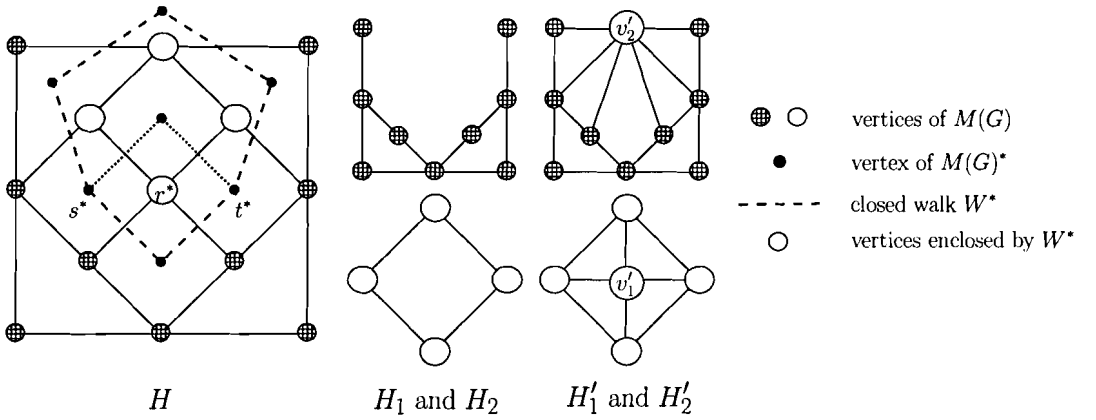


Figure 5.1: Partition graph H into subgraphs by a cycle in H^*.

or T_2. The width of link $\{w_1, w_2\}$ is $|C|$. Thus, T_C has width at most k and is an optimal carving decomposition of $H = M(G)$. By Proposition 5.2.1, T_C can be converted to an optimal branch decomposition of G. □

How to find a valid partition is a key on the efficiency of the divide-and-conquer algorithms. An obvious approach for finding such a partition is to compute a closed walk (cycle) W^* of length at most k in the planar dual $M(G)^*$ of $M(G)$. Let $E^*(W^*)$ be the set of edges in W^*. Let $R^*_{W^*}$ and $V^*_{W^*}$ be the sets of faces and vertices of $M(G)^*$ enclosed by W^*, respectively (see Figure 5.1). Then the set of edges of $M(G)$ corresponding to the edges of $E^*(W^*)$ is a cut set between the subgraph of $M(G)$ with the vertex set and face set corresponding to $R^*_{W^*}$ and $V^*_{W^*}$, respectively, and the rest part of $M(G)$.

In the cycle method [69, 71], a closed walk is computed as follows. First, a face r^* of $M(G)^*$ is selected. Let $E^*_{r^*}$ be the set of edges incident to r^*. Next, a pair of vertices s^* and t^* incident to r^* is selected and a shortest path P^* that does not contain any edge of $E^*_{r^*}$ between s^* and t^* in $M(G)^*$ is computed. A path Q^* between s^* and t^* formed by edges of $E^*_{r^*}$ and path P^* give a closed walk W^* of $M(G)^*$. For a selected face r^*, the cycle method tries every pair of vertices s^* and t^* incident to r^*. If a valid partition is found, then the method is applied recursively, otherwise the edge-contraction method is called.

Similar to the cycle method, our algorithms compute a closed walk W^* formed by paths Q^* and P^* between s^* and t^*. Our algorithms select the vertices s^* and t^* with the consideration on the sizes of subgraphs. Notice that the edges of $E^*_{r^*}$ is a closed walk. For vertices

s^* and t^* incident to r^*, there are two paths Q_1^* and Q_2^* formed by the edges of $E_{r^*}^*$ between s^* and t^*. The partition may be balanced if there is a small difference between the lengths of Q_1^* and Q_2^*. Our first algorithm chooses the vertices s^* and t^* in an order that a smaller difference between the lengths of Q_1^* and Q_2^* is selected with a higher priority. We call this procedure the length-priority algorithm.

The cut set corresponding to a closed walk W^* partitions the input graph in a recursive step into two subgraphs. The size of a subgraph is the number of vertices in the subgraph. The partition is balanced if there is a small difference between the sizes of the two subgraphs. Our second algorithm chooses s^* and t^* in an order that a smaller difference between the sizes of the two subgraphs is selected with a higher priority. We call this procedure the size-priority algorithm.

In both algorithms, we try a constant number of pairs of vertices s^* and t^* incident to r^* in the order defined above. If a valid partition is found then the algorithms are applied recursively, otherwise an edge-contraction method is called. In both algorithms, the constant c in the framework is set to 3 and a subgraph in each partition has at least two vertices.

In the divide-and-conquer algorithms, we partition H into H_i $(i = 1, 2)$ and test if H_i' has carvingwidth at most k by ST Procedure. A test is called *positive* if H_i' has carvingwidth at most k, otherwise *negative*. Similarly, in the edge contraction method, we contract an edge e and test if H/e has carvingwidth at most k by ST Procedure. A test is called *positive* if H/e has carvingwidth at most k, otherwise *negative*.

Theorem 5.5.2 *Both the length-priority and size-priority algorithms compute an optimal branch decomposition of a planar graph G of n vertices in $O(n^3)$ time.*

Proof By Lemma 5.5.1, the algorithms compute an optimal branch decomposition of G. The medial graph $H = M(G)$ has $|E(G)| = O(n)$ vertices. The carvingwidth of H can be computed in $O(n^2 \log n)$ time by ST Procedure (using a binary search). Because the branchwidth of G is $\beta = O(\sqrt{n})$ [57], the carvingwidth of H is $k = 2\beta = O(\sqrt{n})$. Since the carvingwidth of a graph is at least the maximum node degree of the graph, H and the subgraphs in each recursive step have node degree $O(\sqrt{n})$. Therefore, there are $O(n)$ pairs of s^* and t^* incident to a face r^* when we try to find a valid partition. It takes $O(n)$ time to compute a partition for each pair of s^* and t^*. Ordering $O(n)$ partitions takes $O(n \log n)$ time. Thus, both algorithms take $O(n^2)$ time to find and order the partitions for the $O(n)$ pairs of s^* and t^*. ST Procedure takes $O(n^2)$ time to test if a graph of n vertices has

carvingwidth at least k. Since a constant number of partitions are tested by ST Procedure, the total time for deciding whether a valid partition can be found is $O(n^2)$. If a valid partition is not found, the edge contraction method is used to make a progress. This takes $O(n^2)$ time [66]. Let $T(n)$ be the time for computing an optimal carving decomposition of H with n vertices. Then

$$T(n) = \max\{T(n_1) + T(n_2) + O(n^2), T(n-1) + O(n^2)\},$$

where $T(n_i)$ $(i = 1, 2)$ and $T(n-1)$ are the time for computing optimal carving decompositions of H_i' and H/e, respectively. Since $n_1 \leq n-1$, $n_2 \leq n-1$, and $n_1 + n_2 = n+2$, $T(n) = O(n^3)$. It takes $O(n)$ time to get a branch decomposition of G from the carving decomposition of H (Proposition 5.2.1). □

The bound of Theorem 5.5.2 is the worst case time complexity of the divide-and-conquer algorithms. If a valid partition is always found and sizes of the two subgraphs differ only in a constant factor in every recursive step, then the divide-and-conquer algorithms run in $O(n^2 \log n)$ time which is faster than the $O(n^3)$ time edge-contraction algorithm.

We call the length-priority and size-priority algorithms the 2-component method because, the input graph in each recursive step is partitioned into two subgraphs and ST Procedure is used to test the carvingwidth of each subgraph. The 2-component method can be generalized to the 2^i-component method $(i \geq 1)$. Given an input graph, we first choose one pair of s^* and t^* to partition the graph into two subgraphs. We call the subgraphs level-1 subgraphs. A subgraph is called a level-$(j+1)$ subgraph if it is obtained from a partition of a level-j $(j \geq 1)$ subgraph. In the 2^i-component method, we compute the level-i subgraphs (there are 2^i such graphs) by a sequence of partitions of the input graph. During the sequence of partitions, only one pair of s^* and t^* is used for each subgraph. We only check the sizes of the cut sets but do not check the carvingwidth for the level-j subgraphs for $j < i$. We use ST Procedure to check the carvingwidth for every level-i subgraph. If all level-i subgraphs have carvingwidth at most k, then the method is recursively applied to each level-i subgraph. If one level-j $(1 < j \leq i)$ subgraph H' has carvingwidth greater than k then we test the level-$(j-1)$ subgraph from which H' is obtained. If all level-$(j-1)$ subgraphs have carvingwidth at most k then the method is applied recursively (notice that a level-$(j-1)$ subgraph H has carvingwidth at most k if all level-j subgraphs obtained from H have carvingwidth at most k). If a level-1 subgraph has carvingwidth greater than k,

then we give up the current pair of s^* and t^* and apply the method to the input graph on a different pair of s^* and t^*.

This generalization is motivated by the fact that testing the carvingwidth of large graphs by ST Procedure is the most time consuming part in finding the branch decompositions and some observations from the computational study: in most cases, a valid partition can be found in the first try and partitioning the input graph into smaller subgraphs can save the time used by ST Procedure. For constant i, the 2^i-component algorithms have time complexity $O(n^3)$.

The branch decomposition of a graph G which is not 2-connected can be easily constructed from the branch decompositions of its 2-connected components. So, the study of branch decomposition may be concentrated on 2-connected graphs.

5.5.2 Computational Results

We implemented our algorithms and the unaltered cycle method [69, 71]. A number of efficient implementations of ST Procedure are reported in [24]. The implementations of ST Procedure with the best practical performances are used in our algorithms and the cycle method. The implementation of the cycle method is a straightforward one: The pair of vertices s^* and t^* is selected in an arbitrary order. If there are multiple shortest paths P^*'s between s^* and t^* in $M(G)^*$, an arbitrary one is used. Similarly, an arbitrary shortest path P^* is used for the length-priority and size-priority algorithms. We test our implementations on three classes of instances. Class (1) instances include Delaunay triangulations of point sets taken from TSPLIB [107]. The instances are provided by Hicks and are used as test instances in the previous studies [69, 71]. The instances in Class (2) are generated by the LEDA library [1, 90]. LEDA generates two types of planar graphs. One type of the graphs are the randomly generated maximal planar graphs and their subgraphs obtained from deleting some edges. Since the maximal planar graphs generated by LEDA always have branchwidth four, the subgraphs obtained by deleting edges from the maximal graphs have branchwidth at most four. The graphs of this type are not interesting for the study of branch decompositions. The other type of planar graphs are those generated based on some geometric properties, including Delaunay triangulations and triangulations of points uniformly distributed in a two-dimensional plane, and the intersection graphs of segments uniformly distributed in a two-dimensional plane. We report the results on the 2-connected intersection graphs. The instances in Class (3) are generated by the PIGALE library [4].

PIGALE randomly generates one of all possible planar graphs with a given number of edges based on the algorithms of [112]. We report the results on the 2-connected graphs generated by the PIGALE library.

We run the implementations on a computer with Intel(R) Xeon(TM) 3.06GHz CPU, 2GB physical memory and 4GB swap memory. The operating system is SUSE LINUX 10.0, and the programming language we used is C++.

Results for Instances in Class (1)

The computational results for Class (1) instances are reported in Table 5.7. In the table, $|E(G)|$ is the number of edges in the instance and thus the number of vertices in the medial graph $M(G)$ which is the input to the algorithms, bw is the branchwidth of the graph G, NT is the number of negative tests, *Cycle* is the unaltered cycle method, L_P is the length-priority algorithm, S_P is the size-priority algorithm, and $S4$ is the 4-component algorithm with size-priority. For comparison, we include the running time of the $O(n^3)$ time edge-contraction method in column EC_GT (the data is taken from [24] which uses a computer of similar performance to the one we use for the divide-and-conquer algorithms, and the $O(n^3)$ algorithm itself is given in [66]). In the table, an "X" indicates that it requires more than 70,000 seconds to solve the instance and a blank indicates that we did not test the algorithms for that instance.

The data show that all divide-and-conquer algorithms (Cycle, L_P, S_P, and $S4$) are much faster than the edge-contraction algorithm. The length-priority and size-priority algorithms are faster than the edge-contraction method by a factor of $200 \sim 300$ for instances of more than 5,000 edges in this class. It is difficult to compare the data of our algorithms with those of the cycle method reported in previous studies [69, 71], because computers of different speeds and different implementations of ST Procedure are used. To compare our algorithms with the cycle method on a same platform, we give a straightforward implementation of the unaltered cycle method using the same efficient ST Procedure used in our algorithms. Our algorithms are faster than the cycle method by a factor of at least 10 for instances of more than 5,000 edges. Notice that on average the cycle method is faster than the edge-contraction method by a factor of about 10 which is slightly smaller than that ($10 \sim 30$ on average) reported in previous studies [71]. Considering the fact that a more efficient edge-contraction algorithm is used in this study, our implementation of the cycle method has a similar performance as that used in the previous studies and our new

Table 5.7: Computation time (in seconds) of several decomposition algorithms for Class (1) instances. An X in the table indicates that it requires more than 70,000 seconds to solve the instance and a blank indicates that we did not test the algorithms for that instance.

Graphs G	$\|E(G)\|$	bw	EC_GT		Cycle		L_P		S_P		$S4$	
			time	NT	time	NT	time	NT	time	NT	time	NT
pr1002	2972	21	2667	102	369	37	155	34	150	63	271	129
rl1323	3950	22	6879	136	441	0	63	5	189	97	336	200
d1655	4890	29	13529	171	5958	806	295	34	218	28	402	59
rl1889	5631	22	29096	178	1896	527	130	0	115	1	90	2
u2152	6312	31	26092	192	2394	92	156	0	140	0	119	0
pr2392	7125	29	45728	271	5595	210	173	0	153	0	118	0
pcb3038	9101	40			6265	53	490	8	998	17	1899	36
fl3795	11326	25			8954	52	863	3	902	11	1190	22
fnl4461	13359	48			X	X	3795	31	2479	16	2441	16
rl5934	17770	41					2348	2	2585	6	3296	12
pla7397	21865	33					10291	88	3026	10	3376	21
usa13509	40503	63					25956	29	29539	79	50376	160
brd14051	42128	68					10536	19	31554	129	64802	263
d18512	55510	88					22378	44	X	X	X	X

algorithms are faster than the cycle method. For all instances which are solved within the 70,000 seconds time limit, the edge-contraction method is never used by any divide-and-conquer algorithm to make a progress, that is, a valid partition is always found in every recursive step.

There are two factors improving the running time of our algorithms. Both the length-priority and size-priority algorithms find more balanced partitions than the cycle method. This reduces the total running time in the divide-and-conquer approach. The other factor is that our algorithms have a smaller number of negative tests. In finding a valid partition, once a negative test happens, all divide-and-conquer algorithms try a different pair of s^* and t^* and the running time is increased. Also it takes more time for a negative test than a positive one. For Class (1) instances, the length-priority algorithm runs faster than the size-priority algorithm for large graphs while the size-priority algorithm does a better job for smaller graphs. Because the running time of the algorithms depends on both the size of the graphs and the number of negative tests, it may take a longer time to solve some instances than that for a larger graph. For example, Instance usa13509 requires a longer time than Instance brd14051 by the length-priority algorithm.

For Class (1) instances, the number of negative tests is non-trivial, especially for large graphs. This makes the 2^i-component ($i > 1$) algorithms less efficient, because using more than two components generally increases the number of negative tests and thus the total running time. As shown in Table 5.7, the 4-component algorithm is slower than the 2-component algorithms for most instances in this class.

Results for Instances in Classes (2) and (3)

Computational results for Classes (2) and (3) instances are given in Tables 5.8 and 5.9, respectively. In the tables, $S8$ is the 8-component algorithm with the size-priority. An "X" in the tables indicates that it takes more than 150,000 seconds to solve that instance. Similar to results for Class (1) instances, the edge-contraction method is never used by any divide-and-conquer algorithm to make a progress for Classes (2) and (3) instances.

It takes more time to find the branch-decomposition of a Class (2) instance than a Class (1) instance with a similar size by divide-and-conquer algorithms. This may be caused by the fact that Class (2) instances have smaller branchwidth than that of Class (1) instances. A larger branchwidth implies that a longer cycle is used in a valid partition and a longer cycle usually gives a more balanced partition. For Class (2) instances, the size-priority algorithm runs faster than the length-priority algorithm and is faster than the edge-contraction algorithm by a factor of about $50 \sim 150$. Both the length-priority and size-priority algorithms are faster than the cycle method. Since the number of negative tests in the divide-and-conquer algorithms for Class (2) instances is small, the 2^i-component ($i > 1$) algorithms are more efficient than the 2-component ones. Especially, the 8-component algorithm is faster than the edge-contraction, the cycle, and the 2-component size-priority algorithms by factors of about $100 \sim 200$, $5 \sim 8$, and 2, respectively.

It takes more time to find the branch-decomposition of a Class (3) instance than a Class (1) or Class (2) instance with a similar size by the divide-and-conquer algorithms, because Class (3) instances have a smaller branchwidth. As shown in the table, the branchwidth of the instances is small constants and does not increase in the size of the instances. In each valid partition of the divide-and-conquer algorithms, we get a small subgraph of a constant size and a large subgraph for most instances. This limits the speed-up by the 2-component divide-and-conquer algorithms to a constant factor. Similar to the results for Class (2) instances, the number of negative tests in the divide-and-conquer algorithms

Table 5.8: Computation time (in seconds) of several decomposition algorithms for Class (2) instances. An X in the table indicates that it requires more than 150,000 seconds to solve the instance and a blank indicates that we did not test the algorithms for that instance.

Graphs G	$\|E(G)\|$	bw	EC_GT		Cycle		L_P		S_P		$S4$		$S8$	
			time	NT	time	NT	time	NT	time	NT	time	NT	time	NT
rand1160	2081	8	1749	34	53.2	0	46.3	0	29.9	0	23.1	0	18.4	0
rand1672	3047	10	4695	103	137	2	54.6	0	39.7	0	29.7	0	25.7	0
rand2780	5024	10	29073	147	2059	0	727	0	471	0	312	0	249	0
rand3857	7032	11	82409	281	1503	0	810	6	493	6	351	13	292	20
rand5446	10093	11			11474	17	3283	3	2361	3	1532	6	1205	9
rand8098	15031	13			12022	76	2783	1	1864	0	1465	0	1159	0
rand10701	20044	13			11782	9	4368	0	3699	0	2884	0	2475	0
rand15902	30010	14			68809	125	32409	0	19127	14	13240	29	11744	42
rand21178	40190	17			X	X	93897	0	54557	2	33429	4	26910	7
rand26304	50032	19					149570	0	85207	0	59907	0	47039	0

Table 5.9: Computation time (in seconds) of several decomposition algorithms for Class (3) instances. An X in the table indicates that it requires more than 150,000 seconds to solve the instance.

Graphs G	$\|E(G)\|$	bw	EC_GT		Cycle		L_P		S_P		$S4$		$S8$	
			time	NT	time	NT	time	NT	time	NT	time	NT	time	NT
PI855	1434	6	565	61	22.7	0	14.3	0	11.8	0	7.75	0	6.41	0
PI1277	2128	9	1563	101	107	1	47.5	1	25.9	0	17.5	0	14.6	0
PI1467	2511	6	3135	74	304	1	183	0	120	0	72	0	56.7	0
PI2009	3369	7	8127	90	253	0	142	0	115	0	64.8	0	55.7	0
PI2518	4266	8	17807	105	663	0	369	0	206	0	135	0	112	0
PI2968	5031	6	26230	162	2244	9	1235	6	773	3	488	7	423	11
PI3586	6080	8	49108	176	2340	1	1182	0	699	0	443	0	334	0
PI4112	6922	7	70220	132	10808	1	10817	0	5663	0	2973	0	2150	0
PI5940	10016	7	X	X	19770	0	18807	0	9517	0	5205	0	3782	0
PI8950	15097	10	X	X	33862	13	19216	1	11171	0	6871	0	4993	0
PI11974	20071	9	X	X	X	X	X	X	111747	0	61641	0	44479	0

is small and the 2^i-component algorithms are faster than the 2-component ones. The 8-component algorithm is faster than the edge-contraction, the cycle, and the 2-component size-priority algorithms by factors of about $30 \sim 150$, $5 \sim 8$, and 2, respectively.

5.6 Summary

We give efficient implementations of the Seymour and Thomas procedure which, given an integer β, decides whether a planar graph G has the branchwidth at least β or not. We tested our implementations on instances of size up to one hundred thousand edges. The results show that the branchwidth of those instances can be computed within a reasonable time and memory space. This suggests that the required memory may not be a bottleneck for computing branchwidth and optimal branch decompositions of planar graphs in practice. Our implementations without edge re-calculations require $O(n^2)$ bytes memory, although the constant behind the Big-Oh may be small. We have an upper bound $O(n^3)$ on the time complexity of the implementations with re-calculations for edge data. Let p be the maximum number of edges kept in those implementations. This bound is true for any $p \geq 1$. In general, a larger p results in a faster running time of the implementations. It is interesting to prove a better upper bound related to p, say $O(n^2(n/p))$, on the time complexity of those implementations.

We propose divide-and-conquer based algorithms of using ST procedure to compute optimal branch decompositions of planar graphs. Our algorithms have time complexity $O(n^3)$. Computational studies show that our algorithms are much faster than the edge-contraction algorithms and can can compute the optimal branch decompositions for some instances of about 50,000 edges in a practical time. This provides useful tools for applying the branch decomposition based algorithms to practical problems.

Chapter 6

Edge Disjoint Paths Problem in Planar Graphs

Given a set P of pre-routed paths in a graph G, the maximum edge-disjoint paths (MEDP) problem is to find a maximum subset $P' \subseteq P$ of paths such that no two paths in P' share a common edge of G. As we already reviewed in Chapter 2, the MEDP problem has received much attention in the past decades. Most of the previous studies focus on developing performance guaranteed approximation algorithms. These algorithms are, although very important theoretically, often far from optimal. Their practical performances are not known, since there are very little efforts on the implementation of these algorithms. The only implementation we are aware of is given in [50] which implemented the $(\frac{5}{3}+\epsilon)$-approximation algorithm for the MEDP problem in directed trees, where $\epsilon > 0$ is any fixed constant [49].

In this chapter, we study the maximum edge-disjoint paths problem. We are mainly interested in exact algorithms for the problem. We show in Section 6.1 that the maximum edge-disjoint paths problem in planar graphs can be solved optimally, if the carvingwidth of the planar graph is bounded by a small constant. The running time is exponential in the carvingwidth but is polynomial in the number of nodes and edges of the graph, and polynomial in the number of given paths. Our algorithm has two steps: (I) computing an optimal carving decomposition of the planar graph, and (II) computing a maximum set of edge-disjoint paths, using a dynamic programming method based on the carving decomposition computed in Step I. Our algorithm works also for graphs that are close to planar graphs, by first computing a carving decomposition of the planar subgraphs. In

Section 6.2, we show the practical performance of our algorithm. We implement the optimal algorithm and test the implementation on both practical and random generated networks. Our experimental results show that the maximum edge-disjoint paths problem can be solved exactly for graphs with small carvingwidth in a practical time and memory space, when the load of the given set of paths is not too large. We also give an approximation algorithm for the maximum path coloring problem, for which an exact algorithm may not be practical.

6.1 Optimal Algorithm for the MEDP Problem

In this section, we give an algorithm which solves the maximum edge-disjoint paths problem optimally in planar graphs. The input to an instance of the maximum edge-disjoint paths problem consists of two parts: a set P of paths, and a graph G. In Step I of our algorithm, we compute an optimal carving decomposition of the given planar graph, using the divide-and-conquer algorithm given in Section 5.5. For a graph G that is not planar but close to planar, one may compute an optimal carving decomposition T_C for a planar subgraph G' of G, and then use T_C as the carving decomposition of G. (For small graphs, the planar subgraphs may be found by hand. However, for large graphs, one may need to rely on some heuristics to find planar subgraphs.) Recall that the width of a link e in T_C is the number of edges between the two subgraphs obtained by removing e in T_C. Since G has more edges than G', T_C may not be an optimal carving decomposition for the graph G. However, T_C is very close to optimal if G is close to planar. Note that the dynamic programming part of our algorithm does not require the carving decomposition to be optimal, but the running time is exponential in the width of the decomposition. A carving decomposition which is optimal or close to optimal helps to reduce the total running time.

In order to describe Step II of our algorithm, we need some more definitions. Recall that for each link e of a carving decomposition T_C, removing e separates T_C into two subtrees. Let V' and V'' be the sets of leaves of the subtrees. Let C_e be the set of edges of G incident to both a node of V' and a node of V''. Then $|C_e|$ is bounded by the width of T_C. We call a link $e = \{x, y\}$ of T_C a leaf link if one of x and y is a leaf node of T_C, otherwise an internal link.

In Step II, we compute a maximum set of edge-disjoint paths as follows. We first convert the carving decomposition T_C of G into a rooted binary tree by replacing a link $\{x, y\}$ of T_C by three links $\{x, z\}$, $\{z, y\}$, and $\{z, r\}$, where z and r are new nodes added to T_C, r is

the root, and $\{z, r\}$ is an internal link. We call $\{z, r\}$ the root link. For every internal link e of T_C, e has two child links incident to e. For every link e of T_C, let T_e be the subtree of T_C consisting of all descendant links of e. Let G_e be the subgraph of G induced by the nodes at leaf nodes of T_e. For an internal link e of T_C, we use $\mathcal{P}_e$ to denote the set of all subsets of edge-disjoint paths in P on C_e. For a set of edge-disjoint paths $P'_e \subseteq \mathcal{P}_e$, we define $f(e, P'_e)$ as $|Q_e|$, where Q_e is a maximum subset of paths in G_e such that $P'_e \cup Q_e$ is edge-disjoint.

To compute a maximum edge-disjoint paths in G, we find all sets of edge-disjoint paths (solutions, i.e., values $f(e, P'_e)$) of G_e from which a maximum edge-disjoint paths may be constructed for every link e of T_C by a dynamic programming method: the solutions of G_e for each leaf link e is empty and the solutions for an internal link e is computed by merging the solutions for the child links of e.

Initially, $f(e, P'_e)$ is set to 0 for all links e of T_C and for every possible subset $P'_e \subseteq \mathcal{P}_e$. For a leaf link e, no computation is needed. An internal link e has child edges e_1 and e_2 in T_C. The values $f(e, P'_e)$ is computed from $f(e_1, P'_{e_1})$ and $f(e_2, P'_{e_2})$. More specifically, we enumerate all possible subsets P'' of paths such that P'' is edge-disjoint and each path of P'' is on some edges in $C_{e_1} \cup C_{e_2}$. Let $P''_e \subseteq P''$, $P''_{e_1} \subseteq P''$, and $P''_{e_2} \subseteq P''$ be the sets of paths on some edges in C_e, C_{e_1}, C_{e_2}, respectively. Let $P''_{e_1 e_2} = (P''_{e_1} \cup P''_{e_2}) \setminus P''_e$. For every P'', the values $f(e_1, P''_{e_1})$, $f(e_2, P''_{e_2})$, and $|P''_{e_1 e_2}|$ are added up. If this value is greater than the previous value for $f(e, P''_e)$, then $f(e, P''_e)$ is updated. At the root link $e = \{z, r\}$, the maximum value $f(e, P'_e)$ over all P'_e is the solution for the maximum edge-disjoint paths problem.

The running time of the algorithm can be estimated as follows. Step I of our algorithm runs in $O(n^3)$ time (see Section 5.5). Step II is the dominant part for the time complexity. For each internal link e of T_C, C_e is bounded by $cw(G)$, the carvingwidth of G. There are at most $(L+1)^{cw(G)}$ possible subsets Q_e of partial solutions to store, since each edge of C_e has load bounded by L, and each subset Q_e contains at most one path on each edge of C_e. During the merging process, there are at most $(L+1)^{1.5 \cdot cw(G)}$ possible cases to consider: we only need to consider paths on C_e and paths on $C_{e_1} \cap C_{e_2}$, while $|C_e \cup (C_{e_1} \cap C_{e_2})| \leq 1.5 \cdot cw(G)$. Thus, the time complexity for processing one link of T_C is $O((L+1)^{1.5 \cdot cw(G)})$ and the memory requirement is $O((L+1)^{cw(G)})$. The total time complexity for processing all links of T_C is $O((L+1)^{1.5 \cdot cw(G)} \cdot |V(G)|)$ and the memory requirement is $O((L+1)^{cw(G)} \cdot |V(G)|)$. When $cw(G)$ is bounded by a constant, the running time and memory requirement are both polynomial in the input parameters.

Notice that similar approach has been used in [46] to solve the call control problem in bounded degree trees. The highly simplified structure of bounded degree trees makes the problem easier.

The maximum path coloring (Max-PC) problem can be solved in a similar way. Recall that for the Max-PC problem, we are given a set P of paths in a graph G, and we want to color a maximum subset of paths in P using a given number w of colors, such that no two overlapping paths are given the same color. The time complexity would be $O((L+1)^{1.5 \cdot w \cdot cw(G)} \cdot |V(G)|)$ and the memory requirement would be $O((L+1)^{w \cdot cw(G)} \cdot |V(G)|)$. Thus, the maximum path coloring problem can be solved optimally in polynomial time, if the number of colors and the carvingwidth are both bounded by a small constant. The running time and memory space are huge even for very small L, $cw(G)$ and w. Thus, this approach is not practical. One can have a 1.58-approximation iterative greedy algorithm for the Max-PC problem as follows: call the MEDP algorithm to select a set of edge-disjoint paths and color the selected set using one color; remove the colored paths and repeat the procedure on the remaining paths until the colors are used up or there is no remaining path. Obviously, the time complexity of this approach is similar to that of the MEDP algorithm, subject to a polynomial factor of w. It was proved in [120] that this iterative Max-PC algorithm has an approximation ratio of 1.58 if the MEDP algorithm is optimal. If we call the MEDP procedure until all paths are colored, we have an approximation algorithm for the minimum path coloring (Min-PC) problem.

6.2 Computational Results

We implemented our algorithms and tested our implementations on three classes of instances. Class (1) instances are real networks deployed in the US and in Europe. They include a 16-node NSFNET backbone (Figure 6.1), a 20-node European Optical Network (Figure 6.2), a 24-node ARPANET-like network (Figure 6.3), and a 30-node UK Network (Figure 6.4). Note that the 16-node NSFNET backbone, the 24-node ARPANET-like network, and the 30-node UK Network are not planar but very close to planar. In particular, removing the dark edges in Figures 6.1, 6.3, and 6.4 makes these graphs planar. For these non-planar graphs, an optimal carving decomposition T_C is first computed for a planar subgraph (by removing the dark edges in the figure), using the divide-and-conquer algorithm given in Section 5.5. T_C is then used as the carving decomposition of the original

(non-planar) graph G. T_C may not be an optimal carving decomposition for the original graph G because the dark edges are included. The instances in Class (2) are the intersection graphs of segments uniformly distributed in a two-dimensional plane, generated by the LEDA library [1, 90]. The instances in Class (3) are generated by the PIGALE library [4]. PIGALE randomly generates one of all possible planar graphs with a given number of edges based on the algorithms of [112]. We report the results on the 2-connected graphs generated by the PIGALE library. The intersection graphs generated by LEDA and random graphs generated by PIGALE have been used in the branch decomposition studies in Chapter 5.

We generate sets of paths as follows, given a positive integer k and an allowable maximum load L of the k paths. We first generate k source-destination pairs randomly in the given graph, and then connect them using shortest paths. We do not use any pair for which the shortest path has length one. If there are multiple shortest paths between the two end-nodes of a pair, an arbitrary one is used. If the generated set of paths has load more than L, then we discard the generated paths and start over again. Note that for given integers k and L, it may not be possible to generate a set of k paths with maximum load L, if L is small and k is large.

To compute an optimal carving decomposition T_C of a planar graph, we use the length-priority algorithm given in Section 5.5. In Step II, to save memory, we compute the partial solutions for each link e of T_C in the postorder. Once the partial solutions are computed for a link e, the solutions for the child links of e are discarded. The memory requirement may be reduced to $O((L+1)^{cw(G)})$, if the binary tree obtained from T_C is well balanced. Notice that Steps I and II have time complexities $O(|V(G)|^3)$ and $O((L+1)^{1.5 \cdot cw(G)} \cdot |V(G)|)$, respectively.

We run the implementations on a computer with Intel(R) Xeon(TM) 3.06GHz CPU, 2GB physical memory and 4GB swap memory. The operating system is SUSE LINUX 10.0, and the programming language we used is C++.

6.2.1 Results for Instances in Class (1)

The computational results for Class (1) instances are reported in Tables 6.1 - 6.4. In the tables, cw is the width of the carving decomposition for the graph G, n and m are the number of nodes and edges in the instances, respectively. The carving decompositions of the input graphs are computed using the length-priority algorithm given in Section 5.5. The time for computing the carving decompositions is small, thus we do not report it here. $|P|$ is

the number of given paths, $L(P)$ is the maximum load of the paths, $L(P)_{ave}$ is the average load of the edges, $\omega(P)$ is the clique size of the conflict graph of the paths. Notice that $L(P)$ is a lower bound on $\omega(P)$. We compute $\omega(P)$ using the well known *dfmax* program [13]. OPT_{MEDP} is the number of paths in an optimal solution for the maximum edge-disjoint paths problem, SOL_{PC} is the number of colors used by the iterative greedy path coloring algorithm, T_{MEDP} is the time used by the MEDP algorithm (in seconds), T_{PC} is the time used by the iterative greedy path coloring algorithm (in seconds), and Mem is the memory used by the algorithms (in megabytes, or MB).

From the tables, we can see that when the width of the carving decomposition is small (for example, 5 for the 16-node NSFNET backbone), our algorithms can handle a set of paths with load as large as 40. However, when the width of the carving decomposition is large (for example, 10 for the ARPANET-like network), the algorithms can only handle a set of paths with load at most 5. The number of paths cannot be too large. Otherwise the load would be too large since the graphs are small. The maximum set of edge-disjoint paths can usually be computed within several minutes for the tested instances. For larger instances, the program runs out of memory very quickly. The clique sizes of the conflict graphs are usually close to the load of the paths (in many cases, the clique size is equal to the load). This might be due to the simple structure of the networks, and the use of shortest paths routing. The number of colors used by the iterative path coloring algorithm is at most 1.67 times the clique size.

6.2.2 Results for Instances in Classes (2) and (3)

Computational results for Classes (2) and (3) instances are given in Table 6.5 and Table 6.6. The columns are named in the same way as in Class (1) instances. Our algorithms can handle large graphs in these two classes and large number of paths, providing the load is not too large. The maximum set of edge-disjoint paths can usually be computed within several hours for the tested instances. Again, for larger instances, the program runs out of memory very quickly. For some instances, the clique sizes of the conflict graphs can be twice the load of the paths (but within an additive constant of 10). The number of colors used by the iterative greedy path coloring algorithm is at most 2.2 times the clique size.

For a graph with large carvingwidth, Step II is both time and memory consuming, because this step runs exponentially in the carvingwidth. The time and memory increase very quickly for large load and carvingwidth. The memory requirement seems to be the main

hurdle for solving instances with large carvingwidth and large load. The computational results suggest that it may not be practical to solve the MEDP problem for instances in which $(L+1)^{cw} > 10^7$ on a PC with 2GB memory.

6.3 Summary

We have given an optimal algorithm for the maximum edge-disjoint paths problem in planar graphs, and an approximation algorithm for the maximum path coloring problem. Our algorithms use dynamic programming method based on carving decompositions of the input graphs. We also tested the practical performances of the algorithms on both real and randomly generated planar graphs (or graphs close to planar). The computational results coincide with the theoretical analysis of the algorithms: they are efficient for graphs with small carvingwidth when the load is not too large, but may not be practical for graphs with large carvingwidth and for large load.

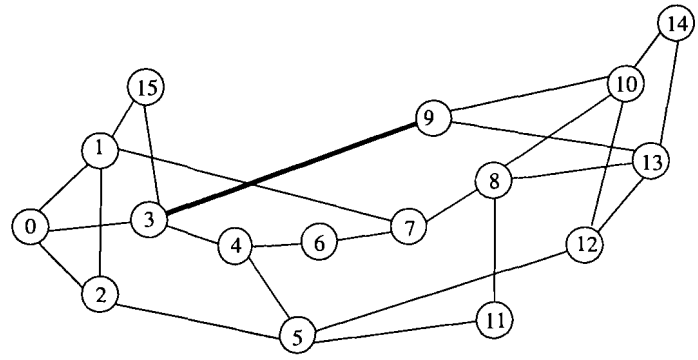


Figure 6.1: A 16-node NSFNET backbone.

Table 6.1: Computation results for a 16-node NSFNET backbone ($n = 16$, $m = 25$, $cw = 5$).

$\|P\|$	$L(P)_{ave}$	$L(P)$	$\omega(P)$	OPT_{MEDP}	SOL_{PC}	T_{MEDP}	T_{PC}	Mem (MB)
100	10	15	18	10	19	1.25	3.97	17
	10	20	20	11	22	2.94	9.06	60
150	14	20	22	11	25	13.4	37.6	112
	16	25	25	11	28	18	57.2	325
	15	30	30	10	32	20	70.8	659
	16	35	35	10	35	22	99.8	1192
200	19	26	28	11	31	77	249	394
	20	30	30	11	35	74	286	780
	20	35	36	11	37	91	374	1634
	20	40	40	11	40	85	346	1600
250	25	35	35	10	43	246	1145	1672

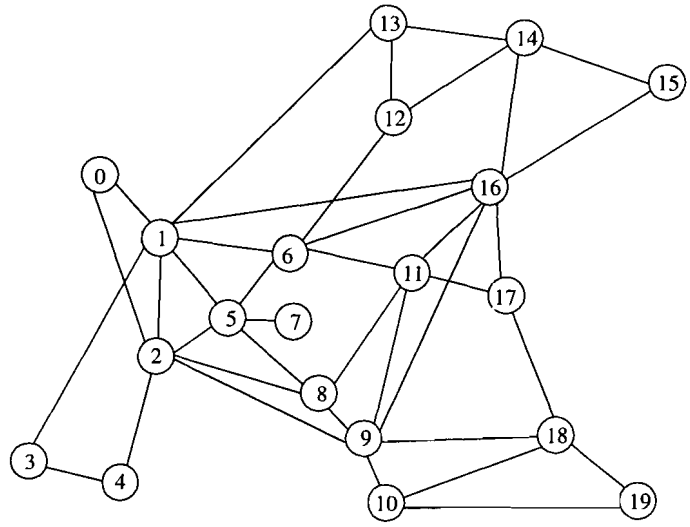


Figure 6.2: A 20-node European Optical Network.

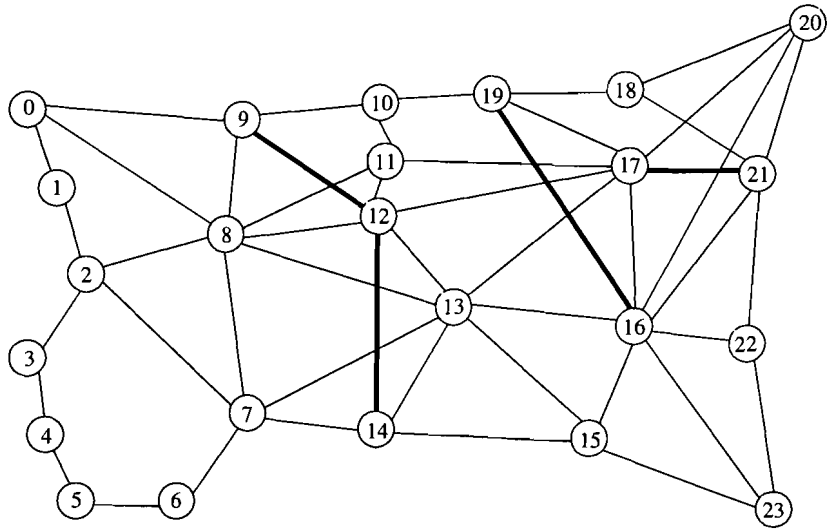


Figure 6.3: A 24-node ARPANET-like network.

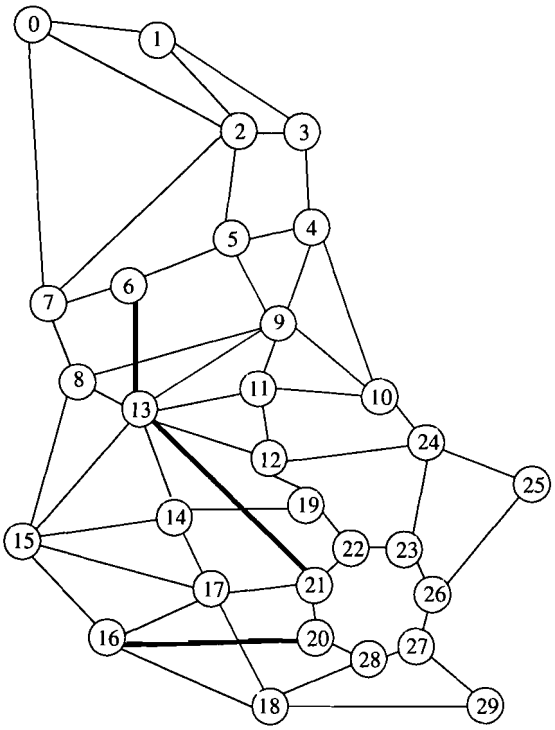


Figure 6.4: A 30-node UK Network.

Table 6.2: Computation results for a 20-node European Optical Network ($n = 20$, $m = 38$, $cw = 8$).

$\|P\|$	$L(P)_{ave}$	$L(P)$	$\omega(P)$	OPT_{MEDP}	SOL_{PC}	T_{MEDP}	T_{PC}	Mem (MB)
40	2.5	4	5	13	6	0.36	0.41	10
	2.7	6	6	11	8	1.47	2.75	244
	2.8	8	8	10	9	1.63	2.41	297
60	3.7	6	7	16	7	5.66	6	245
	3.9	8	8	13	9	10.6	18.8	1791
80	4.9	6	7	15	9	104	121	255
	4.9	7	7	16	9	196	233	713
	5.1	8	8	16	11	58	75.6	1796
100	6.1	8	9	17	12	421	484	1819

Table 6.3: Computation results for a 24-node ARPANET-like network ($n = 24$, $m = 49$, $cw = 10$).

$\|P\|$	$L(P)_{ave}$	$L(P)$	$\omega(P)$	OPT_{MEDP}	SOL_{PC}	T_{MEDP}	T_{PC}	Mem (MB)
30	1.5	2	2	15	3	0.1	0.1	3
	1.7	3	3	12	4	0.14	0.15	17
	1.7	4	4	12	5	0.85	0.99	26
	1.8	5	5	11	6	4	5.03	944
40	2	3	3	15	5	1.18	1.33	17
	2.2	4	4	13	6	1.35	1.43	158
	2.4	5	6	11	7	4.2	5.16	944
50	2.7	4	4	17	6	4.38	4.83	159
	2.8	5	6	15	6	5.61	6.39	944
60	3.1	5	6	18	7	13.7	15.1	945

Table 6.4: Computation results for a 30-node UK Network ($n = 30$, $m = 57$, $cw = 8$).

$\|P\|$	$L(P)_{ave}$	$L(P)$	$\omega(P)$	OPT_{MEDP}	SOL_{PC}	T_{MEDP}	T_{PC}	Mem (MB)
60	3.1	4	5	20	6	1.7	2.09	7
	3.4	8	8	16	9	3.1	6	121
	3.6	10	10	15	10	6.5	9.32	577
80	4.2	6	7	20	9	4.3	5.96	82
	4.4	8	8	20	9	12.2	17.4	591
	4.7	10	10	16	13	23.2	39	1337
100	4.7	6	6	23	8	25.2	39.7	82
	5.2	8	9	23	11	71.1	95.6	591
120	5.7	7	8	23	10	90	123	233
	6.1	8	9	22	11	524	673	591
140	6.7	8	8	22	11	270	392	591

Table 6.5: Computation results for instances generated by LEDA.

n	m	cw	$\|P\|$	$L(P)_{ave}$	$L(P)$	$\omega(P)$	OPT_{MEDP}	SOL_{PC}	T_{MEDP}	T_{PC}	Mem (MB)
104	177	7	200	4.3	5	6	55	9	20.6	25.8	26
				4.7	6	7	52	10	33.7	50	55
				5	7	8	48	12	106	154	138
				5.4	8	9	47	12	87.5	132	300
				5.4	9	11	44	14	159	273	607
				5.7	10	12	40	15	186	343	1125
214	365	8	300	3.2	4	5	101	8	51.6	57.8	32
				3.6	5	7	88	11	186	209	125
				4.1	6	9	78	12	297	412	402
				4.3	7	9	80	11	828	1132	1149
531	922	10	400	1.7	2	3	218	5	9.9	10.4	10
				2.1	3	4	173	7	89.4	100	79
				2.4	4	5	154	8	171	205	578
815	1431	10	600	1.7	2	3	332	5	42.7	44.6	24
				2.1	3	4	268	7	121	141	137
				2.4	4	5	220	10	1904	2087	1187

Table 6.6: Computation results for instances generated by PIGALE.

n	m	cw	$\|P\|$	$L(P)_{ave}$	$L(P)$	$\omega(P)$	OPT_{MEDP}	SOL_{PC}	T_{MEDP}	T_{PC}	Mem (MB)
134	189	5	500	10.1	12	16	67	23	20.7	50	42
				11.7	16	23	63	27	55.7	200	121
				13	20	27	64	35	73.8	256	274
				15.6	30	39	56	43	364	1830	1691
			1000	20.6	24	32	71	43	940	3583	887
1334	1853	7	2000	4.3	5	7	520	15	122	211	63
				4.7	6	10	499	18	403	757	204
				5.1	7	12	470	19	721	1900	352
				5.6	8	12	445	22	1789	4802	659
				5.9	9	13	428	21	4052	9679	1159
				6.2	10	15	409	26	5328	13975	2032
2000	2761	8	2000	3.1	4	7	678	13	279	438	113
				3.6	5	9	597	15	655	1263	226
				4	6	9	553	17	2317	4622	683
				4.3	7	11	531	20	5602	11165	1433
3334	4610	10	2000	1.7	2	4	1072	8	61.7	72	84
				2.1	3	5	843	10	441	625	197
				2.5	4	7	732	15	1915	3652	976
			3000	3	4	7	1020	14	8149	11675	1958
4000	5571	9	2000	2.2	4	8	754	12	623	1077	339
				2.5	5	9	646	15	1995	3412	974
			3000	2.7	4	7	1062	14	1064	1755	340
				3.1	5	10	947	14	3606	8098	1078
5000	6916	10	2000	1.4	2	4	1112	8	48	66	172
				1.7	3	6	866	9	339	469	233
				2.0	4	8	737	12	2392	3371	912

Chapter 7

Conclusion and Future Work

In this thesis, we have studied fundamental routing and channel assignment problems in WDM optical networks with specific topologies and general topologies. Our study on specific topologies includes the Min-PC problem on trees of rings, the Max-RPC problem on rings, the Min-PMC and Max-PMC problems on multifiber trees, and the call control problem on bounded depth trees. We developed a carving-decomposition based method to solve exactly the edge-disjoint paths problem, and to approximate the maximum path coloring problem for networks with more general topologies. The carving-decomposition based approach works for graphs with bounded carvingwidths. In this chapter, we summarize the contributions of this work, and give a few directions for future work.

7.1 Summary of Contributions

We have given an efficient algorithm which solves the Min-PC problem on a tree of rings with an arbitrary (node) degree using at most $3L$ colors and achieves an approximation ratio of 2.75 asymptotically. The $3L$ upper bound is tight even on a tree of rings with degree four. We also give a $3L$ and 2-approximation (resp. 2.5-approximation) algorithm for the Min-PC problem on a tree of rings with degree at most six (resp. eight and ten).

We have shown that the call control problem is NP-hard and MAX SNP-hard even in depth-2 trees. We give optimal algorithms for the call control problem in double-stars and in spiders. We also give 2- and 3-approximation algorithms for the weighted call control problem in depth-2 and depth-3 trees, respectively. We show that the weighted call control problem is solvable in arbitrary trees if all the paths contain a same node of the tree.

We have shown that the Min-PMC and Max-PMC problems are NP-hard in k-fiber (k odd) stars. We give optimal algorithms for the following problems: the Min-PMC and Max-PMC problems in non-uniform stars with even fibers and in k-fiber (k even) spiders. We have given a 1.5-approximation algorithm for the weighted Max-RPC problem in rings.

We give efficient implementations of the Seymour and Thomas procedure which, given an integer β, decides whether a planar graph G has the branchwidth at least β or not. We tested our implementations on instances of size up to one hundred thousand edges. The experimental results show that our implementations run much faster and use less memory than previous implementations, even considering the speed difference of the computers used (see page 127 for details).

We propose divide-and-conquer based algorithms of using Seymour and Thomas procedure to compute optimal branch decompositions of planar graphs. Our algorithms have time complexity $O(n^3)$. Computational studies show that our algorithms are much faster than the edge-contraction algorithms and can compute the optimal branch decompositions for some instances of about 50,000 edges in a practical time. This provides useful tools for applying the branch decomposition based algorithms to practical problems.

We have given an optimal algorithm for the maximum edge-disjoint paths problem in planar graphs, and an approximation algorithm for the maximum path coloring problem. We also tested the practical performances of the algorithms on both real and randomly generated planar graphs (or graphs close to planar). The computational results coincide with the theoretical analysis of the algorithms: they are efficient for graphs with small carvingwidth when the load is not too large, but may not be practical for graphs with large carvingwidth and for large load.

7.2 Future Work

Many research efforts have been devoted to the research problems we studied in this thesis. However, there are still many open problems. In this section, we give a few directions for the future work.

We have given a 2.75-approximation algorithm for the Min-PC problem on trees of rings with arbitrary degrees. An interesting problem is to improve the 2.75-approximation ratio. Our results imply a 3-approximation algorithm for the Min-RPC problem on a tree of rings. It would be challenging to improve the approximation ratio of 3 for the Min-RPC problem.

Our $3L$ algorithm also implies a $6L$ algorithm for the Min-PC problem on directed trees of rings. It would be interesting to improve the approximation ratio for the Min-PC problem on directed trees of rings.

For the optimal branch decomposition of planar graphs, the following problems are unsettled and may be studied in the future:

- Our implementations of ST procedure may require at least $n^2/8$ bytes of memory for a graph of n edges. So they may not be able to solve extremely large instances with a few hundred thousands or more edges within a practical memory space. How to compute the branchwidth of extremely large planar graphs is an interesting open problem.

- Our divide-and-conquer based algorithms can compute an optimal branch decomposition of planar graphs of size up to 50,000 edges in a practical time on a PC with 3GHz CPU. It is still time consuming to compute optimal branch decompositions for planar graphs with more than 50,000 edges. An interesting future work is to design more efficient algorithms for very large planar graphs. Using better approaches to make balanced partitions is one possible direction to get such algorithms.

- All divide-and-conquer algorithms use the edge-contraction method to guarantee the branch decomposition can be found. However, the edge-contraction algorithm has never been called in our computational study. It would be interesting to prove that a valid partition can always be found efficiently in those algorithms.

- It would be interesting to develop a performance guaranteed and yet efficient heuristic for computing a good approximation of the carvingwidth, and for computing a carving decomposition that is close to optimal.

Our optimal algorithm for the maximum edge-disjoint paths problem and approximation algorithm for the maximum path coloring problem are not practical when the carvingwidth of the input graph is large, or when the load of the paths is large. Whether the performance of the algorithms can be improved is an interesting open problem. It is also worth to design heuristic algorithms which can compute a solution for larger instances within a practical time and memory, even if the solution may not be optimal.

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