Sweeping graphs and digraphs

by

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Abstract

Searching a network for an intruder is an interesting and difficult problem. Sweeping is one such search model, in which we "sweep" for intruders along edges. The minimum number of sweepers needed to clear a graph G is known as the sweep number $sw(G)$. The sweep number can be restricted by insisting the sweep be monotonic (once an edge is cleared, it must stay cleared) and connected (new clear edges must be incident with already cleared edges).

We will examine several lower bounds for sweep number, among them minimum degree, clique number, chromatic number, and girth. We will make use of several of these bounds to calculate sweep numbers for several infinite families of graphs. In particular, these families will answer some open problems regarding the relationships between the monotonic sweep number, connected sweep number, and monotonic connected sweep number.

While sweeping originated in simple graphs, the idea may be easily extended to directed graphs, which allow for four different sweep models. We will examine some interesting non-intuitive sweep numbers and look at relations between these models. We also look at bounds on these sweep numbers on digraphs and tournaments.

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Chapter 1

Introduction

> Lark was about to head back to the corridor when he stopped, intrigued to see the Jophur's console was still active. Holo displays flickered, tuned to spectral bands his eyes found murky at best. Still he approached one in curiosity — then growing excitement.
>
> *It's a map!* He recognized the battle cruiser's oblate shape, cut open to expose the ship's mazelike interior. It turned slowly. Varied shadings changed slowly while he watched.
>
> . . . He managed to locate the security section where he and Ling had been imprisoned when they were first brought aboard on Jijo. A deep, festering blue rippled outward from that area and spread gradually "northward" along the ship's main axis, filling one deck at a time.
>
> *A search pattern. They've been driving me into an ever smaller volume...*
>
> Heaven's Reach, David Brin

Imagine you arrive at one of the "big box" department stores that have sprung up around the country with a bag of live garden gnomes. No sooner do you enter the store then you trip, break you glasses, and drop your bag of gnomes. The gnomes scatter throughout the store. Fearing the wrath of the store manager, you begin to look for the gnomes. You are hampered by several problems. First, the shelves in the big box store are very high; too high to pass over. Second, with your broken glasses, the only way you can catch a gnome is by running into it. Third, the gnomes are incredibly quick - while no gnome could walk past you in an aisle, picking the wrong way at a junction would allow them to slip around behind you. Finally (and most embarrassingly), you forgot to count the number of gnomes in the sack, and don't know how many you're looking for!

You begin to look around, walking around one of the large shelves. You complete a full circuit, and think that at least there are no gnomes here... when you realize that a gnome could simply be walking directly in front of you, doing the same circuit! You could walk around the shelf forever! Clearly, you need help, and even then, you need to find a way to walk through the store in a manner to guarantee that you walk around all the shelves (and other nooks and crannies) AND guarantee that no gnomes have "doubled back" on you.

This model of searching was originated by Parsons in [18], though the problem was of interest to spelunkers earlier than that [6]. Parson's original problem dealt with finding a lost spelunker in a system of caves, but the problem has much wider application. We are interested in sweeping as a problem in network security, looking for methods to clean a network of a computer virus, or methods to capture a mobile intruder using software agents. In the literature, sweeping has been linked to pebbling (and hence to computer memory usage) [12], to assuring privacy when using bugged channels [10], and to VLSI (very large-scale integrated) circuit design [9].

An alternate, but equally valid paradigm, to spelunking would be to consider a building filled with poison gas. The job of the sweepers is to remove all the poison gas. Clearly, if a sweeper comes to a junction, at which point the area behind him has been cleared of poison gas, but the area ahead of him has not, great care must be taken, else the cleared area will be recontaminated. Our goal becomes to remove all the poison gas.

In this thesis, we will deal primarily with graphs.

Definition 1.1 A *graph* G is composed of two sets, a finite set of elements $V(G)$ called vertices, and a finite set $E(G)$ of unordered pairs of elements of $V(G)$ called edges. A *reflexive graph* is a graph in which loops are also allowed as edges, and a *multigraph* is a graph in which multiple edge may exist between pairs of vertices.

Definition 1.2 The number of edges incident with a vertex v of a graph G is the *degree* of v, denoted $\deg(v)$.

In this search model, collision between a searcher and an intruder may occur on an edge. We will call this type of search a *sweep*. In Parson's original ***general sweeping*** model, graphs were considered to be embedded in three-space and the motion of sweepers (and intruders) in the graph was described by continuous functions. A successful strategy would be a set of functions for the sweepers such that at some time t, the function value for some sweeper must be the same as that of the intruder.

We will consider a similar, but discrete, model. The specifics of sweeping a reflexive multigraph G are as follows. Initially, all edges of G are *contaminated* (or *dirty*). To sweep G it is necessary to formulate and carry out a sweep strategy. A strategy is a sequence of actions designed so that the final action leaves all edges of G *uncontaminated* (or *cleared*). Two actions are allowed after initially placing sweepers on vertices of G.

1. Move a single sweeper along an edge uv starting at u and ending at v.

2. Move a single sweeper from a single vertex u of G to any other vertex v of G.

Any strategy that uses the above actions will be called a *wormhole sweep strategy*. A strategy that restricts itself to the first will be called a *sweep strategy*. Notice that in a disconnected graph, the number of sweepers needed for a sweep strategy would be much higher than the number needed for a wormhole sweep strategy. In the former, there needs to be sufficiently many sweepers in each component, while in the latter, sweepers could jump from component to component. This thesis will deal solely with connected graphs.

An edge uv in G can be *cleared* in one of two ways.

1. At least two sweepers are placed on vertex u of edge uv, and one of them traverses the edge from u to v while the others remain at u.

2. A sweeper is placed on vertex u, where all edges incident with u, other than uv, are already cleared. Then the sweeper moves from u to v.

Knowing that our goal is a graph where all the edges are cleared, a basic question is: what is the fewest number of sweepers for which a sweep strategy exists? We call this the *sweep number*, denoted $\text{sw}(G)$. We define the *wormhole sweep number* similarly and denote it $\text{wsw}(G)$. In fact, these two numbers are equal for connected graphs [1]. It can be seen that for any graph G the sweep number exists by considering the following strategy. Place a sweeper on each vertex of G, and then use a single extra sweeper to clear all the edges.

Further restrictions may be placed on all strategies. Let $E(i)$ be the set of cleared edges after action i has occurred. A sweep strategy for a graph G for which $E(i) \subseteq E(i+1)$ for all i is said to be *monotonic*. We may then define the *monotonic sweep number* and the *monotonic wormhole sweep number*, denoted $\text{msw}(G)$ and $\text{mwsw}(G)$, respectively. Similarly, a sweep strategy such that $E(i)$ induces a connected subgraph for all i is said to be *connected*, and we may define the *connected sweep number* $\text{ksw}(G)$ and the *connected wormhole sweep number* $\text{kwsw}(G)$. Finally, a sweep strategy may be both connected and monotonic, giving us the *monotonic connected sweep number* $\text{mksw}(G)$ and the *monotonic connected wormhole sweep number* $\text{mkwsw}(G)$.

LaPaugh [13] and Bienstock and Seymour [5] proved that for any connected graph G, $\text{wsw}(G) = \text{mwsw}(G)$. Barrière et al. [4] extended this result, giving the following relations for these numbers.

Lemma 1.3 *For any connected graph G,*

$$\begin{aligned}\text{sw}(G) = \text{wsw}(G) = \text{mwsw}(G) \leqq \text{msw}(G) &\leqq \text{kwsw}(G) = \text{ksw}(G)\\ &\leqq \text{mkwsw}(G) = \text{mksw}(G).\end{aligned}$$

This chain of inequalities suggest several questions. For instance, can equality be achieved? Do graphs exist for which the inequalities are strict?

Definition 1.4 A *path* P_n is a graph on n vertices $v_1, v_2, \ldots, v_n$ where $v_iv_{i+1} \in E(P_n)$ for $1 \leqq i \leqq n-1$. A *cycle* C_n is a graph with $V(C_n) = V(P_n)$ and $E(C_n) = E(P_n) \cup \{v_nv_1\}$.

Definition 1.5 A *tree* is a connected graph that contains no cycles.

Parsons proved the following interesting result about the sweep number of trees [18].

Theorem 1.6 *Let T_1, T_2, and T_3 be vertex-disjoint trees each having at least one edge, where vertex v_j is a vertex of degree one in tree T_j, $1 \leqq j \leqq 3$. Let T be the tree formed by identifying the vertices v_1, v_2, v_3 as a single vertex v. If $\mathrm{sw}(T_i) = k$, for $i = 1, 2, 3$, then $\mathrm{sw}(T) = k + 1$.*

Thus, trees may have arbitrarily large sweep number. Barrière et al. [3, 4] also proved that there are at most two sweep numbers for a given tree. That is, for a tree T, we have

$$\begin{aligned} \mathrm{sw}(T) = \mathrm{wsw}(T) = \mathrm{mwsw}(T) \leqq \mathrm{msw}(T) &= \mathrm{kwsw}(T) = \mathrm{ksw}(T) \\ &= \mathrm{mkwsw}(T) = \mathrm{mksw}(T), \end{aligned}$$

and the inequality may be strict.

We will show that $\mathrm{sw}(K_n) = \mathrm{mksw}(K_n) = n$, where K_n is the complete graph on n vertices. This means that there is exactly one sweep number for complete graphs.

In general, determining the sweep number of a graph G is NP-complete [14]. However, computing the sweep number of a tree is not NP-complete, but can be computed in linear time. As any successful sweep strategy gives an upper bound, our goal becomes first to find the "right" way to clear the graph, using as few sweepers as possible. Once this strategy is found, we must then prove that no fewer sweepers will suffice. Here is where the true difficulty lies: most easily attainable lower bounds are quite poor. We will prove several lower bound results using the graph parameters of minimum degree, girth, and chromatic number, but the parameter we will make the greatest use of is the clique number which occurs repeatedly in our constructions.

Definition 1.7 The *girth* of a graph G, denoted $g(G)$, is the order of the smallest cycle that is a subgraph of G.

Definition 1.8 A *colouring* of a multigraph G is any assignation of colours to the vertices of G. A *proper colouring* is a colouring in which no pair of adjacent vertices receives the same colour. The least number of colours needed to properly colour the vertices of G is called the *chromatic number* of G, denoted $\chi(G)$.

In the case of a reflexive multigraph, a colouring is proper if the corresponding colouring of the multigraph with all loops removed is proper.

Definition 1.9 The *clique number* of the graph G, denoted $\omega(G)$, is the number of vertices in a largest complete subgraph of G.

Finding the "right" strategy is also very hard, but for most of the graphs considered, such strategies will be given explicitly. Work on upper bounds is also vital, and recently an upper bound on the monotonic connected sweep number of planar cubic graphs was found by considering flooding [17].

Returning to Lemma 1.3, we still need to consider the question of strict inequality. An example in [4] shows that the first inequality may be strict. The graph below, which we call the "Y-square", is another example. Moreover, this is an example with fewer vertices and edges. We conjecture that it is the smallest graph that exhibits a strict inequality between sweep number and monotonic sweep number.

In the process of clearing a graph, we will often refer to the condition of its vertices. We introduce the following definitions.

Definition 1.10 A vertex in a graph G is said to be *exposed* if it has edges incident with it that are contaminated as well as edges incident with it that are cleared. Following a sweep strategy S on G, we define $\mathrm{ex}_S(G,i)$ to be the number of exposed vertices after the i-th step. We also define the *maximum number of exposed vertices* to be $\mathrm{mex}_S(G) = \max\limits_i \mathrm{ex}_S(G,i)$.

Definition 1.11 A vertex v is said to be *cleared* if all the edges incident with it are currently uncontaminated.

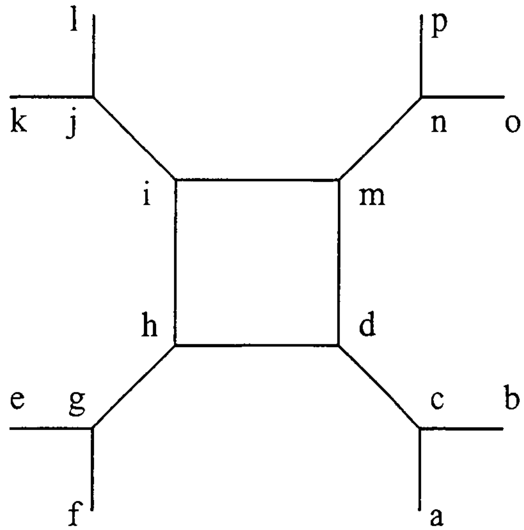

Figure 1.1: The Y-square.

Definition 1.12 A graph H is a *minor* of a graph G if H can be obtained from G through a sequence of edge contractions and deletions.

Theorem 1.13 *The Y-square has sweep number* 3 *and monotonic sweep number* 4.

PROOF. We first show that the Y-square can be cleared by three sweepers. Place a sweeper on each of vertices a, b, and e. Move the sweepers on vertices a and b to c, clearing ac and bc, then clear to d. Move the sweeper on e to g, clearing eg. Leaving one sweeper on d, move the other to h (clearing dh). Then move this sweeper to g. With this move, the edge dh becomes recontaminated (so this sweep is not monotonic). Use this sweeper to clear f, the move the sweeper on g to h. Keeping two sweepers stationed on h and d, use the remaining sweeper to clear dh. Move the sweepers on h and d to i and m, respectively. Use the third sweeper to clear im. Move the sweepers on i and m to j and n, respectively. Finally, use the third sweeper to clear the remaining pendant edges.

Note that by deleting the "Y" starting at h, and then by contracting the 4-cycle to a single point, we form a tree. Since 2 sweepers are necessary to sweep a "Y", by Theorem 1.6, this tree must have sweep number 3. In Theorem 2.15, we will show that the sweep number of a graph is bounded below by the sweep number of any of its minors, and so the sweep number of the Y-square is exactly three.

To see that the Y-square has monotonic sweep number at most 4, we demonstrate a sweep strategy. Place a sweeper on vertices a, b, k, and l. Move the sweepers on a and b to c, then to d. Then move one sweeper on d to m, and the other to h. Then move the sweepers on k and l to j, then to i. Then move one sweeper on i to m, and the other to h. Move the two sweepers on m to n, and then move one of these sweepers to o, and the other to p. Move the two sweepers on h to g, and then move one of these sweepers to e, and the other to f, clearing the Y-square.

Assume that a monotonic sweep strategy exists for three sweepers to clear the Y-square. Consider the first moment that an edge in the 4-cycle is cleared. Without loss of generality, assume the edge dh is the first cleared by a sweeper moving from d to h. Since dm is not cleared, the vertex d is exposed, and must contain another sweeper. If the edge cd is not cleared, then even with a third sweeper at most two more edges can be cleared. Thus, the edge cd must be cleared. If neither ac nor bc is cleared, then c is exposed, and hence the sweeper on c cannot move. Nor can the sweeper on h move. The sweeper on d has only one move available – clearing dm – at which point no sweeper can move without recontaminating an edge. Thus, at least one of ac and bc must be cleared. If only one is cleared, then there must be a sweeper on c. With sweepers on h, c, and d, the sweeper on c may clear ca, but then the sweepers may only clear two more edges before being unable to move with recontaminating an edge. Thus, when the sweeper moves from h to d, the entire "Y" attached at d must be cleared. (This could have been done with 2 sweepers.)

Now, consider the edge gh. If gh is cleared before dh is cleared, there must be a sweeper on h. This sweeper cannot have cleared the "Y" at h alone, so there must be another sweeper in the "Y", as neither of the other sweepers could have gotten to h without clearing the edge dh. But this requires four sweepers. So, the edge gh cannot be cleared. Similarly, the edge mn cannot be cleared.

The only move for the sweeper remaining at d is to clear dm, at which point neither of the sweepers at h or m may move without recontaminating edges. Utilizing the third sweeper, at most two more edges may be cleared, before any move by any sweeper recontaminates an edge. Thus, there is no monotonic sweep strategy with three sweepers, and the monotonic sweep number of the Y-square must be four. ■

The second inequality, msw $\leqq$ ksw, was also proved in [4]. Further, they gave an example showing that the inequality was strict. With this result, they also observed that generally, the monotonic sweep number or connected sweep number of a graph G may be less than the monotonic sweep number or connected sweep number of some minors of G. We prove these results by using large cliques as our building blocks, thereby allowing us to calculate the sweep numbers of the resulting graphs with ease.

Whether the third inequality, ksw $\leqq$ mksw, can be strict was left as an open problem in [4]. We will show that there exists a graph G such that $\mathrm{ksw}(G) < \mathrm{mksw}(G)$, and that, in fact, the difference between these two values can be arbitrarily large.

The concept of sweeping may be extended to directed graphs.

Definition 1.14 A *directed graph* (or digraph) D consists of a finite non-empty set $V(D)$ of vertices and a finite set $A(D)$ of ordered pairs of distinct vertices, called arcs. We denote the arc from z to x by (z, x) and say that z dominates x. We call z the tail of arc (z, x) and x the head of arc (z, x).

Definition 1.15 The *out-degree* $d^+(W) = d^+_D(W)$(respectively, *in-degree,*$d^-(W) = d^-_D(W)$) of a subset W of V is the number of arcs from vertices of W to vertices of $V - W$ (respectively from $V - W$ to W). The *minimum out-degree* and *in-degree* over all vertices x of D are denoted by $\delta^+ = \delta^+(D)$ and $\delta^- = \delta^-(D)$ respectively.

We consider sweeping directed graphs D and the minimum number of sweepers required to clear all of the arcs of D in different situations. If both the sweepers and the intruders must move in the direction of the arcs, we call this a *directed sweep* and the minimum number of sweepers needed to clear D is the *directed sweep number* $\mathrm{sw}_{1,1}(D)$. If both the sweepers and the intruders can move with or against the direction of an arc, we call this an *undirected sweep* and the minimum number of sweepers needed to clear D is the *undirected sweep number* $\mathrm{sw}_{0,0}(D)$. If the intruders must move in the direction of the arcs, but the sweepers need not, we call this a *strong sweep* and the minimum number of sweepers needed to clear D is the *strong* *sweep number* $\mathrm{sw}_{0,1}(D)$. Finally, if the sweepers must move in the direction of the arcs, but the intruders need not, we call this a *weak sweep* and the minimum number of sweepers need to clear D is the *weak sweep number* $\mathrm{sw}_{1,0}(D)$. In all of the directed sweep models we consider, "jumping" is forbidden.

Of the various directed sweep models, perhaps the "least" interesting is the undirected sweep. Any undirected sweep strategy in a digraph D also is a sweep strategy for the undirected graph underlying D (every arc is replaced by an edge). It is included as a model for completeness, but we will concentrate on the other directed sweep models.

The methods by which arcs are cleared also need addressing in these models. In a strong sweep, an arc (u, v) in a digraph D can be cleared in one of three ways.

1. At least two sweepers are placed on vertex u of arc (u, v), and one of them traverses the arc from u to v while the others remain at u.
2. A sweeper is placed on vertex u, where all incoming arcs incident with u already are cleared. Then the sweeper moves from u to v.
3. A sweeper is placed on vertex v, and traverses the arc (u, v) in reverse, from v to u.

In a directed sweep, an arc (u, v) in a digraph D can be cleared in one of two ways.

1. At least two sweepers are placed on vertex u of arc (u, v), and one of them traverses the arc from u to v while the others remain at u.
2. A sweeper is placed on vertex u, where all incoming arcs incident with u already are cleared. Then the sweeper moves from u to v.

Finally, in a weak sweep, an arc (u, v) in a digraph D can be cleared in one of two ways.

1. At least two sweepers are placed on vertex u of arc (u, v), and one of them traverses the arc from u to v while the others at u.

2. A sweeper is placed on vertex u, where all arcs incident with u already are cleared. Then the sweeper moves from u to v.

It also should be mentioned that in a weak sweep, a cleared arc (u, v) may be recontaminated if v is either the tail or head of a contaminated arc and contains no sweeper.

As in the undirected case, we introduce an idea of a vertex being clear.

Definition 1.16 In a strong sweep or a directed sweep, a vertex v of a digraph D is *clear* if all of the incoming arcs with v as head are clear. In an undirected or weak sweep, a vertex v of a digraph D is *clear* if all of the arcs incident with v are clear.

Definition 1.17 An *orientation* of a graph G is a digraph D, where $V(D) = V(G)$ and exactly one of the arcs (u, v) and (v, u) is in $A(D)$ for each edge $uv \in E(G)$.

We often think of an orientation D of G coming from some assignation of direction to each edge in G.

Definition 1.18 A *directed path* $\overrightarrow{P_n}$ on n vertices has an n-element vertex set v_1, v_2, $\ldots$, v_n and arcs (v_i, v_{i+1}) for $1 \leqq i \leqq n-1$. A *directed cycle* $\overrightarrow{C_n}$ on n vertices has an n-element vertex set v_1, v_2, $\ldots$, v_n and arcs (v_i, v_{i+1}) for $1 \leqq i \leqq n-1$, as well as the arc (v_n, v_1).

Certainly, directed paths and directed cycles are orientations of undirected paths and cycles. An interesting and commonly studied class of digraphs is the orientations of the complete graphs.

Definition 1.19 A *tournament* T on n vertices is an orientation of K_n, that is, a directed graph D in which $A(D)$ contains exactly one arc between every pair of distinct vertices.

Definition 1.20 In the case of tournaments, we sometimes call the out-degree of a vertex the *score* of the vertex. The *score sequence* of a tournament is then the sequence of out-degrees of the vertices ordered such that $d^+(v_n) \geqq d^+(v_{n-1}) \geqq \cdots \geqq d^+(v_1)$.

The directed graph analogue of a tree is an acyclic digraph.

Definition 1.21 A digraph D is *acyclic* if it contains no directed cycles. If the vertices of an acyclic digraph D are labelled $v_1, v_2, \ldots, v_n$ such that an arc goes from v_i to v_j only if $i < j$, then this labelling is called an *acyclic ordering* of D.

Definition 1.22 A *transitive tournament* TT_n is a tournament on n vertices that is acyclic. Alternatively, TT_n may be considered the transitive closure of $\overrightarrow{P_n}$.

Definition 1.23 A *minimum feedback vertex set* F in a directed graph D is a smallest set of vertices whose removal leaves $D - F$ acyclic.

Finding such an F is an NP-hard problem [11].

Definition 1.24 A digraph D is *connected* if the underlying undirected graph is connected.

Definition 1.25 A digraph D is *strong* if there exists a directed path from vertex x to vertex y for all possible choices of vertices x and y. A *strong component* of D is a maximal induced subdigraph of D that is strong. A single vertex is taken to be a strong subdigraph.

Definition 1.26 If a tournament is not strong, we say that it is *reducible;* else it is *irreducible.*

Definition 1.27 The *reversibility index* $i_R(T)$ of a strong tournament T is the size of a minimum set of arcs whose reversal changes T into a reducible tournament.

Chapter 2

Some Lower Bounds

2.1 Minimum Degree

For a graph G, we will denote the minimum degree of G by $\delta(G)$. The main result of this section is Theorem 2.2, which will be used to calculate several sweep numbers. We first introduce a simpler version of this theorem.

Theorem 2.1 *If G is a connected graph, then* $\mathrm{sw}(G) \geqq \delta(G)$.

PROOF. Consider the first time a vertex v is cleared. Every edge incident with v is cleared, but every vertex adjacent to v must be exposed. Thus, each vertex adjacent to v must contain a sweeper, and the result follows. ■

Theorem 2.2 *If G is a connected graph and $\delta(G) \geqq 3$, then* $\mathrm{sw}(G) \geqq \delta(G) + 1$.

PROOF. Consider a graph G with minimum degree $\delta(G)$, and a sweep strategy S that clears it. If the first vertex cleared by S is not of minimum degree, then it must have at least $\delta(G) + 1$ vertices adjacent to it. When it is cleared, each of the these vertices must contain a sweeper and $\mathrm{sw}(G) \geqq \delta(G) + 1$.

We now consider the last time that the graph goes from having no cleared vertices to a single cleared vertex u. By the preceding paragraph, we may assume that u is a vertex of minimum degree. We will assume that the strategy S employs at most $\delta = \delta(G)$ sweepers, and arrive at a contradiction. Let the neighbours of u be denoted v_1, v_2, ..., v_δ. Assume, without loss of generality, that uv_1 is the final edge incident with u cleared, and that uv_2 is the penultimate such edge.

Consider the placement of sweepers the moment before uv_1 is cleared. Since each of uv_i, $2 \leqq i \leqq \delta$, is cleared, there must be a sweeper on each end vertex of these edges. But this uses all δ sweepers. Thus the only way that uv_1 can be cleared is if the sweeper at u traverses the edge uv_1 from u to v_1. Thus, all the other edges incident with v_1 must be contaminated. Since $\delta \geqq 3$, the sweeper on v_1 cannot move.

Now consider the placement of sweepers before the penultimate edge uv_2 is cleared. Again, as each of the edges uv_i, $3 \leqq i \leqq \delta$, is cleared, there must be a sweeper on each end vertex of these edges. This accounts for $\delta - 1$ sweepers. Since the next move is to clear uv_2, the single free sweeper must be on either u or v_2. Sweeping from v_2 to u would instantly recontaminate the edge uv_2 which implies the edge must be cleared from u to v_2. This leaves the sweeper at v_2, and all the other edges incident with v_2 must be contaminated. Since $\delta \geqq 3$, the sweeper on v_2 cannot move.

Consider a sweeper on v_i, $3 \leqq i \leqq \delta$. If the vertex v_i is adjacent to v_1 and v_2, then the edges v_1v_i and v_2v_i are contaminated, and the sweeper at v_i cannot move.

If the vertex v_i is adjacent to exactly one of v_1 and v_2, it must also be adjacent to some other vertex w not adjacent to u (as the degree of v_i is at least δ). As there is no sweeper on w, the only way that v_iw can be cleared is if w is a cleared vertex. However, we know that u is the first cleared vertex, so that w is not cleared. Thus, the sweeper at v_i cannot move.

Finally, if the vertex v_i is adjacent to neither v_1 nor v_2, it must be adjacent to two vertices w_1 and w_2 neither of which is adjacent to u. As before, these edges cannot be cleared, and thus the sweeper at v_i cannot move.

As there are still contaminated edges, and none of the δ sweepers can move, we have obtained the required contradiction. ■

Despite the fact that Theorem 2.2 is only a slight improvement on Theorem 2.1, it yields many useful results.

Corollary 2.3 *For a connected graph G, let $\kappa(G)$ be the vertex connectivity and $\kappa'(G)$ be the edge connectivity of G. If $\kappa(G) \geqq 3$, then* $\mathrm{sw}(G) \geqq \kappa'(G) + 1 \geqq \kappa(G) + 1$.

PROOF. Since $\delta(G) \geqq \kappa'(G) \geqq \kappa(G)$ [23], the result follows from Theorem 2.2. ■

Corollary 2.4 *For $n \geqq 4$,* $\mathrm{sw}(K_n) = n$.

PROOF. By Theorem 2.2, we know that $\mathrm{sw}(K_n) \geqq n$. We present the following sweep strategy for K_n using n sweepers. First, clear a vertex v of K_n. This requires $n - 1$ sweepers, leaving one free. This free sweeper may then clear all the edges of K_n that are not incident with v. ■

Corollary 2.4 actually proves a much stronger result, namely, that $\mathrm{mksw}(K_n) \leqq n$, as the sweep strategy described in the proof is in fact both monotonic and connected. From Lemma 1.3 and Corollary 2.4, it follows that there is only one sweep number for the complete graph K_n.

Corollary 2.5 *For all positive integers $n \geqq 4$,*

$$\begin{aligned}\mathrm{sw}(K_n) = \mathrm{wsw}(K_n) &= \mathrm{mwsw}(K_n) = \mathrm{msw}(K_n) = \mathrm{kwsw}(K_n) \\ &= \mathrm{ksw}(K_n) = \mathrm{mkwsw}(K_n) = \mathrm{mksw}(K_n) = n.\end{aligned}$$

Definition 2.6 The *wheel graph* W_n , $n \geqq 3$, is the graph formed by connecting all the vertices of an n-cycle to another vertex v not on the cycle.

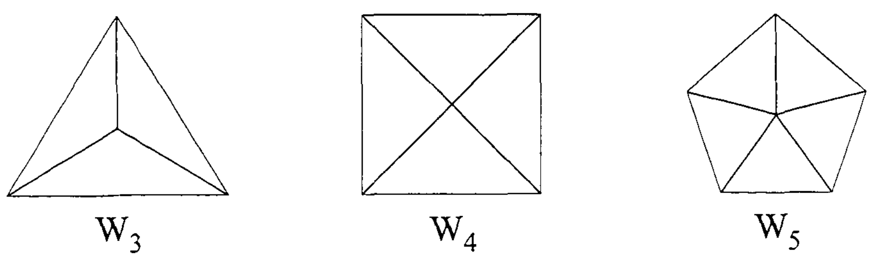

Figure 2.1: The wheel graphs W_3, W_4, and W_5.

Corollary 2.7 *For all $n \geqq 3$,*

$$\begin{aligned}\mathrm{sw}(W_n) = \mathrm{wsw}(W_n) &= \mathrm{mwsw}(W_n) = \mathrm{msw}(W_n) = \mathrm{kwsw}(W_n) \\ &= \mathrm{ksw}(W_n) = \mathrm{mkwsw}(W_n) = \mathrm{mksw}(W_n) = 4.\end{aligned}$$

PROOF. For all $n \geqq 3$, $\delta(W_n) = 3$. Thus, by Theorem 2.2, $4 \leqq \mathrm{sw}(W_4)$. It remains to show that 4 sweepers are sufficient for a monotonic connected sweep of W_n. Label the vertex of degree n with v_0, and the remaining vertices v_1, v_2, …, v_n such that $v_n v_1$ and $v_i v_{i+1}$ are edges for $1 \leqq i \leqq n-1$.

Place sweepers γ_1 and γ_2 at v_0 and sweepers γ_3 and γ_4 at v_1. Move γ_1 along edge v_0v_1, clearing it. Move γ_1 back to v_0. Then move γ_4 to vertex v_2, clearing v_1v_2. Move γ_1 along edge v_0v_2, clearing it. Move γ_1 back to v_0. Then move γ_4 to vertex v_3, clearing v_2v_3. Repeat until all edges of W_n are cleared. Thus, $\mathrm{mksw}(W_n) \leqq 4$, as required. ■

2.2 Minimum Degree and Girth

However, for some graphs, minimum degree by itself is insufficient to easily calculate the lower bound for the sweep number of a graph. For instance, consider the graph $K_{3,3}$. By Theorem 2.2, four sweepers are necessary. However, with four sweepers it is impossible to clear more than two vertices! We introduce the idea of girth to expand our repertoire of lower bounds.

To improve Theorem 2.2, we introduce the following theorem. Since the sweep number of a connected graph is equal to its monotonic wormhole sweep number, we may investigate monotonic wormhole sweep strategies instead. While the wormhole feature is normally not useful, being able to assume a sweep is monotonic is very useful. Moreover, Theorem 2.9 tells us something about how such a sweep strategy may be formulated. However, we must first introduce the following lemma.

Lemma 2.8 *If G is a connected reflexive multigraph, then for any sweep strategy S,* $\mathrm{mwsw}(G) - 1 \leqq \mathrm{mex}_S(G) \leqq \mathrm{mwsw}(G)$.

PROOF. The second inequality is straightforward; every exposed vertex must contain a sweeper, so there cannot be more exposed vertices than sweepers. Let $\mathrm{mwsw}(G) = k$. Assume that for some S, $\mathrm{mex}_S(G) = k - 2$. Following S, we label the sweepers in the order that they move. The first sweeper to move will be labelled γ_1, the second to move will be labelled γ_2, and so on. Finally, if γ_i is the last sweeper to move, and $i < k$, arbitrarily label the remaining sweepers $\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_k$. Now consider the sweeper γ_k. We will first show that given a strategy clearing G with k sweepers, we can construct a strategy S' in which G is cleared with k sweepers and γ_k never clears an edge. Then, over all such strategies where γ_k never clears an edge, we will construct another sweep strategy where γ_k never clears an edge and is never the only sweeper on an exposed vertex.

If γ_k does move, consider the first time (if ever) it clears an edge uv, moving from u to v. Let this occur at step i. We will construct a new strategy S'. Up until step i, S' follows the same strategy as S. At step i, there are at most $k - 2$ exposed vertices. So there are at least two sweepers that can move without recontaminating edges. At worst, one of these is γ_k. Let another be γ_j, located at vertex w. At step i, move γ_j to u. At step $i + 1$, move γ_k to w. For the remainder of S', we follow the strategy S, but whatever moves γ_k makes in S, γ_j makes in S', and vice versa. Thus, there are the same number of moves in S' as in S after the edge uv is cleared, but if γ_k ever clears an edge, it must do so closer to the end. Repeating this process, we eventually end up with a strategy in which γ_k never clears an edge. If γ_k never moves, then it certainly clears no edges.

Over all strategies where γ_k never clears an edge, let S' be a strategy in which γ_k becomes the only sweeper on an exposed vertex the minimum number of times. Assume that this minimum number of times is non-zero. Let step i be the first step that γ_k is the only sweeper on an exposed vertex v. This can only occur by another sweeper γ_j moving away from v at step i. If v was exposed before γ_j moved, then one of the other $k - 2$ sweepers is free to move because there are at most $k - 3$ exposed vertices distinct from v. Let γ_a be a sweeper located at vertex u who is free to move. We construct a new strategy S'' in which γ_k does not become the only sweeper on an exposed vertex. Up until step i, follow S'. At step i, move γ_a to vertex v. Then continue with S' until one of the following occurs: vertex v is cleared; sweeper γ_k moves from vertex v; sweeper γ_a is supposed to become the only sweeper on vertex u, which is exposed; or sweeper γ_a is supposed to move in S'. If the vertex v is cleared, then move sweeper γ_a to the vertex u. If γ_k moves from vertex v in S' but v is not cleared, then in S' another sweeper has moved to vertex v. This also occurs in S'', at which point γ_a can move from v back to vertex u. If γ_a is supposed to become the only vertex on u, an exposed vertex, before v is cleared of γ_k moves, then since there are at most $k-2$ exposed vertices, there is a sweeper other than γ_a and γ_k that can move without recontaminating any edges. This sweeper moves to v and then γ_a returns to vertex u. If, instead, γ_a is supposed to move in S' before v is cleared, before γ_k moves, and before γ_a is supposed to become the only sweeper on an exposed vertex u, then since there are at most $k-2$ exposed vertices, there must be a sweeper other that γ_k and γ_a that can move and not recontaminate an edge. This sweeper moves to v and then γ_a returns to the vertex u. Up until this point, γ_k cannot again become the only sweeper on an exposed vertex. This point must occur; since S' is a sweep strategy, v is eventually cleared. After this point, S'' follows S', and has the same number of moves.

Instead, consider if the vertex v was not exposed before step i. Then the sweeper γ_j must clear an edge vw at step i by moving from v along the edge to w. We consider several cases. If both v and w become exposed, the number of exposed vertices before step i must be at most $k-4$. Thus, there must be a sweeper other than γ_k and γ_j that can move without recontaminating edges at step i. This sweeper moves to vertex v, and behaves in the same fashion as γ_a in the previous case. If w is already exposed before step i, then if it does not become cleared at step i, then the number of exposed vertices before step i must be at most $k-3$ exposed vertices. Thus, there must be a sweeper other than γ_k and γ_j that can move without recontaminating edges at step i. This sweeper moves to vertex v, and behaves in the same fashion as γ_a in the previous case. Finally, if w is already exposed, but moving γ_j from v to w clears w, let γ_b be the sweeper on w. In S'', at step i, move γ_b from w along vw to v, clearing vw. Then move γ_j to vertex w. From this point on, treat γ_b as γ_a in the previous case. As before, we eventually reach a point where the position of vertices and the subsequent moves in S'' are the same as in S'.

Thus, in S'' the sweeper γ_k still never clears an edge, and becomes the only sweeper on an exposed vertex fewer times than in S', contradicting the choice of S'. Thus, there must exist a strategy S' in which γ_k never clears an edge and is never the only sweeper on an exposed vertex. Then, in fact, by removing all of the moves of γ_k from S', we obtain a strategy that clears G using only $k-1$ sweepers. We similarly obtain a contradiction if $\mathrm{mex}_S(G) < k-2$. Thus, $\mathrm{mex}_S(G) \geqq \mathrm{mwsw}(G) - 1$. ■

Theorem 2.9 *If G is a connected reflexive graph with no vertices of degree 2, then there exists a monotonic wormhole sweep S with $\mathrm{mwsw}(G)$ sweepers such that $\mathrm{mex}_S(G) = \mathrm{mwsw}(G) - 1$.*

PROOF. Let G be a connected reflexive graph G with no vertices of degree 2. Assume that for every monotonic wormhole sweep strategy S on G, $\mathrm{mex}_S(G) = \mathrm{mwsw}(G) = k$. Since S is a sweep strategy, there is a moment when the number of exposed vertices becomes $\mathrm{mex}_S(G)$ for the last time. Let S' be a monotonic wormhole sweep strategy which has the minimum number of instances where the number of exposed vertices goes from being less than k to being k and has the minimum number of edge clearings after the last time the number of exposed vertices becomes k. The only action which can increase the number of exposed vertices is clearing an edge, which can expose at most two additional vertices. Let xy be the last edge cleared before the number of exposed vertices becomes $\mathrm{mex}_S(G)$ for the last time. We consider four cases as to how xy can be cleared.

Case 1: The edge xy is a loop, with $x = y$. Since clearing xy can expose at most one additional vertex, the number of exposed vertices must be $k-1$. If x was already exposed, clearing xy would not increase the number of exposed vertices. Thus, x must not have been an exposed vertex. But since there must be a sweeper on each of the $k-1$ exposed vertices, this leaves only one sweeper to clear the loop xy. But a single sweeper cannot clear a loop, a contradiction. Thus, xy cannot be a loop.

Case 2: The number of exposed vertices just before xy is cleared is $k-2$, and at this time neither x nor y is exposed. Label the $k-2$ exposed vertices as $v_1, v_2, \ldots v_{k-2}$, and assume that sweeper γ_i rests on vertex v_i, $1 \leqq i \leqq k-2$. The edge xy must be such that neither x nor y is some v_i. Assume that after xy is cleared, sweeper γ_{k-1} is on x and γ_k is on y.

If there are any pendant edges or loops attached to some v_i that are not cleared, we can use sweeper γ_k to clear these edges first. If this reduces the number of exposed vertices, then at some later action k vertices must be exposed because the number of exposed vertices increasing to k occurs a minimum number of times in S'. This later point must have more cleared edges, contradicting the minimality of S'. Thus, clearing such an edge cannot reduce the number of exposed vertices. But then, clearing xy next would produce a sweep strategy with fewer edges to be cleared after the number of exposed vertices becomes k for the last time, again contradicting the minimality of S'. Similarly, if there are any contaminated edges between v_i and v_j, γ_k may clear these edges first, and then xy, again contradicting the minimality of S'. So we may assume that all edges between the v_i have already been cleared, as have all pendant edges and loops incident with them.

If some vertex v_i is incident with only one contaminated edge, then γ_i may clear that edge first, and then γ_k may clear xy, again contradicting the minimality of S'. Thus, each v_i must have at least two contaminated edges incident with it, and the γ_i, $1 \leqq i \leqq k-2$, must remain where they are as blockers.

Neither x nor y are exposed before xy is cleared so that all edges incident with x and y are contaminated. After xy is cleared, both x and y are exposed. Thus, each of them is incident with a contaminated edge. Since G has no vertices of degree 2, both x and y must have at least two contaminated edges incident with them, and thus neither γ_{k-1} nor γ_k may move, contradicting that S' is a sweep strategy.

Case 3a: The number of exposed vertices just before xy is cleared is $k-1$ and one of the vertices of xy already is an exposed vertex. Label the exposed vertices v_1, v_2, ..., v_{k-1}, and assume that they have sweepers on them, with sweeper γ_i on vertex v_i, $1 \leqq i \leqq k-1$. Without loss of generality, assume that $x = v_{k-1}$. Since the vertex v_{k-1} is still exposed, we may assume that γ_{k-1} stays on v_{k-1}, that the remaining sweeper γ_k clears $v_{k-1}y$ by traversing the edge from v_{k-1} to y, and that there is another contaminated edge $v_{k-1}z$ incident with v_{k-1}.

If there are any pendant edges or loops attached to some v_i that are not cleared, we use the remaining sweeper γ_k to clear these edges first, and then $v_{k-1}y$, contradicting the minimality of S'. In particular, $v_{k-1}z$ is not pendant so that z must have degree at least 3. Similarly, if there are any contaminated edges between v_i and v_j, γ_k may clear these edges first, then $v_{k-1}y$, again contradicting the minimality of S'. So we may assume that all edges between the v_i already have been cleared, as have all pendant edges and loops incident with them.

If some vertex v_i is incident with only one contaminated edge, then γ_i may clear that edge first, then γ_k may clear $v_{k-1}y$, again contradicting the minimality of S'. Thus, each v_i must have at least two contaminated edges incident with it, and all the γ_i, $1 \leqq i \leqq k-2$, must remain where they are as blockers. Note that $\deg(y) > 1$, as otherwise sweeping $v_{k+1}y$ does not expose a new vertex. Since $\deg(y) \geqq 3$, we know that once γ_k clears $v_{k-1}y$, γ_k must remain on y. After $v_{k-1}y$ is cleared, if v_{k-1} has two or more contaminated edges incident with it, then γ_{k-1} must remain at v_{k-1}. Then no sweepers may move, contradicting that S' is a sweep strategy. Thus, the only contaminated edge remaining incident with v_{k-1} must be $v_{k-1}z$. Thus, the next action in S' must be that γ_{k-1} clears $v_{k-1}z$. Since $\deg(z) \geqq 3$, z must have at least two contaminated edges incident with it, and thus γ_{k-1} also cannot move, contradicting that S' is a sweep strategy.

Case 3b: The number of exposed vertices is $k-1$, and neither of the vertices of xy is already exposed. Since the number of exposed vertices increases by 1 after xy is cleared, exactly one of x and y must have degree 1. (If both were degree 1, the graph would be disconnected.) Without loss of generality, assume that $\deg(x) = 1$. Then $\deg(y) \geqq 3$. Assume that the $k-1$ exposed vertices are labelled v_i and that the sweeper γ_i is on v_i, $1 \leqq i \leqq k-1$. Then the sweeper γ_k must clear xy.

As in the previous case, all edges between v_i must be cleared, as must all pendant edges and loops incident with them. Also, each v_i must have at least two contaminated edges incident with it. Thus, none of the γ_i, $1 \leqq i \leqq k-1$, may move. Similarly, since $\deg(y) \geqq 3$, y must have at least two contaminated edges incident with it, meaning that γ_k cannot move. This contradicts that S' is a sweep strategy. ■

This theorem tells us that there exist sweep strategies for some graphs that "save" sweepers, in the sense we may keep a sweeper in reserve, to never be stationed at an exposed vertex, but instead to clear edges between stationed sweepers. If we consider the analogy of a graph filled with gas, we may always keep a sweeper from being exposed, or by rotating sweepers reduce the amount of "exposure" to a toxic substance.

The use of graph instead of multigraph in Theorem 2.9 is intentional. While it is possible that the result may be extended to some multigraphs, this proof does not suffice.

We first introduce a lemma from [21] to be used in the proof of Theorem 2.11.

Lemma 2.10 *If G is a graph and $\delta(G) \geqq 3$, then the number of cycles with pairwise distinct vertex sets is greater than $2^{\frac{\delta}{2}}$.*

So, in particular, any graph with $\delta = 3$ has at least 4 cycles, where the vertex sets of these cycles are pairwise distinct. We can now extend Theorem 2.2 as follows.

Theorem 2.11 *If G is a connected graph and $\delta(G) \geqq 3$ and $g(G) \geqq 3$,*

$$\delta(G) + g(G) - 2 \leqq \mathrm{sw}(G).$$

PROOF. From Lemma 2.10, we know that the girth of G is finite, and that G has at least 4 cycles. Assume that $g = g(G) \geqq 3$. Since $\delta = \delta(G) \geqq 3$, it follows from Theorem 2.9 that there exists a monotonic wormhole sweep S with $\mathrm{mwsw}(G) = \mathrm{sw}(G)$ sweepers such that $\mathrm{mex}_S(G) = \mathrm{mwsw}(G) - 1$. Let $E_0, E_1, ..., E_m$ be the sequence of cleared edge sets corresponding to S. Let G_i be the graph induced by the cleared edges in E_i.

Case 1. $\delta \geqq g = 3$. Consider the smallest i such that G has one cleared vertex u at step i. Since $\deg(u) \geqq \delta$, G must have at least δ exposed vertices adjacent to u. Since S exposes at most $\mathrm{mwsw}(G) - 1$ vertices, $\delta \leqq \mathrm{wsw}(G) - 1$, and thus $\mathrm{wsw}(G) \geqq \delta + 1 = \delta + g - 2$.

Case 2. $\delta \geqq g = 4$. Let i be the least number such that that G has at least two cleared vertices u and v at step i. If u and v are adjacent, they can have no common neighbours, and since $\deg(u) \geqq \delta$ and $\deg(v) \geqq \delta$, they must both be adjacent to at least $\delta - 1$ exposed vertices each. This is $2\delta - 2$ sweepers, and $2\delta - 2 \geqq \delta + g - 2$, as required. If u and v are not adjacent, then they may share common neighbours. At worst, all their neighbours are common. Consider the graph G_{i-1}. Since u and v are not adjacent, only one of them can become cleared by the next move. Assume that v is already cleared at step $i-1$, and u becomes clear at step i. Then v has at least δ exposed vertices adjacent to it, and certainly u itself is exposed at this point. Thus G must have at least $\delta+1$ different exposed vertices at step $i-1$. Since S exposes at most $\text{mwsw}(G)-1$ vertices, $\delta+1 \leqq \text{mwsw}(G)-1$, and thus $\text{mwsw}(G) \geqq \delta+2 = \delta+g-2$.

Case 3. $\delta \geqq g \geqq 5$. Let i be the least number such that G has at least two cleared vertices u and v at step i. If these two vertices are adjacent, then one must have $\delta-1$ exposed vertices adjacent to it, and the other must have at least $\delta-2$ exposed vertices adjacent to it (it may be adjacent to a third cleared vertex). Thus $2\delta-3 \leqq \text{mwsw}(G)-1$, and $\text{mwsw}(G) \geqq 2\delta-2 \geqq \delta+g-2$. If u and v are not adjacent, they have at most one neighbour in common, and hence again must have at least $2\delta-3$ exposed vertices between them. Thus, as above, $\text{mwsw}(G) \geqq \delta+g-2$.

Case 4. $g > \delta = 3$. Consider the smallest i such that G_i contains exactly one cycle C. Then each vertex of this cycle is either exposed or cleared. (Since only one edge was cleared, if G_i contained more than one cycle, then G_{i-1} must have contained a cycle.) Let u be a cleared vertex in C. Consider the graph H obtained when the edges of C are removed from G_i. Certainly, H is a forest, as G_i contained exactly one cycle. Then u is certainly in one of the non-trivial component trees that make up H. Since there are no vertices of degree 1 in G, any vertices of degree 1 in H must be exposed. Thus, there is an exposed vertex in the tree containing u. Further, this exposed vertex cannot be an exposed vertex in C, as this would mean that G_i contains two cycles. Thus, for every clear vertex in C, there is an exposed vertex in G. Certainly, for every exposed vertex in C there is a corresponding exposed vertex (itself), and the number of exposed vertices is at least g. Since the monotonic wormhole sweep strategy S exposes at most $\text{mwsw}(G)-1$ vertices, $g \leqq \text{mwsw}(G)-1$, and thus $\text{mwsw}(G) \geqq g+1 \geqq \delta+g-2$.

Case 5. $g > \delta \geqq 4$. Let i_1 be the smallest i such that G_{i_1} has two or more cycles. Accordingly, we know G_i has at most one cycle for any $i < i_1$. If C_1 and C_2 are two of the cycles formed and are vertex-disjoint, then as before, there is an exposed vertex that corresponds to each vertex in each cycle. But at most one exposed vertex may correspond to a vertex in both cycles. Thus the number of exposed vertices is at least $2g-1$. Thus, $\mathrm{mwsw}(G) \geqq 2g \geqq \delta + g - 2$. If C_1 and C_2 share exactly one common vertex, then there are at least $2g-2$ exposed vertices at step i_2. Again, $\mathrm{mwsw}(G) \geqq 2g-1 \geqq \delta + g - 2$. If C_1 and C_2 share more than one vertex, then G_{i_2} contains exactly three cycles. In this case, we consider step i_2, the first moment that the graph G_i contains four or more cycles.

Let C be the subgraph of G formed by $V(C) = V(C_1) \cup V(C_2)$ and $E(C) = E(C_1) \cup E(C_2)$, as shown in Figure 2.2(i). Let one of the new cycles formed be C_3. If C_3 is vertex-disjoint from C, then G_{i_2} contains two vertex-disjoint cycles, and as before, the number of exposed vertices is at least $2g-1$. Thus, $\mathrm{mwsw}(G) \geqq 2g \geqq \delta + g - 2$. If C_3 and C share exactly one vertex, then there are at least $2g-2$ exposed vertices at step i_2. Again, $\mathrm{mwsw}(G) \geqq 2g-1 \geqq \delta + g - 2$. Otherwise, C and C_3 share two or more vertices. We consider some subcases, as depicted in Figure 2.2.

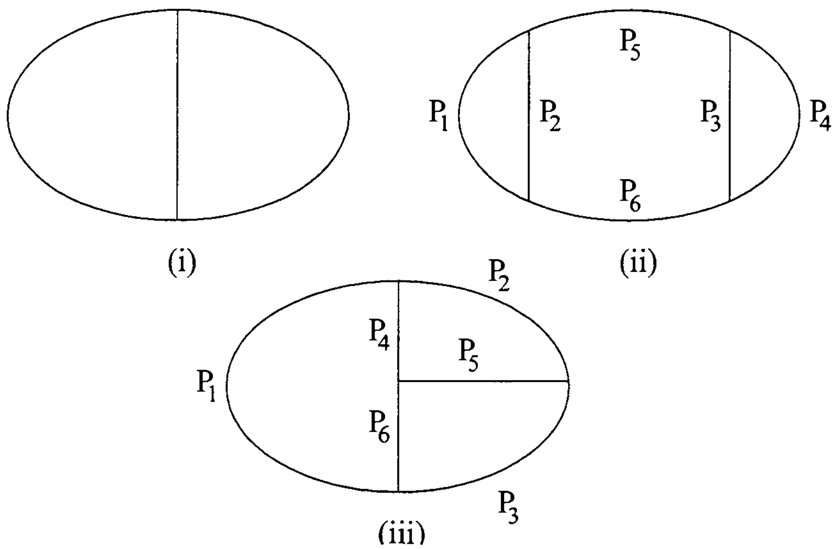

Figure 2.2: (i) The graph C; (ii) Case 5(a); (iii) Case 5(b).

Case 5(a). In this case, we consider four cycles: the cycle induced by the paths P_1 and P_2; the cycle induced by P_2, P_3, P_5, and P_6; the cycle induced by P_3 and P_4; and finally the cycle induced by P_1, P_4, P_5, and P_6. These cycles all have length at least g. We note that either or both of P_5 and P_6 may be paths of length zero. Summing the lengths of the cycles, we see that we count each path, and hence each edge, exactly twice. Thus, in this subgraph G', $E' = E(G') \geqq 2g$. We next consider how many vertices are in $V' = V(G')$. If neither P_5 nor P_6 are paths of length zero, then summing vertex degrees over V' shows that $2(|V'| - 4) + 3 \cdot 4 = 2|E|$, or that $|V'| = |E'| - 2 \geqq 2g - 2$. In this case, every vertex corresponds to an exposed vertex, and so $\mathrm{mwsw}(G) \geqq 2g-1 \geqq \delta+g-2$. If exactly one of P_5 or P_6 is a path of length zero, then summing vertex degrees over V' shows that $2(|V'|-3)+2\cdot 3+4 = 2|E'|$, or that $|V'| = |E'| - 2 \geqq 2g - 2$. All but one of these vertices must correspond to an exposed vertex, so $\mathrm{mwsw}(G) \geqq 2g-2 \geqq \delta+g-2$. Finally, if both P_5 and P_6 are paths of length zero, then summing vertex degrees over V' shows that $2(|V'| - 2) + 2 \cdot 4 = 2|E'|$, or that $|V'| = |E'| - 2$. In this case, however, all but two vertices must correspond to an exposed vertex, so the number of exposed vertices is at least $|E'|-4 \geqq 2g-4 \geqq \delta+g-3$, since $g \geqq \delta + 1$. Thus, $\mathrm{mwsw}(G) \geqq \delta + g - 2$, as required.

Case 5(b). In this case, we again consider four cycles: the cycle induced by the paths P_1, P_4, and P_6; the cycle induced by the paths P_2, P_4, and P_5; the cycle induced by the paths P_3, P_5, and P_6; and the cycle induced by the paths P_1, P_2, and P_3. Each cycle has length at least g. Consider the sum of the lengths of the cycles. Each path is counted twice, as is each edge. Thus, in this subgraph G', the total number of edges $|E'| = |E(G')| \geqq 2g$. We sum the degrees of the vertices, and find that $2(|V'| - 4) + 4 \cdot 3 = 2|E'|$, or that $|V'| = |E'| - 2 \geqq 2g - 2$. Since every vertex in G' corresponds to an exposed vertex, we see that $\mathrm{mwsw}(G) \geqq 2g - 1 \geqq \delta + g - 2$. ■

Definition 2.12 The *complete bipartite graph* $K_{a,b}$ on $a + b$ distinct vertices, where $1 \leqq a \leqq b$, is the graph with vertex set $V(K_{a,b}) = \{v_1, v_2, \ldots, v_a\} \cup \{u_1, u_2, \ldots, u_b\}$ and edge set $E(K_{a,b}) = \{u_i v_j | 1 \leqq i \leqq b, 1 \leqq j \leqq a\}$.

We now have sufficient tools to calculate the sweep number of the complete bipartite graph for all possible values of a and b.

Corollary 2.13 *Let* $1 \leqq a \leqq b$. *If*

1. $a = 1$ *and* $1 \leqq b \leqq 2$, *then* $\mathrm{sw}(K_{a,b}) = 1$;

2. $a = 1$ *and* $b \geqq 3$, *then* $\mathrm{sw}(K_{a,b}) = 2$;

3. $a = b = 2$, *then* $\mathrm{sw}(K_{a,b}) = 2$;

4. $a = 2$ *and* $b \geqq 3$, *then* $\mathrm{sw}(K_{a,b}) = 3$;

5. $3 \leqq a \leqq b$, *then* $\mathrm{sw}(K_{a,b}) = a + 2$.

PROOF. The first and third cases may be easily disposed of, as in the first case, $K_{1,b} = P_b$, and hence has sweep number 1, while in the second case $K_{2,2} = C_4$ and hence has sweep number 2.

Considering the second case, since $\mathrm{sw}(K_{a,b}) = \mathrm{mwsw}(K_{a,b})$, we consider monotonic wormhole sweep strategies. In particular, with a single sweeper it is possible to clear only one pendant edge, at which point the sweeper cannot move along an edge or jump, as this would recontaminate the cleared edge. Thus, $2 \leqq \mathrm{sw}(K_{a,b})$. Two sweepers is certainly sufficient, as we can station one sweeper on the vertex of degree b, and the other sweeper can clear all the edges.

In the fourth case, a similar argument suffices. Again, considering a monotonic wormhole sweep strategy, two sweepers is sufficient to clear two edges, but then the sweepers sit on vertices of degree b with only a single cleared edge incident with those vertices cleared. Thus, the sweepers cannot move without recontaminating the cleared edge, and so $3 \leqq \mathrm{sw}(K_{a,b})$. Three sweepers is sufficient, as a sweeper can be placed on each vertex of degree b, and the third can clear all the edges.

Finally, in the fifth case, since bipartite graphs contain no odd cycles, $g(K_{a,b}) = 4$. By Theorem 2.11, $\mathrm{sw}(K_{a,b}) \geqq a+4-2 = a+2$. In the same way as the previous cases, place a sweeper on each of the a vertices of degree b. Place a single sweeper on one of the vertices of degree a. The remaining sweeper then clears all the edges between these vertices. This is repeated for each vertex of degree a. Thus $a + 2$ sweepers are sufficient. ■

Similarly, the Petersen graph P (as shown in Figure 2.3) is a cubic graph with girth 5. Thus, $\mathrm{sw}(P) \geqq 6$. In fact, 6 sweepers are sufficient. To see this, place a sweeper on each of the vertices a, b, c, d, and e. Use a sixth sweeper to clear the 5-cycle induced by these vertices. Move each sweeper from the vertex it is on along the single remaining contaminated edge incident with it. This leaves sweepers on f, g, h, i, and j, and the sixth sweeper can then clear the 5-cycle induced by these vertices, clearing the graph.

In the same fashion, Theorem 2.11 implies that the Heawood graph and the McGee graph (both of which are pictured in Figure 2.3) which have girths 6 and 7, respectively, must have sweep numbers at least 7 and 8. In fact, it can be shown that these numbers are also sufficient to clear these graphs. The sweep strategies are similar to those of the Petersen graph. First place a sweeper on each vertex of a cycle of minimum length, then use the single remaining sweeper to clear the edges of the cycle. Then each vertex in the cycle has exactly one contaminated edge remaining. Clearing these, the single sweeper then proceeds to clear all possible edges between exposed vertices, the sweepers on these vertices advance, and so on.

The complete graph K_4, complete bipartite graph $K_{3,3}$, the Petersen graph, the Heawood graph, and the McGee graph are all cubic graphs (all vertices have degree three). Further, each has a distinct girth g, and has the fewest vertices of all cubic graphs of their respective girths. Regular graphs with degree d and girth g on the smallest number of vertices possible are (d, g)-cages.

So the $(3, g)$-cage has sweep number $g + 1$ for $3 \leqq g \leqq 7$. However, this does not appear to be a general trend. While Theorem 2.11 implies that the sweep number of the Levi graph (the $(3, 8)$-cage) is bounded below by 9, 9 sweepers do not seem sufficient to clear the graph, though this has not been proved.

2.3 Clique Number

The following lemma is a "junior" version of our eventual goal and comes from [14].

Lemma 2.14 *At the time the first vertex becomes cleared while sweeping the complete graph $K_n, n \geqq 4$, there must be at least $n - 1$ sweepers on the vertices of K_n.*

It is easy to see that $\mathrm{sw}(K_1) = 1$, $\mathrm{sw}(K_2) = 1$, and $\mathrm{sw}(K_3) = 2$, and only slightly more difficult to see that $\mathrm{sw}(K_4) = 4$. The jump of the sweep numbers from 2 for K_3

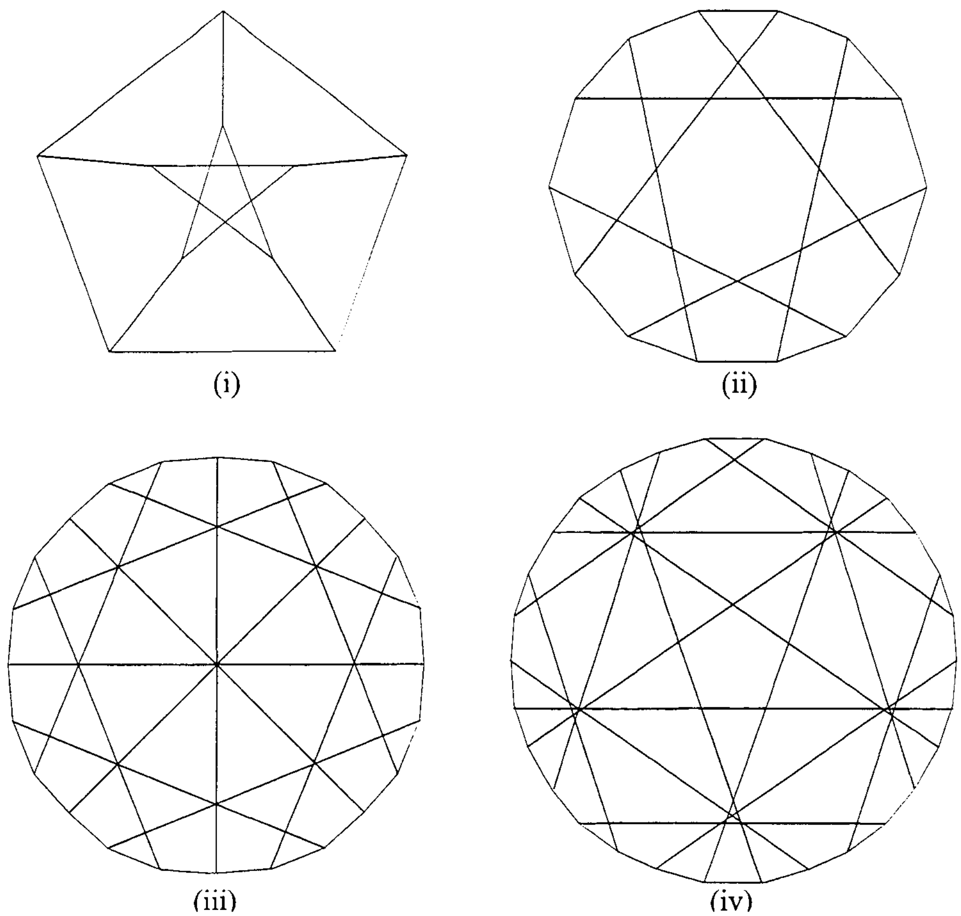

Figure 2.3: (i)The Petersen graph; (ii)the Heawood graph; (iii)the McGee graph; and (iv) the Levi graph.

to 4 for K_4 indicates that obvious methods, such as mathematical induction, will not easily prove a formula for $\mathrm{sw}(K_n)$.

Theorem 2.15 *If a graph H is a minor of a graph G, then* $\mathrm{wsw}(H) \leqq \mathrm{wsw}(G)$.

PROOF. Let $\Phi : V(G) \to V(H)$ denote the function that maps the vertices of G to the corresponding vertices of H that result from vertex identifications that have taken place to form the minor H. Suppose that $\mathrm{wsw}(G) = k$. Whenever a sweeper in G moves from a vertex u along an edge to a vertex v, the corresponding sweeper does nothing in H when $\Phi(u) = \Phi(v)$. If $\Phi(u) \neq \Phi(v)$, then the corresponding sweeper does nothing when $\Phi(u)$ and $\Phi(v)$ are not adjacent in H, but traverses the edge from $\Phi(u)$ to $\Phi(v)$ when they are adjacent in H. It is easy to see that k sweepers clear all of H if they clear G. The result follows. ■

Of course, since $\mathrm{sw}(G) = \mathrm{wsw}(G) = \mathrm{mwsw}(G)$ for any connected graph G, Theorem 2.15 also holds for these other sweep numbers. From Corollary 2.5 and Theorem 2.15, we obtain the following result.

Theorem 2.16 *For any graph G, if $\omega(G) \geqq 4$, then $\omega(G) \leqq \mathrm{sw}(G)$.*

Since trees have clique number 2 and there exist trees with arbitrarily large sweep number, it might appear that the bound presented in Theorem 2.16 is not particularly useful. Nothing could be further from the truth as Theorem 2.16 provides a basis for constructing graphs with easily calculated sweep number. The general idea is to start with a graph G that can be cleared with p sweepers, and use G to construct a cousin G' of G that has K_p as a subgraph, thereby forcing the sweep number of G' to be at least p.

Many graphs for which this bound will be useful will be introduced in Chapter 3.

2.4 Chromatic number

If a graph G has a clique of size k, then at least k colours are required for a proper colouring. Thus, for any graph G, $\omega(G) \leqq \chi(G)$. Since we know that the clique number is a lower bound on the sweep number, it is reasonable to wonder whether Theorem 2.16 can be extended to the chromatic number.

We begin by introducing the homeomorphic reduction of a graph.

Definition 2.17 Let X be a reflexive multigraph. Let $V' = \{u \in V(X) : \deg(u) \neq 2\}$. A *suspended path* in X is a path of length at least 2 joining two vertices of V' such that all internal vertices of the path have degree 2. A *suspended cycle* in X is a cycle of length at least 2 such that exactly one vertex of the cycle is in V' and all other vertices have degree 2.

Definition 2.18 Let X be a reflexive multigraph. Let $V' = \{u \in V(X) : \deg(u) \neq 2\}$. The *homeomorphic reduction* of X is the reflexive multigraph X' obtained from X with vertex set V' and the following edges. Any loop of X incident with a vertex of V' is a loop of X' incident with the same vertex. Any edge of X joining two vertices of V' is an edge of X' joining the same two vertices. Any suspended path of X joining two vertices of V' is replaced by a single edge in X' joining the same two vertices. Any suspended cycle of X containing a vertex u of V' is replaced by a loop in X' incident with u. In the special case that X has connected components that are cycles, these cycles are replaced by loops on a single vertex.

Lemma 2.19 *If X is a connected reflexive multigraph and Y is its homeomorphic reduction, then* $\mathrm{wsw}(X) = \mathrm{wsw}(Y)$.

PROOF. Suppose $\mathrm{wsw}(Y) = k$. Whenever a sweeper clears an edge e of Y, then we can let the corresponding sweeper clear the entire path of X corresponding to e. It is easy to see that k sweepers can clear all the edges of X in this way. Thus $\mathrm{wsw}(X) \leqq \mathrm{wsw}(Y)$.

Suppose that $\mathrm{wsw}(X) = k$. Whenever a sweeper traverses the edge e on vertices u and v by moving from u to v, we have the edge e' in Y on the vertices u' and v' that corresponds to the induced path containing uv (we may have either $u' = u$,v' = v, or both or neither). In Y, we then have the corresponding sweeper jump to u' (if necessary) and traverse the edge $u'v'$. It is clear that this strategy will clear the edges of Y. We conclude that $\mathrm{wsw}(Y) \leqq \mathrm{wsw}(X)$ from which the result follows. ■

To prove a bound on sweep number involving chromatic number, we return to the idea of the maximum number of exposed vertices in a sweep.

Theorem 2.20 *If G is a connected reflexive multigraph with homeomorphic reduction G' and a monotonic wormhole sweep strategy S for G' such that* $\mathrm{mex}_S(G') \geqq 3$, *then* $\chi(G) \leqq \mathrm{mex}_S(G') + 1$.

PROOF. Let $\mathrm{mex}_S(G') = k$. We will show that G is $(k+1)$-colourable. We first show that G' is $(k+1)$-colourable. Following the monotonic wormhole sweep strategy

S that exposes at most k vertices in G', we can design a colouring such that it can colour G' using at most $k+1$ colours.

Initially, sweepers are placed on G'. When a vertex first becomes exposed (or in the case of vertices of degree 1, becomes cleared), the vertex is coloured. This colour cannot be changed or erased in the following sweeping process. We now consider how to colour a vertex v in the moment it becomes exposed (or cleared, in the case of vertices of degree 1). Before this moment, v cannot be adjacent to any cleared vertex. Thus, each coloured vertex that is adjacent to v must be an exposed vertex. Since the number of exposed vertices is less then or equal to k, we can always assign v a colour that is different from the colours of the adjacent vertices of v. Thus, while S clears G', we can assign a colour to each vertex of G' such that any pair of adjacent vertices has different colours. Thus, G' is $(k+1)$-colourable.

We now show that G is $(k+1)$-colourable. For each vertex u in G', assign the colour of u in G' to the corresponding vertex u in G. Any uncoloured vertex in G must be on a suspended path or a suspended cycle. If it is on a suspended cycle, one vertex in this cycle has already been coloured. At most two more colours are needed to colour the remaining vertices of this cycle, but since $k \geqq 3$, we have a sufficient number of colours to do so. Similarly, if the vertex is in a suspended path, the ends of the suspended path have already been coloured. Now at most one more colour is needed to colour the remaining vertices of this path, but again, we have sufficient colours to do so. Hence, G is $(k+1)$-colourable. Therefore, $\chi(G) \leqq k+1$. ∎

Combining Theorem 2.20 with Lemma 2.8, we obtain the following corollary.

Corollary 2.21 *If G is a connected reflexive multigraph and $\mathrm{sw}(G) \geqq 3$, then $\chi(G) \leqq \mathrm{sw}(G)+1$.*

Of course, we can do better. As demonstrated in Theorem 2.9, there are graphs where the maximum number of exposed vertices is one less than the sweep number.

Corollary 2.22 *If G is a connected reflexive graph with the property that no pair of suspended paths have the same end points and $\mathrm{sw}(G) \geqq 3$, then $\chi(G) \leqq \mathrm{sw}(G)$.*

PROOF. Since G is not a multigraph, the homeomorphic reduction can only have multiple edges if two or more suspended paths have the same end points. Forbidding this, the homeomorphic reduction must be a graph with no vertices of degree 2, as required by Theorem 2.9. The result follows. ■

We now demonstrate an infinite family of graphs for which Corollary 2.21 provides a better bound that any of the others demonstrated here. Let P be the graph with vertex set $V(P) = \{v_i\}_{i=1}^{p+1}$, and edge set $E(P) = \{v_iv_j | 1 \leqq i < j \leqq p\} \cup \{v_1v_{p+1}\}$. Thus, the graph P is a complete graph on p vertices with an extra edge incident with a vertex of degree 1.

We will employ the Mycielski construction [15]. Given a graph G, we form the graph $M(G)$, with vertex set $V(M(G)) = V(G) \cup V'(G) \cup \{u\}$, where $V'(G)$ contains the "twins" of $V(G)$. That is, $V'(G) = \{x' | x \in V(G)\}$. The edge set $E(V(M)) = E(G) \cup \{x'y | xy \in E(G)\} \cup \{x'u | x' \in V'(G)\}$. Similarly, we may define an infinite family of graphs by repeatedly applying a Mycielski construction. Define $M^0(G) = G$, and $M^t(G) = M(M^{t-1}(G))$ for $t \geqq 1$.

The Mycielski construction based on the 5-cycle C_5 was introduced in [15] to create an infinite family of triangle-free graphs with arbitrarily large chromatic number. In fact, $\chi(M^t(C_5)) = t+3$ for $t \geqq 0$. More generally, for any graph G, $\omega(M^t(G)) = \omega(G)$, $\delta(M^t(G)) = \delta(G) + t$, and $\chi(M^t(G)) = \chi(G) + t$ for $t \geqq 0$.

Taking the graph P as defined above, it is clear that $\delta(P) = 1$, $\omega(P) = p$, and $\chi(P) = p$. Applying the Mycielski construction, we see that $\delta(M^t(P)) = 1 + t$, $\omega(M^t(P)) = p$, and $\chi(M^t(P)) = p + t$. As well, since P is a subgraph of $M^t(P)$, we know that $g(M^t(P)) = 3$ so long as $p \geqq 3$. So for large p and t, Theorem 2.11 tells us that $\delta(M^t(P)) + 1 \leqq t + 2 \leqq \mathrm{sw}(M^t(P))$. Similarly, Theorem 2.16 tells us that $\omega(M^t(P)) = p \leqq \mathrm{sw}(M^t(P))$. But Corollary 2.21 tells us that $\chi(M^t(P)) - 1 = p + t - 1 \leqq \mathrm{sw}(M^t(P))$, a clear improvement.

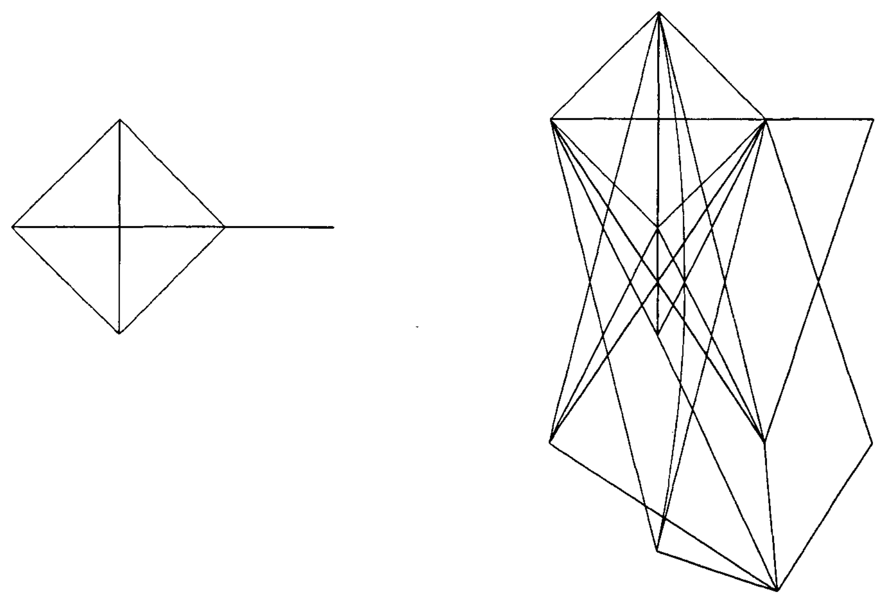

Figure 2.4: A graph P and M(P).

Chapter 3

Some applications of lower bounds

3.1 Some results with cliques

Lemma 3.1 *If G is a connected graph, then* $\mathrm{sw}(G) \leqq \min(|V(G)|, |E(G)|)$.

PROOF. We first clear G with $n = |V(G)|$ sweepers. Pick a vertex $v \in V(G)$ with k neighbours. On every vertex other than v and the neighbours of v, place a sweeper. Place the remaining $k+1$ sweepers on v. Move a sweeper from v to each neighbour u of v, clearing the edge uv. At this point, there is a sweeper on every vertex of G, and the single sweeper on v (which is cleared) may be used to clear all remaining edges.

We next clear G with $m = |E(G)|$ sweepers. Pick a vertex $v \in V(G)$. For each edge xy in G, place a sweeper on whichever of x and y is closer to v. (If both x and y are equidistant, pick one of them.) First clear v; then clear all vertices at distance 1 from v; then those of distance 2, and so on. ■

Theorem 3.2 *For $n \geqq 4$, K_{n+1} is the unique connected supergraph of K_n with the fewest number of edges such that its sweep number is $n+1$.*

PROOF. First, note that K_{n+1} is a supergraph of K_n with sweep number $n+1$, and K_{n+1} has n additional edges. We will show that any other supergraph G containing K_n with at most n additional edges satisfies $\mathrm{sw}(G) = n$.

If G is a supergraph of K_n with only $k < n$ additional edges, consider the connected components of the graph induced by $E(G) - E(K_n)$. Place one sweeper on each of the k edges of these components, and sweep these components as in Lemma 3.1, ending on vertices of K_n. Clear the edges between these vertices, and arbitrarily pick one of these vertices and clear it. This leaves a sweeper on every other vertex of K_n, and one free sweeper. This sweeper may clear all remaining edges. Thus, $\mathrm{sw}(G) = n$.

If G is a supergraph of K_n with n additional edges, and some vertex $v \in K_n$ is not incident with v, then as before, clear the additional edges as in Lemma 3.1, ending with sweepers on vertices of K_n. There are at most $n-1$ such exposed vertices (since v is adjacent to no additional edge) and so there is a free sweeper to clear all edges between these vertices. Then, arbitrarily clear one of these vertices. This leaves a sweeper on every other vertex of K_n, and the remaining sweeper can clear the graph. Thus, $\mathrm{sw}(G) = n$.

Finally, we consider if G is a supergraph of K_n with n additional edges, and every vertex of K_n is incident with exactly one of these edges. If G contains more than one additional vertex than those in $V(K_n)$, let u be one such vertex. To sweep G, first clear u. Use a free sweeper to clear all the edges between neighbours of u, then clear one of the neighbours of u. This leaves a sweeper on every other vertex of K_n, and one free sweeper. Use this sweeper to clear K_n. Then the neighbours of every vertex in $V(G) - V(K_n)$ contains a sweeper, and they may clear these vertices, clearing G. Thus, $\mathrm{sw}(G) = n$. ■

Theorem 3.3 *If $n \geqq 4$, then the graph of order n with the most edges and sweep number $n-1$ is the complete graph K_n with one edge removed.*

PROOF. Let $K_n - \{uv\}$ denote the complete graph of order n with the edge uv deleted.We first use $n-2$ sweepers to clear the vertex v and station these sweepers on the $n-2$ neighbours of v. Then we use one free sweeper to clear all the contaminated edges between the $n-2$ neighbours of v. Finally, we use $n-1$ sweepers to clear all the remaining contaminated edges incident on u. Thus, $K_n - \{uv\}$ is $(n-1)$-sweepable. On the other hand, from Theorem 2.2, we have $\mathrm{sw}(K_n - \{uv\}) \geqq n-1$. Therefore, $\mathrm{sw}(K_n - \{uv\}) = n-1$. ■

In a similar fashion to the previous theorems, we may examine graphs that contain two vertex-disjoint cliques.

Theorem 3.4 *For a connected graph G on $2n$ vertices, $n \geqq 3$, containing two vertex-disjoint copies of K_n and exactly n edges between these two cliques,* $\mathrm{sw}(G) = n + 1$ *if and only if the edges between the cliques saturate the vertices of at least one of the cliques.*

PROOF. Label the two vertex-disjoint cliques K_n and K'_n. Assume the edges between K_n and K'_n do not saturate the vertices of either, let v be a vertex of degree $n-1$ in K_n and v' be a vertex of degree $n-1$ in K'_n. Since K_n is a subgraph of G, we know that $\mathrm{sw}(G) \geqq n$ by Theorem 2.16. To demonstrate that n sweepers are sufficient, we provide a sweep strategy.

Place n sweepers on vertex v. Clear v. At this point, there is a sweeper on every vertex of K_n. Use the single free sweeper on v to clear K_n. Then move the n sweepers across to K'_n, one sweeper to each edge. Since none of these edges connect to v', at least one vertex must contain two sweepers. Move sweepers to every vertex of K'_n except v'. Use the single free sweeper to clear the edges between these vertices. Finally, the $n-1$ other sweepers may clear the edges incident with v'. Thus $\mathrm{sw}(G) = n < n+1$.

Assume, without loss of generality, that the edges between K_n and K'_n saturate the vertices of K_n. Then, contracting K'_n to a single vertex, we see that G contains K_{n+1} as a minor, and hence $\mathrm{sw}(G) \geqq n+1$. Finally, to show that $n+1$ sweepers are sufficient, we provide a sweep strategy. Place one sweeper on each of the vertices of K_n. Use the single remaining sweeper to clear the edges of K_n. Move one sweeper across each of the n edges between the two cliques to K'_n. Move sweepers (if necessary) so that there is at least one sweeper on each vertex of K'_n. Finally, use a free sweeper to clear all the remaining edges. ■

Discussing multiple copies of a graph and the edges between these copies motivates the consideration of graph products. The sweep number of the cartesian product of graphs has also been considered in [20], where the following result is proved.

Definition 3.5 The *cartesian product* of two graphs G and H, denoted $G \square H$, is a graph with vertex set $V(G) \times V(H)$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G\square H$ if $u_1 = u_2$ and $v_1v_2 \in E(H)$ or if $u_1u_2 \in E(H)$ and $v_1 = v_2$.

Theorem 3.6 *For two connected graphs G and H,*

$$\mathrm{sw}(G\square H) \leqq \min\left(|V(G)| \cdot \mathrm{sw}(H), |V(H)| \cdot \mathrm{sw}(G)\right) + 1.$$

Corollary 3.7 *If G is a connected graph and $n \geqq 4$, then*

$$\mathrm{sw}(K_n\square G) \leqq n \cdot \mathrm{sw}(G) + 1.$$

In fact, it is easy to see that we can do better than this when G is also a complete graph.

Corollary 3.8 *For $n \geqq 1$ and $m \geqq 2$,* $\mathrm{mksw}(K_n\square K_m) \leqq n(m-1) + 1$.

PROOF. Label m vertex–disjoint n-cliques of $K_n\square K_m$ by $A_1, A_2, \ldots, A_m$, with $V(A_i) = \{v_{i,j} | 1 \leqq j \leqq m\}$. Place one sweeper on each of the $v_{1,j}$, and the remaining sweepers anywhere on A_1. Use a free sweeper to clear all the edges in A_1. There is a perfect matching between A_1 and A_2. Move n sweepers to clear the perfect matching, $v_{1,j}v_{2,j}$ ending in A_2. Similarly, sweepers can traverse perfect matchings from A_1 to A_i, $3 \leqq i \leqq m$. This leaves $n(m-1)$ sweepers stationed on A_i, $2 \leqq i \leqq m$, and the remaining free sweeper can clear all the edges between these vertices. ∎

In particular, $\mathrm{mksw}(K_n\square K_n) \leqq n(n-1) + 1$. In the special case that exactly one of the complete graphs is K_2, we can say something even more precise. The graph $K_1\square K_2$ is exactly P_2, and hence has monotonic connected sweep number 1. The graph $K_2\square K_2$ is C_4, and hence has monotonic connected sweep number 2. For n greater than two, we have the following corollary.

Corollary 3.9 *If $n \geqq 3$, then* $\mathrm{sw}(K_n\square K_2) = n + 1$.

PROOF. By Theorem 3.6, we know that $\mathrm{sw}(K_n\square K_2) \leqq n + 1$. But as well, $\delta(K_n\square K_2) = n$, so by Theorem 2.2, $\mathrm{sw}(K_n\square K_2) \geqq n + 1$. ∎

Corollary 3.9 is also a direct result of Theorem 3.4.

3.2 Recontamination Helps

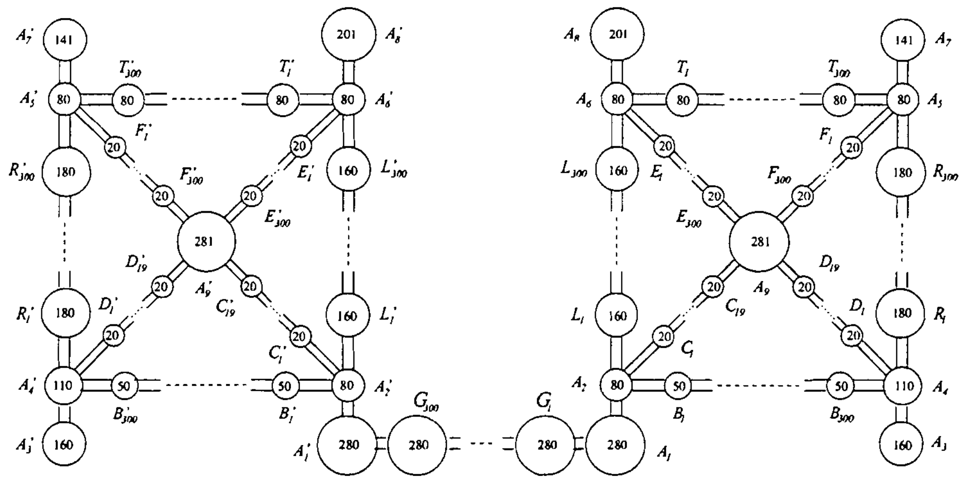

Figure 3.1: The graph W.

We construct the graph W as shown in Figure 3.1. In this figure, a circle represents a complete graph on the indicated number of vertices, and double lines between two cliques A and B indicate a perfect matching either between A and B (if $|A| = |B|$) or between A and a subgraph of B (if $|A| < |B|$). The latter is called a *saturated matching.*

If there is a saturated matching from a graph A to a subgraph of B, we use $B[A]$ to denote the graph induced by those vertices of B adjacent to vertices of A. So $B[A]$ also is a clique.

We construct W such that $A_9[C_{19}]$, $A_9[D_{19}]$, $A_9[E_{300}]$ and $A_9[F_{300}]$ are all vertex-disjoint, and similarly for A_9'. Also, $V(A_2[C_1]) \cap V(A_2[B_1]) = \emptyset$ and $V(A_4[D_1]) \cap V(A_4[B_{300}]) = \emptyset$, and similarly for A_2' and A_4'. Finally, there are 300 cliques between A_1 and A_1', each of which contains 280 vertices.

3.2.1 Computing the connected sweep number of W

Lemma 3.10 *In the process of a connected sweep of W, let v be the first cleared vertex that remains cleared for the remainder of the sweep. If $v \notin V(A_9) \cup V(A_9')$,* *then there are more than* 281 *sweepers.*

PROOF. Let S be a connected sweep strategy such that $v \not\in V(A_9) \cup V(A'_9)$. Certainly, one of A_9 and A'_9 must first obtain a cleared vertex that remains clear for the remainder of the sweep. Without loss of generality, assume it is A_9. Immediately after the action that clears the first vertex of A_9 there must be 280 sweepers on A_9. However, as A'_9 is not cleared, there must be an exposed vertex between A'_9 and v. Thus, none of the 281 sweepers used can move, necessitating at least one more sweeper. ■

Theorem 3.11 *We have* $\mathrm{ksw}(W) = 281$.

PROOF. It follows from Corollary 2.5 that $\mathrm{ksw}(A_9) = \mathrm{ksw}(A'_9) = 281$ and from Theorem 2.16 that we need at least 281 sweepers to clear W. To prove that this number is sufficient, we use the following sweep strategy.

By Lemma 3.10, we must begin in one of A_9 or A'_9. Without loss of generality, we begin by clearing a vertex in A'_9. First, place all 281 sweepers on a single vertex v of $A'_9 \setminus (A'_9[C'_{19}] \cup A'_9[D'_{19}] \cup A'_9[E'_{300}] \cup A'_9[F'_{300}])$. Move 280 of them to the 280 neighbours of v. This clears v, and the single sweeper remaining on v then clears all remaining edges in A'_9.

The sweepers on $A'_9[C'_{19}]$ move to C'_{19} along the perfect matching, and a single free sweeper clears all the edges of C'_{19}. We repeat this process until finally we place 20 sweepers on $A'_2[C'_1]$, and use a single sweeper to clear all edges in $A'_2[C'_1]$. Similarly, we clear the D'_i, E'_i, and F'_i subgraphs, ending with sweepers on $A'_4[D'_1]$, $A'_5[F'_1]$, and $A'_6[E'_1]$. Again we use a single sweeper to clear edges in these subgraphs. We now have 201 free sweepers.

We send these sweepers to a single vertex in A'_8 that is adjacent to a vertex in $A'_6[E'_1]$. Then we clear this vertex leaving a single free sweeper who clears all remaining edges in A'_8. Then the sweepers in $A'_8[A'_6]$ may lift to A'_6, and a single free sweeper may clear all edges in A'_6.

At this point, we have 20 sweepers stationed on each of $A'_2[C'_1]$, $A'_4[D'_1]$, and $A'_5[F'_1]$, as well as 80 stationed on A'_6. This leaves 141 free sweepers. We move all these sweepers to a vertex of A'_7 that is adjacent to $A'_5[F'_1]$. We then clear this vertex, and with the remaining free sweeper, we clear the edges of A'_7. Lifting the sweepers on $A'_7[A'_5]$ along the perfect matching to A'_5, we clear A'_5. We now have 80 sweepers stationed on A'_5, 80 on A'_6, and 20 each on $A'_2[C'_1]$ and $A'_4[D'_1]$ leaving 81 free sweepers.

We clear along the perfect matchings in the T'_i, using the single remaining sweeper to clear each of the edges in the T'_i, and finally clear the edges from T'_1 to A'_6. Now we move the 80 sweepers stationed at A'_6 to $L'_{300}[A'_6]$. We use a single sweeper to clear all the edges in $L'_{300}[A'_6]$, and then move 80 more sweepers to the remaining vertices of L'_{300}. We use the single free sweeper to clear all remaining edges in L'_{300}. Then clear to L'_{299}, and use the single free sweeper to clear the edges in that subgraph, and continue until the edges in L'_1 are cleared. Then send the sweepers from $L'_1[A'_2]$ to A'_2, and use a free sweeper to clear all the edges in A'_2. There are now 80 sweepers stationed at each of A'_2 and A'_5, and 20 stationed on $A'_4[D'_1]$. This leaves 101 free sweepers.

We also clear the R'_i as we cleared the L'_i, first sweeping from A'_5 to $R'_{300}[A'_5]$, then using the 101 free sweepers with these 80 sweepers to clear all the R'_i. Then we move the sweepers from $R'_1[A'_4]$ to A'_4, and use a single sweeper to clear the remaining edges of A'_4. We now have 190 sweepers stationed at A'_4 and A'_2, leaving 91 free sweepers. We use these sweepers to clear the B'_i subgraphs, and then we use 160 sweepers to clear A'_3. This clears the left half of the graph W. We now have only 80 sweepers stationed at A'_2.

We now use 281 sweepers to clear, one by one, the 300 cliques between A'_1 and A_1, followed by A_1 itself. Then we move the sweepers from $A_1[A_2]$ to A_2, and use a free sweeper to clear all edges in A_2. We now have 80 sweepers stationed in A_2, leaving 201 free sweepers.

Pick a vertex in C_1 and clear to this vertex from $A_2[C_1]$. Then move another sweeper along this edge, and to the corresponding vertex in C_2. Then another to the corresponding vertex in C_3, and so on, until finally we have placed a sweeper on the corresponding vertex in $A_9[C_{19}]$. Then move a sweeper to a vertex in $A_9[D_{19}]$, followed by moving a sweeper to a corresponding vertex in D_{19}, then another to a corresponding vertex in D_{18}, and so on, until sweeping to the corresponding vertex in

D_1. Finally, move one sweeper to the corresponding vertex in $A_4[D_1]$. We now have 80 sweepers stationed in A_2, and in total, 41 sweepers along a path through the C_i, through A_9, and finally through the D_i into A_4. This leaves 160 free sweepers.

Move these free sweepers along this path, to the single vertex in A_3 adjacent to the path. Clear this vertex, and then use the single free sweeper to clear A_3. Then the sweepers on $A_3[A_4]$ may clear to A_4, and a free sweeper may clear A_4. With 110 sweepers stationed on A_4, 80 sweepers stationed on A_2, and 40 sweepers strung in that path from A_1 to A_4 through A_9, there are 51 free sweepers. These sweepers can clear the B_i.

We now collapse the path from A_2 to A_4 through A_9, in the following manner. The single sweeper in D_1 moves to the corresponding vertex in D_2. The two sweepers now in D_2 move to the corresponding vertex in D_3. Continue in this way until finally all the sweepers in the path are in C_1, at which point they return to $A_2[C_1]$. By "reeling in" the path in this manner, we preserve the connectedness of this sweep, but the sweep is not monotonic.

We now have 190 sweepers stationed on A_2 and A_4, leaving 91 sweepers free. Of these free sweepers, move 20 from A_4 to D_1, using a free sweeper to clear D_1. Then clear D_2, then D_3, until finally we station 20 sweepers on $A_9[D_{19}]$ and use a free sweeper to clear the edges of $A_9[D_{19}]$. Now move the 110 sweepers on A_4 to $R_1[A_4]$. Use a free sweeper to clear the edges of $R_1[A_4]$. Place the remaining 71 free sweepers on a vertex on $R_1[A_4]$ and clear it. With the single remaining free sweeper, clear R_1. Then move all sweepers in R_1 to R_2, and use the remaining free sweeper to clear R_2. Repeating, clear to R_{300}, finally moving 80 sweepers from $R_{300}[A_5]$ to A_5 and using a single free sweeper to clear A_5. We now have 160 sweepers stationed at A_2 and A_5, and 20 sweepers stationed at $A_9[D_{19}]$. This leaves 101 free sweepers.

As with the D_i, use 21 sweepers to clear the C_i, eventually stationing 20 sweepers at $A_9[C_{19}]$, and using a free sweeper to clear the edges of $A_9[C_{19}]$. Then clear the 80 sweepers from A_2 to $L_1[A_2]$, and use the 81 free sweepers to first clear a vertex in $L_1[A_2]$ and then to clear L_1. We then clear all the L_i, eventually stationing 80 sweepers at A_6 and using a free sweeper to clear the edges of A_6. We now have 80 sweepers at each of A_5 and A_6, and 20 sweepers at each of $A_9[C_{19}]$ and $A_9[D_{19}]$, leaving 81 free sweepers.

We use these sweepers to clear the T_i. Then we clear the F_i, eventually stationing 20 sweepers on $A_9[F_{300}]$, and using a free sweeper to clear the edges of $A_9[F_{300}]$. We then move the 80 sweepers stationed at A_5 to $A_7[A_5]$, and use a free sweeper to clear $A_7[A_5]$. With 80 sweepers at $A_7[A_5]$, 80 at A_6, and 60 in A_9, there are 61 free sweepers. Clear a vertex of $A_7[A_5]$, and then use the single remaining free sweeper to clear A_7.

As before, clear all the E_i by sweeping from A_6, eventually stationing 20 sweepers at $A_9[E_{300}]$, and using a free sweeper to clear the edges of $A_9[E_{300}]$. Then move the 80 sweepers stationed at A_6 to $A_8[A_6]$, using a free sweeper to clear the edges of $A_8[A_6]$. We now have 80 sweepers stationed in A_9, and 80 sweepers stationed in $A_8[A_6]$. The remaining 121 sweepers may be used to clear a vertex in $A_8[A_6]$, and then the single remaining free sweeper may be used to clear the edges of A_8.

Finally, with only 80 sweepers stationed in A_9 (and thus 201 free sweepers), we clear a vertex in A_9, and then use the single remaining free sweeper to clear the edges of A_9, which completes the sweep strategy for W. This strategy, as has been noted, is connected, but is not monotonic, as the edges in the path from A_2 through A_9 to A_4 were allowed to be recontaminated. ■

3.2.2 Computing the monotonic connected sweep number of W

Theorem 3.12 *We have* $\text{mksw}(W) = 290$.

PROOF. We first show that $\text{mksw}(W) \leqq 290$. Starting at A_9', we may sweep as in Theorem 3.11 until we clear all the edges in A_2. At this point, we have 80 sweepers stationed at A_2, and 210 free sweepers. Moving 50 of these sweepers to B_1, we then use another free sweeper to clear the edges in B_1. Then repeat with B_2, B_3, ..., B_{300}. Finally, station 50 sweepers at $A_4[B_{300}]$, and use another sweeper to clear all the edges in $A_4[B_{300}]$. We now have stationed 130 sweepers and have 160 free.

We send these 160 sweepers to a single vertex in A_3 that is adjacent to $A_4[B_{300}]$, and then clear this vertex. Using the single remaining free sweeper, we clear the edges of A_3. Then we lift the sweepers from $A_3[A_4]$ to A_4, and clear all remaining edges in A_4. We now have 110 sweepers stationed at A_4, and 80 stationed at A_2, leaving 100 free sweepers. We use these sweepers to clear the D_i, eventually placing 20 sweepers on $A_9[D_{19}]$. Using another sweeper, we clear the edges of $A_9[D_{19}]$. Then we move all the sweepers on A_4 to $R_1[A_4]$. We have now stationed 80 sweepers on A_2, 20 sweepers on $A_9[D_{19}]$, and 110 on $R_1[A_4]$. This leaves 80 free sweepers.

With these sweepers (and the 110 already in R_1), clear the R_i, and eventually station 80 at A_5, and use another sweeper to clear the edges of A_5. We now have 80 sweepers stationed at each of A_2 and A_5, and 20 stationed at $A_9[D_{19}]$. This leaves 110 free sweepers.

Using these sweepers, clear the C_i, eventually stationing 20 at $A_9[C_{19}]$ and use a free sweeper to clear the edges of $A_9[C_{19}]$. We now have 90 free sweepers. Using these sweepers, and those from A_2, we clear the L_i, eventually stationing 80 sweepers on A_6. With 80 sweepers stationed on each of A_6 and A_5, and 20 stationed on each of $A_9[C_{19}]$ and $A_9[D_{19}]$, we have 90 free sweepers. Use these sweepers to clear the T_i. Then use these sweepers to clear the F_i, eventually stationing 20 sweepers on $A_9[F_{300}]$, using a single free sweeper to clear the edges of $A_9[F_{300}]$.

With 60 sweepers stationed in A_9, and 80 stationed at A_6, we have 150 (including 80 at A_5) with which we can clear A_7. Then these sweepers can clear the E_i, finally stationing 20 sweepers at $A_9[E_{300}]$, and clearing the edges of $A_9[E_{300}]$. With 80 sweepers in A_9, we have 210 sweepers (including 80 at A_6), with which we can clear A_8. Finally, these 210 sweepers move to A_9 and clear it. Thus $\mathrm{mksw}(W) \leqq 290$.

To prove the equality, we will show that $\mathrm{mksw}(W) > 289$. First, assume that W is 289-monotonically connected sweepable. Let S be a monotonic connected sweep strategy using 289 sweepers. Let Y be the subgraph induced by W on the "right side" (starting with A_2 and then to the right and above), and let Z be the subgraph obtained by adding the connecting 280-cliques from A_1' through A_1.

Because of the involution automorphism interchanging the right side and left side, we may assume the first cleared edge lies to the left of the 280-clique G_{150} or has one end vertex in G_{150}.

We will make heavy use of vertex-disjoint paths determined by perfect matchings between successive cliques. The most important family of such paths is $P_1, P_2, \ldots, P_{80}$ consisting of the 80 vertex-disjoint paths having one end vertex in A_2 and the other end vertex in G_{150} along the chain of connecting 280-cliques.

We call a clique pseudo-cleared if it contains exactly one cleared vertex. We are interested in which of A_3, A_8 or A_9 is the first to be pseudo-cleared. Suppose A_9 is the first of the three to be pseudo-cleared. At the moment the first vertex of A_9 is cleared, there must be 280 exposed vertices in A_9. There must be a path Q from G_{150} to A_9 in the subgraph of cleared edges. The path Q must pass through at least 19 20-cliques. Since there are most 9 additional exposed vertices, at least one of the 20-cliques, call it K, through which Q passes is cleared. From K, there are 20 vertex-disjoint paths back to A_2 not passing through A_9. Without loss of generality, assume these 20 vertex-disjoint paths terminate at the end vertices of $P_1, P_2, \ldots, P_{20}$. Call the extensions of $P_1, P_2, \ldots, P_{20}$ to K by $Q_1, Q_2, \ldots, Q_{20}$.

For each Q_i, $1 \leqq i \leqq 20$, we examine what happens as we start working back from K along the path Q_i. Since the clique K is cleared, the last vertex of Q_i is cleared. That is, the last edge of Q_i is cleared. Move to the preceding vertex. If it is not cleared, then we have encountered an exposed vertex on Q_i. If it is cleared, then we move to the preceding vertex on Q_i.

If Q_i passes through either A_6 or A_4, either we encounter an exposed vertex or the vertex u_i of Q_i in A_6 or $A_4$is cleared. But if the latter is the case, then the edge from u_i to A_8 or A_3 is cleared. Since neither A_8 nor A_3 have any cleared vertices, we have found an exposed vertex corresponding to the path Q_i.

If Q_i does not pass through A_4 or A_6, then either we encounter an exposed vertex or we reach the vertex u_i of Q_i in A_2, with u_i cleared. But now we extend a path from u_i through the L_j-cliques to A_8 and we must eventually encounter an exposed vertex.

Therefore, each path Q_i yields a distinct exposed vertex giving us at least 300 exposed vertices. We now see that A_9 cannot be the first pseudo-cleared clique amongst A_3, A_8, and A_9.

We next consider whether A_8 can be the first pseudo-cleared clique amongst the three cliques. Assume this is the case. At the moment the first vertex of A_8 is cleared, there are 200 exposed vertices in A_8. We again know that there is a path Q from G_{150} to A_8. Since Q passes through 150 280-cliques before reaching A_2, one of the 280-cliques must be cleared. This implies that each of the paths $P_1, P_2, \ldots, P_{80}$ contains a cleared vertex before reaching A_2.

First suppose that Q passes through the L_j-cliques. Let $Q_1, Q_2, \ldots, Q_{80}$ be the extensions of $P_1, P_2, \ldots, P_{80}$ through the L_j-cliques with end vertices in A_6. Since there are 300 of the 160-cliques, at least one of the L_j is cleared. This implies that each Q_i has a cleared vertex strictly between A_2 and A_6. Considering a fixed i, as we work from the cleared vertex in Q_i between A_2 and A_6 towards A_6, either we encounter an exposed vertex or the vertex v_i of Q_i in A_6 is cleared. But now we extend from v_i through the T_j-cliques, through A_5, down the R_j-cliques, through A_4 into A_3. We must encounter an exposed vertex at some point on this extension. Thus, each Q_i produces one exposed vertex working towards A_6, or past it as the case may be.

Now work from the cleared vertex of Q_i between A_2 and A_6 towards A_2. Either we encounter an exposed vertex or we reach the vertex u_i of Q_i in A_2. When u_i is one of the 20 vertices adjacent to a vertex of C_1, we extend towards A_9. We must encounter an exposed vertex at some point. Thus, we obtain another 20 exposed vertices giving us 300 altogether. We conclude that the path Q does not use the L_j-cliques.

Let $Q_1, Q_2, \ldots, Q_{80}$ be the same extensions of $P_1, P_2, \ldots, P_{80}$ as used above. Since each Q_i has a cleared vertex before reaching A_2 and there is no path from A_2 to A_6 through the L_j-cliques in the cleared subgraph, every Q_i has at least one exposed vertex on it that is not in A_8. This gives us 280 exposed vertices already. The path Q must pass through either $C_1, \ldots, C_{19}$ or $B_1, \ldots, B_{300}$. If it goes through $C_1, \ldots, C_{19}$, then one of the 20-cliques must be cleared or we have too many exposed vertices. But if one of the 20-cliques is cleared, then upon building 20 vertex-disjoint paths from the 20-clique to A_9, we get at least another 20 exposed vertices.

If Q uses $B_1, \ldots, B_{300}$, then we get a cleared 50-clique leading to another 50 exposed vertices in A_3 using the same kind of path extensions via vertex-disjoint paths. This establishes that A_3 must be the first pseudo-cleared clique amongst A_3, A_8, and A_9.

Assume this is the case. At the moment the first vertex of A_3 is cleared, there are 159 exposed vertices in A_3. We again know that there is a cleared path Q from G_{150} to A_3. Since Q passes through 150 280-cliques before reaching A_2, one of the 280-cliques must be cleared. This implies that each of the paths $P_1, P_2, \ldots, P_{80}$ contains a cleared vertex before reach A_2.

First suppose that Q passes through the R_i-cliques. Since there are 300 R_i-cliques, not every clique can contain an exposed vertex, so one of the cliques must contain none, and hence must be cleared. Call this clique K. Then certainly, there are 20 vertex-disjoint paths that go from K to A_4 and then through the D_i cliques to A_9. Since K is completely cleared, and A_9 contains no cleared vertices, somewhere on each of these paths there must be at least one exposed vertex. Similarly, consider 20 vertex-disjoint paths that start in K, go through the R_i, through A_5, and then through the F_i to A_9. By the same argument, each of these paths must contain an exposed vertex.

If the path Q goes through the T_i, then one of the T_i must be completely cleared. Call this clique K'. From K' there are 80 vertex-disjoint paths through the T_i into A_6, and then to A_8. Again, each of these paths must contain an exposed vertex. Finally, extending the paths P_i, $1 \leqq i \leqq 20$, from A_2, through the C_i, into A_9. Each of these 20 paths must also contain an exposed vertex. But this means that the sweep uses $159 + 20 + 20 + 80 + 20 = 299$ exposed vertices, and hence as many sweepers, a contradiction. Thus, there is no cleared path through the T_i after going through the R_i.

Instead, we consider if the path Q goes through the R_i and not through the T_i. Since the path does not go through the T_i, we consider the 80 vertex-disjoint paths that go from K, through A_5, then through the T_i to T_1. Each of these paths must contain an exposed vertex. Extending the paths P_i, $1 \leqq i \leqq 80$, from A_2, through the L_i into A_6, then to A_9, we must similarly encounter an exposed vertex on each. This means there are at least $159 + 80 + 80 = 319$ exposed vertices, and hence too many sweepers. So, there is no cleared path through the R_i.

Now suppose that Q passes through the T_i cliques. Since there are 300 T_i cliques at least one must be completely cleared. Call this clique K. Consider the 80 vertex-disjoint paths going from K, through the T_i into A_5, then through the R_i to R_{300}. Each of these paths must contain at least one exposed vertex. Similarly, the 80 paths from K through the T_i to A_6 into A_9 must each contain at least one exposed vertex. This means the sweep must use at least $159 + 80 + 80 = 319$ sweepers. So there is no cleared path through the R_i.

Suppose instead that Q passes through the L_i. Since there are 300 L_i-cliques at least one must be cleared. Call it K. From K, through the L_i to A_6 into A_9, there are 80 vertex-disjoint paths each of which must contain an exposed vertex. Since the path Q does not pass through the T_i, it goes through the E_i. Since there are 300 of these cliques, one of the E_i must be cleared. Call it K'. There are 20 vertex-disjoint paths from K' to A_9, each of which must contain an exposed vertex. Similarly the paths P_1 to P_{20} may be extended from A_2 through the C_i to A_9, and each path must contain an exposed vertex. This accounts for $159 + 80 + 20 + 20 = 279$ exposed vertices, and hence as many sweepers. If Q goes through the B_i, then some clique K'' must be cleared, since there are at most 10 sweepers in all the B_i. Consider 50 vertex-disjoint paths that go from K'' through the B_i into A_4, through the R_i to R_{300}. Since there are no connected paths through the R_i, each of the 50 paths must contain an exposed vertex, meaning the sweep must use at least 329 sweepers. Thus Q must go through the D_i. But since there are at most 10 sweepers on this portion of the path, one of the cliques must be cleared. Call this clique K'''. From K''' through the D_i to A_9, there are 20 vertex-disjoint paths, each of which must contain an exposed vertex, and thus this sweep must use 299 sweepers. Thus, Q does not pass through the L_i.

Suppose that Q passes through the C_i-cliques to A_9, then through the D_i into A_4, and finally to A_{160}. If one of the C_i cliques does contain an exposed vertex, than each vertex in that clique must be cleared, and between that clique and $A_9[C_{19}]$ there are 20 vertex-disjoint paths, each of which must contain an exposed vertex and a sweeper. Either way, along the C_i cliques and $A_9[C_{19}]$ there must be 20 sweepers. Similarly, there must be 20 sweepers on the D_i cliques and $A_9[D_{19}]$. Again, the paths P_1 to P_{80} may be extended into vertex-disjoint paths from A_2 through the L_i-cliques to L_{300}. These paths must contain at least 80 exposed vertices. At the moment that A_3 is pseudo-cleared, every vertex in A_3 is exposed or cleared. Thus, there are 110 vertex-disjoint paths from A_3 to A_4, through the R_i to R_{300}. This accounts for $20 + 20 + 80 + 110 = 230$ sweepers.

We consider the next time certain vertices are cleared. Certainly, there is a clique L among the L_i that contains no cleared vertices. Similarly, there is a clique R among the R_i that contain no cleared vertices. We consider which of L, R, or A_9 first contains a cleared vertex. If A_9 is the first that is pseudo-cleared, then there are still 110 vertex-disjoint paths from R to A_3 through the R_i and A_4 that must each contain an exposed vertex. Similarly, the extensions of P_1 through P_{80} from A_2 through the L_i to L must each contain an exposed vertex. But when A_9 first obtains a cleared vertex, there must be at least 280 sweepers in A_9, meaning such a sweep would require $280 + 80 + 110 = 470$ sweepers. If L is pseudo-cleared before A_9 or R, then we still have the 110 exposed vertices as in the previous case. There are also still at least 40 exposed vertices amongst the C_i-cliques, D_i-cliques, $A_9[C_i]$, and $A_9[D_i]$. But when L is pseudo-cleared, it must contain at least 159 sweepers. But this sweep requires $40 + 110 + 159 = 309$ sweepers. Similarly, if R is the first of the three pseudo-cleared, the sweep must require $40 + 80 + 179 = 299$ sweepers.

Thus, the path Q must pass through the B_i. Since there are 300 B_i cliques, one must be completely cleared. Call it K. There are 50 vertex-disjoint paths that go from K, through the B_i cliques to A_4, then through the R_i cliques to R_{300}. Since there are no cleared paths through the R_i, each of these paths must contain an exposed vertex. Similarly, vertex-disjoint extensions of P_1, P_2, $\cdots$, P_{80} from A_2 through the L_i to L_{300} must each contain at least one exposed vertex. With at least 159 sweepers on A_3 when it is pseudo-cleared, this accounts for $50 + 80 + 159 = 289$ sweepers.

If any of the paths contain two or more exposed vertices, the sweep requires more than 289 sweepers. So, each path must contain exactly one exposed vertex. When A_3 is pseudo-cleared, if any cleared vertex in W contains a sweeper, then the sweep uses more than 289 sweepers. If there is more than one sweeper on an exposed vertex, then the sweep uses more than 289 sweepers. Thus, there is exactly one sweeper on every exposed vertex, and no other sweepers. Further, there are no other exposed vertices other than the 289 already considered.

Let v be a vertex in A_4 that is not on one of the 50 vertex-disjoint paths. If v is cleared, then there is another path, vertex-disjoint from the other 50, the goes from v through the R_i to R_{300}. This path must contain an exposed vertex, a contradiction. Thus, every vertex in A_4 that is not on one of the 50 vertex-disjoint paths is incident with no cleared edges (since v cannot be exposed). Thus every vertex in A_4 that is one of the vertex-disjoint paths must be either contaminated or exposed. Certainly, since the path Q passes through A_4, at least one vertex w is exposed.

These paths also pass through B_{300}. If there is an exposed vertex u in B_{299}, consider the path that w sits on. Certainly, the w and u cannot be on the same path, as they are both exposed. Both w and u have counterparts w' and u' in B_{300} on the same vertex-disjoint path. Since w is exposed, w' must be cleared, and hence $u'w'$ is cleared. Since u is exposed, u' must be incident with no cleared edges, a contradiction. Thus, all exposed vertices on these 50 vertex-disjoint paths must be in $A_4[B_{300}]$ or in B_{300}.

Every exposed vertex in A_4 must be incident with at least 60 contaminated edges, so no sweeper on these vertices can move without allowing recontamination. For every exposed vertex v in B_{300}, the edge from that vertex to $A_4[B_{300}]$ is contaminated. If any other edge incident with v is contaminated, the sweeper on v cannot move. If no other vertices are contaminated, the sweeper has only one possible move, to the corresponding vertex in $A_4[B_{300}]$, where it again sits on an exposed vertex and is incident with 109 exposed edges, and cannot move without allowing the graph to be recontaminated.

Since we know that there are no exposed vertices in C_1, there are also no cleared edges. Thus, every vertex in $A_2[C_1]$ is either exposed or is incident with no cleared edges. Since Q passes through A_2, we know that A_2 must contain an exposed vertex, v. Assume that there is an exposed vertex w in G_1. Both w and v have counterparts w' and v' in A_1. Since v is exposed, v' must be clear, and hence $v'w'$ must be a cleared edge. But since w is an exposed vertex w' must have no cleared edge incident with it, a contradiction. Thus, there can be no exposed vertices in G_1. By a similar argument, there can be no exposed vertices in L_2. So, of the 80 exposed vertices on the 80 vertex-disjoint paths from that have been extend from A_2 to L_{300}, all the exposed vertices must be on A_1, A_2, or L_1.

Since B_1 can contain no exposed vertices, all vertices must be clear. (If not, there would be an exposed vertex between B_1 and K.) Then, the vertices of $A_2[B_1]$ must be either clear or exposed. If they are clear, then the vertices in $A_2[C_1]$ must be exposed. The 50 vertices of $L_1[A_2[B_1]]$ must be exposed. Let $A = V(A_2) - (V(A_2[B_1]) \cup V(A_2[C_1])$. Then there are in total 10 exposed vertices in A and $B_1[A]$. Every vertex in $A_2[C_1]$ is incident with two or more contaminated edges (from A_2 to C_1 and from A_2 to L_1), and so the sweepers on these vertices cannot move. The 50 sweepers on the exposed vertices of $L_1[A_2[B_1]]$ likewise cannot move, as each of these vertices is incident with at least 100 contaminated edges. Finally, if v is a vertex of A, and v is exposed, and incident with two or more contaminated edges, then the sweeper on v cannot move. If it is incident with only one contaminated edge, it must be to B_1, and when the sweeper clears that edge it is again on an exposed vertex, this one incident with at least 100 contaminated edges, and hence cannot move. By the same argument, if a vertex in $B_1[A]$ is exposed, it must be incident with at least 100 contaminated edges, and hence its sweeper cannot move.

If the vertices of $A_2[B_1]$ are exposed, and the vertices of $A_2[C_1]$ have no cleared edges incident with them, then the vertices of A are either exposed, or contain no cleared edges. Between A and $A_1[A]$ there must be exactly 10 exposed vertices. Any vertex in A_2 that is exposed is incident with at least 20 contaminated edges, so the sweeper on that vertex cannot move. Any exposed vertex in A_1 if not incident with two contaminated edges, must have a contaminated edge to A_2. Upon clearing this, the sweeper is incident with at least 19 contaminated edges and cannot move.

Finally, if both the vertices of $A_2[B_1]$ and of $A_2[C_1]$ are exposed, then the other 10 exposed vertices are between A, $L_1[A]$ and $A_1[A]$. By similar arguments as before, every sweeper on an exposed vertex either cannot move, or can clear at most one more edge without being able to clear any further.

Since this sweep strategy S is monotonic and connected, the first time there is an exposed vertex in A_4, the 80 sweepers around A_2 and the 50 sweepers on B_{300} or A_4 must already be in place. This leaves exactly 159 sweepers to clear a vertex in A_3. As in Theorem 2.2, once this vertex is cleared, there are at least two contaminated edges incident with each vertex, and hence no sweeper in A_3 can move. Since every other sweeper has at most one more move before being unable to move, there is no way to finish sweeping W, a contradiction. Thus, $\mathrm{mksw}(W) > 289$. ■

3.3 Connected Sweeping and Graph Minors

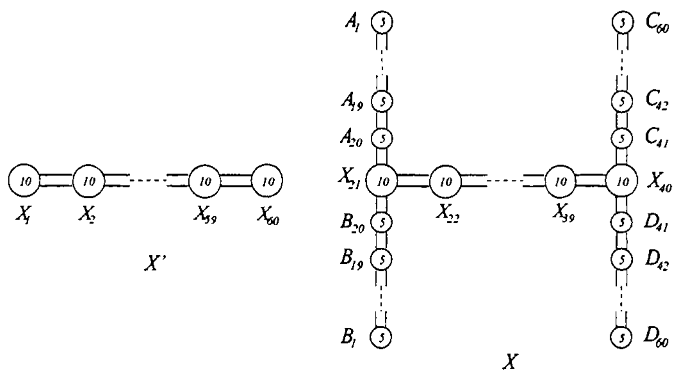

Figure 3.2: The graph X' and its subgraph X.

Referring to Figure 3.2, let $X' = K_{10}\Box P_{60}$. We construct X as pictured, where each circle represents a complete graph on the number of vertices in the circle, and double lines between two cliques indicate a saturated matching.

Also, by construction, $V(X_{21}[A_{20}])\cap V(X_{21}[B_{20}]) = \emptyset = V(X_{40}[C_{41}])\cap V(X_{40}[D_{41}])$. It is easy to see that X is a subgraph of X'. However, we can prove $\mathrm{ksw}(X) > \mathrm{ksw}(X')$.

Theorem 3.13 *For X as pictured in Figure 3.2,*

$$\mathrm{sw}(X) < \mathrm{msw}(X) < \mathrm{ksw}(X).$$

PROOF. Recall from Corollary 3.9 that $\mathrm{sw}(K_{10}\Box K_2) = 11$. Since X contains $K_{10}\Box K_2$ as a minor, by Theorem 2.15 we know that $\mathrm{sw}(X) \geqq 11$. In fact, we can use 11 sweepers to connected clear X, by first placing 5 sweepers on A_1 and 5 sweepers on B_1. Then a single free sweeper can be used to clear all the edges in A_1. Then 5 sweepers move along the perfect matching to A_2, and a single free sweeper clears all the edges in A_2, and so on, finally reaching $X_{21}[A_{20}]$. The single free sweeper moves to B_1, and the process is repeated, clearing the B_i and moving to $X_{21}[B_{20}]$. With 10 sweepers on X_{21}, a single free sweeper may then clear all the edges of X_{21}. These 11 sweepers may then clear the X_i clique by clique, finally reaching X_{40}. Stationing 5 sweepers on $X_{40}[D_{41}]$, the remaining 6 sweepers may clear the C_i. Then the D_i may be cleared. Thus, $\mathrm{sw}(X) = 11$.

The graph X can be cleared by 16 sweepers in a connected sweep. Placing all 16 sweepers on A_1, we may use one free sweeper to clear the edges of A_1. Then 5 sweepers may move to A_2, and a free sweeper may clear the edges of A_2, and so on, until finally $X_{21}[A_{20}]$ is cleared. Then we may place 5 more sweepers on the remaining vertices of X_{21}, and use a free sweeper to clear the remaining edges in X_{21}. Leaving a sweeper on each vertex of X_21, there are 6 free sweepers. These sweepers may clear the B_i clique by clique. Then the 10 sweepers on X_{21} plus another free sweeper may clear the X_i through X_{40}. Finally station 10 sweepers on X_{40}. This leaves 6 free sweepers, who can be used to clear the C_i and the D_i. Thus, $\mathrm{ksw}(X) \leqq 16$. In the same manner as the proof that 289 sweepers are insufficient to clear W in Theorem 3.12, it can be shown that 15 sweepers are insufficient to clear X.

To obtain a monotonic sweep, first place 6 sweepers on A_1, and 6 sweepers on B_1. The 6 sweepers on A_1 can clear the A_i, eventually stationing 5 sweepers on $X_{21}[A_{20}]$. The 6 sweepers on B_1 can clear the B_i, eventually stationing 5 sweepers on $X_{21}[B_{20}]$. Then we may follow the same strategy as the sweep above. Thus $\mathrm{msw}(X) \leqq 12$.

Let S be a monotonic sweep strategy for X. Again, consider the point when the graph goes from having no cleared vertices to exactly 1 cleared vertex v. Since $\mathrm{ksw}(X) = 16$, we know that this sweep must not be connected. Thus, at some point a vertex w is cleared such that there is no clear path from v to w. If v is in some X_i, and w is in X_i, then there are at least 22 exposed vertices, and as many sweepers. If v is in some X_i, and w is in some 5-cliques, then there are at least 16 exposed vertices, and as many sweepers. The same is true if v is in some 5-clique and w is in some 10-clique. Finally, we consider if v is in some 5-clique (without loss of generality, some A_i-clique) and w is in some 5-clique.

If v is in A_j, $2 \leqq j \leqq 20$, then when it is cleared its neighbouring vertices must contain at least 6 sweepers. Consider the subgraph induced by the cleared edges at this point, and the component that v is in. If this component contains exactly 6 sweepers, than as in the proof of Theorem 2.2, each of the exposed vertices adjacent to v must be incident with at least two contaminated edges. Thus none of the six sweepers can move without allowing recontamination. When w is cleared, it has at least 5 exposed neighbours. If it contains no more that 5 sweepers in the component it belongs to of the subgraph induced by the cleared edges, then none of these 5 sweepers may move with allowing recontamination. Thus, there must be another sweeper in the graph.

Instead, we consider if v is in A_1. If w is a vertex that is not in B_1, C_{60}, or D_{60}, then we have the same argument as in the previous case. In the subgraph induced by the cleared edges, each of the components these two vertices belong to must contain at least 5 sweepers. If both have exactly 5, then since these sweepers cannot move without recontaminating edges, there must be at least one more sweeper. Consider the next edge cleared. If the next edge cleared is not incident with a vertex in either of the cleared components containing v or w, since the minimum degree of this graph is 5, two sweepers must be used to clear the edge. This accounts for at least 12 sweepers.

Instead, we assume that the next edge cleared in S must be incident with one of the cleared components. The same argument applies to the edge that is next cleared. If it is not incident with either connected component, then the sweep must use at least 12 sweepers. This continues to be true until there is a connected path between v and w. If, in S, the connected components that contain u and v both continue to "grow" by clearing edges incident with them, then both components must contain at least 6 sweepers. We instead consider if only one of the cleared components continues to clear incident edges. Without loss of generality, assume that this component is the one containing v.

We consider how this component could expand. Before the cleared path from w to v is formed, if the component containing v ever contains a cleared vertex x in some X_i, consider the first time it happens. At the point x is cleared, it must have at least 10 exposed neighbours in X_i-cliques. Each must contain a sweeper. Since that are also at least 5 sweepers in the clique containing w, this accounts for at least 15 sweepers. If the cleared component containing v does not contain a cleared vertex of some X_i-clique before the cleared path from v to w is formed, consider the first time that an edge between X_{20} and B_{20} is cleared. (This must occur before the path from v to w is formed. There are 5 vertex-disjoint paths from $X_{21}[A_{20}]$ to A_1. Since there is a cleared vertex in A_1, but none in X_{21}, each of these paths must contain an exposed vertex, and hence a sweeper. But if an edge is cleared in between X_{21} and B_{20}, there must be a sweeper in B_{20} and another in $X_{21}[B_{20}]$. Since there are at least 5 sweepers in the cleared component containing w, this accounts for at least 12 sweepers. Thus, every monotonic sweep strategy on X must use at least 12 sweepers. ■

We now consider the graph X'.

Lemma 3.14 *For the graphs X and X',* $\mathrm{ksw}(X) > \mathrm{ksw}(X')$.

PROOF. Since X' contains $K_{10} \Box K_2$ as a minor, we know by Theorem 2.15 and Corollary 3.9 that $\mathrm{ksw}(X') \geqq 11$. In fact, we can use 11 sweepers in a connected sweep to clear X', by placing 10 sweepers on X_1, and then using a single free sweeper to clear all the edges in X_1. Then 10 sweepers move along the perfect matching to X_2, and a single free sweeper clears all the edges in X_2, and so on, finally reaching X_{60}. Thus, $\mathrm{ksw}(X') = 11$.

From the proof of Theorem 3.13, we know that $\mathrm{ksw}(X) = 16$, and the result follows. ■

Since X is a subgraph of X', this lemma has an immediate consequence, as observed in [4]. If H is a minor of a graph G, then in contrast to Theorem 2.15, it does not follow that $\mathrm{ksw}(H) \leqq \mathrm{ksw}(G)$.

3.4 Monotonic Sweeping and Graph Minors

Continuing in the same vein, let $Y' = K_{10} \Box P_{120}$, and Y be as pictured in Figure 3.3, where circles and double lines are defined as above. It is easy to see that Y is a subgraph of Y'.

Theorem 3.15 *For graphs Y and Y' as given,* $\mathrm{msw}(Y) > \mathrm{msw}(Y')$.

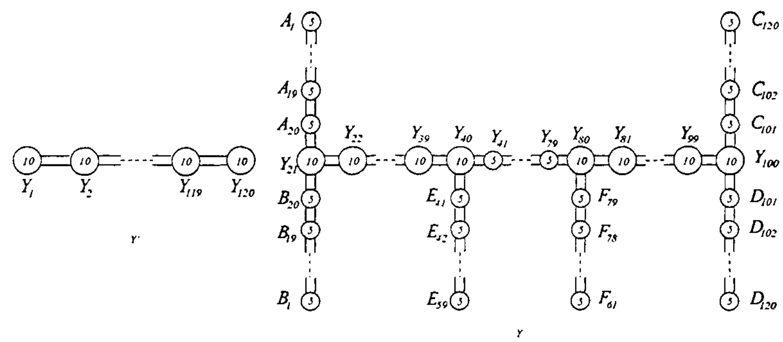

Figure 3.3: The graph Y' and its subgraph Y.

PROOF. We first note that $K_{10} \square K_2$ is a subgraph of Y', and thus $11 \leqq \text{msw}(Y')$. Also, Y' can be cleared using the same strategy as used for X' in Lemma 3.14. Thus, $\text{msw}(Y') = 11$.

The graph Y can be cleared by 16 sweepers in a monotonic fashion. Place 16 sweepers on A_1. Use 6 sweepers to clear the A_i, stationing 10 sweepers on Y_{21}. Use the six remaining sweepers to clear the B_i. Clear to Y_{40}, stationing 10 sweepers on Y_{40}. This leaves 6 free sweepers that can be used to clear the E_i. Then the sweepers may clear to Y_{80}, stationing 10 sweeper there. The remaining 6 free sweepers may clear the F_i. Then all the sweepers may clear to Y_{100}, stationing 10 sweepers there. The 6 remaining sweepers may clear the C_i and then the D_i. Thus, $\text{msw}(Y) \leqq 16$. In the same manner as the proof that 289 sweepers are insufficient to clear W in Theorem 3.12, it can be shown that 15 sweepers are insufficient to clear X. ■

As before, since Y is a subgraph of Y', there is an immediate corollary, as observed in [4]. In contrast to Theorem 2.15, if H is a minor or G, then it does not follow that $\text{msw}(H) \leqq \text{msw}(G)$.

3.5 Inequalities

Theorems 3.11 and 3.12 can be summarized with the following result.

Corollary 3.16 *For the graph W,* $\text{ksw}(W) = 281 < 290 = \text{mksw}(W)$.

Recall that there are three inequalities in Lemma 1.3. Corollary 3.16 shows that the final inequality can be strict, and Theorem 3.13 shows that the first pair may also be strict. This leads us to construct a single graph H for which the three inequalities strictly hold (see Figure 3.4).

Theorem 3.17 *For the graph H as given,* $\mathrm{sw}(H) < \mathrm{msw}(H) < \mathrm{ksw}(H) < \mathrm{mksw}(H)$.

This graph H has $\mathrm{sw}(H) = 561$, $\mathrm{msw}(H) = 570$, $\mathrm{ksw}(H) = 841$, and $\mathrm{mksw}(H) = 850$. The proofs of these claims follow in the same manner as the proofs of Theorems 3.11, 3.12, and 3.13.

In Corollary 3.16 we showed that a graph W existed with $\mathrm{ksw}(W) < \mathrm{mksw}$. The graph W is quite large, containing almost $400,000$ vertices. The large strings of 300 identically sized cliques were constructed so that they would be impossible to "sneak" through, and the C_i and D_i were constructed so that they could be "sneaked" through.

We showed that the difference between $\mathrm{ksw}(W)$ and $\mathrm{mksw}(W)$ was 9, though in fact, the difference can be much smaller. For instance, we could reduce the size of cliques and length of "paths" by an approximate factor of 5 and then prove that $\mathrm{ksw}(W_{\frac{1}{5}}) = 57$ and $\mathrm{mksw}(W_{\frac{1}{5}}) = 58$. (This corresponds to letting $k = \frac{1}{5}$ in the following construction.) However, these results, while valid, are more easily demonstrated by using larger cliques and longer paths to make the difference more believable.

From the graph W, we may define a family W_k of graphs constructed as follows for $k \geqq 1$. In W_k, the 300 cliques of the R_i (and R_i') are replaced by $300k$ cliques of order $180k$. Similarly, the T_i (and T_i') are replaced by $300k$ cliques of order $80k$, the B_i (and B_i') are replaced by $300k$ cliques of order $50k$, the L_i (and L_i') are replaced by $300k$ cliques of order $160k$, the E_i (and E_i') are replaced by $300k$ cliques of order $20k$, the F_i (and F_i') are replaced by $300k$ cliques of order $20k$, and the G_i are replaced by $300k$ cliques of order $280k$. The cliques A_2, A_5, A_6, A_2', A_5', and A_6' are replaced by cliques of order $80k$. The cliques A_4 and A_4' are replaced by cliques of order $110k$. The cliques A_3 and A_3' are replaced by cliques of order $160k$. The cliques A_1 and A_1' are replaced by cliques of order $280k$. The 19 cliques that make up the C_i are replaced by $20k - 1$ cliques of order $20k$. Similarly for the C_i', D_i, and D_i'. The cliques A_8 and A_8' are replaced by

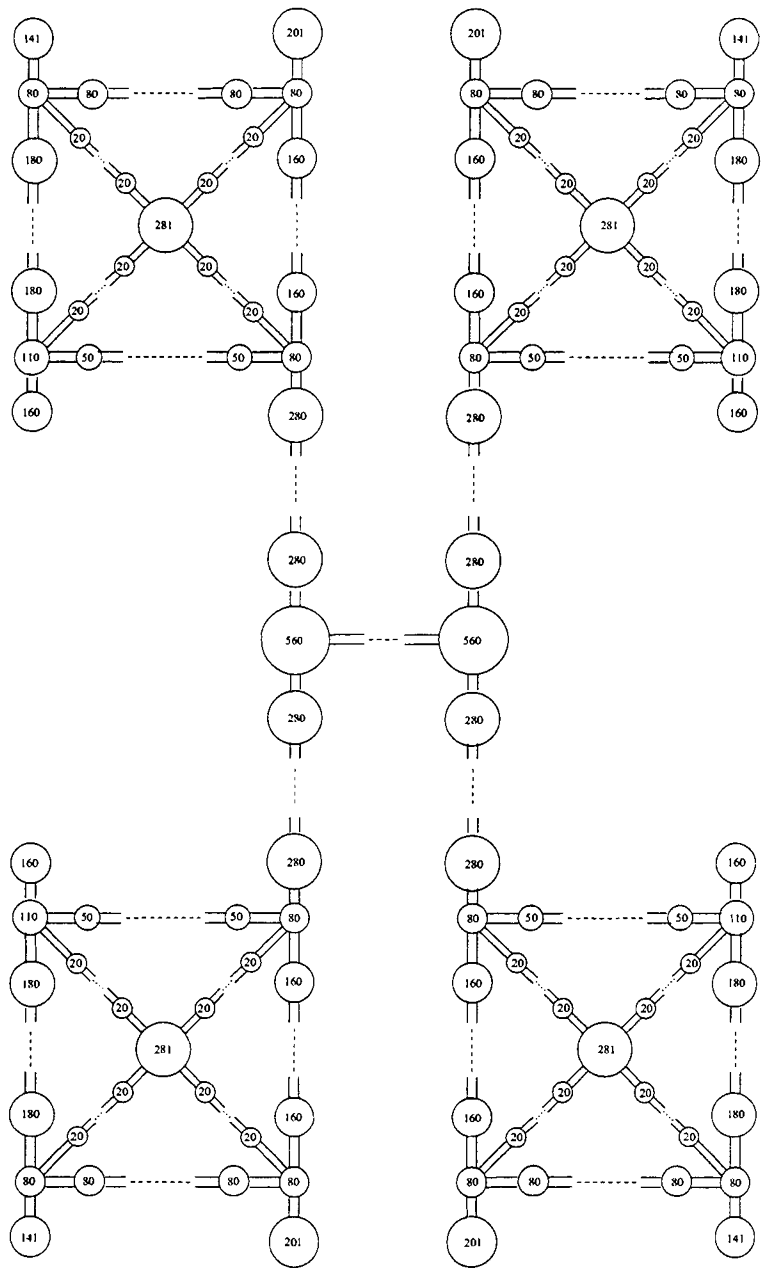

Figure 3.4: The graph H.

cliques of order $200k + 1$, the cliques A_7 and A'_7 are replaced by cliques of order $140k + 1$, and the cliques A_9 and A'_9 are replaced by cliques of order $280k + 1$.

The resulting graphs W_k are "scaled up" version of W, with appropriately changed sweep numbers. Using an argument similar to those in the proofs of Theorems 3.11 and 3.12, we can prove the following theorem.

Theorem 3.18 *For* $k \geqq 1$, $\mathrm{ksw}(W_k) = 280k + 1$, *and* $\mathrm{mksw}(W_k) = 290k$.

So, certainly, the difference between these two values can be arbitrarily large.

In the same manner, we may create families of graphs X_k (and X'_k) and Y_k (and Y'_k) based on X (and X') and Y (and Y'). This is done by replacing cliques of order 5 with cliques of order $5k$ and cliques of order 10 with cliques of order $10k$, and lengthening "paths" of cliques of the same order by a factor of k. (For instance, in X, rather than having 20 cliques of order 10 make up the X_i, they would be replaced by $20k$ cliques of order $10k$.) The results for these families are summarized below.

Theorem 3.19 *For* $k \geqq 1$, $\mathrm{ksw}(X_k) = 15k + 1$; $\mathrm{msw}(X_k) = 10k + 2$; $\mathrm{ksw}(X'_k) = 10k + 1$; $\mathrm{msw}(Y) = 15k + 1$; *and* $\mathrm{msw}(Y') = 10k + 1$.

This result tells us that the difference between the monotonic sweep number of a graph and the connected sweep number can be large. As well, the results tell us that in the case of connected and monotonic sweeps, a subgraph may need arbitrarily more sweepers than a supergraph.

Finally, the graph in Figure 3.5 is a similarly "scaled up" version of the Y-square (pictured in Figure 1.1). Here, edges are replaced by "paths" of cliques, with each path containing k^2 cliques of size k. This increases the sweep number to $3k + 1$, and the monotonic sweep number to $4k$, again showing that the difference in these values can be quite large.

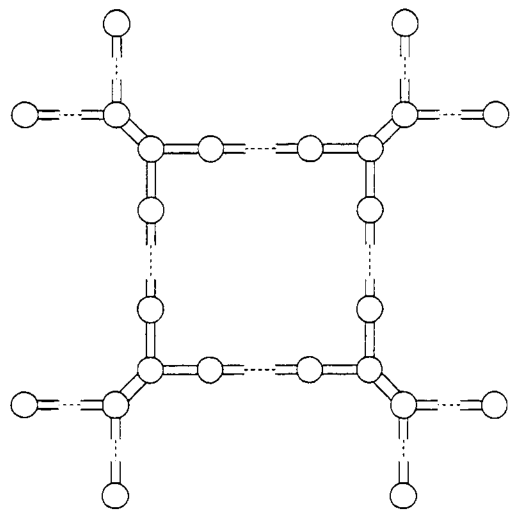

Figure 3.5: The kY-square.

3.6 Variation on the required number of sweepers

We know the maximum number of exposed vertices in a connected graph is at most the sweep number of the graph. Of course, most of the time, the number of exposed vertices is much less than this maximum. In a real world situation, most sweepers not on exposed vertices could "go away," and would only return when needed. So we would be interested in sweep strategies that minimize the number of exposed vertices at each step. For a graph G, the sequence of $\mathrm{ex}_S(G, i)$ for any S could vary greatly. The following theorem illustrates just how great this variance can be.

We first construct a graph Z as pictured in Figure 3.6, where the a_i are positive integers, and $M = \max_{1 \leqq i \leqq n} a_i + 5$. (The value 5 is added as a safety margin.)

Theorem 3.20 *Given a finite sequence of positive integers, $a_1, a_2, \ldots, a_n$, then with Z as given, every monotonic connected sweep strategy S of Z that uses $\mathrm{mksw}(Z)$ sweepers and minimizes the number of exposed vertices at each step has the property that there exist integers $k_1 < k_2 < \cdots < k_n$ such that $\mathrm{ex}_S(Z, k_i) \geqq a_i$.*

PROOF. Since Z contains an M-clique, we know that $\mathrm{sw}(Z) \geqq M$. Further, there is a monotonic connected sweep strategy using M sweepers. First, clear M, then

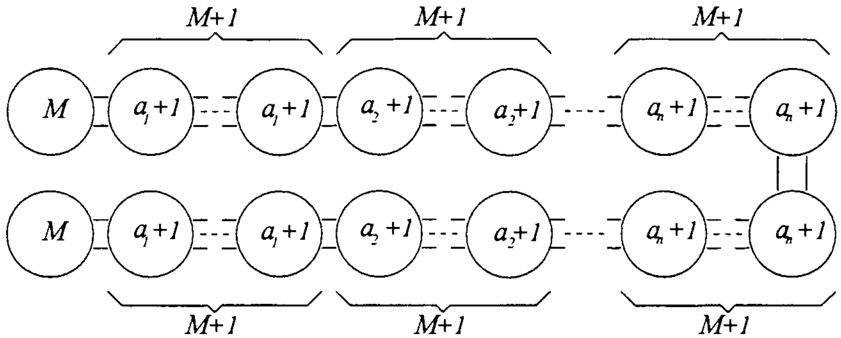

Figure 3.6: The graph Z.

the first $a_1 + 1$ clique, then the next, and so on, moving from left to right. Thus, $\text{mksw}(Z) = M$.

Let v be the first vertex cleared in a monotonic connected sweep strategy S on Z using M sweepers. If v is not in one of the M-cliques, then there is a cleared edge e in some other clique. There are at least two vertex-disjoint paths that pass through the vertices of e to either M. The first time that a vertex is cleared in either M-clique, there are $M - 1$ sweepers on that clique. But at the same time, there are two vertex-disjoint paths from the vertices of e to the other M-clique. These two paths must contain at least one exposed vertex, and hence one sweeper. But this sweep then uses $M + 1$ sweepers. Thus, v must be in one of the M-cliques.

Let $i < j$. Consider a_i and a_j. Assume that no $a_i + 1$ clique obtains a cleared vertex before all the $a_j + 1$ cliques. Let w be a cleared vertex in one of the $a_j + 1$ cliques. Since S is a connected sweep strategy there is a cleared path between v and w. But this pass must pass through the $a_i + 1$ cliques, of which there are $M + 1$. Since these cliques contain no cleared vertices, there must be at least one exposed vertex in each of the $a_i + 1$ cliques, and hence at least $M + 1$ sweepers in the $a_i + 1$ cliques. Since this uses too many sweepers, some $a_i + 1$ clique must contain a cleared vertex before all the $a_j + 1$ cliques do. When the $a_i + 1$ clique first contains a cleared vertex, there are at least a_i exposed vertices. ■

Chapter 4

Sweeping Digraphs

4.1 Elementary bounds

Returning to digraphs, Theorem 4.1 follows directly from the definitions of the sweeping models. We recall that $\mathrm{sw}_{0,1}(D)$ is the strong sweeper number, that $\mathrm{sw}_{1,1}(D)$ is the directed sweep number, that $\mathrm{sw}_{0,0}(D)$ is the undirected sweeper number, and that $\mathrm{sw}_{1,0}(D)$ is the weak sweep number.

Theorem 4.1 *If D is a digraph, then* $\mathrm{sw}_{0,1}(D) \leqq \mathrm{sw}_{1,1}(D) \leqq \mathrm{sw}_{1,0}(D)$ *and* $\mathrm{sw}_{0,1}(D) \leqq \mathrm{sw}_{0,0}(D) \leqq \mathrm{sw}_{1,0}(D)$.

All of these equalities can be achieved; one only need consider a directed path. It is easy to see that for $\overrightarrow{P_n}$, $\mathrm{sw}_{0,1}(\overrightarrow{P_n}) = \mathrm{sw}_{1,0}(\overrightarrow{P_n}) = 1$.

The inequalities can also be strict. Consider the transitive tournament TT_n. We will see in Corollary 4.15 that $\mathrm{sw}_{1,1}(TT_n) = \lfloor \frac{n^2}{4} \rfloor$. Since the undirected graph underlying TT_n is the complete graph K_n on n vertices, $\mathrm{sw}_{0,0}(TT_n) = n$ by Corollary 2.4. By considering the acyclic ordering of TT_n, we see that $\mathrm{sw}_{0,1}(TT_n) = 1$. First, use the single sweeper to clear all the arcs from the source. (A single sweeper suffices because it may return to the source while the intruder cannot.) Then the subtournament induced by the contaminated arcs induces TT_{n-1}, and the process may be repeated.

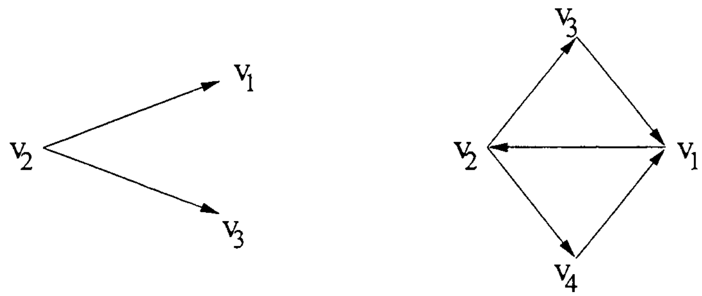

Figure 4.1: The graphs D and E.

Consider the digraph D with vertex set $\{v_1, v_2, v_3\}$ and arc set $\{(v_2, v_1), (v_2, v_3)\}$. The underlying undirected graph is P_3, and thus $\mathrm{sw}_{0,0}(D) = 1$. However, $\mathrm{sw}_{1,0}(D) = 2$.

Let E be the digraph with vertex set $\{v_1, v_2, v_3, v_4\}$ and with arc set $\{(v_1, v_2), (v_2, v_3), (v_3, v_1), (v_2, v_4), (v_4, v_1)\}$. We claim $\mathrm{sw}_{1,1}(E) = 2$. Place a sweeper on v_1, then use another sweeper to clear (v_1, v_2). Use this extra sweeper to clear a directed path from v_2 to v_1, and then clear the other directed path from v_2 to v_1. It is obvious that one sweeper cannot clear E, since one sweeper is insufficient to clear even one arc. But $\mathrm{sw}_{1,0}(E) = 3$. To see that three sweepers are sufficient, place a sweeper on vertices v_1 and v_2, and use a third to clear all the arcs. However, two are not sufficient. Two sweepers can be used to clear two arcs, but cannot clear a third. Thus, three sweepers are necessary to clear E in the weak sweep model.

Furthermore, the directed sweep number and the undirected sweep number are not comparable. Consider the digraph D described above. Remembering that $\mathrm{sw}_{0,0}(D) = 1$, we can also see that in this case $\mathrm{sw}_{1,1}(D) = \mathrm{sw}_{1,0}(D) = 2$. However, consider the digraph $TT_4^\star$ formed from TT_4 by reversing the single arc from source to sink. Then $\mathrm{sw}_{0,0}(TT_4^\star) = 4$, but $\mathrm{sw}_{1,1}(TT_4^\star) = 2$. (In general, $\mathrm{sw}_{0,0}(TT_n^\star) = n$, and $\mathrm{sw}_{1,1}(TT_n^\star) = 2$.)

To make calculation of the various sweep numbers easier, we introduce some elementary lower bounds involving minimum in-degree. This theorem parallels Theorems 2.1 and 2.2.

Theorem 4.2 *If D is a digraph, then* $\mathrm{sw}_{1,1}(D) \geqq \delta^-(D) + 1$, $\mathrm{sw}_{0,1}(D) \geqq \delta^-(D)$, *and*

$\mathrm{sw}_{1,0}(D) \geqq \mathrm{sw}_{0,0}(D) \geqq \delta^-(D) + \delta^+(D)$.

PROOF. Consider a directed sweep strategy that employs a minimum number of sweepers. At some stage we must have the first instance of a clear vertex, call it x. The last move before x is cleared must be that of a sweeper moving along an arc coming in to x, say along the arc (z, x). Since z is not clear, it must be the case that a sweeper remains on z at this stage of the process else we immediately recontaminate the arc (z, x). Similarly, every other vertex with an arc directed towards x must be occupied at this stage, and thus $\mathrm{sw}_{1,1}(D) \geqq \delta^-(D) + 1$.

The second inequality follows in a similar manner. If the arc (z, x) is cleared from z to x, there must be a sweeper on z to prevent recontamination. There also needs to be a sweeper on every other vertex with an arc directed towards x, and so $\mathrm{sw}_{0,1}(D) \geqq \delta^-(D)+1$, and so certainly $\mathrm{sw}_{0,1}(D) \geqq \delta^-(D)$. However, it is possible that the arc (z, x) is cleared by moving a single sweeper from x to z against the direction of the arc. In this case, all we can say is that $\mathrm{sw}_{0,1}(D) \geqq \delta^-(D)$.

The final pair of inequalities comes from Theorem 4.1 and from recalling that the undirected sweep number is equal to the sweep number of the underlying undirected graph. ■

After considering in-degree and out-degree, it is natural to consider the score sequence. However, this is not an interesting parameter to consider, at least from the perspective of the directed sweep number. For example, the almost transitive tournament on 7 vertices, $TT_7^\star$, has score sequence $1, 1, 2, 3, 4, 5, 5$, and $\mathrm{sw}_{1,1}(TT_7^\star) = 2$. Let S be a tournament on the vertices v_i, $1 \leqq i \leqq 7$, and arcs (v_i, v_j) for all $i < j$, except the arcs (v_7, v_5), (v_5, v_3), and (v_3, v_1). Then S also has score sequence $1, 1, 2, 3, 4, 5, 5$, but $\mathrm{sw}_{1,1}(S) = 3$.

4.2 Characterizing 1-sweepable digraphs

In the case of graphs, characterizations are known for graphs that are k-sweepable, for $k \leqq 3$ [14]. It is natural to consider analogous characterizations for digraphs with respect to the various sweep models. We begin by characterizing those digraphs that are 1-sweepable.

Theorem 4.3 *If D' is a subdigraph of a digraph D, and the undirected graph underlying D' is connected, then* $\mathrm{sw}_{0,1}(D') \leqq \mathrm{sw}_{0,1}(D)$.

PROOF. Let $\mathrm{sw}_{0,1}(D) = k$, and let S be a strong sweep strategy that clears D with k sweepers. We will construct a strong sweep strategy S' to sweep D' with k sweepers. Initially, place the sweepers on vertices in $V(D')$ that correspond to vertices in $V(D)$. If a sweeper is supposed to be placed on a vertex of $V(D)$ that is not in $V(D')$, then place the sweeper on the first vertex it is supposed to move to that is in $V(D')$. If no such vertex exists, place the sweeper anywhere in $V(D')$.

Assume that the next action in S is to move a sweeper along the arc (u, v). If neither u nor v is in $V(D')$, do nothing. If u is in $V(D')$ but v is not, then do nothing. If v is in $V(D')$, but u is not, then since the undirected graph underlying D' is connected and sweepers may move against the direction of arcs, move the sweeper from where ever it is to the vertex v. If u and v are both in $V(D')$, and if (u, v) is an arc in $A(D')$, move the sweeper from u to v along the arc. If $(u, v) \not\in A(D')$, then since D' is weakly connected, move the sweeper from u to v along some path from u to v.

Since only arcs (and vertices) are removed, it is clear that when an arc (u, v) exists in both D' and D, then if S clears (u, v) in D, then S' must also clear (u, v) in D'. Similarly, if an arc is ever recontaminated in D' under S' is must also be recontaminated in D under S. ■

Unlike strong sweeping, directed sweeping allows for subdigraphs to have a higher sweep number than their superdigraphs. Recall the almost transitive tournament $TT_4^\star$. The directed sweep number of this digraph is merely 2. But if we remove the reversed arc to obtain a digraph D, it is not hard to prove that $\mathrm{sw}_{1,1}(D) = 3$, even though D is a subgraph of $TT_4^\star$.

We next consider the special case in which D is acyclic (here all of the strong components D_i are of order 1). The *arc digraph* $L(D)$ of a digraph D is the digraph where $V(L(D)) = A(D)$, and $((a, b), (c, d)) \in A(L(D))$ if and only if (a, b) and (c, d)

are arcs of D and $b = c$. The *width* $\mathrm{wd}(D)$ of a digraph D is the minimum number of directed paths in D whose union is $V(D)$. The following result characterizes the weak sweep number for acyclic digraphs [16].

Theorem 4.4 *If D is an acyclic directed graph, then* $\mathrm{sw}_{1,0}(D) = wd(L(D))$.

In sharp contrast to the weak sweep number, the strong sweep number is much smaller for acyclic digraphs.

Theorem 4.5 *If D is an acyclic digraph, then* $\mathrm{sw}_{0,1}(D) = 1$.

PROOF. Consider an acyclic ordering of D, where the vertices are labelled v_1, v_2, $\ldots$, v_n and if $(v_i, v_j) \in A(D)$, then $i < j$. Place the single sweeper on v_1. A single sweeper is sufficient to clear all arcs incident with v_1 as the sweeper may move against the direction of arcs. These arcs cannot be recontaminated, as the intruders cannot move against arc direction. Then repeat with v_2, and so on. ■

Interestingly, acyclic digraphs are not the only digraphs for which a strong sweep strategy exists using exactly one sweeper. For instance, consider the directed cycle $\overrightarrow{C_n}$. Place a single sweeper anywhere on $\overrightarrow{C_n}$, and begin moving against the direction of arcs. This clears the arc, and since the intruder cannot move against arc-direction, the arcs must remain clear. Thus $\mathrm{sw}_{0,1}(\overrightarrow{C_n}) = 1$.

Theorem 4.6 *If D is a strong digraph and $\sum_{v\in V(D)} d^+(v) \geqq n+2$, then* $\mathrm{sw}_{0,1}(D) \geqq 2$.

PROOF. Assume that $\mathrm{sw}_{0,1}(D) = 1$. Since D is strong, every vertex has non-zero in-degree and out-degree. This means that the only way to clear an arc using only one sweeper is to move along the arc in the direction opposite the arc. Also, since $\sum_{v\in V(D)} d^+(v) \geqq n + 2$, either two or more vertices have in-degree at least two, or exactly one vertex has in-degree at least three.

Case 1. The digraph D has at least two vertices with in-degree at least two. Let u and v be two vertices such that $d^-(u) \geqq 2$ and $d^-(v) \geqq 2$. Consider a strong sweep of D that uses exactly one sweeper. If the sweep begins at vertex $w \not\subseteq \{u, v\}$, then the sweeper begins clearing arcs until finally it reaches (without loss of generality) u. At this point, when the sweeper clears any arc incident u, the entire digraph cleared to this point becomes recontaminated. Thus, the sweep must begin at u.

If the sweep begins at u, then the sweeper clears arcs until it reaches v. Then if the sweeper clears any arc incident with u, all of the digraph cleared to this point becomes recontaminated. Thus, there cannot be two vertices with in-degree at least two.

Case 2. The digraph D has exactly one vertex with in-degree at least three. Let u be the vertex with $d^-(u) \geqq 3$. Consider a strong sweep of D that uses exactly one sweeper. If the sweep does not begin at u, then it must eventually reach u, at which point no further arcs can be cleared without recontaminating all arcs previously cleared. Thus, the sweep must begin at u. A sweep beginning at u must eventually reach u again, at which point there are at least two arcs that are contaminated. Neither of these arcs can be cleared without recontaminating all the arcs previously cleared. Thus, there cannot be a vertex with in-degree at least three. ∎

So, for a strong digraph D with $\mathrm{sw}_{0,1}(D) = 1$ that contains arcs, either $\sum_{v \in V(D)} d^+(v) = n$ or $\sum_{v \in V(D)} d^+(v) = n + 1$. If the former holds, since every vertex has non-zero in-degree and out-degree, then D must be an n-cycle. If the latter holds, then there must be exactly one vertex with in-degree two, and exactly one-vertex with out-degree two. Again there are two cases. If the vertex with in-degree two is the same as the vertex with out-degree two, then D is made up of two directed cycles with a single common vertex. If the vertex of in-degree two is distinct from the vertex of out-degree two, then D is a directed cycle $\overrightarrow{C_n}$ with a suspended directed path from one vertex of $\overrightarrow{C_n}$ to a distinct vertex of $\overrightarrow{C_n}$.

The idea of strong components allows us to define a special digraph.

Definition 4.7 The vertices of the strong components $D_1, D_2, \ldots, D_m$ of D partition V into sets and this partition is called the *strong decomposition* of D. The *strong component digraph* $S(D)$ is obtained by contracting each of the strong components of D to a single vertex and deleting any parallel arcs formed.

In particular, the strong component digraph is an acyclic digraph.

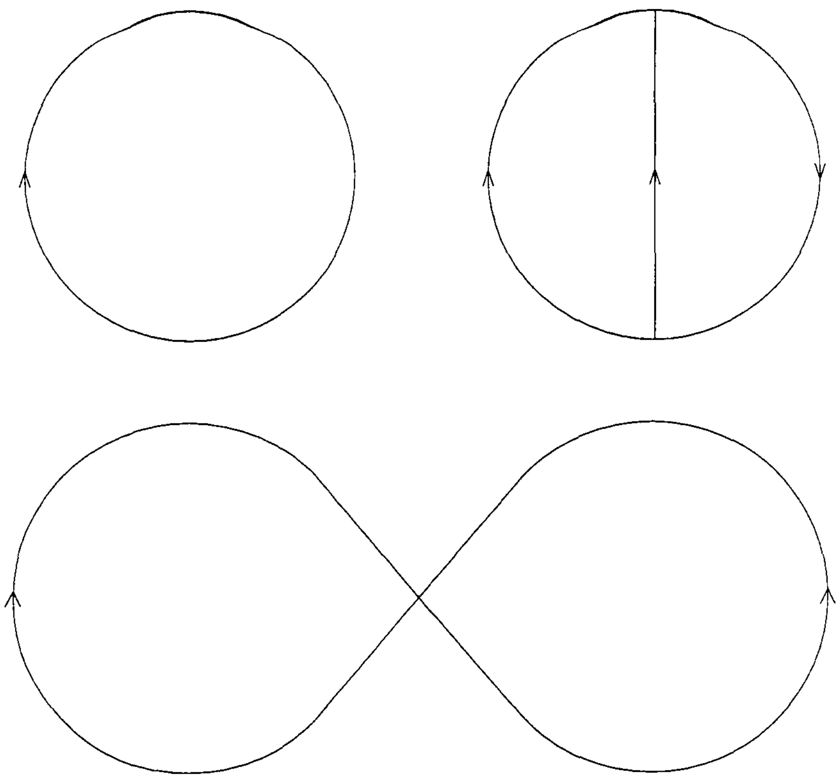

Figure 4.2: The non-trivial strong digraphs that are 1-strong-sweepable.

Theorem 4.8 *A digraph D is 1-strong-sweepable if and only if every strong component of D is one of the three digraphs described above or a single vertex.*

PROOF. If $\mathrm{sw}_{0,1}(D) = 1$ and D is strong, then it is either a vertex or one of the digraphs described above. On the other hand, if D is not strong, then it must have strong components. Since strong components are subdigraphs, by Theorem 4.3 they must have strong sweep number 1, and hence be as described above.

If D is digraph where every strong component is described as above or a single vertex, we consider the strong component digraph $S(D)$. Since $S(D)$ is acyclic, we consider an acyclic ordering, with vertices $D_1, D_2, \ldots, D_m$ ordered such that if (D_i, D_j) is an arc, then $i < j$. Of course, each D_i corresponds to a strong component in D. We construct a strong sweep strategy S for D as follows. If the next arc to be cleared in $S(D)$ is (D_i, D_j), then we clear all the arcs between the components D_i and D_j in D. Then if D_j has all incoming arcs cleared in $S(D)$ (and hence in D), use the single sweeper to clear the component D_j. The strategy S clears D with one sweeper and the result follows. ■

For the other sweep models, a characterization of 1-sweepable proves much easier. From [14], we know that those graphs with sweep number 1 are paths. This immediately gives the following corollary.

Corollary 4.9 *For a digraph D, $\mathrm{sw}_{0,0}(D) = 1$ if and only if D is an orientation of a path.*

In a directed sweep, there are only two ways to clear an arc as described earlier. One of the ways requires two sweepers, so any digraph which is 1-sweepable must only use the other. That is, a digraph that is directed 1-sweepable allows an arc (u, v) to be cleared only when incoming all arcs at u are already cleared.

Theorem 4.10 *For a digraph D, $\mathrm{sw}_{1,1}(D) = 1$ if and only if $D = \overrightarrow{P_n}$.*

PROOF. Certainly, $\mathrm{sw}_{1,1}(\overrightarrow{P_n}) = 1$. Now consider a digraph D that has directed sweep number 1. If (u, v) is the first arc cleared, then $d^-(u) = 0$, by the preceding comments. If any other vertices have in-degree 0, then the single sweeper cannot reach them. Thus, every other vertex must have non-zero in-degree, and u is the unique source of D, and the sweeper must begin at u.

Certainly, the only digraph of order two that has $\mathrm{sw}_{1,1}(D) = 1$ is $\overrightarrow{P_2}$. Proceeding by induction, we consider a digraph D with $\mathrm{sw}_{1,1}(D) = 1$ on n vertices. As argued above, the vertex that the single sweeper starts on must be the unique source u in the digraph. After the first move, there is no way to return to u, so the out-degree of u must be exactly one. Consider the digraph $D - \{u\}$. This is a digraph on $n - 1$ vertices that can be cleared by one sweeper by following the strategy of D. Thus, $D - \{u\}$ must be $\overrightarrow{P_{n-1}}$, and hence $D = \overrightarrow{P_n}$. ■

Finally, since weak sweeping is a restriction of directed sweeping, if a digraph D is such that $\mathrm{sw}_{1,0}(D) = 1$, then necessarily, $\mathrm{sw}_{1,1}(D) = 1$. Since directed paths can also be swept by a single sweeper in the weak sweeping model, we obtain the following corollary.

Corollary 4.11 *For a digraph D, $\mathrm{sw}_{1,0}(D) = 1$ if and only if $D = \overrightarrow{P_n}$.*

4.3 Strong Digraphs

As with digraphs, the strong decomposition of a reducible tournament is a partition of the vertex set into strong subtournaments $T_1, T_2, \ldots, T_m$ in which all vertices in T_i have arcs to all vertices in T_j whenever $i < j$. A special case of this occurs when each T_i is of size $|T_i| = 1$, that is, we have the transitive tournament TT_m on m vertices.

Suppose that for some D we have that $S(D)$ is a directed path P and that D_1, $D_2, \ldots, D_m$, for some $m > 1$, are the strong components of D. Let the weight of the path P be $w(P) = \max_{i,j} (\mathrm{sw}_{1,1}(D_i), d^+(D_j))$.

Theorem 4.12 *If $S(D)$ is a directed path P and $D_1, D_2, \ldots, D_m$, for some $m > 1$, are the strong components of D, then $\mathrm{sw}_{1,1}(D) = w(P)$.*

Proof. Clearly $\mathrm{sw}_{1,1}(D) \geqq \mathrm{sw}_{1,1}(D_i)$ for each i. On the other hand, in the obvious linear ordering of the D_j's, each arc joining a vertex of D_j to a vertex of D_{j+1} requires a different sweeper and $\mathrm{sw}_{1,1}(D) \geqq d^+(D_j)$. Suppose that indeed $w(P) = \mathrm{sw}_{1,1}(D_i)$ for some i, then we may place that many sweepers on D_1, clear it and then place at least as many sweepers on each vertex of D_1 as there are arcs from it to D_2. We then proceed to D_2 by traversing the arcs from D_1 to D_2, clear it and so on, using at most $w(P)$ sweepers in total. If instead for some j, $w(P) = d^+(D_j) > \mathrm{sw}_{1,1}(D_i)$ for all i, then these suffice to clear any D_i and the arcs to D_{i+1}, starting at $i = 1$, and the equality in the theorem follows. ■

We obtain an immediate corollary.

Corollary 4.13 *If $S(D)$ is a directed path P and $D_1, D_2, \ldots, D_m$, for some $m > 1$, are the strong components of D, then $\mathrm{sw}_{0,1}(D) = \max_i (\mathrm{sw}_{0,1}(D_i))$.*

A variation of the above is the following:

Corollary 4.14 *If the vertices of some tournament T can be partitioned into sets A and B in such a way that every vertex in A dominates every vertex in B, that is, T is reducible, then* $\mathrm{sw}_{1,1}(T) \geqq |A||B|$.

In the special case of D being a tournament in which each D_i is of size one we have the following result.

Corollary 4.15 *If $T = TT_n$ is a transitive tournament, then* $\mathrm{sw}_{1,1}(T) = \lfloor \frac{n^2}{4} \rfloor$.

PROOF. If $n = 2t$, then $\mathrm{sw}_{1,1}(T) = t^2$ by choosing A in Corollary 4.14 to be the t vertices of highest score. By sweeping the vertices in non-increasing order of their scores, we see that v_n requires $2t-1$ sweepers to clear all of its outgoing arcs; vertex v_{n-1} requires $2t-2$ sweepers to clear all of its outgoing arcs but a sweeper has been obtained from v_n thus we need only $2t-3$ new sweepers; vertex v_{n-2} requires $2t-3$ sweepers to clear all of its outgoing arcs but 2 sweepers have been obtained from v_n and v_{n-1} thus we need only $2t-5$ new sweepers; and so on. In total the number of sweepers required is t^2. If n is odd the result follows in a similar manner. ∎

Corollary 4.15 is especially interesting when considered in the light of the number of paths in a path decomposition of TT_n. A path decomposition of a digraph D is a partition of the arcs of D into a minimum number of directed paths. We denote this minimum number of paths $m(D)$. It was shown in [2] that for any tournament T, $m(T) \leqq \lfloor \frac{n^2}{4} \rfloor$, with equality holding if T is transitive. Thus, combined with Corollary 4.15, we see that a sweeper may sweep each path in the decomposition, so that every arc is traversed exactly once. Since this would be a strategy that clears the transitive tournament on n vertices with fewest arc traversals and the minimum number of sweepers, this can be considered an optimal strategy.

In marked contrast to Corollary 4.15, we note that for the T_n that differs from TT_n only in that vertex v_1 dominates vertex v_n, we have $\mathrm{sw}_{1,1}(T) = 2$ — we may leave a sweeper on v_n while we clear all of its outgoing arcs with just one other sweeper and then move on to other vertices in order. Corollary 4.15 may also be considered as a special case of the following.

Corollary 4.16 *If T is a reducible tournament with strong decomposition $T_1, T_2, \ldots, T_m$, then* $\mathrm{sw}_{1,1}(T) = \max_j \left(\sum_{i=1}^{j} |T_i|\right) \cdot \left(\sum_{i=j+1}^{m} |T_i|\right)$.

PROOF. By Corollary 4.14 we need only show that this number of sweepers suffice. It is not difficult to show that for any tournament R, $\mathrm{sw}_{1,1}(R) \leqq |R|$. Thus, by partitioning the vertices of T as $A = \sum_{i=1}^{j} |T_i|$ and $B = \sum_{i=j+1}^{m} |T_i|$, there are more than enough sweepers to clear the tournament induced by A first and, after the $|A||B|$ arcs between the parts are cleared, then clear the tournament induced by B. ■

In particular, for acyclic digraphs, every minimum path decomposition corresponds to a set of paths that contain all vertices in $L(D)$, so $\mathrm{wd}(L(D)) \leqq m(D)$.

Theorem 4.17 *If $D_1, D_2, \ldots, D_k$ are the strong components of D for some $k > 1$ and $S(D)$ has a path decomposition of size $m = m(S(D))$ into paths $P_1, P_2, \ldots, P_m$, then* $\mathrm{sw}_{1,1}(D) \leqq \sum_{i=1}^{m} w(P_i)$.

Theorem 4.18 *If T is a strong tournament of order n and has a maximum transitive subtournament TT_k of order k, then* $\mathrm{sw}_{1,1}(T) \leqq n - k + 1$.

PROOF. Place a sweeper on each of the vertices not in the TT_k. Use one extra sweeper to clear the arcs on the induced subtournament of order $n - k$. Next use this sweeper to clear arcs from the T_{n-k} to the TT_k (using the fact that T is strong). Finally, this extra sweeper can clear the vertices of the TT_k in order of non-increasing scores. ■

The order $k(n)$ of a maximum transitive subtournament in a strong tournament of order n has been well studied (dating back to Stearns [19] and Erdös and Moser [8]). Exact values are known only for values of $n \leqq 31$. In general, for $32 \leqq n \leqq 54$ it is known that $k(n)$ satisfies $\log_2\left(\frac{16n}{7}\right) \leqq k(n) \leqq 2\log_2 n + 1$, and for $n \geqq 55$ it is known that $\log_2\left(\frac{n}{5}\right) + 7 \leqq k(n) \leqq 2\log_2 n + 1$. Thus we have a bound on $\mathrm{sw}_{1,1}(T)$ that is better than $n - c$ for some constant c.

Corollary 4.19 *If T is a strong tournament or order n, then* $\mathrm{sw}_{1,1}(T) \leqq n - \log_2 n + 1$.

In contrast to Corollary 4.19, from Theorem 4.2 we know that $\mathrm{sw}_{1,1}(T) \geqq \delta^-(T)+1$. Tournaments exist for which this is large and realizable. For example, consider the family of regular tournaments constructed as follows. The wreath product of two tournaments S and T is obtained by replacing each vertex of S by a copy of T and if the arc uv is in S, then all arcs between the corresponding copies of T are in this same direction. If we repeatedly take the wreath product of the regular graph on three vertices T_3 with itself, we obtain a regular $T = T_N$ on $N = 3^n$ vertices with all scores equal to $\frac{3^n-1}{2}$. Thus, $\mathrm{sw}_{1,1}(T) \geqq \frac{N+1}{2}$. Inductively, we may see that this number of sweepers is sufficient as we may place a sweeper on every vertex of one copy of the $T_{\frac{N}{3}}$, clear all of its arcs with an extra sweeper, use this sweeper to clear all of the arcs out to the neighbouring $T_{\frac{N}{3}}$, use this sweeper and $\frac{3^{n-1}-1}{2}$ others on the second copy of $T_{\frac{N}{3}}$, and finally use these sweepers on the remaining arcs. In total we have used $3^{n-1}+1+\frac{3^{n-1}-1}{2} = \frac{3^n+1}{2}$ sweepers. It seems likely that regular families such as this might be among those that need a maximum number of sweepers when we consider the $\mathrm{sw}_{0,1}(T)$ problem but we have not resolved this at this time. In this example the largest transitive subtournament has order about 2^n so that Theorem 4.18 is not a good fit.

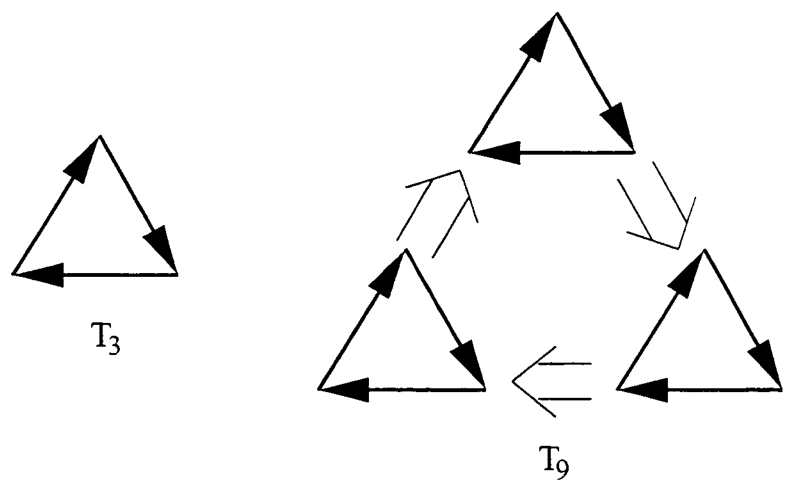

Figure 4.3: An infinite family of tournaments showing Theorem 4.2 can be tight.

Another family of regular tournaments for which there exists a transitive subtournament of order $\frac{n-1}{2}$ may be constructed as follows: Let $n = 2t+1$ and have vertices

0,1,.., $2t$. Let 0 dominate 1,2,.., t; 1 dominate 2,3, ..., $t+1$; 2 dominate 3, 4, ..., $t+2$, etc., where we work modulo n. This gives a t-regular tournament with a TT_k of order $\frac{n+1}{2}$ giving a tight fit to Theorem 4.2.

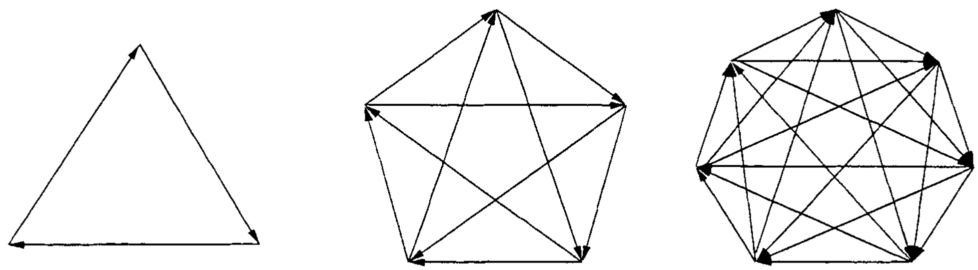

Figure 4.4: An infinite family of tournaments showing Corollary 4.19 can be tight.

Theorem 4.20 *If D is a strong digraph and has a minimum feedback vertex set F of size $n-k$, then* $\mathrm{sw}_{1,1}(D) \leqq n-k+1$.

PROOF. Place a sweeper on each of the vertices in F. Use one extra sweeper to clear the arcs on the F-induced directed graph. Use this sweeper to clear arcs from F to the remaining vertices (using the fact that D is strong). Finally, this extra sweeper can clear the vertices of the acyclic subgraph by sweeping in order of the acyclic ordering. ■

Naturally, F may be quite small in comparison to n. For example, if D is an n-cycle we have $n-k=1$, but since D may in fact be a tournament we cannot say more than Theorem 4.18 without adding much more knowledge of D.

We recall that the reversibility index $i_R(T)$ of a strong tournament T is the size of a minimum set of arcs whose reversal changes T into a reducible tournament. The tournament discussed after Corollary 4.15, which was almost transitive except for the existence of one special arc, is an example of a tournament with $i_R(T)=1$. For a particular strong tournament T, we will let $R(T)$ be a minimal set of vertices whose removal leaves a reducible subtournament T' with, say, strong decomposition $T_1, T_2, \ldots, T_m$ for some $m>1$. We will call this the reducibility vertex set. The set of initial vertices of some $i_R(T)$ arcs whose reversal changes T from being strong is an example of such a reducibility set. We extend the notion seen above in the example with $i_R(T) = 1$ as follows.

Theorem 4.21 *If T is a strong tournament, R is a minimum sized reducibility vertex set for T, and the strong decomposition of T' is $T_1, T_2, \ldots, T_m$ for some $m > 1$, then* $\mathrm{sw}_{1,1}(T) \leqq |R| + \max_j \mathrm{sw}_{1,1}(T_j)$.

PROOF. Place a sweeper on each vertex of R and use an extra sweeper to clear R. Use this sweeper to clear arcs from R to T'. Now $\max_j \mathrm{sw}_{1,1}(T_j)$ sweepers are sufficient to clear the rest of the tournament since T being strong implies that all arcs between successive T_i's can be cleared (sequentially) by a single sweeper once the first of these has been cleared. ■

An interesting question becomes, given a strong tournament T, what can we say about a minimum sized reducibility vertex set?

We may mirror the above for directed graphs. Analogously, we let R be a set of vertices whose removal leaves a graph D' with $S(D')$ having strong decomposition $D_1, D_2, \ldots, D_m$ for some $m > 1$, (a weakening vertex set for D).

Theorem 4.22 *If D is a strong digraph, R is a minimum sized weakening vertex set, and D' has strong decomposition $D_1, D_2, \ldots, D_m$ for some $m > 1$, then* $\mathrm{sw}_{1,1}(D) = |R| + \max_j \mathrm{sw}_{1,1}(D_j)$.

PROOF. Place a sweeper on each vertex of R. Use an extra sweeper to clear R, and then to clear arcs from R to D'. Now $\max_j \mathrm{sw}_{1,1}(D_j)$ are sufficient to clear the rest of the digraph since D being strongly connected implies that all arcs between D_i's can be cleared by a single sweeper — we clear using the acyclic ordering of $S(D')$. ■

Chapter 5

Further directions

There are still an incredible number of open problems in graph sweeping. While we have answered one of the open problems in [4], another remains. We know that all the inequalities in Lemma 1.3 can be strict, and in fact the differences can be arbitrarily large. But what of the ratios? Barriere et al. proved that for any tree T, $\mathrm{mksw}(T) < 2\mathrm{sw}(T) - 2$. Similarly, it seems that the ratio of monotonic connected sweep number to sweep number will be less than 2 in the special case of graphs where are cycles are disjoint. Whether the ratio of monotonic connected sweep number to sweep number will always be less than 2 remains open.

Now that we know each of the inequalities in Lemma 1.3 can be strict, another problem becomes to find the graph(s) of least order that exhibit these properties. Certainly, the graphs X, Y, and W can be scaled down, but even then they may not be the graphs of least order. Similarly, the Y-square is the currently known least order graph with sweep number strictly less than monotonic sweep number (and is smaller than the previously known example), but it may not be the smallest possible. Several things must be true of the graph of least order with sweep number strictly less than monotonic sweep number. For example, the graph must have sweep number of at least 3 (and hence monotonic sweep number of at least 4) and it cannot be a tree.

A related problem is classification of sweep number-critical graph — that is, finding those graphs where the removal of any single edge reduces the sweep number. Certainly, such graphs exist. The complete graph on n vertices is a critical graph, as removing any edge lowers the sweep number. Similarly, the n-cycle is a critical graph, and the literature contains an infinite family of trees that is also critical.

We know that if H is a minor of a graph G, then $\mathrm{sw}(H) \leqq \mathrm{sw}(G)$. As was shown with the graphs X, X', Y, and Y', the same does not hold for either of the monotonic or connected sweep numbers. But, now that we know the monotonic connected sweep number can be strictly greater than the connected sweep number, we may pose the same question: If H is a minor of G, then is $\mathrm{mksw}(H) \leqq \mathrm{mksw}(G)$?

As to digraphs, there is a great deal to be done to parallel the directions already taken with graphs. A characterization of 1-sweepable digraphs was presented here, but for graphs, characterizations are known for 1-,2-, and 3-sweepable. However, from preliminary work, even 2-sweepable looks difficult for digraphs.

As well, the ideas of monotonicity and connectedness also complicate the idea of sweeping directed graphs. For large n, the transitive tournament has directed sweep number $\sim \frac{n^2}{4}$, and in fact the monotonic sweep number is the same. However, the almost transitive tournament on the same number of vertices has directed sweep number 2, but no monotonic sweep strategy exists for two sweepers. As well, introducing the "wormhole jump" into directed sweeping will also complicate matters. The directed wormhole sweep number of any acyclic graph is 1, but is generally much higher without allowing this type of move.

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