AUTOMORPHISM GROUPS OF GRAPHS

by

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B.Sc., Simon Fraser University, 1969

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Mathematics

APPROVAL

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Degree: Master of Science

Title of Thesis: Automorphism Groups of Graphs

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Title of Thesis/Dissertation:

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Monday Jan 21/74

(date)

ABSTRACT

The problem of exhibiting graphs whose group is some given permutation group is examined, and the known answers for certain classes of groups are detailed. In the case of cyclic groups, the (negative) answer has been demonstrated by using a class of graphs here called circulants. This same class has also been shown to contain all graphs with transitive groups of prime degree. Here, by introducing a new class of graphs called 2-circulants, a partial characterization is made of graphs whose groups are transitive permutation groups of degree 2p for any prime p. Cayley graphs are also investigated and some aspects of this type of construction are related to the problem at hand. Included in this work is a corrected version of the published result limiting the existence of graphs with transitive abelian groups, and some additional information relevant to the cases already mentioned. Finally, a summary of the status of the problem is presented, including a statement of some relevant theorems not here proven in detail.

for Joyce

Without her longsuffering and

understanding it could not have

been conceived, let alone written.

Acknowledgment

The author wishes to thank Dr. Brian Alspach who suggested this area of investigation and whose help in correcting the errors of numerous drafts has been invaluable. Special thanks are also due to Mrs. Kathleen Lucas who typed this manuscript and without whose assistance the author would have been hard pressed.

Dr. P. Hell and Dr. L. Berggren also deserve credit for pointing out several errors and helping to re-cast Chapter 3 in its' present form.

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CHAPTER 1

INTRODUCTION

The study of the theory of graphs, both as a mathematical discipline, and as an important aid to many other fields, dates to Euler's well-known 1736 generalization of the problem of the bridges of Königsberg. Nineteenth century investigators included Kirchoff and Cayley who studied them in connection with electrical networks and chemical isomers, respectively. Modern students of the theory have also been concerned with applications to Psychology, Economics, Sociology, Computer Design, and Neurology.

Consonant with the position of graph theory at the (historical) foundation of Topology, much of the work in the field has been concerned with such topological properties as connectivity, transversibility, colourability, and the existence of various kinds of subgraphs. Comparatively less information has been derived concerning algebraic properties, among them the relationship between graphs and groups.

In examining this latter question, this paper will follow the terminology of Harary [17] in graph theory and of Wielandt [45] for permutation groups. Details of this notation are given in Appendix I. Thus the word "graph" is here limited to loopless, undirected structures of points and lines having no multiple lines between any two given points. As a starting point to the discussion, the group of a graph is defined.

Definition 1.1 The (automorphism) group of a graph X, denoted $\mathcal{A}(X)$ is the group of permutations on the vertices of X which preserve the incidence relation.

The object of this thesis is to examine various results obtained to date which are pertinent to a question raised by König [24] in 1936: "When can a given abstract group be represented as the group of a graph, and if possible how can the graph be constructed?" As stated, this question was first answered by Frucht [15], who later [14] gave the answer in the form: "To any abstract group of order $h > 1$ belongs a cubical graph with at most $[2h\ (2 + \log h) \log 2]$ vertices."

Frucht's result was further extended by Sabidussi [32] and Izbicki [28] who showed that one could also stipulate for the required graph any one of: arbitrary connectivity and/or chromatic number, regularity for an arbitrary degree, or the posession of a spanning subgraph homeomorphic to an arbitrary graph. Indeed, the constructions employed produced an infinite class of graphs in each category.

As stated then, it is evident that König's question was not very restrictive. However, if one modifies the statement so that it requires a permutation group to be given and isomorphism as permutation groups (rather than abstractly) between this and the automorphism group of the constructed graph, then the problem is very restrictive indeed and as Kagno [23] and Imrich [20,21] have shown, the answer in this case is often negative.

If a permutation group is intransitive, it can be regarded as a subgroup of the direct product (subdirect product) of its transitive constituents. (Hall [16] p.63) Several researchers have thus restricted their examination to vertex-transitive or point symmetric graphs, among them Sabidussi [33,34], Nowitz [28], Chao [11], Turner [38], and others.

Harary [17] thus poses the related problem of enumerating the (point)-symmetric graphs. Turner [38] solved this problem for point-symmetric graphs on p points, p a prime, and part of this work, together with some consequential results forms Chapter 2 of this thesis. Some of this work is extended in chapter 3; the results on 2p points being examined in particular. Chapter 4 extends these results by examining Cayley graphs and an important theorem of Sabidussi is given together with some consequences relative to chapters 2 and 3. Chapter 5 deals with abelian groups and mentions some other results derived from the work of chapter 4. Finally, chapter 6 summarizes the status of König's question and provides a review of some additional related work without going into great detail. As already mentioned, Appendix I is concerned with notation;

Appendix II contains the statements of several group-theoretic theorems used in the text of this thesis.

CHAPTER 2

CIRCULANTS

The results of this section are derived from the work of Elspas' group at the Stanford Research Institute [13]. Their concern was with cellular interconnection patterns in the organization and fabrication of logical networks for computer systems. They examined a class of graphs which they termed "Star Polygon Graphs" in connection with this work, and these became the vehicle whereby the point-symmetric graphs on a prime number of points were characterized. As in their work, for convenience, these graphs will be referred to as PPS graphs.

These results on PPS graphs were first reported by Turner [38], and these characterize the corresponding passive, or undirected switching systems. Alspach [2], working independently, published the corresponding result for tournaments, and it is in the form of directed graphs that the work has found application to active switching system design. (cf Stone [36]) Here a few changes in terminology are made:

Definition 2.1 A graph X on n points is said to be a circulant if the points of X may be numbered $v_0, v_1, \ldots, v_{n-1}$ such that $[v_i\ v_j] \varepsilon X$ iff $[v_{i+k}\ v_{j+k}] \varepsilon X$ for $k = 1,2,\ldots,n-1$. Here, as in all subsequent similar situations, the subscript addition is taken modulo the order of X.

The fact that, under this definition, the adjacency set of any point is determined by that of v_0 gives rise to the following:

Definition 2.2 The symbol of a circulant X is the set $S = \{j: [v_0\ v_j]\ \varepsilon X\}$.

It is possible to characterize circulants by a particular subgroup of their automorphism group.

Theorem 2.1 A graph X on m points is a circulant iff $Z_m \leq \mathcal{A}(X)$.

Proof: If X is a circulant, then the cycle $\rho = (v_0\ v_1 \ldots v_{m-1})$ is an automorphism of X for $[v_i\ v_j] \varepsilon X$ iff $[v_{i+1}\ v_{j+1}] \varepsilon X$ since X is a circulant. The latter line equals $[\rho(v_i)\ \rho(v_j)]$.

On the other hand if $Z_m \leq \mathcal{A}(X)$, number the points of X so that $\rho = (v_0\ v_1\ \ldots\ v_{m-1}) \varepsilon Z_m$. Now $[v_i\ v_j] \varepsilon X$ iff $[\rho^k(v_i)\ \rho^k(v_j)] = [v_{i+k}\ v_{j+k}] \varepsilon X$ (for each k) since $\rho \varepsilon \mathcal{A}(X)$. But this last statement is just the definition of a circulant, which completes the proof.

Corollary 2.1.1 If X is a graph on m points $(m>2)$ and $Z_m \leq \mathcal{A}(X)$ then $D_m \leq \mathcal{A}(X)$ where D_m denotes the dihedral group of degree m.

Proof: By the theorem, $Z_m \leq \mathcal{A}(X)$ implies X is a circulant.

Now the permutation σ on X defined by $\sigma(v_i) = v_{-i}$ is an automorphism of X for $[v_i\ v_j]\varepsilon X$ iff $[v_o\ v_{j-i}]\varepsilon X$ iff $[v_{-j}\ v_{-i}] = [\sigma(v_j)\ \sigma(v_i)]\varepsilon X$. Also, σ has order 2 and if ρ is as in the theorem, then $\sigma\rho(v_i) = \sigma(v_{i+1}) = v_{-i-1} = \rho^{-1}\sigma(v_i)$ and so $\langle\rho,\sigma\rangle = D_m$.

We are now in a position to give a class of groups for which König's question must be answered in the negative.

Corollary 2.1.2 (Kagno [23]) There is no graph on m points with $\mathcal{A}(X) = Z_m$ for $m > 2$.

Proof: This is immediate from the previous corollary, for once $D_m \leq \mathcal{A}(X)$ we have that $\mathcal{A}(X)$ is at least not regular.

Indeed we have even more, if $Z_m \leq \mathcal{A}(X)$ where $|X| = m > 2$, then $\mathcal{A}(X)$ is nonabelian. We could then ask if there are graph with specified transitive abelian groups. The solution to this question is given in chapter 5; suffice it to say for now that generally the answer is no! We return to our characterization of PPS graphs.

Theorem 2.2 A graph X on a prime number of points is PPS iff it is a circulant.

Proof: If X is a p-point circulant, we have already observed that $Z_m \leq \mathcal{A}(X)$ and so X is point-symmetric for if $v_i, v_j \varepsilon X$ and ρ is as in Theorem 1 then $\rho^{j-i}(v_i) = v_j$.

On the other hand, if X is PPS then since $\mathcal{A}(X)$ is transitive, it contains a p-cycle ρ. (Appendix II-1) Now $\langle\rho\rangle = Z_p \leq \mathcal{A}(X)$ and by Theorem 1 X is a circulant.

In order to enumerate the PPS graphs, it is necessary to have some way of knowing when two such graphs are isomorphic. Although the question of enumeration will not be persued here, the latter problem is interesting in itself, for in general it is a very difficult one but here the symbol is the device which makes a decision possible.

<u>Definition 2.3</u> If X and X' are p-point circulants with corresponding symbols S and S', we say S is <u>equivalent</u> to S' ($S \sim S'$) if there exists an integer q, $1 \leq q \leq \frac{p-1}{2}$, such that $q \cdot S = \{qs_i : s_i \varepsilon S\} = S'$, with the indicated multiplication done within $H_p = \frac{Z_p^*}{\{1, p-1\}}$.

It is obvious what theorem we wish to prove at this point; however, the route to that proof is somewhat indirect.

<u>Definition 2.4</u> If X is any graph with points $v_0\ v_1\ \ldots\ v_{m-1}$ we define the <u>adjacency matrix</u> $A = (a_{ij})$ of X by $a_{ij} = 1$ if $[v_{i-1}\ v_{j-1}] \varepsilon X$ and $a_{ij} = 0$ otherwise.

<u>Lemma 2.3.1</u> Two PPS graphs X and X' are isomorphic iff their respective adjacency matrices A and A' have the same eigenvalues.

<u>Proof</u>: If $X \cong X'$ then $A = PA'P^{-1}$ where P is a permutation matrix [10] and consequently A and A' have the same eigenvalues.

Conversely, we already know that X and X' are circulants. The adjacency matrix of such a graph is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{1m} & a_{11} & a_{12} & \cdots & a_{1\,m-1} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ a_{12} & a_{13} & a_{14} & \cdots & a_{11} \end{pmatrix}$$

in which each row is a cyclic shift of the previous one. Under the name of <u>circulant matrices</u>, these have been investigated by Ablow and Brenner [1] and their eigenvalues are explicitly given by $\alpha_k = a_{11} + a_{12}w^k + \ldots + a_{1m}w^{k(m-1)}$ where w is a primitive m-th root of unity and $k = 0,1,\ldots,m-1$.

Here, we are assuming that $a_1 = a_{12}w + a_{13}w^2 + \ldots + a_{1p}w^{p-1}$ is an eigenvalue of A' as well as of A. Hence there is an integer k, $1 \le k \le p-1$, with $\alpha_1 = a'_{12}w^k + a'_{13}w^{2k} + \ldots + a'_{1p}w^{k(p-1)}$.

Since the primitive roots of unity are linearly independent over the rationals (van der Waerden [41]) we have, equating coefficients that $a'_{1j} = a_{1\,[(j-1)k+1]}$. Now, if we map X' to X by $v'_j \to v_{jk}$, we have that $[v'_o\ v'_j] \varepsilon X$ iff $a'_{1(j+1)} = 1$ iff $a_{1(jk+1)} = 1$ iff $[v_0\ v_{jk}] \varepsilon X$ and so $X \cong X'$.

The next theorem is the one earlier hinted at, and which allows the enumeration of the PPS graphs.

<u>Theorem 2.3</u> Two circulants X and X' are isomorphic iff their respective symbols S and S' are equivalent.

<u>Proof</u>: If $S \sim S'$ then there is an integer q, $1 \leq q \leq \frac{p-1}{2}$ with $q \cdot S = S'$. As in the lemma, the mapping defined by $v_i' \to v_{iq}$ is an isomorphism of X' with X.

If $S \not\sim S'$ and A, A' are the adjacency matrices of X and X' with w a primitive p-th root of unity, then $\alpha = a_{12}w + \ldots + a_{1p}w^{p-1}$ is an eigenvalue of A. Those of A' are $\alpha_k = a_{12}'w^k + \ldots + a_{1p}'w^{(p-1)k}$ for $1 \leq k \leq p-1$. Since the w^k are all the primitive p-th roots of unity and are linearly independent over the rationals, an eigenvalue α_k of A' could equal α only if $a_{1j}' = a_{1\,[(j-1)k+1]}$. Moreover, since $a_{1s} = a_{1\,(p+2-s)}$ for $s = 2, \ldots, \frac{p-1}{2}$ and similarly for the a'_{1j} we have that $a'_{1j} = a_{1[(j-1)k+1]}$ $2 \leq j \leq \frac{p+1}{2}$.

This latter equation holds iff we have that $[v_0' \ v_j'] \varepsilon X'$ iff $[v_0 \ v_{jk}] \varepsilon X$ which holds iff $S' = kS$. Since we have assumed $S \not\sim S'$ however, we have that X and X' are not isomorphic.

Further light will be thrown on the ideas of this section in Chapter 4. In the meantime, we pursue a similar course for graphs on 2p points.

CHAPTER 3

2 CIRCULANTS

The observation made in the previous chapter that the automorphism group of a circulant contains the dihedral group D_m prompts the search for graphs having dihedral groups (isomorphic to D_m) as regular automorphism groups. This is approached by examining graphs consisting of two isomorphic m-point subcirculants joined together in a circulant fashion.

<u>Definition 3.1</u> A <u>2-circulant</u> X is a graph on 2m points, where these can be labeled $v_{0,0}\ v_{0,1} \cdots v_{0,m-1}$ and $v_{1,0}\ v_{1,1} \cdots v_{1,m-1}$ in such a way that

1). $W_0 = \{v_{0,i}\}$ and $W_1 = \{v_{1,j}\}$ are isomorphic subcirculants of X with symbols R_0 and R_1 respectively.

2). $[v_{0,i}\ v_{1,j}] \varepsilon X$ iff $[v_{0,i+t}\ v_{1,j+t}] \varepsilon X\ \forall t$ and

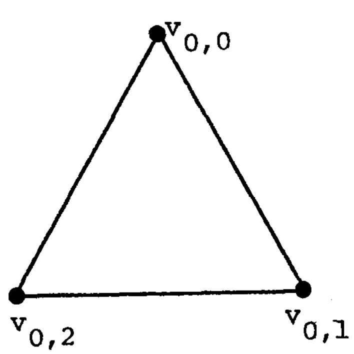
3). there exists an automorphism σ on X mapping W_1 to W_0.

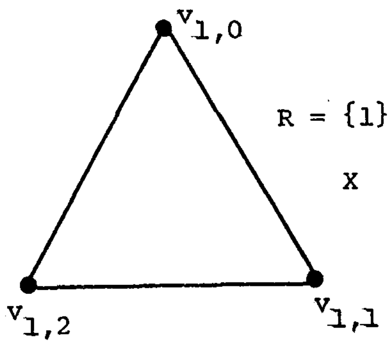
The labelling of X can always be chosen in such a way that $\sigma(v_{1,0}) = v_{0,0}$. From this point on, unless otherwise stated, σ will be assumed to have this property. It is worth noting that σ is not necessarily uniqué. Moreover, since $W_0 \cong W_1$ we have that X is completely determined when the adjacency set of $v_{0,0}$ and the automorphism σ are known. This prompts the following:

Definition 3.2 The symbol $S = \{R, \sigma, F\}$ of a 2-circulant X on 2m points consists of the inner symbol R, which is the symbol of, say, the subcirculant W_0, the automorphism σ and the outer symbol $F = \{j : [v_{0,0}\ v_{1,j}] \varepsilon X\}$.

It is important to note that even if $R_0 = R_1$ whence $W_0 \equiv W_1$ (read W_0 and W_1 are congruent) σ does not necessarily act like the identity on the second coordinates of the subscripts. If it did we would have $[v_{0,0}\ v_{1,j}] \varepsilon X$ iff $[v_{0,j}\ v_{1,0}] \varepsilon X$ which says $j \varepsilon F$ iff $-j \varepsilon F$, a condition for which there is as yet no guarantee. However, when $R_0 = R_1$ we will say that X is of type 1. If X cannot be represented as a type 1 2-circulant we will say that it has type 2.

Example 3.1 If $F = \phi$, then X is disconnected and W_0 and W_1 are components. (If in addition $R = \phi$, X is trivial)



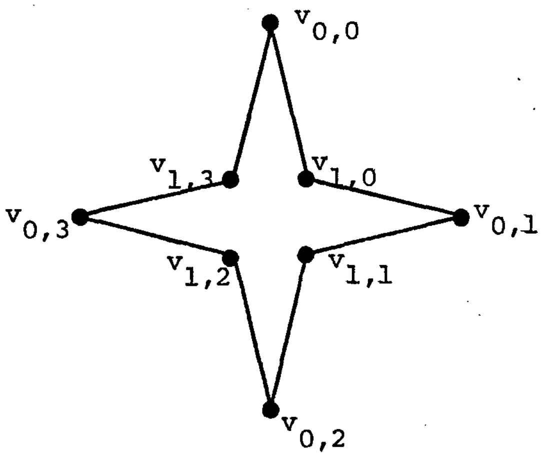


$R = \{1\}$, $F = \phi$,

X is of type 1.

Figure 3.1

<u>Example 3.2</u> If $R = \phi$, X is bipartite



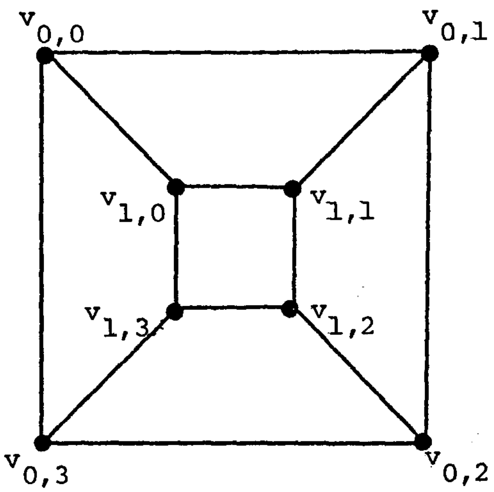
$R = \phi$ $F = \{0\ -1\}$ and

X is of type 1.

Figure 3.2

This example demonstrates more, namely that 2m-cycles are bipartite 2-circulants, hence obviously of type 1. A later theorem will show that 2m-point circulants are all type 1 2-circulants. The converse is false.

<u>Example 3.3</u> Not all type 1 2-circulants are circulants.



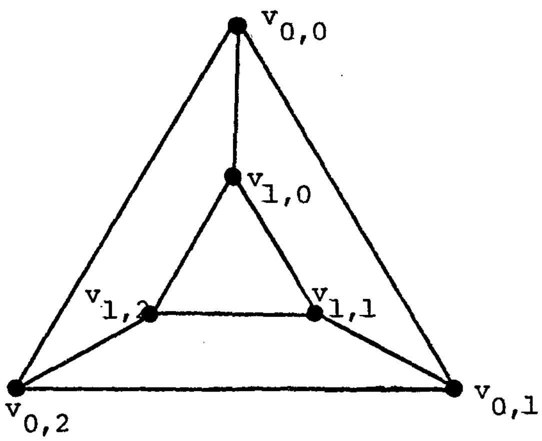
$R = \{1\}$ $F = \{0\}$ and

X is of type 1.

Figure 3.3

This graph is not a circulant because of the two degree three circulants on eight points, one has girth 3 and the other is not planar.

<u>Example 3.4</u> The 6-point graph with the same symbol as the graph of example 3.3 is a circulant, contrary to the statement of Turner [38].



$R = \{1\}$ $F = \{0\}$

X is of type 1.

Figure 3.4

If X is renumbered according to $v_{0,i} \to v_{2i}$, $v_{1,0} \to v_3$

$v_{1,1} \to v_5$ and $v_{1,2} \to v_1$ we obtain

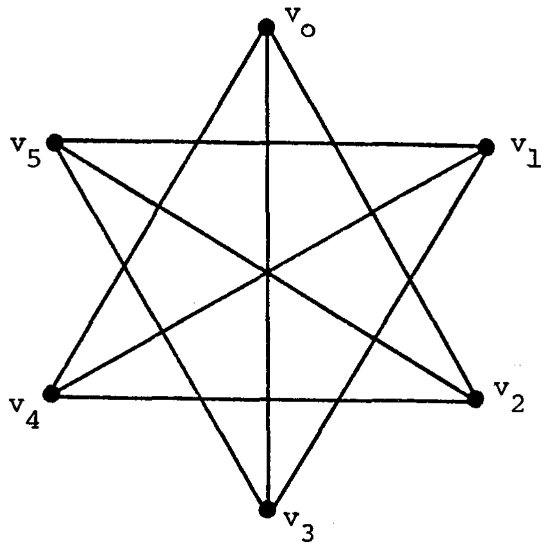


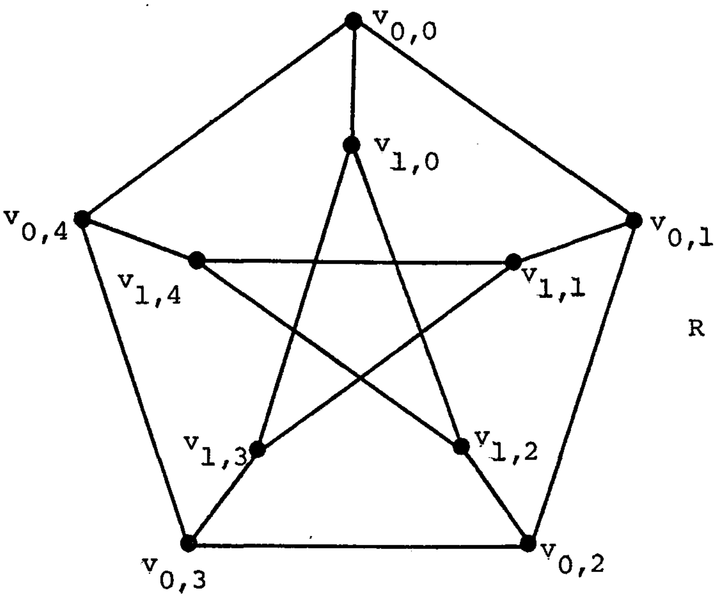
Figure 3.5

and X is a circulant with symbol $S = \{2,3\}$.

Before proceding, two more observations are in order. First, since each of W_0, W_1 is point-symmetric we may choose $v_{1,0}$ to be any point of W_1 we wish, so we may always assume that if $F \neq \phi$ then $0 \in F$, i.e. $v_{1,0}$ is chosen so that $[v_{0,0}\ v_{1,0}] \varepsilon X$ or X is disconnected.

Secondly, if X is of type 1 and not disconnected then X has girth three or four. (The four-cycle $[v_{0,0}\ v_{0,i}\ v_{1,i}\ v_{1,0}] \varepsilon X$ for any $i \in R$)

Example 3.5 Petersen's graph is a 2-circulant which does not have type 1 since it has girth 5 and is also not a circulant because all the degree three circulants on 10 points have girth 4.



$R = \{1\}$ X is not of type 1

$F = \{0\}$

Figure 3.6

We now state the precise relationship between 2-circulants and circulants.

Theorem 3.1 Every circulant on 2m points is a type 1 2-circulant.

Proof: Suppose X is a circulant with symbol S and points $v_o\ v_1 \ldots v_{2m-1}$. Renumber the points according to $v_{2i} \to v_{0,i}$ and $v_{2i+1} \to v_{1,i}$ $i = 0,1,\ldots,m-1$. Then we have $[v_{0,0}\ v_{0,i}] \varepsilon X$ iff $[v_0\ v_{2i}] \varepsilon X$, iff $[v_{2k}\ v_{2i+2k}] \varepsilon X\ \forall k$ iff $[v_{0,k}\ v_{0,i+k}] \varepsilon X\ \forall k$ and similarly $[v_{1,0}\ v_{1,j}] \varepsilon X$ iff $[v_1\ v_{2j+1}] \varepsilon X$ iff $[v_{1+2k}\ v_{2j+2k+1}] \varepsilon X\ \forall k$ iff $[v_{1,k}\ v_{1,j+k}] \varepsilon X\ \forall k$. Hence X has two congruent m-point subcirculants with symbol $R = \{i: 2i \varepsilon S\}$.

Next, $[v_{0,0}\ v_{1,j}] \varepsilon X$ iff $[v_0\ v_{2j+1}] \varepsilon X$ iff $[v_{2k}\ v_{2j+2k+1}] \varepsilon X\ \forall k$ iff $[v_{0,k}\ v_{1,j+k}] \varepsilon X\ \forall k$. Hence X is a type 1 2-circulant with symbol $S' = \{R,F\}$ where $F = \{j: 2j+1\ \varepsilon S\}$.

As shown in example 3.3, the converse is false. However, there is a partial converse which specifies exactly how much symmetry in F is necessary to allow us to rewrite a type 1 2-circulant as a circulant.

<u>Theorem 3.2</u> A type 1 2-circulant X on 2m points is a circulant iff it can be written with symbol $S' = \{R,\sigma,F\}$ where $j \varepsilon F$ iff $-(j+1)\varepsilon F$.

<u>Proof:</u> Let X be a type 1 2-circulant written so as to have symbol $S'=\{R,\sigma,F\}$ such that $j\varepsilon F$ iff $-(j+1)\varepsilon F$. Renumber the points of X according to $v_{0,i}\to v_{2i}$ and $v_{1,j}\to v_{2j+1}$ and let

$S=\{2i: i\varepsilon R\} \cup \{2i+1: i\varepsilon F\}$

<u>Case I</u> $[v_0\ v_{2i}]\varepsilon X$ iff $[v_{0,0}\ v_{0,i}]\varepsilon X$ iff $[v_{0,t}\ v_{0,i+t}]\varepsilon X$ iff $[v_{2t}\ v_{2i+2t}]\varepsilon X$ $\forall t$. Likewise, $[v_1\ v_{2j+1}]\varepsilon X$ iff $[v_{1,0}\ v_{1,j}]\varepsilon X$ iff $[v_{1,t}\ v_{1,j+t}]\varepsilon X$ iff $[v_{2t+1}\ v_{2j+2t+1}]\varepsilon X$ $\forall t$.

So, when renumbered, vertices having subscripts of the same parity are joined in circulant fashion.

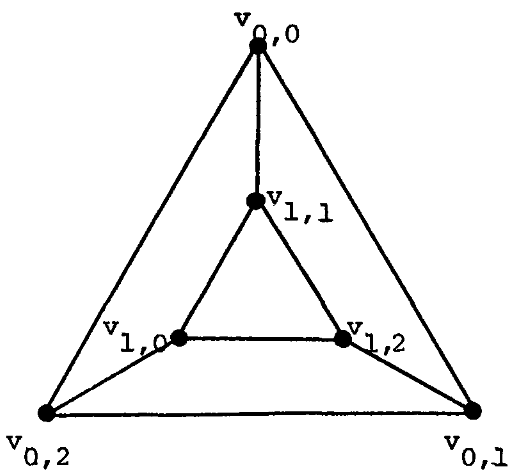
<u>Case II</u> $[v_0\ v_{2j+1}]\varepsilon X$ iff $[v_{0,0}\ v_{1,j}]\varepsilon X$ iff $[v_{0,t}\ v_{1,j+t}]\varepsilon X$ iff $[v_{2t}\ v_{2j+2t+1}]\varepsilon X$ $\forall t$. Moreover $[v_{0,0}\ v_{1,j}]\varepsilon X$ iff $j\varepsilon F$ iff $-(j+1)\varepsilon F$ by hypothesis. This takes place iff $[v_{0,0}\ v_{1,-(j+1)}]\varepsilon X$ iff $[v_{0,j}\ v_{1,-1}]\varepsilon X$ iff $[v_{0,j+t}\ v_{1,-1+t}]\varepsilon X$ iff $[v_{2j+2t}\ v_{2(t-1)+1}]\varepsilon X$ $\forall t$ iff $[v_{2j+1+(2t-1)}\ v_{2t-1}]\varepsilon X$ $\forall t$.

Case II gives the result for vertices of opposite subscript parity. It has now become evident that the set S defined above is the symbol for X written as a circulant.

On the other hand, if X is a type 1 2-circulant with symbol $S=\{R,\sigma,F\}$ whose points may be renumbered so as to rewrite X as a circulant with symbol S', we renumber X again from this circulant numbering by Theorem 3.1 and obtain it once more as a 2-circulant with symbol $S'' = \{R'',\sigma',F'\}$.

Now $j\varepsilon F'$ iff $[v_{0,0}\ v_{1,j}]\varepsilon X$ iff $[v_o\ v_{2j+1}]\varepsilon X$ iff $[v_0\ v_{-(2j+1)}]\varepsilon X$ (since X is a circulant) iff $[v_{0,0}\ v_{1,-(j+1)}]\varepsilon X$ iff $-(j+1)\varepsilon F'$ as asserted.

In view of the above, one might question example 3.4 where $F = \{0\}$. It would seem that Theorem 3.2 requires that $-1\varepsilon F$ also, whereas we have shown the graph of that example to be a circulant. However, if the graph of that example as written in circulant notation were renumbered by Theorem 3.1 we would obtain it once more in 2-circulant fashion; this time with $F=\{1\}$ so that $j=-(j+1)$ (modulo 3).



The graph of example 3.4 rewritten to show compliance with Theorem 3.2.

Figure 3. 7

The above discussion makes the following result obvious, and in view of examples 3.4 and 3.3 it is stated without proof.

Corollary 3.2.1 If X is a 2m-point type 1 2-circulant with symbol $S = \{ \{1\}, \sigma, \{0\} \}$ then X is also a circulant iff m is odd.

The time has now come to return to the main stream of the discussion and show the analogous result to that of Theorem 2.1, characterizing 2-circulants in terms of a particular subgroup of $\mathcal{A}(X)$. Unfortunately, in the general case, the characterization is not as neat as one would like.

Lemma 3.3.1 If X is a 2m-point graph and $\mathcal{A}(X)$ contains a regular group D generated by elements s and t with $s^2 = t^m = 1$, then X is a 2-circulant.

Proof: t is a regular permutation of order m and degree 2m, so is necessarily a product of exactly two disjoint m-cycles. If we number the points of X so that $t = (v_{0,0}\ v_{0,1} \cdots v_{0,m-1})(v_{1,0}\ v_{1,1} \cdots v_{1,m-1})$, we obtain that X has two m-point subcirculants W_0 and W_1. Now $\langle t \rangle$ has index 2 in D and so is normal, though intransitive. Hence, D is imprimitive and the orbits of $\langle t \rangle$ form a complete block system for D (see Appendix II-2) That is, W_0 and W_1 are blocks of D and since D is transitive, s must interchange W_0 and W_1 so those are isomorphic.

Indeed, it is clearly possible to specify that numbering so that $s(v_{1,0}) = v_{0,0}$ and it is now evident that X is a 2-circulant.

If X is of type 1 and we define $s: X \to X$ by $s(v_{0,j}) = v_{1,-j}$ and $s(v_{1,j}) = v_{0,-j}$ for $0 \leq j \leq m-1$, then since $s^2 = 1$ and $D_m \leq \mathcal{Q}(W_0)$ we see that s is acting as an automorphism on the second coordinates as it interchanges W_0 and W_1. In addition, $[v_{0,0}\ v_{1,j}] \varepsilon X$ iff $[v_{1,0}\ v_{0,-j}] \varepsilon X$ since X is a 2-circulant. Therefore, s is in fact an automorphism of X.

This discussion, together with the lemma inplies the following:

<u>Theorem 3.3</u> X is a type 1 2m-point 2-circulant iff $\mathcal{Q}(X)$ contains a regular dihedral subgroup $D \cong D_m$.

<u>Proof</u>: If X is as given, we know immediately that $\mathcal{Q}(X)$ contains the automorphism $t = (v_{0,0}\ v_{0,1} \cdots v_{0,m-1})$ $(v_{1,0}\ v_{1,1} \cdots v_{1,m-1})$ and the automorphism s of the discussion above. Clearly $s^2 = t^m = 1$. Moreover, we have for example that $sts(v_{0,i}) = st(v_{1,-i}) = s(v_{1,-i+1}) = v_{0,i-1} = t^{-1}(v_{o,i})$ so that $D = \langle s,t \rangle$ is dihedral.

On the other hand, if in Lemma 3.3.1 we know that $sts = t^{-1}$, then $ts(v_{0,i}) = st^{-1}(v_{0,i})$ and if we write for convenience

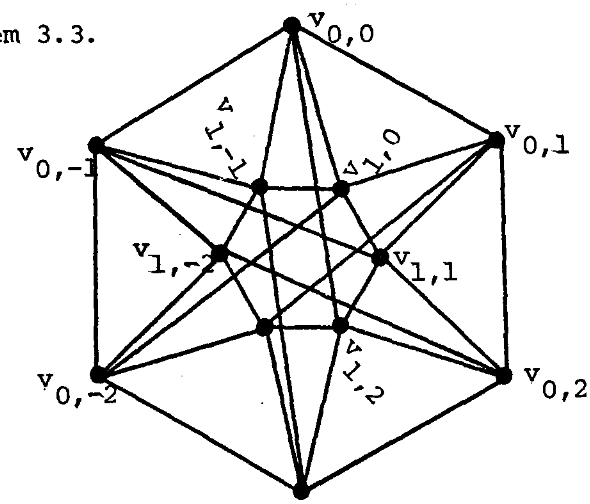
$s(v_{0,i}) = v_{1,\delta(i)}$ this becomes $v_{1,\delta(i)+1} = v_{1,\delta(i-1)}$ which is to say $\delta(i) = \delta(i-1)-1$. We have at once $\delta(1) = -1$, and by recursion $\delta(i) = -i$ so that the element s has in fact the definition $s(v_{0,i}) = v_{1,-i}$ and likewise $s(v_{1,i}) = v_{0,-i}$. It is obvious then that X is of type 1.

We give a slight generalization of Theorem 3.3 in the event that $m = p$ a prime.

Corollary 3.3.1 A 2p point graph X is a type 1 2-circulant iff $\mathcal{A}(X)$ contains a regular subgroup.

Proof: The only groups of order 2p are cyclic and dihedral (see Appendix II-3). In either case, by Theorems 2.1 and 3.1 or by Theorem 3.3, X is a type 1 2-circulant. The converse follows from Theorem 3.3.

Example 3.6 (Watkins) Let X be the type 1 2m-point 2-circulant defined by $S=\{ \{1\}, s, \{0,-1,2\} \}$ for $m \geq 6$ and s as in Theorem 3.3.



The graph X of example 3.6 for m=6

Figure 3.8

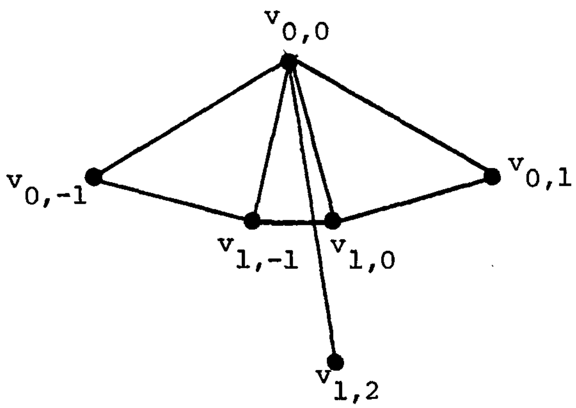


Figure 3.9

The subgraph of X on $v_{0,0}$ and its' neighbours

The set H of neighbours of $v_{0,0}$ contains only the lines $[v_{0,1}\ v_{1,0}]$, $[v_{1,0}\ v_{1,-1}]$ and $[v_{1,0}\ v_{1,-1}]$. If $\phi\varepsilon\mathcal{A}(X)$ fixes $v_{0,0}$ and is not the identity, it must interchange $v_{1,0}$ with $v_{1,-1}$ and $v_{0,1}$ with $v_{0,-1}$ fixing $v_{1,2}$. Now $v_{1,1}$ is adjacent to $v_{1,0}$ but not to $v_{1,-1}$ so must move; say $\phi(v_{1,1}) = Z$. Since $v_{1,1}$ is adjacent to the fixed point $v_{1,2}$, so is Z. Moreover, Z is also adjacent to $v_{1,-1}$ since $v_{1,1}$ is adjacent to $v_{1,0}$. The adjacency sets of $v_{1,2}$ and $v_{1,-1}$ are $\{v_{1,1}\ v_{1,3}\ v_{0,2}\ v_{0,3}\ v_{0,0}\}$ and $\{v_{1,0}\ v_{1,-2}\ v_{0,0}\ v_{0,-1}\ v_{0,-3}\}$ and since $m\geq 6$ $v_{1,3}\neq v_{1,-2}$ and $v_{0,2}\neq v_{0,-3}$, $Z = v_{0,0}$ which is impossible.

Hence ϕ fixes the entire adjacency set of $v_{0,0}$ and so since the graph is a 2-circulant ϕ fixes all of X. We conclude that $\mathcal{A}(X)$ is regular and hence actually equals the group $D \cong D_m$ of Theorem 3.3.1. If $m\varepsilon\{3,4,5\}$ the result does not hold, a fact which will be proven later. In the meantime, it is possible to prove the analogues of several results of the previous chapter.

Theorem 3.4 X is a type 2 2p-point 2-circulant iff the automorphism σ of Defintion 3.1 is of the form (1) $\sigma(v_{0,i}) = v_{1,qi}$ and (2) $\sigma(v_{1,j}) = v_{0,qj}$ where q cannot be 1 or -1.

Proof: Let X be a type 2 2p-point 2-circulant. We can assume that the subcirculants of X are neither complete nor trivial since each of these cases results in the graph having type 1. Hence by our Theorem 2.3 and Theorem 7.3 of [29] the automorphism σ of definition 3.1 must have the form $\sigma(v_{0,j}) = v_{1,rj}$ and $\sigma(v_{1,j}) = v_{0,qj}$ for $q,r \in H_p$. We now show that $q = r$. Now $[v_{0,0}\ v_{1,j}]\varepsilon X$ iff $[\sigma(v_{0,0})\ \sigma(v_{1,j})] = [v_{1,0}\ v_{0,qj}]\varepsilon X$. Hence $[v_{0,k}\ v_{1,j+k}]\varepsilon X$ iff $[v_{1,qk}\ v_{0,qj+qk}]\varepsilon X$ $\forall k$. (as k runs through 1,2,3,...p-1 so does qk). Hence $[\sigma(v_{0,k})\ \sigma(v_{1,j+k})] = [v_{1,rk}\ v_{0,qj+qk}]\varepsilon X$ iff $[v_{1,qk}\ v_{0,qj+qk}]\varepsilon X$ $\forall k = 1,2,\ldots,p-1$. Hence, if $q-r \neq 0$ either $|F| = p-1$ or $|F| = p$. In the former case, we may assume $[v_{0,i}\ v_{1,i}]\not\varepsilon X$ for each i. Hence $[\sigma(v_{0,i})\ \sigma(v_{1,i})] =$

$= [v_{1,ri}\ v_{0,qi}] \notin X$ contradicting $|F| = p-1$. In the latter, X may be rewritten as a type 1 2-circulant by the mapping $v_{1,j} \to v_{1,qj}$ $v_{0,i} \to v_{0,i}$ which is contrary to hypothesis. Hence there exists a q satisfiying (1) and (2). Certainly $q \neq \pm 1$ for then X would have type 1.

On the other hand, if σ has the indicated form on the 2p-point 2-circulant X, then X cannot have type 1 by the proof of Theorem 3.3.

<u>Corollary 3.4.1</u> If X is a type 2 2p point 2-circulant then $\mathcal{A}(X)$ contains a subgroup C isomorphic to the dihedral group D_p.

<u>Proof</u>: Let σ be the automorphism on X discussed in Theorem 3.4. For every odd n $\sigma^n(v_{0,i}) = v_{1,q^n i}$ and $\sigma^n(v_{1,i}) = v_{0,q^n i}$ we have by the preceding theorem that $q^n \neq \pm 1$ for any odd n. Hence there exists an even m such that $\sigma^m(v_{0,i}) = v_{0,-i}$ and $\sigma^m(v_{1,j}) = v_{1,-j}$. Let $\tau = \sigma^m$ and form $C = \langle\tau, t\rangle$ where $t = (v_{0,0}\ v_{0,1} \cdots v_{0,p-1})(v_{1,0}\ v_{1,1} \cdots v_{1,p-1})$. Clearly $\tau^2 = t^p = 1$ and $\tau t \tau = (v_{0,0}\ v_{0,-1} \cdots v_{0,1})(v_{1,0}\ v_{1,-1} \cdots v_{1,1}) =$ $= t^{-1}$ so that C is dihedral.

We have now shown that every 2p-point 2-circulant X has a dihedral subgroup in $\mathcal{A}(X)$. Types 1 and 2 are distinguished by the fact that in the former case this subgroup is regular and in the latter it is not. In addition, in the type 2-case F is invariant under multiplication by -1.

Corollary 3.4.2 If X is a type 2 2p-point 2-circulant then $j \varepsilon F$ iff $-j \varepsilon F$.

Proof: $j \varepsilon F$ iff $[v_{0,0}\ v_{1,j}] \varepsilon X$ iff $[\tau(v_{0,0})\ \tau(v_{1,j})] =$

$= [v_{0,0}\ v_{1,-j}] \varepsilon X$ iff $-j \varepsilon F$.

Example 3.7 Petersen's Graph (Figure 3.6) could be written with symbol $S=\{\{1\}, 2, \{0\}\}$ and here σ is s.t. $\sigma(v_{1,j}) = v_{0,2j}$ $\sigma(v_{0,j}) = v_{1,2j}$ as asserted above.

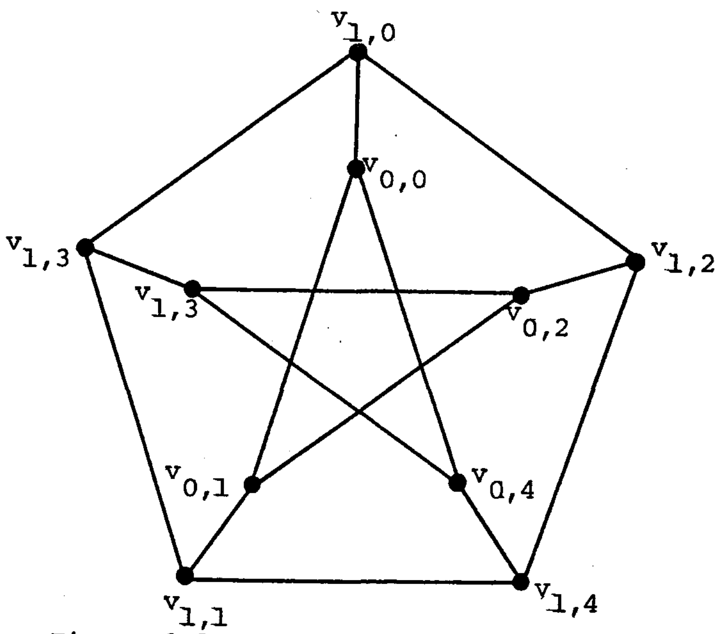


Figure 3-10

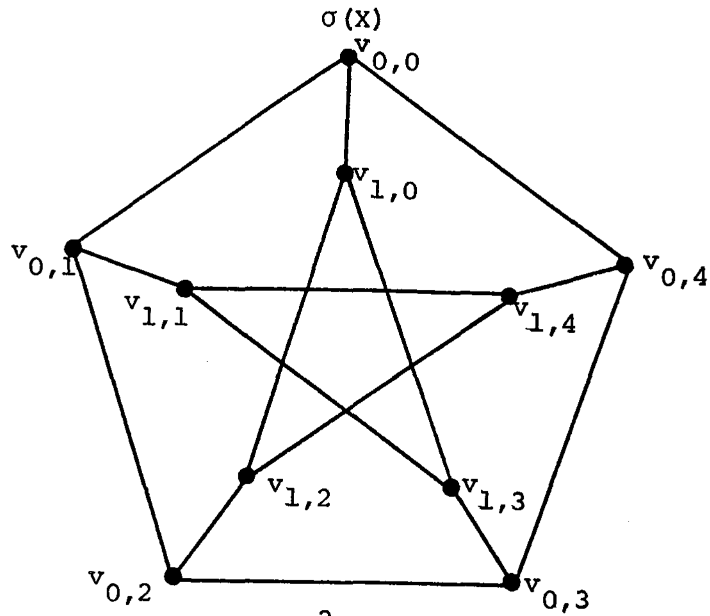


Figure 3-11 $\sigma^2(X)$

The figures show the compliance of Petersen's graph with Theorem 3.4 and Corollary 3.4.1. Clearly $\sigma^4 = 1$.

<u>Definition 3.3</u> If X, X' are 2p-point 2-circulants with symbols $S = \{R,\sigma,F\}$ and $S' = \{R',\sigma,F'\}$ we say that <u>S is equivalent to S'</u> ($S \sim S'$) in case there are integers $q \varepsilon Z_p^*$ and $x \varepsilon Z_p$ such that $q \cdot S' = \{qR',\sigma,qF'+x\} = S$ where $qR' = \{qr: r' \varepsilon R'\}$, $qF'+x = \{qF'+x : f' \varepsilon F'\}$ and the indicated multiplication is within Z_p.

Notice that the definition requires that the same σ be found in each of (X) and (X'). In the following we will assume, that in the type 1 case σ is the involution of Theorem 3.3 or has the form given in Theorem 3.4 for k the smallest possible q for which the Theorem holds. We obtain the following analogue of Turner's theorem.

Theorem 3.5 Two 2p-point 2-circulant X and X' are isomorphic iff they have equivalent symbols.

Proof: If $S \sim S'$ where S and S' are the symbols of X and X' respectively, then there are integers $q, x \in Z_p^*$ such that $S = \{qR', \sigma, qF'+x\}$. Define $f\colon X' \to X$ by: $f(v'_{0,i}) = v_{0,qi}$ and $f(v'_{1,j}) = v_{1,qj+x}$. In the proof of lemma 2.3.1 we showed that such an f wil map W'_0 to W_0 and W'_1 to W_1 as an isomorphism. Moreover, $[v'_{0,0}\ v'_{1,j}] \in X$ iff $j \in F'$ iff $qj + x \in F$ iff $[v_{0,0}\ v_{1,qj+x}] = [f(v'_{0,0})\ f(v'_{1,j})] \in X$ and hence f is an isomorphism of X with X'.

On the other hand, if X and X' are isomorphic 2-circulants with symbols $S = \{R, \sigma, F\}$ and $S' = \{R', \sigma', F'\}$ on 2p points then $\mathcal{A}(X) \cong \mathcal{A}(X')$. In particular the dihedral subgroup (D of Theorem 3.3 or C of Theorem 3.4) of $\mathcal{A}(X')$ is mapped to a dihedral subgroup of $\mathcal{A}(X)$ having the same properties. Hence X and X' have the same type and in the type 2 case the constant k of Theorem 3.4 relating R_0 to R_1 and R'_0 to R'_1 must be the same number. In either case, this implies that $\sigma = \sigma'$ as required if the symbols are to be equivalent.

Moveover, the blocks of $\langle \sigma', t' \rangle \leq \mathcal{A}(X')$ are mapped to blocks of such a subgroup of $\mathcal{A}(X)$ and since W_0', W_1', and W_0 and W_1 are the only p-point blocks involved, we can assume that the isomorphism on X' maps W_0' to W_0 and W_1' to W_1. Hence, since R', R_1', R and R_1 are the symbols of the sub-circulants W_0', W_1', W_0 and W_1 respectively, we have that there are integers $q, q' \varepsilon Z_p^*$ with $qR' = R$ and $q'R_1' = R_1$ (by Theorem 2.3). We have already mentioned that for some $k \varepsilon Z_p^*$ $kR_1' = R'$ and $kR_1 = R$ and these equations give the relationship $q'k = k\,q$. (When X has type 1 use $k = -1$)

$$
\begin{tikzcd}
R & R_1 \arrow[l, "k"'] \\
R' \arrow[u, "q"] & R_1' \arrow[l, "k"] \arrow[u, "q'"']
\end{tikzcd}
$$

Figure 3-12

Since multiplication modulo p is commutative $kqR' = qkR'$ so that $q,$ can be chosen to equal q and the respective subcirculants are related by the same constant q. If the isomorphism we are examining sends $[v'_{0,0}\ v'_{1,0}]$ to $[v_{0,0}\ v_{1,0}]$ we have $j \varepsilon F'$ iff $[v'_{0,0}\ v'_{1,j}] \varepsilon X'$ iff $[v_{0,0}\ v_{1,qj}] \varepsilon X$ iff $qj \varepsilon F$. If $[v'_{0,0}\ v'_{1,0}]$ goes to any other $[v_{0,0}\ v_{1,x}]$ then merely re-label X with symbol $S_1 = \{R,\sigma,F-x\}$ after applying q and the theorem now follows.

CHAPTER 4

CAYLEY GRAPHS

The time has come to examine more general aspects of the relationship between groups and graphs in order to provide needed tools to consider classes of groups other than those of the first two chapters. The basic machinery was provided by Sabidussi [33] and forms this chapter through Theorem 4.2. A number of important consequences are also detailed.

The constructions of Frucht [14,15] produced graphs with semi-regular automorphism groups. The purpose here is to show that the latter fact implies that the graph is of the given construction. We begin with a lemma on disconnected graphs.

Lemma 4.1.1 Let X be a graph with $\mathcal{A}(X)$ semi-regular. Then if X is disconnected it has exactly two isomorphic components X_1 and X_2 such that $\mathcal{A}(X_1) = \mathcal{A}(X_2) = \{1\}$.

Proof: Let the components of X be X_i $i=1,2,\ldots,n$. At least two of the X_i must be isomorphic. Otherwise, $\phi|X_i \varepsilon \mathcal{A}(X_i)$ for every $\phi\varepsilon\mathcal{A}(X)$, $i=1,2,\ldots,m$, and since $\mathcal{A}(X)_x=1$ $\forall x\varepsilon X$, $\mathcal{A}(X)\neq\{1\}$, at least one of the $\mathcal{A}(X_k)\neq\{1\}$ for $1 \leq k \leq m$. Let $\phi_0\varepsilon\mathcal{A}(X_k)$ for a $\phi_0\neq 1$. Define $\phi: X \to X$ by $\phi x = \phi_0 x$ if $x\varepsilon X_k$ and $\phi x = x$ otherwise. Clearly $\phi\varepsilon\mathcal{A}(X)$ but violates the semi-regularity. So, we can assume that $X_1 \cong X_2$.

If X has more than two components and ψ is an isomorphism of X_1 onto X_2, then $\phi': X \to X$ given by $\phi' x = \phi x$ for $x\varepsilon X_1$ $\phi' x = \psi^{-1} x$ for $x \varepsilon X_2$, and $\phi' x = x$ otherwise also provides an automorphism of X which violates semiregularity. Finally if $\mathcal{A}(X_1) \cong \mathcal{A}(X_2) \neq \{1\}$, let $\phi_1\varepsilon\mathcal{A}(X_1)$ for $\phi_1\neq 1$, and define $\phi'': X \to X$ by $\phi'' x = \phi_1 x$ if $x\varepsilon X_1$ and $\phi'' x=x$ if $x\varepsilon X_2$. The same contradiction is reached and this establishes the lemma.

We next establish two results related to the degrees of the points of graphs having $\mathcal{A}(X)$ semi-regular.

<u>Lemma 4.1.2</u> Let X be a graph with $\mathcal{A}(X)$ semi-regular and choose any $x\varepsilon X$. There exist at most two lines of X which are similar and incident with x.

<u>Proof:</u> Let $e_i = [x,x_i]\varepsilon X$ for $i=0,1,2,\ldots,m$ be similar. Then $\exists\phi_i\varepsilon$ (X) with $e_i = \phi_i e_0$ for $i = 1,2$. By semi-regularity

(1) $\phi_i x = x_i$ and (2) $\phi_i x_0 = x$ for $i = 1,2$. Now by (2) $\phi_1=\phi_2$ and so by (1) $x_1=x_2$ so that $e_1=e_2$.

Lemma 4.1.3 Let X be a line-symmetric graph with more than one line and $\mathcal{G}(X)$ semi-regular. Then X is cyclically connected.

Proof: Suppose the contrary and let x be a cutpoint of X. Now line-symmetric graphs must either be point-symmetric or bipartite. (Harary [17] 14.12)

In the first case, by point-symmetry, every point is a cutpoint which is obviously impossible. In the second, every line is incident with a cutpoint. If there is more than one cutpoint and a line joining two, we are back to the first case. Assume then that a pair of cutpoints is joined by a 2-path. By line symmetry, each endpoint that is not a cutpoint is on such a path, and a complete bipartite graph having no cutpoints results, a contradiction. Hence there is only one such cutpoint and every line is incident with it so X is a star, which does not have $\mathcal{G}(X)$ semi-regular, and we again reach a contradiction.

As a consequence of these lemmas, it is now possible to show that the possible line-symmetric graphs having $\mathcal{G}(X)$ semi-regular are severely restricted.

Theorem 4.1 Let X be a nontrivial line-symmetric graph having $\mathcal{G}(X)$ semi-regular. Then X is a 1-path.

Proof: If X has the given properties and more than one line then by Lemma 4.1.1 it is connected, for otherwise $\mathcal{A}(X_1)\cong\mathcal{A}(X_2)=\{1\}$ contradicts the assumption that X is line-symmetric. Lemma 4.1.3 then implies that X is cyclically connected so that $\deg x \geq 2\ \forall x\varepsilon X$. However, Lemma 4.1.2 states that $\deg x \leq 2$. It follows that X is then a cycle, which as we have observed in Corollary 2.1.1 does not have semi-regular automorphism group. Hence, X has only one line, so is a 1-path.

The next step is to introduce the tool whereby graphs with certain properties have been constructed:

Definition 4.1 Given a group G with $H \subset G - \{1\}$, the Cayley graph (group-graph) of G with respect to H is the graph $X_{G,H}$ such that: $V(X_{G,H}) = G$ and $E(X_{G,H}) = \{[a,ab]\colon a\varepsilon G\ b\varepsilon H\}$.

Now if the graph so formed is connected, then for any g, g' there is a path $g\, g_1\, g_2\, g_3 \dots g_n\, g'$ from g to g'. Now since $g_1 = gh_1$, $g_2 = gh_1h_2, \dots$, we have $g' = gh_1h_2\dots h_n$ or $g^{-1}g' = h_1h_2\dots h_n$. Since every product $g^{-1}g'$ can be written as a product of the h_i, we have that H generates G. On the other hand, if $\langle H\rangle=G$ then clearly every such path exists and $X_{G,H}$ is connected. We have proven the following:

Lemma 4.2.1 $X_{G,H}$ is connected iff $\langle H\rangle=G$.

We are leading up to a characterization of all graphs X with regular subgroups contained in $\mathcal{A}(X)$, a property which appeared in both Chapters 2 and 3. The following lemma provides most of the information:

Lemma 4.2.2 Given a graph X, a necessary and sufficient condition for the existence of a group G and an $H \subseteq G$ with $X \cong X_{G,H}$ is that $\mathcal{A}(X)$ contain a regular subgroup G_0. In that case $G = G_0$.

Proof: If $X = X_{G,H}$ define $\eta: G \to \mathcal{A}(X_{G,H})$ by: $(\eta g)g' = gg'$. Clearly η is 1:1 and so $|\text{Im } \eta| = |G|$. Also, it is quite apparent that $\text{Im } \eta$ acts transitively on $V(X) = V(X_{G,H}) = G$. On the other hand, if $G_0 \leq \mathcal{A}(X)$ is regular, we choose an $a \varepsilon X$, numbering it's adjacency set $A = \{a_i \;\; i=1,2,\ldots,n\}$ and the unique automorphisms $H = \{\alpha_1 \;\; i=1,\ldots,n\}$ so that $\alpha_i(a) = a_i$. For any $x \varepsilon X$, let ϕ_x be the automorphism s.t. $\phi_x a = x$. By reqularity, every element of G_0 is a ϕ_x. Form $X_{G_0,H}$ and define the map $\varepsilon X_{G_0,H} \to X$ by $\varepsilon\phi_x = x$.

Now $[\phi_x \;\; \phi_x\alpha_i] \varepsilon X_{G_0,H}$ implies that $[\varepsilon(\phi_x) \;\; \varepsilon(\phi_x\alpha_i)] = [x \;\; \phi_x a_i]$ $= [\phi_x a \;\; \phi_x a_i] \varepsilon X$ since $[a \;\; a_i] \varepsilon X$. Hence ε preserves incidence. Conversely, if $[x \;\; y] \varepsilon X$, then $\phi_x^{-1}[x \;\; y] = [a \;\; \phi_x^{-1}y] \varepsilon X$ and so $\phi_x^{-1}y = a_i = \alpha_i a$ for $1 \leq i \leq n$. Hence $[x \;\; y] = \varepsilon[\phi_x \;\; \phi_x\alpha_i]$

and $[\phi_x \; \phi_x \alpha_i] \varepsilon X_{G_0,H}$ so that ε is onto. Since ε is obviously 1:1, it is an isomorphism of $X_{G_0,H}$ and X.

Theorem 4.2 (Sabidussi [33]) If X is a graph having $\mathcal{A}(X)$ regular, X is either trivial of order 2 or X is isomorphic to the Cayley graph of $\mathcal{A}(X)$ with respect to a set H of generators of $\mathcal{A}(X)$.

Proof: If X is disconnected Lemma 4.1.1 applies. In this case, the point-symmetry of X is incompatible with $\mathcal{A}(X_1) \cong \mathcal{A}(X_2) = \{1\}$ unless X is trivial with two points.

If X is connected, the theorem follows immediately by Lemmas 4.2.1 and 4.2.2.

There are several application to the work of Chapters 2 and 3.

Corollary 4.2.1 If X is a connected graph on a prime number of points, then X is a PPS graph iff it is a Cayley graph.

Proof: By Theorem 2.2 X is PPS iff it is a circulant. By Theorem 2.1 X is a circulant iff $Z_p \leq \mathcal{A}(X)$. Since Z_p is regular, by Lemma 4.2.2 this is iff X is a Cayley graph.

Corollary 4.2.2 If X is a connected graph on 2p points, then X is a 2-circulant iff it is a Cayley graph.

Proof: As above, this is a consequence of Lemma 4.2.2 together with Theorem 3.3.2.

It is also possible to settle the question raised in example 3.6, namely the nonexistence of any graph X having a regular dihedral group $D \cong D_m$ equalling $\mathcal{A}(X)$ for $m=3,4,$or5. By Sabidussi's theorem we may assume such an X would be connected and a Cayley graph with respect to a set H of generators of $\mathcal{A}(X)$.

We may as well note here that H can always be taken to satisfy: $u\varepsilon H \Rightarrow u^{-1}\varepsilon H$ since $[x\ xh]$ and $[xh\ (xh)h^{-1}]$ denote the same edge of $X_{G,H}$.

Suppose then, that $H(=H^{-1}) \subseteq D$ generates D. Then H must contain some element not in $\langle t\rangle$. If there is exactly one such element, in our notation we may assume X is a type 1 2-circulant with $F=\{0\}$. The mapping $\phi: X \to X$ given by $\phi(v_{i,j}) = v_{i,-j}$ is clearly in $\mathcal{A}(X)$ but fixes both $v_{0,0}$ and $v_{1,0}$, violating regularity. If there are two, say j and k and $j-k$ is odd, X can be renumbered so as to have $F=\{i,-i\}$ in which case the same mapping ϕ as above is also in $\mathcal{A}(X)$ violating regularity. If $j-k$ is even, X can be renumbered so that $F=\{-(i+1),i\}$ and in that case X is a circulant by Theorem 3.2 and $\mathcal{A}(X)$ is not regular. We have then that $|F| \geq 3$; this eliminates D_3 at once for here we must have k_6, whence $\mathcal{A}(X) = S_6$ which is not regular.

If $m=4$ or 5 we can apply the above argument to the complement of F. The following theorem has now been established:

<u>Theorem 4.3</u> To every dihedral group D of degree $2m$ $m \geq 6$ where $D \cong D_m$ there corresponds a (Cayley) graph $X_{G,H}$ whose group equals D. For $m < 6$ no such graphs exist.

By making use of the ideas of this chapter we may also return to Chapter 2 and extend those results to graphs which are both point and line-symmetric or just "symmetric". In Theorem 4.4, sufficiency was demonstrated by Turner [38] who also conjectured necessity. The theorem was first proven by Chao [11] using methods developed in [10]. The proof given here is a much simpler one provided by Berggren [6] and is more algebraic in nature.

Theorem 4.4 (Chao, Berggren) A regular graph X of degree n on p points is symmetric iff n is an even divisor of $p-1$ and $X = X_{Z_p,H}$ where H is the unique subgroup of Z_p^* of order n.

Proof: If H is the subgroup of Z_p^* of order n (containing -1) and we form $X = X_{Z_p,H}$, indentifing the points as $v_0, v_1, \ldots, v_{p-1}$ where $[v_i\ v_j] \varepsilon X$ iff $j-i\varepsilon H$, then clearly H is the symbol for X written as a circulant. Moreover, the transformations $\{T_{h,a}: a\varepsilon Z_p, h\varepsilon H\}$ given by $T_{h,a}(v_i) = v_{ai+h}$ form a subgroup of $\mathcal{a}(X)$ which is transitive on the lines of X, so that $\mathcal{a}(X)$ is also line-symmetric, hence symmetric. Clearly n must be an even divisor of $p-1$.

On the other hand if X is as given and $\mathcal{a}(X)$ is doubly transitive then $X=k_p$ and $X = X_{Z_p,H}$ with $|H| = p-1$. If $\mathcal{a}(X)$ is not doubly transitive, then by Burnside's Theorem (see 7.3 of [29]) we may suppose that the points of X are the elements of Z_p and that $\mathcal{A}(X) \leq \{T_{a,b}: a\varepsilon Z_p^* \ b\varepsilon Z_p\} = S$ where $T_{a,b}(x) = ax+b$. As usual, we identify $0,1,2,\ldots,p-1$ with $v_0, v_1, \ldots, v_{p-1}$.

Since $\mathcal{A}(X)$ is transitive on the vertices of X, we have $p \,\big|\, |\mathcal{A}(X)|$; since $K=\{T_{1,b} \ b\varepsilon Z_p\}$ is the subgroup of order p in S, we have $K \leq \mathcal{A}(X)$. Now certainly $H=\{a\varepsilon Z_p^*: T_{a,0}\varepsilon\mathcal{A}(X)\} \leq Z_p^*$ so that $\mathcal{A}(X) = \{T_{a,b}: a\varepsilon H \ b\varepsilon Z_p\}$.

Now for each $i,j \ \varepsilon Z_p$ we have $T_{1,-i-j}\varepsilon \ (X)$ so $[v_i \ v_j]\varepsilon X$ iff $[v_{-i} \ v_{-j}]\varepsilon X$ whence $T_{-1,0}\varepsilon G$ so that $-1\varepsilon H$ and $|H|$ is even. Now $\mathcal{A}(X)_0 = \{T_{a,0}: a\varepsilon H\}$ so that the adjacency set A of 0 is given by $A=Hc_1+\ldots+Hc_r$.

If $r \geq 2$ then there is a $T_{a,b} \ \varepsilon\mathcal{A}(X)$ with $T_{a,b} \ (C_1) = 0$ and $T_{a,b} \ (0) = C_k$ for $1 < k \leq r$. This implies $b = C_k$ and $0 = ac_1+b$ so $ac_1 = -c_k$ or $-ac_1 = c_k$. But $-1,a\varepsilon H$ so $-a\varepsilon H$ and $Hc_1 = Hc_k$ showing that $r = 1$ and $n = |H|$, an even divisor of $p-1$.

We have shown that the lines of X are $\{[v_a \ v_{a+hc_1}]\}$ so if we map $a \to ac_1^{-1}$ we induce a map of X to X' where the points of the latter are identified with those of Z_p

and the lines are $\{[a\ a+h]\}$ for $a\varepsilon Z_p$ and $h\varepsilon H$. Certainly $\mathcal{A}(X') = \mathcal{A}(X)$ for $\mathcal{A}(X') \geq \mathcal{A}(X)$ and since $X \cong X'$, $\mathcal{A}(X') \cong \mathcal{A}(X)$.

A generalization of Theorem 4.4 to graphs of a composite order is not available since the theorem of Burnside is specific to permutation groups of prime degree.

In concluding this chapter it should be noted that while in general a point-symmetric graph X is not a Cayley graph, there is a sense in which some "integral multiple" of X is a Cayley graph. The following provides a summary (without any proofs) of the paper of Sabidussi [34] which introduced the concept.

Definition 4.2 Let X be a graph and N a set of order n, a cardinal. The graph nX is given by $V(nX) = V(X) \times N$ and $E(nX) = \{[(x,\alpha)\ (y,\beta)]:\ \alpha,\beta\varepsilon N,\ [x,y]\varepsilon E(X)\}$.

Theorem 4.5 Let X be a connected point-symmetric graph, G a transitive subgroup of $\mathcal{A}(X)$. Then there is a cardinal n s.t. nX is a Cayley graph of G. If G is finite, n can be chosen as a factor of the order of G.

Definition 4.3 Let X be a connected point-symmetric graph. By the deviation of X is meant the smallest cardinal n s.t. nX is a Cayley graph.

He then went on to give a number of results concerning deviation. In general, it is a difficult function to calculate and Sabidussi was content to prove that there are graphs with arbitrarily large finite deviation. The next chapter provides a very important application of Cayley graphs to the case of Abelian groups.

CHAPTER 5

ABELIAN GROUPS

The existence or nonexistence of graphs corresponding to cyclic and dihedral groups has now been completely detailed, and the time has come to turn attention to other classes of groups, first the abelian ones.

Basing their work on that of Chapter 4, both Chao [9], and Sabidussi [34] thought they had proven the nonexistence of transitive abelian automorphism groups as the groups of graphs for $|X| > 2$. What they did prove forms Theorem 5.1. The proofs contained a similar error, first pointed out by McAndrew [26] who stated the limit $|X| \geq 5$. The first proof of the result was published by Imrich who established it for $|X| \geq 8$ [20] and later [21] completed it to the form presented here. The complete result is here presented in full under one title for the first time, and a gap in Imrich's proof when $|X|=5$ is pointed out.

<u>Theorem 5.1</u> (Chao, Sabidussi) If X is a nontrivial graph with transitive abelian automorphism group, then $\mathcal{A}(X)$ is the direct product of cyclic groups of order 2.

<u>Proof</u>: Such a group is necessarily regular. (see Appendix II-10) By Theorem 4.2 we may assume X is connected and using the same notation as in the proof of Lemma 4.2.2, we take up from the point where we observed that every element of $\mathcal{A}(X)$ is a ϕ_x.

Since by Theorem 4.2 the α_i generate $\mathcal{A}(X)$, a line $[x,y]$ $= [\phi_x(a)\ \phi_y(a)] \varepsilon X$ iff $\exists \alpha_v$ s.t. $\alpha_v \phi_x = \phi_y$ and this is the case exactly when $\phi_x^{-1} = \alpha_v \phi_y^{-1}$, that is when $[\phi_x^{-1} a\ \ \phi_y^{-1} a] \varepsilon X$. Hence the function $\psi\colon X \to X$ defined by $\psi\phi_x(a) = \phi_x^{-1}(a)\ \forall \phi_x \varepsilon \mathcal{A}(X)$ is an automorphism of X. From $\psi(a) = a$, it then follows by the regularity of $\mathcal{A}(X)$ that $\phi_x^{\ 2} = i\ \ \forall \phi_x \varepsilon\, \mathcal{A}(X)$. So, in the abelian group $\mathcal{A}(X)$ every nonidentity element has order 2 and the stated result follows by Prüfers first theorem (see Appendix II-11).

<u>Corollary 5.1.1</u> There is no graph with more than two vertices and having regular primitive automorphism group.

<u>Proof:</u> Every such group is cyclic. (see Appendix II-6) The Corollary now follows immediately from either of Theorems 5.1 or 2.1.

Reconsidering the group $\mathcal{A}(X)$ of Theorem 4.2, it is possible to obtain yet another restriction on graphs having abelian transitive automorphism groups. The set $A=\{\alpha_v: v\epsilon I\}$ was a generating system for $\mathcal{A}(X)$. It therefore contains a maximal independent subsystem B, which is still a generating system of $\mathcal{A}(X)$. That is, every element of $\mathcal{A}(X)$ can be written in the form $\psi=\alpha_{v_1}\alpha_{v_2}\alpha_{v_3}\ldots\alpha_{v_k}$ (*) where $\alpha_{v_i}\epsilon B$.

If $|B|=m$, it is clearly possible to represent each ψ as a vector of length m in which the components $v_1,\ldots,v_k$ appearing in the form (*) are one and the others are zero.

Moreover since the mapping $\psi_x \to x$ is clearly invertible, we may view these (0,1)-vectors of length m as unique representations of the vertices of X. Also, as noted in the proof of Theorem 4.2 $[x,y]\epsilon X$ exactly when $\exists\alpha_i\ \epsilon A$ s.t. $\alpha_i(x)=y$ so that the $\alpha_{v_i}\epsilon B\subseteq A$ determine a subgraph Y of X acording to the rule: There is a line $[x,y]\epsilon Y$ iff the vectors for the points differ in exactly one component.

Such a graph is known as a m-dimensional cube K_m and since B generates $\mathcal{A}(X)$, K_m spans the points of X. Also $\mathcal{A}(K_m)$ for $m>2$ is not regular so $K_m\subset X$. The following result is now established:

<u>Theorem 5.2</u> To every graph X having transitive abelian

automorphism group, there corresponds an m-diminsional cube which is a proper spanning subgraph of X.

The above result may now be used to prove the main theorem of this section.

<u>Theorem 5.3</u> For every natural number n different from 2,3 or 4 there exists a graph X of order 2^n with transitive abelian $\mathcal{A}(X)$ isomorphic to the direct product $\prod_{i=1}^{n} C_2$. For $n = 2,3,4$ no such graphs exist.

<u>Proof</u>: The result is immediate for $n = 1,2$. If X is a graph with $\mathcal{A}(X) \cong \prod_{i=1}^{n} C_2$, then also $\mathcal{A}(-X) \cong \prod_{i=1}^{n} C_2$ so that by Theorem 5.2 both X and $-X$ contain a proper spanning m-dimensional subcube. Since X can have at most half of all possible lines, attention may be restricted in the case $n = 3$ to graphs with at most 14 lines and in the case $n = 4$ to graphs with at most 60 lines.

In the case that $n = 3$ the requirement that X have a proper spanning 3-subcube (which has twelve lines) together with the transitivity of $\mathcal{A}(X)$ implies that X is regular of degree $3 + k > 14$ for some k, which is impossible.

In the case that $n = 4$, one must introduce at most 28 lines into the 4 cube to obtain X, or at most 3 lines at every point of the cube. As in the theorem of Sabidussi, select a particular point $a \varepsilon X$ and let the neighbours of a in k_4 be $H = \{a_1, a_2, a_3$ and $a_4\}$. The ϕ_x: $\phi_x a = x$ are identified with the

$x \varepsilon X$ by Sabidussi's Theorem. Now, at most three new lines of the form $[a, z_i]$ $1 \le i \le 3$ are introduced in K_4, where the z_i are products of two or more of the α_i: $\alpha_i(a) = a_i$, again by the above, for since the cube spans X and is connected, the points adjacent to a in K_4 generate all of $\mathcal{A}(X)$.

Now an automorphism ϕ of $\mathcal{A}(X)$ mapping H to itself can be considered as an automorphism of X fixing a so that $\mathcal{A}(X)$ is not regular if there exists a nontrivial ϕ leaving H invariant. For any $\phi \varepsilon \mathcal{A}(\mathcal{A}(X))$ then, let ϕ_H denote the induced permutation on H and let a <u>factor</u> of an element $g \varepsilon \mathcal{A}(X)$ denote an element a_i in the unique representation of g, with the a_i of course identified with the α_i.

If only one edge is introduced, we can assume that it connects a with a_1a_2, $a_1a_2a_3$ or $a_1a_2a_3a_4$, but in any case the transposition $(a_1\ a_2)$ generates the desired nontrivial automorphism of $\mathcal{A}(X)$.

In the two-edge case, if one of the z_i is $a_1a_2a_3a_4$ we reduce to the one-edge case. If neither is of this form we consider several cases:

<u>Case I</u> z_1 and z_2 each have two factors. If they have a common factor, say $z_1 = a_1a_2$ and $z_2 = a_2a_3$ the the desired ϕ is generated by (a_1a_3); if they do not, and say $z_1 = a_1a_2$ and $z_2 = a_3a_4$ we can take $\phi_H = (a_1a_2)$.

<u>Case 2</u> z_1 has two factors, z_2 three. If they have two factors in common, say $z_1 = a_1 a_2$ $z_2 = a_1a_2a_3$ take $\phi_H=(a_1\ a_2)$ if they have one in common take $z_1 = a_1 a_2$ $z_2 = a_2a_3a_4$ and $\phi_H = (a_3\ a_4)$

<u>Case 3</u> if $z_1 = a_1a_2a_3$ $z_2 = a_2a_3a_4$ then $\phi_H = (a_2\ a_3)$

In the three edge case, again if $z_3=a_1a_2a_3a_4$ we reduce to the two edge case. Otherwise there are four more possibilities.

<u>Case 4</u> All the z_i have two factors. If a_4 does not appear as a factor $z_1 = a_1a_2$ $z_2 = a_1a_3$ $z_3 = a_2a_3$ and $\phi_H = (a_1\ a_2)$. If all four of the a_i appear there are essentially two possibilities: $z_1 = a_1a_2$ $z_2 = a_2a_3$ $z_3 = a_3a_4$ when $\phi_H = (a_1a_4)(a_2a_3)$ or $z_1 = a_1a_2$ $z_2 = a_1a_3$ $z_3 = a_1a_4$ whence $\phi_H = (a_3\ a_4)$.

<u>Case 5</u> $z_1 = a_1a_2a_3$ and z_2, z_3 have two factors. The table gives the possible choices for z_2, z_3 and the ϕ_H which may be used

$a_1a_2,\ a_2a_3;\ \ (a_1\ a_3)$	$a_1a_2,\ a_1a_4;\ \ (a_4,\ a_1a_4)$
$a_1a_2,\ a_3a_4;\ \ (a_1\ a_2)$	$a_1a_4,\ a_2a_4;\ \ (a_1\ a_2)$

<u>Case 6</u> z_1 and z_2 have three factors, z_3 has 2. Again assume $z_1 = a_1a_2a_3$ and tabulate the other possibilities

$a_1a_2a_4,\ a_1a_2;\ \ (a_1\ a_2)$ $\quad a_1a_2a_4,\ a_1a_3;\ \ (a_4\ a_1a_2a_4)$

$a_1a_2a_4$, a_3a_4; (a_3a_4)

Case 7 If all the z_i have three factors, say $z_1 = a_1a_2a_3$, $z_2 = a_1a_2a_4$ and $z_3 = a_1a_3a_4$; take $\phi_H = (a_2a_3a_4)$

In order to show existence of graphs with $\mathcal{A}(X) \cong \prod_{i=1}^{n} (C_2)_i$ for $n \geq 5$, we consider the product of n such cyclic groups of order 2 with the generators a_i $1 \leq i \leq n$ and a subset H of G defined as follows:

$H = \{a_i,\ a_ka_{k+1},\ a_1a_2a_{n-2}a_{n-1},\ a_1a_2a_{n-1}a_n : 1 \leq i \leq n,\ 1 \leq k < n\}$.

Now for any $\alpha\varepsilon G$ the mapping $\phi_\alpha: x \to \alpha x$ for $x\varepsilon V(X_{G,H}) = G$ is an automorphism of $X_{G,H}$. We follow our usual procedure and identify each ϕ_x with x and view G as $\mathcal{A}(X_{G,H})$. It remains to be shown that G is regular, i.e. that there are no nontrivial automorphisms ϕ of G fixing H.

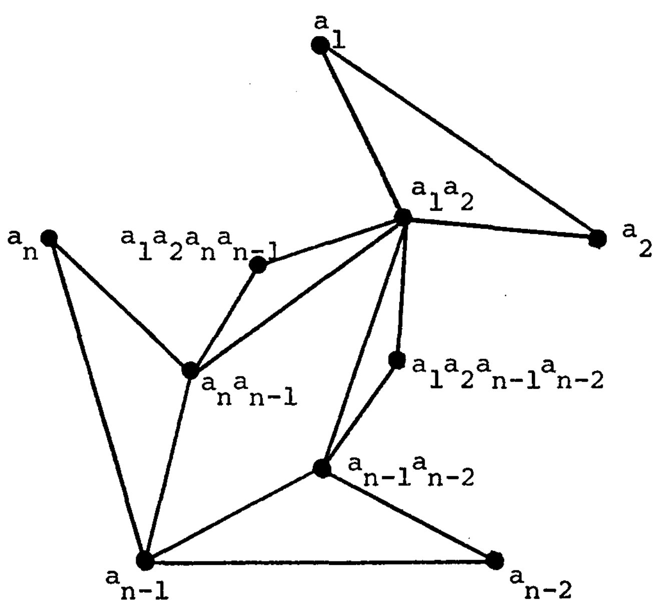


Figure 5.1

The point a_1a_2 has unique degree and so is fixed. Of the degree four points adjacent to a_1a_2 only $a_{n-1}a_{n-2}$ has a single degree two point in it's adjacency set and so is fixed. This forces, $a_1a_2a_{n-1}a_{n-2}$ to be fixed. Suppose some ϕ_H interchanges a_2 and a_na_{n-1}. Then ϕ_H interchanges a_1 and $a_1a_2a_na_{n-1}$, a_2a_3 and a_n and hence a_{n-1} and a_3 which is impossible when $a_3 \neq a_{n-2}$ or $n > 5$. Hence a_2 and a_na_{n-1} are fixed. Now also a_1, a_n, $a_1a_2a_na_{n-1}$ and a_{n-1} are fixed, forcing the n-4 triangles joining a_2 to a_{n-2} to be fixed, which completes the proof whenever $n > 5$.

However if $n = 5$, take $\phi_H = (a_2\ a_4a_5)(a_1\ a_1a_2a_4a_5)\ (a_3\ a_4)(a_2a_3\ a_5)$ and ϕ_H can be extended to a nontrivial automorphism of G fixing H, contrary to the claim of Imrich whose proof contains errors through this section.

A large number of possibilities remain however, for here only 6 points have been added to H whereas up to 10 could be added.

CHAPTER 6

Non-Abelian Case, a Summary. Conclusions

In order to complete this paper, an examination of the known results for non-abelian groups in general is necessary. The available information has been provided by Watkins and Nowitz [28,42,43,44], and what is given here is only a summary of their work (with the proofs omitted) since little would be gained by repeating all their arguments here. Also unlike the approach of Chapters 2 and 3 in which classes of graphs were constructed to completely categorize some classes of groups, here only the existence or nonexistence of graphs with certain regular non-abelian groups is considered. In general then we wish to know whether a group G belongs to

Class I: there exists a graph X with $\mathcal{A}(X) = G$ and acts regularly, or to

Class II: for each $H(=H^{-1})$ with $\langle H\rangle = G$ there exists a nontrival group automorphism ϕ of G with $\phi(H) = H$ and $\phi \neq 1$.

As remarked previously, such $\phi \varepsilon \mathcal{A}(G)$ which fix H have a natural interpretation as automorphisms of $X_{G,H}$ which fix the vertex identified with the identity. Hence, we immediately have the following:

Theorem 6.1 Class I and Class II are disjoint.

The next result, and several more like it are similar in spirit to some of the material of Chapter 3, classifying groups as they do by their generators.

Theorem 6.2 If G is a non-abelian group, the following are equivalent:

A. There exists a non-identity automorphism ϕ of G with $\phi(x) = x$ or $\phi(x) = x^{-1}$ $\forall x \varepsilon G$

B. G is generated by $a_1, \ldots, a_r, b$ where

(i) $b^{-1} a_i b = a_i^{-1}$ $1 \le i \le r$

(ii) $A = \langle a_i \rangle$ is abelian

(iii) $a_k^2 \ne 1$ for some $k = 1, \ldots, r$

(iv) a_1 is of order $2m$ for some m

(v) $b^2 = a_1^m$

Clearly then, groups satisfying B are in class II, and we note that if A is cyclic, G is dicyclic of order $4m$ and if also $m = 2^n$, G is a generalized quaternion group, each of which must then be in class II, a fact first shown by Nowitz [28].

The next theorem completely disposes of the case of the non-abelian groups of order p^3 for an odd prime p. (The fact that the two given sets of generating relations provide all such groups is proven for example by Hall [16, pp 50 - 52])

<u>Theorem 6.3</u> If p is an odd prime and G_p is generated by elements a,b, and c where either

(1) $a^{p^2} = b^p = 1;\quad b^{-1}ab = a^{p+1}$ or

(2) $a^p = b^p = c^p = 1;\quad ab = bac;\quad ac = ca;\quad bc = cb;\ p \geq 5,$

then G_p is in class I. If in case (2) $p = 3$, G_p is in class II.

In his initial paper on the subject, Watkins [42] also showed Class I is closed under direct product.

<u>Theorem 6.4</u> If G_1 and G_2 are in Class I and neither equals C_2, then $G_1 \times G_2$ is in Class I.

The remaining results of this section were the product of joint effort by Watkins and Nowitz [43,44].

<u>Theorem 6.5</u> If the group G is given by $G = \langle a,b\colon a^r=b^s=1;\ b^{-1}ab = a^k\rangle$ where this gives the entire multiplication table for G (so that $|G|= rs$, $(r,k)=1$ and $k^s \equiv 1 \pmod r$) then G is in class II under each of the following conditions:

(1) $k \equiv 1 \pmod r$ (abelian groups)

(2) $s = 2$, $r = 3,4,5$ and $k \equiv -1 \pmod r$ (dihedral groups)

(3) $s = 4$ and $k \equiv -1 \pmod r$ (generalized dicyclic group)

(4) $G = \langle a,b\colon a^8 = b^2 = 1,\ bab = a^5\rangle$

Otherwise, G is in class I.

It is typical of the intricacy of König's question that although (1) through (3) have already been demonstrated, the proof of the last two assertions alone is quite lengthy, this being the main reason that these results are presented here without proof. These same authors present two more general results:

<u>Theorem 6.6</u> Let the non-abelian group G_1 be a cyclic extension of a group G with $[G_1:G] \geq 5$. If G is in class I, so is G_1.

<u>Theorem 6.7</u> Let G be a non-abelian cyclic extension of an abelian group L, and suppose $(|G|, 6) = 1$. Then G is in class I.

In the language of this chapter we note that in chapter 2 we showed that C_m is always in Class II, in Theorm 4.3 that the dihedral groups $D \cong D_m$ are in Class II for $m < 5$ and in Class I otherwise. It is also clear that the imprimitive subgroup E of Theorem 3.4 is not in class I, because it is not regular. The work of chapter 5 demonstrated that abelian groups are in Class I if they are a direct product $\prod_{i=1}^{n}(C_2)_i$ $i \geq 5$ and in Class II otherwise.

This is essentially where König's question stands today, although there are some extensions to the work done here which have not yet been mentioned. For instance, with suitable modifications, the results of chapter 2 can be shown also to hold for directed graphs (see Hemminger [18,19]) and in particular for tournaments. (see Alspach [2,3] Astie [4], Berggren [7], and Moon [27].) An important exception is that the dihedral group is not contained in the automorphism group of a tournament so that in general the results of Chapter 3 will not be available in this case. However it does seem that by modifying the definition of a 2-circulant, the work of Chapter 3 could be extended to consider "k-circulants", with the caveat that less is known about groups of degree np for $n > 2$.

Finally, we mention two conjectures which if correct would plug a few "gaps" in the characterizations presented here. The first is due to Watkins [42] and arises in connection with the material summarized in this chapter.

<u>Conjecture 1</u> Every finite group is either in Class I or in Class II.

The second asserts that the characterization begun in chapter 3 can be completed in the same manner as that of chapter 2.

<u>Conjecture 2</u> A graph X on 2p points is 2PPS iff it is a 2-circulant.

APPENDIX I (Notation)

$\|B\|$	The order of a group, graph or set.
$\mathcal{A}(X)$	The automorphism group of a graph X.
$[v\ w]$	A line or edge in a graph X joining v and w. We write $[v\ w]\varepsilon X$ when this is not ambiguous or $[v,w]\varepsilon E(X)$ if additional clarification is necessary.
Z_m	A cyclic permutation group generated by an m-cycle. (not used abstractly)
D_m	The group of symmetries of the m-gon. (not used abstractly)
$A \leq B$	A is a subgroup of B.
$A \trianglelefteq B$	A is a normal subgroup of B.
Z_p^*	The group of the nonzero elements of Zp under multiplication.
H_p	$Z_p^* / \{1,p-1\}$
$W_0 \equiv W_1$	The two graphs are congruent, i.e. one is a copy of the other.
$\langle a_1,a_2,\ldots,a_n\rangle$	The group generated by $a_1,\ldots,a_n$.
$a\|b$	a divides b
k_r	The complete graph on n points.
$S \sim S'$	S is equivalent to S'.
$A \cong B$	A is isomorphic to B. (Used for groups or graphs)
$\phi\|X_k$	An automorphism of X restricted to a subgraph X_k.

$X_{G,H}$	The Cayley graph of G with respect to H.
$\mathrm{Im}\eta$	The image of a function η.
S_n	The symmetric group on n points.
$T_{a,b}$	A function defined by $T_{a,b}(x) = ax + b$
(a,b)	The greatest common divisor of a and b.
$\mathcal{A}(X)_x$	The subgroup $\mathcal{A}(X)$ which fixes the point x.
$(-X)$	The complement of X.

APPENDIX II

Definitions From Graph Theory The term *graph* was implicitly defined in the introduction. The following refer to certain characteristics, types and properties of, or associated with, graphs.

Order The number of points in a graph.

Degree (of a point) The number of lines incident with a point.

Girth The length of a shortest cycle in a graph.

Regular of degree n Every point has degree n. When n=3 X is *cubic*.

Connected Between any two points of a graph, there exists a path in the graph.

Connectivity The least number of points whose removal results in a disconnected graph.

Cyclically connected - of connectivity two.

Component A maximal connected subgraph of a graph.

Cutpoint A point whose removal increases the number of components in a graph.

Trivial Graph Has one or more points but no lines.

Bipartite Graph The points of the graph may be seperated into two sets so that each set induces a trivial subgraph of the graph.

Star A bipartite graph with all lines incident with one point.

Line Graph Has points representing the lines of an original graph with lines joining the points which represent incident lines of the original graph.

Spanning Subgraph A subgraph containing all the points of the graph.

Similar Lines There exists a $\phi \in \mathcal{G}(X)$ mapping one line to the other.

The definition of a graph may be modified so as to give direction to the lines. The resultant structure is a *directed graph* and its' lines are called arcs. If every pair of points is joined by exactly one arc we call the graph a tournament.

Definitions and Theorems on Permutation Groups and Abstract Groups.

The automorphism group of a graph is defined in the introduction. The following terms and results are used freely in the text of this thesis and are provided here for reference.

Order The number of elements in a group. The least power of a permutation which yields the identity.

Degree The size of the object set on which a particular permuation group acts (nontrivially)

Orbit Set of all points to which some fixed point can be mapped by G.

Transitive A group G is transitive on a set Ω if for every pair of points $a,b\varepsilon\Omega$ $\exists\phi\varepsilon G$ with $\phi(a) = b$.

Theorem AII-1 (Wielandt [45] p.8) In each permutation group whose order is divisible by a given prime number p, there are elements whose cycle decomposition contains a p-cycle.

Semiregular Group For every $k\varepsilon\Omega$ the subgroup of G which fixes k is trivial.

Regular Group Any transitive semiregular group. Equivalently, a transitive group whose order and degree are the same.

Regular Permutation All cycles have the same length.

Block A subset ψ of the object set of G having the property that $\forall\phi\varepsilon G$ $\phi(\Psi)$ equals ψ or is disjoint with ψ.

Primitive Group Has only one-point and $|\Omega|$-point blocks.

Imprimitive Group Has nontrivial blocks, the totality of those conjugate to one particular block being referred to as a complete block system.

Theorem AII-2 (Wielandt [45] p.13) If the transitive permutation group G contains an intransitive permutation subgroup different from 1, then G is imprimitive and the orbits of N form a complete block system for G.

Dihedral Group Any group $G = \langle s,t \rangle$ where $s^2 = t^m = 1$ and $sts = t^{-1}$.

It is important to note that here dihedral groups are defined abstractly and the groups D_m are included but refer to certain specific dihedral groups.

Theorem AII-3 (Rotman [30] p. 92) Any group of order 2p is either cyclic or dihedral.

Theorem AII-4 (Wielandt [45] p. 12) The length of a block of a transitive group G divides the degree of G.

Theorem AII-5 (Wielandt [45] p. 8) In each transitive group of degree $n > 1$ there is an element of degree n.

Theorem AII-6 Every regular primitive permutation group is cyclic.

Proof: By AII-5 G contains an element t of degree $n = |\Omega|$ which must be regular, so of order n also. Now $|\langle t \rangle| = n$ and $\langle t \rangle \leq G$ but $|G| = n$ by regularity so $\langle t \rangle = G$ and G is cyclic.

Theorem AII-7 (Wielandt [45] p. 94) A primitive group of degree $n = 2p$ (p prime) is singly transitive only if n is of the form $n = a^2 + 1$ for an odd positive integer a.

Theorem AII-8 (Scott [35]) In the above theorem, if $a > 3$, a is not a prime.

AII-9 Wielandt's construction [45 p.94] of a uniprimitive group of degree 10.

"Let $G = S^{\Omega}$ be the symetric group on $\Omega = \{1,2,...,5\}$. In addition, let $\overline{\Omega}$ be the set of the 10 unordered pairs $\{a,b\}$ with $a,b \in \Omega$ and $a \neq b$. To each $g \in G$ we assign in a one-to-one manner a permutation $\overline{g}$ on $\overline{\Omega}$ by $\{a,b\}^{\overline{g}} = \{a^g, b^g\}$. In this way we have represented G faithfully as a permutation group $\overline{G}$ or $\overline{\Omega}$. G is not doubly transitive for there is no $\overline{g} \in \overline{G}$ which fixes $\{1,2\}$ and takes $\{1,3\}$ into $\{4,5\}$. On the other hand, $\overline{G}$ is primitive since $\overline{G}_{\{1,2\}}$ is maximal in $\overline{G}$."

Theorem AII-10 (Wielandt [45] p. 9) Every Abelian group G transitive on Ω is regular.

Theorem AII-11 (Kurosh [25] V.1 p. 173) Every primary group in which the orders of the group elements are bounded is a direct sum of cyclic groups.

Direct Product of two groups A and B has elements (a_i, b_i) where $(a_1,b_1)(a_2,b_2) = (a_1a_2, b_1b_2)$.

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