PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' UNDERSTANDING OF IRRATIONALITY

by

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ABSTRACT

This study has investigated prospective secondary mathematics teachers' understanding of the concept of irrationality. It could be understood as an inquiry into the obstacles - epistemological, intuitive, and didactic - which replicate in individual learners as cognitive obstacles, and affect the understanding of irrationality in particular and the notion of real number in general. The concept itself is inherently difficult; yet, understanding of irrational numbers is essential for the extension and reconstruction of the concept of number from the system of rational numbers to the system of real numbers.

Forty-six prospective secondary mathematics teachers, in their final term of studies before certification, participated in the research. The data consists of a written questionnaire followed by a clinical interview conducted with sixteen volunteers from the group. The group of interviewees was chosen to be representative of the entire group in order to capture the diversity of conceptions in their various developmental stages.

The study provides a detailed description and analysis of participants' understanding, both formal and intuitive, and it attempts to identify and explain the possible causes of cognitive obstacles that impede learners in developing a mathematically consistent understanding of irrationality.

With respect to formal knowledge the study examined participants' ability to classify numbers into various number sets, their knowledge of definitions, and their ability to coordinate between various representations. Participants' intuitions and beliefs regarding the relations between the two sets, rational and irrational, were also examined. Three issues were addressed: richness and density of numbers, the fitting of numbers, and operations. The results indicate that there are inconsistencies between participants' intuitions and their formal knowledge. Explanations used by a vast majority of participants relied primarily on considering the infinite non-repeating decimal representations of irrationals, which provided a limited access to issues mentioned above. The results can help teachers understand the difficulties that the concept of irrational number presents to students. Based on the research findings, we propose a number of general recommendations for practice and we provide some specific suggestions on how teachers can help students acquire a more profound understanding of number.

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I dedicate this work to my daughters, Lara and Diana.

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CHAPTER 1

Introduction

People use numbers for three basic purposes in everyday life: counting, ordering, and measuring. However simple this may sound, the notion of what constitutes a number is not so easily grasped. The student of today must master the concept of number in the mere twelve years of formal education, while it took humanity millennia to come up with the modern view of what number is.

As computational needs of people increased through history, new worlds of numbers have emerged. For example, in the world of positive whole numbers the need to subtract a larger number from a smaller could not be met, and so the world of numbers had to be expanded to include negative integers. Now one could freely add, subtract, and multiply but not divide, without crossing the borders. The need for unrestricted division forced the expansion of numbers to include rational numbers.

Rational numbers are numbers suitable for counting. With them we can count things or parts of things. Many people would agree that the ability to count things is the least that should be required from a number, or else it cannot be called a number at all.

The Pythagoreans in ancient Greece believed that all physical reality could be expressed and understood through number, and in particular that all lengths could be expressed in terms of ratios of one another. This seems a perfectly reasonable belief. After all we do it all the time when we express lengths in terms of a chosen unit, always ending up with a finite decimal or fraction.

Rational numbers have a simple geometrical representation. Mark two distinct points A, B on a line. Let A represent 0 and let B represent 1 – the segment AB then represents a unit of length. Positive and negative integers are represented by a set of points on the line spaced at unit intervals apart, the positive integers being represented to the right of A and the negative integers to the left of A. The fractions with the denominator q may then be represented by the points that divide each of the unit intervals into q equal parts. For each rational number there is a unique point on the line. The only reason integers and fractions are capable of counting is that they are evenly spaced on the number line. For example, if the number line is marked in terms of thirds, then one can count thirds. If it is marked in terms of sevenths, then one can count sevenths. In any case, the tick-marks on the number line are evenly spaced, and thus they can count.

To the early mathematicians it seemed evident, as indeed it seems to anyone today who has not yet been initiated into the deeper mysteries of the number line, that all the points on the number line would be in this way used up; ordinary common sense seems to indicate this. But life of human thought is not that simple, and to their great surprise the Greeks and many others made a shattering discovery that there are incommensurable lengths, lengths that seemingly cannot be measured by rational numbers. Probably the first such discovery that was made was that the diagonal of a square is not commensurable with its side length. It must have been a genuine mental shock for humans to learn that there are points on the number line not corresponding to any rational point. According to Eves (1980), this great moment in mathematics occurred sometime in the fifth or sixth century B.C. among the ranks of the Pythagorean brotherhood.

And so we learn, sometime in grade 9 or 10, that there are "numbers" which cannot be used for counting. These are the irrational numbers. Irrational numbers are the numeric expression of incommensurability. We say that irrational numbers are numbers that cannot be written in the form p/q with p and q integers and $q \neq 0$.

Concerning the term "irrational", Klein (1932) gives a concise summary. It comes from the Greek "alogos" which presumably meant "inexpressible" and implied that the new numbers could not be expressed by the ratio of two whole numbers. The Latin word "ratio" that also conveys the meaning "reason", gave to "irrational" the unintended meaning "unreasonable", which seems to cling to the term "irrational number".

It was only in the recent history of mathematics that irrational numbers were given the status of number. Greek mathematicians mostly confined themselves to such irrationalities that could be constructed geometrically using compass and straightedge (those that can be obtained by repeated extraction of square root, ex. $\sqrt{\sqrt{a}+b}$). The general idea of irrational number was not yet known to them.

Much later, in the early 16th century, the "equation-solvers" Cardano, Tartaglia, Ferrari and others, stumbled over them, not via geometry, but via algebra. They dealt with what we call today the "algebraic irrational numbers". Still, these irrational numbers did not have a status of number. In fact, nobody could handle them rigorously, so they were often left in symbolic form. It took nearly another 300 years, as we shall see from the historical account of these matters, before many more irrational numbers, besides the roots, were declared to exist as numbers to make the real number system work properly. These are the transcendental numbers. First transcendental numbers were discovered by Liouville in the mid-1800's (for example, he proved that the number which has zeros - after the decimal point – everywhere else except at the places 1, 2, 6, 24, …, n!, where it has ones, is transcendental). For practical and theoretical reasons, the rational numbers and the irrational numbers have been merged together into the system of "real numbers" despite their different natures.

According to Klein (1932), the general idea of irrational number first appeared in the 16th century as a consequence of the introduction of decimal numbers, in particular, in connection with their use in logarithmic tables. Nevertheless, it was not until the later 1800s that Weierstrass, Cantor, and Dedekind developed a general theory of irrational numbers. Dedekind is said to be the first person who gave a rigorous proof that $\sqrt{2} \times \sqrt{3} = \sqrt{6}$. This part of the historical development will be later discussed in greater detail.

It is worth mentioning that in school mathematics, until quite recently, irrational numbers were named "surds" and used as entities which stood for exact values of incommensurable lengths. This term seems to have lingered in Britain[1] longer than elsewhere. Since the term is archaic, it is usually reserved for algebraic irrational numbers (as those were the only ones known until the 19th century). For example, it would be said "cubic surd", whereas nowadays we usually do not qualify the source of an irrational number in school mathematics – we do not say "cubic irrational number".

Little attention is paid to the irrational numbers in school mathematics despite the fact that understanding of them is essential in order to have a complete concept of the system of real numbers. Furthermore, mathematics education research literature on this topic is scarce. Hence, our interest in how an understanding of irrational numbers is developed.

1.1 Purpose of the Study

The purpose of this study is to provide an account of prospective secondary teachers' understandings and misunderstandings of irrational numbers, to interpret how the understanding

[1] Typing "surd" in a Web search engine will get many references attesting that the term is still alive today. For example,
http://www.projectalevel.co.uk/maths/surds.htm
http://www.bbc.co.uk/education/asguru/maths/12methods/01algebra/02surds/index.shtml
Also in Chrystal's *Algebra, an Elementary Textbook (1886)* there is a section dealing with a definition of surd numbers.

of irrationality is acquired, and to explain how and why difficulties occur. Implications for teaching practices are also proposed.

One can take the view that we have different cognitive schema, related to a given mathematical concept, structured by our previous experience with different representation systems. Acting upon new evidence, we are attempting to reconstruct and coordinate those mental structures, often through inner negotiation of meanings, to once again achieve equilibrium. The idea of human tendency for equilibrium (seeking order and harmony) in the field of consciousness first appeared in Gestalt psychology. It was later adopted in Piaget's theory of equilibration of cognitive structures (Sierpinska, 1994). For Piaget, assimilation and accommodation are two operations of the mind that make the equilibration of cognitive structures possible.

The existence of irrational number is what necessitates the extension of the rational number construct into the real number construct. When irrational numbers are introduced, an individual must reconstruct his or her notion of number to fit the new evidence into the existing cognitive structures. Where once was a fully dense (rational) number line, there seemingly being no space for anything else, now a learner must fit many more numbers whose nature is quite problematic, i.e. it is highly doubtful whether they even deserve the status of number as they cannot be used for counting purposes. This study could be understood as an inquiry into prospective teachers' obstacles (cognitive, intuitive, epistemological, and ontological) affecting the understanding of irrationality in particular, and the notion of real number in general.

The NCTM's Principles and Standards for School Mathematics suggest that "high school students should understand more fully the concept of a number system, how different number systems are related, and whether the properties of one system hold in another system." Furthermore, this document states the following:

> Whereas middle-grades students should have been introduced to irrational numbers, high school students should develop an understanding of the system of real numbers. They should understand that given an origin and a unit of measure, every point on a line corresponds to a real number and vice versa. They should understand that irrational numbers can only be approximated by fractions or by terminating or repeating decimals. They should understand the difference between rational and irrational numbers. Their understanding of irrational numbers needs to extend beyond π and $\sqrt{2}$. (pp. 291-292)

Clearly, the idea is that students obtain an understanding beyond the mere ability to recognize and classify numbers. These skills could be achieved without understanding, by memorizing and attaching the labels without necessarily having an underlying meaning developed. However, we wish that students attain a robust and lasting knowledge which is possible only if they can see meanings and mathematical relationships underneath the surface of labels. If students are to develop these understandings the teachers must first possess them. It is our contention that a close look at the prospective teachers' understanding of the number systems is a good place to start an investigation of these matters.

1.2 Personal Motivation

There are three reasons that I was personally motivated to research this particular topic. Firstly, it has been a pivotal point in my personal struggle to make sense of mathematics. For me, it opened a very special perspective on all of mathematics. The journey has been long and hard, yet intellectually very satisfying. Of course, it did not happen in a instant, but rather bit by bit, surely starting with the Pythagorean Theorem, through the ideas of incommensurability, through various proofs of irrationality, to grappling with infinities of several kinds and seeing the striking abundance of these numbers. I believe that anyone who explores a bit further, what it means to have lengths to which we cannot attach a whole number or even a ratio of whole numbers, is bound to find it a very interesting and exotic concept.

Secondly, research on this topic is very scarce. There is plenty of research on students' learning and understanding of rational numbers. But there is very little research on how learners acquire an understanding of the real number system. Perhaps the reason for this is that the field of mathematics research is a very young field and it is natural that many researchers focus on "first things" first. But what happens after the rational number construct has been mastered by the learner? After all, according to most standard curriculums, it is assumed that this should happen by grade 9. One of the first shocking realizations is that rational numbers comprise a very tiny part of that which is called "number" today. In fact, surprising as it sounds to a novice, when picking a number from a number line at random, the probability of getting a rational number is zero. Of course, that is not to say that rational numbers are any less important – the point I am trying to make is that mathematics teachers need to have knowledge about how people conceptualize irrational numbers so as to not hinder further understanding in students.

Thirdly, I believe that understanding the nature of irrational number, beyond being able to identify it as a non-periodic infinite decimal, is of great importance for seeing the connectedness of mathematics within. While it is impossible to distinguish any characteristics that an irrational might possess by examining its decimal expansion, irrationals do arise in algebraic contexts (as roots of polynomial equations), in geometric contexts (as ratios of lengths in geometric objects), and in analytic contexts (the base of natural logarithms - number *e*; also as limits of infinite decimal expansions of rational numbers). Furthermore, an irrational number might have a natural habitat in several contexts, for example $\sqrt{2}$ is both a diagonal of a unit square as well as an algebraic entity as a solution of $x^2 = 2$.

In addition to helping students see the connections within mathematics, we strive to help students see how mathematics is connected to real world. It is interesting to point out to students that irrational number is where mathematics and physical world manifest dramatic connections. Just look at the simplest forms such as circles, squares, equilateral triangles, … they all involve irrational numbers. Moreover, all empirical evidence suggests that the results of physical measurements are intrinsically irrational. Each time we increase the precision of any instrument, the new digits we discover add to a string that neither terminates nor repeats.

> A vast amount of experimental data spanning several centuries confirms this contention, and there is no empirical evidence to repudiate it. Even quantum numbers, which by theoretical definition are integers or rational fractions, are meaningful only as rational multipliers of measured and irrational physical constants;... for now, my point is that scientific inquiry rides on a thruway paved mostly by irrational numbers. (Zebrowski, pp. 10-11)

Of course, for daily purposes of majority of people rational approximations suffice. However, in my perspective the utilitarian purpose of mathematics is not the main reason why we learn it (that should come as a by-product). To evolve in our thinking capacities, and understand the world we live in is at least as important of a goal. Therefore, the power of mathematics to describe and explain the physical universe should be seen as truly a wonder in itself.

Having said that, I believe we would do a great service to students by helping them gain an appreciation for these numbers and some insight into their origin and nature. The study of irrational number is surely one of the opportunities in school mathematics where a teacher can convey to students that mathematics is a very interesting subject to be understood.

1.3 Thesis Organization

The thesis is organized into nine chapters. Chapter 1 is an introduction. It includes the purpose of and the personal motivation for conducting the study on prospective secondary mathematics teachers' understanding of the concept of irrational numbers.

Chapter 2 provides an examination of the literature in two areas. The first part is a synopsis of the historical development of the concept of irrational numbers. The second part discusses the findings from the mathematics education research literature that are relevant for the study.

Chapter 3 outlines the theoretical considerations guiding the inquiry of the study. Terminology used throughout the thesis is introduced and clarified in this chapter.

Chapter 4 describes the issues inherent in the understanding of the concept of irrationality. The four specific areas of focus are also declared in this chapter.

Chapter 5 details the methodology for the study. It includes a description of the participants and setting, as well as the particulars regarding data collection and data analysis. This chapter also contains a detailed analysis of the tasks used in data collection.

Chapter 6 presents the results and analysis of the participants' responses from the perspective of formal mathematical knowledge of irrational numbers. The formal knowledge analyzed includes classification of numbers into various number sets, ability to translate between various representations of irrational numbers, and the knowledge of definitions.

Chapter 7 describes the results and analysis of the participants' responses from the perspective of intuitive mathematical knowledge, such as the existence and density of irrational numbers and how numbers from the two sets, rational and irrational, fit together on the real number line. Participants' intuitions about the effects of operations between members of various number sets are also analyzed in this chapter.

Chapter 8 presents recommendations for classroom instruction and pedagogical practice. Several "foundational algorithms" aimed at developing a fuller understanding of irrationality are introduced in this chapter.

Lastly, Chapter 9 presents the conclusions and limitations of the study. It finishes by commenting on the direction of further research.

CHAPTER 2

Irrationals in the History of Mathematics and in Educational Research

> Philosophically the irrationals, as a family of numbers, cause ontological problems. Since it is (generally) impossible to describe the exact decimal expansion of such numbers, their very reality can be called into question. In fact, it was not until the mid-1800s that mathematicians confronted this problem directly. The upshot of this confrontation was that the irrational numbers were postulated to exist. In other words, the mathematicians axiomatized their way out of the dilemma by saying that all decimal expansions exist as numbers even if we do not know precisely what they are. This axiom is called the completeness axiom. (Stevenson, p. 214)

The purpose of this chapter is twofold. First, a brief synopsis of the historical development of the notion of irrational numbers is provided. Second, relevant background from the educational research is outlined.

2.1 Historical Background

The focus of the above quote is not on algebraic irrational numbers such as square root of two, the existence of which is undeniable, but on the so-called transcendental numbers, which account for the nondenumerability of the real numbers. Note that algebraic irrational numbers, such as square root of two, are denumerable because they are solutions of polynomial equations with integer coefficients, which are denumerable as well.

A historical account of these developments in mathematics can be found, for example, in the recent work of Lakoff and Núñez, *Where Mathematics Comes From* (2000). For our purpose it suffices to say that in the mid-19th century there were two rival theories. A theory involving infinitesimals[2], believed by some authors (as well as mathematicians) to be more intuitively plausible, has been pushed aside in favour of what we see today as the mainstream theory linking numbers, points on the line, and sets (including infinities) developed by Dedekind, Weierstrass, and Cantor.

Dedekind's work succeeded to explain irrational numbers in terms of rationals, which in turn can be understood in terms of the natural numbers. The motivation for this work comes from the desire to put "calculus on a secure foundation", in other words, to reconceptualize calculus as arithmetic, free from geometric methods (as employed by Leibnitz).

> ...Dedekind was profoundly dissatisfied with the way calculus had been developed – namely, through geometric notions like secants and tangents. Calculus was, after all, part of the subject matter of arithmetic functions and so ought properly to be understood in terms of arithmetic alone and not geometry... How, he asked, could continuity be understood in arithmetic terms? The key, he believed, was the real numbers. (Lakoff &Núñez, pp. 293-294)

The "arithmetization of calculus" (Lakoff & Núñez, p. 293), also referred to as the "discretization program" (ibid., p.292), gave birth to the formal construct known to us today as the "real number line". In short, this is how it happened. Already Pythagoreans observed that there exist incommensurable line segments. That is, when rational numbers, as we understand this term today, are associated with points on the line, there are points not associated with any rational number, which creates a problem. This not only upset their basic assumption that everything depends on whole numbers, but it also upset Pythagorean theory of proportion and it rendered their general theory of similar figures invalid, as their definition of proportion assumed

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[2] It is worth mentioning that there are other notions of real number, besides the Cantor-Weierstrass-Dedekind one. Under the umbrella called "nonstandard analysis" there thrive number systems known as "hyperreal" (A. Robinson, Nonstandard Analysis) and "surreal" (J. Conway), both containing numbers which are infinitely large or small (infinitesimals).

any two like magnitudes to be commensurable (Eves, 1990). For some time $\sqrt{2}$ (or possibly $(\sqrt{5}-1)/2$, which is the ratio of a side to the diagonal of a regular pentagon) was the only known irrational. Later Theodorus of Cyene (ca. 425 B.C.) showed that $\sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \sqrt{15}$, and $\sqrt{17}$ are also irrational. About 370 B.C. the "scandal" was resolved by Eudoxus, a pupil of Plato, who reinvented the theory of proportion to include incommensurable lengths. His treatment appears in the fifth book of Euclid's Elements and coincides essentially with the modern exposition of irrational numbers that was given by Richard Dedekind in 1872 (ibid.). Still, the Greeks confined themselves to the study of such irrationalities as one obtains by a repeated extraction of square root – those which can be constructed geometrically by straightedge and compass. The general idea of irrational number was not yet known to them. More precisely, the Greeks possessed no method for producing or defining, arithmetically, the general irrational number in terms of rational numbers.

The general idea of the irrational number appeared first at the end of the sixteenth century as a consequence of the introduction of decimal fractions, the use of which became established at that time in connection with the appearance of logarithmic tables. When a rational number is represented as a decimal, it may be a finite decimal or an infinite periodic decimal. Now there is nothing to prevent our thinking of an aperiodic decimal whose digits proceed according to any definite rule whatever or according to no rule at all. Anyone would instinctively consider it as a number, however, not a rational one. By this means the general notion of irrational number is established. It arose to a certain extent automatically, by the consideration of decimal fractions. Historically, a similar thing that happened with negative numbers also happened with irrational numbers. Calculation forced the introduction of the new concept, and without being much concerned with its nature or motivation, people simply operated with these numbers as that proved to be extremely useful. Still, there was no satisfactory theory of irrational numbers. The mathematical community felt there was a need for a more precise arithmetic formulation of the foundations of irrational numbers.

Dedekind set out to resolve this problem. The source of this problem is that the line, understood naturally, is continuous, whilst numbers are discrete; that is, each number is an entity. So in order to have a one-one order preserving correspondence between points and numbers, the numbers must be declared continuous too (or the line must be discretized).

There is a guarantee of no "gaps" in the real numbers, by decree. A historical account of how this came about appears in Klein (1932).

> Corresponding to every rational or irrational number there is a point which has this number as abscissa and, conversely, corresponding to every point on the line there is a rational or irrational number, viz., its abscissa. Such a fundamental principle, which stands at the head of a branch of knowledge, and from which all that follows is logically deduced, while itself cannot be logically proved, may properly be called an axiom. Such an axiom will appear intuitively obvious or will be accepted as a more or less arbitrary convention, by each person according to his gifts. This axiom concerning the one-to-one correspondence between real numbers on one hand, and the points of a straight line on the other, is usually called the Cantor axiom because G. Cantor was the first to formulate it specifically in the Mathematische Annalen, vol. 5, 1872. (p. 34)

It is said that the real numbers "exhaust" the (real) number line. This is technically achieved via the so called "Dedekind cut" (any partition of the rationals into an Upper and a Lower Class), which is Dedekind's definition of what "real number" means. In other words, "Dedekind cut" is used to define irrational number as *being* the "cut" between two sets of rationals. This way numbers determine what points are. By completing the set of rational numbers with the set of irrational numbers we get the real numbers.

As we can see, it took about 2500 years from the discovery of irrationals as incommensurable lengths to the construction of the system of real numbers using infinite sets of discrete elements. How do these difficult ideas translate in school mathematics? Do they impact prospective teachers' conceptions of irrationality?

2.2 Educational Research

In this part we present literature review as it relates to the various issues one may encounter in the process of understanding the irrational number construct. As noted earlier, not much research is available on this topic. On the other hand, a great deal of research has been invested in how students acquire the knowledge of rational numbers and, on a closely related matter, how they attain proportional reasoning. Considering the fact that irrational number is defined in school mathematics as a number which is not rational, it is clear that the understanding of irrationality cannot be investigated in the absence of rational number referents. Given that this study involves prospective teachers, and not developing children, we assume solid understanding of rational number is in place. As the understanding of rational numbers is not of our primary concern here, yet it is an essential prerequisite for understanding irrationality, a very brief recapitulation of this research is presented here.

A recent extensive study of children's development of meanings and operations with rational numbers, conducted by Lamon over the span of four years, focuses on how the various representations and interpretations of fractions affect students' understanding of rationality (Lamon, 2001). Other authors have investigated students' abilities to perform both translations of a given idea from one representational system and transformations within a given representational mode in the context of development of the rational number construct (Lesh, Post, Behr, 1987). Still others studied the role of representation (common unit vs. composite) in the process of teaching decimal numbers. For example, researchers analyzed how manipulating representational support during the instruction of procedures (such as marking positions of decimal fractions on a number line) affects gains in the conceptual knowledge. They investigated the effects of varying the representational mode; that is, using the idea of common unit (0.123 seen as one hundred twenty-three thousandths) versus using the idea of composite units (0.123 seen as 1 tenth and 2 hundredths and 3 thousandths) (Rittle-Johnson, Siegler, Alibali, 2001). In summary, research efforts seem to be centering around identifying which representations are more likely to result in students' independent transfer of knowledge to various subconstructs of rational number, and how the learning environment design decisions, especially representational ones, contribute to a robust understanding of rationality (Kaput, 1994).

Further concerning rationality, of greater relevance to this study is the research on how learners think about infinite (periodic) decimal expansions of rational numbers. Several researchers have documented the persisting difficulty of accepting that $0.\overline{9}=1$ (Sierpinska, 1987). Mamona-Downs, who conducted a study on 20 English and 20 Greek students in their final year of high school, found that there are important influences related to the educational background of a pupil affecting his/her approach to this result. Major reasons for the widespread disbelief of this result are reported to be: a) the feeling that a decimal expression for a number should be unique (i.e., if objects have different representations, especially in the same system, then they are different), b) the feeling that $0.\overline{9}$ is an on-going sequential process ruled by time that never actually reaches 1, and c) the infinitesimal reasoning, by which the $0.\overline{9}$ is regarded as a completed process of an infinite procedure where the difference between the two is infinitely small, but not 0 (Mamona-Downs, 2001). On the same issue, Sierpinska (1987) reports that the attitudes towards mathematical knowledge (mathematical validity vs. the absolute truth value) and infinity (potential vs. actual), determine the students' attitudes towards this result. Consistent with Mamona-Downs, Sierpinska found that some students hold models of infinite decimals such that the sequence of decimals is a function of time ($0.\overline{9}$ comes arbitrarily close to 1, but can never reach it), and also that the set of real numbers is dense but not necessarily continuous.

From the perspective of instructional strategies and suggestions for practice, Sinclair (2001) offers a visual calculator with a hundred-digit display to facilitate the perception of rational numbers (and their classes: terminating, periodic, and eventually periodic) and real numbers at the middle-school level. In this environment students experiment and find out for themselves what is interesting or significant to know about fractions; in particular, fractions are seen as patterned objects rather than parts of a whole prompting students' sense-making of the characteristics of rational numbers.

To the best of our knowledge, there are very few studies in the educational research literature that explicitly focus on the concept of irrational numbers. The main objective of the study by Fischbein, Jehiam, and Cohen (1995) was to survey the knowledge that high school students and preservice teachers possess with regard to irrational numbers. This study assumed, on historical and psychological grounds, that the concept of irrational numbers faced two major intuitive obstacles, one related to the incommensurability of irrational magnitudes and the other related to the nondenummerability of the set of real numbers. Contrary to expectations, the study found that these intuitive difficulties did not manifest in the participants' reactions. Instead, it was reported that subjects at all levels were not able to define correctly the concepts of rational, irrational, and real numbers. Many students could not even identify correctly various examples of numbers as being whole, rational, irrational, or real. The study concluded that the two intuitive obstacles mentioned above are not of a primitive nature – they imply a certain intellectual maturity that the subjects of this study did not possess. Fischbein et al's study used a written response questionnaire administered to 62 students (grades 9 and 10) and 29 prospective teachers.

A study of Peled and Hershkovitz (1999), which involved 70 prospective teachers in their second or third year of college mathematics, focused on the difficulties that prevent student teachers from integrating the various knowledge pieces of the concept into a flexible whole. Contrary to Fischbein et al. study, these researchers found that student teachers knew the definitions and characteristics of irrational numbers but failed in tasks that required a flexible use of different representations. They identified the misconceptions related to the limit process as the main source of difficulty.

In their work on using history of mathematics to design pre-service and in-service teacher courses, Arcavi, Bruckheimer and Ben-Zvi (1987) report several findings that are of interest here as they relate to teachers' knowledge, conceptions, and/or misconceptions regarding irrational numbers. One of the most striking discoveries from their study is that there is a widespread belief among teachers that irrationality relies upon decimals. This study was conducted on 84 in-service teachers who attended a summer teacher training program related to a national mathematics curriculum for junior high schools in Israel. Arcavi et al. report that 70% of teachers knew that the first time the concept of irrationality arose was before the Common Era (Greeks). However, although the majority knew "when" it arose, very few also knew "how" it arose. This became particularly apparent when they were asked to order chronologically the appearance of three concepts: negative numbers, decimal fractions, and irrationals. 55% percent of teachers (and an additional 10% did not answer) indicated that decimal fractions preceded irrationals in the historical development. The authors concluded that this not only indicated the lack of knowledge about the relatively recent development of decimals, but more importantly, it indicated that the origin of the concept of irrationality, although associated with the Greeks, is conceived as relying upon decimals, and not connected to geometry as occurred historically (commensurable and incommensurable lengths). Arcavi et al. (1987) point out that " the historical origins of irrationals in general, and the connections to geometry in particular, can provide an insightful understanding of the concept as well as teaching ideas for the introduction of the topic in the classroom" (p. 18).

As a commentary, we find it interesting to point out that the three concepts mentioned in the Arcavi et al. study are generally introduced to students in the reverse order from how they developed historically. The concept of irrationality received its first proper theoretical treatment by Eudoxus around 400 B.C., and it appears in Euclid's *Elements* (Eves, 1990). On the other hand, decimals were introduced by Simon Stevin in his *De Thiende* in 1585 (2000, Schultz). Historically, the first formal introduction to negative numbers appears in *Introduction to Algebra* by Leonard Euler in 1770.

Consistent with the Fischbein et al. study, Arcavi et al. found that many teachers had trouble recognizing numbers as being rational or irrational. Given a list of seven numbers to 60 prospective teachers from various teacher colleges on the pre-test at the start of the course, 60% of the respondents had two or more errors, the most common one being that 22/7 is irrational. This was the question that was administered:

Indicate the irrationals among the following numbers.

(a) $\sqrt{26}$ (b) $-7/3$ (c) $1.010010001\ldots$ (d) $\frac{22}{7}$ (e) π (f) $\frac{\sqrt{2}}{\sqrt{8}}$ (g) $\frac{(\sqrt{2}-1)}{(\sqrt{2}+1)}$

The researchers found that the commonly used rational approximation for π is being confused with the irrational number itself. Although the approximation is good enough for many practical purposes, the distinction should be very clear – certainly to the teacher. Furthermore, as evident from the post-test, the problem of confusing the irrational (π) with one of its rational approximations (22/7) persisted even after the formal instruction, but to a lesser extent. The authors state they are not sure whether these subjects experienced a general confusion between an irrational and its rational approximation or is this something particular connected to π and 22/7. We shall follow up on this later.

On a related matter, as a part of their study on prospective elementary teachers' conceptions of rational numbers involving 147 participants, out of which 121 were non-mathematics majors, Tirosh, Fischbein, Graeber, and Wilson (2003) report on the results on tasks involving class membership identification of numbers, relations between various sets, and definitions of rational and irrational numbers. They found that while the vast majority of the mathematics majors (92%) correctly defined rational and irrational numbers, only 23% of the others were able to do so. Further, 81% of the mathematics majors and 25% of the non-mathematics majors drew an adequate Venn diagram to describe the relations between the natural numbers, the integers, the rational numbers, the irrational numbers, and the real numbers. Concerning the identification of the set membership of various numbers, according to these authors, performance was very poor. For example, only 8% and 22% of the prospective teachers, respectively, knew that 0 and 0.251 were rational numbers and 24% percent argued that 2/0 was an integer. Confusion reigned concerning the notion of real numbers -- the majority of the prospective teachers mistakenly identified "real numbers" with "positive numbers", and consequently argued that all the given numerical expressions, except −8/5 and −3, were real numbers. Others argued that real numbers were "nice numbers", namely that 0.42, 0.251, 23/49, 0/2, 2/0, and -3 were real numbers, while 0.121221222, and 0 were not.

These researchers also report on prospective teachers' thinking about the density of rational numbers. For example, they found that only 24% knew that between 1/4 and 1/5 there are infinitely many numbers, whilst 43% claimed that there are no numbers between 1/4 and 1/5. Moreover, 30% of the participants claimed that 1/4 is the successor of 1/5. With regard to decimals, the results were somewhat better. For instance, 40% knew that between 0.23 and 0.24 there are infinitely many decimal numbers and could even present some of them. In our study we examined to what extent these findings hold true for prospective secondary mathematics teachers.

When thinking about irrational numbers, we cannot avoid thinking about infinity. Whereas Fischbein et al. (1995) suggest that a certain stage of cognitive development must be in place before a person can contemplate the idea of nondenumerability of the set of reals, Tall (2001), on the other hand, maintains that obstacles lie in the differences between the 'natural infinities', a term he uses to describe children's conceptions of infinity which are based on extending everyday finite experience, and 'formal infinities', which arise by selecting different axioms as foundations for infinite concepts. It has been observed that people intuitively conceive the notion of number as a crude kind of measurement rather than in a cardinal sense. Children have no conception of the difference between the 'rational continuum'[3] and the real continuum, which have different cardinalities, nor the fact that real intervals of different length have the same cardinal number (Tall, 1980). For example, when presented with two line segments, one twice as long as the other, they invariably respond that both have infinitely many points with the longer one having twice as many points as the shorter one. Unless they were re-educated to

[3] Tall uses this term when talking about children's perceptions of the rational numbers. Given that between any two rational numbers, no matter how close they are, there are infinitely (albeit denumerably) many rational numbers, the perception is one of it being a dense set, thus "continuous".

accept that such phenomena occur with infinite cardinals, these intuitions about infinity will present a conflict within the cardinal framework. Cantor's cardinal set theoretic interpretation of infinity is purely formal. Yet children do not have access to the formal schemas of mature mathematicians. In other words, the contradiction arises from attempting to impose a framework of interpretation based on experience with finite sets.

It is a common occurrence in the development of mathematical ideas, both in the history of mathematics and in the development of the individual, that within a given context certain facts hold true but break down when the context is broadened. For example, the fact that a proper subset of a set has a smaller number of elements is true for finite sets, but breaks down in the infinite case. Expectations based on experience with finite sets are defying when extrapolated to the infinite sets within the paradigm of the "Theory of Transfinite Numbers". Tall (1980) notes that "such intuitions based on implied truths in a restricted context can cause serious conflicts when the context is broadened" and further that "these conflicts are all the more serious when they are subliminal, unspoken and, as a consequence, unnoticed" (p. 282).

CHAPTER 3

Theoretical Considerations

The purpose of this chapter is to outline the theoretical considerations guiding our inquiry. Theoretical ideas from the existing body of educational research that proved valuable in our struggle to analyze, understand, and organize our findings on how people think about irrational numbers are described and discussed in this chapter. As well, some of the terminology that is used throughout the thesis is introduced and clarified here.

Practice shows there are differences between mathematical theories and cognitive beliefs in many individuals. Learning typically consists of the individual body of knowledge fitting with the collective body of knowledge. Tall and Vinner (1981) use the term *concept image* to describe "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). According to these researchers, concept image grows and changes with experience and its various parts develop at different times and in different ways. What we attempt to do in mathematics is to build up as coherent an image as possible. As learners, we attempt to reconstruct our knowledge to resolve the conflict between old experiences and new evidence.

For any concept, this involves the reconciling of personal *informal image*, to which we will usually refer as "understanding" or "conception", with *formal image*, to which we will commonly refer as "knowledge" (ibid.). Following this, we can say that good understanding of a given concept constitutes knowledge of that concept. For example, we may say that a person has knowledge of a given concept if his or her personal understanding of the concept is in agreement with the shared formal mathematical theory supporting this concept.

3.1 Dimensions of knowledge

For the purpose of our study, we adopt the conceptual framework suggested by Tirosh (2000) in their study of teachers' understanding of rational numbers. The basic assumption of this framework is that learners' mathematical knowledge is embedded in a set of connections among algorithmic, intuitive and formal dimensions of knowledge.

The algorithmic dimension is procedural in nature – it consists of the knowledge of rules and prescriptions with regard to a certain mathematical domain and it involves a person's capability to explain the successive steps involved in various standard operations. The formal dimension of knowledge is represented by definitions of concepts, operations, and structures as well as by theorems and their proofs. The intuitive dimension is composed of our ideas and beliefs about mathematical entities and it includes mental models we use to represent number concepts and operations. As well, intuitive knowledge can be observed through competency in evaluating the adequacy of statements related to arithmetic operations, such as "multiplication always makes bigger". Intuitive knowledge is characterized as the type of knowledge that we tend to accept directly and confidently – it is self-evident, intrinsically necessary and psychologically resistant (Fischbein, 1987). Some characteristics of intuitive knowledge are discussed in greater detail later, in the section 3.4.

Ideally, the three dimensions of knowledge should cooperate in any mathematical activity such as concept acquisition and problem solving. Furthermore, it is well known that both the formal and the algorithmic dimensions can become highly procedural and rote for the learner. Their vitality depends upon the student's constructing consistent connections among algorithms, intuitions, and concepts.

According to Tirosh, Fischbein, Graeber, and Wilson (2003), many prospective teachers', especially those that did not have a major in mathematics, based their conceptions of numbers almost entirely on natural numbers. For instance, there was a wide-spread belief that "division always makes smaller". This belief is grounded in the partitive model of division applied to natural numbers, which is essentially the "sharing" metaphor, most often used in the earliest stages of learning about division. In this model, one is restricted to division by a whole number; for example, a given number of items (say, 24 chocolates) are shared equally among a given number of people (say, 6 children). Each person gets less than the initial amount. The study reports that people tend to transfer the constraints of operations with natural numbers. This is seen as one of the stumbling blocks in the transition from natural to rational numbers. A short summary of the major findings of this study is provided in the previous chapter. We reconsider some of these findings here in light of the presented theoretical framework.

The inadequate model of division discussed in this study seems to linger well after more inclusive higher level models have been taught, and it acts as an intuitive obstacle causing many difficulties, such as the well-documented inability to create representations of operations involving fractions. When dividing by a divisor smaller than one, these teachers would multiply the two numbers instead, or they would reverse the dividend and the divisor, or they would proceed according to a memorized procedure without being able to explain the result (ibid.).

Concerning the algorithmic knowledge of multiplication of decimal numbers, prospective teachers had a great deal of difficulty justifying the placement of the decimal point in the product. Researchers report that they were unable to call upon their knowledge of fractions or their knowledge of multiplication with whole numbers as possible sources for answers, and that they were surprised that there should be a need to explain why certain algorithmic steps are performed the way they are.

In the case of rational numbers, mistakes based on formal knowledge were reported as incorrect performance due to limited conception of fraction as well as inadequate knowledge related to the properties of operations. For example, most participants' conception of fraction rested on the part-whole interpretation, which gives a limited access to interpreting the meaning of operations with fractions. When asked to illustrate 5/3 they were unable to do so. A typical approach was to draw a rectangle, partition it into three sections and then append two equally sized sections. This resulted in a rectangle with five sections, which was consequently evaluated by participants themselves as incorrect and justified as "you can't illustrate 5/3 of a whole" (ibid.). In the realm of operations, there has been a reported "bug" in the formal knowledge, namely that students think that division is commutative and consequently argue that $1 \div (1/2) = 1/2$ because $1 \div (1/2) = (1/2) \div 1 = 1/2$.

Interestingly, mistakes such as these seem to appear inconsistently. For instance, all the subjects who argued that division always makes smaller, correctly computed at least some of the division problems that resulted in a quotient that was greater that the dividend. Nevertheless, these mistakes do stem from systematic line of thinking – they are not just sporadic errors. After all, people *argued* that division always makes smaller. It would seem that depending on how strong the intuitions are, people tend to adapt their formal knowledge and their algorithms to accommodate their beliefs, perhaps as a result of a natural tendency towards consistency. Inconsistencies then, might be the result of the counteraction of the deeply engrained procedures that manifest when the person is not watchful of his or her beliefs, but does things automatically instead.

In light of this discussion, we examined in what ways and to what degree the findings related to rational numbers and demonstrated by the support of this framework apply to our study. As noted, the three dimensions of knowledge are not discrete; they overlap considerably.

However, for the purpose of analyzing subjects' mathematical understanding of irrationality we find it useful to focus on each of them separately, considering relationships between them or lack thereof.

Therefore, we focus on the formal, algorithmic, and intuitive dimensions of knowledge of irrational numbers and we draw connections between the dimensions. According to this framework, inconsistencies between a learner's algorithmic, intuitive and formal knowledge are often the source of misconceptions, cognitive obstacles, and other common difficulties. We strive to spotlight these inconsistencies and explain their causes.

3.2 Process – object duality

Various authors have developed theories about concept acquisition and many of them seem to run the same thread based on the conclusion that abstract mathematical notions, such as number, can be conceived in two fundamentally different, but complementary, ways: *structurally* – as objects, and *operationally* – as processes (Sfard, 1991). We draw on the work of authors such as Sfard, Dubinsky, and Tall in our inquiry of how prospective teachers understand irrationality.

Sfard argues that the ability of seeing a number, or any mathematical concept for that matter, both as a process and as an object is indispensable for a deep understanding of mathematics. She proposes a model of concept formation, in which a certain mathematical notion is regarded as fully developed only if it can be conceived both operationally and structurally. There is a duality, not a dichotomy, between the structural and operational mathematical thinking related to any given concept. Although they are mutually dependent, the structural conception is very difficult to attain as it is much more abstract – it requires a qualitative leap, or an ontological shift, that is, an ability to see the familiar thing in a completely new way. This happens in the *reification* stage of concept development, which is the last stage of the three-stage model proposed by Sfard (we outline this model in the next section). Dubinsky (1991) refers to the attainment of this stage as the "encapsulation."

According to these researchers, being capable of seeing a concept from either of the two viewpoints, as a process and as an object, and flexibly coordinating the two approaches, depending on the situation, appears to be an essential component of mathematical ability. For example, Tall (1995) speaks of this using the idea of *procept,* which refers to the duality of mathematical symbolism to represent both process (such as the subtracting of two numbers 5-2) and the product of that process (the difference 5-2). Proceptual thinking is seen as the ability to flexibly move between this amalgam of process/concept (process/object in Sfard's language) ambiguity. The question is then, how is this quality of thinking acquired.

3.3 Concept formation

Tall (1995), following the ideas of Brunner (1966), addresses the issues of cognitive development. In the context of individual's acquisition of a mathematical concept, cognitive development is seen as a journey starting from enactive interaction with the environment, followed by visual and symbolic representations interacting with one other, finally giving rise to the need for formal definition and proof. There is a sequence of stages where the meaning changes. Objects are initially perceived as physical examples. Visual representations take on successively more subtle meaning until they become "the perfect abstract counterparts of physical experience" (Tall, 1995, p. 3).

According to Tall (2001), in elementary mathematics, given the cognitive structure and representations available to the individual, this is the direction of concept formation: enactive interaction (for example, in its most primitive form, this involves carrying out a physical action to demonstrate the truth of something), followed by (visual or symbolic) representations, followed by (more or less formal) description (which can be seen as surrogate definition). If there is a mismatch, it is the description that is (usually) changed, not that which had been established in previous phases.

In higher mathematics this flow is reversed. Here it is the definition, which comes to have primacy. The formal concept is constructed from the formal definition, and the properties of the formal object are only those which can be deduced from the definition. A huge cognitive struggle is required in order to establish the definition as the basis of concept construction.

Formal image of irrational number, and even more so the formal image of the real number system, build on definition and deduction rather than enactive interaction and intuition. Moreover, this is one of the first instances that this reversal of order between enactive representation and definition happens in school mathematics. In this study, we examine how this affects the understanding of irrationality reached by the prospective secondary teachers.

Note that this is in agreement with the conceptual framework that we described earlier, and that we adopted for the purpose of organizing this work, with the omission of the intuitive dimension of knowledge. Intuitions are not taught explicitly whilst objects and processes are, and thus, we believe, the absence of intuitions from the theories of concept formation. Intuitions come naturally as by-products, and as such they can give insights into a learner's concept image; therefore, we consider them in our analysis of data. At this point, however, we are concerned with the concept acquisition theory because, as we shall see, it seems to happen in very specific stages, to which we refer when interpreting participants' understanding of irrationality.

In order to attain the reification of a concept there must first be a profound insight into the processes underlying mathematical concepts, perhaps even a certain degree of mastery in performing the processes, which form a basis for understanding the concepts (the precedence of enactive interaction, in Tall's terms). Sfard (1991) puts forth a strong case for the operational origins of mathematical concepts, supported by ontological and psychological considerations, as well as on the basis of empirical evidence. That the order of events in the process of learning is really such, she further presents this strong and convincing theoretical argument:

> If the structural approach is more abstract than the operational, if from the philosophical point of view numbers and functions are basically nothing but processes, if doing things is the only way to somehow "get in touch" with abstract constructs – if all this is true, then to expect that a person would arrive at a structural conception without previous operational understanding seems as unreasonable, as hoping that she or he would comprehend the two-dimensional scheme of a cube without being acquainted with its "real-life" model. (Sfard, p. 18)

This model of learning is hierarchical in nature. It assumes three stages corresponding to three "degrees of structuralization". First there must be a process performed on already familiar objects. This is the *interiorization* phase. When a learner becomes fluent at performing this process so that he or she can carry it out through mental representations then we can say that the process has been interiorized. An interiorized process needs no longer to be carried out in order to be considered, analyzed and compared. Second is the phase of *condensation* which is a period of "squeezing" lengthy sequences of operations into more manageable units without feeling an urge to go into details. It manifests itself as a growing easiness to alternate between different representations of a concept. This phase lasts as long as a new entity remains tightly connected to a certain process. Only as the learner becomes capable of conceiving the notion as a fully-fledged object, we say that the concept has been reified. According to Sfard, interiorization and condensation are gradual, quantitative rather than qualitative changes, whereas reification is an instantaneous shift in which a process solidifies into object, into a static structure. The new entity is soon detached from the process, which produced it, and begins to draw its meaning from the fact of its being a member of a certain category.

How and to what extent these ideas apply to understanding irrationality shall become much more apparent as we delve into the analysis of our data. In our study, we strive to determine the stage of the concept development that individual prospective teachers reached with respect to irrationality. As well, it should be pointed out that the implications for teaching practices that we propose in the last chapter rest on the hypothesis of operational origins of mathematical objects presented here.

3.4 Cognitive obstacles

Errors often do not stem only from ignorance, chance, or uncertainty. They could be the effect of a previous piece of knowledge which was interesting and successful, but which in a different, usually extended or higher-level mathematical context become false or simply unadapted. These kinds of errors are seen to be caused not by the lack of a piece of knowledge, but rather by the interference of an existing piece of knowledge or a poorly adapted piece of knowledge. This is the basis of a cognitive obstacle. Herscovics (1989) uses the term "cognitive obstacles" to refer to the obstacles encountered by the individual learner during the process of conceptualization.

Following Bachelard, Sierpinska (1994) focused her attention on the "epistemological obstacles." It is assumed that to learn is to overcome a difficulty. She proposes the historico-empirical approach to understanding in mathematics by which the study of contexts and mental frameworks in the historical development of knowledge can be used both in identifying today's students' difficulties, and in finding ways of dealing with them. In this sense, epistemological obstacles originate in the history of the discipline; however, they may replicate in individual learners as cognitive obstacles. There are similarities in the evolution of students' understandings and the historical understandings because of a certain mathematical and linguistic commonality in these developments.

In this sense, some mathematical concepts are inherently difficult, and reaching a full understanding may be impeded until a radical reconceptualization has occurred. In her study, related to students' understanding of limits, Sierpinska identified epistemological obstacles related to infinity and real number. She reports that "students view the set of reals as dense but not necessarily continuous". The concept of real number as a measure does not exist in some students' intuitive models, and so the difference between 1 and $0.\overline{9}$ fails to be translated as zero, which is seen as "the obstacle of the lack of a uniform concept of real number" (Sierpinska, 1987). The obstacles related to infinity are discussed later in this chapter.

In addition to these, there are two other epistemological obstacles related to the concept of irrational numbers: incommensurability (discovered around 500 BC) and the nondenumerability of irrational numbers (discovered in 1874 by Cantor). Neither one of these notions was easily accepted, historically. For example, the discovery of incommensurability caused a paradigm shift so significant that we now have legends depicting the bewilderment of the ancient scientists over their realization that there exist incommensurable magnitudes. One legend has it that the Pythagorean philosopher Hippasus of Metapontum was thrown overboard into the sea for his "heresies" when he demonstrated that not all lengths can be expressed as ratios of one another (Sfard, 1991). Another tale has it that Pythagoras had one hundred oxen slaughtered and roasted for a huge feast (Hoechsmann, 2003). Although incommensurability is at the heart of the notion of irrationality, it is outside of the scope of this study. On the other hand, issues related to the nondenumerability of real numbers are part of our study. We are interested in the beliefs and intuitions held by the prospective teachers with respect to the abundance of irrational numbers versus rational numbers.

Depending on the origin of an obstacle, Brousseau (1997) differentiates between obstacles of epistemological origin and obstacles of didactical origin. According to Brousseau, obstacles of epistemological origin neither can nor should be escaped because of their formative role in the knowledge being sought. On the other hand, didactical obstacles depend only on a choice or a program within an educational system. Often they can be traced back to an aspect of teaching.

To illustrate this, let us consider decimal numbers. Because of their utility and association with the metric system of measurement, decimal numbers are taught to everyone as soon as possible. In addition, they are related to technical operations with whole numbers. As a result, decimal numbers are seen as "whole numbers with a change of units" and therefore "natural" numbers with a decimal point. For example, 3.25 is 325 with one hundredth as the unit. Consequently, "all topologic relationships will be disturbed, and for a long time, and the child will not be able to find a decimal between 3.25 and 3.26, but on the other hand she will find a predecessor of 3.15, which is 3.14" (ibid, p. 92). This conception, supported by a mechanization by the student, will, right up to university level, be an obstacle to the proper understanding of real numbers. In this study, we investigated the role of epistemological and didactical obstacles in relation to the concept of irrationality. In what ways do these obstacles manifest as cognitive obstacles preventing individuals to acquire a uniform concept of real number?

Lastly, relevant to our study, there is the notion of "intuitive obstacle" introduced by Fischbein (1987). We consider intuitions as developmental phenomena that depend on personal experience and social influences. It is often the case that intuitive ideas are limited compared to their formal counterparts. Being contradictory is part of their nature. They are ideas that seem to be handy in some contexts (take the example of "division always makes smaller", or "a larger number has more factors"), but constitute an obstacle in others. The need for harmonizing intuition and mathematical notions constitutes a basic issue of education. Axioms, definitions and theorems are part of mathematics as much as its ideas and models. There is nothing more dangerous for mathematics learning than neglecting the deep discrepancies between spontaneous thinking, sometimes common sense, and mathematical thinking.

It should be noted that in some cases the distinction between "epistemological obstacle" and "intuitive obstacle" is not clear across research literature. For example, cognitive obstacles related to incommensurability can be viewed as epistemological (Sierpinska) or as intuitive (Fischbein). Epistemological obstacles can be seen as a subset of intuitive obstacles.

In this study, we examine to what degree the epistemological obstacles in the historical acceptance of the idea of irrational numbers are echoed as cognitive obstacles of individual learners. Using the notion of obstacle, we seek to identify the origins of conflicts and discrepancies in the prospective secondary mathematics teachers' understanding of irrational numbers.

CHAPTER 4

Didactical Phenomenology of Irrationality

Didactical phenomenology is the term used by Freudenthal (1983) to capture the idea of describing a mathematical concept in its relation to the phenomena for which it was created and as it concerns the learning process. In other words, it is a content specific analysis dealing with the question of what is there to know or understand about a certain mathematical notion. In this chapter we discuss the didactical phenomenology of the concept of irrationality. Issues related to the conceptualization of irrationality are outlined: What is there to know? Where lie the dangers for conflict? Here we also declare the research focus of our study.

In line with the conceptual framework discussed earlier, we investigated the participants' intuitive, formal, and algorithmic dimensions of knowledge with respect to:

- number sets
- definitions of irrational numbers
- representations of irrational numbers,
- existence, density and fitting of irrationals amongst the rationals, and the effects of operations between members of various number sets.

In the case of irrational numbers it happens that the definition relies on the existence or on the non-existence of a certain, distinguishing representation. Until the exposure to a formal construction of irrational numbers using, for instance, Dedekind cuts, this distinguishing representational feature is used as a working definition of irrational numbers. That is to say, irrational numbers are those numbers that cannot be represented as ratios of integers. An equivalent definition of irrational numbers refers to infinite non-repeating decimal representations. This means that the issues relating to definitions and issues relating to representations of irrational numbers have a significant overlap; therefore, it might seem at times that some part really belongs to a discussion related to representations, and yet it appears in the discussion related to definitions, and vice versa.

We now turn to the aspects of the content knowledge of irrationality that are of our primary interest. Let us consider the question of what is there to know, or to understand, about irrational numbers. As well, here we foreshadow the sources of possible cognitive conflicts and identify the obstacles to learning the irrational number concept. We attend only to components, or aspects, of the mathematical content knowledge relevant to this study:

1. Numbers and their class membership with specific reference to the place of irrational numbers within the set of real numbers, i.e. the set of irrationals as a complement of rationals in reals.
2. The two definitions, or rather the descriptions, of irrational numbers as commonly used in school mathematics and the relationship between these two definitions:
   - Nonexistence of a representation as a/b with a, b integers and $b \neq 0$.
   - Infinite non-repeating decimal representation of irrational numbers.
3. The multiple representations of irrational numbers (decimal, as a number that cannot be expressed as a quotient of two integers, symbolic – for those that even have such representations, geometric – constructible lengths for those that can be constructed or as points on the number line in general).
4. The existence and density of irrational numbers and how they fit among the rational numbers (i.e. denumerable vs. nondenumerable sets). Also, the effects of operations between members of rational and irrational number sets.

It should be noted that the knowledge of this content domain consists of much more than is listed above. For example, this study does not address the issue of incommensurability of irrational magnitudes, proofs of irrationality, different kinds of irrational numbers, rational approximations of irrational numbers via continued fractions, irrational numbers as limits of a sequence of rational approximations, and so on. As well, we do not explicitly investigate the role of the symbolic representation of (algebraic) irrational numbers such as roots of various kinds. Although certain aspects of understanding operations with irrational and real numbers are investigated, this is not central to our study.

In summary, for the purpose of this thesis, we confine our analysis only to the issues mentioned in the above list. It should be noted that borders between these aspects of knowledge are not clear-cut. For example, definitions often play a significant role in deciding on how to classify a number; further, definitions in the case of rational and irrational numbers depend on the existence or the nonexistence of certain kinds of representations of numbers. In the next sections we examine in greater detail what is involved in understanding the aspects of knowledge outlined above.

4.1 Number sets

In the Integrated Resource Package for Principles of Mathematics 10, Number Concepts strand, it is expected that students will "classify numbers as natural, whole, integer, rational or irrational, and show that these number sets are 'nested' within the real number system" (BC Ministry of Education, 2002 p. 136). Further in this document, under suggested instructional strategies we read the following:

(It is suggested that teachers)

- discuss the definitions of the various number systems;
- use a set of nested measuring cups with appropriate labels to illustrate how the number systems fit inside one another;

- illustrate the anomaly of irrational numbers by slipping a piece of paper between the rational-number cup and the real-number cup;
- have students transfer these concepts to Venn diagrams in their notebooks.

It seems that in school mathematics, not much attention is being paid to how these number sets came about historically; in particular, there seems to be no reference to the operational origins that necessitated formal construction of new kind of numbers.

Historically, the notion of number developed through a lengthy cyclic process in which the same sequence of events could be observed again and again whenever a new kind of number was being born. First, there was the preconceptual stage, at which certain operations were performed on the already known numbers. This was followed by a long period of mainly operational approach during which a new kind of number began to emerge out of the familiar processes. For example, roots of negative numbers emerged during Cardano's prescriptions for solving cubic and quartic equations; however, they were regarded as nothing more than abbreviations for certain meaningless by-products. With time, as people became accustomed to these strange but useful kind of computation, the set of numbers had to be broadened again to include the number in question as a fully-fledged mathematical object, in this case the complex numbers.

As discussed earlier, there is a kind of hierarchy, in which what is conceived purely operationally at one level should be conceived structurally at a higher level, both from a historical as well as psychological outlook (Sfard, 1991). This is to say that the natural flow of concept formation, both historically and as it concerns individual learner, is from operational to structural. It should be pointed out that this is an assumption based on personal experience and on observations in our teaching practice. There are also other possibilities for how mathematical conceptions are formed. For example, a skilled teacher might use the driving force of "big questions" to bring a new concept into being, or it could be done formally, by the force of a definition.

Note that at this point in the school curriculum, motivated by the introduction of irrational numbers, students are expected to conceive the real number as new fully-fledged mathematical object in the hierarchical structure of number. In order to understand how the number sets relate to each other, one must rely on a rather structural notion of number. For example, the Venn diagram depicting the structure of the set of real numbers in its relation to other sets of numbers represents a rather sophisticated view of the concept of number. Regardless of their origins, numbers such as natural number, rational number, irrational number are conceived as objects of the same type – real number.

In this study, we are interested in both, the prospective teachers' competence as well as their reasoning in the classification of numbers into various sets. We examined how our group of prospective secondary teachers compared to the prospective elementary teachers from Tirosh et al. (2003) study concerning their ability to identify set membership of various numbers. We examined whether the fact that this topic appears in the secondary rather than elementary mathematics curriculum has any bearing upon prospective teachers' ability to identify correctly the set membership of various numbers.

4.2 Definitions

We examined the role of the two competing definitions in an individual's conceptualization of irrationality.

<u>There are two definitions of irrationals given at a school level:</u>

a) An irrational number is a number whose decimal part is not periodic and has an infinite number of digits.

b) An irrational number is a number that cannot be expressed as a quotient of two integers with a nonzero divisor.

These two definitions, as opposed to the formal mathematical definition by means of Dedekind cuts, are merely the descriptions of, or the introduction to the concept of irrationality as given in high school textbooks guided by didactical considerations given the relative mathematical immaturity of the target student population (Arcavi, 1987). Although the motives behind the existence of these two definitions are considered to be primarily pedagogical, the fact is that these same two definitions are being used by students and by preservice teachers alike in order to classify numbers as either rational or irrational.

It is worthwhile to note that the commonly used student textbooks, such as Mathpower 10, are devoid of any attempt to make the connection between the two definitions explicit. In Chapter 1.1 called "The Real Number System" students are given the following definitions without any explanation of how fractions give rise to repeating decimal expansions:

> An **irrational** number is a number that cannot be expressed as a terminating or repeating decimal. Irrational numbers are non-terminating, non-repeating decimals.
>
> They cannot be expressed in the form *a* / *b*, where *a* and *b* are integers and *b* does not equal 0. The set of irrational numbers is named using the symbol $\overline{\mathrm{Q}}$. (Mathpower 10, Western edition, p. 6)

After this, the textbook gives six examples of irrational numbers. Three of them are both in symbolic and decimal form with nine digits after the decimal point followed by three dots (these examples are π, $\sqrt{2}, -\sqrt{7}$), and the other three examples are transcendental numbers intended to show that non-repeating yet patterned decimals are also irrational. These are the examples provided in the same textbook (spacing is as it appears in the original text):

0.202 002 000 200 002 …

1.121 122 111 222 111 122 221 …

– 0.100 200 300 400 500 …

It seems that the (mis)treatment of this mathematical topic has become chronic – it is just as bad today as it was nearly a century ago. Klein (1932) begins his chapter on irrational numbers with this dismal remark, "Let us not spend any time in discussing how this field is usually treated in the schools, for there one does not get much beyond a few examples" (Klein, p. 31).

The concept of irrational numbers is inherently difficult - historically, these numbers have caused ontological problems and understanding them involves overcoming epistemological obstacles. We examined whether these difficulties compound because of the way irrational numbers are commonly introduced in school mathematics, that is, using two characterizations or representational features as definitions for irrational numbers without an explicit connection made between them. If this were the case, then understanding irrational numbers would require overcoming both, epistemological and didactical obstacles. In this study, we investigated what effect does having the two (unrelated) definitions have on participants' understanding of irrational numbers.

The missing link: Why are the two "definitions" equivalent?

Put bluntly, the reason that we can use the decimal characterization as a "definition" is that a decimal number represents a rational number if and only if it terminates or repeats. Ironically, this characterization has nothing to do with the original motivation for distinguishing between the rational and irrational numbers.

Any fraction whose denominator is not a factor of a power of ten - and only the products of powers of 2 and 5 are - cannot be written as a finite decimal. Consequently, a great number of fractions that students see in school mathematics are those that have infinite decimal expansions. Using the decimal representation it might seem that it would be difficult to distinguish such rational numbers from irrational numbers, except for one interesting thing about infinite decimals that come from fractions - they repeat. It is possible to see, by long division, that the decimal expansion of any fraction a/b with a, b integers and b≠0, necessarily repeats. For a more unified perspective, we can say that the "terminating decimals" are also infinite repeating (having bi-unique infinite expansions). For example, the terminating decimal 2.3 can be seen as an infinite repeating decimal either as 2.2999… or as 2.3000…. The possible remainders on dividing a by b can only be 0, 1, 2, …, (b-1), so with only b possible choices for remainder, the calculations in the long division must eventually start repeating. The claim that every fraction has an infinite repeating decimal expansion can be shown using several examples to clarify why this is so.

It should be within the grasp of a grade 9 student to understand that when we try to convert fractions whose denominators are not powers of 10 into decimals, what stops us from ever finishing is that at some point we end up with a remainder that is the same as one that we got before, so we end up getting the same sequence of digits over and over again. The number of digits in the period will therefore be no greater than 1 less than the denominator. So for example it could take 16 places for the decimal expansion of 1/17 to repeat, but no more than that. This is because when dividing out 1/17 there are only 16 possible remainders (0 does not count because if the remainder where 0 it would be a finite decimal). Therefore, after 16 places all the possible remainders will be used up, so the next remainder will have to be one of the previous ones and from there on the same things will happen over and over again in the long division creating a repeating sequence of digits. Some algorithmic experience is needed for this idea to settle in students' minds.

The converse, that any repeating decimal is a fraction, is much more subtle and difficult to grasp. The general proof of this notion requires the summing of an infinite decreasing geometric progression. Although a formula for the sum of such progression is given in high school, the derivation of such result requires the use of the limit process (usually done in first year university course).

In high school, and even as early as in Grade 7 (in British Columbia, for example) a type of symbolic "juggling" where operations are conducted on infinite decimal expansions is presented to students to convince them of the "iff" relationship. Here is a typical example:

Problem: Convert 0.12121212... to a fraction.

Solution: Let $x = 0.12121212...$, then $100x = 12.121212....$, so $100x-x=12$. But $100x-x$ is also $99x$, so $99x=12$. Dividing both sides by 99, we get $x=12/99=4/33$.

Although resourceful, this juggling is a bit contrived and can possibly leave the student with an impression that there might exist another trick that will turn a non-repeating decimal into a fraction. Compounding this conflict are the fractions which can be "seen to have no repeating pattern", such as 1/257, when displayed on a calculator. Furthermore, there remains the danger of conflict between the theoretical requirement for infinite decimals and the practical experience that finite decimals are both convenient and sufficient. Specifically, we examined to what degree the availability of this juggling affected the understanding of irrationality. In other words, we investigated in what ways does the absence of the link which renders the two definitions equivalent impede the prospective teachers' overall understanding of irrationality.

4.3 Representations

Specifically, we focus here on how irrational numbers can be (or cannot be) represented and how different representations influence participants' responses with respect to irrationality. While the research on representations in mathematics and their role in mathematical learning is extensive (Cuoco, 2001; Goldin & Janvier, 1998, to name just a few collections), and research on irrational numbers is rather slim, there has been no study that investigates understanding of irrational numbers from the perspective of representations.

As a theoretical perspective we use the distinction between transparent and opaque representations, introduced by Lesh, Behr and Post (1987). According to these researchers, a transparent representation has no more and no less meaning than the represented idea(s) or structure(s). An opaque representation emphasizes some aspects of the ideas or structures and de-emphasizes others. Borrowing Lesh's et. al. terminology in drawing the distinction between transparent and opaque representations, Zazkis and Gadowsky (2001) focused on representations of numbers introducing the notion of relative transparency and opaqueness. Namely, they suggested that all representations of numbers are opaque in the sense that they always hide some of the features of a number, although they might reveal other, with respect to which they would be "transparent". For example, representing the number 784 as 28^2 emphasizes that this is a perfect square, but de-emphasizes the divisibility of this number by 98. Representing the same number as $13 \times 60 + 4$ makes it transparent that the remainder of 784 in division by 13 is 4, but de-emphasizes its property of being a perfect square. In general, we say that a representation is transparent with respect to a certain property, if the property can be "seen" or derived from considering the given representation.

Applying the notions of opaqueness and transparency we suggest that infinite non-periodic decimal representation (such as 0.0100100010…, for instance) is a transparent representation of an irrational number (that is, irrationality can be derived from this representation), while representation as a common fraction is a transparent representation of a rational number (that is, rationality is embedded in the representation).

In this study we examined how the availability of certain representations influenced participants' decisions with respect to irrationality.

Infinite non-repeating decimal representation of irrational numbers: Utilized to define, identify, and represent irrationality

Students are told that irrational numbers have infinite non-repeating decimal representation. But how can the learner know that the expansion for $\sqrt{2}$ does not start repeating after, say, a thousand decimal places? The idea of infinity, in contrast with "very large", is not within our primary intuition. Again, we come back to the same missing link that equates the two definitions discussed earlier. In other words, in order to know that the decimals will not start repeating after any finite period, no matter how large, one must know three things. First, one must know that $\sqrt{2}$ cannot be expressed as a ratio of two integers (say by faith or by using a classical proof of irrationality of $\sqrt{2}$). Second, one must know that every repeating decimal number can be expressed as a ratio of two integers. Third, one must know that $\sqrt{2}$ cannot be both rational and irrational (the two sets are mutually exclusive). If one of the pieces is missing, there is leap of faith.

On a related matter, every high school student claims to know that π is irrational. It is clear that such claim can only rest on pure faith, as the knowledge required to show that π is irrational exceeds most third year university students'.

For reader's amusement we present the corresponding prescribed learning outcome (PLO) from the BC Grade 9 curriculum: Students are required to "describe, orally and in writing, whether or not a number is rational" (p. 44). Next, it is says: "The ratio of the circumference to the diameter of any circle is π. Explain whether or not π is a rational number" (p. F-41). This shows that what is to be considered an acceptable explanation to satisfy this curricular requirement can be no other than examining the digits of π as presented on a calculator upon pressing the PI button and then concluding that it must be an irrational number because the digits do not exhibit any repeating pattern. If this kind of argument for determining irrationality is acceptable in case of π, then perhaps it is also acceptable other cases, such as 1/17. Clearly, promoting this kind of argument as a legitimate explanation for the irrationality of π is likely to induce conflicts in subsequent learning.

From the perspective of opaqueness and transparency applied to the decimal representation as displayed on a calculator screen, it is obvious that all displays of irrational numbers are opaque with respect to irrationality and that the majority of displays of rational numbers are opaque with respect to rationality. Note that, on the calculator, the only transparent representations are those of "short" terminating rational numbers; that is, those that use up fewer digits than the calculator display allows.

Even those decimal numbers that can be seen as repeating are not necessarily rational. To illustrate this, we can create an example of irrational numbers that, when entered into the calculator can be seen as "infinite repeating" and therefore indeed seem rational.

Example, using an 8-digit calculator:

Consider the following number, *m:*

$$m = \sqrt{10{,}000{,}000} \div \sqrt{91} \div \sqrt{989{,}011}$$

Entering it in the calculator, we end up with 0.3333333 on the display, which could be interpreted as a demonstration that *m* is rational. However, this is false because each one of the three roots above is irrational (both 91, and 989,011 are prime, and so their square roots are irrational; also $\sqrt{10{,}000{,}000} = 1000\sqrt{2}\sqrt{5}$ is irrational; none of the factors combine to form a perfect square, and so *m* is irrational). The same calculation performed on a 10-digit calculator, yields 0.333333331, which at least casts some doubt. Therefore, using the calculator display to decide on anything is open to errors. The problem is that this method "works" with most if not all of the school examples.

So what use is the decimal characterization in deciding whether or not a number is irrational? The point is, not much, really. Any time we need to perform a computation with a calculator in order to get a number into decimal form, it is of no use, because when the maximum possible digits are displayed, we cannot possibly tell from a calculator whether a decimal repeats or even whether it terminates. It might do either way beyond the number of places that the calculator displays. Remember, a number like 1/17 could take as many as 16 places before it repeats. The only real use for the decimal characterization of rational and irrational numbers is for determining if a number that is already in decimal form is a rational number, and even then some assumptions must be made. For example, the number 0.101101110111101..., assuming the expansion is infinite and the number of ones between the single zeros successively increases in the same manner, can be identified as irrational as this particular number's decimal representation is transparent with respect to irrationality. Likewise, 0.101010101..., and assuming the expansion is infinite repeating, can be identified as rational since its decimal representation is transparent with respect to rationality. It follows that the notion of "infinite non-repeating decimal" representation is useful only for the case of identifying those rare patterned transcendental numbers. If this is all that is gained by using such characterization of irrational number, the question is: "Is it worth it?" A natural side effect, and a realistic danger of introducing this notion is that many people might revert to it as their only working representation for deciding whether or not a number is rational or irrational.

<u>Nonexistence of a representation as a/b with a, b integers and b≠0: Utilized to define, identify, represent, and prove irrationality</u>

Representation that characterizes rational number, becomes, as negation, a definition for irrational number. This is possible to do because the two sets, rational and irrational, are exclusive of each other and at the same time the two sets comprise all of the set of real numbers. Within the set of real numbers there are no gaps and no overlaps between the two sets. However, saying what something is not is not easily accepted as a definition for what something is. Furthermore, to show that a given number is irrational a proof is required – a proof showing that the number cannot be represented as a/b with a, b integers and b≠0. In most cases the proof is indirect - a proof by contradiction. Although direct proofs exist, they are not well known, and they do not appear in standard textbooks.

Contradiction proofs cause problems of acceptance in practice. In the classical proof by contradiction (such as the one given in the standard Addison Wesley Mathematics 10 textbook to prove the irrationality of $\sqrt{2}$) students are required to suppose something is true only to find that such supposition leads to contradiction, and to then conclude on that basis that what was originally supposed to be true must have been false.

Here is the proof that $\sqrt{2}$ is irrational as presented in the Addison-Wesley Mathematics 10 textbook (p. 97).

<u>Step 1</u>

Assume that $\sqrt{2}$ is a rational number.

Then there are natural numbers *m* an *n* such that $\sqrt{2} = \frac{m}{n}$, where *m* an *n* are in lowest terms.

Step 2

Square each side to obtain:

$$(\sqrt{2})^2 = (\tfrac{m}{n})^2$$

Multiply each side by n^2.

$$2 \times n^2 = \tfrac{m^2}{n^2} \times n^2$$

$$2 = \tfrac{m^2}{n^2}$$

Step 3

Since the left side of this equation is even, the right side is even. Hence, *n* must be an even number. Represent this even number by 2*p*. Substitute 2*p* for *m*:

$$2n^2 = (2p)^2$$
$$2n^2 = 4p^2$$
$$n^2 = 2p^2$$

Step 4

Since the right side of this equation is even, the left side is even. Hence, *n* must be an even number. That is, both *m* and *n* are even. This means that the fraction $\tfrac{m}{n}$ is not in lowest terms, although we assumed in Step 1 that it is in lowest terms. This contradicts the assumption in Step 1 that $\sqrt{2}$ can be written as a fraction in lowest terms.

Step 5

The assumption in Step 1 that $\sqrt{2}$ is a rational number is incorrect. Hence, $\sqrt{2}$ is not a rational number.

Learners often feel a sense of emptiness and lack of explanation as to why $\sqrt{2}$ is not rational. Part of the problem may be that the contradiction does not arise by contradicting the simple statement "$\sqrt{2}$ is rational" but rather by contradicting a more sophisticated one, "$\sqrt{2}$ is rational in lowest terms". In the attempt to make this proof as direct as possible, and thus make it more accessible to students, in his study, Tall (1979) proposed an alternative generic approach that rests upon the Fundamental Theorem of Arithmetic.

> We will show that if we start with any rational p/q and square it, then the resulting square cannot be 2. On squaring an integer n, the number of times any prime factor appears in the factorization of n is *doubled* in the factorization of n^2. In the factorization p^2/q^2 we factorize the numerator p^2 and the denominator q^2, canceling common factors where possible. Then each factor either cancels exactly or we are left with an even number of appearances of that factor in the numerator or the denominator. The fraction p^2/q^2 cannot be simplified to give 2/1 because the latter has an odd number of 2s in the numerator. So the square of a rational p/q is never equal to 2. (Tall, 1979, p. 206)

Participants of the study (n=70, students entering university from high school) were presented with two proofs, the classical one and the one shown above. Approximately one half of the participants received the two versions of proof for 2, as shown above, and the other half received the two proofs where 2 was replaced with 5/8. They were asked if they understood either proof or were confused by either proof on the first read through. They were then asked to keep the questionnaires for a few days to see if their attitudes changed and to make a record in case they did. The conclusion was that the alternative proof turned out to be considerably more generalizable than the standard one. The effect on understanding was significantly higher for those students who have seen neither type of proof before. The standard contradiction proof, though mathematically elegant, was found to "lack explanatory power and generalize with difficulty because of linguistic considerations" (Tall, 1979, p. 207).

On a related note, Sierpinska points out that understanding a theorem, such as $\sqrt{2}$ is an irrational number, on the basis of acceptance of the logical soundness of its proof is not the same as understanding the proof and its 'reasons'. The proof does not explain why the fact is so significant – it does not show how incommensurability is related to irrationality nor does it tell us why the decimal expansion of $\sqrt{2}$ should be infinite and non-periodical. (Sierpinska, 1991, p. 77). Understandably so, school mathematics does not go beyond the proof of irrationality of $\sqrt{2}$, and possibly $\sqrt{3}$.

Geometric representation: (Real) number conceived as a point on a (real) number line

A real number can be represented by point on a line. Visually, there is no distinction between a rational number line and a real number line. In a conventional size drawing, it is impossible to distinguish between the point marking $\sqrt{2}$ and the one specifying 1.414. However, not only are they different, but one is irrational while the other is rational, a vital distinction in pure mathematics.

With the introduction of irrational numbers the learner must not only reconstruct the concept of number to include such entities which cannot be used for counting but also his or her conception of the number line must conform to this new evidence to include many more points. Both, the concept of irrational numbers and the concept of the real number line build on formal definition and deduction rather than on enactive interaction and visual representation. As discussed earlier, the real number line is defined as a line such that all real numbers receive a one-one correspondence with points. In that sense it is an "artificial" construct decreed to exist. However, its existence becomes very real, and the need to formulate it can be appreciated, when one realizes that many curves (such as a circle with radius $\sqrt{3}$, for example) would cease to exist without the support of the real number system. Quite strikingly, if coordinate axes contained rational numbers only, the graph of $x^2 + y^2 = 3$ would disappear, as there is not a single point with a rational ordered pair anywhere on this circle. An understanding of this need for continuity of real numbers, and its non-obviousness, comes together with an awareness of the incompleteness of the rational number domain.

Some suggestions for instructional practices regarding the geometric representation of real numbers are given in the Integrated Resource Package for Principles of Mathematics 10 (BC Ministry of Education). It is suggested that teachers:

> Draw a number line on the board and have students place rational numbers in appropriate places, then ask them to place irrational numbers on the line. Point out that a decimal approximation of the number is necessary to allow for the proper placement and that more decimal places lead to greater accuracy. (p. 136)

Note that the second suggestion is false - it applies only to higher roots and transcendental numbers. There are irrational numbers that can be constructed geometrically with compass and straightedge, such as square roots and other irrationalities that can be obtained by repeated extraction of square root. Moreover, those are precisely the irrational numbers most often encountered in school mathematics.

In the analysis of participants' understanding of irrationality, close attention is being given to the degree and nature of the connections between the three forms of representations living in the participants' concept image of irrational number. This is natural, considering that "a mathematical concept is learned and can be applied to the extent that a variety of appropriate representations have been developed together with functioning relationships among them" (Goldin and Shteingold, p. 6). For example, Lamon (2001), who has extensively researched how students come to understand rational numbers, notes that "basing instruction on a single interpretation and selectively introducing only some of its representations in instruction can leave the student with an inadequate foundation to support her or his understanding of the field of rational number" (p. 150). It has been shown that the different representations and interpretations of fractions contribute to a robust understanding of a rational number.

4.4 Relations of two infinite sets: density, fitting, richness, and operations

The idea of irrational number pushes the limits of students' imagination. When dealing with questions such as "Which numbers do we have more of, rational or irrational?" or "Is it possible to find a rational number between any two irrational numbers?" we cannot avoid thinking about infinity. Our intuitions and beliefs about infinity are teased out, and that may or may not be at variance with the formal image. Difficulties in the understanding of irrationality may be revealed through inconsistencies between intuitive and formal dimensions of knowledge. Of course, there is a possibility that one may have been exposed to the formal construction of the concept of infinity in terms of, say, cardinal sets, and that this new evidence forced one to reconstruct one's knowledge in a way that does not involve conflicts. However, there is also a possibility that despite of being exposed to it, one has not absorbed the meaning of it and has not successfully adapted one's personal concept image, or even that one has not been exposed to these notions at all. We assumed that not many prospective teachers would have the formal arguments related to the richness of the two number sets, rational and irrational, in their active repertoire of knowledge. In this sense, we were interested to see whether prospective teachers are capable of supporting well-founded intuitions about these questions independently (i.e., without an external intervention) based solely on their comprehensive and solid formal knowledge about numbers.

Research shows that informal images often persist long after formal ideas are introduced (Fischbein, 1979). Individuals' conceptions may involve essentially contradictory features – the intuitive obstacles, discussed earlier. There are two intuitive obstacles described in literature that are of our concern here. First, the informal image that 'the whole is greater than the part' becomes an intuitive obstacle when the concept of infinity is to be conceived in terms of cardinal infinity. This particular intuitive obstacle has been extensively researched. For instance, Tall speaks about this is terms of *natural infinities* - these are personal conceptions and may contain built-in contradictions, and *formal infinities*. The didactics for overcoming it have been suggested, primarily by varying the representational mode. The number of correct responses increased significantly when the mode of representation used in the question was such that it explicitly exhibited the one-one correspondence (Tsamir & Tirosh,1999).

The second obstacle relates to infinite decimal expansions. It has been documented that learners often see the infinite decimal expansion as a continuous process rather than as an object (Sierpinska, 1991). This is explained in terms of *potential* and *actual infinity* and it seems to closely relate to operational vs. structural conception of decimal number. Specifically, until the concept of decimal number has been reified it remains tied to the process of division, which in the case of recurring decimals never reaches completion. The number is still at the stage where it is seen as being constructed and not as already constructed, as a sequence rather than its limit.

The difficulty in accepting that in the same interval there are infinitely many rational and infinitely many irrational points, is of our interest here. We examined the participants' capability to produce adequate intuitive models for representing irrational number concepts. By "adequate" we mean such that will not create inconsistencies with the other two knowledge dimensions. As well, here we situate participants' difficulties in the context of the system of rational numbers and in the system of real numbers.

We investigated prospective teachers' beliefs and intuitions regarding three distinct threads. First, we explored the beliefs about the relative "sizes" of the two infinite sets. What kind of mental images are used to tackle this question? Second, we looked at participants' intuitions about how the rational and irrational numbers fit together (i.e. the idea of continuity of the set of reals achieved by the "completeness" axiom). How do they reconcile the fact that rational numbers are everywhere dense, that is, between any two rationals, no matter how close they are, there are infinitely many rational numbers, and yet there is still "room" to fit the irrational numbers amongst them. Thirdly, we investigated how preservice teachers respond to questions about the effects of operations between various types of numbers (for example, when is there a closure).

In the following chapter, we turn to the specific tasks that were designed in order to investigate the issues discussed in this chapter.

CHAPTER 5

Methodology

5.1 Participants and Setting

Participants of this study were 46 prospective secondary mathematics teachers (PSTs), 16 male and 30 female, in their final term of studies before certification. Approximately one third of them were mathematics majors, while the rest held a major in science. The data consists of two sources: a written questionnaire (included in Appendix A) completed by all 46 participants and a semi-structured clinical interview conducted with approximately one third of this group (16 of the participants).

The questionnaire was administered during the third session of the secondary mathematics methods course (EDUC 415, Designs for Learning Secondary Mathematics) at Simon Fraser University (SFU), which was held in the summer of 2002. The duration of this course was 13 weeks, with meetings once a week for four hours. The questionnaire was administered in the second half of the session, and prospective teachers were free to leave when they were finished. Therefore, there was no time limit on the written part.

It should be mentioned that at the time of administering the questionnaire prospective teachers were not told that they were asked to do this for research purposes. Instead, they were asked to fill it out to the best of their knowledge, as this information would help the instructors with planning for the course content. They were also asked to put their name on the paper. At the time of writing, participants believed this was a test of their (general) knowledge of mathematics and, although the atmosphere in the room could be described as one filled with anxiety, there was a keen interest to do one's best and to take one's time. Most of the prospective teachers took about 60 to 75 minutes to complete the questionnaire, however some went for as long as 90 minutes.

The following week, at the next meeting, the purpose of the questionnaire was revealed to the participants. We thanked them for their participation, and told them that there is a threefold benefit that could be gained from the experience. First, they gained an insight into the feelings of apprehension that students often feel in a test situation. The anxieties compound in the case of unannounced tests such as pop quizzes and in cases where the content tested does not exactly match what has just been taught, but rather assumes some previous knowledge, which is similar to what they had experienced. There was a lively discussion about whether it was reasonable to expect that a prospective secondary mathematics teacher had his or her repertoire of knowledge on this topic available upon such sudden request. A general conclusion was that if such knowledge could be accessed and could be therefore seen as portable (over time) and transferable (over situations) then this was a good measure that this topic had been understood at the time of its study. Second, there is very little research related to this topic, and through their participation in this study we would gain a better understanding of how the teaching of irrationality could be improved, what the misconceptions related to this topic are, and so on. These insights would benefit prospective teachers in subsequent cohorts. Throughout the course, the course instructor made references to research findings, either verbally or by assigning readings (such as Susan Lamon's research on rational numbers), and so this group of prospective teachers has learned to appreciate having access to this knowledge. Thirdly, there was a lesson towards the end of the course where the preliminary results of this research were shared with this group. They were the first to find out about the issues related to understanding this mathematical domain as revealed through the data from the questionnaire. Therefore, if anyone felt inadequate at the time of writing it, care was taken that their understanding had been "straighten out," and this of course, would be to the benefit of their future students.

After this discussion the participants were given an option to either decline or accept their participation in this study. If they were to decline their paper would be returned to them at this point. If they were to accept they might be approached to participate in the second phase of the data collection, clinical interviews, which were to be conducted in order to probe further ideas and beliefs that participants expressed in the written response. Not a single participant asked to be taken out of the study.

During the next couple of weeks the written data was examined for preliminary results, and 16 prospective teachers were selected for the interview. Item 6 of the written questionnaire (see Appendix A) was used to gauge the preliminary results, because it involves recognition and classification of numbers across a wide range of examples, and could thus be quantified to give a good overview of general performance. All the responses on this item were tabulated and quantified to give a sense of individual as well as overall performance of the group (results appear in Chapter 6.1).

The guiding principle for selecting the interviewees was fair representation. We wished to probe the thinking at all levels, according to the distribution of the results. In accord with this design, the results were such that three interviewees were those that would be considered to have a good understanding of the concepts, nine would be considered to have a fair understanding, and four would be considered to have a poor understanding. In other words, based on the results on Item 6, and in keeping with our strategy to have a fair representation of the whole group, these turned out to be the requirements guiding our selection of candidates for the clinical interview (this point is brought up again in greater detail in Section 6.1.3).

Between the 7th and 10th week that the course was in session all sixteen interviews were conducted, by an appointment outside of class time, with each interview lasting anywhere between 45 to 60 minutes. The interview was semi-structured in the sense that the ideas explored very much depended on the participant's responses to the written part. All the interviews were audio taped and later transcribed for the purpose of analysis.

In the 12th week the preliminary results were shared with the class. Part of the lesson addressed the historical development of the concept of irrationality. Next, misconceptions that this group generated were presented for discussion. The majority of the lesson was dedicated to activities that would help people better understand the concept of irrational numbers, especially the link between the two definitions and the idea of incommensurability.

5.2 Task Analysis

We now turn to particular tasks from the written questionnaire. The entire questionnaire appears in Appendix A. Here we discuss only those items that are relevant to the scope of this work as outlined in the previous chapter. We consider them in the order of issues set out there with the exception of "definitions" and "representations" which have been reversed here (because the methodology for exploring the issues related to definitions references the items related to representations). We discuss the rationale for inclusion of each item, that is, what insights we hoped to gain from the task and how the task relates to the issues identified in the content analysis section described in Chapter 4. As well, here we present what would be considered an acceptable and/or ideal response.

5.2.1 Set membership identification

In order to obtain a general assessment of the participants' formal knowledge of number sets and their relationships, and of their ability to use this knowledge to classify various numbers across sets, we designed Item 6. As mentioned earlier, with its broad scope of 14 sub-items, this item was used to gauge the performance of the group as well as the performance of individual participants in relation to the group. In the written part, we were interested primarily in their competence in classifying a given number into respective sets. During the interviews we also examined the reasons for their decisions. This item helped us identify which candidates would be the most suitable to be invited for the clinical interview so that they would be fairly representative of the group yet revealing of the variety of ways people think about irrational numbers. Here is the question as it appeared in the questionnaire that was administered:

Item 6. For every number listed in the table below check all the attributes that apply. For example, in the case of "cat" it would look like this.

	Animal	Mammal	Reptile
Cat			

	Natural number	**Integer**	**Rational number**	**Irrational number**	**Real number**
0.05755755575…					
5/31					
$-\sqrt{36}$					
0.9999999…					
The solution of the equation $2^x = 3$					

The solution of the equation $x = \cos\frac{\pi}{3}$					
The solution of the equation $x = \sin 60°$					
The solution of the equation $3x + 1 = 0$					
The area of the unit circle					
$\sqrt[4]{0.0016}$					
$\sqrt[3]{0.8}$					
$3\sqrt{8}$					
$\sqrt{\frac{12}{75}}$					
0.012222…					

Correct responses are represented with shaded fields.

	Natural number	**Integer**	**Rational number**	**Irrational number**	**Real number**
0.05755755575…					
5/31					
$-\sqrt{36}$					
0.9999999…					
The solution of the equation $2^x = 3$					
The solution of the equation $x = \cos\frac{\pi}{3}$					
The solution of the equation $x = \sin 60°$					
The solution of the equation $3x + 1 = 0$					
The area of the unit circle					
$\sqrt[4]{0.0016}$					
$\sqrt[3]{0.8}$					

$3\sqrt{8}$					
$\sqrt{\frac{12}{75}}$					
0.012222...					

The difficulty in accepting that 0.9999999... (where the digit 9 repeats infinitely) is 1 has been extensively researched and documented in the education research literature (Sierpinska, 1987; Tall, 1978; Mamona-Downs, 2001). The reason we included this number in our list was to identify those participants that have had a formal exposure to this result or have thought about it extensively and have come to understand the bi-unique decimal representation of terminating decimal numbers. Based on this research, we assumed that a correct set membership identification of this number would likely be an indication of a profound understanding of real number and therefore we included it in our list.

Upon later consideration of the results reported by several researchers (Arcavi, Bruckheimer & Ben-Zvi, 1987; Tirosh, Fischbein, Graeber & Wilson, 2003), and especially after clinical interviews, we regretted that 22/7, and also 0 were not included in our list of numbers to be classified. As already mentioned in the section on literature review, there is a persisting confusion between π and this particular approximation of π, even after instructional intervention (Arcavi et al., 1987).

To get a general overview of how prospective secondary teachers think about the structure of the set of real numbers, we examined:

- what properties of the number guide their decision about whether a number is rational or irrational,
- whether or not they see every number in the list as real number,

- whether or not they see the set of rational numbers as a complement to the set of irrational numbers within the set of real numbers (i.e. the fact that it is impossible for a number to be both rational and irrational).

5.2.2 Representations

To investigate understanding of irrational numbers from the perspective of representations, we designed three questions, each tackling one of the representational features of irrationals:

- Irrational numbers are those that have infinite non-repeating decimal representations.
- Irrational numbers are those that cannot be represented as ratios of integers.
- (Constructible) irrational numbers can be (easily) represented geometrically as points on the number line.

In terms of preference and comfort, when thinking about irrational numbers, people seem to fall into three categories: those who rely on the decimal representation, those who rely on the nonexistence of a representation as a quotient of two integers, and those who use both with equal facility. We refer to them as decimal, fractional and balanced dispositions. Items 1 and 2 (as well as some numbers from the list in Item 6) tend to reveal a person's disposition. In Item 1, we examined how the availability of infinite non-repeating representation influenced participants' decisions with respect to irrationality.

Item 1:	Consider the following number 0.12122122212……. (there is an infinite number of digits where the number of 2's between the 1's keeps increasing by one). Is this a rational number? How do you know?

From the decimal disposition, irrationality of this number is transparent. Suppose that for whatever reason the definition of irrational number as infinite non-repeating decimal is not available to the individual. Then s/he may start from considering what fraction would yield this decimal expansion (fractional disposition). This is a possible line of thinking. Assume such fraction exists, call it m/n, where m and n are whole numbers and n is not 0. Upon dividing m by n we eventually start encountering at most (n-1) different remainders, and so the decimal expansion of m/n is a repeating decimal (terminating decimal can also be seen as repeating, as discussed earlier). Therefore, there will be a difference between this decimal number and the one given above, which means they are not equal. We run into a contradiction, and so the given number cannot be represented as a ratio of integers and is thus irrational. Whether considering it from a decimal or fractional disposition, the irrationality of 0.121221222… is transparent, as it can be derived from considering the given representation in both cases. However, in this case we note that irrationality is more directly and easily seen from a decimal disposition.

In Item 2, we examined how the availability of a representation as a ratio of two integers (with a non-zero divisor) influenced participants' decisions with respect to irrationality.

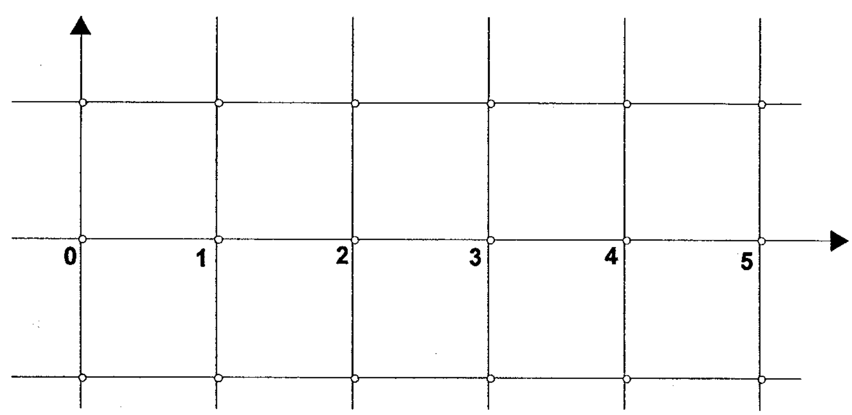
> Item 2: Consider 53 divided by 83. Let's call this quotient M. In performing this division, calculator display shows 0.63855421687.
>
> Is M rational or irrational? Explain.

We agree with the criticism that introducing a calculator display in this question is intentionally misleading. However, it was our goal to check to what degree the participants will be misled. Note that the numbers are carefully chosen so that the repeating is "opaque" on a calculator display. Here the rationality is embedded in the representation – the number is a ratio of two integers. Using the decimal characterization to decide whether 53/83 is rational or irrational is totally missing the point since 53/83 is rational, because it is a ratio of integers, which is the original meaning of rational number. An irrational fraction is a contradiction in terms. It would be like saying that a fraction was not a fraction. Of course, if 53/83 were expanded enough, one would see that its digits repeat (in fact, there are 41 digits in the repeating period). But there is no reason to do that, because for one it is known that this is true for all fractions and for another if it were not true for all fractions that would not be a proper characterization of rational. The whole point of that characterization was for it to be a way to tell if a decimal representation came from a fraction in the first place. A strictly decimal disposition (i.e., a complete absence of considering a fraction as an object) can be identified in learners as they typically do not take the advantage of the transparency of representation with respect to rationality, such as in the above case.

Furthermore, from the way one thinks about this problem, it is possible to discern the developmental stage of the concept of rational number. Consider the notion of decimal number as an object which solidified into a static structure as a result of the process of division of two whole numbers. In the case where one still needs to perform the division in order to decide whether or not the decimals will start repeating, we see the concept in the interiorization phase. If the process of division of two whole numbers had been completely interiorized it would no longer need to be carried out in order for the results of this process to be considered, analyzed and compared. In the condensation phase we would see a growing easiness to alternate between different representations of a concept (decimal number with infinite repeating digits or ratio of two whole numbers), certainly the "missing link" would be non-existent. The reified stage would manifest as the attainment of proceptual thinking about rational numbers – 53/83 would be seen as object drawing its meaning from the fact of its being a member of a certain category, a rational number, detached from the process of division, regardless of the representational mode.

Item 5 was designed in order to investigate understanding of the geometric representation of an irrational number. In particular, we were interested in what means the participants would use in order to locate $\sqrt{5}$ on the number line precisely. It is said that to every real number there corresponds exactly one point on the real number line. One may find this difficult to believe if one has never seen an irrational point located on the number line, especially considering the fact that the number line is everywhere dense with rational numbers.

Item 5: Show how you would find the exact location of $\sqrt{5}$ on the number line.



The number line in this question is intentionally set in the Cartesian plane with a visible grid to simplify the straightedge and compass construction (i.e., there is no need to draw a perpendicular line at 2). It is intended to aid in the invoking of the Pythagorean Theorem in the efforts to construct the required length. The expected response is shown in the figure below.

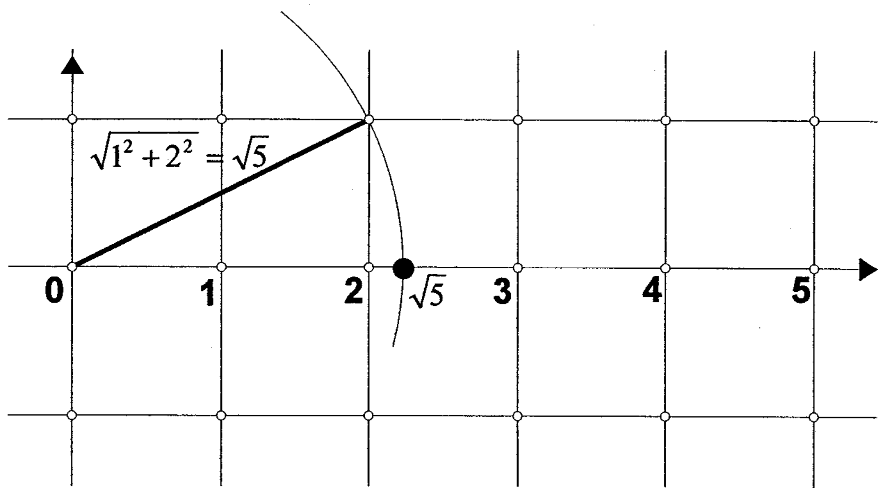


Figure 1: Geometric construction of $\sqrt{5}$

We were interested to see whether participants will use this conventional or a similar approach or whether they will resort to thinking in terms of decimal expansions, such as advised in the Integrated Resource Package for Principles of Mathematics 10 (BC Ministry of Education), and discussed earlier in section 4.3.

5.2.3 Definitions

One of our goals was to assess the role of two competing definitions used to define irrational numbers. We were also interested to see which one of the two is the preferred definition for our group and how the coordination of the two definitions affects the performance. We assumed that people tend to exhibit either a decimal, fractional, or balanced disposition depending on which of the two definitions they adopted or whether they have reconciled the two. We were interested to see whether it is possible to have a balanced disposition without understanding the equivalency of the two definitions. In other words, can one successfully apply either of the two definitions as the situation warrants and still be in the dark as to how one follows from the other? Or would the cognitive conflict that arises in case of such confrontation destabilize one's concept image to the degree where one would be forced to either abandon one of the definitions or reconcile the two in order to once again reach the state of cognitive equilibrium? These were the questions we attempted to answer. Clearly, this is difficult to assess from a written response.

A piece of knowledge is the result of the person's adaptation to a situation which "justifies" this piece of knowledge by making it more or less effective. One can envisage the association of each useful piece of knowledge with a region of effectiveness (and cost). (Brousseau, 1997, p. 98) The question here is how effective is each of the two definitions on its own and what would it take for the learner to become aware of the "missing link".

The issue of definitions and their coordination was explored primarily via the clinical interviews. The interview questions depended on the participant's responses to Items 1 and 2. In general, the interview questions proceeded to establish, first, what definition(s) of irrational number is used by the participant, and second, the completeness and accuracy of the definition(s) being used. Next the participant was confronted with a seemingly conflicting situation which could only be resolved if the "missing link" was not an issue. Such situations are presented in Chapter 6 (Section 6.3). This way we established for each interviewee whether the two definitions are being used in isolation of each other, whether only one of the definitions is being used at all times, or whether the two definitions are seen as equivalent. If they are seen as equivalent, is this by decree (because one cannot have inconsistencies in mathematics) or because there is a mathematical relationship between the two definitions that has been recognized by the participant?

5.2.4 Relations of two infinite sets: density, fitting, richness, and operations

We investigated prospective teachers' capability to produce adequate intuitive models for representing number concepts and operations. Although most of these questions could be answered using formal arguments, most participants resorted to the use of informal intuitive arguments. Therefore, we were forced to analyze their response in light of the intuitive models that they used. In particular, we explored the PSTs' intuitive dimension of knowledge of irrational numbers in the context of real numbers. Furthermore, we were interested in the ways in which PSTs strive to harmonize their intuitions with what they formally know to be true about the two types of numbers, the rationals and the irrationals, with respect to their abundance, density, fitting, and operations between the members of the two sets.

Existence, density, and richness of irrational numbers

In order to investigate how PSTs think about the abundance of irrational numbers within the set of real numbers, we designed the following two closely related items:

Item 3: What set do you think is "richer," rationals or irrationals (i.e. which do we have more of)?

Item 4: Suppose you pick a number at random from [0,1] interval (on the number line of reals). What is the probability of getting a rational number?

The correct answer to Item 4 is 0. One way to see this is by the use of Cantor's diagonalization proof of denumerability of rational numbers combined with a reasoning on limits. According to the proof, all positive rational numbers can be enumerated; that is, they can be brought to a one-one correspondence with natural numbers. We omit the proof, as it is well known (interested reader may look it up, for example, in Serge Lang's *Math! Encounters with high school students*, p. 116). This implies that all rational numbers on the [0,1] interval can be enumerated as well, so that there will be the first, second, third, ... without missing any of them. Of course they will not be in order of magnitude, but rather in order that comes from the above mentioned diagonalization process, where all the rational numbers not in the [0,1] interval are omitted from the enumeration process. Let us call these enumerated rationals $q_1, q_2, q_3, \ldots$

Next, we can "capture" each one of the enumerated rationals inside intervals of ever decreasing size according to this model:

$q_1 \rightarrow 10^{-(n)}$

$q_2 \rightarrow 10^{-(n+1)}$

$q_3 \rightarrow 10^{-(n+2)}$

...

where n is a natural number. The sum of all these intervals in which all the rational numbers in the [0,1] have been captured is then equal to: $\sum_{i=0}^{\infty} 10^{-(n+i)}$.

For example, if we choose n = 5, then all rational numbers are captured in the interval of size 0.000011111..., or $1/9 \times 10^{-4}$. In general, all the rational numbers are captured in the interval of size $1/9 \times 10^{-n+1}$. This interval can be made as small as we wish (by making n sufficiently large). At its limit it decreases to 0, that is to say, no positive, however small, interval size can be assigned. $\lim_{n \to \infty} \sum_{i=0}^{\infty} 10^{-(n+i)} = 0$.

Since all the rational numbers can be "squeezed" into an interval of size 0, the probability of picking a rational number is 0. In other words, the rationals can be made to fit into a union of intervals whose total length is arbitrarily small. Hence no positive probability can be assigned.

It was not expected that PSTs come up with a proof such as the one above. We include this proof for the sake of completeness only. Our aim was to examine what intuitions our participants had regarding the density of irrational numbers and whether or not the idea that there are various degrees of infinity was a part of their knowledge repertoire.

Fitting

The following four questions from Item 7 are related to how rational and irrational numbers fit together. These questions were designed to both further our understanding about the intuitive models of number concepts used by the PSTs, as well as to investigate their formal understanding given that the tools for a correct derivation are accessible.

Item 7: Determine whether the claim is true (T) of false (F) and explain your thinking.

e) It is always possible to find a rational number between any two irrational numbers.

f) It is always possible to find an irrational number between any two irrational numbers.

g) It is always possible to find an irrational number between any two rational numbers.

h) Between any two rational numbers there is always another rational number.

All of the above statements are true, which can be shown, for example, by using existence proofs. How does one reconcile the idea that for any two irrationals, no matter how close they are, there is always a rational in between, with the idea that irrationals are so much more abundant that the probability of hitting a rational on any interval on the number line, no matter how large the interval may be, is 0? It is not easy to imagine that between any two numbers whatsoever, regardless of how close they are, there is a countable infinity of rational and an uncountable infinity of irrational numbers. To adapt to the evidence of this result, one must force oneself to abandon thinking from finite experience when considering questions such as these (see the discussion on formal versus natural infinities in 3.2.4). We examined the formal, algorithmic, and intuitive arguments that were used by the prospective secondary teachers to address these questions. Let us now consider what would be the expected responses concerning the fitting of numbers:

e) It is always possible to find a rational number between any two irrational numbers.

The claim is true. To see this, let the two irrational numbers be $a = a_1a_2a_3a_4\ldots$ and $b = b_1b_2b_3b_4\ldots$, where a_i and b_i are the i-th digits in the decimal expansions of numbers a, b respectively. Without loss of generality, we can assume that there is some first position $i = k$ for which $a_k > b_k$ (i.e., for $i < k$ we have $a_i = b_i$; according to this assumption we also have $a > b$). Then there exists a rational number $c = a_1a_2a_3a_4\ldots a_k$, with a property that $b < c < a$. In other words, cutting off all the digits after the first digit in which the two numbers differ from each other from the larger one will produce a rational number with the required property.

f) It is always possible to find an irrational number between any two irrational numbers.

The claim is true. We can construct such an irrational number similarly as in the question before, or, alternatively, we can proceed by looking at the difference d between the two given irrational numbers a, b. Without loss of generality, assume $a > b$. The difference $d = a - b$ can be rational or irrational. If it is rational, take one half of it, and add that to the smaller of the two irrationals. The resulting number is irrational which is exactly in the middle (arithmetic mean). If the difference d is irrational, construct a rational number c by cutting off all the digits after the first non-zero digit of d. Clearly, $c < d$. Now add c to the smaller of the two given irrationals. The result is an irrational number which is greater than b (this is because it is equal to $b + c$ where c is a positive rational) and it is smaller than a (because $a = b + d$ and $c < d$ so $b + c < a$). The reason we did not take the arithmetic mean of a, b in the case when their difference is irrational is because it is not guaranteed that the arithmetic mean of two irrationals is necessarily irrational. In fact, it is easy to construct two irrationals, such that their mean is a whole number.

g) It is always possible to find an irrational number between any two rational numbers.

The truth of this claim can again be shown by demonstrating how this can be achieved in general, for any two rational numbers a, b. Without loss of generality, assume $a > b$. Now take the difference $d = a - b$ and let $e = d/\pi$. Note that e is irrational and $e < d$. An irrational number between a and b is $b + e$. This proof takes into account that d/π is irrational, that is, it assumes that rational over irrational is irrational. This is easy to justify using proof by contradiction (i.e., rational over irrational equals rational is impossible according to the justification provided for question d).

h) Between any two rational numbers there is always another rational number.

The truth of this claim is easy to show. Let $a = c/d$ and $b = e/f$ be two rational numbers with $c, d, e, f \in Z$ and $d, f \neq 0$. Then $\frac{1}{2}(c/d + e/f) = (cf + ed)/(2df)$ is also a rational number and it is between a and b (the arithmetic mean).

Operations between members of the two complementary number sets

To investigate how PSTs think about number operations in context of the rational and irrational number sets, we designed the following questions:

Item 7: Determine whether the claim is true (T) or false (F) and explain your thinking:
a) If you add two positive irrational numbers the result is always irrational.
b) If you add a rational number to an irrational number the result is always irrational.
c) If you multiply two different irrational numbers together the result is always irrational.
d) If you multiply a rational number by an irrational number the result is always irrational.
i) A product of two rational numbers can sometimes be irrational.

It was our assumption that these questions could aid in revealing the stage of development of the concepts of rational, irrational and real number. This assumption is based on what has been discussed earlier regarding operational versus structural conceptions and the attainment of proceptual thinking with respect to number operations. As well, we expected to see a variety of ways of tackling these questions. This is because they are accessible in the sense that anybody with high school knowledge of number concepts can successfully attempt them, yet they are non-standard and as such they lend themselves to a variety of approaches. In particular, we were interested in the dispositions that might be revealed (see 4.2.2 for discussion on dispositions), such as, for example, whether the arguments given by the participants are based on considering the decimal, fractional or symbolic representations. We now offer some possible correct responses to the above questions on number operations:

<u>a) If you add two positive irrational numbers the result is always irrational.</u>

This claim is false. We can show this using an example. Take the irrational number 0.1212212221222212… where the number of 2's between the 1's keeps increasing by one ad infinitum. Adding this number to a "matching transcendental" 0.8787787778777787… … where the number of 7's between the 8's keeps increasing by one ad infinitum, we get $0.\overline{9} = 1$. We would consider this argument as indicative of decimal disposition. On the other hand, one could say, consider two irrational numbers a and b, such that $a = 5 - \pi$ and $b = 5 + \pi$. Clearly, a and b are both positive and irrational. Their sum $a + b = 5 - \pi + 5 + \pi = 10$, so it is rational. This kind of argument would be considered as indicative of symbolic disposition; moreover, it would also be an indication that the individual has attained a proceptual level of thinking about number concepts. It is no longer relying on the decimal representation, but rather makes use of the additive inverse property of numbers.

b) If you add a rational number to an irrational number the result is always irrational.

This claim is true. One possible way to explain this is by assuming the contrary, and then showing it leads to a contradiction. Assume that the sum of a rational number q and an irrational number p is rational – call it c. From q + p = c it follows that c – p = q. The left side is rational, because the difference of any two rationals is necessarily rational (let c = a/b and p = e/f, where a, b, e, f are integers, and b, f are not 0. Then c – p = (af – eb)/(bf). The difference c – p is also a ratio of two integers with non-zero denominator, and hence it is rational.), while the right side is the irrational q. This is a contradiction; therefore, the sum c must be irrational.

c) If you multiply two different irrational numbers the result is always irrational.

The falsehood of this claim can be shown in various ways; however, it suffices to say that, since real numbers are a field, there exists a multiplicative inverse for every number except 0. For example, multiplicative inverse of π is $1/\pi$. We would need to show that $1/\pi$ is irrational (assuming that π is irrational, this follows from question (d) below, so we omit it here), and then we could claim that the product of two irrationals, π and $1/\pi$ is rational. In addition to this argument, one could say that the procedure commonly known as "rationalizing the denominator" is in fact a case where we multiply an irrational number by another irrational number, namely its conjugate, to obtain a rational number (usually in the denominator). Take for example $(5 - \sqrt{2})(5 + \sqrt{2}) = 23$ (note that according to (b) above, and given that $\sqrt{2}$ is irrational, also $5 + \sqrt{2}$, and similarly $5 - \sqrt{2}$, are irrational). Notice that in order to produce arguments of this kind, a proceptual level of thinking about numbers such as $5 - \sqrt{2}$ would have to be attained.

Alternatively, it is possible to use the knowledge of factors of whole numbers together with the rule that $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ to arrange that the product of two irrationals is rational, such as in $\sqrt{75}\,\sqrt{3} = 15$.

<u>d) If you multiply a rational number by an irrational number the result is always irrational.</u>

This claim is true. We show this by contradiction. Take the product of an irrational number p and a rational number a/b where a, b are non-zero integers, and assume that this product is rational, that is, $p\times(a/b) = (c/d)$ where c, d are non-zero integers. Multiplying both sides by b/a we obtain $p=(b/a)(c/d)=(bc)/(ad)$, which is a contradiction since p is irrational and thus cannot be expressed as a ratio of two integers.

<u>i) A product of two rational numbers can sometimes be irrational.</u>

This claim is false - a product of rational numbers is necessarily rational, because the set of rationals together with the four arithmetic operations form a field. The product (a/b)(c/d), where a, b, c, d are integers and $b, d \neq 0$ is expressible as a ratio of two integers, (ac)/(bd).

In summary, we analyzed the data related to the relations between the two infinite sets from the perspective of intuitive dimension of knowledge for three reasons: first, in order to better understand how prospective teachers think about irrational numbers in the context of rational and real numbers; second, in order to identify the stage in the life of concept development, and third, in order to assess the degree to which the three intuitive obstacles mentioned in the previous chapter manifest in the prospective teachers' concept images.

CHAPTER 6

Results and Analysis of Formal Knowledge

In this chapter we present the results and analysis of PSTs' formal knowledge of irrational numbers as revealed through their understanding of number sets, representations, and definitions.

6.1 Set membership identification

As a starting point we look at the PSTs formal knowledge related to irrational numbers and their place within the number system. Using the results from the data collected on Item 6 (see Appendix A) of the written questionnaire, we obtained a broad view of PSTs' knowledge of the definitions of the various number sets, such as natural numbers, integers, rational numbers, irrational numbers, real numbers, and their hierarchical structure within the set of real numbers. As described earlier in Section 5.2.1, this question contained 14 items presented in the table format. Participants were asked to checkmark *all* the attributes that apply. An example was presented to ensure correct interpretation of instructions. The results are taken to be indicative of the knowledge of definitions and relationships between the number sets.

Two of the participants left the entire table blank, and were thus not included in the quantification of results that follows. In other words, the table below and the discussion that follows applies to the results for n=44 participants. All the checkmarks from the participants' response sheets were tallied. The shaded fields represent correct responses. The frequency report, both as raw score and as percentage is as follows:

	Natural number	Integer	Rational number	Irrational number	Real number
0.05755755575...	2 (4.5%)	0 (0%)	7 (15.9%)	35 (79.5%)	37 (84.1%)
5/31	3 (6.8%)	1 (2.3%)	34 (77.3%)	8 (18.2%)	36 (81.8)
$-\sqrt{36}$	3 (6.8%)	35 (79.5%)	31 70.5%)	3 (6.8%)	38 (86.4)
0.9999999...	3 (6.8%)	2 (4.5%)	32 (72.7%)	7 (15.9%)	38 (86.4)
The solution of the equation $2^x = 3$	3 (6.8%)	2 (4.5%)	7 (15.9%)	21 (47.7%)	31 (70.5%)
The solution of the equation $x = \cos\frac{\pi}{3}$	3 (6.8%)	1 (2.3%)	26 59.1%)	11 (25%)	35 (79.5%)
The solution of the equation $x = \sin 60°$	5 (11.4%)	4 (9.1%)	15	21 (47.7%)	33 (75%)
The solution of the equation $3x+1=0$	1 (2.3%)	5 (11.4%)	35 (79.5%)	3 (6.8%)	37 (84.1%)
The area of the unit circle	5 (11.4%)	4 (9.1%)	11 (25%)	26 (59.1%)	34 (77.3%)
$\sqrt[4]{0.0016}$	4 (9.1%)	1 (2.3%)	36 (81.8%)	2 (4.5%)	38 (86.4%)
$\sqrt[3]{0.8}$	4 (9.1%)	2 (4.5%)	11 (25%)	24 (54.5)	34 (77.3%)
$3\sqrt{8}$	6 (13.6%)	6 (13.6%)	9 (20.5%)	25 (56.8%)	33 (75%)
$\sqrt{\frac{12}{75}}$	3 (6.8%)	1 (2.3%)	32 (72.7%)	4 (9.1%)	36 (81.8%)
0.012222...	3 (6.8%)	1 (2.3%)	29 (65.9%)	10 (22.7%)	36 (81.8%)

Table 1: Frequencies and percentages (n=44) of responses to Question #6 of the written questionnaire.

The results presented in this table are discussed in greater detail below. Here we would like to point out that this format of accounting for responses proved very useful in identifying trends. In particular, it served as an aid for bringing the issues related to mathematical understanding of the system of real numbers in general, and set recognition in particular to our attention. However, it should be noted that the group is too small to do a meaningful statistical analysis on these data; we present the results with an aim to bring up some points that we found interesting.

6.1.1 Structure of the real number set

One of the things we noticed, is that the structure of the set of real numbers as commonly depicted by a Venn diagram in high school textbooks, and as suggested by the IRPs, was not a part of the knowledge repertoire of all PSTs. Several participants did not make use of the fact that some sets are mutually exclusive while others are subsets of other sets, so that a member that belongs to such a subset automatically belongs to the "parent set". For these participants the definitions were not firmly grasped, and consequently inconsistencies of this nature did not seem to bother them.

Structure versus arbitrary labels

In some cases the lacking in the formal knowledge, in this case of definitions, turned out to be so severe as to render the concepts of "natural", "integer", "rational", "irrational", and "real" devoid of any meaning or purpose beyond arbitrary labeling. The excerpt from the interview with Connie exemplifies this view.

Interviewer:	What would you say is a real number, what do we call a real number?
Connie:	A real number. . .
Interviewer:	Yeah, what's a real number?
Connie:	**A real number, it exists**, like 1, 2, 3, **negative is not real**, so, **real number exists**, 1,2,3…
Interviewer:	And what's natural number then?
Connie:	Natural number would be, oh, I don't know, natural like 1,2,3,4,5,6. . .
Interviewer:	So, natural and real are the same?
Connie:	It shouldn't be the same, but I can't remember. . .

It seems that Connie takes literally the "real" in the "real number" and interprets it as "exists" as opposed to negative number, which is only fictional for her. Later on, in the discussion about what is the distinction between rational and irrational numbers, Connie repeatedly states that she is "muddled" on her definitions. She can provide some examples of each, but cannot express what is unifying about each of these categories of numbers. Interestingly, Connie finds no real benefit to get cleared up on her definitions; in her view, there is no practical benefit to being able to categorize numbers. In the absence of underlying concepts, her perception is that it is a game of mere labeling.

Interviewer:	What makes you say this is rational and that is not rational, is there any point in distinguishing rationals and irrationals?
Connie:	Square root of 2 is irrational. . .
Interviewer:	Okay. . .
Connie:	I guess this is like **tops of things that I think I remember**, I think I remember what is rational and irrational, **but I don't really know why it is irrational, why it is rational**.
Interviewer:	Okay, I see. Um, is there any point, like should the students know about these things, what do you think?
Connie:	I don't think there is, it's like I don't really, I don't need it, even if you do your math, you know square root 2 is a square root 2, whether it is rational or not rational, you're using it in the calculation or in proving, or whatever. You still use it. So **it doesn't serve any purpose**, like **why do we want to categorize them as rational and irrational**, why do you want to learn it, you know, so...

Out of the 16 PSTs interviewed, two had an extremely limited knowledge of the concept or irrational number. When investigating the reason behind this, we found that a simple conflict in their understanding proved to have powerful effects, and discouraged them from believing they could ever understand what seemed so nonsensical. Judging from the attitude Connie expressed, it seems this conflict must have been a part of her concept image for a long time. The conflict we speak of consists of a person's simultaneous holding of two beliefs: first, that "fraction is rational" and second, that "infinite decimal is irrational". Clearly, this is very problematic, as she too recognizes that decimal expansions of many fractions, such as 1/3, are infinite. Interestingly, she keeps holding on to both of these beliefs, and rather than releasing one of them, or making an adjustment in her understanding, she simply gives up. In the end, she is even willing to accept that a number could be both rational and irrational.

Connie:	I think fraction is a rational, okay, I kind of remember something, so it's a rational number, but when we divide it out, then it's never ending, and then it is an irrational, but it doesn't like uh, it's like, if I'm not using it, and I'm not using it, then you know, it's just classification and it doesn't mean anything, so it's just, I never bothered to really find out.
Interviewer:	So can some numbers maybe be at the same time irrational and irrational? Could that be (pause) that some numbers are both?
Connie:	(pause) If I can see it my way, then yes...

An understanding of the structure of the system of real numbers cannot be reached if a person is prevented from drawing a distinction between members of the two complementary sets, rational and irrational. However, from the other 14 participants we were able to learn more about what the underlying issues in the conceptualization of irrationality are.

Subsets, exclusive sets, and complementary sets

The structural relations between number sets were not firmly grasped amongst a number of participants. The most well understood relation is that rational and irrational are complementary sets while the least understood are the various subset relations, in particular those related to the real number set.

For example, an integer is not automatically a rational number – there were 12 PSTs with at least one case such that a number being identified as integer was not identified as rational. Similarly, a rational or irrational number is not automatically real – there were 10 PSTs in our group that identified a number as rational or irrational but not real. In addition, we found 9 PSTs who checked "real", but checked neither "rational" nor "irrational". It is possible, at least in some cases that a participant was certain that the number was real, but could not decide what type of real number it was. Some of these errors could be seen as accidental omissions, but for the most part they were systemic across many items, and in several cases even accompanied by an outright statement, such as "don't know definition of natural number", written on the response sheet. Two PSTs left the entire column "real" unchecked, which we interpret as the absence of the definition of real number in their concept image. One participant marked $-\sqrt{36}$ as both integer and irrational, perhaps being misled by the square root symbol; however, such response is possible only in the absence of understanding that these are mutually exclusive sets.

On the other hand, all participants seem to have known that a number cannot be both rational and irrational. There were only two cases of an item having both "rational" and "irrational" check-marked (in the case of $\sqrt[4]{0.0016}$ and $x = \sin 60°$), which we explain as the participant's uncertainty about what to choose and then choosing both (just to be safe) rather than his/her belief that it could be both.

To sum it up, we found that a large proportion of the participants in our group were unclear about the definitions of various number sets to the degree where it significantly impaired their ability to classify numbers into various numbers sets. It seems that most participants relied on incomplete verbal definitions, memory and examples when deciding on which attributes apply to a given number, rather than on a schematic diagram, for example. This is consistent with the study of Tirosh, Fischbein, Graeber, and Wilson (2003) study on prospective elementary teacher's understanding of rational numbers which found that 81% of mathematics majors were able to draw an adequate Venn diagram to describe the relations between the natural numbers, the integers, the rational numbers, the irrational numbers, and the real numbers. On the other hand, only 25% of the non-mathematics majors were able to do so (Tirosh et. al, 2003).

Given that the ratio of mathematics majors to non-mathematics majors in our group of participants was 1:2, it would be expected that about 43% of them would have an adequate knowledge of the definitions of various number sets so as to be able to reproduce the Venn diagram (on the basis of Tirosh at al. study). When drawing the comparison, several issues should be taken into account. Firstly, our study did not ask the participants to draw the Venn diagram; therefore, we can only make inferences on the basis of data pertaining to Item 6. Secondly, the prospective teachers in our study are prospective secondary as opposed to primary mathematics teachers, and so we might expect the results to be better than the expected 43%. However, considering the fact that the knowledge of definitions of the subsets of the real number set is normally expected to be in place long before taking advanced mathematics courses, this fact may not be very significant (i.e., the knowledge we are trying to asses is not taught in senior level university courses). Lastly, the ability to reproduce the Venn diagram of number sets only partially serves as an indicator of the knowledge of definitions. Of course, examining the participants' ability to classify examples of numbers into various sets also gives only a partial picture. However, if the examples of numbers to be classified are varied sufficiently, the consistency or inconsistency of responses in such set membership identification task may serve to discern the level of mastery of the definitions. Section 6.1.3 discusses these findings in greater detail. At this point we wish to mention that the overall performance of the group was lower than the expected 43%.

Number sets versus number systems

We were interested in what ways those participants who demonstrated a proficiency on Item 6 differ from those whose responses were flawed with errors. Although this would be useful, it is beyond the scope of this work to provide a fine-grained analysis of how their conceptions were acquired or even what the attributes of these conceptions are. However, we wish to point out that their conceptions do not rest solely on definitions. Rather, they operate from a much richer platform that sometimes bares reference to the motivation behind the extending of the number sets in order to accommodate operations that are not possible in the lower sets.

In this sense we speak of a number system rather than a number set. Under this view, the set of natural numbers, for example, is just that, a "bare" set with no structure, just a bunch of elements. In contrast, we may think of the naturals as a *structur*e, in which case we may need to specify what kind of structure we are viewing the natural numbers as. We may be interested in the natural numbers as an *ordered set*; so we have the bare objects with the < relation, which is a minimal structure. We may be interested in the natural numbers closed[4] under the standard addition and multiplication. Attending to these structural properties of sets these participants seemed to have an advantage over those that had to rely on memory of definitions, or even on memory of the Venn diagram. Note that the Venn diagram of number sets does not explain anything about number systems in that sense.

It seems that attending to number systems as opposed to number sets contributes to developing a more robust understanding. One of the participants of the study with a high success on Item 6, Claire, explains how she thinks of the number sets and how she would teach this to her students. What we note in Claire's elaborate discussion is that she is grounding the concepts of "natural", "integer", and so on in operations. This supports the view that in the process of concept formation, operational conceptions precede the structural.

Interviewer: How do you define a real number, what is real number to you?

Claire: A real number, it's a set of numbers which um include all numbers natural, whole numbers, natural numbers, integers, rational and irrational. And um, in which almost all operations are possible, or defined, yes addition is possible always, subtraction, multiplication, division. What is not possible is square root of a negative, because this is the how you introduce the complex numbers, so **each set of numbers is introduced, in my opinion, from this perspective of operations**.

[4] By the idea of closure we mean the following. Let U be a set with certain operations defined on it. A subset B is said to be closed under the operations if the result of applying the operations to elements of B gives an object in B.

Interviewer: Oh, right. How is that?

Claire: So in order to introduce, for example, the integers, we talk to the student, you know, if you want to subtract 5 - 2, for example, is 3. The answer is a natural number, but if you want to subtract 5 - 9, what number is it, and you said, oh, oh, it's not a number. . ., yes it is, we define it, a new class, a new set of numbers, it's called whatever, and you introduce integers.

Interviewer: So you provide a motivation for introducing the new sets of numbers?

Claire: Why, so the why, I provide them always the why. And then the next level, I provide the why because the division is not always possible in fact. It is for some numbers, and so on, you know, but not in general, in the set of integers. So from natural then you go to Z, then rational Q, yes, and we have real. So the whole numbers are included in the real numbers, and also in the rationals.

Here we see Claire jumping a little bit too fast from rational to real numbers, so we questioned her on that. It turns out the road is not as smooth as in previous cases, even for her, and that the operation of taking a square root is perceived as a stumbling block and likely to present a didactical problem.

Interviewer: Okay. And then um, as you said, there's always a widening set so that you can incorporate new operations, what will be then the motivation, why, how, why have we gone to reals?

Claire: In order to solve the equation $x^2 = 2$, yes we need the square root, it's not possible, the square root is not possible in um, just a sec, the value, now yeah, I never thought about this question, and see how a student can get them, yeah, as long as you don't have square root as operation in other sets of numbers, of course that square root of 16, it's natural it's 4, it is a natural number then, because the symbol square root does not even exist as operation in natural, or Z or Q. . .Not many students understand why, it's very hard to understand, but you can make them, how I taught, I can tell you, for example I always prove **as a model square root of 2 and then by examples**. Because **it's not a continuous set**, **they cannot see it like um, for example natural numbers, on the number line, like nicely one after another one**, so you have to explain them, okay I give you models.

What we find interesting in this reply is the claim that irrational numbers are not continuous and that they need to be built through models, perhaps meaning geometric models. From the excerpt it would seem that Claire's conception of irrational is confined to square roots: however; this is not evident from her written response. It is more likely that this is how she would teach the concept, which is not necessarily identical to how she personally understands it.

6.1.2 Analysis of sub items

We now examine the degree of success on each of the sub items given in Item 6. We tallied the correct responses for each of these numbers in order to find out which were the most and which the least problematic numbers to classify and what we can learn from that. That is to say, only those numbers that were correctly classified across *all* fields (natural, integer, rational, irrational, real) are included in this count. The frequencies appear in the table below.

Item	**Number of entirely correct responses**	**[%]**
0.05755755575...	29	[65.9%]
5/31	27	[61.4%]
$-\sqrt{36}$	24	[54.5%]
0.9999999...	1	[2.3%]
The solution of the equation $2^x = 3$	18	[40.9%]
The solution of the equation $x = \cos\frac{\pi}{3}$	23	[52.3%]
The solution of the equation $x = \sin 60°$	15	[34.1%]
The solution of the equation $3x + 1 = 0$	31	[70.5%]
The area of the unit circle	20	[45.5%]
$\sqrt[4]{0.0016}$	30	[68.2%]
$\sqrt[3]{0.8}$	20	[45.5%]
$3\sqrt{8}$	24	[54.5%]
$\sqrt{\frac{12}{75}}$	30	[68.2%]
0.012222...	24	[54.5%]

Table 2: Frequencies and percentages (n=44) of responses to the sub items of Question #6 for which *all* the attributes that apply have been correctly identified.

As expected, the most problematic number to classify was the classic example of 0.9999999…, which, as we mentioned, served primarily to identify those candidates from which we can learn a great deal about how they acquired their mathematical knowledge of number. Only one person responded correctly to this item.

What came to us as a surprise is the unexpectedly low performance on the "the area of a unit circle." Upon examining the reasons behind this erroneous error, we found that it is most likely the word *unit* that triggered a mechanical response in 5 of the participants who checked "natural", "integer", "rational", and "real". Six other PSTs left this item unanswered, perhaps wondering how an area can be a number.

Similarly, x = sin 60° seems to have triggered an unexpectedly high response stating that x should be rational. This item was the most problematic after the 0.9999999… The connection between the height of an equilateral triangle whose side length equals 1 and sin 60° does not seem to be readily available to most PSTs. Instead, what seems to be the case here, is the thinking that since 60 is a nice whole number, sin 60° should be rational too. In fact, a good number of participants who identified sin 60° as rational also identified the cos (π/3) as irrational. That is to say, it was the argument of the trigonometric function that they were focusing on rather than the value of the function at that argument.

The next in the group of most problematic items, with the performance still well bellow 50% of the PSTs was the solution to $2^x = 3$. Although there exists a simple and elegant way to see that x must be irrational this kind of reasoning seemed to be out of reach for most of the PSTs. Most of those that answered correctly based their decision on their experience with logarithms. The prevailing thinking is that in a great majority of cases logarithms do not end up being "nice" numbers, so it is most likely that such is the case here as well. Many participants left this item blank, perhaps because they did not have a calculator available (please note that calculator was allowed, but not provided) that would help them decide. Quite a few others chose rational or even natural, failing to see that it cannot be. Assuming x = m/n with m, n integers and raising both sides of the equation to the power of n, we get $2^m = 3^n$, an impossibility according to the Fundamental Theorem of Arithmetic. Therefore, x is irrational.

Also surprising is the poor performance on $-\sqrt{36}$. The types of errors ranged from the omission of "rational" (this was the most common error in this item) to classifying it as irrational, to classifying it as natural.

On the other hand, the most successful item was the solution to the liner equation, 3x+1=0. Perhaps due to familiarity with linear equations as well as with fractions such as –1/3, despite the negative sign, 31 out of 44 PSTs were able to correctly identify all the applicable attributes. We note the relatively high performance on both $\sqrt[4]{0.0016}$ and $\sqrt{\frac{12}{75}}$ which seems to be an indication that PSTs are well trained in the algorithms of simplifying a radical.

6.1.3 A snapshot of the PSTs' performance as a group

We counted how many items were incorrectly classified by each of the participants. The bar graph below is representative of the general performance of the group. The vertical axis represents the number of participants, and the horizontal axis represents the number of incorrect responses. We see that only one person correctly classified all 14 numbers given in Item 6. Next, we see that 4 people correctly classified 13 out of the 14 items. In other words, they had only one incorrectly classified item, that being the problematic 0.9999999…. Another 4 people correctly classified 12 out of the 14 items, with only one other error besides the 0.9999999… (each participant incorrectly classified a different item: #1, #6, #7, #12). Looking at the next bar in the bar graph, we see that nobody had exactly 3 incorrect items, but there were five people who incorrectly classified 4 out of the 14 items, and so on. There were 3 people that had errors in all

14 items. The median performance of the group was 8 correctly classified items out of a total of 14 items. More precisely, over 2/3 of the participants incorrectly classified 6 or more items from this table.

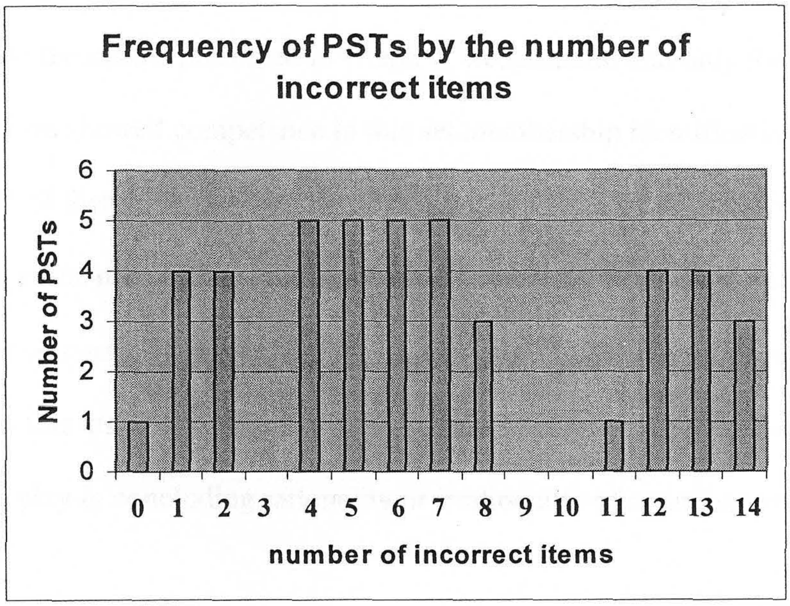


Figure 2: Distribution of participants by the number of incorrectly classified items from Question #6 (n=44).

Judging from this table that contains 14 items, it seems that the prospective teachers from this group fall into three distinct groups. Although the distribution is roughly natural, we can say that the three groups are formed by the top nine (20.4%) with at most two incorrect items, the middle twenty-three (52.3%) with four to eight incorrect items, and the low twelve (27.3%) with eleven or more incorrect items. Let us call these groups A, B, C respectively for easier reference later on.

The participants' performance on these items was used to identify which candidates would be interesting to choose for the interview to gain a fuller insight into how people think about irrationality. It should be noted that from the 16 prospective teachers that participated in the clinical interview 3 come from group A, 9 from group B, and 4 from the high end of group C.

In other words, the prospective teachers who participated in the interview were representative of the entire group (this has been briefly discussed in the part describing the design of the study, Section 5.1). In Appendix B the reader may find the list of the interviewed participants by their pseudonyms and their respective group.

Based on the results presented in Graph 1, we maintain that only 9 out of 46 prospective secondary teachers showed competence in this set membership identification task. That is to say, only about 20% of them demonstrated an adequate ability or skill in classifying the given numbers into various number sets, so that they achieved 12/14 or higher on Item 6 (i.e., 12 out of 14 items classified correctly across all attributes). Naturally, we wanted to account for such weak performance on this task, which lead us to dig beneath the surface. Attending to the role that representations play in concluding rationality or irrationality of a number provided some key answers.

6.2 Representations

Using Items 1, 2, and 3 we analyzed the participants' reliance on the representational features of numbers affecting their decisions with respect to irrationality. We first present the quantitative summary of written responses for the first two items. We then focus on the details of several interviews, identifying some common trends in participants' approaches to the presented questions. We also discuss some common erroneous beliefs found amongst the PSTs and we attempt to identify their sources.

> Item 1: Consider the following number 0.12122122212… (there is an infinite number of digits where the number of 2's between the 1's keeps increasing by one). Is this a rational or irrational number? How do you know?

Response category	Number of participants [%]	
Correct answer with correct justification	27	[58.7%]
Correct answer with incorrect justification (such as, "this number is irrational because there is an infinite number of digits")	7	[15.2%]
Correct answer with no justification	1	[2.2%]
Incorrect answer	6	[13%]
No answer	5	[10.9%]

Table 3: Quantification of results for Item 1 – considering 0.12122122212... (n=46).

Item 2: Consider 53/83. Let's call this number M. In performing this division, the calculator display shows 0.63855421687. Is M a rational or an irrational number? Explain.

Response category	Number of participants [%]	
Correct answer with correct justification	31	[67.4%]
Correct answer with incorrect justification (such as, "this number is rational because the digits terminate")	7	[15.2%]
Correct answer with no justification	2	[4.3%]
Incorrect answer	5	[10.9%]
No answer	1	[2.2%]

Table 4: Quantification of results for Item 2 – considering 53/83 (n=46).

As shown in these tables, over 40% of the participants did not recognize the non-repeating decimal representation as a representation of an irrational number. Further, over 30% of the participants either failed to recognize a number represented as a common fraction as being rational or provided incorrect justifications for their claim. It is evident that for a significant number of participants, the definitions of rational and irrational numbers were not a part of their active repertoire of knowledge. We examine the issue of definitions in greater detail later in this chapter.

Let us consider the responses of one participant, Steve, that shed light on the possible sources of errors and misconceptions.

Steve: (claiming 0.121221222... is irrational) Um hm, I would say because it's not a common, there's not a **common element** repeating there that it would make it a rational...

Interviewer:	How about this one, 0.0122222...with 2 repeating endlessly, is this rational of irrational?
Steve:	Okay, I would have to say that's irrational real number.
Interviewer:	Irrational or rational, I couldn't hear you.
Steve:	Irrational. Well oh, the 2 repeats, no but it has to be, then it repeats, even though the 2 repeats, it has to be a **common pattern**, so I would say it's irrational.
Interviewer:	Okay, so 0.01222... repeating infinitely is irrational.
Steve:	I think so, but I forget if the fact that that, if the 1 there changes, I would have thought it would have to be **012, 012,** ... but if it starts repeating later, yeah I can't remember if it starts repeating later, I'm pretty sure it's irrational, but I could be mistaken.
Interviewer:	How about the second question, when you consider 53 divided by 83...
Steve:	Um hm...
Interviewer:	And let's call this quotient M, and if you perform this division on the calculator the display shows this number, 0.63855421687.
Steve:	And I assume it keeps going, that's just what fits on your calculator...
Interviewer:	Yeah, that's what the calculator shows, that's right. So is M rational or irrational?
Steve:	So this is the **quotient** M, yeah I would say it's **irrational**.
Interviewer:	Because?
Steve:	Because **we can't see a repeating decimal**.
Interviewer:	But maybe later, down the road it starts repeating.
Steve:	Well that's true, it's possible...
Interviewer:	So we can't really determine?
Steve:	Well I guess we don't, we **wouldn't know** for sure just **from looking at that number on the calculator**, **but chances are that if it hasn't repeated that quickly, then it would be irrational.** I haven't seen a lot of examples where they start repeating with 10 digits or more. I'm sure there are some but....
Interviewer:	Okay, and the fact that it comes from dividing 53 by 83, does that not qualify it as rational?
Steve:	Oh so that is a fraction, it's 53/83?
Interviewer:	Yeah we, that's how we got this number, so we divided 53 by 83 and called this M...
Steve:	**53/83 as it's written would be rational**, but yeah, I see what you mean, if you took that decimal, yeah, I guess that's a good point. I see what you're, you're saying that fact that it's 53/83 that is A/B, so that is rational, but then when you take, **if you started dividing... It would just go on and on and on and on, so that you would think is irrational.** Yeah, I must say I don't know the answer to that.

In the beginning of the interview Steve claims correctly that an infinite non-repeating decimal fraction represents an irrational number. However, his use of the words "common element" prompts an inquiry into his perception of "common". This perception is clarified in Steve's incorrect claim that 0.0122222... is also irrational. Steve is looking for a common *pattern*, and the repeating digit of 2 does not seem to fit his perception of a pattern. For the next question Steve is presented with a fraction 53/83 and distracted by its display on a calculator. Focusing on the decimal representation rather than the common fraction representation, his first response – this quotient is irrational – presents an oxymoron. It is based on the inability to "see" the repeating pattern. The underlying assumption here is that a repeating pattern, if it exists, has a short and easily detectable repeating cycle. This perception is confronted by the interviewer in directing Steve's attention to the number representation as a fraction, 53/83. From his reply it appears that Steve believes that whether the number is rational or irrational depends on how it is written; that is, a common fraction represents a rational number, but its equivalent decimal representation could be irrational.

6.2.1 Decimal disposition

We found that decimal representation is the preferred representation in deciding the rationality or irrationality of a number for many PSTs. As in the case of Steve, there was a tendency to ignore the transparency of rationality inherent in the fractional representation, and instead rely on the truncated decimal representation offered by the calculator display.

Interviewer:	How did you decide that 5/31 is irrational, what makes you believe that?
Ed:	Well I punched it into the calculator so yeah. . .
Interviewer:	Oh okay. . .
Ed:	Yeah, but I mean even if I hadn't, I would have looked at it and said that's, I mean that's got to be, must be irrational right, but I didn't want to, I wasn't sure if you wanted us to kind of, you know, just do it in our heads, but um. . .

We found that one of the reasons for this state of affairs is the purely operational approach to rationals. That is to say, the division of integers is still only a process, and so 53/83 cannot yet be seen at will as a static entity. From our theoretical perspective, we would say that the concept of fraction has not been reified. We learn from one of the interviewees, Amy, why it is so unnatural for her to consider the representation as a common fraction in deciding irrationality or rationality of a number.

> Amy: (After interviewer directed her attention to the number representation as a fraction) Well okay, what I see from, thinking about it that way, because **I don't see this as being a number**, like if you're going to decide about something. Like this isn't a number, **53/83 isn't a number**, but 0.638.., and does this continue? Is that ..., or is that ?

Amy cannot decide simply by looking at 53/83; she thinks she must divide it out first. Amy interprets a/b as an instruction for division. Unlike Steve, however, Amy is not convinced that periods should always be short. She considers the conflict that arises from judging rationality of a number based on the digits displayed on the calculator. It may lead her to a wrong conclusion, because "the decimals could start repeating".

> Amy: It seems like a contradiction (laugh). But how do you know, like how far, like maybe the pattern happens and it's harder to see. Like maybe there's something that's, maybe this does repeat at some point, I don't know. Yeah, I don't know. . . I mean you could always long divide it out with long division, and keep going and going and going and look for a pattern, um, because you get more numbers, or with a computer that you can set up to look at more decimal places and look for a pattern.

It is most likely that Amy has not had much experience with division in her life, because if she had, she could have known that the decimals must start repeating. We designate this as the case of "missing link", and we discuss it further in the section on definitions.

Verbal obstacles

If one is to determine rationality or irrationality of a number from its decimal representation, one must have a reliable way to do this. The way we speak about things influences the way we think about things. Although the usage of the word "pattern" with respect to decimal expansion of a number is not used as a guiding principle in discerning rationality or irrationality of a number in any of the standard mathematics high school textbooks, we find that is a common practice nevertheless. Nearly all the PSTs who exhibited decimal disposition, such as Steve, Ed and Amy, used this word, either in the stricter sense of "repeating pattern" or in the looser sense without the crucial *repeating* part. Regardless of the sense in which they used it, we find this language to be vague and prone to personal interpretation. We thus consider it a verbal obstacle.

According to Brousseau (1997), an obstacle has its domain of validity (but in another domain it is false, ineffective, and a source of errors), which makes it so resistant. All PSTs who rely on "pattern" are able to successfully identify as rational the *immediately repeating decimals* (such as 0.121212…) but not necessarily the *eventually repeating decimals*, such as 0.012222…). The following excerpt demonstrates this.

Interviewer:	How about this one, 0.0122222 with 2 repeating endlessly, is this number rational or irrational?
Amy:	Like why is it irrational, why did I say that?
Interviewer:	Yeah. . .
Amy:	Um, because there isn't um, **it repeats without a pattern**. **I think that the pattern should have to start right after the decimal point, not anywhere in the middle**. . .
Interviewer:	Okay. . .
Amy:	Because .01 wouldn't come up again, just the 2, if that's written the way, like the way that I would see it is like the low line over the 2, to say only that the 2 repeats.
Interviewer:	Okay, so if you have 0.012 with a little line indicating that only 2 repeats endlessly then you would slot it as irrational?
Amy:	Right. . .
Interviewer:	Because the repeating doesn't have the starting right from the decimal point. . .
Amy:	Yeah, yeah, I think the **whole decimal has to be part of the pattern for it to be rational**.

This is the same misconception as presented earlier; however, in contrast with Steve, Amy is very sure of her rule while Steve admits he could be mistaken. The point is, why would one ever want to memorize such strange and insignificant rules, to think that if the decimal starts repeating right away then it is a rational number, while if it starts repeating later it is irrational?

This has nothing to do with rationality or irrationality of numbers. Cases such as these speak of why it is that for so many, mathematics is nothing but a collection of arbitrary rules to be memorized.

Patterned transcendentals

We have demonstrated how some PSTs applied their notion of "pattern" to conclude that the eventually repeating decimals are irrational. Interestingly, this verbal obstacle works the other way as well. It allows people to claim that irrational numbers are rational. More specifically, the transcendental irrationals with patterned decimal representations, such as the one in Item 1, have been identified by some PSTs as rational numbers. Interview with Matthew exemplifies this.

Interviewer:	So can you please tell me how you recognize an irrational number?
Matthew:	It's a number that um doesn't have any sort of pattern, or doesn't, or it doesn't terminate ever, or as far as you can see. And it **doesn't have any sort of pattern**...
Interviewer:	So number like 0.12 122 1222, where the number of 2's between every successive 1's keeps increasing by 1, would that be a rational or irrational number?
Matthew:	Rational...
Interviewer:	So to you, what is the definition of a rational number?
Matthew:	A number that either does terminate, or doesn't terminate and **has a definite pattern**.
Interviewer:	Well, so you consider this to be a pattern . . .
Matthew:	Yes, it's definitely a pattern... So it is rational.

We see this omission of the requirement for *repeating* despite the fact that examples of such transcendental numbers are given in high school textbooks with an explicit intention to clarify for the student what is to be taken as pattern and what not when determining whether a number is irrational. Again, the true distinction between Q and $\overline{Q}$ is not understood – it often becomes just a game of analyzing decimals and nothing more. This unfortunate misinterpretation of repeating pattern as simply "pattern" and the reliance on this interpretation, as we have demonstrated, serves only to complicate the matters.

Irrationals as unreasonable numbers

The theme of the requirement for pattern also came up in the interview with Ed, who holds quite a sophisticated conception of rational and irrational numbers. He was a very successful student, one of the best in the class. For example, on Item 6 he identified entirely correctly 10 out of 14 numbers, demonstrating that the region of effectiveness across the sub items was rather high. However, his conceptions lead to incorrect conclusions for both Items 1 and 2.

Interviewer:	So tell me what does it mean to you for a number to be irrational?
Ed:	Um oh god, um irrational number, um. . .
Interviewer:	There's no rush, you can take time and think.
Ed:	Well I mean, it's, **the word irrational obviously means it can do something that we can't understand**, right, it's kind of like you have to shake your head and you can't figure it out right. . .
Interviewer:	The word? And as a number?
Ed:	As a, yeah, so it's the same kind of thing right, so you think of it as like **a number that you can't really explain**, it's just sort of, you know, it goes on and on and **there's no system to it**. . .
Interviewer:	How do you identify them though? How do you distinguish them from rational numbers?
Ed:	Um, well my understanding is that like **with a rational number you always have some kind of a pattern**, there's always something that you can sort of, okay like at this point we can, we can tell you what all the numbers will be because we know the rule that the number follows, whatever that, no matter how long it is, whatever, we know what they're going to be, all the digits, whereas irrational number it's like you can't really, you can't know its digits. You have to actually sit down like with pi right, **you don't know ahead of time** like, okay what's the 1,000th digit, I can't just quickly tell you right. Of course the computer will tell you right, so it's a complicated process to find out what that would be, there's no like systematic way of finding the answer right.
Interviewer:	Okay, and you can tell easily what the 1,000th digit will be for any rational number, no problem?
Ed:	That's my feeling yeah.
Interviewer:	How exactly do you tell for a rational?
Ed:	Uh, well like I'm saying, my understanding is that **there is a system** so I guess um like if the number, for example, goes 12 and it goes, like because of the decimal places right, 0.

	121221222..., right, there's one 2 and there's two 2's and there's three 2's and four 2's and five 2's um I mean I'm assuming there's a quick way to turn that into sort of a formula or whatever where you can actually establish what that 1,000th digit, whether it would be a 2 or would it be a 1 right, . . .
Interviewer:	So is this number that you just mentioned a rational or irrational?
Ed:	I think it would be rational. . .
Interviewer:	Rational. . .
Ed:	Because **there seems to be a system to it** like, like um, you know, you could, I mean it is interminable, like it goes on forever, but at the same time like **you can easily explain how it's working** right. Like that's my feeling, so in an irrational number it's impossible to pin it down like that, I mean you can say okay well this came, from this divided by this or whatever, right, but um **you can't really explain the number itself**, like the result of that division, you can't explain what is, why are these digits the way they are without just multiplying, or sorry without doing the division so (laugh).
Interviewer:	Okay, so what would you say for 53 divided by 83. Is this number rational or irrational?
Ed:	Yeah, I think it's irrational. . .
Interviewer:	You think it's irrational. . .
Ed:	Just by looking, I mean **I have no idea what the result would be just by looking at the two, 53 and 83 right**, by looking at the answer right, probably doing it on your calculator, I mean, it just looks like it would be an irrational number, right. I mean that's, like the thing is, you know, my understanding of math isn't huge, so I just look at a number and like "that looks irrational to me" right, then I was thinking at the time like if I could, if I really want to **I could turn on my computer** and put it in there, one of those it just displays all the numbers, it doesn't limit you to 10 digits, ?? so I mean if it. .
Interviewer:	So you're aware that you're limited by 10 digits here. . .
Ed:	Yeah, yeah. . . Yeah, I **mean I just have to assume one or the other and it looks like a disorganized number so (laugh), it looks an irrational to me**.

As evident, Ed thinks rational numbers are organized, they have a system to them, and you can predict their digits – they are reasonable. On the other hand, irrational numbers are something we cannot explain, they are disorganized and unpredictable. Ed's conception of irrational number hinges on the lay meaning of the word irrational as "unreasonable" to which he adds a somewhat more mathematical connotation as "disorganized". In contrast with Connie, to him, "rational" and "irrational" are not just meaningless labels - there is rather a tangible quality of numbers that is captured by this distinction. For example, rational numbers have an "appealing nature". What is more, due to the huge restrictions imposed by what Ed considers to be an organized number, rational numbers are seen to be kind of scarce.

Ed: But like if I have, like you think about numbers right, like think about numbers that are rational numbers right, and they look, I mean they have a certain kind of **appealing nature to them right**, because they're nice like .2222 right, or something like that right, .2121 and just **very nice** right, but like it, you know. **It tends to be that there's very few things that work out that way, it's just that because we kind of manipulate the situations right.** We always think like, okay let's make our math textbook, or whatever right, let's make sure everything is a rational number right, because it just looks better in print, or whatever right.

Ed thinks of rationality as some kind of quality of numbers that can easily be destroyed, especially with actions such as division and other manipulations. This explains why Ed expects most fractions to be irrational - there are just very few ratios that end up looking "nice". And one can pretty much rely on the calculator to see if the number is nice and organized. Later on in the interview this conception of the volatile nature of rational numbers is confirmed.

Ed: But just imagine if you have a rational number and **you change one digit** like, you know, a million places down the line whatever, increase it by 1 right, um, automatically I think **you destroy that it's a rational number**, it's no longer a rational number, so that a number that's very close to a rational number it's like can't be rational, you know.

Interviewer: Let me ask you something, what if you have a number like um 0.33333 a hundred of those 3's and then a 1 on the 101st place and then again continues just with 3's. . .

Ed: And then comes another 1. . .

Interviewer: No never again, is that a rational or irrational?

Ed: I'm thinking right now that that's, I mean that seems irrational, because um it looks nice right, but why is that 1 there right, why is it not coming back, I mean if a number comes back right, then there's something, there's not really, well yeah **there's no real pattern** I guess.

Ed's comment that increasing the value of one digit in a rational number would destroy its rationality prompts and inquiry in how he thinks about the eventually repeating decimals. Since the very last number of Item 6, that is 0.01222..., was answered correctly by Ed, as it seemed to fit his view of organized number, the interviewer offered and exaggerated eventually repeating number instead. Here Ed becomes confused because of the clash between the two requirements imposed by his definition – that digits should be predictable, and that the number should be organized. Ed's concept image seems to be very demanding in that it requires of him very difficult mental processes in handling or at least considering all the infinitely many digits at once in case one of them should misbehave. From our perspective, we would say Ed displayed an accentuated decimal disposition. In summary, from the perspective of rational as reasonable instead of ratio, one is doomed to rely on calculator to judge the reasonableness of digits. From this perspective, most fractions turn out to be unreasonable, thus irrational. We consider this another verbal obstacle.

Infinite decimals versus terminating decimals

Although we were aware that there exists a belief among students that terminating decimals are rational and infinite decimals are irrational, we did not expect to find this belief among our group of participants, given their educational background. One prospective teacher, Katie, who has been out of school for a while and decided to take her teaching degree, held this belief.

Interviewer:	Okay, and how do you distinguish rational numbers from irrational numbers?
Katie:	Um, (pause) Irrational numbers are something which you don't know the exact value. If you divide something and then it gives 0.3333..., and **you don't know what the exact number is**, that would be irrational, but .5 would be rational.
Interviewer:	Because?
Katie:	It terminates and then it's a, then I can see if that's the real number and otherwise if it's let's say 0.3333 ... and **it just repeats and then there's no end, then I can't really tell what the actual number is**.
Interviewer:	So if it repeats and there's no end to it, you would say that it is irrational. . .
Katie:	Um hm. . .
Interviewer:	Okay. Um, now here on question 2 if you remember, when you divide 53 by 83 and we said let's call this quotient M, but when we perform this division on the calculator, this is what the display shows, so the question is, is M rational or irrational? What would you say?
Katie:	It ends there right? So the number ends there, and when I multiplied, I don't know what I was thinking, but I thought that when I multiplied this number with 83 I do get 53, so I think that's why I put down it's a rational number.
Interviewer:	Oh, okay I see, because it's a decimal that terminates. . .
Katie:	Yeah. . .

Interviewer: I see.

Katie: This is what I was thinking, **it's infinite numbers and it's finite numbers**. . .

Interviewer: So am I right if I say that you distinguish irrational from irrationals just by looking at whether it is. . .

Katie: Whether they terminate or not, that's what I was looking really. . .

Interviewer: Yeah. . .

Katie: Yeah. If, let's say **if it's a ratio, and I try to express it in decimal form and if it doesn't terminate, then I would say it's an irrational and if it terminates, then I would say it's rational**.

From her response we see that Katie is troubled by the infinite decimals, she feels the number is somehow constructed in time and she cannot know what the exact number is since there is no end to its digits. From her perspective, the number is still in the making; therefore, impossible to be conceived of as an object. We see this as yet another case of the concept of rational number in its early stage of development (interiorization phase).

Interesting enough, despite her erroneous belief, Katie answered both Items 1 and 2 correctly, and from the justifications on the written response it would be hard to tell that she maintains this view. For example, she claims that 0.12122122212… is irrational because "it cannot be expressed as a ratio *and* has infinite number of entries after the decimal point". The way she uses *and* was not clear until the probing of her thinking occurred during the clinical interview where we find out that whether a number can be expressed as a ratio of two integers is in fact irrelevant for Katie.

Now, it should be pointed out that later on in the interview with Katie some teaching from the interviewer occurred, and some ideas became clarified. This aided us to see how Katie thought about rational and irrational numbers, as she was able to become the observer of her thinking which helped explain how it is that she correctly answered Item 2 despite her problematic concept image.

On Item 2, her thinking was that M and the number on the calculator display were identical, and since the number on the display was a terminating decimal, M must be rational. In the written response, she justifies this by writing "M×83=53". What we do not find out until the interview is that she is in fact aided by the calculator in confirming her thinking that M = "calculator display".

Interviewer:	Yeah, okay. Is there anything that still sort of puzzles you or bugs you that you would like to address here or tell me more about?
Katie:	Okay, if I have 0.33333… and plugged it into my computer and multiplied it by 3, would I get answer 1?
Interviewer:	Your calculator doesn't hold infinitely many digits, you know, so you, you have a round off error, but still most calculators adjust for that and do give you the answer 1.
Katie:	Okay, **there's another way I was testing whether I should get irrational or not**. . .
Interviewer:	How is that?
Katie:	Uh huh, so because then I was looking at this one **and I was trying to test whether it would be rational or not. I was trying to cross multiply M with 83 and see if I can come up with 53**. . .
Interviewer:	And you did?
Katie:	And I did, so I figured this is rational because, this is another thing that I remembered that **if you could multiply and then come up with the numerator then it's rational**.

In fact, Katie was using the calculator to *test* out whether the digits of the resulting quotient continue or whether they terminate. When she used the finite decimal and multiplied it by the denominator, and got the numerator – this confirmed that the division of 53 by 83 yields a finite decimal, thus M was concluded to be rational. As we can see, Katie had a good reason, not just a plain guess, to believe that what the calculator showed was indeed equal to M. How could she have known that she had been tricked by the sophisticated rounding feature built in most of today's calculators? Had she entered the digits of M, rather than arriving at them as a result of division, the result could have been different.

From the perspective of decimal disposition, it seems reasonable to distinguish between infinite and finite decimals. It is certainly the most prominent feature of a number when looking at its decimal form. But to bunch the infinite repeating decimals with terminating decimals and to call that rational, and to call all the other infinite decimals that are left (the non-repeating) irrational, does not make much sense for many students. If anything, one should have a three-way distinction: terminating, infinite-repeating, and infinite non-repeating. The concept of irrationality is confused with one of its representations. In the climate of overemphasis on decimals in school mathematics, this view would be a plausible view to adopt. However, as we shall see, this view may be sometimes due to the confusion between π and one of its approximations.

<u>π as a special case</u>

In our inquiry into how Katie acquired her understanding of infinite and finite decimal expansions to correspond to irrational and rational numbers respectively, we found that it is the result of an adaptation of her concept image to fit the evidence concerning the number π. Namely, Katie believes that π is irrational. Of course she would, everyone knows that – this fact does not escape anyone who was schooled. The problem is, Katie *also* believes that π *equals* 22/7. Necessarily, these two "pieces of knowledge" fuse together to bring about the kind of understanding that Katie holds. The need for reshaping one's cognitive schema to ensure some consistency in one's mathematical thinking about these things is a pressing matter, and one does the best one can do from the basis of such faulty information.

Interviewer:	With pi, your teacher said it's. . .
Katie:	Irrational, I had heard that. Like I was trying to, when talking real and rational, irrationals, I had to close my eyes and reflect back so many years, and one of the numbers that came from my memory bank as irrational was pi and I based the whole exam on that.
Interviewer:	Okay, so when you based your, that's not an exam, you know, . . .
Katie:	No, no, not an exam, I mean the whole questionnnaire here, I was trying to answer based on that limited knowledge, because I didn't have the depth of knowledge. I had two strings that I was hanging on to, one was . . .Pi and yes pi is there. . .
Interviewer:	So you remembered that pi is irrational. . .

Katie: Yeah, and Pi is normally expressed as 22/7 and I knew pi is irrational, and then I thought when I divide 22/7 the number repeats and never ends, so I thought a number that never ends is irrational.

Interviewer: It makes very, very good sense to me, you know, the way you thought about it. I can see your point.

Katie: Yeah, this is what I thought, so this is the wrong way of looking at rational and irrational numbers. . .

Summary of written responses stemming from decimal disposition

In what follows we demonstrate several frequent erroneous beliefs expressed by the participants in their written questionnaires, some of which have been exemplified in the excerpts from various interviews. Given that several of the PSTs we interviewed operated from a decimal disposition, we see that conflicts arise from applying incorrect or incomplete characterization of decimal expansions in deciding the irrationality of numbers. However, the source of the conflict is poor understanding of the relationship between fractions and their decimal representations. It seems that the present day didactical choice is that somehow we can short-circuit the need for understanding this relationship and give the student a substitute for understanding instead - a recipe on how to look at the decimal expansion of a number to decide whether it is rational ("if the digits terminate or have a repeating pattern") or irrational ("if the digits are non-repeating and non-terminating). The following examples summarize the major themes that were brought to light with respect to incorrect usage and over reliance on this recipe.

- If there is a pattern, then the number is rational. Therefore 0.12122122212… is rational, (similarly, 0.100200300… is rational, but 0.745555… is not, because there is no pattern).
- 53/83 is irrational because there is no pattern in the decimal 0.63855421687.
- 53/83 is rational because it terminates (calculator shows 0.63855421687)
- 53/83 could be rational or irrational – I cannot tell whether digits will repeat because too few digits are shown. They might repeat or they might not.

- There is no way of telling if 53/83 is rational - unless you actually do the division which could take you forever. Digits might terminate at a millionth place or they might start repeating after a millionth place.
- It is possible that a number is rational and irrational at the same time. For example, there are fractions that have non-repeating non-terminating decimals, yet they can be represented as a/b.

The first illustration above echoes Ed's reliance on a personal interpretation of "pattern", which ignores the requirement for the repetition of digits. The other three responses demonstrate participants' dependence on a calculator and preference towards decimal representation, which is misinterpreted as either terminating or having no repeating pattern, or treated as ambiguous. The last response involves a contradiction in terms ("irrational fractions"), resulting from a warped understanding of the two competing definitions of irrational number used in school mathematics.

These approaches are mostly procedural in their focus on carrying out the operation of division or performing conversion, rather than attending to the structure of the given representation. It is apparent that the connection between fractions and repeating decimals is not recognized. In all these cases the person is prevented from attending to the given transparent representation as the ratio of two integers to conclude the number's rationality. The root of the problem that causes these conflicts in understanding is a weak or non-existing understanding about the interrelations between fractions and repeating decimals. We discuss this issue in greater detail in the section on definitions.

6.2.2 Fractional disposition

We now consider the reactions of those PSTs that hold a primarily fractional disposition. We consider a person exhibiting fractional disposition as someone who primarily bases his/her understanding of rational number as a number that can be expressed as a ratio of two integers with a nonzero denominator, and an irrational number as a number that cannot be expressed in this way. Fractionally disposed participants had an advantage in that they automatically concluded rationality of common fractions as it was natural for them to attend to the transparent feature of number represented in this form. However, some fractionally disposed PSTs had difficulty concluding rationality of infinite repeating decimals, beyond simple cases such as 0.33333… Interview with Paul demonstrates this.

Paul:	(claiming that 0.12122122212… is irrational because its digits go on forever)
Interviewer:	Yeah, okay. So what about if the question was about 0.12121212, like this infinitely, is that also, is that a rational or not a rational number?
Paul:	I would say it's (pause) not a rational number because it. . .Rational number can be written one over the other... as the ratio of two whole numbers. .
Interviewer:	That's right. So you're saying that 0.121212 with 12 repeating cannot be written as a ratio of two whole numbers?
Paul:	Hmm, the way I see it is like this, you'd have to go . . (pause) you have, I can't imagine this number, **<u>I can't imagine 0.121212. . .being multiplied by any number to get a whole number</u>**. . .
Interviewer:	I see, that's the way you think about it, okay. Um, hmm, interesting point. But how about 0.3333 repeating, is that rational?
Paul:	0.333. . .
Interviewer:	Yeah. . .
Paul:	(pause) This, this number **<u>0.333… it comes up like a lot, when, when you just do your math. . .over the years, and I can see it's, when multiplied by 3, you get 1</u>**. . .

Among the participants that we interviewed we found only two fractionally disposed. Although it may seem that this view would be a more desirable one than the decimal disposition, we find that simply having this disposition does not immunize one to erroneous beliefs.

<u>Rational as a ratio of two numbers: ignoring the requirement for integers</u>

When we speak of probing someone's understanding via a clinical interview then we think of the interviewee describing the state of the object of understanding, in this case his/her conception of irrational number, as if it were frozen in time for the duration of the interview. However, this is rarely the case, even if much care is taken that no teaching occurs. Interview questions provoke the person to perform mental actions on the object of understanding that may cause the transforming of this object. This is especially true if the concept is still in its early stage of evolution (i.e. when it had certainly not been reified, or condensed, or perhaps not even internalized). We suggest that the more embryonic the conception is, the more dramatic this "search for equilibration", which manifests as "changing the way one thinks", will be – perhaps this instability can even be used as a measure of the stage of concept development.

In our interview with Anna we witnessed an instance of a dramatic shift in the conceptual schema of an individual. When probing her understanding of irrational number, we found out about the dangers of operating from an incomplete definition of a rational number as a ratio, omitting the essential requirement that it must be a ratio of two integers. When asked about how she distinguishes between rational and irrational numbers, this is what Anna offers:

Anna: Well like I've heard before that, like the way that I was taught in school is how that **rational numbers can be written as a fraction, and when they're put into a decimal, they either, they repeat in some way, or else they terminate and so irrational are all the non-repeating, continuous decimals.**

On the surface we would judge this as a balanced disposition; that is, it would seem that Anna would be able to coordinate the two representations as needed. Also, she is successful on both, Item 1 and 2. Only later in the interview, after Anna repeatedly makes statements such as these: "...any number divided by any other number is how, is like where we find our rational number", or "... an irrational number is never uh the result of an operation that we do, like if the operation is dividing, the result wouldn't be, an irrational", we learn that she is missing the essential part of the definition, namely the requirement that the two numbers must be integers. She bases all her reasoning on the "nonexistence of a representation as a ratio" as the defining characteristic of an irrational; therefore, we would consider this to be a case of fractional disposition. In addition, she is entirely focused on the process of division.

In the excerpt that follows, Anna is considering whether $3\sqrt{8}$ should be rational or irrational. There is a conflict – basing her whole argumentation on the interpretation that irrationals can never be a result of a division, disregarding that it is meant the division of integers, she now thinks that they shouldn't be obtainable by multiplication either. In the written part she identified 3 times root 8 as rational and so we inquired why this would be so.

Anna:	(thinking that 3 root 8 should be rational because "you cannot produce irrationals by multiplication")
Interviewer:	What do you mean by produce, like um, how does 3 root 8 um produce, like...
Anna:	The two **operations or actions** that we're taking on 8 is taking the square root of it, and then multiplying that by 3.
Interviewer:	Okay...
Anna :	So what I was saying sort of then is that taking, like dividing 12 by 75 would never produce, any division, if the **operation is operation of division, that will never produce an irrational number**...
Interviewer:	Okay...
Anna:	I guess I was thinking that the same thing would consider, but I just don't think that by multiplying, like I was saying **dividing will never give you an irrational number**...
Interviewer:	Okay...
Anna:	Because **if you divide something by something else, that means you can put it in a fraction**, **because of what a fraction is, something divided by something**...
Interviewer:	Right...
Anna:	But a **fraction is also like the inverse of multiplying**, kind of in a way, so this is why I would say false...
Interviewer:	Because? I'm not sure I understand.
Anna:	Because I don't think that um **any product can be irrational**, just the way I was saying that **any quotient can't be irrational, you can never have an irrational quotient. Now I'm thinking you can never have an irrational product either**.
Interviewer:	So, for example, taking a square root of 5 and dividing it by a square root of 2 would that be in your opinion a rational or an irrational?
Anna:	Oh (laugh) I think that would be rational, yeah...
Interviewer:	By your argument...
Anna:	That the, by the fraction argument, yeah. I just think **the fraction argument extends to be able to say that, because a fraction argument produces a quotient right**, ...
Interviewer:	Um hm...

Anna: **<u>So the quotient came from a fraction, so the quotient can never be irrational</u>**. . .

Interviewer: So whatever can be put as a quotient is by default, by definition not irrational. . .

Anna: Right.

Interviewer: Okay, how about pi then, because pi is a quotient of, what is it, circumference and diameter. .

Anna: (laugh) . . .

Interviewer: What would you say to that?

(At this point Anna becomes very confused. She mumbles something, and the fragments of her inaudible speech are impossible to discern. All we can gather is that she is desparatly trying to make sense of it all.)

Interviewer: It's a hard question. . .

Anna: I know (laugh)

Interviewer: I'm sorry you have to endure that, but we want to teach ...

Anna: So I don't know now. . .

Interviewer: Did I confuse you about pi now?

Anna: No, you didn't confuse me, but it is sort of what I had held to be true, like using this whole idea, if pi is a quotient, it pi, which is what circumference over diameter . . . ., is that what it is?

Interviewer: Yeah. . .

Anna: Then you can reflect, **<u>produce it as a fraction</u>**, like **<u>circumference over diameter is pi, that's a fraction, so you can show pi as a fraction, therefore according to the fraction rule it just destroys my whole theory</u>**.

Interviewer: (laughter) What are we going to do now?

Anna: I don't know.

Interviewer: Okay, uh, all these things will be completely clear to you, you know, two weeks from now, when we talk about it in your course. . .

Anna: That's exciting.

Compounding the problem of incomplete definition is that fraction is seen as an instruction for division, the result of which is the quotient which "can never be irrational". We see this as the concept of rational number in the interiorization developmental stage, that is to say it is still very much tied to its operational origins. Based on her own conception, Anna is driven to conclude that π is rational, and because that cannot be, it "destroys her whole theory", as she acknowledges herself. It is interesting that Anna did not run into these inconsistencies before, because if she did, she would be forced to adapt her concept image and refine her definition so that it would not admit numbers such as π into the set of rationals. We did not question this, but it would be interesting to know how Anna would explain that every irrational *p* can be written as *p*/1 – would the fact that we can produce it from dividing it by 1 make it rational? Abiding by her incomplete definition she would have to conclude that every irrational is rational. In the interview with Anna we learned that one has to be aware that what we have access to are only symbols and representations (verbal, mostly), and it is hard to know how close or distant the conceptions might be from their formal counterparts. At first all seemed to be in order, with the exception of somewhat loose wording when casting her definitions. However, we did not question that in much detail at the onset of the interview, given that she correctly identified the numbers in Items 1 and 2. However, as we scratched a little through these symbols and verbal representations, we got to the conceptions that were hidden behind. We learned that conceptions stemming from the incomplete and unquestioned definition can survive for a very long time despite the many contradictions that this may cause.

6.2.3. Balanced disposition

A balanced disposition assumes a flexible use of both fractional and decimal representation of a number. We consider it the most desirable one. Seven of the 16 PSTs that were interviewed demonstrated this disposition. We are interested in how a balanced disposition is acquired, and how in manifests. To relate to the reader what we consider a balanced disposition, we present an excerpt from the interview with Dave. He answered both Items 1 and 2 correctly, and he had a success of 8 out of 14 correctly classified numbers on Item 6, putting him in the midst of Group B.

Interviewer: How do you distinguish rationals from irrationals?

Dave: Just the, you know, it's rational if you can write it in the form A/B where A and B are integers, and B is not 0, (laugh) but I didn't start using that till like, I saw it in the number theory course up here at Math 342. . .

Interviewer: Oh okay, and in high school what kind of notion did you have then?

Dave: Um, I think I was told just, you had to be able to write it as a fraction, that's all. That was the definition, like if you can write it as a fraction then its rational, if not it is irrational. . .

Interviewer: And when you see a number that is in a decimal form, do you have a way of telling. . .

Dave: Um, in the number theory course we talk about how you can predict like, um, when you see a bunch of decimals, there should be a section that repeats and then you can calculate how long the section would be and then, it boils down if you see a pattern and it's probably like a pattern that repeats, not the one that keeps getting bigger, then it's um rational.

Interviewer: Okay. Could you tell me what do you think is the connection between the repeating and the being able to express it as A/B, what's the connection here?

Dave: Oh, like. . .If you see something repeating and if it's a rational number. . .

Interviewer: Yes, rational by your definition so that it's possible to express it as A/B, where A and B are integers and B is not 0. . .

Dave: What would the connection be, oooh (laugh), good question, um. . .

Interviewer: Are these two equivalent? Because people use them, yeah people use both of these as definitions of a rational number, so I was just wondering, are they two separate definitions, or is there a connection between them. . .

Dave: Well I think **there probably would be a connection**, but I'm not sure how to describe it, but you, if there's something that looks like it's there, it probably is (laugh). I think, like if you say yeah, **if there is a pattern, there is a very good chance that it would be rational**. . .

Interviewer: Very good chance. . .

Dave: Yeah, (laugh). . .

Interviewer: But sometimes it might happen that it's not. . .

Dave: Yeah, like maybe you just didn't look far enough right, and actually it does repeat or something. .

Interviewer: Oh okay, so if it repeats, you are sure that every time it's going to be a rational number. . .

Dave: Pretty sure (laugh), **I wouldn't say all the time**, I'm not that confident in it, but. . .yeah. .

Although Dave is able to attend to the transparent properties of a number to conclude its rationality or irrationality, regardless of how it is represented, he is unsure of what it is that makes the two definitions equivalent. It seems that the number theory course taken at university that he refers to aided in his conviction that "almost always" a repeating decimal would be a rational number. In our analysis on moving between the two representations, we would consider this to be a case of "leap of faith". That is to say, the knowing that a decimal number represents a fraction if and only if that decimal number terminates or repeats, is based on some authority and not on one's own understanding. We found that all of the 7 interviewees who held a balanced disposition held it more or less on the basis of "leap of faith". We wish to point out that "balanced" does not necessarily imply the understanding of the connection between fractions and decimals; rather, we use the term to describe the situation where both views are employed. In fact, we did not find a single PST that would be able to explain the connection between the repeating decimals and a ratio of two integers. We analyzed why this is so and we present our findings in greater detail in Section 6.3.

6.2.4 Irrationals and the number line

The geometric representation of irrational numbers was strangely absent from the concept images of many participants. The common conception of real number line appears to be limited to rational number line, or even more strictly, to decimal rational number line where only finite decimals receive their representations as "points on the number line". This is in agreement with the practical experience that finite decimal approximations are both convenient and sufficient, which could be the source of these conflicts.

However, in mathematics there is a theoretical requirement for irrational numbers, for example, "completeness" of the real number system is required for calculus. And with irrational numbers one is faced with infinite decimal numbers of a special kind – numbers that cannot be written down or known fully. On this note, Stewart (1995) challenges the wisdom of calling irrational numbers *real*; that is, how can something be real if it cannot be even written down fully? In this sense, geometric representation should come almost as a relief in the process of learning about irrationals. To be able to capture infinite decimals with something finite and concrete, and as simple as a point on the number line, even if this is only possible for a certain category of irrationals (constructible lengths), should help in taming the difficult notion of irrationality. Moreover, the geometric representation of irrational number may well turn out to be a very powerful and indispensable teaching tool for encapsulating a process into an object, especially in the case where the learner is on the verge of the reification stage in the development of the concept of irrationality. It is both accessible to the learner (required is the knowledge of the Pythagorean Theorem) and yet revealing of the idea that to every number there corresponds a (single) point on the number line.

The following table summarizes the results of the written responses to Item 5.

Item 5: Show how you would find the exact location of $\sqrt{5}$ on the number line.

Response category	**Number of participants [%]**	
Exact, using Pythagorean Theorem	9	[19.6%]
Decimal approximation using one or more digits after the decimal point	18	[39.2%]
Very rough approximation, i.e. "between 2 and 3"	6	[13%]
Other response (for example, using graphs of f(x)= $\sqrt{x}$ or f(x) = $x^2 - 5$)	6	[13%]
Responses arguing "you can't"	4	[8.7%]
No response	3	[6.5%]

Table 5: Quantification of results for Item 5 – Geometric construction (n=46):

The responses fall into five distinct categories: an exact location of the point using the knowledge of Pythagorean Theorem, more or less fine decimal approximation, very rough approximation (between 2 and 3), responses related to graphing of a related function, and an outright claim that this is impossible to do. Next we examine some representatives of each category.

Geometric approaches

In section 5.2.2 we presented what could be considered a conventional geometric approach. Indeed, it appeared in the work of four participants. This is an example of such response:

- The length of the hypotenuse shown is $\sqrt{5}$. Just rotate the segment so it falls on the number line, then move it up on the line (horizontal translation 1 unit to the left).

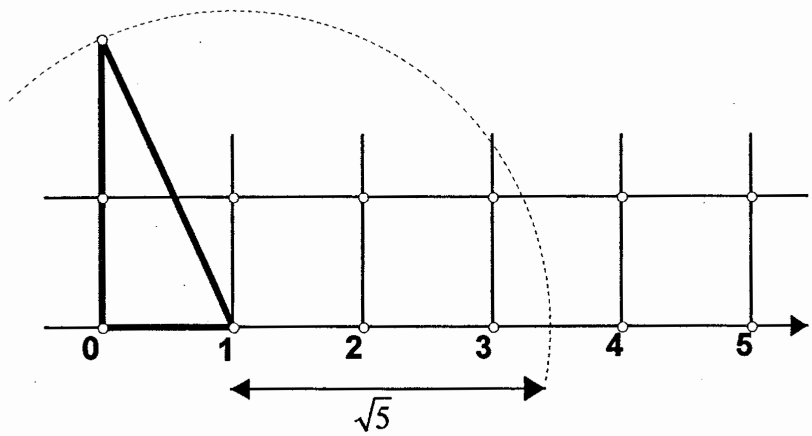


Figure 3: Geometric approach to construct $\sqrt{5}$

Two other valid geometric approaches were found. One of them is a slight variation of the previous response. Instead of by construction it uses a "ready made" right triangle with the side lengths of 1 and 2. Four participants gave the response such as this.

- Make the hypotenuse $h = \sqrt{1^2 + 2^2} = \sqrt{5}$ lie on the number line.

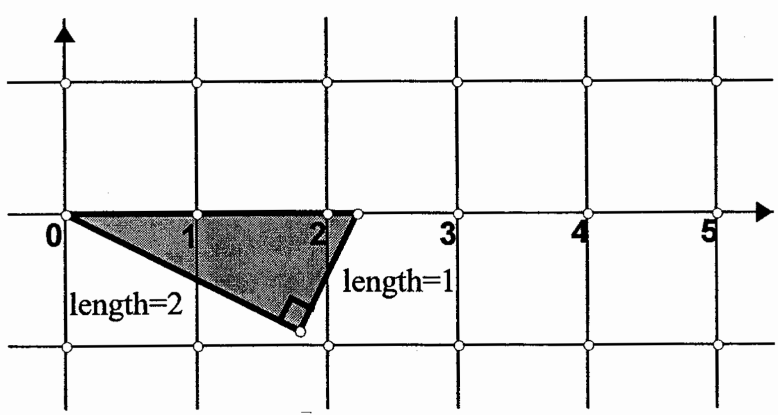


Figure 4: Locating $\sqrt{5}$ by a "ready-made" right triangle

The other valid geometric approach is the familiar spiral of right triangles constructed by the successive application of Pythagorean Theorem with one of the legs always equal to 1 and the other leg equal to the hypotenuse of the previously constructed triangle. This construction is a more generalized version of the conventional geometric approach in the sense that a square root of any whole number can be constructed in this way. It might not be the most efficient construction, but it spares one from having to think about what two perfect squares add up to the required square of the length of the hypotenuse. Only one participant used this approach.

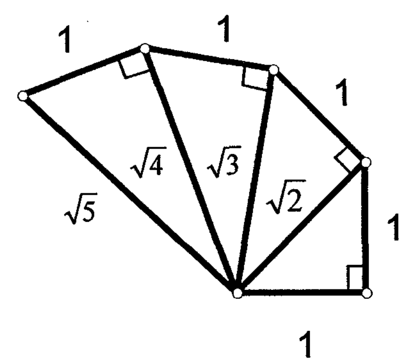


Figure 5: Construction of $\sqrt{5}$ using successive triangles

Next response is interesting. It seems to involve "eye-balling" of when the partial pieces in square A will make a whole squared unit.

Area A = Area B, where A is a square. $\sqrt{5} \times \sqrt{5} = 5 \times 1$

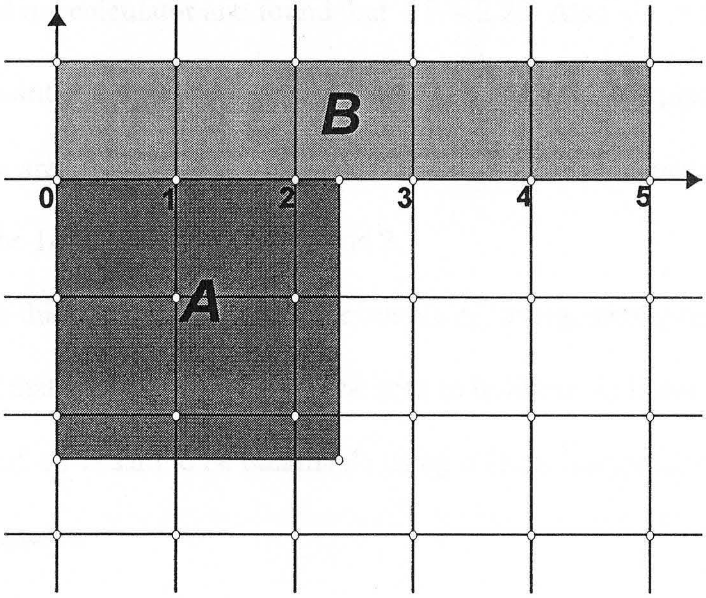


Figure 6: Locating $\sqrt{5}$ by "eye-balling" the areas

Numerical approaches

Next we present a range of responses from the written part, arranged by the degree of accuracy. Twenty-four participants (over 52%) offered an approach based on the decimal expansion of $\sqrt{5}$. We start with those who offered a very rough approximation, and end with those who demonstrated a genuine striving for accuracy.

- Some participants circled a "big blob" around the area of expected location and said "somewhere around here".

- $\underbrace{\sqrt{4}}_{=2}\sqrt{5}\sqrt{6}\sqrt{7}\sqrt{8}\underbrace{\sqrt{9}}_{=3}$

  Therefore, between 2 and 3.

- Somewhere between 2 and 3. I have no idea of the exact location, but it's closer to 2 than to 3.

- I used my calculator and found that $\sqrt{5} \approx 2.23$. Also $\sqrt{5} = 5^{1/2}$. To plot the point I found the midpoint between 2 and 3, then between 2 and 2.5, then plotted $\sqrt{5}$ roughly at 2.25.
- There are 5 whole numbers between 4 and 9 (perfect squares), and since 5 comes after 4 it will be 1/5 the way between 2 and 3.

In this response we note an example of 'overgeneralization of linearity' (Matz, 1982), a response that stems from what can be seen to hold true in linear relationships. In particular, the location of $\sqrt{5}$ is said to be obtainable using a linear interpolation between the two neighbouring perfect squares.

- Divide the section between 2 and 3 into 10 equal parts, find the two neighbouring tick-marks that correspond to just below and just above 5 when squared. Then divide this segment into 10 parts and repeat the process until you get better and better approximation.
- Closest perfect square is 4, $\sqrt{4} = 2$, so it is a little over 2.

  For greater accuracy, we would try more digits.

2.3×2.3	$= 5.29$	(too high)
2.2×2.2	$= 4.84$	(too low)
2.23×2.23	$= 4.9729$	(too low)
2.24×2.24	$= 5.0176$	(too high)
2.238×2.238	$= 5.008644$	(too high)
2.237×2.237	$= 5.004169$	(still too high)
2.236×2.236	$= 4.99696$	(too low)

Function-graph approach

This type of response was found among three participants. These approaches assume what is to be found; that is to say, they assume the availability of an accurate graph, from which the required length would be simply read off, instead of finding a way to construct such length. It should be noted that one of the three participants who offered this kind of response admitted his doubts about the validity of such approach.

- Using functions, such as a sketch of $f(x) = x^2 - 5$ and then looking at the zero of this function $x^2 - 5 = 0$. A statement "if my graph is absolutely accurate, I will find the exact location" accompanied this approach.
- Similar as above, only using $f(x) = \sqrt{x}$ and then looking at the value of this function at $x = 5$ on the graph (the ordinate distance).

Impossible?

Some participants questioned the validity of the assignment. Most likely the word "exact" triggered these kinds of responses.

- $\sqrt{1} = 1, \sqrt{2} \approx 1.4, \sqrt{3} \approx 1.7, \sqrt{4} = 2, \sqrt{5} \approx 2.3$

  I don't think you can find the exact location of $\sqrt{5}$ looking at the number line because it is a huge decimal form number. I do believe there is a way by using calculus, but I'm not sure how to do it.
- This is a trick question, as $\sqrt{5}$ is irrational, it cannot be placed exactly on the number line, because its digits are infinite.
- Can I find the exact location without knowing the rest of ∞ digits?
- You can't.
- Divide on calculator. There is no exact point like that.

Real number line versus rational number line

Since only 9 out of 46 prospective teachers (19.6%) were able to locate the $\sqrt{5}$ on the number line accurately, we investigated what may be the reason for these difficulties. A rather striking observation is that the vast majority of participants perceive the number line as a *rational* number line. It turns out that those arguing "you can't" and those that used a more or less fine decimal approximation hold this perception. This can be concluded from the interviews where we probed for a precise, not approximate, location of $\sqrt{5}$. Under such demand, all participants that previously offered a decimal approximation later concluded it can not be done. In other words, the common opinion was that it must be rounded before it can be located.

Next, we look at a range of responses from the clinical interviews that may shed some light on why locating $\sqrt{5}$ is perceived to be so problematic.

(responding to the question about whether $\sqrt{5}$ can be found on the number line precisely)

Anna: No, because we don't know the exact value, because .0 bigillion numbers ending with 5 is smaller than .0 bigillion numbers ending with 6. They're two different numbers, right, so **because it never ends we can never know the exact value**.

Kyra: Yeah, yeah, like you would never be able to finally say okay, this is where it is, because **there are still more numbers that you're reading off your irrational number**. But if you're using this scale of, you know, 1, between 1 and 2 is 2 cm or something, there's only so much precision that you can make with that point that you draw on there, **like I can't make it as precise as an irrational number** or, you know. . .

Finding the precise location of rational numbers

From these excerpts it is evident that part of the difficulty lies in the infinite digits. To confirm that it is not the irrationality itself, but the fact that there are infinitely many digits in the decimal expansion, the interviewer inquired about the precise placement of rational numbers.

Interviewer: How about 1/3, can you find the location of 1/3 on a number line?

Anna: On a number line?

Interviewer: Yeah. . .

Anna: Yeah, it would be, well okay you could divide, 1 divided by 3 and get that standard 0.3 repeating. . ., oh but that doesn't end either. Okay, (pause) um, I think because we know that the 3 will never change, do we really know, I don't know, because it repeats. Like how do we not know that in the one millionth decimal place it's a 4 or something, or 0 or another number, I don't know. But because that we assume that 3 repeats always, we can like sort of cut it off and round it.

Interviewer: Does that mean that we can't really find the exact location?

Anna: No, it's going to be somewhere in between .3 repeated and .3 repeated and then 4.

Interviewer: Somewhere in between?

Anna: But, no (laugh) I guess not, because it is a different number, like by stopping the repetition of a decimal you're like cutting off its value. Like you're assuming it has a specific value, when in actuality it doesn't have, in reality it doesn't.

Similary, the interview with William suggests that the number line is perceived in a limited sense, as containing only terminating decimals. That is to say, the number line is reduced to the common ruler as used in everyday life.

William: I can find approximate position probably, exact position like I'd probably have to round it off at some point, and then come up with an approximate position, 0.334, something like that, depending on how I could, you want it, let's say you want it accurate to the ten hundredth place, a 1,000th place, **I would round it off to that place** and. . .

Further in the interview there is a discussion about how this would be done, which leads into an inquiry about what it is that makes the breaking of the unit into 10 equal pieces easier than breaking it into 3 pieces.

William: That I know, I can put a ruler there and I know, that's easy. 10 can be done with a ruler, I can also do it with the compass… (here William tries showing that a unit can be broken into 10 equal parts using the compass, but does not succeed) … I don't know how, but I think there is a distinct possibility. The ruler is the simplest, and on ruler you don't see the, let's say 1 cm divided into 3 parts, that's again divided into 10 parts. Anytime I have to do that like 3.33, I would, I normally approximate, just approximate the 3. . . .

It should be noted that William's understanding of irrational numbers was one of the weakest of all the prospective teachers that we interviewed. It would be very difficult to build the concept of irrationality from William's concept image of rational numbers. Although rational approximations are often sufficient for most practical applications, we see this as an extreme example of the number line being reduced to an ordinary ruler, where common fractions that have infinite repeating decimals seize to exist.

From numerical to geometric approach

As noted earlier the most common approach was using decimal approximation. Pythagorean Theorem was seldom invoked by the question. We were curious to find out if this is just because it did not come to mind at the time the written part was administered, or whether there is a deeper issue. It turns out that although the prospective teachers are well acquainted with the theorem they would generally use it only for finding the unknown length in a given right triangle, and not for the purpose of constructing a desired length. In the excerpt that follows, the interviewer prompts Steve to consider a more geometric approach, and even shows how this can be done in the case of $\sqrt{2}$.

Interviewer: Okay, and next question. Um, how would you find the exact location of square root of 5 on the number line?

Steve: Okay, so again without using a calculator?

Interviewer: Yeah, without.

Steve: Um, what roughly find the, the two closest perfect squares so root 4 is 2, and root 9 is 3, so it's going to be somewhere between 2 and 3, so I guess I would then try like 2.2 and multiply it together to see whether it's 5, or whether it's lower, so I guess I'd just try different numbers, try multiplying different numbers together, and see how close to root 5. . .

Interviewer: That would be quite tedious without a calculator, right?

Steve: Yeah, yeah.

Interviewer: How about a more geometric approach?

[interviewer introduces the idea of finding $\sqrt{2}$ as a hypotenuse of an isosceles right-angle triangle with side of 1]

Steve: Oh okay, oh that's interesting.

Interviewer: Um hm, so I'm just trying to see if we can also do something geometric to find the exact location of square root of 5, because the other method would work perfectly fine, but it would be an approximation only and it would be quite tedious.

Steve: Um hm, um hm, so how can you come up with a square root of 5, um, (pause)

Interviewer: Always just say, you know, I'll want to skip that. . .

Steve: Well it's not that I want to skip, it would just take me a long time to think about number combinations that come to root 5. . .

Interviewer: In what way combinations are you talking about?

Steve: Well, you know, that works for the 45-45-90 triangle, root 2 does and you know, root 3 can work for the 60-30-90, but I'd have to, I guess I'd have to find out a **ratio that used root 5**. Yeah, I couldn't figure out the answer just looking at it. . . It would be really hard for me to do without a calculator.

Upon the prompting, Steve invokes the trigonometric ratios for some commonly used right triangles that students are expected to memorize in high school, failing to recognize that these trigonometric ratios have been derived using the Pythagorean Theorem in the first place. The fact is, only 20% of the participants were able to invoke their knowledge of the theorem in order to satisfy the requirement of Item 5. Furthermore, the eliciting questions at the time of the interviews still did not draw out or assist in evoking the theorem from the participants' concept image.

On this basis, we suggest that the knowledge of the Pythagorean Theorem is an inert kind of knowledge for a great majority of our prospective secondary mathematics teachers. We see this as a symptom of two general issues surrounding the present state of mathematics education: one, the trend of weakening of geometry in school curriculum, and two, the fragmentation of the curriculum. For example, the Pythagorean Theorem is commonly taught in Grade 8 in British Columbia. At that time it gets its due share of curricular time, perhaps 2 to 5 lessons, and then students move on to the next topic. The theorem becomes a dormant piece of knowledge. What is more, the common practice is that many results in geometry that could be derived using the theorem are not derived at all (just as is the case with our "missing link"). Instead they are being presented as ready-made, thus depriving students from making the necessary connections. Examples are many. To mention just a few, let us take a look at the so-called "special triangles" mentioned by Steve in the interview. They refer to lengths and angles of the frequently used right triangles, and they are often presented to students without requiring from them that they derive these facts for themselves. These special triangles, which would be best seen as half of the equilateral triangle with side length of 1 and half of the unit square with an application of an appropriate scaling factor, can be found in textbooks under the headings “30-60-90 property” and “45-45-90 property” (Addison-Wesley, Mathematics 10, p. 109). Students are encouraged to memorize these “properties”, these properties being the side-lengths. Other examples are various problems or even general formulas involving distance, area and volume (such as in triangle, trapezoid, parallelogram, cone, pyramid, and even sphere – which could be derived using the method of Archimedes, but would require the use of the Pythagorean Theorem). For example, when students learn how to find the area of, say a regular trapeziod, the height of the trapezoid would systematically be given instead of asking them to find it first. It is would be interesting to know how many students see the Pythagorean Theorem lurking in the equation of a circle, or the trigonometric identity $\sin^2 + \cos^2 = 1$. It seems that the constraints of time, textbook, and teacher subject matter knowledge do not allow for such luxury.

In summary, we see the issues discussed above as instances of a larger problem, which has been repeatedly identified, namely that of the fragmentation of the curriculum. In other words, if the unit of study is, say, volumes of solids, then, in practice, teachers and problems in the textbook alike, generally avoid the requirement for application of Pythagorean Theorem. The theorem already received its due share in the curriculum, and after that it no longer needs to occupy pupil’s time or minds. A limited exposure to geometry coupled with such infrequent need to apply the theorem may be responsible that the desired approach in responding to the construction question was found to be so rare.

Precise location: What can be gained?

Among those participants who were able to find the precise location of $\sqrt{5}$ we found there was a sense of security that such number indeed existed. Their understanding seemed much more robust. Perhaps we could even say that the availability of a geometric representation aided them in the life cycle of concept development towards its final stage of encapsulation. This is in contrast with many others who offered the decimal approximation approach, where the number was seen as a process, stuck in its making forever. The following excerpt with Stephanie exemplifies this view.

Stephanie: Yeah. Okay, what I am thinking of, because somehow you can build this triangle and this triangle exists, this is another interpretation of the irrational number, so this segment represents the length of that hypotenuse, represents square root of 5, because this triangle exists. So it should be something what is, like we can touch, I don't know.

Finally, we present an excerpt from the interview with Claire, who communicated to us why she thinks teachers should not be satisfied with approximations.

Claire: Now, of course a point does not have dimensions. So on the number line you don't have actually the led of the pencil, it's still a dimension, although it's not. So intuitively you can say yes, it's there, a number can be represented in this way … As an answer, if you have the construction with a compass, yes you assume that construction is exact and precise, yes, $1^2 + 1^2 = 2$ and square root of 2 is the exact representation of square root of 2 irrational number, not how we are used to say 1.41, which is an estimation, and approximate answer.

Interviewer: And what do you think is there, what's the importance of us um having a student understand this, you know, exact and approximate, when they always work with approximation? What is the value for you? Do you think they should learn about these things?

Claire: I still tend to believe that it's better to work with the exact value, rather than an estimation, instead of, I'm the person that I like to speak with the terminology in math, so saying that pi is 3.14 ends up, if you don't insist in elementary school in grade 7, 8 whatever, saying that it's not, it's only estimation of the number, but you explain the pi like being, you know, some, the lengths of the circle and whatever, I think it's very important the terminology here, to understand that they have a specific value.

Interviewer: Okay…

Claire: So I agree with not being careless about this. When it's exact value, it's exact value, when it's a rounding of a number, it's a rounding of a number in estimation.

At some point students need to become aware that there is a distinction between the exact value of an irrational number and its rational approximation. We suggest this is better done sooner than later (think of Katie, who did not understand the distinction between π and $\frac{22}{7}$ at the time of the interview, suggesting that the misconception may remain permanent). As well, students need to be aware of the effects of premature substitution of irrational values by their rational approximations in partial results during calculations, both in the sense that this complicates the calculations and creates problems of cumulative error. It is our contention that placing more emphasis on the geometric representation of irrational numbers can aid students in two ways. Firstly, they are likely to become more sensitive to the distinction between the irrational number and its rational approximation. Secondly, it is likely to help them encapsulate the concept of irrationality by drawing their attention to yet another representation of the object (point on the number line, an irrational distance from 0) and away from the never-ending process of construction in time, as often perceived through the infinite decimal representation.

6.3 Definitions and coordination of definitions

As demonstrated in previous section (6.2), transparent features of the given representations were often either not recognized or not attended to. We shall argue that the main reason for this is that the equivalence of the two definitions of irrational numbers given in school mathematics – the nonexistence of representation as a/b, where a is an integer and b is a nonzero integer and the infinite non-repeating decimal representation – is not recognized. We consider this as a *missing link* that is rooted in understanding of rational numbers, that is, the understanding of how and when the division of whole numbers gives rise to repeating decimals, and conversely, that every repeating decimal can be represented as a ratio of two integers. With this in mind, we examined the role of the two competing definitions in conceptualizing irrationality.

6.3.1 Superficial coordination of definitions

As mentioned earlier, only 7 out of the 16 interviewed PSTs exhibited a balanced disposition; that is to say, they were able to flexibly use either of the two definitions, depending on the situation. From our theoretical perspective, we would say that they attended to the transparent features of the given representations.

Of the other 9 participants that we interviewed, only 2 were identified as fractionally disposed (see Section 6.2.2). One was Paul, who held a rather operational view. He could not see how "an infinite repeating decimal number could be multiplied by a whole number and give a whole-number result", except in the case of $0.\bar{3}$. The other was Anna, who was using an incomplete definition. We identified her as "fractionally disposed" (although we had some doubts about it at first) because of her prominent focus on fractional representation – however, it was an entirely operational kind of focus (i.e., "division will never produce an irrational number"). The remaining 7 participants relied primarily on the decimal representation when trying to discern whether a number is irrational or not.

We assumed that the 7 PSTs who demonstrated a flexible use of both definitions did so because they understood their equivalency. This assumption was probed during the clinical interviews since on the basis of written responses alone this is impossible to judge. Our assumption was challenged time and again during the interviews. We were forced to conclude, on the basis of evidence, that the equivalence of the two definitions is not common knowledge even among those prospective teachers who displayed a balanced disposition and who performed well on the classification tasks, such as Item 6 of the questionnaire. Interview with Erica tells another part of the story why this might be so.

Interviewer:	Could you tell me how you understand the difference between rational and irrational numbers?
Erica:	I just, I know that a **rational number is a number that has, it can be expressed as a ratio** and it has like an, um, a **repeating definable quantity**. Whereas an **irrational number is a number that can't, be expressed as a ratio, and it has no pattern to it, it's a non-repeating decimal, and it has infinite amount of digits.**
Interviewer:	Has infinite amount of digits, and rationals, can they have infinite amount of digits too?
Erica:	**Yes, but they'd have a pattern or a repeating, a repeating pattern to them. Pattern is not the right word, because their pattern could be non-repeating, so whether they repeat or not.**
Interviewer:	So that is the determining factor, whether they repeat or not, or the pattern, which word is more important, repeating or pattern?
Erica:	Repeating...
Interviewer:	Oh okay. And you mentioned that rational numbers can be expressed as a ratio...
Erica:	Yes...
Interviewer:	And irrational sometimes?
Erica:	**An irrational number, can it be as a ratio, um, no it cannot be, it is not a ratio.**
Interviewer:	I have heard of a definition of pi that it is a ratio between circumference and diameter, would that make pi rational? Because it's a ratio . . .
Erica:	No, **because pi is irrational, that's (pause) not a ratio of two numbers, a ratio of 2 factors, I don't know** (laugh).
Interviewer:	Okay. Um, maybe we can refine more, a ratio of what, is meant here?
Erica:	**A ratio of real numbers.**
Interviewer:	What is a real number to you?
Erica:	Real number is any, (pause) um. . .Well there's 4 classifications of numbers, right, there's the um, there's the numbers that are only the positive whole, positive numbers, then there's the real numbers which is both positive and negative numbers, and then there's also the rational numbers which can be expressed as a ratio and the irrational numbers that cannot. I keep it in my head...
Interviewer:	So um, what's the relationship then between real numbers and irrational and rational numbers?
Erica:	Uh real numbers would include all irrational and rational numbers...

Erica correctly answered both Items 1 and 2; in Item 6 she correctly attributed 8 of the 14 numbers given (this was the median performance on this item), and she demonstrated a flexible use of both definitions. Her relative success can be attributed to the fact that she transcended the verbal obstacle of "pattern" which we discussed earlier – this alone allows for a complete success in discriminating rationals from irrationals given their decimal representation. However, Erica's definition using the "nonexistence of a ratio" is incomplete. It is missing the essential requirement that the two numbers be integers. It echoes the same problem as discussed earlier in case of Anna (Section 6.2.2), who eventually was driven to discard her definition after concluding π would have to be rational under her definition. Interestingly, the inconsistencies that spring up from harboring this kind of incomplete definition do not seem to prompt the person to seek to remedy the situation. It seems that often people just adapt to it (note Erica's adjustment of this definition to exclude π), and carry on, most likely believing that mathematics really does not make much sense. Although Erica attended to the transparent property and recognized 53/83 as rational "because it is a ratio", this alone does not imply that the equivalence of the two competing definitions has been recognized. Rather, we would say, it demonstrates a kind of opportunistic thinking, which is not concerned that both definitions yield a consistent result – as long as the number fits one or the other definition, that definition can be applied to conclude rationality or irrationality. Therefore, while on the surface it may seem that a balanced disposition presupposes that the equivalence of the two definitions had been recognized, we found that this is rarely the case.

6.3.2 Which of the two definitions do PSTs really use?

While for Erica the recognition of the equivalence of the two definitions was unattainable due to incompleteness of one of the definitions, for many others, as shown earlier, it was unattainable because the concept of rational number had not yet been encapsulated (i.e. a fraction is not being seen as a number). Seven of the 16 PSTs that were interviewed relied on the decimal representation, and used the corresponding definition exclusively. We believe this is the major reason for the preference of the decimal representation over the fractional when deciding on rationality or irrationality of a number. Moreover, if one is to be convinced that the two definitions are indeed equivalent, both directions of the missing link must be considered: one, that every ratio of two integers is a repeating decimal (taking a broad interpretation of repeating decimals, allowing biunique representation, as discussed earlier in section 4.2), and the converse, that every repeating decimal is a ratio of two integers. It would seem that the first direction is more easily conceived as one commonly experiences how fractions transform into repeating decimals as a result of division. However, we found that this type of generalization was rare to come across. For example, Ed exhibits that such is indeed the case.

> Ed: It seems like a contradiction (laugh). But how do you know, like how far, like maybe the pattern happens and it's harder to see. Like maybe there's something that's, maybe this does repeat at some point, I don't know. Yeah, I don't know. . . It could take a million digits before it starts repeating. I mean you could devote your life to looking at that and, but how does anybody know, I mean why, I don't know…

6.3.3 One direction in the "missing link": from fraction to repeating decimal

There are three reasons for the difficulty in reaching a general conclusion on why any ratio of two integers results in repeating decimal number (taken broadly). The extension of the common experience of division of two whole numbers is needed. First, part of the problem, we believe, lays in the lack of algorithmic experience with division. Most likely this is because calculators become commonly used before this insight is given a chance to develop. Second, fractions that have long periods further complicate the issue, as not many people have ever taken the time to reach the full expansion of the entire period of such fractions (i.e., lack of experience with long periods). Third, the separation into terminating and repeating decimals further compounds the issue. While finite decimals are indeed the first kind met, it would be beneficial later on to support the acquiring of a perspective that terminating decimals can be seen as repeating decimals, as we indicated earlier. Consequently, there would be no need to consider terminating decimals as a separate case, which intuitively people are likely to do. Instead, we could point out that terminating decimals are just a special case of repeating decimals. Furthermore, most participants could not even tell how one were to know whether a decimal that came from a fraction repeated or terminated. Neither could they tell why it should repeat at all. Next, we consider the response of Kathryn, after she was being probed on whether 53/83 could have a non-repeating non-terminating decimal expansion.

Kathryn:	It might not be repeating, but we can't tell from, from this... **we could never tell because we can never see every single digit in the number and see if it repeated some millions of digits down the line.**
Interviewer:	Um hm...
Kathryn:	Um, but from that information there, um, is it too large to be on the calculator screen, that's why it's rounded off at the end?
Interviewer:	I don't know, that's what the calculator shows...
Kathryn:	Well if that's the number of digits that the calculator screen can hold...
Interviewer:	Yeah...
Kathryn:	Then I would say, based on that, I would say irrational, but if there was one space left on the calculator screen or something like that, then I would say, oh it's a rational number, because it terminates...
Interviewer:	But you know it comes from 53 divided by 83, 53/83, so it is a ratio of two whole numbers...
Kathryn:	Oh, I see what you're saying, yeah...
Interviewer:	So what's the connection, like it seems contradictory to some, if you look at this, you would say it's rational, but if you look at that you would say it's irrational, so how do you reconcile these two sort of different definitions of a rational number ...
Kathryn:	I don't know, maybe, **I know that if it continues on without repeating ever, then that is definitely an irrational number**, so this, I could have this part wrong I guess. That if it can be expressed as a ratio of two whole numbers, **maybe that's something wrong in my understanding...**
Interviewer:	So if you had to vote, if you had to put your money on one of those two, which one would you choose as a judge, for a rational...
Kathryn:	The decimal form...
Interviewer:	The repeating or terminating decimal, rather than ratio of two whole numbers...
Kathryn:	Um hm... I guess so.

Kathryn's response echoes the response of Connie (Section 6.1.1), who admitted that if she could see it her way, then it would make sense to say that some numbers can be both rational and irrational. However, Kathryn's understanding that the two sets, rational and irrational, are exclusive of each other, seems more stable. When challenged to either reconcile the two definitions or abandon one of them, Kathryn chooses to stick with the decimal definition. These beliefs cannot coexist: that a ratio of two whole numbers can be an infinite non-repeating decimal (we cannot know as we cannot see all the decimals), that a ratio of two whole numbers represents a rational number, that an infinite non-repeating decimal is irrational, and that a number is either rational or irrational but not both. The least stable one of these conceptions is forced to leave the arena so that others can exist in harmony. In the age of dominance of decimals and calculators, we find the definition of irrational number as a ratio of two integers to be the least stable one of these conceptions.

Leap of faith versus conviction from understanding division

We did not find a single participant who could explain this direction of the missing link, that is, that every division of two whole numbers must yield a repeating decimal. However, it should be noted that we found two participants who firmly believed that this must be the case. Not because they came to understand the connection, but because they trusted their correct recall of both definitions and they believed that in mathematics there is no place for inconsistencies of this sort to happen. In these two cases we would say that the equivalence of the two definitions was reached not by genuine understanding of the underlying concepts, but by a leap of faith. Stephanie, who we found to have the most complete understanding of irrational number of all the participants (the only one with perfect performance on Item 6) exemplified this kind of rendering of the two competing definitions as equivalent via a "leap of faith".

Stephanie: (considering why 53/83 is rational and what does that have to do with repeating or terminating decimals) Because it's a ratio of integers. There is a pattern, it might just be down here, or it might be a repeating of 50 digits and you just don't have the whole repeating. . .

Interviewer: Oh so we just have a partial view because of the calculator display?

Stephanie: Yes.

Interviewer: But are you sure this is going to repeat?

Stephanie: Yes I am.

Interviewer: What makes you say that?

Stephanie: Um, (pause) because okay, um, prime numbers, like some rational numbers cut off right, in a certain time, but like it's something to do with the prime numbers so it doesn't um. .

Interviewer: OK. . .

Stephanie: I don't know, I don't know, it's, anyway I'm pretty sure it will repeat, I'm not, I don't think. .

Interviewer: But there's no chance it will just keep going randomly?

Stephanie: No. No. So it might for the first 300 digits and then start repeating. So. . .

Interviewer: Is there any way to tell after which digit it will start repeating?

Stephanie: Not that I know of, probably though ...

Interviewer: I talked to someone earlier today and this person told me that it is possible that this keeps going without repeating. . .

Stephanie: Okay. . .

Interviewer: What would you say to that?

Stephanie: No I don't think so, it either terminates or repeats. . .

Stephanie is sure that the decimals of 53/83 either terminate or repeat, but she does not have a mathematical explanation for this. However, her faith is extremely stable, she does not need to abandon either of the definitions to assure consistency of her thinking and a flexible usage of both definitions. We found this kind of faith very rare. We are not suggesting that learners should be able to reproduce the proof instantly. In mathematics we often use the results that we have once been convinced of without having to go back to the first principles. However, the equivalence of the two definitions together with the reasons for it should be certainly clear to a secondary school mathematics teacher.

6.3.4 Other direction in the "missing link": from repeating decimal to fraction

The converse, that a repeating decimal represents a fraction, was also not commonly found in the knowledge repertoire of the participants of this study. Most common responses regarding this were:

It is easy to turn a fraction into a decimal. But there is no easy, general way of turning a decimal into a fraction. Looking at a decimal, unless it is a terminating decimal, you cannot tell if it is rational or not.

0.012222… is not rational. I cannot think of any two numbers to divide to get that decimal.

Furthermore, not a single participant was able to reproduce the "symbolic juggling on infinite decimals" to show how a repeating decimal can be transformed into a fraction (this method is presented in Section 4.2). Often, upon requesting that a repeating decimal number be converted into a fraction, we received the outright claim that this cannot be done. Excerpt from our interview with Erica exemplifies this view.

Erica: See it's, I think it's for a student and for me, it's virtually impossible to look at a decimal and put it into a fraction, like to go that direction. To go from a fraction to a decimal no problem, but to go the other way, is impossible. . .

Considering the evidence of this study, we find it difficult to justify the teaching of this method to students as early as in Grade 7 (for decimals with a single digit repeating) and again in Grade 9 (for any repeating decimal), in British Columbia. Although the intention behind the inclusion of this juggling method in the curriculum is probably just to convince students that every repeating decimal can be turned into a fraction, we maintain that it hardly serves the purpose. Amongst the participants of this study, we found only three individuals who believed that every repeating decimal can be represented as a fraction. Again, this belief was largely based on faith in the consistency of mathematics and not on the actual ability to perform this conversion. Interview with Claire, who was one of the best students, reveals that although she trusts that any repeating decimal can be transformed into a fraction, she does not believe this can be done using elementary mathematics.

Interviewer:	So, so okay, so if you see a number that does have a infinite repeating decimals, can you at least say from that, that it can always be turned into such a representation as A/B?
Claire:	Yeah if you know different, yeah you can, you can turn it in different ways, knowing rules, or doing different things yeah.
Interviewer:	As long as you have infinite repeating decimals, you say then it's for sure you can turn it into a fraction form?
Claire:	No I can't...
Interviewer:	Then no? But if there's a repeating you can do it, right?
Claire:	Yeah, if it's like that, um, and, you know continues and continues forever, **I can't, I cannot. Probably with more calculus, who knows, I forgot all that**.
Interviewer:	Calculus ?
Claire:	Yeah I was thinking if you rewrite it like um, infinite um, (pause) infinite something, let's see what happens, **no I don't want to think so far. I want to put myself in the skin of the student**..

In our assessment of the accessibility, portability and transferability of the knowledge of this symbolic juggling, we found that more than anything it tends to leave people mystified and it is difficult to retrieve. Even the very best students could not remember or use the method despite having been exposed to it. On top of this, they see no practical benefit for knowing it; that is to say, it in not perceived as critical piece of knowledge for the development of further concepts. We argue that if the very best amongst the prospective teachers in our study could not recall or make use of this method, and if the exposure to this method did not succeed in establishing the converse direction in the "missing link", then we are safe to conclude that it does not serve the intended function. We now present the excerpts from interviews with Dave and Stephanie, who were the only two interviewees that had any reference at all to the method of symbolic juggling performed on infinite decimals.

Dave: (considering the question on how we can know that every infinite repeating decimal has a representation as a/b where a, b are integers, and b is nonzero)...

Yeah, yeah, from what I remember, like **if I remember correctly from my number theory course**, like the repeating part will, it doesn't matter where it starts, as long as there is one. . . I remember um our professor, he, like say there's a decimal then a bunch of random, looks like random numbers and then the pattern, he was able to do it, to convert it to a fraction. . .

Interviewer: Okay, every time?

Dave: Yeah pretty, yeah pretty, like for examples he had, you could do it, like it took**, I remember I did some work and I couldn't really understand how he did it, like I followed it, but I couldn't remember**. . .

Interviewer: Yeah. . .

Dave: But I didn't look at it long enough to really learn what he was doing, I just go okay, it works, I wasn't, because **I was thinking oh, after this course, you know, I'm not going to use this too often. I couldn't see it being applied to anything**, except for, and just doing those kind of problems, although I thought they were very interesting, because it took a while for the brain power to try to make sense of it (laugh), so. . .

For Dave the method of symbolic juggling on infinite repeating decimals succeeded in convincing him that every repeating decimal can be represented as a ratio. His understanding is more robust and it does not suffer from the instability caused by the "missing link". In his reply we note that he had been recently exposed to this method in his number theory course at the university. We wonder what is the benefit, and the cost, of exposing middle school and high school students to it. Similarly, the only other participant who had any reference to the method, Stephanie, was also a recent mathematics major graduate in the honors program. She referred to the method as a "trick", and could not reproduce it either; however, the "trick" did the job of convincing her that every repeating decimal can be transformed into a fraction.

Interviewer: Why repeating, what's so special about repeating?

Stephanie: Um, it helps us put them in a ratio, if digits repeat, we can manipulate them and put into a ratio of integers which is a rational number.

Interviewer: How, is there a technique or something that you are referring to?

Stephanie: Um, uh there's a technique, I know they teach it in grade 10 or 11 or something and you can manipulate it in some way, but um, that's. . . yeah and it doesn't matter where it starts repeating, as long as it would repeat.

Interviewer: Any number that repeats can be manipulated in that way and put into a ratio?

Stephanie: Well there's a trick that you can show that it is a ratio of integers ...

As mentioned, out of the 16 PSTs that we interviewed, Stephanie, Dave, and Claire were the only ones for whom the missing link did not present a significant conflict in understanding the equivalence of the two definitions. Either there was a trust that the two definitions must give a consistent result, or they had been convinced by the juggling method sometime in the recent past, as university students.

Undoubtedly, the quality and depth of understanding are affected if the equivalence of the two definitions and the reasons for it are not recognized. When directly confronted with the issue, the majority of PSTs' recognized that their notions of these matters are inextricably muddled. We present Kyra's response, provoked by drawing her explicit attention to the problem of the "missing link".

Kyra:	Umm, I just thought like, by saying that a rational can be expressed as a fraction means or implies that every rational is a quotient, right, it's the answer of something divided by something else. And it's just really difficult for kids to go back, or take a quotient and find out what the two dividends would have been, so I think that's why we use the second criteria, but I have no idea who figured that out, or why that rule, does that rule support the other one. Is there proof. . .
Interviewer:	Does it always support it?
Kyra:	And that's something I've, yeah, I have never even considered it.

6.3.5 Confronting the two definitions: exposing the "missing link"

It is interesting to note that people can pass largely unaware of the problem of the "missing link", particularly if they have a balanced disposition. This is probably the best possible outcome of the present didactical choice – to have people fluently use either of the definitions as the situation warrants, never questioning their equivalency. In the next excerpt, we present Erica's confrontation with the "missing link", provoked by the use of an interviewing technique referred to as the "probing of the strength of belief" (Ginsburg, 1997).

Interviewer:	Um, you know, I talked to someone else before, and this question on, you know consider 53/83, and let's call this quotient M, and then when we enter it into the calculator display, this

	is what we see. Do you think that it terminates here or there's more, but we just don't see it because of the limited calculator display?
Erica:	Um, (pause) I think, rational or irrational explain, well **I said it was rational because it's a ratio, I'm, I don't know whether I even considered whether it terminates or not. Which means my definitions aren't very strong, are they**. Um, yeah, I don't know. If it, looking at again, if it terminates then I would say that it's a rational number, but if it doesn't and it doesn't repeat, then it would be considered an irrational number.
Interviewer:	See I had a student who had similar kind of reasoning, he said um, this, he believed, continues, and it doesn't show any repeats so this decimal, in his opinion, would be, you know, what he has as a definition of irrational number, yet 53/83 looking at another definition of what a rational number is, would be considered rational, so he said well there are some numbers obviously, that can be both, rational and irrational, I mean both at the same time. They can be rational and irrational, do you think that it is possible...
Erica:	(laugh) So I think **my definition maybe isn't clear**, um, because that, my understanding is, it's either going to be a rational number or an irrational number, not both...
Interviewer:	So it can't be both...
Erica:	So I would say that this would be an example of my definition not being strong enough for either one. Um, so maybe my understanding of one or the other isn't correct, because obviously there needs to be some way of deciding...
Interviewer:	One or the other, what do you mean by one or the other...
Erica:	Rational or irrational, like my definition of what I, **how I define a rational number or an irrational number is not complete perhaps**, which is why I'm not identifying, like, because if I simply say a rational number is any number that's a ratio, yet **a ratio can give a number that has a non-repeating decimal** and I say a non-repeating decimal is irrational. **There's got to be something else that's going to distinguish between the two**, because otherwise it's going to overlap too much, so **I'm thinking that my definition is missing a part**, rather than say that some are both, is my fence.

As discussed earlier, Erica's definition is indeed incomplete (missing the requirement that the ratio be a ratio of integers), but that does not prevent her from identifying common fractions as rational numbers. Only after her attention is drawn to the conflicts arising under the condition of an unresolved "missing link", her mathematical notions of rational and irrational numbers are cast into shades of doubt. While Erica experienced this consciously, we believe a similar kind of insecurity is experienced subconsciously in many learners. What is more, we believe that this problem could be avoided, or at least reduced to a lesser extent by a more prudent didactical choice.

6.3.6 Equivalence of the two definitions is not obvious

Interview is indeed a special situation where the mind is extremely active, in contrast with the classroom situation where "students tend to be passive in their processes of understanding, taking things as they are, solving problems as they are given, often strictly following some model solution, never asking themselves questions that are not already in the book" (Sierpinska, 1991, p. 103). In some cases, the interview can be seen as a simulation of a learning situation at its best. If an individual is unable to make the relevant connections right then and there, independently, then we are safe to assume that these crucial connections are even less likely to happen in a classroom situation. Instead, they should be taught explicitly. They are clearly not obvious. Such is the case of the "missing link", the part of understanding that allows the learner to see why the two definitions are equivalent. We suggest that the "missing link" is the major cause for many of the cognitive obstacles we witnessed in prospective teachers' understanding of irrational numbers.

As shown, there is a conflict between concepts of "fraction" and "decimal" in the vast majority of PSTs preventing them from building an understanding, such that there exist smooth paths linking one thought to another without the stress and instability. As demonstrated, this serious cognitive conflict caused by the "missing link" is unlikely to get resolved on its own. Our argumentation is based on the assumption that in the process of mathematical learning, there are certain concepts that either need to be taught explicitly, without black-boxes and short-cuts, or not cast upon students at all. The "missing link" is presented here as one such area in the curriculum. According to Tall,

> "Most of the mathematics met in secondary school consists of sophisticated ideas conceived by intelligent adults translated into suitable form to teach to developing children. This translation process contains two opposing dangers. On the one hand, taking a subtle high level concept and

talking it down can mean the loss of precision and an actual increase in conceptual difficulty. On the other, the informal language of the translation may contain unintended shades of colloquial meaning." (Tall, 1978)

In Chapter 8 we present some ideas on how to link the two interpretations of irrationals with aim of supporting a balanced and connected perception.

CHAPTER 7

Results and Analysis of Intuitive Knowledge

In the same way that it is impossible to conceive of mathematics deprived of its theoretical organization (axioms, definitions and theorems), so one cannot conceive of a theory devoid of intuitive meaning (ideas, models). Often theory and intuition are distant, conflicting, and difficult to reconcile. This report is a story of the ingenious ways in which participants strive to harmonize their intuitions with what they formally know to be true.

In this chapter we focus on the results and analysis from the perspective of intuitive dimension of knowledge. We examined the prospective secondary teachers' intuitions and beliefs regarding the relations between the two sets, rational and irrational, such as density, richness, and how numbers fit together, as well as the operations between members of these sets. In our analysis of the participants' intuitions of these notions, we detected various inconsistencies in relation to the other two dimensions of knowledge, algorithmic and formal. These inconsistencies are often revealing of misconceptions, cognitive obstacles, and other common difficulties; as such, our goal here is to describe them and to attempt to identify their sources.

7.1 Intuitions on richness and density

First, we explore PSTs' beliefs about the relative "sizes" of the two infinite sets. What kind of mental images are used to tackle the questions about the abundance and density of rational versus irrational numbers? Participants' intuitions about the order of infinity of rational numbers versus irrational numbers (denumerable versus non-denumerable set) were investigated using Items 3 and 4. The tables below show the quantitative summary of written responses.

Item 3: What set do you think is "richer," rationals or irrationals (i.e. which do we have more of)?

Response category	Number of participants	[%]
Irrationals	22	[47.8]
Neither	11	[23.9]
Rationals	10	[21.8]
Other (not specified or no answer)	3	[6.5]

Table 6: Quantification of results for Item 3 – what set is "richer", rationals or irrationals... (n=46).

the set of real numbers, we designed the following two closely related items:

Item 4: Suppose you pick a number at random from [0,1] interval (on the number line of reals).

What is the probability of getting a rational number?

Response category	Number of participants	[%]
"Equal to 0"	2	[4.3]
"Close to 0"	9	[19.6]
"Close or equal to 50%"	10	[21.7]
"Close or equal to 100%"	8	[17.4]
"Undefined"	1	[2.2]
No answer	16	[43.8]

Table 7: Quantification of results for Item 4 – probability of picking a rational number from [0,1] interval. (n=46).

7.1.1 Intuitions regarding the abundance of irrational numbers

Next we present a collection of written responses to Items 3 and 4, categorized according to the proximity of the intuitive reasoning to formal results.

Irrationals are much richer than rationals

These are some examples of responses from the written questionnaire that came the closest to the formal theory of cardinal sets:

- Irrationals are richer, because they are not countable while rationals are a countable set. Bijection exists between integers and rationals; bijection does not exist between integers and irrationals.
- Irrational numbers are richer. Since every irrational number has an infinite number of digits, there are many more possible irrational numbers. For example, a number that is 100 digits long can be rearranged to form 10^{100} numbers.

In the next excerpt, we present Ed's response. We find it interesting because he arrives very close to the correct result without any reference to the formal theory of cardinal sets.

Interviewer:	You say that you think that irrationals are richer, meaning we have more of irrationals, how can you justify that thinking.
Ed:	Because there's always going to be more, if, if you were to just **take random digits**, but anywhere, **and pull them out of a hat**, like whatever, chances are those numbers, like say up to 1,000 digits or whatever, **chances are those numbers are going to have no pattern, there's much bigger chance**. Like if you think about it, there's no way you're going to, of course you're occasionally going to get one that has an exact pattern, but that's less likely, it's just in nature, in your environment, you do see more rational numbers then irrational numbers. But in the actual numbers themselves, if you, there's probably, in my opinion how I think of it, there'd probably be way more irrationals.

From Ed we learned that it is possible to intuitively know that there are many more irrationals without ever having seen Cantor's diagonalization proofs or knowing about the possibility of infinities of different order. Of course, this simple and intuitively sound reasoning is more likely to occur in those who see irrationals primarily as infinite non-repeating decimals. As mentioned earlier, Ed displayed an accentuated decimal disposition, which seems to have contributed in devising this type of reasoning to show that the number of irrationals must be far greater than the number of rationals.

Note that there is an inconsistency between participants' responses in this category across Item 3 and Item 4. On Item 3, 22 people claimed that irrationals are "richer," yet only eleven people maintained that the probability of picking a rational number at random from the given interval was equal or close to zero. Looking at the written data, we see that many of the PSTs who said that irrationals are richer abstained from answering Item 4 altogether. We interpret that this pattern of responses is indicative of having heard that irrationals are much more abundant than rational numbers without having an understanding of why this may be. When probed more deeply as to how much more abundant, some PSTs could not to respond.

Arguments revealing misconceptions

Some people see rationals as *terminating* decimals and consequently claim that the probability of picking a rational is very low, close to 0. This misconception was already described in detail in previous chapter, and we will not dwell on it here. However, it may be related to another misconception that was exposed in response to these items, namely that there is a finite number of rationals. The following are some responses to Item 4 in the written questionnaire. Though the decision of which set is richer is correct, the reasoning is flawed.

- Probability of picking a rational is 0 because we have an infinite number of irrationals between 0 and 1, but we only have a finite number of rationals.
- There is a finite number of rationals but an infinite number of irrationals, so the probability of getting a rational is very, very low.
- Rational numbers are defined as a number with a ratio. It seems there would be a finite amount of rational numbers and an infinite number of irrational numbers. Probability of getting a rational number is very small – say 1%.
- If the probability of picking any number is $1/\infty$, the probability of picking any rational number would have to be (all the rational numbers)/∞, which is very small because anything divided by ∞ is very small.

The misconception lurking from the first three responses came as a surprise to us. It is obvious to everyone, even from a very early age on, that there are infinitely many natural numbers. It is also well known, even to most high school students, that natural numbers are a subset of rational numbers. In light of this, the thinking that there is a finite number of rationals seems absurd; however, we propose a hypothesis that needs to be examine in further research that this thinking may be the result of advanced mathematical training when the required background is missing. We see it as an individual's abandonment of common sense and that which would seem intuitively obvious in order to accommodate higher knowledge, especially if this knowledge is counterintuitive. This misconception, we believe, may have developed in these individuals as a consequence of exposure to cardinal infinities, in a situation where the underlying conceptions of rational, irrational and real number were underdeveloped at the time to begin with. Moreover, it could hardly be said that this misconception was an isolated case. We came upon it several times, both in the written responses as well as in the interviews. However, it is not clear for every participant whether it applied to rational numbers in general or specifically to the interval [0,1]. This topic is discussed further in at the end of this section.

<u>Arguments involving mapping</u>

In this category we placed intuitive responses that involve a mapping using either the operation of addition or multiplication to transform every rational number into an irrational number. This is intended to show that the set of irrationals is richer. Here are some examples.

- Irrationals are richer. If we take each element of Q and add $\sqrt{2}$ to each, all of those numbers are irrational. Then we could take each element of Q and add π to it. Already we have twice the amount of irrationals as rationals. We could do this forever, so the set of irrationals is much richer.

- Irrationals are richer because all irrational × rational = irrational (eg. $\sqrt{2}$ =irrat, 2$\sqrt{2}$ =irrat, 3$\sqrt{2}$ =irrat, …).

Although these intuitions lead to a correct conclusion, they are not formally correct, unless it is already known that the set of irrationals in nondenumerable. Instead, these arguments seem to imply that $\aleph_0 \times \aleph_0 \neq \aleph_0$ which is not the case according to the proof of denumerability of the set of rational numbers. These responses reflect the application of finite experience to infinite sets, in particular that part is smaller than the whole or that infinity plus infinity is twice as large as the original infinity. In the interview with Dave, who is responding to how he knows that the set if irrational numbers is richer than the set of rational numbers, we see an example of this additional view.

Dave: So what I did was, in order to, like I could, I could take um one irrational number and I could add all of the rational numbers to it one at a time so I could have like pi plus 1, pi plus 2, pi plus 4, and I would have some set of numbers that has the same cardinality as the rationals. Then I can take another irrational number, like root 5 and add again all the rational numbers to it, so right there we got twice as many irrationals as rational numbers, so I can continue to do that with all of the irrational numbers that I can possibly think of. As well, um, no that's pretty much it..

We found only two instances of this particular approach. Although it is not indicative of the knowledge of cardinal sets, we found it to appear only in those participants whose notion of number was at a rather mature developmental stage where it could be conceived not only operationally but also structurally.

Vague arguments

This category of responses we found difficult to interpret, yet they are correct.

- Irrationals are richer. They fill in all the "gaps" on the number line.
- Irrationals are richer – because of all the other figures out there (log, ln, e, sin, cos)

The two sets are equally abundant

The most common response in Item 4 was that the two sets are equally abundant. About the same number of people chose that neither of the two sets is richer in Item 3. From looking at individual questionnaires, we see the consistency of this belief across both items. In the following list, we present some common justifications for this response:

- I think there is an infinite number of rationals and an infinite number of irrationals. You can't have one infinity greater than other infinity. So both sets are equally rich!
- Since we have an infinite number of both, neither is "richer".
- Neither is richer. There are infinite number of rationals and irrationals.
- For every rational there will be an irrational to follow, so the probability of picking a rational is equal to 50%.
- Probability of picking a rational is 1 in 2. For every rational number there is an irrational.

The intuition that there should be one irrational number for each rational number, as if the numbers were nicely packed like that, following this kind of order on the number line, was detected again in responding to the items on how numbers fit together. A more detailed analysis of these intuitions is presented in the next section. However, it is interesting to note that we found the belief that the two sets have equal cardinality among some of the very best PSTs in the group. We suggest that this is the most natural intuition, and that it is mast likely the most common belief before one is exposed to the theory of cardinal sets. Moreover, it is likely that even those with basic familiarity with different infinities tend to fall back to their naive intuition after some time of not having much use for this knowledge. We found this to be the case in two of the participants that we interviewed. Finally, for the purpose of secondary school teaching, the knowledge of cardinalities of infinite sets does not seem to be very applicable. We would agree with Claire that it would be appropriate if high school students knew nothing about infinities of various kinds, as this can be very confusing for children and even for adults.

Interviewer:	Which ones do you think we have more of, rationals or irrationals?
Claire:	Okay, my opinion is that none, if you give to a student, but we discussed a little bit about that, and I remember that indeed I did in some of my algebra, not calculus, I did in my algebra course it's something with the numerable and not numerable...
Interviewer:	Um hm, ...
Claire:	Yeah, countable, here we call them numerable and not numerable, larger, little bit, or, and yeah I remember something that irrational are, but I cannot tell to a student, so I think for a student, if I can explain they are going on and on and on forever that infinity, which you cannot reach it, it's somewhere there like a symbol, because even in a high school level of calculus, saying that infinite plus infinite is still infinite, how can you decide which is larger, you understand what I am saying...
Interviewer:	Um hm, yeah...
Claire:	So I would stop with that and say okay, infinite...
Interviewer:	Yeah, we are just exploring your intuition about this, you know, it's not something that we teach in high school, definitely not.
Claire:	Yeah and I would tell them only yeah, you go on and on, you cannot reach it, you cannot say that one is richer. How much richer? Because if you tell them it's richer, they will ask you by how much richer, and how can you explain to them that those numbers, no they understand sometimes that, that the real number, for example, are richer than, and I'm not sure now it came to me, I'm not sure, I think we learned about rational numbers, those are countable, I'm not quite sure, numerable, rational, and real or not, something like that. So far away, I cannot remember.

In passing, we noticed that Claire frequently talks about her potential students as if her personal knowledge is irrelevant beyond what there is to teach. It is possible that some teachers resort to this behaviour as a defense for what they perceive as a deficiency within their own understanding.

Rationals are more abundant

The response that rationals are more abundant was quite consistent across both Item 3 and Item 4. In the following list, we present some common reasonings.

- Rational set is richer. Because any integer divided by another integer repeats and is rational. Each integer can be divided by infinitely many other integers.

- Rationals are richer. Because I cannot remember many numbers similar to π.
- The chance of picking a rational number is pretty good, since some numbers that seem like irrational numbers can be written as rational numbers.
- Wouldn't all the numbers between 0 and 1 be rational? Probability of choosing a rational is 1.
- Aren't we choosing a fraction every time? Probability is 100%.
- Probability is 100% because I don't know of many numbers like Pi.

In the interview, Amy initially expressed the belief that all numbers in the given interval should be rational, but soon rectified this view upon finding the evidence that it could not be so. We find it interesting that she accessed the fact that there are some irrational numbers in the interval not by considering the decimal representation but rather by considering a symbolic representation.

Interviewer:	If you pick at random a number on the closed interval 0 to 1, what is the probability that you will pick a rational number?
Amy:	Any number I get, I would say (pause) 1, I would say any number is rational. . .
Interviewer:	Any number is rational, so you would say there are no irrational numbers on an interval between 0 and 1. . .
Amy:	Mmm… (pause) It should be because I would say square root of 2 divided by 2 is between 0, no it's not, square root of 2 divided by, yes it is, it is, . . .
Interviewer:	So there are a few, you would say there are some irrational numbers, but there's way more rational numbers, is that your intuition?
Amy:	0 and 1, lots, lots of numbers, rational numbers and in between each of them we can find again and again and again more rational numbers. But square root of 2 divided by 2 is there too…
Interviewer:	So would you change your answer now?
Amy:	I would say yes. I have to, now I'm interested in, you know, about irrational and rational numbers.

Unclassified responses

There were some responses that could not be fitted into any of the above themes but we include them in the list below for completeness.

- I don't think anybody knows which set is richer.
- I would imagine that we have quite a few of both types of numbers.

7.1.2 Summary of intuitions on richness and density

As we can see, in Item 3 almost half of the PSTs identified irrationals as the richer of the two sets. Yet, in Item 4 we note an inconsistently large drop in the corresponding response category. In addition, over 40% of participants provided no response to this item. Only about a quarter of PSTs declared that the probability of picking a rational number on the interval [0,1] is 0 or close to 0. It is possible that this drop occurred as a result of insecurity in this belief, or it may have had something to do with the limited interval. Furthermore, we believe that the correct response is more often a result of natural intuitive reasoning, such as Ed's, than a result of advanced training and formal exposure to cardinal infinities. This is not to say that the participants of this study did not have the formal exposure to these concepts in their background, but only that, for the most part, they did not bring this knowledge into the foreground in any notable way. In addition, as noted earlier, many of the "correct" responses were found to be incidental, due to a misconception.

Although many PSTs who responded that the set of irrationals is richer did not justify their thinking (15), of those that did (7), only three made an explicit reference to denumerable and non-denumerable infinities, either in the written response or during the interview, whereas the rest of them used informal intuitive reasoning to justify their choice. Three of the 7 PSTs who responded that irrationals are more abundant than rationals and also justified their choice, did so for incorrect reasons, such as "irrationals have infinite decimal expansions, while rationals are finite decimals – this means there are infinitely many irrationals and only finitely many rationals".

Nearly one quarter of the participants of this study had no knowledge about the two degrees of infinity and consequently claimed the two sets should be equally rich. Of those, many argued that both sets are infinite, and that "you can't have one infinity greater than another".

Furthermore, about a fifth of the participants of our study were not aware of the existence of irrational numbers beyond π, e, and some commonly seen square roots. Although many PSTs abstained from answering Item 4, there was a much greater consistency between the two items with respect to later two response categories, namely that the two sets are approximately equal in size, or that there are very few irrational numbers, if any, on the interval [0,1]. It seems that intuitions and beliefs of participants who fall into one of these categories are more resistant. That is to say, these participants who either expressed the view of equal cardinality, or thought that rational numbers by far outnumber the irrationals, were much more likely to sustain the same belief across both items in comparison to those who expressed an opposing view.

7.2 Intuitions regarding the fitting of numbers

We looked at participants' intuitions about how the rational and irrational numbers fit together (i.e. the idea of continuity of the set of reals achieved by the "completeness" axiom). How do they reconcile the fact that rational numbers are everywhere dense, that is, between any two rationals, no matter how close they may be, there are infinitely many rational numbers, and yet it is still possible to fit the irrational numbers amongst them.

We present our analysis of the responses related to the questions (e), (f), (g), and (h) of Item 7. These questions were designed to investigate both PSTs' intuitive models and their formal knowledge about number concepts and relations between members of the two infinite sets, in particular how rational and irrational numbers fit together. As noted in Section 5.2.4., we anticipated to investigate participants' knowledge as the tools for a correct derivation were accessible to everyone. However, the answers were mostly intuitively based so our analysis focuses on participants' intuitive models. Next we present the quantification of the responses (shaded fields signal correct responses).

Part of Item 7:

Determine whether the claim is true (T) of false (F) and explain your thinking.

e) It is always possible to find a rational number between any two irrational numbers.

f) It is always possible to find an irrational number between any two irrational numbers.

g) It is always possible to find an irrational number between any two rational numbers.

h) Between any two rational numbers there is always another rational number.

Item	False	[%]	True	[%]	No answer	[%]
(e) rational between two irrationals	12	[26.1]	24	[52.2]	10	[21.7]
(f) irrational between two irrationals	5	[10.9]	32	[69.5]	9	[19.6]
(g) irrational between two rationals	3	[6.5]	33	[71.6]	11	[23.9]
(h) rational between two rationals	10	[21.7]	24	[52.2]	12	[26.1]

Table 8: Quantification of responses to Item 7: e,f,g,h – how numbers fit together. (n=46).

Although majority of the participants correctly responded to all these questions, we see this majority is marginal for items (e) and (h). What we find very interesting here is that as much as one quarter of the participants expressed a belief that there are some closest irrational numbers such that no rational number could be found between them. In one of the interviews a participant referred to "consecutive irrationals" to describe his idea of the absence of gaps between irrational numbers. Even more interesting is the unexpectedly high frequency of belief that there exist some closest two rational numbers, such that no other rational number could be found in between. More than one fifth of the participants expressed this view, while over a quarter of them abstained from answering this question altogether. With this fact being so elementary, and the proof of it being so within reach, especially for this group of participants given their educational background, we wanted to find an explanation for this. Possible source of error is presented in the excerpt from the interview with Erica below.

It should be noted that amongst those who answered question (h) correctly, there were very few (4) who either used a general symbolic argument or verbalized that the "arithmetic mean of two rationals is also rational". Most of the explanations provided by PSTs relied almost entirely on the decimal representation of numbers. This was prominent across all four items, as can be seen from the following collection of justifications. We present both those explanations that are mathematically valid and those that are not.

7.2.1 Justifications inconsistent with the formal dimension of knowledge

- Between at least some rational numbers there are only irrational ones. This must be the case because there are many more irrational than rational numbers.
- There aren't as many rationals - irrationals fill in the gaps between rationals (justification for why (h) should be False).
- Irrational numbers are so dense, you can find two that do not have a rational in between.
- Spaces between irrational numbers can be infinitely small.
- There will be two irrational numbers that are closest to one another.
- I believe numbers alternate: rational, irrational, rational, irrational, … So there will be some closest rational numbers where only an irrational will be found. Similarly, between any two closest irrationals, you'd find a rational, not an irrational.
- Two non-patterned decimals can exist without a number that has a pattern existing between them. The two irrational numbers can be very close, but not the same.

7.2.2 Justifications consistent with the formal dimension of knowledge

- You can always find an irrational between two irrationals because you could change the last little bit to make a new number between the two given ones.
- Let a, b € Irrational and a≠b. There must exist (a+b)/2 which could be rounded to some nearby rational number so that this number would fall between a and b.
- If two rational numbers exist then there is certainly a midpoint between them, which would be found by adding the numbers and dividing by 2. This yields a rational number.
- I can easily generate another ratio between two ratios.
- You can find a rational number between any two irrationals by terminating the decimal expansion of the larger number such that you create a number bigger than one and smaller than the other.
- There should always be a terminating decimal between any two infinite non-repeating decimals.
- It is always possible to find an irrational number between any two rationals: just expand the decimal expansion so that it neither terminates or repeats *and* it is bigger than one and smaller than the other.

In examining the participants' justifications, we found several cases where an example was shown, and then on the basis of this example a claim was made that the statement is *always* true, that is, for *any* such pair of numbers. For example, one of the participants in response to whether it is always possible to find a rational number between any to irrational numbers, wrote "Take $\sqrt{2} \approx 1.414$ and $\sqrt{3} \approx 1.732$; in between there is 1.6 =16/10 , i.e. can be written in form m/n where n≠0; therefore this statement is True". The same individual also believes that it is always possible to find an irrational number between any two rational numbers, because "between 1 and 2 there is $\sqrt{2}$, $\sqrt{3}$ and between 2 and 3 there are $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$". This kind of non-mathematical argumentation, attempting to reach a generalized statement on the basis of a few cases, was found amongst three participants. It is possible that these individuals misinterpreted the questions, ignoring the requirements for "always" and "any".

In the following passage, let us consider an excerpt from the interview with Erica, that may help explain the unexpected discrepancy in questions (e) and (h) mentioned earlier. Erica was one of the participants who held the belief that there are finitely many rationals and infinitely many irrationals. We interpret this belief as a distorted remnant of her encounter with the theory of cardinal infinities; in particular, we see it as the confusion between denumerable set and finite set, and between non-denumerable set and infinite set. She changed her mind as the interview unfolded. This was likely due to the stimulus arising from the questions, giving her a chance to rethink the absurdity of maintaining that the set of rationals were finite in size. Still, in the end she is left puzzled, but certainly equipped to resolve the conflict.

Interviewer:	How about this, is it always possible to find a rational number between any two irrational numbers?
Erica:	No...
Interviewer:	So, sometimes not?
Erica:	Sometimes not...
Interviewer:	Can you explain please...
Erica:	<u>**Back to my idea that there is an infinite number of irrational numbers, but a finite number of rational numbers, and if that holds true, then there mustn't, there can't possibly be a rational number that fits between every two irrationals**</u>...
Interviewer:	And vice versa, is it always possible to find an irrational number between two rational numbers?
Erica:	Yeah, because if there's an infinite number of them, then there must, (pause) <u>**okay this logic is based on my idea of, of there being an infinite number of irrational numbers and a finite number of rational numbers**</u>...
Interviewer:	And how about this, between any two rational numbers there's always another rational number, you say this is false, because?
Erica:	Any two rational numbers there's always another..., well again that will be <u>**based on my assumption that there's a finite number of rational numbers**</u>, therefore, you have to at some point have gone as far as you can go, <u>**but I'm starting to think that that may not be**</u>

true (laugh), I'm starting to think that how could you actually have a finite number of rational numbers, because even though rational number is a repeating or terminating decimal, you can still take it to an infinite number of decimal places, I mean you can still divide that space on that number line into an infinitely small subdivisions, so (laugh) .. my thinking on that at all. . .

Interviewer: That's okay. . .

Erica: Um, **I think that my whole idea of infinite and finite number of either one is something that I was either told, or somehow thought I was told and put into my brain and just took it as face value and didn't really actually think about what that meant, or what that would look like**.

Interviewer: Um hm, but you believe that there is an infinite number of natural numbers. . .

Erica: Yeah. . .

Interviewer: You do, right?

Erica: Um hm. . .

Interviewer: And you told me before, that rational numbers include natural numbers. . .

Erica: Yeah. . .

Interviewer: Already from that. . .

Erica: It doesn't, it's, **there must be an infinite number of both**, (pause) … going back to trying to like as I said, I need to kind of picture it, see it . . .

Interviewer: Maybe you meant in a given interval. . .

Erica: In a given interval, there's still an infinite amount of numbers that can be both rational and irrational.

Interviewer: And they never overlap?

Erica: Uh, they never overlap, no they don't overlap. . .

Interviewer: Yet we have infinitely many of both kinds?

Erica: Yeah (laugh). . .

Interviewer: That is interesting. . .

Erica: (laugh) It is. . ., yeah I don't know how that's possible but that's what um, yeah. . .

Interviewer: Okay. So, so you would change it now again, this one too?

Erica: Uh, between any two rational numbers there's always another rational number, yeah, I don't, um I'm not sure, but I'm, **my thoughts right now are leading me to believe that my whole conception of rational numbers having a finite number and there being a finite number of rational numbers is not, is false**, that there's also an infinite number of rational numbers and if that's the case, then there's going to be a rational number in between two rational numbers… that's possible … I can't picture any of it, so it's very hard for me to…

Interviewer: You can't picture any. . .

Erica: For that number, I just, I keep seeing this string of numbers that just keep going into infinity that. . .

Interviewer: You mean you can't picture an irrational number, am I right in saying that?

Erica: Yeah, or even that there's an infinite number of rational numbers like.

Interviewer: Um hm. . .

Erica: I just, this too, it's not something to understand, I, I can't, I don't know, yeah.

According to common sense, "countable" means "what can be counted", and it implies that what is countable must necessarily be finite. Although the usage of "countable" in Cantor's theory of infinite sets is entirely different, meaning that the elements of the set can be put into one-to-one correspondence with the set of natural numbers, it is possible that Erica adopted this more colloquial meaning of the word. Further research could examine whether this is indeed a "verbal obstacle" and, if so, what its extent is.

7.2.3 The sources of misconceptions regarding the fitting of numbers

We suggest that the reasons for many of these ill intuitions regarding the fitting of numbers lie in the non-intuitive character of the infinite; for example, that rational numbers, given their dense order compared to the discrete order in the natural numbers, are a proper subset of natural numbers. Or, that the rational numbers, in spite of being everywhere dense, are in fact very sparse in comparison to irrationals. All this is non-obvious, and often not convincing. Cantor's proofs are both simple yet very sophisticated, leaving many who have contemplated them still in doubts. Sometimes these doubts come not from what they show us, but from what new questions they open up for us, and leave unanswered. For example, one may be left wondering why the famous Cantor's diagonal argument cannot be applied to show that the rationals are non-denumerable too (the proof that another real number can always be found, different from all the ones in the supposedly complete list of all the real numbers presented as infinite decimal expansions).

Therefore, there are epistemological obstacles that may account for the difficulties preventing learners from concluding that there is a rational number between any two irrationals, and also that there is a rational number between any two rational numbers. Rational numbers are seen to be both very dense, and very scarce, and moving between these two conflicting ideas may cause inconsistencies to erupt. Furthermore, the formal knowledge that the irrationals by far outnumber the rationals, encourages the thinking that there must be some closest, neighbouring irrationals between which no rational number can be found. Excerpts from interviews with Kyra and Kathryn exemplify this.

(...responding to whether a rational number can always be found between two irrationals..)

Kyra: No, just because it's so, it's so dense, **the amount of irrational numbers is so dense**, I don't think, I don't think in every case you would find, because if you could find a rational number between any two irrational numbers, that would mean that the richness, that wouldn't hold, it would have to be equal richness, in order to find one, so to be consistent, I would have to say no. . .

Kathryn: Not always though, because I mean there are going to be, if you look at your irrational numbers, they're, in one case **there will be an irrational number that's right beside another one**, right. . .**So there can't be anything between those two**. . .Like if you think of um 1234, if you think of those as the only numbers that exist, then you can't put anything between 1 and 2, right. . .So in the same way, there has to be, like you have to go down far enough that there will be two irrational numbers that are right next to each other right. . .So between those two, you can't put anything else, but between any two that you pick sort of arbitrarily, then you should be able to.

What we see here, is the mind's desperate effort to accommodate new evidence brought about by the exposure to new formal knowledge, such as that of cardinal infinities. Sometimes consistent connections fail to be created, and some more basic conceptions fall apart; for example, the understanding that there is always a rational number between two rational numbers may be lost. This can be seen as a failure to integrate different items properly, and reorganize one's knowledge, to reach a better understanding of the subject following this exposure. As it is, there are inherent difficulties in contemplating the infinite. Beyond the naïve notion as "something that goes on and on", infinity is difficult to imagine. Compounding the problem is that a great majority of PSTs based their thinking entirely on a single type of representation, namely the decimal.

One of the participants of the study, Dave, who initially reached the same conclusion as Kyra was later convinced by another student of the fallacy of such thinking. Still, he admits he feels troubled by the contradiction this acceptance purports. We present this excerpt because it was the most successful, albeit still unsatisfying for Dave, resolution of this conflict.

Interviewer: Is it always possible to find a rational number between any two irrational numbers?

Dave: Between any two irrational numbers, okay at first I thought no, so I put false, but I talked to Jody about it later and he came up with a really good example and I thought that convinced me. So what he said was, he says, so let's say I have two irrational numbers, and one of them is obviously, one on the top is bigger than the other one, so what I do is I take the larger one, or I take the two of them and line them up and pair up the places or match up the places, as soon as there's a place value that they differ the larger one I can just chop off the remaining and that gives me a rational number that then is smaller than the larger one, but bigger than the smaller one that I had. . .

Interviewer: Wow!

Dave: This is a convincing argument for me, so I'm going with that from now on (laugh).

Interviewer: Okay, but then how do you reconcile that with your previous discussion here, in that irrationals are so much richer, that there's way more irrational numbers and yet here you're just showing me how you can always insert a rational number between two irrationals. . .

Dave: Good question, I don't know, I had thought no way, I thought the irrationals are so dense, there are so many more of them that I probably could find two, that there wasn't a rational number in between, but then Jody said that to me and I thought, yeah that seems right, that seems like you can do it, so I don't know. I was torn. But he was convincing, it was a very convincing argument but I don't know.

Dave has a method – he knows how to do it, and that is convincing enough. At the same time, he is also convinced that irrationals by far outnumber the rationals (in fact, he showed us a "proof" for that too, one using the mapping argument). He is torn, because he believes both are true, and yet they seem to contradict each other. However, most individuals could not sustain this contradiction, and thus consciously or subconsciously resolved the conflict in one of the two ways. They either decided that both infinities are equipotent, reflecting the intuition of infinity as absolute; that is, there cannot be many kinds of infinity, and if two sets are infinite then they have an equal number of elements, that is an "infinity" of them. Or, it was concluded that there are some closest irrationals between which no other number could be inserted.

7.3 Intuitions on operations

We investigated PSTs' intuitions regarding the effects of operations between various types of numbers, using the following questions from Item 7:

Determine whether the claim is true (T) of false (F) and explain your thinking.

a) If you add two positive irrational numbers the result is always irrational.

b) If you add a rational number to an irrational number the result is always irrational.

c) If you multiply two different irrational numbers together the result is always irrational.

d) If you multiply a rational number by an irrational number the result is always irrational.

i) A product of two rational numbers can sometimes be irrational.

Item	False	[%]	True	[%]	No answer	[%]
(a) irrational + irrational = irrational (always?)	19	[41.3]	23	[50]	4	[8.7]
(b) rational + irrational = irrational (always?)	2	[4.3]	35	[76.1]	9	[19.6]
(c) irrational × irrational = irrational (always?)	16	[34.8]	21	[45.6]	9	[19.6]
(d) rational × irrational = irrational (always?)	5	[10.9]	32	[69.5]	9	[19.6]
(i) rational × rational = irrational (possible?)	30	[65.2]	3	[6.5]	13	[28.3]

Table 9: Quantification of responses to Item 7: a, b, c, d, i – Operations. (n=46).

Looking at these results, what really stood out was that the majority of responses to question a and c were incorrect. From our theoretical perspective, we would suggest that this is due to participants' difficulty in conceiving irrational numbers as an object. That is to say, the concept of irrational number has not been encapsulated; instead, an irrational number is viewed as stuck in the process of becoming by an endless summation of its decimals. Despite the fact that all the participants were competent at algorithms such as rationalizing the denominator, as much as 45.6% did not invoke this knowledge when asked whether the product of two irrational numbers could ever be rational (question c). Instead, they used intuitive reasoning, which almost entirely consisted of considering decimal representations of numbers. Below are examples of such reasoning.

7.3.1 Justifications inconsistent with the formal dimension of knowledge

1. The multiplying of a rational and irrational number will not stop the decimals from continuing.
2. When you multiply two numbers each with an infinite number of digits together, the result will still be a number with an infinite number of digits.
3. I use decimal representations. Because the decimal representations of irrational numbers cannot be terminated, the sum of such numbers will be a decimal that cannot be terminated.
4. You cannot add $\sqrt{2}+\pi$, but you can add their decimal representations. The sum cannot be a terminating decimal.
5. Two numbers that have an infinite number of non-repeating digits to the right of the decimal will have an infinite number of non-repeating digits in their sum.
6. If we think of the product of two irrational numbers as an irrational number of irrational things, the question becomes "will these numbers somehow add up to rational?" I don't think so.
7. The sum of two irrational numbers is irrational because $2\sqrt{2}+3\sqrt{2}=5\sqrt{2}$.
8. No pattern × no pattern = no pattern. You can't create a pattern through multiplication. You are just increasing the numbers, not changing the relationship.

9. You cannot add two irrational numbers because they both continue forever so you would be adding infinitely.

Moreover, of those individuals that did answer question (c) correctly, many of them ignored the requirement that the two irrational numbers must be different and gave examples such as $\sqrt{2} \times \sqrt{2} = 2$. Hidden in these results on the table as well are questions that were answered correctly in terms of true or false but involved errors of fundamental logic in their justification. Some students just showed one example which held true in order to prove that all possibilities are true. For instance, one of the participants wrote $\sqrt{3} + \sqrt{2}$ as a justification that it is true that the sum of two irrational numbers is necessarily an irrational number.

Many of the examples above show the over-reliance on infinite decimals when considering the results of operations involving irrational and rational numbers. Although the thinking in the first response above leads to a correct conclusion about the product of rational and irrational number, a similar kind of thinking is misleading in considering questions (a) and (c), and may account for the surprisingly weak performance. The last comment in our list is an example of perceiving a number with infinite decimals as being constructed in time (potential versus actual infinity), not as an already made object.

We suggest that one of the main reasons for difficulties is the disposition towards closure of operations within a number set, in this case, the set of irrational numbers. The intuitive belief that the sum of two irrationals is irrational, and that the product likewise, was expressed many times either implicitly of explicitly. The following three responses from the interviews exemplify this.

(In response to whether the sum of two irrationals is necessarily irrational.)

Steve: I still believe that that's the case. Um, because if you cannot, I think this is how I thought of this one, I was again thinking in fraction form, so if I had two numbers that cannot be in a fraction form, I don't see how I could all of a sudden put in a fraction form. Because when we add fractions, we always look at the denominator and add it, find the common denominator and all that, but here I don't think it was possible you can do that because you can't put them in, in the fraction form. So that the addition of them would always have to be irrational.

Claire: No,... simply because if you have 2 irrational yes, a and b, irrational numbers, and you add them, addition is closed, addition is stable in that set, in irrational numbers, yeah the answer is still um, it's still irrational.

(In response to whether the product of two irrationals is necessarily irrational.)

Kathryn: You can't multiply two different numbers that cannot be expressed as fractions, and get fractions. I think you might be able to do it, but it's going to take until Christmas to find an example...

(Note: the interview took place in June)

7.3.2 Justifications consistent with the formal dimension of knowledge

In contrast, the following responses utilize the formal knowledge of operations with irrational numbers. We number them for the convenience of future reference. The letter in brackets indicates the question for which the answer was given.

1. $(2+\sqrt{3})+(2-\sqrt{3})=4$ (a)
2. $\sqrt{3}\times 2\sqrt{3}=2\times 3=6$ (c)
3. $\sqrt{5}\times\frac{1}{\sqrt{5}}=1$ (c)
4. Proof by Contradiction: Suppose a is rational, b is irrational, a+b=c and c is rational then b=c-a is rational. Therefore, b is rational and it contradicts the original assumption so, in conclusion, c cannot be rational. (b)
5. There must be two irrational numbers whose digits will "cancel" when added to result in a rational number. (a)
6. You cannot multiply two different numbers that cannot be expressed as fractions and get a fraction. (c)

7. Multiplying a rational by an irrational is just scaling the irrational by a certain factor. This will not change the product from being irrational. (d)

8. Proof by counterexample. You can find two irrational numbers that create a repeating decimal expansion. (a)

$$\begin{array}{ll} \text{For example,} & 0.12122122212222\ldots \\ & \underline{+\,0.21211211121111\ldots} \\ & \underline{0.33333333333333\ldots} \end{array}$$

9. Rational numbers can be written as ratios. We multiply two ratios and we still get another ratio. (i)

The first three justifications demonstrate a high level of concept development; that is to say, $(2+\sqrt{3})$ or $2\sqrt{3}$ is conceived as an object and not only as an instruction for adding or multiplying two numbers. On the basis of both written questionnaire and clinical interviews, we found only six instances of PSTs exhibiting the attainment of such proceptual thinking with respect to irrational numbers. In the interview, Claire expresses her views about the challenges and the importance of helping students attain a proceptual thinking about numbers.

Interviewer: If you multiply two different, okay, so we want to have different irrational numbers, together, the result is always irrational. Is this true or false?

Claire: It's false, because you can take one number being 5 square root of 2 times 7 square root of 2, this is an irrational number, a product between, so I think, no I think, I'm, this is an irrational. So I think it should be set, these kind of questions should be, when you teach irrational, you have to show to the students, you know what, when you have. . ., I heard many, and I think they're wrong, saying that 7 square root of 2 are two numbers. . .

Interviewer: Hmm. . .

Claire: You understand? And the student is confused with those two numbers. No it's only one number, you have to see it like a symbol, 7 square root of 2 a number, don't see it like two, as long as you see it, but it should be somewhere in the definition or somewhere when you teach the lesson, it should be pointed out that this is a number, instead of saying two different numbers. Like, you know, 1a, 2a, 1 square root of 2, 2 root 2, yes, it's a slightly difference, and you don't have to, you have to be very careful how you, you say it in front of the students. Otherwise they will come and say, oh those two numbers I don't know. . .

It is interesting to note the inconsistency between Claire's responses to questions (a) and (c) . Although she clearly uses proceptual thinking in case of multiplication, she does not do so in case of addition. We wonder whether it has to do with the writing of the number, that is $7\sqrt{2}$ is more likely to be thought of as "one thing" in comparison to $7+\sqrt{2}$ In other words, the operation of multiplication is implicit in $7\sqrt{2}$ while the operation of addition is explicit in $7+\sqrt{2}$. It could be that as a result of this, she maintains that irratioanals are closed under the operation of addition. As the vast majority of other PSTs, she falls into the trap of considering the decimal representations without the recourse to a symbolic representation.

The fourth example in our list of justifications uses the proof by contradiction. Only one participant from the group used such formal and rather sophisticated argument in judging the effects of operations between members of the two sets.

In conclusion, we found that there was a great reliance on decimal representation (even when a symbolic representation would be more appropriate and revealing) and a general lack of competency in evaluating the adequacy of statements related to operations with irrationals. An ability to flexibly move between representations in considering the truth of these statements was exhibited by as few as 4 participants. Vast majority of participants incorrectly argued that adding two positive irrational numbers will always produce an irrational number, and likewise, that multiplying two different irrational numbers must result in an irrational number. Interestingly, there was not a single case of drawing upon a standard procedure, such as the commonly used "rationalizing the denominator". This reveals that algorithmic knowledge can become highly procedural and rote for the learner to the extent where the very purpose of using such procedures may be completely lost. It indicates there is a problem in the integration of algorithmic, formal and intuitive knowledge. We interpret the strikingly poor performance on item #6 (a) and (c) as an indication that the notion of irrational number, such as $5 + \sqrt{2}$ for example, is commonly conceived operationally (as a process) rather than structurally (as an object) (Sfard, 1991).

In this chapter, we centered our attention on the complex notion of intuition as manifested in the participants' responses regarding the relations between the two infinite sets (rationals and irrationals) that comprise the set of real numbers. Our findings indicate that underdeveloped intuitions are often related to a weak formal knowledge and to the lack of algorithmic experience. Constructing consistent connections among algorithms, intuitions and concepts is essential for having a vital (as opposed to rote) knowledge of any mathematical domain, and therefore also for understanding irrationality. It is clear that intuitions cannot develop in a vacuum. What is often missing, particularly in this domain, is the attention to algorithmic dimension. In the next chapter, we present several ideas on how to address this lacuna.

CHAPTER 8

Recommendations for Teaching Practices

It is our viewpoint that structural and operational conceptions are mutually dependent, and that the development of a concept often involves an individual's journey starting from an operational conception and ending with a structural conception. There is a growing interest in how concepts and procedures are related, both from ontological and psychological angle (Sfard, 1991). In our work, we find it important to distinguish between standard procedures or algorithms and those procedures or algorithms that are used for pedagogical purposes in order to build a foundation for a given concept. We will refer to the later as "pedagogical procedures" or "pedagogical algorithms". Whilst standard procedures are efficient ways of doing things that have developed over long periods of time, pedagogical procedures are designed only for the didactical purpose of illuminating the concept in question. This implies that a pedagogical procedure is abandoned when the purpose has been achieved, i.e. when the concept has been understood. Standard procedures, on the other hand, are not always sufficient in teaching a concept, as they often obscure rather than illuminate the concept (just think of the multi-digit multiplication, for example).

In this chapter, we propose several pedagogical algorithms to address what we have found to be the shortcomings of the present-day didactical choice in the teaching of irrational numbers. The ideas behind these algorithms were developed by Professor Emeritus Klaus Hoechsmann ("Division by Chunks", "Squaring up a Rectangle") and by Professor Andrew Adler ("Revealing the Digits Beyond") at the University of British Columbia (UBC), and they come from the many years of their teaching experience and interest in mathematics education. I have first encountered them in the courses "Mathematics by Inquiry – MATH 336", offered jointly by the departments of mathematics and education at UBC in the year 1998 (instructed by Cyntia Nicol and Klaus Hoechsmann) and "Mathematical Demonstrations – MATH 414" offered by mathematics department at UBC in the year 1999 (instructed by Andrew Adler). What is presented in the remainder of this chapter is an application or an adaptation of these ideas for pedagogical and remediation purposes. Discussed is the purpose behind each of the algorithms, how it relates to the findings of this study, and what we hope to achieve by it in an instructional setting.

This research study found that the equivalency of the two competing definitions is not established within the concept image of most prospective secondary mathematics teachers, and this is seen as one of the major obstacles to the understanding of irrationality. The first two algorithms presented in this chapter, "Division by chunks" and "Using unitized decimal expansions to convert decimals into fractions" are aimed at facilitating the recognition of the equivalency of the two definitions. By establishing the understanding that a decimal number can be written as a fraction if and only if it is an infinite repeating decimal (taken broadly, as discussed in Section 4.2), these two algorithms directly target the issue of "missing link". That is to say, these two algorithms, in conjunction, are meant to deepen the part of understanding of rational number that is necessary for the development of the concept of irrational number, in particular its decimal representation. This work should come prior to the exposure to the concept of irrational number.

The next two algorithms, "Squaring up a Rectangle" and "Cubing up a Brick" are aimed at developing the concept of irrational number. In accordance with the view that the genesis of structural conceptions rests on performing operations on the already existing lower level objects, we suggest that the concept of irrational number should be developed starting with the square root. Square roots and then higher roots can offer a first-hand experience of what it means to say that a number is irrational (unlike π, which until a senior level university course is irrational only by "hear-say"). In this sense, more consideration should be given to the formative role that square roots and higher roots play in the development of the concept of irrationality (unlike the "patterned transcendentals", from which this kind of insight is unlikely to develop).

The final algorithm presented in this chapter, "Revealing the Digits Beyond", aims at just that – revealing more digits of a square root than a calculator can display. This algorithm can be used when teaching the procedure of "rationalizing the denominator" to show that often rationalizing the numerator can be useful too. In addition, it can help with motivating the need for the proof of irrationality $\sqrt{2}$.

8.1 Attending to the "missing link"

Having observed the problem of the "missing link" discussed in detail in Chapter 6, we claim that a good number of troubles responsible for the limited understanding of the concept of irrational number stems from the poor didactical choice of casting out upon students two competing definitions of "irrational". The relationship between the two definitions is often not recognized. Out of the 46 participants of our study, we have not found a single individual that could explain this relationship. The non-understanding of this relationship was found to perpetuate misconceptions and cognitive conflicts. For this relationship to be recognized, one needs to understand,

a) how fractions give rise to repeating decimals,

b) that every fraction is a repeating (or terminating) decimal, and

c) that every repeating (or terminating) decimal can be represented as a fraction.

To understand the first two ideas, one needs to have experienced division beyond what many students get to experience in schools. As for the last, we have been convinced by the results of this study that the method of "symbolic juggling of infinite decimals" does not achieve the intended goal. For these reasons, we propose instructional methods, which we believe are useful in practice.

8.1.1 From fractions to repeating decimals

As shown in Section 6.3, how fractions give rise to repeating decimals is not common knowledge. Nor is it common knowledge that fractions always and necessarily either terminate or repeat their decimals. We think two interrelated factors contribute to this state of affairs:

a) lack of experience with division by hand (it's tedious, calculators do it for us anyway),

b) over-reliance on calculator's display and the inclination to draw conclusions from partial information displayed.

Even a simple task, such as expanding the digits of $\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{5}{7},$ and $\frac{6}{7}$ can serve to build an understanding of how remainders necessitate the repeating of digits.

In what follows, we present an algorithm, which we call "Division by chunks". It is intended for the taming of large periods; that is, to facilitate the display of digits that generally cannot be seen on a calculator display. Of course, one could always do it by hand, but this is prone to error and quite discouraging for many students. This algorithm, beyond revealing the interesting laws governing the behaviour of repeating digits, such as cyclic permutations of digits, also reinforces the understanding of division and place value.

"Division by chunks"

Investigate: When you divide 60 by 19, is it going to be a repeating decimal? Using the "division paper" and a calculator, divide 60 by 19 using division by chunks. Option: work in periods of 3 digits, or work in periods of 6 digits.

In considering $19\overline{)60}$, the student is supposed to decompose the remainders into units of smaller value unconventionally. Taking three digits at a time (i.e. decomposing first into thousandths, then millionths, then billionths, …), dividing by calculator, and, keeping track of the result, the students needs to find the remainder each time. To start, the calculator entry is $60000 \div 19$. The display shows 3157.894737. We take the integral part of this, 3157, and multiply it by 19. The result, 59983, is then subtracted from 60000, and this gives a remainder of 17 (i.e., 17 thousandths remain when 60000 thousandths are divided by 19). Next we repeat the steps using 17000 as a dividend, that is, we decompose the remaining 17 thousandths into 17000 millionths, and continue with the process. Once the same remainder has been encountered, everything after that repeats.

	3	1	5	7	8	9	4	7	3	6	8	4	2	1	0	5	2	6	3	1	5	7			
6	0	0	0	0																					
			1	7	0	0	0																		
						1	4	0	0	0															
									1	6	0	0	0												
													2	0	0	0									
																5	0	0	0						
																			3	0	0	0			
																					1	7			

Decimal point

In the above example, the repeating period has 18 digits, which are 157894736842105263. Going by 6 digits at a time (decomposing into millionths, then trillionths, ...), speeds up the process twofold, and can thus be used to investigate fractions with very long periods.

	3	1	5	7	8	9	4	7	3	6	8	4	2	1	0	5	2	6	3	1	5	7	8	9	4
6	0	0	0	0	0	0	0																		
						1	4	0	0	0	0	0	0												
													2	0	0	0	0	0	0						
																			3	0	0	0	0	0	0
																								1	4

Decimal point

8.1.2 Converting decimals into fractions

The converse, that any repeating decimal is a fraction is even more difficult to grasp. The "juggling" method presented in Section 4.2 has been found ineffective.

Excerpts from interviews speak of the ineffectiveness of this method intended to convince students that every repeating decimal can be represented as a common fraction. In fact, in our study we have not found a single participant who could reproduce or demonstrate this method. Two participants of this study had a distant memory of having seen some kind of "trick" that convinced them that this conversion can be done (recall Dave and Stephanie from Section 6.3).

For these reasons, we suggest an approach, which, instead of relying on the "juggling" method, uses the idea of "unitized infinite expansions". The method relies on the fact that every repeating decimal can be "undone", step-wise if necessary, to the point where the unitized expansion is visible. It provides an algorithmic experience which also teaches numeracy and encourages fluency in operating with fractions. In what follows, we outline a series of exercises intended to lead students to understand not only that every repeating decimal can be converted into a fraction, but also how this can be done.

"Unitized Infinite Expansions"

An investigation similar to this could be presented to students:

> For each of the given fractions, find its decimal equivalent (using long division).
>
> $\frac{1}{9} =$
>
> $\frac{1}{99} =$
>
> $\frac{1}{999} =$
>
> $\frac{1}{9,999,999} =$

Now find the fraction form of the given decimal number.

$= 0.55555555...$

$= 0.12121212...$

$= 0.012012012...$

$= 0.0111111...$

$= 2.00121212...$

$= 4.056785678...$

In this activity, students are expected to make connections between the repeating periods of a given infinite repeating decimal and the corresponding unitized infinite expansion. That is to say, they need to make conclusions about what operations might have been performed on the appropriate unitized infinite expansion to result in the given infinite decimal. In this sense, the method involves "working backwards". For example, 0.55555... can be seen as $5\times0.11111...$, which is equal to $5\times\frac{1}{9}$. Therefore, $0.5555...=\frac{5}{9}$.

As a side note, the task above should be preceded by a task that requires the converting of a terminating decimal. Fractions without repeating digits are converted easily. The most that needs to be done is to reduce them. Of course, this could be followed up by an investigation into what characterizes fractions that have terminating decimal expansions.

Example: Express 1.75 as a fraction.

$$1.75=\frac{175}{100}=\frac{\cancel{5}\cdot\cancel{5}\cdot7}{\cancel{5}\cdot\cancel{5}\cdot4}=\frac{7}{4}$$

Fractions with repeating digits, however, require some creativity. For the student, the idea starts building on a simple fact: that is, it is easy to see that the unitized infinite decimal expansions never "resolve". They always give a remainder of one (one tenth, one tenth of a tenth, one tenth of a tenth of a tenth; for others we can just replace the word "tenth" by hundredth, thousandth, ten-thousandth):

$$\left.\begin{aligned} \frac{1}{9} &= 0.11111\ldots = 0.\bar{1} \\ \frac{1}{99} &= 0.010101\ldots = 0.\overline{01} \\ \frac{1}{999} &= 0.001001001\ldots = 0.\overline{001} \\ \frac{1}{9999} &= 0.000100010001\ldots = 0.\overline{0001} \end{aligned}\right\}$$

Let us call these "unit expansions"

After students have verified these results for themselves, a sequence of tasks such as the following can be presented:

Using these results, write $43 \times \frac{1}{99}$ in decimal form.

Learners are expected to use the results obtained from the expansion of $\frac{1}{99}$ to conclude that

$43 \times \frac{1}{99} = 43 \times 0.\overline{01} = 0.\overline{43}$.

Can you guess what would be the fraction form of 0.5555…?

Here we would expect the students to think along these lines, "Since I know that

$0.\bar{5} = 5 \times 0.\bar{1} = 5 \times \frac{1}{9} = \frac{5}{9}$, it must be that 0.5555… is five ninths."

The exercises could then evolve to include the varying of the period.

Now try finding the fraction equivalent of 0.123123123…

Again, working backwards, the student would have to conclude that

$0.\overline{123} = 123 \times 0.\overline{001} = 123 \times \frac{1}{999} = \frac{123}{999}$

Often decimal numbers have parts that repeat as well as parts that do not repeat. The following exercise could be given to students to develop an understanding of what is implied by such "shifts" and how we can deal with such cases.

Try finding the fraction form of 0.033333…

Students are expected to see that this is related to the unit expansion of one ninth – it has the same period but it is ten times smaller. Therefore we have:

$$0.0\bar{3} = \frac{1}{10} \cdot \frac{1}{9} = \frac{1}{90}$$

The next exercise could involve a shift (multiplication by a tenth, hundredth, …) and a sum.

Now convert 0.488888… into a fraction.

Here four tenths have been added to a tenth of eight times our unit expansion of 0.11111… Therefore, the process of conversion involves figuring out what makes the number different from the corresponding unit expansion. Learners are expected to notice that after systematic disassembling of the decimal number, they can simply translate all the parts into fractions linked by appropriate operations. For example:

Since $\quad 0.4\bar{8} = 0.4 + 0.0\bar{8} = 0.4 + 8 \times 0.0\bar{1} = 0.4 + 8 \times 0.1 \times 0.\bar{1}$

it holds that $\quad 0.4\bar{8} = \frac{4}{10} + 8 \cdot \frac{1}{10} \cdot \frac{1}{9} = \frac{44}{90} = \frac{22}{45}$

After a series of exercises such as these we would expect students to be able to find the fraction equivalent of any repeating decimal number, both "immediately repeating" and "eventually repeating", and even more importantly, to understand that this can always be done.

8.2 On concept acquisition

The genesis of the notion of irrational must be grounded in some process. We have good reasons to believe this, based on psychological and ontological grounds. Sfard (1991) suggests that the building of a new mathematical object, such as the concept of irrational number, starts by carrying out processes on lower objects, which are previously constructed structural conceptions. A profound insight into the processes underlying the new notion is a necessary foundation for the resulting entity to be accepted as a new kind of mathematical object. In our research we have seen a number of cases, where participants developed a debased, quasi-structural approach which demonstrated as a tendency to identify the concept of irrational number with "infinite, non-repeating decimal". It seems that the textbook writers think that the concept of irrational number can be brought into being just by the force of an appropriate definition (or two, to be safe). However, our findings indicate that this is far removed from the truth.

8.2.1 The genesis of the concept of "Square Root"

Even for prospective secondary teachers, the absence of the mediation of computational processes in the process of acquiring a structural version of the concept can be a serious obstacle. There is a tendency when a concept is introduced purely structurally, that the definition is interpreted in an operational way. Recall the case of Anna who considered irrational as "cannot be produced by division", and who was consequently led to conclude, on the basis this definition, that π was rational.

An excerpt from the interview below further suggests that greater attention needs to be paid in designing instruction so that students may come to understand in the first place what a square root is. Again, we turn to the interview with Anna. In our efforts to explain the cause for such limited understanding as seen in her case, we came to a conclusion that the reason may be that she never came to understand the meaning of square root. On the basis of historical development, we maintain that square root is the first and most important pillar on which the learning of the concept of irrational number ought to be based. If a student cannot get passed that, it is unlikely that he or she will develop any meaningful notion of the concept of irrational number. The case of Anna supports this view.

Anna: Um, I think, because I think that like, irrational numbers can never be a quotient, or they can never be a result or a product, it can be used um as um, you know, a multiple or a divident, or um, or and not even as a sum either, so **they can never be the result of any equation**, I don't think. . .

Interviewer: Never be the result of any equation… What do you mean, can you please explain?

Anna: No, what I meant was um, **when I said that an irrational number is never uh the result of an operation that we do, like if the operation is multiplying, the result wouldn't be, if the operation was dividing, the result wouldn't be, an irrational.** (pause) But if the result, if the operation was taking a square root of something. (pause) **But I'm like really not even sure of what taking the square root of something is, in the same way that I understand multiplying and dividing anyway.**

Interviewer: What is the closest to what you're understanding is. . .

Anna: Of taking the square root?

Interviewer: Um hm. . .

Anna: Well, it's the opposite of squaring something, like a step forward, a step back. . . .

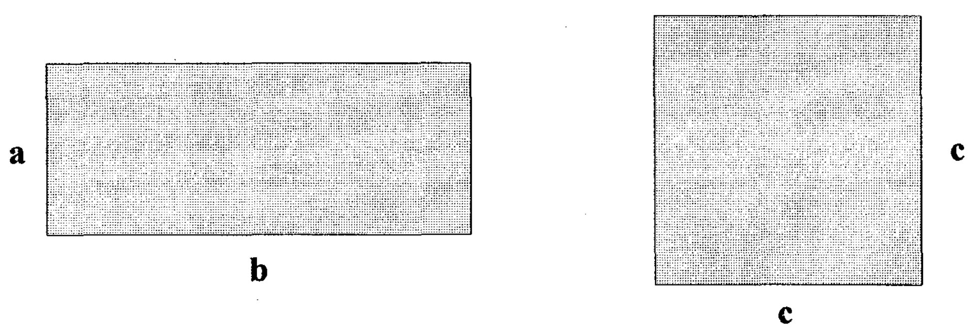
Interviewer: Okay. . .

Anna: Right, so when you're squaring something, you're multiplying it by itself, or not, yeah, a number of times, right, so then **taking a square root should be dividing something by itself**, if you were to just take that concept and mirror it. I'm just exploring this right now, ?? I really never thought of it, you know like 4, if you were going to, if you had 2^3, you'd go forward twice, multiply it by itself once, multiply it by itself again, to take the square root, then you'd be taking two steps backwards, but you can't divide, you can't go more extra steps backwards, you can't go into the negative. You can divide 8 by 2 and get 4 and divide 4 by 2 and get 2, but you can't divide 2 by 2 and get the square root of 2, you'd get 1. **So I'm not sure what taking a square root of something is, it must now be the opposite of square**. (Laugh)

It is evident that Anna confuses the square root with repeated division. She has a hard time making sense of how the squaring of a number relates to the square root, although she intuitively believes they must be "opposites" in some sense. To prevent people from failing to understand the square root (and other roots), which form the basis of understanding the irrational number, we suggest that every student should experience the creation of a square root via an algorithm, which we call "Squaring up a Rectangle". In what follows we present this foundational algorithm with an aim to help students build the notion of square root. Note that this algorithm is in keeping with the idea that operations on lower objects, which student has already encapsulated, give birth to higher-level mathematical conceptions, and as such it honors the nature of learning mathematics.

"Squaring up a Rectangle"

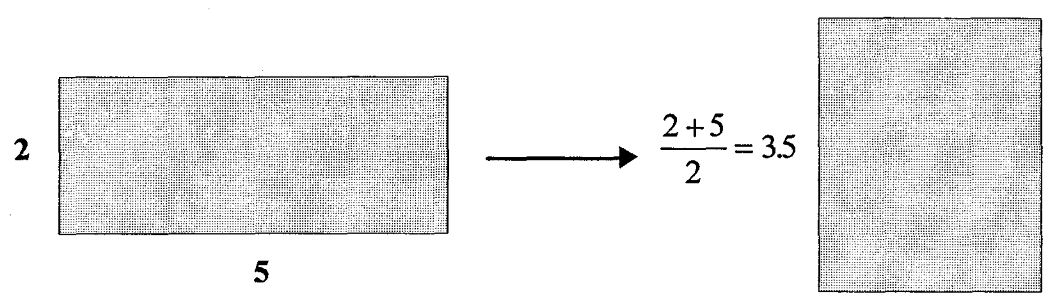
On the left-hand side you see a rectangle with side lengths a = 2 and b = 5. Can you devise a way of transforming this rectangle into a square so that the area remains unchanged? How big is the side of the square?



The role of the teacher would be to help students to think of a series of systematic steps - an algorithm - that will produce the desired result. Clearly, side a must be increased, and side b must be decreased if we are to end up with a square whose area is equal to the area of the rectangle. Since it is quite obvious that the side of the square will end up being somewhere between 2 and 5, a reasonable suggestion usually made by students is to start by taking the average of the two sides. This approach makes sense because it is systematic and can be repeated using a simple rule (one that can be programmed for a computer).

For the other side, there is no choice - the area must be kept unchanged at all times. This means the area of 10 needs to be divided by the newly obtained side length to arrive to the result for the other side length. Students should do the averaging and the division using a calculator, writing down all the digits that the calculator displays. The goal is to know the side of the square exactly.

The situation would look like this:



$$\frac{2+5}{2} = 3.5$$

$$\frac{10}{3.5} = 2.857142857$$

Algorithm:

Step 1: Average the two sides $(\frac{a_i + b_i}{2})$

Step 2: Let that be the new side *a* $(a_{i+1} = \frac{a_i + b_i}{2})$

Step 3: Find the other side keeping the area unchanged. $(b_{i+1} = \frac{10}{a_{i+1}})$

Students would be required to apply the algorithm for i = 0,1,2,3… until needed, that is, until the two sides appear to be equal, at least according to the calculator. In this example, a "square" is obtained after five iterations. Both side lengths, according to the calculator are equal to 3.162277660.

STEP	a	b
0	2	5
1	3.5	2.857142857
2	3.178571429	3.146067415
3	3.162319422	3.162235899
4	3.162277661	3.162277659
5	3.162277660	3.162277660

This approach gives the student a concrete representation of the square root as the side length of a square with a given area. For example, a square with an area equal to 10 has a side length of 3.162277660. This can be seen as just another way of saying that the square root of 10 is 3.162277660. The reason we chose 10 is not accidental. As students are very familiar with multiplying by 10, the result could be used to launch an inquiry into its accuracy. That is, have we really found the precise length of such square? In the next section we present a possible didactical approach for introducing the proof of irrationality of $\sqrt{10}$.

It is worth mentioning that using the same algorithm, we can find the side lengths both as common fractions and as decimals. We present an example, this time starting with different side lengths for the rectangle with area 10.

Width		Length	
Fraction	Decimal	Fraction	Decimal
$\frac{3}{1}$	3.0000000000000000000	$\frac{10}{3}$	3.3333333333333333333
$\frac{19}{6}$	3.1666666666666666666	$\frac{60}{19}$	3.1578947368421052631
$\frac{721}{228}$	3.1622807017543859649	$\frac{2280}{721}$	3.1622746185852981969

$\frac{1039681}{328776}$	3.16227766016984 20808	$\frac{3287760}{1039681}$	3.1622776601669165830
$\frac{2161873163521}{683644320912}$	3.1622776601683793319	$\frac{6836443209120}{2161873163521}$	3.1622776601683793319

What is interesting for students to notice upon arriving to these results, is that the decimal version surely looks the same. In fact this happens already at the fourth iteration of the algorithm. Hopefully students will ask themselves, "What about fractions? Shouldn't they be equal too?"

Introducing the proof of irrationality of $\sqrt{10}$

When checking out on the calculator whether 3.162277660 × 3.162277660 equals 10, some calculators report exactly 10, while others report 9.999999999. This is a good place to start the discussion about whether this product could ever be exactly 10. In a natural way, we come to the need for proof.

Let us suppose that the two fractions that represent the side length of the square, which we obtained in the last step of the algorithm discussed earlier, are equal.

$$\frac{2161873163521}{683644320912}=\frac{6836443209120}{2161873163521}$$

This gives the following, impossible result

$$(2161873163521)^2 = 10\cdot(683644320912)^2$$

The right side of the equation is a number terminating with zero, while the left side of the equation is a number terminating with one – so the two sides cannot be equal.

Is there a "perfect" square root of 10 (as 3 is the square root of 9)? Of course it could not be a whole number, but it might be a common fraction, a different one than the one obtained by our algorithm. At this point perhaps students would be able to accept a more formal argument, namely the proof of irrationality of $\sqrt{10}$, which is a didactical variation of the classic proof of irrationality of $\sqrt{2}$, presented in Section 4.3.

Let us suppose that there is some common fraction *m*/*n* that is a perfect square root of 10.

$$\frac{m}{n} = \sqrt{10}$$

Upon squaring both sides, we get

$$\frac{m^2}{n^2} = 10$$

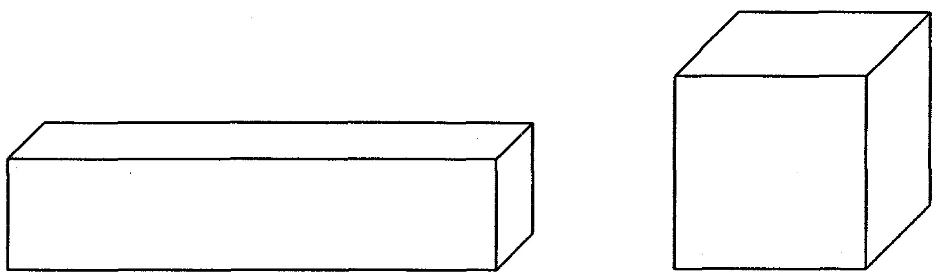
It follows that $m^2 = 10n^2$.

With *m* and *n* being whole numbers, the right side of this equation is a number with zero as its last digit. This can only happen if *m* ends with a zero too (as no other digit except zero, when multiplied by itself, can end with zero). But then the left side of the equation will have an even number of trailing zeros. Can there be an even number of zeros on the right side?

There is no such common fraction. And as we have found earlier, all decimals with repeating digits can be transformed into fractions (as well as those decimals that terminate, of course). We conclude there are such numbers, which neither terminate nor repeat decimals - these we call "irrational numbers". Furthermore, this algorithm can be extended to build the notion of higher roots as well.

Extending the algorithmic experience: "Cubing up a brick"

Without losing any substance, transform the "soft" brick with a volume of 9, on the left-hand side of the diagram, into a cube. What will be the side of the cube?



One may decide to start with 1×1×9 or 3×3×1 prism, or any other rectangular prism with a volume of 9.

Since all three sides must end up having the same size, it makes sense to start with a brick with two equal sides. Then we can keep two sides equal and vary only the third side accordingly to keep the volume unchanged. Of course, when averaging, we take all three sides into account. This method is the most efficient one. Here are the results of the process described.

STEP	**a**	**b**	$a_{next} = \frac{2a+b}{3}$
0	1	9	3.666666667
1	3.66666667	0.669421487	2.66758494
2	2.66758494	1.264753808	2.199974563
3	2.199974563	1.18595471133	2.086498753
4	2.086498753	2.067313071	2.080103526
5	2.080103526	2.080044418	2.080083823
6	2.080083823	2.080083823	

As demonstrated here, the algorithm of "Squaring up a Rectangle" can easily be extended beyond square roots and cube roots to include any root. Students will find it amusing, and delightful, after such amount of hard work, to find out that there is a button on the calculator that one can press, and find the same result.

$$10 \ \boxed{\sqrt{\ }}$$

The symbolic representation, for example $\sqrt{10}$, is introduced through the need to convey that we mean not the approximation, but rather the exact length of the square whose area is 10. The students can now, after this experience, without fear of the unknown, find the approximation of $\sqrt{10}$ by pressing the square root button on their calculator, and know exactly what it means, and how it can be found by hand. To say the least, this is very reassuring, and hopefully nobody will be left in the dark as Anna regarding the meaning of square root.

What remains to be explored with students with respect to irrational numbers is that although such length is inexpressible as either a common or decimal fraction, it indeed exists. This is where the geometric representation of an irrational length segment, via a construction using the Pythagorean Theorem, becomes indispensable. Although irrational numbers cannot be "captured" numerically, many of them can be "captured" geometrically. This notion should facilitate the reification of the concept of irrational number. As such, the concept of real number, in particular the idea that every real number has its place on the number line may become more easily understood. Although very scarce, there are suggestions in the literature, which promote the teaching of irrational numbers from the perspective of geometric representation (Coffey, 2001); therefore, we omit this from our discussion of suggested teaching practices.

Our final algorithm presented here is meant for students who already understand the concept of square root, and who are perhaps studying the idea of rationalizing the denominator. It may seem more of a mathematical curiosity; however, we include it because it is not commonly known, and yet it shows to the student that there are more digits following the decimal expansion of, say $\sqrt{2}$, than presented on the calculator screen.

Algorithmic experience: "Revealing the digits beyond"

This algorithm uses conjugates to reveal more digits $\sqrt{2}$ than the calculator can display. It is suitable for students who are learning the algorithm of "rationalizing the denominator", as it points out that sometimes "rationalizing the numerator" can be interesting too.

Problem: Find $\sqrt{2}$ to 11 decimal places using a cheap calculator.

Solution:

Take $\sqrt{2000000}$ and get 1414.---- on the calculator.

$$\sqrt{2000000} = 1414.\_\_\_\_$$
$$\sqrt{2000000} - 1414 = 0.\_\_\_\_$$

Now look at $\sqrt{2000000} - 1414$ and "rationalize the numerator". That is, multiply it by its conjugate. In fact, we will multiply it by 1, where 1 is taken in the form $\dfrac{\sqrt{2000000}+1414}{\sqrt{2000000}+1414}$. We get the following:

$$\sqrt{2000000}-1414=\frac{(\sqrt{2000000}-1414)(\sqrt{2000000}+1414)}{(\sqrt{2000000}+1414)}=\frac{2000000-1414^2}{\sqrt{2000000}+1414}$$

This can be handled by calculator. The display shows 0.2135623, therefore a more precise approximation is obtained by tacking on the extra digits, i.e. $\sqrt{2}\approx 0.14142135623$.

In this chapter we outlined a systematic approach to teaching the concept of irrational number, with special attention to the areas of understanding that this study found to be the most problematic. Several pedagogical algorithms for helping students grasp the concept were presented starting from those intended to create the necessary background related to rational numbers, continuing by the construction of the concept of square root, and ending with those suggestions for teaching that may provide extension, enrichment, and further insight for students.

CHAPTER 9

Conclusion

Here we provide a brief summary of the major results of the study, and we discuss its limitations. Next we discuss the practical recommendations. That is, what are the pedagogical implications of this study, and what changes should take effect immediately. Open questions for further research that we think would be worthwhile pursuing in order to make the concept of irrationality more accessible to students are also discussed.

9.1 Summary of Results

The study found that the majority of the prospective secondary mathematics teachers who participated in this study possessed a limited knowledge of irrationality. With respect to their formal knowledge, the study found that vast majority of participants used the same criteria or characteristics as presented in standard high school textbooks as the only working definitions for irrational number. This suggests that the mathematical knowledge in this particular area did not advance much after high school for most individuals. Generalizing this, it seems that what people learn in high school about irrational numbers becomes somehow "cemented", or stuck at the level at which it is typically presented to high school students.

Three major areas of formal knowledge were examined: a) ability to classify numbers into various number sets such as natural, integer, rational, irrational, and real; b) knowledge of the various representations of irrational numbers and the ability to translate from one representation to another; and c) knowledge of definitions and the ability to coordinate them. The study found that the major flaws in the understanding ranged from having an incomplete definition (such as the ratio of two numbers without the requirement that they be integers), to maintaining a collection of memorized rules for the purpose of identification of numbers (such as "immediately repeating decimals" are rational but the "eventually repeating decimals" are irrational), and sometimes hinging on a single representational feature as "an infinite non-repeating decimal" in lieu of the concept itself. It was not uncommon that a participant of the study needed to transform a given common fraction into a decimal number using a calculator in order to conclude that the number is irrational because no repeating could be seen. Particularly disturbing was performance on the item involving geometric representation of a constructible irrational length, with only 20% of the participants achieving an acceptable solution. A common belief was that such length cannot be represented geometrically on a number line because there are infinitely many digits in the decimal representation of the number; therefore, the process of making finer and finer decimal approximations would never conclude. Furthermore, this was not found to be a problem confined to irrational number alone; in general, the idea of number as a measure was found to be quite alien to most participants.

In the realm of participants' intuitive knowledge, the study investigated intuitions on richness and density, intuitions regarding the fitting of numbers, and intuitions on operations. The data show that there are inconsistencies between the intuitive and formal knowledge, which is suggestive of obstacles. There are also inconsistencies between the intuitive and algorithmic knowledge, which suggests that algorithmic knowledge can sometimes become rote and disconnected from the system of concepts it is supposed to support. Such is the case with the algorithm of "rationalizing the denominator". Although every prospective teacher most certainly knows how to execute it, very few seem to understand what it means and why we do it, as can be inferred by the commonly held belief that the product of two irrationals must necessarily be irrational. Other obstacles are of epistemological character (as can be seen by comparing historical obstacles with obstacles to individual understanding) but most are simply due to weaknesses in the formal understanding of number concepts. We found that if the formal knowledge has been secured, learners are capable of seeking acceptable explanations, such that would not violate their formal knowledge. However, we found that "ill" intuitions are much more prevalent.

What was particularly noticeable to the researcher was the participants' resorting to intuitive arguments when formal arguments were perfectly accessible, at least at this level of background training in mathematics. This was especially true for tasks that asked of participants to explain the effects of operations between the members of the two number sets, rational and irrational. As anticipated, intuitions and beliefs that individuals held revealed a great deal about their understanding of number in general, and also about their formal knowledge of irrational number in particular. This is not surprising given that most of the questions posed in these items can only be considered after the concepts of irrational and real numbers have been solidified into objects and seen as new members in the category of number. Only in such state one can meaningfully investigate general properties of various sets of numbers and relations between their representatives, or solve problems involving finding all the instances of the category which fulfill a given condition. The evidence suggests that only about 10% of participants achieved a reified stage of the concept development.

9.2 Pedagogical implications

Here we wish to make our concluding remarks regarding the present state of the common practice of the teaching of this topic. The most screaming issue is the issue of using two competing definitions to define irrational number. This is a didactical choice supported, implied, or perpetuated by curricular documents, such as IRP, and textbooks, such as MathPower. Based on the results of our research, we suggest that it would be better to teach one definition of irrational only, namely, that an irrational number is a number that cannot be expressed as a ratio of two integers with a non-zero divisor, and completely omit the other one (decimal). If the second "definition" is to be taught, then is should be derived from the first one, in which case it would be seen as a representational property of an irrational number when expressed as a decimal, and not as a definition. Furthermore, the focus on decimal representation, together with the non-understanding of how fractions give rise to repeating decimals, encourages a personal interpretation of the "decimal definition" and is prone to verbal obstacles leading to a host of misconceptions. For example, the common practice of identifying rational numbers as those numbers that have a "repeating pattern" often leads to confusion in cases where the decimal expansion exhibits a pattern, yet it is aperiodic, and also in cases of long and not easily detected periods. In addition, having the "decimal definition" as the only working definition results in a perception that all this is just a game of useless labeling. The concept of irrationality remains hidden.

Thus we see the practice of defining irrational numbers via two competing definitions as very problematic, even irresponsible. Such practice is promoted by some most commonly used textbooks (as described in Section 4.2), and followed by many teachers. Results of this study indicate that it cannot be assumed that learners will see the connection between the two definitions on their own. As demonstrated, almost none of the prospective secondary mathematics teachers that participated in our study recognized the equivalence of the two definitions and none of the participants could explain why the two definitions were equivalent. At the very best, the equivalence of the two definitions was taken by faith. We cannot expect students to make this leap on their own. The instructional practice of building the notion of irrational number using two competing definitions causes cognitive conflicts and often leads to insurmountable increase in conceptual difficulty (see discussions on the issue of the "missing link"). This is a didactical obstacle to learning, which could easily be avoided.

Next big issue in the practice of teaching irrationality, also screaming for attention, is the fact that students must build the concept starting from definition only (together with seeing a few examples of irrational numbers). This is very difficult. What is missing is the attention to the algorithmic dimension of knowledge. We maintain that the development of the algorithmic dimension of knowledge of irrationality has been much neglected in school mathematics. Attention to algorithmic dimension is necessary because it provides the basis for the operational understanding of a concept, which in turn provides the basis for structural understanding of the concept. Structural and operational conceptions are mutually dependent, and so it is very difficult to bring a new concept into being without attaching it to the mathematical objects that students already know and can experiment with. Pedagogical ideas on how to do this can come from the historical development of the concept. For example, the concept of irrational number can be fully developed through geometry using the phenomenon of incommensurability. This implies that first students would have to acquire a fairly intimate understanding of the idea of "unit" and what it means for two lengths to be commensurable with each other. This in itself is a worthwhile pursuit as it allows students to see, from a different perspective, why it is that when adding two fractions with different denominators we must always find the common denominator first, before they can be added. It is just a matter of finding the common unit for the two lengths given by these fractions. In addition, the idea of "unit" and the ability to perform the shifting of referent unit is so far reaching and so fundamental at the same time that it warrants a greater attention and emphasis throughout the curriculum. If the idea of unit is well developed it can provide an access point for students into understanding the world of irrational numbers. Other suggestions for enhancing the operational understanding of the concept of irrationality have been provided in Chapter 8.

On a related note, we suggest that for the teaching of this topic it would be much more effective to build the notion of irrational number gradually, starting with square roots, then other constructible lengths, then higher (nonconstructible) roots. Only after all this knowledge had been secured, and after the relationships between the decimal representation and the corresponding geometric representation as incommensurable length and the corresponding nonexistence of a representation as a common fraction had been recognized, is an introduction to (patterned or non-patterned) transcendental numbers warranted. No harm would be done if students knew nothing about these numbers until and perhaps even beyond university, except for those that intend to become mathematicians. As it stands now, the practice of using transcendental numbers to show to students what kinds of decimal numbers are or are not irrational adds no value to understanding the concept of irrationality. On the contrary, putting this material into textbooks without explaining the reason why these patterned infinite, yet non-repeating, decimals are irrational, and how we can know that that they cannot be transformed into a common fraction, seems to be adding confusion rather than revealing something of importance. It adds confusion by developing in learners the tendency to equate the concept of irrational number with "infinite, non-repeating decimal", promoting a pseudo structural approach devoid of the concept itself. Lumping all irrational numbers into the same bag like that does not honour the natural development of these matters. It does not support the individual learner in coming to terms with epistemological obstacles that seem to be unavoidable in the conceptualization of irrationality.

Lastly on pedagogical implications comes our commentary about π and what it is that teachers can and cannot, or rather should not, expect of students. We do not agree with the IRP curriculum document which states that students should be able to, "Explain whether or not π is a rational number" (p. F-41). When interpreting this instruction, there is a tendency in taking it to mean that the digits of π, when displayed on the calculator, show no repeating, and that that suffices as an explanation. We suggest that it would be far better if students are given a direct statement that π is irrational without having to explain it. They could be asked to accept it on the basis of faith, and reassured that a proper explanation will follow later on in university when they will have acquired the knowledge that they need to see this. They can trust that this missing bit will not in any way compromise their ability to work with π for the purposes for which it is used throughout school mathematics. It is unfair to ask students to "explain whether or nor π is irrational" because they simply cannot explain this without compromising the integrity of mathematics as a science of reasoning that is meant to be understood. Furthermore, accepting that the digits on the calculator display do not repeat as a legitimate explanation induces conflicts for subsequent learning and issues a license for many erroneous errors, such as claiming that 1/257 and most other fractions are irrational for the same reasons.

9.3 Limitations of the study and questions for further research

There are several limitations to this study regarding $\sqrt{9}$ different areas: participants, scope of the study, and the experience of the researcher.

Although a conscious attempt was made to identify the stage of concept development for each individual that participated in the interview stage of data collection, this information was found to be of limited value. What would be more interesting to know is what is the relation between the stage and quality of participants' understanding and the way they were taught. Have they had the luxury of being exposed to the ideas of irrational constructible lengths using geometric representation of numbers? Have they had the opportunity to build the notion from an operational platform, at least in the initial stage, or were they forced to bring the concept into being solely by the force of definition? Although participants at times disclosed what they learned from their teachers and how, it is difficult to make any conclusions as the concept of irrational number had been acquired too far in the past to rely on this information.

On a related issue, this study did not explore the correlation between participants' mathematical backgrounds and their understanding of irrationality. The number and the kind of university level mathematics courses taken were not considered. Incidentally, however, the three participants in the clinical interviews that demonstrated the most solid and evolved understanding of irrationality were all educated abroad (Group A participants). Although all the participants were in progress to become secondary teachers of mathematics, this group clearly demonstrated a higher level of preparedness to teach the topic. It may well be that teaching traditions in other regions of the world had some effect on this, but again, it would be too far fetched to make such claims. To gain an insight into how instructional methodology and didactical choice influence the formation of knowledge related to irrationality a teaching experiment type of study is needed. The methods proposed in Chapter 8 could be tested for that purpose.

The scope of the study was limited. It did not venture into some of the key areas of participants' understanding, such as the incommensurability of irrational magnitudes or the proof of irrationality of various square roots. The concept of irrationality proved to be too complex to address these areas. It should be mentioned that there was a item on incommensurability included in the questionnaire (see Appendix A, question 8b). However, there were hardly any responses to the question indicating that the idea was very alien to most of the participants in this group.

Lastly, as the data were being collected, the researcher was still learning the art of interviewing. The conceptions that were revealed through the interviews, especially at the beginning, were often not anticipated, leaving the author baffled.

9.4 Open issues

This study found that most participants' subject matter knowledge of irrationality was problematic to a degree where it is questionable whether they can teach it to others, let alone teach it effectively. Compounding the issue is the perception that this knowledge is not really useful. For example, one can work with a square root of 2 without knowing anything about irrational numbers. Often, the perception is that this part of the curriculum is only about memorizing definitions, and that the sole purpose for this knowledge is to be able to classify numbers and identify them as belonging to one set or another. This seems a rather mindless task and, for most learners, of interest only at the time of the test. Although this is not the researcher's position, it would be interesting to re-examine, by conducting further research, whether the topic should even be taught to high school students, given that many prospective teachers according to this study did not understand the concept themselves. Would there be anything lost if students did not hear about irrational numbers until they reached university? Admittedly, this is a rather big and overriding question, which is not easily answered.

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Appendix A

The written questionnaire administered to pre-service secondary mathematics teachers (n=46)

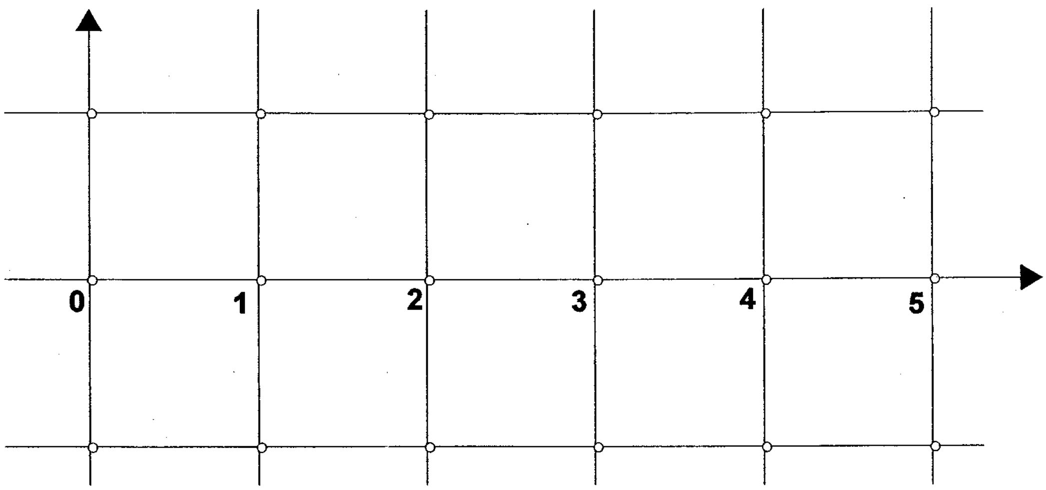
1. Consider the following number 0.12122122212……. (there is an infinite number of digits where the number of 2's between the 1's keeps increasing by one).
   Is this a rational number? How do you know?

2. Consider 53 divided by 83. Let's call this quotient M. In performing this division, calculator display shows 0.63855421687.
   Is M rational or irrational? Explain.

3. What set do you think is "richer", rationals or irrationals (i.e. which do we have more of)?

4. Suppose you pick a number at random from [0,1] interval (on the number line of reals). What is the probability of getting a rational number?

5. Show how you would find the exact location of $\sqrt{5}$ on the number line.



6. For every number listed in the table below check <u>all the attributes</u> that apply.

For example, in the case of "cat" it would look like this.

	Animal	Mammal	Reptile
Cat			

	Natural number	Integer	Rational number	Irrational number	Real number
0.05755755575...					
5/31					
$-\sqrt{36}$					
0.9999999...					
The solution of the equation $2^x = 3$					
The solution of the equation $x = \cos\frac{\pi}{3}$					
The solution of the equation $x = \sin 60°$					
The solution of the equation $3x+1=0$					
The area of the unit circle					
$\sqrt[4]{0.0016}$					
$\sqrt[3]{0.8}$					
$3\sqrt{8}$					
$\sqrt{\frac{12}{75}}$					
0.012222...					

7. For each question a-h determine whether the claim is true (T) of false (F) and explain your thinking.

a) If you add two irrational numbers the result is always irrational. ___

b) If you add a rational number to an irrational number the result is always irrational. ___

c) If you multiply two different irrational numbers together the result is always irrational. ____

d) If you multiply a rational number by an irrational number the result is always irrational. ____

e) It is always possible to find a rational number between any two irrational numbers. ____

f) It is always possible to find an irrational number between any two irrational numbers.

___

g) It is always possible to find an irrational number between any two rational numbers.

___

h) Between any two rational numbers there is always another rational number.

___

i) A product of two rational numbers can sometimes be irrational. ____

8. Suppose you have a square whose diagonal is 20 units long.
   a) How big is the side of this square?
   b) A common unit of two lengths is a unit that fits into each of the lengths a whole number of times. Is it possible to find a common unit (which can be very, very small) that would measure both the diagonal and the side? Explain your thinking.

9. Consider the equation $x^2 + y^2 = c$ for integral values of c = {1, 2, 3, 4, 5}
   Suppose the number axes contained only rational points. For what value(s) of c would the graph of this equation exist*? Explain your thinking.

---

* It is easy to show that any curve in the coordinate system has either infinitely many rational points or none at all. Reason: Suppose there is one rational point P on the curve. There exist infinitely many lines l_i with rational slopes that intersect the given curve at point P and at another point T_i. All these points are necessarily rational.

Appendix B

Participants' pseudonyms and their respective Groups

This Appendix gives the Group information for all the 16 interviewees. The Group is determined on the basis of the performance on Item 6. This item consists of 14 sub-items, which are to be identified as belonging to various number sets (natural, integer, rational, irrational, real). For the sub-item to be considered correct, it must be correctly identified across all fields that apply (i.e., there is no credit for a partially correct response).

Group A : those scoring 11 to14

Group B: those scoring 5 to 10

Group C: those scoring 0 to 4

Our participants fell into these groups quite distinctly as can be seen from Figure 2. The number in brackets indicates the individual's score on Item 6.

Group A	**Group B**	**Group C**
Stephanie (14)	Ed (10)	Anna (3)
Claire (12)	Kathryn (10)	Katie (3)
Thomas* (12)	Kyra (9)	William (2)
	Dave (8)	Connie (1)
	Erica (8)	
	Paul (7)	
	Steve (7)	
	Amy (6)	
	Matthew (6)	

* Thomas does not appear in the text of the thesis, as all the themes that emerged in the clinical interview with him have been already exemplified with excerpts from other interviews.