

HOMOMORPHISMS OF INFINITE DIRECTED GRAPHS

by

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Abstract

In this thesis we will study various aspects of homomorphisms of infinite digraphs with emphasis on cores and compactness. We begin by looking at several reasonable generalizations of the definition of the core of a digraph. Various equivalent definitions of this term have been applied to finite digraphs for some time. However, when these definitions are applied to infinite digraphs they are no longer equivalent. We examine several properties of infinite digraphs which are possible definitions of the core of an infinite digraph. We determine the logical relationships between the different properties and answer some natural questions regarding invariance of the properties over homomorphic equivalence classes. We argue that a core should be defined to be a digraph all of whose endomorphisms are automorphisms.

We define the property of homomorphic compactness for digraphs in the same spirit as the compactness property of formal logic. We subsequently show that our notion of homomorphic compactness is also related to topological compactness. We prove that homomorphic compactness of a digraph is a sufficient condition for the digraph to contain a core. We also examine a weakened compactness condition and show that when this condition is assumed compactness is equivalent to containing a core. We use this result to prove a bound for the maximum size of the smallest certificate of non-compactness for a digraph. Some large classes of digraphs are then shown to be compact.

We determine exactly the cardinality of the set of homomorphic equivalence classes of compact digraphs, and give some results regarding the maximum cardinality of a compact digraph. We also define a notion of finite equivalence for digraphs, and examine the properties of classes of finitely equivalent digraphs. In particular, we

determine for a given digraph the size of a maximum collection of pairwise inequivalent but finitely equivalent digraphs.

Finally, we examine the notion of compactness for various types of list-homomorphisms. Characterizations are given of digraphs which are compact with respect to certain types of list-assignments.

Many of our results apply directly to relational structures in general. Other results admit analogous statements applying to structures. In yet other cases we obtain different and more interesting results when we examine structures. These generalizations are informally discussed.

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Chapter 1

Introduction

1.1 Definitions

In this section we will present the basic definitions that will be used throughout this thesis.

1.1.1 Graph Theory

We first give the basic definitions for the theory of directed graphs. Refer also to [14].

A *digraph* G is an ordered pair $(V(G), E(G))$ where $V(G)$ is a non-empty, possibly infinite, set whose elements are called the vertices of G , and $E(G) \subseteq V(G) \times V(G)$. The elements of $E(G)$ are called the *edges* of G . We will generally write uv to denote the edge (u, v) . Note that we allow edges of the form vv , called *loops*. If a digraph has no edges of this form it is called *loopless*. A vertex which occurs in no edge of G is called an *isolated* vertex. The *in-neighbourhood* of a vertex $v \in V(G)$ is the set $\{u : uv \in E(G)\}$ and is denoted $N^-(v)$. The *out-neighbourhood* of v is the set $N^+(v) = \{u : vu \in E(G)\}$. We define an equivalence relation \equiv on the vertex-set of a digraph by $u \equiv v$ if and only if $N^+(u) = N^+(v)$ and $N^-(u) = N^-(v)$. If S is a subset of $V(G)$ then $N^+(u) \cap S$ will be referred to as the *out-neighbourhood of u in S* , denoted by $N_S^+(u)$. The in-neighbourhood of u in S is similarly defined. If both $N^-(v)$ and $N^+(v)$ are finite for each $v \in V(G)$ then we say that G is *locally finite*. An

independent set S in G is a subset of $V(G)$ such that $uv \notin E(G)$ for each $u, v \in S$.

By convention we will consider the *size* (or *cardinality*) of a digraph to be the size of its vertex-set, so $|G|$ is defined to be equal to $|V(G)|$.

If G and H are digraphs with $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$ then G is called a *subdigraph* of H and we write $G \subseteq H$. If G is a digraph and $S \subseteq V(G)$, then we denote by $G[S]$ the digraph with vertex-set S and edge-set $E(G) \cap (S \times S)$. This is referred to as the *subdigraph of G induced by S* .

An *oriented walk* W of length n in a digraph G is a sequence of $n + 1$ vertices v_0, \dots, v_n in $V(G)$ and n edges e_0, \dots, e_{n-1} in $E(G)$ such that either $e_i = v_i v_{i+1}$ or $e_i = v_{i+1} v_i$ for each i with $0 \leq i < n$. When $v_0 = v_n$ the walk is said to be *closed*. If all of the v_i are distinct then W is an *oriented path* of length n in G . If all of the edges are of the form $v_i v_{i+1}$ then W is called a *directed walk* or *directed path*, respectively. A *bidirected path* of length n in a digraph G is a sequence of distinct vertices v_0, \dots, v_n together with a set of $2n$ edges consisting of $v_i v_{i+1}$ and $v_{i+1} v_i$ for $0 \leq i < n$.

An *oriented cycle* C of length n in a digraph G is a sequence v_0, \dots, v_{n-1} of distinct vertices together with a set of n edges containing exactly one of either $v_i v_{i+1}$ or $v_{i+1} v_i$ for each i between 0 and $n - 1$, with subscripts reduced modulo n . These will be called *forward* and *backward* edges of the cycle, respectively. If each edge of C is of the form $v_i v_{i+1}$ then C is called a *directed cycle*. The *net length* of an oriented cycle C , denoted $net(C)$, is defined to be the absolute value of the difference between the number of forward and backward edges of C . Note that the net length of an oriented cycle does not depend on which vertex in the cycle is chosen to be v_0 or on which direction the cycle is traversed.

A digraph G is said to be *connected* if for all $u, v \in V(G)$ there is an oriented path from u to v in G . The digraph G is said to be *strongly connected* if for all $u, v \in V(G)$ there is a directed path from u to v . A (*strong*) *component* of G is a maximal (strongly) connected induced subdigraph of G . Given two vertices u and v which are in the same component of a digraph G , the *distance* from u to v , denoted $d(u, v)$, is defined to be the length of a shortest oriented path from u to v . If u is a vertex and S is a set of vertices, all of which are in the same component of G , we define the distance $d(u, S)$ to be $\min_{v \in S} d(u, v)$.

Let G and H be digraphs. A *homomorphism* from G to H is a mapping $f : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$ for all $u, v \in V(G)$. We will often use the notation $f : G \rightarrow H$ when f is a homomorphism from G to H . We write $G \rightarrow H$ to indicate that a homomorphism from G to H exists. We denote by $f(V(G))$ the set $\{f(v) : v \in V(G)\}$. The set $f(V(G))$ is also called the *range of f* or *range(f)*. We denote by $f(G)$ the digraph with vertex-set $f(V(G))$ and edge-set $\{f(u)f(v) : uv \in E(G)\}$. We call $f(G)$ the *image of G under f* . A homomorphism is said to *preserve non-edges* if $uv \notin E(G)$ implies $f(u)f(v) \notin E(H)$ for all $u, v \in V(G)$. If a homomorphism f is a bijection and preserves non-edges then f is called an *isomorphism*. If $G \rightarrow H$ and $H \rightarrow G$ then we write $G \leftrightarrow H$ and say that G and H are *homomorphically equivalent* or simply *equivalent*. Obviously if $G \rightarrow H$ and $H \rightarrow K$ then $G \rightarrow K$. If $G \not\rightarrow H$ and $H \not\rightarrow G$ then G and H are said to be *incompatible*. A collection of digraphs is called *mutually incompatible* if the digraphs in the collection are pairwise incompatible.

A homomorphism from a digraph G to itself is called an *endomorphism of G* . An endomorphism which is not a surjection is called a *proper endomorphism*. An isomorphism from G to itself is called an *automorphism*. We use the standard notation $f|_S$ to indicate the restriction of a function f to a subset S of its domain. If H is a subdigraph of G and $f : G \rightarrow H$ is a homomorphism such that $f|_{V(H)}$ is the identity mapping, then f is called a *retraction* and H is called a *retract of G* . Observe that in this case $G[V(H)] = f(G)$. The retract H is a *proper retract of G* if $V(H)$ is a proper subset of $V(G)$.

Note that since homomorphisms are mappings of vertices, a homomorphism $f : G \rightarrow H$ is a surjection when every vertex of H has a pre-image, although there may be edges of H without pre-images, *i.e.* f may be a surjection and yet there may be an edge $uv \in E(H)$ such that $uv \neq f(r)f(s)$ for any $rs \in E(G)$. If a homomorphism $f : G \rightarrow H$ is a surjection and furthermore every edge of H has a pre-image in G we will say that f is an *edge-surjection*.

1.1.2 Graphs, Hypergraphs, and Structures

The edges of a digraph are defined by a single binary relation. A natural way to generalize the notion of a digraph is to allow the edges to be defined by several different relations of possibly different arities. In this section we will formally define the notion of a *structure*. Refer also to [33].

An n -ary relation on a set S is a subset of S^n , that is, a collection of n -tuples over S . A *relational language* \mathcal{L} is a non-empty ordered set of relation symbols $(R_i)_{i \in I}$, for some index set I , together with their associated arities. A language may contain infinitely many different relations. The languages in this thesis will all have finite arity and we disallow unary relations. A *structure* G for \mathcal{L} is an ordered $(|\mathcal{L}| + 1)$ -tuple $(V(G), (R_i)_{i \in I})$ where $V(G)$ is a set called the *vertex-set of G* , and each R_i is a relation on $V(G)$ of the appropriate arity. If R is an n -ary relation in \mathcal{L} and $v_1, \dots, v_n \in V(G)$ then we will write $R(v_1, \dots, v_n)$ to indicate that $(v_1, \dots, v_n) \in R$. The set $\{(v_1, \dots, v_n) : R(v_1, \dots, v_n)\}$ will be referred to as the set of *edges* of G of *type R* or simply the set of *R -edges* of G , and denoted by $R(G)$. If (v_1, v_2, \dots, v_n) is an *R -edge* and $v_1 = v_2 = \dots = v_n$ then it is called an *R -loop*.

As before, $|G|$ is taken to mean $|V(G)|$. Also $|\mathcal{L}|$ is defined to be the number of relation symbols in \mathcal{L} .

A structure G is said to be *finitely induced* if $|V(G)|$ is finite. If in addition $\sum_{R \in \mathcal{L}} |R(G)|$ is finite then G is called *finite*. Obviously this distinction is only important when \mathcal{L} is infinite.

Structures generalize many standard graph-theoretic objects. If \mathcal{L} consists of a single binary relation, then the structures for \mathcal{L} are digraphs. If this relation is symmetric on a given structure then the structure is a graph. More generally, if \mathcal{L} contains at most one n -ary relation for each $n \geq 2$, and each of these relations is symmetric for a given structure, then the structure is a hypergraph. If \mathcal{L} contains a single n -ary relation which is symmetric for a given structure, then the structure is an n -uniform hypergraph. When G is a graph or digraph we will always use the symbol E for the single binary relation on $V(G)$.

We emphasize here that there is a subtle but important difference between digraphs

and graphs, in that the collection of all digraphs is equal to the collection of all structures for a certain language. The class of graphs is a subclass of the class of all digraphs, and does not equal the class of structures for any relational language. By its nature a graph can be regarded as a set of vertices together with a set of *unordered* pairs of vertices as the edges of the graph. However, we will always take graphs to be a special type of digraph.

A path or cycle in a graph is defined to be a bidirected path or cycle in the associated digraph, with the exception that cycles of length two are disallowed. The *girth* of a graph G is the length of a shortest cycle in G . The *odd girth* of a graph G , denoted $og(G)$, is the length of a shortest cycle of odd length in G .

For any set S , we will denote an n -tuple (v_1, \dots, v_n) , $v_i \in S$, by (\bar{v}) . When we use this notation the value of n will usually not be given explicitly, but will be assumed to be appropriate for the context. If f is a function we will denote $(f(v_1), \dots, f(v_n))$ by $(f(\bar{v}))$. The n -tuple obtained by replacing every occurrence of x in (\bar{v}) by y will be denoted $(\bar{v})_{y/x}$. Observe that if \bar{u} and \bar{v} are any n -tuples then $\bar{u}_{y/x} = \bar{v}_{y/x}$ if and only if for each $1 \leq i \leq n$ we have $u_i = v_i$, or $u_i \in \{x, y\}$ and $v_i \in \{x, y\}$.

We now define an equivalence relation \equiv on the vertices of a structure in a manner similar to the definition for digraphs. For a structure G and vertices $x, y \in V(G)$, we say that $x \equiv y$ if for all relations $R \in \mathcal{L}$, all \bar{v} , and for each \bar{u} which satisfies $(\bar{u})_{y/x} = (\bar{v})_{y/x}$ we have $\bar{u} \in R(G)$ if and only if $\bar{v} \in R(G)$. This condition states that in some sense the relation R cannot distinguish between x and y . Observe that when this definition is applied to digraphs it is equivalent to the definition of \equiv given in the section on digraphs.

The definition of a homomorphism generalizes to structures in a natural way. If G and H are structures for a relational language \mathcal{L} , a homomorphism from G to H is a mapping $f : V(G) \rightarrow V(H)$ such that for all $R \in \mathcal{L}$ and all (\bar{v}) , $R(\bar{v})$ implies $R(f(\bar{v}))$. All other definitions relating to homomorphisms of digraphs generalize similarly. Note particularly that when we say that a homomorphism $f : G \rightarrow H$ preserves non-edges, we mean that for all $R \in \mathcal{L}$ and all \bar{v} , if $\bar{v} \notin R(G)$ then $f(\bar{v}) \notin R(H)$. Also, we say that a homomorphism between structures $f : G \rightarrow H$ is an edge-surjection when it is a surjection and every edge of H has a pre-image of the same type in G .

One class of structures that will prove particularly useful in some of our constructions are those we obtain by taking \mathcal{L} to contain binary relations only. In this case our structures are known as *edge-coloured* digraphs. These are like digraphs, except that instead of a single edge-set, such a structure has a different edge-set for each binary relation $R \in \mathcal{L}$. The edge-set defined for a relation R is also referred to as the set of *edges of colour R* . Note that these relations need not be disjoint, so the same pair of vertices may occur as an edge of several different colours.

1.1.3 Set Theory

In this section we present various set-theoretic definitions and related concepts.

The sets we refer to in this thesis are those of Zermelo-Fraenkel (**ZF**) set theory [56], and so every element of a set is itself a set. By a *class* we mean any well-defined collection of sets [56], *e.g.* the class of all digraphs. Such a class may or may not be a set. A class which is not a set is called a *proper class*.

We now present a brief introduction to the theory of ordinals and cardinals. A more extensive treatment may be found in [44]. Proofs of the claims we make in this section may also be found in [44].

A *partial ordering* (\leq) on a set S is a binary relation on S which is reflexive, antisymmetric, and transitive. A *linear ordering* on S is a partial ordering in which for any $x, y \in S$, either $x \leq y$ or $y \leq x$. The relation \leq is a *well-ordering* on S if \leq is a linear ordering and every nonempty subset of S contains a minimum element with respect to \leq . A *strict partial ordering* $<$ on a set S is a binary relation on S which is antireflexive, antisymmetric, and transitive. If \leq is a partial ordering then there is a natural strict partial ordering $<$ corresponding to \leq , *i.e.*, $x < y$ if and only if $x \leq y$ and $x \neq y$. Similarly, given a strict partial ordering $<$ we may define a partial ordering \leq by $x \leq y$ if and only if $x < y$ or $x = y$. Whenever we define a partial ordering we will assume the corresponding strict partial ordering to be defined as well, and *vice versa*. We will say that a strict partial ordering is a *strict linear ordering* or a *strict well-ordering* if the corresponding partial ordering is a linear ordering or well-ordering, respectively.

We say that a set S is *transitive* if $x \in S$ implies $x \subseteq S$. We say that a set S is an *ordinal number* or *ordinal* if S is transitive and \in defines a strict well-ordering on S . We will always use lowercase Greek letters to denote ordinal numbers.

If α and β are ordinals then we say $\alpha < \beta$ if $\alpha \in \beta$. It can be shown that for any ordinals α , β , and γ exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\alpha > \beta$ is true, and that if $\alpha < \beta$ and $\beta < \gamma$ then $\alpha < \gamma$. Furthermore, any class of ordinals contains an element which is least with respect to $<$. Thus, $<$ satisfies the properties of a well-ordering, although the class of all ordinals is not a set.

Two sets are *equipotent* if there is a bijective mapping from one to the other. An ordinal α is called an *initial* ordinal if there is no ordinal β equipotent to α with $\beta < \alpha$, i.e. α is the least element of the class of mutually equipotent ordinals which contains α . The Axiom of Choice [75] implies that every set S is equipotent with some initial ordinal, which we call the *cardinal number* of S . Initial ordinals will be referred to as cardinal numbers or cardinals.

If α is an ordinal then $\alpha = \{\beta : \beta < \alpha\}$. Also, we denote by $\alpha + 1$ the ordinal $\alpha \cup \{\alpha\}$. The ordinal $\alpha + 1$ is always the least ordinal larger than α . If $\alpha = \beta + 1$ for some β then α is called a *successor* ordinal. Otherwise α is called a *limit* ordinal. Observe that the empty set is an ordinal, which we will denote as 0.

If κ is a cardinal we will denote the least cardinal larger than κ by κ^+ .

Many of our results will be proved using the following principle [44].

Transfinite Induction Principle

Let $P(\alpha)$ be a property of ordinals such that

- $P(0)$ holds,
- $P(\alpha)$ implies $P(\alpha + 1)$ for all ordinals α , and
- for all limit ordinals $\alpha \neq 0$, if $P(\beta)$ holds for all $\beta < \alpha$, then $P(\alpha)$ holds.

Then $P(\alpha)$ holds for all ordinals α .

We will assume the Axiom of Choice [45, 66, 75] to hold throughout this thesis. In particular we will make extensive use of the Well-Ordering Theorem due to Zermelo [75] which states that any set can be well-ordered, and which is equivalent to the

Axiom of Choice. We will also make extensive use of the Tychonoff product theorem [70], which states that the product of compact topological spaces is compact. This is also equivalent to the axiom of choice [46, 65, 70].

As stated earlier, every set can be put into a one-to-one correspondence with some cardinal. On the other hand, no proper class can be put into a one-to-one correspondence with any cardinal [56]. This provides a means of distinguishing between sets and classes which we will make use of later.

1.1.4 Colourings

If G is a digraph and κ is a cardinal, a κ -colouring of G is a function $f : G \rightarrow \kappa$. If $uv \in E(G)$ and $f(u) = f(v)$ then the edge uv is said to be *monochromatic*. If G has no monochromatic edges under the action of f then f is a *proper* κ -colouring of G . The least κ such that G admits a proper κ -colouring is the *chromatic number* of G , denoted $\chi(G)$.

Observe that if there is a loop $vv \in E(G)$ then vv will be a monochromatic edge under any colouring of G . In this case the chromatic number of G is undefined.

We define κ -colourings for structures analogously. Let G be a structure and let κ be a cardinal. A κ -colouring of G is a function $f : G \rightarrow \kappa$. If (v_1, \dots, v_n) is an edge of G (of any type) and $f(v_1) = \dots = f(v_n)$ then the edge (v_1, \dots, v_n) is called monochromatic. A *proper* κ -colouring of G is again defined to be a κ -colouring in which there is no monochromatic edge. The chromatic number of G is defined to be the least κ such that G admits a proper κ -colouring. In this case the chromatic number is undefined if there is a vertex $v \in V(G)$ such that (v, v, \dots, v) is an edge of G .

1.2 Background and Overview

The origins of our current work may ultimately be traced back to the study of the computational complexity of certain homomorphism problems. We refer the reader to [30] for the relevant definitions. In [54], Maurer *et al* defined the H -colouring

problem for digraphs. If H is a fixed digraph, the H -colouring problem is the problem of determining whether a finite input digraph admits a homomorphism to H . The problem was originally considered only for the case where H is finite digraph or graph. For undirected graphs it was determined by Hell and Nešetřil in [39] that H -colouring has polynomial complexity when H is bipartite, and otherwise is NP-complete.

The case for finite digraphs has been studied in [5, 6, 7, 34, 39, 40, 41, 42, 52, 54], among others, but has not been completely solved. In [54], Maurer *et al* showed that H -colouring is polynomial when H is a directed path, a directed cycle, or a transitive tournament. The result for directed paths was improved upon by Gutjahr *et al* in [34], where it was shown that H -colouring is polynomial whenever H is an oriented path. Those authors also gave an example of an oriented tree T for which T -colouring is NP-complete. The complexity of H -colouring for oriented trees was studied further by Hell *et al* in [41, 42]. In [6], Bang-Jensen *et al* examined the H -colouring problem in the case where H is a semicomplete digraph, *i.e.* for every $u, v \in V(H)$ at least one of uv or vu is an edge of H . The result in this case was that H -colouring is NP-complete if H contains at least two directed cycles, and is polynomial otherwise. A similar result is shown to hold for semicomplete bipartite digraphs by Bang-Jensen and Hell in [5]. This paper also contains some results relating to sparse digraphs which contain exactly two directed cycles. The authors conjecture that for digraphs H without sources or sinks, if each component of the core of H (*cf.* chapter 3) is a directed cycle then H -colouring is polynomial, and otherwise it is NP-complete.

The case where H is an edge-coloured undirected graph is studied by Brewster in [15, 16]. In [15] the author defines an edge-coloured undirected graph H to be a structure for a language \mathcal{L} containing only binary relations, and for which each relation is symmetric. The *underlying graph* G of H is defined by $V(G) = V(H)$ and $E(G) = \cup_{R \in \mathcal{L}} R(H)$. In [15] the author determines the complexity of H -colouring for each edge-coloured graph on three vertices with two edge-colours. Some results on 2-coloured cycles are also given. In [16] he proves that if the underlying graph of H is a path, then H -colouring is polynomial, and gives an example of a structure H whose underlying graph is a tree, and for which H -colouring is NP-complete.

In [8, 9], the present author upped the ante somewhat by considering H -colouring

problems where H is allowed to be countably infinite, but the input graph is finite. Here the main question considered was whether the H -colouring problem is solvable for a fixed graph H . It is shown in [8] that there exist recursive graphs H for which H -colouring is unsolvable. In [9] it is shown that there exist vertex-transitive graphs H for which H -colouring is unsolvable, but that H -colouring is solvable whenever the vertex-transitive graph H is recursive and locally finite.

One might generalize this problem still further by allowing H to be a countably infinite digraph and allowing countably infinite digraphs as input. In this case the input would have to be in the form of a decision procedure for the edges of the input digraph. We will refer to this as the inf- H -colouring problem. If H contains a countable clique the problem is trivial. Unfortunately, for any other H we strongly suspect that inf- H -colouring is not recursively enumerable, much less solvable, since to even verify that a given mapping from one infinite digraph to another is a homomorphism can probably not be performed in finitely many operations. However, we might be able to prove that a digraph G does not map to H by examining only a finite portion of G . This naturally gives rise to the notion of homomorphic compactness. In this thesis we will investigate those digraphs H which have the property that whenever a digraph G does not admit a homomorphism to H there exists some finite subdigraph of G which does not admit a homomorphism to H . Such a digraph is called *compact*. If a digraph H has the above property with respect to all digraphs G with $|G| \leq \kappa$, then we will say that H is κ -compact. If we have a compact digraph H for which H -colouring is solvable, then inf- H -colouring will be co-recursively enumerable.

The study of compactness as we have defined it is not without precedent. A well-known result of de Bruijn and Erdős [17] states that for any finite n , an infinite graph is n -colourable if and only if all of its finite subgraphs are n -colourable. This is equivalent to the statement that each finite clique K_n is homomorphically compact.

In a similar spirit, it is shown in Hell [37] that an infinite graph G has a retraction to a given finite subgraph H if and only if every finite subgraph of G which contains H admits a retraction to H . This turns out to be equivalent to the statement that every finite graph is homomorphically compact *cf.* section 4.4.

In the study of homomorphic properties of digraphs, the notion of the *core* of a

digraph has historically proven to be very useful. The core of a finite digraph has on different occasions been defined in various ways. In [28, 38, 74] a core is defined to be a digraph H such that H has no proper endomorphisms, and a core of G is a core H such that H is a subdigraph of G and $G \rightarrow H$. In [5, 47, 48, 52, 59] a core is defined to be a digraph H such that H has no proper retracts, and a core H is said to be a core of a digraph G if H is a retract of G . This apparent inconsistency is not a problem in finite graph theory, as it is easily shown *cf.* [38] that these two definitions are equivalent for finite digraphs. It is also easy to prove *cf.* [38] that every finite digraph has a core, and that this core is unique up to isomorphism.

The notion of a core is a useful one because many questions pertaining to homomorphisms of digraphs can be answered by considering only cores. The core of a digraph preserves many of the homomorphic properties of the original digraph, but contains none of the extraneous vertices which may have existed in the original digraph. In particular, if H is a core of G then $H \leftrightarrow G$, and so for any digraph K we have $K \rightarrow H$ if and only if $K \rightarrow G$ and $H \rightarrow K$ if and only if $G \rightarrow K$. We therefore began our investigations by attempting to generalize this notion to apply to infinite digraphs.

One of our first observations was that the above definitions are not equivalent for infinite digraphs. Furthermore, under either of the these definitions an infinite digraph may have no core at all, or may have many nonisomorphic cores.

In Chapter 3 we consider several properties which are potential definitions of the core of an infinite digraph, including the two given above. We will examine the relationships between these properties, and examine some characteristics of the individual properties. Ultimately we will choose to define a core to be a digraph G for which any endomorphism of G is an automorphism. This is equivalent to the above definitions for finite digraphs.

An obvious question one might ask is, given a digraph G , is G a core? Or in the case where G is infinite, does G have a core? In [38], Hell and Nešetřil show that determining whether a finite undirected graph G is a core is NP-hard if $\chi(G) > 2$. A bipartite graph G is a core if and only if $G = K_2$. It follows that determining whether a digraph is a core is NP-hard for digraphs with chromatic number at least three.

In the same paper a polynomially verifiable characterization is given for cores with independence number at most two.

In chapter 4 we investigate some partial characterizations of infinite digraphs which have cores. Early on in our study of compactness in digraphs, we noticed that non-compact digraphs tended not to have cores. Our first major result in this chapter states that every compact digraph has a core. On the other hand, we give an example of a core which is not compact, and so we obtain a necessary but insufficient condition for a digraph to have a core. We subsequently show that if H is a core and H is $|H|$ -compact, then H is compact, and so if we restrict our attention to digraphs H which are $|H|$ -compact, compactness is a necessary and sufficient condition for a digraph to have a core. Another interesting consequence of this result is that if a digraph H is $|H|^+$ -compact then H is compact.

The remainder of chapter 4 is dedicated to proving the existence of large families of compact digraphs. We first use Tychonoff's theorem to prove a very general result showing that a certain condition, involving both homomorphic and topological properties, is sufficient to guarantee that a digraph is compact. We use this result to show that compact metric spaces can be used to generate compact digraphs, providing a satisfying connection between homomorphic and topological compactness. We apply this technique to show that if D is the digraph obtained by setting $V(D)$ to be the set of points in the plane and $E(D)$ to be the set of all pairs of points of unit distance, then D is compact. Furthermore, we show that D is a core. The graph D itself has been studied extensively in other contexts [20, 35]. In particular, the chromatic number of D remains unknown, although it is easily shown to be at least four and no greater than seven *cf.* [35].

In chapter 5 we take a more global view. We begin by examining the class of all compact digraphs. We show that up to homomorphic equivalence there are exactly 2^{\aleph_0} compact digraphs. In light of the results of chapter three this is the same as saying that there are exactly 2^{\aleph_0} compact cores, and so the class of compact cores is a set. In fact we show that there are this many compact cores of countable size, and so the question naturally arises as to whether there are any compact cores of uncountable

cardinality. The digraph D defined in chapter 4 yields an example of a compact core with $|D| = 2^{\aleph_0}$. This leads us to wonder whether there might be compact cores of still greater cardinality. Erdős *et al* show in [24] that every compact metric space has cardinality no greater than 2^{\aleph_0} , and so we cannot use metric spaces to generate compact cores of cardinality greater than 2^{\aleph_0} . The determination of the maximum cardinality of a compact core remains an open problem.

The fact that the collection of all compact cores is a set, as opposed to a proper class, implies that there is some cardinal κ such that all compact cores have size no greater than κ . However, in [61], Pultr and Trnková show that for every cardinal κ there is a core of size κ . This provides a non-constructive proof that there exist non-compact cores.

We might also ask, in the same vein, how large the chromatic number of a compact digraph can be. We can easily construct a compact digraph with a countably infinite chromatic number, but at the present time we have no examples of compact digraphs with uncountable chromatic numbers. This question is related to a long-standing open problem for undirected graphs, posed in Taylor [68]: suppose H is a graph with uncountable chromatic number, and κ is any cardinal. Does there exist a κ -chromatic graph G with the same set of finite subgraphs as H ? If H can be chosen to be compact, then no such G can exist, since every finite subgraph of G would admit a homomorphism to H , and so G would admit a homomorphism to H , implying that the chromatic number of G is no greater than that of H .

In [21], Duffus and Sauer examine the class of all digraphs, modulo homomorphic equivalence, and show that it satisfies the properties of a distributive lattice. We show that the set of all compact cores is also a distributive lattice, with the same meet and join operators as in the class of all digraphs.

In chapter 5 we also define the notion of finite equivalence. Two digraphs are said to be finitely equivalent if any finite subdigraph of one admits a homomorphism to the other. For a fixed digraph G we define $\mathcal{F}(G)$ to be the class of all digraphs finitely equivalent to G , modulo homomorphic equivalence. We find that $\mathcal{F}(G)$ also satisfies the properties of a distributive lattice, except that it does not always contain a maximum element. In fact, we show that $\mathcal{F}(G)$ contains a maximum element exactly

when it contains a compact digraph, and furthermore that this compact digraph is the maximum element of $\mathcal{F}(G)$. We then characterize exactly when $\mathcal{F}(G)$ consists of a single element, and show that in most other cases $\mathcal{F}(G)$ is a proper class. It is still not known whether there are any digraphs G for which $\mathcal{F}(G)$ is a set with cardinality greater than one. However, we know that such a G must be finitely equivalent to a compact core in which every infinite component is an acyclic bipartite digraph. The case for undirected graphs is somewhat simpler, and we can show that for any undirected graph G , either $\mathcal{F}(G)$ has cardinality one or it is a proper class. This last result has somewhat the same flavour as the elementary theorem of measure theory which states that every σ -algebra is either finite or uncountable [64].

Our proofs of these last results rely on certain density properties of digraphs. We say that collection \mathcal{G} of digraphs has the *density property* if whenever G and H are any two digraphs in \mathcal{G} such that $G \rightarrow H$ and $H \not\rightarrow G$, then there is a digraph K in \mathcal{G} such that $G \rightarrow K \rightarrow H$ and $H \not\rightarrow K \not\rightarrow G$. Welzl shows in [74] that the collection of all finite graphs of chromatic number at least two has the density property. We will make use of a more recent and very elegant proof due to Perles [personal communication], which applies to infinite digraphs and structures as well. We show that if $\mathcal{F}(G)$ contains two inequivalent digraphs and has the density property then it is a proper class. We then use the method of Perles to show that in many cases these conditions are satisfied.

In our final chapter we examine compactness properties of list-homomorphisms. List-homomorphisms generalize list-colourings in the same way that homomorphisms generalize colouring. A proper list-colouring of a graph is a proper vertex-colouring of the digraph in which the colour of each vertex must be chosen from a list of colours specified for that vertex. A graph H is said to be *k-list-colourable* or *k-choosable* if a proper list-colouring of H exists for every possible choice of lists of size k for the vertices of H . The list-chromatic number of H is the least k such that H is k -list-colourable. List-colourings were defined independently in both of [26, 71]. In [26] Erdős *et al* show that the list-chromatic number of a graph H cannot in general be bounded in terms of the chromatic number of H . In fact, the authors give a construction for bipartite graphs with arbitrarily large list-chromatic numbers.

Other results in [26] include a characterization of 2-list-chromatic graphs, the exact value of the list-chromatic number of the complement of a perfect matching, and that determining the list-chromatic number of a graph is NP-hard. The authors also conjecture that every planar graph is 5-list-chromatic but that there exist planar graphs which are not 4-list-chromatic. The second half of this conjecture is settled by Voigt in [73], where the author gives an example of a planar graph which is not 4-choosable. More recently, Thomassen [69] completed the solution by proving that every planar graph is 5-list-chromatic. The general question of determining the list-chromatic number of a graph is studied in [2, 3, 4].

The complexity of the list-colouring problem is studied by Kratochvíl and Tuza in [50]. The authors show that, given a graph G and a list of colours for each vertex of G , it is NP-complete to determine whether a list-colouring of G exists. This remains true even when each list has at most three elements, each colour occurs in at most three lists, each vertex of G has degree at most three, and G is planar. The authors also show that the list-colouring problem has polynomial complexity if each list has at most two elements, each colour occurs in at most two lists, or each vertex of G has degree at most two.

A similar notion can be defined for edge-colourings of a graph [71]. A graph G is said to be *k-edge-list-colourable* or, somewhat confusingly, simply *k-list-colourable*, if for every possible assignment of a list of k colours to each edge of G there is a proper edge-colouring of G in which the colour of each edge is chosen from its assigned list. The *list-chromatic-index* of G , denoted $\chi'_l(G)$, is the least k such that G is *k-edge-list-colourable*. Unlike the situation for vertex-colouring, it is possible to bound the list-chromatic index of a graph in terms of the chromatic index χ' of the graph. In fact, it has been conjectured that the list-chromatic index of a graph is never greater than its chromatic index [12]. This problem has been investigated in [12, 19, 29, 49].

A list-homomorphism from a digraph G to a digraph H is a homomorphism from G to H in which the image of each vertex of G must be chosen from a list of vertices of H specified for that vertex. In [27] Feder and Hell define the list-homomorphism problem for a fixed digraph H to be the problem of determining, for a given input digraph G and a given set of lists for the vertices of G , whether there is a list-homomorphism

from G to H . The authors restrict their attention to graphs in which every vertex has a loop. They prove that for undirected graphs, the list-homomorphism problem has polynomial complexity when H is an interval graph *cf.* [31], and is NP-complete otherwise. They also examine the case where the input is restricted so that the list assigned to each vertex of G must induce a connected subgraph of H , and show that in this case the problem is polynomial when H is a chordal graph *cf.* [31], and is NP-complete otherwise.

We examine list-homomorphisms from the perspective of homomorphic compactness. We show that essentially only finite digraphs are compact with respect to list-homomorphisms. We obtain some characterizations of richer classes of graphs which are compact with respect to list-homomorphisms when the types of lists permitted are restricted in various ways.

Many of our results apply to more general types of relational structures as well. For simplicity of exposition we will prove most results for digraphs only. Our proofs usually generalize to apply to the case for relational structures with only trivial modifications. In some cases richer behaviour can be observed in the more general case. At the end of each chapter we discuss generalizations of our results to relational structures as well as restrictions to graphs.

Chapter 2

Tools

In this chapter we will introduce some of the basic concepts that will be used in subsequent chapters. We will also prove some technical lemmas.

It is well known that homomorphisms are a natural generalization of graph colouring. If \mathcal{L} is a language and κ is a cardinal, then we may define a structure K_κ for \mathcal{L} as follows: $V(K_\kappa) = \kappa$ and for each $R \in \mathcal{L}, R(v_1, \dots, v_n)$ if and only if not all v_i are equal. If G is a structure for \mathcal{L} , then the definition of a homomorphism from G to K_κ is identical to the definition of a proper κ -colouring of G in section 1.1.4. This fact provides an immediate proof of our first lemma.

Lemma 1 *If G and H are structures and $\chi(G) > \chi(H)$ then $G \not\rightarrow H$.*

Proof: Let $\kappa = \chi(H)$. If $G \rightarrow H$ then by transitivity of \rightarrow we have $G \rightarrow K_\kappa$, which contradicts $\chi(G) > \kappa$. ■

In subsequent chapters, we will often want to give counterexamples to various propositions. We will also wish to generalize our results to apply to structures for an arbitrary language. Our first construction will allow us to construct a structure for an arbitrary language from a digraph, while preserving many of its properties.

Construction A:

Let G be a digraph. Let \mathcal{L} be a language and let $R \in \mathcal{L}$ be an n -ary relation for some $n \geq 2$. We define a structure G^R for \mathcal{L} as follows:

$$\begin{aligned} V(G^R) &= V(G) \cup \{v_3, \dots, v_n\} \\ R(G^R) &= \{(u_1, u_2, v_3, \dots, v_n) : (u_1, u_2) \in E(G)\} \\ R'(G^R) &= \emptyset \quad \text{for all } R' \in \mathcal{L}, R' \neq R \end{aligned}$$

This construction is actually a restricted version of a construction found in [61](p. 57). Note that if $n = 2$ in the above construction then G^R is identical to G . Our next two lemmas describe the properties of G^R .

Lemma 2 *Let G and H be digraphs without isolated vertices. Let \mathcal{L} be a language and let $R \in \mathcal{L}$ be an n -ary relation. Define G^R and H^R according to construction A. Then any homomorphism $f : G^R \rightarrow H^R$ has the following properties:*

1. $f(v_i) = v_i$ for $3 \leq i \leq n$.
2. $f|_{V(G)}$ is a homomorphism from G to H ,
3. f preserves non-relations if and only if $f|_{V(G)}$ preserves non-relations,
4. f is a surjection if and only if $f|_{V(G)}$ is a surjection,
5. f is an injection if and only if $f|_{V(G)}$ is an injection,

Proof: Let $f : G^R \rightarrow H^R$ be a homomorphism. For each i with $3 \leq i \leq n$, v_i occurs only in the i^{th} position of any n -tuple in $R(G^R)$. Also, v_i occurs in some R -edge of G^R . Thus, $f(v_i)$ must occur in the i^{th} position of some n -tuple in $R(H^R)$, and so $f(v_i)$ must equal v_i . Similarly, since any $u \in V(G)$ occurs in the first two co-ordinates of some edge of G^R , $f(u)$ must occur in the first two co-ordinates and of some edge of H^R , and so $f(u) \in V(H)$. It now follows immediately from the definitions of $R(G^R)$ and $R(H^R)$ that $f|_{V(G)}$ is a homomorphism from G to H . It is also clear that a non- R -edge of G^R can only map to an R -edge of H^R if the corresponding non-edge in G maps to an edge of H . Claims (4) and (5) follow immediately from (1). ■

Corollary 3 Let \mathcal{L} be a language and let $R \in \mathcal{L}$. If G and H are digraphs then $G^R \rightarrow H^R$ if and only if $G \rightarrow H$. Furthermore, a homomorphism $f : G^R \rightarrow H^R$ is an edge-surjection if and only if the homomorphism $f|_{V(G)} : G \rightarrow H$ is an edge-surjection.

■

Proof: By lemma 2 we know that if $f : V(G^R) \rightarrow V(H^R)$ is a homomorphism then $f|_{V(G)}$ is a homomorphism from G to H . On the other hand, if $f : G \rightarrow H$ is a homomorphism then we may define a mapping $g : V(G^R) \rightarrow V(H^R)$ by $g|_{V(G)} = f$ and $g(v_i) = v_i$ for each $3 \leq i \leq n$. Clearly g is a homomorphism from G^R to H^R .

The second claim is a direct result of our construction. ■

In order to apply our results to undirected graphs, we will use the following construction, which may be found in [61](p. 68).

Construction B: Let G be a digraph. We define an undirected graph G^U by replacing each $uv \in E(G)$ by the undirected graph in figure 2.1, identifying the marked vertices with u and v as indicated.

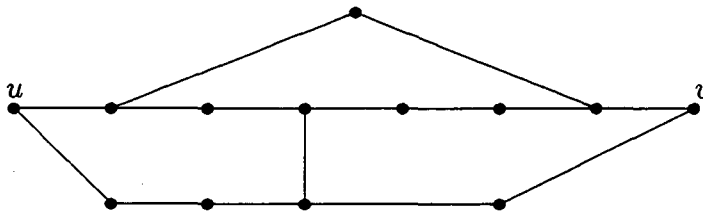


Figure 2.1

We will refer to the copy of this graph which replaces the edge uv as the uv -supersedege of G^U , and to u and v as the *endpoints* of the supersedege. If G and H are digraphs, $f : V(G^U) \rightarrow V(H^U)$ is any mapping, $uv \in E(G)$ and $xy \in E(H)$ then we say that f maps the uv -supersedege of G^U *identically* onto the xy -supersedege of H^U if f is the unique isomorphism between the two supersedes which satisfies $f(u) = x$ and $f(v) = y$.

Lemma 4 (cf. [61]) Let G and H be digraphs and let G^U and H^U be obtained by applying construction B to G and H . Then $f : V(G^U) \rightarrow V(H^U)$ is a homomorphism

if and only if $f|_{V(G)} : V(G) \rightarrow V(H)$ is a homomorphism and for all $uv \in E(G)$ f maps the uv -superedge of G^U identically onto the $f(u)f(v)$ -superedge of H^U .

Proof: We will give an outline of the proof: a detailed version may be found in [61].

Let $f : V(G^U) \rightarrow V(H^U)$ be a mapping such that $f|_{V(G)}$ is a homomorphism from G to H . Then $f|_{V(G)}$ preserves edges, so for each $uv \in E(G)$ we must have $f(u)f(v) \in E(H)$. And so if for each $uv \in E(G)$ it is the case that f maps the uv -superedge of G^U identically onto the $f(u)f(v)$ -superedge of H , then clearly f will be a homomorphism from G^U to H^U .

Now observe that each superedge of G^U is a union of three cycles of length seven. Any homomorphic image of a 7-cycle must either contain a cycle of length less than seven or must be a 7-cycle. However, H^U contains no cycles of length less than seven and each 7-cycle in H^U is contained within a single superedge of H^U . Furthermore, if two 7-cycles of G^U share an edge in common then their images must share an edge in common, and so each superedge E of G^U must map into a single superedge F of H^U . Straightforward verification now shows that a homomorphism from the superedge E to the superedge F must be an isomorphism, and that each endpoint of E must map to the corresponding endpoint of F . The lemma now follows immediately. ■

Corollary 5 *If G and H are digraphs then $G \rightarrow H$ if and only if $G^U \rightarrow H^U$. Furthermore, a homomorphism $f : G^U \rightarrow H^U$ is an edge-surjection if and only if the homomorphism $f|_{V(G)} : G \rightarrow H$ is an edge-surjection.* ■

In many of our results we will want to employ large families of mutually incompatible structures. Our next few lemmas will demonstrate the existence of such families. A straightforward property of oriented cycles which will be used here as well as in later chapters is the following. Recall that $net(C)$ denotes the net length of an oriented cycle C .

Lemma 6 *Let G and H be oriented cycles. Then $G \rightarrow H$ only if $net(H)|net(G)$. If H is a directed cycle then $G \rightarrow H$ if and only if $net(H)|net(G)$.*

The proof of this is straightforward *cf.* [36]. ■

Lemma 7 *Let \mathcal{L} be any language. There exists a countably infinite mutually incompatible family of finite structures over \mathcal{L} .*

Proof: We will first prove the result for digraphs. Denote by D_n a directed cycle of length n . By the previous lemma $D_n \rightarrow D_m$ if and only if $m|n$. Thus, the set $\mathcal{P}_2 = \{D_p : p \text{ prime}\}$ is an infinite mutually incompatible family of digraphs.

Now let \mathcal{L} be an arbitrary language, and let R be an n -ary relation in \mathcal{L} for some $n \geq 2$. Let D_p^n denote the graph obtained by applying construction A to D_p . By lemma 3 the set $\mathcal{P}_n = \{D_p^n : p \text{ prime}\}$ is an infinite mutually incompatible family of structures over \mathcal{L} . ■

It will also sometimes be helpful to have access to a countable mutually incompatible family of graphs. Such a family can be obtained immediately by applying construction B to the family \mathcal{P}_2 . We will construct another such family which is particularly useful.

For each $i \geq 0$ we define a graph G_i to be any graph such that $\text{girth}(G_i) = 5 + 2i$ and $\chi(G_i) = 3 + i$. That such graphs exist is proved in [22, 51, 58]. We define a family of graphs \mathcal{B} by $\mathcal{B} = \{G_i : i \geq 0\}$.

Lemma 8 *The set \mathcal{B} is an infinite mutually incompatible family of graphs.*

Proof: By lemma 1 we know that if $i > j$ then $G_i \not\rightarrow G_j$. The homomorphic image of an odd cycle must contain an odd cycle of lesser or equal size, so if $i < j$ then $G_i \not\rightarrow G_j$. ■

One final construction which we will want to use is given below. This construction will allow us to convert edge-coloured digraphs to digraphs while retaining important properties.

Construction C: Let G be an edge-coloured digraph with at most countably many edge-colours. Denote the edge-colours of G by $\{c_1, c_2, \dots\}$. We define a digraph G^C by replacing each edge uv of colour c_i in G by the digraph in figure 2.2, where p_i is the i^{th} odd prime and, as before, D_{p_i} is a directed cycle of length p_i . We will refer to this as the uv -superedge of colour c_i of G^C .

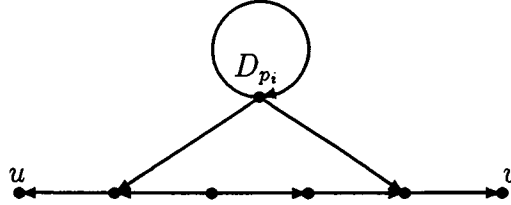


Figure 2.2

As before, we say that a uv -superedge E maps identically onto some xy -superedge F under a mapping f if the restriction of f to E is the unique isomorphism from E to F with $f(u) = x$ and $f(v) = y$.

Lemma 9 *Let G and H be edge-coloured digraphs with countably many edge-colours $\{c_1, c_2, \dots\}$ and let G^C and H^C be obtained by applying construction C to G and H . Then $f : V(G^C) \rightarrow V(H^C)$ is a homomorphism if and only if $f|_{V(G)} : V(G) \rightarrow V(H)$ is a homomorphism and for all $uv \in E(G)$ the uv -superedge of colour c_i of G^C maps identically onto the $f(u)f(v)$ -superedge of colour c_i of H^C .*

Proof: Clearly if f is a mapping from $V(G^C)$ to $V(H^C)$ such that $f|_{V(G)}$ is a homomorphism from G to H and f maps superedges identically onto superedges then f is a homomorphism from G^C to H^C .

Now suppose that $f : V(G^C) \rightarrow V(H^C)$ is a homomorphism. Let E be a superedge of G^C of colour c_i . Then E contains a directed cycle C of length p_i . Under any homomorphism from G^C to H^C the image of C must be a closed directed walk of length p_i . However, the only closed directed walks in H^C are inside the cycles within the superedges of H^C , and traverse such a cycle an integral number of times. Since p_i is prime, the only closed directed walks of length p_i in H^C are the directed cycles D_{p_i} in the superedges of colour c_i in H^C . Thus, the directed cycle D_{p_i} in E must map to the directed cycle D_{c_i} in some superedge F of colour c_i in H^C . It is now straightforward to verify that E must map identically onto F , and so if E is a uv -superedge then F is an $f(u)f(v)$ -superedge. It follows immediately that $f|_{V(G)}$ is a homomorphism from G to H . ■

Corollary 10 *If G and H are edge-coloured digraphs with countably many edge colours then $G \rightarrow H$ if and only if $G^C \rightarrow H^C$. Furthermore, a homomorphism $f : G^C \rightarrow H^C$ is an edge-surjection if and only if the homomorphism $f|_{V(G)} : G \rightarrow H$ is an edge-surjection. ■*

Chapter 3

Core-like Properties of Digraphs

3.1 Definitions

The notion of a core has been used in finite graph theory for some time now. Cores have appeared under various names, such as *unretractive graphs* or *minimal graphs*, and with different definitions [6, 28, 38, 47, 57, 59, 74], all of which turn out to be equivalent. The two most common of these are:

- a core is a digraph such that any endomorphism is a surjection,
- a core is a digraph with no proper retracts.

Other reasonable definitions which are readily seen to be equivalent for finite digraphs are:

- a core is a digraph such that any endomorphism is an injection,
- a core is a digraph such that any endomorphism is an automorphism.

If G is a finite digraph, then a subdigraph H of G is said to be a *core of G* if $G \rightarrow H$ and H is a core. Such an H always exists when G is finite, as we may take successive proper endomorphisms of G until we obtain a digraph that admits no further proper endomorphism. In fact, if G is a finite digraph then all cores of G are isomorphic. To see this, observe that any two cores H_1 and H_2 of G must be

homomorphically equivalent, and so must have the same number of vertices, since if $|H_1| > |H_2|$ then composing a homomorphism from H_1 to H_2 with a homomorphism from H_2 to H_1 would yield a proper endomorphism of H_1 . Now it follows by a similar argument that any homomorphism from H_1 to H_2 must be a bijection. Similarly any homomorphism from H_2 to H_1 must also be a bijection. The existence of a bijective homomorphism from H_1 to H_2 clearly implies that $|E(H_1)| \leq |E(H_2)|$, and similarly we obtain $|E(H_2)| \leq |E(H_1)|$, and so H_1 and H_2 have the same number of edges. Now it becomes clear that a bijective homomorphism from H_1 to H_2 must preserve non-edges, and so it is an isomorphism. The same argument shows that if G and H are homomorphically equivalent then their cores are isomorphic.

When G is allowed to be infinite, the situation deteriorates. For infinite digraphs these definitions are not equivalent. Furthermore, under some definitions G may have several (even infinitely many) non-isomorphic cores, and there exist digraphs which do not have a core at all under any of these definitions. In addition, under certain of these definitions it is possible for digraphs G and H to be homomorphically equivalent, yet for G to have a core and for H not to have a core. In this chapter we will examine these and other core-like properties of infinite digraphs and determine the relationships among them. The last definition, that a core is a digraph all of whose endomorphisms are automorphisms, is the one we will ultimately adopt as our definition of a core. We will show that this definition has several very nice properties. For example, under this definition we will see that homomorphically equivalent cores must be isomorphic, which is not true under any of the other definitions we will examine.

To begin our exploration into these matters we will define some predicates for digraphs. Each of the following predicates is a potential definition of what it means for G to be a core:

Definition 11 *Let G be a digraph.*

- $I(G)$ holds if every endomorphism of G is an injection.
- $S(G)$ holds if every endomorphism of G is a surjection.
- $N(G)$ holds if every endomorphism of G preserves non-relations.

- $\mathbf{R}(G)$ holds if G has no proper retracts.

We also define a corresponding collection of predicates relating to the subdigraphs of a digraph. Each of these should be thought of as being analogous to the property of having a core:

Definition 12 *Let G be a digraph.*

- $\mathbf{i}(G)$ holds if G contains a subdigraph H such that $G \rightarrow H$ and $\mathbf{I}(H)$ holds.
- $\mathbf{s}(G)$ holds if G contains a subdigraph H such that $G \rightarrow H$ and $\mathbf{S}(H)$ holds.
- $\mathbf{n}(G)$ holds if G contains a subdigraph H such that $G \rightarrow H$ and $\mathbf{N}(H)$ holds.
- $\mathbf{r}(G)$ holds if G contains a retract H such that $\mathbf{R}(H)$ holds.

We also define combinations of properties \mathbf{I}, \mathbf{S} and \mathbf{N} (or their lowercase versions). We will write $\mathbf{IS}(G)$ to assert $\mathbf{I}(G)$ and $\mathbf{S}(G)$, and similarly for all of the other uppercase combinations. For the lowercase versions, we will use this notation to denote the assertion that the same subdigraph of G has the required properties, *e.g.* $\mathbf{is}(G)$ indicates that G contains a subdigraph H such that $G \rightarrow H$ and $\mathbf{I}(H)$ and $\mathbf{S}(H)$ hold. We will use $\mathbf{p}, \mathbf{q}, \mathbf{P},$ and \mathbf{Q} to denote arbitrary lowercase and uppercase properties or combinations thereof.

When some lowercase property \mathbf{p} holds for a digraph G , we will say that a subdigraph H of G , such that $G \rightarrow H$ and for which \mathbf{P} holds is a *certificate* of $\mathbf{p}(G)$, or that H *certifies* $\mathbf{p}(G)$.

3.2 Relationships Among Properties

We shall describe the logical relationships among these properties and combinations of properties. We first present some infinite digraphs with useful properties. First we define the Ray as in figure 3.1.



Figure 3.1: The Ray

Lemma 13 *Every endomorphism of the Ray is of the form $f(v_n) = v_{n+k}$, for some fixed $k \geq 0$. Furthermore, for each $k \geq 0$ the mapping $f(v_n) = v_{n+k}$ is an endomorphism of the Ray.*

Proof: If $f(v_0) = v_k$, then clearly $f(v_1) = v_{k+1}$, $f(v_2) = v_{k+2}$, etc., since the directions of arcs must be preserved. Clearly the given mapping is an endomorphism of the Ray for each $k \geq 0$. ■

Corollary 14 *The Ray satisfies the properties **IN**, **R**, **in**, and **r** but not **S** or **s**.* ■

We define the Line similarly:

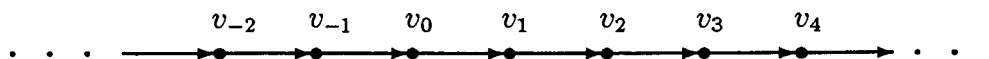


Figure 3.2: The Line

Lemma 15 *Every endomorphism of the Line is of the form $f(v_n) = v_{n+k}$, for some fixed $k \in \mathbf{Z}$. Also, for each $k \in \mathbf{Z}$ the mapping $f(v_n) = v_{n+k}$ is an endomorphism of the Line.*

Proof: Again, the homomorphism is uniquely determined by $f(v_0)$. It is also clear that any mapping of the given form is an endomorphism. ■

Corollary 16 *The Line satisfies all of the properties **INS**, **R**, **isn**, and **r**.* ■

These two digraphs are particularly useful in constructions since if a digraph G contains a Ray (Line) but G contains no directed cycle, then the image of the Ray (Line) must be a Ray (Line), although not necessarily an induced one. Our subsequent examples all exploit this fact.

The 5-3-Line is defined in Figure 3.3.

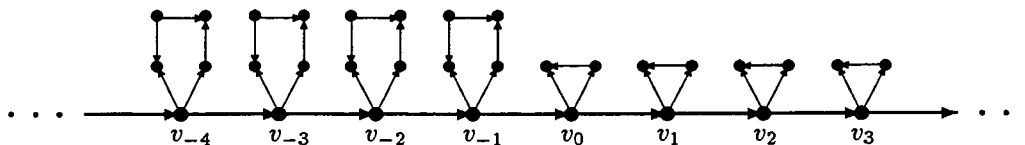


Figure 3.3: The 5-3-Line

Let $A(v_i)$ denote the oriented 5-cycle or 3-cycle that contains v_i in the 5-3-Line shown in figure 3.3.

Lemma 17 *Every endomorphism f of the 5-3-Line has the following properties:*

- $f(v_n) = v_{n+k}$, for some fixed $k \geq 0$,
- f maps $A(v_n)$ onto $A(f(v_n))$.

Furthermore, for each $k \geq 0$ there exists an endomorphism of the 5-3-Line of this form.

Proof: The 5-3-Line contains exactly one subdigraph isomorphic to the Line and no directed cycles. Thus, this copy of the Line must map to itself. However, if $f(v_0) = v_k$, where $k < 0$, then clearly $A(v_0)$ will have no possible image, so $k \geq 0$. Since the image of an odd oriented cycle must be an odd oriented closed walk, it follows that $A(v_n)$ must map onto $A(f(v_n))$. ■

Corollary 18 *The 5-3-Line satisfies the properties S , R , s and r , but none of I , N , i , or n .* ■

We now define the 5-Line as follows:

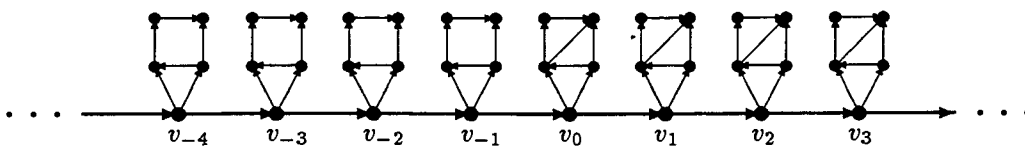


Figure 3.4: The 5-line

Let $A(v_i)$ denote the five-vertex subdigraph of the 5-line containing v_i .

Lemma 19 *Any endomorphism f of the 5-line has the following properties:*

- $f(v_n) = v_{n+k}$, for some fixed $k \geq 0$,
- $A(v_i)$ maps onto $A(f(v_i))$.

Furthermore, for each $k \geq 0$ there exists such an endomorphism of the 5-Line.

Proof: As above, the unique copy of the Line contained in the 5-Line must map to itself. However, $A(v_0)$ has no possible image if $f(v_0) = v_k$ for any $k < 0$. Again it is a simple matter to verify that f must map $A(v_i)$ onto $A(f(v_i))$. ■

Corollary 20 *The 5-Line satisfies the properties **IS**, **R**, **is** and **r**, but not **N** or **n**.*

■

Finally, we define the Onestep Line:

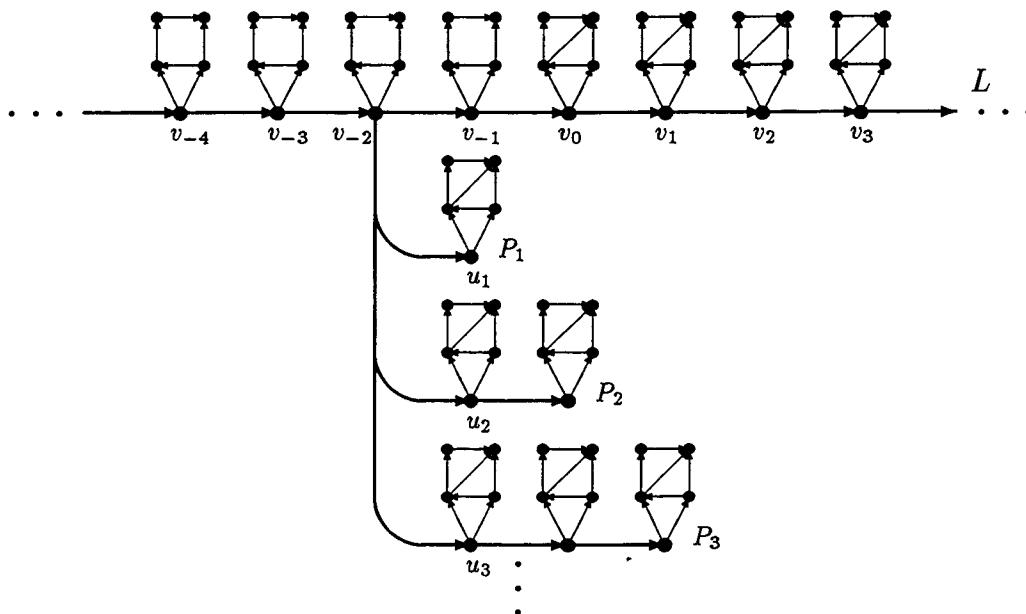


Figure 3.5: The Onestep Line

Let L denote the unique copy of the 5-Line contained in the Onestep Line, and let P_i denote the directed path on i vertices beginning at u_i , together with its associated 5-vertex subdigraphs. Also if $w = v_i$ or $w = u_i$ for some i , let $A(w)$ denote the 5-vertex subdigraph containing w .

Lemma 21 *Any endomorphism f of the Onestep Line has the following properties:*

- $f(v_n) = v_{n+k}$, $k \geq 0$,
- if $f(v_0) = v_0$ then for each $i \geq 1$, $f(P_i) \subseteq P_j$ for some $j \geq i$,
- if $f(v_0) \neq v_0$ then $f(P_i) \subseteq L$ for all $i \geq 1$.

Also, for each $k \geq 0$ there is an endomorphism of the Onestep Line satisfying these conditions.

Proof: This digraph also contains a unique Line and no directed cycles, so the Line must map to itself. The subdigraphs $A(v_i)$ again force v_0 to map to some v_k with $k \geq 0$. Suppose some endomorphism f has $f(v_0) = v_0$. Then $f|_L$ must be the identity map. The mapping f cannot map the paths P_i into the line, since then $f(u_i) = v_{-1}$, and so $A(u_i)$ would map to $A(v_{-1})$, which cannot occur. Straightforward verification now shows that f cannot map P_i to any P_j where $j < i$, so each P_i must map into some $P_j, j \geq i$.

On the other hand, if $f(v_0) = v_k$ for some $k > 0$, then $f(v_{-2}) = v_j$ for some $j \geq -1$. Thus for all $i \geq 1$, $f(u_i) = v_{j+1}$, and so all of the P_i are clearly forced to map into L . ■

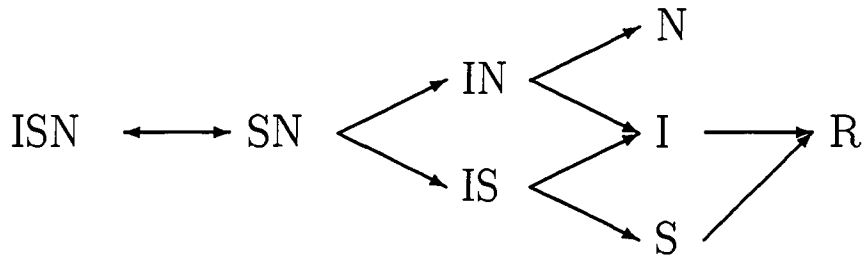
Corollary 22 *The Onestep Line satisfies the property is , but none of I, S, N, R, n or r .* ■

We say that a property P implies a property Q if whenever $P(G)$ holds for a digraph G , then $Q(G)$ holds for G as well. Our next theorem describes all the relationships that hold among the uppercase properties we have defined. We first prove a small but useful lemma. Recall the equivalence relation \equiv defined on $V(G)$ by $u \equiv v$ whenever $N^+(u) = N^+(v)$ and $N^-(u) = N^-(v)$.

Lemma 23 *Let G be a digraph satisfying $N(G)$, and let f be an endomorphism of G . If $f(u) = f(v)$ for some $u, v \in V(G)$, then $u \equiv v$.*

Proof: Suppose $u \not\equiv v$ and in particular that $N^+(u) \neq N^+(v)$ (the case where $N^-(u) \neq N^-(v)$ is similar). We may assume without loss of generality that there is a vertex $w \in V(G)$ such that $uw \in E(G)$ but $vw \notin E(G)$. Then $f(u)f(w) \in E(G)$ since f is a homomorphism, but $f(v)f(w) \notin E(G)$ because f preserves non-edges. Therefore $f(u) \neq f(v)$. ■

Theorem 24 *The following implications, and no others (except those implied by transitivity), hold among the properties I , S , N , R and combinations thereof.*



Proof: Clearly $ISN(G)$ implies $SN(G)$. Suppose $SN(G)$ holds. Then $N(G)$ holds, and so by lemma 23 under any endomorphism f of G , $f(u) = f(v)$ implies that $u \equiv v$. But if $u \neq v$ and $u \equiv v$ then we may define an endomorphism g of G by $g(u) = g(v) = v$, and $g(w) = w$ for $w \neq u, v$, which is not a surjection. Thus, f must be an injection. Therefore, $ISN(G)$ holds, and so $SN(G)$ also implies $IN(G)$ and $IS(G)$.

Suppose $IN(G)$ holds. Clearly $I(G)$ and $N(G)$ hold. However, if G is the Ray then $IN(G)$ holds but $S(G)$ is false, and so $IN(G)$ implies none of $S(G)$, $IS(G)$, or $SN(G)$.

Suppose $IS(G)$ holds. Clearly $I(G)$ and $S(G)$ hold. However, if G is the 5-Line then $IS(G)$ holds but $N(G)$ is false, and so $IS(G)$ implies none of $N(G)$, $IN(G)$, or $SN(G)$.

Let G be defined by $V(G) = \{u, v, w\}, E(G) = \{uv, uw\}$. Then $\mathbf{N}(G)$ holds but $\mathbf{I}(G)$, $\mathbf{S}(G)$, and $\mathbf{R}(G)$ do not. Thus, $\mathbf{N}(G)$ implies no other uppercase property.

Suppose $\mathbf{I}(G)$ holds. If $\mathbf{R}(G)$ does not hold then G has a proper retract H . Let $f : G \rightarrow H$ be a retraction, and let v be a vertex of G not in $V(H)$, so obviously $f(v) \neq v$. But $f(v) \in V(H)$ and so $f(f(v)) = f(v)$ since f is a retraction. But then f is not an injection, contradicting $\mathbf{I}(G)$. Thus, $\mathbf{R}(G)$ must hold. If G is the Ray, then $\mathbf{I}(G)$ holds but $\mathbf{S}(G)$ is false. Also, if G is the 5-line, then $\mathbf{I}(G)$ holds but $\mathbf{N}(G)$ is false, and so $\mathbf{I}(G)$ implies only $\mathbf{R}(G)$.

Suppose $\mathbf{S}(G)$ holds. Then $\mathbf{R}(G)$ must hold since a proper retraction is not a surjection. If G is the 5-3-line then $\mathbf{S}(G)$ holds but $\mathbf{I}(G)$ is false. If G is the 5-line then $\mathbf{S}(G)$ holds but $\mathbf{N}(G)$ is false, and so $\mathbf{S}(G)$ implies only $\mathbf{R}(G)$.

If G is the 5-3-Line then $\mathbf{R}(G)$ holds but $\mathbf{I}(G)$ and $\mathbf{N}(G)$ are false. If G is the Ray then $\mathbf{R}(G)$ holds but $\mathbf{S}(G)$ is false. Thus $\mathbf{R}(G)$ implies no other uppercase property.

■

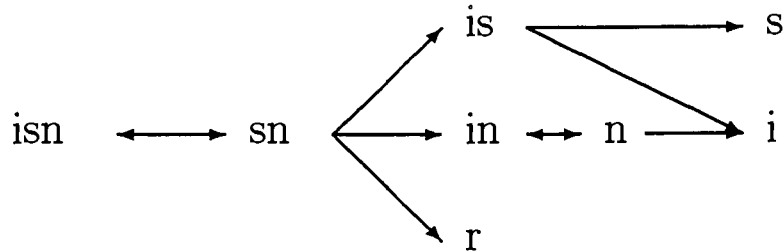
Let \mathbf{P} and \mathbf{Q} be arbitrary uppercase properties other than \mathbf{R} , or combinations thereof. Let \mathbf{p} and \mathbf{q} be the corresponding lowercase properties. The following lemma will be useful in subsequent proofs.

Lemma 25 *If $\mathbf{P}(G)$ implies $\mathbf{q}(G)$ then $\mathbf{p}(G)$ implies $\mathbf{q}(G)$.*

Proof: If $\mathbf{p}(G)$ holds then let H be a certificate of $\mathbf{p}(G)$. By assumption $\mathbf{q}(H)$ holds and so H contains a subdigraph K which certifies $\mathbf{q}(H)$. However, K also certifies $\mathbf{q}(G)$, so $\mathbf{q}(G)$ holds. ■

Note that since $\mathbf{Q}(G)$ trivially implies $\mathbf{q}(G)$, this lemma proves that any implication that holds for uppercase properties other than \mathbf{R} must also hold for the corresponding lowercase properties.

Theorem 26 *The following implications, and no others (except those implied by transitivity), hold among the properties i , s , n , r and combinations thereof.*



Proof: As mentioned above, lemma 25 shows that all implications among properties other than R in theorem 24 must also be true for the lowercase versions of those properties. We shall first verify that no other implications are true except those indicated above, and then we will check the implications involving the property r .

The Ray shows that in does not imply s , and so in does not imply sn either. The 5-line shows that is does not imply n or sn . The 5-3-Line shows that s does not imply i or n , or any property involving i or n . The Ray also shows that i does not imply s , and the 5-Line shows that i does not imply n , so i implies no combination of properties either. The Ray shows that n does not imply s .

To see that n implies in , we note that by lemma 25 it is sufficient to show that N implies in . Suppose $N(G)$ is true. As in the proof of theorem 24, we use the equivalence relation \equiv defined on the vertices of G . Let $[w]$ denote the equivalence class of a vertex w , and let $v_{[w]}$ be an fixed representative of $[w]$. Define a mapping $f : V(G) \rightarrow V(G)$, by $f(u) = v_{[u]}$. Straightforward verification shows that this is an endomorphism, and clearly $f(v_{[u]}) = v_{[u]}$, so f is a retraction. Also note that the retract $f(G)$ has the property that no two vertices of $f(G)$ are equivalent, and that $N(f(G))$ holds. Also, $I(f(G))$ holds, since any endomorphism that is not an injection would map two inequivalent vertices to the same image, contradicting lemma 23. Clearly $f(G) \subseteq G$ and $G \rightarrow f(G)$, so we conclude that $in(G)$ holds.

Now suppose $sn(G)$ holds. Then G has a subdigraph H such that $G \rightarrow H$ and $SN(H)$ holds. By theorem 24 we know that $ISN(H)$ holds as well, and so any

endomorphism of H is an automorphism. Thus, if $f : G \rightarrow H$ is a homomorphism, then $f|_H$ is an automorphism of H . If we let g be the inverse of $f|_H$, then $g \circ f$ is a homomorphism from G to H and $(g \circ f)|_H$ is the identity map, so $g \circ f$ is a retraction and H is a retract of G . Of course, H has no proper retracts because $\mathbf{S}(H)$ holds, and so we conclude that $\mathbf{r}(G)$ holds.

If G is the Onestep Line and H is the 5-Line, then H is a subdigraph of G , $G \rightarrow H$ and $\mathbf{IS}(H)$ holds. Thus $\mathbf{is}(G)$ holds. On the other hand, $\mathbf{r}(G)$ is false. If G is the subdigraph of the Onestep Line obtained by deleting all $A(v_i)$ with $i < -2$, and H is the subdigraph of G induced by $\cup_{i \geq 0} A(v_i)$, then $G \rightarrow H$ and $\mathbf{IN}(H)$ holds. Hence, $\mathbf{in}(G)$ holds. However, $\mathbf{r}(G)$ is false for the same reason that \mathbf{r} is false for the Onestep Line.

If G is the 5-3-Line then $\mathbf{r}(G)$ holds but $\mathbf{i}(G)$ and $\mathbf{n}(G)$ are false. If G is the Ray then $\mathbf{r}(G)$ holds but $\mathbf{s}(G)$ is false, so \mathbf{r} implies no other property. ■

Theorem 27 *The following implications, those implied by transitivity and theorems 24 and 26, and no others, hold between the uppercase and lowercase properties.*

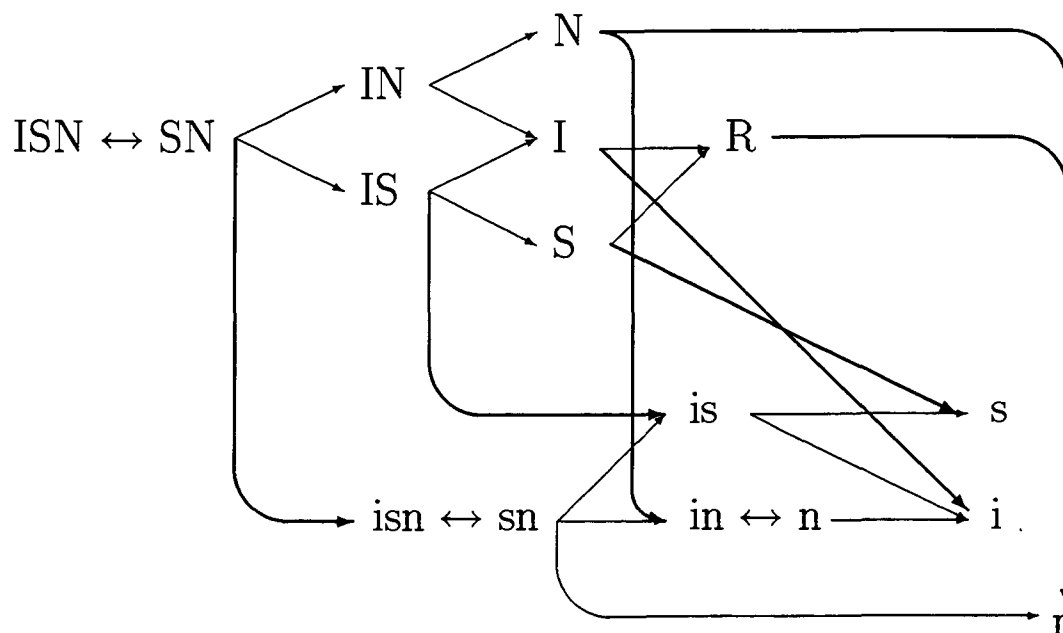
- $\mathbf{SN}(G) \rightarrow \mathbf{isn}(G)$,
- $\mathbf{IS}(G) \rightarrow \mathbf{is}(G)$,
- $\mathbf{S}(G) \rightarrow \mathbf{s}(G)$,
- $\mathbf{I}(G) \rightarrow \mathbf{i}(G)$,
- $\mathbf{N}(G) \rightarrow \mathbf{in}(G)$,
- $\mathbf{N}(G) \rightarrow \mathbf{r}(G)$,
- $\mathbf{R}(G) \rightarrow \mathbf{r}(G)$,

Proof: We first observe that no lowercase property can imply any uppercase property, for if we define a digraph G by $V(G) = \{u, v, w\}$ and $E(G) = \{uv\}$, then obviously every lowercase property holds for G , but G satisfies no uppercase property. On the other hand, any uppercase property trivially implies its lowercase version. Also, if \mathbf{P}

and Q are arbitrary properties other than R , and p and q are their lowercase versions, then lemma 25 implies that if p does not imply q , then P does not imply q . Thus, we need only verify that N implies r to complete the proof.

Recall from the proof of theorem 26 that if $N(G)$ holds, then G has a retract H such that no two vertices of H are equivalent under \equiv , and $I(H)$ holds. But $I(H)$ implies $R(H)$ by theorem 24, and so $r(G)$ holds. ■

All of the above results are summarized in the following diagram.



Having established the relationships among the various core-like properties we have defined, we will next examine certain characteristics of the individual properties. In the study of homomorphisms of digraphs the notion of homomorphic equivalence is very natural, and we would hope that digraphs which are homomorphically equivalent might share many of the same properties. Our next two theorems demonstrate the extent to which the core-like properties we have defined are invariant over homomorphic equivalence classes.

We say that a lowercase property p is *invariant* if whenever G and H are homomorphically equivalent digraphs and $p(G)$ holds, then $p(H)$ holds as well. It is a trivial observation that no uppercase property is invariant in the corresponding sense, as the reader may observe that any uppercase property may be destroyed simply by adding an isolated vertex to a digraph. Theorem 28 describes the situation for lowercase properties.

Theorem 28 *The properties i , n , in , is , sn and isn are invariant. The properties s and r are not.*

Proof: Let G and H be homomorphically equivalent digraphs, and suppose $i(G)$ holds. Let G' be a certificate for $i(G)$. Then $G' \subseteq G$, so $G' \rightarrow H$. Let $f : G' \rightarrow H$ be a homomorphism. Observe that G' , $f(G')$, G and H are all homomorphically equivalent. So let $g : f(G') \rightarrow G'$ be a homomorphism. Now suppose there is a homomorphism $h : f(G') \rightarrow f(G')$ which is not an injection. We claim that $(g \circ h \circ f) : G' \rightarrow G'$ is not an injection. Since h is not an injection, there exist distinct $u, v \in V(f(G'))$ such that $h(u) = h(v)$. But $V(f(G')) = \text{range}(f)$, so u and v have pre-images in G' under f . Thus, there exist distinct $x, y \in V(G')$ such that $(h \circ f)(x) = (h \circ f)(y)$, and so $(g \circ h \circ f)(x) = (g \circ h \circ f)(y)$. However, this contradicts $I(G')$, and so we conclude that $I(f(G'))$ holds, and so $i(H)$ holds.

The proof that n is invariant is similar to the above: Let G and H be homomorphically equivalent digraphs, and suppose $n(G)$ holds. Let G' be a certificate for $n(G)$, and let $f : G' \rightarrow H$ be a homomorphism. Observe again that G' , $f(G')$, G and H are all homomorphically equivalent, and let $g : f(G') \rightarrow G'$ be a homomorphism. Now suppose there is a homomorphism $h : f(G') \rightarrow f(G')$ which does not preserve non-edges. Since h does not preserve non-edges, there exist $u, v \in V(f(G'))$ such that $uv \notin E(f(G'))$ but $h(u)h(v) \in E(f(G'))$. But again u and v have pre-images in G' under f , and so there exist $x, y \in V(G')$ such that $xy \notin E(G')$ but $(h \circ f)(x)(h \circ f)(y) \in E(f(G'))$, and so $(g \circ h \circ f)(x)(g \circ h \circ f)(y) \in E(G')$. This contradicts $N(G')$, and so we conclude that $N(f(G'))$ holds, and so $n(H)$ holds.

The fact that in is invariant follows from the above facts: Let G and H be homomorphically equivalent digraphs, and suppose $in(G)$ holds. Let G' be a certificate for

$\mathbf{in}(G)$, and let $f : G' \rightarrow H$ be a homomorphism. The digraph G' is a certificate for $\mathbf{I}(G)$, and so $\mathbf{I}(f(G'))$ holds exactly as in the proof of the invariance of \mathbf{i} . Furthermore, G' is a certificate for $\mathbf{N}(G)$, and so $\mathbf{N}(f(G'))$ holds exactly as in the proof of the invariance of \mathbf{n} . Therefore, $\mathbf{IN}(f(G'))$ holds, and so $\mathbf{in}(H)$ holds.

Now let G and H be equivalent, and now suppose $\mathbf{is}(G)$ holds. Let G' be a certificate for $\mathbf{is}(G)$. Let $f : G' \rightarrow H$ and $g : f(G') \rightarrow G'$ be homomorphisms. Also, let h be any endomorphism of $f(G')$. Observe that we have defined the domain of g to be the range of f , so f and g must both be injections, for otherwise the homomorphism $(g \circ f) : G' \rightarrow G'$ would fail to be an injection. Similarly, h must be an injection, for otherwise the homomorphism $(g \circ h \circ f) : G' \rightarrow G'$ would not be an injection.

We claim that h must be a bijection. If h is not a surjection, then there is a vertex $u \in f(G')$ with no pre-image in $f(G')$ under h . In other words, $u \notin \text{range}(h \circ f)$. However, the homomorphism g is an injection, so u is the unique pre-image of $g(u)$ in $f(G')$ under g , and so $g(u) \notin \text{range}(g \circ h \circ f)(G')$. But now $(g \circ h \circ f) : G' \rightarrow G'$ is not a surjection, contradicting $\mathbf{IS}(G')$. Thus h must be a bijection. We may therefore conclude that $\mathbf{IS}(f(G'))$ holds, and so $\mathbf{is}(H)$ holds.

Recall that $\mathbf{sn}(G)$ and $\mathbf{isn}(G)$ are logically equivalent, so to show that both are invariant it suffices to show that $\mathbf{isn}(G)$ is invariant.

Let G and H be equivalent, and suppose $\mathbf{isn}(G)$ holds. Let G' be a certificate for $\mathbf{isn}(G)$. In particular $\mathbf{IS}(G')$ and $\mathbf{N}(G')$ hold. Let $f : G' \rightarrow H$ and let $h : f(G') \rightarrow f(G')$. As in the proof for \mathbf{is} , we must have $\mathbf{IS}(f(G'))$. Also, we must have $\mathbf{N}(f(G'))$ as in the proof for \mathbf{n} . Therefore, $\mathbf{ISN}(f(G'))$ holds, so $\mathbf{isn}(H)$ holds.

To see that \mathbf{r} is not invariant, let G be the Onestep Line and let H be the 5-Line. Then G and H are homomorphically equivalent but $\mathbf{r}(H)$ holds and $\mathbf{r}(G)$ is false.

The counterexample for \mathbf{s} is more complex. We begin with some definitions. By \mathbf{P}^* we mean all finite strings of non-negative integers, and by \mathbf{P}^n all such strings of length n . We denote the empty string by ϵ . The length of a string α is denoted by $|\alpha|$, and we will always assume $\alpha = \alpha_0, \dots, \alpha_{|\alpha|-1}$. If $|\alpha| = 1$ then we will sometimes represent α by a numeric variable such as i . By $\alpha\beta$ we mean the concatenation of the strings α and β . If $\alpha = \beta\gamma$ for some string γ we say that β is a *prefix* of α . In this

case if $\gamma \neq \epsilon$ then β is a *proper prefix* of α . We say that a string α is *lexicographically less than* a string β and write $\alpha < \beta$ if either α is a proper prefix of β ; or β is not a prefix of α and $\alpha_i < \beta_i$, where i is the least index such that $\alpha_i \neq \beta_i$.

Let $A = \{(i, j) : i, j \geq 0\}$. For each $\alpha \in \mathbf{P}^*$ we define a set $S_\alpha \subseteq A$ and a sequence T_α of positive integers as follows:

- $S_\alpha = \{(i, j) : 0 \leq i < |\alpha|, 0 \leq j \leq \alpha_i; \text{ or } i \geq |\alpha|, j \geq 0\}$
- $T_\alpha = (t_\alpha(i) : i \geq 0)$, where $t_\alpha(i) = 3 + 2 \sum_{j=0}^i \alpha_j$, where α_j is considered to be zero for $j \geq |\alpha|$.

We will write $T_\alpha \leq T_\beta$ if $t_\alpha(i) \leq t_\beta(i)$ for each i .

Example: $S_{201} = \{ (0, 0), (0, 1), (0, 2),$
 $(1, 0),$
 $(2, 0), (2, 1),$
 $(3, 0), (3, 1), (3, 2), (3, 3), \dots$
 $(4, 0), (4, 1), (4, 2), (4, 3), \dots$
 $\dots \quad \quad \quad \}$

$$T_{201} = (7, 7, 9, 9, 9, \dots)$$

Claim 1 *The sets S_α have the following properties:*

- if β is a prefix of α then $S_\alpha \subseteq S_\beta$,
- for any fixed $n \geq 1$, we have $\bigcup_{\gamma \in \mathbf{P}^n} S_{\alpha\gamma} = S_\alpha$,
- if β is a proper prefix of α then $S_\beta \not\subseteq S_\alpha$.
- if α is not a proper prefix of β and $\alpha < \beta$, then $S_\beta \not\subseteq S_\alpha$.

Proof of Claim: The first part of the claim is an immediate consequence of the definition of S_α . It is clear from the definition of S_α that $\bigcup_{i=0}^{\infty} S_{\alpha_i} = S_\alpha$. Induction on n proves the second part of the claim. For the third part, if β is a proper prefix of α then $|\alpha| > |\beta|$ so let $n = |\beta|$. Then $(n+1, \alpha_{n+1}+1) \in S_\beta$ but $(n+1, \alpha_{n+1}+1) \notin S_\alpha$. To prove the last part, observe that if $\alpha < \beta$ and α is not a prefix of β , then there exists an $i \geq 0$ with $i \leq |\alpha|$ and $i \leq |\beta|$, and $\alpha_i < \beta_i$. Hence, $(i, \beta_i) \in S_\beta$ but $(i, \beta_i) \notin S_\alpha$. \square

Claim 2 *The sequences T_α have the following properties:*

- if β is a prefix of α then $T_\beta \leq T_\alpha$.
- if $\alpha < \beta$ and $\beta \neq \alpha 0^n$ for any $n \geq 1$ then $T_\alpha \not\leq T_\beta$.

Proof of Claim: The first part follows immediately from the definition of T_α . Now suppose $\alpha < \beta$ and $\beta \neq \alpha 0^n$. If α is a proper prefix of β then there exists an $i \geq 0$ such that $i > |\alpha|$ and $\beta_i \neq 0$. Then clearly $t_\alpha(i) < t_\beta(i)$. If α is not a proper prefix of β then let i be the least index such that $\alpha_i \neq \beta_i$, and so $\alpha_i < \beta_i$. Again we have $t_\alpha(i) < t_\beta(i)$, and so $T_\alpha \not\leq T_\beta$. \square

We will now construct a digraph G such that $\mathcal{S}(G)$ holds. Our construction will be greatly simplified by taking G to be an edge-coloured digraph. We will subsequently apply lemma 9 to show that G can be transformed into an uncoloured digraph.

Our construction will require countably many edge-colours. We will have one colour for each element of the set A defined above, another colour for each $i \geq 0$, and a special colour called c . The colours for the elements $(i, j) \in A$ and the non-negative integers i will simply be called (i, j) and i , respectively.

For each $\alpha \in \mathbf{P}^*$, we define an edge-coloured digraph P_α . We begin the construction of P_α with three vertices $\{x_\alpha, y_\alpha, z_\alpha\}$. Now for each $i \geq 0$ we add a bidirected path of colour i and length $t_\alpha(i)$ with endpoints x_α and y_α . Finally, we add an edge from y_α to z_α of colour (i, j) for each $(i, j) \in S_\alpha$.

Example: P_{201}

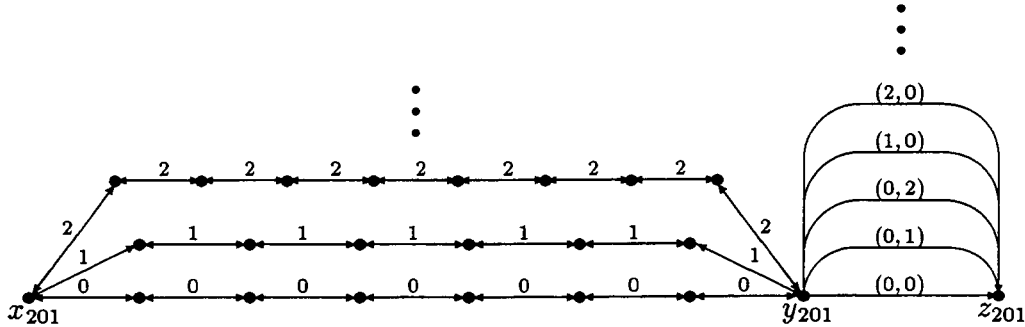


Figure 3.6

Claim 3 For each α and β we have $P_\alpha \rightarrow P_\beta$ if and only if $S_\alpha \subseteq S_\beta$ and $T_\alpha \geq T_\beta$.

Proof of Claim: Suppose there is a homomorphism $f : P_\alpha \rightarrow P_\beta$. In both P_α and P_β , the vertices y_α and y_β are the only vertices which are incident to edges of colour i for each $i \geq 0$ and edges of colour (i, j) for some nonempty set of pairs (i, j) . Therefore, it must be the case that $f(y_\alpha) = y_\beta$. Now the vertex x_α is incident with edges of each colour $i \geq 0$, but $f(x_\alpha) \neq y_\beta$ because this would force P_β to contain an odd-length directed cycle in each colour i with $i \geq 0$, but no such cycle exists. Thus, $f(x_\alpha) = x_\beta$, since x_β is the only other vertex in P_β which is incident to edges of each colour i with $i \geq 0$. Similarly we see that $f(z_\alpha) = z_\beta$.

Now suppose $(i, j) \in S_\alpha$. Then $y_\alpha z_\alpha$ is an (i, j) -edge of P_α , and so the pair $f(y_\alpha)f(z_\alpha)$, i.e. the pair $y_\beta z_\beta$, must be an (i, j) -edge of P_β . And so $(i, j) \in S_\beta$, and therefore $S_\alpha \subseteq S_\beta$.

If $t_\alpha(i) = k$ then P_α contains a bidirected path of length k and of colour i with endpoints x_α and y_α . The image of this path in P_β must contain a bidirected path of colour i and length no greater than k with endpoints x_β and y_β . But this exists only if $t_\beta(i) \leq k$. Thus, $T_\alpha \geq T_\beta$.

On the other hand, if we know that $S_\alpha \subseteq S_\beta$ and $T_\alpha \geq T_\beta$, then in light of the above arguments there is an obvious homomorphism from P_α to P_β , and so our claim is proved. \square

It is clear from the above proof that if $f : P_\alpha \rightarrow P_\beta$ is a homomorphism then f is a surjection, each edge of P_β of any colour i with $i \geq 0$ has a pre-image in P_α , and each edge of colour (i, j) of P_β has a pre-image in P_α exactly when $(i, j) \in S_\alpha$.

We now define G by taking a Line with vertices v_i for $i \in \mathbf{Z}$, and where $v_i v_{i-1}$ is an edge of color c for each $i \in \mathbf{Z}$. We adjoin digraphs P_α to this Line as follows:

- for each $v_i, i \leq 0$ in the Line, add a copy of P_c and an edge $v_i x_c$ of colour c ,
- For each $v_i, i > 0$ in the Line, add a copy of each $P_\alpha, |\alpha| = i$, and an edge $v_i x_\alpha$ of colour c for each P_α .

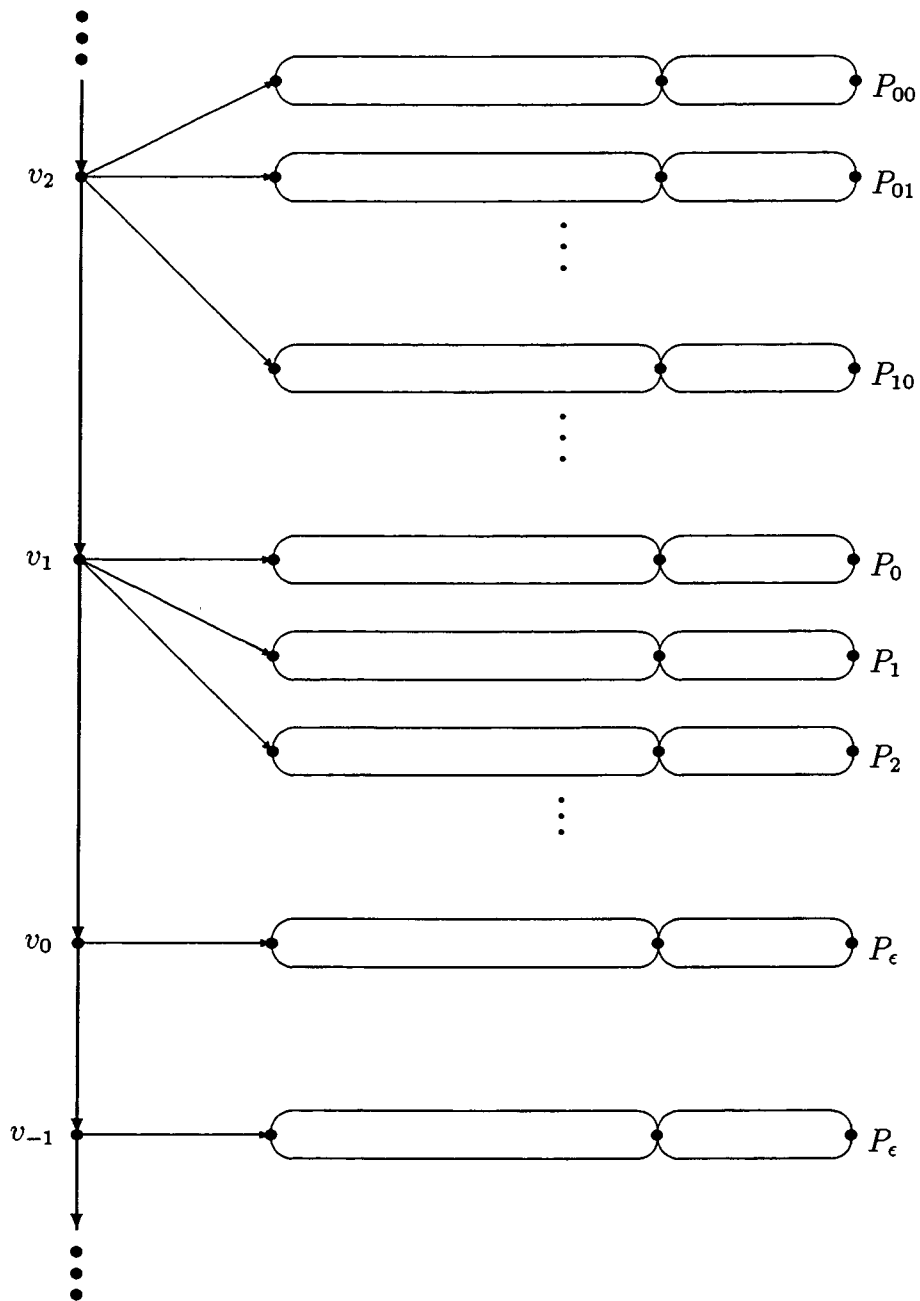


Figure 3.7

Claim 4 *The edge-coloured digraph G has the property $\mathcal{S}(G)$. Furthermore, any endomorphism of G is an edge-surjection.*

Proof of Claim: The Line of colour c in G must map to itself since it is the only Line of colour c in G , so if f is an endomorphism of G then $f(v_i) = v_{i+k}$ for all i and some fixed $k \in \mathbb{Z}$. Furthermore, since each P_α is joined to the Line by a single edge of colour c , each P_α joined to some v_i must map into a single P_β joined to v_{i+k} .

No endomorphism of G can map v_0 to any v_i with $i > 0$, since by claim 1, $S_\epsilon \not\subseteq S_\alpha$ for any $\alpha \neq \epsilon$, so by claim 3 we know $P_\epsilon \not\rightarrow P_\alpha$ when $\alpha \neq \epsilon$.

If v_0 maps to itself, then each P_α must map to a P_β with $|\alpha| = |\beta|$. If $\alpha \neq \beta$, then assume without loss of generality that $\alpha < \beta$ lexicographically. Then $S_\beta \not\subseteq S_\alpha$ by claim 1, and so $P_\beta \not\rightarrow P_\alpha$ by claim 3. Also $T_\alpha \not\subseteq T_\beta$ by claim 2, and so $P_\alpha \not\rightarrow P_\beta$ by claim 3. Thus, each P_α must map onto itself, and in fact the endomorphism f must be the identity.

Finally, suppose v_0 maps to some v_{-i} with $i > 0$. Then each vertex v_k in the Line must map to v_{k-i} . We now show that f is an edge-surjection. Let k be any integer and let P_α be any subdigraph joined to v_k . Observe that if some subdigraph P_β is mapped into P_α by f , then each edge of P_α of any colour $j \geq 0$ will have a pre-image in P_β under f . This follows immediately from the fact that f must map the endpoints of the bidirected path of colour j in P_β to the endpoints of the bidirected path of colour j in P_α , as shown in the proof of claim 3. Thus, to show that f is an edge-surjection, it suffices to show that for each $(x, y) \in S_\alpha$, there is a P_β joined to v_{k+i} such that f maps P_β into P_α and $(x, y) \in S_\beta$. Since all of the sets S_α are non-empty, the preceding statement implies that for each P_α there is a P_β which is mapped onto P_α by f .

Let (x, y) be an element of S_α .

Suppose $k \leq 0$, so $\alpha = \epsilon$. If $k + i \leq 0$ then there is also a copy of P_ϵ joined to v_{k+i} which must map onto the copy of P_ϵ joined to v_k . By definition $(x, y) \in S_\epsilon$. If $k + i > 0$ then all of the digraphs P_β joined to v_{k+i} must map onto the copy of P_ϵ joined to v_k . By claim 1 we know that $\bigcup_{\gamma \in \mathbf{P}^{k+i}} S_\gamma = S_\epsilon$, and so there is some P_β joined to v_{k+i} which is mapped to P_α by f and which has $(x, y) \in S_\beta$.

Suppose $k > 0$, so $|\alpha| = k$. Let β be any string of length $k + i$ such that α is a prefix of β . Then there is a copy of P_β joined to v_{k+i} , and this copy of P_β must map onto some P_γ joined to v_k . However, $|\gamma| = |\alpha|$, so if $\gamma \neq \alpha$ then γ is not a prefix of β , and so there exists a least $i \geq 0$ with $i \leq |\beta|$ and $i \leq |\gamma|$, and $\beta_i \neq \gamma_i$. If $\beta_i < \gamma_i$

then $\beta < \gamma$, and β is not a prefix of γ so by lemma 1 we know $S_\beta \not\subseteq S_\gamma$. If $\gamma_i < \beta_i$ then $\gamma < \beta$ and $\beta \neq \gamma 0^n$ for any $n \geq 1$ so $T_\gamma \not\geq T_\beta$. In either case $P_\beta \not\rightarrow P_\gamma$ by claim 3. Thus f must map P_β onto P_α .

Now by claim 1 we know that $\bigcup_{\gamma \in \mathbf{P}^{k+i}} S_{\alpha\gamma} = S_\alpha$, and so for any $(i, j) \in S_\alpha$ there is a P_β joined to v_{k+i} such that $(i, j) \in S_\beta$ and P_β is mapped onto P_α by f .

In every possible case f is an edge-surjection, and so our claim is proved. \square

We will now define another edge-coloured digraph H which is equivalent to G , but for which $\mathfrak{s}(H)$ is false. For each $\alpha \in \mathbf{P}^*$ we define an infinite sequence $T'_\alpha = (t'_\alpha(i) : i \geq 0)$, defined by $T'_{\alpha_0 \dots \alpha_{|\alpha|-1}} = T_{\alpha_0 \dots \alpha_{|\alpha|-2}}$. We now construct a digraph P'_α for each α exactly as we constructed P_α , except that the length of the bidirected path of colour i in P'_α is given by $t'_\alpha(i)$ rather than $t_\alpha(i)$. Also, we will rename the vertices $x_\alpha, y_\alpha, z_\alpha$ as $x'_\alpha, y'_\alpha, z'_\alpha$. We now obtain H by replacing each copy of some P_α in G by a copy of P'_α .

Claim 5 *The edge-coloured digraph H is equivalent to G , and $\mathfrak{s}(H)$ is false.*

Proof of Claim: We first observe that, as in claim 3, for any α and β we have $P'_\alpha \rightarrow P'_\beta$ if and only if $S_\alpha \subseteq S_\beta$ and $T'_\alpha \geq T'_\beta$. Furthermore, $P'_\alpha \rightarrow P_\beta$ if and only if $S_\alpha \subseteq S_\beta$ and $T'_\alpha \geq T_\beta$, and $P_\alpha \rightarrow P'_\beta$ if and only if $S_\alpha \subseteq S_\beta$ and $T_\alpha \geq T'_\beta$.

To see that $G \rightarrow H$, observe that for all α , we have $T_\alpha \geq T'_\alpha$ by Claim 2. Thus, $P_\alpha \rightarrow P'_\alpha$ for all α , and so to map G to H apply the identity map to the Line in G and map each P_α to P'_α .

To show that $H \rightarrow G$, recall that for all $n \geq 0$ we have $S_{\alpha n} \subseteq S_\alpha$, and observe that for each $n \geq 0$, $T'_{\alpha n} = T_{\alpha 0} = T_\alpha$ for all $n \geq 0$. Thus $P'_{\alpha n} \rightarrow P_\alpha$ for all α and all $n \geq 0$. Thus, to map H homomorphically to G , we map the vertex v_i in the Line in H to the vertex v_{i-1} in the Line in G . We map each subdigraph $P'_{\alpha n}$ incident to $v_{|\alpha|+1}$ to the subdigraph P_α incident to $v_{|\alpha|}$, and for each $i \leq 0$ we map the subdigraph P'_ϵ incident to v_i to the subdigraph P_ϵ incident to v_{i-1} .

We must now demonstrate that $\mathfrak{s}(H)$ is false. Suppose $\mathfrak{s}(H)$ holds. Let K be a certificate of $\mathfrak{s}(H)$ and let $f : H \rightarrow K$ be a homomorphism. Then K must contain the Line, and to each vertex v_i in the Line must be joined a (possibly empty) subdigraph

of each P'_α incident to v_i in H . Since every P'_α with $|\alpha| > 0$ occurs exactly once in H , we can refer to the subdigraph of such a P'_α which is present in K as Q_α without ambiguity. Observe that under any homomorphism $g : P'_\alpha \rightarrow P'_\beta$ it must be the case that $g(x'_\alpha) = x'_\beta$, $g(y'_\alpha) = y'_\beta$, and $g(z'_\alpha) = z'_\beta$. Also, the bidirected path of colour i in P'_α must map onto the bidirected path of colour i in P'_β . Thus, if f maps P'_α to Q_β , then Q_β must contain x'_β , y'_β , and z'_β , as well as each of the bidirected paths in P'_β . And so Q_β is identical to P'_β except that it may lack some of the edges of P'_β given by S_β .

Consider the vertex v_1 in K , and its pre-image v_n in H under f . Note that in any endomorphism of H , as was the case for G , the vertex v_0 must map to some v_k with $k \leq 0$. Since $K \subseteq H$, this implies that $n \geq 1$.

Every P'_α joined to v_n in H must map under f to a Q_i incident to v_1 in K . Let a be the smallest index such that f maps some P'_β to Q_a . By claims 1 and 3 we know that no P'_α with $\alpha_1 = a + 1$ admits a homomorphism to Q_a , so f must map each such P'_α to some Q_b with $b > a$. Since f maps P'_β to Q_a , we know that $\beta_1 \leq a$. Let $\beta' = (a + 1)\beta_2 \dots \beta_n$, so $S_\beta \subseteq S_{\beta'}$. Also, note that $T'_i = T'_j$ for all $i, j \geq 0$. Thus, for any $k \geq 0$, if $P'_{\beta'} \rightarrow Q_k$ then $P'_\beta \rightarrow Q_k$. However, f must map $P'_{\beta'}$ to some Q_b with $b > a$, so there must be a homomorphism $f' : H \rightarrow K$ which maps any P'_β to some Q_b with $b > a$. Therefore $H \rightarrow K - Q_a$, so $K \rightarrow K - Q_a$, and so $\mathcal{S}(K)$ is false. Hence $\mathcal{s}(H)$ is false as well. \square

To complete the proof of theorem 28, we apply construction C to G and H to obtain uncoloured digraphs G^C and H^C . Since any endomorphism of G is an edge-surjection, corollary 10 guarantees that $\mathcal{S}(G^C)$ holds and so $\mathcal{s}(G^C)$ holds. However, $\mathcal{s}(H^C)$ is false. \blacksquare

The above result is somewhat curious, in that the two properties which fail to be invariant, namely \mathcal{s} and \mathcal{r} , are the two most commonly used definitions of a core in finite graph theory. This indicates that these may not be the most appropriate definitions of the core of an infinite digraph.

Our final theorem for this chapter also deals with a kind of invariance. Let \mathbf{P} be an arbitrary uppercase property. We will say that the property \mathbf{P} is *strongly invariant*

if whenever G and H are digraphs such that both $\mathbf{P}(G)$ and $\mathbf{P}(H)$ hold and G and H are homomorphically equivalent, then G and H are isomorphic. It is a simple observation that no lowercase property is strongly invariant in this sense, and so we have restricted our attention to uppercase properties. If a property \mathbf{P} is strongly invariant, we can readily see that whenever $\mathbf{p}(G)$ holds there is, up to isomorphism, a unique subdigraph H of G such that $G \rightarrow H$ and $\mathbf{P}(H)$.

Theorem 29 *The properties \mathbf{ISN} and \mathbf{SN} are strongly invariant. The properties \mathbf{I} , \mathbf{S} , \mathbf{N} , \mathbf{IS} , \mathbf{IN} and \mathbf{R} are not.*

Proof: Let G and H be homomorphically equivalent digraphs such that $\mathbf{ISN}(G)$ and $\mathbf{ISN}(H)$ both hold. Let $r : G \rightarrow H$ and $s : H \rightarrow G$ be homomorphisms. If r is either not injective or does not preserve non-edges, then $s \circ r$ is an endomorphism of G lacking the same property. If r is not surjective then $r \circ s$ is a non-surjective endomorphism of H . Hence, r must be an isomorphism.

Since \mathbf{SN} is logically equivalent to \mathbf{ISN} it is also strongly invariant.

Let G be obtained from the 5-line by replacing A_1 with a copy of A_{-1} . This shares the properties \mathbf{IS} and \mathbf{R} with the 5-line and is homomorphically equivalent to it, but is not isomorphic. Thus none of the properties \mathbf{R} , \mathbf{I} , \mathbf{S} , or \mathbf{IS} are strongly invariant.

Let G_1 be the subdigraph of the 5-line induced by the vertices $v_i, i \geq 0$, and their respective A_i . Let G_2 be obtained from G_1 by replacing A_0 by the triangle consisting of v_0 and its two neighbors in A_0 . These digraphs are homomorphically equivalent and share the property \mathbf{IN} , but are non-isomorphic. Thus neither \mathbf{N} nor \mathbf{IN} are strongly invariant. ■

Recall that earlier we claimed that \mathbf{ISN} was the property with the nicest behavior for infinite digraphs. We have shown that the two most common definitions of core for finite digraphs, \mathbf{s} and \mathbf{r} , are not invariant over homomorphic equivalence classes. The property \mathbf{ISN} , in addition to being strongly invariant, is also the strongest of the properties under discussion, while still being equivalent to \mathbf{R} and \mathbf{S} for finite structures. We conclude this section by reiterating our claim that \mathbf{ISN} is the most reasonable definition for the core of an infinite structure. We now formally adopt this definition.

Definition 30 A digraph G is a core if and only if $ISN(G)$ holds. A digraph H is a core of G if and only if H certifies $isn(G)$.

The following is now a corollary to theorem 29.

Corollary 31 If a digraph G has a core, then the core of G is unique up to isomorphism. ■

3.3 Structures and Graphs

The major results in this chapter all generalize nicely to arbitrary structures and graphs. The definitions of the properties I, S, N , and R , as well as their combinations and lowercase versions, may be applied to structures and graphs without modification.

Recall the generalized definition of \equiv for structures. Lemma 23 can be restated for structures and remains true.

Lemma 23' Let G be a structure satisfying $N(G)$, and let f be an endomorphism of G . If $f(u) = f(v)$ for some $u, v \in V(G)$, then $u \equiv v$.

Proof: Suppose $N(G)$ holds and let f be any endomorphism of G . Suppose $f(u) = f(v)$ for some $u, v \in V(G)$. If $u \not\equiv v$ then there must exist a relation R and edges $\bar{u}, \bar{v} \in R(G)$ such that $\bar{u}_{v/u} = \bar{v}_{v/u}$ and $\bar{u} \in R(G)$ but $\bar{v} \notin R(G)$. We know that f is an endomorphism, and so $(f(\bar{u})) \in R(G)$. But $f(u) = f(v)$, so $(f(\bar{u})) = (f(\bar{u}_{v/u})) = (f(\bar{v}_{v/u})) = (f(\bar{v}))$. Hence we have $(f(\bar{v})) \in R(G)$, contradicting $N(G)$. ■

We can see that this proof is essentially identical to the proof of lemma 23, except that it is restated in the terminology of structures. We will generally not rewrite subsequent proofs where this is the case.

Lemma 23 may also be restated in the obvious way for graphs, and the proof is essentially identical.

Theorems 24, 26, and 27 all have analogues applying to structures. Let P and Q be arbitrary uppercase properties, and let \mathcal{L} be a language. We say that P implies Q in \mathcal{L} if whenever G is a structure for \mathcal{L} and $P(G)$ holds, then $Q(G)$ holds as well. It turns out that the implications between properties are the same for all languages.

Theorem 24' *Let \mathcal{L} be any language, and let P and Q be arbitrary uppercase properties. Then P implies Q in \mathcal{L} if and only if P implies Q for digraphs.*

We show that a given property implies another using arguments essentially identical to those in the proof of theorem 24. To obtain counterexamples to show that a given property does not imply another, we apply lemma 2 to the counterexamples used in the proof of theorem 24.

Analogues of theorems 26 and 27 are proved by the same method.

Analogues to these three theorems also hold for graphs. For example, we have the following result.

Theorem 24'' *Let P and Q be arbitrary uppercase properties. Then P implies Q for graphs if and only if P implies Q for digraphs.*

Again we may prove that a given property implies another just as in the proof of theorem 24. We now show that these are the only implications that hold. Referring back to the proof of theorem 24, we observe that all of the counterexamples used in that theorem can be transformed into undirected graphs using construction B. The constructed graphs will provide counterexamples for the same implications. The only exceptions to this are the cases where we used the 5-line as a counterexample, since the 5-line has property S , but the graph obtained by applying construction B to the 5-line does not have property S . We will therefore explicitly construct an undirected graph G with the same properties as the 5-line, *i.e.* $IS(G)$ and $R(G)$ will hold but $N(G)$ will not hold. The graph G is shown in figure 3.8. The directed edges are replaced as in construction B, while the undirected edges will remain unaltered.

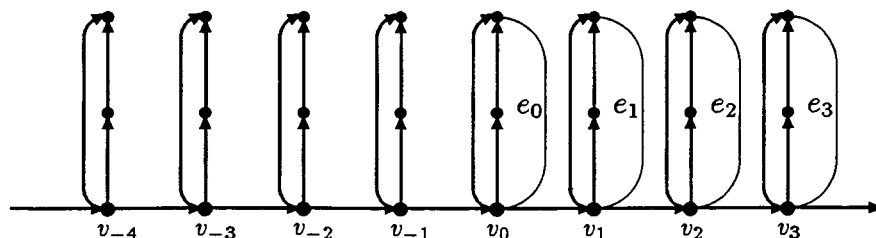


Figure 3.8

We claim that this graph has properties I, S , and R , but not N . Let us first

examine the directed graph G' consisting of the vertices and directed edges of G prior to applying construction B. Clearly any endomorphism of G' is an automorphism. If we apply construction B to G' to obtain G'' , then any endomorphism of G'' must also be an automorphism. Now $V(G'') = V(G)$ and $E(G'') = E(G) - \{e_0, e_1, \dots\}$. Observe that none of the edges $e_i \in E(G)$ are contained in any cycle of G of length less than or equal to seven in G . But each edge of G'' lies in a 7-cycle. Thus, if f is any endomorphism of G and $uv \in E(G) - \{e_i : i \geq 0\}$ then $f(u)f(v) \in E(G) - \{e_i : i \geq 0\}$. Thus, f is also an endomorphism of G'' . Therefore f must be an injection and a surjection, and f is not a proper retraction. However, if $f(v_0) = v_1$ then f does not preserve non-edges of G . ■

Analogues of theorems 26 and 27 also hold for graphs. In all cases the proofs that the implications hold are the same as for directed graphs. In cases not involving the property S , undirected counterexamples to implications can be obtained by applying construction B to the directed counterexamples. In all other cases counterexamples can be constructed by methods similar those used in the above theorem.

Analogues of theorems 28 and 29 also hold for graphs and structures. In all cases the proofs are essentially identical to the proofs for digraphs, and counterexamples may be constructed using constructions A and B.

Chapter 4

Compactness of Digraphs

4.1 Definitions

Let G and H be digraphs. We say that H is *compact with respect to* G if either $G \rightarrow H$ or there exists a finite subdigraph G' of G such that $G' \not\rightarrow H$. We say that H is α -*compact* if H is compact with respect to G for all G with $|V(G)| \leq \alpha$. If H is compact with respect to every G , then we say that H is *compact*. Equivalently, we might say that H is compact if for every digraph G , we have $G \rightarrow H$ if and only if $G' \rightarrow H$ for every finite subdigraph G' of G . If every finite subdigraph of G admits a homomorphism to H , but $G \not\rightarrow H$, then we say G is a *certificate of non-compactness* for H . Observe that if G is a certificate of non-compactness for H then some connected component of G is a certificate of non-compactness for H .

If two digraphs H and H' are homomorphically equivalent and G is any digraph, then clearly H is compact with respect to G if and only if H' is compact with respect to G . Also note that if H is α -compact and $\beta < \alpha$ then H is β -compact.

Example 32 *The Line is compact.*

A digraph G admits a homomorphism to the Line if and only if every cycle in G has net length zero. If G has a cycle of non-zero net length, then this cycle is a finite subdigraph of G admitting no homomorphism to the Line. That the Line is compact also follows from results proved later in this chapter.

Example 33 *The Ray is not compact.*

The Line is a certificate of non-compactness for the Ray, since every finite subdigraph of the Line admits a homomorphism to the Ray, but clearly the Line admits no homomorphism to the Ray. In fact, the Line demonstrates that the Ray is not \aleph_0 -compact.

A very important class of non-compact digraphs are infinite tournaments. A *tournament* is a digraph G where for each distinct $u, v \in V(G)$ exactly one of uv or vu is an edge of G . A tournament is *transitive* if whenever $uv \in E(G)$ and $vw \in E(G)$, then $uw \in E(G)$. Equivalently, we may define a transitive tournament to be any digraph G where $V(G)$ is linearly ordered and $uv \in E(G)$ if and only if $u < v$.

Tournaments are the subject of our first lemma.

Lemma 34 *If T is an infinite tournament and G is a loopless digraph such that $T \subseteq G$ then G is not compact.*

Proof: A frequently-used application of Ramsey's theorem shows that every infinite tournament contains an infinite transitive tournament (for example see [53]). Let T' be a transitive tournament on $|G|^+$ vertices. Clearly every finite subdigraph of T' admits a homomorphism to G . However, as G contains no loops, any homomorphism from T' to G must be an injection, which is impossible since $|T'| > |G|$. ■

Example 35 *If T is an infinite transitive tournament then T is $|T|$ -compact but not $|T|^+$ -compact.*

Any digraph G with $|G| \leq |T|$ admits a homomorphism to T if and only if G contains no directed cycle. A directed cycle in G is a finite subdigraph of G which admits no homomorphism to T , and so T is $|T|$ -compact. As in lemma 34 a transitive tournament on $|T|^+$ vertices suffices to show that T is not $|T|^+$ -compact.

In the next section we will show that every compact digraph contains a core. In pursuit of this result we will also give some results relating to other core-like digraphs necessarily contained in compact digraphs.

In section 4.3 we examine digraphs H which are $|H|$ -compact. We prove that if H is a $|H|$ -compact core then H is compact. We use this result to prove that a digraph H is compact if and only if it is $|H|^+$ -compact.

In section 4.4 we will describe some large classes of compact digraphs, and thereby also describe classes of infinite digraphs with cores.

4.2 Compact Digraphs have Cores

The problem of characterizing cores is a difficult one even in the case of finite digraphs, and has been solved only for undirected graphs with independence number at most two [38]. In this section we provide a partial result by showing that every compact digraph contains a core, which by corollary 31 must be unique up to isomorphism. Along the way we will prove uniqueness and existence results for some other core-like subdigraphs of compact digraphs. Our first lemma is of a slightly different type, but will be useful in proving further results.

Lemma 36 *If H is an $|H|$ -compact digraph then there exists a digraph $H' \supseteq H$ with $V(H') = V(H)$ and such that $H' \rightarrow H$ and $\mathcal{N}(H')$ holds.*

Proof: Suppose that H is $|H|$ -compact but no such H' exists. We define a transfinite sequence $\{H_\alpha\}$ of superdigraphs of H on the same vertex-set $V(H)$. Each of these will have the property that $H_\alpha \rightarrow H$. Let $H_0 = H$, and if H_α is defined for some ordinal α , define $H_{\alpha+1}$ as follows. Since $H_\alpha \rightarrow H$, by assumption there is an endomorphism f of H_α which does not preserve non-edges.

Define $H_{\alpha+1}$ by

$$V(H_{\alpha+1}) = V(H_\alpha)$$

$$E(H_{\alpha+1}) = \{uv : f(u)f(v) \in E(H_\alpha)\}$$

Clearly $f : H_{\alpha+1} \rightarrow H_\alpha$ is a homomorphism, and so by transitivity we have $H_{\alpha+1} \rightarrow H$. Also, since f does not preserve non-edges in H_α , it must be the case that $E(H_\alpha)$ is properly contained in $E(H_{\alpha+1})$.

If λ is a limit ordinal and H_α is defined, with $V(H_\alpha) = V(H)$, for all $\alpha < \lambda$, then we define H_λ by

$$\begin{aligned} V(H_\lambda) &= V(H) \\ E(H_\lambda) &= \bigcup_{\alpha < \lambda} E(H_\alpha). \end{aligned}$$

To show that $H_\lambda \rightarrow H$, let K be any finite subdigraph of H_λ . Then K has only finitely many edges, so $K \subseteq H_\alpha$ for some $\alpha < \lambda$, and so $K \rightarrow H$. Since $|H_\lambda| = |H|$ and H is $|H|$ -compact, we have $H_\lambda \rightarrow H$.

However, for each α , the cardinality of $E(H_\alpha)$ can be no more than $|H_\alpha| = |H|$. At each step in the above induction we add at least one new edge to $H_{\alpha+1}$, and we never remove edges once added. Hence for some $\alpha < |H|^+$ we must have an H_α for which $E(H_\alpha) = \{uv : u, v \in V(H_\alpha)\}$, i.e. all possible edges are present. But $H_\alpha \rightarrow H$ and so by assumption there is an endomorphism of H_α which does not preserve non-edges. This is clearly impossible as H_α is without non-edges.

Thus, there must be some superdigraph H' of H such that $\mathbf{N}(H')$ is true. \blacksquare

We now apply this lemma to obtain some useful results relating to core-like subdigraphs of compact digraphs.

Lemma 37 *Let H be a digraph which is $|H|$ -compact. If G certifies $\mathbf{s}(H)$, then G is a core of H .*

Proof: Suppose G is not a core, i.e., $\mathbf{ISN}(G)$ is false. Recall that $\mathbf{ISN}(G)$ holds if and only if $\mathbf{SN}(G)$ holds, and certainly $\mathbf{S}(G)$ is true, and so $\mathbf{N}(G)$ must be false.

We will show that $\mathbf{N}(H')$ is false for every superdigraph H' of H where $H' \rightarrow H$ to obtain a contradiction to lemma 36.

Let H' be a superdigraph of H which admits a homomorphism to H . Then $\mathbf{s}(H')$ is true, since $H' \rightarrow G$ and $\mathbf{S}(G)$ holds, and so G certifies $\mathbf{s}(H')$.

Now let $h : H' \rightarrow G$ be a homomorphism, and let f be an endomorphism of G which does not preserve non-edges. Then there exist $u, v \in V(G)$ such that $uv \notin E(G)$ but $f(u)f(v) \in E(G)$. Since $G \subseteq H'$, the mapping h must be a surjection, and so both u and v have pre-images u' and v' under h in H' . Now $u'v'$ cannot be an edge of H' since then h would not be a homomorphism, but $(f \circ h)(u')(f \circ h)(v')$ is an edge

of H' . Thus the composition $f \circ h$ does not preserve non-edges. Since $f \circ h$ is an endomorphism of H' , we have shown that $N(H')$ is false. ■

Corollary 38 *Let H be a digraph which is $|H|$ -compact. If $\mathfrak{s}(H)$ holds, then H contains exactly one certificate for $\mathfrak{s}(H)$ (up to isomorphism).*

Proof: Suppose H is $|H|$ -compact, and let G_1 and G_2 be certificates for $\mathfrak{s}(H)$. Lemma 37 guarantees that both G_1 and G_2 are cores of H , and theorem 29 implies that G_1 and G_2 are isomorphic. ■

According to the above lemma, we need only show that a compact digraph H satisfies $\mathfrak{s}(H)$ in order to guarantee that H has a unique core. Our next two lemmas will show that a compact digraph does indeed satisfy the property \mathfrak{s} .

We will require the following definition. Recall the equivalence relation \equiv defined in section 1.1.1.

Definition 39 Let K be a digraph. Let S be a set containing one vertex from each equivalence class of $V(K)$. We define a new digraph K^r to be the subdigraph of K induced by S . The digraph K^r is called the *reduced* digraph of K .

It is a simple matter to verify that K^r is a retract of K , and that no two vertices of K^r are equivalent. We will refer to the retraction which maps every vertex of K to the representative of its equivalence class as the *canonical* retraction from K to K^r . Note also that it makes no difference which vertex is chosen from an equivalence class.

Lemma 40 *Every $|H|$ -compact digraph H satisfies $\mathfrak{in}(H)$.*

Proof: Suppose H is $|H|$ -compact but $\mathfrak{in}(H)$ is false. As in the previous lemma we will show that no superdigraph H' of H which admits a homomorphism to H satisfies $N(H')$ to obtain a contradiction.

Let H' be a superdigraph of H such that $H' \rightarrow H$, and let $f : H' \rightarrow H$ be a homomorphism. Let $G = (f(H'))^r$. By definition f is a surjection from H' to $f(H')$, and the canonical retraction from $f(H')$ to G is a surjection as well. Thus

the composition of these homomorphisms is a surjection from H' to G . Call this composition f' .

Now $G \subseteq H$ and $H \rightarrow G$ because $H \subseteq H'$ and $H' \rightarrow G$, so by assumption $\mathbf{IN}(G)$ is false. However, observe that if $\mathbf{I}(G)$ is false then $\mathbf{N}(G)$ is also false. For suppose that g is an endomorphism of G which is not an injection. Then there exist $u, v \in V(G)$ such that $g(u) = g(v)$. But in G no two vertices are equivalent, and so lemma 23 implies that $\mathbf{N}(G)$ is false.

Let g be an endomorphism of G which does not preserve non-edges. Look at the composition $g \circ f' : H' \rightarrow G$. This homomorphism does not preserve non-edges since g does not preserve non-edges and f' is a surjection. Also, since $G \subseteq H \subseteq H'$, $g \circ f'$ is an endomorphism of H' which does not preserve non-edges. Thus, $\mathbf{N}(H')$ is false.

■

This lemma is used to prove the following important result.

Lemma 41 *Any $|H|^+$ -compact digraph H satisfies $\mathfrak{s}(H)$.*

Proof: Let H be a $|H|^+$ -compact digraph, and suppose $\mathfrak{s}(H)$ is false. Since H is also $|H|$ -compact, by the preceding lemma we know that H contains a subdigraph H_0 which certifies $\mathbf{in}(H)$. We will show that H_0 also certifies $\mathfrak{s}(H)$.

Suppose that $\mathfrak{S}(H_0)$ is false. Note that H_0 is $|H_0|^+$ -compact, since $H_0 \leftrightarrow H$ and $|H_0|^+ \leq |H|^+$. As in lemma 36 we will obtain a contradiction by constructing a transfinite sequence of digraphs $\{H_\alpha\}$, for $\alpha \leq |H_0|^+$. For each $\alpha > 0$ the digraph H_α will be a proper superdigraph of H_0 . The H_α will all satisfy $\mathbf{IN}(H_\alpha)$, $|H_\alpha| = |H_0|$, and each H_α will be isomorphic to a subdigraph of H_0 . This last property is somewhat counter-intuitive, but is quite possible when dealing with infinite digraphs. In the construction we are about to present, unlike that in lemma 36, the vertex-set will not remain constant.

We have already defined H_0 , which satisfies the above conditions by assumption.

Now suppose H_α is defined, $\mathbf{IN}(H_\alpha)$ holds, $|H_\alpha| = |H_0|$, and H_α is isomorphic to a subdigraph of H_0 . We claim that there must exist a non-surjective endomorphism of H_α . When $\alpha = 0$ this is true by assumption. If $\alpha > 0$ then we know that there is

a homomorphism $h : H_\alpha \rightarrow H_0$ because H_α is isomorphic to a subdigraph of H_0 . But H_0 is a proper subdigraph of H_α , so h is a non-surjective endomorphism of H_α .

We now define $H_{\alpha+1}$.

Let $I = \{v \in V(H_\alpha) : h^{-1}(v) = \emptyset\}$, and let $I' = \{v' : v \in I\}$ be a set of new vertices. We will define $H_{\alpha+1}$ in two steps. First we define

$$V(H_{\alpha+1}) = V(H_\alpha) \cup I'.$$

Clearly $|H_{\alpha+1}| = |H_\alpha| = |H_0|$. Also, since I is nonempty we know that $H_{\alpha+1}$ is a proper superdigraph of H_0 . We now define a mapping $h' : V(H_{\alpha+1}) \rightarrow V(H_\alpha)$ by $h'|_{V(H_\alpha)} = h$, and $h'(v') = v$ for all $v' \in I'$. We may now define

$$E(H_{\alpha+1}) = \{uv : h'(u)h'(v) \in E(H_\alpha)\}.$$

It follows immediately from the definition of $H_{\alpha+1}$ that h' is a homomorphism from $H_{\alpha+1}$ to H_α .

Since $\mathbf{IN}(H_\alpha)$ holds, the homomorphism h is an injection and preserves non-edges. The homomorphism h' is also an injection, since no two vertices in I' get mapped to the same vertex, and no vertex in I' is mapped to a vertex that is the image of a vertex under h . Also, h' preserves non-edges, by definition of $E(H_{\alpha+1})$. Furthermore, h' is a surjection, since we explicitly added a pre-image of each vertex of H_α to $H_{\alpha+1}$. Thus, h' is an isomorphism between $H_{\alpha+1}$ and H_α , and so $H_{\alpha+1}$ is also isomorphic to a subdigraph of H_0 and $\mathbf{IN}(H_{\alpha+1})$ holds, since these properties are preserved by isomorphism.

If $\lambda \leq |H_0|^+$ is a limit ordinal and H_α is defined for all $\alpha < \lambda$, then we define a digraph G by

$$V(G) = \bigcup_{\alpha < \lambda} V(H_\alpha)$$

$$E(G) = \bigcup_{\alpha < \lambda} E(H_\alpha).$$

Note that $\{V(H_\alpha)\}_{\alpha < \lambda}$ is an increasing nested sequence of sets, all of cardinality $|H_0|$, and so $|V(G)| \leq |H_0|^+$. Thus, H_0 is compact with respect to G . Any finite subdigraph G' of G must be contained in some H_α with $\alpha < \lambda$, and so $G' \rightarrow H_0$. And so $G \rightarrow H_0$.

We will now show that either G satisfies the conditions of our inductive hypothesis, so we may set $H_\lambda = G$, or we will obtain a contradiction, thus completing our proof.

We first show that $\mathbf{IN}(G)$ holds. Suppose that f is an endomorphism of G , and suppose that either f is not an injection or f does not preserve non-edges. Then there exist $u, v \in V(G)$ such that $f(u) = f(v)$ or $uv \notin E(G)$ but $f(u)f(v) \in E(G)$. However, there must exist some H_α with $\alpha < \lambda$ such that $u, v \in V(H_\alpha)$.

Now look at $f|_{H_\alpha} : H_\alpha \rightarrow G$. Since $u, v \in V(H_\alpha)$, $f|_{H_\alpha}$ is either not an injection or does not preserve non-edges. We also know that there exists a homomorphism $g : G \rightarrow H_0$, and $H_0 \subseteq H_\alpha$. Thus, the composition $g \circ f|_{H_\alpha} : H_\alpha \rightarrow H_\alpha$ is an endomorphism of H_α which is either not an injection or does not preserve non-edges. This contradicts $\mathbf{IN}(H_\alpha)$, and so f must be an injection and preserve non-edges. Thus, $\mathbf{IN}(G)$ must hold. Furthermore, g is an endomorphism of G , since $H_0 \subseteq G$, and so it is an injection and preserves non-edges. Hence, $g(G)$ is a subdigraph of H_0 which is isomorphic to G . It follows that $|G| = |H_0|$, and so we may define $H_\lambda = G$.

But at each inductive step in our construction at least one new vertex is added to the digraph. Eventually, for some limit ordinal $\lambda \leq |H_0|^+$, it must be the case that $|H_\lambda| \geq |H_0|^+$. Thus we obtain a contradiction, and may conclude that $\mathfrak{s}(H)$ holds. ■

Corollary 42 *Let H be a digraph. If H is $|H|^+$ -compact, then H has a core.*

Proof: By lemma 41 H contains a subdigraph G which certifies $\mathfrak{s}(H)$. By lemma 37 we see that G is a core of H . ■

In particular every compact digraph has a core.

4.3 $|G|$ -compact Digraphs

One might hope that the converse of the corollary at the end of the preceding section might also be true. Unfortunately, this is not the case. Consider, for example, the following digraph. We first define a sequence of finite digraphs $H_i, i \geq 1$. Let p_n denote the n^{th} odd prime, and C_n the directed cycle of length p_n . As observed in Chapter 2, $\{C_n : n \geq 1\}$ is a mutually incompatible family of digraphs, and in

fact each C_n is a core. To construct H_i , begin with $V(H_i) = \{v_0, \dots, v_{i+1}\}$ and $E(H_i) = \{v_0v_j : 1 \leq j \leq i+1\}$. Now for $1 \leq j \leq i$, attach a copy of C_{2j} to v_j by identifying one of the vertices of the cycle with v_j . Finally, attach a copy of C_{2i+1} to v_{i+1} in like manner.

We claim that each H_i is a core and that $\{H_i : i \geq 1\}$ is a mutually incompatible family of digraphs. Choose $i \geq 1$ and let v be a vertex of H_i which occurs in a copy of some $C_n \subseteq H_i$. Then v has in-degree at least one. However, v_0 has in-degree 0, so no endomorphism of H_i can map v to v_0 . Hence, any endomorphism of H_i must map each $C_n \subseteq H_i$ into some $C_m \subseteq H_i$, and so C_n must map onto itself. Since $i \geq 1$ the vertex v_0 must map to itself, and so H_i is a core.

Similarly, if $f : H_i \rightarrow H_j$ is a homomorphism for some $i \neq j$, then f must map each $C_n \subseteq H_i$ into some $C_m \subseteq H_j$. Again, this is possible if and only if $n = m$, and so $H_i \rightarrow H_j$ if and only if every C_n contained in H_i is also contained in H_j . However, if $i \neq j$ then H_i contains a copy of C_{2i+1} , but H_j does not, and so H_i and H_j are mutually incompatible.

We now define H to be the disjoint union of the digraphs H_i . Then H is a core, since each component of H is a core and there is no homomorphism from any component of H to any other. However, H is not compact. Consider the digraph G obtained by taking $V(G) = \{v_0, v_1, v_2, \dots\}$ and $E(G) = \{v_0v_i : 1 \leq i\}$, and attaching a copy of C_{2i} to v_i for each $i \geq 1$. Any finite subdigraph G' of G will admit a homomorphism to H , since it can contain vertices from at most finitely many of the cycles in G , and therefore will be a subdigraph of some H_i . However, $G \not\rightarrow H$, since G is connected and no component of H contains cycles of all lengths p_{2i} .

Note that in this example the digraph H is not $|H|$ -compact. In fact in this case both H and G were countable. The property of being $|H|$ -compact turns out to be quite strong, and in this section we will show that several properties of digraphs are equivalent for $|H|$ -compact digraphs.

We begin with some definitions, leading up to a very useful lemma.

Let G and H be digraphs. Let $l : V(G) \rightarrow \mathcal{P}(V(H))$ be a mapping from $V(G)$ to the power set of $V(H)$, called a *list-assignment* for G (with respect to H). An *l -list-homomorphism* $f : G \rightarrow H$ is a homomorphism from G to H such that for

each $v \in V(G)$ we have $f(v) \in l(v)$. We will often wish to apply the same lists to subdigraphs G' of G . By convention we will say that $f : G' \rightarrow H$ is an l -list-homomorphism if f is an $l|_{V(G')}$ -list-homomorphism.

We will say that a list-assignment $l : V(G) \rightarrow \mathcal{P}(V(H))$ has property R if for each $v \in V(G)$ either $|l(v)| = 1$ or $l(v) = V(H)$. We will say that a digraph H is α - R -list-compact if for every digraph G with $|G| \leq \alpha$ and every list-assignment l for G with respect to H for which R holds, either there is an l -list-homomorphism from G to H , or there is a finite subdigraph $G' \subseteq G$ for which no l -list homomorphism exists. If H is α - R -list-compact for every cardinal α then H is R -list-compact.

An l -list-homomorphism where l has the property R is more commonly known as a *precolouring-extension* [11]. However, we will regard them as a special type of list-homomorphism. In chapter 6 we will examine list-homomorphisms in much greater detail. For our present purposes, the following lemma will be very useful.

Lemma 43 *Let H be a core and $\alpha \geq |H|$ be a cardinal. Then H is α - R -list-compact if and only if H is α -compact.*

Proof: If H is α - R -list-compact then H is α -compact, since for any input digraph G we may define a list-assignment l by $l(v) = V(H)$ for each $v \in V(G)$.

Now suppose H is an α -compact core with $\alpha \geq |H|$, and let G be any digraph with $|G| \leq \alpha$. Let l be any list-assignment for G with respect to H such that R holds for l , and let $S = \{v \in V(G) : |l(v)| = 1\}$. Suppose also that every finite subdigraph of G maps to H subject to l . We construct a new digraph G^* by taking a copy of G and a copy of H , and identifying $w \in V(H)$ with all $v \in V(G)$ such that $l(v) = \{w\}$. It will be useful to formally define G^* as follows. We define a mapping $s : V(G) \cup V(H) \rightarrow V(G) \cup V(H)$ by setting $s(v) = l(v)$ for all $v \in S$, and $s(v) = v$ otherwise. Now we define $V(G^*) = s(V(H) \cup V(G))$ and $E(G^*) = \{s(u)s(v) : uv \in E(G) \cup E(H)\}$. Note that $|G^*| \leq \alpha$.

We say that a vertex $v \in V(G^*)$ is a G -vertex if $v = s(u)$ for some $u \in V(G)$. We say that an edge $uv \in E(G^*)$ is a G -edge if $uv = s(x)s(y)$ for some $xy \in E(G)$.

We will show that $G^* \rightarrow H$. Let K be a finite subdigraph of G^* . Of course K contains only finitely many G -vertices and G -edges. Thus, there is a finite subset A

of $V(G)$ such that every G -edge of K and every G -vertex of K has a pre-image in A under s . Let B be the set of all vertices of K which are not G -vertices. Note that each $v \in B$ is its own unique pre-image under s so $V(K) \subseteq s(A \cup B)$.

Let $G' = G[A] \cup H[B]$. By assumption there exists an l -list-homomorphism $f : G[A] \rightarrow H$, which we can extend to a homomorphism $g : G' \rightarrow H$ by applying the identity mapping to all $v \in B$.

We now use the homomorphism g to define a homomorphism $h : K \rightarrow H$. Let v be a vertex of K . Let $v' \in A \cup B$ be some pre-image of v under s , and define $h(v) = g(v')$. Note that h is independent of the choice of v' , since if $v \in s(S)$ then $g(v') = l(v')$ for all pre-images v' of v , and if $v \notin s(S)$ then v has a unique pre-image under s .

If uv is an edge of K then there exist $u', v' \in V(G')$ such that $s(u') = u, s(v') = v$, and $u'v' \in E(G')$. We know that $h(u) = g(u')$ and $h(v) = g(v')$ by the above remark. But g is a homomorphism so $g(u')g(v') \in E(H)$, and so $h(u)h(v) \in E(H)$. Thus, $h : K \rightarrow H$ is a homomorphism, and so by α -compactness of H we know that $G^* \rightarrow H$.

Now let $f : G^* \rightarrow H$ be a homomorphism. Since H is a core, $f|_{V(H)}$ is an automorphism of H . Let $g = (f|_{V(H)})^{-1}$. Then $(g \circ f) : G^* \rightarrow H$ is a homomorphism and $(g \circ f)|_{V(H)}$ is the identity. It is obvious from the definition of s that $s : (G \cup H) \rightarrow G^*$ is a homomorphism and $s|_{V(H)}$ is the identity. So $(g \circ f \circ s) : (G \cup H) \rightarrow H$ is a homomorphism from $G \cup H$ to H . In particular $(g \circ f \circ s)|_{V(G)} : G \rightarrow H$. But $s(v) = l(v)$ for all $v \in S$, and $l(v) \in V(H)$ for all $v \in S$, so $(g \circ f)(l(v)) = l(v)$ for all $v \in S$. Thus, $(g \circ f \circ s)(v) = l(v)$ for all $v \in S$, and so $(g \circ f \circ s)|_{V(G)}$ is an l -list-homomorphism from G to H . ■

We will use this lemma to prove a very interesting sufficient condition for compactness. First we require a definition. Let G, H , and K be digraphs with $G \subseteq H$, and let $g : G \rightarrow K$ and $h : H \rightarrow K$ be homomorphisms. We say that h is an *extension* of g if $h|_G = g$.

Theorem 44 *Let H be a core. If H is $|H|$ -compact then H is compact.*

Proof: Suppose that H is a $|H|$ -compact core and H is not compact. Let κ be

the least cardinal such that H is not κ -compact, and let G be a certificate of non-compactness for H with $|G| = \kappa$. We assume $V(G) = \{\alpha : \alpha < \kappa\}$, and for each ordinal $\alpha < \kappa$ we define G_α to be the subdigraph of G induced by $\{\beta : \beta \leq \alpha\}$. For each G_α we have $|G_\alpha| = |\alpha| + 1$, and $|\alpha| + 1 < \kappa$ since κ is an initial ordinal. Thus, G_α has fewer than κ vertices, and so each G_α admits a homomorphism to H . For each ordinal $\alpha < \kappa$ we will construct a homomorphism $f_\alpha : G_\alpha \rightarrow H$ which will be an extension of each f_β with $\beta < \alpha$. Also, every f_α will have the property that for each β with $\alpha \leq \beta < \kappa$, there exists a homomorphism $g_\beta : G_\beta \rightarrow H$ which is an extension of f_α .

We will define the f_α inductively. Let γ be an ordinal smaller than κ and suppose that we have defined a homomorphism f_α satisfying the required properties for each $\alpha < \gamma$. Note that this condition is trivially satisfied when $\gamma = 0$. We proceed to define f_γ .

We claim that there exists a vertex $v_0 \in V(H)$ such that for all β with $\gamma \leq \beta < \kappa$, there exists a homomorphism $g_\beta : G_\beta \rightarrow H$ such that g_β is an extension of f_α for all $\alpha < \gamma$, and $g_\beta(\gamma) = v_0$.

If no such vertex exists then for all $v \in V(H)$ there must exist a β with $\gamma \leq \beta < \kappa$ such that there is no homomorphism $g_\beta : G_\beta \rightarrow H$ which is an extension of every f_α with $\alpha < \gamma$, and for which $g_\beta(\gamma) = v$.

We can rephrase this last statement in terms of list-homomorphisms. Let v be an arbitrary vertex of H . We will define a list-assignment r_v for G with respect to H . Let $r_v(\alpha) = \{f_\alpha(\alpha)\}$ for $0 \leq \alpha < \gamma$, $r_v(\gamma) = \{v\}$, and $r_v(\alpha) = V(H)$ for each α with $\gamma < \alpha < \kappa$. The above statement is equivalent to the assertion that for some β with $\gamma \leq \beta < \kappa$ there is no r_v -list-homomorphism from G_β to H . But H is $|\beta|$ -compact, so by lemma 43 H is $|\beta|$ - R -list-compact. Also, R holds for r_v . Therefore, there must be some finite subdigraph G_v of G_β such that there is no r_v -list-homomorphism from G_v to H . Note that $\gamma \in V(G_v)$, since for each $\alpha < \gamma$ there is an r_v -list-homomorphism from G_α to H .

We now define $G^* = \bigcup_{v \in V(H)} G_v$. We claim that for each $\alpha < \gamma$ there is a homomorphism from G^* to H which is an extension of f_α . The digraph H is $|H|$ -compact

so by lemma 43 H is $|H|$ - R -list-compact. Clearly $|G^*| \leq |H|$, and so H is R -list-compact with respect to G^* . Now if G' is any finite subdigraph of G^* , then G' is a finite subdigraph of G , and so $G' \subseteq G_\beta$ for some $\beta < \kappa$. For each $\alpha < \gamma$ we define a list assignment l_α for G with respect to H by $l_\alpha(\delta) = \{f_\alpha(\delta)\}$ for $\delta \leq \alpha$, and $l_\alpha(\delta) = V(H)$ otherwise. By assumption, for each $\alpha < \gamma$ there is an l_α -list-homomorphism from G_β to H , and so for each $\alpha < \gamma$ there is an l_α -list-homomorphism from G' to H . Thus, since H is $|H|$ - R -list-compact, for each $\alpha < \gamma$ there is an l_α -list-homomorphism from G^* to H . Clearly such a homomorphism is an extension of f_α .

We claim that in fact there is one homomorphism $g : G^* \rightarrow H$ such that g is an l_α -list-homomorphism for all $\alpha < \gamma$. Let us define a list-assignment s for G^* with respect to H by $s(\alpha) = \{f_\alpha(\alpha)\}$ for $\alpha < \gamma$ and $s(\alpha) = V(H)$ otherwise. We will show that there is an s -list-homomorphism from G^* to H , which clearly will be an extension of each f_α for $\alpha < \gamma$. Let G' be a finite subdigraph of G^* . We claim there is an s -list-homomorphism from G' to H . Since G' is finite, there must be some $\alpha < \gamma$ such that for each $\delta \in V(G')$, either $\delta \leq \alpha$ or $\delta \geq \gamma$. Also, there is some $\beta < \kappa$ such that $G' \subseteq G_\beta$. Thus we may find a homomorphism $f : G_\beta \rightarrow H$ which is an extension of f_α , and so $f|_{G'}$ is an s -list-homomorphism from G' to H . Thus, by $|H|$ - R -list-compactness of H , there is an s -list-homomorphism from G^* to H . This is the required homomorphism g .

Now let $v = g(\gamma)$. Then $g|_{G_v}$ is an r_v -list-homomorphism from G_v to H , a contradiction. Hence, our claim is proven. And so there must be some $v_0 \in V(H)$ such that for all β with $\gamma \leq \beta < \kappa$, there exists a homomorphism $g_\beta : G_\beta \rightarrow H$ such that g_β is an extension of f_α for each $\alpha < \gamma$, and $g_\beta(\gamma) = v_0$. Thus, we simply define f_γ by $f_\gamma(\alpha) = f_\alpha(\alpha)$ for each $\alpha < \gamma$ and $f_\gamma(\gamma) = v_0$.

By this method we construct the functions f_α for each $\alpha < \kappa$.

To complete the proof, we define a mapping $h : G \rightarrow H$ by $h(\alpha) = f_\alpha(\alpha)$ for each $\alpha < \kappa$. This is clearly a homomorphism, contradicting the choice of G . We may therefore conclude that H is compact. ■

Corollary 45 *Let H be a digraph. If H is $|H|^+$ -compact then H is compact.*

Proof: Let H be a $|H|^+$ -compact digraph. By corollary 42 H has a core K . Clearly

$|K| \leq |H|$ and $K \leftrightarrow H$ so K is $|H|^+$ -compact. Also, $|K| \leq |H|$ so K is certainly $|K|$ -compact, and so by theorem 44, K is compact. But $H \leftrightarrow K$ so H is also compact.

■

These results are somewhat surprising. They implies that if a digraph is not compact, then we need only look at digraphs of the next larger cardinality to find a certificate of non-compactness. Furthermore, if a core is not compact, then there is a certificate of non-compactness of the same cardinality. For digraphs which are not cores this is false. For example, a transitive tournament with vertex-set κ is κ -compact but not κ^+ -compact.

4.4 Families of Compact Digraphs

By now I'm sure you will all agree that compact digraphs are quite an interesting type of object. The problem now is to determine what kinds of digraphs are compact. In particular, we would like to construct some broad families of compact digraphs, or perhaps give some general sufficient conditions for a digraph to be compact.

In this section we will first prove a very general and rather unappealing technical lemma, which will subsequently prove its worth through a series of elegant corollaries. Before we begin we will need some terminology.

We will first generalize the notion of a list-homomorphism. Let G and H be digraphs, and let $l : V(G) \rightarrow \mathcal{P}(V(H))$ be a list-assignment for G with respect to H . An l -list-mapping is a mapping $f : G \rightarrow H$ such that $f(v) \in l(v)$ for each $v \in V(G)$. An l -list-homomorphism, then, is an l -list-mapping which is also a homomorphism.

Suppose G , H , and l are as above. We define $\mathcal{S} = \prod_{v \in V(G)} l(v)$, i.e. \mathcal{S} is the product of the sets $l(v)$. There is an obvious one-to-one correspondence between l -list-mappings and elements of \mathcal{S} . We will therefore consider such a mapping to be identical to the corresponding element of \mathcal{S} .

We will make use of a classic result in topology; which we state now.

Theorem 46 (Tychonoff) *The product of compact topological spaces is compact.*

The original proof of this may be found in [70], but the reader may prefer [45]. The property of compact topological spaces which we use is the following: given any collection \mathcal{C} of closed sets in a compact topological space, if the intersection of any finite subcollection of \mathcal{C} is nonempty, then the intersection of all of the sets in \mathcal{C} is nonempty. We may now state our result.

Lemma 47 *Let G and H be digraphs. Suppose there exists a function $l : V(G) \rightarrow \mathcal{P}(V(H))$, and for each $v \in V(G)$ a compact topology \mathcal{T}_v on $l(v)$, such that for every finite subdigraph $G' \subseteq G$:*

- *there exists an l -list-homomorphism $f : G' \rightarrow H$, and*
- *the set $\{g : g \text{ is a mapping from } V(G) \text{ to } V(H) \text{ and } g|_{G'} \text{ is an } l\text{-list-homomorphism from } G' \text{ to } H\}$ is closed in the product topology $\mathcal{T} = \prod_{v \in V(G)} \mathcal{T}_v$ on $\mathcal{S} = \prod_{v \in V(G)} l(v)$.*

Then $G \rightarrow H$.

Proof: Suppose that the conditions of the lemma are satisfied. Let \mathcal{T} be the product topology on \mathcal{S} . By Tychonoff's theorem \mathcal{T} is compact. For each finite subdigraph $G' \subseteq G$, let $F_{G'} \subseteq \mathcal{S}$ be the set of all mappings $h : V(G) \rightarrow V(H)$ such that $h|_{G'}$ is an l -list-homomorphism from G' to H . Each $F_{G'}$ is a nonempty closed set in the topological space $(\mathcal{S}, \mathcal{T})$. We claim that the intersection of the collection $\{F_{G'} : G' \text{ is a finite subdigraph of } G\}$ is nonempty. Since \mathcal{T} is compact it suffices to show that for any finite collection G_1, \dots, G_n of finite subdigraphs of G , the intersection $\bigcap_{i=1}^n F_{G_i}$ is nonempty. But given any finite collection G_1, \dots, G_n of finite subdigraphs of G , the digraph $G' = \bigcup_{i=1}^n G_i$ is a finite subdigraph of G , and so there is an l -list-homomorphism $f : G' \rightarrow H$. Let g be any mapping from G to H such that $g|_{G'} = f$. Then it will be the case that $g|_{G_i}$ is an l -list-homomorphism for each $1 \leq i \leq n$. Therefore $\bigcap_{i=1}^n F_{G_i}$ is non-empty. And so our claim is proved.

Now any element of the intersection of the collection $\{F_{G'} : G' \text{ is a finite subdigraph of } G\}$ is an l -list-homomorphism from G to H , and so we conclude that $G \rightarrow H$. ■

Tychonoff's theorem is an extremely useful tool in proving many different types of compactness theorems. In fact, the above lemma can be regarded as a generalization

of compactness results such as those found in [32, 62, 63]. These papers exploit the fact that if l is a finite set and \mathcal{T} is the discrete topology on l , then (l, \mathcal{T}) is compact and every subset of l is closed. This same property of finite sets will be the basis of the proofs of our first two corollaries.

Corollary 48 *Any finite digraph is compact.*

Proof: Suppose H is finite, and let G be any digraph such that all finite subdigraphs $G' \subseteq G$ admit homomorphisms to H . For each $v \in V(G)$ let $l(v) = V(H)$ and let \mathcal{T}_v be the discrete topology on $l(v)$. It is a simple matter to verify that the conditions of lemma 47 are satisfied. Thus $G \rightarrow H$ and so H is compact. ■

This result is not particularly surprising, and is a generalization of the well known compactness theorem for chromatic number [17], which states that a graph (or digraph) is n -colourable if and only if each of its finite subdigraphs is n -colourable. It can also be proved using a similar result found in [37], which states that a finite subdigraph H of a digraph G is a retract of G if and only if H is a retract of every finite subdigraph of G which contains H . Our next result characterizes a large class of infinite digraphs which are compact. We denote by $Aut(H)$ the automorphism group of H . We say that H is *locally finite* if for all $v \in V(H)$, both $N^+(v)$ and $N^-(v)$ are finite.

Theorem 49 *Let H be a locally finite digraph. If there are only finitely many orbits in $Aut(H)$ then H is compact.*

Proof: Let H be a digraph satisfying the conditions above, and let G be any digraph such that every finite $G' \subseteq G$ admits a homomorphism to H . We may assume without loss of generality that G is connected. Define $A \subseteq V(H)$ to be a set containing one vertex from each orbit of $Aut(H)$, so A is finite. Let v_0 be some fixed vertex in $V(G)$ and define $l(v_0) = A$. Clearly for any subdigraph $G' \subseteq G$ (not necessarily finite), there is a homomorphism $f : G' \rightarrow H$ if and only if there is such a homomorphism with $f(v_0) \in l(v_0)$.

We now define a finite set $l(u) \subseteq V(H)$ for each $u \in V(G)$. Given $u \in V(G)$, let $l(u) = \{y \in V(H) : d(y, A) \leq d(u, v_0)\}$. Since H is locally finite, $l(u)$ will be finite.

We will show that every finite subdigraph of G admits an l -list-homomorphism to H . Thus, let G' be a finite subdigraph of G . Let G'' be a finite subdigraph of G containing G' , and such that for any $u, v \in V(G')$, the distance between u and v in G'' is the same as their distance in G . Such a digraph is easily constructed by adding to G' a shortest oriented path from u to v in G , for each $u, v \in V(G')$. Since G'' is finite we have $G'' \rightarrow H$. And so if we choose a homomorphism $f : G'' \rightarrow H$ such that $f(v_0) \in A$, then $f(u) \in l(u)$ for all $u \in V(G')$. So $f|_{G'}$ will be an l -list-homomorphism from G' to H .

If we now assign the discrete topology to each set $l(v)$, we may apply lemma 47 to conclude $G \rightarrow H$, and so H is compact. ■

As we mentioned before, both of these results exploit finitary properties of the digraphs in question. In the first case the digraphs are finite, and in the second case the digraphs are locally finite. Our next result deals with digraphs that do not possess any such finitary property, except of course in the sense that compactness is itself a finitary property.

Before stating the result we note that applying lemma 47 is often simplified by the following observation. Let $G, H, l(v)$ and \mathcal{T}_v be given as usual. Suppose that G' is a finite subdigraph of G , and let $\mathcal{S}' = \prod_{v \in V(G')} l(v)$ and $\mathcal{T}' = \prod_{v \in V(G')} \mathcal{T}_v$. It is a simple matter to verify that the set $\{g : g \text{ is a mapping from } V(G) \text{ to } V(H) \text{ and } g|_{G'} \text{ is an } l\text{-list-homomorphism}\}$ is closed in $(\mathcal{S}, \mathcal{T})$ if and only if the set $X = \{g : g \text{ is an } l\text{-list-homomorphism from } G' \text{ to } H\}$ is closed in $(\mathcal{S}', \mathcal{T}')$. But X is closed if and only if its complement X^c is open. An open set in $(\mathcal{S}', \mathcal{T}')$ is just a product $\prod_{v \in V(G')} O_v$ where each O_v is open in $(l(v), \mathcal{T}_v)$. Thus, it is sufficient to show that if $f : V(G') \rightarrow V(H)$ is not a homomorphism, then for each $v \in V(G')$, there is an open set N_v in $l(v)$ containing $f(v)$ such that no $g \in \prod_{v \in V(G')} N_v$ is a homomorphism.

We use \mathbb{R} to denote the set of real numbers.

Theorem 50 *Let $\mathcal{M} = (M, d)$ be a metric space, and let C be a compact subset of \mathbb{R} . Define a digraph H by $V(H) = M$ and $E(H) = \{uv : d(u, v) \in C\}$. If either*

i) \mathcal{M} is compact, or

ii) every closed and bounded subspace of \mathcal{M} is compact and $\text{Aut}(H)$ has only finitely many orbits,

then H is compact.

Proof: Let H be a digraph as defined above and let G be any digraph. Assume without loss of generality that G is connected. Suppose that every finite subdigraph of G admits a homomorphism to H . We will define for each $v \in V(G)$ a set $l(v) \subseteq V(H)$ and a compact topology \mathcal{T}_v on $l(v)$ so that every finite subdigraph of G admits an l -list-homomorphism to H .

Case 1: (i) holds.

For all $v \in V(G)$ let $l(v) = V(H)$ and let \mathcal{T}_v be the metric topology given by \mathcal{M} on $l(v)$. Every finite subdigraph of G clearly admits an l -list-homomorphism to H .

Case 2: (ii) holds.

Let $A \subseteq V(H)$ contain exactly one element from each orbit of $\text{Aut}(G)$. Choose some arbitrary $v_0 \in V(G)$ and define $l(v_0) = A$. Now note that the set C must be closed and bounded, so let $r = \max\{x : x \in C\}$. For any $v \in V(G) - \{v_0\}$, let k be the length of a shortest oriented path from v to v_0 in G . Let $l(v) = \{w : d(w, A) \leq kr\}$. Now for all $v \in V(G)$ let \mathcal{T}_v be the metric topology given by \mathcal{M} on $l(v)$. Since $l(v)$ is closed and bounded, the topology \mathcal{T}_v is compact. It is clear that every finite subdigraph of G admits an l -list-homomorphism to H .

Having defined our lists and topologies in one of the above ways, it remains only to show that for any finite $G' \subseteq G$, the set of mappings $f : G \rightarrow H$ such that $f|_{G'}$ is an l -list-homomorphism is a closed subset of $\mathcal{S} = \prod_{v \in V(G)} l(v)$ under the product topology $\mathcal{T} = \prod_{v \in V(G)} \mathcal{T}_v$. To do this, we will show that given G' and a mapping $f : V(G') \rightarrow V(H)$ which is not a homomorphism, there exists a neighbourhood $N_v \subseteq l(v)$ of $f(v)$ such that no $g \in \prod_{v \in V(G')} N_v$ is a homomorphism.

Suppose that $f : V(G') \rightarrow V(H)$ is not a homomorphism. Then there exist $u, v \in V(G')$ such that $uv \in E(G')$ but $f(u)f(v) \notin E(H')$. Then, recalling that $f(u)$ and $f(v)$ are points in the metric space \mathcal{M} , we know that $d(f(u), f(v)) \notin C$. But C is closed, so there exists an $\epsilon > 0$ such that for all $x \in \mathfrak{R}$, $|x - d(f(u), f(v))| < \epsilon$ implies that $x \notin C$. Therefore, for all $r, s \in M$, if $d(f(u), r) < \epsilon/2$ and $d(f(v), s) < \epsilon/2$, it

must be the case that $|d(r, s) - d(f(u), f(v))| < \epsilon$, applying the triangle inequality. So if we let N_u and N_v be the neighbourhoods of radius $\epsilon/2$ around $f(u)$ and $f(v)$, respectively, in \mathcal{T}_u and \mathcal{T}_v , then no vertex in N_u is adjacent to any vertex in N_v in H . Now define $N_w = l(w)$ for all $w \neq u, v$ in $V(G')$. No $g \in \prod_{w \in V(G')} N_w$ will be a homomorphism, since $g(u)g(v) \notin E(H)$. It follows that the set $\{f \in \mathcal{S} : f|_{G'} \text{ is an } l\text{-list-homomorphism}\}$ is closed in $(\mathcal{S}, \mathcal{T})$, and so lemma 47 applies. We conclude that $G \rightarrow H$. ■

Note that since the distance function d is symmetric, all relations in these digraphs will be symmetric, and so they may be considered to be graphs. We will continue to regard them as digraphs, although for simplicity in our diagrams we will often draw pairs of directed edges as a single undirected edge.

This last result allows us to construct some particularly interesting compact digraphs. Define a digraph D by $V(D) = \mathfrak{R}^2$, *i.e.* points in the plane, and $E(D) = \{\{u, v\} : d(u, v) = 1\}$, where d is the usual metric on \mathfrak{R}^2 . This digraph has been studied extensively in the literature [20, 35], and has several interesting open problems associated with it. For example, it is quite simple to show that $4 \leq \chi(D) \leq 7$, but no improvement on these bounds is known. The properties of this digraph which are of interest to us are given by the following theorem.

Theorem 51 *D is a compact core.*

Proof: That D is compact is a simple corollary of theorem 50, since $\{1\}$ is a compact subset of \mathfrak{R} , D is vertex-transitive, and closed bounded sets in \mathfrak{R}^2 are compact. The fact that D is a core is more difficult to prove. We will prove the stronger claim that any endomorphism of D is a rigid transformation of \mathfrak{R}^2 .

For any three vertices $\{v_1, v_2, v_3\} \subset V(D)$, the vertices $\{v_1, v_2, v_3\}$ induce a K_3 in D if and only if the corresponding points in \mathfrak{R}^2 are the vertices of an equilateral triangle with side length one. Since the homomorphic image of K_3 must be another K_3 , any endomorphism of D must be a rigid transformation of these three points. Thus, to prove our claim it suffices to show that any endomorphism of D which fixes $\{v_1, v_2, v_3\}$ pointwise must be the identity.

Let f be an endomorphism of D which fixes $\{v_1, v_2, v_3\}$. We will first show that D must fix the vertices of the triangular lattice containing $\{v_1, v_2, v_3\}$ (see figure 4.1).

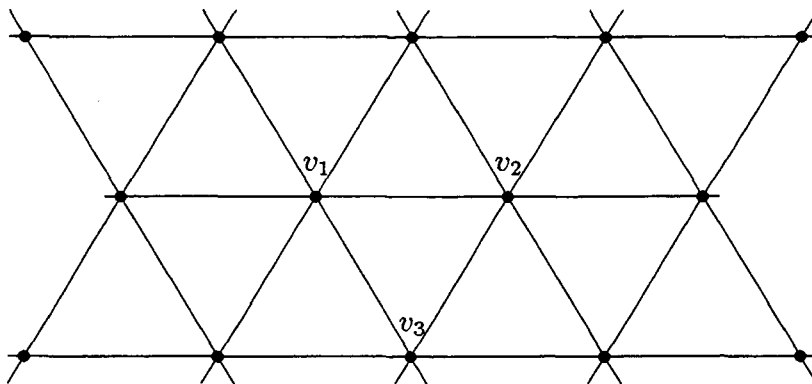


Figure 4.1

We will do this by showing that if any triangle $\{u_1, u_2, u_3\}$ in the lattice is fixed, then the lattice point which is the unique common neighbour of u_2 and u_3 other than u_1 must also be fixed. A moment's reflection is all that is required to see that this will force every vertex in the lattice to be fixed.

Suppose that a triangle $\{u_1, u_2, u_3\}$ is fixed by f . There is a subdigraph of D containing $\{u_1, u_2, u_3\}$ as indicated in figure 4.2. Here u is the common neighbour of u_2 and u_3 not equal to u_1 . Since $\{u_1, u_2, u_3\}$ are fixed, $f(u) = u$ or $f(u) = u_1$, so either $d(u_1, f(u)) = \sqrt{3}$ or $d(u_1, f(u)) = 0$.

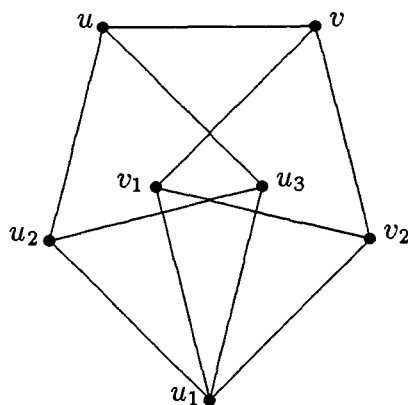


Figure 4.2

The triangles $\{u_1, v_1, v_2\}$ and $\{v, v_1, v_2\}$ must map to triangles with the edge v_1v_2 in common, and so $d(u_1, f(v)) = \sqrt{3}$ or $d(u_1, f(v)) = 0$. So if $f(u) = u_1$ then it must be the case that either $d(f(u), f(v)) = \sqrt{3}$ or $d(f(u), f(v)) = 0$. But $d(f(u), f(v))$ must be 1. Thus, $f(u) = u$, and so we may conclude that f fixes the entire lattice.

We must now show that f fixes every point in the plane. Suppose there is a vertex v such that $f(v) \neq v$. We claim that there is a lattice vertex w and an integer k such that $d(w, v) \leq k$ and $d(w, f(v)) > k$. To see this, let l be a line which is perpendicular to some edge of the triangular lattice, and contains a lattice point. Then l contains infinitely many lattice points which are spaced at a distance of $\sqrt{3}$ from each other. Furthermore, choose l so that the perpendicular projections of v and $f(v)$ on l are distinct (figure 4.3).

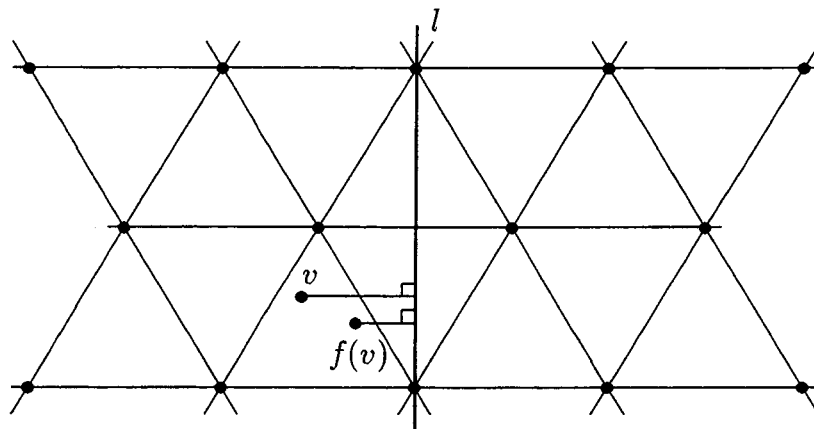


Figure 4.3

Let p_v and $p_{f(v)}$ be the perpendicular projections of v and $f(v)$ onto l . Let p be a point on l midway between p_v and $p_{f(v)}$ and let $r = d(p, p_v)$. Let W be the set of lattice points on l which are outside the interval $[p_v, p_{f(v)}]$ and are on the same side of the interval as p_v (figure 4.4).

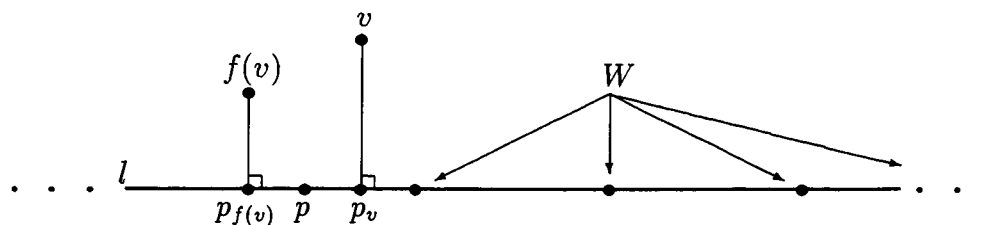


Figure 4.4

Every point in W will have distance $c + k\sqrt{3}$ from p , where c is a positive constant and k ranges over the non negative integers. Hereafter we will use $\langle x \rangle$ to denote the fractional part of a real number x , i.e. $\langle x \rangle = x - [x]$. Since $\sqrt{3}$ is irrational, the set $\{c + k\sqrt{3}\}$, where k ranges over the non-negative integers, is dense in $[0, 1]$. Thus, for every $\varepsilon > 0$ there exist infinitely many $w \in W$ such that $\langle d(w, p) \rangle < \varepsilon$, and so there exist such w arbitrarily far from p , p_v , and $p_{f(v)}$. For each $w \in W$ it is clearly true that $d(w, p_v) < d(w, v)$ and $d(w, p_{f(v)}) < d(w, f(v))$.

Consider any fixed $\varepsilon > 0$. For each $w \in W$ which is sufficiently far from p_v and $p_{f(v)}$, we have $d(w, v) - d(w, p_v) < \varepsilon$. Thus, there is a $w \in W$ such that $\langle d(w, p) \rangle < \varepsilon$,

$|d(w, v) - d(w, p_v)| < \varepsilon$ and $|d(w, f(v)) - d(w, p_{f(v)})| < \varepsilon$. We now take $\varepsilon = r/2$, and so there exists a $w \in W$ such that

$$\begin{aligned} d(w, v) &< d(w, p_v) + r/2 = d(w, p) - r/2 \\ &< d(w, p) - [d(w, p) - \lfloor d(w, p) \rfloor] = \lfloor d(w, p) \rfloor \end{aligned}$$

and

$$d(w, f(v)) > d(w, p_{f(v)}) > d(w, p) > \lfloor d(w, p) \rfloor.$$

Thus, if we set $k = \lfloor d(w, p) \rfloor$ we have $d(w, v) < k$ and $d(w, f(v)) > k$, and so our claim is proved.

Now since $d(w, v) < k$ it is easily seen that there is a directed path of length k from w to v in D . Thus, there must be a directed walk of length k from $f(w)$ to $f(v)$ in D . But this is impossible, as $f(w) = w$ and $d(w, f(v)) > k$.

We conclude that f is the identity mapping, and so any endomorphism of D is a rigid transformation of the plane. Obviously a rigid transformation of the plane is an automorphism of D , and so D is a core. ■

We may construct higher-dimensional analogues of D in a natural way. We simply let the vertex-set be \mathfrak{R}^n , and define the edge-set exactly as we did for D . The proof of the preceding theorem will also generalize to these digraphs, the major difference being that the triangles in the graph in figure 4.2 will be replaced by copies of K_{n+1} .

In general the digraphs constructed using the theorems in this section need not be cores. The preceding example is particularly interesting because D is a compact core with $|V(D)| = 2^{\aleph_0}$ (cf. theorem 52).

4.5 Structures and Graphs

All of the results from sections 4.2 and 4.3 apply to structures and graphs, and the proofs are essentially the same.

We may prove a result similar to lemma 47 in section 4.4. However, note that we have been forced to replace the term 'finite' in the statement of the lemma by

‘finitely induced’ to account for the possibility that our structures may be defined for an infinite language.

Lemma 47' *Let G and H be structures. Suppose there exists a function $l : V(G) \rightarrow \mathcal{P}(V(H))$, and a compact topology \mathcal{T}_v on $l(v)$, for each $v \in V(G)$, with the following properties for every finitely induced substructure $G' \subseteq G$:*

- *there exists an l -list-homomorphism $f : G' \rightarrow H$, and*
- *the set of mappings $\{(g : G \rightarrow H) : g|_{G'} \text{ is an } S\text{-list-homomorphism}\}$ is closed in the product topology $\mathcal{T} = \prod_{v \in V(G)} \mathcal{T}_v$ on $\mathcal{S} = \prod_{v \in V(G)} (l(v))$.*

Then $G \rightarrow H$.

The proof of lemma 47' is now identical to the proof of lemma 47, with the term ‘finite’ uniformly replaced by ‘finitely induced’.

The original statement of lemma 47 applies to graphs without modification.

We may prove a stronger version of corollary 48 for structures. The original statement holds for graphs.

Corollary 48' *Any finitely induced structure is compact.*

Proof: If R_1 and R_2 are two relations defined on a structure G , we will say that R_1 and R_2 are equivalent if $R_1(\bar{v})$ if and only if $R_2(\bar{v})$ for all \bar{v} . We first claim that it is sufficient to prove that any finite structure is compact, for if G is a finitely induced structure, then only finitely many pairwise inequivalent relations can be defined on $V(G)$. If G is not compact, then let H be a certificate of noncompactness for G . We will construct a noncompact finite structure G' and a certificate of noncompactness H' . Let $V(G') = V(G)$, $V(H') = V(H)$ and for each equivalence class $\{R_i : i \in I\}$ of relations on G we select one representative R and define $R(G') = R(G)$, $R(H') = \bigcup_{i \in I} R_i(H)$. Now G' is finite, since $V(G)$ is finite and only finitely many relations are defined on G' . Also, any finitely induced substructure of H' is finite. Finally, it is clear that given any set $S \subseteq V(H)$ and any mapping $f : S \rightarrow V(G)$, f is a homomorphism from $H[S]$ to G if and only if f is a homomorphism from $H'[S]$ to G' . But every finitely induced substructure of H admits a homomorphism to G , and so every finitely induced

(and therefore finite) substructure of H' admits a homomorphism to G' . However, $H \not\rightarrow G$ so $H' \not\rightarrow G'$, and so the claim is proven. ■

We may also restate theorem 49 for structures. In fact, in this case the theorem statement is identical, but we must define what it means for a structures to be locally finite. We will say that a structure G is locally finite if for each $v \in V(G)$ and each $R \in \mathcal{L}$, v occurs in only finitely many R -edges of G . Note that if \mathcal{L} is infinite then v may occur in infinitely many edges of different types.

The proof of theorem 49 is now easily modified to apply to structures. Again the original proof applies directly to graphs.

Chapter 5

Classes of Digraphs

5.1 Definitions

Up until now we have been examining properties of individual digraphs. We now turn our attention to properties of broad classes of digraphs. In the first section of this chapter we will examine the class of all compact digraphs. We will determine exactly how many compact cores there are, and show that they may be partially ordered in a natural way to produce a distributive lattice.

In all sections of this chapter, we will use \rightarrow to impose a binary relation on the digraphs under discussion, *i.e.* G is related to H if $G \rightarrow H$. This relation is reflexive and transitive, and so defines a natural partial order on homomorphic equivalence classes of digraphs. Many interesting properties of this ordering are examined in [21]. One very special digraph which will occur occasionally in this chapter consists of a single vertex v with a loop vv . This digraph is named *one* because it is maximum with respect to \rightarrow , that is, $G \rightarrow one$ for all digraphs G . This immediately implies that *one* is compact.

We now define three operations which we will use throughout this chapter. Let G and H be digraphs. As usual, we denote by $G \cup H$ the disjoint union of G and H . We denote by $G \times H$ the categorical product of G and H , that is, $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(r, u)(s, v) : rs \in E(G) \text{ and } uv \in E(H)\}$. We denote by H^G the digraph defined by $V(H^G) = \{f : f \text{ is a mapping from } V(G) \text{ to } V(H)\}$ and

$E(H^G) = \{fg : uv \in E(G) \text{ implies } f(u)g(v) \in E(H)\}$. This digraph has also been called the *map-graph* [36].

It is a simple exercise to verify the following facts [21]:

- $G \rightarrow G \cup H$ and $H \rightarrow G \cup H$,
- if $G \rightarrow K$ and $H \rightarrow K$ then $G \cup H \rightarrow K$,
- $G \times H \rightarrow G$ and $G \times H \rightarrow H$,
- if $K \rightarrow G$ and $K \rightarrow H$ then $K \rightarrow G \times H$,
- $one \rightarrow H^G$ if and only if $G \rightarrow H$.

For any digraph G we define $[G]$ to be the class of all digraphs which are homomorphically equivalent to G . It is easy to verify that if $[G_1] = [G_2]$ and $[H_1] = [H_2]$, then $[G_1 + H_1] = [G_2 + H_2]$, $[G_1 \times H_1] = [G_2 \times H_2]$, and $[G_1^{H_1}] = [G_2^{H_2}]$. Thus, we may define \cup , \times , and exponentiation for two equivalence classes of digraphs by applying the given operation to two arbitrary representatives of the classes. Wherever it is appropriate to the context we will take G to mean $[G]$.

5.2 The Class of Compact Digraphs

We define \mathcal{C} to be the class of all compact digraphs. Observe that \mathcal{C} is always a proper class, since for any cardinal κ , an independent set of size κ is a compact digraph. However, we may still hope to reduce \mathcal{C} to a more reasonable size by partitioning it into homomorphic equivalence classes. We define \mathcal{E} to be the class of all homomorphic equivalence classes of compact digraphs. Strictly speaking, we define \mathcal{E} to be a class containing one representative from each homomorphic equivalence class of compact digraphs, since a class cannot be an element of a class. Since every compact digraph contains a unique core, and the cores of homomorphically equivalent digraphs are isomorphic, we know that every homomorphic equivalence class of digraphs contains exactly one core.

We will denote by $S(G)$ the set of all finite subdigraphs of G , sometimes called the *age* of G [18].

Our first result shows that \mathcal{E} is a set.

Theorem 52 *The class \mathcal{E} is a set and $|\mathcal{E}| = 2^{\aleph_0}$.*

Proof: Observe that if G and H are compact digraphs such that $S(G) = S(H)$, then $G \leftrightarrow H$. In other words, if $G \not\leftrightarrow H$ then $S(G) \neq S(H)$. Thus, there can be no more equivalence classes of compact digraphs than there are distinct sets of finite digraphs.

Clearly there are exactly \aleph_0 distinct finite digraphs and so there are no more than 2^{\aleph_0} equivalence classes of compact digraphs.

We must now show that there are at least this many distinct equivalence classes. It suffices to show that there exists a set of 2^{\aleph_0} pairwise inequivalent compact digraphs. Let \mathcal{P}_2 be the infinite incompatible set of digraphs defined in the proof of lemma 7, *i.e.*, the set \mathcal{P}_2 is the set of all directed cycles of prime length. We claim that for any nonempty subset $\mathcal{G} \subseteq \mathcal{P}_2$, the digraph $H_{\mathcal{G}} = \bigcup_{G \in \mathcal{G}} G$ is compact. Let G be any digraph and suppose that every finite subdigraph $G' \subseteq G$ admits a homomorphism to $H_{\mathcal{G}}$. Assume without loss of generality that G is connected.

Observe that $H_{\mathcal{G}}$ is a disjoint union of directed cycles of distinct prime lengths. If G contains no cycle C with $net(C) > 0$ then it is a simple matter to verify that $G \rightarrow D_n$ for any directed cycle D_n , and so $G \rightarrow H_{\mathcal{G}}$. On the other hand, if G contains a cycle C with $net(C) = k > 0$, then by lemma 6 we know that $C \rightarrow D_n$ if and only if $n|k$. This trivially implies that $n \leq k$, and so there are only finitely many components $\{D_{p_1}, \dots, D_{p_r}\}$ of $H_{\mathcal{G}}$ such that $C \rightarrow D_{p_i}$. Let $D = \bigcup_{i=1}^r D_{p_i}$. Since G is connected we know that $G \rightarrow H_{\mathcal{G}}$ if and only if $G \rightarrow D$. But D is finite, and so is compact. Furthermore, every finite subdigraph of G admits a homomorphism to D , since every finite subdigraph of G is contained in a finite connected subdigraph of G which also contains C . Thus, $G \rightarrow H_{\mathcal{G}}$.

Lemma 6 also guarantees that if $\mathcal{G}_1 \neq \mathcal{G}_2$ then $H_{\mathcal{G}_1} \not\leftrightarrow H_{\mathcal{G}_2}$. Since \mathcal{P}_2 , is countably infinite, it has 2^{\aleph_0} nonempty subsets, and so we are done. ■

Corollary 53 *There are exactly 2^{\aleph_0} compact cores.* ■

Note that in the above proof the 2^{\aleph_0} inequivalent digraphs we construct all have countable vertex-sets. In other words, cardinality arguments give us no reason to believe that compact cores of uncountable size exist. However, the digraph D constructed in theorem 51 demonstrates that such objects do exist.

Open Problem 54 *What is the maximum cardinality of a compact core?*

We now know that \mathcal{E} is a set, so if we impose the partial order \rightarrow on \mathcal{E} we obtain a partially ordered set. Our next few results, in the spirit of [21], show that the partially ordered set $(\mathcal{E}, \rightarrow)$ is in fact a distributive lattice with exponentiation.

We claim that \cup and \times define join and meet operations, respectively, for $(\mathcal{E}, \rightarrow)$, and so we must show that the class of compact digraphs is closed under \cup and \times .

Lemma 55 *If G and H are compact digraphs then $G \cup H$ and $G \times H$ are compact.*

Proof: Let G and H be compact digraphs. It suffices to show that $G \cup H$ and $G \times H$ are compact with respect to every connected digraph. Thus, let K be a connected digraph such that all finite subdigraphs $K' \subseteq K$ admit a homomorphism to $G \cup H$. It cannot be the case that there exist finite subdigraphs $K', K'' \subseteq K$ such that $K' \not\rightarrow G$ and $K'' \not\rightarrow H$, for then there would be a finite connected subdigraph of K containing K' and K'' . This finite subdigraph of K would admit no homomorphism to $G \cup H$.

Thus it must be the case that every finite subdigraph K' of K admits a homomorphism to G , or every finite subdigraph K' of K admits a homomorphism to H . By compactness of G and H we have $K \rightarrow G$ or $K \rightarrow H$, so $K \rightarrow G \cup H$, and so we conclude that $G \cup H$ is compact.

Suppose now that all finite $K' \subseteq K$ admit a homomorphism to $G \times H$. Then since $G \times H \rightarrow G$ and $G \times H \rightarrow H$, all such finite K' admit homomorphisms to both G and H . And so by compactness of G and H there exist homomorphisms $f : K \rightarrow G$ and $g : K \rightarrow H$. We now define $h : K \rightarrow G \times H$ by $h(v) = (f(v), g(v))$. It is straightforward to verify that h is a homomorphism. ■

At this point it is not difficult to verify that \cup and \times satisfy the properties of the join and meet operations in $(\mathcal{E}, \rightarrow)$. It is also a simple matter to show that \cup and \times satisfy the distributive laws.

We may show that H^G defines an exponentiation operation for \mathcal{E} . To do so we must prove that if G and H are compact then H^G is also compact. We will in fact prove the following stronger result.

Lemma 56 *A digraph H is compact if and only if H^G is compact for every digraph G .*

Proof: We refer the reader to [21] for proofs of the following facts: If G and H are any digraphs, then $H^G \times G \rightarrow H$, and for every digraph K , if $K \times G \rightarrow H$ then $K \rightarrow H^G$.

Suppose that H is compact and let G and Z be arbitrary digraphs such that every finite subdigraph of Z admits a homomorphism to H^G . Let W be an arbitrary finite subdigraph of $Z \times G$. Then for some finite $Z' \subseteq Z$ and $G' \subseteq G$ we must have $W \subseteq Z' \times G'$. But $Z' \rightarrow H^G$ and $G' \rightarrow G$ so $Z' \times G' \rightarrow H^G \times G \rightarrow H$. Thus, every finite subdigraph of $Z \times G$ admits a homomorphism to H , and so $Z \times G \rightarrow H$. Therefore $Z \rightarrow H^G$.

Now suppose that H is not compact. Then $H^{one} \leftrightarrow H$ is not compact. It is also easy to prove the more interesting fact that if G is a certificate of non-compactness for H , then H^G is not compact. ■

Observe that the lattice $(\mathcal{E}, \rightarrow)$ has a maximum element, namely *one*, and a minimum element, namely the trivial digraph with a single vertex and no edges.

5.3 Finite Equivalence

In the study of compact digraphs, we are interested in studying the connection between the existence of homomorphisms of infinite digraphs and the existence of homomorphisms of finite digraphs. Within this context, it may be useful to examine a class of digraphs with the property that any finite subdigraph of a digraph in the class admits a homomorphism to any other digraph in the class. We say that two digraphs G and H are *finitely equivalent* if for any finite $G' \subseteq G$ we have $G' \rightarrow H$, and *vice versa*. This is easily seen to be an equivalence relation.

Digraphs which are homomorphically equivalent are certainly finitely equivalent, but finitely equivalent digraphs may not be equivalent (consider, for example, the Ray and the Line). In this section we will examine classes of digraphs which are finitely equivalent. Let $\mathcal{C}(G)$ be the class of all digraphs which are finitely equivalent to G , and let $\mathcal{F}(G)$ be the class of homomorphic equivalence classes of $\mathcal{C}(G)$. Again, we should strictly consider $\mathcal{F}(G)$ to be a class containing a representative from each homomorphic equivalence class of $\mathcal{C}(G)$. We partially order $\mathcal{F}(G)$ by \rightarrow .

We will discover in this section that $\mathcal{F}(G)$ can be a proper class. In other words, it is possible for there to exist a proper class of pairwise inequivalent digraphs which are pairwise finitely equivalent. In fact this will turn out to be true for ‘most’ digraphs.

We will also show that $(\mathcal{F}(G), \rightarrow)$ has the properties of a distributive lattice. It is not, strictly speaking, a lattice because a lattice must be defined on a set rather than a class. However, by slight abuse of terminology we will call an object a lattice if its ordering satisfies the properties of a lattice, even if it is defined over a proper class.

We again define the join and meet operations by using \cup and \times , respectively.

Theorem 57 *For any digraph G , $(\mathcal{F}(G), \rightarrow)$ is a distributive lattice.*

Proof: It suffices to show that $\mathcal{C}(G)$ is closed under \cup and \times . If $H_1, H_2 \in \mathcal{C}(G)$, then certainly any finite subdigraph of H_1 admits a homomorphism to $H_1 \cup H_2$. If K is a finite subdigraph of $H_1 \cup H_2$, on the other hand, then any component of K must be a finite subdigraph of either H_1 or H_2 , and so in either case admits a homomorphism to H_1 . Thus, $H_1 \cup H_2$ is finitely equivalent to H_1 . Note that this argument applies equally well to arbitrary unions of digraphs.

Now, any finite subdigraph of H_1 admits a homomorphism to both H_1 and H_2 , and so admits a homomorphism to $H_1 \times H_2$. Also, any finite subdigraph of $H_1 \times H_2$ admits a homomorphism to H_1 , since $H_1 \times H_2 \rightarrow H_1$. Thus, $H_1 \times H_2$ is finitely equivalent to H_1 , and, by a symmetric argument, to H_2 . ■

Unlike \mathcal{E} in the previous section, $\mathcal{F}(G)$ is not generally closed under exponentiation. For example, given any digraph G , $G^G \leftrightarrow one$ since $G \rightarrow G$, but $one \notin \mathcal{F}(G)$ unless G contains a loop.

The class $\mathcal{F}(G)$ is also unlike \mathcal{E} in that it does not always contain a maximum element with respect to \rightarrow , although it does always have a minimum element. Our next theorem proves this and more. We will denote by S_G the digraph which is the disjoint union of all finite subdigraphs of G .

Theorem 58 *For any G , the lattice $(\mathcal{F}(G), \rightarrow)$ has a maximal element if and only if $\mathcal{F}(G)$ contains a compact digraph. Furthermore, this compact digraph is the maximum element of $(\mathcal{F}(G), \rightarrow)$. The lattice $(\mathcal{F}(G), \rightarrow)$ always has a minimum element.*

Proof: A moment's reflection should convince the reader that G is compact if and only if G is the maximum element of $\mathcal{F}(G)$. Formally: if $(\mathcal{F}(G), \rightarrow)$ contains a maximal element G , then G must be maximum, since if $H \not\rightarrow G$ for some $H \in \mathcal{F}(G)$ then $G \cup H$ is strictly greater than G . If G is maximum then given any H , all of whose finite subdigraphs admit homomorphisms to G , the digraph $H \cup S_G$ will be finitely equivalent to G , so $H \cup S_G \rightarrow G$, and therefore $H \rightarrow G$. Thus G is compact. If G is compact, then clearly $H \rightarrow G$ for any H finitely equivalent to G .

The digraph S_G is always a minimum element of $(\mathcal{F}(G), \rightarrow)$, since each component of S_G is finite, and so maps to every other element of $\mathcal{F}(G)$. ■

We mentioned before that $\mathcal{F}(G)$ may not be a set. Our next series of results deal with the possible sizes of $\mathcal{F}(G)$. We determine that in most cases $\mathcal{F}(G)$ is a proper class and that in many of the remaining cases it consists of a single element.

Lemma 59 *The class $\mathcal{F}(G)$ consists of a single homomorphic equivalence class if and only if there exists a digraph H such that $H \leftrightarrow G$, H is compact, and H is a disjoint union of finite digraphs.*

Proof: Suppose that $|\mathcal{F}(G)| = 1$. Let $H = S_G$, so H is a disjoint union of finite digraphs. Obviously H is finitely equivalent to G , and so $H \leftrightarrow G$. Also, H is trivially maximum in $\mathcal{F}(G)$, and so H is compact.

Now suppose there exists a compact digraph H which is a disjoint union of finite digraphs and is homomorphically equivalent to G . Clearly $H \leftrightarrow S_H$, and so H is minimum in $\mathcal{F}(G)$. If $|\mathcal{F}(G)| > 1$ then H is not maximum in $\mathcal{F}(G)$, and so H is not compact, a contradiction. ■

Obviously any finite digraph satisfies the conditions of this lemma. We may also construct infinite compact cores which are disjoint unions of finite digraphs.

Example 60 *Let H be the disjoint union of all directed cycles of prime length. Then H is a compact core.*

Clearly every component of H is a finite core, and no component of H admits a homomorphism to any other component of H , so H is a core. Also, in the proof of theorem 52 we showed that any disjoint union of directed cycles of prime length is compact.

In the remainder of this section we will prove the rather impressive fact that in most cases where $|\mathcal{F}(G)| \neq 1$, it is a proper class. Our first result treats all cases where G is not finitely equivalent to a compact digraph.

Lemma 61 *If $\mathcal{F}(G)$ does not contain a compact digraph, then it is a proper class.*

Proof: Suppose that $\mathcal{F}(G)$ is not a proper class. Then $\mathcal{F}(G)$ is a set, so we may define a new digraph G^* by $G^* = \bigcup_{H \in \mathcal{F}(G)} H$. The digraph G^* is finitely equivalent to G , and any $H \in \mathcal{F}(G)$ will obviously map to G^* . Thus G^* is a maximum element of $\mathcal{F}(G)$, and so is compact. ■

This result allows us to restrict our attention to digraphs G which are finitely equivalent to some compact digraph H . Since in this case $\mathcal{F}(G) = \mathcal{F}(H)$, it will be sufficient for our purposes to restrict our attention to the behavior of $\mathcal{F}(H)$ when H is a compact digraph.

Further results rely on the following lemma, in which we show that a certain density property is sufficient for $\mathcal{F}(G)$ to be a proper class.

Lemma 62 *Let H be a compact core which has an infinite component C , and let H' be a digraph finitely equivalent to H such that C is a component of H' and such that $C \not\rightarrow H' - C$. Suppose that for any $G \in \mathcal{F}(H)$ such that $G \rightarrow H'$ and $H' \not\rightarrow G$, there exists a K such that $G \rightarrow K \rightarrow H'$ but $H' \not\rightarrow K \not\rightarrow G$. Then $|\mathcal{F}(H)| = 1$ or $\mathcal{F}(H)$ is a proper class.*

Proof: Suppose that $\mathcal{F}(H)$ has the density property described above, and that $\mathcal{F}(H)$ is a set with $|\mathcal{F}(H)| > 1$. Let C and H' be as stated above. Note that C is a core, since it is a component of a core. Let $\mathcal{U} = \{G \in \mathcal{F}(H) : G \rightarrow H'\}$.

We define a digraph $H_C = \bigcup_{K \in \mathcal{U}}^{C \not\rightarrow K} K$. Observe that this union is nonempty, since $S_H \in \mathcal{U}$ and $C \not\rightarrow S_H$. Since H_C is a union of digraphs which are finitely equivalent to H' , the digraph H_C is finitely equivalent to H' . Clearly $H_C \rightarrow H'$. Also, since C is connected and C does not admit a homomorphism to any component of H_C we know that $C \not\rightarrow H_C$, and so $H' \not\rightarrow H_C$. By the density property, there must exist a digraph K such that $H_C \rightarrow K \rightarrow H'$ but $H' \not\rightarrow K \not\rightarrow H_C$.

Let K be such a digraph. Every finite subdigraph of K admits a homomorphism to H' , and every finite subdigraph of H' admits a homomorphism to H_C , and therefore to K . Thus K is finitely equivalent to H' , and so $K \in \mathcal{U}$. If $C \not\rightarrow K$ then $K \rightarrow H_C$ a contradiction. On the other hand, if $C \rightarrow K$ then let $H'' = (H' - C) \cup S_C$, i.e., replace C by the union of its finite subdigraphs. We know that $C \not\rightarrow S_C$, since C is an infinite core, and $C \not\rightarrow H' - C$, so $C \not\rightarrow H''$. But clearly $H'' \in \mathcal{U}$, and so $H'' \rightarrow H_C$. Hence $H'' \rightarrow K$. But H'' contains every component of H' except C , and so every component of H' other than C admits a homomorphism to K . We have also assumed that $C \rightarrow K$, and so $H' \rightarrow K$, a contradiction. Thus, K cannot exist. ■

Corollary 63 *Let H be a compact digraph such that $|\mathcal{F}(H)| > 1$. Suppose that for each $G \in \mathcal{F}(H)$ such that $H \not\rightarrow G$, there exists a K such that $G \rightarrow K \rightarrow H$ but $H \not\rightarrow K \not\rightarrow G$. Then $\mathcal{F}(H)$ is a proper class.*

Proof: We may assume that H is a core. Since $|\mathcal{F}(H)| > 1$, we know that H is not a disjoint union of finite digraphs by lemma 59. Thus, let C be an infinite component of H . Clearly $C \not\rightarrow H - C$, so we may take H' to be H and apply lemma 62. ■

It is interesting to note that the key to this lemma and its corollary is the simple fact that we may take the union of a set of digraphs, but not of a proper class of digraphs. It is also quite surprising that a very simple density condition is sufficient to force $\mathcal{F}(G)$ to be a proper class.

The following lemma, and its use in proving density results, are due to Perles [60].

Lemma 64 *Let G and H be digraphs such that $G \rightarrow H$ but $H \not\rightarrow G$. If there exists a digraph K such that $H \not\rightarrow K \cup G$ and $K \not\rightarrow G^H$, then there exists a digraph K' such that $G \rightarrow K' \rightarrow H$ and $H \not\rightarrow K' \not\rightarrow G$.*

Proof: Suppose that such a K exists for a given G and H . Since $K \not\rightarrow G^H$, we have $K \times H \not\rightarrow G$. Let $K' = (K \times H) \cup G$. Then $K' \not\rightarrow G$ and $G \rightarrow K'$. Furthermore, $K \times H \rightarrow H$ so $K' \rightarrow H$. The fact that $K \times H \rightarrow K$ also implies that $K' \rightarrow K \cup G$. Therefore $H \not\rightarrow K'$, since otherwise $H \rightarrow K \cup G$. ■

Corollary 65 *Let G be a digraph and let H be a connected digraph such that $G \rightarrow H$ but $H \not\rightarrow G$. If there exists a digraph K such that $H \not\rightarrow K$ and $K \not\rightarrow G^H$, then there exists a digraph K' such that $G \rightarrow K' \rightarrow H$ and $H \not\rightarrow K' \not\rightarrow G$.* ■

The conditions of lemma 64 require that $H \not\rightarrow G$, so G^H is loopless. The chromatic number of G^H is therefore always defined, and so we will often choose K to have a higher chromatic number than G^H to ensure that $K \not\rightarrow G^H$.

We now begin to apply these results to show that for many digraphs G , the class $\mathcal{F}(G)$ is a proper class.

Lemma 66 *Let H be a compact core, and suppose H has an infinite component C satisfying at least one of the following:*

- C contains an oriented odd cycle,
- C contains a directed cycle (including 2-cycles),
- C contains the Line.

Then either $|\mathcal{F}(H)| = 1$ or $\mathcal{F}(H)$ is a proper class.

Proof: Let H be an infinite connected compact core such that $|\mathcal{F}(H)| > 1$, and let C be an infinite component of H satisfying one of the above conditions.

Suppose C contains an oriented odd cycle. Let n be the length of some oriented odd cycle in H . Let G be any digraph in $\mathcal{F}(G)$ such that $G \rightarrow C$ and $C \not\rightarrow G$. Let

K be a *graph* with $og(K) > n$ and $\chi(K) > \chi(G^C)$. The existence of such is shown in [22, 51, 58] when $\chi(G^C)$ is finite and in [23, 25] when $\chi(G^C)$ is infinite. Let K' be any orientation of K . Then $C \not\rightarrow K'$ and $K' \not\rightarrow G^C$. Applying corollary 65 and lemma 62 we see that $\mathcal{F}(H)$ is a proper class.

Suppose that C contains a directed cycle or the Line. Let G be a digraph in $\mathcal{F}(H)$ such that $G \rightarrow C$ and $C \not\rightarrow G$. Let $\kappa = \chi(G^C)$ and let K be the transitive tournament with $V(K) = \{\alpha : \alpha < \kappa^+\}$ and $E(K) = \{\alpha\beta : \alpha < \beta < \kappa^+\}$. Then $C \not\rightarrow K$ since K contains no Line and no directed cycle, and $K \not\rightarrow G^C$ since $\chi(K) > \chi(G^C)$. Thus, applying corollary 65 and lemma 62 we see that $\mathcal{F}(H)$ is a proper class. ■

Thus, if H is a compact core, we know that $|\mathcal{F}(H)| = 1$ or $\mathcal{F}(H)$ is a proper class unless every infinite component of H is an acyclic bipartite digraph. Furthermore, we know that if H is any digraph which is not finitely equivalent to some compact core, then $\mathcal{F}(H)$ is a proper class.

Open Problem 67 *Determine the size of $\mathcal{F}(H)$ when H is a compact core and every infinite component of H is an acyclic bipartite digraph.*

Conjecture 68 *For every digraph G either $|\mathcal{F}(G)| = 1$ or $\mathcal{F}(G)$ is a proper class.*

5.4 Structures and Graphs

The results in this chapter generalize nicely and in some cases nontrivially to structures and graphs. We will begin by re-examining theorem 52.

Given a language \mathcal{L} , we define $\mathcal{C}_{\mathcal{L}}$ to be the class of all compact structures on \mathcal{L} , and we define $\mathcal{E}_{\mathcal{L}}$ to be the class of all homomorphic equivalence classes of compact structures on \mathcal{L} . Our next result will show that for any language \mathcal{L} , $\mathcal{E}_{\mathcal{L}}$ is a set. However, if \mathcal{L} is large, $\mathcal{E}_{\mathcal{L}}$ will also be large.

Theorem 52' *Let \mathcal{L} be a language. Then $\mathcal{E}_{\mathcal{L}}$ is a set and $|\mathcal{E}_{\mathcal{L}}| = 2^{\max\{\aleph_0, |\mathcal{L}|\}}$.*

Proof: As in the proof of theorem 52, we observe that if there are exactly κ distinct finite structures on \mathcal{L} , then there can be no more than 2^κ equivalence classes of compact structures on \mathcal{L} .

Case 1: $|\mathcal{L}| \leq \aleph_0$

In this case there are \aleph_0 distinct finite structures on \mathcal{L} and so there are no more than 2^{\aleph_0} equivalence classes of compact structures on \mathcal{L} .

We must now show that there are at least this many inequivalent compact structures on \mathcal{L} . Choose some $n > 1$ such that \mathcal{L} contains an n -ary relation, and let \mathcal{P}_n be the infinite mutually incompatible family of structures defined in lemma 7. We claim that for any nonempty subset $\mathcal{G} \subseteq \mathcal{P}_n$, the structure $H_{\mathcal{G}} = \bigcup_{G \in \mathcal{G}} G$ is compact. Let G be any structure and suppose that every finite substructure $G' \subseteq G$ admits a homomorphism to $H_{\mathcal{G}}$. Assume without loss of generality that G is connected and has no isolated vertices.

If $n = 2$ then we prove the claim exactly as in theorem 52. Suppose $n > 2$. If G contains a vertex v and edges E_1, E_2 such that v occurs in the i^{th} coordinate of E_1 and the j^{th} coordinate of E_2 with $i \neq j$ and $i \geq 3$, then the finite substructure of G induced by E_1, E_2 admits no homomorphism to $H_{\mathcal{G}}$, a contradiction. Thus, any $v \in V(G)$ must either always occur in the i^{th} coordinate of edges in which it occurs, or it must always appear in one of the first two coordinates of edges in which it occurs. The same restrictions apply to any substructure of G with no isolated vertices. Now for any substructure $G' \subseteq G$, we define a digraph G'_2 by $V(G'_2) = \{v \in V(G') : v \text{ occurs in the first two coordinates of edges}\}$ and $(u, v) \in E(G'_2)$ if and only if $(u, v, w_3, \dots, w_n) \in E(G')$. We construct H_2 from $H_{\mathcal{G}}$ similarly. Now $G' \rightarrow H_{\mathcal{G}}$ if and only if $G'_2 \rightarrow H_2$. But H_2 is a disjoint union of directed cycles, by construction of \mathcal{P}_n , and so is compact by the argument given for the case $n = 2$. Hence, $H_{\mathcal{G}}$ is compact.

By lemma 3, the set $\{H_{\mathcal{G}} : \mathcal{G} \subseteq \mathcal{P}_n\}$ is a pairwise inequivalent family of structures.

Case 2: $|\mathcal{L}| > \aleph_0$

In this case there are exactly $|\mathcal{L}|$ finite structures on \mathcal{L} . Hence there are no more than $2^{|\mathcal{L}|}$ equivalence classes of compact structures on \mathcal{L} .

To show that there are at least this many equivalence classes, we will first construct a specific set \mathcal{G} of finite structures with $|\mathcal{G}| = |\mathcal{L}|$. For each $R \in \mathcal{L}$, define a finite structure G_R consisting of a single R -edge on the appropriate number of vertices. Let $\mathcal{G} = \{G_R : R \in \mathcal{L}\}$. Now given a set $X \subseteq \mathcal{G}$ we define G_X to be the disjoint union of all $G_R, R \in X$. For any $X \subseteq \mathcal{G}$, G_X will be compact by theorem 49, as it will

contain only finitely many edges of each type. Also, distinct subsets X and Y of \mathcal{G} yield inequivalent structures, since there will be some $R \in \mathcal{L}$ such that G_X contains an R -edge and G_Y does not, or *vice versa*. Thus we obtain $2^{|\mathcal{L}|}$ inequivalent compact structures on \mathcal{L} . ■

Similar arguments show that the number of equivalence classes of compact graphs is exactly 2^{\aleph_0} . In this case we would use the infinite mutually incompatible family \mathcal{B} from lemma 8 in the construction.

Lemmas 55 and 56 are true for arbitrary structures, and the proofs are identical to the proofs for digraphs.

The notion of finite equivalence generalizes naturally to structures and graphs. However, when dealing with the class of structures over an infinite language we must take care to recall the distinction between finite structures and finitely induced structures. We say that two structures are finitely equivalent when any finite substructure of one admits a homomorphism to the other. An analogous notion could be defined using finitely induced substructures, but we choose not to do so at this time.

All of lemmas 57, 58, 59, 61, 62, 64, and corollaries 63 and 65 are true for structures and graphs, and the proofs are identical to those for digraphs. Note, however, that in the proof of lemma 63, the infinite component C may be finitely induced.

If G and H are structures for a language \mathcal{L} , and $R \in \mathcal{L}$, then it is clear from the definition of H^G that H^G contains an R -loop if and only if there is a mapping $f : V(G) \rightarrow V(H)$ which preserves all R -edges of G . As was the case for digraphs, $H^G \leftrightarrow \text{one}$ if and only if $G \rightarrow H$.

Lemma 66 relates to digraphs only. However, we can prove results in the same spirit for structures and graphs.

Theorem 69 *Let \mathcal{L} be an infinite language, and let H be a finitely induced structure for \mathcal{L} , where H has at least one R -edge for each $R \in \mathcal{L}$. Then $\mathcal{F}(H)$ is a proper class.*

Proof: First observe that H is compact, since H is finitely induced, and so we may assume that H is a core. Thus, H is maximum in $(\mathcal{F}(H), \rightarrow)$. We will show that the density condition of corollary 63 holds for $\mathcal{F}(H)$.

We begin by defining a class of structures which will be useful in our proof. If R is an n -ary relation and κ is a cardinal, we define T_κ^R by $V(T_\kappa^R) = \{\alpha : \alpha < \kappa\}$ and $R(\alpha_1, \alpha_2, \dots, \alpha_n)$ if and only if $\alpha_1 < \alpha_2 < \dots < \alpha_n$. Such a structure is called a *transitive k -tournament* [10].

Now let G be any structure in $\mathcal{F}(H)$ such that $H \not\rightarrow G$. There must be some component C of H such that $C \not\rightarrow G$. Clearly C must contain edges of infinitely many types. We will now examine two cases.

Case 1: For some $R \in \mathcal{L}$ there is no R -edge-preserving mapping $f : V(C) \rightarrow V(T_\kappa^R)$ for any cardinal κ .

Let R be such a relation, and let n be the arity of R . Let $\kappa = |V(G^H)|^+$. We construct a structure K with $V(K) = \{\alpha : \alpha < \kappa\}$ as follows. Let $R(K) = R(T_\kappa^R)$, and for $R' \neq R$ let $R'(K) = \{(\alpha, \alpha, \dots, \alpha) : \alpha < \kappa\}$. Clearly $C \not\rightarrow K$ because of our choice of R , and so $H \not\rightarrow K \cup G$.

We claim $K \not\rightarrow G^H$. If $f : K \rightarrow G^H$ is a homomorphism, then since $|K| > |G^H|$ there must be some $v \in V(G^H)$ and some infinite collection $v_1, v_2, \dots \in V(K)$ such that $f(v_i) = v$ for all $i \geq 1$. We require only v_1, v_2, \dots, v_n . Assume that $v_1 < v_2 < \dots < v_n$. Now (v_1, v_2, \dots, v_n) is an R -edge of K , so v must have an R -loop. But v must also have an R' -loop for every $R' \neq R$, and so v has an R' -loop for every $R' \in \mathcal{L}$. But this can only occur if $H \rightarrow G$, which is not the case.

Thus, by lemma 64, there exists a structure K' such that $G \rightarrow K' \rightarrow H$ and $H \not\rightarrow K' \not\rightarrow G$.

Case 2: For every $R \in \mathcal{L}$ there is an R -edge-preserving mapping $f : V(C) \rightarrow V(T_\kappa^R)$ for some cardinal κ .

In particular this implies that if (v_1, v_2, \dots, v_n) is an R -edge of C , then all v_i are distinct. Since $V(C)$ is finite and C contains edges of infinitely many types, there must exist distinct $u, v \in V(C)$ and distinct $R, S \in \mathcal{L}$ such that u and v both occur in some R -edge of C , and both occur in some S -edge of C .

We now define K exactly as above, with $\kappa = |V(G^H)|^+$, $V(K) = \{\alpha : \alpha < \kappa\}$, $R(K) = R(T_\kappa^R)$, and $R'(K) = \{(\alpha, \alpha, \dots, \alpha) : \alpha < \kappa\}$ for each $R' \neq R$. As in case 1 we see that $K \not\rightarrow G^H$.

We now claim $C \not\rightarrow K$. Suppose $f : C \rightarrow K$ is a homomorphism. All S -edges of K are S -loops, and u and v occur together in an S -edge of C , so we must have $f(u) = f(v)$. But all vertices are distinct within each R -edge of K , and u and v occur together in an R -edge of H , so $f(u) \neq f(v)$, a contradiction. Thus, $H \not\rightarrow K \cup G$.

Again, by lemma 64, there exists a structure K' such that $G \rightarrow K' \rightarrow H$ and $H \not\rightarrow K' \not\rightarrow G$.

Thus, by corollary 63, we conclude that in both cases $\mathcal{F}(H)$ is a proper class. ■

We obtain a particular nice result for undirected graphs.

Theorem 70 *If G is a graph then $|\mathcal{F}(G)| = 1$ or $\mathcal{F}(G)$ is a proper class.*

Proof: If G is not finitely equivalent to a compact core, then $\mathcal{F}(G)$ is a proper class, so let H be the compact core which is finitely equivalent to G . If $\chi(H) \leq 2$ then H retracts to a vertex or an edge, and so $|\mathcal{F}(G)| = 1$. Suppose that $\chi(H) \geq 3$ and $|\mathcal{F}(G)| > 1$. Let G' be any graph in $\mathcal{F}(G)$ such that $G' \rightarrow H$ but $H \not\rightarrow G'$. Then there is a component C of H such that $C \not\rightarrow G'$. Clearly $\chi(C) \geq 3$, and so C contains an odd cycle. Let K be a graph with $og(K) > og(C)$ and $\chi(K) > \chi(G'^H)$. The existence of such is shown when $\chi(G'^H)$ is finite in [22, 51, 58] and when $\chi(G'^H)$ is infinite in [23, 25]. Then $C \not\rightarrow K \cup G'$, so $H \not\rightarrow K \cup G'$. Furthermore, $K \not\rightarrow G'^H$. By lemma 64 and corollary 63 we see that $\mathcal{F}(H)$ is a proper class. ■

Chapter 6

List-Homomorphisms

6.1 Definitions

In this chapter we will define and examine homomorphisms in which the set of possible images of a vertex is subject to various types of constraints. The reader may recall that in chapter 4 we defined the notion of an l -list-homomorphism. We begin this chapter by reiterating the definition of a list-homomorphism, along with some new related notions.

Let G and H be digraphs. Let $l : V(G) \rightarrow \mathcal{P}(V(H))$ be a mapping from $V(G)$ to the power set of $V(H)$, called a *list-assignment* for G (with respect to H). An *l -list-homomorphism* $f : G \rightarrow H$ is a homomorphism from G to H such that for each $v \in V(G)$ we have $f(v) \in l(v)$. We will often wish to apply the same lists to subdigraphs G' of G . By convention we will say that $f : G' \rightarrow H$ is an l -list-homomorphism if f is an $l|_{V(G')}$ -list-homomorphism. We say that a digraph H is *list-compact* with respect to a digraph G if for every list-assignment l for G with respect to H , either there exists an l -list-homomorphism $f : G \rightarrow H$ or there is a finite subdigraph $G' \subseteq G$ for which no l -list-homomorphism $f : G' \rightarrow H$ exists. If H is list-compact with respect to every G with $|G| \leq \alpha$ then H is α -*list-compact*. If H is α -list-compact for every ordinal α then we say H is *list-compact*. A *certificate of non-list-compactness* for a digraph H is a digraph G and a list assignment l for G where every finite subdigraph of G admits an l -list-homomorphism to H but G does

not.

6.2 List-Compactness

Our first major result in this chapter will show that essentially only finite digraphs are list-compact. Later on we will place restrictions on the types of list-assignments allowed and obtain richer classes of compact digraphs.

Recall once again the equivalence relation \equiv defined on the vertices of a digraph by $u \equiv v$ if and only if u and v have the same in-neighbourhoods and out-neighbourhoods. We may simplify our later results by considering only digraphs containing no pair of equivalent vertices. Our first lemma shows that this causes no loss of generality. Let G and H be digraphs and let l be a list-assignment for G with respect to H . We define the reduced digraph H^r as in definition 39 in chapter 4. Let h be the canonical retraction from H to H^r , so if $w \in V(H^r)$ and $u, v \in V(H)$ are pre-images of w under h , then $u \equiv v$. We will define a list-assignment l^r for G with respect to H^r . For each $v \in V(G)$ let $l^r(v) = \{h(w) : w \in l(v)\}$. Note that the definition of $l^r(v)$ involves only $l(v)$ and h , and is independent of the other vertices of G and their lists. So if G' is a subdigraph of G , and $l'(v) = l(v)$ for all $v \in V(G')$, then $(l')^r = l^r|_{V(G')}$. Thus, we can regard l^r as a list-assignment for G' without ambiguity.

It will be useful for later results to note that when we define l^r as above, then $l^r(v) = V(H^r)$ whenever $l(v) = V(H)$; and $l^r(v)$ is finite whenever $l(v)$ is finite.

Lemma 71 *Let G and H be digraphs and let l be a list-assignment for G . Then G admits an l -list-homomorphism to H if and only if G admits an l^r -list-homomorphism to the reduced digraph H^r .*

Proof: If $f : G \rightarrow H$ is an l -list-homomorphism and h is the canonical retraction from H to H^r , then clearly $h \circ f$ is an l^r -list-homomorphism from G to H^r .

On the other hand, suppose $f : G \rightarrow H^r$ is an l^r -list-homomorphism. We will define a new mapping $g : G \rightarrow H$. For each $v \in V(G)$ we know $f(v) \in l^r(v)$, and any element $w \in l^r(v)$ must have some pre-image under h in $l(v)$. Choose $g(v) \in l(v)$ to be any vertex such that $h(g(v)) = f(v)$. It remains only to show that g is a

homomorphism. If $uv \in E(G)$, then $f(u)f(v) \in E(H^r)$. Since $g(u)$ and $g(v)$ are pre-images of $f(u)$ and $f(v)$, respectively, under h , then it must be the case that $g(u)g(v) \in E(H)$, since $g(u) \equiv f(u)$ and $g(v) \equiv f(v)$ by definition of h . ■

Corollary 72 *A digraph H is list-compact if and only if its reduced digraph H^r is list-compact.*

Proof: If H^r is not list-compact then let G with list-assignment l be a certificate of non-list-compactness for H^r . Note that l is also a list-assignment for G with respect to H , since H^r is an induced subdigraph of H . Clearly G with l is also a certificate of non-list-compactness for H .

If H is not list-compact then let G with list-assignment l be a certificate of non-list-compactness for H . We claim that G with list-assignment l^r is a certificate of non-list-compactness for H^r . By lemma 71 there is no l^r -list-homomorphism from G to H^r . However, if G' is a finite subdigraph of G , then there is an l -list-homomorphism from G' to H , and so by lemma 71 there is an l^r -list-homomorphism from G' to H . ■

Given any digraph H , the reduced digraph H^r contains no pair of equivalent vertices, so this corollary shows that for the purposes of characterizing list-compact digraphs it is sufficient to consider reduced digraphs. The following set-theoretic lemma, due to Aharoni [1], will help us exploit this fact. The symbol \subset denotes *proper* containment.

Lemma 73 *Let $\mathcal{A} = \{A_i : i \geq 0\}$ be a collection of countably many distinct sets. Then there exists a subcollection of sets $\{B_i : i \geq 0\} \subseteq \mathcal{A}$ such that either*

- $B_0 \subset (B_0 \cup B_1) \subset (B_0 \cup B_1 \cup B_2) \cup \dots$, or
- $B_0 \supset (B_0 \cap B_1) \supset (B_0 \cap B_1 \cap B_2) \cap \dots$

Proof: Let $A = \bigcup_{i=0}^{\infty} A_i$. We may assume without loss of generality that $\bigcap_{i=0}^{\infty} A_i = \emptyset$, since we may ignore any vertices which occur in all of the A_i . We may also assume that there do not exist any $x, y \in A$ such that $x \in A_i$ if and only if $y \in A_i$ for all $i \geq 0$. If such vertices do exist, then we may define an equivalence relation \sim on A by

$x \sim y$ if $x \in A_i$ exactly when $y \in A_i$. As usual $[x]$ will denote the equivalence class containing x . We may now construct a new family of sets $\mathcal{A}' = \{A'_i : i \geq 0\}$ by taking $A'_i = \{[x] : [x] \subseteq A_i\}$. Now clearly a subcollection $\{B_i : i \geq 0\}$ of \mathcal{A} will satisfy one of the above nesting properties if and only if the corresponding subcollection $\{B'_i : i \geq 0\}$ of \mathcal{A}' satisfies the same property. And clearly no two distinct equivalence classes $[x]$ and $[y]$ will occur in exactly the same sets A'_i .

Now let S be any subset of A , and let us first assume that the set $\{S \cap A_i : i \geq 0\}$ is finite, that is, $S \cap A_i$ takes on only finitely many (say n) distinct values when all $i \geq 0$ are considered.

We define an equivalence relation on S by $x \sim y$ whenever $x \in S \cap A_i$ if and only if $y \in S \cap A_i$ for all $i \geq 0$. If x and y are distinct elements of S and $x \sim y$, then $x \in A_i$ if and only if $y \in A_i$ for all $i \geq 0$, but by assumption this cannot occur. Thus $x \not\sim y$ for all distinct $x, y \in S$. But there are only n distinct values of $S \cap A_i$, so S has at most 2^n equivalence classes under \sim , and so S must be finite.

Thus, if S is infinite, $S \cap A_i$ must take on infinitely many distinct values. We now examine two cases.

Case 1: There exists an infinite set $S \subseteq A$ such that for all $i \geq 0$ we have $S - (S \cap A_i)$ finite or $S \cap A_i = \emptyset$.

We know that $S \cap A_i$ takes on infinitely many distinct values, so there must be infinitely many $i \geq 0$ such that $S \cap A_i \neq \emptyset$. In these cases $S - (S \cap A_i)$ must be finite. This implies that $|S - (S \cap A_i)|$ must take on arbitrarily large finite values. Thus, we choose some $i \geq 0$ such that $S - (S \cap A_i)$ is finite, and let $B_0 = A_i$. Now inductively, if we have defined B_0, B_1, \dots, B_n , we can find some $i \geq 0$ such that $|S - (S \cap A_i)| > |S - \bigcap_{j=0}^n (S \cap A_j)|$. Let $B_{n+1} = A_i$.

Thus, $\bigcap_{i=0}^n (S \cap B_i) \supset \bigcap_{i=0}^{n+1} (S \cap B_i)$, and so $\bigcap_{i=0}^n B_i \supset \bigcap_{i=0}^{n+1} B_i$.

Case 2: For every infinite set $S \subseteq A$ there exists an $i \geq 0$ such that $S - (S \cap A_i)$ is infinite and $S \cap A_i \neq \emptyset$.

Let $S = A$ and choose some $i \geq 0$ satisfying the above condition. Let $B_0 = A_i$. Then $A - B_0 = A - (A \cap B_0) = S - (S \cap A_i)$ so $A - B_0$ is infinite.

Now suppose we have defined B_0, B_1, \dots, B_n in such a way that $A - \cup_{i=0}^n B_i$ is infinite. Let $S = A - \cup_{i=0}^n B_i$. Then S is an infinite subset of A , so we may find an $i \geq 0$ such that $S - (S \cap A_i)$ is infinite and $S \cap A_i \neq \emptyset$. Let $B_{n+1} = A_i$. Then $A - \cup_{i=0}^{n+1} B_i$ is infinite since it is equal to $S - (S \cap A_i)$. Also, since $B_{n+1} \cap S \neq \emptyset$ we have $\cup_{i=0}^n B_i \subset \cup_{i=0}^{n+1} B_i$. ■

Before proceeding with our first major result we had best explain the conventions followed by the diagrams in this chapter. We will often want to specify that certain edges are present in a digraph and certain edges are not present, while still other edges may or may not be present. When we say that a diagram is a *schema* for a digraph, we mean that wherever there is a solid arrow in the diagram there is an edge present in the digraph, wherever there is a dashed arrow in the diagram the corresponding edge is absent in the digraph, and if no solid or dashed arrow is present, then the corresponding edge may or may not be present in the digraph. Note that an arrow indicating the presence or absence of an edge uv has no bearing on the existence of the edge vu .

At this point we will define some digraphs which will appear often in the remainder of this chapter. Note that the edge-sets of these digraphs are not completely specified in most of the following definitions, so that many non-isomorphic digraphs will satisfy each of the definitions. In particular, we do not specify whether any or all of the vertices in these digraphs have loops.

A *complete digraph* G is one in which $uv \in E(G)$ for every distinct $u, v \in V(G)$.

A *transitive tournament* T is a digraph with a linearly ordered vertex-set where for all distinct $u, v \in V(T)$ we have $uv \in E(T)$ if and only if $u < v$.

An *increasing digraph* G is a digraph whose vertices can be partitioned into two countably infinite sets $U(G) = \{u_1, u_2, \dots\}$ and $L(G) = \{v_1, v_2, \dots\}$ such that for each $i \geq 1$ we have $u_i v_j \in E(G)$ for all $j \leq i$, but $u_i v_j \notin E(G)$ for any $j > i$. Figure 6.1 is a schema for an increasing digraph.

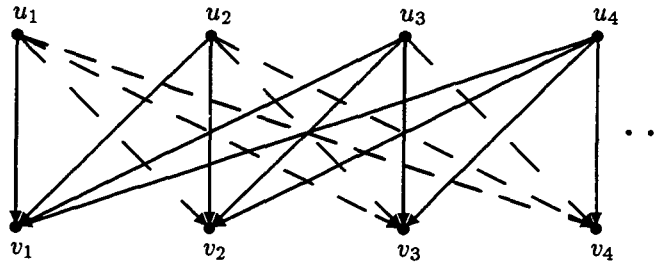


Figure 6.1

A *decreasing digraph* G is a digraph whose vertices can be partitioned into two countably infinite sets $U(G) = \{u_1, u_2, \dots\}$ and $L(G) = \{v_1, v_2, \dots\}$ such that for each $i \geq 1$ we have $u_i v_j \in E(G)$ for all $j \geq i$, but $u_i v_j \notin E(G)$ for any $j < i$. Figure 6.2 is a schema for a decreasing digraph.

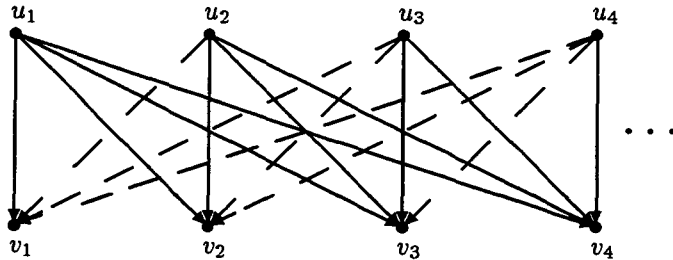


Figure 6.2

A *cocktail-party digraph* G is a digraph whose vertices can be partitioned into two countably infinite sets $U(G) = \{u_1, u_2, \dots\}$ and $L(G) = \{v_1, v_2, \dots\}$ such that for each $i \geq 1$ we have $u_i v_j \in E(G)$ if and only if $i \neq j$. Figure 6.3 is a schema for a cocktail-party digraph.

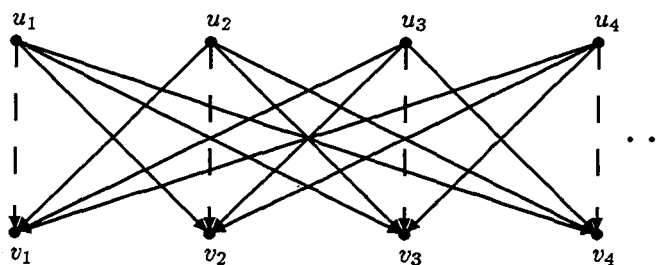


Figure 6.3

A *matching digraph* G is a digraph whose vertices may be partitioned into two countably infinite sets $U(G) = \{u_1, u_2, \dots\}$ and $L(G) = \{v_1, v_2, \dots\}$ so that $u_i v_j \in E(G)$ if and only if $i = j$. Figure 6.4 is a schema for a matching digraph.

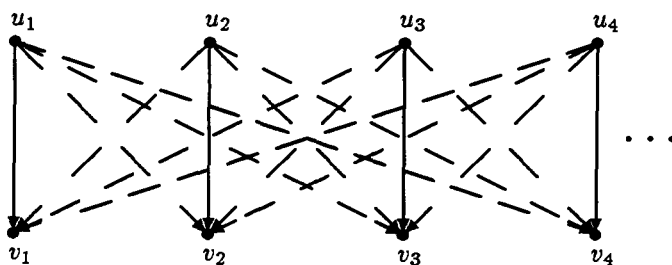


Figure 6.4

An *independent digraph* G is a digraph in which $uv \notin E(G)$ whenever $u \neq v$. Note that we still allow loops.

A *star digraph* is a countably infinite digraph G whose vertices may be labelled $\{v_0, v_1, \dots\}$ in such a way that either $E(G) = \{v_0 v_i : i \geq 1\}$ (an *out-directed star*) or $E(G) = \{v_i v_0 : i \geq 1\}$ (an *in-directed star*). The vertex v_0 is called the *center* of G . Figure 6.5 shows an out-directed star. Note that figure 6.5 is not a schema, *i.e.*, it contains only the edges which are marked.

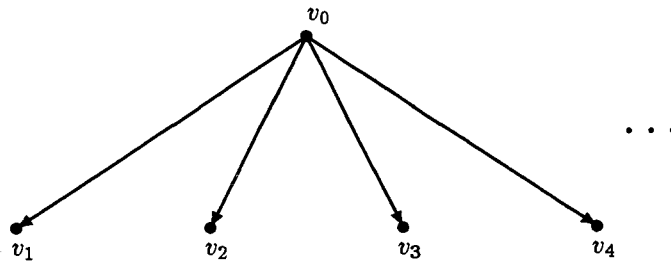


Figure 6.5

Recall that if $vv \notin E(G)$ for all $v \in V(G)$ then G is called loopless, and if $vv \in E(G)$ for all $v \in V(G)$ then G is called a digraph with loops.

Most of the proofs in the remainder of this chapter will work by demonstrating the existence of certain induced subdigraphs in a digraph with some specified property. Our next lemma gives a list of subdigraphs, at least one of which must occur in any infinite reduced digraph. This will allow us to give short proofs of two nice characterizations of list-compact digraphs.

Lemma 74 *Let H be an infinite reduced digraph. Then H contains at least one of the following as an induced subdigraph:*

- *a loopless infinite complete digraph,*
- *an increasing digraph,*
- *a decreasing digraph,*
- *a cocktail-party digraph,*
- *a matching digraph,*
- *an infinite independent digraph with loops.*

Proof: Let H be an infinite reduced digraph, so no two vertices in $V(H)$ are equivalent. Let S be any countable subset of $V(H)$, and denote the members of S by the positive integers $\{1, 2, 3, \dots\}$. We will partition the unordered pairs $\{i, j\}$, where

$i, j \in S$ and $i < j$, into four parts, P_1, P_2, P_3, P_4 , with the intention of applying Ramsey's theorem. The parts are defined as follows:

$$P_1 = \{\{i, j\} : ij \in E(H) \text{ and } ji \in E(H)\}$$

$$P_2 = \{\{i, j\} : ij \notin E(H) \text{ and } ji \in E(H)\}$$

$$P_3 = \{\{i, j\} : ij \in E(H) \text{ and } ji \notin E(H)\}$$

$$P_4 = \{\{i, j\} : ij \notin E(H) \text{ and } ji \notin E(H)\}.$$

By Ramsey's theorem S contains an infinite subset R such that all pairs $\{i, j\}$ with $i, j \in R$ and $i < j$ are in the same part P_i . We now consider the four cases. In each case we will show that H contains one of the required subdigraphs. In some cases we will do so directly, while in other cases we will show that R contains an infinite subset Q satisfying the following property: Given any two vertices $u, v \in Q$, we have $N^+(u) - Q \neq N^+(v) - Q$ or $N^-(u) - Q \neq N^-(v) - Q$. In other words, every two vertices in Q have distinct neighbourhoods in $V(H) - Q$. We will subsequently show that such a set allows us to find one of the required subdigraphs in H .

Case 1: All pairs of elements of R are in P_1 (R induces a complete digraph).

If R contains infinitely many vertices with loops, say a_1, a_2, \dots , then we proceed as follows: let $Q = \{a_1, a_2, \dots\}$. Then $N^+(a_i) \cap Q = N^-(a_i) \cap Q = Q$ for all $i > 0$. But no two a_i are equivalent, so for all $i, j > 0$ with $i \neq j$, we have $N^+(a_i) - Q \neq N^+(a_j) - Q$ or $N^-(a_i) - Q \neq N^-(a_j) - Q$.

On the other hand, if R contains infinitely many vertices without loops, then these vertices induce a loopless infinite complete digraph.

Case 2: All pairs of elements of R are in P_2 (R induces a transitive tournament).

Label the elements of R by a_1, a_2, \dots , where for $i \neq j$, $a_i a_j \in E(H)$ if and only if $i < j$. A particular a_i may or may not have a loop. Let $Q = \{a_1, a_3, a_5, \dots\}$. Then for all $n > 0$ we have $a_{2n} \in N^-(a_{2n-1})$ but $a_{2n} \notin N^-(a_{2m-1})$ for any $m > n$. It is easy to verify that Q now satisfies the required properties.

Case 3: All pairs of elements of R are in P_3 (R induces a transitive tournament).

We define Q exactly as above. Now for all $n > 0$ we have $a_{2n} \in N^+(a_{2n-1})$ but $a_{2n} \notin N^+(a_{2m-1})$ for any $m > n$. Again Q satisfies the required properties.

Case 4: All pairs of elements of R are in P_4 (R is an independent set).

If there are infinitely many vertices in R without loops, then let $Q = \{a_1, a_2, \dots\}$ be such a set. Now $N^+(a_i) \cap Q = N^-(a_i) \cap Q = \emptyset$ for all $i > 0$. But no two a_i are equivalent, so for all $i, j > 0$ with $i \neq j$, we have $N^+(a_i) - Q \neq N^+(a_j) - Q$ or $N^-(a_i) - Q \neq N^-(a_j) - Q$.

If there are infinitely many vertices in R with loops, then these vertices induce an infinite independent digraph with loops.

We now use the set Q to show that H contains one of the required subdigraphs. Recall that every two vertices in Q have distinct in-neighbourhoods or out-neighbourhoods in $V(H) - Q$. Clearly there must exist an infinite subset P of Q such that $N^+(u) - Q \neq N^+(v) - Q$ for all $u, v \in P$, or $N^-(u) - Q \neq N^-(v) - Q$ for all $u, v \in P$. Let us assume the former case holds, as the latter case is essentially identical.

For $v \in P$, let us define $A(v) = N^+(v) - P$. The sets $N^+(v) - Q$ with $v \in P$ are distinct, and $P \subseteq Q$, so the sets $A(v)$ with $v \in P$ are distinct. We may therefore apply lemma 73 to obtain a set $\{u_0, u_1, \dots\} \subseteq P$ such that one of the following formulas holds:

$$A(u_0) \subset A(u_0) \cup A(u_1) \subset A(u_0) \cup A(u_1) \cup A(u_2) \subset \dots \quad (6.1)$$

$$A(u_0) \supset A(u_0) \cap A(u_1) \supset A(u_0) \cap A(u_1) \cap A(u_2) \supset \dots \quad (6.2)$$

We will now define a set of vertices $P' = \{v_1, v_2, \dots\}$ in one of two ways. Note that we have intentionally indexed the u_i starting at 0 and the v_i at 1. The reason for this is that $A(u_0)$ may be the empty set in 6.1, or it may contain all of $V(H) - P$ in 6.2. In either of these special cases our definition of v_0 would fail.

If 6.1 holds then for $i \geq 1$ let $v_i \in V(H) - P$ be a vertex such that $u_i v_i \in E(H)$ but for each $j < i$ we have $u_j v_i \notin E(H)$. Formula 6.1 guarantees the existence of such a vertex.

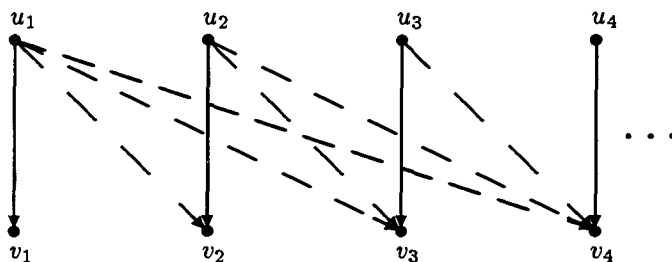


Figure 6.6

If 6.1 does not hold, then 6.2 holds. So for $i \geq 1$ let $v_i \in V(H) - P$ be a vertex such that $u_i v_i \notin E(H)$ but for each $j < i$ we have $u_j v_i \in E(H)$. Formula 6.2 guarantees the existence of such a vertex.

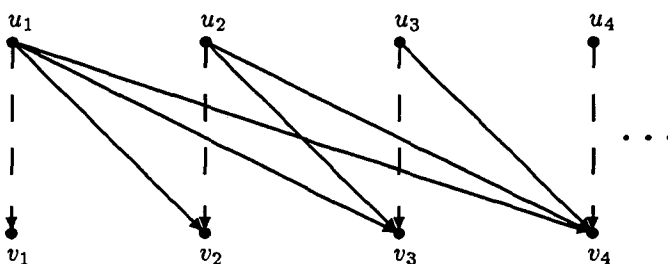


Figure 6.7

We now prepare for another application of Ramsey's theorem. We partition the unordered pairs $\{i, j\}$, where i and j are positive integers with $i < j$, into parts P_1, P_2, P_3, P_4 according to the following rules:

$$P_1 = \{\{i, j\} : u_i v_j \in E(H) \text{ and } u_j v_i \in E(H)\}$$

$$P_2 = \{\{i, j\} : u_i v_j \notin E(H) \text{ and } u_j v_i \in E(H)\}$$

$$P_3 = \{\{i, j\} : u_i v_j \in E(H) \text{ and } u_j v_i \notin E(H)\}$$

$$P_4 = \{\{i, j\} : u_i v_j \notin E(H) \text{ and } u_j v_i \notin E(H)\}.$$

By Ramsey's theorem, there must be an infinite set T of positive integers such that all pairs of elements of T are in the same part P_i . We will now examine the subgraph of H induced by the vertices u_i and v_i , for $i \in T$. Let us write T as

$\{a_1, a_2, \dots\}$ and define $x_i = u_{a_i}$ and $y_i = v_{a_i}$ for each $i \geq 1$. We find ourselves once again with four cases.

Case 1: All pairs of elements of T are in P_1 .

Then 6.1 cannot hold: Look at any $i, j \in T$ with $i < j$. By definition of the u_k and v_k we have $u_i v_j \notin E(H)$. This contradicts the definition of P_1 .

Thus it must be the case that 6.2 holds. Then $x_i y_i \notin E(H)$ for all $i \geq 1$ but $x_i y_j \in E(H)$ whenever $i \neq j$ and $i, j \geq 1$. So the vertices x_k and y_k , $k \geq 1$, induce a cocktail-party digraph.

Case 2: All pairs of elements of T are in P_2 .

Then 6.2 cannot hold, since in that case $x_i y_j \in E(H)$ whenever $i < j$, contradicting the definition of P_2 , and so 6.1 must hold. In this case the vertices x_k and y_k , $k \geq 1$, induce an increasing digraph.

Case 3: All pairs of elements of T are in P_3 .

Then 6.1 cannot hold, for then $x_i y_j \notin E(H)$ whenever $i < j$, which contradicts the definition of P_3 . So 6.2 must hold, and the vertices x_k , $k \geq 1$ and y_k , $k \geq 2$, induce a decreasing digraph.

Case 4: All pairs of elements of T are in P_4 .

Once again we see that 6.2 cannot hold, since then $x_i y_j \in E(H)$ whenever $i < j$, contradicting the definition of P_4 , and so 6.1 must hold. The vertices x_k and y_k , $k \geq 1$, induce a matching digraph in this case.

Thus, in each case H contains one of the required subdigraphs, and so the lemma is proved. ■

Observe that we have actually proved the following slightly stronger assertion, which will be more useful for later results.

Corollary 75 *Let H be an infinite reduced digraph and let S be a countable subset of $V(H)$. Then H contains at least one of the following as an induced subdigraph:*

- a loopless infinite complete digraph G with $V(G) \subseteq S$,
- an increasing digraph G with $U(G) \subseteq S$ or $L(G) \subseteq S$,

- a decreasing digraph G with $U(G) \subseteq S$ or $L(G) \subseteq S$,
- a cocktail-party digraph G with $U(G) \subseteq S$ or $L(G) \subseteq S$,
- a matching digraph G with $U(G) \subseteq S$ or $L(G) \subseteq S$,
- an infinite independent digraph G with loops and with $V(G) \subseteq S$.

■

We now use lemma 74 to prove the following characterization of list-compact digraphs.

Theorem 76 *Let H be a reduced digraph. Then H is list-compact if and only if H is finite.*

Proof: If H is finite, then lemma 47 guarantees that H is list-compact.

On the other hand, suppose H is not finite. Then H contains a loopless infinite complete digraph, an increasing digraph, a decreasing digraph, a cocktail-party digraph, a matching digraph, or an infinite independent digraph with loops as an induced subdigraph.

Suppose $S \subseteq V(H)$ is a countable set of vertices which induces a loopless infinite complete digraph. Let $S = \{a_1, a_2, \dots\}$, and define a digraph G with $V(G) = \{v_0, v_1, \dots\}$, and $uv \in E(G)$ for all distinct $u, v \in V(G)$. Define a list-assignment l for G by $l(v_0) = \{a_1, a_2, \dots\}$ and $l(v_i) = a_i$ for all $i \geq 1$. Now there can be no l -list-homomorphism $f : G \rightarrow H$, since $f(v_0) = a_i$ for some $i \geq 1$, but also $f(v_i) = a_i$, and $a_i a_i \notin E(H)$ so f does not preserve the edge $v_0 v_i$. However, if G' is a finite subdigraph of G , then we may define an l -list-homomorphism $f : G' \rightarrow H$ by $f(v_i) = a_i$, for each $v_i \in V(G')$ with $i \geq 1$; and if $v_0 \in V(G')$ we define $f(v_0) = a_j$ for some j such that $v_j \notin V(G')$. Thus G with l is a certificate of non-list-compactness for H .

Suppose H contains an increasing digraph as shown in the schema in figure 6.8 (a). We define an out-directed star digraph G with $V(G) = \{w_0, w_1, w_2, \dots\}$ and $E(G) = \{w_0 w_i : i \geq 1\}$. We define a list-assignment l for G by $l(w_0) = \{x_i : i \geq 1\}$

and $l(w_i) = y_i$ for each $i \geq 1$. The digraph G is shown in figure 6.8 (b). It is a simple matter to verify that G with l is a certificate of non-list-compactness for H .

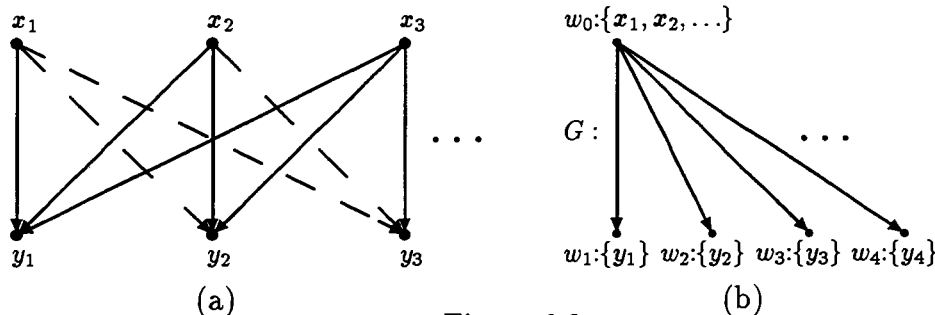


Figure 6.8

Suppose H contains a decreasing digraph as shown in the schema in figure 6.9 (a). We define an in-directed star digraph G (figure 6.9 (b)) with $V(G) = \{w_0, w_1, w_2, \dots\}$ and $E(G) = \{w_i w_0 : i \geq 1\}$. We define a list-assignment l for G by $l(w_0) = \{y_i : i \geq 1\}$ and $l(w_i) = x_i$ for each $i \geq 1$. As in the previous case we see that G with l is a certificate of non-list-compactness for H .

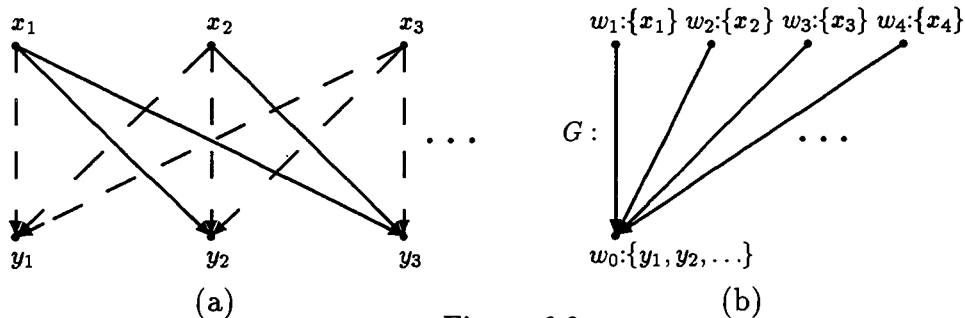


Figure 6.9

Suppose H contains a cocktail-party digraph as shown in the schema in figure 6.10 (a). We define an out-directed star digraph G (figure 6.10 (b)) with $V(G) = \{w_0, w_1, w_2, \dots\}$ and $E(G) = \{w_0 w_i : i \geq 1\}$. We define a list-assignment l for G by $l(w_0) = \{x_i : i \geq 1\}$ and $l(w_i) = y_i$ for each $i \geq 1$. Clearly G with l is a certificate of non-list-compactness for H .

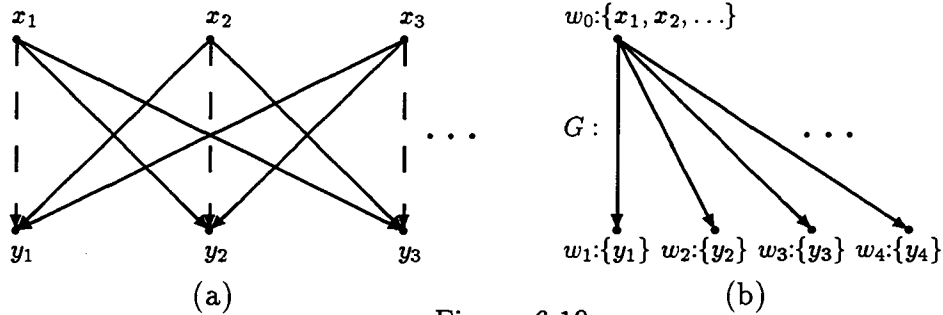


Figure 6.10

Suppose H contains a matching digraph as shown in the schema in figure 6.11 (a). In this case we define a digraph G (figure 6.11 (b)) by $V(G) = \{w_1, w_2, \dots\}$ and $E(G) = \{w_{2i-1}w_{2i} : i \geq 1\} \cup \{w_{2i+1}w_{2i} : i \geq 1\}$. We define a list-assignment l for G by $l(w_{2i}) = \{y_i, y_{i+1}, \dots\}$ and $l(w_{2i-1}) = \{x_i, x_{i+1}, \dots\}$ for each $i \geq 1$. The reader may again observe that G with l is a certificate of non-list-compactness for H .

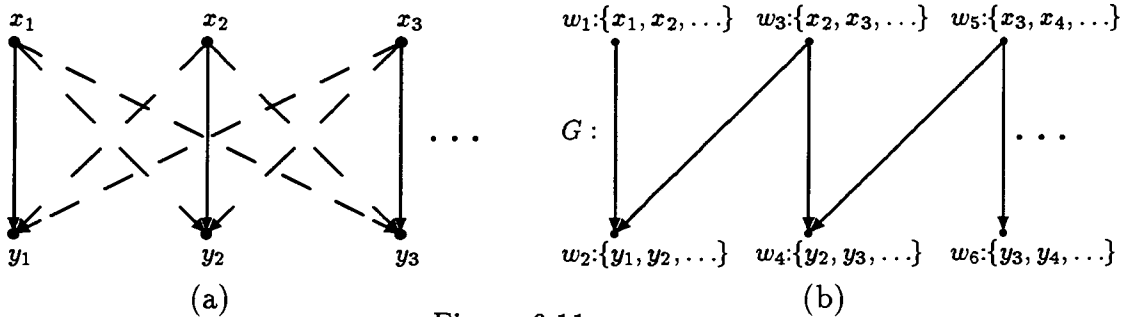


Figure 6.11

If there is an infinite independent digraph with loops in H , then let $S = \{a_1, a_2, \dots\}$ induce such a digraph. We define a digraph G by $V(G) = \{v_1, v_2, \dots\}$ and $E(G) = \{v_i v_{i+1} : i \geq 1\}$, i.e. G is the Ray. We define a list-assignment l for G by $l(v_i) = \{a_i, a_{i+1}, a_{i+2}, \dots\}$. A moments reflection will convince the reader that G with l is a certificate of non-list-compactness for H .

We have seen that in all possible cases H is not list-compact, and so our result is proved. ■

We may restate this result as follows.

Corollary 77 *A digraph H is list-compact if and only if its reduced digraph H^r is finite.* ■

The astute reader may have observed that all of the certificates of non-list-compactness in the above proof were countable, so we have in fact proved the stronger claim that if H^r is not finite then H is not \aleph_0 -list-compact.

6.3 Restricted Lists

We may obtain more interesting characterizations by restricting the types of list-assignments allowed. Let P be some property that applies to list-assignments. For example, if $l : V(G) \rightarrow \mathcal{P}(V(H))$ is a list-assignment, we might say that $P(l)$ is true if $l(v) = V(H)$ for every $v \in V(G)$. We will say that a digraph H is *P -list-compact* if for every digraph G and every list-assignment l for G with respect to H where $P(l)$ holds, either there is an l -list-homomorphism from G to H , or there is a finite subdigraph $G' \subseteq G$ for which no such homomorphism exists.

If P is defined as in the above example, then clearly the class of P -list-compact digraphs is exactly the class of compact digraphs. Our next result is straightforward. We use the standard notation $\text{dom}(l)$ to indicate the domain of the function l .

Theorem 78 *Let $A(l)$ be the property that $l(v)$ is finite for every $v \in \text{dom}(l)$. Then every digraph H is A -list-compact.*

Proof: Let H be given, and let G be any digraph. Let l be any list-assignment for G such that $l(v)$ is finite for each $v \in V(G)$. Also, assume that every finite subdigraph of G admits an l -list-homomorphism to H . Then let \mathcal{T}_v be the discrete topology on $l(v)$. Lemma 47 guarantees the existence of an l -list-homomorphism from G to H . ■

In a moment we will examine some less trivial types of lists. However, first we will prove a useful technical lemma. It shows that for our purposes, any finite list can be assumed to be a singleton list.

Lemma 79 *Let G and H be digraphs, and l a list-assignment for G with respect to H . If G with l is a certificate of non-compactness for H , then there exists a list-assignment l' for G with respect to H such that for all $v \in V(G)$*

- *if $l(v)$ is finite then $l'(v)$ is a singleton,*
- *if $l(v)$ is infinite then $l'(v) = l(v)$,*

and such that G with l' is also a certificate of non-compactness for H .

Proof: Let G , H , and l be given. It suffices to show that there exists a list-assignment l' such that l' agrees with l when $l(v)$ is infinite, $l'(v)$ is a single element of $l(v)$ when $l(v)$ is finite, and for each finite subdigraph $G' \subseteq G$ there is an l' -list-homomorphism from G' to H . Clearly in this case there can be no l' -list-homomorphism from G to H , and so G with l' will also be a certificate of non-list-compactness for H .

We will again apply Tychonoff's theorem. Let $S = \{v \in V(G) : l(v) \text{ is finite}\}$. For each $v \in S$ let T_v be the discrete topology on $l(v)$. For each $v \in S$ the set $l(v)$ is finite, so T_v is a compact topology. Thus, the product topology $\mathcal{T} = \prod_{v \in S} T_v$, defined on $\mathcal{S} = \prod_{v \in S} l(v)$, is compact.

There is a natural correspondence between elements of \mathcal{S} and mappings $f : S \rightarrow H$ which satisfy $f(v) \in l(v)$ for all $v \in S$. Hence, we will consider each element of \mathcal{S} to be such a mapping.

Let G' be a finite subdigraph of G . We define a set $C(G') \subseteq \mathcal{S}$ to be the set of all mappings $f \in \mathcal{S}$ such that $f|_{(G' \cap S)}$ can be extended to an l -list-homomorphism from G' to H .

The set $C(G')$ is non-empty since by assumption there is an l -list-homomorphism $g : G' \rightarrow H$, and so any $f \in \mathcal{S}$ satisfying $f|_{(G' \cap S)} = g|_{(G' \cap S)}$ will be in $C(G')$. Furthermore, we claim that the set $C(G')$ is closed. For any $v \in S$, each subset of $l(v)$ is closed in the topology T_v , since $l(v)$ is finite. Also, the set $S \cap V(G')$ is finite, so the set $\{f|_{S \cap V(G')} : f \in C(G')\}$ is closed in the topology $\prod_{v \in S \cap V(G')} T_v$. The set $\{f|_{S - V(G')} : f \in C(G')\}$ is simply $\prod_{v \in S - V(G')} l(v)$. But this is the definition of a closed set in the product topology, so our claim is proved.

If G_1, \dots, G_n are finite subdigraphs of G , then we know that $C(G_1) \cap \dots \cap C(G_n)$ is nonempty, since clearly $G_1 \cup \dots \cup G_n$ is finite and $C(G_1 \cup \dots \cup G_n) \subseteq C(G_1) \cap \dots \cap C(G_n)$. We may therefore conclude, by compactness, that the intersection of $C(G')$ over all finite subdigraphs $G' \subseteq G$ is nonempty. So choose $f \in \bigcap_{G' \subseteq G} C(G')$. Then f has the property that given any finite subdigraph $G' \subseteq G$, there is an l -list-homomorphism $g : G' \rightarrow H$ such that $f(v) = g(v)$ for all $v \in S \cap G'$.

Thus, we define l' by $l'(v) = \{f(v)\}$ for each $v \in S$, and $l'(v) = l(v)$ for $v \notin S$. By the above arguments it is clear that every finite subdigraph of G admits an l' -list-homomorphism to H , and so we are done. ■

This result will simplify our subsequent proofs, since singleton lists are much easier to work with than other finite lists. It also yields another proof of theorem 78, since obviously every digraph is list-compact when we require all lists to be singletons.

Our next result deals with list-assignments where ‘almost all’ of the lists are finite. Specifically, we define a property B for lists where $B(l)$ is true of the list-assignment l if and only if $l(v)$ is finite for all but finitely many of the vertices in $\text{dom}(l)$, i.e. only finitely many vertices are assigned infinite lists. The reader may observe that most of the certificates of non-list-compactness used in the various cases in the proof of theorem 76 used list-assignments which did in fact satisfy the property B . In particular, we saw that whenever a digraph H contains a loopless complete digraph, an increasing digraph, a decreasing digraph or a cocktail-party digraph as an induced subdigraph, then there is a digraph G and a list-assignment l for G with respect to H such that $B(l)$ holds and G with l is a certificate of non-list-compactness for H . Our next theorem shows that it is exactly when H contains one of these digraphs as an induced subdigraph that H fails to be B -list-compact, and so we will obtain a forbidden-subdigraph type characterization of B -list-compact digraphs.

We first prove a lemma which will simplify our proof, and is also of some interest itself.

Lemma 80 *If a digraph H is not B -list-compact then there is a certificate of non-list-compactness G with list-assignment l such that G is a star with center v , $l(v)$ is countably infinite, and $l(w)$ is a singleton for each $w \neq v$.*

Proof: Suppose that H is not B -list-compact. Applying lemma 79 we may obtain a certificate of non-list-compactness for H , call it G with list-assignment l , where all but finitely many of the vertices of G have singleton lists, and the rest have infinite lists. Let $\kappa = |G|$ and let $n = |\{v \in V(G) : l(v) \text{ is infinite}\}|$. We choose G so that κ is minimum, and amongst those digraphs G of minimum cardinality we choose G and l so that n is minimized. We will refer to the vertices which are assigned infinite lists as *free* vertices, and to those which are assigned singleton lists as *fixed* vertices.

We begin by labelling the vertices of G with the ordinals smaller than κ , and we require that the first n ordinals be used to label the free vertices. We will denote by G_α the subdigraph of G induced by $\{0, 1, \dots, \alpha\}$. Then for each $\alpha < \kappa$ there exists an l -list-homomorphism $f_\alpha : G_\alpha \rightarrow H$. Of course if v is a fixed vertex then $f_\alpha(v) = f_\beta(v)$ for all $\alpha, \beta < \kappa$. However, if v is a free vertex, then v must take on many different values. We will formalize this idea now.

For each $\alpha < \kappa$, let $F_\alpha = \{f : f \text{ is an } l\text{-list-homomorphism from } G_\alpha \text{ to } H\}$. Note that if $\beta < \alpha$ and $f \in F_\alpha$ then $f|_{G_\beta} \in F_\beta$. Let v be a particular free vertex of G . We claim that if I is any collection of ordinals smaller than κ with $\bigcup_{\alpha \in I} \alpha = \kappa$, then there can be no collection of l -list-homomorphisms $\{f_\alpha\}_{\alpha \in I}$ where $f_\alpha \in F_\alpha$ and $f_\alpha(v) = f_\beta(v)$ for all $\alpha, \beta \in I$. For if such a sequence did exist, then for each $\alpha < \kappa$ there would be an l -list-homomorphism $g_\alpha : G_\alpha \rightarrow H$ where $g_\alpha(v) = g_\beta(v)$ for all $\alpha, \beta < \kappa$. But then by replacing $l(v)$ by $\{g_\alpha(v)\}$ for some $\alpha < \kappa$, we would obtain a certificate of non-list-compactness for H with fewer free vertices. Hence, for any free vertex $v \in V(G)$, any $\alpha < \kappa$ and any $f_\alpha \in F_\alpha$, there exists a $\beta < \kappa$ such that for all γ with $\beta \leq \gamma < \kappa$ and all $f_\gamma \in F_\gamma$ we have $f_\gamma(v) \neq f_\alpha(v)$.

Now let α be any ordinal with $n \leq \alpha < \kappa$ (*i.e.* include all free vertices) and let f_α be any l -list-homomorphism from G_α to H . For each free vertex $v_i, 0 \leq i < n$, we may find an ordinal $\alpha_i < \kappa$ such that for all ordinals γ with $\alpha_i \leq \gamma < \kappa$, there is no $f_\gamma \in F_\gamma$ with $f_\gamma(v_i) = f_\alpha(v_i)$. If we let $\beta = \bigcup_{i=0}^{n-1} \alpha_i$, then $\beta < \kappa$, and for all ordinals γ with $\beta \leq \gamma < \kappa$ there is no $f_\gamma \in F_\gamma$ such that $f_\gamma(v_i) = f_\alpha(v_i)$ for any $i < n$.

Thus, we may find a countable sequence $\alpha_1, \alpha_2, \dots$ of ordinals with $n \leq \alpha_i < \kappa$ and a sequence of l -list-homomorphisms $f_{\alpha_i} \in F_{\alpha_i}$ such that whenever $i < j$ there is no l -list-homomorphism in F_{α_j} which agrees with f_{α_i} on any free vertex.

Now let i and j be arbitrary positive integers with $i < j$. We will find specific vertices which prevent the $f \in F_{\alpha_j}$ from agreeing with f_{α_i} on the free vertices of G . There must exist some free vertex $v_{i,j}$ and some fixed vertex $w_{i,j} \in G_{\alpha_j}$ such that $v_{i,j}w_{i,j} \in E(G)$ (or $w_{i,j}v_{i,j} \in E(G)$), but $f_{\alpha_i}(v_{i,j})f_{\alpha_j}(w_{i,j}) \notin E(H)$. (or $f_{\alpha_j}(w_{i,j})f_{\alpha_i}(v_{i,j}) \notin E(H)$). If there were no such $v_{i,j}$ and $w_{i,j}$ then there would be a homomorphism $g \in F_{\alpha_j}$ such that $g|_{G_{\alpha_i}} = f_{\alpha_i}$. But by definition of f_{α_i} no such homomorphism exists.

We will now apply Ramsey's theorem to find a single free vertex of G which we will use to construct a new certificate of non-compactness for H . We will colour the unordered pairs of natural numbers $\{\{i, j\} : 1 \leq i < j\}$. For each free vertex $v \in V(G)$ we will define two colours v^+ and v^- . Since G contains only finitely many free vertices we will obtain only finitely many colours. To colour the pair $\{i, j\}$, choose any free vertex $v_{i,j}$ and any fixed vertex $w_{i,j}$ satisfying the conditions given above. If $v_{i,j}w_{i,j} \in V(G)$ but $f_{\alpha_i}(v_{i,j})f_{\alpha_j}(w_{i,j}) \notin E(H)$ then $\{i, j\}$ gets colour $v_{i,j}^+$. Otherwise it must be the case that $w_{i,j}v_{i,j} \in E(G)$, but $f_{\alpha_j}(w_{i,j})f_{\alpha_i}(v_{i,j}) \notin E(H)$, and we colour it with $v_{i,j}^-$. Now by Ramsey's theorem there must be a countably infinite set A of positive integers such that the colour of $\{i, j\}$ is the same for all $i, j \in A$.

Suppose first that for each $i, j \in A$, the colour of $\{i, j\}$ is v^+ for some free vertex v of G .

We define an out-directed star S with $V(S) = \{s_0\} \cup \{s_{i,j} : 1 \leq i < j\}$ and $E(S) = \{s_0s_i : i \geq 1\}$. We define a list-assignment l' for S with respect to H by $l'(s_0) = \{f_{\alpha_i}(v) : i \in A\}$ and $l'(s_{i,j}) = \{f_{\alpha_j}(w_{i,j})\}$ for all $1 \leq i < j$.

Now given any finite subdigraph S' of S , we may choose some $k \geq 1$ large enough so that $w_{i,j} \in V(G_{\alpha_k})$ whenever $s_{i,j} \in V(S')$. Thus there is a natural l' -list-homomorphism $g : S' \rightarrow H$ given by $g(s_0) = f_{\alpha_k}(v)$ and $g(s_{i,j}) = f_{\alpha_k}(w_{i,j})$. However, for each $k \geq 1$ there exists an i and j with $1 \leq i < j$ such that $f_{\alpha_k}(v)f_{\alpha_k}(w_{i,j}) \notin E(H)$, and so there is no k such that $f_{\alpha_k}(v)f_{\alpha_k}(w_{i,j}) \in E(H)$ for all $1 \leq i < j$, and hence there is no l' -list-homomorphism from S to H . Thus, S with l' is a certificate of non-list-compactness for H .

In the case where the colour of $\{i, j\}$ is v^- for each $i, j \in A$, the proof is identical

to the previous case except that S will be an in-directed star. ■

Theorem 81 *Let B be as defined above. A digraph H is B -list-compact if and only if it contains none of the following as an induced subdigraph:*

- *an infinite loopless complete digraph,*
- *an increasing digraph,*
- *a decreasing digraph,*
- *an infinite cocktail-party digraph.*

Proof: Suppose that H contains one of the above as an induced subdigraph. We have already seen certificates of non-list-compactness for each of these cases, using list-assignments satisfying B , in the proof of theorem 76. Thus H is not B -list-compact.

On the other hand, suppose that H is not B -list-compact. We must show that H contains one of the above digraphs as an induced subdigraph. Applying lemma 80 we may obtain a certificate of non-list-compactness for H , call it G with list-assignment l , where G is a star and $l(v)$ is a singleton for all $v \in V(G)$ except the center of G . When $l(v)$ is a singleton we will abuse notation and consider $l(v)$ to be the vertex which is the unique element of $l(v)$.

Case A: G is an out-directed star.

We begin by labelling the vertices of G as $\{s_0, s_1, \dots\}$ so that s_0 is the center of G . We denote by G_n the subdigraph of G induced by $\{s_0, s_1, \dots, s_n\}$.

For each $n \geq 1$, let $F_n = \{v \in l(s_0) : \text{there exists an } l\text{-list-homomorphism } f : G_n \rightarrow H \text{ with } f(s_0) = v\}$. Clearly $F_{n+1} \subseteq F_n$ for all $n \geq 0$, and $\bigcap_{n=1}^{\infty} F_n = \emptyset$, but all F_n are nonempty. Thus, we may find an increasing sequence of positive integers a_1, a_2, \dots such that for each $i \geq 1$ there exists a vertex v_i such that $v_i \in F_{a_i}$ but $v_i \notin F_{a_j}$ for any $j > i$.

Now for each $i, j \geq 1$ with $i < j$, there must be a fixed vertex s_k with $a_i < k \leq a_j$ such that $v_i l(s_k) \notin E(H)$ but $v_j l(s_k) \in E(H)$, since otherwise $v_i \in F_{a_j}$. We will

denote the vertex s_k as $w_{i,j}$ and the vertex $l(s_k)$ as $z_{i,j}$. Observe that if $i < j$ then $v_j z_{k,i} \in E(H)$ for any $k < i$.

Let $X = \{v_i : i \geq 1\}$ and $Y = \{z_{i,j} : 1 \leq i < j\}$. Note that X and Y are not necessarily disjoint. We now restrict our attention to the subdigraph H' of H induced by $X \cup Y$. Note that the v_i all have distinct out-neighbourhoods in Y , and so the v_i are all distinct. However, the $z_{i,j}$ might not be distinct, and it is also possible for some v_i to be identical to some $z_{j,k}$.

The remainder of our proof is quite similar to the proof of lemma 74. However, the set X we have constructed has additional properties not possessed by the arbitrary set of vertices with distinct neighbourhoods we examined in the proof of theorem 76. These properties will allow us to avoid the cases in the proof of theorem 76 which required certificates of non-list-compactness which did not satisfy the property B .

Our goal in the next section of the proof is to either find a forbidden subdigraph in H' directly, or to find an infinite subset Q of X such that for all $u, v \in Q$ we have $N_Y^+(u) - Q \neq N_Y^+(v) - Q$. In the final section of the proof we will use such a set Q to find a forbidden subdigraph of H' .

If $X - Y$ is infinite, then we simply let $Q = X - Y$. Since Q includes no vertex of Y , we know that $N_Y^-(u) - Q \neq N_Y^-(v) - Q$ for all $u, v \in Q$.

If $X - Y$ is finite, then $X \cap Y$ must be infinite. We will use Ramsey's theorem to obtain the desired result. We partition the unordered pairs $\{i, j\}$, where $1 \leq i < j$, into four parts, P_1, P_2, P_3, P_4 , defined as follows:

$$P_1 = \{\{i, j\} : v_i v_j \in E(H') \text{ and } v_j v_i \in E(H')\}$$

$$P_2 = \{\{i, j\} : v_i v_j \notin E(H') \text{ and } v_j v_i \in E(H')\}$$

$$P_3 = \{\{i, j\} : v_i v_j \in E(H') \text{ and } v_j v_i \notin E(H')\}$$

$$P_4 = \{\{i, j\} : v_i v_j \notin E(H') \text{ and } v_j v_i \notin E(H')\}.$$

By Ramsey's theorem there is an infinite set S of natural numbers such that all pairs $\{i, j\}$ with $i, j \in S$ are in the same part P_k . Let $V_S = \{v_i : i \in S\}$. Note that $V_S \cap Y$ must be infinite. We now consider the four cases.

Case 1: All pairs of elements of S are in P_1 .

If there are infinitely many loopless vertices in V_S then these vertices induce an infinite loopless complete digraph in H' , and so we are done.

If there are infinitely many vertices in V_S with loops, then let $Q = \{a_1, a_2, \dots\}$ be an infinite set of vertices with loops in $V_S \cap Y$. Then $N_Y^+(a_i) \cap Q = Q$ for all $i \geq 1$. But no two a_i have the same out-neighbourhood in Y , so for all $i, j > 0$ with $i \neq j$, we have $N_Y^+(a_i) - Q \neq N_Y^+(a_j) - Q$.

Case 2: All pairs of elements of S are in P_2 .

Label the elements of $V_S \cap Y$ by a_1, a_2, \dots , where for $i \neq j$, $a_i a_j \in E(H)$ if and only if $i > j$. A particular a_i may or may not have a loop. Let $Q = \{a_1, a_3, a_5, \dots\}$. Then for all $n \geq 1$ we have $a_{2n} \notin N_Y^+(a_{2n-1})$ but $a_{2n} \in N_Y^+(a_{2m-1})$ for all $m > n$. Clearly Q has the required properties.

Case 3: All pairs of elements of S are in P_3 .

We define Q as in the previous case and discover that for all $n > 0$ we have $a_{2n} \in N_Y^+(a_{2n-1})$ but $a_{2n} \notin N_Y^+(a_{2m-1})$ for any $m > n$. Again it is clear that Q satisfies the required properties.

Case 4: All pairs of elements of S are in P_4 .

We claim that this case cannot occur. Let v be any element of $V_S \cap Y$. Then $v = z_{i,j}$ for some $i, j \geq 1$. But V_S is an infinite subset of X so it must contain v_k for arbitrarily large values of k . If we choose some $k > i$ such that $v_k \in V_S$, then $v_k v \in E(H')$, contradicting the definition of P_4 .

We must now show that we can find a forbidden subdigraph in H' using the set Q .

We claim that we may find an infinite collection of pairs (q_i, r_i) for $i \geq 1$ such that all q_i and all r_i are distinct, $q_i \in Q$ and $r_i \in Y - Q$ for all $i \geq 1$, and $q_i r_i \notin E(H')$.

A simple 'greedy' algorithm will be sufficient to find such a set. Suppose we have chosen $(q_1, r_1), (q_2, r_2), \dots, (q_n, r_n)$ satisfying the above conditions for some $n \geq 0$ (if $n = 0$ we have chosen nothing yet). Suppose there is no pair (q_{n+1}, r_{n+1}) which satisfies the required properties. Then for all $v \in Q - \{q_1, q_2, \dots, q_n\}$ it must be the case that v is adjacent to every $w \in Y - Q - \{r_1, r_2, \dots, r_n\}$. But this is clearly inconsistent

with the fact that the infinitely many vertices in Q have distinct neighbourhoods in $Y - Q$. Thus, such a pair must exist. By continuing this process we obtain our infinite collection of pairs.

We now prepare for another application of Ramsey's theorem. We partition the unordered pairs of natural numbers $\{i, j\}$, where $i < j$, into parts P_1, P_2, P_3, P_4 according to the following rules:

$$P_1 = \{\{i, j\} : q_i r_j \in E(H) \text{ and } q_j r_i \in E(H)\}$$

$$P_2 = \{\{i, j\} : q_i r_j \notin E(H) \text{ and } q_j r_i \in E(H)\}$$

$$P_3 = \{\{i, j\} : q_i r_j \in E(H) \text{ and } q_j r_i \notin E(H)\}$$

$$P_4 = \{\{i, j\} : q_i r_j \notin E(H) \text{ and } q_j r_i \notin E(H)\}.$$

By Ramsey's theorem, there must be an infinite subset T of natural numbers such that all pairs of elements from T are in the same part P_i . We find ourselves once again with four cases.

Case 1: All pairs of elements of T are in P_1 .

In this case the vertices $\{q_i : i \geq 1\} \cup \{r_i : i \in T\}$ induce an infinite cocktail-party digraph in H' .

Case 2: All pairs of elements of T are in P_2 .

The vertices $\{q_i : i \geq 2\} \cup \{r_i : i \in T\}$ induce an infinite increasing digraph in H' .

Case 3:

Now the vertices $\{q_i : i \geq 1\} \cup \{r_i : i \in T\}$ induce an infinite decreasing digraph in H' .

Case 4: All pairs of elements of T are in P_4 .

This case cannot occur, since for any $r_i, i \in T$, $r_i = z_{j,k}$ for some $1 \leq j < k$. But there must be some $q_n, n \in T$, such that $q_n = v_m$ with $m > j$. But then $q_n r_i \in E(H')$, a contradiction.

This concludes case A.

Case B: G is an in-directed star.

In this case our argument is essentially identical, except that the directions of the edges in the subdigraphs we find are reversed. For a given digraph K we define the *reverse* of K , denoted K^- , by $V(K^-) = V(K)$ and $E(K^-) = \{vu : uv \in E(K)\}$. We complete the proof with the following simple observations.

- If K is a loopless complete digraph then K^- is a loopless complete digraph.
- If K is an increasing digraph then K^- is a decreasing digraph.
- If K is an decreasing digraph then K^- is a increasing digraph.
- If K is a cocktail-party digraph then K^- is a cocktail-party digraph.

■

Another way of restricting list-assignments, which we have already seen in an example, is to require some of the lists to be equal to $V(H)$. We have already noted that if C is the property that every vertex in $\text{dom}(l)$ is equal to $V(H)$ then a digraph is C -list-compact if and only if it is compact.

We may define another natural property D by saying that a list-assignment l has property D if $l(v)$ is either finite or is equal to $V(H)$ for every $v \in \text{dom}(l)$. Recall that in chapter 4 we defined a property R for lists by saying that a list-assignment l has property R if $l(v)$ is a singleton or $l(v) = V(H)$ for every $v \in \text{dom}(l)$. Using lemma 79 we see that a digraph is D -list-compact if and only if it is R -list-compact. We immediately obtain the following result.

Theorem 82 *The class of D -list-compact digraphs is a superclass of the class of compact cores and a subclass of the class of compact digraphs.*

Proof: In chapter 4 we showed that being a compact core was a sufficient condition for a digraph to be R -list-compact. Obviously compactness is a necessary condition for R -list-compactness, since we can set $l(v) = V(H)$ for all $v \in \text{dom}(l)$. ■

Unfortunately, it appears to be difficult to say more than this.

Yet another possible property similar to the above, and in the spirit of theorem 81, might be the following. We say that a list-assignment l has property E if $l(v) = V(H)$

for only finitely many $v \in \text{dom}(l)$, and $l(v)$ is finite for every other $v \in \text{dom}(l)$. Since E is a strengthening of the property B from theorem 81, we know that every B -list-compact digraph is also E -list-compact. Thus, if H contains none of the forbidden subdigraphs from theorem 81 then H is E -list-compact. However, the converse is not true, as we will soon see. In fact, unlike in our two major results so far, there are non- E -list-compact digraphs for which there are no countable certificates of non-list-compactness.

For example, let us define a digraph H by $V(H) = \{u_\alpha : \alpha < \aleph_1\} \cup \{v_\alpha : \alpha < \aleph_1\}$ and $E(H) = \{u_\alpha v_\beta : \beta \leq \alpha\}$. We may think of H as an uncountable analogue of an increasing digraph. Certainly H contains an induced increasing subdigraph. The reader should have little difficulty convincing himself that there is no countable certificate of non- E -list-compactness for H , yet there is an obvious uncountable certificate.

We can also find significant differences between E -list-compactness and B -list-compactness within countable digraphs. In particular, it is not sufficient to look only at stars as potential certificates of non-list-compactness when our list-assignments have property E .

Let us define a digraph H as follows. Let $V(H) = \{u_1, u_2, \dots\} \cup \{v_1, v_2, \dots\} \cup \{w_1, w_2, \dots\} \cup \{z\}$, and $E(H) = \{u_i w_j : j \leq i\} \cup \{v_i w_j : j \leq i\} \cup \{u_i v_i : i \geq 1\} \cup \{z w_i : i \geq 1\}$. In this case we may think of H as an increasing digraph with the vertices in $U(H)$ replaced by edges, and with an extra vertex z which dominates $L(H)$. If G is an out-directed star and l is a list-assignment for G with respect to H such that $E(l)$ holds, it is not difficult to see that G admits an l -list-homomorphism to H whenever all of its finite subdigraphs admit such a homomorphism. However, if we replace the vertex which is the center of a star G with an edge in the obvious way, there is a natural list-assignment for G with respect to H which satisfies E and provides a certificate of non-list-compactness for H .

Our final result is a particularly nice characterization. We define a property F for list-assignments as follows: a list-assignment $l : V(G) \rightarrow \mathcal{P}(V(H))$ has property F if in each component of G there is at least one vertex v such that $l(v)$ is finite. In particular if G is connected then l has property F if at least one vertex of G is assigned a finite list. We obtain the following characterization.

Theorem 83 *A digraph H is F -list-compact if and only if the reduced digraph H^r is locally finite.*

Proof: Observe first that if a list-assignment l for a digraph G with respect to H has property F , then the reduced list-assignment l^r for G with respect to H^r also has property F , and so if H is not F -list-compact then H^r is not F -list-compact. If H^r is not F -list-compact then H is not F -list-compact either (*c.f.* proof of corollary 72), so it suffices to show that a reduced digraph is F -list-compact if and only if it is locally finite.

Let H be a locally finite reduced digraph. Let G be any digraph and l be a list-assignment for G with respect to H such that $F(l)$ holds. Suppose that every finite subdigraph of G admits an l -list-homomorphism to H . We will use lemma 47 to show that G admits an l -list-homomorphism to H . We will define a new list-assignment l' for G with respect to H such that for each $v \in V(G)$, $l'(v)$ is a finite subset of $l(v)$, and every finite subdigraph of G admits an l' -list-homomorphism to H .

For each component C of G , we define $l'|_{V(C)}$ as follows: choose some $v \in V(C)$ such that $l(v)$ is finite, and let $l'(v) = l(v)$. Now for each $w \in V(C)$ we define $l'(w) = \{u \in l(w) : d(u, l'(v)) \leq d(w, v)\}$. As H is locally finite we know that $l'(w)$ is finite, and we can see that every finite subdigraph of G admits an l' -list-homomorphism to H exactly as we did in the proof of theorem 49.

We now let \mathcal{T}_v be the discrete topology on $l'(v)$ for each $v \in V(G)$. By lemma 47, G admits an l' -list-homomorphism to H , but the same homomorphism is an l -list-homomorphism from G to H , and so H is F -list-compact.

Now suppose H is not locally finite. Then there exists some $v \in V(H)$ such that v has either infinite in-degree or infinite out-degree. Assume that v has infinite out-degree, as the other case is essentially identical. The component C of H containing v must obviously be infinite. Furthermore, there must be an infinite subset S of $N^+(v)$ such that either $uv \in E(H)$ for all $u \in S$, or $uv \notin E(H)$ for all $u \in S$. Thus, any two vertices $x, y \in S$ have distinct in-neighbourhoods or distinct out-neighbourhoods in $C - \{v\}$, and so the reduced digraph $(C - \{v\})^r$ is infinite, and S is an infinite subset of $V((C - \{v\})^r)$.

Now by corollary 75 the digraph $(C - \{v\})^r$ must contain a subdigraph K which is one of the following: a loopless infinite complete digraph or an infinite independent digraph with loops, where $V(K) \subseteq S$; or an increasing digraph, a decreasing digraph, a cocktail-party digraph or a matching digraph with $U(K) \subseteq S$ or $L(K) \subseteq S$. Of course $(C - \{v\})^r$ is an induced subdigraph of H , so let K be an induced subdigraph of H of one of the above types.

In the proof of theorem 76 we constructed certificates of non-list-compactness for each of the above types of digraph. We will modify those certificates to obtain a certificate of non-list-compactness for H . Recall that the vertex v dominates K . Our modification consists of adding a new vertex u and appropriate edges to a certificate from theorem 76. In all of our list-assignments we will define $l(u) = v$, and our modified certificates will all be connected, so $F(l)$ will hold. Two examples will suffice to make our method clear.

Case 1: K is a loopless infinite complete digraph with $V(K) \subseteq S$.

Let $V(K) = \{a_1, a_2, \dots\}$, and define a digraph G by $V(G) = \{u, v_0, v_1, v_2, \dots\}$, and $E(G) = \{v_i v_j : i, j \geq 0\} \cup \{u v_i : i \geq 0\}$. We define a list-assignment l for G with respect to H by $l(v_0) = V(K)$, $l(v_i) = \{a_i\}$ for each $i \geq 1$ and $l(u) = v$. The digraph G with list-assignment l is easily seen to be a certificate of non- F -list-compactness for H .

Case 2: K is an increasing digraph with $U(K) \subseteq S$.

The schema for the subdigraph of H induced by $V(K) \cup \{v\}$ is shown in figure 6.12 (a). We define a digraph G by $V(G) = \{u, w_0, w_1, w_2, \dots\}$, and $E(G) = \{w_0 w_i : i \geq 1\} \cup \{u w_0\}$. We define a list-assignment l for G by $l(w_0) = U(K)$, $l(w_i) = y_i$ for each $i \geq 1$, and $l(u) = v$ (figure 6.12 (b)). Again it is obvious that G with l is a certificate of non- F -list-compactness for H .

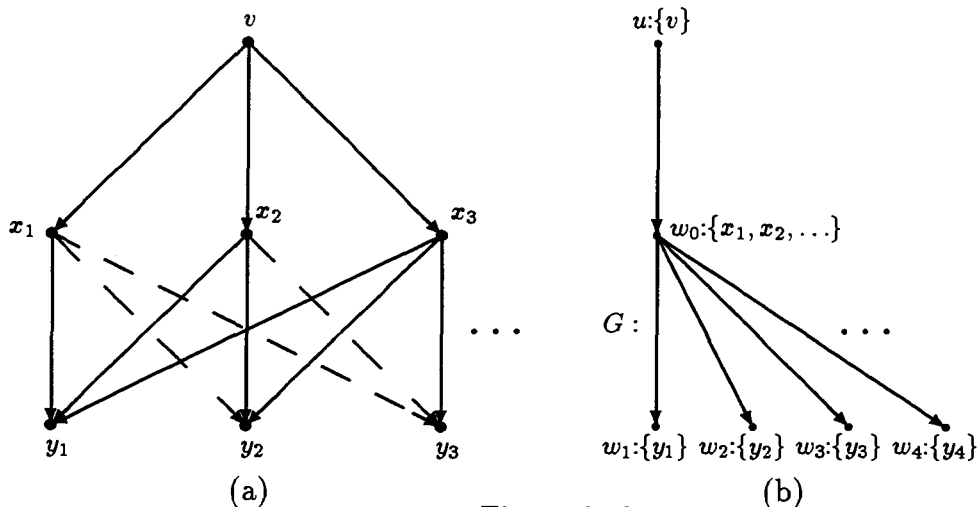


Figure 6.12

Certificates of non- F -list-compactness for H can be constructed for the remaining cases by similar methods. ■

6.4 Structures and Graphs

The results in this section do not easily generalize to structures, and we have not attempted to do so. However, all of these results do hold for undirected graphs. In the case of theorem 81, the forbidden subgraphs for the undirected case are just the underlying undirected graphs of the forbidden subdigraphs for the directed case. Observe that this yields only three classes of forbidden subgraphs, as the underlying undirected graphs of an increasing digraph and a decreasing digraph are the same.

Bibliography

- [1] R. Aharoni, personal communication.
- [2] N. Alon, Restricted colourings of graphs, *London Mathematical Society Lecture Note Series* **187** (1993) 1–34.
- [3] N. Alon, Choice numbers of graphs: a probabilistic approach, *Combinatorics, probability and computing* **1** (1992) 107–114.
- [4] N. Alon and M. Tarsi, Colourings and Orientations of graphs, *Combinatorica* **12** (1992) 125–134.
- [5] J. Bang-Jensen and P. Hell, The effect of two cycles on the complexity of colourings by directed graphs, *Discrete Applied Math.* **26** (1990) 1–23.
- [6] J. Bang-Jensen, P. Hell, and G. MacGillivray, The complexity of colourings by semi-complete digraphs, *SIAM Journal on Discrete Math.* **1** (1988) 281–298.
- [7] J. Bang-Jensen, P. Hell, and G. MacGillivray, On the complexity of colouring by superdigraphs of bipartite graphs, Simon Fraser University CSS/LCCR TR90-01, 1990
- [8] B. Bauslaugh The complexity of infinite H-colouring, *J. Comb. Th. Ser. B* **61** (1994) 141–154.
- [9] B. Bauslaugh The complexity of colouring by infinite vertex-transitive graphs, *Ann. Disc. Math.*, **27**, 1993 1–10.

- [10] A. Bialostocki, An application of the Ramsey theorem to ordered r -tournaments, *Discrete Math.* **61** (1986) 325–328.
- [11] M. Biró, M. Hujter and Zs. Tuza, Precoloring extension. I. Interval graphs, *Discrete Math.* **100** (1992) 267–279.
- [12] B. Bollobás and A. J. Harris, List-colourings of graphs, *Graphs Combinat.* **1** (1985) 115–127.
- [13] B. Bollobás and H. R. Hind, A new upper bound for the list chromatic number, *Discrete Math.* **74** (1989) 65–75.
- [14] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Elsevier Science Publishing, New York (1976).
- [15] R. Brewster, *Vertex colourings of edge coloured graphs*, Ph. D. thesis, Simon Fraser University, 1993.
- [16] R. Brewster, The complexity of colouring symmetric relational systems, *Discrete Applied Math.* **49** (1994) 95–105.
- [17] N. G. de Bruijn and P. Erdős, A colour problem for infinite graphs and a problem in the theory of relations, *Indagationes Math.* **13** (1951) 369–373.
- [18] P. J. Cameron, The age of a relational structure, *Discrete Math.* **95** (1991) 49–67.
- [19] A. Chetwynd and R. Häggvist, A note on list-colourings, *Journal of Graph Theory* **13** (1989) 87–95.
- [20] K. B. Chilakommarri, The unit-distance graph problem: A brief survey and some new results, *Bull. Inst. Combin. Math.* **8** (1993) 39–60.
- [21] D. Duffus and N. Sauer, A categorical version of Hedetniemi’s conjecture, *Discrete Math.*, to appear.
- [22] P. Erdős, Graph theory and probability, *Canadian Journal of Mathematics* **11** (1959) 34–38.

- [23] P. Erdős and A. Hajnal, On chromatic number of graphs and set-systems, *Acta Mathematica Scientiarum Hungaricae* **17** (1966) 61–99.
- [24] P. Erdős, A. Hajnal, A. Máté, and R. Rado, *Combinatorial Set Theory: Partition relations for cardinals*, North-Holland, 1984.
- [25] P. Erdős and R. Rado, A construction of graphs without triangles having pre-assigned order and chromatic number, *The Journal of the London Mathematical Society* **35** (1960) 445–448.
- [26] P. Erdős, A. L. Rubin, and H. Taylor, Choosability in graphs, *Congressus Numerantium* **26** (1979) 125–157.
- [27] T. Feder and P. Hell, List homomorphisms to reflexive graphs, manuscript, 1994.
- [28] W.D. Fellner, On minimal graphs, *Theoretical Computer Science* **17** (1982) 103–110.
- [29] F. Galvin, A proof of the Dinitz conjecture, preprint 1994.
- [30] M. R. Garey and D. S. Johnson, *Computers and Intractability*, W. H. Freeman and Company, 1979.
- [31] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, 1980.
- [32] W. H. Gottschalk, Choice functions and Tychonoff's theorem, *Proceedings of the American Mathematical Society* **2** (1951) 172.
- [33] G. Grätzer, *Universal Algebra*, Springer-Verlag, 1979.
- [34] W. Gutjahr, E. Welzl, and G. Woeginger, Polynomial graph-colourings, Tech. Rept. B-88-06, Freie Universität Berlin, 1988.
- [35] H. Hadwiger, Ungelöste probleme No. 40, *Elemente des Math.* **16** (1961) 103–104.
- [36] R. Haggkvist, P. Hell, D.J. Miller, and V. Neumann-Lara, On multiplicative graphs and the product conjecture, *Combinatorica* **8**(1) (1988), 63–74.

- [37] P. Hell, *Rétractions de graphes*, Ph. D. thesis, Université de Montréal, 1972.
- [38] P. Hell and J. Nešetřil, The core of a graph, *Discrete Math.* **109** (1992), 117–126.
- [39] P. Hell and J. Nešetřil, On the complexity of H -colouring, *Journal of Combinatorial Theory, Series B* **48** (1990), 92–110.
- [40] P. Hell, J. Nešetřil and X. Zhu, Duality of graph homomorphisms, *Combinatorics, Paul Erdős is Eighty* **2** (1994).
- [41] P. Hell, J. Nešetřil and X. Zhu, Complexity of tree homomorphisms, *Journal of Comb. Theory Ser. B*, to appear.
- [42] P. Hell, J. Nešetřil and X. Zhu, Duality and polynomial testing of tree homomorphisms, *Trans. Amer. Math. Soc*, accepted.
- [43] H. Herrlich and G. E. Strecker, *Category Theory*, Allyn and Bacon, 1973.
- [44] K. Hrbacek and T. Jech, *Introduction to Set Theory*, Marcel Decker, New York, 1984.
- [45] T. J. Jech, *The Axiom of Choice*, North-Holland, London, 1973.
- [46] J. L. Kelley, The Tychonoff product theorem implies the Axiom of Choice, *Fund. Math.* **37** (1950) 75–76.
- [47] U. Knauer, Unretractive and S-unretractive joins and lexicographic products of graphs, *Journal of Graph Theory* **11** (1987) 429–440.
- [48] U. Knauer, Endomorphisms of graphs II., Various unretractive graphs, *Arch. Math* **55** (1990) 193–203.
- [49] A. V. Kostochka, List edge chromatic numbers of graphs with large girth, *Discrete Math.* **101** (1992) 189–201.
- [50] J. Kratochvíl and Z. Tuza, Algorithmic complexity of list colourings, *Disc. App. Math*, to appear.

- [51] L. Lovász, On chromatic number of finite set-systems, *Acta Mathematica Scientiarum Hungaricae* **19** (1968) 59–67.
- [52] G. MacGillivray, Colouring by vertex-transitive digraphs, *SIAM Journal of Discrete Math* **4** (1991), 297–408.
- [53] S. Marshall, *Properties of k -tournaments*, Ph.D. thesis, Simon Fraser University, 1994.
- [54] H. A. Maurer, J.H. Sudborough and E. Welzl, On the complexity of the general colouring problem, *Information and Control* **51** (1981) 123–145.
- [55] J.R. Munkres, *Topology*, Prentice-Hall, 1975.
- [56] G. H. Müller, *Sets and Classes*, North-Holland, 1976.
- [57] J. Nešetřil and A. Pultr, On class of relations and graphs defined by subobjects and factorobjects, *Discrete Math.* **22** (1979) 187–200.
- [58] J. Nešetřil and A. Rödl, A short proof of the existence of highly chromatic hypergraphs without short cycles, *J. Comb. Th. Ser. B* **27** (1979) 225–227.
- [59] R. Nowakowski and I. Rival, Fixed edge theorems for graphs with loops, *Journal of Graph Theory* **3** (1979) 339–350.
- [60] M. Perles, personal communication.
- [61] A. Pultr and V. Trnková, *Combinatorial Algebraic and Topological Representations of Groups Semigroups and Categories*, North-Holland, 1980.
- [62] R. Rado, Axiomatic treatment of rank in infinite sets, *Canadian Journal of Mathematics* **1** (1949) 337–343.
- [63] R. Rado, A selection lemma, *Journal of Combinatorial Theory Ser. A* **10** (1971) 176–177.
- [64] H. L. Royden, *Real Analysis*, MacMillan Publishing Co. Inc., 1968.

- [65] H. Rubin and J. Rubin, *Equivalents of the Axiom of Choice*, North-Holland, 1963.
- [66] B. Russel, On some difficulties in the theory of transfinite numbers and order types, *Proceedings of the London Mathematical Society, Ser. 2*, **4** (1906), 29–53.
- [67] J.R. Shoenfield, *Mathematical Logic*, Addison-Wesley, Reading, MA, 1967.
- [68] W. Taylor, Atomic compactness and elementary equivalence, *Fund. Math.* **71** (1971) 103–112.
- [69] C. Thomassen, Planar graphs are 5-choosable, preprint 1994.
- [70] A. Tychonoff, Über einen functionenraum, *Math. Annalen* **111** (1935) 762–766.
- [71] V. G. Vizing, Vertex colouring with given colours (Russian), *Discrete Analysis* **29** (1976) 3–10.
- [72] V. G. Vizing, On an estimate of the chromatic class of a p -graph (Russian), *Discrete Analysis* **3** 25–30.
- [73] M. Voigt, List colourings of planar graphs, *Discrete Math.* **120** (1993) 215–219.
- [74] E. Welzl, Color families are dense, *Theoretical Computer Science* **17** (1982) 29–41.
- [75] E. Zermelo, Beweis dass jede Menge wohlgeordnet werden kann, *Math. Ann.*, **59** (1904), 514–516.