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CANADIAN THESES ON MICROFICHE

THÈSES CANADIENNES SUR MICROFICHE

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TITLE OF THESIS/*TITRE DE LA THÈSE* DISCONTINUOUS HOMEOMORPHISM GROUPS OF SURFACES

UNIVERSITY/*UNIVERSITÉ* SIMON FRASER

DEGREE FOR WHICH THESIS WAS PRESENTED/ *GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE* Ph. D

YEAR THIS DEGREE CONFERRED/*ANNÉE D'OBTENTION DE CE DEGRÉ* 1981

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DATED/*DATÉ* 16 - 10 - 1980 SIGNED/*SIGNÉ*

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DISCONTINUOUS HOMEOMORPHISM GROUPS OF SURFACES

by

Erich Durnberger

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in the Department

of

Mathematics

SIMON FRASER UNIVERSITY

September 1980

APPROVAL

Name: Erich Durnberger

Degree: Doctor of Philosophy (Mathematics)

Title of Thesis: Discontinuous homeomorphism groups of surfaces

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ABSTRACT

A group Γ of homeomorphisms of a topological space X is said to act discontinuously on X (or to be a discontinuous group of X) if for any two points x and y of X there are neighbourhoods V_x of x and V_y of y such that there are only finitely many elements γ of Γ for which $\gamma(V_x)$ and V_y intersect.

In Chapter 1 we discuss discontinuous groups Γ in general and later restrict our attention to discontinuous groups acting on a surface S. The local action of Γ on S is closely related to groups of isometries of the unit disc. Among the most important of the many examples of discontinuous groups are (a) finite homeomorphism groups (b) discontinuous groups of Möbius transformations (c) translations, reflected translations, contractions and reflected contractions of the euclidean plane.

In Chapter 2 we show that for any discontinuous group Γ of a surface S there is a Γ-invariant triangulation of S. For this construction knowledge of the local action of the group is of basic importance.

In Chapter 3 we introduce the notions of abstract polyhedron, isomorphism between polyhedra, boundaries, one and two-sidedness of cycles and 2-infinite paths as a combinatorial description of a topological polyhedron introduced in Chapter 1 and as a means of investigating discontinuous groups. It is shown that a two-sided path or cycle partitions a polyhedron into two parts. One-sidedness of paths in planar polyhedra can be characterized in terms of sepa-

ration by a cycle. Moreover, it is shown that isomorphisms preserve boundaries and the separation properties of cycles and paths.

In Chapter 4 the concepts developed in Chapter 3 are applied to investigate groups Γ of automorphisms of polyhedra. In the first section we investigate the set of fixed vertices and edges of Γ, especially if the polyhedron is planar. The second deals with the two types of automorphisms of infinite order of a planar polyhedron, yet the methods used also apply to orientation preserving automorphisms of finite order. If α is of type 1, then there is an α-invariant, pairwise disjoint collection $\{C_i \mid i \in Z\}$ of β-invariant ($\beta = \alpha$ or $\beta = \alpha^2$) 2-infinite and two-sided paths partitioning the polyhedron. If α is of type 2, then there is a disjoint and α-invariant collection $\{C_i \mid i \in Z\}$ of "concentric" cycles on which α acts transitively. Thus, if α is of type 1, it can appropriately be described as a translation or reflected translation and if it is of type 2, as a contraction or reflected contraction. The finite order orientation preserving automorphisms can be appropriately named rotations (see Theorems 4.1.14 and 4.2.20). In the third section we state some known theorems about groups Γ of automorphisms of special planar infinite polyhedra for which there is a finite set of boundaries representing each Γ-orbit.

In Chapter 5 we establish the link between the topological and algebraic concepts of polyhedron and discontinuous groups and the combinatorial concept of polyhedron and automophism groups. There is a 1-1 correspondence (up to topological equivalence) between discontinuous homeomorphism groups acting on surfaces and groups of automorphisms of abstract polyhedra. This allows us to study (in Chapter 4) algebraic and other properties of discontinuous groups without referring to the underlying topological space. In fact, these properties only depend on the combinatorial properties of the surface. We finally topologically characterize (that is, give a picture of) orientation preserving elements of finite order and elements of infinite order in discontinuous homeomorphism groups of planar surfaces by showing that they are topologically equivalent to certain types of elementary mappings of the sphere or euclidean plane. We also characterize discontinuous homeomorphism groups Γ of the euclidean plane with compact fundamental domain(that is, for which there is a compact set whose Γ-images cover the plane), by showing that they are topologically equivalent to groups of isometries of the euclidean or non-euclidean plane.

To my uncle Johann

ACKNOWLEDGEMENTS

I take this opportunity to express my gratitude to Dr. Brian Alspach for his continued patience, encouragement and support. His guidance during the preparation of this thesis has been inspiring and he tirelessly and critically read the final draft. Dr. Alspach has been helpful in other matters whenever needed.

My very special thanks go to Dr. Wilfried Imrich who initiated me to the topic and whose personal publications, suggestions and interest in my work were of great benefit. He also performed the difficult task of reading my handwritten manuscripts.

I remain thankful to the Department of Mathematics of the University of Newcastle where I was permitted to type my thesis and who made my stay in Australia very pleasant.

Finally, I wish to extend my thanks to the faculty and staff members of the Mathematics Department at Simon Fraser University.

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CHAPTER 1. DISCONTINUOUS HOMEOMORPHISM GROUPS

A family $(A_i | i \in I)$ of subsets of a topological space X is locally finite if every point of X has a neighbourhood which meets A_i for only finitely many $i \in I$. Based on this concept it is natural to define "discontinuity" of the action of a group on the space X. A group Γ acting on X is said to act discontinuously on X if any two points x and y of X have neighbourhoods V_x and V_y such that $\gamma(V_x)$ and V_y are disjoint except for finitely many $\gamma \in \Gamma$. This concept of discontinuity generalizes the concept of finite groups Γ and the concept of discontinuity for Moebius transformations (see Section 3, Example 1.3.4) which has been studied extensively (see [11], [17], [24], etc.).

Section 1 of this chapter contains an overall introduction to this thesis and in Section 2 we give some of the basic definitions used throughout Chapter 1 and 2. In Section 3 we give some examples of discontinuous groups, mostly restricted to discontinuous homeomorphism groups acting on surfaces. In Section 4 we investigate the basic properties of local finiteness and of X-graphs frequently used in later sections.

In Section 5 we study general properties of discontinuous groups on arbitrary spaces. Any group Γ acting on the topological space X may be assigned a topology expressing the fact that with "little modification" of $\gamma \in \Gamma$ the values of γ will be displaced only a "little bit". We show that with the topology usually considered any discontinuous homeomorphism group acting on a locally compact Hausdorff space has the discrete topology (see Theorem 1.5.1). This fact is very useful when working with groups which parameterized, for example, groups of Moebius transformations. In Proposition 1.5.2 we show that a discontinuous group Γ of isometries of a metric space X is characterized by the property that for any $x \in X$ the family $(\{\gamma(x)\} | \gamma \in \Gamma)$ is locally finite. This shows that groups of isometries of the euclidean plane or non-euclidean plane which are discontinuous in the usual way are discontinuous in the sense defined by us. Another observation about discontinuous groups is that the size of a group depends on the size of the space, that is, the size of the smallest cover by compact sets. Thus in particular every discontinuous group Γ of a compact space is finite and if the space has a countable cover by bounded sets, Γ is countable. Another nice property of discontinuity is its topological invariance and the fact that any restriction of the group to invariant subspaces is again discontinuous. Moreover, subgroups of discontinuous groups are discontinuous.

In Section 6 we finally discuss the local action of discontinuous groups of surfaces. Here the most important observation is Theorem 1.6.3 which says that given any discontinuous homeomorphism group Γ acting on the surface S and any $x \in S$, x has a neighbourhood base consisting of discs (homeomorphic images of the unit disc in R^2) with the property that $\gamma(V)$ and V are disjoint for all γ in Γ which are not in the stabilizer of V. Thus to understand the local action we have to consider finite homeomorphism groups acting on discs and we know these groups very well (see Theorem 1.6.4).

§1. INTRODUCTION

This thesis was motivated on the one hand by the problem (posed to me by Dr. W. Imrich) of describing homeomorphism groups of infinite order of the euclidean plane E which induce automorphisms of infinite order on a plane graph whose vertex-set and edge-set are locally finite in E. On the other hand, it was motivated by a paper [3] of Babai, Imrich and Lovász on finite homeomorphism groups of the 2-sphere, in connection with some investigations of automorphisms of a finite planar graph. It is well known that any finite 3-connected planar graph G can be embedded into the 2-sphere S^2 in such a way that all automorphisms of G are induced by congruences of S^2 (see [3, p. 62, Corollary]). Thus the finite groups of congruences of S^2 (which are well understood) are used to "visualize" the action of automorphism groups of finite planar, 3-connected graphs. Vice versa, given any finite homeomorphism group Γ of S^2, there is a 3-connected finite graph embedded in S^2 which is invariant under the action of Γ (see [3, Corollary, p. 67]). This fact is used in [3] to prove that Γ is topologically equivalent to a group of congruences of S^2. The construction of the above mentioned graph [3, p. 67] is possible by the following property of Γ: For any two points x and y of $S = S^2$ there are "small" neighbourhoods V_x and V_y such that $\gamma(V_x)$ and V_y meet for only finitely many $\gamma \in \Gamma$. Let this property define the discontinuity of a group of bijections of a topological space. Discontinuity thus seems to be an important notion for the following reasons. Firstly, it is of purely topological nature; secondly, it generalizes the well known and thoroughly investigated notion of discontinuity (as defined by Ford [11]) for Moebius transformations, as shown in Example 1.3.4; thirdly, there is topologically no distinction between certain discontinuous groups of Moebius transformations and discontinuous homeomorphism groups of the euclidean plane (see Theorem 5.4.1) and these groups therefore are isomorphic and can explicitly be described in terms of generators and relations (see Theorem 4.3.5).

We shall restrict our attention almost exclusively to discontinuous homeomorphism groups Γ of planar surfaces S. We try to "visualize" the action of

(1) elements of Γ of infinite order (see Theorem 5.3.1),

(2) orientation preserving elements of Γ of finite order (see Theorem 5.3.1) and

(3) Γ, if Γ acts on the euclidean plane E and has compact fundamental domain, that is, there is a bounded set, all of whose images cover E. (See Theorem 5.4.1)

The most important step towards this goal is the Triangulation Theorem: If Γ is a discontinuous homeomorphism group acting on a surface S then there is a triangulation of S, called a 3-polyhedron, which is invariant under Γ.

This makes it possible to discard any topology from the study of discontinuous groups and instead deal with automorphism groups of abstract polyhedra.

§2. DEFINITIONS AND NOTATION

We shall use here the terminology of topology commonly used in any standard text, for example, by Willard [27]. However we shall give a few definitions which will be used frequently throughout and are not in common usage.

A *topological space* shall be denoted by (X,τ), where X is the underlying set and τ is the topology. The notation for a *metric space* will be (X,d), where X is the underlying set and d is the metric. When not necessary we shall not mention the topology τ (respectively, metric d) and simply refer to X as the topological space (respectively, metric space).

We shall write $\bar{A}$ for the *closure* of A, A° for the *interior* of A and fr(A) for the *frontier* of A. The *r-disc* about x, $\{z\in X \mid d(z,x)<r\}$, shall be denoted by $V_{x,r}$. A subset A of the topological space X is *bounded* if $\bar{A}$ is compact. X is *locally compact* if every point in X has a neighbourhood base consisting of bounded (or compact) sets. A collection $\{A_i \mid i\in I\}$ of subsets of X is *locally finite* if every $x\in X$ has a neighbourhood meeting only finitely many A_i's. A family $(A_i \mid i\in I)$ of subsets of X is locally finite if for each $x\in X$ there is a neighbourhood U of x such that $\{i\in I \mid A_i\cap U \neq \phi\}$ is finite. A subset Z of X is locally finite if $\{\{z\} \mid z\in Z\}$ is locally finite. Two subsets A and B of X are *compatible* if $A\cap B$ is locally finite and the two families $(A_i \mid i\in I)$ and $(B_j \mid j\in J)$ of subsets of X are compatible if A_i is compatible with B_j for any $i\in I$ and any $j\in J$. $(A_i \mid i\in I)$ is compatible if $(A_i \mid i\in I)$ is compatible with $(A_i \mid i\in I)$.

A subspace D of X is called an *n-disc* (an *open n-disc*) in X if it is homeomorphic to the closed (open) unit disc in R^n. (By the closed unit disc in R^n we mean $\{x\in R^n \mid |x|\le 1\}$ and by the open unit disc in R^n we mean $\{x\in R^n \mid |x|<1\}$.) Its boundary is denoted as ∂D. A *simple closed curve* in X is a subspace homeomorphic to the unit circle. An *arc* in X is a subspace homeomorphic to the closed unit interval $[0,1]$. Similarly, we define an *open* (*half open*) *arc* in X to be a subspace homeomorphic to $(0,1)$ ($[0,1)$). Whenever it is desirable to indicate the end points of an arc we shall use the notation $e_{[x,y]}$ for an arc with end points x and y and $e_{(x,y)}$, respectively, $e_{(x,y]}$ or $e_{[x,y)}$ for $e_{[x,y]}-\{x,y\}$, respectively, $e_{[x,y]}-\{x\}$ or $e_{[x,y]}-\{y\}$. Sometimes we shall denote the set of end points $\{x,y\}$ of $e_{[x,y]}$ by ∂e.

If $e_{[x,y_1]},\ldots\ldots,e_{[x,y_n]}$, where $n\ge 1$, are arcs in X then $\cup\{e_{[x,y_i]} \mid 1\le i\le n\}$ is called a(n) (*n-*)*star* in X if $e_{(x,y_i]}\cap e_{(x,y_j]} = \phi$, for $i \neq j$ and $1\le i,j\le n$.

An *n-manifold* M is a connected Hausdorff space in which every point has an open neighbourhood which is an open n-disc. Every n-manifold is locally compact and each of its points has a neighbourhood base consisting of n-discs. If M is second countable then by Urysohn's metrization theorem [27] M is metrizable. A *surface* is a second countable 2-manifold. The letter S without subscripts shall be used exclusively to denote a surface. We note that each point of a surface S has a neighbourhood base consisting of discs. A surface is *planar* if it is homeomorphic to an open subset of the unit sphere $S^2=\{x\in R^3 \mid |x| = 1\}$. We shall not prove either of the next two results about surfaces. A complete classification of all surfaces has been given by I. Richards [23].

THEOREM 1.2.1. *Every surface* S *is homeomorphic to a surface formed from a sphere* O *by first removing a closed totally disconnected set* X *from* O *, then removing the interiors of a finite or infinite sequence* $D_1, D_2, \ldots$ *of nonoverlapping discs in* O-X*, and finally suitably identifying the boundaries of these discs in pairs. It may be necessary to identify the boundary of one disc with itself to produce an odd "crosscap". The sequence* $D_1, D_2, \ldots$ *"approaches* X*" in the sense that, for any open set* U *in* O *containing* X *, all but a finite number of the* D_i*'s are contained in* U.

Another interesting theorem [20] about surfaces is now stated.

THEOREM 1.2.2. *Every noncompact surface is homeomorphic to a subset of* R^3.

A *graph* G is a triple (V,E,i), where V and E are disjoint sets and i is a mapping from E into $[V]^2$, where $[V]^2$ denotes the set of all subsets of V with two elements. The set V is called the *vertex-set* of G, also denoted by V(G), and E is the *edge-set* of G, denoted by E(G). The mapping i describes the incidence between vertices and

and edges. We shall use the graph theoretical terminology of standard texts in graph theory, such as, Bondy and Murty [5], Behzad, Chartrand and Lesniak-Foster [4] or Harary [13]. However, we shall give a few definitions which are not in common usage.

Let X be a Hausdorff space and $G = (V,E,i)$ be a graph. G is called an *X-graph*, if

(1) $V \subset X$,

(2) E is a collection of arcs in X,

(3) $i(e) = \partial e$ for all $e \in E$,

(4) $(e_1 - i(e_1)) \cap (e_2 - i(e_2)) = \phi$, if $e_1, e_2 \in E$ and $e_1 \neq e_2$,

(5) $\cup\{e - i(e) \mid e \in E\} \cap V = \phi$, and

(6) $V \cup E$ is locally finite in X.

$\cup\{e \mid e \in E\} \cup V$ is called the *point set* of G and will be denoted ps(G). Note that every object in $V \cup E$ is closed in X, since X is a Hausdorff space. We shall be concerned most of the time with the case where $X = S$ is a surface. Each component of S-ps(G) is called a *domain* or *region* of G (in S) and its closure is called a *face* of G. Let $F(S,G)$ be the collection of all faces of G. The subgraph of G whose point set is the (topological) boundary of the region C is called the *boundary* of the face $\bar{C}$. A face F of G is a *polygon* if F is a disc and its boundary is a cycle in G; its edges and vertices are the edges and vertices of the polygon. An *n-gon* is a polygon with n edges. A *3-gon* shall be called a *triangle*.

An S-graph G is a *triangulation* of S if

(1) every face of G is a triangle and

(2) any two faces have empty intersection or exactly one common vertex or one common edge.

According to Radó [22] a 2-manifold can be triangulated if and only if it is a surface.

A *(topological) polyhedron* is a pair (S,G) consisting of a surface S and an S-graph G such that

(1) G is connected and

(2) every face of G is a polygon.

The vertices, edges and faces of (S,G) are the vertices, edges and faces of G in S. If G triangulates S, then (S,G) is a *(topological) 3-polyhedron* or *triangular polyhedron*. We say (S,G) is *orientable* if S is orientable (for a definition of orientability of a surface see [1]), and *planar* if S is planar.

Two families $(G_i | i \in I)$ and $(H_j | j \in J)$ of X-graphs are compatible if $(ps(G_i) | i \in I)$ and $(ps(H_j) | j \in J)$ are compatible. $(G_i | i \in I)$ is compatible if $(ps(G_i) | i \in I)$ is compatible. Similarly, we define compatibility for collections of X-graphs. We shall show in Proposition 1.4.10 that for any locally finite and compatible collection $\{G_i | i \in I\}$ of X-graphs a *union* $\vee\{G_i | i \in I\}$ can be defined uniquely.

Let G = (V,E,i) be an X-graph and U be a locally finite set in X. We shall call the X-graph H = (V',E',i') the *subdivision* of G by U if $V' = V \cup U$, E' is the collection of arcs $e_{[x,y]}$ in ps(G) such that $e - \partial e \subset ps(G) - V'$, $\partial e \subset V'$ and $i'(e) = \partial e$ for all $e \in E'$. For further explanation see Proposition 1.4.9.

If Γ is a group of bijections of a set X (where the group operation is composition of mappings), we say Γ acts on X. We shall denote the identity mapping on X by ι. The point $x \in X$ is a *fixed point* of $\gamma \in \Gamma$ if $\gamma(x) = x$. We let F_γ denote the set of all fixed points of γ, F_Γ denote $\cup\{F_\gamma | \gamma \in \Gamma - \{\iota\}\}$, and $\Gamma_K = \{\gamma \in \Gamma | \gamma(K) = K\}$ is called the *stabilizer* of K. Let $\Delta K = \cup\{\delta(K) | \delta \in \Delta\}$, where $\Delta \subset \Gamma$, $K \subset X$. If $Y \subset X$ and $\gamma(Y) = Y$ for all $\gamma \in \Gamma$ then $\Gamma|_Y = \{\gamma|_Y | \gamma \in \Gamma\}$.

Let $A = \{A_i | i \in I\}$ be a collection of subsets of X and let Γ act on X. We say that A is Γ-*invariant* if for each $i \in I$ and $\gamma \in \Gamma$, $\gamma(A_i) \in A$. We say A, $B \subset X$ are Γ-*equivalent* or *equivalent modulo* Γ if $\gamma(A) = B$ for some $\gamma \in \Gamma$.

Let $G = (V,E,i)$ be an X-graph and Γ be a group of homeomorphisms acting on X. We say G is Γ-*invariant*, if

(1) V is Γ-invariant and

(2) E is Γ-invariant.

If $G = (V,E,i)$ is an X-graph and $\gamma: X \to Y$ is a homeomorphism, then $\gamma(G)$ shall be the Y-graph with $V(\gamma(G)) = \gamma(V)$, $E(\gamma(G)) = \{\gamma(e) | e \in E\}$ and $i_{\gamma(G)}(\gamma(e)) = \gamma(i_G(e))$, for $e \in E$; see Proposition 5.4.1.
It follows that G is Γ-invariant if and only if $\gamma(G) = G$ for all $\gamma \in \Gamma$. The collection $G = \{G_i | i \in I\}$ of X-graphs is called Γ-invariant if for every $\gamma \in \Gamma$ and $i \in I$, $\gamma(G_i) \in G$.

If $\phi: S_1 \to S_2$ is a homeomorphism of the surfaces S_1 and S_2, then we shall call ϕ a homeomorphism of the polyhedrons (S_1,G_1) and (S_2,G_2) and write $\phi: (S_1,G_1) \to (S_2,G_2)$ in case $\phi(G_1) = G_2$. If $(S_1, G_1) = (S_2,G_2) = (S,G)$ then ϕ is called a homeomorphism of (S,G). If Γ is a group of homeomorphisms of the polyhedron (S,G) then we shall say that (S,G) is Γ-invariant.

Let G_i be X_i-graphs, $1\leq i\leq 2$, and let $\tilde{\gamma}:X_1\to X_2$ be a homeomorphism such that $\gamma(G_1) = G_2$. Then $\tilde{\gamma}$ induces an isomorphism $\gamma:G_1\to G_2$ given by $\gamma(x) = \tilde{\gamma}(x)$ for all $x\in V(G_1)\cup E(G_1)$ and γ is called the induced isomorphism. Note that if $\tilde{\Gamma}$ is a group of homeomorphisms of the topological space X leaving the X-graph G invariant, then the homo morphism $\Phi:\tilde{\Gamma}\to\mathrm{Aut}(G)$ assigning to each $\tilde{\gamma}\in\tilde{\Gamma}$ the induced automorphism $\Phi(\tilde{\gamma})\in\mathrm{Aut}(G)$ is called the *canonical* homomorphism from $\tilde{\Gamma}$ into Aut(G). We call $\Phi(\tilde{\Gamma})$ the induced group of automorphisms.

Let Γ_i act on the topological space X_i, $1\leq i\leq 2$. Then Γ_1 and Γ_2 are called *topologically equivalent* if there is a homeomorphism $\phi:X_1\to X_2$ such that $\Gamma_2 = \phi\Gamma_1\phi^{-1} = \{\phi\gamma\phi^{-1}|\gamma\in\Gamma_1\}$.

Given a group Γ acting on the topological space X, we say Γ acts *discontinuously* on X if for any $x,y\in X$ there are neighbourhoods U of x and V of y such that $\{\gamma\in\Gamma|\gamma(U)\cap V = \phi\}$ is finite. We shall call Γ a *discontinuous group* on X. We see without difficulty that subgroups of discontinuous groups are discontinuous.

§3. SOME EXAMPLES OF DISCONTINUOUS GROUPS

We recall that by Proposition 1.5.7 it suffices to study discontinuous groups on one specimen of each class of homeomorphic spaces. We shall restrict ourselves exclusively to examples of discontinuous groups acting on surfaces.

EXAMPLE 1.3.1. Every finite group acting on a topological space is discontinuous, and conversely, by Proposition 1.5.3 any discontinuous group acting on a compact space is finite. Thus for compact spaces the discontinuous groups are exactly the finite groups.

The finite groups of homeomorphisms of the sphere (and therefore as well the euclidean plane) are topologically well characterized by the following theorem [3].

THEOREM 1.3.2. *Each finite group of homeomorphisms of the sphere is topologically equivalent to a finite group of congruences (that is, isometries) of the sphere. That is, to a subgroup of the group of congruences of a regular polyhedron or a regular prism.*

This was found by Kerékjártó [15] in 1919. He in turn pointed out that this also follows from a result of Brower [6] .

We are not aware of what is known if anything about finite homeomorphism groups of compact nonplanar surfaces, for example, the torus or the projective plane.

EXAMPLE 1.3.3. For each real number δ let $\rho_\delta : S^2 \to S^2$ be the rotation of the sphere S^2 given by $\rho_\delta(x_1,x_2,x_3) = (y_1,y_2,y_3)$ where y_1 and y_2 are the real, respectively imaginary, part of $(x_1 + ix_2)e^{i\delta}$ and $x_3 = y_3$ for all $(x_1,x_2,x_3) \in S^2$. Moreover, let $a = (1,0)$ and $s = 1/2$ and let $\bar{x} = (x_1,-x_2)$ for every $x = (x_1,x_2) \in R^2$. We define the homeomorphisms $\tau,\bar{\tau},\sigma,\bar{\sigma}$ of R^2 by setting $\tau(x) = x + a$, $\bar{\tau}(x) = \bar{x} + a$, $\sigma(x) = sx$ and $\bar{\sigma}(x) = s\bar{x}$. The mappings $\tau,\bar{\tau},\sigma$ and $\bar{\sigma}$ are called *translation, gleitreflection, contraction and reflected contraction,* respectively. Let $M = \{\sigma,\bar{\sigma},\tau,\bar{\tau}\}$.

We note that ρ_δ has finite order if and only if δ is a rational multiple of π. In fact, if ρ_δ has infinite order, then $\Delta = \langle\rho_\delta\rangle$ doesn't act discontinuously on any ρ_δ-invariant subspace of S^2 since the Δ-orbit of any point $x \in S^2$ is dense on the circle of S^2 containing it.

We shall now give four simple but important examples of discontinuous groups. It is easy to see that the groups $\langle\tau\rangle$ and $\langle\bar{\tau}\rangle$ act discontinuously on R^2 whereas the groups $\langle\sigma\rangle$ and $\langle\bar{\sigma}\rangle$ don't since the origin is a fixed point for σ and $\bar{\sigma}$. Yet $\langle\sigma|_S\rangle$ and $\langle\bar{\sigma}|_S\rangle$, where $S = R^2 - \{(0,0)\}$, act discontinuously on S. The mappings $\tau, \bar{\tau}, \sigma,\bar{\sigma}$ weren't chosen as random examples but they turn out to be the only types of elements of infinite order in any discontinuous homeomorphism group of a planar surface S as will be shown in Theorem 5.3.1. Similarly, ρ_δ is the only type of orientation preserving elements of finite order as is well known if S is the sphere S^2 (see [7] and [16]).

EXAMPLE 1.3.4. Let us be given a group Γ of Möbius transformations, that is, mappings $\gamma : z \to (az+b)/(cz+d)$ of the extended complex plane $\bar{C} = C \cup \{\infty\}$. (See [11], [18] and [24].) The group Γ is called properly discontinuous by Ford [11,p. 35] if there is a point $z_o \in \bar{C}$ and a neighbourhood U of z_o such that all elements of Γ, except the identity transformation, carry z_o outside U. A point $z \in \bar{C}$ is a limit point of Γ if there is an $x \in \bar{C}$ such that every neighbourhood of z meets $\gamma(x)$ for infinitely many $\gamma \in \Gamma$. A non-limit point is called an ordinary point. It is a well known fact that the set O of ordinary points of Γ is a Γ-invariant open set in $\bar{C}$ whose boundary consists of the set L of limit points of Γ. A fundamental region R of Γ is an open subset of O no two of whose points are Γ-equivalent (or equivalently, $\gamma(R) \cap \delta(R) = \emptyset$ if $\gamma \neq \delta$) and so that any neighbourhood of any point on the boundary of R contains points from outside R which are Γ-equivalent to points of R (see [11, p. 37]). A description of a fundamental region R of a properly discontinuous group Γ is given in [11, p. 44]. From the discussion of the boundary of R [11, p. 47], it follows that Γ acts discontinuously in O in the sense defined in this thesis. In order to get some feeling for what O looks like we note that the images of the region R under Γ form a set of regions which extend into the neighbourhood of every point in $\bar{C}$ (see [11, Theorem 6, p. 44]). More precisely, any compact set A not containing limit points of the group Γ is "covered" by a finite number of these regions (images of R) which fit together without lacunae ([11, Theorem 8, p. 46]). Moreover, within any neighbourhood of a limit point of the group, there are an infinite number of images of R [11, Theorem 9, p. 46]. It is noted here that the construction of R and the proof of its properties cannot be accomplished with purely topological means but uses analytical methods.

The complex analytical theory of discontinuous groups of Möbius transformations is too voluminous to be described here in a few words and we therefore refer to [11], [17] and [24] for reference. Moreover, it is beyond the scope of our purely topological study.

For reasons of Theorem 5.4.1 we shall now consider a special case of discontinuous groups of Möbius transformations. It is well known that the most general conformal mappings of $\bar{C}$ which fix the open unit disc $D_o = \{z \in C \mid |z| < 1\}$ can be written as $z \to (\bar{a}z + \bar{b})/(bz + a)$ and $z \to (\bar{a}\bar{z} + \bar{b})/(b\bar{z} + a)$, where $a\bar{a} - b\bar{b} = 1$. The first is orientation preserving, the second orientation reversing. Moreover, (see [24, p. 16 ff]) D_o can be given a metric that induces a geometry which is known as the Poincaré model of plane non-euclidean geometry. Moreover, the set of isometries with respect to this metric turns out to be precisely the mappings mentioned above, restricted to D_o.
Let us denote D_o with this metric as NE. In [24, p. 30] a group Γ of isometries of NE is defined as acting discontinuously on NE if for any $x \in NE$ the family $(\gamma(x) \mid \gamma \in \Gamma)$ is (in our terminology) locally finite in NE. Thus by Proposition 1.5.2 such groups act discontinuously on NE in the way we have defined.

Similarly, a group Γ of isometries of the euclidean plane E is usually said to act discontinuously on E [24] if $(\gamma(x) \mid \gamma \in \Gamma)$ is locally finite in E for all $x \in E$ and thus by Proposition 1.5.2 is discontinuous in our sense.

§4. SOME PRELIMINARY OBSERVATIONS ABOUT X-GRAPHS

The proofs of the Propositions 1.4.1 up to 1.4.4 are straightforward and are therefore omitted.

PROPOSITION 1.4.1. *Let* $\phi:X\to Y$ *be a homeomorphism between the topological spaces* X *and* Y . *The family* $(A_i|i\in I)$ *of subsets of* X *is locally finite in* X *if and only if* $(\phi(A_i)|i\in I)$ *is locally finite in* Y.

Proposition 1.4.1 gives the justification for the definition of $\gamma(G)$, where G is an X-graph and γ is a homeomorphism of X.

PROPOSITION 1.4.2. *Let* X *and* Y *be topological spaces and let* $f:X\to Y$ *be a mapping. Moreover, let* $\{A_i|i\in I\}$ *be a locally finite collection of closed subsets of* X *so that* $\cup\{A_i|i\in I\} = X$. *Then* f *is continuous if and only if* $f|_{A_i}$ *is continuous for every* $i \in I$.

PROPOSITION 1.4.3. *Let* X *be a topological space and* Y *be a subspace of* X .

(1) *If the family* $(A_i|i\in I)$ *is locally finite in* X , *then* $(A_i\cap Y|i\in I)$ *is locally finite in* Y .

(2) *If* Y *is closed in* X *and* $(A_i|i\in I)$ *is locally finite in* Y , *then* $(A_i|i\in I)$ *is locally finite in* X .

PROPOSITION 1.4.4. *Given a topological space* X *and a locally finite family* $(A_i|i\in I)$ *of closed subsets of X, then* $\cup\{A_i|i\in I\}$ *is closed.*

COROLLARY 1.4.5. *Given an* X-*graph* G = (V,E,i), *then* ps(G) *is closed in* X.

Proof. Now $ps(G) = \cup\{e|e\in E\}\cup V$ and $V\cup E$ is a locally finite collection of closed subsets of X. This follows because X is Hausdorff so that every compact subset is closed in X. In particular, every edge of G is closed in X. Thus by Proposition 1.4.4 ps(G) is closed in X. □

PROPOSITION 1.4.6. *Let* X *be a locally compact space. The family* $(A_i|i\in I)$ *of subsets of* X *is locally finite if and only if* $\{i\in I|A_i\cap B \neq \phi\}$ *is finite for every bounded subset* B *of* X .

Proof. To prove necessity, assume $(A_i|i\in I)$ is locally finite in X and let B be a bounded subset of X. Then C = cl(B) is compact; thus it has a covering by finitely many (open) sets each of which meets A_i for only finitely many $i\in I$. Thus $\{i\in I|B\cap A_i \neq \phi\}$ is finite. To prove sufficiency observe that at each point of X the bounded neighbourhoods form a neighbourhood base. □

The Propositions 1.4.1, 1.4.3, 1.4.4 and 1.4.6 also hold for collections $\{A_i|i\in I\}$.

COROLLARY 1.4.7. *Every* S-*graph is countable and each one of its vertices has finite degree.*

Proof. This immediately follows from Theorem 2.1.1 and Proposition 1.4.6. □

The next proposition expresses the natural relationship between connectedness in X-graphs and arcwise connectedness of their point sets.

PROPOSITION 1.4.8. *Let* G *be an* X-*graph. Then* H *is a component of* G *if and only if* ps(H) *is an (arc)component of* ps(G).

Proof. Every arc in ps(G) joining two vertices of G is the point set of a path in G joining the same two vertices. Thus for $s, t \in V(G) \cup E(G)$, s and t belong to the same component of G if and only if there is a path or cycle containing them, and the latter is the case if and only if there is an arc or simple closed curve in ps(G) which contains them. Thus s and t belong to the same component of G if and only if they are in the same arccomponent of ps(G). Thus H is a component of G if and only if ps(H) is an arccomponent of ps(G). □

The following proposition gives a local picture of an S-graph. Henceforth, whenever we use the word disc it shall refer to a 2-disc.

PROPOSITION 1.4.9. *Given an* S-*graph* G *and a point* $v \in ps(G)$, *there is a disc* D *such that* $v \in D^{\circ}$ *and either* $D \cap ps(G) = \{v\}$ *or* $D \cap ps(G)$ *is an n-star* $Y = e_{[v,x_1]} \cup \ldots \cup e_{[v,x_n]}$ *with* $\{x_1, \ldots, x_n\} \subset fr(D)$ *and* $Y - \{x_1, \ldots, x_n\} \subset int(D)$. *Thus at each vertex* v *the edges incident to* v *are arranged in cyclical order in exactly two opposite ways which we shall refer to as the rotations at* v *induced by* S.

Proof. Each point of S has a disc shaped neighbourhood and thus we may refer to [21, p.169-174]. □

In the next proposition we shall explain the notion of a subdivision of an X-graph. Let G = (V,E,i) be an X-graph and U be a locally finite subset of X. Let $V' = V \cup U$, $E' = \{e_{[x,y]} \subset ps(G) \mid e_{(x,y)} \cap V' = \emptyset$ and $x,y \in V'\}$ and $i'(e) = \partial e$ for $e \in E'$.

PROPOSITION 1.4.10. H = (V',E',i') *is an* X-*graph.*

Proof. Clearly V' is locally finite. We observe that if $e \in E'$, then $e \subset a \in E$. For any $a \in E$, $a \cap V'$ is finite by Proposition 1.4.6. Thus the number of arcs of E' in a is finite. Hence E' is locally finite since E is locally finite. All other conditions which make (V',E',i') an X-graph are easily verified. □

We call H the *subdivision* of G by U. In the next proposition we shall explain the notion of union of X-graphs. Let $\mathcal{G} = \{G_j \mid j \in J\}$ be a locally finite and compatible collection of X-graphs. Let $W = \cup\{ps(G_j)|j\in J\}$, $Z = \cup\{ps(G_i)\cap ps(G_j)|i,j\in J \text{ and } i \neq j\}$, $V = \cup\{V(G_j)|j\in J\}\cup Z$ and $E = \{e_{[u,v]}|e_{[u,v]}\subset e \in \cup\{E(G_j)|j\in J\},\ e_{(u,v)}\cap V = \phi \text{ and } u,v\in V\}$.

PROPOSITION 1.4.11. *Let* Γ *be a group of homeomorphisms acting discontinuously on* X *and let* $G = \{G_j = (V_j,E_j,i_j)|j\in J\}$ *be a locally finite, compatible and* Γ*-invariant collection of* X*-graphs. Then there is a* Γ*-invariant* X*-graph* G *with* $V(G) = V$, $E(G) = E$, $ps(G) = W$ *and* $i_G(e) = \partial e$ *for* $e\in E$, *where* V,E *and* W *are as defined above.*

Proof. It suffices to prove the following five statements.

(1) V is locally finite ,

(2) $\cup\{V_j\cup E_j|j\in J\}$ is locally finite ,

(3) E is locally finite ,

(4) $\cup\{e|e\in E\}\cup V = W$, and

(5) For all $\gamma\in\Gamma$ and $e\in E$, $\gamma(e)\in E$ and $\gamma(V) = V$.

Proof of (1). Let $x\in X$ be given. Since $\{G_j|j\in J\}$ is locally finite, there is a neighbourhood O of x such that $J_o = \{j\in J|ps(G_j)\cap O \neq \emptyset\}$ is finite. Now $\cup\{V_j|j\in J_o\}$ is locally finite so that there is a neighbourhood $Q\subset O$ of x such that $\cup\{V_j|j\in J\}\cap Q = \cup\{V_j|j\in J_o\}\cap Q$ is finite. Thus, $\cup\{V_j|j\in J\}$ is locally finite. Now $\cup\{ps(G_i)\cap ps(G_j)|i,j\in J_o \text{ and } i \neq j\}$ is locally finite and hence for some neighbourhood $Q'\subset O$ of x, $Z\cap Q' = \cup\{ps(G_i)\cap ps(G_j)|i,j\in J_o \text{ and } i \neq j\}\cap Q'$ is finite. Hence Z is locally finite so that V is locally finite.

Proof of (2). Let x, O, J_o be as in (1). For $I \subset J$ let $X_I = \cup\{V_j \cup E_j | j \in I\}$. Now $X_{\{j\}}$ is locally finite for every $j \in J$. Thus X_{J_o} is locally finite and there is a neighbourhood $Q \subset O$ of x such that $\{x \in X_J | x \cap Q \neq \emptyset\} = \{x \in X_{J_o} | x \cap Q \neq \emptyset\}$ is finite. Hence X_J is locally finite.

Proof of (3). We observe that if $a, e \in A = \cup\{E_j | j \in J\}$ then $\partial e \subset V$ and $e \cap a \subset V$ and $a \cap e$ is finite. Let $x \in X$. We shall show that there is a neighbourhood O of x such that $C = \{c \in E | c \cap O \neq \emptyset\}$ is finite. A is locally finite and thus for some neighbourhood O of x, $e_1, \ldots, e_n$ are all the elements of A which meet O. Now V is locally finite so that $\cup\{e_i \cap V | 1 \leq i \leq n\}$ is finite. Then C is a subcollection of the collection of arcs into which V partitions $\cup\{e_i | 1 \leq i \leq n\}$. Thus C is finite.

So far we know that $V \cup E$ is locally finite and (V,E,i) is an S-graph. It is easy to see that (4) is satisfied.

Proof of (5). First let us show that $\gamma(V) = V$ for $\gamma \in \Gamma$. Let $v \in V$. If $v \in V_j$ then $\gamma(v) \in \gamma(V(G_j)) = V(\gamma(G_j)) \subset V$, since $\gamma(G_j) \in G$. If $v \in ps(G_i) \cap ps(G_j)$ for $i \neq j$, then $\gamma(v) \in \gamma(ps(G_i) \cap ps(G_j)) = ps(\gamma(G_i)) \cap ps(\gamma(G_j)) \subset V$. Thus $\gamma(V) \subset V$ for all $\gamma \in \Gamma$. It follows that $\gamma(V) = V$ for all $\gamma \in \Gamma$. Let $e \in E$. Then for some $j \in J$ we have $e \subset a \in E_j$. Thus $\gamma(e) \subset \gamma(a) \in E(\gamma(G_j))$, $\partial e \subset V$ and $(e - \partial e) \cap V = \emptyset$. Thus $\partial(\gamma(e)) \subset V$ and $(\gamma(e) - \partial(\gamma(e))) \cap V = \emptyset$. Hence $\gamma(e) \in E$. □

We shall call the graph G the *union* of the collection $G = \{G_j | j \in J\}$ and denote it by $\vee\{G_j | j \in J\}$. Obviously this union operation is associative and commutative.

§5. SOME GENERAL OBSERVATIONS ABOUT DISCONTINUOUS GROUPS

Given a topological space X and a group Γ acting on X it is possible to assign a topology to Γ which expresses the fact that with "little modifications" of $\gamma \in \Gamma$ the the values of γ will be displaced only "a little bit". One topology τ which is frequently considered for this purpose has as a subbase the collection of sets $S = \{\{\gamma \in \Gamma \mid \gamma(K) \subset U\} \mid K \text{ is compact and } U \text{ is open in } X\}$. It is clear that with this topology τ the subgroups of Γ have the induced or subspace topology. The reason for this topology to be considered is that for certain spaces Γ of Moebius transformations (with topology τ) this topology is identical with the natural topology associated with the parameters of Γ [28, p.142].

We shall now give a theorem describing the topology τ as defined above for discontinuous homeomorphism groups acting on a locally compact Hausdorff space.

THEOREM 1.5.1. *Let Γ be a discontinuous homeomorphism group acting on a locally compact Hausdorff space* X. *Then*

(1) for every $\gamma \in \Gamma$ there is a finite set $\Delta \in S$ with $\gamma \in \Delta$ and

(2) if $X - F_\Gamma \neq \emptyset$, then τ is the discrete topology on Γ.

Proof of (1). Let $\gamma \in \Gamma$ and $x \in X$ and let U be a bounded open set with $\gamma(x) \in U$. Then $\gamma \in \Delta = \{\delta \in \Gamma \mid \delta(x) \in U\} \in S$, and Δ is finite by Proposition 1.5.4 which will be proved shortly.

Proof of (2). Let $x \in X - F_\Gamma$; let $\gamma \in \Gamma$ and U be an open bounded set containing $\gamma(x)$. Then $\{\delta \in \Gamma \mid \delta(x) \in U\}$ is finite, and since $x \notin F_\Gamma$ and X is Hausdorff, there exists a bounded open set V such that $\Delta = \{\delta \in \Gamma \mid \delta(x) \in V\} = \{\gamma\}$. Thus $\{\gamma\} \in S$ and therefore $\tau \supset \{\{\gamma\} \mid \gamma \in \Gamma\}$ so that τ is the discrete topology. □

Theorem 1.5.1 tells us that we cannot learn much about discontinuous homeomorphism groups from the topology τ. Thus we shall forget about τ and develop other methods to study these groups. In the following we shall give two more characterizations of discontinuous groups.

PROPOSITION 1.5.2. *A group* Γ *of isometries of a metric space* X *acts discontinuously on* X *if and only if for each* $x \in X$ *the family* $(\{\gamma(x)\} \mid \gamma \in \Gamma)$ *is locally finite.*

Proof. To prove the necessity assume that Γ acts discontinuously on X. Let $x,z \in X$. Then there exist discs $V_{x,r}$ and $V_{z,s}$ such that both $\Delta = \{\gamma \in \Gamma \mid \gamma(V_{x,r}) \cap V_{z,s} \neq \emptyset\}$ and $\Delta' = \{\gamma \in \Gamma \mid \gamma(x) \in V_{z,s}\}$ are finite. Since z was arbitrary, we have that $\{\gamma(x)\} \mid \gamma \in \Gamma\}$ is locally finite.

For sufficiency, assume that for each $x \in X$ the family $(\{\gamma(x)\} \mid \gamma \in \Gamma)$ is locally finite. Let $x,z \in X$. There is a disc $V_{z,r}$ such that $\Delta = \{\gamma \in \Gamma \mid \gamma(x) \in V_{z,r}\}$ is finite. Now choose s with $0 < 2s < d(z,\Gamma(x) - \{z\})$, where $d(A,B) = \inf\{d(a,b) \mid a \in A, b \in B\}$. Then $\{\gamma \in \Gamma \mid V_{z,s} \cap \gamma(V_{x,s}) \neq \emptyset\} = \{\gamma \in \Gamma \mid \gamma(x) = z\} \subset \Delta$ and hence is finite. (In the last equality we used the fact that Γ is a group of isometries). Thus Γ is discontinuous. □

PROPOSITION 1.5.3. *Let Γ be a group of bijections acting on a topological space X. Then* :

(1) If Γ acts discontinuously, then for any two nonempty bounded sets $J,K \subset X$ the set $\Delta = \{\gamma \in \Gamma \mid K \cap \gamma(J) \neq \emptyset\}$ is finite;

(2) If X is locally compact and for any two nonempty bounded sets $J,K \subset X$ the set $\Delta = \{\gamma \in \Gamma \mid K \cap \gamma(J) \neq \emptyset\}$ is finite, then Γ acts discontinuously.

Proof of (1). Assume Γ acts discontinuously. Let J,K be bounded, nonempty subsets of X. Without loss of generality we may assume that J and K are compact. Fix $x \in J$. Then for all $z \in K$ there are open neighbourhoods V_z of x and U_z of z such that $\{\gamma \in \Gamma \mid \gamma(V_z) \cap U_z \neq \emptyset\}$ is finite. Now $\{U_z \mid z \in K\}$ is an open cover of K and because K is compact there are $z_1,\ldots,z_n \in K$ such that $\cup\{U_{z_i} \mid 1 \le i \le n\} \supset K$. Also $W_x = \cap\{V_{z_i} \mid 1 \le i \le n\}$ is an open neighbourhood of x and $\Delta_x = \{\gamma \in \Gamma \mid \gamma(W_x) \cap K \neq \emptyset\} \subset \{\gamma \in \Gamma \mid \gamma(W_x) \cap \cup\{U_{z_i} \mid 1 \le i \le n\} \neq \emptyset\} \subset \cup\{\gamma \in \Gamma \mid \gamma(V_{z_i}) \cap U_{z_i} \neq \emptyset\}$. Thus Δ_x is finite. Hence for each $x \in J$ there is an open neighbourhood W_x of x such that Δ_x is finite. Since J is compact, there are points $x_1,\ldots,x_m \in J$ such that $\cup\{W_{x_i} \mid 1 \le i \le m\} \supset J$. Now $\Delta = \{\gamma \in \Gamma \mid \gamma(J) \cap K \neq \emptyset\} \subset \{\gamma \in \Gamma \mid \gamma(\cup\{W_{x_i} \mid 1 \le i \le m\}) \cap K \neq \emptyset\} \subset \cup\{\Delta_{x_i} \mid 1 \le i \le m\}$. Thus Δ is finite.

Proof of (2). Since X is locally compact, at each point of X the bounded neighbourhoods form a neighbourhood base. Thus from the assumptions in (2) it follows immediately that Γ is discontinuous. □

PROPOSITION 1.5.4. *Let Γ be a group of bijections acting discontinuously on the topological space X and let K be bounded in X. Then*

(1) $\Delta = \{\gamma \in \Gamma \mid$ *there is an* $x \in K$ *with* $\gamma(x) = x\}$ *is finite,*

(2) for every $x \in X$ *the set* $\{\gamma \in \Gamma \mid \gamma(x) \in K\}$ *is finite, and*

(3) $\mathrm{card}(\Gamma(x)) \leq \mathrm{card}(\Gamma) \leq kb$, *for all* $x \in X$, *where* b *is the cardinality of a cover of* X *by bounded sets and* k *is some finite cardinal.*

Proof of (1). If $\gamma \in \Gamma$, $x \in K$ and $\gamma(x) = x$, then $K \cap \gamma(K) \neq \emptyset$. Thus $\Delta \subset \{\gamma \in \Gamma \mid K \cap \gamma(K) \neq \emptyset\}$, and the latter set is finite by Proposition 1.5.3.

Proof of (2). Let $x \in X$ and $\Delta = \{\gamma \in \Gamma \mid \gamma(x) \in K\}$. For any $\delta \in \Delta$ we have $\Delta\delta^{-1} \subset \{\gamma \in \Gamma \mid K \cap \gamma(K) \neq \emptyset\}$ and thus $\mathrm{card}(\Delta\delta^{-1}) = \mathrm{card}(\Delta)$ is finite.

Proof of (3). Now $\mathrm{card}(\Gamma(x)) \leq \mathrm{card}(\Gamma)$ as shown by the mapping $\gamma \to \gamma(x)$ which is from Γ onto $\Gamma(x)$. Let $\{B_i \mid i \in I\}$ be a cover of X by bounded sets with $\mathrm{card}(I) = b$. Then $\mathrm{card}(\Gamma) = \mathrm{card}(\{\gamma \in \Gamma \mid \gamma(x) \in X\}) = \mathrm{card}(\cup\{\{\gamma \in \Gamma \mid \gamma(x) \in B_i\} \mid i \in I\}) \leq ab$, if $b \geq a$, and $\leq kb$, where a is the smallest infinite cardinal and k is some finite cardinal, if $b < a$. □

COROLLARY 1.5.5. *If* Γ *is a group of bijections acting discontinuously on a surface* S, *then* Γ *is finite if* S *is compact and countable if* S *is not compact.*

Proof. By Theorem 2.1.1 every noncompact surface has a countable cover by bounded sets, and Proposition 1.5.4 establishes the result. □

COROLLARY 1.5.6. *If* Γ *is group of bijections acting discontinuously on a locally compact space* X *then the family* $(F_\gamma \mid \gamma \in \Gamma)$ *is locally finite.*

Proof. This follows from Proposition 1.5.4. □

The following three propositions are an immediate consequence of the definitions.

PROPOSITION 1.5.7. *Let* Γ_i *be groups acting on the topological spaces* X_i, $1 \leq i \leq 2$. *If* Γ_1 *and* Γ_2 *are topologically equivalent, then* Γ_1 *is discontinuous if and only if* Γ_2 *is discontinuous.*

PROPOSITION 1.5.8. *If* Γ *acts discontinuously on the topological space* X, *and* $Y \subset X$ *is* Γ*-invariant, then* $\Gamma|_Y$ *acts discontinuously on the subspace* Y *of* X.

PROPOSITION 1.5.9. *Every subgroup of a discontinuous group is discontinuous.*

§6. THE LOCAL ACTION OF DISCONTINUOUS HOMEOMORPHISM GROUPS

The following lemma is a simple consequence of Kerékjartó's Theorem (see [21, p.168]). We derive from it Lemma 1.6.2 which is essential in the proof of the next theorem and of Theorem 2.1.1. A Jordan domain is an open, connected subset of the sphere S^2 whose frontier is a simple closed curve. Thus the closure of a Jordan domain is a disc and its frontier is the frontier of the disc.

LEMMA 1.6.1. *If any two of* $n \geq 2$ *simple closed curves* $J_1, \ldots, J_n$ *on the sphere* S^2 *have at least two common points then the components of* $S^2 \setminus (J_1 \cup \ldots \cup J_n)$ *are Jordan domains.*

LEMMA 1.6.2. *Let* S *be a surface and* $x \in S$ *and let* Γ *be a finite group of homeomorphisms of* $\overset{\circ}{S}$ *which fixes* x. *If* $U, V \subset S$ *are discs such that* $x \in V^\circ$ *and* $\Gamma(V)^\circ \subset U$, *then the component of* $U^\circ - \Gamma(\partial V)$ *which contains* x *is a* Γ*-invariant Jordan domain having a* Γ*-invariant frontier.*

Proof. We note that $x \in \gamma(V)^\circ$ for all $\gamma \in \Gamma$. We claim that $|\gamma_1(\partial V) \cap \gamma_2(\partial V)| \geq 2$ for $\gamma_1, \gamma_2 \in \Gamma$. Suppose $|\gamma_1(\partial V) \cap \gamma_2(\partial V)| \leq 1$. Then $\gamma_1(V) \not\subset \gamma_2(V)$ or vice versa. This implies that Γ is infinite, a contradiction. By Lemma 1.6.1 the domain containing x is a Jordan domain, and it is Γ-invariant for Γ fixes x. Moreover, its frontier is Γ-invariant and contained in $\Gamma(\partial V)$. □

THEOREM 1.6.3. *Let* Γ *be a discontinuous homeomorphism group acting on a surface* S *and let* $x \in S$. *Then* x *has a neighbourhood base consisting of discs* V *with the property that* $\gamma(V) \cap V = \emptyset$ *for* $\gamma \in \Gamma - \Gamma_V$.

Proof. The point x has a neighbourhood base consisting of discs as first observed in § 1. Let U be a disc with $x \in U^\circ$. The set $\Delta = \{ \gamma \in \Gamma \mid U \cap \gamma(U) \neq \emptyset \}$ is finite. Using the Hausdorff property and the continuity of γ, $\gamma \in \Delta$, we can find a disc neighbourhood W of x such that $\gamma(W) \cap W = \emptyset$ for $\gamma \in \Gamma - \Gamma_x$ and $\gamma(W) \subset U^\circ$ for $\gamma \in \Gamma_x$. Since Γ_x is finite, by Lemma 1.6.2 there is a Γ_x-invariant disc neighbourhood V of x. Moreover, $V \subset W \subset U^\circ$ and $\gamma(V) \cap V = \emptyset$ for $\gamma \in \Gamma - \Gamma_V$. □

The following theorem follows by a result of Kerékjartó [16] and Eilenberg [7]. For a proof we refer to [3].

THEOREM 1.6.4. *Given a finite homeomorphism group* Γ *acting on the disc* D_1 *then there is a homeomorphism* μ_2 *of* D_1 *onto the unit disc* $D = \{x \in R^2 \mid |x| \leq 1\}$ *such that* $\mu_2 \Gamma \mu_2^{-1}$ *is a group of congruences of* D. *Given a homeomorphism* μ_1 *of the boundary* ∂D_1 *of* D_1 *onto the boundary* ∂D *of* D *such that* $\mu_1 \Gamma \mu_1^{-1}$ *is a group of congruences of* ∂D *, then* μ_2 *can be chosen so that* $\mu_2 |_{\partial D_1} = \mu_1$.

COROLLARY 1.6.5. *Let* $\tilde{\Gamma}_i$ *be finite homeomorphism groups acting on the discs* D_i , $1 \le i \le 2$. *Suppose there is a homeomorphism* $\mu_1 : \partial D_1 \to \partial D_2$ *so that* $\mu_1(\tilde{\Gamma}_1|_{\partial D_1})\mu_1^{-1} = \tilde{\Gamma}_2|_{\partial D_2}$, *then there is a homeomorphism* $\mu_2 : D_1 \to D_2$ *such that*

(1) $\mu_2|_{\partial D_1} = \mu_1$ *and* *(2)* $\mu_2 \tilde{\Gamma}_1 \mu_2^{-1} = \tilde{\Gamma}_2$.

Proof. In view of Theorem 1.6.4 there are homeomorphisms $\kappa_i : D_i \to D$ and $\lambda_i : \partial D_i \to \partial D$, $1 \le i \le 2$, such that the following conditions are met:

(1) $\lambda_i = \kappa_i|_{\partial D_i}$, $1 \le i \le 2$,

(2) $\lambda_2 \circ \mu_1 = \lambda_1$,

(3) $\lambda_1(\tilde{\Gamma}_1|_{\partial D_1})\lambda_1^{-1} = \lambda_2(\tilde{\Gamma}_2|_{\partial D_2})\lambda_2^{-1}$, and

(4) $\kappa_i \tilde{\Gamma}_i \kappa_i^{-1} = \Delta_i$ are groups of congruences of D , $1 \le i \le 2$.

By (3) it follows that $\Delta_1|_{\partial D} = \Delta_2|_{\partial D}$ and thus $\Delta_1 = \Delta_2$.

Thus we have $\kappa_1 \tilde{\Gamma}_1 \kappa_1^{-1} = \kappa_2 \tilde{\Gamma}_2 \kappa_2^{-1}$ or $\tilde{\Gamma}_2 = \kappa_2^{-1}\kappa_1 \tilde{\Gamma}_1 \kappa_1^{-1}\kappa_2$. Taking $\mu_2 = \kappa_2^{-1}\kappa_1$ we have $\mu_2|_{\partial D_1} = \lambda_2^{-1}\lambda_1 = \mu_1$ and $\tilde{\Gamma}_2 = \mu_2 \tilde{\Gamma}_1 \mu_2^{-1}$. □

COROLLARY 1.6.6. *Let us be given two discs* D_1 *and* D_2 *and finite homeomorphism groups* Δ_1 *and* Δ_2 *acting on* ∂D_1 , *respectively,* ∂D_2. *If there is a homeomorphism* $\mu_1 : \partial D_1 \to \partial D_2$ *so that* $\Delta_2 = \mu_1 \Delta_1 \mu_1^{-1}$, *then there are finite homeomorphism groups* Γ_1 *and* Γ_2 *acting on* D_1 , *respectively* , D_2 *and a homeomorphism* $\mu_2 : D_1 \to D_2$ *such that* $\mu_2|_{\partial D_1} = \mu_1$, $\mu_2 \Gamma_1 \mu_2^{-1} = \Gamma_2$ *and* $\Gamma_i|_{\partial D_i} = \Delta_i$, $1 \le i \le 2$.

Proof. Each Δ_i can easily be extended to a homeomorphism group Γ_i acting on D_i. The corollary now follows from Corollary 1.6.5. □

COROLLARY 1.6.7. *If* Γ *is a finite homeomorphism group acting on the disc* D_1 *then* Γ *either is a cyclic group of orientation preserving homeomorphisms topologically equivalent to rotations of* D *or a dihedral group with a cyclic subgroup* Γ_o *of orientation preserving elements of* Γ *and* $|\Gamma : \Gamma_o| = 2$. *The orientation reversing elements of* Γ *are topologically equivalent to reflections of* D.

Proof. For a proof of this corollary we refer to [3 , p.64] . □

We shall call the orientation preserving elements of Γ *rotations* and the orientation reversing ones *reflections*. If Γ contains rotations then let R_Γ be their common fixed point.

COROLLARY 1.6.8. *Let two discs* D_1, D_2 *and a homeomorphism* $\phi: D_1 \to D_2$ *be given. If* γ *is a rotation of* D_1 *with fixed point* x, *then* $\phi\gamma\phi^{-1}$ *is a rotation of* D_2 *with fixed point* $\phi(x)$. *If* γ *is a reflection of* D_1 *with* l *as the axis of reflection, then* $\phi\gamma\phi^{-1}$ *is a reflection of* D_2 *with axis* $\phi(l)$.

Proof. This corollary immediately follows from Corollary 1.6.7. □

NOTE 1.6.9. We observe that if $\Gamma = \langle\gamma\rangle$ in Corollary 1.6.7 consists only of orientation preserving elements, then F_Γ is either empty (in case $\Gamma = \{\iota\}$) or a singleton. If Γ is dihedral and is generated by the rotation ρ and the reflection σ, where ρ is of order $n \geq 1$, then $\Gamma = \langle \rho^0, \rho^1, \ldots, \rho^{n-1}, \sigma, \rho\sigma, \ldots, \rho^{n-1}\sigma \rangle$. Let l be the axis of σ; then $\rho^i(l)$ is the axis for $\rho^i\sigma$, $0 \leq i \leq n-1$. If $n > 1$, then $\rho^i(l) \cap \rho^j(l) = \{x\}$, for $i \neq j$, $0 \leq i,j \leq n-1$, and x is the fixed point of ρ. Thus F_Γ in this case is an n-star, if n is even and a 2n-star if n is odd.

The local action of a discontinuous homeomorphism group nearby a point x of a surface S is summarized in the following corollary.

COROLLARY 1.6.10. *Let* Γ *be a discontinuous homeomorphism group acting on a surface* S *and let* $x \in S$ *and* V *be a disc neighbourhood such that* $\gamma(V) = V$ *for all* $\gamma \in \Gamma_x$. *Then* $\Gamma_x|_V$ *is topologically equivalent to a group af congruences of the unit disc.*

CHAPTER 2 . THE CONSTRUCTION OF A Γ-INVARIANT TRIANGULATION

In this chapter we shall prove the existence of a Γ-invariant triangulation for every surface S and every discontinuous homeomorphism group Γ acting on S. There are two possible ways of proving this result. The first way is as follows. We consider the quotient space $S' = S/_{\Gamma}$ of S modulo Γ. Using Theorems 1.6.3 and 1.6.4, we show that S' is a surface (possibly with boundaries). Now we suitably triangulate S' by using a triangulation theorem for surfaces with boundaries and then lift up the triangulation to S with the help of the canonical projection $\Pi : S \to S/_{\Gamma}$. However, since we realized this way only later we have choosen to give our original proof. Its structure is essentially the same as that of the proof of Rado's Theorem given in [1]. The only new feature is the action of Γ. The Theorems 1.6.3 and 1.6.4 which describe the local action of discontinuous homeomorphism groups are of basic importance for this proof.

The proof is in three parts. In Theorem 2.1.1 we prove the existence of coverings of S by discs having desirable properties. In Theorem 2.2.1 we improve those coverings by some additional properties. Lemma 2.3.1 is a triangulation lemma which makes it possible to triangulate any given Γ-invariant polyhedron (S,G).

§1. THE PROOF OF THEOREM 2.1.1

THEOREM 2.1.1. *Given a surface* S *with a metric* d *for its topology and a group* Γ *of homeomorphisms acting discontinuously on* S, *then there are finite or infinite sequences* (X_n) *and* (Y_n) *of discs with the following properties:*

(1) $X_n \subset Y_n^o$ *for every* n ;

(2) $\cup X_n = S$;

(3) (Y_n) *is locally finite;*

(4) *For all* n *we have* $\Gamma_{X_n} = \Gamma_{Y_n}$ *and for* $\gamma \in \Gamma - \Gamma_{Y_n}$ *we have* $\gamma(Y_n) \cap Y_n = \emptyset$;

(5) (X_n) *and* (Y_n) *are* Γ*-invariant; and*

(6) *If* S *is compact, then for some given positive real number* a *we have* $\operatorname{diam}(Y_n) < a$ *for all* n.

Proof. Given an arbitrary positive number b, by Theorem 1.6.3 there is a collection $\{D_i \mid i \in I\}$ of discs in S such that $B_o = \{D_i^\circ \mid i \in I\}$ is a base for S, $\operatorname{diam}(D_i) < b$ for all $i \in I$, and for all $\gamma \in \Gamma - \Gamma_{D_i}$ we have $D_i \cap \gamma(D_i) = \emptyset$. S also has a countable base. Therefore S has a countable base $B_1 = \{B_i \mid i \in N\}$ such that for every B_i there is some disc $D \subset S$ with $\operatorname{diam}(D) < b$, $B_i \subset D$, and $\gamma(D) \cap D = \emptyset$ for $\gamma \in \Gamma - \Gamma_D$.

LEMMA 2.1.2. *Given an open subset* O *of* S *contained in the disc* $D \subset S$, *there is a countable collection* U_i *of discs such that*

(a) $U_i \subset O$,

(b) $O = \cup U_i^\circ$, *and*

(c) $\delta(U_i) \cap U_i = \emptyset$ *for* $\delta \in \Gamma_D - \Gamma_{U_i}$.

Proof of Lemma 2.1.2. Let $\Theta = \Gamma_D|_D$ so that Θ is a finite group of homeomorphisms of D. Hence there is a homeomorphism $\phi: D \to D_o = \{x \in R^2 \mid |x| \leq 1\}$ such that $\Delta = \phi\Theta\phi^{-1}$ is a group of congruences of D_o. We note that $O \subset D^\circ$ and therefore $\phi(O) \subset \{x \in R^2 \mid |x| < 1\}$. D_o and F_Δ are separable spaces so that D_o contains a countable subset A with $R_\Delta \subset A$, A dense in D_o and $A \cap F_\Delta$ dense in F_Δ. We can choose discs in D_o with rational radius and center in A to obtain a countable collection $\{V_i\}$ satisfying

(a') $V_i \subset \phi(O)$,

(b') $\phi(O) = \cup V_i^\circ$, and

(c') $V_i \cap \delta(V_i) = \emptyset$ for all i and all $\delta \in \Delta - \Delta_{V_i}$.

Thus, $U_i = \phi^{-1}(V_i)$ are discs in O satisfying (a), (b) and (c). □

LEMMA 2.1.3. *There are countable sequences* $(U_n)_{n \in \mathbf{N}}$ *and* $(V_n)_{n \in \mathbf{N}}$ *of discs in* S *such that*

(a) $B_2 = \{U_n^\circ \mid n \in \mathbf{N}\}$ *and* $B_3 = \{V_n^\circ \mid n \in \mathbf{N}\}$ *are bases for* S,

(b) *every open set* $O \subset S$ *is the union of sets* U_n° *where* $U_n \subset V_n^\circ \subset V_n \subset O$,

(c) *for all* n, $\Gamma_{U_n} = \Gamma_{V_n}$ *and for* $\gamma \in \Gamma - \Gamma_{V_n}$, $\gamma(V_n) \cap V_n = \emptyset$, *and*

(d) $\mathrm{diam}(V_n) < b$ *for all* $n \in \mathbf{N}$.

Proof of Lemma 2.1.3. By Lemma 2.1.2, for every $B_i \in B_1$ there is a countable sequence (U_{ij}) of discs such that $B_i = \cup\{U^\circ_{ij} | \text{ all } j\}$, $U_{ij} \subset B_i$ and $\gamma(U_{ij}) \cap U_{ij} = \emptyset$ for $\gamma \in \Gamma - \Gamma_{U_{ij}}$. Similarly, for each U_{ij} there is a countable sequence (U_{ijk}) of discs with $U^\circ_{ij} = \cup\{U^\circ_{ijk} | \text{ all } k\}$, $U_{ijk} \subset U^\circ_{ij}$ and $\gamma(U_{ijk}) \cap U_{ijk} = \emptyset$ for $\gamma \in \Gamma - \Gamma_{U_{ijk}}$. We observe that $\operatorname{diam}(U_{ijk}) \le \operatorname{diam}(U_{ij}) < b$ for all i, j, k. Now we rearrange the U_{ijk}'s in a sequence (U_n) and set $V'_n = U_{ij}$ if $U_n = U_{ijk}$. The sequences (U_n) and (V'_n) satisfy (a), (b) and (d). We note that for all n, $U_n \subset V'^\circ_n$, $\Gamma_{U_n} \subset \Gamma_{V'_n}$, $\gamma(V'_n) \cap V'_n = \emptyset$ for $\gamma \in \Gamma - \Gamma_{V'_n}$ and $\gamma(U_n) \cap U_n = \emptyset$ for $\gamma \in \Gamma - \Gamma_{U_n}$.

Let n be given. If $\Gamma_{U_n} = \Gamma_{V'_n}$, then set $V_n = V'_n$. If $\Gamma_{U_n} \neq \Gamma_{V'_n}$, then let $\gamma_o(U_n) = U_n$, $\gamma_1(U_n), \ldots, \gamma_m(U_n)$ be all the (pairwise disjoint) images of U_n in V'°_n under the action of $\Gamma_{V'_n}$. We choose $m+1$ pairwise disjoint discs $D_o, \ldots, D_m$ such that $\gamma_j(U_n) \subset D^\circ_j \subset D_j \subset V'^\circ_n$. Then $U_n \subset \gamma\gamma_j^{-1}(D_j) \subset V'^\circ_n$ for all $\gamma \in \Gamma_{U_n}$ and $0 \le j \le m$. $\Gamma_{V'_n}$ is finite so that by Lemma 1.6.2 there is a Γ_{U_n}-invariant disc D such that $\gamma_j(U_n) \subset \gamma_j(D^\circ) \subset D_j \subset V'^\circ_n$ for $0 \le j \le m$. Thus $\Gamma_{U_n} = \Gamma_D$ and $\gamma(D) \cap D = \emptyset$ for $\gamma \in \Gamma - \Gamma_D$. Define $V_n = D$. Thus (U_n) and (V_n) satisfy (a), (b), (c) and (d) as well as properties (1), (2) and (4).

We shall go on to construct sequences which also satisfy (3), (5) and (6). Define the sequence (n_k) of integers as follows.
We define $n_1 = 1$ and n_k, $k > 1$, to be the least integer such that $\Gamma(U_1 \cup \ldots \cup U_{n_{k-1}}) \subset \Gamma(U^\circ_1 \cup \ldots \cup U^\circ_{n_k})$. We observe that such an integer n_k always exists since $U_1 \cup \ldots \cup U_{n_{k-1}}$ is compact and (U°_i) is an open cover of S. By Corollary 1.5.5 Γ is countable. Let (γ_i) be an enumeration

of Γ. □

LEMMA 2.1.4. *If* $n_k \leq n_{k-1}$ *for some* k *, then* $\Gamma(U_1 \cup \ldots \cup U_{n_{k-1}}) = S$ *and the sequences* $(\gamma_i(U_j))_{i,j}$ *and* $(\gamma_i(V_j))_{i,j}$, $1 \leq j \leq n_{k-1}$ *, properly relabeled as* (X_n) *and* (Y_n) *such that* $Y_n = \gamma_i(V_j)$ *if* $X_n = \gamma_i(U_j)$ *, satisfy (1)-(5).*

Proof. Assume $n_k \leq n_{k-1}$. Then $\Gamma(U_1 \cup \ldots \cup U_{n_{k-1}}) \subset \Gamma(U_1^\circ \cup \ldots \cup U_{n_{k-1}}^\circ) \subset \Gamma(U_1 \cup \ldots \cup U_{n_{k-1}})$. Now $U_1 \cup \ldots \cup U_{n_{k-1}}$ is compact and hence closed in S. The family $(\gamma(U_1 \cup \ldots \cup U_{n_{k-1}}) \mid \gamma \in \Gamma)$ is a locally finite family of closed sets. Thus $\Gamma(U_1 \cup \ldots \cup U_{n_{k-1}})$ is closed in S, but it is also open in S and hence equal to S since S is connected. The remainder of the lemma is an obvious consequence of the choices of (X_n) and (Y_n). □

LEMMA 2.1.5. *If* S *is compact, then* $n_k \leq n_{k-1}$ *for some* k *and* (Y_n) *can be choosen to satisfy (6).*

Proof of Lemma 2.1.5. If S is compact, then for some l we have $U_1^\circ \cup \ldots \cup U_l^\circ = S$ so that $\Gamma(U_1 \cup \ldots \cup U_l) \subset \Gamma(U_1^\circ \cup \ldots \cup U_l^\circ) = S$. Therefore $n_k \leq l$ for all k. It follows that $n_k \leq n_{k-1}$ for some k. Furthermore, Γ is finite. Each $\gamma \in \Gamma$ is uniformly continuous and hence there is a positive number c such that for all $\gamma \in \Gamma$, $d(\gamma(x),\gamma(y)) < a$ if $d(x,y) < c$. Let us choose b so that $b \leq c$ in which case $\mathrm{diam}(V_n) < c$ for all n. Then $\mathrm{diam}(\gamma_i(V_n)) \leq a$ for all i and n. This proves that Y_n can be choosen to satisfy (6).

We shall now consider the case that (n_k) is strictly increasing which implies that S is not compact. We put $A_{-1} = A_0 = \emptyset$ and $A_k = U_1^\circ \cup \dots \cup U_{n_k}^\circ$, for $k \geq 1$. Then $\bar{A}_k = U_1 \cup \dots \cup U_{n_k}$. We observe that $\bar{A}_k$ is compact and $\cup\{A_k \mid k \geq 0\} = S$. By definition $\Gamma(\bar{A}_k) \subset \Gamma(A_{k+1})$. Given $k \geq 0$, we write $F_k = \Gamma(\bar{A}_{k+1}) - \Gamma(A_k) = \Gamma(\bar{A}_{k+1} - \Gamma(A_k))$. The set F_k is closed and Γ-invariant while $\Gamma(A_{k+2}) - \Gamma(\bar{A}_{k-1}) = \Gamma(A_{k+2} - \Gamma(\bar{A}_{k-1}))$ is open, Γ-invariant and contains F_k. Moreover, since $F_k = \Gamma(\bar{A}_{k+1} - \Gamma(A_k)) \subset \Gamma(\bar{A}_{k+1} - A_k)$, we have $F_k = F_k \cap \Gamma(\bar{A}_{k+1} - A_k) = \cup\{\gamma(\bar{A}_{k+1} - A_k) \cap F_k \mid \gamma \in \Gamma\} = \cup\{\gamma(\bar{A}_{k+1} - A_k) \cap \gamma(F_k) \mid \gamma \in \Gamma\} = \Gamma((\bar{A}_{k+1} - A_k) \cap F_k)$.

Now $(\bar{A}_{k+1} - A_k) \cap F_k$ is compact so that there is a finite subsequence (U_{kl}) of (U_i) such that $U_{kl} \subset V_{kl}^\circ \subset V_{kl} \subset \Gamma(A_{k+2}) - \Gamma(\bar{A}_{k-1})$ and $\cup U_{kl}^\circ \supset F_k \cap (\bar{A}_{k+1} - A_k)$. Thus the sequence $(\gamma_i(U_{kl}^\circ))_{i,l}$ covers F_k and $\gamma_i(U_{kl}) \subset \gamma_i(V_{kl}^\circ) \subset \gamma_i(V_{kl}) \subset \Gamma(A_{k+2}) - \Gamma(\bar{A}_{k-1})$. □

LEMMA 2.1.6. *The sequences* $(\gamma_i(U_{kl}))_{i,k,l}$ *and* $(\gamma_i(V_{kl}))_{i,k,l}$, *properly relabeled as* (X_n) *and* (Y_n) *such that* $Y_n = \gamma_i(V_{kl})$ *if* $X_n = \gamma_i(U_{kl})$, *satisfy (1)-(6).*

Proof of Lemma 2.1.6. The only conditions whose truth is not immediately obvious are (2) and (3). To show (2) we observe that $\cup\{F_k \mid k \geq 0\} = \cup\{\Gamma(\bar{A}_{k+1}) - \Gamma(A_k) \mid k \geq 0\} \supset \{\Gamma(\bar{A}_k) \mid k \geq 0\} = S$. Thus $(\gamma_i(U_{kl}))_{i,k,l}$ covers S. Next we shall show that $(\gamma_i(V_{kl}))$ is locally finite. Assume $0 \leq m \leq k-3$ and $\gamma_r, \gamma_s \in \Gamma$. Then $\gamma_r(V_{m_j}) \subset \Gamma(A_{m+2}) \subset \Gamma(\bar{A}_{k-1})$ and $\gamma_s(V_{kl}) \subset \Gamma(A_{k+2}) - \Gamma(\bar{A}_{k-1})$. Let $x \in S$.

Then $x \in F_k \subset \Gamma(A_{k+2}) - \Gamma(\bar{A}_{k-1}) = T_k$ for some k. The set T_k is open and doesn't meet $\gamma_i(V_{ml})$ whenever $m \leq k-3$ or $m \geq k+3$. The family $(\gamma_i(V_{ml}))_{i,l,m}$, $k-3 \leq m \leq k+3$, is locally finite. Thus x has a neighbourhood in T_k which meets $\gamma_i(V_{ml})$ for only finitely many i, m and l. Hence, $(\gamma_i(V_{ml}))_{i,m,l}$ is locally finite. □

§2. THE PROOF OF THEOREM 2.2.1

THEOREM 2.2.1. *Given a group* Γ *of homeomorphisms acting discontinuously on a surface* S, *then there is a countable collection* $\mathcal{D}$ *of discs with the following properties:*

(1) $\cup\{D^\circ \mid D \in \mathcal{D}\} = S$;

(2) $\mathcal{D}$ *is locally finite;*

(3) If $D_1, D_2 \in \mathcal{D}$, $D_1 \neq D_2$, *then* $D_1 \not\subset D_2$;

(4) If $D_1, D_2 \in \mathcal{D}$, $D_1 \neq D_2$, *then* $\partial D_1 \not\subset D_2$;

(5) $\{\partial D \mid D \in \mathcal{D}\}$ *is compatible;*

(6) $\cup\{\partial D \mid D \in \mathcal{D}\}$ *is compatible with* F_Γ;

(7) $\mathcal{D}$ *is* Γ*-invariant; and*

(8) For $D \in \mathcal{D}$ *and* $\gamma \in \Gamma - \Gamma_D$, $\gamma(D) \cap D = \emptyset$.

Proof. Let $(X_n)_{n \in J}$ and $(Y_n)_{n \in J}$ be the sequences of discs of Theorem 2.1.1. We can choose a subsequence (X_{n_i}) of $(X_n)_{n \in J}$ such that X_{n_i} and X_{n_j} are not Γ-equivalent if $i \neq j$, and each X_m is Γ-equivalent to X_{n_i} for some i. We shall now construct collections $\mathcal{D}_i$ of discs as follows.

Let $\mathcal{D}_o = \{\gamma(Y_{n_1}) \mid \gamma \in \Gamma\}$. Suppose $k \geq 0$ and locally finite and Γ-invariant collections $\mathcal{D}_o, \ldots, \mathcal{D}_k$ of discs have been constructed so that Y_{n_l}, $l \geq k+1$, is met by at most finitely many of the discs in $\mathcal{D}_o \cup \cdots \cup \mathcal{D}_k$. Then $\mathcal{D}_{k+1}$ is constructed as follows. By [1, 46 D,E] there is a disc D_1 such that $X_{n_{k+1}} \subset D_1^\circ \subset Y_{n_{k+1}}$ and ∂D_1 meets F = $\cup\{\partial D \mid D \in \mathcal{D}_o \cup \cdots \cup \mathcal{D}_k\}$ in at most finitely many points. We know

$\Gamma_{Y_{n_{k+1}}} = \Gamma_{X_{n_{k+1}}} = \Delta$ is finite and $\Delta(\partial D_1) \cap F$ is then finite too because F is Γ-invariant. By Lemma 1.6.2 there is a Δ-invariant disc D so that $X_{n_{k+1}} \subset D^{\circ} \subset Y_{n_{k+1}}$ and $\partial D \cap F \subset \Delta(\partial D_1) \cap F$. Thus ∂D is compatible with F.

We define $D_{k+1} = \{\gamma(D) \mid \gamma \in \Gamma\}$. We observe that D_{k+1} is Γ-invariant, locally finite and $\{\partial D \mid D \in D_{k+1}\}$ is itself compatible as well as compatible with $\{\partial D \mid D \in D_0 \cup \ldots \cup D_k\}$. Further, for $\gamma \in \Gamma - \Gamma_D$ and $D \in D_{k+1}$, $\gamma(D) \cap D = \emptyset$. By construction, Y_{n_l}, $l \geq k+2$, meets $\gamma(Y_{n_{k+1}})$ for at most finitely many $\gamma \in \Gamma$ and we have $D \subset Y_{n_{k+1}}$. It follows that Y_{n_l} meets at most finitely many sets in D_{k+1}.

We claim that the collection $D' = \cup\{D_i \mid i \geq 0\}$ of discs satisfies the requirements (1), (2), (5), (6), (7) and (8). To prove (1) recall that $\cup\{X_n \mid \text{all } n\} = S$. Given n, X_n is Γ-equivalent to X_{n_j} for some j and thus by construction, there is some $D \in D_j$ with $X_{n_j} \subset D^{\circ} \subset Y_{n_j}$. Hence $X_n \subset \cup\{\gamma(D^{\circ}) \mid \gamma \in \Gamma\} \subset \cup\{D \mid D \in D_j\}$. It follows that $S = \cup\{D \mid D \in D'\}$. To prove (2) we observe that by our construction of D' there is a mapping ϕ from J onto D' with $X_n \subset \phi(n)^{\circ} \subset Y_n$ for all $n \in J$. (J is the index set of the sequence $(X_n)_{n \in J}$). Since (Y_n) is locally finite, D' is locally finite.

For (5), assume $D_1 \in D_i$, $D_2 \in D_j$, $i \leq j$, and $D_1 \neq D_2$. If $i < j$, then by construction ∂D_2 is compatible with ∂D_1. If $i = j$, then $D_1 \cap D_2 = \emptyset$ and thus they are compatible. The latter observation also shows that (8) is obvious.

To establish (6) let $D \in D'$. If $x \in F_\Gamma \cap D$, then $x \in F_{\Gamma_D}$ by (8). Thus $F_\Gamma \cap \partial D = F_{\Gamma_D|D} \cap \partial D$ is finite. Hence ∂D is compatible with F_Γ and thus by (2), $\cup\{\partial D \mid D \in \mathcal{D}\}$ is compatible with F_Γ. Clearly $\mathcal{D}'$ is Γ-invariant since D_i is Γ-invariant for all i thus proving (7).

Let us now partially order $\mathcal{D}'$ by inclusion. $\mathcal{D}'$ and with it the order defined on $\mathcal{D}'$ are Γ-invariant, that is, $D_1 \subset D_2$ if and only if $\gamma(D_1) \subset \gamma(D_2)$ for all $\gamma \in \Gamma$. $\mathcal{D}'$ is locally finite so that every chain in $\mathcal{D}'$ is finite. Thus the collection $\mathcal{D}$ of maximal elements of $\mathcal{D}'$ is nonempty and satisfies all properties except possibly (4).

We observe that if $\partial D_1 \subset D_2$, for $D_1, D_2 \in \mathcal{D}$, $D_1 \neq D_2$, then $D_1 \cup D_2 = S$ and S is homeomorphic to the sphere S^2. Thus in this case we only have to show that $\mathcal{D}$ can be choosen to satisfy (4). According to Proposition 1.5.7 we may assume that $S = S^2$. We endow S^2 with the euclidean metric of R^3. Thus if a in Theorem 2.1.1 is small enough and $\operatorname{diam}(D) \leq a$ for all $D \in \mathcal{D}$, then there are no two discs in $\mathcal{D}$ which cover S. Hence $\mathcal{D}$ satisfies (4). □

§3. THE PROOF OF LEMMA 2.3.1

Let (S,G) be a polyhedron each of whose faces is a polygon. A polyhedron (S,H) is called a *barycentric subdivision* of (S,G) if it is obtained from (S,G) by the following two steps:

(1) Each edge of G is subdivided into two edges thus yielding the polyhedron (S,G'); and

(2) There is a subdivision of each n-gon face of (S,G') into n triangles by n arcs joining a point $v \in F^\circ$ with the n vertices of the face whenever $n \geq 2$.

We note that the barycentric subdivision (S,H) is a triangular polyhedron. Before proving the lemma about the existence of a Γ-invariant barycentric subdivision, we briefly discuss finite homeomorphism groups of arcs.

We note that if Γ is a finite homeomorphism group of an arc $e = e_{[u,v]}$, then either $\Gamma = \{\iota\}$ or $\Gamma = \langle\gamma\rangle$, where γ is of order 2, interchanges u and v and has exactly one fixed point $x \in e_{(u,v)}$. The proof of this statement is straightforward and will be omitted.

LEMMA 2.3.1. *Let Γ be a discontinuous homeomorphism group of the polyhedron* (S,G). *Then there is a Γ-invariant barycentric subdivision* (S,H) *of* (S,G) *with the following properties:*

(1) $F_\Gamma \subset ps(H)$; *and*

(2) If $\gamma(x) = x$ *for* $x \in E(H) \cup F(S,H)$ *and* $\gamma \in \Gamma$, *then* $\gamma|_x = \iota|_x$.

Proof. In order to obtain a barycentric subdivision we first subdivide the edges of G in an appropriate way to be described below. We note that Γ_e is finite if $e \in E(G)$ and according to the above observation $\Gamma_e|_e$ either consists of the identity mapping only or $\Gamma_e|_e = \langle\gamma\rangle$, where γ is a homeomorphism of e of order 2 which has a unique fixed point $w_e \in e - i_G(e)$. Let $E_o \subset E(G)$ consist of exactly one edge from each Γ-orbit and let $W' = \{w_e |\ e \in E_o\}$, where $w_e \in e - i_G(e)$ and $\gamma(w_e) = w_e$ for all $\gamma \in \Gamma_e$. Let $W = \Gamma(W')$. Then W is Γ-invariant, every edge $e \in E(G)$ contains exactly one point w_e of W and $\gamma(w_e) = w_e$ for $\gamma \in \Gamma_e$. Thus the subdivision G' of G by W provides the first step of a barycentric subdivision. We also note that $\gamma(e) = e$ implies $\gamma|_e = \iota|_e$ for $e \in E(G')$ and $\gamma \in \Gamma_e$.

Let $F \in F(S,G') = F$ and let $V(F) = F \cap V(G')$. The group $\Gamma_F|_F$ is topologically equivalent to a group of congruences of the unit disc. Thus in view of the previous remark it follows that $V(F) \supset F_{\Gamma_F} \cap \partial F$ and $F_\Gamma \cap F^\circ \subset F_{\Gamma_F|F}$. Let $F_o \subset F$ consist of exactly one face from each Γ-orbit. Then there is a collection $K_o = \{K_F |\ F \in F_o\}$ of S-graphs dividing the faces $F \in F_o$ into triangles and satisfying the following properties:

(a) $ps(K_F) = e_{[v,v_1]} \cup \ldots \cup e_{[v,v_n]}$ is a star, where $\{v_1,\ldots,v_n\} = V(F)$;

(b) $V(K_F) = \{v,v_1,\ldots,v_n\} \supset F_{\Gamma_F} \cap \partial F$;

(c) $F^\circ \supset ps(K_F) - V(F) \supset F_\Gamma \cap F^\circ$; and

(d) K_F is Γ_F-invariant.

We now define $\mathcal{K} = \{K_F \mid F \in \mathcal{F}\}$ as follows.
For $F \in \mathcal{F}$ we shall define $K_F = \gamma(K_{F_o})$ where $F_o \in \mathcal{F}_o$ and $\gamma \in \Gamma$ with $\gamma(F_o) = F$.
Then K_F is well-defined as $F = \gamma(F_o) = \delta(F_o)$ implies that $\gamma^{-1}\delta \in \Gamma_{F_o}$.
Hence $\gamma^{-1}\delta(K_{F_o}) = K_{F_o}$ and $\gamma(K_{F_o}) = \delta(K_{F_o})$.

There are some immediate consequences of the construction of G' and $\mathcal{K}$. We have that $\mathcal{K}$ is a locally finite, compatible and Γ-invariant collection of S-graphs each of which satisfies (a)-(d).
Also, $K = \cup\{K_F \mid F \in \mathcal{F}\}$ is a Γ-invariant S-graph and K is compatible with G' as $ps(K) \cap ps(G') = V(G')$. It follows that (S,H), where $H = K \cup G'$, is a Γ-invariant barycentric subdivision of (S,G). We have $E(H) = E(G') \cup E(K)$ and $\gamma(e) = e$ implies $\gamma|_e = \iota|_e$ for all $\gamma \in \Gamma$ and $e \in E(H)$.
Moreover, since $ps(H) \supset F_\Gamma$ it follows that $F^\circ \cap F_\Gamma = \emptyset$ for every $F \in \mathcal{F}(S,H)$ and therefore $\gamma|_F = \iota|_F$ for every $\gamma \in \Gamma_F$. □

§4. THE PROOF OF THE TRIANGULATION THEOREM

THEOREM 2.4.1. *Given a group* Γ *of homeomorphisms acting discontinuously on the surface* S , *there is a* Γ*-invariant triangular polyhedron* (S,G) *satisfying the following two properties:*

(1) $ps(G) \supset F_\Gamma$ *; and*

(2) If $\gamma(x) = x$ *for* $\gamma \in \Gamma$ *and* $x \in E(G) \cup F(S,G)$, *then* $\gamma|_x = \iota|_x$ *(where* ι *is the identity mapping on* S*).*

Proof. As a first step we choose a collection $\mathcal{D}$ of discs as in Theorem 2.2.1 and construct a Γ-invariant graph H with $ps(H) = \cup\{\partial D \mid D \in \mathcal{D}\}$. Let $\mathcal{D}_o \subset \mathcal{D}$ consist of exactly one representative from each Γ-equivalence class of $\mathcal{D}$. For each $D \in \mathcal{D}_o$ we choose a finite set V_D with $F_{\Gamma_D} \cap \partial D \subset V_D \subset \partial D$, $\Gamma_D(V_D) = V_D$ and $card(V_D) \geq 3$. Let H_D be the S-graph with vertex-set V_D and point set ∂D so that H_D is Γ_D-invariant. For $D \in \mathcal{D}$ define $H_D = \gamma(H_{D_1})$ where $D_1 \in \mathcal{D}_o$ and $\gamma \in \Gamma$ with $\gamma(D_1) = D$. H_D is well defined since $\gamma(D_1) = \delta(D_1) = D$ implies $\gamma^{-1}\delta \in \Gamma_{D_1}$ and hence $\gamma^{-1}\delta(H_{D_1}) = H_{D_1}$ or $\gamma(H_{D_1}) = \delta(H_{D_1})$. The following observations (a) and (b) are immediate consequences of the definition of H_D.

(a) $\mathcal{H} = \{H_D \mid D \in \mathcal{D}\}$ is a compatible, locally finite and Γ-invariant collection of S-graphs; and

(b) $H = \cup\{K \mid K \in \mathcal{H}\}$ is a Γ-invariant S-graph with $ps(H) = \cup\{\partial D \mid D \in \mathcal{D}\}$ and $V(H) \supset F_\Gamma \cap ps(H)$.

We claim that the graph H is connected. It suffices to prove that ps(H) is arcwise connected. Suppose ps(H) is not arcwise connected. Then there is an arc $e_{[u,v]} \subset S$ such that $e_{(u,v)} \cap ps(H) = \emptyset$, where u and v are in distinct components of ps(H). Now $S = \cup\{D^\circ \mid D \in \mathcal{D}\}$ so that there are discs $D, D_1, D_2 \in \mathcal{D}$ with $u \in D^\circ$, $u \in \partial D_1$ and $v \in \partial D_2$. Since $\partial D \subset ps(H)$ we have $e_{[u,v]} \subset D$ and $v \in \partial D$ or $v \in D^\circ$. In any case, $\partial D \cap \partial D_2 \neq \emptyset$ since either $v \in \partial D \cap \partial D_2$ or $v \in D^\circ$ and $\partial D_2 \not\subset D$. We have $\partial D \cap \partial D_1 \neq \emptyset$ since $\partial D_1 \not\subset D$. Therefore u and v belong to the same component of ps(H) which is a contradiction of our assumption.

We claim that every face of H is a polygon. Let F be a face of H in S. Then there is some disc $D \in \mathcal{D}$ with $F \subset D$. Consider the subgraph K of H contained in D and let L be the cycle in K whose point set is ∂D. We shall show that K is 2-connected. This implies that every face of H within D is a polygon. Given any vertex $v \in V(K) - V(L)$, there is a path P in K containing v and joining two distinct vertices of L. Thus the removal of any point of $V(K) - V(L)$ results in a connected graph. Thus K is 2-connected.

An application of Lemma 2.3.1 to the polyhedron (S,H) completes the proof. $\square$

CHAPTER 3. ABSTRACT POLYHEDRA

By Theorem 2.4.1 it is possible to substitute the study of discontinuous groups on surfaces by the study of groups of automorphisms of abstract polyhedra. The notion of a topological polyhedron will be replaced by an equivalent notion (see Theorems 5.1.1 and 5.1.3) of an abstract polyhedron. An abstract polyhedron shall consist of a locally finite (possibly infinite) graph and a boundary tour scheme, which in the theory of graph embeddings is commonly called a generalized embedding scheme [25], [26]. Moreover, instead of homeomorphism between topological polyhedra we shall be talking about isomorphisms of abstract polyhedra (see Theorem 5.2.2). (If one is concerned with embeddings of graphs the problem is to determine when two embedding schemes produce the same embedding.) The question of when two abstract polyhedra are isomorphic or when a polyhedron is orientable can easily be answered by looking at their boundary tour schemes (see Propositions 3.1.3 and 3.1.4 and Corollary 3.1.8). In view of Theorem 2.4.1 we may restrict our attention mostly to automorphisms of abstract 3-polyhedra, that is, polyhedra all of whose boundaries are triangles.

We define one-sidedness and two-sidedness of cycles and 2-infinite paths. Thus a two-sided cycle or path C partitions an abstract polyhedron into two parts whose intersection is C. Isomorphisms naturally preserve boundaries as well as the properties of one- and two-sidedness and also the sides. A 3-polyhedron is defined to be planar if all of its cycles are two-sided. We shall see in Theorem 5.1.6 that abstract planar 3-polyhedra are exactly the 3-polyhedra which correspond to planar topological 3-polyhedra. One- and two-sided 2-infinite paths shall be important tools in the study of automorphisms of infinite order in planar polyhedra (Chapter 4).

§1. AUTOMORPHISMS OF ABSTRACT POLYHEDRA

Let G be a connected and locally finite graph (locally finite for graphs means that every vertex has finite degree) without multiple edges. We shall write $H \subset K$ if H is a subgraph of K, $H \cap K$ for the subgraph induced by $V(H) \cap V(K)$ and $E_v(G)$ shall denote the set of edges of G incident to v. We shall write $d_G(x,y)$, respectively, $d_G(K,L)$ to denote the distance in G between the vertices x and y of G, respectively, the subgraphs K and L of G. A *two way infinite* (abbreviated as 2-infinite) path is a connected, infinite graph which is regular of degree 2. A *one way infinite* (abbreviated as 1-infinite) path is a connected, infinite graph with maximum degree 2 and minimum degree 1. If H is a path or cycle in G and $u,v \in V(H)$, then $H_{(u,v)}$ shall denote a path on H joining u and v. If there is no unique such path it shall be clear from the context which one is meant.

A *rotation system* P for G is a set $\{P_v \mid v \in V(G)\}$ where P_v is a cyclic permutation of $E_v(G)$. Although the permutations P_v and P_v^{-1} are identical as mappings in case $1 \leq |E_v(G)| \leq 2$, we shall consider them as distinct objects. If $\{e_1,\ldots,e_n\} \subset E_v(G)$ and $n \geq 3$ then we shall write $(e_1,\ldots,e_n) \subset P_v$ if $P_v^{\varepsilon_1}(e_1) = e_2$, $P_v^{\varepsilon_2}(e_2) = e_3$, ..., $P_v^{\varepsilon_{n-1}}(e_{n-1}) = e_n$, where $\varepsilon_1,\ldots,\varepsilon_{n-1}$ are positive integers and $\varepsilon_1 + \ldots + \varepsilon_{n-1} < |E_v(G)|$. We call $(e_1,\ldots,e_n)$ the induced cyclical arrangement on $\{e_1,\ldots,e_n\}$.

A *boundary tour scheme* for G is a pair (P,λ) consisting of a rotation system P and a mapping $\lambda: E(G) \to Z_2 = \{1,-1\}$. The triple

(G,P,λ) is called an *(abstract) polyhedron.* We shall write (G,P) instead of (G,P,λ) if $\lambda(e) = 1$ for all $e \in E(G)$. For any finite subgraph H of G we define $\lambda(H) = \Pi\{\lambda(e) \mid e \in E(H)\}$, where the product is defined to be 1 if $E(H) = \emptyset$. We say H is λ-trivial if $\lambda(H) = 1$ (see [25, p.48]).

Given two polyhedra (G,P,λ) and (H,Q,μ) and an isomorphism $\phi : G \to H$, we shall say that ϕ is an *isomorphism* from (G,P,λ) to (H,Q,μ) and write $\phi : (G,P,\lambda) \to (H,Q,\mu)$ in case

(1) $\phi P_u \phi^{-1} \in \{Q^{\delta}_{\phi(u)} \mid \delta \in Z_2\}$ for all $u \in V(G)$ and

(2) if $e \in E(G)$, $i_G(e) = \{u,v\}$ and $\phi P_u \phi^{-1} = Q^{\delta}_{\phi(u)}$, $\phi P_v^{\lambda(e)} \phi^{-1} = Q^{\varepsilon}_{\phi(v)}$ imply $\varepsilon = \delta\mu(\phi(e))$.

(We shall adopt the convention of denoting the mapping of the edges induced by an automorphism by the same symbol.)

An *automorphism* of (G,P,λ) is an isomorphism of (G,P,λ) onto itself. We use $\mathrm{Aut}(G,P,\lambda)$ to denote the set of all automorphisms of (G,P,λ). The automorphism $\alpha \in \mathrm{Aut}(G,P)$ is called *orientation preserving* if $\alpha P_u \alpha^{-1} = P_{\alpha(u)}$ for some (or equivalently, for all) $u \in V(G)$, and *orientation reversing* otherwise. The polyhedron (G,P,λ) is *orientable* if the identity isomorphism $\iota : G \to G$ is an isomorphism from (G,P,λ) to (G,Q).

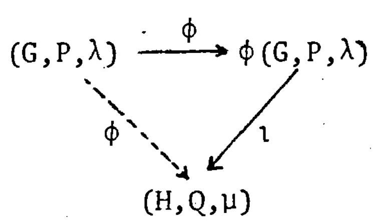
The proofs of the following two statements are easy exercises and therefore omitted.

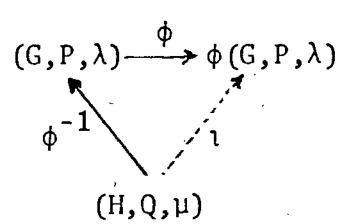
PROPOSITION 3.1.1. *If* $\phi : (G,P,\lambda) \to (H,Q,\mu)$ *and* $\psi : (H,Q,\mu) \to (K,R,\nu)$ *are isomorphisms, then* $\phi^{-1} : (H,Q,\mu) \to (G,P,\lambda)$ *and* $\psi\circ\phi : (G,P,\lambda) \to (K,R,\nu)$ *are isomorphisms.*

COROLLARY 3.1.2. *The set* Aut(G,P,λ) *of all automorphisms of* (G,P,λ) *form a group with composition of mappings as group operation.*

PROPOSITION 3.1.3. *Let the polyhedra* (G,P,λ) *and* (H,Q,μ) *and an isomorphism* $\phi: G \to H$ *be given. Let* $\phi(G,P,\lambda)$ *denote the polyhedron* $(H,\phi P\phi^{-1},\nu)$ *where* $\phi P\phi^{-1} = \{\phi P_v \phi^{-1} \mid v \in V(G)\}$ *and* $\nu(\phi(e)) = \lambda(e)$ *for all* $e \in E(G)$. *Then* ϕ *is an isomorphism from* (G,P,λ) *to* (H,Q,μ) *if and only if the identity isomorphism* $\iota: H \to H$ *is an isomorphism of* $\phi(G,P,\lambda)$ *and* (H,Q,μ).

Proof. We prove this proposition by means of diagrams. Full arrows denote given isomorphisms and dotted arrows are for implied isomorphisms.





□

The next statement gives a simple criterion for determining whether the identity $\iota: G \to G$ is an isomorphism between the polyhedra (G,P,λ) and (H,Q,μ).

PROPOSITION 3.1.4. *The identity isomorphism* $\iota: G \to G$ *is an isomorphism of the polyhedra* (G,P,λ) *and* (H,Q,μ) *if and only if*

(1) $P_u \in \{Q_u^{\delta} \mid \delta \in Z_2\}$ *for all* $u \in V(G)$ *and*

(2) for all $e \in E(G)$, $\lambda(e) \neq \mu(e)$ *if and only if exactly one element of* $i_G(e)$ *belongs to* $U = \{v \in V(G) \mid P_v = Q_v^{-1}\}$.

Proof. First we prove the "only if" part. If $\iota: (G,P,\lambda) \to (G,Q,\mu)$ is an isomorphism of the polyhedra then by definition $P_u \in \{ Q_u^{\delta} \mid \delta \in Z_2\}$ for all $u \in V(G)$. Now let $e \in E(G)$ and $i_G(e) = \{u,v\}$. It follows that $Q_v^{\delta\mu(e)} = P_v^{\lambda(e)}$ if $P_u = Q_u^{\delta}$. We conclude that $\lambda(e) \neq \mu(e)$ if and only if either $u \in U$ or $v \in U$ but not both hold.

We now prove the "if" part. Assume (1) and (2) hold and let $e \in E(G)$, $i_G(e) = \{u,v\}$. By (1), $P_u = Q_u^{\delta}$ and $P_v^{\lambda(e)} = Q_v^{\varepsilon}$, $\varepsilon,\delta \in Z_2$, and by (2) it immediately follows that $\varepsilon = \delta\mu(e)$. Thus $\iota: (G,P,\lambda) \to (H,Q,\mu)$ is an isomorphism. □

COROLLARY 3.1.5. *If* $\iota : (G,P,\lambda) \to (G,Q,\mu)$ *is an isomorphism, then in every cycle* C *of* G *the number of edges* e *with* $\lambda(e) \neq \mu(e)$ *is even.*

Proof. Let C be a cycle and $E' = \{e \in E(C) \mid \lambda(e) \neq \mu(e)\}$ and $E'' = E(C) - E'$. The endpoints of an edge in E" both belong to U or both don't belong to U, and exactly one endpoint of an edge in E' belongs to U. Thus alternating "blocks" in C of vertices of U and $V(G) - U$ are separated by edges in E'. It follows that $|E'|$ is even. □

The following statement is an immediate consequence of Corollary 3.1.5. and definitions.

COROLLARY 3.1.6. *If* (G,P,λ) *is orientable, then every cycle in* G *is* λ*-trivial.*

COROLLARY 3.1.7. *If every cycle of the polyhedron* (G,P,λ) *is* λ*-trivial, then* (G,P,λ) *is orientable.*

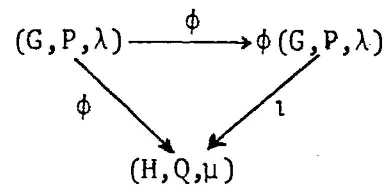
Proof. Fix $u \in V(G)$. For arbitrary $v \in V(G)$ define $Q_v = P_v$ if there is a λ-trivial path from u to v and $Q_v = P_v^{-1}$ if there is a path from u to v which is not λ-trivial. Then $Q = \{Q_v \mid v \in V(G)\}$ is well-defined as every cycle of (G,P,λ) is λ-trivial. Moreover, if $i_G(e) = \{v_1,v_2\}$ it follows that $\lambda(e) = -1$ if and only if exactly one of v_1 or v_2 belongs to $\{w \in V(G) \mid P_w = Q_w^{-1}\}$. Thus $\iota : (G,P,\lambda) \to (G,Q)$ is an isomorphism and (G,P,λ) is therefore orientable. □

Corollaries 3.1.6 and 3.1.7 imply the following criterion for orientability of polyhedra. (See also [25, Theorem 5].)

COROLLARY 3.1.8. (G,P,λ) *is orientable if and only if every cycle of* (G,P,λ) *is* λ*-trivial.*

COROLLARY 3.1.9. *If* $\phi : (G,P,\lambda) \to (H,Q,\mu)$ *is an isomorphism and* C *is a* λ*-trivial cycle in* G, *then* $\phi(C)$ *is* μ*-trivial.*

Proof. By Proposition 3.1.3 we have the following diagram.



Clearly, C is a λ-trivial cycle in (G,P,λ) if and only if $\lambda(C)$ is μ-trivial in $\phi(G,P,\lambda)$. Thus it suffices to prove the corollary in the case where $G = H$ and $\phi = \iota$. Let C be a λ-trivial cycle. In view of Corollary 3.1.5, C has an even number of edges e with $\lambda(e) \neq \mu(e)$. It follows that $|\{e \in E(C) \mid \mu(e) = -1 \text{ and } \lambda(e) = 1\}|$ and $|\{e \in E(C) \mid \mu(e) = 1 \text{ and } \lambda(e) = -1\}|$ have the same parity. We conclude that $|\{e \in E(C) \mid \mu(e) = -1\}|$ and $|\{e \in E(C) \mid \lambda(e) = -1\}|$ are of the same parity, and therefore $\phi(C)$ is μ-trivial. □

The following corollary immediately follows from the Corollaries 3.1.8 and 3.1.9.

COROLLARY 3.1.10. *If* (G,P,λ) *and* (H,Q,μ) *are isomorphic and* (G,P,λ) *is orientable, then* (H,Q,μ) *is orientable.*

§2. BOUNDARIES OF POLYHEDRA

For any graph G the term *arc* shall denote an edge with a specific choice of orientation. We shall write $\vec{e}$ for an arc corresponding to an edge $e \in E(G)$ and $\overleftarrow{e}$ (or occasionally $\vec{e}^{\,-}$) for the reverse of $\vec{e}$, which is the arc whose orientation is opposite to that of $\vec{e}$. Let $\vec{E}(G)$, respectively $\vec{E}_v(G)$, denote the set of arcs of G, respectively, the set of arcs of G terminating at v. Let (G,P,λ) be a polyhedron, let $\vec{P}_v$ be the permutation of $\vec{E}_v(G)$ induced by P_v in the obvious way and let $\lambda(\vec{e}) = \lambda(e)$ for all $\vec{e} \in \vec{E}(G)$. We define a permutation P^* of $\vec{E}(G) \times Z_2$ as follows. We put $P^*((\vec{e},\varepsilon)) = ((\vec{P}_v^{\varepsilon\lambda(\vec{e})}(\vec{e}))^-, \varepsilon\lambda(\vec{e}))$, for $e \in \vec{E}_v(G)$ and $\varepsilon \in Z_2$. This definition agrees with the one given in [8 , p.14]. An inspection of the orbits of P^* reveals that they occur in pairs, that is, if $(\ldots(\vec{e}_1,\varepsilon_1)(\vec{e}_2,\varepsilon_2)\ldots(\vec{e}_n,\varepsilon_n)\ldots)$ is an orbit then $(\ldots(\overleftarrow{e}_n,-\varepsilon_n)\ldots(\overleftarrow{e}_2,-\varepsilon_2)(\overleftarrow{e}_1,-\varepsilon_1)\ldots)$ also is an orbit, [25, Lemma 1]. The subgraphs of G naturally induced by the orbits of P^* are called the *boundaries* of the polyhedron (G,P,λ).

Let $(\vec{e},\delta) \in \vec{E}(G) \times Z_2$, where e has initial vertex u, and let $\phi : (G,P,\lambda) \to (H,Q,\mu)$ be an isomorphism. We put $\phi(\vec{e},\delta) = (\phi(\vec{e}),\varepsilon)$ where ε is chosen such that $\phi P_u^{\delta} \phi^{-1} = Q_{\phi(u)}^{\varepsilon}$. It naturally follows from the definition of isomorphism and boundaries that isomorphisms preserve boundaries.

PROPOSITION 3.2.1. *If* $\phi : (G,P,\lambda) \to (H,Q,\mu)$ *is an isomorphism, then* $(\ldots(\vec{e}_i,\varepsilon_i)(\vec{e}_{i+1},\varepsilon_{i+1})\ldots)$ *is an orbit of* P^* *if and only if* $(\ldots\phi(\vec{e}_i,\varepsilon_i)\cdot\phi(\vec{e}_{i+1},\varepsilon_{i+1})\ldots)$ *is an orbit of* Q^*. *Moreover, it follows that* B *is a boundary of* (G,P,λ) *if and only if* $\phi(B)$ *is is a boundary of* (H,Q,μ).

Proof. The second claim is a trivial consequence of the first claim. In order to prove the first one we note that $P^*(\vec{e},\delta) = ((P_v^{\delta\lambda(\vec{e})}(\vec{e}))^-, \delta\lambda(\vec{e}))$ and $Q^*(\phi(\vec{e}),\varepsilon) = ((Q_{\phi(v)}^{\varepsilon\mu(\phi(\vec{e}))}(\phi(\vec{e})))^-, \varepsilon\mu(\phi(\vec{e})))$ assuming that $\vec{e}$ has inital vertex u and terminal vertex v. Given that $\phi(\vec{e},\delta) = (\phi(\vec{e}),\varepsilon)$ we conclude that $Q_{\phi(v)}^{\varepsilon\mu(\phi(\vec{e}))} = \phi P_v^{\delta\lambda(\vec{e})}\phi^{-1}$. Thus $\phi(P^*(\vec{e},\delta)) = Q^*(\phi(\vec{e},\delta))$ for $(\vec{e},\delta) \in E(G)\times Z_2$, and by applying the same considerations to ϕ^{-1} it follows that $P^*(\phi^{-1}(\vec{e},\delta)) = \phi^{-1}Q^*(\vec{e},\delta)$ for $(\vec{e},\delta) \in E(H)\times Z_2$. □

In view of Theorem 2.4.1 we shall be concerned exclusively with polyhedra all of whose boundaries are triangles. We shall call them *3-polyhedra* or *triangular polyhedra.*

PROPOSITION 3.2.2. *The boundaries of a 3-polyhedron* (G,P,λ) *are* λ*-trivial.*

Proof. Let H be the boundary induced by the orbit $(\ldots(\vec{e}_1,\varepsilon_1)(\vec{e}_2,\varepsilon_2)(\vec{e}_3,\varepsilon_3)\ldots)$ as shown in Figure 3.2.1. Then $P_{v_i}^{\varepsilon_i}(\vec{e}_{i-1}) = \overleftarrow{e}_i$

and $\varepsilon_i \lambda(\vec{e}_i) = \varepsilon_{i+1}$, where $1 \le i \le 3$ and $\vec{e}_0 = \vec{e}_3$ and $\vec{e}_4 = \vec{e}_1$. Thus $\lambda(\vec{e}_1)\lambda(\vec{e}_2)\lambda(\vec{e}_3) = 1$.

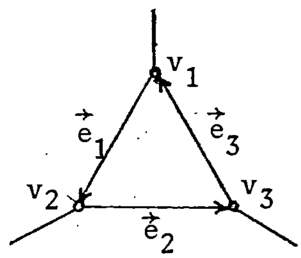


figure 3.2.1

□

PROPOSITION 3.2.3. *If* (G,P,λ) *is a 3-polyhedron, then* G *is 3-connected.*

Proof. Now $e_1, e_2 \in E_v(G)$ belong to the same boundary of (G,P,λ) if and only if $P_v^{\delta}(e_1) = e_2$, for some $\delta \in Z_2$. Thus the set $\{v_1,\ldots,v_n\}$ of all vertices adjacent to v is contained in an n-cycle of G. Consequently, if u and v are adjacent and $U = \{u_1,\ldots,u_m\}$ are all vertices of G distinct from u and v but adjacent to u or v, then U induces a connected subgraph of G. It follows that the omission of two (adjacent or non-adjacent) vertices never disconnects G, that is, G is 3-connected. □

Note 3.2.4. (a) The edges $e_1, e_2 \in E_v(G)$ belong to the same boundary of the 3-polyhedron (G,P,λ) if and only if $P_v^{\delta}(e_1) = e_2$, for some $\delta \in Z_2$, as follows from the definition of the boundaries.

(b) Any edge of G belongs to exactly two distinct boundaries, and any two distinct boundaries are disjoint or share exactly one vertex or share exactly two vertices and the edge joining them.

(c) If $\{a_1,a_2,e\}$ and $\{e_1,e_2,e\}$ are the edge sets of two distinct boundaries as in Figure 3.2.2, then $(e_1,e\,,\,a_1) \in P_u^{\delta}$ implies $(a_2,e\,,\,e_2) \subset P_v^{\delta\lambda(e)}$.

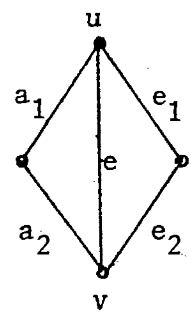


figure 3.2.2

§3. ONE-SIDEDNESS AND TWO-SIDEDNESS OF CYCLES AND 2-INFINITE PATHS

Suppose $H = (v_o, e_1, v_1, \ldots, v_{n-1}, e_n, v_n = v_o)$ is a λ-trivial cycle or $H = (\ldots, v_{i-1}, e_i, v_i, \ldots)$ is a 2-infinite path in the 3-polyhedron (G,P,λ). Let $u \in V(H)$. We define relations $\omega, \omega_u \in E(G) \times \{0,1,-1\}$ as follows:

(1) If $e \in E(H)$ or $i_G(e) \cap V(H) = \emptyset$, then let $(e,0) \in \omega \cap \omega_u$ while if $e \notin E_u(G)$, then let $(e,0) \in \omega_u$; and

(2) If $v_k \in i_G(e) \cap V(H)$ and $e \notin E(H)$, then let $(e,\varepsilon) \in \omega$, and $(e,\varepsilon) \in \omega_u$ if $u = v_k$, in case $(e_k, e, e_{k+1}) \subset P_{v_k}^{\varepsilon\delta_k}$ with $\delta_k = \lambda(H(v_o,v_k))$, where $\varepsilon \in Z_2$, $0 \le k \le n-1$, $e_0 = e_n$ and $H(v_o,v_k) = (v_o,e_1,v_1,\ldots,v_k)$ if H is a cycle.

We define $-\omega$ as the relation for which $(e,\varepsilon) \in -\omega$ if and only if $(e,-\varepsilon) \in \omega$. We define $-\omega_u$ in a similar fashion. If K is a subgraph of G, then $\omega(K)$ is the set of all $\varepsilon \in \{0,1,-1\}$ with $(e,\varepsilon) \in \omega$ for some $e \in E(K)$. $\omega_u(K)$ is defined similarly.

NOTE 3.3.1. ω_u is a function with domain $E(G)$ and range $\{0,1,-1\}$. If $i_G(e) \cap V(H) = \emptyset$ or $e \in E(H)$ and $i_G(e) = \{u,v\}$, then $\omega_u(e) = \omega_v(e) = \omega(e) = 0$. If $e \notin E(H)$ but $i_G(e) \cap V(H) \ne \emptyset$, then $\omega(e) = \cup\{\omega_x(e) \mid x \in i_G(e)\}$. We emphasize that the definitions of ω and ω_u depend on the particular labelling of the vertices and edges of H.

PROPOSITION 3.3.2. *If* H *is a λ-trivial cycle or a 2-infinite path of the 3-polyhedron* (G,P,λ) *and* ω_1 *and* ω_2 *are derived as described above, then either* $\omega_{1_u} = \omega_{2_u}$ *or* $\omega_{1_u} = -\omega_{2_u}$, *for all* $u \in V(H)$.

Proof. We shall restrict the proof to the case where H is a cycle as the other case is similar. We claim that if $H = (v_o, e_1, \ldots, e_n, v_n = v_o)$ and $H = (u_o, f_1, \ldots, f_n, u_n = u_o)$ with $u_i = v_{n-i}$, for $0 \le i \le n-1$, and if the derived relations are ω_1 and ω_2, respectively, then $\omega_{2_u} = -\omega_{1_u}$ for every $u \in V(H)$. To prove this claim let $e \in E(G) - E(H)$ and $u = v_k = u_{n-k} \in i_G(e)$. H is λ-trivial so that $\lambda(H(v_o, v_k)) = \delta = \lambda(H(u_o, u_{n-k}))$. We conclude that $(f_{n-k}, e, f_{n-k+1}) \subset P_{v_k}^{-\delta\varepsilon}$ if and only if $(e_k, e, e_{k+1}) \subset P_{v_k}^{\delta\varepsilon}$. Thus $(e, \varepsilon) \in \omega_{1_u}$ if and only if $(e, -\varepsilon) \in \omega_{2_u}$.

Next we claim that if $H = (v_o, e_1, \ldots, e_n, v_n = v_o)$ and $H = (u_o, f_1, \ldots, f_n, u_n = u_o)$ with $u_i = v_{i+k}$ for $0 \le i \le n-1$ and fixed k, $0 \le k \le n-1$, and if the derived relations are ω_1 and ω_2, then either $\omega_{1_u} = \omega_{2_u}$ for all $u \in V(H)$ or $\omega_{1_u} = -\omega_{2_u}$ for all $u \in V(H)$, depending on the value of $\lambda(H(v_o, v_k))$. To prove the claim we note that $\lambda(H(v_o, v_{l+k})) = \lambda(H(u_o, u_l)) \times \lambda(H(v_o, v_k))$ since H is λ-trivial. Moreover, $(e_{l+k}, e, e_{l+k+1}) = (f_l, e, f_{l+1})$ for $1 \le l \le n$. Thus if $\lambda(H(v_o, v_k)) = 1$, then $\omega_{1_u} = \omega_{2_u}$ and otherwise $-\omega_{2_u} = \omega_{1_u}$.

These two claims establish the proof of the proposition as we can get all labelings from a particular one by suitable "rotations" and "reflections". □

We shall reserve the letter ω to denote these relations derived from cycles or 2-infinite paths. All properties of ω ever used hold for ω if and only if they hold for $-\omega$. Thus we never need to specify which ω we use. For example $\omega(K) \supset \{-1,1\}$ if and only if $-\omega(K) \supset \{-1,1\}$.

PROPOSITION 3.3.3. *Let* H *be a* λ*-trivial cycle or a 2-infinite path with associated relation* ω *and let* $u \in i_G(a) \cap V(H)$, $v \in i_G(b) \cap V(H)$ *and* $a,b \in E(G) - E(H)$.

(a) *If* $u = v$, $a \neq b$, $E_u(H) = \{e_1, e_2\}$, then $\omega_u(a) \neq \omega_u(b)$ *if and only if* $(a, e_1, b, e_2) \subset P_u^{\delta}$, for $\delta \bullet Z_2$, *and the latter holds if and only if* $(e_1, b, e_2) \subset P_u^{\delta}$ *and* $(e_1, a, e_2) \subset P_u^{-\delta}$ for $\delta \in Z_2$.

(b) *If* $u \neq v$ *and* $H(u,v) = (u = u_o, e_1, \ldots, e_n, u_n = v)$ *is a path joining* u *and* v *in* H *and* $e \in E_u(H) - \{e_1\}$, $f \in E_v(H) - \{e_n\}$, *then* $\omega_u(a) = \omega_u(b)$ *if and only if* $(e, a, e_1) \subset P_u^{\delta}$ *and* $(e_n, b, f) \subset P_v^{\delta\lambda(H(u,v))}$.

Proof. This immediately follows from the definition of ω. □

If H is a cycle or 2-infinite path in the 3-polyhedron (G,P,λ), then it is called *one-sided* if, in case it is a cycle, H is not λ-trivial, or there is a walk $W = (u_o, a_1, \ldots, u_m)$ such that

(1) $u_o, u_m \in V(H)$ and $u_k \notin V(H)$ for $1 \leq k \leq m-1$ and

(2) $\omega(E(W)) \supset \{-1, 1\}$, where ω is derived from H, or equivalently,

$\omega_{u_o}(E(W)) \neq \omega_{u_m}(E(W))$.

If H is not one-sided then it is said to be *two-sided*. We note that by definition, if H is two-sided and $e \notin E(H)$ but $i_G(e) \cap V(H) \neq \emptyset$, for $x \in i_G(e)$ $\omega(e) = \omega_x(e)$. The reason for calling a cycle or 2-infinite path two-sided is given in the following theorem.

THEOREM 3.3.4. *If* H *is a two-sided cycle or 2-infinite path in the 3-polyhedron* (G,P,λ)*, then there is a unique decomposition of* G *into connected subgraphs* H_1 *and* H_2 *such that* $H_1 \cup H_2 = G$, $H_1 \cap H_2 = H$ *and* $\omega(H_1) \neq \omega(H_2)$.

Proof. Given $i \in \{1,2\}$, let H_i be the subgraph of G induced by the walks W in G for which $V(W) \cap V(H) \neq \emptyset$ and $\omega(E(W)) \subset \{0,(-1)^i\}$. It is obvious that $H_1 \cup H_2 = G$, $H \subset H_1 \cap H_2$ and H_i is connected for $1 \leq i \leq 2$. We need to verify that $V(H_1) \cap V(H_2) = V(H)$ and $E(H_1) \cap E(H_2) = E(H)$. Assume to the contrary that $v \in V(H_1) \cap V(H_2) - V(H)$. Then there are shortest paths P_1 in H_1 and P_2 in H_2 joining v with H. Now $P_1 \cup P_2$ contains a walk W which has only its endvertices in H yet $\omega(E(W)) \supset \{-1,1\}$ which is absurd since H is two-sided. Thus $V(H_1) \cap V(H_2) = V(H)$. If $i_G(e) \cap V(H_1) - V(H_2) \neq \emptyset$ or $i_G(e) \cap V(H_2) - V(H_1) \neq \emptyset$, then clearly $e \in (E(H_2) - E(H_1)) \cup (E(H_1) - E(H_2)) = E$. If $i_G(e) \subset V(H)$ and $e \notin E(H)$, then $\omega(e) = 1$ or $\omega(e) = -1$. Hence $e \in E$. It follows that $H_1 \cap H_2 = H$. □

We shall call H_1 and H_2 the *sides* of H in (G,P,λ). If K is a subgraph of G which is not in H_1 or in H_2, then we shall say that H *separates* K. Similarly, we say that H *separates* the subgraphs K and L of G if $K \subset H_1$ and $L \subset H_2$ or vice versa.

LEMMA 3.3.5. *Let* $\phi:(G,P,\lambda)\to(H,Q,\mu)$ *be an isomorphism of the 3-polyhedra, let* $L=(u=v_o,e_1,\ldots,e_n,v_n=v)$ *be a path in* G *and let* $a,e\in E_u(G)-\{e_1\}$, $b,f\in E_v(G)-\{e_n\}$. *If* $(e,a,e_1)\subset P_u^{\delta}$ *and* $(e_n,b,f)\subset P_v^{\eta\delta\lambda(L)}$, *then* $(\phi(e),\phi(a),\phi(e_1))\subset Q_{\phi(u)}^{\varepsilon}$ *and* $(\phi(e_n),\phi(b),\phi(f))\subset Q^{\eta\varepsilon\mu(\phi(L))}$, *for some* $\varepsilon\in Z_2$.

Proof. We prove the proposition by induction on the length l of L. For $l=0$ and $l=1$ it follows immediately from the fact that ϕ is an isomorphism. Assume $l=k+1$ and assume the statement is true for all paths of length less than or equal to k. Let $L=(v_o,e_1,\ldots,v_{k+1})$ be a path of length $k+1$ as shown in Figure 3.3.1 and let $c\in E_{v_1}(G)-E(L)$. We apply the induction hypothesis to the subpaths $L(v_o,v_1)$ and the edges e,a,c,e_2, and to the subpath $L(v_1,v_{k+1})$ and the edges e_1,c,b, and f. This shows that the lemma holds for L,a,b. □

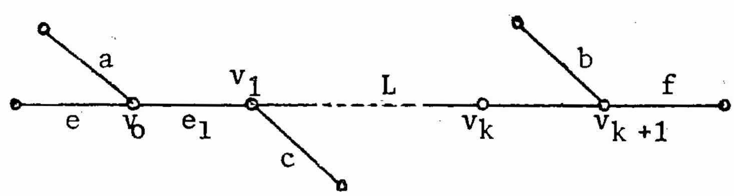


figure 3.3.1

THEOREM 3.3.6. *Let* $\phi:(G,P,\lambda)\to(H,Q,\mu)$ *be an isomorphism of the 3-polyhedra. If* C *is a two-sided cycle or 2-infinite path in* G, *then* $\phi(C)$ *is two-sided. Moreover, if* C' *and* C" *are the sides of* C *in* (G,P,λ), *then* $\phi(C')$ *and* $\phi(C'')$ *are the sides of* $\phi(C)$ *in* (H,Q,μ).

Proof. We shall provide a proof for the case that C is a cycle as the other case is similar. Let $C = (v_o, e_1, \ldots, e_n, v_n = v_o)$ and $\phi(C) = (\phi(v_o), \phi(e_1), \ldots, \phi(v_n) = \phi(v_o))$ and let ω and $\tilde{\omega}$ be the relations derived from C and $\phi(C)$. By Corollary 3.1.8 $\phi(C)$ is μ-trivial.

As a direct consequence of Lemma 3.3.5, if $e, f \in E(G) - E(C)$ and $x \in i_G(e) \cap V(C)$, $y \in i_G(f) \cap V(C)$, then $\omega_x(e) = \omega_y(f)$ if and only if $\tilde{\omega}_{\phi(x)}(\phi(e)) = \tilde{\omega}_{\phi(y)}(\phi(f))$. Thus it follows that $\phi(C)$ is two-sided if and only if C is and that $\phi(C')$ and $\phi(C'')$ are the sides of $\phi(C)$ in (H, Q, μ). □

The proof of the following Corollary is implied by the proof of Theorem 3.3.6.

COROLLARY 3.3.7. *Let* C *be a two-sided cycle or a two-sided 2-infinite path in the 3-polyhedron* (G,P) *and let* $\alpha \in \mathrm{Aut}(G,P) - \{\iota\}$ *satisfy*

(1) $\alpha(C) = C$,

(2) *If* $C = (\ldots, v_{i-1}, e_i, v_i, \ldots)$ *is a 2-infinite path, then* $\alpha(v_i) = v_{i+k}$ *for all* $i \in Z$ *and some* $k \in Z$, *and*

(3) *If* $C = (v_o, e_1, \ldots, e_{n-1}, v_n = v_o)$ *is a cycle, then* $\alpha(v_i) = v_{i+k}$ *for some* $k \in Z_n$ *and all* $i \in Z_n$.

Then α *is orientation reversing if and only if* α *interchanges the sides of* C.

THEOREM 3.3.8. *Let* L *, respectively* M*, be a two-sided cycle or 2-infinite path with sides* L' *and* L'' *, respectively* M' *and* M''*, in the 3-polyhedron* (G,P,λ)*. If* $L \subset M'$*, then* $M \subset L'$ *or* $M \subset L''$ *while if* $L \neq M$ *and* $L \subset M'$ *and* $M \subset L'$ *, then for any* $v \in V(L) \cap V(M)$ *we have* $E_v(M'') \subset E_v(L')$ *and* $M'' \subset L'$.

Proof. We shall prove the first statement by a contrapositive argument. Let $L = (\ldots,v_{i-1},e_i,v_i,\ldots)$, $M = (\ldots,u_{i-1},a_i,u_i,\ldots)$ and assume $M \not\subset L'$ and $M \not\subset L''$. Without loss of generality we may assume L is labeled so that it contains a subpath $M(u_o,u_j)$ of M such that the edges a_{-1} and a_{j+1} belong to different sides of L but not to L. We may also assume that $u_i = v_i$ for $0 \le i \le j$ so that the edges e_{-1} and e_{j+1} of L are not contained in M. Let ω and $\tilde{\omega}$ be the relations associated with the labelings of L and M, respectively. In view of Proposition 3.3.3 we see that $\omega_{u_o}(a_{-1}) \neq \omega_{u_j}(a_{j+1})$ if and only if $\tilde{\omega}_{u_o}(e_{-1}) \neq \tilde{\omega}_{u_j}(e_{j+1})$. It follows that e_{-1} and e_{j+1} are on different sides of M, that is, $L \not\subset M'$ and $L \not\subset M''$.

To prove the second assertion we assume $L \subset M'$, $M \subset L'$ and $M \neq L$. Let $v \in V(L) \cap V(M)$. If $E_v(M'') = E_v(M)$, then obviously $E_v(M'') \subset E_v(L')$. If $e \in E_v(M'') - E_v(M)$, then e and L are on distinct sides of M. Hence if $E_v(L) \neq E_v(M)$ we immediately see that $E_v(M'') \subset E_v(L')$. On the other hand, if $E_v(L) = E_v(M)$, then we choose a subpath $M(u,v) \subset L$ such that there is some edge $a \in E_u(L) - E_u(M)$. Now e and a are on different sides of M and thus it follows as above that $E_u(M)$ and e and therefore e and M are in L'. It follows that $E_v(M'') \subset E_v(L')$ for all $v \in V(L) \cap V(M)$

and therefore $M'' \subset L'$. □

Let L, M, L', M' be as in Theorem 3.3.8 and assume that $L \subset M'$ and $M \subset L'$. If $K \subset L' \cap M'$, then we say K lies *between* L and M.

THEOREM 3.3.9. *Let* H *be a finite 2-connected subgraph of the 3-polyhedron* (G,P,λ) *and assume that every cycle in* H *is two-sided. Moreover, assume* $K \subset G$, $K \not\subset H$, *and* K *is not separated by any cycle of* H. *Then there is a cycle* $C \subset H$ *such that* K *and* H *are on different sides of* C.

Proof. We may assume $K \neq \emptyset$ since $K \not\subset H$. If C is a cycle in H let F_C denote the side of C containing K. We choose a cycle C in H minimizing $k = |(E(F_C) - E(C)) \cap E(H)|$ and claim $k = 0$.
Assume $k > 0$. Because H is 2-connected there is a path R in $F_C \cap H$ with $u,v \in V(C)$ such that R contains no vertices of C other than u and v. Now u and v partition C into two paths which form together with R the two cycles C_1 and C_2. Let F_i denote the side of C_i which doesn't contain C. Then $F_1 \cup F_2 = F_C$ and either F_1 or F_2 contains K. Moreover, $|(E(F_i) - E(C_i)) \cap E(H)| < k$, which contradicts the choice of k. Thus $k = 0$ and both $F_C \cap H = C$ and $K \subset F_C$ hold. □

The following three statements are concerned with the relation between two-sided cycles or paths and boundaries. Their proofs are straightforward and therefore omitted.

PROPOSITION 3.3.10. *If* H *is a boundary of the* 3-*polyhedron* (G,P,λ), *then* H *is two-sided and the sides of* H *in* (G,P,λ) *are* H *and* G.

COROLLARY 3.3.11. *If* B *is a boundary of the* 3-*polyhedron* (G,P,λ) *and* H *is a two-sided cycle or 2-infinite path, then* B *is contained in one side of* H.

COROLLARY 3.3.12. *If* H *is a two-sided cycle or 2-infinite path of the* 3-*polyhedron* (G,P,λ) *and* $e \in E(H)$, *then the two distinct boundaries containing* e *lie an distinct sides of* H.

If every cycle in the 3-polyhedron (G,P,λ) is two-sided then we shall call the polyhedron *planar*. We shall justify the use of the term planar in Theorem 5.1.6. The following theorem gives a criterion of "one-sidedness" for 2-infinite paths in planar 3-polyhedra which is close to but more readily applicable than the definition.

THEOREM 3.3.13. *The 2-infinite path* R *of the planar* 3-*polyhedron* (G,P,λ) *is one-sided if and only if there is a cycle* $C \subset G$ *and* 1-*infinite subpaths* R_1 *and* R_2 *of* R *which are on different sides of* C.

Proof. To prove the "only if" part assume R is one-sided. Then there is a path $Q = (u_o,\ldots,u_m)$ with $u_o, u_m \in V(R)$, $u_i \notin V(R)$ if $1 \le i \le m-1$ and $\omega(Q) \supset \{-1,1\}$ where ω is a relation associated with R. If C is the cycle $Q \cup R(u_o,u_m)$ and $\tilde{\omega}$ is associated with C, then $\tilde{\omega}(R) \supset \{-1,1\}$ by Proposition 3.3.3, and therefore the two 1-infinite subpaths of $R - R(u_o,u_m)$ are on different sides of C.

For the "if" part assume that R has 1-infinite subpaths on different sides of the cycle C. We choose $u,v \in V(R) \cap V(C)$ such that $V(R) \cap V(C) \subset V(R(u,v))$ and denote the 1-infinite subpaths of R on different sides of C and ending in u, respectively v, by R_u, respectively R_v. The graph $H = C \cup R(u,v)$ clearly is 2-connected. By Theorem 3.3.9 there is a cycle C' in H such that R_u, respectively $R_v \cup H$, is on the side F_1, respectively F_2, of C'. If $u = v$, then R_u and R_v are on distinct sides of C' so that $\omega(C') \supset \{-1,1\}$ and R is one-sided (see Figure 3.3.2).

If $u \neq v$, consider the subgraph K of C' obtained by omitting all edges of $E = E(R) \cap E(C')$ and all vertices incident with two edges of E. Let $L_1,\ldots,L_k$ be the maximal subpaths of K containing exactly two vertices of R, let this be the cyclic order in which they are encountered on C' and let $u \in V(L_k)$. If $u \in V(L_1)$, then we have the situation depicted in Figure 3.3.3. Since $E_u(R)$ is not on one side of C', $\delta \in \{-1,1\} \cap \omega_u(L_1)$ implies that $-\delta \in \omega_u(L_k)$. If $u \notin V(L_1)$ and L_1 is separated from u by a subpath of $R \cap C'$ (as depicted in Figure 3.3.4) and if $\delta \in \{-1,1\} \cap \omega_W(L_1)$, then $-\delta \in \omega_u(L_k)$ because R_u and R_w are on different sides of C'. Thus we conclude that $\delta \in \{-1,1\} \cap \omega(L_1)$ implies $-\delta \in \omega(L_k)$. Now let $1 \leq i \leq k-1$. If $x \in V(L_i) \cap V(L_{i+1}) - \{u\}$ as depicted in Figure 3.3.5, then $\omega_x(L_i) = \omega_x(L_{i+1})$ because $E_x(R)$ is in one side of C'. If L_i is separated from L_{i+1} by a subpath of R as in Figure 3.3.6, then $\omega_x(L_i) = \omega_y(L_{i+1})$ since $E_x(R)$, $R(x,y)$ and $E_y(R)$ are on the same side of C'.

We conclude that $\omega(L_i) \not\supset \{-1,1\}$ implies $\omega(L_i) - \{0\} \subset \omega(L_{i+1})$ for $1 \le i \le k-1$, and $\delta \in \{-1,1\} \cap \omega(L_1)$ implies $-\delta \in \omega(L_k)$. Therefore, $\omega(L_i) \supset \{-1,1\}$ for some i, that is, R is one-sided by definition. □

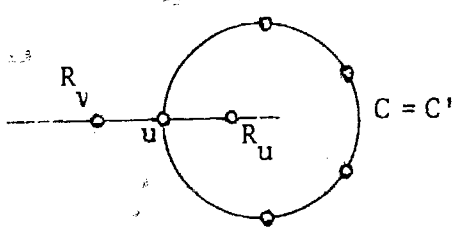


figure 3.3.2

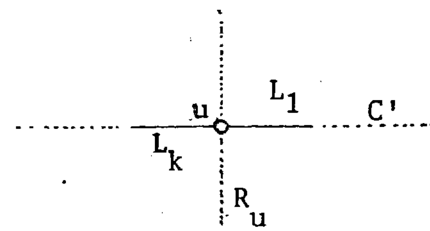


figure 3.3.3

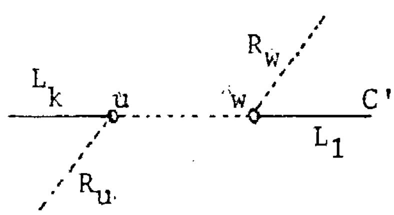


figure 3.3.4

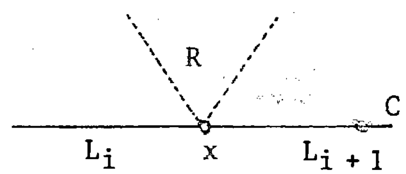


figure 3.3.5

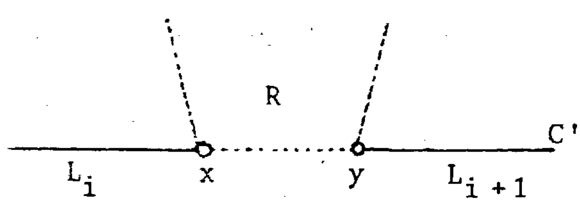


figure 3.3.6

We say that the 1-infinite paths Q_1 and Q_2 of a graph G are *cofinal* if for every finite subset $U \subset V(G)$ some component of $G - U$ contains 1-infinite subpaths of Q_1 and Q_2. The 2-infinite paths Q_1 and Q_2 are cofinal in G if there are disjoint 1-infinite subpaths Q_i' and Q_i'' of Q_i such that the pairs Q_1', Q_2' and Q_1'', Q_2'' are cofinal in G.

COROLLARY 3.3.14. *If* (G,P,λ) *is a planar* 3-*polyhedron each of whose cycles has a finite side, then every 2-infinite path is two-sided.*

COROLLARY 3.3.15. *If* (G,P,λ) *is a planar* 3-*polyhedron and* Q *is a 2-infinite path in* G *which has disjoint, cofinal 1-infinite subpaths* Q_1 *and* Q_2, *then* Q *is two-sided.*

COROLLARY 3.3.16. *Given two cofinal 2-infinite paths* H_1 *and* H_2 *in the planar* 3-*polyhedron* (G,P,λ), *then* H_1 *is one-sided if and only if* H_2 *is one-sided.*

§4. THE BARYCENTRIC SUBDIVISION OF A 3-POLYHEDRON (G,P,λ).

Let us be given a 3-polyhedron (G,P,λ). In Chapter 2, §3 we introduced the notion of barycentric subdivision of a 3-polyhedron (S,G). We shall now discuss the corresponding combinatorial concept.

The *first order* (or simply) *barycentric subdivision* $(\tilde{G},\tilde{P},\tilde{\lambda})$ of (G,P,λ) is a 3-polyhedron derived from (G,P,λ) as follows.

(1) Every edge of G is subdivided by a new vertex into two edges.

(2) For each boundary B of (G,P,λ) there is a new vertex v_B and six new edges joining v_B to the six vertices $v_1,\ldots,v_6$ of the subdivided boundary B' (as illustrated in Figure 3.4.1). We shall denote the resulting graph $\tilde{G}$.

(3) The rotation system $\tilde{P} = \{\tilde{P}_v \mid v \in V(\tilde{G})\}$ for $\tilde{G}$ is such that $P_v \subset \tilde{P}_v$ for every $v \in E(G)$; $\tilde{P}_v^{\delta}(a_i) = a_{i+1}$ for $v = v_B$, $1 \le i \le 6$ and some $\delta \in Z_2$; and finally $\tilde{P}_{v_i}^{\delta}(b_{i-1}) = a_i$, $\tilde{P}_{v_i}^{\delta}(a_i) = b_i$ for $1 \le i \le 6$ and suitable $\delta \in Z_2$, where a_i, b_i are as illustrated in Figure 3.4.1.

(4) We choose $\tilde{\lambda} : E(\tilde{G}) \to Z_2$ in order to obtain the boundaries (a_i, b_i, a_{i+1}) for $1 \le i \le 6$.

It is easily seen that $\tilde{\lambda}(\tilde{e}) = \lambda(e)$ for every $e \in E(G)$, where $\tilde{e}$ is the path in $\tilde{G}$ resulting from subdividing e. Thus, a cycle or 2-infinite path C in G is two-sided in (G,P,λ) if and only if the subdivided cycle or path $\tilde{C}$ is two-sided in $(\tilde{G},\tilde{P},\tilde{\lambda})$. Moreover, it is easily seen that we get the sides $\tilde{C}^{(i)}$ of $\tilde{C}$ from the sides $C^{(i)}$ of C by subdividing each boundary $B \subset C^{(i)}$ as illustrated in Figure 3.4.1. It is easy to see that there is a canonical monomorphism $\Sigma:\mathrm{Aut}(G,P,\lambda) \to \mathrm{Aut}(\tilde{G},\tilde{P},\tilde{\lambda})$ assigning to each automorphism $\gamma \in \mathrm{Aut}(G,P,\lambda)$ the automorphism $\Sigma(\gamma) = \tilde{\gamma} \in \mathrm{Aut}(\tilde{G},\tilde{P},\tilde{\lambda})$ which is defined by $\tilde{\gamma}(v) = \gamma(v)$ for all $v \in V(G)$ and $\tilde{\gamma}(v_B) = v_{B'}$, if $\gamma(B) = B'$, where B and B' are any boundaries of (G,P,λ). Let us be given a discontinuous homeomorphism group of the 3-polyhedron (S,G) with the boundary tour scheme (P,λ), let $(S,\tilde{G})$ be the barycentric subdivision constructed in Chapter 2, §3 and let $(\tilde{P},\tilde{\lambda})$ be as constructed above. Suppose Φ and $\tilde{\Phi}$ are the canonical monomorphisms from Γ into $\mathrm{Aut}(G,P,\lambda)$, respectively, $\mathrm{Aut}(\tilde{G},\tilde{P},\tilde{\lambda})$ then $\tilde{\Phi} = \Sigma \circ \Phi$.

The n-*th order* barycentric subdivision of the 3-polyhedron (G,P,λ) is the 3-polyhedron $(\tilde{G},\tilde{P},\tilde{\lambda})$ obtained by n successive applications of a (first order) barycentric subdivision to (G,P,λ).

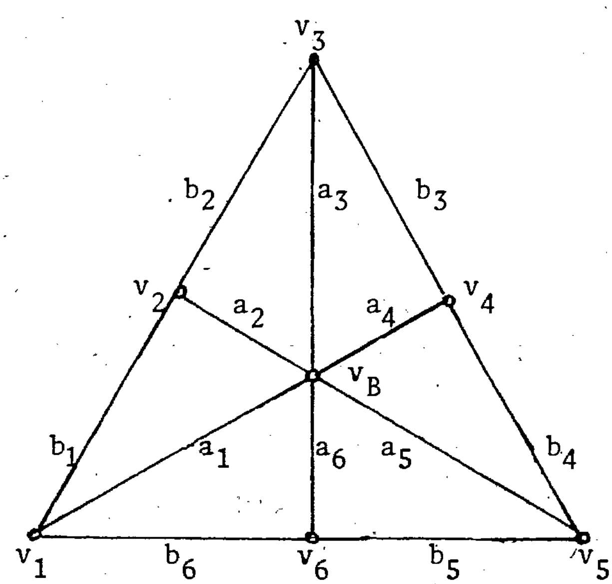


figure 3.4.1

CHAPTER 4. AUTOMORPHISMS OF PLANAR 3-POLYHEDRA

There are simple examples of infinite trees which show that in general the automorphism group of a locally finite connected graph G may be uncountable. In fact, Halin [12, Theorem 6] shows that Aut(G) is uncountable if and only if for every finite $F \subset G$ there is an automorphism $\phi \in \mathrm{Aut}(G) - \{\iota\}$ such that $\phi(v) = v$ for all $v \in V(F)$. Then, in particular, for each finite subgraph there are infinitely many automorphisms fixing it. Yet this cannot occur in the automorphism group of a 3-polyhedron, in fact, of an arbitrary polyhedron, due to the structure imposed by the boundary tour scheme. It follows from Proposition 4.1.2 that any vertex can be fixed by only a finite number of elements in Aut(G,P,λ). Thus Aut(G,P,λ) is countable since V(G) is countable.

In section 1 of this chapter we discuss the fixed vertices and fixed edges of Aut(G,P,λ). Among other things we show that the subgraph induced by the fixed edges of an automorphism is regular of degree 2 and that an orientation preserving automorphism of finite order of a planar 3-polyhedron contains at most two fixed vertices.

In section 2 we discuss automorphisms of infinite order and orientation preserving automorphisms of finite order of a planar 3-polyhedron (G,P). We distinguish exactly two types of automorphisms of infinite order. The first type could appropriately be described as translations and reflected translations and the second type as contractions and reflected contractions. An object α of type one is distinguished from an object β of type two by the existence of an α-invariant, 2-infinite and two-sided path. We shall show that if α is of type one there is a countable disjoint collection of α-invariant two-sided paths (see Theorem 4.2.17). On the other hand, if β is of type 2, then there is a countable collection of disjoint and "concentric" cycles on which β acts transitively (see Theorem 4.2.9). The finite order orientation preserving automorphisms can be appropriately named rotations (see Theorems 4.1.14 and 4.2.20).

In Section 3 we restrict our attention to automorphisms of a special kind of planar 3-polyhedra (G,P), namely infinite polyhedra, in which every cycle has one finite side. Such polyhedra can also be considered as simplicial complexes induced by their boundaries, and are therefore "Ebene Netze", as defined by Zieschang (see [28, p.55]). The algebraic structure of any subgroup Γ of Aut(G,P) with compact fundamental domain (that is, groups for which there is a finite maximal collection of boundaries of (G,P) which are not Γ-equivalent) can be determined by first obtaining a canonical fundamental domain (see Theorem 4.5.4) and then using it to derive a set of generators and defining relations for Γ (see Theorem 4.5.5). The canonical fundamental domains also are essential for the proof of Theorem 5.5.1.

§1. FIXED VERTICES AND EDGES OF A SUBGROUP Γ OF AUT(G,P,λ)

In the following we shall make a few simple observations about the set of fixed vertices and edges of a subgroup Γ of the automorphism group of a 3-polyhedron (G,P,λ). If H is a graph and Δ is a subgroup of Aut(H), respectively $\gamma \in$ Aut(H), then we shall call $x \in V(H) \cup E(H)$ a *fixed vertex* or a *fixed edge* of Δ, respectively γ, if $\delta(x) = x$ for some $\delta \in \Delta - \{\iota\}$, respectively $\gamma(x) = x$.

Assumptions 4.1.1. Throughout this section we shall assume that (G,P,λ) is a 3-polyhedron and Γ is a subgroup of Aut(G,P,λ) meeting the following conditions:

(1) If $\gamma \in \Gamma$ fixes the edge e, then it also fixes the vertices incident to e; and

(2) If $\gamma \in \Gamma$ fixes a boundary B of (G,P,λ), then $\gamma = \iota$.

These assumptions are made in view of Theorem 2.4.1 and Corollary 4.1.3.

PROPOSITION 4.1.2. *Let* $\gamma \in$ Aut(G,P,λ) *and let* $\gamma(v) = v$ *and* $\gamma(e) = e$ *for some* $v \in V(G)$ *and all* $e \in E_v(G)$. Then $\gamma = \iota$.

Proof. In order to show that $\gamma(u) = u$ for all $u \in V(G)$ we first prove that if u and w are adjacent, $\gamma(u) = u$ and $\gamma(e) = e$ for all $e \in E_u(G)$, then $\gamma(w) = w$ and $\gamma(e) = e$ for all $e \in E_w(G)$. To see this let $i_G(e) = \{u,w\}$ and let $\{e,e_1,e_2\}$ be the edges of a boundary so that $u \in i_G(e_1)$. Now γ fixes u, e_1 and e and therefore also e_2. Thus γ fixes w and the edges $e_2, e \in E_w(G)$. It follows that $\gamma P_w \gamma^{-1} = P_w$, and therefore $\gamma(e) = e$ for all $e \in E_w(G)$. The conclusion follows by induction on the distance from v. □

COROLLARY 4.1.3. *If* $\gamma \in \mathrm{Aut}(G,P,\lambda)$ *fixes* v *and* $e \in E_v(G)$ *and if* $\gamma P_v \gamma^{-1} = P_v$, *then* $\gamma = \iota$.

COROLLARY 4.1.4. *If* $\gamma \in \mathrm{Aut}(G,P,\lambda)$ *fixes* $e \in E(G)$ *and* $u \in i_G(e)$, *then* $\gamma^2 = \iota$.

Proof. We have $\gamma P_u \gamma^{-1} = P_u^{\varepsilon}$, where $u \in i_G(e)$ and $\varepsilon \in Z_2$ so that $\gamma^2 P_u \gamma^{-2} = P_u$. By Corollary 4.1.3, $\gamma^2 = \iota$. □

COROLLARY 4.1.5. *If* $\alpha \in \mathrm{Aut}(G,P,\lambda)$ *has infinite order, then* $\langle\alpha\rangle$ *has no fixed vertex or edge.*

For $\gamma \in \Gamma - \{\iota\}$ we shall define S_γ as the subgraph of G induced by all edges which are fixed by γ.

PROPOSITION 4.1.6. *If* $S_\gamma \neq \emptyset$, *then it is a regular subgraph of* G *of degree* 2.

Proof. Let $e_o \in E(S_\gamma)$ and $u \in i_G(e_o)$. Then $\gamma P_u \gamma^{-1} = P_u^{-1}$ since $\gamma \neq \iota$. Hence, if $P_u = (e_o, e_1, \ldots, e_{n-1})$, then $\gamma(e_i) = e_{n-i}$ for $1 \leq i \leq n-1$. By Assumption 4.1.1.(2) γ does not fix any boundary of (G,P,λ). Therefore n is even and $\gamma(e_{n/2}) = e_{n/2}$. □

PROPOSITION 4.1.7. *If* S_{γ_1} *and* S_{γ_2} *have a common edge, then* $\gamma_1 = \gamma_2$.

Proof. Let $e \in E(S_{\gamma_1}) \cap E(S_{\gamma_2})$ and $v \in i_G(e)$. Then $\gamma_i P_v \gamma_i^{-1} = P_v^{-1}$ so that $\gamma_1\gamma_2 P_v (\gamma_1\gamma_2)^{-1} = P_v$ and $\gamma_1\gamma_2(e) = e$. By Corollary 4.1.3, $\gamma_1\gamma_2 = \iota$ and thus $\gamma_1 = \gamma_2$ by Corollary 4.1.4. □

COROLLARY 4.1.8. *If* $\gamma, \delta \in \Gamma - \{\iota\}$, $\gamma \neq \delta$, $E_v(S_\gamma) = \{b_1, b_2\}$ and $E_v(S_\delta) = \{a_1, a_2\}$, then $(a_1, b_1, a_2, b_2) \subset P_v^{\varepsilon}$, where $\varepsilon \in Z_2$.

PROPOSITION 4.1.9. *If* C *is either a two-sided cycle or a 2-infinite two-sided path and* C *is contained in* S_γ, *then* $S_\gamma = C$.

Proof. Let $v \in V(C)$. Then $\gamma P_v \gamma^{-1} = P_v^{-1}$ and therefore γ changes the sides of C. Thus $S_\gamma = C$ since every vertex in S_γ is fixed by γ. □

For the rest of this section let (G,P,λ) be orientable and $\lambda(e) = 1$ for all $e \in E(G)$. Let V_Δ, respectively V_δ, denote the set of fixed vertices of the subgroup $\Delta \subset \mathrm{Aut}(G)$, respectively $\delta \in \mathrm{Aut}(G) - \{\iota\}$.

LEMMA 4.1.10. *Let* $\gamma \in \Gamma - \{\iota\}$ *and* $v \in V_\gamma$. *If* γ *is orientation reversing, then* $v \in V(S_\gamma)$.

Proof. Let $P_v = (a_0, \ldots a_{n-1})$. We have $\gamma P_v \gamma^{-1} = P_v^{-1}$, since γ is orientation reversing. Thus if $\gamma(a_0) = a_k$, then $\gamma(a_i) = a_{k-i}$

for $0 \le i \le n-1$, where the indices are calculated modulo n. No boundary is fixed by γ so that for some j we have $\gamma(a_j) = a_j$, that is $v \in V(S_\gamma)$. □

COROLLARY 4.1.11. *Let* $\gamma \in \Gamma - \{\iota\}$ *and* $V_\gamma \ne \emptyset$. *Then* γ *is orientation reversing if and only if* $S_\gamma \ne \emptyset$.

Proof. The "only if" part follows from Lemma 4.1.10 and the "if" part follows from Corollary 4.1.3 and Proposition 4.1.6. □

COROLLARY 4.1.12. *Let* $\gamma \in \Gamma - \{\iota\}$. *If* $S_\gamma \ne \emptyset$, *then* $V(S_\gamma) = V_{<\gamma>}$.

Proof. By Corollary 4.1.4 γ has order 2. Hence $V_{<\gamma>} = V_\gamma$. Clearly we have $V(S_\gamma) \subseteq V_{<\gamma>}$ but, on the other hand by Lemma 4.1.10 we have $V_{<\gamma>} \subset V(S_\gamma)$. □

The following lemma shall be used frequently throughout the rest of Chapter 4. For Lemma 4.1.13 we shall make the following assumptions. Let H_i, $1 \le i \le 2$, be a cycle or a singleton vertex of the planar 3-polyhedron (G,P) and assume that $V(H_1) \cap V(H_2) = \emptyset$. Let H be the subgraph of G between H_1 and H_2 which is defined to be $H_1' \cap H_2'$, where H_i' is the side of H_i containing H_{3-i}, if H_1 and H_2 are cycles, or the side of H_i containing H_{3-i} if H_i is a cycle and H_{3-i} a vertex, or G if H_1 and H_2 are both vertices. Let P be a shortest path joining any vertex $v_1 \in V(H_1)$ with any vertex $v_2 \in V(H_2)$.

LEMMA 4.1.13. *Let* H_1, H_2, H *and* P *be as descibed above. Let* $\alpha \in \Gamma - \{\iota\}$ *be an orientation preserving automorphism of finite order such that* $\alpha(H_i) = H_i$, *for* $1 \le i \le 2$. *Then* $H_3 = H - (H_1 \cup H_2 \cup \{\alpha_i(P) \mid 0 \le i \le |\alpha| - 1\})$ *is not connected and has no* α^i *invariant component, for* $1 \le i \le |\alpha| - 1$, *where* $|\alpha|$ *is the order of* α. $\langle\alpha\rangle$ *acts transitive on the set of components of* H_3.

Proof. Suppose the statement is not true. Then we choose H_1 and H_2 to be vertex disjoint and of minimum distance apart so that H_i, $1 \le i \le 2$, is fixed by some orientation preserving automorphism $\alpha \in \Gamma - \{\iota\}$ but such that there is some α^i-invariant component in H_3 with $1 \le i \le |\alpha| - 1$. In view of Assumptions 4.1.1 there is no fixed edge of $\langle\alpha\rangle$ in G.

We claim that if $x \in V(H_1 \cup H_2 \cup P)$ is a fixed vertex of $\langle\alpha\rangle$, then $x \in V(H_1 \cup H_2)$. Moreover, if $x \in V(H_i)$ then H_i consists of x alone. To prove this claim suppose $x \in V(P) - V(H_1 \cup H_2)$ is a fixed vertex of $\alpha^n \ne \iota$. Without loss of generality we may assume there is no other fixed vertex of $\langle\alpha\rangle$ on the path $P_1 = P_{xy}$ joining x with $y \in V(H_2)$. Now x and H_2 are of shorter distance than H_1 and H_2 and are both fixed by α^n. Let us define H^* and H_3^* for $H_1^* = \{x\}$, H_2, P_1, α^n similar to H and H_3 for H_1, H_2, P, α. It follows that $H^* \supset H$, H_3^* is not connected and no component of H_3^* ~~is~~ α^{ni}-invariant, for $1 \le i \le |\alpha^n| - 1$. But this contradicts the fact that H_1 is α^{ni}-invariant and a subgraph of H_3^*.

Now assume that $x \in V(H_i)$ is a fixed vertex of α and assume H_i is a cycle. Since α^n doesn't interchange the sides of H_i it follows that α^n is the identity which is a contradiction. This proves the claim.

Now consider the α-invariant subgraph K generated by H_1, H_2 and P. It is easy to see that K is 2-connected. Let e be an edge of H_3. By Theorem 3.3.9 there is a cycle C in K separating K from e and it is easy to see that $C \neq H_1$ and $C \neq H_2$. Let C'' be the side of C containing e and C' the side containing K. Thus $C = H_1(u,\alpha^i(u)) \cup H_2(v,\alpha^i(v)) \cup P \cup \alpha^i(P)$ for some $\alpha^i \neq \iota$, where u and v are the endpoints of P on H_1 and H_2. Since α fixes H_j and is orientation preserving it follows that $\alpha^i(C'') \subset C'$ for all $\alpha^i \neq \iota$. It is easy to see that α acts transitive on the set of components of H_3. □

THEOREM 4.1.14. *Let* (G,P) *be planar and let* $\gamma \in \Gamma - \{\iota\}$ *be of finite order and orientation preserving. Then* $V_\gamma = V_{<\gamma>}$ *and* $|V_\gamma| \leq 2$.

Proof. As a first step, we shall prove that $V_\gamma = V_{<\gamma>}$ and $|V_\gamma| \leq 2$ in case $V_\gamma \neq \emptyset$. It is clearly the case that $V_\gamma \subset V_{<\gamma>}$. Now suppose $u \in V_\gamma$ and v is a vertex in $V_{<\gamma>} - \{u\}$ nearest to u. Let Q be a shortest path joining u and v. It follows that $V(\gamma^{ki}(Q)) \cap V(\gamma^{kj}(Q)) = \{u,v\}$ and $E(\gamma^{ki}(Q)) \cap E(\gamma^{kj}(Q)) = \emptyset$ whenever $\gamma^{ki} \neq \gamma^{kj}$ and $v \in V_{\gamma^k}$. In order to show that $v \in V_\gamma$ consider the graph $H_k = \cup\{\gamma^{kj}(Q) \mid j \in Z\}$. It is 2-connected and by Lemma 4.1.13 none of the components of $G - H_k$ is γ^k-invariant. We note that $\gamma(v) \in V_{\gamma^k} - \{u\} \subset V_{<\gamma>} - \{u\}$. Hence $\gamma(v) \in V(H_k)$ and by choice of v, $\gamma(v) = v$, that is, $v \in V_\gamma$.
By Lemma 4.1.13 none of the components of $G - H_1$ is γ^i-invariant, where $\gamma^i \neq \iota$ and it follows that $V(G - H_1) \cap V_{<\gamma>} = \emptyset$. Thus $V_{<\gamma>} = \{u,v\} = V_\gamma$. This concludes the first step.

In the second step we shall show that $V_{<\gamma>} = \emptyset$ if $V_\gamma = \emptyset$. Assume $V_\gamma = \emptyset$ but $V_{<\gamma>} \neq \emptyset$. Then let i be the least positive integer so that $v \in V_{\gamma^i}$ for some v. Then $i \geq 2$ since $V_\gamma = \emptyset$. From the first part it follows that $i \leq 2$ since $v, \gamma(v), \ldots, \gamma^{i-1}(v)$ are distinct members of V_{γ^i}. Hence $i = 2$ and $\{v, \gamma(v)\} = V_{\gamma^2} = V_{<\gamma^2>} \subset V_{<\gamma>}$. We claim that $V_{<\gamma^2>} = V_{<\gamma>}$. Suppose $W = V_{<\gamma>} - V_{<\gamma^2>} \neq \emptyset$. Clearly $<\gamma>$ has even order since $<\gamma^2> \neq <\gamma>$. Then there is a smallest odd positive integer j, $3 \leq j \leq |\gamma| - 1$, where $|\gamma|$ is the order of γ, so that $\gamma^j(u) = u$ for $u \in V_{\gamma^j}$. We have $u, \gamma(u), \ldots, \gamma^{j-1}(u)$ are fixed by γ^j. In view of the first part there are distinct integers k, l, where $0 \leq k < l \leq j - 1$ so that $\gamma^k(u) = \gamma^l(u)$, that is, $\gamma^{l-k}(u) = u$ and $1 \leq l - k \leq j - 1$. Now $l - k$ is odd as $u \notin V_{<\gamma^2>}$ but $l - k < j$, a contradiction to the choice of j. Thus we have proved that $V_{<\gamma>} = V_{<\gamma^2>}$.

Since G is 3-connected, $K = G - V_{<\gamma>}$ is connected (see Proposition 3.2.3). By Theorem 4.2.12 K contains a γ-invariant cycle C which contains no fixed vertex or fixed edge of $<\gamma>$. We shall now show that γ interchanges the sides of C and therefore is orientation reversing by Corollary 3.3.7 which contradicts our original assumption about γ and thus implies that $V_{<\gamma>} = \emptyset$. Suppose v and $\gamma(v)$ are on the same side, say C', of C. Without loss of generality we may assume v is nearer to C and join v to C by a shortest path Q.
Then $V(\gamma^{2i}(Q)) \cap V(\gamma^{2j}(Q)) = \{v\}$, $E(\gamma^{2i}(Q)) \cap E(\gamma^{2j}(Q)) = \emptyset$ in case $\gamma^{2i} \neq \gamma^{2j}$. By Lemma 4.1.13 no component of $H = C' - (\cup\{\gamma^{2j}(Q) \mid j \in Z\} \cup C)$ is γ^2-invariant. Hence $\gamma(v) \notin V(H)$. It follows that $\gamma(v) = v$, that is, $v \in V_\gamma$ which is a contradiction. □

COROLLARY 4.1.15. *Let* (G,P) *be planar and let* $\gamma,\delta \in \Gamma - \{\iota\}$, *where* $\delta \neq \gamma$. *Then* $|V(S_\gamma) \cap V(S_\delta)| \leq 2$.

Proof. Let $U = V(S_\gamma) \cap V(S_\delta)$. If $U \neq \emptyset$, then $U \subset V_{\gamma\delta}$, $\gamma\delta \neq \iota$ and $\gamma\delta$ is of finite order. Moreover, since $\gamma\delta$ is orientation preserving it follows from Theorem 4.1.14 that $|U| \leq |V_{\gamma\delta}| \leq 2$. □

§2. AUTOMORPHISMS OF INFINITE ORDER

The special case of the following Lemma when α has infinite order is due to Halin [12,p.268].

LEMMA 4.2.1. *Let* G *be a connected and locally finite graph and assume that* $\langle\alpha\rangle$, $\alpha \in \mathrm{Aut}(G)$, *has no fixed vertex or fixed edge in* G. *Then there are* n, $n \geq 1$, *pairwise disjoint graphs* $H_o, \ldots, H_{n-1}$ *so that*

(1) $\alpha(H_i) = H_{i+1}$ *for all* $i \in Z$ *(where the indices are chosen by using proper modulo arithmetic) and*

(2) H_i *is a cycle if* α *has finite order and a 2-infinite path otherwise.*

Proof. We select a vertex $u \in V(G)$ which minimizes $f(x) = \min\{d_G(\alpha^i(x),\alpha^j(x)) \mid \alpha^i \neq \alpha^j\}$. We know $f(u) > 0$ as $\langle\alpha\rangle$ has no fixed vertex. Since α acts transitively on $\{\alpha^i(u) \mid i \in Z\}$, for some $n > 0$ there is a path L of length f(u) joining u and $\alpha^n(u) = v$. We claim that if $\alpha^i \neq \alpha^j$, then $\alpha^i(L)$ and $\alpha^j(L)$ have no common edge and meet each other at most in terminal vertices. To prove this we suppose $x \in V(\alpha^i(L)) \cap V(\alpha^j(L)) - \alpha^i\{u,v\}$. Then $\{x,\alpha^{i-j}(x)\} \subset V(\alpha^i(L)) - \alpha^i\{u,v\}$. Since $\langle\alpha\rangle$ has no fixed vertex, $x \neq \alpha^{i-j}(x)$ so that $f(x) < f(u)$ which is a contradiction. If $\alpha^i(L)$ and $\alpha^j(L)$ had a common edge e, then they would either have a common non terminal vertex or $\langle\alpha\rangle$ would have a fixed edge e both of which are impossible.

The terminal vertices of $\alpha^k(L)$ are $\alpha^k(u)$ and $\alpha^{k+n}(u)$. Hence, if $\alpha^l \neq \alpha^k$ and $V(\alpha^k(L)) \cap V(\alpha^l(L)) \neq \emptyset$, then $\alpha^l = \alpha^{k+n}$ or $\alpha^l = \alpha^{k-n}$, in view of the fact that $\langle\alpha\rangle$ has no fixed vertex. It follows that $H_i = \cup\{\alpha^{i+nl}(L) \mid l \in Z\}$, $0 \leq i < n$, are n pairwise disjoint cycles if α has finite order or 2-infinite paths if α has infinite order. Obviously $\alpha(H_i) = H_{i+1}$. □

COROLLARY 4.2.2. *Let* G *be a* 2*-infinite path and* $\alpha \in \mathrm{Aut}(G)$ *be of infinite order. Then*

(1) *there is a positive integer* d *such that* $d = f(x) = d_G(x,\alpha(x))$ *for all* $x \in V(G)$ *(where* f(x) *is as defined at the beginning of the proof of Lemma* 4.2.1 *) and*

(2) $G = \cup\{G(\alpha^l(x),\alpha^{l+1}(x)) \mid l \in Z\}$ *for all* $x \in V(G)$.

COROLLARY 4.2.3. *Let* G *and* α *be as in Lemma* 4.2.1 *If* α *has order 2, then there is an* α*-invariant cycle in* G.

Proof. This follows from the proof of Lemma 4.2.1 in the way each H_i is defined. □

THEOREM 4.2.4. *Let* (G,P) *be a planar* 3*-polyhedron, let* $\alpha \in \mathrm{Aut}(G,P)$ *be of infinite order and let* K *be an* α*-invariant connected subgraph of* G. *If* $H_0,\ldots,H_{n-1}$ *are* n *pairwise disjoint, two-sided, 2-infinite paths in* K *such that* $\alpha(H_i) = H_{i+1}$ *for* $0 \leq i \leq n-1$, *where* $H_0 = H_n$, *then* $n \leq 2$.

Proof. There is some H_k with the property that one of its sides in (G,P) contains $H_1 \cup \cdots \cup H_{n-1}$ and since α acts transitively on the collection $H_o, \ldots, H_{n-1}$, each H_i has this property. Assume $n \geq 3$. We choose H_k and H_j such that $d_K(H_k, H_j)$ is minimum. Let $H_m \neq H_k$ or H_j. As H_i is α^n-invariant for $0 \leq i \leq n-1$ and H_m doesn't meet any shortest path in K between H_k and H_j it follows from Lemma 4.2.12 that H_m is not between H_k and H_j. Thus either H_m and H_j are on different sides of H_k or H_m and H_k are on different sides of H_j both of which are contradictions. □

LEMMA 4.2.5. *Let* (G,P) *be a planar 3-polyhedron and* $\Gamma \subset \mathrm{Aut}(G,P)$ *be a subgroup satisfying Assumptions 4.1.1. Let* $\alpha \in \Gamma - \{\iota\}$ *be orientation preserving and let* K *be an* α*-invariant connected subgraph of* G *containing no fixed vertex of* $\langle\alpha\rangle$ *and with the property that the subgraph of* G *between two cycles of* K *belongs to* K. *If* $H_o, \ldots, H_{n-1}$ *are pairwise disjoint cycles in* K *such that* $\alpha(H_i) = H_{i+1}$ *for* $0 \leq i \leq n-1$, *where* $H_n = H_o$, *and if* $\alpha^n \neq \iota$, *then* $n \leq 2$.

Proof. We claim that each H_i has the property that $H_o \cup \cdots \cup H_{n-1}$ is on one side of H_i. Suppose H_i, H_j and H_k are distinct and H_i and H_k are on distinct sides of H_j. Let H_i' be the side of H_i which contains H_j and H_k. We note that $\alpha^{j-i}(H_i) = H_j$ and $\alpha^{k-i}(H_i) = H_k$. If $\alpha^{j-i}(H_i')$ is contained in H_i' or $\alpha^{k-i}(H_i')$ is contained in H_i', then α has infinite order. If on the other hand $\alpha^{j-i}(H_i')$ and $\alpha^{k-i}(H_i')$ contain H_i, then $\alpha^{k-i}(H_i')$ contains $\alpha^{j-i}(H_i')$ so that $\alpha^{k-j}(H_i')$ contains H_i'. Thus α has infinite order. If α has infinite order, then by Corollary 4.1.5 there are no finite α^k-invariant subgraphs for all $k \neq 0$. This contradicts our assumption that $\alpha^n(H_o) = H_o$ and establishes the claim.

We now claim that $n \leq 2$. To prove this let $d = d_K(H_k, H_l) = \min\{d_K(H_i, H_j) \mid i \neq j\}$ and let L be a path of length d joining H_k and H_l in K. We note that $Q = \cup\{H_i - (H_k \cup H_l) \mid 0 \leq i \leq n-1\}$ lies between H_k and H_l, and $V(Q) \cap V(\cup\{\alpha^{nj}(L) \mid j \in Z\}) = \emptyset$. We recall that $\alpha^n(H_i) = H_i$ for $0 \leq i \leq n-1$ and that $\alpha^n \neq \iota$. Let M be the α^n-invariant subgraph containing H_k, H_l and L and let H be the subgraph of G between H_k and H_l. By Lemma 4.1.12 no component of $H - M$ is α^n-invariant, contradicting the fact that H_i lies between H_k and H_l and is connected and α^n-invariant. □

PROPOSITION 4.2.6. *Let* G *be a connected and locally finite graph and let* $\alpha \in \mathrm{Aut}(G)$ *be of infinite order such that* $\langle\alpha\rangle$ *has no fixed vertex in* G. *If* H_1 *and* H_2 *are* α^k*-invariant 2-infinite paths, where* $k \neq 0$, *then they are cofinal.*

Proof. Let L be a shortest path joining a vertex $u_1 \in V(H_1)$ with a vertex $u_2 \in V(H_2)$. L consists of a single vertex if $V(H_1) \cap V(H_2) \neq \emptyset$. Since $\langle\alpha\rangle$ has no fixed vertices, $\alpha^{ik}(L)$ and $\alpha^{jk}(L)$ are disjoint whenever $i \neq j$ and thus any finite set of vertices is met by only finitely many of them. Thus any 1-infinite subpath of H_1 containing $\{\alpha^{lk}(u_1) \mid l \geq 0\}$ is cofinal to any 1-infinite subpath of H_2 containing $\{\alpha^{lk}(u_2) \mid l \geq 0\}$. The same holds for the 1-infinite subpaths of H_1 and H_2 containing $\{\alpha^{lk}(u_1) \mid l \leq 0\}$, respectively, $\{\alpha^{lk}(u_2) \mid l \leq 0\}$. Therfore H_1 and H_2 are cofinal. □

PROPOSITION 4.2.7. *Let* (G,P) *be a planar 3-polyhedron and* $\alpha \in \mathrm{Aut}(G,P)$ *be of infinite order. If for some non-zero integer* k *there is an* α^k*-invariant, two-sided 2-infinite path in* G*, then every* α^n*-invariant,* $n \neq 0$*, 2-infinite path is two-sided.*

Proof. We note that $\alpha^i(x) \neq \alpha^j(x)$ if $i \neq j$ and $x \in V(G)$. Let H_1 be an α^k-invariant, two-sided 2-infinite path and H_2 be an α^n-invariant 2-infinite path. Now H_1 and H_2 are α^{nk}-invariant and thus cofinal in G by Proposition 4.2.6. Hence by Corollary 3.3.16 they are both two-sided. □

LEMMA 4.2.8. *Let* (G,P) *be a planar 3-polyhedron and let* $\alpha \in \mathrm{Aut}(G,P)$ *be of infinite order. If for every* $k \in \mathbb{Z} - \{0\}$ *there is no* α^k*-invariant, 2-infinite and two-sided path, then there is a cycle* $B \subset G$ *with sides* B' *and* B'' *which has the property that for every finite subgraph* L *of* G *there is a positive integer* i_o *so that* $\cup\{\alpha^j(B) \mid j \leq -i_o\}$ *is contained in some component of* $B'' - L$ *and* $\cup\{\alpha^j(B) \mid j \geq i_o\}$ *is contained in some component of* $B' - L$.

Proof. By Lemma 4.2.1 and Proposition 4.2.7 there is a positive integer n and pairwise disjoint 2-infinite, one-sided cofinal paths $H_o,\ldots,H_{n-1}$ so that $\alpha(H_i) = H_{i+1}$, where $0 \leq i \leq n-1$ and $H_o = H_n$. Because H_o is one-sided, by Theorem 3.3.13 there is a cycle B in G

with sides B' and B'' so that if L is a finite subgraph of G containing B, then B' - L contains the 1-infinite path $H_o(\alpha^{j_o}(v),\alpha^{j_o+n}(v),\dots) = H_o'$ and B'' - L contains the 1-infinite path $H_o(\alpha^{-j_o}(v),\alpha^{-j_o-n}(v),\dots) = H_o''$, where $v \in V(H_o)$ and $j_o \geq 0$. Let K be a finite connected subgraph of G containing B and $\{v,\alpha(v),\dots,\alpha^{n-1}(v)\}$. Since α has infinite order, by Corollary 4.1.5 there is an $i_o \geq j_o$ such that $\alpha^i(K) \cap L = \emptyset$ for $|i| \geq i_o$. Then $\cup\{\alpha^i(K) \mid i \geq i_o\} \cup H_o'$ and $\cup\{\alpha^i(K) \mid i \leq -i_o\} \cup H_o''$ are connected subgraphs of B' - L, respectively B'' - L, and contain $\cup\{\alpha^i(B) \mid i \geq i_o\}$, respectively $\cup\{\alpha^i(B) \mid i \leq -i_o\}$. □

THEOREM 4.2.9. *Let* (G,P) *be a planar* 3*-polyhedron, let* $\alpha \in \mathrm{Aut}(G,P)$ *be of infinite order and assume that for every* $k \in Z - \{0\}$ *there is no 2-infinite, two-sided* α^k*-invariant path. Then for suitable* $k \geq 0$ *the* k*-th order barycentric subdivision* $(\tilde{G},\tilde{P})$ *of* (G,P) *contains a pairwise disjoint collection* $\{C_i \mid i \in Z\}$ *of cycles with the following properties:*

(1) if $i < j < k$ *then* C_j *separates* C_i *and* C_k *; and*

(2) $\alpha(C_i) = C_{i+1}$ *for all* $i \in Z$.

Proof. By Lemma 4.2.8 there is a cycle $B \subset G$ with the sides B' and B'' which has the property that for every finite subgraph L of G there is a positive integer i_o such that $\cup\{\alpha^j(B) \mid j \leq -i_o\}$ is contained in some component of B'' - L and $\cup\{\alpha^j(B) \mid j \geq i_o\}$ is contained in some component of B' - L.

We now claim that $\alpha^j(B) \not\subset B''$ for every positive integer j. To prove the claim let L = B and i_o be chosen accordingly. Now suppose that $\alpha^j(B) \subset B''$ for some $j > 0$. Then $\alpha^j(B'') \supset \alpha^i(B)$ for $i \leq -i_o + j$. It follows that $\alpha^j(B'') \subset B''$ and therefore $\alpha^{mj}(B) \subset B''$ for every positive integer m which is a contradiction. This proves the claim.

Now consider $I = \{i > 0 \mid \alpha^i(B) \not\subset B'\}$. By the preceding claim, $|V(\alpha^i(B)) \cap V(B)| \geq 2$ for every $i \in I$. By Corollary 4.1.5 I is finite. Moreover, $K = \cup\{\alpha^i(B) \mid i \in I\} \cup B$ is a finite 2-connected subgraph of G. In view of the choice of B and of Theorem 3.3.9 there is a cycle $C \subset K$ and an integer k_o with the property that C separates ~~K from~~ $\alpha^i(B)$ for $i \leq k_o$. Let C' and C'' be the sides of C and assume that K is contained in C' and $\alpha^i(B) \subset C''$ for $i \leq k_o$. This implies that $B' \subset C'$.

We now claim that $\alpha(C') \subset C'$. To prove this we note that $C \subset \cup\{\alpha^i(B) \mid 0 \leq i \leq n\}$ for some n, $\alpha^i(B) \subset K$ if $i \in I$ and $\alpha^i(B) \subset B'$ if $i \geq 0$ and $i \notin I$. Hence $\cup\{\alpha^i(C) \mid i \geq 0\} \subset \cup\{\alpha^i(B) \mid i \geq 0\} \subset C'$ and moreover $\cup\{\alpha^i(C) \mid j \leq k_o - n\} \subset C''$. It follows that $\alpha(C') \subset C'$ since otherwise $\alpha(C') \supset C'' \supset \cup\{\alpha^i(C) \mid j \leq k_o - n_o\}$, that is, $C' \supset \cup\{\alpha^j(C) \mid j \leq k_o - n_o - 1\}$ which is impossible. This proves the second claim.

We now claim that for some non-negative integer n, the n-th order barycentric subdivision $(\tilde{G},\tilde{P})$ of (G,P) contains a cycle H with the properties

(1') H and $\alpha(H)$ are edge disjoint,

(2') $\alpha(H)$ is in one side of H, and

(3') $\alpha(H') \subset H'$ where H' is the side of H containing $\alpha(H)$.

To prove this take C,C' and C'' from above and assume C and $\alpha(C)$ are not edge disjoint. Then we shall detach C and $\alpha(C)$ at their common edges as shall be described now. Define an order relation $\leq$ on E(C) by defining $y \leq x$ if $\alpha^i(x) = y$ for some $i \geq 0$. It is clear that this relation is reflexive and transitive. By Corollary 4.1.5 it follows that it is anti-symmetric. For each maximal $e \in E(C)$ define m_e to be the largest integer i for which $\alpha^i(e) \in E(C)$. Let $E^* = \{e \in E(C) \mid e$ is maximal and $m_e \geq 1\}$ and let $m = \max\{m_e + 1 \mid e \in E^*\}$.

For $e \in E^*$ let $i_G(e) = \{u_e, v_e\}$ and let B_e be the boundary in C" which contains e. Without loss of generality we may assume that B_e and C, respectively B_e and E^*, have at most one common edge. If not we subdivide (G,P) barycentrically to obtain this property. (See Chapter 3, §4). Let $(\tilde{G},\tilde{P})$ be the m-th order barycentric subdivision of (G,P). For each subgraph K of G let $\tilde{K}$ denote its subdivision in $\tilde{G}$. Moreover, let F_e denote the side of $\tilde{B}_e$ not containing V(G). It is easy to see (by induction on the order of the subdivision) that for every $e \in E^*$ there are at least $k = 2^m - 1$ paths P_i^e, $0 \leq i \leq k-1$, with the following properties:

(a) P_i^e is a path in F_e which joins u_e and v_e;

(b) $V(P_i^e) \cap V(P_j^e) = \{u_e, v_e\}$, for $i \neq j$;

(c) $V(P_i^e) \cap V(\tilde{B}_e) = \{u_e, v_e\}$; and

(d) $i < j$ if and only if $R_i^e \subset R_j^e$, where R_s^e is the side of $\tilde{e} \cup P_s^e$ which doesn't contain F_e.

We note that $\alpha^i(B_e) \subset C''$ whenever $\alpha^i(e) \in E(C)$ for some $e \in E^*$ and $i \geq 0$ or else $C'' \subset \alpha^i(C'') \subset C'$, which is impossible. For each $e \in E^*$ and i, where $0 \leq i \leq m_e$, we substitute the subpath $\alpha^i(\tilde{e})$ of $\tilde{C}$ by $\alpha^i(P_i^e)$ if and only if $\alpha^i(e) \in E(C)$, (for an illustration of the case $m_e = 2$, $E^* = \{e\}$ see the Figures 4.2.1 and 4.2.2) and it is easily seen that the resulting cycle H has the property that H and $\alpha(H)$ are edge disjoint and $\alpha(H)$ is not separated by H. Moreover, if H' is the side of H containing $\alpha(H)$, then $\alpha(H') \subset H'$.

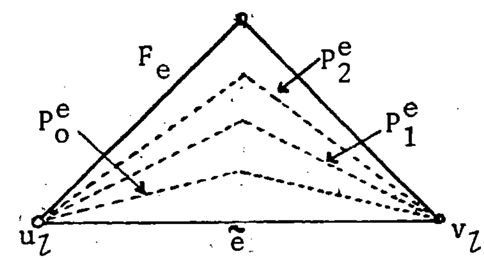


figure 4.2.1

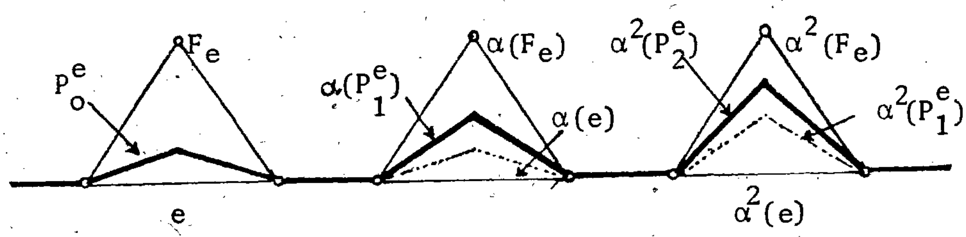


figure 4.2.2

We now claim that for some nonnegative integer n the n-th order barycentric subdivision (G',P') of (G,P) contains a cycle H so that

(1") H and $\alpha(H)$ are vertex disjoint, and

(2") $\alpha(H') \subset H'$ where H' is the side of H containing $\alpha(H)$.

By the third claim we may assume that the cycle C and $\alpha(C)$ of (G,P) constructed originally are edge disjoint. Let C' and C" be the sides of C and assume $\alpha(C') \subset C'$. If C and $\alpha(C)$ are not vertex disjoint, we shall detach their common vertices as described below. We define a partial order on V(C) similar to the one defined on E(C) in the third claim. Let V* consist of all minimal vertices of V(C) which are contained in some chain of length at least two. Thus for any $v \in V^*$, $\alpha(v) \in C' - C$. Let (G',P') be the first order barycentric subdivision of (G,P). We denote the resulting subdivision of C,C' and C" again by C,C' and C". Let $v \in V^*$ and let N_v be the set of vertices in C" adjacent to v. It is easily seen that N_v induces a path L which meets C in its endpoints u,w only. Moreover, $V(\alpha(C) \cup \alpha(L)) \cap V(L) = \emptyset$ since $\alpha(v) \in C' - C$. Thus by substituting the path $C_{u,v,w}$ with L we obtain a cycle H_1^* which doesn't separate $\alpha(H_1)$ and satisfies (2") and H_1 and $\alpha(H_1)$ have fewer common vertices than C and $\alpha(C)$. By repeating this separation procedure a finite number of times we finally obtain the desired cycle H. We define $C_i = \alpha^i(H)$ for $i \in Z$. It is immediately obvious that the collection $\{C_i \mid i \in Z\}$ satisfies (1) and (2). □

NOTE 4.2.10. Let (G,P) be a planar 3-polyhedron and let $\alpha \in Aut(G,P)$ be of infinite order. Assume there is a collection of cycles $\{C_i \mid i \in Z\}$ with the properties (1) and (2) of Theorem 4.2.9. It is easy to see that for suitable m the m-th order barycentric subdivision $(\tilde{G},\tilde{P})$ of (G,P) contains disjoint paths L_i meeting $\tilde{C}_0$ and $\tilde{C}_1$ ($\tilde{C}_i$ is the induced subdivision of C_i) only at v_i and $\alpha(v_i)$ for $1 \le i \le 2$. Now let H_1 be the smallest α-invariant subgraph containing $\tilde{C}_0, L_1$ and L_2. (See Figure 4.2.3). Let $\Gamma_1 = \langle\alpha\rangle$. It is easy to see that H_1 satisfies Assumptions 5.2.10,(1),(2) and (3). Moreover, H_1 can be drawn in the plane punctured through the origin 0 to yield an isomorphic $\tilde{\Gamma}_2$-invariant $(R^2 - \{0\})$-graph

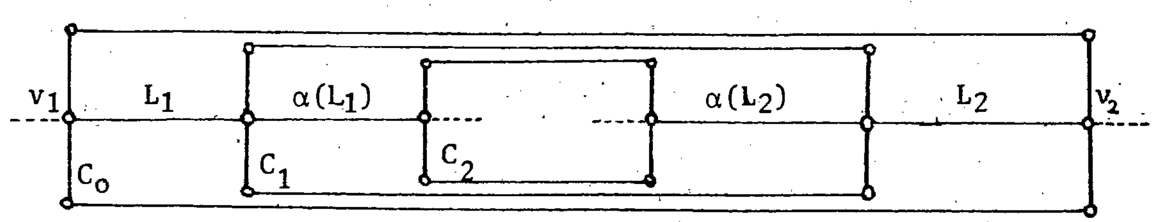


figure 4.2.3

H_2 (where $\tilde{\Gamma}_2 = \langle\sigma\rangle$, respectively $\langle\bar{\sigma}\rangle$, depending if α is orientation preserving or orientation reversing, and $\sigma, \bar{\sigma}$ are the mappings defined on p. 13) such that conditions (4) and (5) of Lemma 5.2.12 are satisfied. These observations are of basic importance in the proof of Theorem 5.3.1.

COROLLARY 4.2.11. *Let* (G,P) *and* α *be as in Theorem* 4.2.9. *Then for suitable* $k \geq 0$ *the* k*-th order barycentric subdivision* $(\tilde{G},\tilde{P})$ *contains an* α*-invariant, one-sided 2-infinite path.*

LEMMA 4.2.12. *Let* (G,P) *be a planar 3-polyhedron and* $\alpha \in \mathrm{Aut}(G,P)$ *be of infinite order. Let* H_1 *and* H_2 *be disjoint* α*-invariant, 2-infinite and two-sided paths and let* H *be the subgraph of* G *between* H_1 *and* H_2. *If* P *is a shortest path joining any* $v_1 \in V(H_1)$ *with any* $v_2 \in V(H_2)$ *then no component of* $H_3 = H - (\cup\{\alpha^i(P) \mid i \in Z\} \cup H_1 \cup H_2)$ *is* α^i*-invariant for any* $i \in Z - \{0\}$.

Proof. Let C be the cycle $H_1(v_1,\alpha(v_1)) \cup H_2(v_2,\alpha(v_2)) \cup P \cup \alpha(P)$. H_1 and H_2 are two-sided and by Theorem 3.3.13 they are not separated by C. Moreover, by Proposition 4.2.6 H_1 and H_2 are cofinal. Hence they are on the same side of C, say in C'. Let C'' be the other side of C. We conclude the proof by noting that $\alpha^i(C'') \subset C'$ for every $i \in Z - \{0\}$ and $\cup\{\alpha^i(C'') \mid i \in Z\} = H$. $\square$

THEOREM 4.2.13. *Let (G,P) be a planar 3-polyhedron and let* $\alpha \in \Gamma - \{\iota\} \subset \mathrm{Aut}(G,P)$ *where* Γ *satisfies Assumptions 4.1.1. If* α *has finite order then let* α *be orientation preserving. Moreover, assume* K *is a connected,* α*-invariant subgraph of* G *with the property that the subgraph of* G *between any two cycles or any two two-sided 2-infinite paths belongs to* K. *If* α *has infinite order and* K *contains an* α^k*-invariant* $(k \neq 0)$, *2-infinite two-sided path, then it has an* α*-invariant one. If* α *has finite order and there are no fixed vertices of* $\langle\alpha\rangle$ *in* K , *then* K *contains an* α*-invariant cycle.*

Proof. Considering the two cases simultaneously we assume that K doesn't have an α-invariant 2-infinite path in case α has infinite order and has no α-invariant cycle if α is of finite order. By Lemma 4.2.1, Theorem 4.2.4 and Lemma 4.2.5, there are two disjoint subgraphs H_o and H_1 of K satisfying the following conditions:

(1) $D = d_K(H_o,H_1)$ is minimum among all pairs satisfying (2),(3) and (4) below;

(2) $\alpha(H_o) = H_1$ and $\alpha(H_1) = H_o$;

(3) H_o and H_1 are disjoint 2-infinite paths if α has infinite order; and

(4) H_o and H_1 are disjoint cycles if α has finite order and $\alpha^2 \neq \iota$.

We note that in case $\alpha^2 = \iota$ the construction in Lemma 4.2.1 yields an α-invariant cycle; also see Corollary 4.2.3. Therefore we can assume $\alpha^2 \neq \iota$ in (4). Let H be the graph between H_o and H_1. H is connected and α-invariant and $H \subset K$. Let L be a path of length D joining u in H_o

with v in H_1 and put $d_L = |E_u(H)| + |E_v(H)|$. We additionally require that we choose H_o, H_1 and L such that d_L is minimum.

We claim that if we let $A = \cup\{\alpha^i(E_u(H_o) \cup E_v(H_1)) | i \in Z\}$, then $H - A$ contains an α-invariant connected subgraph. In order to prove this claim we consider the subgraph M of H induced by $E(H) - A$ and show that it is connected and α-invariant. Clearly it is α-invariant as H and A are. Let B_H denote the set of all boundaries of (G,P) contained in H. Let $U \subseteq V(H_o) \cup V(H_1)$ consist of all vertices w for which $E_w(H_i) \subset A$ and $E_w(H_i)$ is contained in some boundary $B \in B_H$. Then $U \cap V(M) = \emptyset$. Let $w_1, w_2 \in V(M)$ and Q be a path in H joining them. Any vertex w of $U \cap V(Q)$ and its incident edges of A can be replaced by an edge of M in the boundary $B \in B_H$ containing w and any edge e of A in Q which is not incident to a vertex of U can be replaced by the other two edges, say e_1 and e_2, of the boundary $B \in B_H$ containing e. Moreover $e_1, e_2 \notin A$. Thus w_1 and w_2 can be joined by a path in M. That is, M is connected and this establishes the claim.

M does not contain an α-invariant 2-infinite path or cycle, respectively, but by Lemma 4.2.1 and Theorem 4.2.4 it contains subgraphs H_o' and H_1' satisfying properties (2), (3) and (4). Now H_o', H_1', H_o and H_1 are α^2-invariant. By Lemma 4.2.11, respectively, Lemma 4.1.13, it follows that $V(H_i') \cap V(\alpha^j(L)) \neq \emptyset$ for $0 \leq i \leq 1$ and $j \in Z$. Thus by choice of D we may assume that $\alpha^j(u) \in V(H_o')$ and $\alpha^j(V) \in V(H_1')$ for all $j \in Z$. Again by the above lemmas it follows that H_o' separates $E_u(L)$ and $E_u(H_o)$. Thus H_o' separates H_o and H_1. Similarly H_1' separates H_o and H_1. If H' is the subgraph of G between H_o' and H_1' it follows that $H' \subset K \cap H$, $(E_u(H_o) \cup$

$E_v(H_1)) \cap E(H') = \emptyset$ and $E_x(H') \subset E_x(H)$ for $x \in \{u,v\}$. Thus $|E_u(H')| + |E_v(H')| < d_L$ which is a contradiction. Thus the claim of the theorem is true. □

LEMMA 4.2.14. *Let* (G,P) *be a planar 3-polyhedron and let* $\alpha \in \mathrm{Aut}(G,P)$ *be orientation preserving and of infinite order. Let* H *be a 2-infinite, two-sided* α*-invariant path and* H' *be either one of the sides of* H *in* (G,P) . *Then* $V(H') - V(H) \neq \emptyset$.

Proof. First we note that $H' \neq H$ since H is not a boundary and that $\alpha(H') = H'$ since α is orientation preserving. We claim that if $e \in E(H')$ and $i_G(e) = \{x,y\} \subset V(H)$, then $d_H(x,y) \leq d_H(x,\alpha(x)) = d$. To prove this we first note that by Corollary 4.2.2, $d = d_H(z,\alpha(z))$ for all $z \in V(H)$. Now assume to the contrary that $d_H(x,y) > d_H(x,\alpha(x))$. Then $e \notin E(H)$ and we may assume without loss of generality that $\alpha(x) \in V(H(x,y)) - \{x,y\}$, $\alpha(y) \notin V(H(x,y))$. Thus the edge $\alpha(e)$ is on both sides of the cycle C formed by H(x,y) and e, but $\alpha(e)$ is not on C which is a contradiction and proves the claim.

We now claim there is a vertex $u \in V(H') - V(H)$ and vertices $x,y \in V(H)$ adjacent to u such that $d_H(x,y) \geq d_H(v,w)$ for every $e \in E(H')$ with $i_G(e) = \{v,w\} \subset V(H)$. We now prove this claim. If there is no edge $e \in E(H') - E(H)$ such that $i_G(e) \subset V(H)$, then any edge $e \in E(H)$ is contained in a boundary B one of whose vertices belongs to $V(H') - V(H)$ and our claim is satisfied trivially. On the other hand, if there is such an edge we pick one for which $d_H(x,y)$ is maximum where $\{x,y\} = i_G(e)$.

Now we consider the cycle K induced by H(x,y) and the edge e. The edge e is contained in a boundary $B \subset H'$ on the same side of C as H. One vertex of B is not in V(H) by maximality of $d_H(x,y)$ and hence it is in $V(H') - V(H)$. □

LEMMA 4.2.15. *If* $(G,P), \alpha$, H *and* H' *are as in Lemma 4.2.14, then* $\tilde{H} = H' - H$ *contains an* α*-invariant component.*

Proof. For each component K of $\tilde{H}$ let V_K denote the set of vertices on H which are adjacent in H' to some vertex of K. We claim that if there is some component K of $\tilde{H}$ and vertices $u,v \in V_K$ with $d_H(u,v) > d_H(u,\alpha(u))$, then K is α-invariant. To prove this claim assume $u,v \in V_K$ and $d_H(u,v) > d_H(u,\alpha(u))$. Moreover, we may assume that $\alpha(u) \in H(u,v) - \{u,v\}$ and $\alpha(v) \notin H(u,v)$. There are vertices $u',v' \in V(K)$ adjacent to u and v, respectively, and a path L joining u' and v' in K. Let C be the cycle consisting of L, H(u,v) and edges e_u and e_v joining u to u' and v to v', respectively. Now $\alpha(e_u)$ and $\alpha(e_v)$ are on distinct sides of C and thus the path $\alpha(L)$ joining $\alpha(u')$ to $\alpha(v')$ meets L. Hence $\alpha(K)$ meets K so that $\alpha(K) = K$ and the claim is proved.

It now suffices to show that there is some component K of $\tilde{H}$ and vertices $u,v \in V_K$ with $d_H(u,v) > d_H(u,\alpha(u))$. Suppose there is no such component. Then let K be a component of $\tilde{H}$ for which the H-diameter d of V_K in H, defined by $d = \max\{d_H(u,v) \mid u,v \in V_K\}$, is maximum. We pick $u,v \in V_K$ with $d = d_H(u,v)$. By the second claim in Lemma 4.2.14, $u \neq v$. Let M be the subgraph of H' consisting of all cycles $C \subset H'$ for which $C \cap H = H(u,v)$. Since M is 2-connected, it follows from Theorem 3.3.9 that there is a cycle C containing H(u,v) and separating M and H. From the second claim in Lemma 4.2.14, it follows that there is a vertex $u' \in V(C) - V(H(u,v))$ adjacent to u where e is the edge joining u and u'. Let B be the boundary containing e on the same side of C as H and let $w \in V(B) - \{u,v\}$. We know $w \notin V(H) - V(H(u,v))$ by the choice of u and v. Hence $B \subset M$ but $B \neq C$. On the other hand, B and H are on the same side of C which contradicts the fact that C separates H and M. ∎

COROLLARY 4.2.16. *If* (G,P), α, H *and* H' *are as in Lemma 4.2.14, then* $\tilde{H}$ *contains an* α*-invariant 2-infinite path.*

Proof. This follows immediately from Lemma 4.2.1 and Theorem 4.2.13. □

THEOREM 4.2.17. *Let* (G,P) *be a planar 3-polyhedron, let* $\alpha \in \mathrm{Aut}(G,P)$ *be of infinite order and assume that there is an* α*-invariant, 2-infinite and two-sided path. Then there is a collection* $\{H_i \mid i \in Z\}$ *of pairwise disjoint, 2-infinite and two-sided paths with the following properties:*

(1) If α *is orientation preserving, then for all* $i \in Z$ $\alpha(H_i) = H_i$;

(2) If α *is orientation reversing, then for all* $i \in Z$ $\alpha(H_i) = H_{-i}$; *and*

(3) If $i < j < k$, *then* H_i *and* H_k *are separated by* H_j.

Proof. By Theorem 4.2.13 there is an α-invariant, 2-infinite and two-sided path H_o. Let H_o' and H_o'' be its sides. If α is orientation preserving, then $\alpha(H_o') = H_o'$ and if α is orientation reversing then $\alpha(H_o') = H_o''$. In the first case put $\beta = \alpha$ and in the second $\beta = \alpha^2$ so that β is orientation preserving and $\beta(H_o') = H_o'$ and $\beta(H_o'') = H_o''$.

We now proceed by induction. Suppose we have found pairwise disjoint, β-invariant, 2-infinite and two-sided paths $H_o, \ldots, H_n$ in H_o' such that (3) holds whenever $0 \le i < j < k \le n$. In view of Theorem 4.2.13 there is a β-invariant, 2-infinite and two-sided path H_{n+1} disjoint from H_n which is separated from H_i, $0 \le i \le n-1$, by H_n. It follows that (3) holds for $0 \le i < j < k \le n+1$.

Now consider the collection $\{H_i \mid i = 0,1,2,\ldots\}$ constructed in the above fashion. If $\alpha = \beta$, we similarly construct $\{H_i \mid i = -1,-2,\ldots\}$ in H_o'' but if $\beta = \alpha^2$, then we put $H_i = \alpha(H_{-i})$ for $i = 0,-1,-2,\ldots$. It immediately follows that the collection $\{H_i \mid i \in Z\}$ satisfies (1)-(3). ∎

NOTE 4.2.18. Let (G,P), $\{H_i \mid i \in Z\}$ and α be as in Theorem 4.2.17 and let L_i be a shortest path joining H_i and H_{i+1}, for all integers i if α is orientation preserving, respectively, for all nonnegative i's if α is orientation reversing. It is easily seen that the paths $\alpha^j(L_i)$ and $\alpha^k(L_i)$ are disjoint for $j \ne k$. We now consider the smallest subgraph H^1 of G which is α-invariant and contains H_i and L_j for all i and j (for illustration see Figure 4.2.4). Let $\Gamma_1 = \langle\alpha\rangle$. By Lemma 4.2.12 H^1 satisfies the Assumptions 5.2.10, (1), (2) and (3). Moreover, H^1 can be drawn in the plane R^2 to yield an isomorphic, $\tilde{\Gamma}_2$-invariant R^2-graph H^2

(where $\tilde{\Gamma}_2 = \langle\tau\rangle$, respectively $\langle\bar{\tau}\rangle$, depending if α is orientation preserving or orientation reversing, and τ and $\bar{\tau}$ are the translation, respectively gleitreflection, defined on p. 13) such that conditions (4) and (5) of Lemma 5.2.12 are satisfied. These observations are of basic importance in the proof of Theorem 5.3.1.

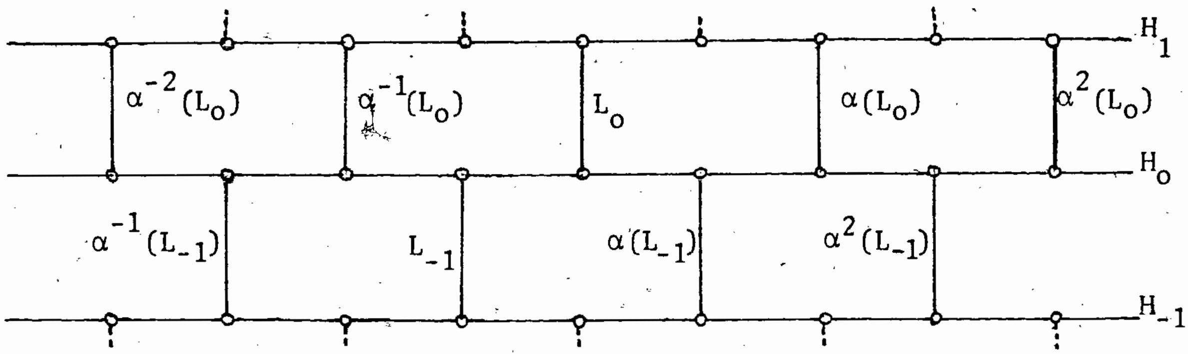


figure 4.2.4

LEMMA 4.2.19 *Let* H *be a cycle and* H' *be one of its sides in the planar 3-polyhedron* (G,P) . *Let* $\alpha \in \Gamma - \{\iota\}$ *be orientation preserving and let* Γ *meet Assumptions 4.1.1. If* α *fixes* H' *and if there is no fixed vertex of* $\langle\alpha\rangle$ *in* H' , *then* H' - H *contains a nonempty* α*-invariant component.*

Proof. An edge $e \in E(H') - E(H)$ with $i_G(e) = \{x,y\} \subset V(H)$ is called a chord of H' and $d_H(x,y)$ is called its length. If e_1, e_2 are chords of H', $i_G(e_1) = \{x,y\}$ and $i_G(e_2) = \{u,v\}$, then $\{x,y\}$ doesn't separate $\{u,v\}$ on H. Moreover, G has no multiple edges and no edge is fixed by α so that every chord of H' has length less than $|V(H)|/2$.

We partially order the chords of H' as follows: $e_1 \le e_2$ if $H(x_1,y_1) \subset H(x_2,y_2)$, where $i_G(e_i) = \{x_i,y_i\}$ and $H(x_i,y_i)$ is a subpath of

H of length $d_H(x_i, y_i)$. Since α acts order preserving the collection C of all maximal chords is α-invariant. Moreover $|C| = 0$ or $|C| \geq 2$. If $C \neq \emptyset$, then C together with all paths in H between endpoints of distinct maximal chords forms an α-invariant cycle K in H'. Let K' be the side of K not containing H. We know $K' \subset H'$ and no non-maximal chord of H' belongs to K'. Hence K' doesn't have any chords. Since $\alpha(K') = K'$, K is not a boundary and therefore $V' = V(K') - V(K) \neq \emptyset$. Moreover, $K' - K$ is connected since K' has no chords. Also, $K' - K$ is α-invariant. Thus, $K' - K$ is an α-invariant component of $H' - H$. □

We recall that by Theorem 4.1.14, $|V_{<\gamma>}| \leq 2$ where $V_{<\gamma>}$ is the set of fixed vertices of $<\gamma>$. The following theorem is of basic importance in the proof of Theorem 5.3.1.

THEOREM 4.2.20. *Assume that the subgroup* Δ *of* Aut(G,P) *satisfies Assumptions 4.1.1 and let* $\alpha \in \Delta - \{1\}$ *be orientation preserving and of finite order. If* $V_{<\alpha>} = \{u\}$, *respectively* $V_{<\alpha>} = \emptyset$, *then there is a collection* $\{C_i \mid i = 1,2,\ldots\}$, *respectively* $\{C_i \mid i \in \mathbb{Z}\}$, *of pairwise disjoint* α*-invariant cycles with the following properties.*

(1) $V(C_i) \cap V_{<\alpha>} = \emptyset$ *for all* i.

(2) If $V_{<\alpha>} = \{u\}$, *then with* $C_0 = \{u\}$ C_i *separates* C_{i-1} *from* C_{i+1} *for all* $i \in \{1,2,\ldots\}$.

(3) If $V_{<\alpha>} = \emptyset$, *then* C_i *separates* C_{i-1} *from* C_{i+1} *for all* $i \in \mathbb{Z}$.

Proof. We note that $G - V_{<\alpha>}$ is connected since G is 3-connected. Using Lemma 4.2.19 and Theorem 4.2.13 the remainder of the proof follows by induction and is similar to the proof of Theorem 4.2.17. □

NOTE 4.2.21. Let Δ and α be as in Theorem 4.2.20. If $V_{<\alpha>} = \{u,v\}$, then by Proposition 3.2.3 $G_1 = G - \{u,v\}$ is connected and by Theorem 4.2.13 has an α-invariant cycle K. By Lemma 4.1.13 it follows that u and v are separated by K. Let L_u and L_v be shortest paths joining u and v to K and let H_1 be the smallest α-invariant subgraph of G containing K, L_u and L_v. The subgraphs of H_1 in the two sides of K are "wheels" (as illustrated in Figure 4.2.5) since the only fixed vertices of α are u and v.

In case $|V_{<\alpha>}| \leq 1$ consider the subgraphs C_i constructed in Theorem 4.2.19. We join C_i and C_{i+1} by a shortest path L_i for all i and consider the smallest α-invariant subgraph H_1 containing C_i and L_i for all i. We note that $\alpha^k(L_i)$ and $\alpha^j(L_i)$ are disjoint whenever $\alpha^k \neq \alpha^j$ except possibly when i = 0 in which case their intersection is contained in $V_{<\alpha>}$ (see Figures 4.2.6 and 4.2.7 for illustration).

Now set $\Gamma_1 = <\alpha>$. It is easy to see (see Lemma 4.1.13) that H_1 satisfies Assumptions 5.2.10, (1), (2) and (3) in both cases considered above. Moreover, it can be drawn on $S = S^2$ if Γ_1 has two fixed vertices; on $S = S^2 - \{(0,0,1)\}$ if Γ_1 has one fixed vertex; and on $S = S^2 - \{(0,0,1), (0,0,-1)\}$ if Γ_1 has no fixed vertex. This produces an isomorphic $\tilde{\Gamma}_2$-invariant S-graph H_1 (where $\tilde{\Gamma}_2 = <\rho_\delta>$, with ρ_δ the rotation defined on p.13) such that conditions (4) and (5) of Lemma 5.2.12 are satisfied. These observations will be used in the proof of Theorem 5.3.1.

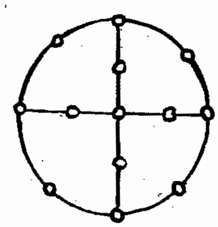


figure 4.2.5

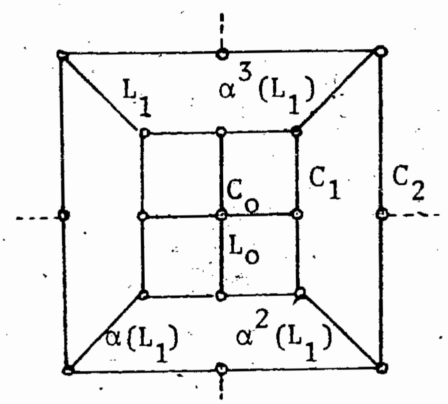


figure 4.2.6

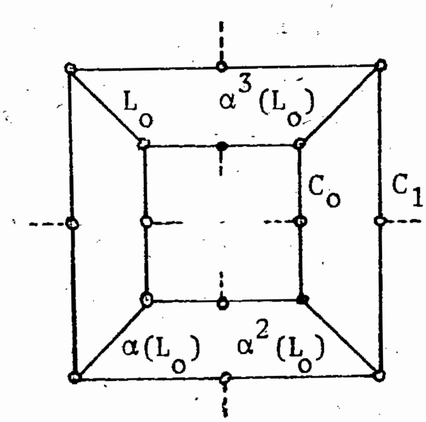


figure 4.2.7

§3. GROUPS OF AUTOMORPHISMS OF (G,P,λ) WITH COMPACT FUNDAMENTAL DOMAIN

Let $\tilde{G} = (G,P,\lambda)$ be a 3-polyhedron and let Γ be a subgroup of $\mathrm{Aut}(G,P,\lambda)$ satisfying Assumptions 4.1.1. We note that barycentric subdivisions of $\tilde{G}$ again satisfy Assumptions 4.1.1 and Γ naturally acts on them (see Chapter 3, §4). The quotient graph of $\tilde{G}$ modulo Γ, denoted as G^* or $G/_\Gamma$, is a graph whose vertices and edges are the Γ-orbits of vertices and edges of G, respectively. The incidence relation i_{G^*} is defined by setting $i_{G^*}(e^*) = \{u^*,v^*\}$ where x^* is the Γ-orbit containing x for every $x \in V(G) \cup E(G)$. G^* is connected and locally finite and possibly has loops or multiple edges. A loop in G^* is caused by two adjacent vertices of G in the same Γ-orbit. By applying a first order barycentric subdivision to $\tilde{G}$ we eliminate the loops from G^* and it is easy to see that another first order barycentric subdivision also eliminates multiple edges from G^*. Hence we may assume G^* has no loops or multiple edges. Let Π denote the canonical projection mapping from G to G^* assigning to each vertex and edge of G its Γ-orbit.

PROPOSITION 4.3.1. *The collection* $\{\Pi(B) \mid B$ *is a boundary of* $(G,P,\lambda)\}$ *consisting of triangles in* G^*, *called the boundaries of* G^*, *has the following properties.*

(1) *Each edge of* G^* *is contained in at least one but no more than two boundaries of* G^*.

(2) *At each vertex* v^* *of* G^* *the edges incident to* v^* *can be arranged in linear order* $e^*_o, e^*_1, \ldots, e^*_{n-1}$ *or in cyclical order* $(e^*_o, e^*_1, \ldots, e^*_{n-1})$ *such that* e^*_i *and* e^*_{i+1} *belong to the same boundary of* G^*, *for* $0 \le i \le n-2$, *respectively* $0 \le i \le n-1$, *where* $e^*_o = e^*_n$.

Proof. The first part of the claim follows from the fact that each edge of G is contained in exactly two boundaries of $\tilde{G}$. To prove the second part let $v \in V(G)$ and $P_v = (e_o, \ldots, e_{n-1})$. If $v \notin V_\Gamma$, then no two edges of $E_v(G)$ are in the same Γ-orbit since no two adjacent vertices of G are in the same Γ-orbit. Hence $(e^*_o, \ldots, e^*_{n-1})$ is the desired cyclical arrangement of $E_{v^*}(G^*)$. Now assume that $v \in V_\Gamma - V(S_\Gamma)$ (where S_Γ is the subgraph of G induced by the fixed edges of Γ). By Lemma 4.1.10, $\gamma P_v \gamma^{-1} = P_v$ for all $\gamma \in \Gamma_v$ and therefore for all $\gamma \in \Gamma_v$ there is a k depending on γ such that $\gamma(e_i) = e_{i+k}$ for $0 \le i \le n-1$, where $i+k$ is calculated modulo n. Let k be the smallest positive integer with $\gamma(e_i) = e_{i+k}$ for some $\gamma \in \Gamma_v - \{\iota\}$. We note that by Corollary 4.1.3, $\Gamma_v = \langle\gamma\rangle$, Moreover, $e^*_o, \ldots, e^*_{k-1}$ are distinct and $(e^*_o, \ldots, e^*_{k-1})$ is the desired cyclical arrangement of $E_{v^*}(G^*)$. Now we assume $v \in V(S_\Gamma)$. We recall from Lemma 4.1.10 that $v \in V(S_\Gamma)$ if and only if $\gamma P_v \gamma^{-1} = P_v^{-1}$. Thus $\gamma(e_i) = e_{k-i}$, for $0 \le i \le n-1$, where $k-i$ is calculated modulo n. By Corollary 4.1.8 we may assume that $e_o \in E(S_\Gamma)$ and $e_k \in E(S_\delta)$ but $e_i \notin E(S_\Gamma)$ for $0 < i < k$. It follows that $e^*_o, \ldots, e^*_k$ are distinct, e^*_i and e^*_{i+1} are in a boundary of G^* for $0 \le i \le k-1$ and $E_{v^*}(G^*) = \{e^*_o, \ldots, e^*_k\}$. Yet there is at most one boundary of G^* containing both e_o^* and e_k^*. □

From the previous lemma it follows that G^* has the simplicial structure of a triangulated (bordered) surface. (For a definition of a bordered surface see [20]. A simplicial complex is a set K equipped with a collection $\mathcal{C}$ of non-empty subsets of K which has the property that every non-empty subset of a set of $\mathcal{C}$ belongs to $\mathcal{C}$). From Proposition 4.3.1 it follows that the borders of G^*, that is, the subgraph of G^* consisting of all edges which are in only one boundary of G^*, is the image of S_Γ under Π and that each border component is a cycle or 2-infinite path.

Now let us assume that $(G,P,\lambda) = (G,P)$ is a planar 3-polyhedron and every cycle of G^* has at least one finite side. Then every 2-infinite path in G is two-sided. From Proposition 4.1.9 it follows that S_Γ is a cycle or a two-sided and 2-infinite path if S_Γ is not empty. Hence G is finite if S_γ is a cycle since γ switches the sides of S_γ. If S_γ and S_δ are distinct cycles, then it follows from the finiteness of G that S_γ meets S_δ, more precisely, S_γ and S_δ meet in exactly two vertices (see Corollary 4.1.8 and Theorem 4.1.14). If $\gamma \in \Gamma - \{\iota\}$ is orientation preserving and fixes two vertices, then by Lemma 4.1.13, G is finite. Conversely, if G is finite and $\gamma \in \Gamma - \{\iota\}$ is orientation preserving then by Proposition 4.1.2 γ has finite order and by Theorem 4.1.14 has two fixed vertices. If G is infinite and $\gamma \in \Gamma - \{\iota\}$ is orientation preserving and of finite order, then by Theorem 4.2.20 γ has exactly one fixed vertex. It follows that if S_γ and S_δ are distinct 2-infinite paths, then they have at most one common vertex.

We now additionally assume that G is infinite and $G^* = G/_\Gamma$ is finite. (If G* is infinite then the simplicial complex induced by the boundaries of (G,P) is called an "Ebenes Netz" by Zieschang [28, p. 55].) Then every one of the finitely many borders of G* is a cycle and G* has only finitely many r-points, where an r-point of G* is the image under Π of an r-point of G and the latter has the property that $\gamma(v) = v$ and $\gamma P_v \gamma^{-1} = P_v$ for some $\gamma \in \Gamma - \{\iota\}$. A fundamental domain F for Γ is a simplicial complex induced by a maximal collection of non-equivalent boundaries of (G,P). Since G* is finite, every fundamental domain F for Γ is finite. We say that F is connected if for any two boundaries B' and B'' of F there are finitely many boundaries $B_o, \ldots, B_n$ in F such that $B' = B_o$, $B'' = B_n$ and B_i shares an edge with B_{i+1}. A connected fundamental domain for Γ is obtained from a maximal set $B_1, \ldots, B_n$ of non-equivalent boundaries of (G,P) with the property that B_i has a common edge with the simplicial complex formed by $B_1, \ldots, B_{i-1}$, for $1 \leq i \leq n$.

If C is a cycle in G, then the finite side of C is called the *inside* and the other side the *outside* of C. The next proposition is the equivalent of Satz IV.6 of [28, p. 66] and a remark following after its proof. It shall be stated without proof.

PROPOSITION 4.3.2. *If* F *is a connected fundamental domain for* Γ*, then there is a cycle* C *in* F *such that* F *is the inside of* C*. If* $\alpha, \beta \in \Gamma$ *are distinct, then* $\alpha(F)$ *and* $\beta(F)$ *are disjoint or intersect in a path. Moreover,* $\cup\{\alpha(F) \mid \alpha \in \Gamma\} = G$.

The following proposition is Satz IV.7 of [28, p.67] and its proof is omitted.

PROPOSITION 4.3.3. *Let* F *and* C *be as in Proposition 4.3.2 and let* $\vec{C}$ *and* $\overleftarrow{C}$ *be the two orientations of* C *and* $E(\vec{C}) = \{\vec{e}_1,\ldots,\vec{e}_n\}$. *Then* $E^* = E(\vec{C}) \cup E(\overleftarrow{C})$ *is partitioned into orbits modulo* Γ *in such a way that*

(1) each orbit has size less than or equal to 2,

(2) for no i $\vec{e}_i$ *and* $\overleftarrow{e}_i$ *are in the same orbit,*

(3) if $\gamma(\vec{e}_i) = \overleftarrow{e}_j$ *then* γ *is orientation preserving, and*

(4) if $\gamma(\vec{e}_i) = \vec{e}_j$ *and* $\gamma \neq \iota$ *then* γ *is orientation reversing.*

It is clear that a connected fundamental domain F contains a vertex, an edge, an r-point and a fixed edge of Γ from each of their respective orbits. Moreover, by the last proposition every r-point or fixed edge in F lies on the border C of F. In addition, no two vertices or edges of F other than on C are equivalent. Thus we also obtain the complex G* from F by identifying, through application of Π, vertices and edges of C from the same orbit.

Let g,q,m be the genus, the number of boundaries, the number of non-border r-points of G*. These parameters sufficiently describe the bordered surface G* (if we also consider the m r-points as borders).

According to [28, p. 65], a connected fundamental domain can be transformed, using the method of "cutting and pasting" described in [28, p. 46], into a canonical fundamental domain which can be thought of being obtained by cutting $G/_\Gamma$ along a canonical system of curves. This fact is stated without proof in Theorem 4.3.4. (Theorem 4.3.4 is an equivalent version of Satz IV.5 in [28, p. 65].)

THEOREM 4.3.4. *Let* (G,P) *be an infinite, planar 3-polyhedron and assume the subgroup* Γ *of* Aut(G,P) *satisfies Assumptions 4.1.1. Moreover, assume* $G^* = G/_\Gamma$ *is finite and every cycle* K *in* (G,P) *has one finite side. Then in some barycentric subdivision* (G_1,P_1) *of* (G,P) *of sufficiently high order there is a connected fundamental domain* F *of* Γ *which has an oriented boundary cycle* $\vec{C}$ *which has one of the following two canonical forms;*

(a) $\prod_{i=1}^{m} s_i' s_i^{-1} \prod_{j=1}^{g} t_j' u_j^{-1} t_j u_j \prod_{k=1}^{q} e_k' c_{k,1} \cdots c_{k,m_k+1} e_k^{-1}$ *or*

(b) $\prod_{i=1}^{m} s_i' s_i^{-1} \prod_{j=1}^{g} v_j v_j' \prod_{k=1}^{q} e_k' c_{k,1} \cdots c_{k,m_{k+1}} e_k^{-1}$

where

(c) $m,g,q \geq 0$, $m+g+q \geq 1$ *and* $m=0$, $g=0$ *or* $q=0$ *means that the corresponding term in (a) or (b) is missing.*

(d) s_i, s_i', t_i, t_i', *etc. are oriented paths on* C *and paths denoted by the same symbol (for example,* s_i *and* s_i' *or* t_i *and* t_i'*) are* Γ*-equivalent. The terminal vertices of* s_i' *are r-points,* $1 \leq i \leq m$. *If* $m_k > 0$, *then the terminal vertices of* $c_{k,i}$, $1 \leq i \leq m_k$ *also are r-points. If* $m_k = 0$, *then the initial vertex of* $c_{k,1}$ *is not an r-point. Any two of these r-points and any two paths which are not denoted by the same symbol*

are not Γ*-equivalent.*

If (G,P) and Γ are as in Theorem 4.3.4, then as shown in [28, §IV.5] Γ has one of the following two algebraic structures, given in terms of generators and defining relations and stated in Theorem 4.3.5, that is, [28, Satz IV.8].

THEOREM 4.3.5. *If* (G,P) *and* Γ *are as in Theorem 4.3.4, then it has one of the following algebraic structures described in (A) and (B).*

(A)

Generators:

(a) $\sigma_1, \ldots, \sigma_m,\ m \geq 0$;

(b) $\tau_1, \mu_1, \ldots, \tau_g, \mu_g,\ g \geq 0$;

(c) $\eta_1, \ldots, \eta_q,\ q \geq 0$; *and*

(d) $\gamma_{1,1}, \ldots, \gamma_{1,m_1+1}, \ldots, \gamma_{q,m_q+1},\ m_k \geq 0$ *for* $1 \leq k \leq q$.

Defining relations:

(e) $\sigma_i^{-h_i} = \iota,\ 1 \leq i \leq m$;

(f) $\gamma_{i,j}^2 = \iota,\ 1 \leq i \leq q,\ 1 \leq j \leq m_i$;

(g) $(\gamma_{i,j+1}\gamma_{i,j})^{h_{i,j}} = \iota,\ 1 \leq i \leq q,\ 1 \leq j \leq m_i$;

$\gamma_{i,1}\eta_i\gamma_{i,m_i+1}\eta_i^{-1} = \iota,\ 1 \leq i \leq q$; *and*

(h) $\prod_{i=1}^{m} \sigma_i \prod_{i=1}^{g} \tau_i\mu_i\tau_i^{-1}\mu_i^{-1} \prod_{i=1}^{q} \eta_i = \iota$.

(B) *The same as (A) except that (b) is sustituted by (b') and (h) by (h'):*

(b') $\nu_1, \ldots \nu_g$, $g \geq 0$ *and*

(h') $$\prod_{i=1}^{m} \sigma_i \prod_{j=1}^{g} \nu_j^2 \prod_{k=1}^{q} \eta_k = \iota.$$

Note 4.3.6. Let (G_1, P_1), Γ and the fundamental domain F with boundary C be as in Theorem 4.3.4. Let H_1 be the smallest Γ-invariant subgraph containing C. Then H_1 is infinite and thus by Proposition 4.3.2 satisfies Assumptions 5.2.10. From the proof of Satz VI.8 of [28, p. 145] it follows that H_1 can be drawn in the euclidean plane S = E or non-euclidean plane S = NE (see Example 1.3.4) to yield an isomorphic, $\tilde{\Gamma}_2$-invariant S-graph H_2, where $\tilde{\Gamma}_2$ is a group of isometries of S, such that conditions (4) and (5) of Lemma 5.2.12 are satisfied. These observations are used in the proof of Theorem 5.4.1.

CHAPTER 5. TOPOLOGICAL CHARACTERIZATION OF CERTAIN DISCONTINUOUS GROUPS

In Chapter 1 we introduced the notions of topological polyhedron, discontinuous homeomorphism groups and in Chapter 2 we showed that every discontinuous homeomorphism group acting on the surface S can be thought of as acting on a topological 3-polyhedron. This fact is important since it allows us to study algebraic and other properties of discontinuous groups without refering to the underlying topological space (see Chapter 4). In fact these properties only depend on the combinatorial properties of the surface. Thus instead of discontinuous groups we study groups of automorphisms of abstract polyhedra.

In Chapter 5 we shall establish the link between the topological and algebraic concepts of polyhedra and discontinuous groups and the combinatorial concept of polyhedra and automorphism groups.

In Section 1, Theorem 5.1.1 together with Theorem 5.1.3 is an infinite version of the Embedding Theorem [25, p. 43] for graphs on surfaces. We do not prove a full generalization of that theorem, since we have no need for it. In Theorem 5.1.1 we show that any topological 3-polyhedron (S,G) can be equipped with a boundary tour scheme describing the boundaries of the faces of (S,G) and vice versa by Theorem 5.1.3 for every abstract 3-polyhedron (G,P,λ) there is a topological 3-polyhedron $(\tilde{S},\tilde{G})$ with boundary tour scheme $(\tilde{P},\tilde{\lambda})$ such that (G,P,λ) and $(\tilde{G},\tilde{P},\tilde{\lambda})$ are isomorphic, that is, (G,P,λ) can be triangularly embedded in a surface. Moreover, (G,P,λ) is orientable if and only if $\tilde{S}$ is orientable and planar if and only if $\tilde{S}$ is planar. If (S,G) is a topological 3-polyhedron with boundary tour scheme (P,λ), then a cycle or path C in G is two-sided if and only if S - ps(C) is disconnected.

Section 2 deals with the relationship of discontinuous groups and groups of automorphisms of an abstract 3-polyhedron. Every discontinuous homeomorphism group of a topological 3-polyhedron (S,G) induces an isomorphic subgroup of $\mathrm{Aut}(G,P,\lambda)$ where (P,λ) is a boundary tour scheme for (S,G). Conversely, if Γ is a subgroup of $\mathrm{Aut}(G,P,\lambda)$ then there is a discontinuous group $\tilde{\Gamma}$ of (S,G) inducing Γ. Let $\tilde{\Gamma}_i$ be two discontinuous groups acting on the 3-polyhedra (S_i,G_i) and let Γ_i be the induced subgroups of $\mathrm{Aut}(G_i,P_i,\Gamma_i)$ (where (P_i,Γ_i) is a boundary tour scheme for (S_i,G_i)), $1 \le i \le 2$. Then $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are topologically equivalent if and only if $\phi\Gamma_1\phi^{-1} = \Gamma_2$ for some isomorphism ϕ from (G_1,P_1,λ_1) to (G_2,P_2,λ_2). At the end of this section we shall prove a lemma which is used in the proof of Theorem 5.4.1.

In Section 3 we topologically characterize orientation preserving elements of finite order and elements of infinite order in a discontinuous homeomorphism group of a planar surface by showing that they are topologically equivalent to certain types of elementary mappings of the sphere or euclidean plane.

In Section 4 we topologically characterize discontinuous homeomorphism groups with compact fundamental domain of the euclidean plane by showing that they are topologically equivalent to groups of isometries of the euclidean or non-euclidean plane.

§1. TOPOLOGICAL POLYHEDRA AND ABSTRACT POLYHEDRA

THEOREM 5.1.1. *Given a 3-polyhedron* (S,G) *there is a boundary tour scheme* (P,λ) *for* G *such that*

(1) P_v *is a rotation at* v *induced by* S *for all* $v \in V(G)$ *and*

(2) (G,P,λ) *is a(n) (abstract) 3-polyhedron and the boundaries of* (G,P,λ) *are identical with the boundaries of the faces of* (S,G).

We shall call (P,λ) a boundary tour scheme for the topological polyhedron (S,G).

Proof. Let $P = \{P_v \mid v \in V(G)\}$ be any rotation system such that P_v is induced by S for all $v \in V(G)$. Given $e \in E(G)$ there are (unique) triangular faces F_1 and F_2 meeting along e (see Figure 5.1.1). Now $F_1 \cup F_2 = F$ is a disc. We note that $P_u^\delta(a_1) = e$, $P_u^\delta(e) = e_1$, $P_v^\varepsilon(e_2) = e$, $P_v^\varepsilon(e) = a_2$, for some $\delta, \varepsilon \in Z_2$, and (a_1,e,e_1) and (e_2,e,a_2) have the same (clockwise or counter clockwise) sense with respect to some clockwise orientation of F. There is a unique $\lambda(e) \in Z_2$ with $\varepsilon = \delta\lambda(e)$ and thus $(a_1,e,e_1) \subset P_u^\delta$ and $(e_2,e,a_2) \subset P_v^{\delta\lambda(e)}$. If S is orientable and endowed with an orientation and if P_u and P_v have the same clockwise or counter-clockwise sense, then $\delta = \varepsilon$ and $\lambda(e) = 1$ (see Corollary 5.1.2). By determining $\lambda(e)$ as described above for all $e \in E(G)$, we get a polyhedron (G,P,λ).

Let F be a face of (S,G) with boundary arcs $\vec{a}, \vec{b}$ and $\vec{c}$ oriented as shown in Figure 5.1.2. By the definition of λ we have $P_y^{\delta}(\vec{a}) = \overleftarrow{b}$, $P_z^{\delta\lambda(\vec{b})}(\vec{b}) = \overleftarrow{c}$, $P_x^{\delta\lambda(\vec{b})\lambda(\vec{c})}(\vec{c}) = \overleftarrow{a}$ and $P_y^{\delta\lambda(\vec{b})\lambda(\vec{c})\lambda(\vec{a})}(\vec{a}) = \overleftarrow{b}$. Hence $\lambda(\vec{a})\lambda(\vec{b})\lambda(\vec{c}) = 1$ and $((\vec{b},\delta)(\vec{c},\delta\lambda(\vec{b}))(\vec{a},\delta\lambda(\vec{b})\lambda(\vec{c})))$ is an orbit of P*. Thus the edges a,b and c form a boundary of (G,P,λ).

In the other direction let $(\ldots(\vec{a},\delta)(\vec{b},\delta\lambda(\vec{a}))(\vec{c},\delta\lambda(\vec{a})\lambda(\vec{b}))\ldots)$ be an orbit of P*. From the way λ was defined we conclude that a,b and c bound a face of (S,G) and also form a boundary in (G,P,λ). Therefore, (G,P,λ) is a 3-polyhedron and each boundary of (S,G) is a boundary of (G,P,λ) and viceversa. □

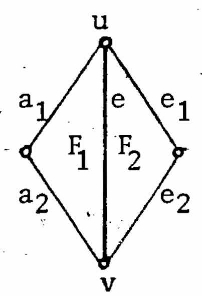


figure 5.1.1

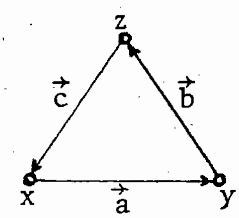


figure 5.1.2

COROLLARY 5.1.2. *If* (S,G) *is orientable and* S *is endowed with a clockwise sense of orientation, then* (P,λ) *can be chosen such that*

(1) P_v *is a clockwise rotation for each* $v \in V(G)$ *and*

(2) $\lambda(e) = 1$ *for each* $e \in E(G)$.

THEOREM 5.1.3. *Given a 3-polyhedron* (G,P,λ) *there is a topological 3-polyhedron* (S,$\tilde{G}$) *and a boundary tour scheme* ($\tilde{P}$,$\tilde{\lambda}$) *for* (S,$\tilde{G}$) *such that* (G,P,λ) *and* ($\tilde{G}$,$\tilde{P}$,$\tilde{\lambda}$) *are isomorphic.*

Proof. We choose a collection of pairwise disjoint triangles corresponding to the boundaries of (G,P,λ) and label their vertices and edges appropriately by the vertices and edges of the corresponding boundaries. Every edge of G is in exactly two distinct boundaries. Therefore this scheme labels exactly two edges belonging to distinct triangles. Also, if $v \in V(G)$ and $P_v = (a_1,\ldots,a_n)$ is the rotation at v, then a_i and a_{i+1} belong to the same boundary, for $1 \le i \le n$. Hence the classical side identification process used to identify identically labelled vertices and edges of the triangles yields a surface S and an S-graph $\tilde{G}$ whose edges and vertices are the edges and vertices of the identified triangles. The mapping ϕ which assigns a vertex $v \in V(G)$ to the vertex of $\tilde{G}$ labelled by v is an isomorphism of G and $\tilde{G}$. Moreover, $\phi P_v \phi^{-1} = \tilde{P}_{\phi(v)}$ is a rotation at $\phi(v)$ induced by S and if we define $\tilde{\lambda}(\phi(e)) = \lambda(e)$ for all $e \in E(G)$, then $(\tilde{P},\tilde{\lambda})$ is a boundary tour scheme for $(S,\tilde{G})$. It is then trivial that $\phi\colon (G,P,\lambda) \to (\tilde{G},\tilde{P},\tilde{\lambda})$ is an isomorphism. □

COROLLARY 5.1.4. *Let* (G,P) *be an orientable 3-polyhedron and* $(S,\tilde{G})$ *be the topological 3-polyhedron with the boundary tour scheme* $\tilde{P}$ *constructed in Theorem 5.1.3. Then* $(S,\tilde{G})$ *is orientable.*

Proof. We note that $\tilde{P}^*(E(G) \times \{1\}) = E(G) \times \{1\}$. Moreover, each face of $(S,\tilde{G})$ corresponds to an orbit of $\tilde{P}^*$ in $E(G) \times \{1\}$ and the orientations induced on them by those orbits are coherent. That is, the orientations induced on any edge by those of the adjacent faces are opposite. Therefore $(S,\tilde{G})$ is orientable. □

LEMMA 5.1.5. *Let* (G,P) *be a planar 3-polyhedron,* C *a cycle or 2-infinite two-sided path of* G *and let* $c \subset S^2$ *be a simple closed curve. Then there is a planar topological 3-polyhedron* $(\tilde{S},\tilde{G})$ *with boundary tour scheme* $\tilde{P}$ *so that*

(1) (G,P) *is isomorphic to* $(\tilde{G},\tilde{P})$ *and*

(2) if $\tilde{C}$ *is the subgraph of* $\tilde{G}$ *corresponding to* C *(under the isomorphism), then* $ps(\tilde{C}) = c$ *if* C *is a cycle, and* $ps(\tilde{C}) = c - \{x\}, x \in c$*, if* C *is a path.*

Proof. Let $(e_i)_{i \geq 1}$ be an enumeration of $E(G)$. (We know G is locally finite and connected so that $V(G) \cup E(G)$ is countable.) We shall recursively construct a sequence $(\tilde{G}_i)_{i \geq 0}$ of graphs and derive from it the required 3-polyhedron $(\tilde{S},\tilde{G})$. Given an isomorphism between two graphs H_1 and H_2 we shall denote the image of a subgraph K of H_1 by $\tilde{K}$.

Set $G_o = C$. If C is a cycle, let $\tilde{G}_o$ be an S-cycle, where $S = S^2$, is the sphere which is isomorphic to G_o and whose pointset is c. If C is a 2-infinite path, then let $\tilde{G}_o$ be an S-graph, where $S = S^2 - \{x\}$, which is isomorphic to G_o and whose pointset is $c - \{x\}$.

Now suppose $i \geq 0$ and we have constructed a subgraph G_i of G and an isomorphic S-graph $\tilde{G}_i$ meeting the following conditions:

(1) $G_i - G_o$ is finite;

(2) For all $e \in E(G) - E(G_i)$ there is a (unique) cycle or 2-in-

finite two-sided path in G_i separating e from G_i and, moreover, $\tilde{K}$ separates S into two regions both having $ps(\tilde{K})$ as their boundary;

(3) $\{e_1,\ldots,e_i\} \subset E(G_i)$; and

(4) If $\{a_1,\ldots,a_n\} = E_v(G_i)$, where $v \in V(G_i)$ and $n \geq 3$, and $(a_1,\ldots,a_n) \subset P_v$, then $(\tilde{a}_1,\ldots,\tilde{a}_n)$ is the clockwise rotation induced by S on $\{\tilde{a}_1,\ldots,\tilde{a}_n\} = E_{\tilde{v}}(\tilde{G}_i)$.

These assumptions trivially hold for $i = 0$. We shall now determine G_{i+1} and $\tilde{G}_{i+1}$. If $e_{i+1} \in E(G_i)$, then we set $G_{i+1} = G_i$ and $\tilde{G}_{i+1} = \tilde{G}_i$. Clearly (1) - (4) hold for $i+1$. If $e_{i+1} \notin E(G_i)$, then by (4) there is a cyle or 2-infinite two-sided path K in G_i separating e_{i+1} from G_i. Let K_i, $1 \leq i \leq 2$, be its sides and assume $e_{i+1} \subset K_1$ and $G_i \subset K_2$. Since G is 3-connected, there is a "chordal" path B in K_1 containing e_{i+1} and joining $u,v \in V(K)$. Let $K = (\ldots,u_o,a_1,u_1,\ldots)$ and assume $u = u_o$, $v = u_k$. There is a $\delta \in Z_2$ such that $(a_j,e,a_{j+1}) \subset P^{\delta}_{u_j}$ for $e \in E_{u_j}(G_i) - E(K)$ and $(a_o,a,a_1) \subset P^{-\delta}_{u_o}$, $(a_k,b,a_{k+1}) \subset P^{-\delta}_{u_k}$ for $a \in E_u(B)$, $b \in E_v(B)$. (We note that by (1) K contains disjoint 1-infinite subpaths of $G_o = C$ if K is a 2-infinite path. Moreover, $ps(\tilde{K}) \cup \{x\}$ (recall $x \in c$) is a simple closed curve in S^2 and thus divides S into two Jordan regions R_1 and R_2.) Now $\tilde{K} = (\ldots,\tilde{u}_o,\tilde{a}_1,\tilde{u}_1,\ldots)$ divides S into two regions R_1 and R_2 both having $ps(\tilde{K})$ as their boundary. Moreover, by (4), $ps(\tilde{G}_i) - ps(\tilde{K})$ is either in R_1 or R_2. Let us draw an isomorphic copy $\tilde{B}$ of B as a "crosscut" of R_1 or R_2 joining $\tilde{u}$ with $\tilde{v}$ so that $(\tilde{a}_o,\tilde{a},\tilde{a}_1)$ and $(\tilde{a}_k,\tilde{b},\tilde{a}_{k+1})$ are clockwise if $\delta = -1$ and counter-clockwise if $\delta = 1$. Then $ps(\tilde{G}_i) \subset \bar{R}_{3-j}$ if $ps(\tilde{B}) \subset \bar{R}_j$, by (4) and the choice of δ and $\tilde{B}$.

Therefore $\tilde{G}_{i+1} = \tilde{G}_i \cup \tilde{B}$ is an S-graph isomorphic to $G_{i+1} = G_i \cup B$ and (1) - (4) are easily seen to be satisfied for $i+1$.

Let G be the graph whose vertexset and edgeset are $\cup\{V(G_i) \mid 0 \le i < \infty\}$ and $\cup\{E(G_i) \mid 0 \le i < \infty\}$, respectively, and whose incidence relation is defined by $i_{\tilde{G}}(e) = i_{\tilde{G}_i}(e)$ for $e \in E(\tilde{G}_i)$. $\tilde{G}$ is isomorphic to G. Moreover, if $v \in V(G)$ and $(a_1,\ldots,a_n) = P_v$, then $(\tilde{a}_1,\ldots,\tilde{a}_n) = \tilde{P}_{\tilde{v}}$ is the clockwise rotation induced by S on $\{\tilde{a}_1,\ldots,\tilde{a}_n\} = E_{\tilde{v}}(\tilde{G})$. Thus (G,P) and $(\tilde{G},\tilde{P})$ are isomorphic.

We note that $\tilde{G}$ need not be an S-graph but we shall show that there is a subsurface $\tilde{S} \subset S$ containing $ps(\tilde{G})$ so that $(\tilde{S},\tilde{G})$ is a 3-polyhedron.

Each boundary $\tilde{B}$ of $(\tilde{G},\tilde{P})$ bounds a disc $F \subset S^2$ where $F \cap ps(\tilde{G}) = ps(\tilde{B})$. We shall call F a face of $\tilde{G}$ in S^2. We note that if $\tilde{v} \in V(\tilde{G})$ and $\tilde{P}_{\tilde{v}} = (\tilde{e}_1,\ldots,\tilde{e}_n)$, then there are faces F_i containing $\tilde{a}_i$ and $\tilde{a}_{i+1}$ so that $F_i \cap F_{i+1} = \{e_{i+1}\}$ and $F_1 \cup \ldots \cup F_n = F$ is a disc containing v in its interior. Thus the union $\tilde{S}$ of all faces of $\tilde{G}$ in S^2 is a surface and $(\tilde{S},\tilde{G})$ is a 3-polyhedron. □

THEOREM 5.1.6. *Let* (G,P) *be a planar 3-polyhedron. Then there is a planar topological 3-polyhedron* $(\tilde{S},\tilde{G})$ *with boundary tour scheme* $\tilde{P}$ *so that* (G,P) *is isomorphic to* $(\tilde{G},\tilde{P})$.

Proof. This follows immediately from Lemma 5.1.5 and concludes this section. □

LEMMA 5.1.7. *Let* (S,G) *be a 3-polyhedron with a boundary tour scheme* (P,λ) *and let* C *be a cycle or 2-infinite path in* G. *If* $E_v(C) = \{e_1,e_2\}$, $a_1,a_2 \in E_v(G) - E(C)$, $(e_1,a_1,e_2) \subset P_v^\delta$ *and* $(e_1,a_2,e_2) \subset P_v^\delta$, *then* $a_1 - i_G(a_1)$ *and* $a_2 - i_G(a_2)$ *belong to the same component of* $S - ps(C)$.

Proof. Clearly it suffices to prove the lemma in the case that $P_v^\delta(a_1) = a_2$. But in this case the statement is trivial as a_1 and a_2 are in the boundary of a face. □

LEMMA 5.1.8. *Let* (S,G), (P,λ) *and* C *as in Lemma 5.1.7, let* $Q = (v_o,e_1,v_1,\ldots,v_{n-1},e_n,v_n) \neq C$ *be a subpath of* C *and let* $e_o \in E_{v_o}(C) - E(Q)$, $e_{n+1} \in E_{v_n}(C) - E(Q)$. *If* $a \in E_{v_o}(G) - \{e_o,e_1\}$ *and* $b \in E_{v_n}(G) - \{e_n,e_{n+1}\}$ *and if* $(e_o,a,e_1) \subset P_{v_o}^\delta$ *and* $(e_n,b,e_{n+1}) \subset P_{v_n}^{\delta\lambda(Q)}$, *then* $a - i_G(a)$ *and* $b - i_G(b)$ *belong to the same component of* $S - ps(C)$.

Proof. We prove the lemma by induction on the length n of Q. If $n = 1$, then in view of Lemma 5.1.7 we may assume $P_{v_o}^\delta(a) = e_1$ and $P_{v_1}^{\delta\lambda(Q)}(e_1) = b$, that is, a,e_1 and b form a boundary. The lemma then is trivially true. Now let $n = 2$. In case there is some $c \in E_{v_1}(G) - E(C)$ with $(e_1,c,e_2) \subset P_{v_1}^{\delta\lambda(e_1)}$, the lemma follows from the case $n = 1$. Otherwise, in view of the previous lemma, the existence of an edge a with $P_{v_o}^\delta(a) = e_1$, $P_{v_1}^{\delta\lambda(e_1)}(e_1) = e_2$ and $P_{v_2}^{\delta\lambda(e_1)\lambda(e_2)}(e_2) = a$ (see Figure 5.1.4) settles the case $n = 2$.

Now assume Q has length $n+1$ and that the claim is true for every path of length less than or equal to n. Let $Q = (v_o, e_1, \ldots, e_{n+1}, v_{n+1})$ and $a \in E_{v_o}(G) - E(C)$, $b \in E_{v_{n+1}}(G) - E(C)$ such that $(e_o, a, e_1) \subset P^{\delta}_{v_o}$ and $(e_{n+1}, b, e_{n+2}) \subset P^{\delta\lambda(Q)}_{v_{n+1}}$. Since every boundary is triangular there is a $c \in E_{v_i}(G) - E(C)$ with $(e_i, c, e_{i+1}) \subset P^{\delta\lambda(Q')}_{v_i}$ where $i \in \{n-1, n\}$, $Q' = (v_o, e_1, \ldots, v_i)$ and $Q'' = (v_i, \ldots, e_{n+1}, v_{n+1})$. We conclude the induction proof by applying the hypothesis to the paths Q' and Q" which are both of length at most n. □

LEMMA 5.1.9. *Let* (S,G), (P,λ) *and* C *be as in Lemma 5.1.7. If* C *is one-sided, then* S - ps(C) *is connected.*

Proof. First we consider the case that C is a cycle which is not λ-trivial. Let $u, v \in V(C)$, $u \neq v$ and $a \in E_u(G) - E(C)$, $b \in E_v(G) - E(C)$. We choose edges e_1, e_2, c_1, c_2 and paths Q_1, Q_2 as shown in Figure 5.1.3.

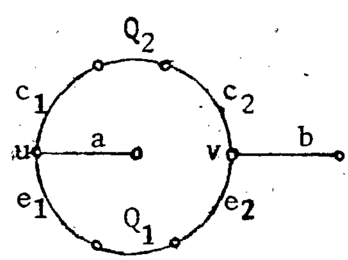


figure 5.1.3

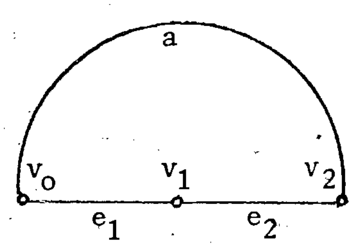


figure 5.1.4

If $(e_1, a, c_1) \subset P^{\delta}_u$ and $(c_2, b, e_2) \subset P^{\varepsilon}_v$, where $\delta, \varepsilon \in Z_2$, then $\varepsilon = \delta\lambda(Q_1)$ or $\varepsilon = \delta\lambda(Q_2)$ since $\lambda(Q_1) \neq \lambda(Q_2)$. Therefore, $a - i_G(a)$ and $b - i_G(b)$ belong to the same component of S - ps(C), as follows from Lemma 5.1.8 and, moreover, ps(G) - ps(C) is contained in the same component of

$S - ps(C)$.

Now consider the case where C is a λ-trivial cycle or a 2-infinite path with associated relation ω. Since C is one-sided there are edges $a,b \in E(G)$, and $i \in Z_2$ such that $i \in \omega(a)$, $-i \in \omega(b)$ and $a - i_G(a)$ and $b - i_G(b)$ belong to the same component of $S - ps(C)$. On the other hand, Lemmas 5.1.7 and 5.1.8 imply that $e - i_G(e)$ and $f - i_G(f)$ belong to the same component of $S - ps(C)$ if $Z_2 \cap \omega(e) \cap \omega(f) \neq \emptyset$. We conclude that $ps(G) - ps(C)$ is contained in the same component of $S - ps(C) = S'$. C is not a boundary and therefore S' is connected. □

LEMMA 5.1.10. *Let* (S,G), (P,λ) *and* C *be as in Lemma 5.1.7. If* C *is two-sided, then* $S - ps(C)$ *is disconnected.*

Proof. Assume C is two-sided and C_1, C_2 are the sides of C in (G,P,λ). Each boundary of (G,P,λ) is in C_1 or in C_2. Let O_i' be the union of all faces whose boundary is in C_i and $O_i = O_i' - ps(C)$. We claim that $S - ps(C)$ is disconnected. To prove this let e be an arc in S joining $u_1 \in O_1$ with $u_2 \in O_2$. The collection of faces of (S,G) is locally finite and each face is compact. Hence there is a last point w on the arc e from u_1 to u_2 which lies in the face F_1 contained in O_1'. Then w lies on the boundary B_1 of F_1 and also on the boundary B_2 of a face $F_2 \subset O_2'$. Thus w is contained in a common vertex or edge of B_1 and B_2 so that $w \in ps(C)$. It follows that $e \cap ps(C) \neq \emptyset$, that is, O_1 and O_2 are in distinct components of $S' = S - ps(C)$. This shows that S' is disconnected. □

THEOREM 5.1.11. *Let* (S,G), (P,λ) *and* C *be as in Lemma 5.1.7. Then* C *is two-sided if and only if* $S - ps(C)$ *is disconnected.*

Proof. This follows from Lemma 5.1.9 and 5.1.10. □

THEOREM 5.1.12. *Let* (S,G) *be a topological 3-polyhedron with boundary tour scheme* (P,λ). (S,G) *is planar if and only if* (G,P,λ) *is planar.*

Proof. Assume (S,G) is planar. If C is a cycle in G, then $S - ps(C)$ is disconnected (by Jordan's Theorem). By Theorem 5.1.11 C is two-sided in (G,P,λ). It follows that (G,P,λ) is planar.

Now assume that (G,P,λ) is planar. (G,P,λ) is orientable and therefore isomorphic to the 3-polyhedron (G,Q). By Theorem 5.1.6 there is a planar topological 3-polyhedron $(\tilde{S},\tilde{G})$ with the boundary tour scheme $\tilde{Q}$ such that (G,Q) and $(\tilde{G},\tilde{Q})$ are isomorphic. In view of Theorem 5.2.2, $(\tilde{S},\tilde{G})$ and (S,G) are homeomorphic. Therefore (S,G) is planar. □

§2. DISCONTINUOUS HOMEOMORPHISM GROUPS AND AUTOMORPHISMS OF POLYHEDRA.

LEMMA 5.2.1. *Let* (S_i, G_i) *be topological 3-polyhedra with boundary tour schemes* (P_i, λ_i) , *for* $1 \le i \le 2$. *If* $\tilde{\phi} : (S_1, G_1) \to (S_2, G_2)$ *is a homeomorphism, then the induced isomorphism* $\phi : G_1 \to G_2$ *is an isomorphism from* (G_1, P_1, λ_1) *to* (G_2, P_2, λ_2). *Conversely, if* ϕ *is an isomorphism from* (G_1, P_1, λ_1) *to* (G_2, P_2, λ_2), *then there is a homeomorphism* $\tilde{\phi} : (S_1, G_1) \to (S_2, G_2)$ *inducing* ϕ.

Proof. In order to prove the first statement, let $\tilde{\phi} : (S_1, G_1) \to (S_2, G_2)$ be a homeomorphism, and let $\phi : G_1 \to G_2$ be the induced isomorphism, that is, $\tilde{\phi}(x) = \phi(x)$ for all $x \in V(G_1) \cup E(G_1)$. Now $\tilde{\phi}$ preserves the rotations P_{i_v} so that $\phi P_{1_v} \phi^{-1} \in \{P^{\delta}_{2_{\phi(v)}} \mid \delta \in Z_2\}$ for all $v \in V(G_1)$. Let $e \in E(G_1)$, $i_{G_1}(e) = \{u, v\}$, $\phi P_{1_u} \phi^{-1} = P^{\delta}_{2_{\phi(u)}}$ and $\phi P^{\lambda(e)}_{1_v} \phi^{-1} = P^{\varepsilon}_{2_{\phi(v)}}$. We shall show that $\varepsilon = \delta\mu(\phi(e))$ and thus conclude that ϕ is an isomorphism. Let e_1 and e_2 be edges such that $P_{1_u}(e_1) = e$ and $P^{\lambda(e)}_{1_v}(e) = e_2$. Thus e, e_1, e_2 bound a face, say F, (see Figure 5.2.1) and $\phi(e) = e'$, $\phi(e_1) = e'_1$ and $\phi(e_2) = e'_2$ bound the face $\tilde{\phi}(F)$. Moreover, $P^{\delta}_{2_{u'}}(e'_1) = e'$ and $P^{\varepsilon}_{2_{v'}}(e') = e'_2$ so that $\varepsilon = \delta\lambda_2(e')$ because (P_2, λ_2) is a boundary tour scheme for (S_2, G_2). This proves the first statement.

Now assume the isomorphism $\phi : (G_1, P_1, \lambda_1) \to (G_2, P_2, \lambda_2)$ is given. It is easy to see that there is a homeomorphism $\tilde{\phi}_1 : ps(G_1) \to ps(G_2)$ which induces ϕ and since ϕ maps boundaries of faces of (S_1, G_1) into boundaries of faces of (S_2, G_2), $\tilde{\phi}_1$ can be extended to a homeomorphism

$\tilde{\phi} : (S_1,G_1) \to (S_2,G_2)$ by applying Corollary 1.6.6 to each face F of (S_1,G_1) and the corresponding face $\phi(F)$ of (S_2,G_2). □

THEOREM 5.2.2. *Let* (S_i,G_i) *be topological 3-polyhedra with boundary tour scheme* (P_i,λ_i) , *for* $1 \le i \le 2$. *Then* (S_1,G_1) *and* (S_2,G_2) *are homeomorphic if and only if* (G_1,P_1,λ_1) *and* (G_2,P_2,λ_2) *are isomorphic.*

Proof. This theorem is a direct consequence of Lemma 5.2.1. □

COROLLARY 5.2.3. *Let* (S,G) *be a 3-polyhedron with boundary tour scheme* (P,λ). *Then* (S,G) *is orientable if and only if* (G,P,λ) *is orientable.*

Proof. If (S,G) is orientable then by Corollary 5.1.2 it also has a boundary tour scheme Q. By Theorem 5.2.2, (G,P,λ) and (G,Q) are isomorphic. Thus (G,P,λ) is orientable. If (G,P,λ) is orientable, then there is some boundary tour scheme Q such that (G,P,λ) and (G,Q) are isomorphic. According to Corollary 5.1.2 there is some orientable topological 3-polyhedron $(\tilde{S},\tilde{G})$ with boundary tour scheme $\tilde{Q}$ such that $(\tilde{G},\tilde{Q})$ and (G,Q) are isomorphic. By Theorem 5.2.2, (S,G) and $(\tilde{S},\tilde{G})$ are homeomorphic which implies that S is orientable. □

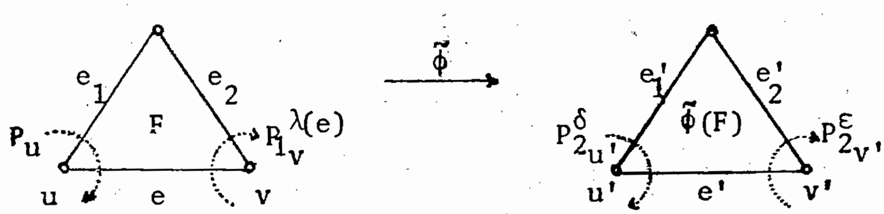


figure 5.2.1

THEOREM 5.2.4. *Let* (S,G) *be a polyhedron and assume* G *is not a cycle. Given a group* $\tilde{\Gamma}$ *of homeomorphisms of* (S,G) *, the canonical homomorphism* $\phi:\tilde{\Gamma} \to \mathrm{Aut}(G)$ *is a monomorphism if* $\tilde{\Gamma}$ *acts discontinuously on* S.

Proof. Assume $\tilde{\Gamma}$ acts discontinuously on S. Let $\tilde{\gamma} \in \tilde{\Gamma}$ and assume $\phi(\tilde{\gamma}) = \iota$. Then $\tilde{\gamma}$ fixes every vertex and thus every edge of G. Moreover, for every $e \in E(G)$, $\langle\tilde{\gamma}\rangle|_e$ is a discontinuous group of the arc e fixing its endpoints. As noted in Chapter 2, §3, $\tilde{\gamma}|_e = \iota|_e$, and therefore $\tilde{\gamma}|_{ps(G)} = \iota|_{ps(G)}$. Now $\tilde{\gamma}$ fixes every face F of (S,G) since G is not a cycle. Thus $\langle\tilde{\gamma}\rangle|_F$ is a discontinuous homeomorphism group of the disc F fixing its boundary pointwise. By Corollary 1.6.5, $\tilde{\gamma}|_F = \iota|_F$ and therefore $\tilde{\gamma} = \iota$. Hence ϕ is a monomorphism. □

We shall also prove a partial converse for Theorem 5.2.4.

THEOREM 5.2.5. *Let* (S,G) *be a 3-polyhedron with boundary tour scheme* (P,λ) *and let* $\tilde{\Gamma}$ *be a group of homeomorphisms of* (S,G). *If the canonical homomorphism* $\phi:\tilde{\Gamma} \to \mathrm{Aut}(G,P,\lambda)$ *is 1-1, then* $\tilde{\Gamma}$ *acts discontinuously on* S.

Proof. We shall prove this theorem by a contrapositive argument. Assume $\tilde{\Gamma}$ does not act discontinuously on S. Then there are points x and y in S such that $\{\tilde{\gamma} \in \tilde{\Gamma} \mid \tilde{\gamma}(V_x) \cap V_y \neq \emptyset\}$ is infinite for all neighbourhoods V_x of x and V_y of y. For all $z \in S$ let U_z be the neighbourhood consisting of the union of all faces containing z (there are finitely many such faces). Thus $\{\tilde{\gamma} \in \tilde{\Gamma} \mid \tilde{\gamma}(U_x) \cap U_y \neq \emptyset\}$ is infinite. It follows that there are two faces F_1 and F_2 such that $\{\tilde{\gamma} \in \tilde{\Gamma} \mid \tilde{\gamma}(F_1) \cap F_2 \neq \emptyset\}$ is infinite and since each face meets only finitely many other faces, there is a face F such that $\tilde{\Gamma}_F = \{\tilde{\gamma} \in \tilde{\Gamma} \mid \tilde{\gamma}(F) = F\}$ is infinite. On the other hand, $\Phi(\tilde{\Gamma}_F)$ is finite by Proposition 4.1.2. Hence Φ is not 1-1 which is a contradiction. □

COROLLARY 5.2.6. *Let* (S,G), (P,λ) *and* $\tilde{\Gamma}$ *be as in Theorem 5.2.4. If* Φ *is not a monomorphism, then the kernel of* Φ *is infinite.*

THEOREM 5.2.7. *Let* (S,G) *be a 3-polyhedron with boundary tour scheme* (P,λ). *Let* Γ *be a subgroup of* Aut(G,P,λ) *and assume that*

(a) $\gamma(e) = e$ *implies* $\gamma(u) = u$ *for* $u \in i_G(e)$ *where* $\gamma \in \Gamma$ *and* $e \in E(G)$ *and*

(b) $\gamma(B) = B$ *implies* $\gamma = \iota$ *where* B *is a boundary of* (G,P,λ) *and* $\gamma \in \Gamma$.

Then there is a discontinuous homeomorphism group $\tilde{\Gamma}$ *of* (S,G) *with* $\Phi(\tilde{\Gamma}) = \Gamma$.

Proof. Condition (a) is equivalent to (a'): $\gamma_1|_{i_G(e)} = \gamma_2|_{i_G(e)}$ whenever $\gamma_1, \gamma_2 \in \Gamma$ and $\gamma_1(e) = \gamma_2(e)$. Condition (b) is equivalent to (b'): $\gamma_1(B) = \gamma_2(B)$ implies $\gamma_1 = \gamma_2$ where B is a boundary of (S,G) and $\gamma_1, \gamma_2 \in \Gamma$. We introduce some notation. Let $X_o = \{\{v\} \mid v \in V(G)\}$,

$X_1 = E(G)$ and $X_2 = F(S,G)$. Given $x \in X_2$ with boundary B and $\gamma \in \Gamma$, then let $\gamma(x)$ denote the face with boundary $\gamma(B)$. If $x \in X_i$, let $\partial x = \cup\{y \mid y \subset x,\ y \in X_j,\ 0 \le j < i\}$, that is $\partial x = \emptyset$ if $x \in X_0$, $\partial x = i_G(x)$ for $x \in X_1$ and ∂x is the frontier of x in S if $x \in X_2$. Let $Y_i \subset X_i$ be a set consisting of one object from each Γ-orbit. Let Z_i be the set of all pairs $(x,y) \in X_i^2$ of Γ-equivalent objects x,y.

We shall recursively construct a system $\{\chi_{xy} \mid (x,y) \in Z_2\}$ of homeomorphisms and use it to define a homeomorphism $\tilde{\gamma}$ for each $\gamma \in \Gamma$. Let $0 \le k \le 1$ and assume we have constructed a system $\{\chi_{xy} \mid (x,y) \in Z_j,\ 0 \le j \le k\}$ of homeomorphisms $\chi_{xy} : x \to y$ such that the following conditions are met.

(1) If $(x,y) \in Z_j$, then $\chi_{xy}|_u = \chi_{u\delta(u)}$ whenever $u \subset x$, $u \in X_i$, $0 \le i \le j$ and $\delta(x) = y$ for $\delta \in \Gamma$.

(2) $\chi_{xx}(u) = u$ for all $u \in x$ and all $x \in X_j$.

(3) $\chi_{xy} = \chi_{yx}^{-1}$ if $(x,y) \in Z_j$.

(4) $\chi_{xz} = \chi_{yz} \circ \chi_{xy}$ if $(x,y),(y,z) \in Z_j$.

(5) $\chi_{x\gamma(x)}|_{x \cap y} = \chi_{y\gamma(y)}|_{x \cap y}$ for all $x,y \in X_j$ and $\gamma \in \Gamma$.

For k = 0 these assumptions hold trivially if we define $\chi_{\{x\}\{\gamma(x)\}}(x) = \gamma(x)$ for every $x \in X_0$ and $\gamma \in \Gamma$. We shall now construct a system $\{\chi_{xy} \mid (x,y) \in Z_{k+1}\}$ of homeomorphisms $\chi_{xy} : x \to y$ such that (1) - (5) hold with $k+1$ instead of k.

We start by defining $\chi_{xx}(u) = u$ for all $x \in X_{k+1}$ and all $u \in x$, noting that this definition agrees with (2) - (5). In order to show that it also agrees with (1), we recall from (a') and (b') that $\delta(x) = x$ implies $\delta(u) = u$ for all $u \in X_i$ with $u \subset x$ and $i \le k+1$. Thus $\chi_{xx}|_u =$

$\chi_{uu} = \chi_{u\delta(u)}$.

We now claim that for all $x \in Y_{k+1}$ and $\gamma \in \Gamma$ there is a homeomorphism $\chi_{x\gamma(x)}: x \to \gamma(x)$ such that $\chi_{x\gamma(x)}|_u = \chi_{u\delta(u)}$ for $u \in X_j$, $u \subset x$, $j < k+1$ and for all $\delta \in \Gamma$ with $\gamma(x) = \delta(x)$. We distinguish the cases $k = 0$ and $k = 1$. In case $k = 0$, $\delta(u) = \gamma(u)$ if $\delta(x) = \gamma(x)$, and $u \in X_o$, $u \subset x \in Y_1$, by (a'). Hence there is a homeomorphism $\chi_{x\gamma(x)}: x \to \gamma(x)$ with the desired property. In case $k = 1$ we recall that for $x \in X_2$, $\gamma(x) = \delta(x)$ implies $\gamma = \delta$. Let x_1, x_2, x_3 be the edges of the face x (as in Figure 5.2.2).

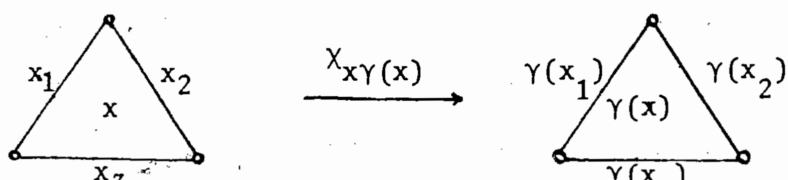


figure 5.2.2

By assumption, $\chi_{x_i\gamma(x_i)}(u) = \chi_{u\gamma(u)} = \gamma(u)$ for $u \subset x_i$, $u \in X_o$. Hence there is a homeomorphism $\chi': \partial x \to \partial\gamma(x)$ such that $\chi'|_z = \chi_{z\gamma(z)}$ for all $z \in X_j$, $0 \le j \le k = 1$, with $z \subset \partial x$. By Corollary 1.6.5 there is a homeomorphism $\chi_{x\gamma(x)}: x \to \gamma(x)$ such that $\chi_{x\gamma(x)}|_z = \chi_{z\gamma(z)}$ for $z \in X_j$, $0 \le j \le k+1$, with $z \subset x$. This proves the claim.

Having thus defined $\chi_{x\gamma(x)}$ for all $x \in Y_{k+1}$ and $\gamma \in \Gamma$ we define $\chi_{\gamma(x)x} = \chi^{-1}_{x\gamma(x)}$. We note that χ_{xy} has so far been defined in case $x = y$ and in case $(x,y) \in Z_{k+1}$ and $\{x,y\} \cap Y_{k+1} \neq \emptyset$ and this definition is in agreement with conditions (1) - (5).

If $(x,y) \in Z_{k+1}$ and $x,y \notin Y_{k+1}$, we define $\chi_{xy} = \chi_{yz} \circ \chi_{xz}$, where $(x,z) \in Z_{k+1}$. We now claim that the collection $\{\chi_{xy} \mid (x,y) \in Z_{k+1}\}$

defined satisfies conditions (1) - (5) with $k+1$ instead of k. Notice that conditions (1) - (5) only need to be checked for $j = k+1$ as they are true for $j \leq k$ by assumption. Now (2) is true by definition. Condition (3) needs only be checked for $(x,y) \in Z_{k+1}$, $x,y \notin Y_{k+1}$ as for all other pairs it is already known to be true. We have $\chi_{xy} = \chi_{zy} \circ \chi_{xz}$, where $(x,z) \in Z_{k+1}$, $z \in Y_{k+1}$ and $\chi_{yx} = \chi_{zx} \circ \chi_{yz} = \chi_{xz}^{-1} \circ \chi_{zy}^{-1} = (\chi_{zy} \circ \chi_{xz})^{-1} = \chi_{xy}^{-1}$.

To prove (4) let $(x,y),(y,z) \in Z_{k+1}$ and we distinguish two cases. In case 1, $x,y,z \notin Y_{k+1}$ and case 2 is the negation of case 1.

Case 1: If $x,y,z \notin Y_{k+1}$, then $(x,u) \in Z_{k+1}$ for $u \in Y_{k+1}$. Thus $\chi_{xy} = \chi_{uy} \circ \chi_{xu}$, $\chi_{yz} = \chi_{uz} \circ \chi_{yu}$ and $\chi_{xz} = \chi_{uz} \circ \chi_{xu}$ by definition. Hence $\chi_{yz} \circ \chi_{xy} = \chi_{uz} \circ (\chi_{yu} \circ \chi_{uy}) \circ \chi_{xu} = \chi_{uz} \circ \chi_{uu} \circ \chi_{xu} = \chi_{uz} \circ \chi_{xu} = \chi_{xz}$.

Case 2: $\{x,y,z\} \cap Y_{k+1} \neq \emptyset$. If $x \in Y_{k+1}$, then $\chi_{yz} = \chi_{xz} \circ \chi_{yz}$ by definition. Hence $\chi_{yz} \circ \chi_{xy} = \chi_{xz}$. A similiar argument works if $y \in Y_{k+1}$ or $z \in Y_{k+1}$.

To prove (1) let $(x,y),(y,z) \in Z_{k+1}$, $z \in Y_{k+1}$, $x,y \notin Y_{k+1}$, and let $\gamma(z) = x$, $\delta(z) = y$. Hence $\delta\gamma^{-1}(x) = y$. It suffices to prove that $\chi_{xy}|_u = \chi_{u\beta(u)}$, for all $u \in X_k$, $u \subset x$ and $\beta \in \Gamma$ with $\beta(x) = y$. $\chi_{xz}|_u = \chi_{u\gamma^{-1}(u)}$ and $\chi_{zy}|_{\gamma^{-1}(u)} = \chi_{\gamma^{-1}(u)\delta\gamma^{-1}(u)}$ for all $u \in X_k$ with $u \subset x$ by the above note. Thus $\chi_{xy}|_u = \chi_{u\delta\gamma^{-1}(u)}$ and by assumption $\chi_{u\delta\gamma^{-1}(u)} = \chi_{u\beta(u)}$ for all $\beta \in \Gamma$ with $\beta(x) = y$.

To prove (5) let $x,y \in X_{k+1}$, $x \neq y$, $x \cap y \neq \emptyset$ and let $\gamma \in \Gamma$. Now $x \cap y = \cup\{z \mid z \subset x \cap y,\ z \in \cup\{X_j \mid j \leq k\}\}$. By condition (1), $\chi_{x\gamma(x)}|_z =$

$\chi_{z\gamma(z)} = \chi_{y\gamma(y)}|_z$ for all $z \in \cup\{X_j \mid j \le k\}$ with $z \subset x \cap y$. Thus $\chi_{x\gamma(x)}|_{x\cap y} = \chi_{y\gamma(y)}|_{x\cap y}$.

In view of condition (5) for $\{\chi_{xy} \mid (x,y) \in Z_2\}$ it is possible to define a mapping $\tilde{\gamma}: S \to S$ for each $\gamma \in \Gamma$ through its restriction at the faces of (S,G), that is, if $x \in X_2$, define $\tilde{\gamma}|_x = \chi_{x\gamma(x)}$. It follows that if $\gamma = \iota$, then $\tilde{\gamma}$ is the identity mapping on S. It is clear that $\tilde{\gamma}$ is onto S. Moreover, since $\tilde{\gamma}|_x$ is 1-1, for $x \in X_2$, and $\tilde{\gamma}(x) = \gamma(x) \ne \gamma(y) = \tilde{\gamma}(y)$ if $x \ne y \in X_2$, it follows that $\tilde{\gamma}$ is 1-1.

Let $\tilde{\Gamma} = \{\tilde{\gamma} \mid \gamma \in \Gamma\}$. We now claim that:

(a) $\tilde{\gamma}$ is a homeomorphism of (S,G) and $\Phi(\tilde{\gamma}) = \gamma$ (where $\Phi(\tilde{\gamma})$ is the induced isomorphism);

(b) $\tilde{\Gamma}$ is a group, and $\psi: \Gamma \to \tilde{\Gamma}$, defined by $\psi(\gamma) = \tilde{\gamma}$ is an isomorphism; and

(c) $\tilde{\Gamma}$ is discontinuous.

To prove (a) we note that $\tilde{\gamma}|_{\bar{x}}$ and $\tilde{\gamma}^{-1}|_y$ are continuous for all $x,y \in X_2$. Now X_2 is a locally finite collection of closed sets whose union is S. By Proposition 1.4.2 $\tilde{\gamma}$ and $\tilde{\gamma}^{-1}$ are continuous, that is, $\tilde{\gamma}$ is a homeomorphism of S. Let $u \in X_o \cup X_1$, $u \subset x \in X_2$ and $\gamma \in \Gamma$. Then $\tilde{\gamma}(u) = (\tilde{\gamma}|_x)(u) = \chi_{x\gamma(x)}(u) = \chi_{u\gamma(u)}(u) = \gamma(u)$ so that $\Phi(\tilde{\gamma}) = \gamma$.

To prove (b) we know ψ is onto by definition. Moreover, since γ_i is induced by $\tilde{\gamma}_i$, $\gamma_1 \ne \gamma_2$ implies $\tilde{\gamma}_1 \ne \tilde{\gamma}_2$ and hence ψ is 1-1. Thus we only have to show that ψ is a homomorphism. Let $\gamma, \gamma_1, \gamma_2 \in \Gamma$ and $x \in X_2$. Now $\widetilde{\gamma_2\gamma_1}|_x = \chi_{\gamma_1(x)\gamma_2\gamma_1(x)} \circ \chi_{x\gamma_1(x)} = \tilde{\gamma}_2|_{\gamma_1(x)} \circ \tilde{\gamma}_1|_x$. Thus $\widetilde{\gamma_2\gamma_1} = \tilde{\gamma}_2 \circ \tilde{\gamma}_1$. Now $\gamma\gamma^{-1} = \iota$ and hence $\widetilde{\gamma\gamma^{-1}} = \tilde{\gamma}\widetilde{\gamma^{-1}} = \tau$. It follows that

$\tilde{\gamma}^{-1} = \widetilde{\gamma^{-1}}$. Thus ψ is a homomorphism and since ψ is onto $\tilde{\Gamma}$, $\tilde{\Gamma}$ is a group.

We note that $\Phi = \psi^{-1}$ is a monomorphism. Hence by Theorem 5.2.5, $\tilde{\Gamma}$ is discontinuous which proves (c).

COROLLARY 5.2.8. *Let* (S,G) *be a 3-polyhedron with boundary tour scheme* (P,λ) *and let* Γ *be a subgroup of* Aut(G,P,λ) . *Then there is a discontinuous homeomorphism group* $\tilde{\Gamma}$ *of* (S,G) *inducing* Γ.

Proof. Let (S,G') be the first order barycentric subdivision of (S,G) and let (P',λ') be its boundary tour scheme. The group Γ induces a subgroup Γ' of Aut(G',P',λ') isomorphic to Γ which satisfies conditions (a) and (b) of Theorem 5.2.7. We construct a discontinuous group $\tilde{\Gamma}$ according to Theorem 5.2.7 and it follows that $\tilde{\Gamma}$ induces Γ. □

THEOREM 5.2.9. *For* $1 \le i \le 2$, *let* $\tilde{\Gamma}_i$ *be a discontinuous homeomorphism group of the 3-polyhedron* (S_i,G_i) *with boundary tour schemes* (P_i,λ_i) *and let* $\Gamma_i \subset \mathrm{Aut}(G_i,P_i,\lambda_i)$ *be the induced groups. If there is an isomorphism* $\sigma : (G_1,P_1,\lambda_1) \to (G_2,P_2,\lambda_2)$ *such that* $\sigma\Gamma_1\sigma^{-1} = \Gamma_2$, *then* $\tilde{\Gamma}_1$ *and* $\tilde{\Gamma}_2$ *are topologically equivalent.*

Proof. It is sufficient to show that there is a homeomorphism $\Sigma : (S_1,G_1) \to (S_2,G_2)$ such that $\tilde{\Gamma}_2 = \Sigma\tilde{\Gamma}_1\Sigma^{-1}$. Before going ahead with the proof we introduce some notation. Let $X_i^o = \{\{v\} \mid v \in V(G_i)\}$, $X_i^1 = E(G_i)$, $X_i^2 = F(S_i,G_i)$ and $Z_i^j = \cup\{x \mid x \in X_i^j\}$ for $1 \le i \le 2$, $0 \le j \le 2$. We note that $Z_i^o = V(G_i)$, $Z_i^1 = ps(G_i)$ and $Z_i^2 = S_i$. If $x \in X_i^k$, then let $\partial x = \cup\{y \mid y \subset x,\ y \in X_i^j,\ 0 \le j < k\}$. Let Y_1^j, $1 \le j \le 2$, be a subset of X_1^j containing exactly one object from each orbit with respect to Γ_1. Set $Y_2^j = \{\sigma(y) \mid y \in Y_1^j\}$. Then $Y_2^j \subset X_2^j$ and Y_2^j contains exactly one object from each orbit with respect to Γ_2. We shall use the symbols γ_i, δ_i etc., respectively $\tilde{\gamma}_i, \tilde{\delta}_i$ etc., to denote the objects of Γ_i, respectively $\tilde{\Gamma}_i$. Moreover, we shall assume that $\Phi_i(\tilde{\gamma}_i) = \gamma_i$, $\Phi_i(\tilde{\delta}_i) = \delta_i$ etc., and $\gamma_2 = \sigma\gamma_1\sigma^{-1}$, $\delta_2 = \sigma\delta_1\sigma^{-1}$ (where $\Phi_i : \tilde{\Gamma}_i \to \Gamma_i$ is the canonical monomorphism).

We note that if $x \in X_1^j$ and $y = \sigma(x)$, then $\tilde{\gamma}_1 \in \tilde{\Gamma}_{1_x}$ if and only if $\gamma_1 \in \Gamma_{1_x}$ if and only if $\gamma_2 \in \Gamma_{2_y}$ if and only if $\tilde{\gamma}_2 \in \tilde{\Gamma}_{2_y}$. Also, for simplicity of notation we shall not distinguish distinct functions with the same domain if they take identical values.

We shall construct the homeomorphism Σ recursively. Assume that $0 \le k \le 1$, and that for each j, $0 \le j \le k$, we have defined a homeomorphism $\Sigma_j : Z_1^j \to Z_2^j$ such that

(i) $\Sigma_j|_{Z_1^i} = \Sigma_i$ for $0 \le i \le j$ and

(ii) $\tilde{\gamma}_2|_{Z_2^j} = \Sigma_j \tilde{\gamma}_1|_{Z_1^j} \Sigma_j^{-1}$, for all $\tilde{\gamma}_1 \in \tilde{\Gamma}_1$ (or equivalently for all $\tilde{\gamma}_2 \in \tilde{\Gamma}_2$).

This assumption holds for $k = 0$ by defining $\Sigma_o : Z_1^o \to Z_2^o$ by $\Sigma_o(x) = \sigma(x)$ for all $x \in Z_1^o$. Then (i) is trivially satisfied and (ii) holds since $\Sigma_o = \sigma$, $\tilde{\gamma}_1|_{Z_1^o} = \gamma_1$ and $\tilde{\gamma}_2|_{Z_2^o} = \gamma_2$. Thus $\tilde{\gamma}_2|_{Z_2^o} = \gamma_2 = \sigma\gamma_1\sigma^{-1} = \Sigma_o\tilde{\gamma}_1|_{Z_1^o}\Sigma_o^{-1}$.

We shall construct a homeomorphism $\Sigma_{k+1} : Z_1^{k+1} \to Z_2^{k+1}$ satisfying conditions (i) and (ii) with $k+1$ instead of k. Pick $x \in Y_1^{k+1}$ and put $y = \sigma(x)$. Then $\partial x \subset Z_1^k$, $\partial y \subset Z_2^k$ and $\Sigma_k(\partial x) = \partial y$. In view of the note above and (ii), $\tilde{\gamma}_1 \in \tilde{\Gamma}_{1_x}$ if and only if $\tilde{\gamma}_2 \in \tilde{\Gamma}_{2_y}$ and $\tilde{\gamma}_2|_{\partial y} = \Sigma_k|_{\partial x}\tilde{\gamma}_1|_{\partial x}\Sigma_k^{-1}|_{\partial y}$. This implies $\tilde{\Gamma}_{2_y}|_{\partial y} = \Sigma_k|_{\partial x}\tilde{\Gamma}_{1_x}|_{\partial x}\Sigma_k^{-1}|_{\partial y}$. By Corollary 1.5.6 there is a homeomorphism $\Sigma_x : x \to y$ such that

(a) $\Sigma_x|_{\partial x} = \Sigma_k|_{\partial x}$ and

(b) $\tilde{\gamma}_2|_y = \Sigma_x\tilde{\gamma}_1|_x\Sigma_x^{-1}$ for all $\tilde{\gamma}_1 \in \tilde{\Gamma}_{1_x}$, or equivalently, for all $\tilde{\gamma}_2 \in \tilde{\Gamma}_{2_y}$ both hold.

Having defined Σ_x satisfying (a) and (b) for all $x \in Y_1^{k+1}$, we proceed to define Σ_x for all $x \in X_1^{k+1} - Y_1^{k+1}$. Assume $x \in X_1^{k+1}$, $x_o \in Y_1^{k+1}$, $x_o \neq x$, and assume x and x_o are in the same orbit with respect to Γ_1. Set $\sigma(x) = y$ and $\sigma(x_o) = y_o$. We note that if $\tilde{\gamma}_1\tilde{\delta}_1 \in \tilde{\Gamma}_1$ and $\tilde{\gamma}_1(x) = \tilde{\delta}_1(x) = x_o$, then $\tilde{\gamma}_2(y) = \tilde{\delta}_2(y) = y_o$.

We now claim that if $\tilde{\gamma}_1(x) = \tilde{\delta}_1(x) = x$, then $\tilde{\gamma}_2^{-1}\Sigma_{x_o}\tilde{\gamma}_1|_x = \tilde{\delta}_2^{-1}\Sigma_{x_o}\tilde{\delta}_1|_x$. We have $\tilde{\gamma}_1\tilde{\delta}_1^{-1} \in \tilde{\Gamma}_{1_{x_o}}$ and therefore $\Sigma_{x_o}\tilde{\gamma}_1\tilde{\delta}_1^{-1}|_{x_o}\Sigma_{x_o}^{-1} = \tilde{\gamma}_2\tilde{\delta}_2^{-1}|_{y_o}$ by choice of Σ_{x_o} (see (b)). It follows that $\tilde{\gamma}_2^{-1}\Sigma_{x_o}\tilde{\gamma}_1|_x = \tilde{\delta}_2^{-1}\Sigma_{x_o}\tilde{\delta}_1|_x$ which proves the claim.

By the preceeding claim, we unambiguously define a homeomorphism $\Sigma_x : x \to y$ by setting $\Sigma_x(u) = \tilde{\gamma}_2^{-1}\Sigma_{x_o}\tilde{\gamma}_1(u)$ for all $u \in x$. Thus we have defined a homeomorphism $\Sigma_x : x \to \sigma(x) = y$ for all $x \in X_1^{k+1}$ and we note:

(a') If $x \in X_1^{k+1}$, then $\Sigma_x|_{\partial x} = \Sigma_k|_{\partial x}$; and (b') If $x,y \in X_1^{k+1}$ and $x \neq y$, then $\Sigma_x|_{x\cap y} = \Sigma_y|_{x\cap y}$ as $x \cap y \in X_1^j$ for $0 \le j \le k$, or $x \cap y = \emptyset$.

It is therefore possible to define a mapping $\Sigma_{k+1}: Z_1^{k+1} \to Z_2^{k+1}$ by setting $\Sigma_{k+1}|_x(u) = \Sigma_x(u)$ for all $u \in x$ and all $x \in X_1^{k+1}$.

We claim that:

(1) $\Sigma_{k+1}|_{Z_1^j} = \Sigma_j$ for $0 \le j \le k+1$;

(2) Σ_{k+1} is 1-1 and onto;

(3) Σ_{k+1} is a homeomorphism; and

(4) $\tilde{\gamma}_2|_{Z_2^{k+1}} = \Sigma_{k+1}\tilde{\gamma}_1|_{Z_1^{k+1}}\Sigma_{k+1}^{-1}$ for all $\tilde{\gamma}_1 \in \tilde{\Gamma}_1$ (or equivalently for all $\tilde{\gamma}_2 \in \tilde{\Gamma}_2$).

To prove (1) observe that if $x \in X_1^{k+1}$, then $\Sigma_{k+1}|_{\partial x} = \Sigma_k|_{\partial x}$. Moreover, $Z_1^k = \cup\{\partial x \mid x \in X_1^{k+1}\}$ and therefore $\Sigma_{k+1}|_{Z_1^k} = \Sigma_k$ and $\Sigma_{k+1}|_{Z_1^j} = \Sigma_{k+1}|_{Z_1^k}|_{Z_1^j} = \Sigma_k|_{Z_1^j} = \Sigma_j$, with the last equality given by (i).

Now Σ_{k+1} is onto, as $\cup\{\sigma(x) \mid x \in X_1^{k+1}\} = Z_2^{k+1}$ and $\sigma(x) = \Sigma_x(x) = \Sigma_{k+1}(x)$ for all $x \in X_1^{k+1}$. Also Σ_{k+1} is 1-1 since $\Sigma_{k+1}|_{Z_1^k} = \Sigma_k$ is 1-1, $\Sigma_{k+1}|_x$ is 1-1 and, moreover, $\Sigma_{k+1}(x - \partial x) \cap \Sigma_{k+1}(z - \partial z) = (\sigma(x) - \partial\sigma(x)) \cap (\sigma(z) - \partial\sigma(z)) = \emptyset$ if $x \neq z$ and $x,z \in X_1^{k+1}$. This proves (2).

Now X_i^{k+1} is a locally finite collection of closed sets such that $\cup\{x \mid x \in X_i^{k+1}\} = Z_i^{k+1}$ and both $\Sigma_{k+1}|_x$ and $\Sigma_{k+1}^{-1}|_y$ are continuous for all $x \in X_1^{k+1}$ and $y \in X_2^{k+1}$. By Proposition 1.4.2 Σ_{k+1} and Σ_{k+1}^{-1} are continuous and hence Σ_{k+1} is a homeomorphism. This proves (3).

To prove (4) it suffices to prove that $\tilde{\gamma}_2|_y = \Sigma_{\tilde{\gamma}_1(x)}\tilde{\gamma}_1|_x\Sigma_x^{-1}|_y$ for all $x \in X_1^{k+1}$ where $y = \sigma(x)$. Let $x \in X_1^{k+1}$, $x_o \in Y_1^{k+1}$, $y_o = \sigma(x_o)$,

$y = \sigma(x)$ and $\tilde{\delta}_1(x) = x_o$ for $\tilde{\delta}_1 \in \tilde{\Gamma}_1$. Then $\tilde{\delta}_2(y) = y_o$. We have $\Sigma_x = \tilde{\delta}_2^{-1}\Sigma_{x_o}\tilde{\delta}_1|_x$ and $\Sigma_{\tilde{\gamma}_1(x)} = \tilde{\gamma}_2\tilde{\delta}_2^{-1}\Sigma_{x_o}\tilde{\delta}_1\tilde{\gamma}_1^{-1}|_{\tilde{\gamma}_1(x)}$ by definition. Therefore,

$\Sigma_{\tilde{\gamma}_1(x)}\tilde{\gamma}_1|_x\Sigma_x^{-1}|_y = \tilde{\gamma}_2|_y$ as can be seen by substituting Σ_x and $\Sigma_{\tilde{\gamma}_1(x)}$.

Thus we have shown that the homeomorphism Σ_{k+1} satisfies (i) and (ii) with $k+1$ instead of k. So $\Sigma = \Sigma_2$ is the desired homeomorphism proving the theorem. □

The following Lemma is used in the proof of the Theorems 5.3.1 and 5.4.1 . As the assumptions for it are very lengthy we shall state them separately.

Assumptions 5.2.10. Let (G_1,P_1) be a planar 3-polyhedron and let Γ_1 be a subgroup of $\mathrm{Aut}(G_1,P_1)$. Assume that H_1 is a Γ_1-invariant subgraph of G_1 satisfying the following properties:

(1) H_1 is not a cycle.

(2) For every edge $x \in E(G_1) - E(H_1)$ there is a cycle C of H_1 separating x from H_1. We shall call the side of C in (G_1,P_1) not containing H_1 a face of H_1 in (G_1,P_1) and C its boundary. Similarly, if $C \subset H_1$ is a boundary of (G_1,P_1), we shall call C a face of H_1 in (G_1,P_1). Let $\mathcal{F}$ be the collection of all faces of H_1 in (G_1,P_1).

(3) For every $\gamma \in \Gamma_1 - \{\iota\}$ and every $F \in \mathcal{F}$, $\gamma(F) \neq F$.

Let $Q_1 = \{Q_{1_v} \mid v \in V(H_1)\}$ be a rotation system for H_1 so that $Q_{1_v} \subset P_{1_v}$ for all $v \in V(H_1)$. We note that $\mathcal{F}$ is Γ_1-invariant. In view of (1) and (3) the canonical homomorphism which assigns to each automorphism of

(G_1,P_1) the induced automorphism of (H_1,Q_1) is 1-1. Let Γ_1' be the subgroup of $Aut(H_1,Q_1)$ thus induced by $\tilde{\Gamma}_1$.

Assumption 5.2.11. Let $\tilde{\Gamma}_2$ be a discontinuous group of homeomorphisms of the planar and clockwise oriented polyhedron (S_1,H_2). Let Q_2 be the induced clockwise rotation scheme. By Theorem 5.2.4 $\tilde{\Gamma}_2$ induces a subgroup Γ_2' of $Aut(H_2,Q_2)$ which is isomorphic to $\tilde{\Gamma}_2$.

LEMMA 5.2.12. *If under the Assumptions 5.2.10 and 5.2.11* $\psi:(H_1,Q_1)\to(H_2,Q_2)$ *is an isomorphism so that*

(4) $\psi Q_{1_u}\psi^{-1} = Q_{2_{\psi(u)}}$ *for some (and thus for all)* $v\in V(H_1)$ *and*

(5) $\psi\Gamma_1'\psi^{-1} = \Gamma_2'$,

then there is a 3-polyhedron $(\tilde{S},G_2)$ *so that*

(6) $\tilde{S}\subseteq S$ *and* $(\tilde{S},G_2)$ *is* $\tilde{\Gamma}_2$*-invariant,*

(7) $H_2\subset G_2$ *and* $Q_{2_v}\subset P_{2_v}$ *for all* $v\in V(H_2)$ *where* P_2 *is the induced clockwise rotation scheme on* G_2 *and*

(8) *there is an isomorphism* $\Psi:(G_1,P_1)\to(G_2,P_2)$ *so that* $\psi = \Psi|_{V(H_1)}$ *and* $\Psi\Gamma_1\Psi^{-1} = \Gamma_2$ *where* Γ_2 *is the subgroup of* $Aut(G_2,P_2)$ *induced by* $\tilde{\Gamma}_2$.

Proof. Let $F_1\in F$ be a face with the boundary C_1 and let $C_2 = \psi(C_1)$. In view of (4) there is a unique face F_2^* of (S,H_2) which is bounded by C_2. In fact, there is a 1-1 correspondence between F and the set $F(S,H_2)$ of faces of (S,H_2). By Theorem 5.1.6, F_1 can be drawn on F_2^* resulting in a graph F_2 with the following properties:

(9) $C_2 \subset F_2$ and $ps(F_2) \subset F_2^*$; and

(10) There is an isomorphism $\chi = \Psi_{F_1}$, $\chi: F_1 \to F_2$, so that $\chi|_{V(C_1)} = \psi|_{V(C_1)}$ and $\chi R_v \chi^{-1} = T_{\chi(v)}$ for all $v \in V(F_1)$ where R_v is induced on $E_v(F_1)$ by P_1 and $T_{\chi(v)}$ is the clockwise rotation induced on $E_{\chi(v)}(F_2)$ by S.

Let $F_o \subset F$ be a minimal set containing an object from each Γ_1^*-orbit of F. For each $F_1 \in F_o$ let us define F_2 and $\Psi_{F_1}: F_1 \to F_2$ satisfying (9) and (10). Let $B_1 \in F - F_o$ have boundary C_1 and let B_2^* be the face of (S,H_2) bounded by $C_2 = \psi(C_1)$. We shall now define B_2 and $\psi_{B_1}: B_1 \to B_2$. In view of Assumptions 5.2.10 and 5.2.11, there is an isomorphism from $\tilde{\Gamma}_1$ to $\tilde{\Gamma}_2$ assigning to each $\gamma_1 \in \Gamma_1$ the unique $\tilde{\gamma}_2 \in \tilde{\Gamma}_2$ for which $\psi\gamma_1'\psi^{-1} = \gamma_2'$ where γ_1' and $\tilde{\gamma}_2'$ are the automorphisms of (H_1,Q_1), respectively (H_2,Q_2), induced by γ_1, respectively $\tilde{\gamma}_2$. Thus if $\gamma_1, \delta_1 \in \Gamma_1$, then let $\tilde{\gamma}_2, \tilde{\delta}_2$ denote the corresponding elements of $\tilde{\Gamma}_2$. By Assumption 5.2.10 (3) there is a unique $\gamma_1 \in \Gamma_1$ and a unique $F_1 \in F_o$ with $\gamma_1(F_1) = B_1$. Hence $\tilde{\gamma}_2(F_2^*) = B_2^*$. We define $B_2 = \tilde{\gamma}_2(F_2)$, $\psi_{B_1} = \tilde{\gamma}_2\psi_{F_1}\gamma_1^{-1}|_{B_1}$ and see without difficulty that (9) and (10) hold for B_1 and B_2 instead of F_1 and F_2. Now consider the graph $G_2 = \cup\{\psi_F(F) \mid F \in F\}$. It follows from (9) and (10) that $H_2 \subset G_2$ and $Q_{2_v} \subset P_{2_v}$ for all $v \in V(H_2)$ where P_2 is the clockwise rotation induced on G_2 by S. According to our construction G_2 is $\tilde{\Gamma}_2$-invariant. Moreover, by Theorem 5.2.4, $\tilde{\Gamma}_2$ induces a subgroup Γ_2 of $\mathrm{Aut}(G_2,P_2)$ which is isomorphic to $\tilde{\Gamma}_2$. Let the isomorphism $\Psi: G_1 \to G_2$ be defined by $\Psi|_F = \psi_F$ for all $F \in F$. From (10) it follows that $\Psi: (G_1,P_1) \to (G_2,P_2)$ is an isomorphism and that $\Psi|_{V(H_1)} = \psi$. In order to show that $\Psi\Gamma_1\Psi^{-1} =$

Γ_2 let $\gamma_1 \in \Gamma_1$ and $\tilde{\gamma}_2 \in \tilde{\Gamma}_2$ induce $\gamma_2 \in \Gamma_2$. It suffices to show that $\Psi_{A_1}\gamma_1\Psi_{B_1}^{-1} = \gamma_2|_{B_2}$ for any $B_1 \in F$ where $A_1 = \gamma_1(B_1)$ and $\Psi_{B_1}(B_1) = B_2$. Let $F_1 \in F_o$ and $\delta_1 \in \Gamma_1$ with $\delta_1(F_1) = B_1$. By definition, $\Psi_{A_1} = \tilde{\gamma}_2\tilde{\delta}_2\psi_{F_1}(\gamma_1\delta_1)^{-1}|_{A_1} = \gamma_2\delta_2\psi_{F_1}(\gamma_1\delta_1)^{-1}$. Hence $\psi_{A_1}\gamma_1(\delta_1\psi_{F_1}^{-1}\delta_2^{-1}|_{B_2}) = \gamma_2|_{B_2}$, that is, $\Psi_{A_1}\gamma_1\Psi_{B_1}^{-1}|_{B_2} = \gamma_2|_{B_2}$. We see that the collection of faces of G_2 in S (defined at the end of the proof of Lemma 5.1.5) form a $\tilde{\Gamma}_2$-invariant subsurface $\tilde{S}$ of S. Hence $(\tilde{S},G_2)$ is a $\tilde{\Gamma}_2$-invariant polyhedron. □

§3. TOPOLOGICAL CHARACTERIZATION OF INDIVIDUAL ELEMENTS OF A DISCONTINUOUS GROUP OF A PLANAR SURFACE.

Let ρ_δ, τ, $\bar{\tau}$, σ, and $\bar{\sigma}$ be the mappings defined in Chapter 1, §3, Example 2, and let $M = \{\tau,\bar{\tau},\sigma,\bar{\sigma}\}$.

THEOREM 5.3.1. *Let* S *be a planar surface and* $\tilde{\Gamma}$ *be a discontinuous homeomorphism group acting on* S.

(1) *If* $\tilde{\alpha} \in \tilde{\Gamma} - \{\iota\}$ *has finite order and is orientation preserving, then for some real number* δ *there is some* ρ_δ*-invariant subsurface* $S_o \subset S^2$ *such that* $\tilde{\alpha}$ *is topologically equivalent to* $\rho_\delta|S_o$.

(2) *If* $\tilde{\alpha} \in \tilde{\Gamma}$ *has infinite order, then there is a* $\gamma \in M$ *and a* γ*-invariant subsurface* S_o *of* R^2 *such that* $\tilde{\alpha}$ *is topologically equivalent to* $\gamma|_{S_o}$.

Proof. By Theorem 2.4.1 there is a $\tilde{\Gamma}$-invariant 3-polyhedron (S,G). Let P be a boundary tour scheme for (S,G). Let $\Phi:\tilde{\Gamma} \to \mathrm{Aut}(G,P)$ be the canonical monomorphism (see Theorem 5.2.4) and let $\Phi(\tilde{\Gamma}) = \Gamma$ and $\Phi(\tilde{\gamma}) = \gamma$ for all $\tilde{\gamma} \in \tilde{\Gamma}$. In view of Theorem 2.4.1 we may assume that Γ satisfies Assumptions 4.1.1.

If $\alpha \in \tilde{\Gamma} - \{\iota\}$ has finite order and is orientation preserving, then α acts orientation preserving on (G,P). In view of Note 4.2.21, Lemma 5.2.12, and finally Theorem 5.2.9, there is a planar ρ_δ-invariant subsurface $S_o \subset S^2$ such that $\tilde{\alpha}$ is topologically equivalent to $\rho_\delta|_{S_o}$.

If $\tilde{\alpha} \in \tilde{\Gamma}$ has infinite order, then in view of Note 4.2.10 or Note 4.2.18 (depending if (G,P) has an α-invariant, two-sided and 2-infinite path), Lemma 5.2.12 and Theorem 5.2.9, there is a (planar) γ-invariant subsurface $S_o \subset R^2$, where $\gamma \in M$, such that $\tilde{\alpha}$ is topologically equivalent to $\gamma|_{S_o}$. □

§4. DISCONTINUOUS HOMEOMORPHISM GROUPS OF THE EUCLIDEAN PLANE E WITH COMPACT FUNDAMENTAL DOMAIN.

THEOREM 5.4.1. *Let* $\tilde{\Gamma}$ *be a discontinuous group of homeomorphisms of the euclidean plane* E *which has compact fundamental domain, that is, there is a bounded set* B *in* E *such that* $\tilde{\Gamma}B = E$. *Then* $\tilde{\Gamma}$ *is topologically equivalent to a group of isometries of the euclidean plane* E *or the non-euclidean plane* NE.

Proof. By Theorem 2.4.1 there is a $\tilde{\Gamma}$-invariant 3-polyhedron (E,G). Let P be a boundary tour scheme for (E,G). Then (G,P) is planar, by Theorem 5.1.12 and every cycle in (G,P) has one finite side. Let $\Phi:\tilde{\Gamma} \to \mathrm{Aut}(G,P)$ be the canonical monomorphism (see Theorem 5.2.4) and let $\Phi(\tilde{\Gamma}) = \Gamma$ and $\Phi(\tilde{\gamma}) = \gamma$ for all $\tilde{\gamma} \in \tilde{\Gamma}$. In view of Lemma 2.3.1 and Theorem 2.4.1 we may assume Γ satisfies Assumptions 4.1.1. The fact that $\tilde{\Gamma}$ has compact fundamental domain implies that $G^* = G/_{\Gamma}$ is finite. It follows from Note 4.3.6, Lemma 5.2.12 and Theorem 5.2.9 that $\tilde{\Gamma}$ is topologically equivalent to a group of isometries of E or NE. □

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