

**ASSESSING CDOS UNDER ALTERNATIVE COPULAS**

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## Abstract

Synthetic collateralized debt obligations are popular vehicles for trading portfolios of credit risks. We present a copula based Monte Carlo simulation procedure for pricing them. Using the Gaussian copula of joint default times, we assess the risks of CDOs and their sensitivity to model parameters. Joint defaults are rare; many studies suggest Gaussian copula has limited ability to capture extreme events. We use the  $t$  copula to assess the risks of misspecifying tail dependence. The choice of copula is shown to significantly affect tranche prices.

**Keywords:** credit modeling; copulas; risk assessing

**Subject Terms:** risk management; copulas (mathematical statistics)

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# 1 Introduction

Credit derivatives are financial instruments whose value derives from the creditworthiness of underlying reference. They allow banks, and other financial institutions to efficiently manage and transfer credit risks.

There are two main groups of credit derivatives. Single name instruments: contracts whose payoff depend on the creditworthiness of underlying reference entity. Most common example of such instruments is credit default swaps, total return swaps. Many of the single name instruments are liquid. Multi-name instruments: constructs with payoffs contingent on the creditworthiness of a number of reference entities. Important examples are  $k$ -th to default basket swap and collateralized debt obligations (CDOs). These instruments are usually less liquid than the single name ones.

Credit default swaps (CDS), as a key component of the credit derivatives market in terms of volume, have seen substantial growth, with around 42 trillion US notional outstanding in June 2007 as highlighted by the Bank of international Settlements. At the same time, other instruments based on the CDS have seen substantial growth as well, such as basket default swaps, collateralized debt obligations (CDOs). One of the most important multi-name credit derivatives is the CDO. In cash CDO, the underlying collateral is usually a portfolio of corporate bonds or bank loans. The cash flow structure of a CDO passes payments from the collateral pool to a prioritized collection of securities called tranches. There are also synthetic CDOs, which use a pool of CDSs or other instruments such as total return swap (TRS).



Determining the price of each CDO tranche for which the investors is ready to support the risk associated with that tranche is the key challenge. The payoff is driven by default dependency of pair wise firms in the reference portfolio. There are two main traditional approaches to model default: the structural approach and the reduced form approach. Merton's model (1974) is considered the first structural model. In Merton's model, the assets value of the firm follows a geometric Brownian motion. The firm defaults if its assets are below its outstanding debt. Merton's model has been extended in many ways; among them are Black and Cox (1976), Schonbucher (1996), and Zhou (1997). The structural approach is intuitive and has a clear interpretation. However it is difficult to calibrate the current market data, which are the spreads of the CDSs. Therefore, its pricing applicability is limited. The reduced form model does not explain why the firm defaults. Instead, it models the default time using the default intensity, which can be inferred from market data. This approach was developed by Jarrow and Turnbull (1995), Duffie and Singleton (1999), Lando (1994, 1998). However, the most accepted pricing model in the industry is the Gaussian copula as discussed in Li (2000). This technique is also used as a core instrument in Credit Metrics. The computation of the Gaussian copula model requires the Monte Carlo simulation work and allows one to specify a joint default time distribution by combining individual credit spreads in the portfolio with their pair wise correlation. The modeling of joint default times is difficult because the joint defaults are rare. In Li's model, the marginal default times are assumed to be exponential, with marginal distributions tied together by the Gaussian copula. However, as Embrechts (2001) shows, the main pitfall of Gaussian copula is the small probability of extreme joint events. Based on that, other copula based models have been

developed such as  $t$  copula (Mashal and Naldi 2001) or Clayton copula (Rogge and Schonbucher 2003).

The rest part of the paper proceeds as follows: In section 1, we provide mathematical background on modeling marginal default times and the pricing principle for single name credit derivatives. In section 2, we focus on using copulas to model dependence structure. We define common copulas and some related properties. In section 3, we give some applications of copulas in pricing basket credit derivatives. In section 4, we assess the risk of CDO tranches with respect to several risk factors under the Gaussian copula. In section 5, we assess tranche prices and risks under the  $t$  copula and reveal some potential problems of the Gaussian model.

## 2 Modeling Default Time

As mentioned above, two types of models have been developed to characterize the default time: structural models, developed by Merton (1974), and reduced form models. In this section, we focus on concepts which are essential to understanding the reduced form models.

### 2.1 Poisson and Cox Process

The Poisson distribution is a discrete probability distribution that describes the probability of the number of events occurring in a fixed period of time. If the expected number of occurrences on a unit interval is  $\lambda$ , the probability of  $k$  occurrences in the unit

interval is defined as:  $f(k, \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$

The Poisson Process is a collection  $\{N(t) : t \geq 0\}$  of random variables, where  $N(t)$  is the number of event that have occurred up to time  $t$ . The number of events between time interval  $[a, b]$  is  $N(a) - N(b)$  and has a Poisson distribution.

Now, we consider a Poisson process with constant rate  $\lambda$  per unit time and the random variable  $\tau$ , which is the time one must wait to see the first event occur. The following two events are equivalent  $\{\tau > t\} = \{N(t) = 0\}$ . Hence,  $P(\tau > t) = e^{-\lambda t}$ . The cumulative distribution of  $\tau$  is  $F(t) = P(\tau \leq t) = 1 - e^{-\lambda t}$ . Furthermore, the density function of  $\tau$  can be found by differentiating  $F$  respect to  $t$ . We obtain  $f(t) = \lambda e^{-\lambda t}$ . For  $t > 0$ , this is the density function of exponential distribution.

Another important concept is the hazard rate. Let  $\tau$  be the time of first default.

The hazard rate function is defined as:  $h(t) = \lim_{\Delta t \rightarrow 0} P(t < \tau \leq t + \Delta t | \tau > t)$ . Rewriting the

above expression, we get some important relations among  $h(t)$ ,  $f(t)$  and  $F(t)$ :

$$h(t) = \lim_{\Delta t \rightarrow 0} \frac{P(t < \tau \leq t + \Delta t, \tau > t)}{P(\tau > t)} = \lim_{\Delta t \rightarrow 0} \frac{\int_t^{t+\Delta t} f(u) du}{\int_t^{\infty} f(u) du} = \frac{f(t)}{1 - F(t)} = -\frac{\partial}{\partial t} \log(1 - F(t))$$

And, solving the differential equation, we have

$$F(t) = 1 - \exp\left(-\int_0^t h(u) du\right)$$

From this point, we will use exponential distribution to denote the hazard rate.

Assuming constant hazard rate, the exponential distribution can be used to model the default time. If extending the setting to allow  $\lambda$  being a non-negative stochastic process, then  $N(t)$  defines a Cox process with intensity  $\lambda(t)$ . The survival probability function under the Cox process is:

$$S(t) = P(\tau > t) = 1 - F(t) = E\left[\exp\left(-\int_0^t \lambda(u) du\right)\right]$$

## 1.2 Pricing of Credit Default Swap (CDS)

The Credit default swap is one of the most important instruments in the credit derivative market. A CDS is a bilateral contract through which two counterparties can trade the default risk of particular reference entity. Under a CDS contract, the protection buyer pays a premium until the maturity time, or the reference default time if it happens before the maturity. When a credit event triggered, the protection seller either takes

delivery of a bond for the par value or pays the protection buyer the difference between the par value and the recovery value of the bond.

Given the model in the last section, we can model the default time by specify the hazard rate function. One of the common methods adopted for estimating  $\lambda(t)$  is inferring it from the market CDS quotes. The basic idea is to derive the implied default probability from a CDS with one maturity time, and then use this risk neutral probability estimate to value other derivatives based on same reference. Here, we are not concentrating on the estimation part; instead, we illustrate the pricing mechanism.

We can define the premium and default leg for a CDS. Assume constant risk-free interest rate and the premium is paid continuously. When default happens, the protection seller pays the protection buyer the difference between the par value and the recovery value of the bond. The recovery rate, which is the residual value of the bond when default happens, is assumed to be  $R$ . Then value of the premium leg is:

$$PL = E \left[ s \int_0^T S(t) B(0, t) dt \right] = E \left[ s \int_0^T e^{-\int_0^t \lambda(u) du} e^{-rt} dt \right]$$

where  $s$  is the continuous premium that the protection buyer pays until the minimum of maturity time and the default time.  $r$  is the risk-free interest rate.  $B(0, t)$  is the discount factor which discounts value from time  $t$  to present, and the expectation is over  $\lambda$  paths under the risk neutral probability distribution. Value of the default leg is:

$$DL = E \left[ \int_0^T (1 - R) f(t) B(0, t) dt \right] = E \left[ \int_0^T (1 - R) \lambda(t) e^{-\int_0^t \lambda(u) du} e^{-rt} dt \right]$$

Here  $f(t)$  is the probability density of default time. The fair price (spread)  $s^*$  of CDS is defined as:

$$s^* \Rightarrow PL(s^*) - DL(s^*) = 0 \quad \text{and we obtain: } s^* = \frac{E \left[ \int_0^T (1-R)\lambda(t) e^{-\int_0^t \lambda(u) du} e^{-rt} dt \right]}{E \left[ \int_0^T e^{-\int_0^t \lambda(u) du} e^{-rt} dt \right]}$$

From above formula, we can see the spread depends on both the recovery rate  $R$  and the default probability through the  $\lambda(t)$ . Therefore, assuming some empirical estimate of the value of  $R$ , it is possible to calibrate the hazard rate function from the CDS spreads with different maturities.

### 3 Dependence and Copula Functions

Specifying a default time distribution for individual credit is often straightforward. But deciding what dependencies should exist in a portfolio of credits may not be. Modeling dependence between default events is one of the major components in credit risk modeling. The most obvious reason is that dependence affects the loss distribution of the portfolio. Using either empirical data or current market data, we can derive the marginal distribution of survival time for each individual credit in portfolio. If credits in the portfolio are independent, we can study the loss distribution without further modeling. However, realistically, when the economy is booming, the default rates tend to be lower. When the economy is in a recession, default rates tend to be higher and bonds tend to default together. It implies that the default events should depend on some common macroeconomic factors; in other words, there exist some positive relationships among those default events. Therefore it is reasonable to incorporate positive correlations in our modeling.

It can be difficult to actually generate random variables with dependence relation when they have distributions not from a standard multivariate distribution. Further, some of the standard multivariate distributions can model only very limited types of dependence. In this paper, we used the copula function approach to model the dependence relationships among portfolio of bonds.

### 3.1 Definition and Basic Properties of Copula Function

In the context of credit risk modeling, copula functions are mainly used for creating families of multivariate distributions with given marginal distributions.

Briefly, a  $d$ -dimensional copula is a distribution function on  $[0,1]^d$  with standard uniform marginal distributions. The following theorem stated in Galiani (2003) constitutes the most relevant result in a copula framework.

**Sklar's Theorem:** Let  $G$  be an  $n$ -dimensional distribution function with marginal Distribution functions  $F_1, F_2, \dots, F_n$ . Then there exists an  $n$ -dimensional copula  $C$  such that for  $x \in \mathbb{R}^n$  we have

$$G(x_1, x_2, \dots, x_n) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n)).$$

Moreover, if  $F_1, F_2, \dots, F_n$  are continuous, then  $C$  is unique.

Sklar's theorem describes how the marginal distributions are tied together in the joint distribution. In another word, the joint distribution can be decomposed into marginal and the copula. Given we can estimate the marginal, the dependence structure can be captured by the copula.

### 3.2 Different Types of Copulas

Here, we introduce two common copulas used in the later simulations from the elliptical family.

#### **Multivariate Gaussian copula**

Let  $\Sigma$  be a symmetric, positive definite matrix with diagonal elements equal to



one and let  $\phi_{\Sigma}$  be the standardized multivariate normal distribution function with correlation matrix  $\Sigma$ . Then the multivariate Gaussian copula is defined as

$$C(u_1, u_2, \dots, u_n; \Sigma) = \phi_{\Sigma}(\phi^{-1}(u_1), \phi^{-1}(u_2), \dots, \phi^{-1}(u_n))$$

where the  $\phi^{-1}(u)$  denotes the inverse of the normal cumulative distribution function.

Figure 1 shows the scatter plots of simulated standard uniform random values for various levels of rho with Gaussian copulas.

### **Multivariate Student's $t$ copula**

Let  $\Sigma$  be a symmetric, positive definite matrix with diagonal elements equal to one and let  $T_{\Sigma, \nu}$  be the standardized multivariate Student's  $t$  distribution with correlation matrix  $\Sigma$  and  $\nu$  degree of freedom. Then the multivariate Student's  $t$  copula is defined as

$$C(u_1, u_2, \dots, u_n; \Sigma, \nu) = T_{\Sigma, \nu}(t_{\nu}^{-1}(u_1), t_{\nu}^{-1}(u_2), \dots, t_{\nu}^{-1}(u_n))$$

where  $t_{\nu}^{-1}(u)$  denotes the inverse of the Student's cumulative distribution function.

Figure 2 shows the scatter plots of simulated uniform random values with the Gaussian and  $t$  copula functions. Under the same rho value, we can see that the  $t$  copula have more extreme co-movements than the Gaussian copula.

### **3.3 Tail Dependence**

The concept of tail dependence measures the amount of dependence in the tail of joint distribution. It turn out that the tail dependence between two variables is a copula

property. Then the amount of tail dependence is invariant under strictly increasing transformations of variables.

From Mashal, R. and M. Naldi (2001), the upper tail dependence coefficient of  $X$  and  $Y$  is defined as:

$$\theta_U = \lim_{\mu \rightarrow 1} P(Y > G^{-1}(\mu) | X > F^{-1}(\mu))$$

provided that the limit of  $\theta_U \in [0,1]$  exists, where  $X$  and  $Y$  are random variables with continuous distributions functions  $F$  and  $G$  respectively.

If  $\theta_U \in (0,1]$ , then  $X$  and  $Y$  are said to be asymptotically dependent in the upper tail; if  $\theta_U = 0$ ,  $X$  and  $Y$  are said to be asymptotically independent in the upper tail.

Then From Embrechts (2001), we can write tail dependence for the Gaussian copula  $C_\rho^{Ga}$  over  $[0,1]^2$  with correlation  $\Sigma_{12} = \rho$  as:

$$\theta_U = 2 \lim_{x \rightarrow \infty} (1 - \Phi(x\sqrt{1-\rho} / \sqrt{1+\rho})).$$

Thus the Gaussian copula gives asymptotic independence when  $\rho < 1$ . Regardless of how high a correlation we choose, if we go far enough to the tail, extreme events will occur independently. However, the  $t$  distribution provides an interesting contrast. The tail dependence for the bivariate  $t_{v,\Sigma}$  copula with correlation  $\rho$  can be shown to be:

$$\theta_U = 2 \left( 1 - t_{v+1} \left( \frac{\sqrt{v+1}\sqrt{1-\rho}}{\sqrt{1+\rho}} \right) \right)$$

Here  $\nu$  is the degrees of freedom. The above formula indicates that for random variables that linked by a  $t$  copula we can expect joint extreme movements to occur with non-negligible probability even when the correlation is small. For example, given

$\nu=4$  and  $\rho = 0.5$ , we have  $\theta_U = 0.25$ . Moreover, even if  $\rho = 0$ , we still have  $\theta_U = 0.08$ .

The strength of this dependence increases as  $\nu$  decreases and the marginal distribution become heavier-tailed. This is also one of the fundamental differences between the  $t$  and Gaussian dependence structure.

## 4 Basket Credit Derivatives

### 4.1 Pricing of $k$ -th to Default Basket Swaps

We will use the following notation in both CDS and CDOs pricing:

- $N$  is the number of reference entities included in the collateral pool.
- $A_i$  is the notional amount of the  $i$ -th reference obligation.
- $M$  is the total notional amount, which is the summation of the notional amounts of all credits in the reference portfolio.
- $R_i$  is the recovery rate of the  $i$ -th reference obligation.
- $T$  is the maturity of the contract.
- $\tau_{(k)}$  is the  $k$ -th default time for reference portfolio.
- $D(0,t)$  is the risk-free discount factor.
- $S$  is the fair spread of the basket swap contract, expressed as a percentage of  $M$ , to be paid by the protection seller until either  $T$  or  $\tau_{(k)} < T$  in case of payout.
- $L_i$  is the loss for  $i$ -th reference entity, and equal to  $A_i(1 - R_i)$ .

The fair market price of the  $k$ -th to default basket swaps is calculated by equating the expected value of the loss leg and the expected value of premium leg under the risk neutral distribution. Formally, we can define  $F_{(k)}(t) = P(\tau_{(k)} \leq t)$  the distribution function of the  $k$ -th default time. With homogenous notional amounts  $A$ , the value of loss leg can be shown to equal:

$$TL = E^* \left[ \int_0^T A(1 - R) f_{(k)}(t) D(0, t) dt \right]$$

Assuming continuous payment until the earlier of  $k$ -th default time and maturity date, the premium leg can be shown to equal:

$$PL = E^* \left[ SM \int_0^T (1 - F_{(k)}(t)) D(0, t) dt \right]$$

Hence, the fair price of the  $k$ -th to default basket swaps is the spread  $S^*$  such that the loss leg equal to the premium leg and can be written as:

$$S^* = \frac{E^* \left[ \int_0^T A(1 - R) f_{(k)}(t) D(0, t) dt \right]}{E^* \left[ M \int_0^T (1 - F_{(k)}(t)) D(0, t) dt \right]}$$

### 4.2 Pricing of Collateralized Debt Obligations

The framework for pricing  $k$ -th to default basket swaps can be extended to price the reference pool with more individual credits. Collateralized debt obligations are an important class of portfolio product. They normally contain more than 100 credits.

Considering a CDO with  $N$  obligors, credit  $i$  has nominal amounts  $A_i$ , recovery rates  $R_i$ . The maturity time is  $T$  years and interest rate is assumed to be constant. We use the random variable  $Q_i(t) = 1_{\{\tau_i < t\}}$  as default indicator at time  $t$  for the  $i$ -th obligor.

The accumulated loss in the portfolio can be shown to be:

$$L(t) = \sum_{i=1}^N L_i Q_i(t).$$

The loss distribution for each tranche depends on the structure of the CDO. If we use  $Y$  and  $\Lambda$  to denote the lower and upper bound of a given tranche, the equity tranche

will have a  $Y=0$  and the senior tranche will have a  $\Lambda = \sum_{i=1}^N A_i$ . The cumulative loss on a given tranche will be zero if  $L(t)$  is less than  $Y$ , equal to  $L(t) - Y$  if  $Y \leq L(t) \leq \Lambda$ , and  $\Lambda - Y$  if  $L(t) \geq \Lambda$ . Then, we can write the general loss function for any given tranches as:

$$L(t)^{Y,\Lambda} = [L(t) - Y]1_{\{Y \leq L(t) < \Lambda\}} + (\Lambda - Y)1_{\{L(t) \geq \Lambda\}}.$$

Assuming the default payment is made at the maturity time, the fair price of each tranche can be defined as the ratio of the expected accumulated default payment for that tranche to the maximum possible payment, which is the tranche notional amount.

In this paper we use a Monte Carlo procedure to calculate the tranche prices. In the simulation, the ratios of loss to tranche notional are computed for every scenario individually, and the price of CDO tranches is estimated by the average of the present values. Under a Gaussian copula, the pricing of the CDO tranches with the Monte Carlo method is accomplished by the following steps:

Step 1: Simulate the  $m_i$  correlated  $N(0,1)$  random variables,  $i = 1, \dots, N$ .

Step 2: Transform to uniform variables  $U_i = \Phi(m_i)$  where  $\Phi$  is the cumulative normal distribution function.

Step 3: Transform to exponential default times for each credit:  $\tau_i = \frac{-\ln U_i}{\lambda}$ .

Step 4: From the simulated default times, compute the loss at maturity time  $T$  for each tranche as:  $L(T)^{Y,\Lambda} = [L(T) - Y]1_{\{Y \leq L(T) < \Lambda\}} + (\Lambda - Y)1_{\{L(T) \geq \Lambda\}}$ .

Step 5: Repeat steps above until the required number of scenarios  $m$  have been simulated.

The estimator of expected loss at maturity can be computed as the average of all

scenarios: 
$$E[L(T)^{Y,\Lambda}] = \frac{1}{m} \sum_{k=1}^m L(T)_k^{Y,\Lambda}.$$

Step 6: Discount the tranche notional amounts and expected losses at maturity. The tranche price per dollar of notional can be computed as the ratio of the present value of expected tranche loss to tranche notional value.

## 5 Numerical Example of a Synthetic CDO

In financial markets, the collateralized debt obligations are a type of structured asset-backed products. CDOs have exposure to a portfolio of fixed income assets and they divide the risk among different tranches: senior, mezzanine, and equity. Losses are applied in reverse order of seniority. The equity tranche offers a higher return, assuming no defaults, to compensate for the additional risk.

Cash CDO vehicle actually purchase a reference portfolio of corporate bonds or loans. In contrast, a synthetic CDO is an instrument where the underlying collateral is a set of credit default swaps (CDS). Furthermore, synthetic CDOs can be funded or unfunded, depending on whether the tranche investors are required to fund their credit exposures at origination. The risk of loss on the reference portfolio is divided into tranches of increasing seniority. Losses first affect the equity tranche, next the mezzanine tranche, and finally the senior tranche. CDO investors take credit risk on different tranches by taking the protection seller position on the reference portfolio via single-name credit default swaps. The protection buyers from the CDS are essentially the “sellers” of credit risk to the CDO investors, with the CDO issuer as the intermediary.

To illustrate the risk of the CDO tranches, we consider an example of a hypothetical \$1 billion synthetic CDO with three tranches. The equity tranche bears the first \$30 million of losses, the mezzanine tranche bears the next \$70 million, and the senior tranche bears any losses above \$100 million. In this example, all the three tranches are unfunded. There are 100 credits in the reference portfolio. All have the following base case characteristics:



- Individual CDS credit spread: 100 basis points
- Individual notional amount: 50% 5 million and 50% 15 million
- Recovery rate: mean of 40%
- Default time correlation: 20%
- Interest rate: 5% (constant)

Risk of synthetic CDO tranches are subject to mark-to-market movements as well as the defaults over time. For example, if someone holds a long position on the mezzanine tranche, an increase in the credit spread of an individual credit may cause a loss as the expected loss of the portfolio has increased. An increase in the correlation in the underlying credit may also change the value of mezzanine tranche. Consequently, understanding of the sensitivities to those risk factors is the key element in the risk management of CDO tranches.

Figures 3-5 show the price surfaces of the three tranches with different assumptions on the credit spreads and correlations. In the example, we use the ratio of expected value of the default payment and the maximum possible payment of each tranche to describe the 'price' of the CDO tranches. The figures show that there are different relationships between correlations and the CDO tranche prices. For the equity tranche, there is a negative relationship between the correlations and tranche prices, while the opposite relationship can be found in the senior tranche. For the Mezzanine tranche, the relationship is ambiguous. In the later section, we will examine the dependence of the mezzanine price on the tranche structure in details. With regard to the credit spread, all of the three tranche prices increase with the portfolio-wide credit spread.

## 5.1 Default Rate Sensitivity

One of the most important factors that drive the risk of synthetic CDO is the credit quality of the underlying reference portfolio. Given the credit spread reflects the market perceptions on the creditworthiness of the company; the wider the spread the greater is the risk of default. However, the risk depends not only on the level of the spread but also on the sensitivity to changes the credit spread.

To measure the sensitivity of the price to the spread, we use the ratio of the change of the tranche price per basis point shift in the CDS spread.

Figure 6 presents a measure that shows the impact of increasing one individual credit spread on the value of tranches. The figure shows that for the different tranches levels, the sensitivity is higher for the tranche with the lower subordination. This implies that the equity tranche is much more sensitive to individual spread movements than the mezzanine and senior tranches.

Furthermore, the sensitivity also depends on the credit quality of the reference entity. This allows us to identify which credit is riskier than another in the portfolio.

In addition to individual credit risk, we also look at systematic risk. Figure 7 displays the price change of the CDO tranches per basis point increase in spread of all credits in the reference portfolio. The sensitivity is highest for the equity tranche when the systematic spread is low. However, the differences in the sensitivity shrink as systematic spread increases. The mezzanine tranche is more sensitive than the equity tranche once the systematic spread exceeds 0.25. In the extreme, the senior tranche is the riskiest since the equity and mezzanine tranches will almost be wiped out. Any further increase in the spread will not influence the tranche price.

Although the change of in tranche price with respect to credit spread is a good measure for the sensitivity of a CDO tranche, it is inadequate if we want to hedge against the movements in the CDS spread. We define the tranche delta for a specific underlying credit as the ratio of the price change of the CDO tranche per one basis point rise in that credit's CDS spread to the change in the value of the CDS for the same credit and term to maturity. This tells us the notional amount of single name CDS we would have to purchase to hedge against fluctuations in that name's spread.

We examine the relations between the tranche delta and the following factors:

- The credit spread of the underlying credit
- Tranche structure
- Time to maturity

As shown above, the equity tranche is the riskiest. This implies we need a higher notional amount of CDS to hedge the tranche against individual spread movements. We assume the spread movement is from a CDS with notional amount of fifteen million and with other assumptions as in the base case.

## 5.2 Tranche Structure

Changes in subordination level change the tranche loss distributions as well as their risk profiles. To see the dependency of price on subordination level, we choose the mezzanine tranche as an example. Figure 9 shows that the mezzanine delta decreases as subordination increases. This is an intuitive result. A mezzanine tranche with less subordination is more vulnerable to default as there is less protection. Furthermore,

Figure 10 shows that the width of the tranche also matters. A wider tranche means a less sensitivity of the mezzanine tranche value to changes in the underlying credit spread.

### 5.3 Time to Maturity

As maturity time decreases, the tranche deltas also change. Figure 11 shows that the equity tranche becomes riskier compared to the other tranches as there is less time for defaults to hit the mezzanine or senior tranches. In the extreme case, when the maturity time approaches zero, the senior and mezzanine tranche deltas converge to zero because of the negligible of being hit by defaults.

### 5.4 Correlation and Recovery Rate

Tranche deltas are also influenced by the underlying default correlations. As the correlation increases, the underlying credits tend to default together; risk is shifted from the equity tranche to higher subordination tranches. So the senior tranche delta increases and the equity tranche delta decreases. In a later section, this property is illustrated in with different measurement.

### 5.5 Correlation Sensitivity

Another factor affecting the tranche prices is the correlation of underlying credits. A synthetic CDO is a portfolio product, which means that the loss distribution depends on the correlation of the underlying reference pool. Figure 12 shows how the values of the three tranches vary with the correlations among the credits in the reference portfolio. In the figure, we vary the correlation from 0 to 1.

The effect of correlation on the CDO tranches is clear. Assuming no recovery, correlation one implies that the three tranches will all be wiped out together whenever any one of the credits defaults. Thus the senior tranche should have the same 'price' as the mezzanine and equity tranches under perfect correlation. As correlation decreases, the price of the senior decreases and vice versa for the equity. The mezzanine tranche is less sensitive to correlation, but the exact relation depends on the CDO structure.

## 6 Effect of Copula Choice

As Embrechts (2001) shows, there are many pitfalls to the normality assumption. The main pitfall that we are concerned with is the small probability of extreme joint events. Alternative dependence models have been proposed in the literature; the Clayton, Frank, Gumbel and  $t$  are the most notable. Mashal and Zeevi (2003) compares the different copulas above in the context of modeling joint financial returns behavior. In a formal statistical test, they find the  $t$  copula provides a better fit than others. Dobrić (2005) finds similar results using a chi-square test. In this section, we generalize the commonly used Gaussian copula to the  $t$  copula. The  $t$  dependence structure puts more probability mass on joint extreme events. This might be manifest in the events of recent credit market crisis.

In our simulation study, we first examine the portfolio loss distribution by varying portfolio size, portfolio-wide credit spread and degrees of freedom  $\nu$  of the  $t$  copula. At the same time, the covariance matrix for the copula is kept fixed as  $\nu$  changes. There are three groups of increases in the credit spread, which we label A, B and C.  $N$  represents the portfolio size. All the results are from 100 thousand scenarios.

**Table 6.1 Quantile of Portfolio Loss Distribution**

N	Group	$Q_{0.99}$			$Q_{0.995}$		
		$\nu = \infty$	$\nu = 20$	$\nu = 3$	$\nu = \infty$	$\nu = 20$	$\nu = 3$
10	A ( $\lambda = 0.01$ )	27.5846	27.5733	30.8549	45.1517	44.6827	59.8645
10	B ( $\lambda = 0.05$ )	64.9501	64.6886	68.7832	88.1165	88.0458	93.2451
10	C ( $\lambda = 0.07$ )	76.1707	76.1557	78.1776	98.4072	98.4074	99.6238
100	A ( $\lambda = 0.01$ )	16.5336	16.9090	25.2218	26.9236	27.5344	46.9878
100	B ( $\lambda = 0.05$ )	50.4477	50.9046	55.9205	64.6973	65.3064	72.4350
100	C ( $\lambda = 0.07$ )	60.6667	60.6252	63.6878	74.7410	74.2988	78.1421

From the simulation, it is not difficult to see that the expected portfolio loss is very close to the summation of the individual expected loss. However, the more interesting point here is the high quantiles of the loss distribution, which give a better indication of the extreme risk in the model. We record the 95 and 99 percent quantiles of the loss distribution measured as a percentage of the total notional amount. Here, we can clearly see that  $\nu$  has a massive influence on these risk measures, especially for groups with larger portfolio sizes which we can assume to be well diversified. If we only specify the copula and do not fix the degrees of freedom, the risk measure is subject to huge model risk.

**Table 6.2 Quantile of Tranche Loss Distribution**

	$Q_{0.95}$			$Q_{0.99}$			Mean		
	$\nu = 3$	$\nu = 20$	$\nu = \infty$	$\nu = 3$	$\nu = 20$	$\nu = \infty$	$\nu = 3$	$\nu = 20$	$\nu = \infty$
AE	15.3534	86.9927	99.2239	100	100	100	42.1217	61.6057	66.3601
BE	100	100	100	100	100	100	81.6158	91.1528	92.7123
CE	100	100	100	100	100	100	94.8954	97.5274	97.8763
AM	0	0	0	100	100	100	22.6364	27.2673	28.3986
BM	83.8126	100	100	100	100	100	58.6995	67.9189	70.0697
CM	100	100	100	100	100	100	80.9591	86.2198	87.1742
AS	0	0	0	16.9312	8.9023	7.1712	2.2833	1.2257	0.9773
BS	0	0.8715	1.2534	38.9326	30.9831	29.6014	8.1920	7.1560	6.9213
CS	9.2122	10.1687	10.4434	51.1123	45.7978	44.8665	15.0697	14.6078	14.5244

Note: The table gives both the mean and quantiles for the three tranches; the portfolio size is 100; A, B, C represent three groups of credit quality in Table 1; AE represent equity tranche from group A.

Table 6.2 shows the tranche loss distribution under the Gaussian and  $t$  copulas. The results show the significant impact of dependence structure on the loss distributions across tranches. Given the same credit spreads, the expected losses are clearly redistributed from the equity tranche to senior tranche as the degree of freedom  $\nu$  decreases. In the case of extreme co-movement, the Gaussian assumption may

underestimate the fair compensation for senior tranche investors and overestimate the fair value for the equity tranche holders. Furthermore, the Gaussian copula risk assessments seem to be more optimistic for the senior tranche. This is one of the criticisms of many senior tranches being assigned high investment grades, since they have turned out to be much riskier than similarly rated corporate bonds. Table 6.2 also shows that the risk measure doubles in some of senior tranches when a  $t$  copula is used with low degrees of freedom compared to the Gaussian.

Figure 13 plots the tranche deltas under the Gaussian and  $t$  copulas. It reveals that we need a higher hedge ratio for the equity tranche and a lower ratio for the mezzanine tranche under the  $t$  copula dependence structure.



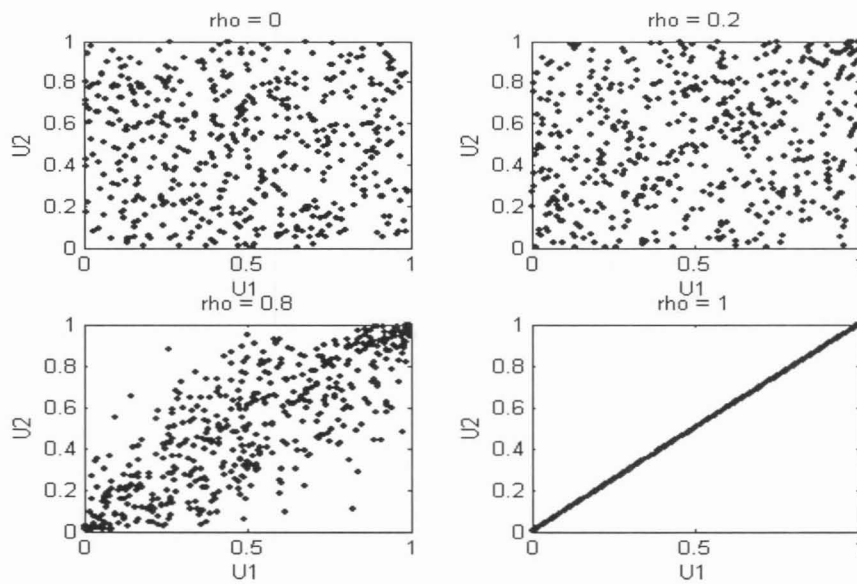
## 7 Conclusion

To sum up, this paper presents a simulation-based methodology for multi-name credit derivatives. It shows how CDOs and the other basket credit derivatives can be valued in the copula framework. We assess how different factors, such as credit quality, correlation and tranche structure affect tranche risks. As expected, the equity tranche is the riskiest under 'normal' conditions. When correlation increases, risk shifts from the equity tranche to the senior tranche. The effect of correlation on the mezzanine tranche is ambiguous in our study. The risk profiles of the tranches also depend on tranche structure and the maturity time. To accommodate the heavy tail effect, the  $t$  copula is chosen. Compared with the Gaussian copula, the equity tranche prices are lower and senior tranche prices are higher. Considering the recent credit market meltdown, we conclude that risk assessment using the  $t$  copula may be more appropriate. Future research should focus on model testing and calibration using time series and market price data.

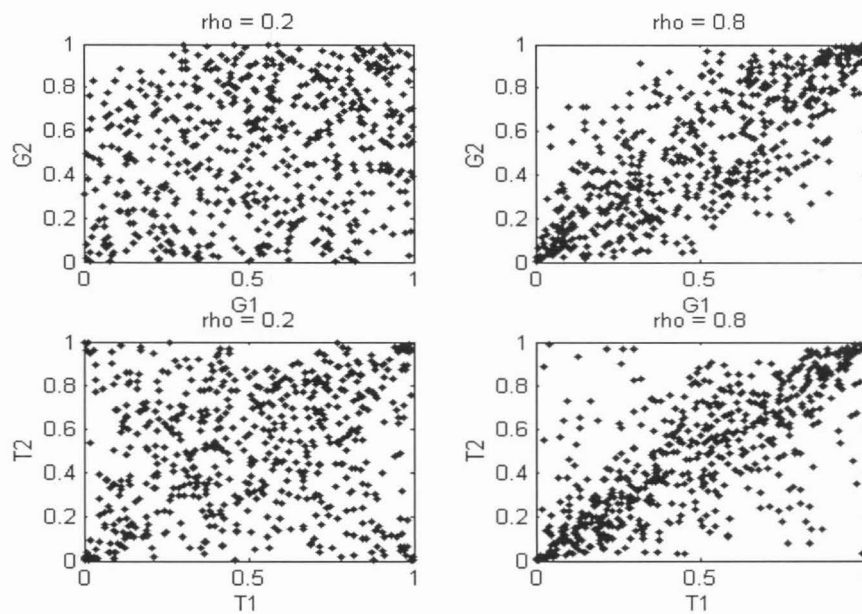
# Appendices

## Appendix A: Figures

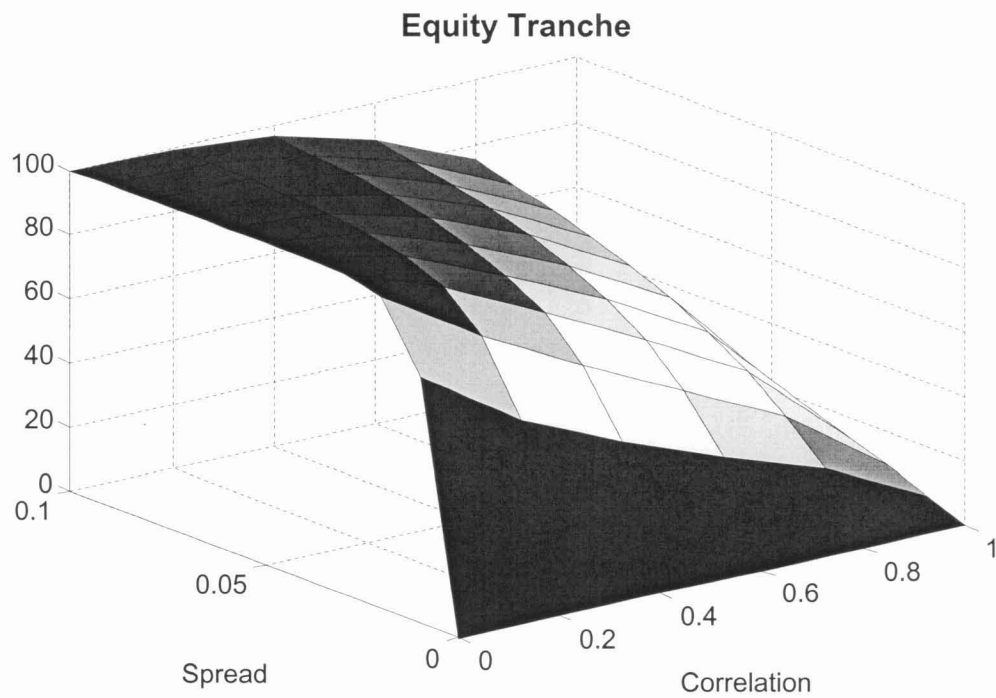
**Figure 1: 500 simulated random variables under Gaussian copula with different rho values**



**Figure 2: 500 simulated random variables under Gaussian and t copula**



**Figure 3: Equity CDO tranche price with Gaussian copula**



**Figure 4: Mezzanine CDO tranche price with Gaussian copula**

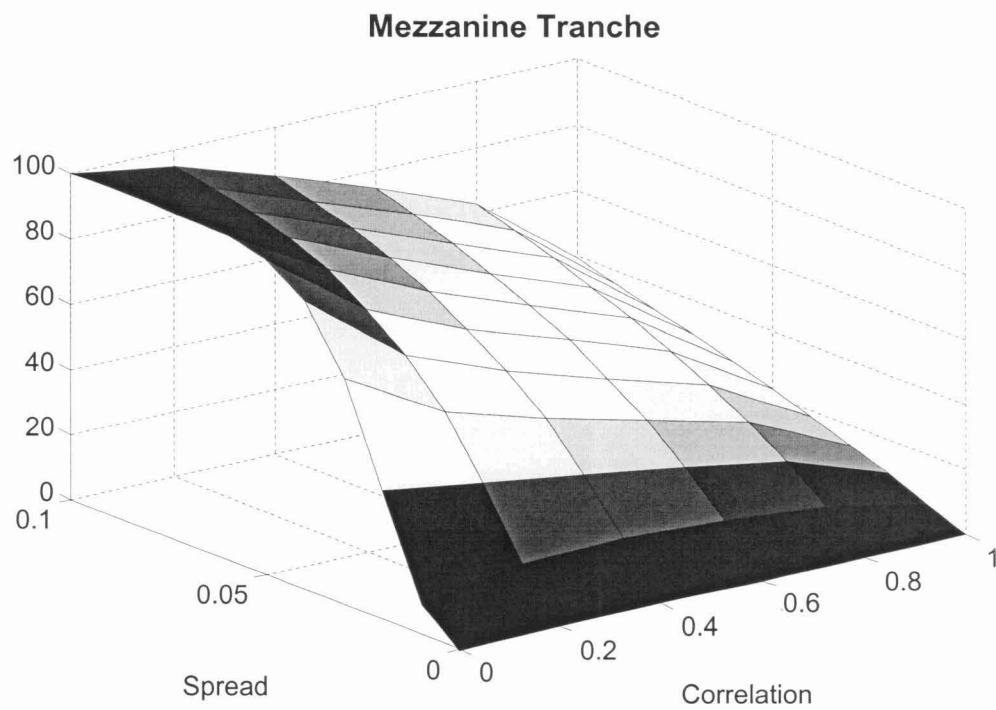


Figure 5: Senior CDO tranche price with Gaussian copula

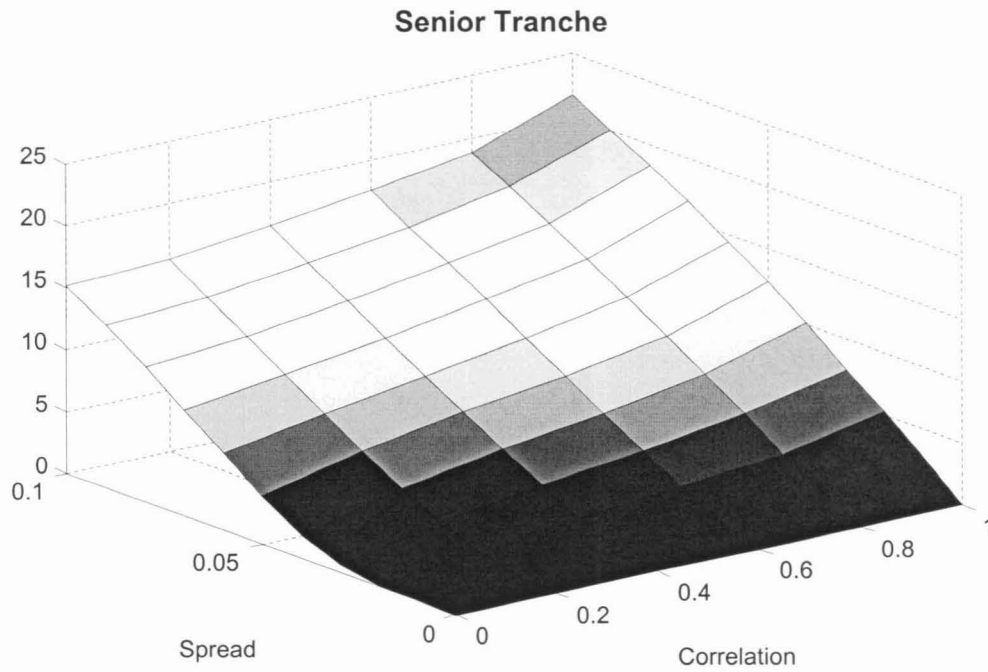
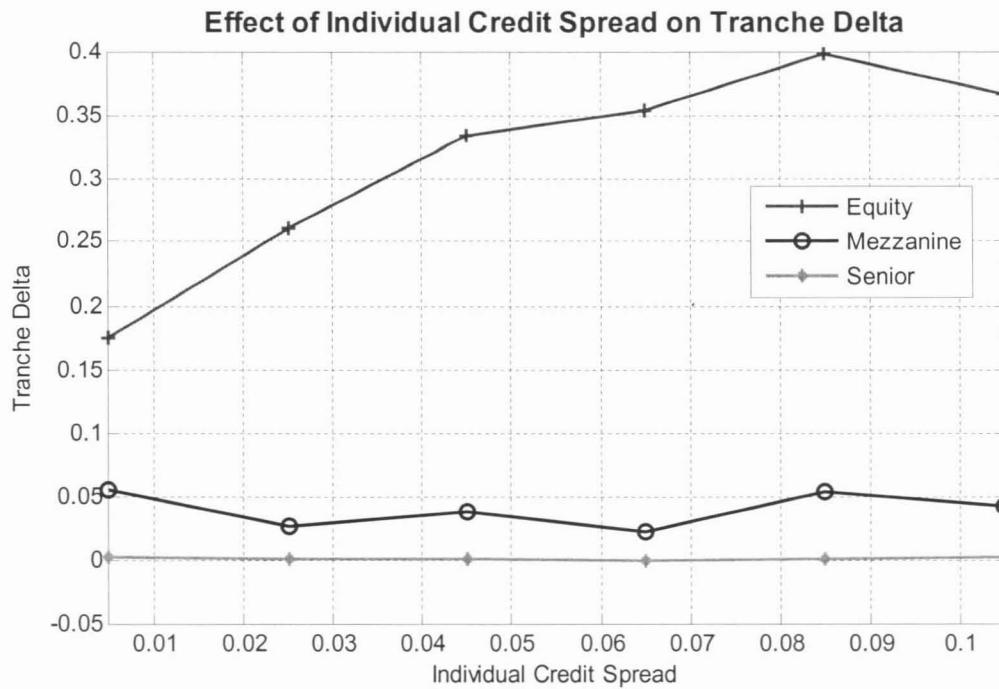
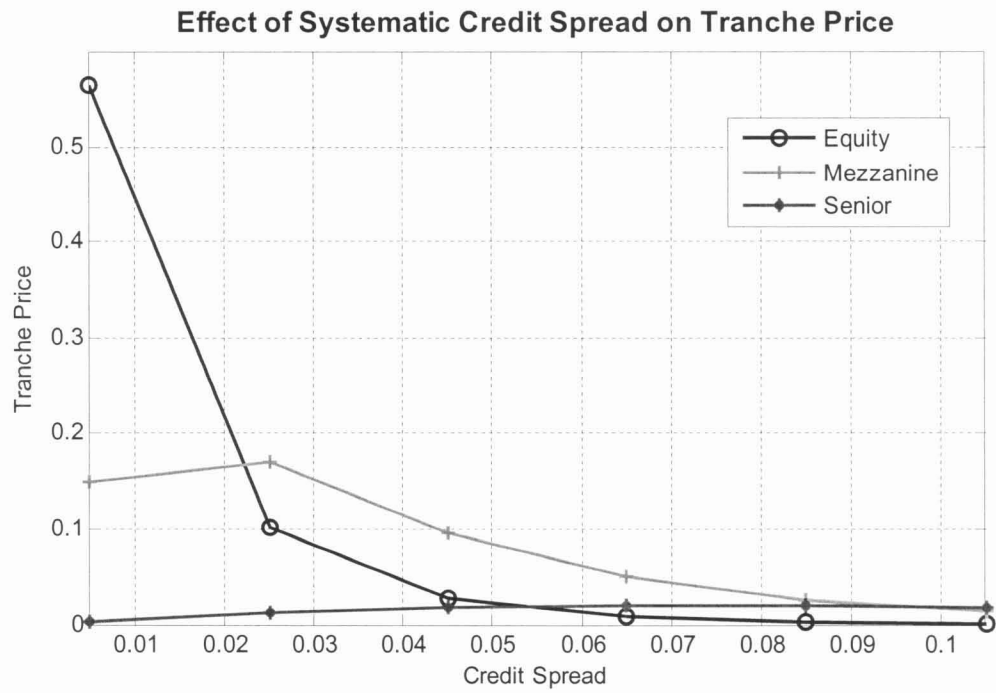


Figure 6



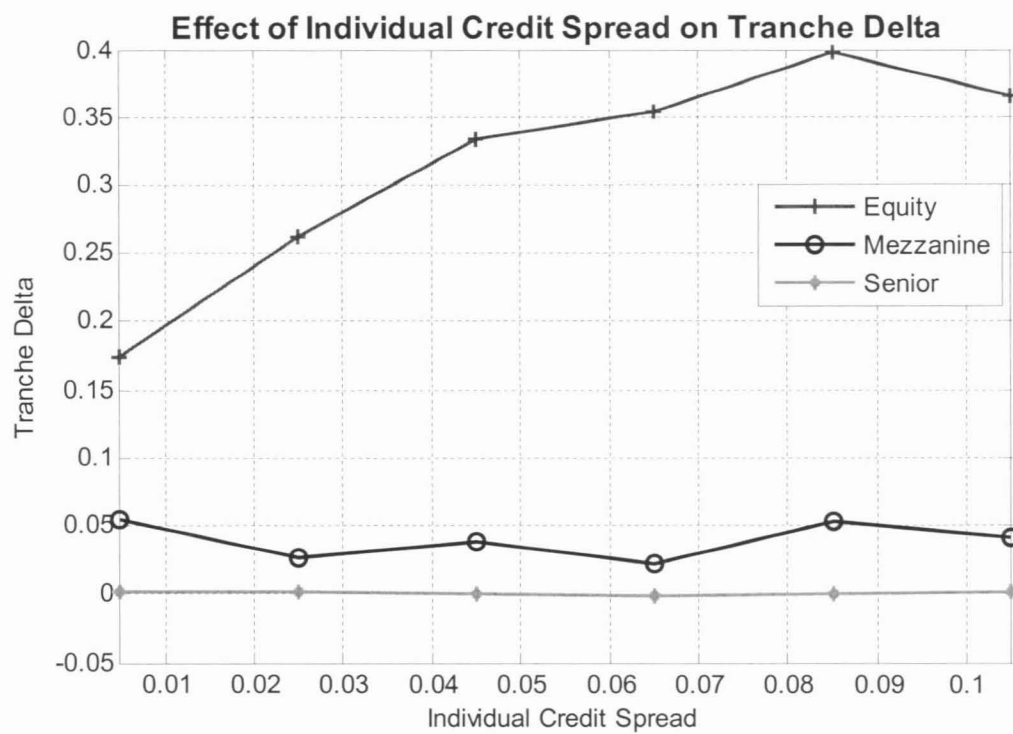
Note: The tranche price is defined as percentage of tranche notional amount.

Figure 7



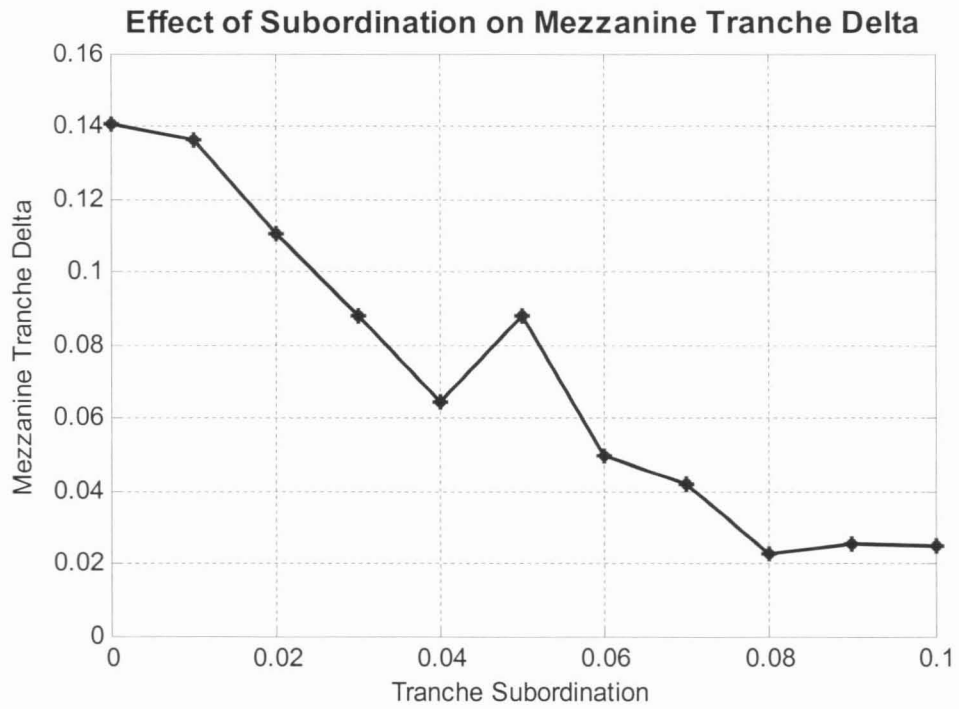
Note: The tranche price is defined as percentage of tranche notional amount. Credit Spread is the portfolio wide spread.

Figure 8



Note: To hedge individual credit spread change, tranche delta represents the notional amount of CDS need to hold for per dollar tranche notional holding.

**Figure 9**



Note: Tranche subordination is the width of equity tranche in this CDO structure.

**Figure 10**

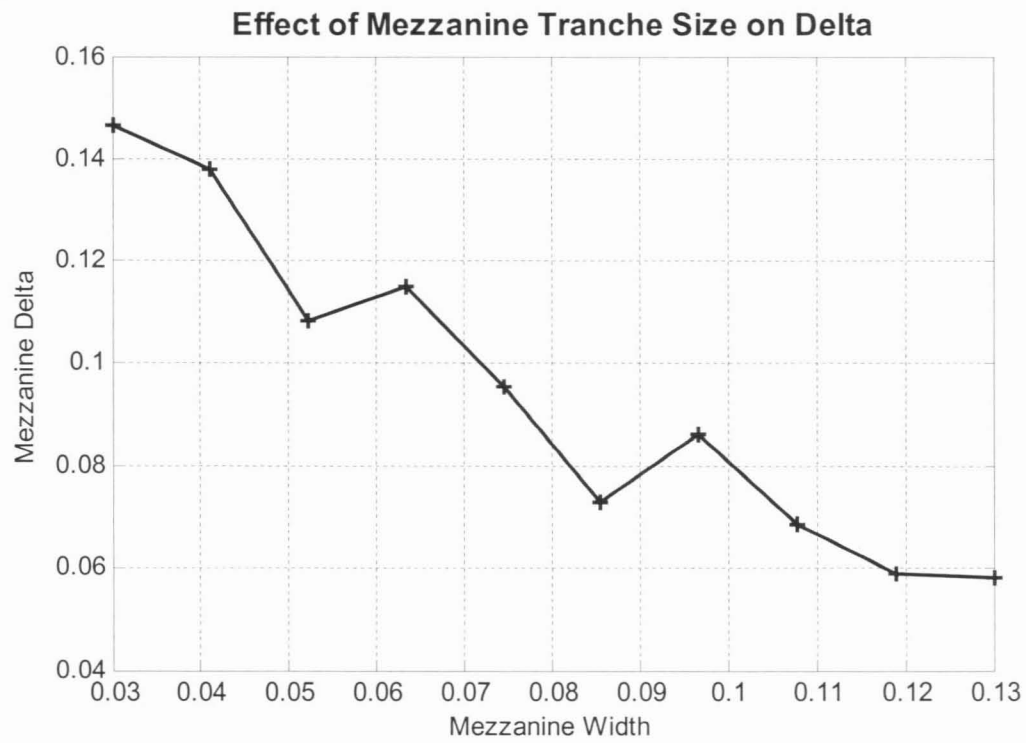


Figure 11

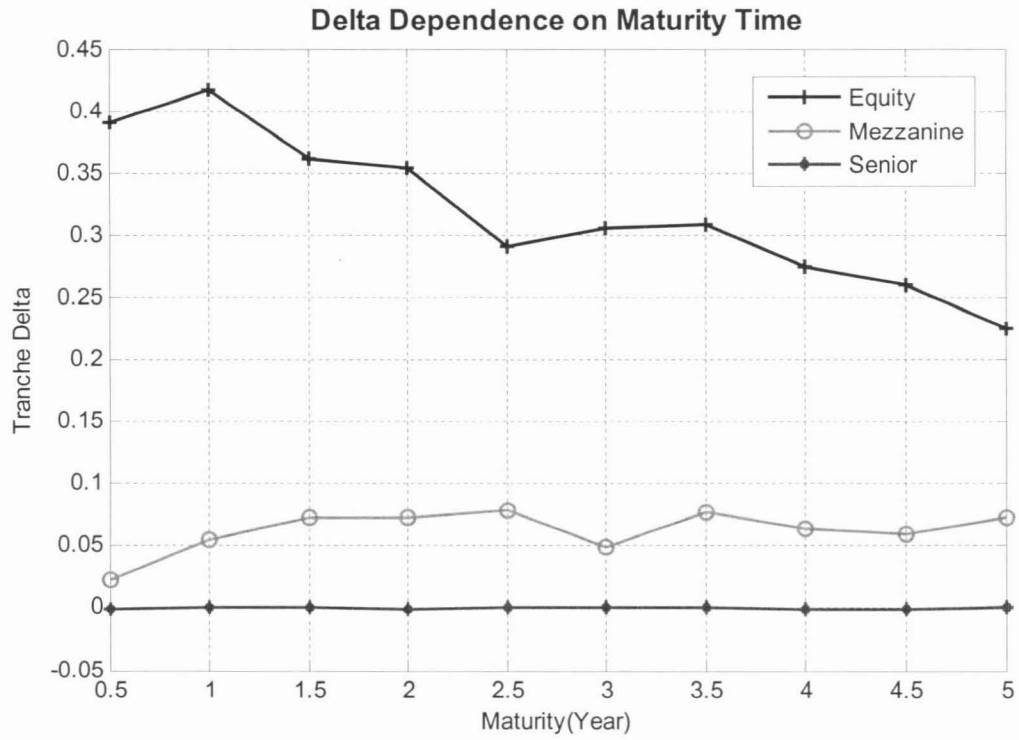


Figure 12

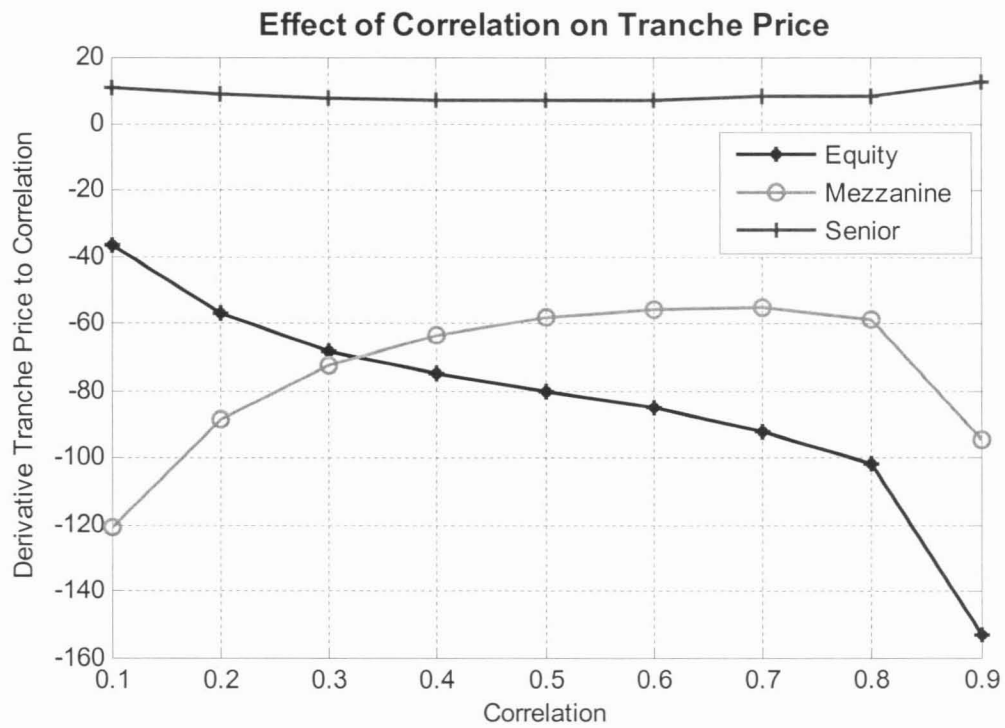
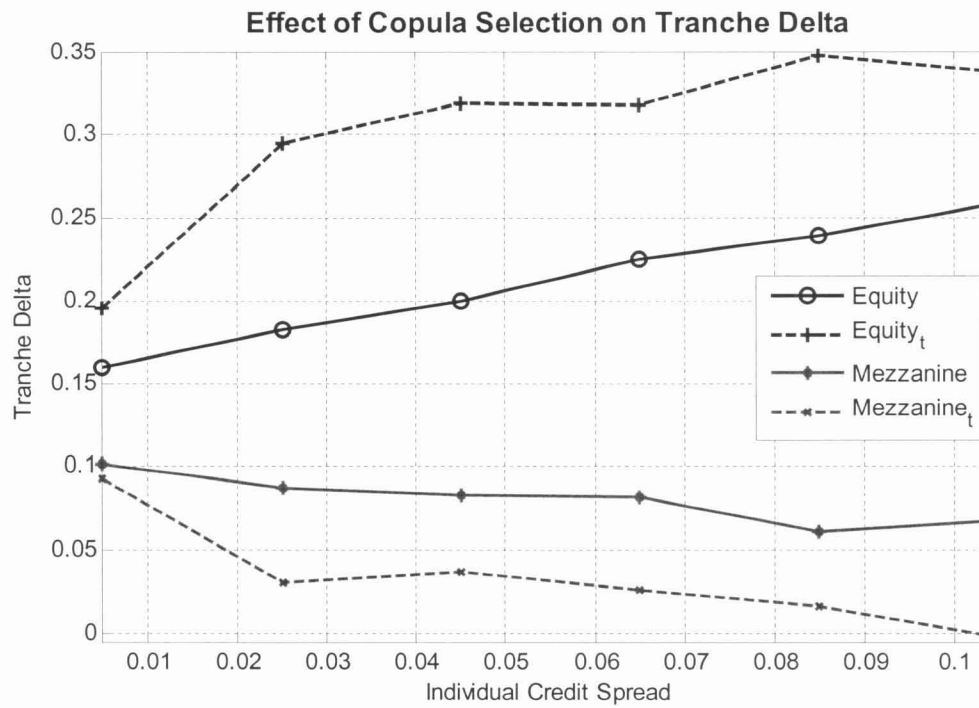




Figure 11



## Appendix B: Programming Codes

```
Code for tranche prices and standard deviations under t copula
```

```
f=50;
```

```
s=50;
```

```
%creat a vector contain the notional amount of the bonds in the cdo
```

```
for m=1:f
```

```
    notiv(m,1)=5000000;
```

```
end
```

```
for n=f+1:f+s
```

```
    notiv(n,1)=15000000;
```

```
end
```

```
N=100000;%Sample size we need to generate
```

```
M=f+s; %Number of random variables we need to generate
```

```
% contact parameter
```

```
r=0.05; %risk free interest rat
```

```
total_notional=f*5000000+s*15000000;
```

```
tmatcdo=5;
```

```
%initial value in simulation
```

```
tloss=zeros(N,1);
```

```
actpayment_senior=zeros(N,1);
```

```
actpayment_mezzanine=zeros(N,1);
```

```
actpayment_equity=zeros(N,1);
```

```
%parameters for cdo
```

```
percent_equity=0.1;
```

```
percent_mezzanine=0.3;
```

```
percent_senior=1-percent_equity-percent_mezzanine;
```

```
deductible_senior=(percent_equity+percent_mezzanine)*total_notional;
```

```
deductible_mezzanine=percent_equity*total_notional;
```

```
totalnotion_mezzanine=percent_mezzanine*total_notional;
```

```
maxpayment_equity=deductible_mezzanine*exp(-r*tmatcdo);
```

```
maxpayment_mezzanine=totalnotion_mezzanine*exp(-r*tmatcdo);
```

```
maxpayment_senior=(percent_senior*total_notional)*exp(-r*tmatcdo);
```

```
disc=100*exp(-r*tmatcdo);
```

```
disc_std=(100*exp(-r*tmatcdo))/sqrt(N);
```

```
Dof=3;
```

```
lamda=0;
```

```

cor_rho=0;
maxrho=1;
maxlam=.1;
k=12;
y=6;
cor_step=maxrho/(y-1);
lam_step=maxlam/(k-2);

for z=1:y %index for correlation
    cor_rho=(z-1)*cor_step-.000000001;
for s=1:k %index for default rate

    if s==k;
        lamda=1;
    else
        lamda=(s-1)*lam_step+.000000001;
    end

    rho=cor_rho*ones(M);

for i=1:M
    rho(i,i)=1;
end

rho =((Dof-2)/Dof)*rho;
x = mvtrnd(rho, Dof, N/2);
u = tcdf(x,Dof);

Tao1=-log(1-u)/lamda;
Tao=-log(u)/lamda; % the default times
Tao=[Tao;Tao1];%combine the two default times

for i=1:N
    for j=1 : M
        if Tao(i,j)<=tmatcdo;
            indicator(1,j)=1;
        else
            indicator(1,j)=0;
        end
    end
    tloss(i,1)=indicator*(notiv);
    indicator=zeros(1,M);

    actpayment_equity(i,1)=min(deductible_mezzanine,tloss(i,1));

```

```

if tloss(i,1)>deductible_mezzanine;

    actpayment_mezzanine(i,1)=min(totalnotion_mezzanine,tloss(i,1)-
deductible_mezzanine);
    else
        actpayment_mezzanine(i,1)=0;
    end

    actpayment_senior(i,1)=max(0,tloss(i,1)-deductible_senior);

end

Premium_equity(s,z)=disc*mean(actpayment_equity)/maxpayment_equity;
Premium_mezzanine(s,z)=disc*mean(actpayment_mezzanine)/maxpayment_mezzanine;
Premium_senior(s,z)=disc*mean(actpayment_senior)/maxpayment_senior;

Premium_equity_std(s,z)=disc_std*std(actpayment_equity)/maxpayment_equity;
Premium_mezzanine_std(s,z)=disc_std*std(actpayment_mezzanine)/maxpayment_mezza
nine;
Premium_senior_std(s,z)=disc_std*std(actpayment_senior)/maxpayment_senior;

end
end

%output
Premium_equity
Premium_mezzanine
Premium_senior

Premium_equity_std
Premium_mezzanine_std
Premium_senior_std

```

```

Code for tranche delta and individual credit spread movement

```

```

f=50;
g=50;

```

```

%creat a vector contain the notional amount of the bonds in the cdo
for m=1:f
    notiv(m,1)=15000000;
end
for n=f+1:f+g
    notiv(n,1)=5000000;
end

```

```

N=1000;%Sample size we need to generate for each sub run
M=f+g; %Number of random variables we need to generate
P=1000; %Mumber of iteration for the NbyM matrix
SS=N*P; %Total sample size

```

```

% contact parameter
r=0.05; %risk free interest rate
total_notional=5000000*f+15000000*g;
tmatcdo=5;

```

```

%parameters for cdo
percent_equity=.03;
percent_mezzanine=.07;
percent_senior=.9;
deductible_senior=(percent_equity+percent_mezzanine)*total_notional;
totalnotion_mezzanine=percent_mezzanine*total_notional;
deductible_mezzanine=percent_equity*total_notional;
maxpayment_equity=deductible_mezzanine*exp(-r*tmatcdo);
maxpayment_mezzanine=percent_mezzanine*total_notional*exp(-r*tmatcdo);
maxpayment_senior=percent_senior*total_notional*exp(-r*tmatcdo);
disc=100*exp(-r*tmatcdo);
disc_std=(100*exp(-r*tmatcdo))/sqrt(SS);

```

```

%parameters for simulation loop
k=2;
y=6;
Dof=3;
lamda=0.01;
lamda1=0.005;
cor_rho=0.2;
maxlam=.1;

```

```

lam_step=maxlam/(y-1);

%initial values for simulation results
tloss=zeros(N,1);
actpayment_senior=zeros(N,1);
actpayment_mezzanine=zeros(N,1);
actpayment_equity=zeros(N,1);

Tactpayment_senior=zeros(N,1);
Tactpayment_mezzanine=zeros(N,1);
Tactpayment_equity=zeros(N,1);

SQTactpayment_senior=zeros(N,1);
SQTactpayment_mezzanine=zeros(N,1);
SQTactpayment_equity=zeros(N,1);

Premium_equity=zeros(k,y);
Premium_mezzanine=zeros(k,y);
Premium_senior=zeros(k,y);

Premium_equity_std=zeros(k,y);
Premium_mezzanine_std=zeros(k,y);
Premium_senior_std=zeros(k,y);

for z=1:y %index for correlation

    lamda1=(z-1)*lam_step+0.01;

for s=1:k %index for default rate

    lamda1=lamda1+0.005*(s-1); %credit spread increase by 50 basis points

cdsp(s,z)=0.6*(1-exp(-(r+lamda1)*tmatcdo))*(lamda1/(r + lamda1));

rho=cor_rho*ones(M);

for i=1:M
    rho(i,i)=1;
end

for iter=1:P

u = Gaussrnd(rho,N/2,M);% call gaussian fuction

Tao1=-log(1-u)/lamda;

```

```

Tao=-log(u)/lamda; % the default times
Tao=[Tao;Tao1];%combine the two default times
Tao(:,1)=(Tao(:,1)*lamda)/lamda1;

%Tao is an NbyM matrix is the number of trails is the number of firms we simulate

for i=1:N
    for j=1 : M
        if Tao(i,j)<=tmatcdo;
            indicator(1,j)=1;
        else
            indicator(1,j)=0;
        end
    end
    tloss(i,1)=(indicator.*(0.2+0.8*unifrnd(0,1,1,M)))*(notiv);

    actpayment_equity(i,1)=min(deductible_mezzanine,tloss(i,1));

    if tloss(i,1)>deductible_mezzanine;

        actpayment_mezzanine(i,1)=min(totalnotion_mezzanine,tloss(i,1)-
deductible_mezzanine);
    else
        actpayment_mezzanine(i,1)=0;
    end

    actpayment_senior(i,1)=max(0,tloss(i,1)-deductible_senior);

end

Tactpayment_senior=Tactpayment_senior+actpayment_senior;
Tactpayment_mezzanine=Tactpayment_mezzanine+actpayment_mezzanine;
Tactpayment_equity=Tactpayment_equity+actpayment_equity;

end

Premium_equity(s,z)=disc*mean(Tactpayment_equity)/(maxpayment_equity*P);
Premium_mezzanine(s,z)=disc*mean(Tactpayment_mezzanine)/(maxpayment_mezzanin
e*P);
Premium_senior(s,z)=disc*mean(Tactpayment_senior)/(maxpayment_senior*P);

Tactpayment_senior=zeros(N,1);
Tactpayment_mezzanine=zeros(N,1);
Tactpayment_equity=zeros(N,1);

end

```

```
end
```

```
%output
```

```
for i=1:y
```

```
equity(i,1)=Premium_equality(2,i)-Premium_equality(1,i);
```

```
mezzanine(i,1)=Premium_mezzanine(2,i)-Premium_mezzanine(1,i);
```

```
senior(i,1)=Premium_senior(2,i)-Premium_senior(1,i);
```

```
cds(i,1)=(cdsp(2,i)-cdsp(1,i));
```

```
spread(i,1)=(i-1)*lam_step+0.01;
```

```
end
```

```
DT=exp(-r*tmatcdo);
```

```
equity_delta=(DT/100)*equity./cds;
```

```
mezzanine_delta=(DT/100)*mezzanine./cds;
```

```
senior_delta=(DT/100)*senior./cds;
```



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