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Integrability of Moufang Foundations

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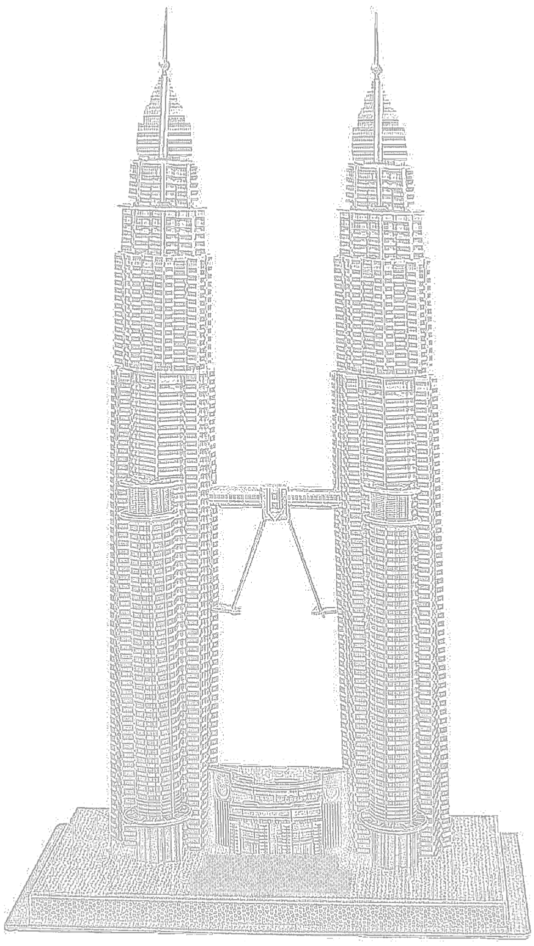
Wer fragt, ist ein Narr für fünf Minuten. Wer nicht fragt, bleibt ein Narr für immer.

Aus China

Integrability of Moufang Foundations

A Contribution to the Classification of Twin Buildings

Dedicated to My Siblings



Die Zukunft gehört denjenigen, die an die Wahrhaftigkeit ihrer Träume glauben.

Eleanor Roosevelt

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Introduction

Historical and Theoretical Context

The description below closely follows those given in [MLoc] and [AB].

Twin Buildings

Buildings have been introduced by J. Tits in order to study semi-simple algebraic groups from a geometrical point of view. One of the most important results in the theory of buildings is the classification of irreducible spherical buildings of rank at least 3 in [T74]. Meanwhile, there is a simplified proof in [TW] which makes use of the classification of Moufang polygons.

About 25 years ago, M. Ronan and J. Tits defined a new class of buildings, which generalize spherical buildings in a natural way, namely the class of twin buildings. The motivation of their definition is provided by the theory of Kac-Moody groups, and we refer to [T92] for further general information about twin buildings.

The sense in which twin buildings generalize spherical buildings is the following: Given a building of spherical type, there is a natural *opposition relation* on the set of its chambers. This relation restricts the structure of spherical buildings essentially. The classification of irreducible spherical buildings of rank at least 3 mentioned above is in fact based on this opposition relation. The idea of a twin building is to introduce a symmetric relation between the chambers of two different buildings of the same type which has properties similar to the opposition relation of a spherical building. Thus a twin building is a triple consisting of two buildings of the same type and an opposition relation between the chambers of the two "halves" of the twin building.

The Classification Program for 2-Spherical Twin Buildings

In view of the classification of spherical buildings, it is natural to ask whether it is possible to classify higher rank twin buildings. A large part of [T92] deals with this question. As a first observation, it turns out that such a classification seems only to be feasible under the additional assumption that the entries in the corresponding Coxeter matrices are all finite. We call these buildings *2-spherical*. The classification program described in [T92] is based on the conjecture that there is a bijective correspondence between twin buildings of type M and certain Moufang foundations of type M for each 2-spherical Coxeter diagram of type M.

Foundations have been introduced by M. Ronan and J. Tits in [RT] in order to describe chamber systems which are candidates for being the local structure of a building. Roughly speaking, foundations can be seen as amalgams of rank 2 buildings which are glued along certain rank 1 residues. Given a chamber c of a building $\mathcal{B}$ of type M, the union $E_2(c)$ of the rank 2 residues which contain this chamber constitutes a foundation of type M, the *foundation of $\mathcal{B}$ at c*. Thus the term "local structure" above has to be understood as a kind of 2-neighbourhood of a given chamber of a building.

It is a (not completely trivial) fact that if two chambers are contained in the same half of a twin building, the foundations at these chamber are isomorphic. Moreover, if one knows the isomorphism class of the foundation of one half of a twin building, then the isomorphism class of the foundations of the other half is uniquely determined. Conversely, a generalization of Tits' extension theorem by B. Mühlherr and M. Ronan in [MR] states that a twin building is uniquely determined by the foundation of one of its halves in almost all cases, cf. (5.10), (*5.11), (*9.11) and (*9.12) of [AB] for a summary. Thus the foundation at a chamber of a twin building is a classifying invariant of the corresponding twin building if the following condition is satisfied:

(CO) No rank 2 residue is isomorphic to one of the buildings which are associated with the groups $B_2(2)$, $G_2(2)$, $G_2(3)$ and ${}^2F_4(2)$.

This condition guarantees that for every chamber $c \in \mathcal{B}_\epsilon$ ($\epsilon \in \{\pm\}$), the set c^o of chambers opposite c is a gallery-connected subset of $\mathcal{B}_{-\epsilon}$.

In view of what has been mentioned so far, the classification of 2-spherical twin buildings reduces to the classification of all foundations which can be realized as the local structures of a twin building. We call such a foundation *integrable*. In order to determine the integrable foundations, one proceeds in two steps.

Step 1: Exclude Non-Integrable Foundations

It is proved in [T92] that an integrable foundation is Moufang, which means that the rank 2 buildings in the foundation are Moufang, i.e., they are Moufang polygons, and that the glueings are compatible with the Moufang structures induced on the rank 1 residues. Thus a first necessary condition for the integrability of a foundation is that it is Moufang.

As a consequence, the classification of Moufang polygons in [TW] and the solution of the isomorphism problem for Moufang sets are essential to work out which Moufang polygons fit together in order to form a foundation. Moreover, one can reduce the list of possibly integrable foundations by considering certain automorphisms of the twin building, the so-called *Hua automorphisms*, which are closely related to the double μ-maps of the appearing Moufang sets.

Step 2: Existence / Integrability Proof

Finally, one has to prove that each of the remaining candidates is in fact integrable, i.e., realized by a twin building, which is then unique up to isomorphism. In [MLoc] and his Habilitationsschrift [MHab], B. Mühlherr developed techniques which produce certain twin buildings as fixed point structures in twin buildings coming from Kac-Moody groups. He, H. Petersson and R. Weiss actually prepare a book which provides further well-founded background.

Goals and Main Results

The present thesis contributes to establish complete lists of integrable foundations for certain types of diagrams. We closely follow the approach for the classification of spherical buildings in [TW]. However, we have to refine the techniques, since in general, foundations don't only depend on the diagram and the defining field. For example, there may be several non-isomorphic foundations of type $\tilde{A}_n$ with respect to a given skew-field $\mathbb{A}$: Automorphisms of $\mathbb{A}$ are involved as well, which represents the fact that there are several possibilities for glueing Moufang polygons along a rank 1 residue.

The main question is how to parametrize sequences of Moufang polygons with respect to the usual commutator relations in order to make the glueings visible. The crucial subtlety is the following: Each Moufang polygon is parametrized twice, once for each direction in which the underlying root group sequence can be read. As a consequence, we obtain glueings between directed Moufang polygons, and it's a difference whether we look at $\mathrm{id}_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$ or $\mathrm{id}_{\mathbb{A}}^o : \mathbb{A} \to \mathbb{A}^o$, where $\mathbb{A}^o$ is the opposite with respect to $\mathbb{A}$: The former is an isomorphism, while the latter is an anti-isomorphism of skew-fields.

As mentioned above, excluding non-integrable foundations is closely related to the investigation of Moufang sets and their isomorphisms. Therefore, a large part deals with the introduction of underlying parameter systems and, in the sequel, with the solution of the isomorphism problem for Moufang sets. Many questions have already been answered, cf. [K], but we need to refine and extend the existing results for our purposes and translate their proofs into our setup.

Simply Laced Foundations

The main result of this thesis is the complete classification of simply laced twin buildings via their foundations. Of course, the basic requirement for a foundation to be integrable is that it is Moufang: Its glueings are Jordan isomorphisms, i.e., they preserve the Jordan product xyx.

A powerful tool is Hua's theorem, cf. [H] for a reference, which answers the isomorphism problem for Moufang sets of skew-fields: Each Jordan isomorphism is in fact an iso- or anti-isomorphism of skew-fields. However, the class of parameter systems for Moufang triangles additionally includes octonion division algebras, which cause a lot of trouble due to the lack of associativity. A byproduct is the existence of Jordan isomorphisms which are neither iso- nor anti-isomorphisms of alternative rings. The most sophisticated part is the handling of the exceptional cases where octonions occur.

We give an overview of the restriction process and point out the main ideas.

The following observations yield the first restriction of possibilities:

(1) Each Moufang triangle is defined over the same alternative division ring $\mathbb{A}$.

(2) An integrable foundation of type A_3 is necessarily defined over a skew-field, and the glueing is necessarily an isomorphism of skew-fields.

Thus the crucial step is the classification of integrable foundations of type $\tilde{A}_2$ since these are the smallest ones which allow weird "non-standard" things to appear. The theory of affine and Bruhat-Tits buildings and the theory of composition algebras which are complete with respect to a discrete valuation enable us to get further restrictions:

(3) Given an octonion division algebra $\mathbb{O}$, there is only one twin building of type $\tilde{A}_2$ with respect to $\mathbb{O}$.

(4) An integrable foundation of type $\tilde{A}_2$ whose glueings are anti-isomorphisms is necessarily defined over a quaternion division algebra, and given a quaternion division algebra $\mathbb{H}$, there is only one such "positive" twin building of type $\tilde{A}_2$ with respect to $\mathbb{H}$.

A closer look at the group of Jordan automorphisms of octonion division algebras completes the classification of integrable foundations which are defined over octonions:

(5) There are no integrable foundations over octonions such that the corresponding graph is a tetrahedron. In particular, up to isomorphism, the only integrable foundations with respect to an octonion division algebra $\mathbb{O}$ are $\mathcal{A}_2(\mathbb{O}) = \mathcal{T}(\mathbb{O})$ and $\tilde{\mathcal{A}}_2(\mathbb{O})$.

Finally, in connection with (4), the following observation heavily restricts the list of integrable foundations over non-commutative skew-fields which are not quaternion division algebras:

(6) An integrable foundation of type D_4 is necessarily defined over a field.

Kac-Moody theory provides the integrability proofs as the corresponding Coxeter diagram is a tree. The remaining integrability proofs rely on techniques developed by B. Mühlherr.

Jordan Automorphisms of Alternative Division Rings

In view of Hua's theorem

$$\mathrm{Aut}_J(\mathbb{D}) = \mathrm{Aut}(\mathbb{D}) \cup \mathrm{Aut}^o(\mathbb{D})$$

for any skew-field $\mathbb{D}$, its group $\mathrm{Aut}_J(\mathbb{D})$ of Jordan automorphisms, its subgroup $\mathrm{Aut}(\mathbb{D})$ of automorphisms and its set $\mathrm{Aut}^o(\mathbb{D})$ of anti-automorphisms, the question arises whether it is possible to get a similar result for octonion division algebras.

In the proof that integrable tetrahedron-foundations over an octonion division algebra $\mathbb{O}$ do not exist, we define a subset $\Gamma \subseteq \mathrm{Aut}_J(\mathbb{O})$ which turns out to not contain the standard involution σ_s. The elements of Γ are automorphisms of $\mathbb{O}$ multiplied with one of the "exceptional" Jordan automorphisms as defined in [TW], which fix a quaternion subalgebra $\mathbb{H}$ pointwise and which act on the orthogonal complement of $\mathbb{H}$ as conjugation.

The fact that Γ is a subgroup of $\mathrm{Aut}_J(\mathbb{O})$ can be deduced from the knowledge about the automorphism group of the corresponding Moufang triangle $\mathcal{T}(\mathbb{O})$. This subgroup Γ corresponds to the subgroup $\mathrm{Aut}(\mathbb{D})$ in Hua's theorem, i.e., we obtain

$$\mathrm{Aut}_J(\mathbb{O}) = \langle \sigma_s, \Gamma \rangle = \Gamma \cup \sigma_s \Gamma \ .$$

The strategy for the proof is as follows:

(1) Jordan automorphisms restricted to subfields are monomorphisms of rings, i.e., the image of a subfield is again a subfield.

(2) As an immediate consequence, Jordan automorphisms of octonions are norm similarities.

(3) The results of [Sp] allow us to restrict to isometries which fix a quaternion subalgebra pointwise.

(4) Hua's theorem and the Skolem-Noether theorem allow us to show that any Jordan automorphism is indeed a product in $\langle \sigma_s, \Gamma \rangle$.

443-Foundations

The second result in connection with the classification of twin buildings is the completion of step 1 for 443-foundations whose diagram is a triangle and whose Moufang polygons are two quadrangles and one triangle. Although we only deal with a single diagram in this case, there is a rich variety of integrable 443-foundations as there are six families of Moufang quadrangles which often fit together in this configuration. Nevertheless, quadrangles of type E_n, of type F_4 and of indifferent type don't appear since their Moufang sets are not of linear type, i.e., they aren't projective lines.

The same holds for Moufang sets of pseudo-quadratic form and involutory type, but the second panel of the corresponding *unitary* quadrangle is of linear type so that there is exactly one possibility for the orientation of the quadrangles. The solution of the isomorphism problem for the appearing Moufang sets and the knowledge about the automorphism group of a unitary quadrangle allow us to show the following:

(1) The appearing pseudo-quadratic spaces are defined over a quaternion division algebra $\mathbb{H}$ or over a separable quadratic extension $\mathbb{E}$.

(2) In the former case, there is exactly one possibly integrable 443-foundation with respect to such a pseudo-quadratic space Ξ.

(3) In the latter case, the isomorphism class of a possibly integrable 443-foundations additionally depends on an automorphism $\gamma \in \mathrm{Aut}(\mathbb{E})$.

(4) The appearing involutory sets are defined over a quaternion division algebra $\mathbb{H}$, and there is exactly one possibly integrable 443-foundation with respect to such an involutory set Ξ.

Finally, quadrangles of quadratic form type are the most flexible ones since there are Moufang sets which are both of quadratic form type and of linear type so that they can glued together in any orientation. Furthermore, there is one point where we need to restrict to proper quadratic spaces as parametrizing structures to exclude characteristic 2 phenomenons in order to obtain a satisfying description.

In contrast to the classification of integrable simply laced foundations however, we omit step 2 in the classification program as the proofs require different kinds of techniques, established by B. Mühlherr, H. Petersson and R. Weiss. As before, there are two possibilities how to show the integrability of a given foundation: Either the universal cover is isomorphic to a canonical foundation, which is a foundation such that each glueing is the identity map and thus integrable if it the corresponding diagram is a tree, or the foundation can be obtained as a fixed point structure via a Tits index. The former method applies to 443-foundations with quadrangles of quadratic form type, while the latter applies to 443-foundations with unitary quadrangles.

Jordan Isomorphisms of Pseudo-Quadratic Spaces

Hua's theorem is essential for the classification of integrable simply laced foundations. In the same spirit, the solution of the isomorphism problem for the appearing Moufang sets is essential for the classification of integrable 443-foundations. As mentioned above, R. Knop handles a lot of cases in his PhD thesis [K]. However, he only deals with commutative Moufang sets. Thus we need to establish the corresponding results for Moufang sets of pseudo-quadratic form type.

We obtain that Jordan isomorphism between two Moufang sets of pseudo-quadratic form type are induced by isomorphisms of the corresponding pseudo-quadratic spaces in almost all cases, i.e., whenever the dimension is at least 3 or the involved involutory set is proper. As a consequence, exceptions necessarily involve pseudo-quadratic spaces of small dimensions which are defined over a quaternion division algebra or over a separable quadratic extension. Luckily, these exceptional cases don't occur in the classification of integrable 443-foundations. As a consequence, both the quadrangles are defined over the same pseudo-quadratic space Ξ.

Outlook and Open Problems

Jordan Isomorphisms

In the theory of Moufang sets, the μ-maps and the Hua maps play a central role as they carry a lot of information. As a consequence, Jordan isomorphisms – which are additive isomorphisms preserving the Hua maps – are closely related to isomorphisms of Moufang sets. In fact, each isomorphism of Moufang sets is a Jordan isomorphism since the Hua maps can be expressed in terms of sums and the permutation τ.

In this context, the following question naturally arises: Is each Jordan isomorphism an isomorphism of Moufang sets? Of course, the Hua maps of sharply 2-transitive Moufang sets are trivial. Therefore, the question has to be answered negatively for these “non-proper” Moufang sets. But experts in the area such as R. Weiss and T. De Medts are optimistic that both the definitions are equivalent if we restrict to proper Moufang sets.

The Classification Program

The main conjecture in connection with the classification program is the following, cf. page 5 in [MHab]:

> A Moufang foundation of 2-spherical type is integrable if and only if each of its rank 3 residues is integrable.

In his Habilitationsschrift [MHab], B. Mühlherr indicates that one could prove the conjecture under the additional assumption that all rank 3 residues are spherical, which is of course a severe restriction. However, there isn't any written proof yet.

Once one has proved the conjecture, the classification program reduces to the classification of integrable Moufang foundations of rank 3. Most of them can be handled with the methods established in [MHab] and [MLoc]. However, there are some exceptions, the most complicated of which are foundations of type $\tilde{C}_2$, $\tilde{A}_2$ and 443-foundations. The $\tilde{A}_2$- and the 443-case are solved in the present thesis, while there are (unpublished) partial results for the $\tilde{C}_2$-case by T. De Medts, B. Mühlherr, H. Van Maldeghem and R. Weiss.

The Classification of Simply Laced Twin Buildings

Although the classification of integrable simply laced foundations is complete, we don't make any statement whether two given foundations in our list are isomorphic. By taking classifying invariants into account and introducing suitable parameters, one could create a list with pairwise non-isomorphic foundations.

If the underlying Coxeter diagram $\mathcal{G}_F$ is a tree, the foundations $\mathcal{F}$ depends only on the defining field. Circles in the diagram cause an additional dependence on “twists", i.e., on automorphisms of the defining field $\mathbb{A}$. More precisely:

- If $\mathbb{A}$ is a field, an integrable foundation $\mathcal{F}$ is uniquely determined by $\mathcal{G}_F$ and a homomorphism $\varphi : \Pi_1(\mathcal{G}_F) \to \mathrm{Aut}(\mathbb{A})/\mathrm{Inn}(\mathbb{A}) \cong \mathrm{Aut}(\mathbb{A})$, where $\Pi_1(\mathcal{G}_F)$ is the fundamental group of $\mathcal{G}_F$.

- If $\mathbb{A}$ is a skew-field distinct from a quaternion division algebra and $\mathcal{F}$ is an integrable foundation of type $\tilde{A}_n$, the foundation is uniquely determined by n and an element of $\mathrm{Aut}(\mathbb{A})/\mathrm{Inn}(\mathbb{A})$.

- If the defining field is a quaternion division algebra, then a similar result as in the field case holds.

Moreover, the integrability proofs might be improved at some points as soon as the applied theory is developed properly by B. Mühlherr, H. Petersson and R. Weiss.

Finite Moufang Foundations

The introduced terminology and the methods of [MHab] can be used to show that each locally finite twin building of 2-spherical type is the fixed point building of a Galois action in the sense of B. Rémy, which means that it is of algebraic origin.

Acknowledgments

I would like to express my gratitude towards my primary advisor Bernhard Mühlherr for drawing my attention to the interesting field of Moufang foundations and their Moufang sets: It was a pleasure to contribute to the classification program for twin buildings. Many fruitful discussions showed me the right direction, i.e., the assertions that one should be able to prove and the way how to achieve it. His intuition is tremendous.

Furthermore, I would like to thank Richard Weiss, who raised the question for the generalization of Hua's theorem and who laid the foundation for this thesis with his wonderful and detailed work on Moufang polygons, spherical and affine buildings as well as their classification. Many little questions could be answered with the aid of [TW], and if not, he always had the right idea where to look for the solution.

Ralf Köhl enhanced the work group with many nice people and gave me the opportunity to take part in a research project about compact subgroups of Kac-Moody groups. His enthusiasm and his dedication are impressive and build the basis for our prospering group.

Thanks go also to the following people: Tom De Medts who provided me a pleasant stay in Ghent during which we worked on the isomorphism problem for Moufang sets of pseudo-quadratic form type. And besides my family finally, there are so many friends who enriched my life with time, conversations and activities that were an essential contrast to the "abstract nonsense" called mathematics.

The most important factor which made the last five years a wonderful time of my life is the following: Bernhard and Ralf both endowed me with maximal flexibility in my academic work. In particular, this allowed me and two old friends of mine, Steffen Presse and Joram Gornowitz, to realize our ambitious cinema project.

Part I

Preliminaries

As we will have to reconstruct structure out of some given identities and as we will make use of several structures at the same time, it is important to have exact definitions and notations and to make precise statements, e.g., it is important to know whether we talk about an isomorphism as an isomorphism of algebras or as an isomorphism of vector spaces, and we avoid “identification”.

We start with giving the definitions of elementary structures such as vector spaces, algebras and graphs, then we proceed by introducing Coxeter matrices and Coxeter diagrams before we turn to twin buildings themselves.

Chapter 1 Notations

We fix some notations.

(1.1) Definition (Vector Spaces)

- A *(right) vector space* is a pair $(V, \mathbb{K})$ consisting of a commutative group V and a skew-field $\mathbb{K}$ together with a *scalar multiplication* $\cdot : V \times \mathbb{K} \to V$ satisfying
$$\forall\, v \in V : \; v \cdot 1_{\mathbb{K}} = v \;, \qquad \forall\, v \in V,\; s,t \in \mathbb{K} : \; (v \cdot s) \cdot t = v \cdot (st)$$
and
$$\forall\, v,w \in V,\; s,t \in \mathbb{K} : \qquad (v+w) \cdot s = v \cdot s + w \cdot s \;, \qquad v \cdot (s+t) = v \cdot s + v \cdot t \;.$$
- If $(V, \mathbb{K})$ is a vector space, we say that V *is a* $\mathbb{K}$*-vector space* or that V *is a vector space over* $\mathbb{K}$.
- Two vector spaces $(V, \mathbb{K})$ and $(\tilde{V}, \tilde{\mathbb{K}})$ are *isomorphic* if there is a pair (φ, ϕ) of isomorphisms $\varphi : V \to \tilde{V}$ and $\phi : \mathbb{K} \to \tilde{\mathbb{K}}$ of groups and skew-fields, resp., satisfying
$$\forall\, s \in \mathbb{K},\; v \in V : \qquad \varphi(v \cdot s) = \varphi(v) \cdot \phi(s) \;.$$
- If $(\varphi, \phi) : (V, \mathbb{K}) \to (\tilde{V}, \tilde{\mathbb{K}})$ is an isomorphism of vector spaces, we say that φ is a ϕ*-isomorphism*.
- Let $(V, \mathbb{K})$ be a vector space.
  - An automorphism (φ, ϕ) of $(V, \mathbb{K})$ with $\phi = \mathrm{id}_{\mathbb{K}}$ is *linear*.
  - We denote the *group of (semi-linear) automorphisms of* $(V, \mathbb{K})$ by $\Gamma L(V, \mathbb{K})$.
  - We denote the *group of linear automorphisms of* $(V, \mathbb{K})$ by $GL(V, \mathbb{K})$.

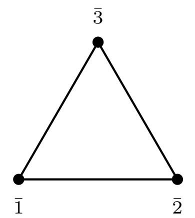
(1.2) Definition (Algebras)

- An *algebra* is a pair $(A, \mathbb{K})$ such that A is a vector space over a field $\mathbb{K}$ together with a $\mathbb{K}$-bilinear map $\cdot : A \times A \to A$.
- If $(A, \mathbb{K})$ is an algebra, we say that A *is a* $\mathbb{K}$*-algebra* or that $\mathbb{A}$ *is an algebra over* $\mathbb{K}$.
- An algebra $(A, \mathbb{K})$ is *associative* if the map $\cdot : A \times A \to A$ is associative.
- Two algebras $(A, \mathbb{K})$ and $(\tilde{A}, \tilde{\mathbb{K}})$ are *isomorphic* if there is an isomorphism (φ, ϕ) of vector spaces satisfying
$$\forall\, x,y \in A : \qquad \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) \;.$$
- If $(\varphi, \phi) : (A, \mathbb{K}) \to (\tilde{A}, \tilde{\mathbb{K}})$ is an isomorphism of algebras, we say that φ is a ϕ*-isomorphism*.
- Let $(A, \mathbb{K})$ be an algebra.
  - An automorphism (φ, ϕ) of $(A, \mathbb{K})$ with $\phi = \mathrm{id}_{\mathbb{K}}$ is *linear*.
  - We denote the *group of (semi-linear) automorphisms of* $(A, \mathbb{K})$ by $\mathrm{Aut}(A, \mathbb{K})$.
  - We denote the *group of linear automorphisms of* $(A, \mathbb{K})$ by $\mathrm{Aut}_{\mathbb{K}}(A, \mathbb{K})$.

(1.3) Definition (**Graphs**)

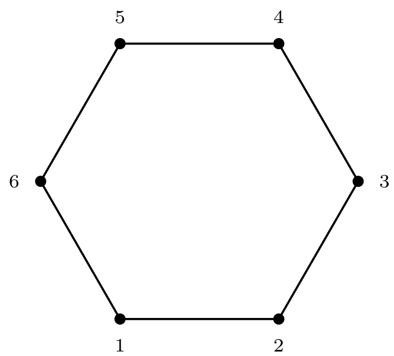
- A *graph* is a pair $\mathcal{G} = (V, E) = \big(V(\mathcal{G}), E(\mathcal{G})\big)$ consisting of a set of *vertices* V and a set of *edges*
$$E \subseteq \binom{V}{2} := \{X \subseteq V \mid |X| = 2\} \ .$$
- Given two graphs $\mathcal{G} = (V, E)$, $\tilde{\mathcal{G}} = (\tilde{V}, \tilde{E})$, a *morphism* $\varphi : \mathcal{G} \to \tilde{\mathcal{G}}$ *of graphs* is a map $\varphi : V \to \tilde{V}$ such that
$$\forall\ v, w \in V : \qquad \{v, w\} \in E \ \Rightarrow\ \{\varphi(v), \varphi(w)\} \in \tilde{E} \ .$$
- Given a graph $\mathcal{G} = (V, E)$, we set
$$A(\mathcal{G}) := \{(i, j) \in V^2 \mid \{i, j\} \in E\} \ ,$$
which is the set of directed edges, and
$$G(\mathcal{G}) := \{(i, j, k) \in V^3 \mid (i, j) \neq (k, j) \in A(\mathcal{G})\} \ .$$
- Given a graph $\mathcal{G} = (V, E)$ and a vertex $v \in V$, the *set of neighbours of* v is
$$B_1(v) := \{w \in V \mid \{v, w\} \in E\} \ .$$
- Given a graph $\mathcal{G} = (V, E)$, a *cover of* $\mathcal{G}$ is a pair $(\tilde{\mathcal{G}}, \varphi)$ consisting of a graph $\tilde{\mathcal{G}} = (\tilde{V}, \tilde{E})$ and an epimorphism $\varphi : \tilde{V} \to V$ of graphs such that for each $v \in \tilde{V}$, the map
$$\varphi_{|B_1(v)} : B_1(v) \to B_1\big(\varphi(v)\big)$$
is a bijection.

(1.4) Example Given the graph

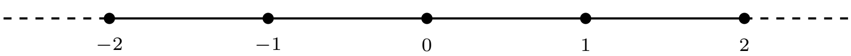


,

the graphs



and



are covers, where
$$\varphi : \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}, \ z \mapsto \bar{z}$$
is the natural homomorphism in both cases.

(1.5) Remark We only deal with Coxeter matrices such that $m_{ij} \neq \infty$ for all $i, j \in I$.

(1.6) Definition (**Coxeter Matrices**)

- A *(2-spherical) Coxeter matrix over (an index set)* I is a map $M : I \times I \to \mathbb{N}^*$ such that
$$\forall\ i \in I :\ m_{ii} = 1\ , \qquad \forall\ i \neq j \in I :\ m_{ji} = m_{ij} > 1\ ,$$
where $m_{ij} := M(i,j)$ for all $i, j \in I$.

- Given two Coxeter matrices M over I and $\tilde{M}$ over $\tilde{I}$, a *morphism* $\varphi : M \to \tilde{M}$ *of Coxeter matrices* is a map $\varphi : I \to \tilde{I}$ such that
$$\forall\ i, j \in I : \qquad \tilde{m}_{\varphi(i)\varphi(j)} = m_{ij}\ .$$

- Given a Coxeter matrix M over I and a subset $J \subseteq I$, we set $M_J := M_{|J \times J}$.

(1.7) Definition (**Coxeter Diagrams**)

- A *(2-spherical) Coxeter diagram* is a pair $(\mathcal{G}, \nu)$ consisting of a graph $\mathcal{G}$ and a map $\nu : E(\mathcal{G}) \to \mathbb{N}_{\geq 3}$.

- Given two Coxeter diagrams $(\mathcal{G}, \nu)$ and $(\tilde{\mathcal{G}}, \tilde{\nu})$, a *morphism* $\varphi : (\mathcal{G}, \nu) \to (\tilde{\mathcal{G}}, \tilde{\nu})$ *of Coxeter diagrams* is a morphism $\varphi : \mathcal{G} \to \tilde{\mathcal{G}}$ of graphs such that
$$\forall\ \{i,j\} \in E(\mathcal{G}) : \qquad \tilde{\nu}(\{\varphi(i), \varphi(j)\}) = \nu(\{i,j\})\ .$$

(1.8) Remark

(a) Given a Coxeter diagram $(\mathcal{G}, \nu)$, we indicate an edge such that $\nu(\{i,j\}) = 3$ by a single edge, an edge $\{i,j\}$ such that $\nu(\{i,j\}) = 4$ by a double edge, an edge $\{i,j\}$ such that $\nu(\{i,j\}) = 6$ by a triple edge and an edge $\{i,j\}$ such that $\nu(\{i,j\}) = 8$ by a quadruple edge.

(b) Given a Coxeter matrix M over I, the corresponding *Coxeter diagram* is $\Pi_M := \big(\mathcal{G}_M, \nu_M\big)$ with $V(\mathcal{G}_M) := I$ and
$$\forall\ i, j \in I :\ \{i,j\} \in E(\mathcal{G}_M)\ :\Leftrightarrow\ m_{ij} \geq 3\ , \qquad \forall\ \{i,j\} \in E(\mathcal{G}_M) :\ \nu_M(\{i,j\}) := m_{ij}\ .$$
We set
$$V(\mathcal{G}) := V(\mathcal{G}_M)\ , \qquad E(\mathcal{G}) := E(\mathcal{G}_M)\ , \qquad A(\mathcal{G}) := A(\mathcal{G}_M)\ , \qquad G(\mathcal{G}) := G(\mathcal{G}_M)\ .$$

(c) Let CM be the set of Coxeter matrices and let CD be the set of Coxeter diagrams. Then the map
$$\Pi : CM \to CD,\ M \mapsto \Pi_M$$
is a bijection such that $M \cong \tilde{M}\ \Leftrightarrow\ \Pi_M \cong \Pi_{\tilde{M}}$.

(1.9) Definition (**Coxeter Systems**) Let M be a Coxeter matrix over I.

- The *Coxeter group of type* M is the group
$$W_M := \langle \{r_i \mid i \in I\} \mid \{(r_i r_j)^{m_{ij}} = 1 \mid i, j \in J\} \rangle\ .$$

- The *Coxeter system of type* M is the pair (W_M, r), where
$$r : \mathrm{Mon}(I) \to W_M,\ f \mapsto r_f$$
is the unique extension of the map $r : I \to W_M,\ i \mapsto r_i$ to a homomorphism from the free monoid $\mathrm{Mon}(I)$ on I to W_M.

Chapter 2 Twin Buildings

As the main issue of this thesis is the classification of twin buildings, we give a rough overview of the main concepts and the main results which allow us to pass from the whole building to its local structure without loss of information. By theorem (2.26) and (2.27) below, each irreducible residue of rank 2 of an irreducible twin building of rank at least 3 is a Moufang polygon. As a consequence, we may exploit the classification of Moufang polygons in [TW].

(2.1) Definition (**Chamber Systems**)

- A *chamber system over an index set* I is a set Δ (whose elements are called *chambers*) together with an equivalence relation $\sim_i$ on Δ (called *i-equivalence*) for each $i \in I$.
- Given a chamber system Δ over I and $i \in I$, an *i-panel* is an i-equivalence class, and a *panel* is an i-panel for some $i \in I$.
- Given a chamber system Δ over I and $i \in I$, two distinct chambers $x, y \in \Delta$ such that $x \sim_i y$ are called *i-adjacent*, and they are *adjacent* if they are i-adjacent for some $i \in I$.
- Given a chamber system Δ over I, chambers $x, y \in \Delta$ and $J \subseteq I$, a *J-gallery of length k from x to y* is a sequence $\gamma = (x_0, \dots, x_k) \subseteq \Delta^{k+1}$ for some $k \in \mathbb{N}$ such that
$$x_0 = x \ , \qquad x_k = y \ , \qquad \forall\, j \in \{1, \dots, k\}\ \exists\, i_j \in J :\ x_{j-1} \sim_{i_j} x_j \ \wedge\ x_{j-1} \neq x_j \ ,$$
a *gallery from x to y* is an I-gallery from x to y, and we write $x \sim_J y$ if there is a J-gallery from x to y.
- Given a chamber system Δ and chambers $x, y \in \Delta$, the *distance* $\mathrm{dist}(x, y)$ *from x to y* is the length of a shortest gallery from x to y if there is one and ∞ otherwise.
- Given a chamber system Δ over I, chambers $x, y \in \Delta$ and a gallery $\gamma = (x_0, \dots, x_k)$ from x to y, the *type of* γ is the word $i_1 \cdots i_k \in \mathrm{Mon}(I)$.
- Given a chamber system Δ over I, a chamber $x \in \Delta$ and $J \subseteq I$, the *J-residue of* x is
$$\Delta_J(x) := \{y \in \Delta \mid x \sim_J y\} \ .$$
A *residue* is a J-residue $\Delta_J(x)$ for some chamber $x \in \Delta$ and some $J \subset I$.

(2.2) Definition (**Buildings**) Let M be a Coxeter diagram over I and let (W_M, r) be the corresponding Coxeter system. A *building of type* M is pair $\mathcal{B} = (\Delta, \delta)$, where Δ is a chamber system over I endowed with a function $\delta : \Delta \times \Delta \to W_M$ such that the following holds:

(B1) Each panel contains at least two chambers.

(B2) For each reduced word $f \in \mathrm{Mon}(I)$ and for each ordered pair (x, y) of chambers, we have $\delta(x, y) = r_f$ if and only if there is a gallery of type f from x to y.

(2.3) Remark Cf. definition (39.10) of [TW] for the definition of a reduced word.

(2.4) Definition (**Standard Thin Buildings**) Let M be a Coxeter Matrix. Then the building $\Sigma(M) := (W_M, \delta_{W_M})$ with
$$\delta_{W_M} : W_M \times W_M \to W_M,\ (w_1, w_2) \mapsto w_1^{-1} w_2$$
is the *standard thin building of type* M.

(2.5) Remark Cf. example (5.7) of [AB] that $\Sigma(M)$ is a building of type M.

(2.6) Definition (Apartments) Let $\mathcal{B} = (\Delta, \delta)$ be a building of type M and let $X \subseteq W_M$.

- An *isometry from X to $\mathcal{B}$* is a map $\pi : X \to \Delta$ such that

$$\forall\, x, y \in X : \qquad \delta(x^\pi, y^\pi) = x^{-1}y \ .$$

- An *apartment of $\mathcal{B}$* is the image Σ of some isometry $\pi : W_M \to \Delta$.

(2.7) Theorem (*J*-Residues) Let M be a Coxeter matrix over I, let $\mathcal{B} = (\Delta, \delta)$ be a building of type M, let $x \in \Delta$ and let $J \subseteq I$. Then the following holds:

(a) The *J-residue*

$$\mathcal{B}_J(x) := \big(\Delta_J(x), \delta_{|\Delta_J(x)\times\Delta_J(x)}\big)$$

is a building of type M_J.

(b) If Σ is an apartment of $\mathcal{B}$ such that $\Sigma \cap \mathcal{B}_J(x) \neq \emptyset$, then $\Sigma_J := \mathcal{B}_J(x) \cap \Sigma$ is an apartment of $\mathcal{B}_J(x)$.

(c) If Σ_J is an apartment of $\mathcal{B}_J(x)$, then we have $\Sigma_J = \mathcal{B}_J(x) \cap \Sigma$ for some apartment Σ of $\mathcal{B}$.

Proof
This results from (39.52) of [TW]. □

(2.8) Definition (Roots) Let $\mathcal{B}$ be a building of type M, let Σ be an apartment of $\mathcal{B}$ and let c be a chamber of Σ.

- A *root of Σ* is a subset $\alpha \subset \Sigma$ such that

$$\alpha = \{w \in \Sigma \mid \operatorname{dist}(w, x) < \operatorname{dist}(w, y)\}$$

for some ordered pair (x, y) of adjacent chambers. We denote the set of roots of Σ by $\Phi(\mathcal{B}, \Sigma)$.

- A *root of $\mathcal{B}$* is a root of some apartment $\Sigma \subseteq \mathcal{B}$.

- Given $i \in I$, the *simple root α_i with respect to* (Σ, c) is the root

$$\alpha_i := \{w \in \Sigma \mid \operatorname{dist}(w, c) < \operatorname{dist}(w, c_i)\} \ ,$$

where c_i is the unique chamber of Σ which is i-adjacent to c. We write $\Phi(\mathcal{B}, \Sigma, c)$ instead of $\Phi(\mathcal{B}, \Sigma)$ if we additionally take the simple roots with respect to (Σ, c) into account.

(2.9) Definition (Standard Root Systems) Let M be a Coxeter Matrix.

- The set $\Phi(M) := \Phi\big(\Sigma(M), \Sigma(M), 1_{W_M}\big)$ of roots of $\Sigma(M)$ is the *standard root system of type M*.

- Given $\alpha, \beta \in \Phi(M)$, the pair $\{\alpha, \beta\}$ is *prenilpotent* if we have

$$\alpha \cap \beta \neq \emptyset \neq (-\alpha) \cap (-\beta) \ .$$

In this case, we set

$$[\alpha, \beta] := \{\gamma \in \Phi(M) \mid \alpha \cap \beta \subseteq \gamma,\ (-\alpha) \cap (-\beta) \subseteq -\gamma\} \ , \qquad (\alpha, \beta) := [\alpha, \beta] \setminus \{\alpha, \beta\} \ .$$

(2.10) Remark Let M be a Coxeter matrix over I, let $\Phi(M)$ be the standard root system of type M and let $\alpha \in \Phi(M)$. Then we have

$$\alpha = v\alpha_i = \{v \cdot w \mid w \in \alpha_i\}$$

for some $i \in I$ and some $v \in W_M$, cf. proposition (5.81) of [AB].

(2.11) Definition (Twin Buildings) Let M be a Coxeter diagram over I. A *twin building of type* M is a triple $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$, where each *half* $\mathcal{B}_\epsilon = (\Delta_\epsilon, \delta_\epsilon)$ with $\epsilon \in \{\pm\}$ is a building of type M and

$$\delta^* : (\Delta_+ \times \Delta_-) \cup (\Delta_- \times \Delta_+) \to W_M$$

is a *codistance*, i.e., given $\epsilon \in \{\pm\}$, $x \in \Delta_\epsilon$, $y \in \Delta_{-\epsilon}$ and $w := \delta^*(x, y)$, the following holds:

(C1) We have $\delta^*(y, x) = w^{-1}$.

(C2) Given $z \in \Delta_{-\epsilon}$, $i \in I$ such that $\delta_{-\epsilon}(y, z) = r_i$ and $l(wr_i) = l(w) - 1$, we have $\delta^*(x, z) = wr_i$.

(C3) Given $i \in I$, there exists a chamber $z \in \Delta_{-\epsilon}$ such that $\delta_{-\epsilon}(y, z) = r_i$ and $\delta^*(x, z) = wr_i$.

Here $l : W \to \mathbb{N}^*$ is the length function with respect to the set $\{r_i \mid i \in I\}$ of generators.

(2.12) Definition (Opposite Chambers) Let M be a Coxeter diagram over I, let $\mathcal{B}$ be a twin building of type M, let $J \subseteq I$ and let $\epsilon \in \{\pm\}$.

- Two chambers $x \in \mathcal{B}_\epsilon$ and $y \in \mathcal{B}_{-\epsilon}$ such that $\delta^*(x, y) = 1$ are called *opposite*. We set

$$\mathcal{O}_B := \{(x, y) \in \mathcal{B}_+ \times \mathcal{B}_- \mid \delta^*(x, y) = 1\} \ .$$

- Two residues $\mathcal{R}_+ \subseteq \mathcal{B}_+$ and $\mathcal{R}_- \subseteq \mathcal{B}_-$ such that

$$\mathcal{R}_+ \times \mathcal{R}_{-\epsilon} \cap \mathcal{O}_B \neq \emptyset$$

  are called *opposite*.

(2.13) Lemma Let $\mathcal{B}$ be a twin building, let $\epsilon \in \{\pm\}$ and let $x \in \mathcal{B}_\epsilon$. Then there exists a chamber $y \in \mathcal{B}_{-\epsilon}$ such that $\delta^*(x, y) = 1$.

Proof
This results from corollary (5.141) of [AB]. □

(2.14) Theorem (*J*-Residues) Let M be a Coxeter diagram over I, let $\mathcal{B}$ be a twin building of type M, let $J \subseteq I$, let $(x, y) \in \mathcal{O}_B$ and let $\mathcal{B}_J(x) := (\mathcal{B}_+)_J(x)$, $\mathcal{B}_J(y) := (\mathcal{B}_-)_J(y)$. Then the *$J$-residue*

$$\mathcal{B}_J(x, y) := \left(\mathcal{B}_J(x), \mathcal{B}_J(y), \delta^*_{|(\mathcal{B}_J(x) \times \mathcal{B}_J(y)) \cup (\mathcal{B}_J(y) \times \mathcal{B}_J(y))}\right)$$

is a twin building of type M_J.

Proof
By lemma (5.148) of [AB], we have

$$\delta(\bar{x}, \bar{y}) \in W_{M_J} \delta^*(x, y) W_{M_j} = W_{M_J} \cdot 1 \cdot W_{M_J} = W_{M_J}$$

for all $\bar{x} \in \mathcal{B}_J(x)$, $\bar{y} \in \mathcal{B}_J(y)$. □

(2.15) Corollary Let M be a Coxeter diagram over I, let $\mathcal{B}$ be a twin building of type M, let $\epsilon \in \{\pm\}$, let $x \in \mathcal{B}_\epsilon$ and let $J \subseteq I$. Then $(\mathcal{B}_\epsilon)_J(x)$ is the half of a twin building.

Proof
By lemma (2.13), there is a chamber $y \in \mathcal{B}_{-\epsilon}$ such that $\delta^*(x, y) = 1$, thus $(\mathcal{B}_\epsilon)_J(x)$ is the half of a twin building by theorem (2.14). □

(2.16) Notation Let M be a Coxeter matrix over I and let $\mathcal{B}$ be a building of type M. Given a chamber $c \in \mathcal{B}$, we define

$$E_2(c) := \{\mathcal{B}_{\{i,j\}}(c) \mid i \neq j \in I\} \ .$$

(2.17) Definition (Twin Apartments) Let $\mathcal{B}$ be a twin building.

- A *twin apartment of* $\mathcal{B}$ is a pair $\Sigma = (\Sigma_+, \Sigma_i)$ of apartments Σ_ϵ of $\mathcal{B}_\epsilon$ such that each chamber of $\Sigma_+ \cup \Sigma_-$ is opposite precisely one other chamber $\mathrm{op}_\Sigma(c) \in \Sigma_+ \cup \Sigma_-$.
- Given a twin apartment Σ, the map

$$\mathrm{op}_\Sigma : \Sigma_+ \cup \Sigma_- \to \Sigma_+ \cup \Sigma_-,\ c \mapsto \mathrm{op}_\Sigma(c)$$

  is the *opposition involution with respect to* Σ.

(2.18) Definition (Twin Roots) Let $\mathcal{B}$ be a twin building and let Σ be a twin apartment of $\mathcal{B}$.

- A *twin root of* Σ is a pair $\alpha = (\alpha_+, \alpha_-)$ of roots α_ϵ of Σ_ϵ such that

$$\mathrm{op}_\Sigma(\alpha) = -\alpha\ .$$

- We denote the set of twin roots of Σ by $\Phi(\mathcal{B}, \Sigma)$.

(2.19) Remark Let $\mathcal{B}$ be a twin building. Given a twin apartment Σ of $\mathcal{B}$ and a root α_+ of Σ_+, then $\alpha_- := -\mathrm{op}_\Sigma(\alpha_+)$ is the unique root of Σ_- such that $\alpha := (\alpha_+, \alpha_-)$ is a twin root of Σ. As a consequence, the map

$$f : \Phi(\mathcal{B}, \Sigma) \to \Phi(\mathcal{B}_+, \Sigma_+),\ (\alpha_+, \alpha_-) \to \alpha_+$$

is a bijection.

(2.20) Definition (Isometries and Automorphisms) Let $\mathcal{B} = (\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ be a twin building of type M and let $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}_+, \tilde{\mathcal{B}}_-, \tilde{\delta}^*)$ be a twin building of type $\tilde{M}$.

- An *isometry of twin buildings* is a triple $\phi = (\sigma, \phi_+, \phi_-)$ consisting of an isomorphism $\sigma : M \to \tilde{M}$ of Coxeter diagrams and maps $\phi_\epsilon : \mathcal{B}_\epsilon \to \mathcal{B}_\epsilon$ such that

$$\forall\ c_\epsilon, d_\epsilon \in \mathcal{B}_\epsilon : \qquad \tilde{\delta}_\epsilon\big(\phi_\epsilon(c_\epsilon), \phi_\epsilon(d_\epsilon)\big) = \sigma\big(\delta_\epsilon(c_\epsilon, d_\epsilon)\big)$$

  and

$$\forall\ c_\epsilon \in \mathcal{B}_\epsilon, c_{-\epsilon} \in \mathcal{B}_{-\epsilon} : \qquad \tilde{\delta}^*\big(\phi_\epsilon(c_\epsilon), \phi_{-\epsilon}(c_{-\epsilon})\big) = \sigma\big(\delta^*(c_\epsilon, c_{-\epsilon})\big)$$

- An *isomorphism of twin buildings* is a surjective isometry $\phi : \mathcal{B} \to \tilde{\mathcal{B}}$.
- An *automorphism of* $\mathcal{B}$ is an isomorphism $\phi : \mathcal{B} \to \mathcal{B}$. We denote the group of automorphisms of $\mathcal{B}$ by $\mathrm{Aut}(\mathcal{B})$.
- An automorphism $\phi \in \mathrm{Aut}(\mathcal{B})$ is *special* if we have $\sigma = \mathrm{id}_M$. We denote the group of special automorphisms of $\mathcal{B}$ by $\mathrm{Aut}_0(\mathcal{B})$.

(2.21) Definition (Strongly Transitive Actions) Let $\mathcal{B}$ be a twin building and let G be a group.

- An *action of* G *on* $\mathcal{B}$ is a homomorphism

$$\varphi : G \to \mathrm{Aut}(\mathcal{B})\ .$$

- An action $\varphi : G \to \mathrm{Aut}(\mathcal{B})$ is *strongly transitive* if the action is transitive on the set

$$\{(\Sigma, c) \mid \Sigma \text{ is a twin apartment of } \mathcal{B},\ c \in \mathcal{O}_\Sigma\}\ .$$

(2.22) Theorem Let $\mathcal{B}$ be a Moufang twin building. Then $\mathrm{Aut}_0(\mathcal{B})$ acts strongly transitively on $\mathcal{B}$.

Proof
This results from proposition (8.19) of [AB]. □

(2.23) Theorem (Extension Theorem) Let $\mathcal{B}$, $\tilde{\mathcal{B}}$ be thick (2-spherical) twin buildings of the same type. Assume that $\mathcal{B}$ and $\tilde{\mathcal{B}}$ satisfy condition (CO). Given $c \in \mathcal{O}_{\mathcal{B}}$ and $\tilde{c} \in \mathcal{O}_{\tilde{\mathcal{B}}}$ and a surjective isometry

$$\phi : E_2(c_+) \cup \{c_-\} \to E_2(\tilde{c}_+) \cup \{\tilde{c}_-\} \ ,$$

there is a unique extension of ϕ to an isomorphism $\phi : \mathcal{B} \to \tilde{\mathcal{B}}$ of twin buildings. (A building is *thick* if each panel contains at least three chambers.)

Proof
Cf. theorem (5.213) of [AB]. Notice that our buildings are 2-spherical by definition. □

(2.24) Definition (Root Groups and the Moufang Property) Let $\mathcal{B}$ be a twin building of rank at least 2, where the *rank of a building of type M* is just $|V(M)|$.

- Given a twin root α, the corresponding *root group* is

$$U_\alpha := \{g \in \mathrm{Aut}(\mathcal{B}) \mid g \text{ acts trivially on each panel of } \alpha^o\} \ ,$$

  where α^o is the set of all panels of $\mathcal{B}$ which contain at least two chambers in α.

- The building $\mathcal{B}$ is *Moufang* if it is thick and if for each root α of $\mathcal{B}$, the root group U_α acts transitively on the set of twin apartments containing α.

- The building $\mathcal{B}$ is *strictly Moufang* if the actions of the root groups are simply transitive.

(2.25) Remark A Moufang twin building whose Coxeter diagram has no isolated nodes is strictly Moufang, cf. p. 455 of [AB].

(2.26) Theorem Every thick, irreducible twin building of rank at least 3 that satisfies condition (CO) is Moufang.

Proof
This is theorem (8.27) of [AB]. Notice that our buildings are 2-spherical by definition. □

(2.27) Theorem Let $\mathcal{B}$ be a Moufang twin building. Then every spherical residue of $\mathcal{B}$ is Moufang.

Proof
This is proposition (8.21) of [AB].

(2.28) Corollary Let $\mathcal{B}$ be a thick, irreducible twin building of rank at least 3. Then each residue of rank 2 is also Moufang. In particular, the irreducible residues of rank 2 are Moufang polygons.

Proof
This results from remark (8.30)(a) of [AB]. Notice that our buildings are 2-spherical by definition. □

(2.29) Definition (RGD System) Let M be a Coxeter matrix, let $\Phi := \Phi(M)$, let G be a group, let $(U_\alpha)_{\alpha\in\Phi}$ be a family of non-trivial subgroups of G and let

$$T := \bigcap_{\alpha\in\Phi} N_G(U_\alpha) \ .$$

Then the pair $\big(G,(U_\alpha)_{\alpha\in\Phi}\big)$ is an *RGD system of type M* if the following holds:

(RGD1) We have

$$[U_\alpha, U_\beta] \le U_{(\alpha,\beta)}$$

for all $\alpha \neq \beta$ such that $\{\alpha,\beta\}$ is prenilpotent.

(RGD2) Given $i \in I$, there is a function $\mu : U^*_{\alpha_i} \to G$ such that we have

$$\forall\ u \in U^*_{\alpha_i} :\ \mu(u) \in U_{-\alpha_i} u U_{-\alpha_i} \ , \quad \forall\ u \in U^*_{\alpha_i},\ \alpha \in \Phi :\ \mu(u) U_{\alpha_i} \mu(u)^{-1} = U_{r_i \alpha_i} \ .$$

(RGD3) Given $i \in I$, we have

$$U_{-\alpha_i} \not\le U_+ \ ,$$

where

$$U_\epsilon := \langle U_\alpha \mid \alpha \in \Phi_\epsilon \rangle \ , \qquad \Phi_+ := \{\alpha \in \Phi \mid 1 \in \alpha\} \ , \qquad \Phi_- := \{\alpha \in \Phi \mid 1 \notin \alpha\} \ .$$

(RGD4) We have

$$G = T\langle U_\alpha \mid \alpha \in \Phi \rangle \ .$$

(2.30) Theorem The following holds:

(a) Each RGD system $\big(G,(U_\alpha)_{\alpha\in\Phi}\big)$ gives rise to a twin building $\mathcal{B}\big(G,(U_\alpha)_{\alpha\in\Phi}\big)$.

(b) Let $\mathcal{B}$ be a twin building of type M, let Σ be a twin apartment of $\mathcal{B}$, let $c \in \mathcal{O}_\Sigma$, let $G := \mathrm{Aut}(\mathcal{B})$, let $\Phi := \Phi(\mathcal{B},\Sigma,c)$ and let $(U_\alpha)_{\alpha\in\Phi}$ be the family of root groups with respect to (Σ, c). Then the pair $\big(G,(U_\alpha)_{\alpha\in\Phi}\big)$ is an RGD system of type M, and $\mathcal{B}$ is uniquely determined by this RGD system, i.e., we have

$$\mathcal{B} \cong \mathcal{B}\big(G,(U_\alpha)_{\alpha\in\Phi}\big) \ .$$

Proof

(a) This is theorem (8.81) of [AB].

(b) The first statement is example (8.47)(a) of [AB] while the second one results from theorem (8.9) of [AB].

□

(2.31) Proposition Let M be a Coxeter matrix over I, let $\mathcal{B}$ be a strictly Moufang twin building of type M, let Σ be a twin apartment of $\mathcal{B}$ and let $c \in \mathcal{O}_\Sigma$. Let $J \subseteq I$ be such that M_J has no isolated nodes, let $\mathcal{B}_J := \mathcal{B}_J(c)$ and let α_J be a root of $\Sigma_J = \Sigma \cap \mathcal{B}_J$. Then there is a unique root α of Σ such that $\alpha_J = \alpha \cap \Sigma_J$, and the restriction map

$$\rho : U_\alpha \to U_{\alpha_J}$$

is an isomorphism of groups.

Proof
For spherical buildings this is proposition (7.32) of [AB], and the arguments given in its proof go through in the twin case without much change. □

(2.32) Theorem Let M be an irreducible Coxeter matrix over I such that $|I| \geq 3$, let $\mathcal{B}$ be a thick Moufang twin building of type M, let Σ be a twin apartment of $\mathcal{B}$, let $c \in \mathcal{O}_\Sigma$ and let $\Phi := \Phi(\mathcal{B}, \Sigma, c)$. Let $(i,j) \in A(M)$, $\mathcal{B}_{ij} := \mathcal{B}_{\{i,j\}}(c)$ and $n := m_{ij}$. Then the following holds:

(a) The residue $\mathcal{B}_{ij}$ is a Moufang n-gon.

(b) The intersection $\Sigma_{ij} := \Sigma \cap \mathcal{B}_{ij}$ is an apartment of $\mathcal{B}_{ij}$, and the roots $\alpha_i \cap \mathcal{B}_{ij}$ and $\alpha_j \cap \mathcal{B}_{ij}$ form a root basis of $\mathcal{B}_{ij}$.

(c) Let

$$(\bar{\omega}_1 = \alpha_i \cap \mathcal{B}_{ij}, \bar{\omega}_2, \ldots, \bar{\omega}_{n-1}, \bar{\omega}_n = \alpha_j \cap \mathcal{B}_{ij})$$

be the root sequence of $\mathcal{B}_{ij}$ from $\alpha_i \cap \mathcal{B}_{ij}$ to $\alpha_j \cap \mathcal{B}_{ij}$. Then there are exactly n roots $\omega_1 = \alpha_i, \omega_2, \ldots, \omega_n = \alpha_j$ of Σ such that

$$\forall\ 1 \leq i \leq n : \qquad \bar{\omega}_i = \omega_i \cap \mathcal{B}_{ij}\ .$$

(d) For $i = 1, \ldots, n$ let $U_i := U_{\omega_i}$, let $U_{[\alpha_i,\alpha_j]} := U_1 \cdots U_n$ and let

$$\Theta_{(i,j)} := \big(U_{[\alpha_i,\alpha_j]}, U_1, \ldots, U_n\big)\ .$$

Then $\Theta_{(i,j)}$ is isomorphic to the root group sequence of $\mathcal{B}_{ij}$ from $\alpha_i \cap \mathcal{B}_{ij}$ to $\alpha_j \cap \mathcal{B}_{ij}$.

Proof

(a) This is corollary (2.28).

(b) The first assertion results from theorem (2.7) (b), and by definition, the roots $\alpha_i \cap \mathcal{B}_{ij}$ and $\alpha_j \cap \mathcal{B}_{ij}$ are simple roots of Σ_{ij}.

(c) This results from proposition (2.31).

(d) This results from proposition (2.31).

□

(2.33) Definition (Double μ-Maps) Let M be a Coxeter matrix over I, let $\mathcal{B}$ be a Moufang twin building of type M, let Σ be a twin apartment of $\mathcal{B}$, let $c \in \mathcal{O}_\Sigma$ and let $\Phi := \Phi(\mathcal{B}, \Sigma, c)$. Let $i \in I$, let $a, b \in U_{\alpha_i}^*$ and let $\mu(a), \mu(b)$ be as in (RGD2). Then the map

$$h_{a,b} : \langle U_\alpha \mid \alpha \in \Phi \rangle \to \langle U_\alpha \mid \alpha \in \Phi \rangle, u \mapsto \mu(a)^{-1}\mu(b)u\mu(b)^{-1}\mu(a)$$

is the *double μ-map with respect to a, b.*

(2.34) Remark

(a) We have $h_{a,b} \in T$ for all $a, b \in U_{\alpha_i}^*$.

(b) By (3) in section (7.8.2) of [AB], the μ-maps in (RGD2) are uniquely determined.

(2.35) Theorem Let M be an irreducible Coxeter matrix over I such that $|I| \geq 3$, let $\mathcal{B}$ be a thick Moufang twin building of type M, let Σ be a twin apartment of $\mathcal{B}$, let $c \in \mathcal{O}_\Sigma$ and let $\Phi := \Phi(\mathcal{B}, \Sigma, c)$. Let $(i,j) \in A(M)$, let $\mathcal{B}_{ij} := \mathcal{B}_{\{i,j\}}(c)$, let $\Sigma_{ij} := \Sigma \cap \mathcal{B}_{ij}$ and let $\Phi_{ij} := \Phi(\mathcal{B}_{ij}, \Sigma_{ij}, c)$. Then we have

$$\forall\ a \in U_{\alpha_i}^* : \qquad \mu^{\mathcal{B}_{ij}}\big(\rho(a)\big) = \rho\big(\mu^{\mathcal{B}}(a)\big)\ ,$$

where $\rho : U_{[\alpha_i,\alpha_j]} \to \mathrm{Aut}(\mathcal{B}_{ij})$ is the restriction homomorphism. In particular, we have

$$\forall\ a, b \in U_{\alpha_i}^* : \qquad h^{\mathcal{B}_{ij}}_{\rho(a),\rho(b)} = \rho \circ h^{\mathcal{B}}_{a,b} \circ \rho^{-1}\ .$$

Proof
Given a root $\bar{\alpha} \in \Phi_{ij}$, let $\alpha \in \Phi$ be the unique root of Σ such that $\bar{\alpha} = \alpha \cap \Sigma_{ij}$. By proposition (2.31), the map $\rho : U_\alpha \to U_{\bar{\alpha}}$ is an isomorphism of groups for each root $\bar{\alpha} \in \Phi_{ij}$. Given $a \in U^*_{\alpha_i}$, we have

$$\rho\big(\mu^{\mathcal{B}}(a)\big) \in \rho(U_{-\alpha_i} a U_{-\alpha_i}) = U_{-\bar{\alpha}_i} \rho(a) U_{-\bar{\alpha}_i}$$

and

$$\forall\ \alpha \in \Phi_{ij} : \qquad \rho\big(\mu^{\mathcal{B}}(a)\big) U_{\bar{\alpha}} \rho\big(\mu^{\mathcal{B}}(a)\big)^{-1} = \rho\big(\mu^{\mathcal{B}}(a) U_\alpha \mu^{\mathcal{B}}(a)^{-1}\big) = \rho(U_{r_i \alpha}) = U_{r_i \bar{\alpha}}\ .$$

and thus

$$\mu^{\mathcal{B}_{ij}}\big(\rho(a)\big) = \rho\big(\mu^{\mathcal{B}}(a)\big)$$

by remark (2.34) (b). □

Part II

Parameter Systems

In this part, we introduce the parameter systems which arise in the description of Moufang triangles and the six families of Moufang quadrangles. Moreover, we collect the basic results which will be needed for the classification of twin buildings, and, closely related, the solution of the isomorphism problem for Moufang sets.

For a detailed reference on these subjects, see [TW]. Concerning alternative rings, we additionally refer to [RSch].

Chapter 3 Alternative Rings

Alternative division rings are the parametrizing structures for Moufang triangles, the building bricks for simply laced twin buildings.

§ 3.1 Basic Definitions and Basic Properties

(3.1) Definition An *alternative ring* is a triple $(\mathbb{A}, +, \cdot)$ such that the following holds:

(A1) The pair $(\mathbb{A}, +)$ is a commutative group.

(A2) The multiplication $\cdot : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is biadditive.

(A3) The multiplication $\cdot : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is *alternative*, i.e., it satisfies

$$\forall\, x, y \in \mathbb{A} : \qquad [x, x, y] = 0_{\mathbb{A}} = [y, x, x] \ .$$

where $[x, y, z] := (xy)z - x(yz)$ is the *associator of* $x, y, z \in \mathbb{A}$

(A4) There is an identity element $1_{\mathbb{A}}$.

(3.2) Lemma An alternative ring $\mathbb{A}$ is *flexible*, i.e., given $x, y \in \mathbb{A}$, we have

$$[x, y, x] = 0_{\mathbb{A}} \ .$$

Proof
Cf. page 27 of [RSch]. □

(3.3) Lemma (**Moufang Identities**) An alternative ring $\mathbb{A}$ satisfies the *Moufang identities*, i.e., given $x, y, z \in \mathbb{A}$, we have

$$(xyx)z = x\big(y(xz)\big) \ , \qquad z(xyx) = \big((zx)y\big)x \ , \qquad (xy)(zx) = x(yz)x \ .$$

Proof
Cf. page 28 of [RSch]. □

(3.4) Definition An alternative ring $\mathbb{A}$ is an *alternative division ring* if the maps

$$\rho_w : \mathbb{A} \to \mathbb{A},\ x \mapsto xw \ , \qquad \lambda_w : \mathbb{A} \to \mathbb{A},\ x \mapsto wx$$

are bijective for each $w \in \mathbb{A}^*$.

(3.5) Remark Let $\mathbb{A}$ be an alternative division ring. Given $x \in \mathbb{A}^*$, there are unique elements $x^{-l}, x^{-r} \in \mathbb{A}^*$ such that

$$x^{-l} \cdot x = 1_{\mathbb{A}} = x \cdot x^{-r} \ .$$

By lemma (3.2), we have

$$\lambda_{x^{-l}}(xx^{-l}) = x^{-l} \cdot xx^{-l} = x^{-l}x \cdot x^{-l} = 1_{\mathbb{A}} \cdot x^{-l} = x^{-l} = \lambda_{x^{-l}}(1_{\mathbb{A}}) \ , \qquad xx^{-l} = 1_{\mathbb{A}}$$

and therefore $x^{-1} := x^{-l} = x^{-r}$.

(3.6) Lemma An alternative division ring $\mathbb{A}$ has the *inverse properties*, i.e., given $x, y \in \mathbb{A}^*$, we have

$$x^{-1}(xy) = y \ , \qquad (yx)x^{-1} = y \ , \qquad (xy)^{-1} = y^{-1}x^{-1} \ .$$

Proof
This results from the Moufang identities. □

(3.7) Definition Let $(\mathbb{A}, +, \cdot)$ be an alternative ring. Then $(\mathbb{A}, +, \circ)$ with the multiplication

$$\circ : \mathbb{A} \times \mathbb{A} \to \mathbb{A}, \ \ x \circ y := y \cdot x$$

is the *opposite alternative ring*.

(3.8) Definition Let $\mathbb{A}, \tilde{\mathbb{A}}$ be alternative rings.

- An *(anti-)isomorphism of alternative rings* is an additive isomorphism $\gamma : \mathbb{A} \to \tilde{\mathbb{A}}$ such that

$$\forall \ x, y \in \mathbb{A} : \qquad \gamma(xy) = \gamma(x)\gamma(y) \qquad \big(\ \gamma(xy) = \gamma(y)\gamma(x) \ \big) \ .$$

- A *Jordan homomorphism* is an additive monomorphism $\gamma : \mathbb{A} \to \tilde{\mathbb{A}}$ such that

$$\gamma(1_{\mathbb{A}}) = 1_{\tilde{\mathbb{A}}} \ , \qquad \forall \ x, y \in \mathbb{A} : \ \gamma(xyx) = \gamma(x)\gamma(y)\gamma(x) \ .$$

(3.9) Notation Let $\mathbb{A}$ be an alternative division ring.

- We denote the *group of automorphisms of* $\mathbb{A}$ by $\mathrm{Aut}(\mathbb{A})$.
- We denote the *set of anti-automorphisms of* $\mathbb{A}$ by $\mathrm{Aut}^o(\mathbb{A})$.
- Given $w \in \mathbb{A}$, we set

$$\lambda_w : \mathbb{A} \to \mathbb{A}, \ x \mapsto wx \ , \qquad \rho_w : \mathbb{A} \to \mathbb{A}, \ x \mapsto xw \ , \qquad \gamma_w : \mathbb{A} \to \mathbb{A}, \ x \mapsto w^{-1}xw \ .$$

  Notice that the conjugation map γ_w is well-defined by (9.23)(ii) of [TW] with $c := a$.
- We denote the *opposite alternative division ring* by $\mathbb{A}^o$.
- We denote the *group of Jordan automorphisms of* $\mathbb{A}$ by $\mathrm{Aut}_J(\mathbb{A})$.

(3.10) Definition The *center* of an alternative ring $\mathbb{A}$ is

$$Z(\mathbb{A}) := \{x \in \mathbb{A} \mid [x, \mathbb{A}, \mathbb{A}] = [x, \mathbb{A}] = 0_{\mathbb{A}}\} \ ,$$

where $[x, y] := xy - yx$ is the *commutator of* $x, y \in \mathbb{A}$.

(3.11) Lemma Let $\mathbb{A}$ be an alternative division ring. Then $\mathbb{K} := Z(\mathbb{A})$ is a field and $\mathbb{A}$ is an algebra over $\mathbb{K}$.

Proof
This results from (9.18) and (9.23) of [TW]. □

§ 3.2 Octonion Division Algebras

The Bruck-Kleinfeld theorem states that a non-associative alternative division ring is an octonion division algebra. First of all we give the exact definition of such an algebra before we collect some basic concepts, including the doubling process.

(3.12) Remark The construction here is taken from [TW].

(3.13) Definition Let $\mathbb{E}/\mathbb{K}$ be a separable quadratic extension and let σ be the non-trivial element of $\mathrm{Gal}(\mathbb{E}/\mathbb{K})$.

- Given $x \in \mathbb{E}$, we write $\bar{x} := \sigma(x)$.
- We denote the norm map and the trace map of $\mathbb{E}/\mathbb{K}$ by N and T, respectively.
- Given $\beta \in \mathbb{K}^*$, we set

$$(\mathbb{E}/\mathbb{K}, \beta) := \left\{ \begin{pmatrix} x & \beta\bar{y} \\ y & \bar{x} \end{pmatrix} \mid x, y \in \mathbb{E} \right\} \subseteq M_2(\mathbb{E}) \ , \qquad e := \begin{pmatrix} 0_{\mathbb{E}} & \beta \\ 1_{\mathbb{E}} & 0_{\mathbb{E}} \end{pmatrix} \in (\mathbb{E}/\mathbb{K}, \beta) \ .$$

(3.14) Lemma Let $\mathbb{E}/\mathbb{K}$ be a separable quadratic extension and let $\beta \in \mathbb{K}^* \setminus N(\mathbb{E})$. Then the following holds:

(a) The set

$$\mathbb{H} := (\mathbb{E}/\mathbb{K}, \beta) \subseteq M_2(\mathbb{E})$$

is an associative division ring.

(b) We have

$$\mathbb{H} = 1_{\mathbb{E}} \cdot \mathbb{E} + e \cdot \mathbb{E} \ .$$

(c) The map

$$\sigma_s : \mathbb{H} \to \mathbb{H}, \ x + e \cdot y \mapsto \bar{x} - e \cdot y$$

is an involution of $\mathbb{H}$ extending σ.

(d) The maps

$$N : \mathbb{H} \to \mathbb{K}, \ x + e \cdot y \mapsto N(x) - \beta \cdot N(y) \ , \qquad T : \mathbb{H} \to \mathbb{K}, \ x + e \cdot y \mapsto T(x)$$

are extensions of N and T.

Proof
Cf. (9.2), (9.3) and (9.4) of [TW]. □

(3.15) Definition A *quaternion division algebra* is an algebra $\mathbb{H}$ isomorphic to $(\mathbb{E}/\mathbb{K}, \beta)$ for some separable quadratic extension $\mathbb{E}/\mathbb{K}$ and some $\beta \in \mathbb{K}^* \setminus N(\mathbb{E})$. The map σ_s is the *standard involution of* $\mathbb{H}$.

(3.16) Definition Let $\mathbb{H}$ be a quaternion division algebra.

- Given $x \in \mathbb{H}$, we write $\bar{x} := \sigma_s(x)$.
- Given $\beta \in \mathbb{K}^*$, we set

$$(\mathbb{H}, \beta) := \left\{ \begin{pmatrix} x & \beta\bar{y} \\ y & \bar{x} \end{pmatrix} \mid x, y \in \mathbb{H} \right\} \subseteq M_2(\mathbb{H}) \ , \qquad e := \begin{pmatrix} 0_{\mathbb{H}} & \beta \\ 1_{\mathbb{H}} & 0_{\mathbb{H}} \end{pmatrix} \in (\mathbb{H}, \beta) \ .$$

- We define a multiplication on $(\mathbb{H}, \beta)$ by

$$\begin{pmatrix} x & \beta\bar{y} \\ y & \bar{x} \end{pmatrix} \cdot \begin{pmatrix} u & \beta\bar{v} \\ v & \bar{u} \end{pmatrix} := \begin{pmatrix} xu + \beta v\bar{y} & \beta(\bar{v}x + \bar{y}\bar{u}) \\ \bar{x}v + uy & \bar{u}\bar{x} + \beta y\bar{v} \end{pmatrix} \ ,$$

which is non-associative.

(3.17) Lemma Let $\mathbb{H}$ be a quaternion division algebra and let $\beta \in \mathbb{K}^* \setminus N(\mathbb{H})$. Then the following holds:

(a) With the ordinary matrix addition and the above multiplication, the set

$$\mathbb{O} := (\mathbb{H}, \beta) \subseteq M_2(\mathbb{H})$$

is an alternative division ring.

(b) We have

$$\mathbb{O} = 1_{\mathbb{H}} \cdot \mathbb{H} + e \cdot \mathbb{H} \ .$$

(c) The map

$$\sigma_s : \mathbb{O} \to \mathbb{O}, \ x + e \cdot y \mapsto \bar{x} - e \cdot y$$

is an involution of $\mathbb{O}$ extending σ_s.

(d) The maps

$$N : \mathbb{O} \to \mathbb{K}, \ x + e \cdot y \mapsto N(x) - \beta \cdot N(y) \ , \qquad T : \mathbb{O} \to \mathbb{K}, \ x + e \cdot y \mapsto T(x)$$

are extensions of N and T.

Proof
Cf. (9.8) of [TW]. □

(3.18) Definition An *octonion division algebra* is an algebra $\mathbb{O}$ isomorphic to $(\mathbb{H}, \beta)$ for some quaternion division algebra $\mathbb{H}$ and some $\beta \in \mathbb{K}^* \setminus N(\mathbb{H})$. The map σ_s is the *standard involution of* $\mathbb{O}$.

(3.19) Remark In the following, we list the basic properties of an octonion division algebra $\mathbb{O}$ which will be needed in the sequel. Since each subalgebra of $\mathbb{O}$ is a division algebra by (20.8) of [TW], we will omit the term "division" whenever we deal with an octonion division algebra and its subalgebras.

(3.20) Notation Throughout the rest of this paragraph, $\mathbb{O}$ denotes an octonion division algebra.

(3.21) Remark The norm $N : \mathbb{O} \to \mathbb{K}$ is a quadratic form with associated bilinear form

$$\langle \cdot, \cdot \rangle : \mathbb{O} \times \mathbb{O} \to \mathbb{K}, \ (x, y) \mapsto x\bar{y} + y\bar{x} = T(x\bar{y}) \ .$$

(3.22) Definition Let V be a vector space over $\mathbb{K}$. A quadratic form $q : V \to \mathbb{K}$ is *non-defective* if the associated bilinear form

$$f_q : V \to \mathbb{K}, \ (x, y) \mapsto q(x + y) - q(x) - q(y)$$

is non-degenerate, i.e., we have $V^\perp = \{0_V\}$, cf. definition (4.30). Otherwise, it is *defective*.

(3.23) Lemma There exists an element $x \in \mathbb{O}$ such that $\bar{x} \neq x$.

Proof
Cf. (20.15) of [TW]. □

(3.24) Corollary We have

$$\langle \cdot, \cdot \rangle \not\equiv 0_{\mathbb{K}} \ .$$

As a consequence, $\langle \cdot, \cdot \rangle$ is non-degenerate, hence N is non-defective.

Proof

- Char $\mathbb{O} \neq 2$: In this case, we have
$$\langle 1_\mathbb{O}, 1_\mathbb{O} \rangle = 2 \cdot N(1_\mathbb{O}) = 2 \cdot 1_\mathbb{O} \neq 0_\mathbb{O} \ .$$
- Char $\mathbb{O} = 2$: By lemma (3.23), there is an element $x \in \mathbb{O}$ such that $x \neq \bar{x}$. We obtain
$$\langle x, 1_\mathbb{O} \rangle = x + \bar{x} \neq 0_\mathbb{O} \ .$$

Now the map $\langle \cdot, \cdot \rangle = \bar{T}$ is non-degenerate by (20.16) of [TW], cf. definition (20.12) of [TW]. □

(3.25) Lemma Let $x, y \in \mathbb{O}$. Then the following holds:

(a) There is an associative subalgebra $\mathbb{A}$ containing both x and y.

(b) If we have $\bar{x} \neq x$, then $\mathbb{A}$ can be chosen to be a quaternion subalgebra $\mathbb{H}$.

(c) There exists a quaternion subalgebra $\mathbb{H}$ containing x.

Proof
Parts (a) and (b) result from the proof of (20.22) in [TW]. Part (c) is (20.23) of [TW]. □

(3.26) Lemma (Doubling Process) Let $\mathbb{A}$ be a subalgebra such that $\mathbb{A}^\perp \nsubseteq \mathbb{A}$, let $e \in \mathbb{A}^\perp \setminus \mathbb{A}$ and let $u := -N(e)$. Then the following holds:

(a) The set $\tilde{\mathbb{A}} := \mathbb{A} + e \cdot \mathbb{A}$ is a subalgebra.

(b) We have
$$\bar{e} = -e \ , \qquad e \cdot \mathbb{A} \subseteq \mathbb{A}^\perp \ , \qquad \mathbb{A} \cap e \cdot \mathbb{A} = \{0_\mathbb{O}\} \ .$$

(c) Given $x, y \in \mathbb{A}$, we have
$$(e \cdot x)(e \cdot y) = u(y\bar{x}) \ , \qquad (e \cdot x)y = e \cdot yx \ , \qquad x(e \cdot y) = e \cdot \bar{x}y \ .$$

Proof
This is (20.17) of [TW]. □

(3.27) Remark Let $\mathbb{A}$ be an alternative division ring. By definition, we have
$$Z(\mathbb{A}) = \{x \in \mathbb{A} \mid [x, \mathbb{A}] = [x, \mathbb{A}, \mathbb{A}] = 0_\mathbb{A}\} \ .$$

(3.28) Lemma Let $\mathbb{A}$ be an alternative division ring. Then we have
$$Z(\mathbb{A}) = \{x \in \mathbb{A} \mid [x, \mathbb{A}] = 0_\mathbb{A}\} \ .$$

Proof
The assertion is clearly true if $\mathbb{A}$ is a skew-field, thus we may suppose $\mathbb{O} := \mathbb{A}$ to be an octonion division algebra. By proposition (1.9.2) of [Sp], we have
$$\{x \in \mathbb{O} \mid [x, \mathbb{O}, \mathbb{O}] = 0_\mathbb{O}\} \subseteq \{x \in \mathbb{O} \mid [x, \mathbb{O}] = 0_\mathbb{O}\}$$
and therefore
$$\begin{aligned} Z(\mathbb{O}) &= \{x \in \mathbb{O} \mid [x, \mathbb{O}] = [x, \mathbb{O}, \mathbb{O}] = 0_\mathbb{O}\} \\ &\subseteq \{x \in \mathbb{O} \mid [x, \mathbb{O}, \mathbb{O}] = 0_\mathbb{O}\} \subseteq \{x \in \mathbb{O} \mid [x, \mathbb{O}] = [x, \mathbb{O}, \mathbb{O}] = 0_\mathbb{O}\} = Z(\mathbb{O}) \ . \end{aligned}$$
□

§ 3.3 The Bruck-Kleinfeld Theorem

On the one hand, we will need the minimum equation in §28.2, and on the other hand, we will need the classification of alternative division rings which are quadratic over a subfield of its center in §21.5. The main steps in the proof of the Bruck-Kleinfeld theorem in [TW] involve those algebras and the corresponding classification result, thus we mention them at this point and dedicate a short paragraph to this fundamental theorem.

(3.29) Definition Let $\mathbb{A}$ be an alternative division ring, let $\mathbb{K} := Z(\mathbb{A})$ and let $\mathbb{F}$ be a subfield of $\mathbb{K}$. Then $\mathbb{A}$ is *quadratic over* $\mathbb{F}$ if there are maps $T = T^{\mathbb{A}}_{\mathbb{F}}, N = N^{\mathbb{A}}_{\mathbb{F}} : \mathbb{A} \to \mathbb{F}$ such that

$$\forall\ a \in \mathbb{A} :\ a^2 - T(a)a + N(a) = 0_{\mathbb{A}}\ , \qquad \forall\ a \in \mathbb{F} :\ T(a) = 2a,\ N(a) = a^2\ . \tag{3.1}$$

The maps T and N are the *trace* and the *norm*, respectively.

(3.30) Remark Trace and norm are uniquely determined by the *minimum equation* (3.1).

(3.31) Proposition A non-associative alternative division ring $\mathbb{A}$ is quadratic over its center. In particular, an octonion division algebra $\mathbb{O}$ is quadratic over its center.

Proof
This is theorem (20.2) of [TW]. □

(3.32) Proposition Let $\mathbb{A}$ be an alternative division ring which is quadratic over some subfield $\mathbb{F}$ of its center $\mathbb{K} := Z(\mathbb{A})$, let T and N be the trace and the norm, respectively, and let

$$\sigma : \mathbb{A} \to \mathbb{A},\ x \mapsto T(x) - x\ .$$

Then exactly one of the following holds:

(i) $\mathbb{A} = \mathbb{K},\ \operatorname{Char}\mathbb{K} = 2,\ \mathbb{K}^2 \subseteq \mathbb{F} \neq \mathbb{K}$ and $\sigma = \mathrm{id}_{\mathbb{A}}$.

(ii) $\mathbb{A} = \mathbb{K} = \mathbb{F}$ and $\sigma = \mathrm{id}_{\mathbb{A}}$.

(iii) $\mathbb{A} = \mathbb{K}$, $\mathbb{K}/\mathbb{F}$ is a separable quadratic extension and $\langle\sigma\rangle = \mathrm{Gal}(\mathbb{K}/\mathbb{F})$.

(iv) $\mathbb{A}$ is a quaternion division algebra over $\mathbb{K}$, $\mathbb{F} = \mathbb{K}$ and $\sigma = \sigma_s$.

(v) $\mathbb{A}$ is an octonion division algebra over $\mathbb{K}$, $\mathbb{F} = \mathbb{K}$ and $\sigma = \sigma_s$.

In each case, we have

$$N(x) = xx^{\sigma} = x^{\sigma}x$$

for each $x \in \mathbb{A}$.

Proof
This is theorem (20.3) of [TW]. □

(3.33) Theorem (Bruck-Kleinfeld-Theorem) A non-associative alternative division ring is an octonion division algebra.

Proof
This is a consequence of proposition (3.31) and proposition (3.32). □

§ 3.4 Discrete Valuations and Composition Algebras

Given a Bruhat-Tits building, the defining field for the building at infinity is complete with respect to a discrete valuation. In particular, we will have to deal with octonions and thus composition algebras which are complete with respect to a discrete valuation.

(3.34) Definition Let $\mathbb{A}$ be an alternative division ring. A *discrete valuation of* $\mathbb{A}$ is a map $\nu : \mathbb{A}^* \to \mathbb{Z}$ such that

$$\nu(xy) = \nu(x) + \nu(y) \ , \qquad\qquad \nu(x+y) \geq \min\{\nu(x), \nu(y)\}$$

for all $x, y \in \mathbb{A}^*$. A *ν-uniformizer* is an element $\pi \in \mathbb{A}^*$ such that $\langle \nu(\pi) \rangle = \nu(\mathbb{A}^*)$.

(3.35) Lemma Let $\mathbb{A}$ be an alternative division ring with discrete valuation ν. Then the map

$$\delta_\nu : \mathbb{A} \times \mathbb{A} \to \mathbb{R}, \ (x,y) \mapsto \begin{cases} 2^{-\nu(x-y)} & , \ x \neq y \\ 0 & , \ x = y \end{cases}$$

is a metric.

Proof
This is lemma (9.18) of [W]. □

(3.36) Definition Let $\mathbb{A}$ be an alternative division ring with discrete valuation ν. Then $\mathbb{A}$ is *complete with respect to* ν if $(\mathbb{A}, \delta_\nu)$ is a complete metric space.

(3.37) Definition A *composition algebra over a field* $\mathbb{K}$ is a unital algebra $\mathbb{A}$ over $\mathbb{K}$ together with a non-defective quadratic form $N : \mathbb{A} \to \mathbb{K}$ which *permits composition*, i.e., we have

$$\forall \ x, y \in \mathbb{A} : \qquad N(xy) = N(x)N(y) \ .$$

(3.38) Lemma An octonion division algebra $\mathbb{O}$ is a composition algebra over $\mathbb{K} := Z(\mathbb{O})$.

Proof
The norm N is non-defective by corollary (3.24) and multiplicative by (9.9)(iii) of [TW]. □

(3.39) Lemma Let $\mathbb{A}$ be an alternative division ring with discrete valuation ν. Then the following holds:

(a) The algebra $\mathbb{A}$ is complete with respect to ν if and only if the center $Z(\mathbb{A})$ is complete with respect to $\nu_{|Z(\mathbb{A})}$.

(b) If $\mathbb{A}$ is a composition algebra which is complete with respect to ν, we have

$$\forall \ x \in \mathbb{A} : \qquad \nu(x) = \frac{\nu\big(N(x)\big)}{2} \left(= \frac{\nu\big(N(-x)\big)}{2} = \nu(-x) \right) \ .$$

Proof

(a) This is proposition (23.14) of [W].

(b) This results from proposition 1 of [P]. □

(3.40) Definition Let $\mathbb{A}$ be a composition division algebra over $\mathbb{K}$ which is complete with respect to a discrete valuation ν and such that its residue field $\bar{\mathbb{A}}$ is a composition algebra over the residue field $\bar{\mathbb{K}}$. Then $\mathbb{A}$ is *unramified* if we have $\nu(\mathbb{A}) = \nu(\mathbb{K})$, and *ramified* otherwise.

Chapter 4 Quadratic Spaces

Quadrangles of quadratic form type are parametrized by quadratic spaces.

§ 4.1 Basic Definitions and Basic Properties

(4.1) Definition

- An *(anisotropic) quadratic space* is a triple $(L_0, \mathbb{K}, q)$ such that $\mathbb{K}$ is a field, L_0 is a right vector space over $\mathbb{K}$ and q is an *(anisotropic) quadratic form on L_0*, i.e., $q : L_0 \to \mathbb{K}$ is a map such that the following holds:

  (Q1) $\forall\ a \in L_0,\ t \in \mathbb{K} :\ q(at) = q(a)t^2$.

  (Q2) The map
  $$f_q : L_0 \times L_0 \to \mathbb{K},\ (a, b) \mapsto q(a + b) - q(a) - q(b)$$
  is bilinear.

  (Q3) $\forall\ a \in L_0 :\ q(a) = 0_{\mathbb{K}} \ \Leftrightarrow\ a = 0_{L_0}$.

- A quadratic space $(L_0, \mathbb{K}, q)$ is *proper* if we have $f_q \not\equiv 0_{\mathbb{K}}$.

- Two quadratic spaces $(L_0, \mathbb{K}, q)$ and $(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ are *isomorphic* if there is an isomorphism
  $$\Phi = (\varphi, \phi) : (L_0, \mathbb{K}) \to (\tilde{L}_0, \tilde{\mathbb{K}})$$
  of vector spaces such that
  $$\tilde{q} \circ \varphi = \phi \circ q\ .$$

- A quadratic space $(L_0, \mathbb{K}, q)$ is *unital* if there is a *basepoint* $\epsilon \in L_0$ such that $q(\epsilon) = 1_{\mathbb{K}}$.

- Given a quadratic space $(L_0, \mathbb{K}, q)$ with basepoint ϵ, we set
  $$T : L_0 \to \mathbb{K},\ x \mapsto f_q(\epsilon, x) = f_q(x, \epsilon)\ , \qquad \sigma : L_0 \to L_0,\ x \mapsto \bar{x} := \epsilon \cdot T(x) - x\ .$$
  Given $a \in L_0^*$, we set
  $$\pi_a : L_0 \to L_0,\ v \mapsto v - a \cdot f_q(a, v)/q(a)\ , \qquad h_a : L_0 \to L_0,\ v \mapsto \pi_a \pi_\epsilon(v) \cdot q(a)\ .$$

(4.2) Lemma Let $(L_0, \mathbb{K}, q)$ and $(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ be quadratic spaces with basepoints ϵ and $\tilde{\epsilon}$, respectively, and let $(\varphi, \phi) : (L_0, \mathbb{K}, q) \to (\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ be an isomorphism of quadratic spaces such that $\varphi(\epsilon) = \tilde{\epsilon}$. Then we have
$$\tilde{T} \circ \varphi = \phi \circ T\ , \qquad \tilde{\sigma} \circ \varphi = \varphi \circ \sigma\ .$$

Proof
Given $x, y \in L_0$, we have
$$\begin{aligned} f_{\tilde{q}}\big(\varphi(x), \varphi(y)\big) &= \tilde{q}\big(\varphi(x) + \varphi(y)\big) - \tilde{q}\big(\varphi(x)\big) - \tilde{q}\big(\varphi(y)\big) \\ &= \phi\big(q(x + y) - q(x) - q(y)\big) = \phi\big(f_q(x, y)\big)\ . \end{aligned}$$
In particular, we have
$$\tilde{T}\big(\varphi(x)\big) = f_{\tilde{q}}\big(\varphi(x), \tilde{\epsilon}\big) = \phi\big(f_q(x, \epsilon)\big) = \phi\big(T(x)\big)$$
and thus
$$\varphi(x)^{\tilde{\sigma}} = \tilde{\epsilon} \cdot \tilde{T}\big(\varphi(x)\big) - \varphi(x) = \tilde{\epsilon} \cdot \phi\big(T(x)\big) - \varphi(x) = \varphi\big(\epsilon \cdot T(x) - x\big) = \varphi(x^\sigma)$$
for each $x \in L_0$. □

(4.3) Notation Throughout this paragraph, $(L_0, \mathbb{K}, q)$ is a quadratic space with basepoint ϵ.

(4.4) Lemma The maps $T : L_0 \to \mathbb{K}$ and $\sigma : L_0 \to L_0$ are $\mathbb{K}$-linear.

Proof
Given $x \in L_0$ and $s \in \mathbb{K}$, we have

$$T(x \cdot s) = f_q(\epsilon, x \cdot s) = f_q(\epsilon, x)s = T(x)s$$

and thus

$$(x \cdot s)^\sigma = \epsilon \cdot T(x \cdot s) - x \cdot s = \epsilon \cdot T(x)s - x \cdot s = \big(\epsilon \cdot T(x) - x\big) \cdot s = x^\sigma \cdot s \ .$$

Given $x, y \in L_0$, we have

$$T(x + y) = f_q(\epsilon, x + y) = f_q(\epsilon, x) + f_q(\epsilon, y) = T(x) + T(y)$$

and thus

$$(x + y)^\sigma = \epsilon \cdot T(x + y) - (x + y) = \epsilon \cdot T(x) - x + \epsilon \cdot T(y) - y = x^\sigma + y^\sigma \ .$$

□

(4.5) Lemma Given $x \in L_0$, $a \in L_0^*$, we have

$$h_a(x) = a \cdot f_q(a, x^\sigma) - x^\sigma \cdot q(a) \ .$$

Proof
Given $x \in L_0$, $a \in L_0^*$, we have

$$\begin{aligned} h_a(x) &= \pi_a \pi_\epsilon(x) \cdot q(a) = \pi_a\big(x - \epsilon \cdot f_q(\epsilon, x)\big) \cdot q(a) \\ &= -\pi_a(x^\sigma) \cdot q(a) = -\big(x^\sigma - a \cdot f_q(a, x^\sigma)/q(a)\big)q(a) = a \cdot f_q(a, x^\sigma) - x^\sigma \cdot q(a) \ . \end{aligned}$$

□

(4.6) Corollary Given $a \in L_0^*$, the corresponding *Hua map* h_a is $\mathbb{K}$-linear.

Proof
Let $a \in L_0^*$, $x, y \in L_0$ and $s \in \mathbb{K}$. By lemma (4.5) and lemma (4.4), we have

$$h_a(x \cdot s) = a \cdot f_q(a, x^\sigma \cdot s) - x^\sigma \cdot sq(a) = \big(a \cdot f_q(a, x^\sigma) - x^\sigma \cdot q(a)\big) \cdot s = h_a(x) \cdot s$$

and

$$\begin{aligned} h_a(x + y) &= a \cdot f_q\big(a, (x + y)^\sigma\big) - (x + y)^\sigma \cdot q(a) \\ &= a \cdot \big(f_q(a, x^\sigma) + f_q(a, y^\sigma)\big) - x^\sigma \cdot q(a) - y^\sigma \cdot q(a) = h_a(x) + h_a(y) \ . \end{aligned}$$

□

(4.7) Lemma Given $x \in L_0$, $a \in L_0^*$ and $s \in \mathbb{K}$, we have

$$h_{a \cdot s}(x) = h_a(x \cdot s^2) = h_a(x) \cdot s^2 \ , \qquad\qquad h_{a \cdot s} = s^2 \cdot h_a \ .$$

Proof
Let $x \in L_0$, $a \in L_0^*$ and $s \in \mathbb{K}$. By lemma (4.5), lemma (4.4) and corollary (4.6), we have

$$\begin{aligned} h_{a \cdot s}(x) &= a \cdot s f_q(a \cdot s, x^\sigma) - x^\sigma \cdot q(a \cdot s) = a \cdot f_q(a, x^\sigma)s^2 - x^\sigma \cdot q(a)s^2 \\ &= a \cdot f_q\big(a, (xs^2)^\sigma\big) - (xs^2)^\sigma \cdot q(a) = h_a(x \cdot s^2) = h_a(x) \cdot s^2 \ . \end{aligned}$$

□

(4.8) Lemma Given $x, y \in L_0$, we have

$$f_q(x^\sigma, y) = f_q(y^\sigma, x) = f_q(x, y^\sigma) \ .$$

Proof
Given $x, y \in L_0$, we have

$$f_q(x^\sigma, y) = f_q\big(\epsilon \cdot T(x) - x, y\big) = -f_q(x, y) + f_q(\epsilon, y)T(x) = -f_q(x, y) + T(x)T(y) \ ,$$

which is symmetric in x and y. □

(4.9) Remark

(a) Given $x \in L_0$, we have

$$f_q(x, x) = q(2x) - q(x) - q(x) = 4q(x) - 2q(x) = 2q(x) \ .$$

(b) We have

$$\epsilon^\sigma = \epsilon \cdot f_q(\epsilon, \epsilon) - \epsilon = \epsilon \cdot 2 - \epsilon = \epsilon \ .$$

(4.10) Corollary Given $x \in L_0$, we have

$$T(x^\sigma) = T(x) \ .$$

Proof
Let $x \in L_0$. By remark (4.9) (b) and lemma (4.8), we have

$$T(x) = f_q(\epsilon, x^\sigma) = f_q(\epsilon^\sigma, x^\sigma) = f_q\big(\epsilon, (x^\sigma)^\sigma\big) = T(x^\sigma) \ .$$

□

(4.11) Corollary We have $\sigma^2 = \mathrm{id}_{L_0}$.

Proof
Given $x \in L_0$, we have

$$(x^\sigma)^\sigma = \epsilon \cdot T(x^\sigma) - x^\sigma = \epsilon \cdot T(x) - \big(\epsilon \cdot T(x) - x\big) = x \ .$$

□

§ 4.2 Small Dimensions

Quadratic spaces of small dimension are in fact quadratic spaces corresponding to fields.

(4.12) Theorem Let $(L_0, \mathbb{K}, q)$ be a quadratic space with basepoint ϵ such that $\dim_{\mathbb{K}} L_0 \leq 2$. Then there is a unique multiplication $* : L_0 \times L_0 \to L_0$ such that the following holds:

(i) The triple $\tilde{\mathbb{F}} := (L_0, +, *)$ is a field.

(ii) The subspace $\tilde{\mathbb{K}} := \langle \epsilon \rangle_{\mathbb{K}}$ is a subfield of $\tilde{\mathbb{F}}$.

(iii) The map $\phi : \mathbb{K} \to \tilde{\mathbb{K}},\ s \mapsto \epsilon \cdot s$ is an isomorphism of fields.

(iv) Given $x \in L_0$, we have

$$\phi\big(q(x)\big) = x * x^\sigma \ .$$

Proof
Condition (iii) forces

$$\forall\ s,t \in \mathbb{K}: \qquad (\epsilon \cdot s) * (\epsilon \cdot t) = \phi(s) * \phi(t) = \phi(st) = \epsilon \cdot st\ ,$$

which makes $\tilde{\mathbb{K}}$ really into a field, hence (ii) and (iii) hold.

- If we have $\dim_{\mathbb{K}} L_0 = 1$, then $\tilde{\mathbb{F}} = \tilde{\mathbb{K}}$ is a field. By lemma (4.4) and remark (4.9) (b), we have
$$\forall\ s \in \mathbb{K}: \qquad (\epsilon \cdot s)^\sigma = \epsilon^\sigma \cdot s = \epsilon \cdot s$$
and thus
$$\forall\ s \in \mathbb{K}: \qquad \phi\big(q(\epsilon \cdot s)\big) = \phi(s^2) = \epsilon \cdot s^2 = (\epsilon \cdot s) * (\epsilon \cdot s) = (\epsilon \cdot s) * (\epsilon \cdot s)^\sigma\ .$$
- If we have $\dim_{\mathbb{K}} L_0 = 2$, there is an element $\tilde{x} \in L_0 \setminus \tilde{\mathbb{K}}$. Conditions (i) and (ii) force $\tilde{x} * \epsilon := \tilde{x} =: \epsilon * \tilde{x}$, and condition (iv) forces
$$\tilde{x} \cdot T(\tilde{x}) - \tilde{x} * \tilde{x} = \tilde{x} * (\epsilon \cdot T(\tilde{x}) - \tilde{x}) = \tilde{x} * \tilde{x}^\sigma = \phi\big(q(\tilde{x})\big) = \epsilon \cdot q(\tilde{x})$$
and thus $\tilde{x} * \tilde{x} := \tilde{x} \cdot T(\tilde{x}) - \epsilon \cdot q(\tilde{x})$. Given $s, t, \tilde{s}, \tilde{t} \in \mathbb{K}$, we set
$$(\epsilon \cdot s + \tilde{x} \cdot t) * (\epsilon \cdot \tilde{s} + \tilde{x} \cdot \tilde{t}) := \epsilon \cdot (s\tilde{s} - q(\tilde{x})t\tilde{t}) + \tilde{x} \cdot \big(s\tilde{t} + \tilde{s}t + T(\tilde{x})t\tilde{t}\big)\ .$$
Let $f := y^2 - yT(\tilde{x}) + q(\tilde{x}) \in \mathbb{K}[y]$. Given $s \in \mathbb{K}$, we have
$$s^2 - sT(\tilde{x}) + q(\tilde{x}) = q\big(\epsilon\cdot(s)\big) - sf_q(\epsilon, \tilde{x}) + q(\tilde{x}) = q\big(\epsilon\cdot(s)\big) + q(\tilde{x}) - f_q(\epsilon\cdot s, \tilde{x}) = q(\epsilon\cdot s + \tilde{x}) \neq 0_{\mathbb{K}}\ .$$
Therefore, the polynomial $f \in \mathbb{K}[y]$ is irreducible. Let x be an element in the algebraic closure of $\mathbb{K}$ such that $f(x) = 0_{\mathbb{K}}$. Then $\mathbb{F} := \mathbb{K}(x)$ is a field with $\dim_{\mathbb{K}} \mathbb{F} = \deg(f) = 2$ and multiplication given by
$$x^2 = xT(\tilde{x}) - q(\tilde{x})\ .$$
Therefore, the map
$$\Phi : \mathbb{F} \to \tilde{\mathbb{F}},\ \ s + xt \mapsto \epsilon \cdot s + \tilde{x} \cdot t$$
is an isomorphism of fields, hence (i) holds. Given $s, t \in \mathbb{K}$, we finally have
$$\begin{aligned}\phi\big(q(\epsilon \cdot s + \tilde{x} \cdot t)\big) &= \epsilon \cdot \big(s^2 + q(\tilde{x})t^2 + f_q(\epsilon \cdot s, \tilde{x} \cdot t)\big) = \epsilon \cdot \big(s^2 + f_q(\epsilon, \tilde{x})st + q(\tilde{x})t^2\big) \\ &= \epsilon \cdot (s^2 + T(\tilde{x})st + q(\tilde{x})t^2) + \tilde{x} \cdot \big(-st + st + T(\tilde{x})t^2 - T(\tilde{x})t^2\big) \\ &= \big(\epsilon \cdot s + \tilde{x} \cdot t\big) * \big(\epsilon \cdot s + \epsilon \cdot T(\tilde{x} \cdot t) - \tilde{x} \cdot t\big) \\ &= (\epsilon \cdot s + \tilde{x} \cdot t) * (\epsilon \cdot s + \tilde{x} \cdot t)^\sigma\ ,\end{aligned}$$
hence (iv) holds.

□

(4.13) Definition Given a quadratic space $(L_0, \mathbb{K}, q)$ with basepoint ϵ and $\dim_{\mathbb{K}} L_0 \leq 2$, we set
$$\mathbb{F}(L_0, \mathbb{K}, q) := \tilde{\mathbb{F}}\ ,$$
where $\tilde{\mathbb{F}}$ is as in theorem (4.12).

(4.14) Corollary Let $(L_0, \mathbb{K}, q)$ be a quadratic space with basepoint ϵ and $\dim_{\mathbb{K}} L_0 \leq 2$, let $\tilde{\mathbb{F}} := \mathbb{F}(L_0, \mathbb{K}, q)$, let $\tilde{\mathbb{K}} := \langle \epsilon \rangle_{\mathbb{K}}$ and let $\phi : \mathbb{K} \to \tilde{\mathbb{K}},\ s \mapsto \epsilon \cdot s$. Then $\tilde{\mathbb{F}}$ is quadratic over $\tilde{\mathbb{K}}$, we have $N^{\tilde{\mathbb{F}}}_{\tilde{\mathbb{K}}} = \phi \circ q$, the map $(\mathrm{id}_{L_0}, \phi) : (L_0, \mathbb{K}, q) \to (\tilde{\mathbb{F}}, \tilde{\mathbb{K}}, N^{\tilde{\mathbb{F}}}_{\tilde{\mathbb{K}}})$ is an isomorphism of quadratic spaces, and exactly one of the following holds:

(i) $\operatorname{Char} \tilde{\mathbb{F}} = 2$, $\tilde{\mathbb{F}}^2 \subseteq \tilde{\mathbb{K}} \neq \tilde{\mathbb{F}}$ and $\sigma = \mathrm{id}_{\tilde{\mathbb{F}}}$, which means that $\tilde{\mathbb{F}}/\tilde{\mathbb{K}}$ is inseparable.

(ii) $\tilde{\mathbb{K}} = \tilde{\mathbb{F}}$ and $\sigma = \mathrm{id}_{\tilde{\mathbb{F}}}$.

(iii) $\tilde{\mathbb{F}}/\tilde{\mathbb{K}}$ is a separable quadratic extension and $\langle \sigma \rangle = \mathrm{Gal}(\tilde{\mathbb{F}}/\tilde{\mathbb{K}})$.

Proof

By construction, we have

$$\forall\, x \in \tilde{\mathbb{F}}: \qquad x * x - x * \big(\epsilon \cdot T(x)\big) + \epsilon \cdot q(x) = 0_{\tilde{\mathbb{F}}}\ ,$$

thus $\tilde{\mathbb{F}}$ is quadratic over $\tilde{\mathbb{K}} = \langle \epsilon \rangle_{\mathbb{K}}$ with $N_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{F}}} = \phi \circ q$, $T_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{F}}} = \phi \circ T$ and

$$\forall\, x \in L_0,\ s \in \mathbb{K}: \qquad \mathrm{id}_{L_0}(x \cdot s) = x \cdot s = x * (\epsilon \cdot s) = \mathrm{id}_{L_0}(x) * \phi(s)$$

(which shows that $(\mathrm{id}_{L_0}, \phi)$ is an isomorphism of quadratic spaces), and we have

$$\forall\, x \in \tilde{\mathbb{F}}: \qquad \sigma(x) = \epsilon \cdot T(x) - x = T_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{F}}}(x) - x\ ,$$

which is just the map σ in proposition (3.32) (which we apply). We have $\dim_{\tilde{\mathbb{K}}} \tilde{\mathbb{F}} \leq 2$, thus $(\tilde{\mathbb{F}}, \tilde{\mathbb{K}}, \sigma)$ is neither of type (iv) nor of type (v). □

(4.15) Lemma Let $\mathbb{A}$ be an alternative division ring which is quadratic over a subfield $\mathbb{F}$ of its center. Then $(\mathbb{A}, \mathbb{F}, N)$ with $N := N_{\mathbb{F}}^{\mathbb{A}}$ is a quadratic space.

Proof

By proposition (3.32), we have $N(x) = xx^\sigma$ for each $x \in \mathbb{A}$.

(Q1) Given $s \in \mathbb{F}$, we have

$$N(x \cdot s) = N(x)N(s) = N(x)s^2\ .$$

(Q2) Given $x, y \in \mathbb{A}$, we have

$$\begin{aligned} f_q(x,y) &= N(x+y) - N(x) - N(y) \\ &= (x+y)(x+y)^\sigma - xx^\sigma - yy^\sigma = xy^\sigma + yx^\sigma = xy^\sigma + (xy^\sigma)^\sigma = T(xy^\sigma)\ , \end{aligned}$$

which is $\mathbb{K}$-linear in x and y by lemma (4.4).

(Q3) Given $x \in \mathbb{A}$, we have

$$N(x) \neq 0_{\mathbb{A}} \ \Leftrightarrow\ x \in U_{\mathbb{A}} = \mathbb{A}^*\ .$$

□

(4.16) Definition A quadratic space $(L_0, \mathbb{K}, q)$ with basepoint ϵ is *(linear) of type (m)* if we have $(L_0, \mathbb{K}, q) \cong (\mathbb{A}, \mathbb{F}, N_{\mathbb{F}}^{\mathbb{A}})$ as in (m) of proposition (3.32), i.e., the quadratic space $(L_0, \mathbb{K}, q)$ is of type

(i) if $\mathbb{F} := L_0$ is a field with $\mathrm{Char}\,\mathbb{F} = 2$, $\mathbb{F}^2 \subseteq \mathbb{K} \neq \mathbb{F}$, $\sigma = \mathrm{id}_{\mathbb{F}}$ and $q = N_{\mathbb{K}}^{\mathbb{F}}$,

(ii) if we have $L_0 = \mathbb{K}$, $\sigma = \mathrm{id}_{\mathbb{K}}$ and $q = N_{\mathbb{K}}^{\mathbb{K}}$,

(iii) if $\mathbb{E} := L_0$ is a field, $\mathbb{E}/\mathbb{K}$ is a separable quadratic extension, $\langle \sigma \rangle = \mathrm{Gal}(\mathbb{E}/\mathbb{K})$ and $q = N_{\mathbb{K}}^{\mathbb{E}}$,

(iv) if $\mathbb{H} := L_0$ is a quaternion division algebra over $\mathbb{K}$, $\sigma = \sigma_s$ and $q = N_{\mathbb{K}}^{\mathbb{H}}$,

(v) if $\mathbb{O} := L_0$ is an octonion division algebra over $\mathbb{K}$, $\sigma = \sigma_s$ and $q = N_{\mathbb{K}}^{\mathbb{O}}$.

(4.17) Remark These are exactly the quadratic spaces such that the corresponding Moufang set of quadratic form type is isomorphic to a Moufang set of linear type, cf. theorem (31.7) and lemma (31.23). This explains the terminology.

§ 4.3 Clifford Algebras and the Clifford Invariant

(4.18) Definition

- A *central simple algebra* is a finite-dimensional associative algebra $(A, \mathbb{K})$ which is simple as a ring and such that $Z(A) \cong \mathbb{K}$ as fields.
- Two central simple algebras $(A, \mathbb{K})$ and $(\tilde{A}, \tilde{\mathbb{K}})$ are *isomorphic* if there is an isomorphism $(\gamma, \phi) : (A, \mathbb{K}) \to (\tilde{A}, \tilde{\mathbb{K}})$ of vector spaces such that γ is an isomorphism of rings.
- Let $(L_0, \mathbb{K}, q)$ be a quadratic space, let $T(L_0)$ be the tensor algebra of L_0 and let

  $$I(q) := \langle u \otimes u - 1_{\mathbb{K}} \cdot q(u) \mid u \in L_0 \rangle \ .$$

  Then $C(q) := T(L_0)/I(q)$ is the *Clifford algebra of* q.

(4.19) Lemma Let $(L_0, \mathbb{K}, q)$ be a non-defective quadratic space such that $\dim_{\mathbb{K}} L_0$ is even. Then $\big(C(q), \mathbb{K}\big)$ is a central simple algebra.

Proof
This results from proposition (11.6) of [EKM]. □

(4.20) Theorem (Wedderburn) Given a central simple algebra $(A, \mathbb{K})$, there are a unique natural number $n \in \mathbb{N}^*$ and an associative division algebra $(\mathbb{D}, \mathbb{K})$ such that $(A, \mathbb{K}) \cong \big(M_n(\mathbb{D}), \mathbb{K}\big)$ as algebras. The algebra $(\mathbb{D}, \mathbb{K})$ is unique up to isomorphism of algebras.

Proof
Cf. theorem (1.1) of [KMRT]. □

(4.21) Definition

- Given a central simple algebra $(A, \mathbb{K})$, we set $\mathrm{S}(A, \mathbb{K}) := [(\mathbb{D}, \mathbb{K})]$, where $(\mathbb{D}, \mathbb{K})$ is a division algebra as in theorem (4.20) and $[(\mathbb{D}, \mathbb{K})]$ denotes its isomorphism class.
- Let $(L_0, \mathbb{K}, q)$ be a non-defective quadratic space such that $\dim_{\mathbb{K}} L_0$ is even. Then

  $$\mathrm{Clif}(q) := \mathrm{S}\big(C(q), \mathbb{K}\big)$$

  is the *Clifford invariant of* q.

(4.22) Lemma Given two isomorphic central simple algebras $(A, \mathbb{K})$ and $(\tilde{A}, \tilde{\mathbb{K}})$, we have

$$\mathrm{S}(A, \mathbb{K}) = S(\tilde{A}, \tilde{\mathbb{K}}) \ .$$

Proof
Let $n, \tilde{n}$ and $(\mathbb{D}, \mathbb{K}), (\tilde{\mathbb{D}}, \tilde{\mathbb{K}})$ be as in theorem (4.20). Then we have

$$\big(M_n(\mathbb{D}), \mathbb{K}\big) \cong (A, \mathbb{K}) \cong (\tilde{A}, \tilde{\mathbb{K}}) \cong \big(M_{\tilde{n}}(\tilde{\mathbb{D}}), \tilde{\mathbb{K}}\big)$$

as algebras and thus $(\mathbb{D}, \mathbb{K}) \cong (\tilde{\mathbb{D}}, \tilde{\mathbb{K}})$ by theorem (4.20). □

(4.23) Lemma Let $(L_0, \mathbb{K}, q)$, $(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ be isomorphic quadratic spaces. Then we have

$$\big(C(q), \mathbb{K}\big) \cong \big(C(\tilde{q}), \tilde{\mathbb{K}}\big)$$

as algebras. In particular, we have $\mathrm{Clif}(q) = \mathrm{Clif}(\tilde{q})$ if the dimensions are even and at least one (and thus both) quadratic spaces are non-defective.

Proof
This results from (12.23) of [TW]. In particular, we may apply lemma (4.22). □

§ 4.4 Norm Splittings

(4.24) Definition Let $(L_0, \mathbb{K}, q)$ be a quadratic space. A *norm splitting of* q is a triple $(\mathbb{E}, \cdot, \{v_1, \ldots, v_d\})$ such that the following holds:

(N1) $\mathbb{E}/\mathbb{K}$ is a separable quadratic extension,

(N2) $\cdot : L_0 \times \mathbb{E} \to L_0$ is a scalar multiplication extending the scalar multiplication $\cdot : L_0 \times \mathbb{K} \to L_0$,

(N3) $\{v_1, \ldots, v_d\}$ is an $\mathbb{E}$-Basis of L_0 with

$$\forall\ t_1, \ldots, t_d \in \mathbb{E} : \qquad q\Big(\sum_{i=1}^{d} v_i t_i\Big) = \sum_{i=1}^{d} s_i N(t_i)\ ,$$

where $s_i = q(v_i)$ for each $i \in \{1, \ldots, n\}$ and $N = N_{\mathbb{K}}^{\mathbb{E}}$.

The elements $s_1, \ldots, s_d \in \mathbb{K}$ are called the *constants of the norm splitting*.

(4.25) Lemma Let $\mathbb{O}$ be an octonion division algebra with center $\mathbb{K} := Z(\mathbb{O})$ and let $\mathbb{E}$ be a subfield such that $\mathbb{E}/\mathbb{K}$ is a separable quadratic extension (which exists by lemma (3.23) and (20.19) of [TW]). Then there are $v_1, \ldots, v_4 \in \mathbb{O}$ such that $(\mathbb{E}, \cdot, \{v_1, \ldots, v_4\})$ is a norm splitting of $(\mathbb{O}, \mathbb{K}, N)$, satisfying $s_1 \cdots s_4 \in N(\mathbb{E})$.

Proof
Let $v_1 := 1_{\mathbb{O}}$. By (20.20) of [TW], there is an element $v_2 \in \mathbb{E}^{\perp} \setminus \mathbb{E}$, and $\mathbb{H}_2 := \mathbb{E} + v_2 \cdot \mathbb{E}$ is a quaternion division algebra. By (20.21) of [TW], there is an element $v_3 \in \mathbb{H}_2^{\perp} \setminus \mathbb{H}_2$. Finally, let $v_4 := v_2 v_3 \in \mathbb{H}_2^{\perp} \setminus \mathbb{H}_2$. Then $\{v_1, \ldots, v_4\}$ is an $\mathbb{E}$-Basis of $\mathbb{O}$. By construction and (20.20) of [TW] again,

$$\mathbb{H}_i := \mathbb{E} + v_i \cdot \mathbb{E}$$

is a quaternion division algebra for each $i \in \{2, 3, 4\}$, satisfying $v_j \in \mathbb{H}_i^{\perp} \setminus \mathbb{H}_i$ for all $i \neq j \in \{2, 3, 4\}$. By lemma (3.26), we have

$$\begin{aligned}
N\Big(\sum_{i=1}^{4} t_i v_i\Big) &= (t_1 v_1 + t_2 v_2 + t_3 v_3 + t_4 v_4) \cdot (\bar{v}_1 \bar{t}_1 + \bar{v}_2 \bar{t}_2 + \bar{v}_3 \bar{t}_3 + \bar{v}_4 \bar{t}_4) \\
&= \sum_{i=1}^{4} t_i v_i \bar{v}_i \bar{t}_i + \sum_{i \neq j} \big((t_i v_i)(\bar{v}_j \bar{t}_j) + (t_j v_j)(\bar{v}_i \bar{t}_i)\big) \\
&= \sum_{i=1}^{4} N(v_i) N(t_i) + \sum_{i \neq j} \big(\bar{v}_j (\bar{v}_i \bar{t}_i \bar{t}_j) + (v_j \bar{t}_j)(\bar{v}_i \bar{t}_i)\big) \\
&= \sum_{i=1}^{4} N(v_i) N(t_i) + \sum_{i \neq j} \big(\bar{v}_j (\bar{v}_i \bar{t}_i \bar{t}_j) + v_j (\bar{v}_i \bar{t}_i \bar{t}_j)\big) \\
&= \sum_{i=1}^{4} N(v_i) N(t_i) + \sum_{i \neq j} \big((\bar{v}_j + v_j)(\bar{v}_i \bar{t}_i \bar{t}_j)\big) \\
&= \sum_{i=1}^{4} N(v_i) N(t_i) + \sum_{i \neq j} \big((-v_j + v_j)(\bar{v}_i \bar{t}_i \bar{t}_j)\big) = \sum_{i=1}^{4} N(v_i) N(t_i)\ .
\end{aligned}$$

In particular, we have $s_i = N(v_i)$ for each $i \in \{1, \ldots, 4\}$, and thus, by lemma (3.26) again,

$$s_1 \cdots s_4 = N(v_1) \cdots N(v_4) = N(v_1 \cdots v_4) = N(v_2 v_3 v_2 v_3) = N(-v_2^2 v_3^2) \in N(\mathbb{K}) \subseteq N(\mathbb{E})\ .$$

□

(4.26) Definition A quadratic space $(L_0, \mathbb{K}, q)$ is

- *of type E_6* if $\dim_{\mathbb{K}} L_0 = 6$ and q has a norm splitting,
- *of type E_7* if $\dim_{\mathbb{K}} L_0 = 8$ and q has a norm splitting $(\mathbb{E}, \cdot, \{v_1, \ldots, v_4\})$ such that
$$s_1 \cdots s_4 \notin N(\mathbb{E}) \ ,$$
- *of type E_8* if $\dim_{\mathbb{K}} L_0 = 12$ and q has a norm splitting $(\mathbb{E}, \cdot, \{v_1, \ldots, v_6\})$ such that
$$-s_1 \cdots s_6 \in N(\mathbb{E}) \ .$$

(4.27) Remark As we only deal with anisotropic quadratic spaces and since each anisotropic space having a norm splitting is automatically non-defective by remark (12.12) of [TW] (and thus proper), we may reformulate remark (12.30) of [TW] as follows:

(4.28) Lemma Let $(L_0, \mathbb{K}, q)$ be a quadratic space having a norm splitting $(\mathbb{E}, \cdot, \{v_1, \ldots, v_d\})$ with constants $s_1, \ldots, s_d$ and let
$$\gamma := (-1)^{[d/2]} s_1 \cdots s_d \ .$$
Then the following holds:

(a) We have $C(q) \cong M_{2^d}(\mathbb{K})$ if $\gamma \in N(\mathbb{E})$ and thus $\mathrm{Clif}(q) = [(\mathbb{K}, \mathbb{K})]$.

(b) We have $C(q) \cong M_{2^{d-1}}(\mathbb{H})$ if $\gamma \notin N(\mathbb{E})$ and thus $\mathrm{Clif}(q) = [(\mathbb{H}, \mathbb{K})]$, where $\mathbb{H} = (\mathbb{E}/\mathbb{K}, \gamma)$.

Proof
This is remark (12.30) of [TW]. □

(4.29) Corollary Let $\mathbb{O}$ be an octonion division algebra with norm N and center $\mathbb{K} := Z(\mathbb{O})$ and let $(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ be a quadratic space of type E_7. Then the following holds:

(a) We have $\mathrm{Clif}(N) = [(\mathbb{K}, \mathbb{K})]$.

(b) We have $\mathrm{Clif}(\tilde{q}) = [(\tilde{\mathbb{H}}, \tilde{\mathbb{K}})]$ for some quaternion division algebra $\tilde{\mathbb{H}}$ with center $\tilde{\mathbb{K}}$.

(c) We have $(\mathbb{O}, \mathbb{K}, N) \ncong (\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ as quadratic spaces.

Proof

(a) By lemma (4.25), N has a norm splitting $(\mathbb{E}, \cdot, \{v_1, \ldots, v_4\})$ such that
$$\gamma = s_1 \cdots s_4 \in N(\mathbb{E}) \ ,$$
hence $\mathrm{Clif}(N) = [(\mathbb{K}, \mathbb{K})]$ by lemma (4.28) (a).

(b) By definition, q has a norm splitting $(\tilde{\mathbb{E}}, \cdot, \{v_1, \ldots, v_4\})$ such that
$$\gamma = s_1 \cdots s_4 \notin N(\tilde{\mathbb{E}}) \ ,$$
hence we may apply lemma (4.28) (b).

(c) Since $\mathbb{K}$ is a field and $\tilde{\mathbb{H}}$ is non-commutative, we have $(\mathbb{K}, \mathbb{K}) \ncong (\tilde{\mathbb{H}}, \tilde{\mathbb{K}})$ as algebras and thus
$$\mathrm{Clif}(N) \neq \mathrm{Clif}(q) \ .$$
Now the quadratic spaces can't be isomorphic by lemma (4.23).

□

§ 4.5 Quadratic Spaces of Type F_4

(4.30) Definition Let $(L_0, \mathbb{K}, q)$ be a quadratic space.

- Given a subset $W \subseteq L_0$, we set

$$W^\perp := \{v \in L_0 \mid f_q(v, W) = 0_\mathbb{K}\} .$$

- The subspace $\mathrm{Def(q)} := L_0^\perp$ is the *defect of q*.
- The quadratic space $(L_0, \mathbb{K}, q)$ is *non-defective* if $\mathrm{Def}(q) = \{0_{L_0}\}$, and *defective* otherwise.

(4.31) Remark Let $(L_0, \mathbb{K}, q)$ be a non-proper quadratic space, i.e., we have $f_q \equiv 0_\mathbb{K}$. Then we have $\mathrm{Def}(q) = L_0$, and $(L_0, \mathbb{K}, q)$ is defective. In particular, a quadratic space is proper if it is non-defective.

(4.32) Definition Let $(L_0, \mathbb{K}, q)$ be a quadratic space and let $R_0 := \mathrm{Def}(q)$. Then $(L_0, \mathbb{K}, q)$ is a quadratic space *of type F_4* if we have $\mathrm{Char}\, \mathbb{K} = 2$ and the following holds:

- $q(R_0)/q(\rho)$ is a subfield of $\mathbb{K}$ for some $\rho \in R_0^*$.
- For some complement S_0 of R_0 in L_0, the restriction of q to S_0 has a norm splitting $(\mathbb{E}, \cdot, \{v_1, v_2\})$ with constants s_1, s_2 such that $s_1 s_2 \in q(R_0)/q(\rho)$.

(4.33) Remark By (14.2) of [TW], the field $q(R_0)/q(\rho)$ is independent of the choice of $\rho \in R_0^*$. In particular, we have

$$\mathbb{F} := g(R_0)/q(\epsilon) = q(R_0)$$

if $(L_0, \mathbb{K}, q)$ is a quadratic space with basepoint $\epsilon \in R_0^*$.

(4.34) Notation Let $(L_0, \mathbb{K}, q)$ be a quadratic space of type F_4 and let $\rho \in R_0^*$. We set

$$\mathbb{F} := q(R_0)/q(\rho) .$$

(4.35) Lemma A quadratic space $(L_0, \mathbb{K}, q)$ of type F_4 is proper.

Proof
By definition, there is a complement S_0 of R_0 in L_0 such that the restriction of q to S_0 has a norm splitting $(\mathbb{E}, \cdot, \{v_1, v_2\})$ with constants s_1, s_2 such that $s_1 s_2 \in \mathbb{F}$. Moreover, q is anisotropic, hence $(S_0, \mathbb{K}, q_{|S_0})$ is non-defective by (12.12) of [TW]. In particular, $(S_0, \mathbb{K}, q_{|S_0})$ and thus $(L_0, \mathbb{K}, q)$ is proper. □

(4.36) Lemma Let $(L_0, \mathbb{K}, q)$ be a quadratic space of type F_4 and let $(\tilde{\mathbb{A}}, \tilde{\mathbb{F}}, \tilde{N})$ be a quadratic space of type (m). Then we have

$$(L_0, \mathbb{K}, q) \ncong (\tilde{\mathbb{A}}, \tilde{\mathbb{F}}, \tilde{N})$$

as quadratic spaces.

Proof
Assume $(L_0, \mathbb{K}, q) \cong (\tilde{\mathbb{A}}, \tilde{\mathbb{F}}, \tilde{N})$. Since $(L_0, \mathbb{K}, q)$ is proper by lemma (4.35), $(\tilde{\mathbb{F}}, \tilde{\mathbb{A}}, \tilde{N})$ is proper as well and thus (m)$\notin$ {(i),(ii)} since we have $\mathrm{Char}\, \mathbb{K} = 2$. But a quadratic space of type (m)$\in$ {(iii),. . . ,(v)} is non-defective by (20.15) of [TW] and the proof of corollary (3.24), while a quadratic space of type F_4 is defective by definition since $\mathbb{F} = q(R_0)/q(\rho)$ is a field. □

Chapter 5 Involutory Sets

Quadrangles of purely involutory type are parametrized by proper involutory sets.

§ 5.1 Basic Definitions

(5.1) Definition

- An *involutory set* is a triple $(\mathbb{K}, \mathbb{K}_0, \sigma)$, where $\mathbb{K}$ is a skew-field, σ is an involution of $\mathbb{K}$ and $\mathbb{K}_0$ is an additive subgroup of $\mathbb{K}$ such that

$$1_{\mathbb{K}} \in \mathbb{K}_0 \ , \qquad \{a + a^\sigma \mid a \in \mathbb{K}\} =: \mathbb{K}_\sigma \subseteq \mathbb{K}_0 \subseteq \mathrm{Fix}(\sigma) \ , \qquad \forall\, a \in \mathbb{K} : \ a^\sigma \mathbb{K}_0 a \subseteq \mathbb{K}_0 \ .$$

- An involutory set $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is *proper* if we have $\sigma \neq \mathrm{id}_{\mathbb{K}}$ and $\langle \mathbb{K}_0 \rangle = \mathbb{K}$ as rings.

- Two involutory sets $(\mathbb{K}, \mathbb{K}_0, \sigma)$ and $(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ are *isomorphic* if there is an isomorphism $\phi : \mathbb{K} \to \tilde{\mathbb{K}}$ of skew-fields such that

$$\phi(\mathbb{K}_0) = \tilde{\mathbb{K}}_0 \ , \qquad\qquad \phi \circ \sigma = \tilde{\sigma} \circ \phi \ .$$

- Let $(\mathbb{K}, \mathbb{K}_0, \sigma)$, $(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ be two involutory sets. A *Jordan homomorphism* is an additive monomorphism $\gamma : \mathbb{K}_0 \to \tilde{\mathbb{K}}_0$ such that

$$\gamma(1_{\mathbb{K}}) = 1_{\tilde{\mathbb{K}}} \ , \qquad\qquad \forall\, x, y \in \mathbb{K}_0 : \ \gamma(xyx) = \gamma(x)\gamma(y)\gamma(x) \ .$$

(5.2) Lemma Let $(\mathbb{K}, \mathbb{K}_0, \sigma)$ be an involutory set. If $\mathbb{K}_0$ is commutative, then $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is non-proper. In particular, $\mathbb{K}$ is non-commutative if $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is proper.

Proof
Suppose that we have $\langle \mathbb{K}_0 \rangle = \mathbb{K}$ and $\sigma \neq \mathrm{id}_{\mathbb{K}}$. Then $\mathbb{K}$ is commutative and $\mathbb{K}_0 = \mathrm{Fix}(\sigma) \subsetneq \mathbb{K}$ is a field, cf. remark (11.3) of [TW]. But then we have

$$\mathbb{K} = \langle \mathbb{K}_0 \rangle = \mathbb{K}_0 \neq \mathbb{K} \qquad \lightning \ .$$

$\square$

§ 5.2 Jordan Isomorphisms of Involutory Sets

The following result is essential for the classification of Jordan isomorphisms of pseudo-quadratic spaces on the one hand, and on the other hand, it is essential for the classification of a certain class of 443-foundations.

Parts of the solution can be deduced from paragraph 4.10 of [K]. However, there are still some details which have to be worked out. But for reasons of brevity, we don't try to complete the proof at this point and suppose the result to be true.

(5.3) Theorem Let $(\mathbb{K}, \mathbb{K}_0, \sigma)$ be a proper involutory set, let $(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ be an involutory set and let $\gamma : \mathbb{K}_0 \to \tilde{\mathbb{K}}_0$ be a Jordan isomorphism such that $\gamma(1_{\mathbb{K}}) = 1_{\tilde{\mathbb{K}}}$. Then γ is induced by an isomorphism

$$\phi : (\mathbb{K}, \mathbb{K}_0, \sigma) \to (\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma}) \ .$$

In particular, $(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ is proper as well.

§ 5.3 Involutory Sets of Quadratic Type

Quadrangles parametrized by non-proper involutory sets can equally described by quadratic spaces, cf. chapter 38 of [TW]. Although they don't appear explicitly in the description of the six families, they occur as substructures of proper pseudo-quadratic spaces, the parametrizing structures for quadrangles of purely pseudo-quadratic form type.

(5.4) Definition Let $(\mathbb{A}, \mathbb{F}, \sigma)$ be an involutory set (with $\mathbb{A}$ possibly an alternative division ring) and $\mathbb{K} := Z(\mathbb{A})$. Then the involutory set is *quadratic of type*

(i) if we have $\mathbb{A} = \mathbb{K}$, $\operatorname{Char}\mathbb{K} = 2$, $\mathbb{K}^2 \subseteq \mathbb{F} \neq \mathbb{K}$ and $\sigma = \mathrm{id}_{\mathbb{A}}$,

(ii) if we have $\mathbb{A} = \mathbb{K} = \mathbb{F}$ and $\sigma = \mathrm{id}_{\mathbb{A}}$,

(iii) if we have $\mathbb{A} = \mathbb{K}$, $\mathbb{K}/\mathbb{F}$ is a separable quadratic extension and $\langle\sigma\rangle = \mathrm{Gal}(\mathbb{K}/\mathbb{F})$,

(iv) if $\mathbb{A}$ is a quaternion division algebra over $\mathbb{K}$, $\mathbb{F} = \mathbb{K}$ and $\sigma = \sigma_s$,

(v) if $\mathbb{A}$ is an octonion division algebra over $\mathbb{K}$, $\mathbb{F} = \mathbb{K}$ and $\sigma = \sigma_s$,

where σ_s denotes the standard involution, cf. proposition (3.32). We denote the corresponding norm by N and the corresponding trace by T.

(5.5) Remark The following lemma gives a criterion when an involutory set is of quadratic type. We will need it for the classification of a certain class of 443-foundations.

(5.6) Lemma Let $(\mathbb{K}, \mathbb{K}_0, \sigma)$ be an involutory set, let $\mathbb{F}$ be a subfield of the center $Z(\mathbb{K})$ and let $\gamma \in \mathrm{Aut}(\mathbb{K}, +)$ such that

$$\gamma(1_{\mathbb{K}}) = 1_{\mathbb{K}} \ , \qquad\qquad \forall\, x \in \mathbb{K}: \ \gamma(x)x \in \mathbb{F} \ .$$

Then the following holds:

(a) $\mathbb{K}$ is quadratic over $\mathbb{F}$.

(b) If $\mathbb{K}$ is non-commutative, then $\mathbb{K}$ is a quaternion division algebra and we have

$$\mathbb{F} = Z(\mathbb{K}) \ , \qquad\qquad \gamma = \sigma_s \ .$$

Proof

(a) Given $x \in \mathbb{K}$, we have

$$\begin{aligned} \gamma(x) + x &= 1_{\mathbb{K}} + \gamma(x) + x + \gamma(x)x - 1_{\mathbb{K}} - \gamma(x)x \\ &= \big(1_{\mathbb{K}} + \gamma(x)\big)(1_{\mathbb{K}} + x) - 1_{\mathbb{K}} - \gamma(x)x \\ &= \gamma(1_{\mathbb{K}} + x)(1_{\mathbb{K}} + x) - 1_{\mathbb{K}} - \gamma(x)x \in \mathbb{F} \ . \end{aligned}$$

If we set

$$m_x := t^2 - \big(\gamma(x) + x\big) \cdot t + \gamma(x)x \in \mathbb{F}[t] \ ,$$

we have

$$m_x(x) = x^2 - \gamma(x)x - x^2 + \gamma(x)x = 0_{\mathbb{K}} \ ,$$

thus $\mathbb{K}$ is quadratic over $\mathbb{F}$.

(b) If $\mathbb{K}$ is non-commutative, proposition (3.32) implies that $\mathbb{K}$ is a quaternion division algebra with $\mathbb{F} = Z(\mathbb{K})$. Moreover, we have

$$\forall\, x \in \mathbb{K}: \qquad \sigma_s(x) = T(x) - x = \gamma(x) + x - x = \gamma(x)$$

and therefore

$$\gamma = \sigma_s \ .$$

□

(5.7) Remark The following results will be needed for the solution of the isomorphism problem for Moufang sets of pseudo-quadratic form type in §11.

(5.8) Lemma Let $\mathbb{D}$ be a skew-field and let $\mathbb{E}$ be a maximal subfield of $\mathbb{D}$. Then we have

$$C_{\mathbb{D}}(\mathbb{E}) = \mathbb{E} \ .$$

Proof
This results from corollary (4.9) in chapter 8 of [WSch]. □

(5.9) Notation Let $\mathbb{H}$ be a quaternion division algebra with center $\mathbb{K}$. Given $x \in \mathbb{H} \setminus \mathbb{K}$, let $\mathbb{E}_x := \langle 1_{\mathbb{H}}, x \rangle_{\mathbb{K}}$ be the quadratic subfield of $\mathbb{H}$ generated by $1_{\mathbb{H}}$ and x, cf. (20.9) of [TW].

(5.10) Corollary Let $\mathbb{H}$ be a quaternion division algebra and let $x, y \in \mathbb{H} \setminus Z(\mathbb{H})$. Then we have

$$xy = yx \ \Rightarrow \ \mathbb{E}_x = \mathbb{E}_y \ .$$

Proof
By lemma (5.8), we have

$$\mathbb{E}_x = \langle 1, x \rangle_{Z(\mathbb{H})} \subseteq C_{\mathbb{H}}(\mathbb{E}_y) = \mathbb{E}_y \ .$$

□

(5.11) Lemma For $i = 1, 2$, let $(\mathbb{A}_i, \mathbb{F}_i, \sigma_i)$ be quadratic of type (iii), (iv) or (v) with corresponding norms and traces N_i and T_i, respectively, and let $\phi : \mathbb{A}_1 \to \mathbb{A}_2$ be an isomorphism of alternative rings such that $\phi(\mathbb{F}_1) = \mathbb{F}_2$. Then we have

$$\forall \ x \in \mathbb{A} : \qquad \phi\big(N_1(x)\big) = N_2\big(\phi(x)\big) \ , \qquad \phi\big(T_1(x)\big) = T_2\big(\phi(x)\big) \ .$$

Proof
Given $x \in \mathbb{A}_i$, we have

$$x^2 - T_1(x)x + N_1(x) = 0_{\mathbb{A}_1} \ , \qquad \phi(x)^2 - T_2\big(\phi(x)\big)\phi(x) + N_2\big(\phi(x)\big) = 0_{\mathbb{A}_2}$$

and hence

$$\begin{aligned} 0_{\mathbb{A}_1} &= \phi^{-1}\big(\phi(x)^2 - T_2(\phi(x))\phi(x) + N_2(\phi(x))\big) \\ &= \phi^{-1}\phi\big(x^2 - \phi^{-1}(T_2(\phi(x)))x + \phi^{-1}(N_2(\phi(x)))\big) = x^2 - \phi^{-1}\big(T_2(\phi(x))\big)x + \phi^{-1}\big(N_2(\phi(x))\big) \ . \end{aligned}$$

As the maps T_1 and N_1 are uniquely determined by the minimum equation, we obtain

$$\begin{aligned} N_1(x) &= \phi^{-1}\big(N_2(\phi(x))\big) \ , \qquad & \phi\big(N_1(x)\big) &= N_2\big(\phi(x)\big) \ , \\ T_1(x) &= \phi^{-1}\big(T_2(\phi(x))\big) \ , \qquad & \phi\big(T_1(x)\big) &= T_2\big(\phi(x)\big) \ . \end{aligned}$$

□

(5.12) Corollary For $i = 1, 2$, let $(\mathbb{A}_i, \mathbb{F}_i, \sigma_i)$ be quadratic of type (iii), (iv) or (v) with corresponding norms and traces N_i and T_i, respectively, and let $\phi : \mathbb{A}_1 \to \mathbb{A}_2$ be an isomorphism of alternative rings such that $\phi(\mathbb{F}_1) = \mathbb{F}_2$. Then we have

$$\phi \circ \sigma_1 = \sigma_2 \circ \phi \ .$$

Proof
Given $x \in \mathbb{A}_1$, we have

$$\phi\sigma_1(x) = \phi(\bar{x}) = \phi\big(N_1(x) \cdot x^{-1}\big) = \phi\big(N_1(x)\big) \cdot \phi(x^{-1}) = N_2\big(\phi(x)\big) \cdot \phi(x)^{-1} = \overline{\phi(x)} = \sigma_2\phi(x) \ .$$

□

(5.13) Lemma Let $\mathbb{H}$ be a quaternion division algebra, let $\mathbb{E}$ be a separable quadratic subfield and let $y \in \mathbb{H} \setminus \mathbb{E}$. Then we have

$$y \in \mathbb{E}^{\perp} \;\Leftrightarrow\; \forall\, x \in \mathbb{E} :\; xy = y\bar{x} \;.$$

Proof

"$\Rightarrow$" This holds by lemma (3.26) (c).

"$\Leftarrow$" Let $e \in \mathbb{E}^{\perp}$ and $y_1, y_2 \in \mathbb{E}$ such that $y = y_1 + ey_2$. Given $x \in \mathbb{E} \setminus Z(\mathbb{H})$, we have

$$y_1 x + ey_2\bar{x} = y_1 x + e\bar{x}y_2 = x(y_1 + e \cdot y_2) = xy = y\bar{x} = y_1\bar{x} + ey_2\bar{x} \;,$$

hence

$$y_1 x = y_1\bar{x} \;, \qquad y_1(x - \bar{x}) = 0_{\mathbb{H}} \;, \qquad y_1 = 0_{\mathbb{H}} \;, \qquad y = ey_2 \in \mathbb{E}^{\perp} \;.$$

Notice that we use lemma (3.26) several times.

□

(5.14) Remark The following results give a description for some extensions of isomorphisms between subfields of two given composition algebras.

(5.15) Lemma For $i = 1, 2$, let $\mathbb{E}_i/\mathbb{K}_i$ be a separable quadratic extension and let $t_i \in \mathbb{E}_i \setminus \mathbb{K}_i$. Let $\phi : \mathbb{K}_1 \to \mathbb{K}_2$ be an isomorphism of fields. Then the map

$$\tilde{\phi} : \mathbb{E}_1 \to \mathbb{E}_2,\; x + t_1 \cdot y \mapsto \phi(x) + t_2 \cdot \phi(y)$$

is an isomorphism of fields if and only if we have

$$\phi\big(N_1(t_1)\big) = N_2(t_2) \;, \qquad \phi\big(T_1(t_1)\big) = T_2(t_2) \;.$$

Proof

"$\Rightarrow$" This holds by lemma (5.11).

"$\Leftarrow$" This is a direct calculation using the minimum equation.

□

(5.16) Lemma For $i = 1, 2$, let $\mathbb{H}_i = (\mathbb{E}_i/\mathbb{K}_i, \beta_i)$ be a quaternion division algebra and let $t_i \in \mathbb{E}_i^{\perp}$. Let $\phi : \mathbb{E}_1 \to \mathbb{E}_2$ be an isomorphism of fields. Then the map

$$\tilde{\phi} : \mathbb{H}_1 \to \mathbb{H}_2,\; x + t_1 \cdot y \mapsto \phi(x) + t_2 \cdot \phi(y)$$

is an isomorphism of skew-fields if and only if we have

$$\phi\big(N_1(t_1)\big) = N_2(t_2) \;.$$

Proof

"$\Rightarrow$" This holds by lemma (5.11).

"$\Leftarrow$" This is a direct calculation using the minimum equation and lemma (5.13).

□

(5.17) Remark We finally recall the list of quadratic spaces corresponding to involutory sets of quadratic type.

(5.18) Definition A quadratic space $(L_0, \mathbb{K}, q)$ is *of type*

(i) if $\mathbb{F} := L_0$ is a field with $\operatorname{Char}\mathbb{F} = 2$, $\mathbb{F}^2 \subseteq \mathbb{K} \neq \mathbb{F}$, $\sigma = \mathrm{id}_\mathbb{F}$ and $q = N_\mathbb{K}^\mathbb{F}$,

(ii) if we have $L_0 = \mathbb{K}$, $\sigma = \mathrm{id}_\mathbb{K}$ and $q = N_\mathbb{K}^\mathbb{K}$,

(iii) if $\mathbb{E} := L_0$ is a field, $\mathbb{E}/\mathbb{K}$ is a separable quadratic extension, $\langle\sigma\rangle = \mathrm{Gal}(\mathbb{E}/\mathbb{K})$ and $q = N_\mathbb{K}^\mathbb{E}$,

(iv) if $\mathbb{H} := L_0$ is a quaternion division algebra over $\mathbb{K}$, $\sigma = \sigma_s$ and $q = N_\mathbb{K}^\mathbb{H}$,

(v) if $\mathbb{O} := L_0$ is an octonion division algebra over $\mathbb{K}$, $\sigma = \sigma_s$ and $q = N_\mathbb{K}^\mathbb{O}$.

(5.19) Remark

(a) By corollary (4.14), a quadratic space $(L_0, \mathbb{K}, q)$ with $\dim_\mathbb{K} L_0 \leq 2$ is of type (i)-(iii).

(b) We will see that the Moufang sets of a quadratic space $(\mathbb{A}, \mathbb{F}, N_\mathbb{F}^\mathbb{A})$ of type (m) and of the corresponding alternative division ring $\mathbb{A}$ coincide, cf. lemma (31.23).

Chapter 6 Indifferent Sets

Quadrangles of purely indifferent type are parametrized by proper indifferent sets.

(6.1) Definition

- An *indifferent set* is a triple $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$, where $\mathbb{K}$ is a field with $\operatorname{Char}\mathbb{K} = 2$ and $\mathbb{K}_0$ and $\mathbb{L}_0$ are additive subgroups of $\mathbb{K}$ containing $1_\mathbb{K}$ such that

$$\mathbb{K}_0^2\mathbb{L}_0 \subseteq \mathbb{L}_0 \ , \qquad \mathbb{L}_0\mathbb{K}_0 \subseteq \mathbb{K}_0 \ , \qquad \langle\mathbb{K}_0\rangle = \mathbb{K} \text{ as rings} \ .$$

- An indifferent set is *proper* if we have $\mathbb{K}_0 \neq \mathbb{K}$ and $\mathbb{L}_0 \neq \mathbb{L} := \langle\mathbb{L}_0\rangle$.
- Two indifferent sets $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$ and $(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\mathbb{L}}_0)$ are *isomorphic* if there is an isomorphism $\phi : \mathbb{K} \to \tilde{\mathbb{K}}$ of fields such that

$$\phi(\mathbb{K}_0) = \tilde{\mathbb{K}}_0 \ , \qquad \phi(\mathbb{L}_0) = \tilde{\mathbb{L}}_0 \ .$$

(6.2) Lemma Let $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$ be an indifferent set. Then $(\mathbb{L}, \mathbb{L}_0, \mathbb{K}_0^2)$ is an indifferent set.

Proof
This is (10.2) of [TW]. □

(6.3) Definition The *opposite of an indifferent set* $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$ is the indifferent set $(\mathbb{L}, \mathbb{L}_0, \mathbb{K}_0^2)$.

(6.4) Lemma Given a proper indifferent set $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$, its opposite $(\mathbb{L}, \mathbb{L}_0, \mathbb{K}_0^2)$ is proper.

Proof
Since $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$ is proper, we have $\mathbb{L}_0 \neq \mathbb{L}$. We have to show $\mathbb{K}_0^2 \neq \langle\mathbb{K}_0^2\rangle$. By remark (10.8) of [TW], the opposite of $(\mathbb{L}, \mathbb{L}_0, \mathbb{K}_0^2)$ is $(\mathbb{K}^2, \mathbb{K}_0^2, \mathbb{L}_0^2)$, and we have

$$(\mathbb{K}^2, \mathbb{K}_0^2, \mathbb{L}_0^2) \cong (\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$$

as indifferent sets. In particular, we have $\langle\mathbb{K}_0^2\rangle = \mathbb{K}^2$. Moreover, we have $\mathbb{K}_0^2 \neq \mathbb{K}^2$ since the map $\mathrm{Fr} : \mathbb{K} \to \mathbb{K},\ x \mapsto x^2$ is injective and $\mathbb{K}_0 \neq \mathbb{K}$ by assumption. We finally obtain

$$\mathbb{K}_0^2 \neq \mathbb{K}^2 = \langle\mathbb{K}_0^2\rangle \ .$$

□

Chapter 7 Pseudo-Quadratic Spaces

Quadrangles of purely pseudo-quadratic form type are parametrized by proper pseudo-quadratic spaces.

§ 7.1 Basic Definitions and Basic Properties

First of all we give the basic definitions and introduce the Moufang set of pseudo-quadratic form type, more precisely, its associated group, corresponding to a pseudo-quadratic space.

(7.1) Definition

- An *(anisotropic) right (resp. left) pseudo-quadratic space* is a quintuple $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma, L_0, q)$ such that $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is an involutory set, L_0 is a right (resp. left) vector space over $\mathbb{K}$ and q is an *(anisotropic) pseudo-quadratic form on L_0 with respect to σ*, i.e., there is a skew-hermitian form f on L_0 such that the following holds:

  (P1) $\forall\ a, b \in L_0 :\ q(a+b) \equiv q(a) + q(b) + f(a,b) \mod \mathbb{K}_0,$

  (P2) $\forall\ a \in L_0,\ t \in \mathbb{K} :\ q(at) \equiv t^\sigma q(a) t \mod \mathbb{K}_0$ (resp. $q(ta) \equiv tq(a)t^\sigma \mod \mathbb{K}_0$),

  (P3) $q(a) \equiv 0_{\mathbb{K}} \mod \mathbb{K}_0 \ \Leftrightarrow\ a = 0_{L_0}$.

- A pseudo-quadratic space $(\mathbb{K}, \mathbb{K}_0, \sigma, L_0, q)$ is *proper* if we have $\sigma \neq \mathrm{id}_{\mathbb{K}}$, $L_0 \neq \{0\}$ and if the associated skew-hermitian form f is non-degenerate.

- Two pseudo-quadratic space Ξ and $\tilde{\Xi}$ are *isomorphic* if there is an isomorphism

$$\Phi = (\varphi, \phi) : (L_0, \mathbb{K}) \to (\tilde{L}_0, \tilde{\mathbb{K}})$$

  of vector spaces such that $\phi : (\mathbb{K}, \mathbb{K}_0, \sigma) \to (\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ is an isomorphism of involutory sets and such that

$$\phi \circ q \equiv \tilde{q} \circ \varphi \mod \tilde{\mathbb{K}}_0 \ .$$

(7.2) Lemma The skew-hermitian form f is uniquely determined by (7.1) (P1) and satisfies

$$\forall\ a \in L_0 : \qquad f(a,a) = q(a) - q(a)^\sigma \ .$$

Proof
This is (11.19) of [TW]. Notice that we have $\mathbb{K}_0 \neq \mathbb{K}$ since Ξ is proper. □

(7.3) Corollary Given an isomorphism $\Phi = (\varphi, \phi) : \Xi \to \tilde{\Xi}$ of pseudo-quadratic spaces, we have

$$\forall\ a, b \in L_0 : \qquad \tilde{f}\big(\varphi(a), \varphi(b)\big) = \phi\big(f(a,b)\big) \ .$$

Proof
We have to show that

$$\forall\ a, b \in L_0 : \qquad \phi^{-1}\big(\tilde{f}(\varphi(a), \varphi(b))\big) \equiv q(a+b) - q(a) - q(b) \mod \mathbb{K}_0 \ .$$

Given $a, b \in L_0$, we have

$$\begin{aligned}
\phi^{-1}\big(\tilde{f}(\varphi(a), \varphi(b))\big) &\in \phi^{-1}\big(\tilde{q}(\varphi(a+b)) - \tilde{q}(\varphi(a)) - \tilde{q}(\varphi(b)) + \tilde{\mathbb{K}}_0\big) \\
&= \phi^{-1}\big(\phi(q(a+b)) - \phi(q(a)) - \phi(q(b)) + \tilde{\mathbb{K}}_0\big) \\
&= q(a+b) - q(a) - q(b) + \mathbb{K}_0 \ .
\end{aligned}$$

□

(7.4) Remark Let Ξ be a pseudo-quadratic space and let $a \in L_0$.

- Assume $\operatorname{Char}\mathbb{K} \neq 2$. Then we have

$$4q(a) \equiv q(2a) \equiv 2q(a) + f(a,a) \mod \mathbb{K}_0 \ , \qquad q(a) \equiv \frac{f(a,a)}{2} \mod \mathbb{K}_0 \ .$$

- Assume $\operatorname{Char}\mathbb{K} = 2$. Then we have $f(a,a) = q(a) + q(a)^\sigma \in \mathbb{K}_0$.

(7.5) Definition Given a pseudo-quadratic space Ξ, we set

$$T := T(\Xi) := \{(a,t) \in L_0 \times \mathbb{K} \mid q(a) - t \in \mathbb{K}_0\} \ .$$

(7.6) Notation Throughout the rest of this chapter, let Ξ be a proper pseudo-quadratic space and let T be the corresponding set as in definition (7.5).

(7.7) Lemma Given $(a,t) \in T,\ k \in \mathbb{K}$, we have

$$(a, t+k) \in T \ \Leftrightarrow\ k \in \mathbb{K}_0 \ .$$

Proof
Given $(a,t) \in T$ and $k \in \mathbb{K}$, we have

$$(a,t+k) \in T \ \Leftrightarrow\ q(a) - t - k \in \mathbb{K}_0 \ \Leftrightarrow\ k \in q(a) - t + \mathbb{K}_0 = \mathbb{K}_0 \ .$$

□

(7.8) Corollary Given $(a,t) \in T$, we have $f(a,a) = t - t^\sigma$.

Proof
Let $(a,t) \in T$. By lemma (7.7), there is an element $k \in \mathbb{K}_0$ such that $t = q(a) + k$, hence

$$t - t^\sigma = q(a) + k - q(a)^\sigma - k^\sigma = q(a) - q(a)^\sigma = f(a,a)$$

by lemma (7.2). □

(7.9) Corollary Let

$$\cdot : T \times T \to T,\ (a,t)\cdot(b,v) := \big(a+b, t+v+f(b,a)\big) \ .$$

Then $(T,\cdot)$ is a group with $Z(T) = \{(0_{L_0}, t) \mid t \in \mathbb{K}_0\} \cong \mathbb{K}_0$ and $(a,t)^{-1} = (-a, -t^\sigma)$ for each $(a,t) \in T$.

Proof
This results from (11.24) and (38.10) of [TW]. Notice that Ξ is proper. □

(7.10) Remark In the following, we don't distinguish between $Z(T)$ and $\mathbb{K}_0$, i.e., we consider $\mathbb{K}_0$ to be a subset of T via the above identification.

(7.11) Lemma Given $(a,t),(b,v) \in T$, we have

$$(a,t)\cdot(b,v) \in \mathbb{K}_0 \ \Leftrightarrow\ a = -b \ .$$

Proof
Given $(a,t),(b,v) \in T$, we have

$$(a,t)\cdot(b,v) \in \mathbb{K}_0 \ \Leftrightarrow\ \big(a+b, t+v+f(b,a)\big) \in \mathbb{K}_0 \ \Leftrightarrow\ a+b = 0_{L_0} \ \Leftrightarrow\ a = -b \ .$$

□

§ 7.2 Jordan Isomorphisms

We collect some first results about isomorphisms preserving the Moufang set structure.

(7.12) Notation Throughout this paragraph, let $\tilde{\Xi}$ be an additional pseudo-quadratic space, let $\tilde{T}$ be the corresponding group as in definition (7.5) and let $\gamma : T \to \tilde{T}$ be an isomorphism of groups.

(7.13) Remark As T is non-abelian and γ is an isomorphism of groups, we have $\tilde{\sigma} \neq \mathrm{id}_{\tilde{\mathbb{K}}}$, $\tilde{L}_0 \neq \{0\}$, and the associated skew-hermitian form is not identically zero, i.e., $\tilde{\Xi}$ is pre-proper. Moreover, it is proper if we have $\operatorname{Char} \tilde{\mathbb{K}} \neq 2$, and, if $\operatorname{Char} \tilde{\mathbb{K}} = 2$, we may replace $\tilde{\Xi}$ by a proper pseudo-quadratic space, cf. definition (35.5) of [TW].

Thus we get a satisfying solution of the isomorphism problem for Moufang sets of pseudo-quadratic form type if we restrict to the case of two proper pseudo-quadratic spaces.

(7.14) Corollary We have $\gamma(\mathbb{K}_0) = \tilde{\mathbb{K}}_0$.

Proof
We have $\gamma(\mathbb{K}_0) = \gamma\big(Z(T)\big) = Z(\tilde{T}) = \tilde{\mathbb{K}}_0$. □

(7.15) Lemma Let $\varphi_1 : T \to \tilde{L}_0,\ \varphi_2 : T \to \tilde{\mathbb{K}}$ defined by

$$\gamma(a,t) = \big(\varphi_1(a,t), \varphi_2(a,t)\big) \ .$$

Then we have

$$\forall\ (a,t) \in T : \qquad \varphi_1(a,t) = \varphi_1(a) \ .$$

Moreover, the $\varphi_1 : L_0 \to \tilde{L}_0$ is an isomorphism of groups.

Proof

- Given $(a,t),(a,u) \in T$, we have

  $$(a,t) \cdot (a,u)^{-1} = (a,t) \cdot (-a, -u^{\sigma}) \in \mathbb{K}_0 = Z(T)$$

  by lemma (7.11); therefore,

  $$\begin{aligned}\big(\varphi_1(a,t), \varphi_2(a,t)\big) \cdot \big(-\varphi_1(a,u), -\varphi_2(a,u)^{\tilde{\sigma}}\big) &= \gamma_2(a,t) \cdot \gamma_2(a,u)^{-1} \\ &= \gamma_2\big((a,t)\cdot(a,u)^{-1}\big) \in Z(\tilde{T}) = \tilde{\mathbb{K}}_0\end{aligned}$$

  and thus $\varphi_1(a,t) = \varphi_1(a,u)$ by lemma (7.11) again.

- We have $\big(a, q(a)\big) \in T$ for each $a \in L_0$, hence φ_1 is well-defined.

- As the multiplication in T is additive in the first component, φ_1 is additive.

- Let $(a,t) \in T$ such that $a \in \operatorname{Ker} \varphi_1$. Then we have

  $$\gamma(a,t) = \big(0_{L_0}, \varphi_2(a,t)\big) \in \tilde{\mathbb{K}}_0 \subseteq Z(\tilde{T}) \ ,$$

  hence $(a,t) \in Z(T) = \mathbb{K}_0$ by corollary (7.9) and thus $a = 0_{L_0}$.

- We have $\big(a, \tilde{q}(a)\big) \in \tilde{T}$ for each $a \in \tilde{L}_0$. As γ is surjective, φ_1 is surjective as well.

□

(7.16) Definition A *Jordan isomorphism* is an isomorphism of groups $\gamma : T \to \tilde{T}$ with $\gamma(0, 1_{\mathbb{K}}) = (0, 1_{\tilde{\mathbb{K}}})$ such that the maps $\varphi_1 : T \to \tilde{L}_0,\ \varphi_2 : T \to \tilde{\mathbb{K}}$ defined by

$$\gamma(a,t) = \big(\varphi_1(a,t) = \varphi_1(a), \varphi_2(a,t)\big)$$

satisfy

$$\varphi_1\big(bt^\sigma - at^{-1}f(a,b)t^\sigma\big) = \varphi_1(b)\varphi_2(a,t)^{\tilde{\sigma}} - \varphi_1(a)\varphi_2(a,t)^{-1}\tilde{f}\big(\varphi_1(a),\varphi_1(b)\big)\varphi_2(a,t)^{\tilde{\sigma}} \ , \tag{7.1}$$

$$\varphi_2\big(bt^\sigma - at^{-1}f(a,b)t^\sigma, tvt^\sigma\big) = \varphi_2(a,t)\varphi_2(b,v)\varphi_2(a,t)^{\tilde{\sigma}} \tag{7.2}$$

for all $(a,t),(b,v) \in T$.

(7.17) Definition Given $(a,t) \in T^*$, the *Hua-map with respect to* (a,t) is

$$h_{(a,t)} : T \to T,\ (b,v) \mapsto \big(bt^\sigma - at^{-1}f(a,b)t^\sigma, tvt^\sigma\big) \ .$$

(7.18) Remark A Jordan isomorphism could equally be defined as an isomorphism of groups $\gamma : T \to \tilde{T}$ satisfying $\gamma(0,1_{\mathbb{K}}) = (0, 1_{\tilde{\mathbb{K}}})$ and preserving the Hua-maps, i.e, given $(a,t),(b,v) \in T^*$, we have

$$\gamma\big(h_{(a,t)}(b,v)\big) = \tilde{h}_{\gamma(a,t)}\big(\gamma(b,v)\big) \ .$$

(7.19) Lemma We have $h_{(a,t)} \in \mathrm{Aut}(T)$ for each $(a,t) \in T^*$.

Proof
This is theorem 2 of [DW]. □

(7.20) Notation Since the first component in $h_{(a,t)}(b,v)$ is independent of v, we may restrict $h_{(a,t)}$ to the first component, i.e., given $b \in L_0$, we set

$$h_{(a,t)}(b) := bt^\sigma - at^{-1}f(a,b)t^\sigma \ .$$

(7.21) Lemma Let $\gamma : T \to \tilde{T}$ be a Jordan isomorphism. Then the following holds:

(a) Given $b \in L_0,\ (0_{L_0}, t) \in \mathbb{K}_0$, we have $\varphi_1(b \cdot t) = \varphi_1(b) \cdot \varphi_2(0_{L_0}, t)$.

(b) Given $(a,t) \in T,\ (0_{L_0}, v) \in \mathbb{K}_0$, we have $\varphi_2(0_{L_0}, tvt^\sigma) = \varphi_2(a,t) \cdot \varphi_2(0_{L_0}, v) \cdot \varphi_2(a,t)^{\tilde{\sigma}}$.

(c) The map $\phi : \mathbb{K}_0 \to \tilde{\mathbb{K}}_0,\ t \mapsto \varphi_2(0_{L_0}, t)$ is a Jordan isomorphism as in definition (5.1).

(d) Given $(a,t) \in T$ and $s \in \mathbb{K}_0$, we have $\varphi_2(a, t+s) = \varphi_2(a,t) + \phi(s)$.

Proof

(a) This results from identity (7.1) with $a = 0_{L_0}$. Notice that we have

$$\mathbb{K}_0 \subseteq \mathrm{Fix}(\sigma) \ , \qquad \gamma(\mathbb{K}_0) = \tilde{\mathbb{K}}_0 \subseteq \mathrm{Fix}(\tilde{\sigma}) \ .$$

(b) This results from identity (7.2) with $b = 0_{L_0}$.

(c) By corollary (7.14), ϕ is an isomorphism of groups. Given $(0_{L_0}, t), (0_{L_0}, v) \in \mathbb{K}_0$, we have

$$\varphi_2(0, tvt) = \varphi_2(0, tvt^\sigma) = \varphi_2(0,t) \cdot \varphi_2(0,v) \cdot \varphi_2(0,t)^{\tilde{\sigma}} = \varphi_2(0,t) \cdot \varphi_2(0,v) \cdot \varphi_2(0,t) \ .$$

(d) Given $(a,t) \in T$ and $s \in \mathbb{K}_0$, we have

$$\begin{aligned}\big(\varphi_1(a), \varphi_2(a, t+s)\big) &= \gamma(a, t+s) = \gamma(a,t) \cdot \gamma(0,s) \\ &= \big(\varphi_1(a), \varphi_2(a,t)\big) \cdot \big(0, \phi(s)\big) = \big(\varphi_1(a), \varphi_2(a,t) + \phi(s)\big) \ .\end{aligned}$$

□

Part III

Jordan Isomorphisms of Pseudo-Quadratic Spaces

For the classification of 443-foundations in part IX, we will need a part of the solution of the isomorphism problem for Moufang sets of pseudo-quadratic form type. We state a version of the result at this point and break up the proof into several steps. The main point is that under certain conditions, a Jordan isomorphism of pseudo-quadratic spaces is induced by an isomorphism of the corresponding pseudo-quadratic spaces, and we handle the possible cases one by one.

Then we have a closer look at the exceptions, which only occur in small dimensions, before we finally show that each of the appearing maps really induces a Jordan isomorphism.

Chapter 8 A Partial Result

(8.1) Theorem Let Ξ and $\tilde{\Xi}$ be proper pseudo-quadratic spaces, let $\gamma : T \to \tilde{T}$ be a Jordan isomorphism and suppose that one of the following holds:

(i) The involutory set $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is proper.

(ii) The involutory set $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is quadratic of type (iii) or (iv) and $\dim_{\mathbb{K}} L_0 \geq 3$.

(iii) The involutory set $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is quadratic of type (iii) or (iv), $\dim_{\mathbb{K}} L_0 \leq 2$ and $\tilde{\mathbb{K}} \cong \mathbb{K} \ncong \mathbb{F}_4$.

Then γ is induced by an isomorphism $\Phi : \Xi \to \tilde{\Xi}$ of pseudo-quadratic spaces.

Proof

This results from theorem (9.8), theorem (10.38), theorem (11.11) and theorem (12.18). □

(8.2) Notation Throughout this part, let Ξ and $\tilde{\Xi}$ be proper pseudo-quadratic spaces, let $\gamma : T \to \tilde{T}$ be a Jordan isomorphism and let $\phi : \mathbb{K}_0 \to \tilde{\mathbb{K}}_0,\ t \mapsto \varphi_2(0_{L_0}, t)$.

Chapter 9 The Involutory Set Is Proper

The first case is that of a proper involutory set $(\mathbb{K}, \mathbb{K}_0, \sigma)$. The solution of the isomorphism problem for proper involutory sets yields an isomorphism $\phi : \mathbb{K} \to \tilde{\mathbb{K}}$. From identity (7.1) and the fact that $\mathbb{K}_0$ generates $\mathbb{K}$ as a ring, we deduce that $(\varphi_1, \phi) : (L_0, \mathbb{K}) \to (\tilde{L}_0, \tilde{\mathbb{K}})$ is an isomorphism of vector spaces.

Finally, we show that the second component of γ is induced by ϕ, using identity (7.2) and the fact that the dimension of $\langle \mathbb{K}_0 \rangle_{Z(\mathbb{K})}$ over $Z(\mathbb{K})$ is at least 2.

(9.1) Notation Throughout this chapter, let $(\mathbb{K}, \mathbb{K}_0, \sigma)$ be proper.

(9.2) Lemma Let $(\hat{\mathbb{K}}, \hat{\mathbb{K}}_0, \hat{\sigma})$ be a proper involutory set and let $s, t \in \hat{\mathbb{K}}$ be such that

$$\forall\ u \in \hat{\mathbb{K}}_0 : \qquad sus^{\hat{\sigma}} = tut^{\hat{\sigma}}\ .$$

Then we have $t^{-1}s \in Z(\hat{\mathbb{K}})$.

Proof

First of all we notice that

$$ss^{\hat{\sigma}} = s \cdot 1_{\hat{\mathbb{K}}} \cdot s^{\hat{\sigma}} = t \cdot 1_{\hat{\mathbb{K}}} \cdot t^{\hat{\sigma}} = tt^{\hat{\sigma}}$$

and thus

$$\forall\ u \in \hat{\mathbb{K}}_0 : \qquad sus^{-1} = sus^{\hat{\sigma}}(ss^{\hat{\sigma}})^{-1} = tut^{\hat{\sigma}}(tt^{\hat{\sigma}})^{-1} = tut^{-1}\ .$$

It follows that

$$\forall\ u \in \langle \hat{\mathbb{K}}_0 \rangle = \hat{\mathbb{K}} : \qquad t^{-1}su(t^{-1}s) = u\ ,$$

hence $t^{-1}s \in Z(\hat{\mathbb{K}})$. □

(9.3) Remark As $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is proper, the map $\phi : \mathbb{K}_0 \to \tilde{\mathbb{K}}_0$ as defined in notation (8.2) is induced by an isomorphism $\phi : (\mathbb{K}, \mathbb{K}_0, \sigma) \to (\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ of involutory sets, and $(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ is proper, cf. theorem (5.3).

(9.4) Corollary Let $(a, t) \in T$. Then there is an element $\lambda \in Z(\tilde{\mathbb{K}})$ such that

$$\varphi_2(a, t) = \lambda \cdot \phi(t) \ .$$

Proof
By lemma (7.21) (b), we have

$$\phi(t)\phi(v)\phi(t)^{\tilde{\sigma}} = \phi(tvt^{\sigma}) = \varphi_2(0, tvt^{\sigma}) = \varphi_2(a, t)\varphi_2(0, v)\varphi_2(a, t)^{\tilde{\sigma}} = \varphi_2(a, t)\phi(v)\varphi_2(a, t)^{\tilde{\sigma}}$$

for all $(a, t) \in T$, $v \in \mathbb{K}_0$ and thus

$$\phi(t) \cdot \tilde{v} \cdot \phi(t)^{\tilde{\sigma}} = \varphi_2(a, t) \cdot \tilde{v} \cdot \varphi_2(a, t)^{\tilde{\sigma}}$$

for all $(a, t) \in T$, $\tilde{v} \in \tilde{\mathbb{K}}_0$. Now the assertion results from lemma (9.2). □

(9.5) Lemma We have

$$\dim_{Z(\mathbb{K})} \langle \mathbb{K}_0 \rangle_{Z(\mathbb{K})} \geq 2 \ .$$

Proof
As $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is proper, there is an element $x \in \mathbb{K}_0 \setminus Z(\mathbb{K})$ by lemma (5.2). □

(9.6) Corollary Let $(a, s) \in T$. Then there is an element $t \in \mathbb{K}$ such that

$$(a, t) \in T \ , \qquad\qquad s \notin \langle t^{\sigma} \rangle_{Z(\mathbb{K})} \ .$$

Proof
We have $(a, -s^{\sigma}) \in T$. By lemma (9.5), there is an element $u \in \mathbb{K}_0$ such that $s \notin \langle u \rangle_{Z(\mathbb{K})}$, thus

$$s \notin \langle -s + u \rangle_{Z(\mathbb{K})} = \langle (-s + u)^{\sigma^2} \rangle_{Z(\mathbb{K})} = \langle (-s^{\sigma} + u)^{\sigma} \rangle_{Z(\mathbb{K})} \ ,$$

and by lemma (7.7), we have $(a, -s^{\sigma} + u) \in T$. □

(9.7) Lemma Given $(a, s) \in T$, we have

$$\varphi_2(a, s) = \phi(s) \ .$$

Proof
Let $(a, s) \in T$ and let $t \in \mathbb{K}$ be as in corollary (9.6). Notice that $s + t^{\sigma} \in \mathbb{K}_0$. By corollary (9.4), there are $\lambda_1, \lambda_2 \in Z(\tilde{\mathbb{K}})$ such that

$$\varphi_2(a, s) = \lambda_1 \cdot \phi(s) \ , \qquad\qquad \varphi_2(-a, t) = \lambda_2 \cdot \phi(t) \ .$$

Observing corollary (7.8), we obtain

$$\begin{aligned}
\big(0, \phi(s) + \phi(t^{\sigma})\big) &= \gamma(0, s + t^{\sigma}) = \gamma\big(0, s + t - f(a, a)\big) \\
&= \gamma(a, s) \cdot \gamma(-a, t) = \big(\varphi_1(a), \varphi_2(a, s)\big) \cdot \big(\varphi_1(-a), \varphi_2(-a, t)\big) \\
&= \big(0, \varphi_2(a, s) + \varphi_2(-a, t) - \tilde{f}(\varphi_1(-a), \varphi_1(-a))\big) \\
&= \big(0, \varphi_2(a, s) + \varphi_2(-a, t)^{\tilde{\sigma}}\big) = \big(0, \lambda_1 \cdot \phi(s) + \lambda_2 \cdot \phi(t^{\sigma})\big)
\end{aligned}$$

and thus $\lambda_1 = 1_{\tilde{\mathbb{K}}}$ by the linear independence of $\phi(s)$ and $\phi(t^{\sigma})$, cf. remark (9.3). □

(9.8) Theorem If $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is proper, the map $\Phi : \Xi \to \tilde{\Xi}$ defined by

$$\Phi := (\varphi_1, \phi) : (L_0, \mathbb{K}) \to (\tilde{L}, \tilde{\mathbb{K}}),\ (a, t) \mapsto \big(\varphi_1(a), \phi(t)\big)$$

is an isomorphism of pseudo-quadratic spaces satisfying

$$\forall\ (a, t) \in T : \qquad \Phi(a, t) = \gamma(a, t)\ .$$

Proof

- By remark (9.3), the map $\phi : (\mathbb{K}, \mathbb{K}_0, \sigma) \to (\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ is an isomorphism of involutory sets.
- By lemma (7.15), the map $\varphi_1 : L_0 \to \tilde{L}_0$ is an isomorphism of groups. Given $a \in L_0$ and $t_1, \ldots, t_n \in \mathbb{K}_0$, we have

$$\varphi_1\big(a \cdot t_1 \cdots t_n\big) = \varphi_1(a) \cdot \phi(t_1) \cdots \phi(t_n) = \varphi_1(a) \cdot \phi(t_1 \cdots t_n)$$

  by lemma (7.21) (a), thus (φ_1, ϕ) is an isomorphism of vector spaces as we have $\langle \mathbb{K}_0 \rangle = \mathbb{K}$.
- Given $(a, t) \in T$, we have

$$\tilde{q}\big(\varphi_1(a)\big) \in \varphi_2(a, t) + \tilde{\mathbb{K}}_0 = \phi(t) + \tilde{\mathbb{K}}_0 = \phi\big(t + \mathbb{K}_0\big) = \phi\big(q(a) + \mathbb{K}_0\big) = \phi\big(q(a)\big) + \tilde{\mathbb{K}}_0$$

  by lemma (9.7).
- Given $(a, t) \in T$, we have

$$\Phi(a, t) = \big(\varphi_1(a), \phi(t)\big) = \big(\varphi_1(a), \varphi_2(a, t)\big) = \gamma(a, t)\ .$$

$\square$

(9.9) Remark If $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is non-proper, then $(\mathbb{A}, \mathbb{F}, \sigma) := (\mathbb{K}, \mathbb{K}_0, \sigma)$ is quadratic of type (iii) or (iv), cf. (38.14) of [TW]. It would be nice to have the same result in this case. However, this is false in general, thus we will need some additional assumptions.

Chapter 10 The Involutory Set Is of Quadratic Type

The second case is that of an involutory set which is of quadratic type and $\dim_{\mathbb{K}} L_0 \geq 3$. The crucial step is to show that φ_1 maps orthogonal vectors to orthogonal vectors and hence subspaces to subspaces. In particular, we may apply the fundamental theorem of projective geometry to get an isomorphism $(\varphi_1, \phi) : (L_0, \mathbb{K}) \to (\tilde{L}_0, \mathbb{K}_0)$ of vector spaces.

In order to prove this, we introduce a technical condition concerning orthogonality which is preserved by the Hua-maps. However, this condition is only enough to handle separable elements which results in a separate treatment of inseparable elements.

Finally, we show that the second component of γ is induced by ϕ, using identity (7.1) and the fact that the dimension of L_0 over $\mathbb{K}$ is at least 3, which ensures the existence of enough orthogonal vectors.

(10.1) Notation Throughout this chapter, we suppose $(\mathbb{A}, \mathbb{F}, \sigma) := (\mathbb{K}, \mathbb{K}_0, \sigma)$ $\big($and therefore $(\tilde{\mathbb{A}}, \tilde{\mathbb{F}}, \tilde{\sigma}) := (\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})\big)$ to be quadratic of type (iii) or (iv).

(10.2) Definition

- An element $a \in L_0$ with $f(a, a) = 0_{\mathbb{A}}$ is called *inseparable*.
- Otherwise, it is called *separable*.

(10.3) Remark

(a) If $a \in L_0$ is separable, we have

$$q(a) - q(a)^\sigma = f(a,a) \neq 0_{\mathbb{A}} \ , \qquad\qquad q(a)^\sigma \neq q(a) \ .$$

(b) If $a \in L_0^*$ is inseparable, we have

$$q(a)^\sigma = q(a) \ , \qquad\qquad q(a) \in \mathrm{Fix}(\sigma) \setminus \mathbb{F} \ ,$$

which implies $\mathrm{Char}\, \mathbb{A} = 2$. Moreover, $(\mathbb{A}, \mathbb{F}, \sigma)$ is quadratic of type (iv) in this case.

(10.4) Lemma An element $a \in L_0^*$ is inseparable iff we have $(a,t)^2 = 0_T$ for each $(a,t) \in T$.

Proof
Notice that we have $\mathrm{Char}\, \mathbb{A} = 2$ by remark (10.3) (b). Given $(a,t) \in T$, we have

$$0_T = (a,t)^2 = \big(a + a, t + t + f(a,a)\big) = \big(0_{L_0}, f(a,a)\big) \ \Leftrightarrow \ f(a,a) = 0_{\mathbb{A}} \ .$$

□

(10.5) Corollary An element $a \in L_0^*$ is inseparable if and only if $\varphi_1(a) \in \tilde{L}_0^*$ is inseparable.

Proof
Given $(a,t) \in T$, we have

$$(a,t)^2 = 0_T \ \Leftrightarrow \ \gamma(a,t)^2 = 0_{\tilde{T}} \ .$$

□

(10.6) Notation

- We set
$$g : L_0 \times L_0 \to \mathbb{F}, \ (a,b) \mapsto f(b,a) - q(a+b) + q(a) + q(b) \ . \tag{10.1}$$
- Given $x \in \mathbb{A} \setminus \mathbb{F}$, let $\mathbb{E}_x$ be the quadratic subfield of $\mathbb{A}$ generated by $1_{\mathbb{A}}$ and x.
- Given $a \in L_0^*$, we set $\mathbb{E}_a := \mathbb{E}_{q(a)}$. It is a separable quadratic subfield iff a is separable.
- Given $a \in L_0^*$, we set
$$R_a := \langle a, a \cdot q(a) \rangle_{\mathbb{F}} = a \cdot \mathbb{E}_a \ ,$$
so that R_a is a 2-dimensional $\mathbb{F}$-subspace of L_0.
- Suppose $(\mathbb{A}, \mathbb{F}, \sigma)$ to be quadratic of type (iv). Given a separable element $a \in L_0$, let e_a be an element of $\mathbb{A}$ orthogonal to $\mathbb{E}_a$ (with respect to the standard trace of $\mathbb{A}$). For each $x \in \mathbb{A}$, let $\alpha_a(x)$ and $\beta_a(x)$ be the unique elements of $\mathbb{E}_a$ such that
$$x = \alpha_a(x) + e_a \beta_a(x) \ .$$ [1]
- Given $a \in L_0^*$, we set
$$X_a := \{t \in \mathbb{K} \mid (a,t) \in T\} = \{s + q(a) \mid s \in \mathbb{F}\} \subseteq \mathbb{E}_a \ .$$

(10.7) Remark Given $a \in L_0^*$, we have $X_a \subseteq \mathbb{E}_a \setminus \mathbb{F}$ which implies

$$\forall \ t \in X_a : \qquad \mathbb{E}_t = \mathbb{E}_a \ .$$

In particular, we have $\mathbb{E}_a = \mathbb{E}_t$ for each $(a,t) \in T$ such that $a \neq 0_{L_0}$.

[1] We assume $e_{a'} = e_a$ for each $a' \in R_a$, hence $\alpha_{a'} = \alpha_a$ and $\beta_{a'} = \beta_a$ for each $a' \in R_a$

(10.8) Lemma Let $a \in L_0^*$ be such that $N(t) = T(t)^2$ for each $t \in X_a$. Then we have $|\mathbb{F}| = 2$.

Proof
Let $t \in X_a$. Given $s \in \mathbb{F}$, we have

$$0_{\mathbb{A}} = T(t)^2 - N(t) = T(t+s)^2 - N(t) = N(t+s) - N(t) = sT(t) + N(s) = s\big(T(t) + s\big)$$

and therefore

$$\forall\, s \in \mathbb{F}^* : \qquad s = -T(t) \ ,$$

which implies $|\mathbb{F}| = 2$. □

(10.9) Lemma Let $a \in L_0^*$ and $|\mathbb{F}| \geq 3$. Then we have

$$\{t^\sigma - t^{-1} f(a,a) t^\sigma \mid t \in X_a\} \not\subseteq \mathbb{F} \ .$$

Proof
Notice that we have

$$\{t^\sigma - t^{-1} f(a,a) t^\sigma \mid t \in X_a\} = \{\big(1_{\mathbb{A}} - t^{-1}(t - t^\sigma)\big) t^\sigma \mid t \in X_a\} = \{t^{-1}(t^\sigma)^2 \mid t \in X_a\} \ . \tag{10.2}$$

Assume $\{t^{-1}(t^\sigma)^2 \mid t \in X_a\} \subseteq \mathbb{F}$. Then we have

$$\forall\, t \in X_a : \qquad (t^\sigma)^3 = t^{-1}(t^\sigma)^2 \cdot N(t) \in \mathbb{F} \ , \qquad t^3 \in \mathbb{F} \ .$$

Therefore, we have

$$\big(T(t)^2 - N(t)\big) \cdot t - T(t)N(t) = \big(T(t)t - N(t)\big) \cdot t = t^3 \in \mathbb{F} \tag{10.3}$$

and thus $T(t)^2 = N(t)$ for each $t \in X_a$. Now lemma (10.8) yields $|\mathbb{F}| = 2$. □

(10.10) Remark

(a) The map

$$\phi : \mathbb{F} \to \tilde{\mathbb{F}}, \ t \mapsto \varphi_2(0,t)$$

is a Jordan isomorphism by lemma (7.21) (c), hence an isomorphism of fields by Hua's theorem, cf. theorem (19.31). Therefore, the map $\varphi_1 : L_0 \to \tilde{L}_0$ is an isomorphism of vector spaces over $\mathbb{F}$ by lemma (7.21) (a).

(b) Given $(a,t) \in T$, we have $\big(\varphi_1(a), \varphi_2(a,t)\big) \in \tilde{T}$, hence $\tilde{\mathbb{E}}_{\varphi_2(a,t)} = \tilde{\mathbb{E}}_{\varphi_1(a)}$ by remark (10.7).

(10.11) Corollary Given $a \in L_0^*$, we have

$$\varphi_1(R_a) = \tilde{R}_{\varphi_1(a)} \ .$$

Proof

- Assume $|\mathbb{F}| \geq 3$. By lemma (10.9), we have

$$\langle 1_{\mathbb{A}}, \{t^\sigma - t^{-1} f(a,a) t^\sigma \mid t \in X_a\}\rangle_{\mathbb{F}} = \mathbb{E}_a \ ,$$

hence

$$\varphi_1(R_a) = \varphi_1(a \cdot \mathbb{E}_a) \subseteq \varphi_1(a) \cdot \tilde{\mathbb{E}}_{\varphi_1(a)} = \tilde{R}_{\varphi_1(a)}$$

by remark (10.10) (a), identity (7.1) with $(b,v) := (a,t)$, corollary (7.8) and remark (10.10) (b).

- It remains to consider the case $|\mathbb{F}| = 2$ which implies $\mathbb{A} = \mathbb{E}_a \cong \mathbb{F}_4 \cong \tilde{\mathbb{E}}_{\varphi_1(a)} = \tilde{\mathbb{A}}$. Given $a \in L_0^*$ and $t \in X_a$, we have

$$N(t) = 1_{\mathbb{A}} = T(t) = f(a,a) \ ,$$

hence

$$(at,t) = (at, t^\sigma t t) \in T \ .$$

Substituting (a,t) by (at,t) and (b,v) by (a,t) in identity (7.1) yields

$$\begin{aligned}\varphi_1(a) &= \varphi_1\big(a \cdot (t^\sigma + t)\big) = \varphi_1\big(a \cdot t^\sigma + (at) \cdot t^{-1} f(at,a) t^\sigma\big) \\ &= \varphi_1(a) \cdot \varphi_2(at,t)^{\tilde{\sigma}} + \varphi_1(at) \cdot \varphi_2(at,t)^{-1} \tilde{f}\big(\varphi_1(at), \varphi_1(a)\big) \varphi_2(at,t)^{\tilde{\sigma}} \ .\end{aligned}$$

Because of $\varphi_2(at,t) \notin \tilde{\mathbb{F}}$ we have

$$\varphi_2(at,t)^{-1} \tilde{f}\big(\varphi_1(at), \varphi_1(a)\big) \varphi_2(at,t)^{\tilde{\sigma}} \neq 0_{\tilde{\mathbb{A}}}$$

and thus

$$\varphi_1(at) = \big(\varphi_1(a) + \varphi_1(a) \cdot \varphi_2(at,t)^\sigma\big) \cdot \big(\varphi_2(at,t)^{-1} \tilde{f}(\varphi_1(at), \varphi_1(a)) \varphi_2(at,t)^{\tilde{\sigma}}\big)^{-1} \in \tilde{R}_{\varphi_1(a)} \ .$$

$\square$

(10.12) Remark The following lemma is due to Tom De Medts.

(10.13) Lemma Let $x \in \mathbb{A}$ with $x^\sigma \neq x$. Then the set

$$S := \big\{(s+x)^{-1}(s+x)^\sigma \mid s \in \mathbb{F}\big\}$$

cannot be completely contained in a one-dimensional $\mathbb{F}$-subspace of $\mathbb{A}$.

Proof
Suppose that $|\mathbb{F}| = 2$. Then we have $\mathbb{A} \cong \mathbb{F}_4$, hence

$$S = \{x, x + 1_{\mathbb{A}}\} \ ,$$

and the assertion is true. So assume $|\mathbb{F}| \geq 3$ and $S \subseteq y \cdot \mathbb{F}$ for some $y \in \mathbb{A}^*$. Then for each $s \in \mathbb{F}$, there is an element $t_s \in \mathbb{F}$ such that $(s+x)^\sigma = (s+x)yt_s$. In particular, $x^\sigma = xyt_0$, and hence we also have $(s+x)^\sigma = s + x^\sigma = s + xyt_0$. Therefore, $(s+x)yt_s = s + xyt_0$ for all $s \in \mathbb{F}$. Multiplying on the right by $s^{-1}y^{-1}$ yields

$$s^{-1}\big((s+x)t_s - xt_0\big) = y^{-1} \tag{10.4}$$

for all $s \in \mathbb{F}^*$. It follows that

$$r^{-1}\big((r+x)t_r - xt_0\big) = s^{-1}\big((s+x)t_s - xt_0\big)$$

for all $r, s \in \mathbb{F}^*$. This can be rearranged to get

$$rs(t_s - t_r) = x(st_r - st_0 - rt_s + rt_0)$$

for all $r, s \in \mathbb{F}^*$. Since $x \notin \mathbb{F}$, this can only happen if both sides are zero. Hence $t_s = t_r$ for all $r, s \in \mathbb{F}^*$, and substituting this in the right hand side gives $(s-r)(t_r - t_0) = 0_{\mathbb{A}}$ for all $r, s \in \mathbb{F}^*$ and hence $t_r = t_0$ for all $r \in \mathbb{F}$, where we use the fact that $\mathbb{F}^*$ has at least two elements. Substituting this in equation (10.4) yields $t_0 = y^{-1}$, but then $x^\sigma = xyt_0 = x$. This contradiction finishes the proof. $\square$

(10.14) Remark Let $d \in L_0^*$ be separable. Then for all $s \in \mathbb{F}$ and $x \in \mathbb{A}$, we have

$$\big(s+q(d)\big)^{-1} x \big(s+q(d)\big)^{\sigma} = \alpha_d(x) \big(s+q(d)\big)^{-1} \big(s+q(d)\big)^{\sigma} + e_d \beta_d(x) \ ,$$

and by lemma (10.13), the set

$$S := \Big\{ \big(s+q(d)\big)^{-1} \big(s+q(d)\big)^{\sigma} \mid s \in \mathbb{F} \Big\}$$

cannot be completely contained in a one-dimensional $\mathbb{F}$-subspace of $\mathbb{A}$.

(10.15) Lemma Let $\mathbb{A}$ be a quaternion division algebra and let $a, b \in L_0$ be separable. Then there are $a' \in R_a$ and $b' \in R_b$ such that $a' + b'$ and $a' \cdot q(a') + b'$ are separable.

Proof

- Let $\operatorname{Char} \mathbb{A} \neq 2$. By remark (10.3) (b), it is enough to choose $a' := a$ and $s \in \mathbb{F}$ such that

$$b' := b \cdot s \notin \{-a, -a \cdot q(a)\} \ .$$

- Let $\operatorname{Char} \mathbb{A} = 2$. Given $x \in L$, we have

$$f(x,x) = q(x) + q(x)^{\sigma} \in \mathbb{F}$$

and thus

$$\begin{aligned} F_x(s) := f(x + b \cdot s, x + b \cdot s) &= s^2 \cdot f(b,b) + s \cdot \big(f(x,b) + f(b,x)\big) + f(x,x) \\ &= s^2 \cdot f(b,b) + s \cdot \big(f(x,b) + f(x,b)^{\sigma}\big) + f(x,x) \in \mathbb{F}[s] \ . \end{aligned}$$

Since $\mathbb{A}$ is non-commutative, we have $|\mathbb{F}| = \infty$. As a consequence, there is an element $s \in \mathbb{F}$ not contained in the set of zeroes of $F_a(s)$ and $F_{a \cdot q(a)}(s)$. Then $a' := a$ and $b' = b \cdot s$ satisfy the required conditions.

□

(10.16) Notation Given $a, b \in L_0^*, c \in L_0$ and $r, s, t \in \mathbb{F}$, we set

$$M_{(a,s),(b,t),r}(c) := h_{(a,s+q(a)) \cdot (b,t+q(b)) \cdot (0,r)}(c) - h_{(a,s+q(a))}(c) - h_{(b,t+q(b))}(c) \ .$$

(10.17) Remark Given $a, b, c \in L_0^*$, $r, s, t \in \mathbb{F}$ and $r' := r + s + t + g(a,b)$, we have

$$\big(a, s+q(a)\big) \cdot \big(b, t+q(b)\big) \cdot (0, r) = \big(a+b, r' + q(a+b)\big) \ .$$

Now it follows from equation (10.1) and definition (7.17) that we have

$$\begin{aligned} M_{(a,s),(b,t),r}(c) = c \cdot \big(r - f(a,b)\big) & \\ + a \cdot \Big(& \big(s+q(a)\big)^{-1} f(a,c) \big(s+q(a)\big)^{\sigma} \\ & \qquad - \big(r' + q(a+b)\big)^{-1} f(a+b,c) \big(r' + q(a+b)\big)^{\sigma} \Big) \\ + b \cdot \Big(& \big(t+q(b)\big)^{-1} f(b,c) \big(t+q(b)\big)^{\sigma} \\ & \qquad - \big(r' + q(a+b)\big)^{-1} f(a+b,c) \big(r' + q(a+b)\big)^{\sigma} \Big) \ . \end{aligned}$$

(10.18) Remark The following lemma shows that orthogonality can be encoded in a condition that is preserved by Jordan isomorphisms. Notice that we need three pairwise orthogonal vectors. This is the point where the assumption about the dimension will come in.

(10.19) Lemma Let $x, y, z \in L_0^*$. Suppose that $f(x,y) = f(x,z) = f(y,z) = 0_{\mathbb{A}}$. Then we have

$$\forall\, r, s, t \in \mathbb{F},\ a \in R_{a'}, b \in R_{b'}, c \in R_{c'} : \qquad M_{(a,s),(b,t),r}(c) = c \cdot r \tag{10.5}$$

for each permutation (a', b', c') of (x, y, z).

Proof
For each permutation (a', b', c') of (x, y, z) and for all $a \in R_{a'}$, $b \in R_{b'}$ and $c \in R_{c'}$, we have

$$f(a,b) = f(a,c) = f(b,c) = 0_{\mathbb{A}}$$

as well, and hence $M_{(a,s),(b,t),r}(c) = c \cdot r$ for all $r, s, t \in \mathbb{F}$, which shows (10.5). □

(10.20) Remark Let $a \in L_0^*$. Then we have

$$\forall\, t \in X_a : \qquad \varphi_2(a,t) \in \tilde{q}\big(\varphi_1(a)\big) + \tilde{\mathbb{F}} = \tilde{X}_{\varphi_1(a)} \ .$$

Since γ is surjective, it follows that

$$\{\varphi_2(a,t) \mid t \in X_a\} = \tilde{X}_{\varphi_1(a)} \ .$$

(10.21) Corollary Let $x, y, z \in L_0^*$. Suppose that

$$f(x,y) = f(x,z) = f(y,z) = 0_{\mathbb{A}} \ .$$

Then (10.5) holds for each permutation (a', b', c') of $\big(\varphi_1(x), \varphi_1(y), \varphi_1(z)\big)$.

Proof
By lemma (10.19), (10.5) holds for each permutation (a', b', c') of (x, y, z). Since γ preserves the Hua-maps and $\varphi_1 : L_0 \to \tilde{L}_0$ is an isomorphism of vector spaces over $\mathbb{F}$ by remark (10.10) (a), we have

$$\tilde{M}_{(\varphi_1(a),\varphi_2(a,s)),(\varphi_1(b),\varphi_2(b,t)),\phi(r)}(\varphi_1(c)) = \varphi_1\big(M_{(a,s),(b,t),r}(c)\big) = \varphi_1(c \cdot r) = \varphi_1(c) \cdot \phi(r)$$

for all $a \in R_{a'}, b \in R_{b'}, c \in R_{c'}$, $r, s, t \in \mathbb{F}$ and for each permutation (a', b', c') of (x, y, z). Corollary (10.11) and remark (10.20) yield

$$\forall\, a \in \tilde{R}_{a'}, b \in \tilde{R}_{b'}, c \in \tilde{R}_{c'},\ r, s, t \in \tilde{\mathbb{F}} : \qquad \tilde{M}_{(a,s),(b,t),r}(c) = c \cdot r$$

for each permutation (a', b', c') of $\big(\varphi_1(x), \varphi_1(y), \varphi_1(z)\big)$. □

(10.22) Remark The following lemma is essentially due to Tom De Medts. It shows that we can reconstruct the orthogonality from the above condition if we suppose all appearing elements to be separable. In this situation, we know that we have a one-dimensional $\mathbb{F}$-subspace on the right side of (10.5), but on the left side we have terms which are not contained in a one-dimensional $\mathbb{F}$-subspace if we vary the coefficients. As a consequence, the occurring scalar products necessarily vanish.

Afterwards we will have a closer look at inseparable elements. In order to obtain the same result in this case, we need to establish a connection between the corresponding quadratic extensions. For this purpose, it is convenient to use an inseparable element in (10.5) twice so that we can deduce more information about the occurring terms.

(10.23) Lemma Let $x, y, z \in L_0^*$. Suppose that (10.5) holds for each permutation (a', b', c') of (x, y, z). If x, y, z are separable, we have

$$f(x,y) = f(x,z) = f(y,z) = 0_{\mathbb{A}} \ .$$

Proof
Assume that (10.5) holds for each permutation (a', b', c') of (x, y, z). In particular, for all $a \in R_{a'}$, $b \in R_{b'}$ and $c \in R_{c'}$, the set
$$\{M_{(a,s),(b,t),r}(c) \mid r, s, t \in \mathbb{F}\}$$
is a one-dimensional $\mathbb{F}$-subspace of L_0.

- Suppose that $\mathbb{A}$ is a quaternion division algebra. By lemma (10.15), there are $a \in R_{a'}, b \in R_{b'}$ such that $a + b$ and $a \cdot q(a) + b$ are separable. By varying s over $\mathbb{F}$ in $M_{(a,s),(b,t),r}(c)$ but keeping $r' := r + s + t + g(a, b)$ and t invariant (by the right choice for r), remark (10.14) yields[2] $\alpha_a\big(f(a, c)\big) = 0_{\mathbb{A}}$. Similarly,

  $$\alpha_b\big(f(b, c)\big) = 0_{\mathbb{A}} = \alpha_{a+b}\big(f(a + b, c)\big)\ .$$

  Now the expression for $M_{(a,s),(b,t),r}(c)$ can be simplified[3] to

  $$M_{(a,s),(b,t),r}(c) = c \cdot \big(r - f(a, b)\big) - a \cdot f(b, c) - b \cdot f(a, c)$$

  for all $c \in R_{c'}$, $r, s, t \in \mathbb{F}$. By assumption, we have

  $$c \cdot f(a, b) + a \cdot f(b, c) + b \cdot f(a, c) = 0_{\mathbb{A}} \tag{10.6}$$

  for each $c \in R_{c'}$.

  Now suppose that $f(a, b) \neq 0_{\mathbb{A}}$; we will derive a contradiction. If we interchange a and b in equation (10.6), we get $f(a, b) = f(b, a)$, and if we replace a by $a \cdot q(a) \in R_{a'}$, then $f\big(a \cdot q(a), b\big) = f\big(b, a \cdot q(a)\big)$ and hence

  $$q(a)^\sigma f(a, b) = f(a, b)q(a)\ . \tag{10.7}$$

  Now let t be an arbitrary element of $\mathbb{E}_c$. On the one hand, we can multiply equation (10.6) by t; on the other hand, we can replace c by $ct \in R_c$. Comparing these two resulting equations yields $tf(a, b) = f(a, b)t$ for all $t \in \mathbb{E}_c$, which implies $f(a, b) \in C_{\mathbb{A}}(\mathbb{E}_c) = \mathbb{E}_c$. If we now replace a by $a \cdot q(a) \in R_a = a \cdot \mathbb{E}_a$, we get $\mathbb{E}_a f(a, b) \subseteq \mathbb{E}_c$, which implies $\mathbb{E}_a = \mathbb{E}_c$ and hence $f(a, b) \in \mathbb{E}_a$. But then $q(a)f(a, b) = f(a, b)q(a)$, and comparing this with equation (10.7) yields $q(a) = q(a)^\sigma$, which gives us the required contradiction and hence

  $$f(a', b') = f(a, b) = 0_{\mathbb{A}}\ .$$

  Permuting a', b' and c' now yields

  $$f(x, y) = f(x, z) = f(y, z) = 0_{\mathbb{A}}\ .$$

- If $\mathbb{A}$ is commutative, we immediately obtain

  $$f(a, c) = 0_{\mathbb{A}} = f(b, c)$$

  by the same arguments, followed by

  $$f(a + b, c) = f(a, c) + f(b, c) = 0_{\mathbb{A}}$$

  and finally $f(a, b) = 0_{\mathbb{A}}$.

□

[2]Notice that we need a, b and $a + b$ to be separable so that we can apply Lemma (10.13) at this point. This follows from the assumption that x, y, z are separable and from the choice of a, b. As we will replace a by $a \cdot q(a)$, we additionally need $a \cdot q(a) + b$ to be separable.

[3]Notice that we have $e_a\beta_a\big(f(a, c)\big) = \alpha_a\big(f(a, c)\big) + e_a\beta_a\big(f(a, c)\big) = f(a, c)$ etc.

(10.24) Remark Lemma (10.23) is enough to handle the cases where $\operatorname{Char}\mathbb{A} \neq 2$ or where $(\mathbb{A}, \mathbb{F}, \sigma)$ is quadratic of type (iii) since there are no inseparable elements in this situation. But with some technical effort, we can handle the remaining case as well.

(10.25) Notation Until proposition (10.30), we suppose $(\mathbb{A}, \mathbb{F}, \sigma)$ to be quadratic of type (iv) with $\operatorname{Char}\mathbb{A} = 2$.

(10.26) Remark Let $x \in L_0$ be inseparable and let $y \in L_0$ be such that $f(x,y) = f(x,y)^\sigma$. Then we have

$$f(x+y, x+y) = 0_{\mathbb{A}} \;\Leftrightarrow\; f(x,x) + f(x,y) + f(y,x) + f(y,y) = 0_{\mathbb{A}} \;\Leftrightarrow\; f(y,y) = 0_{\mathbb{A}} \;.$$

(10.27) Lemma Let $x, y, z \in L_0^*$ be such that (10.5) holds for each permutation (a', b', c') of (x, y, z). If $x = y$ is inseparable and $f(x,z) \neq 0_{\mathbb{A}}$, we have $f(x,z) \in \mathbb{E}_x = \mathbb{E}_{x+z}$, and z is inseparable.

Proof
Since x is inseparable and $f(x,z) \neq 0_{\mathbb{A}}$, we have $z \notin \langle x \rangle_{\mathbb{A}}$. Putting $a' = c' = x,\; b' = z$ and $a = c$ in (10.5) and comparing the coefficients of $a = c$ yield

$$f(a,b) = \big(r' + q(a+b)\big)^{-1} f(b,a) \big(r' + q(a+b)\big)^{\sigma}$$

for all $a \in R_x, b \in R_z$, and putting $a' = z,\; b' = c' = x$ and $b = c$ in (10.5) and comparing the coefficients of $b = c$ yield

$$f(a,b) = \big(r' + q(a+b)\big)^{-1} f(a,b) \big(r' + q(a+b)\big)^{\sigma} \tag{10.8}$$

for all $a \in R_z, b \in R_x$, hence

$$\forall\; a \in R_x, b \in R_z : \qquad f(a,b) = f(b,a) = f(a,b)^{\sigma} \;.$$

In particular, we have

$$f(x,z) = f(z,x) \;, \qquad q(x) f(x,z) = f(x,z) q(x) \;, \qquad f(x,z) \in C_{\mathbb{A}}\big(q(x)\big) = \mathbb{E}_x$$

Putting $a' = z,\; b' = c' = x,\; a = z,\; b = x$ and $c = x \cdot q(x)$ in (10.5) and comparing the coefficients of x yield

$$q(x) f(z,x) = q(x+z)^{-1} f(z,x) q(x) q(x+z)^{\sigma} = q(x+z)^{-1} q(x) f(z,x) q(x+z)^{\sigma} \;.$$

Using equation (10.8) with $a = z$ and $b = c = x$ and multiplying by $q(x)$ yield

$$q(x) f(z,x) = q(x) q(x+z)^{-1} f(z,x) q(x+z)^{\sigma} \;.$$

As a consequence, we have

$$q(x) \in C_{\mathbb{A}}\big(q(x+z)\big) = \mathbb{E}_{x+z} \;, \qquad\qquad \mathbb{E}_x = \mathbb{E}_{x+z} \;.$$

Since x is inseparable, $x + z$ has to be inseparable as well. Finally, z is inseparable by remark (10.26). □

(10.28) Corollary Let $x \in L_0^*$ be inseparable and let $y \in x^{\perp}$ be separable. Then we have

$$\varphi_1(x) \in \varphi_1(y)^{\perp} \;.$$

Proof
By corollary (10.21), identity (10.5) holds for each permutation (a', b', c') of $\big(\varphi_1(x), \varphi_1(x), \varphi_1(y)\big)$. By corollary (10.5), $\varphi_1(x)$ is inseparable and $\varphi_1(y)$ is separable, thus lemma (10.27) yields

$$\big(\varphi_1(x), \varphi_1(y)\big) = 0_{\tilde{\mathbb{A}}} \;.$$

□

(10.29) Lemma Let $x \in L_0^*$ and $y \in x^\perp \setminus \langle x \rangle_{\mathbb{A}}$ both be inseparable. Then we have

$$\varphi_1(x) \in \varphi_1(y)^\perp \ .$$

Proof
By corollary (10.21), identity (10.5) holds for each permutation (a', b', c') of $\big(\varphi_1(x), \varphi_1(x), \varphi_1(y)\big)$ and $\big(\varphi_1(x), \varphi_1(y), \varphi_1(y)\big)$, respectively. Suppose that $\tilde{f}\big(\varphi_1(x), \varphi_1(y)\big) \neq 0_{\tilde{\mathbb{A}}}$. By corollary (10.5), $\varphi_1(x)$ and $\varphi_1(y)$ are inseparable, hence lemma (10.27) yields

$$\tilde{f}\big(\varphi_1(x), \varphi_1(y)\big) \in \tilde{\mathbb{E}}_{\varphi_1(x)} = \tilde{\mathbb{E}}_{\varphi_1(x+y)} = \tilde{\mathbb{E}}_{\varphi_1(y)} \ .$$

By identity (7.1) with $b := y$ and $(a, t) := \big(x, q(x)\big)$, we have

$$\varphi_1\big(y \cdot q(x)^\sigma\big) = \varphi_1(y)\varphi_2\big(x, q(x)\big)^{\tilde{\sigma}} - \varphi_1(x) \cdot \varphi_2\big(x, q(x)\big)^{-1} \tilde{f}\big(\varphi_1(x), \varphi_1(y)\big)\varphi_2\big(x, q(x)\big)^{\tilde{\sigma}}$$

with

$$\varphi_2\big(x, q(x)\big)^{\tilde{\sigma}} \in \tilde{\mathbb{E}}_{\varphi_1(x)} = \tilde{\mathbb{E}}_{\varphi_1(y)} \ , \qquad \varphi_2\big(x, q(x)\big)^{-1} \tilde{f}\big(\varphi_1(x), \varphi_1(y)\big)\varphi_2\big(x, q(x)\big)^{\tilde{\sigma}} \in \tilde{\mathbb{E}}_{\varphi_1(x)} \ .$$

By corollary (10.11), there are elements $s, t \in \mathbb{A}$ such that

$$\varphi_1(y \cdot s) = \varphi_1(y)\varphi_2\big(x, q(x)\big)^{\tilde{\sigma}} \ , \quad \varphi_1(x \cdot t) = \varphi_1(x) \cdot \varphi_2\big(x, q(x)\big)^{-1} \tilde{f}\big(\varphi_1(x), \varphi_1(y)\big)\varphi_2\big(x, q(x)\big)^{\tilde{\sigma}} \ ,$$

which yields

$$y \cdot q(x)^\sigma + y \cdot s + x \cdot t = 0_{L_0} \ , \qquad\qquad s = q(x)^\sigma, \ t = 0_{\mathbb{A}}$$

and finally

$$0_{L_0} = \varphi_1(x \cdot t) = \varphi_1(x) \cdot \varphi_2\big(x, q(x)\big)^{-1} \tilde{f}\big(\varphi_1(x), \varphi_1(y)\big)\varphi_2\big(x, q(x)\big)^{\tilde{\sigma}} \ , \quad \tilde{f}\big(\varphi_1(x), \varphi_1(y)\big) = 0_{\tilde{\mathbb{A}}} \ .$$

□

(10.30) Proposition Let $x \in L_0^*$ be inseparable and let $y \in x^\perp$. If $\dim_{\mathbb{A}} L_0 \geq 3$, we have

$$\varphi_1(x) \in \varphi_1(y)^\perp \ .$$

Proof
If y is separable, we may apply lemma (10.28), and if x and y are linearly independent over $\mathbb{A}$, we may apply lemma (10.29) so that we may assume $y \in \langle x \rangle_{\mathbb{A}}$. By assumption, there is an element $z \in x^\perp \setminus \langle x \rangle_{\mathbb{A}}$. As a consequence, we have

$$y + z \in x^\perp \setminus \langle x \rangle_{\mathbb{A}} \ .$$

Now corollary (10.28) and lemma (10.29), respectively, yield

$$\tilde{f}\big(\varphi_1(x), \varphi_1(y)\big) = \tilde{f}\big(\varphi_1(x), \varphi_1(y+z)\big) + \tilde{f}\big(\varphi_1(x), \varphi_1(z)\big) = 0_{\tilde{\mathbb{A}}} + 0_{\tilde{\mathbb{A}}} = 0_{\tilde{\mathbb{A}}} \ .$$

□

(10.31) Lemma Assume $\dim_{\mathbb{A}} L_0 \geq 3$ and let $a, b \in L_0^*$. Then $f(a, b) = 0_{\mathbb{A}}$ if and only if there is an element $c \in L_0^*$ such that

$$f(a, b) = f(a, c) = f(b, c) = 0_{\mathbb{A}} \ .$$

Proof
If a or b is inseparable, we may choose $c := a$ or $c := b$, respectively. So assume that a and b are separable and $f(a, b) = 0_{\mathbb{A}}$. Since $\dim_{\mathbb{A}} L_0 \geq 3$ and $\dim_{\mathbb{A}} \langle a, b \rangle_{\mathbb{A}} \leq 2$, there is some element $d \in X \setminus \langle a, b \rangle_{\mathbb{A}}$. Then

$$c := d - a \cdot f(a, a)^{-1} f(a, d) - b \cdot f(b, b)^{-1} f(b, d) \neq 0_{\mathbb{A}}$$

satisfies the required conditions. □

(10.32) Remark We return to the general case.

(10.33) Proposition Let $x \in L_0$ be separable and let $y \in x^\perp$. If $\dim_{\mathbb{A}} L_0 \geq 3$, we have
$$\varphi_1(x) \in \varphi_1(y)^\perp \ .$$

Proof

- If y is inseparable, we may apply proposition (10.30).
- If y is separable, lemma (10.31) yields an element $z \in L_0^*$ such that
$$f(x,z) = 0_{\mathbb{A}} = f(y,z) \ ,$$
thus (10.5) holds for each permutation (a', b', c') of $\bigl(\varphi_1(x), \varphi_1(y), \varphi_1(z)\bigr)$ by corollary (10.21).
  - If z is inseparable, proposition (10.30) yields
$$\tilde{f}\bigl(\varphi_1(x), \varphi_1(z)\bigr) = 0_{\tilde{\mathbb{A}}} = \tilde{f}\bigl(\varphi_1(y), \varphi_1(z)\bigr) \ ,$$
followed by
$$\tilde{f}\bigl(\varphi_1(x+y), \varphi_1(z)\bigr) = 0_{\tilde{\mathbb{A}}}$$
and finally $\tilde{f}\bigl(\varphi_1(x), \varphi_1(y)\bigr) = 0_{\tilde{\mathbb{A}}}$ by (10.5).
  - If z is separable, then $\varphi_1(x), \varphi_1(y), \varphi_1(z)$ are separable by corollary (10.5), thus we may apply lemma (10.23) to obtain
$$\tilde{f}\bigl(\varphi_1(x), \varphi_1(y)\bigr) = 0_{\tilde{\mathbb{A}}} \ .$$

□

(10.34) Corollary If we have $\dim_{\mathbb{A}} L_0 \geq 3$, the map $\varphi_1 : L_0 \to \tilde{L}_0$ is an isomorphism of vector spaces.

Proof
By proposition (10.30) and proposition (10.33), we have
$$\forall\ a \in L_0 : \qquad \varphi_1(a^\perp) = \varphi_1(a)^\perp \ .$$
Since f is non-degenerate, this implies that we have
$$\forall\ a \in L_0 : \qquad \varphi_1(\langle a \rangle_{\mathbb{A}}) = \varphi_1(a^{\perp\perp}) = \varphi_1(a)^{\perp\perp} = \langle \varphi_1(a) \rangle_{\tilde{\mathbb{A}}} \ .$$
Now the assertion results from the fundamental theorem of projective geometry. □

(10.35) Notation Let $\phi : \mathbb{A} \to \tilde{\mathbb{A}}$ be the isomorphism of skew-fields defined by
$$\forall\ a \in L_0,\ t \in \mathbb{A} : \qquad \varphi_1(a \cdot t) = \varphi_1(a) \cdot \phi(t) \ .$$

(10.36) Remark Notice that ϕ is an extension of the isomorphism $\phi : \mathbb{F} \to \tilde{\mathbb{F}}$ of fields.

(10.37) Remark We state the theorem using the general notation.

(10.38) Theorem Let $(\mathbb{K}, \mathbb{K}_0, \sigma)$ be quadratic of type (iii) or (iv) and let $\dim_{\mathbb{K}} L_0 \geq 3$. Then the map $\Phi : \Xi \to \tilde{\Xi}$ defined by
$$\Phi := (\varphi_1, \phi) : (L_0, \mathbb{K}) \to (\tilde{L}_0, \tilde{\mathbb{K}}),\ (a,t) \mapsto \bigl(\varphi_1(a), \phi(t)\bigr)$$
is an isomorphism of pseudo-quadratic spaces satisfying
$$\forall\ (a,t) \in T : \qquad \Phi(a,t) = \gamma(a,t) \ .$$

Proof

- By corollary (10.34) and notation (10.35), the map $(\varphi_1, \phi) : (L_0, \mathbb{K}) \to (\tilde{L}_0, \tilde{\mathbb{K}})$ is an isomorphism of vector spaces.
- By remark (10.36), we have $\phi(\mathbb{K}_0) = \tilde{\mathbb{K}}_0$.
- By corollary (5.12), we have
$$\forall\, x \in \mathbb{K} : \qquad \phi(x^\sigma) = \phi(x)^{\tilde{\sigma}} \ .$$
- Let $(a,t) \in T$ and $0_{L_0} \neq b \in a^\perp$. Then we have
$$\varphi_1(b) \cdot \phi(t)^{\tilde{\sigma}} = \varphi_1(b) \cdot \phi(t^\sigma) = \varphi_1(b \cdot t^\sigma) = \varphi_1(b) \cdot \varphi_2(a,t)^{\tilde{\sigma}}$$
by identity (7.1), proposition (10.30) and proposition (10.33), thus
$$\varphi_2(a,t) = \phi(t)$$
and therefore
$$\tilde{q}\big(\varphi_1(a)\big) \in \varphi_2(a,t) + \tilde{\mathbb{K}}_0 = \phi(t) + \phi(\mathbb{K}_0) = \phi\big(t + \mathbb{K}_0\big) = \phi\big(q(a) + \mathbb{K}_0\big) = \phi\big(q(a)\big) + \tilde{\mathbb{K}}_0$$
as well as
$$\Phi(a,t) = \big(\varphi_1(a), \phi(t)\big) = \big(\varphi_1(a), \varphi_2(a,t)\big) = \gamma(a,t) \ .$$

□

Chapter 11 Small Dimensions I

First we refine some results of the previous chapter, without assuming additional assumptions. The first step is to show that the map $\varphi_1 : R_a \to R_{\varphi_1(a)}$ is an isomorphism of vector spaces for each $a \in L_0$. We manage to do this by proving that the map $\varphi_2 : X_a \to X_{\varphi_1(a)},\ t \mapsto \varphi_2(a,t)$ is induced by an isomorphism between the associated separable extensions.

Once we have done this, it is easy to prove theorem (8.1) (iii) if the involutory sets are quadratic of type (iii). Afterwards we will need some more considerations to handle the case of involutory sets which are quadratic of type (iv), cf. chapter 12.

(11.1) Notation Throughout this chapter, the involutory sets $(\mathbb{A}, \mathbb{F}, \sigma)$ and $(\tilde{\mathbb{A}}, \tilde{\mathbb{F}}, \tilde{\sigma})$ are still quadratic of type (iii) or (iv).

(11.2) Lemma Let $a \in L_0$. Then the following holds:

(a) $\forall\, t \in X_a :\ \phi\big(N(t)\big) = \tilde{N}\big(\varphi_2(a,t)\big)$.

(b) $\forall\, t \in X_a :\ \phi\big(T(t)\big) = \tilde{T}\big(\varphi_2(a,t)\big)$.

Proof

(a) By lemma (7.21) (b) with $(b,v) := (0_{L_0}, 1_{\mathbb{A}})$, we have
$$\phi(tt^\sigma) = \varphi_2(0_{L_0}, t \cdot 1_{\mathbb{A}} \cdot t^\sigma) = \varphi_2(a,t) \cdot \varphi_2(0_{L_0}, 1_{\mathbb{A}}) \cdot \varphi_2(a,t)^{\tilde{\sigma}} = \tilde{N}\big(\varphi_2(a,t)\big) \ .$$

(b) By part (a) and lemma (7.21) (d), we have
$$\begin{aligned} \phi\big(N(t)\big) + \phi\big(T(t)\big) + 1_{\tilde{A}} &= \phi\big((t+1_{\mathbb{A}})(t+1_{\mathbb{A}})^\sigma\big) = \phi\big(N(t+1_{\mathbb{A}})\big) \\ &= \tilde{N}\big(\varphi_2(a,t+1_{\mathbb{A}})\big) = \tilde{N}\big(\varphi_2(a,t) + \phi(1_{\mathbb{A}})\big) \\ &= \tilde{N}\big(\varphi_2(a,t)\big) + \tilde{T}\big(\varphi_2(a,t)\big) + 1_{\tilde{\mathbb{A}}} \ , \end{aligned}$$
hence
$$\phi\big(T(t)\big) = \tilde{T}\big(\varphi_2(a,t)\big)$$
by part (a) again. □

(11.3) Lemma Let $a \in L_0^*$ and $t \in X_a$. Then the map

$$\phi_{(a,t)} : \mathbb{E}_a \to \tilde{\mathbb{E}}_{\varphi_1(a)},\ x + t \cdot y \mapsto \phi(x) + \varphi_2(a,t) \cdot \phi(y) \qquad (x, y \in \mathbb{F})$$

is an isomorphism of fields such that

$$\forall\ u \in X_a : \qquad \varphi_2(a,u) = \phi_{(a,t)}(u)\ .$$

Proof
By lemma (11.2), we have

$$\phi\big(N(t)\big) = \tilde{N}\big(\varphi_2(a,t)\big)\ , \qquad\qquad \phi\big(T(t)\big) = \tilde{T}\big(\varphi_2(a,t)\big)$$

so that we may apply lemma (5.15). By lemma (7.21) (d), we have

$$\forall\ s \in \mathbb{F} : \qquad \varphi_2(a, t+s) = \varphi_2(a,t) + \phi(s) = \phi_{(a,t)}(t) + \phi_{(a,t)}(s) = \phi_{(a,t)}(t+s)\ .$$

□

(11.4) Corollary Let $a \in L_0^*$. Given $t, u \in X_a$, we have

$$\phi_a := \phi_{(a,t)} = \phi_{(a,u)}\ .$$

Proof
We have $\phi_{(a,u)}(u) = \varphi_2(a,u) = \phi_{(a,t)}(u)$ and $u \notin \mathbb{F}$. □

(11.5) Remark Let $a \in L_0^*$.

(a) Notice that ϕ_a is an extension of the isomorphism $\phi : \mathbb{F} \to \tilde{\mathbb{F}}$ of fields.

(b) We have $\phi_a(\mathbb{E}_a) = \tilde{\mathbb{E}}_{\varphi_1(a)}$.

(c) Given $t \in \mathbb{A}$, we have

$$\begin{aligned} t^{-1}(t^\sigma)^2 &= \frac{(t^\sigma)^3}{N(t)} \overset{(10.3)}{=} \frac{\big(T(t)^2 - N(t)\big) \cdot t^\sigma - T(t)N(t)}{N(t)} \\ &= \frac{\big(T(t)^2 - N(t)\big)\big(T(t) - t\big) - T(t)N(t)}{N(t)} = \frac{T(t)^3 - 2T(t)N(t)}{N(t)} + \frac{N(t) - T(t)^2}{N(t)} \cdot t\ . \end{aligned}$$

(d) Assume $|\mathbb{F}| \geq 3$. By lemma (10.8), there is an element $t \in X_a$ such that $T(t)^2 \neq N(t)$.

(11.6) Lemma Assume $|\mathbb{F}| \geq 3$. Let $a \in L_0^*$ and $t \in X_a$ be as in remark (11.5) (d), i.e., we have $N(t) \neq T(t)^2$ and thus $\phi\big(N(t)\big) \neq \phi\big(T(t)\big)^2$. Then we have

$$\varphi_1(a \cdot t) = \varphi_1(a) \cdot \varphi_2(a,t)\ .$$

Proof
By remark (11.5) (c), equation (10.2) and identity (7.1) with $b = a$, we have

$$\begin{aligned} &\varphi_1(a) \cdot \frac{\phi\big(T(t)\big)^3 - 2\phi\big(T(t)\big)\phi\big(N(t)\big)}{\phi\big(N(t)\big)} + \varphi_1(a \cdot t) \cdot \frac{\phi\big(N(t)\big) - \phi\big(T(t)\big)^2}{\phi\big(N(t)\big)} \\ \overset{(10.10)\ (a)}{=}\ &\varphi_1\left(a \cdot \left(\frac{T(t)^3 - 2T(t)N(t)}{N(t)} + \frac{N(t) - T(t)^2}{N(t)} \cdot t\right)\right) \\ \overset{\substack{(11.5)\ (c) \\ (10.2),(7.1)}}{=}\ &\varphi_1(a) \cdot \left(\frac{\tilde{T}\big(\varphi_2(a,t)\big)^3 - 2\tilde{T}\big(\varphi_2(a,t)\big)\tilde{N}\big(\varphi_2(a,t)\big)}{\tilde{N}\big(\varphi_2(a,t)\big)} + \frac{\tilde{N}\big(\varphi_2(a,t)\big) - \tilde{T}\big(\varphi_2(a,t)\big)^2}{\tilde{N}\big(\varphi_2(a,t)\big)} \cdot \varphi_2(a,t)\right) \\ \overset{(11.2)}{=}\ &\varphi_1(a) \cdot \frac{\phi\big(T(t)\big)^3 - 2\phi\big(T(t)\big)\phi\big(N(t)\big)}{\phi\big(N(t)\big)} + \varphi_1(a) \cdot \varphi_2(a,t) \cdot \frac{\phi\big(N(t)\big) - \phi\big(T(t)\big)^2}{\phi\big(N(t)\big)} \end{aligned}$$

Notice that we may cancel scalars by assumption. □

(11.7) Remark By corollary (10.11), we have $\varphi_1(R_a) = \tilde{R}_{\varphi_1(a)}$ even in the case $|\mathbb{F}| = 2$. Notice that we have

$$\mathbb{A} = \mathbb{E}_a \cong \mathbb{F}_4 \cong \tilde{\mathbb{E}}_{\varphi_1(a)} = \tilde{\mathbb{A}}$$

in this situation which shows that we have $\varphi_1(\langle a\rangle_{\mathbb{A}}) = \langle \varphi_1(a)\rangle_{\tilde{\mathbb{A}}}$. As a consequence, the map $\phi_a : \mathbb{A} \to \tilde{\mathbb{A}}$ defined by

$$\forall\, t \in \mathbb{A} : \qquad \varphi_1(a \cdot t) = \varphi_1(a) \cdot \phi_a(t)$$

is an isomorphism of skew-fields. However, it doesn't necessarily satisfy $\phi_a(t) = \varphi_2(a,t)$ for each $t \in X_a$. We will discuss this case later and go on with assuming $|\mathbb{F}| \geq 3$.

(11.8) Proposition Given $a \in L_0^*$, the map

$$(\varphi_1, \phi_a) : (R_a, \mathbb{E}_a) \to (\tilde{R}_{\varphi_1(a)}, \tilde{\mathbb{E}}_{\varphi_1(a)})$$

is an isomorphism of vector spaces such that $\phi_a(t) = \varphi_2(a,t)$ for each $t \in X_a$.

Proof
Let $t \in X_a$ be as in remark (11.5) (d). By remark (10.10) (a), lemma (11.6) and lemma (11.3), we have

$$\forall\, x, y \in \mathbb{F} : \qquad \varphi_1\big(a \cdot (x + ty)\big) = \varphi_1(a) \cdot \big(\phi(x) + \varphi_2(a,t)\phi(y)\big) = \varphi_1(a) \cdot \phi_a(x + ty)\ .$$

The second assertion results from lemma (11.3). □

(11.9) Lemma Let $(\mathbb{A}, \mathbb{F}, \sigma)$ and $(\tilde{\mathbb{A}}, \tilde{\mathbb{F}}, \tilde{\sigma})$ both be quadratic of type (iii). Then we have

$$\forall\, a, b \in L_0^* : \qquad \phi := \phi_a = \phi_b\ .$$

Proof
By assumption, we have $\mathbb{A} = \mathbb{E}_a = \mathbb{E}_b$ and $\tilde{\mathbb{A}} = \tilde{\mathbb{E}}_{\varphi_1(a)} = \tilde{\mathbb{E}}_{\varphi_1(b)}$ for all $a, b \in L_0^*$.

- Let $b = a \cdot s$ for some $s \in \mathbb{A}^*$. Given $t \in \mathbb{A}$, we have

$$\begin{aligned}\varphi_1(a) \cdot \phi_a(t) &= \varphi_1(a \cdot t) = \varphi_1(as \cdot s^{-1}t)\\ &= \varphi_1(b) \cdot \phi_b(s^{-1}t) = \varphi_1(b) \cdot \phi_b(s^{-1}) \cdot \phi_b(t) = \varphi_1(a) \cdot \phi_b(t)\ .\end{aligned}$$

- Let $b \notin \langle a\rangle_{\mathbb{A}} = R_a$, hence $\varphi_1(b) \notin \tilde{R}_{\varphi_1(a)} = \langle \varphi_1(a)\rangle_{\tilde{\mathbb{A}}}$ by proposition (11.8). Given $t \in \mathbb{A}$, we have

$$\begin{aligned}\varphi_1(a) \cdot \phi_a(t) + \varphi_1(b) \cdot \phi_b(t) &= \varphi_1(a \cdot t) + \varphi_1(b \cdot t) = \varphi_1\big((a+b) \cdot t\big)\\ &= \varphi_1(a+b) \cdot \phi_{a+b}(t) = \varphi_1(a) \cdot \phi_{a+b}(t) + \varphi_1(b) \cdot \phi_{a+b}(t)\ ,\end{aligned}$$

  thus

$$\phi_a(t) = \phi_{a+b}(t) = \phi_b(t)$$

□

(11.10) Remark

(a) The proof shows that we have

$$\forall\, t \in \mathbb{E}_a : \qquad \phi_a = \phi_{at}\ ,$$

independent of the type of $(\mathbb{A}, \mathbb{F}, \sigma)$.

(b) Notice that ϕ is an extension of the (Jordan) isomorphism $\phi : \mathbb{F} \to \tilde{\mathbb{F}}$.

(c) We state the theorem using the general notation.

(11.11) Theorem Let $(\mathbb{K}, \mathbb{K}_0, \sigma)$ and $(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ both be quadratic of type (iii) such that $|\mathbb{K}_0| \geq 3$. Then the map $\Phi : \Xi \to \tilde{\Xi}$ defined by

$$\Phi := (\varphi_1, \phi) : (L_0, \mathbb{K}) \to (\tilde{L}, \tilde{\mathbb{K}}),\ (a,t) \mapsto \big(\varphi_1(a), \phi(t)\big)$$

is an isomorphism of pseudo-quadratic spaces satisfying

$$\forall\ (a,t) \in T : \qquad \Phi(a,t) = \gamma(a,t)\ .$$

In particular, we have $\mathbb{K} \cong \tilde{\mathbb{K}}$.

Proof

- By proposition (11.8) and lemma (11.9), the map $(\varphi_1, \phi) : (L_0, \mathbb{K}) \to (\tilde{L}_0, \tilde{\mathbb{K}})$ is an isomorphism of vector spaces.
- By remark (11.10) (b), we have $\phi(\mathbb{K}_0) = \tilde{\mathbb{K}}_0$.
- By corollary (5.12), we have $\phi \circ \sigma = \tilde{\sigma} \circ \phi$.
- Let $a \in L_0^*$ and $t \in X_a$. By proposition (11.8), we have

  $$\varphi_2(a,t) = \phi_a(t) = \phi(t)\ ,$$

  hence

  $$\tilde{q}\big(\varphi_1(a)\big) \in \varphi_2(a,t) + \tilde{\mathbb{K}}_0 = \phi(t) + \phi(\mathbb{K}_0) = \phi\big(t + \mathbb{K}_0\big) = \phi\big(q(a) + \mathbb{K}_0\big) = \phi\big(q(a)\big) + \tilde{\mathbb{K}}_0\ .$$

- Given $(a,t) \in T$, we have

  $$\Phi(a,t) = \big(\varphi_1(a), \phi(t)\big) = \big(\varphi_1(a), \varphi_2(a,t)\big) = \gamma(a,t)\ .$$

□

Chapter 12 Small Dimensions II

Now we prove theorem (8.1) (iii) for the case that $(\mathbb{A}, \mathbb{F}, \sigma)$ is quadratic of type (iv). As in the previous paragraph, we exploit the identities (7.1) and (7.2) to show that $\phi_a : \mathbb{E}_a \to \tilde{\mathbb{E}}_{\varphi_1(a)}$ is induced by an isomorphism $\phi_{a,e} : \mathbb{A} \to \bar{\mathbb{A}}$ of skew-fields, where $a \in L_0^*$ is a separable element and $e \in \mathbb{E}_a^\perp$.

Notice that we restrict to a separable element $a \in L_0^*$ because of the helpful decomposition $\mathbb{A} = \mathbb{E}_a \oplus e\mathbb{E}_a$ with $e\mathbb{E}_a = \mathbb{E}_a^\perp$. If we assume $\varphi_1(a \cdot e) = \varphi_1(a) \cdot \tilde{e}$ for some $\tilde{e} \in \tilde{\mathbb{A}}$, the isomorphisms ϕ_a and ϕ_{ae} of fields induce a map

$$\phi_{a,e} : \mathbb{A} \to \tilde{\mathbb{A}},\ s + et \mapsto \phi_a(s) + \tilde{e}\phi_{ae}(t) \qquad (s,t \in \mathbb{E}_a)$$

which turns out to be an isomorphism of skew-fields. As a consequence, the two isomorphisms $\varphi_1 : R_a \to \tilde{R}_{\varphi_1(a)}$ and $\varphi_1 : R_{ae} \to \tilde{R}_{\varphi_1(ae)}$ of vector spaces over $\mathbb{E}_a$ are induced by an isomorphism $\varphi_1 : \langle a \rangle_{\mathbb{A}} \to \langle \varphi_1(a) \rangle_{\tilde{\mathbb{A}}}$ of vector spaces over $\mathbb{A}$.

(12.1) Notation Throughout this chapter, let $(\mathbb{A}, \mathbb{F}, \sigma)$ be quadratic of type (iv), and let $(\tilde{\mathbb{A}}, \tilde{\mathbb{F}}, \tilde{\sigma})$ be quadratic of type (iii) or (iv). Until corollary (12.12), let $a \in L_0^*$ be separable and let $e \in \mathbb{E}_a^\perp$.

(12.2) Lemma Given $t \in X_a \subseteq \mathbb{E}_a$, we have

$$ae \cdot t = ae \cdot t^\sigma - a \cdot t^{-1} f(a, ae) t^\sigma\ .$$

Proof
We have

$$\begin{aligned} ae \cdot t^{\sigma} - a \cdot t^{-1} f(a, ae) t^{\sigma} &= ae \cdot t^{\sigma} - a \cdot t^{-1} f(a, a) e t^{\sigma} \\ &= ae \cdot t^{\sigma} - a \cdot t^{-1}(t - t^{\sigma}) e t^{\sigma} = ae \cdot t^{\sigma} - ae \cdot (t^{\sigma} - t) t^{-\sigma} t^{\sigma} = ae \cdot t \ . \end{aligned}$$

□

(12.3) Lemma We have

$$X_{ae} = -N(e) \cdot X_a \ .$$

Proof
We have

$$q(ae) \equiv e^{\sigma} q(a) e = N(e) q(a)^{\sigma} \mod \mathbb{F} \ ,$$

hence

$$\begin{aligned} -N(e) \cdot X_a &= -N(e) \cdot \{s + q(a) \mid s \in \mathbb{F}\} = -N(e) \cdot \{s - q(a)^{\sigma} \mid s \in \mathbb{F}\} \\ &= \{-N(e)s + N(e) q(a)^{\sigma} \mid s \in \mathbb{F}\} = \{s + q(ae) \mid s \in \mathbb{F}\} = X_{ae} \ . \end{aligned}$$

□

(12.4) Lemma We have

$$\mathbb{E}_{ae} = \mathbb{E}_a \ .$$

Proof
We have

$$q(ae) \in N(e) q(a)^{\sigma} + \mathbb{F} \subseteq \mathbb{E}_a \ .$$

□

(12.5) Lemma The isomorphisms ϕ_{ae} and ϕ_a of fields as in remark (11.4) satisfy

$$\phi_{ae}(\mathbb{E}_a) = \phi_a(\mathbb{E}_a) \ .$$

Proof
Let $t \in X_a \subseteq \mathbb{E}_a = \mathbb{E}_{ae}$. Then we have $-N(e)t \in X_{ae}$ by lemma (12.3), hence

$$(a, t) \in T \ , \qquad \big(ae, -N(e)t\big) \in T \ .$$

Notice that we have

$$\phi_a(t)\phi_a(t)^{\tilde{\sigma}} = \tilde{N}\big(\phi_a(t)\big) = \tilde{N}\big(\varphi_2(a, t)\big) = \phi\big(N(t)\big)$$

by lemma (11.2) (a). Remarks (11.5) (a), (11.10) (a) and identity (7.2) with $(b, v) := \big(ae, -N(e)t\big)$ yield

$$\begin{aligned} \phi_{ae}(t)\phi_a(t)\phi_a(t)^{\tilde{\sigma}}\phi\big(-N(e)\big) &= \phi_{ae}(t)\phi\big(N(t)\big)\phi\big(-N(e)\big) \\ &= \phi_{ae}(t)\phi_{ae}\big(-N(e)\big)\phi_{ae}\big(N(t)\big) = \phi_{ae}\big(t(-N(e))N(t)\big) \\ &= \phi_{aet}\big(t \cdot (-N(e)t) \cdot t^{\sigma}\big) = \varphi_2\big(ae \cdot t, t \cdot (-N(e)t) \cdot t^{\sigma}\big) \\ &= \varphi_2\big(ae \cdot t^{\sigma} - a \cdot t^{-1} f(a, ae) t^{\sigma}, t \cdot (-N(e)t) \cdot t^{\sigma}\big) \\ &= \varphi_2(a, t)\varphi_2\big(ae, -N(e)t\big)\varphi_2(a, t)^{\tilde{\sigma}} = \phi_a(t)\phi_{ae}\big(-N(e)t\big)\phi_a(t)^{\tilde{\sigma}} \\ &= \phi_a(t)\phi_{ae}(t)\phi_{ae}\big(-N(e)\big)\phi_a(t)^{\tilde{\sigma}} = \phi_a(t)\phi_{ae}(t)\phi_a(t)^{\tilde{\sigma}}\phi\big(-N(e)\big) \ , \end{aligned}$$

which implies $\phi_{ae}(t)\phi_a(t) = \phi_a(t)\phi_{ae}(t)$ and thus

$$\phi_{ae}(\mathbb{E}_a) = \phi_{ae}(\mathbb{E}_{ae}) = \tilde{\mathbb{E}}_{\varphi_1(ae)} = \tilde{\mathbb{E}}_{\phi_{ae}(t)} = \tilde{\mathbb{E}}_{\phi_a(t)} = \tilde{\mathbb{E}}_{\varphi_1(a)} = \phi_a(\mathbb{E}_a) \ ,$$

cf. lemma (12.4), remark (11.5) (b), lemma (12.3), remark (10.10) (b) and corollary (5.10). □

(12.6) Corollary We have
$$\phi_{ae} \in \{\phi_a, \tilde{\sigma} \circ \phi_a\} \ .$$

Proof
The field $\tilde{\mathbb{E}} := \phi_{ae}(\mathbb{E}_a) = \phi_a(\mathbb{E}_a)$ is a quadratic separable extension with $\mathrm{Aut}(\tilde{\mathbb{E}} : \tilde{\mathbb{F}}) = \langle \tilde{\sigma} \rangle$, and by remark (11.5) (a), we have
$$\phi_{ae} \circ \phi_a^{-1} \in \mathrm{Aut}(\tilde{\mathbb{E}} : \tilde{\mathbb{F}}) \ .$$
$\square$

(12.7) Remark The following lemma provides the assumptions which are necessary to show that we can extend the isomorphism $\phi_a : \mathbb{E}_a \to \tilde{\mathbb{E}}_{\varphi_1(a)}$ of fields to an isomorphism $\phi : \mathbb{A} \to \tilde{\mathbb{A}}$ of skew-fields which induces ϕ_{ae}. As a consequence, the map $\varphi_1 : \langle a \rangle_{\mathbb{A}} \to \langle \varphi_1(a) \rangle_{\tilde{\mathbb{A}}}$ is an isomorphism of vector spaces.

(12.8) Lemma Suppose that we have $\varphi_1(ae) = \varphi_1(a)\tilde{e}$ for some $\tilde{e} \in \tilde{\mathbb{A}}$ and let $t \in X_a$. Then the following holds:

(a) We have $\phi := \phi_{ae} = \phi_a$.

(b) We have
$$\tilde{f}\big(\varphi_1(ae), \varphi_1(ae)\big) = -\phi\big(N(e)\big)\tilde{f}\big(\varphi_1(a), \varphi_1(a)\big) \ .$$

(c) We have
$$\tilde{f}\big(\varphi_1(a), \varphi_1(a)\big)\tilde{e} = -\tilde{e}\tilde{f}\big(\varphi_1(a), \varphi_1(a)\big) \ .$$

(d) We have $\tilde{N}(\tilde{e}) = \phi\big(N(e)\big)$.

(e) We have $\tilde{e} \in \tilde{\mathbb{E}}^{\perp}_{\varphi_1(a)}$.

Proof

(a) By corollary (10.5), $\varphi_1(a)$ is separable, hence
$$\tilde{f}\big(\varphi_1(a), \varphi_1(ae)\big) = \tilde{f}\big(\varphi_1(a), \varphi_1(a)\big)\tilde{e} \neq 0_{\tilde{\mathbb{A}}} \ .$$
By identity (7.1) with $(b, v) := \big(ae, -N(e)t\big)$ and lemma (12.2), we have
$$\begin{aligned} \varphi_1(ae)\phi_{ae}(t) &= \varphi_1(ae \cdot t) = \varphi_1\big(ae \cdot t^{\sigma} - a \cdot t^{-1} f(a, ae) t^{\sigma}\big) \\ &= \varphi_1(ae) \cdot \varphi_2(a,t)^{\tilde{\sigma}} - \varphi_1(a) \cdot \varphi_2(a,t)^{-1} \tilde{f}\big(\varphi_1(a), \varphi_1(ae)\big)\varphi_2(a,t)^{\tilde{\sigma}} \\ &= \varphi_1(ae) \cdot \phi_a(t)^{\tilde{\sigma}} - \varphi_1(a) \cdot \underbrace{\phi_a(t)^{-1} \tilde{f}\big(\varphi_1(a), \varphi_1(ae)\big)\phi_a(t)^{\tilde{\sigma}}}_{\neq 0} \ , \end{aligned} \tag{12.1}$$
hence
$$\phi_{ae}(t) \neq \phi_a(t)^{\tilde{\sigma}}$$
for each $t \in X_a$ and thus $\phi_{ae} \neq \tilde{\sigma} \circ \phi_a$. Now corollary (12.6) yields
$$\phi_{ae} = \phi_a \ .$$

(b) By lemma (12.3), we have $-N(e)t \in X_{ae}$, hence
$$\begin{aligned} \tilde{f}\big(\varphi_1(ae), \varphi_1(ae)\big) &= \phi_{ae}\big(- N(e)t\big) - \phi_{ae}\big(- N(e)t\big)^{\tilde{\sigma}} \\ &= -\phi\big(N(e)\big)\big(\phi_a(t) - \phi_a(t)^{\tilde{\sigma}}\big) = -\phi\big(N(e)\big)\tilde{f}\big(\varphi_1(a), \varphi_1(a)\big) \ , \end{aligned}$$
cf. remark (10.20) and proposition (11.8).

(c) By (a), we have

$$\begin{aligned}
\varphi_1(a)\tilde{e}\cdot\phi(t) = \varphi_1(ae)\cdot\phi(t) &\overset{(12.1)}{=} \varphi_1(ae)\cdot\phi(t)^{\tilde{\sigma}} - \varphi_1(a)\cdot\phi(t)^{-1}\tilde{f}\big(\varphi_1(a),\varphi_1(ae)\big)\phi(t)^{\tilde{\sigma}} \\
&= \varphi_1(ae)\cdot\phi(t)^{\tilde{\sigma}} - \varphi_1(a)\cdot\phi(t)^{-1}\tilde{f}\big(\varphi_1(a),\varphi_1(a)\big)\tilde{e}\phi(t)^{\tilde{\sigma}} \\
&= \varphi_1(a)\tilde{e}\cdot\phi(t)^{\tilde{\sigma}} - \varphi_1(a)\cdot\phi(t)^{-1}\big(\phi(t)-\phi(t)^{\tilde{\sigma}}\big)\tilde{e}\phi(t)^{\tilde{\sigma}} \\
&= \varphi_1(a)\cdot\phi(t)^{-1}\phi(t)^{\tilde{\sigma}}\tilde{e}\phi(t)^{\tilde{\sigma}}\ ,
\end{aligned} \tag{12.2}$$

hence

$$\tilde{e}\phi(t) = \phi(t)^{-1}\phi(t)^{\tilde{\sigma}}\tilde{e}\phi(t)^{\tilde{\sigma}}\ , \qquad\qquad \phi(t)\tilde{e}\phi(t) = \phi(t)^{\tilde{\sigma}}\tilde{e}\phi(t)^{\tilde{\sigma}}$$

Since $t \in X_a$ is arbitrary, we may replace t by $t+1_{\mathbb{A}}$ to obtain

$$\tilde{e}\phi(t)+\phi(t)\tilde{e} = \tilde{e}\phi(t)^{\tilde{\sigma}}+\phi(t)^{\tilde{\sigma}}\tilde{e}\ , \qquad \tilde{e}\big(\phi(t)-\phi(t)^{\tilde{\sigma}}\big) = -\big(\phi(t)-\phi(t)^{\tilde{\sigma}}\big)\tilde{e}\ .$$

(d) We have

$$\begin{aligned}
-\tilde{e}\tilde{N}(\tilde{e})^{-1}\phi\big(N(e)\big)\tilde{f}\big(\varphi_1(a),\varphi_1(a)\big) &\overset{(b)}{=} \tilde{e}^{-\tilde{\sigma}}\tilde{f}\big(\varphi_1(ae),\varphi_1(ae)\big) = \tilde{f}\big(\varphi_1(ae)\tilde{e}^{-1},\varphi_1(ae)\big) \\
&= \tilde{f}\big(\varphi_1(a),\varphi_1(ae)\big) = \tilde{f}\big(\varphi_1(a),\varphi_1(a)\big)\tilde{e} \\
&\overset{(c)}{=} -\tilde{e}\tilde{f}\big(\varphi_1(a),\varphi_1(a)\big)\ ,
\end{aligned}$$

hence

$$\tilde{N}(\tilde{e})^{-1}\phi\big(N(e)\big) = 1_{\tilde{\mathbb{A}}}\ , \qquad\qquad \tilde{N}(\tilde{e}) = \phi\big(N(e)\big)\ .$$

(e) We have

$$\varphi_1(a)\cdot\tilde{e}\phi(t) \overset{(12.2)}{=} \varphi_1(a)\cdot\tilde{e}\phi(t)^{\tilde{\sigma}} - \varphi_1(a)\cdot\phi(t)^{-1}\tilde{f}\big(\varphi_1(a),\varphi_1(a)\big)\tilde{e}\phi(t)^{\tilde{\sigma}}\ ,$$

hence

$$\begin{aligned}
\varphi_1(a)\cdot\tilde{e}\tilde{f}\big(\varphi_1(a),\varphi_1(a)\big) &= -\varphi_1(a)\cdot\phi(t)^{-1}\tilde{f}\big(\varphi_1(a),\varphi_1(a)\big)\tilde{e}\phi(t)^{\tilde{\sigma}} \\
&\overset{(c)}{=} \varphi_1(a)\cdot\phi(t)^{-1}\tilde{e}\phi(t)^{\tilde{\sigma}}\tilde{f}\big(\varphi_1(a),\varphi_1(a)\big)
\end{aligned}$$

which yields

$$\tilde{e} = \phi(t)^{-1}\tilde{e}\phi(t)^{\tilde{\sigma}}\ , \qquad\qquad \phi(t)\tilde{e} = \tilde{e}\phi(t)^{\tilde{\sigma}}\ .$$

Because of $\phi(t) \in \tilde{\mathbb{E}}_{\varphi_1(a)} \setminus \tilde{\mathbb{F}}$, we have $\tilde{e} \in \tilde{\mathbb{E}}^{\perp}_{\varphi_1(a)}$ by lemma (5.13).

□

(12.9) Remark Suppose that we have $\varphi_1(ae) = \varphi_1(a)\tilde{e}$ for some $\tilde{e} \in \tilde{\mathbb{A}}$. Since we have $\varphi_1(R_a) = \tilde{R}_{\varphi_1(a)}$ and $a\cdot e \notin R_a$, the skew-field $\tilde{\mathbb{A}}$ is necessarily a quaternion division algebra since $(\tilde{\mathbb{A}},\tilde{\mathbb{F}},\tilde{\sigma})$ is quadratic of type (iii) or (iv). Moreover, we have $\varphi_1(\langle a\rangle_{\mathbb{A}}) = \langle\varphi_1(a)\rangle_{\tilde{\mathbb{A}}}$.

(12.10) Lemma Suppose that we have $\varphi_1(ae) = \varphi_1(a)\tilde{e}$ for some $\tilde{e} \in \tilde{\mathbb{A}}$. Then the map

$$\phi_{a,e} : \mathbb{A} \to \tilde{\mathbb{A}},\ x+ey \mapsto \phi(x)+\tilde{e}\phi(y) \qquad (x,y \in \mathbb{E}_a)$$

is an isomorphism of skew-fields such that

$$\forall\ t \in X_a : \qquad \varphi_2(a,t) = \phi_{a,e}(t)\ .$$

Proof
By lemma (12.8), we have

$$\phi\big(N(e)\big) = \tilde{N}(\tilde{e}) \ , \qquad\qquad \tilde{e} \in \tilde{\mathbb{E}}^{\perp}_{\varphi_1(a)}$$

so that we may apply lemma (5.16). By lemma (11.3), we have

$$\forall\, t \in X_a \subseteq \mathbb{E}_a : \qquad \varphi_2(a,t) = \phi_a(t) = \phi_{a,e}(t) \ .$$

□

(12.11) Proposition Suppose that we have $\varphi_1(ae) = \varphi_1(a)\tilde{e}$ for some $\tilde{e} \in \tilde{\mathbb{A}}$. Then the map

$$(\varphi_1, \phi_{a,e}) : (\langle a\rangle_{\mathbb{A}}, \mathbb{A}) \to (\langle \varphi_1(a)\rangle_{\tilde{\mathbb{A}}}, \tilde{\mathbb{A}})$$

is an isomorphism of vector spaces.

Proof
By lemma (12.4) and lemma (12.8) (a), we have

$$\forall\, x, y \in \mathbb{E}_a : \qquad \varphi_1\big(a \cdot (x + ey)\big) = \varphi_1(a) \cdot \big(\phi(x) + \tilde{e}\phi(y)\big) = \varphi_1(a) \cdot \phi_{a,e}(x + ey) \ .$$

□

(12.12) Corollary Suppose that we have $\varphi_1(ae) = \varphi_1(a)\tilde{e}$ for some $\tilde{e} \in \tilde{\mathbb{A}}$ and let $f \in \mathbb{E}_a^{\perp}$. Then we have $\varphi_1(af) = \varphi_1(a)\tilde{f}$ for some $\tilde{f} \in \tilde{\mathbb{A}}$ and $\phi_a := \phi_{a,e} = \phi_{a,f}$.

Proof
By remark (12.9), the first assertion holds, hence $\phi_{a,f}$ is well-defined. Given $x \in \mathbb{A}$, we have

$$\varphi_1(a) \cdot \phi_{a,f}(x) = \varphi_1(a \cdot x) = \varphi_1(a) \cdot \phi_{a,e}(x) \ .$$

□

(12.13) Lemma Suppose that we have $\varphi_1(\langle a\rangle_{\mathbb{A}}) = \langle \varphi_1(a)\rangle_{\tilde{\mathbb{A}}}$ for each separable element $a \in L_0^*$ and let $a, b \in L_0^*$ be separable. Then we have

$$\phi := \phi_a = \phi_b \ .$$

Proof

- Let $b = a \cdot s$ for some $s \in \mathbb{A}^*$. Given $t \in \mathbb{A}$, we have

$$\begin{aligned}\varphi_1(a) \cdot \phi_a(t) &= \varphi_1(a \cdot t) = \varphi_1(as \cdot s^{-1}t) \\ &= \varphi_1(b) \cdot \phi_b(s^{-1}t) = \varphi_1(b) \cdot \phi_b(s^{-1}) \cdot \phi_b(t) = \varphi_1(a) \cdot \phi_b(t) \ .\end{aligned}$$

- Let $b \notin \langle a\rangle_{\mathbb{A}}$, hence $\varphi_1(b) \notin \langle \varphi_1(a)\rangle_{\tilde{\mathbb{A}}}$. By lemma (10.15), there are $a' \in R_a$ and $b' \in R_b$ such that $a' + b'$ is separable. Given $t \in \mathbb{A}$, we have

$$\begin{aligned}\varphi_1(a') \cdot \phi_a(t) + \varphi_1(b') \cdot \phi_b(t) &= \varphi_1(a') \cdot \phi_{a'}(t) + \varphi_1(b') \cdot \phi_{b'}(t) = \varphi_1(a' \cdot t) + \varphi_1(b' \cdot t) \\ &= \varphi_1\big((a' + b') \cdot t\big) = \varphi_1(a' + b') \cdot \phi_{a'+b'}(t) \\ &= \varphi_1(a') \cdot \phi_{a'+b'}(t) + \varphi_1(b') \cdot \phi_{a'+b'}(t) \ ,\end{aligned}$$

thus

$$\phi_a(t) = \phi_{a'+b'}(t) = \phi_b(t) \ .$$

□

(12.14) Proposition Suppose that we have $\varphi_1(\langle a\rangle_{\mathbb{A}}) = \langle \varphi_1(a)\rangle_{\tilde{\mathbb{A}}}$ for each separable element $a \in L_0^*$ and that L_0 has a basis $\{a_i \mid i \in I\}$ of separable elements. Then the map

$$(\varphi_1, \phi) : (L_0, \mathbb{A}) \to (\tilde{L}_0, \tilde{\mathbb{A}})$$

is an isomorphism of vector spaces.

Proof
Let $a = \sum_{i\in I} a_i\lambda_i \in L_0$ and $t \in \mathbb{A}$. By lemma (12.13), we have

$$\begin{aligned}\varphi_1(a \cdot t) &= \varphi_1\Big(\big(\sum_{i\in I} a_i\lambda_i \cdot t\big)\Big) = \sum_{i\in I} \varphi_1(a_i\lambda_i \cdot t) \\ &= \sum_{i\in I} \varphi_1(a_i\lambda_i) \cdot \phi_{a_i\lambda_i}(t) = \sum_{i\in I} \varphi_1(a_i\lambda_i) \cdot \phi(t) = \Big(\sum_{i\in I} \varphi_1(a_i\lambda_i)\Big) \cdot \phi(t) = \varphi_1(a) \cdot \phi(t) \ .\end{aligned}$$

□

(12.15) Remark Let $\dim_{\mathbb{A}} L_0 = 2$ and $a \in L_0^*$. Then $\langle a\rangle_{\mathbb{A}}$ is of the form $L_{a,b} = \langle R_a, R_b\rangle_{\mathbb{F}}$ for some element $b \in L_0$, and for each $c \in L_{a,b}$, we have $R_c \subseteq L_{a,b}$. Conversely, if we have a subspace $L_{a,b} = \langle R_a, R_b\rangle_{\mathbb{F}}$ such that $R_c \subseteq L_{a,b}$ for each $c \in L_{a,b}$, we have $L_{a,b} = \langle a\rangle_{\mathbb{A}_i}$ for one of the three pairwise non-isomorphic quaternion division algebras $\mathbb{A} =: \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ mentioned in [D], cf. proposition (3.2) in [D].

(12.16) Lemma Let $\dim_{\mathbb{A}} L_0 = 2$, let $a \in L_0^*$ be separable and let $\tilde{\mathbb{A}}$ be a quaternion division algebra. Then we have

$$\varphi_1(\langle a\rangle_{\mathbb{A}}) = \langle \varphi_1(a)\rangle_{\tilde{\mathbb{A}}_i}$$

for one of the three pairwise non-isomorphic quaternion division algebras $\tilde{\mathbb{A}} =: \tilde{\mathbb{A}}_1, \tilde{\mathbb{A}}_2, \tilde{\mathbb{A}}_3$ mentioned in [D], and the map $(\varphi_1, \phi_a) : (\langle a\rangle_{\mathbb{A}}, \mathbb{A}) \to (\langle \varphi_1(a)\rangle_{\tilde{\mathbb{A}}_i}, \tilde{\mathbb{A}}_i)$ is an isomorphism of vector spaces. In particular, we have $\tilde{\mathbb{A}}_i = \tilde{\mathbb{A}}$ if we assume $\tilde{\mathbb{A}} \cong \mathbb{A}$.

Proof
By remark (12.15), the subspace $\langle a\rangle_{\mathbb{A}}$ is of the form $L_{a,b} = \langle R_a, R_b\rangle_{\mathbb{F}}$ for some element $b \in L_0$, and for each $c \in L_{a,b}$, we have $R_c \subseteq L_{a,b}$. As $\varphi_1 : L_0 \to \tilde{L}_0$ is $\mathbb{F}$-linear, we have

$$\varphi_1(L_{a,b}) = \varphi_1(\langle R_a, R_b\rangle_{\mathbb{F}}) = \langle \tilde{R}_{\varphi_1(a)}, \tilde{R}_{\varphi_1(b)}\rangle_{\tilde{\mathbb{F}}} = \tilde{L}_{\varphi_1(a),\varphi_1(b)}$$

by corollary (10.11). Now let $\tilde{c} \in \tilde{L}_{\varphi_1(a),\varphi_1(b)}$. Because of $c := \varphi_1^{-1}(\tilde{c}) \in L_{a,b}$ we have $R_c \subseteq L_{a,b}$, hence

$$\tilde{R}_{\tilde{c}} = \varphi_1(R_c) \subseteq \varphi_1(L_{a,b}) = \tilde{L}_{\varphi_1(a),\varphi_1(b)} \ ,$$

which shows that the first assertion holds. The second assertion holds by proposition (12.11). In particular, we have $\tilde{\mathbb{A}}_i \cong \mathbb{A}$. As $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ are pairwise non-isomorphic, we must have $\tilde{\mathbb{A}}_i = \tilde{\mathbb{A}}$ if we assume $\tilde{\mathbb{A}} \cong \mathbb{A}$. □

(12.17) Remark Once again we switch to the general notation.

(12.18) Theorem Let $(\mathbb{K}, \mathbb{K}_0, \sigma)$ be quadratic of type (iv), let $\dim_{\mathbb{K}} L_0 \leq 2$ and $\tilde{\mathbb{K}} \cong \mathbb{K}$. Then the map $\Phi : \Xi \to \tilde{\Xi}$ defined by

$$\Phi := (\varphi_1, \phi) : (L_0, \mathbb{K}) \to (\tilde{L}, \tilde{\mathbb{K}}), \ (a, t) \mapsto \big(\varphi_1(a), \phi(t)\big)$$

is an isomorphism of pseudo-quadratic spaces satisfying

$$\forall\ (a, t) \in T : \qquad \Phi(a, t) = \gamma(a, t) \ .$$

Proof

- In the case $\dim_{\mathbb{K}} L_0 = 1$, each $a \in L_0^*$ is separable, and by assumption, we have
$$\varphi_1(\langle a\rangle_{\mathbb{K}}) = \varphi_1(L_0) = \tilde{L}_0 = \langle \varphi_1(a)\rangle_{\tilde{\mathbb{K}}}$$
so that we may apply proposition (12.14). Now assume $\dim_{\mathbb{K}} L_0 = 2$. By theorem (6.3) in chapter 7 of [WSch], L_0 has an orthogonal basis, and lemma (12.16) implies that we have $\varphi_1(\langle a\rangle_{\mathbb{K}}) = \langle \varphi_1(a)\rangle_{\tilde{\mathbb{K}}}$ for each separable element $a \in L_0^*$ so that we may apply proposition (12.14). In both cases, the map $(\varphi_1, \phi) : (L_0, \mathbb{K}) \to (\tilde{L}_0, \tilde{\mathbb{K}})$ is an isomorphism of vector spaces.
- By remark (11.10) (b), we have $\phi(\mathbb{K}_0) = \tilde{\mathbb{K}}_0$.
- By corollary (5.12), we have $\phi \circ \sigma = \tilde{\sigma} \circ \phi$.
- Let $a \in L_0^*$ and let $t \in X_a$ be as in remark (11.5) (d). By proposition (11.8), we have
$$\phi(t) = \phi_a(t) = \varphi_2(a,t) \ ,$$
hence
$$\tilde{q}\big(\varphi_1(a)\big) \in \varphi_2(a,t) + \tilde{\mathbb{K}}_0 = \phi(t) + \phi(\mathbb{K}_0) = \phi\big(t + \mathbb{K}_0\big) = \phi\big(q(a) + \mathbb{K}_0\big) = \phi\big(q(a)\big) + \tilde{\mathbb{K}}_0$$
- Given $(a,t) \in T$, we have[1]
$$\Phi(a,t) = \big(\varphi_1(a), \phi(t)\big) = \big(\varphi_1(a), \varphi_2(a,t)\big) = \gamma(a,t) \ .$$

Chapter 13 Exceptional Isomorphisms I

Now we drop the condition $\tilde{\mathbb{A}} \cong \mathbb{A}$, i.e., we assume $\tilde{\mathbb{A}} \ncong \mathbb{A}$. By theorem (10.38), this can only occur in small dimensions. As a consequence, there aren't many possibilities for those exceptional isomorphisms which are, of course, not induced by an isomorphism of pseudo-quadratic spaces.

(13.1) Lemma Suppose that $\tilde{\mathbb{A}} \ncong \mathbb{A}$. Then exactly one of the following holds:

(i) The involutory sets $(\mathbb{A}, \mathbb{F}, \sigma)$ and $(\tilde{\mathbb{A}}, \tilde{\mathbb{F}}, \tilde{\sigma})$ both are quadratic of type (iv) and we have
$$\dim_{\mathbb{A}} L_0 = 2 = \dim_{\tilde{\mathbb{A}}} \tilde{L}_0 \ .$$

(ii) The involutory sets $(\mathbb{A}, \mathbb{F}, \sigma)$ and $(\tilde{\mathbb{A}}, \tilde{\mathbb{F}}, \tilde{\sigma})$ are quadratic of type (iv) and (iii), respectively, and we we have
$$\dim_{\mathbb{A}} L_0 = 1 \ , \qquad \dim_{\tilde{\mathbb{A}}} \tilde{L}_0 = 2 \ .$$

Proof
By theorem (11.11) and remark (11.7), we have $\mathbb{A} \cong \tilde{\mathbb{A}}$ if $(\mathbb{A}, \mathbb{F}, \sigma)$ and $(\tilde{\mathbb{A}}, \tilde{\mathbb{F}}, \tilde{\sigma})$ both are quadratic of type (iii). Thus we may suppose $(\mathbb{A}, \mathbb{F}, \sigma)$ to be quadratic of type (iv). By theorem (10.38), we have $\dim_{\mathbb{A}} L_0 \leq 2$ and $\dim_{\tilde{\mathbb{A}}} \tilde{L}_0 \leq 2$. Notice that we have
$$\dim_{\mathbb{F}} L_0 = \dim_{\tilde{\mathbb{F}}} \tilde{L}_0 \ .$$

(i) In the case $\dim_{\mathbb{A}} L_0 = 1 = \dim_{\tilde{\mathbb{A}}} \tilde{L}_0$, we may apply proposition (12.14) so that there is an isomorphism $\phi : \mathbb{A} \to \tilde{\mathbb{A}}$ of skew-fields $\lightning$.

(ii) As we have $\dim_{\mathbb{F}} L_0 \geq 4$ and $\dim_{\tilde{\mathbb{F}}} \tilde{L}_0 \leq 4$, the assertion follows immediately.

$\square$

[1] Although we obtained ϕ only by taking separable elements into account, ϕ of course extends the isomorphism associated with inseparable elements as well since $\varphi_1 : L_0 \to \tilde{L}_0$ is an extension of $\varphi_1 : R_a \to \tilde{R}_{\varphi_1(a)}$.

(13.2) Notation Throughout this chapter, let $(\mathbb{A}, \mathbb{F}, \sigma)$ be quadratic of type (iv).

(13.3) Remark By lemma (13.1), there are two cases left. First of all, we deal with the case where both the involutory sets are quadratic of type (iv). We do this by using lemma (12.16) in a suitable way. The appearing isomorphisms turn out to be almost isomorphisms of pseudo-quadratic spaces, but modified by switching the parametrizing space.

(13.4) Notation Let $(\mathbb{A}, \mathbb{F}, \sigma)$ be quadratic of type (iv) and suppose that $\dim_{\mathbb{A}} L_0 = 2$. By [D], there are exactly three pseudo-quadratic spaces

$$(\mathbb{A}, \mathbb{F}, \sigma, L_0, q) = (\mathbb{A}_1, \mathbb{F}, \sigma, L_0, q_1) = \Xi_1 \ , \quad (\mathbb{A}_2, \mathbb{F}, \sigma, L_0, q_2) = \Xi_2 \ , \quad (\mathbb{A}_3, \mathbb{F}, \sigma, L_0, q_3) = \Xi_3$$

with pairwise non-isomorphic quaternion division algebras $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ which define the group T. When we switch between the parametrizing pseudo-quadratic spaces, we indicate this by the map

$$\mathrm{id}_T^i : T \to T, \ (a,t) \mapsto (a,t) \ ,$$

i.e., after applying id_T^i, we consider T to be defined by Ξ_i.

(13.5) Proposition Let $\tilde{\mathbb{A}}$ be quadratic of type (iv) and suppose that $\dim_{\mathbb{A}} L_0 = 2$. Then there are an $i \in \{1,2,3\}$ and an isomorphism $\Phi : \Xi \to \tilde{\Xi}_i$ of pseudo-quadratic spaces such that γ is induced by $(\mathrm{id}_{\tilde{T}}^i)^{-1} \circ \Phi$.

Proof
Let $a \in L_0^*$ be separable. By lemma (12.16), there is an $i \in \{1,2,3\}$ such that

$$\varphi_1(\langle a \rangle_{\mathbb{A}}) = \langle \varphi_1(a) \rangle_{\tilde{\mathbb{A}}_i} \ ,$$

and the map $(\varphi_1, \phi_a) : (\langle a \rangle_{\mathbb{A}}, \mathbb{A}) \to (\langle \varphi_1(a) \rangle_{\tilde{\mathbb{A}}_i}, \tilde{\mathbb{A}}_i)$ is an isomorphism of vector spaces. In particular, we have $\tilde{\mathbb{A}}_i \cong \mathbb{A}$, thus $i \in \{1,2,3\}$ is independent of the choice of a. Now the Jordan isomorphism

$$\mathrm{id}_{\tilde{T}}^i \circ \gamma : T \to \tilde{T}$$

is induced by an isomorphism $\Phi : \Xi \to \tilde{\Xi}_i$ of pseudo-quadratic spaces by theorem (12.18). □

(13.6) Remark Now we consider the last case, where a quaternion division algebra "splits" into two separable quadratic extensions.

(13.7) Lemma Let $a \in L_0^*$ be separable and let $e \in \mathbb{E}_a^{\perp}$. Suppose that $\varphi_1(ae) \notin \langle \varphi_1(a) \rangle_{\tilde{\mathbb{A}}}$. Then we have

$$\tilde{f}\big(\varphi_1(ae), \varphi_1(a)\big) = 0_{\tilde{\mathbb{A}}} \ , \qquad \phi_{ae} = \tilde{\sigma} \circ \phi_a \ .$$

Proof
By equation (12.1), we have

$$\varphi_1(ae) \cdot \phi_{ae}(t) = \varphi_1(ae) \cdot \varphi_2(a,t)^{\tilde{\sigma}} - \varphi_1(a) \cdot \varphi_2(a,t)^{-1} \tilde{f}\big(\varphi_1(a), \varphi_1(ae)\big) \varphi_2(a,t)^{\tilde{\sigma}}$$

for each $t \in X_a$, hence

$$\varphi_2(a,t)^{-1} \tilde{f}\big(\varphi_1(a), \varphi_1(ae)\big) \varphi_2(a,t)^{\tilde{\sigma}} = 0_{\tilde{\mathbb{A}}} \ , \qquad \forall \ t \in X_a : \ \phi_{ae}(t) = \varphi_2(a,t)^{\tilde{\sigma}} = \phi_a(t)^{\tilde{\sigma}} \ .$$

□

(13.8) Notation Throughout the rest of this chapter, we suppose $\tilde{\mathbb{A}}$ to be quadratic of type (iii).

(13.9) Remark By lemma (13.1), we have $\dim_{\mathbb{A}} L_0 = 1$ and $\dim_{\tilde{\mathbb{A}}} \tilde{L}_0 = 2$. As a consequence, each element $a \in L_0^*$ is separable. Moreover, we have

$$\varphi_1(ae) \notin \tilde{R}_{\varphi_1(a)} = \langle \varphi_1(a) \rangle_{\tilde{\mathbb{A}}}$$

for each element $e \in \mathbb{E}_a^\perp$. As a consequence, lemma (13.7) applies.

(13.10) Notation

- Throughout the rest of this chapter, let $a \in L_0^*$ (which is separable) and let $e \in \mathbb{E}_a^\perp$.
- We set

$$\tilde{a} := \varphi_1(a) \ , \qquad \tilde{b} := \varphi_1(ae) \in \tilde{a}^\perp \ , \qquad \phi := \phi_a \ .$$

(13.11) Remark Given $x \in \mathbb{A}$, we have

$$q(a \cdot x) \equiv x^\sigma q(a) x \mod \mathbb{F} \ ,$$

cf. definition (7.1) (P2), hence

$$\big(a \cdot x, x^\sigma q(a) x\big) \in T \ .$$

(13.12) Lemma Given $x = s + et \in \mathbb{A}$, we have

$$\gamma\big(a \cdot x, x^\sigma q(a)x\big) = \big(\tilde{a} \cdot \phi(s) + \tilde{b} \cdot \phi(t)^{\tilde{\sigma}}, \phi\big(N(x)q(a)\big)\big) \ .$$

Proof

- We have

$$\varphi_1(a \cdot x) = \varphi_1(a \cdot s + ae \cdot t) = \varphi_1(a) \cdot \phi(s) + \varphi_1(ae) \cdot \phi(t)^{\tilde{\sigma}} = \tilde{a} \cdot \phi(s) + \tilde{b} \cdot \phi(t)^{\tilde{\sigma}}$$

  by lemma (13.7).

- By Proposition (11.8), we have

$$\varphi_1\big(ax \cdot x^\sigma q(x) x\big) = \varphi_1(ax) \cdot \varphi_2(ax, x^\sigma q(a) x) \ .$$

  On the other hand, we have

$$\begin{aligned}
\varphi_1\big(a \cdot x x^\sigma q(a) x\big) &= \varphi_1\big(a \cdot N(x) q(a)(s + et)\big) \\
&= \varphi_1\big(as \cdot N(x)q(a) + aet \cdot N(x) q(a)^\sigma\big) \\
&= \varphi_1(as) \cdot \phi\big(N(x)q(a)\big) + \varphi_1(aet) \cdot \phi\big(N(x)q(a)^\sigma\big)^{\tilde{\sigma}} \\
&= \varphi_1(as + aet) \cdot \phi\big(N(x)q(a)\big) \\
&= \varphi_1(ax) \cdot \phi\big(N(x)q(a)\big) \ .
\end{aligned}$$

□

(13.13) Proposition Given $x = s + et \in \mathbb{A}$ and $u \in \mathbb{F}$, we have

$$\gamma(a \cdot x, x^\sigma q(a) x + u) = \big(\tilde{a} \cdot \phi(s) + \tilde{b} \cdot \phi(t)^{\tilde{\sigma}}, \phi\big(N(x)q(a) + u\big)\big) \ .$$

Proof
This results from lemma (13.12) and lemma (7.21) (d). □

(13.14) Remark This describes γ completely.

(13.15) Theorem Suppose that $\tilde{\mathbb{K}} \ncong \mathbb{K}$. Then exactly one of the following holds:

(i) The involutory sets $(\mathbb{K}, \mathbb{K}_0, \sigma)$ and $(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ both are quadratic of type (iv), we have $\dim_{\mathbb{K}} L_0 = 2 = \dim_{\tilde{\mathbb{K}}} \tilde{L}_0$ and there are an $i \in \{2,3\}$ and an isomorphism $\Phi : \Xi \to \tilde{\Xi}_i$ of pseudo-quadratic spaces such that γ is induced by $(\mathrm{id}_{\tilde{T}}^i)^{-1} \circ \Phi$, where $\mathrm{id}_{\tilde{T}}^i$ and $\tilde{\Xi} =: \tilde{\Xi}_1, \tilde{\Xi}_2, \tilde{\Xi}_3$ are as in notation (13.4).

(ii) The involutory sets $(\mathbb{K}, \mathbb{K}_0, \sigma)$ and $(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ are quadratic of type (iv) and (iii), respectively, we have $\dim_{\mathbb{K}} L_0 = 1$, $\dim_{\tilde{\mathbb{K}}} \tilde{L}_0 = 2$ and γ can be described by

$$\forall\, x = s{+}et \in \mathbb{K},\ u \in \mathbb{K}_0 :\ \gamma\big(ax, x^\sigma q(a)x{+}u\big) = \big(\varphi_1(a)\phi(s){+}\varphi_1(ae)\phi(t)^{\tilde{\sigma}}, \phi\big(N(x)q(a){+}u\big)\big)\ ,$$

where $a \in L_0^*$ is arbitrary, $\phi = \phi_a$, $e \in \mathbb{E}_a^\perp$ and $\varphi_1(ae) \in \varphi_1(a)^\perp$.

Proof
This results from lemma (13.1), proposition (13.5), lemma (13.7) and proposition (13.13). Notice that we have $i \neq 1$ in (i) as we have $\mathbb{K} \ncong \tilde{\mathbb{K}}$. □

Chapter 14 The Reverse Direction

Now we consider the reverse direction, i.e., we prove that each map as above is a Jordan isomorphism.

(14.1) Remark Notice that we don't assume γ to be a Jordan isomorphism any longer.

(14.2) Theorem Let Ξ and $\tilde{\Xi}$ be proper pseudo-quadratic spaces and let $\gamma : T \to \tilde{T}$ be a map that is induced by an isomorphism $\Phi = (\varphi, \phi) : \Xi \to \tilde{\Xi}$ of pseudo-quadratic spaces. Then γ is a Jordan isomorphism.

Proof

- We have
$$\gamma(0_{L_0}, 1_{\mathbb{K}}) = \big(\varphi(0_{L_0}), \phi(1_{\mathbb{K}})\big) = (0_{\tilde{L}_0}, 1_{\tilde{\mathbb{K}}})\ .$$

- Given $(a,t),(b,v) \in T$, we have
$$\begin{aligned}\gamma\big((a,t)\cdot(b,v)\big) &= \gamma\big(a+b, t+v+f(b,a)\big) = \big(\varphi(a+b), \phi(t+v+f(b,a))\big)\\ &= \big(\varphi(a)+\varphi(b), \phi(t)+\phi(v)+\phi(f(a,b))\big)\\ &= \big(\varphi(a)+\varphi(b), \phi(t)+\phi(v)+\tilde{f}\big(\varphi(a),\varphi(b)\big)\big)\\ &= \big(\varphi(a),\phi(t)\big)\cdot\big(\varphi(b),\phi(v)\big) = \gamma\big((a,t)\big)\cdot\gamma\big((b,v)\big)\ .\end{aligned}$$

- Let $(a,t),(b,v) \in T$. By corollary (7.3), we have
$$\begin{aligned}\gamma\big(h_{(a,t)}(b,v)\big) &= \gamma\big(b\cdot t^\sigma - a\cdot t^{-1}f(a,b)t^\sigma, tvt^\sigma\big)\\ &= \big(\varphi(b\cdot t^\sigma - a\cdot t^{-1}f(a,b)t^\sigma), \phi(tvt^\sigma)\big)\\ &= \big(\varphi(b)\cdot\phi(t^\sigma) - \varphi(a)\cdot\phi(t)^{-1}\phi\big(f(a,b)\big)\phi(t^\sigma), \phi(t)\phi(v)\phi(t^\sigma)\big)\\ &= \big(\varphi(b)\cdot\phi(t)^{\tilde{\sigma}} - \varphi(a)\cdot\phi(t^{-1})\tilde{f}\big(\varphi(a),\varphi(b)\big)\phi(t)^{\tilde{\sigma}}, \phi(t)\phi(v)\phi(t)^{\tilde{\sigma}}\big)\\ &= \tilde{h}_{(\varphi(a),\phi(t))}\big((\varphi(b),\phi(v))\big) = \tilde{h}_{\gamma(a,t)}\big(\gamma(b,v)\big)\ .\end{aligned}$$

□

(14.3) Remark This shows that each map as in proposition (13.5) is a Jordan isomorphism since id_T^i is a Jordan isomorphism for each $i \in \{1,2,3\}$.

(14.4) Notation

- Throughout the rest of this chapter, let Ξ and $\tilde{\Xi}$ be proper pseudo-quadratic spaces, let $(\mathbb{A}, \mathbb{F}, \sigma)$ and $(\tilde{\mathbb{A}}, \tilde{\mathbb{F}}, \tilde{\sigma})$ be quadratic of type (iv) and (iii), respectively, let
$$\dim_{\mathbb{A}} L_0 = 1 \ , \qquad\qquad \dim_{\tilde{\mathbb{A}}} \tilde{L}_0 = 2 \ ,$$
let $a \in L_0^*$ and $e \in \mathbb{E}_a^{\perp}$, and let $\phi : \mathbb{E}_a \to \tilde{\mathbb{A}}$ be an isomorphism of fields.
- Moreover, let $\gamma : T \to \tilde{T}$ be a map such that
$$\forall\ x = s + et \in \mathbb{A},\ u \in \mathbb{F} : \qquad \gamma(ax, x^\sigma q(a)x + u) = \big(\tilde{a}\phi(s) + \tilde{b}\phi(t)^{\tilde{\sigma}}, \phi\big(N(x)q(a) + u\big)\big)$$
for some $\tilde{a}, \tilde{b} \in \tilde{L}_0$ such that $\tilde{f}(\tilde{a}, \tilde{b}) = 0_{\tilde{\mathbb{A}}}$.
- We set $b := ae$ and
$$A := \{(a \cdot x, t) \mid x \in \mathbb{E}_a, t \in X_{ax}\} \leq T \ , \qquad B := \{(b \cdot x, t) \mid x \in \mathbb{E}_a, t \in X_{bx}\} \leq T \ ,$$
$$\tilde{A} := \{(\tilde{a} \cdot x, t) \mid x \in \tilde{\mathbb{A}}, t \in \tilde{X}_{\tilde{a}x}\} \leq \tilde{T} \ , \qquad \tilde{B} := \{(\tilde{b} \cdot x, t) \mid x \in \tilde{\mathbb{A}}, t \in \tilde{X}_{\tilde{b}x}\} \leq \tilde{T} \ .$$

(14.5) Lemma Given $x, y \in \mathbb{E}_a$, we have
$$f(a \cdot x, b \cdot y) = f(b \cdot y, a \cdot x) \ .$$

Proof
Given $x, y \in \mathbb{E}_a$, we have
$$\begin{aligned} f(a \cdot x, b \cdot y) &= f(a \cdot x, ae \cdot y) = x^\sigma f(a, a)ey = eyxf(a, a)^\sigma \\ &= -y^\sigma e f(a, a)x = (ey)^\sigma f(a, a)x = f(ae \cdot y, a \cdot x) = f(b \cdot y, a \cdot x) \ . \end{aligned}$$
□

(14.6) Lemma Given $(a, t), (b, v) \in T$, we have
$$(a, t) \in C_T\big((b, v)\big) \ \Leftrightarrow\ f(a, b) = f(b, a) \ .$$

Proof
We have
$$\begin{aligned} (a, t)(b, v) = (b, v)(a, t) \ &\Leftrightarrow\ \big(a + b, t + v + f(b, a)\big) = \big(b + a, v + t + f(a, b)\big) \\ &\Leftrightarrow\ f(b, a) = f(a, b) \ . \end{aligned}$$
□

(14.7) Corollary We have
$$A \subseteq C_T(B) \ , \qquad\qquad \tilde{A} \subseteq C_{\tilde{T}}(\tilde{B}) \ .$$

Proof
The first assertion results from lemma (14.5), and given $x, y \in \tilde{\mathbb{A}}$, we have
$$f(\tilde{a} \cdot x, \tilde{b} \cdot y) = 0_{\tilde{\mathbb{A}}} = f(\tilde{b} \cdot y, \tilde{a} \cdot x) \ .$$
□

(14.8) Lemma We have
$$T = AB \ , \qquad\qquad \tilde{T} = \tilde{A}\tilde{B} \ .$$

Proof
Let $(a \cdot x + b \cdot y, t) \in T$ where $x, y \in \mathbb{E}_a$. Then we have

$$t \equiv q(a \cdot x + b \cdot y) \equiv q(a \cdot x) + q(b \cdot y) + f(a \cdot x, b \cdot y) \mod \mathbb{F} ,$$

hence

$$\tilde{t} := t - q(a \cdot x) - q(b \cdot y) - f(a \cdot x, b \cdot y) \in \mathbb{F} .$$

Observe that we have $f(a \cdot x, b \cdot y) = f(b \cdot y, a \cdot x)$ by lemma (14.5), thus

$$\begin{aligned}(a \cdot x + b \cdot y, t) &= (a \cdot x + b \cdot y, q(a \cdot x) + q(b \cdot y) + f(a \cdot x, b \cdot y) + \tilde{t}) \\ &= (a \cdot x + b \cdot y, q(a \cdot x) + q(b \cdot y) + f(b \cdot y, a \cdot x) + \tilde{t}) \\ &= \big(a \cdot x, q(a \cdot x)\big) \cdot \big(b \cdot y, q(b \cdot y) + \tilde{t}\big) \in AB .\end{aligned}$$

The second assertion follows analogously. □

(14.9) Lemma The maps

$$\begin{aligned}\Phi_1 &: (\mathbb{E}_a, \mathbb{F}, \sigma, R_a, q) \to (\tilde{\mathbb{A}}, \tilde{\mathbb{F}}, \tilde{\sigma}, \langle \tilde{a} \rangle_{\tilde{\mathbb{A}}}, \tilde{q}),\ (ax, t) \mapsto \big(\tilde{a}\phi(x), \phi(t)\big) , \\ \Phi_2 &: (\mathbb{E}_a, \mathbb{F}, \sigma, R_b, q) \to (\tilde{\mathbb{A}}, \tilde{\mathbb{F}}, \tilde{\sigma}, \langle \tilde{b} \rangle_{\tilde{\mathbb{A}}}, \tilde{q}),\ (bx, t) \mapsto \big(\tilde{b}\phi(x)^{\tilde{\sigma}}, \phi(t)^{\tilde{\sigma}}\big)\end{aligned}$$

are isomorphisms of pseudo-quadratic spaces inducing $\gamma_A := \gamma_{|A} : A \to \tilde{A}$ and $\gamma_B := \gamma_{|B} : B \to \tilde{B}$, respectively.

Proof
We consider Φ_2, the first assertion follows analogously.

- Given $x = et \in \mathbb{E}_a^\perp$ and $u \in \mathbb{F}$, we have

$$\begin{aligned}\gamma_B\big(a \cdot x, x^\sigma q(a) x + u\big) &= \big(\tilde{b} \cdot \phi(t)^{\tilde{\sigma}}, \phi(N(x)q(a) + u)\big) \\ &= \big(\tilde{b} \cdot \phi(t)^{\tilde{\sigma}}, \phi\big((x^\sigma x q(a)^{\tilde{\sigma}} + u\big)^{\tilde{\sigma}}\big)\big) \\ &= \big(\tilde{b} \cdot \phi(t)^{\tilde{\sigma}}, \phi(x^\sigma q(a) x + u)^{\tilde{\sigma}}\big) = \Phi_2(a \cdot x, x^\sigma q(a) x + u) .\end{aligned}$$

- Given $x \in \mathbb{E}_a$, we have

$$\varphi_1(b \cdot x) = \tilde{b} \cdot \phi(x)^{\tilde{\sigma}} ,$$

  thus $(\varphi_{1|R_b}, \tilde{\sigma} \circ \phi) : (R_b, \mathbb{E}_a) \to (\langle \tilde{b} \rangle_{\tilde{\mathbb{A}}}, \tilde{\mathbb{A}})$ is an isomorphism of vector spaces.

- The map $\tilde{\sigma} \circ \phi : \mathbb{E}_a \to \tilde{\mathbb{A}}$ is clearly an isomorphism of involutory sets, cf. corollary (5.12).

- Given $x = et \in \mathbb{E}_a^\perp$, we have $\big(\varphi_1(a \cdot x), \tilde{q}(\varphi_1(a \cdot x))\big) \in \tilde{B}$ and $\big(\varphi_1(a \cdot x), \phi\big(N(x)q(a)\big)\big) \in \tilde{B}$, thus

$$\begin{aligned}\tilde{q}\big(\varphi_1(x)\big) \in \phi\big(N(x)q(a)\big) + \tilde{\mathbb{F}} = \phi\big(x^\sigma q(a)^\sigma x + \mathbb{F}\big) &= \phi\big((x^\sigma q(a) x)^\sigma + \mathbb{F}\big) \\ &= \phi\big(q(a \cdot x) + \mathbb{F}\big)^{\tilde{\sigma}} = \phi\big(q(a \cdot x)\big)^{\tilde{\sigma}} + \tilde{\mathbb{F}} .\end{aligned}$$

□

(14.10) Proposition The map $\gamma : T \to \tilde{T}$ is an isomorphism of groups.

Proof
By corollary (14.7), it is enough to consider the following two cases:

- By lemma (14.9) and theorem (14.2), $\gamma_A : A \to \tilde{A}$ and $\gamma_B : B \to \tilde{B}$ are Jordan isomorphisms. In particular, they are isomorphisms of groups.

- Given $x \in \mathbb{E}_a,\ y = et \in \mathbb{E}_a^{\perp}$ and $u, v \in \mathbb{F}$, we have

$$\begin{aligned}
&\gamma\big((a \cdot x, x^\sigma q(a)x + u) \cdot (a \cdot y, y^\sigma q(a)y + v)\big)\\
&\quad = \gamma\big(a \cdot (x+y), x^\sigma q(a)x + y^\sigma q(a)y + f(ax, ay) + u + v\big)\\
&\quad = \gamma\big(a \cdot (x+y), x^\sigma q(a)x + y^\sigma q(a)y + x^\sigma f(a,a)y + u + v\big)\\
&\quad = \gamma\big(a \cdot (x+y), x^\sigma q(a)x + y^\sigma q(a)y + x^\sigma q(a)y - x^\sigma q(a)^\sigma y + u + v\big)\\
&\quad = \gamma\big(a \cdot (x+y), x^\sigma q(a)x + y^\sigma q(a)y + x^\sigma q(a)y + y^\sigma q(a)x + u + v\big)\\
&\quad = \gamma\big(a \cdot (x+y), (x+y)^\sigma q(a)(x+y) + u + v\big)\\
&\quad = \big(\tilde{a} \cdot \phi(x) + \tilde{b} \cdot \phi(t)^{\tilde{\sigma}}, \phi(N(x+y)q(a) + u + v)\big)\\
&\quad = \big(\tilde{a} \cdot \phi(x) + \tilde{b} \cdot \phi(t)^{\tilde{\sigma}}, \phi(N(x)q(a) + N(y)q(a) + u + v)\big)\\
&\quad = \big(\tilde{a} \cdot \phi(x), \phi(N(x)q(a) + u)\big) \cdot \big(\tilde{b} \cdot \phi(t)^{\tilde{\sigma}}, \phi(N(y)q(a) + v)\big)\\
&\qquad\qquad = \gamma\big((a \cdot x, x^\sigma q(a)x + u)\big) \cdot \gamma\big((a \cdot y, y^\sigma ay + v)\big)\ .
\end{aligned}$$

□

(14.11) Lemma Given $x \in \mathbb{A},\ t := x^\sigma q(a)x + s \in X_{ax}$ and $\tilde{t} := N(x)q(a) + s \in \mathbb{E}_a$, we have

$$N(\tilde{t}) = N(t)\ .$$

Proof
We have

$$\begin{aligned}
N\big(\tilde{t}\big) - N(t) &= N\big(N(x)q(a) + s\big) - N\big(x^\sigma q(a)x + s\big)\\
&= N\big(N(x)q(a)\big) + N(s) + T\big(N(x)q(a)s\big) - N\big(x^\sigma q(a)x\big) - N(s) - T\big(x^\sigma q(a)xs\big)\\
&= N(x)^2 N\big(q(a)\big) - N(x)N\big(q(a)\big)N(x) + s\big(T(N(x)q(a)) - T(x^\sigma q(a)x)\big)\\
&= s\big(x^\sigma x(q(a) + q(a)^\sigma) - x^\sigma(q(a) + q(a)^\sigma)x\big) = sx^\sigma\big(xT(q(a)) - T(q(a))x\big) = 0_{\mathbb{A}}\ .
\end{aligned}$$

□

(14.12) Lemma Given $x \in \mathbb{A},\ t := x^\sigma q(a)x + s \in X_{ax}$ and $\tilde{t} := N(x)q(a) + s \in \mathbb{E}_a$, we have

$$xt^{-1}t^\sigma x^{-1} = \tilde{t}^{-1}\tilde{t}^\sigma \in \mathbb{E}_a\ .$$

Proof
By lemma (14.11), we have

$$xt^{-1}t^\sigma x^{-1} = (xt^\sigma x^{-1})^2 \cdot N(t)^{-1} = (\tilde{t}^\sigma)^2 \cdot N(\tilde{t})^{-1} = \tilde{t}^{-1}\tilde{t}^\sigma \in \mathbb{E}_a\ .$$

□

(14.13) Corollary Given $x \in \mathbb{A}$ and $t := x^\sigma q(a)x + s \in X_{ax}$, we have

$$(x^{-\sigma}tt^{-\sigma}x^\sigma)q(a)(xt^{-1}t^\sigma x^{-1}) = q(a)\ .$$

Proof
First of all observe that we have

$$x^{-\sigma}tt^{-\sigma}x^\sigma = N(x)^{-1}xt^{-\sigma}tx^{-1}N(x) = (xt^{-1}t^\sigma x^{-1})^{-1}\ .$$

Now we have $xt^{-1}t^\sigma x^{-1} \in \mathbb{E}_a = C_{\mathbb{A}}(\mathbb{E}_a)$ by lemma (14.12). □

(14.14) Lemma Given $x, y \in \mathbb{A},\ t := x^\sigma q(a)x + s \in X_{ax}$ and $u \in \mathbb{F}$, we have

$$\gamma\big(h_{(ax,t)}\big(a\cdot y, y^\sigma q(a)y + u\big)\big) = \big(\varphi_1(a\cdot z), \phi\big(N(t)N(y)q(a) + N(t)u\big)\big)\ ,$$

where $z = xt^{-1}t^\sigma x^{-1}yt^\sigma$.

Proof
The first component of $h_{(ax,t)}\big(a\cdot y, y^\sigma q(a)y + u\big)$ is

$$\begin{aligned} ayt^\sigma - axt^{-1}f(ax, ay)t^\sigma &= ayt^\sigma - axt^{-1}f(ax, ax)x^{-1}yt^\sigma = ayt^\sigma - axt^{-1}(t - t^\sigma)x^{-1}yt^\sigma \\ &= ayt^\sigma - axt^{-1}tx^{-1}yt^\sigma + axt^{-1}t^\sigma x^{-1}yt^\sigma = axt^{-1}t^\sigma x^{-1}yt^\sigma\ , \end{aligned}$$

and the second component of $h_{(ax,t)}\big(ay, y^\sigma q(a)y + u\big)$ is

$$t(y^\sigma q(a)y + u)t^\sigma = ty^\sigma q(a)yt^\sigma + N(t)u\ .$$

By corollary (14.13), we have

$$\begin{aligned} z^\sigma q(a)z &= \big(xt^{-1}t^\sigma x^{-1}yt^\sigma\big)^\sigma q(a)\big(xt^{-1}t^\sigma x^{-1}yt^\sigma\big) \\ &= ty^\sigma(x^{-\sigma}tt^{-\sigma}x^\sigma)q(a)(xt^{-1}t^\sigma x^{-1})yt^\sigma = ty^\sigma q(a)yt^\sigma\ . \end{aligned}$$

Therefore, we have

$$\begin{aligned} \gamma\big(h_{(ax,t)}\big(a\cdot y, y^\sigma q(a)y + u\big)\big) &= \gamma\big((a\cdot z, z^\sigma q(a)z + N(t)u)\big) = \big(\varphi_1(a\cdot z), \phi\big(N(z)q(a) + N(t)u\big)\big) \\ &= \big(\varphi_1(a\cdot z), \phi\big(N(t)N(y)q(a) + N(t)u\big)\big) \end{aligned}$$

□

(14.15) Proposition The map γ preserves the second component of the Hua-maps.

Proof
Let $x, y \in \mathbb{A},\ t := x^\sigma q(a)x + s \in X_{ax},\ \tilde{t} := N(x)q(a) + s \in \mathbb{E}_a$ and $u \in \mathbb{F}$. The second component of

$$\tilde{h}_{\gamma(ax,t)}\big(\gamma((a\cdot y, y^\sigma q(a)y + u))\big) = \tilde{h}_{(\varphi_1(ax),\phi(\tilde{t}))}\big(\varphi_1(a\cdot y), \phi(N(y)q(a) + u)\big)$$

is

$$\begin{aligned} \phi(\tilde{t})\phi\big(N(y)q(a) + u\big)\phi(\tilde{t})^{\tilde{\sigma}} &= \phi\big(N(\tilde{t})\cdot(N(y)q(a) + u)\big) \\ &= \phi\big(N(t)\cdot(N(y)q(a) + u)\big) = \phi\big(N(t)N(y)q(a) + N(t)u\big) \end{aligned}$$

by lemma (14.11). Now the assertion results from Lemma (14.14). □

(14.16) Proposition Let $x = \lambda + e\mu \in \mathbb{A},\ t := x^\sigma q(a)x + s \in A_{ax}$ and $\tilde{t} := N(x)q(a) + s \in \mathbb{E}_a$. Then the following holds:

(a) Given $(a\cdot y, u) \in A$, we have

$$\gamma\big(h_{(ax,t)}(a\cdot y, u)\big) = \tilde{h}_{\gamma(ax,t)}\big(\gamma(a\cdot y, u)\big)\ .$$

(b) Given $(b\cdot y, u) \in B$, we have

$$\gamma\big(h_{(ax,t)}(b\cdot y, u)\big) = \tilde{h}_{\gamma(ax,t)}\big(\gamma(b\cdot y, u)\big)\ .$$

Proof
By proposition (14.15), it remains to check the first component.

(a) By lemma (14.14) and lemma (14.12), the first component of $h_{(ax,t)}(a \cdot y, u)$ is

$$a \cdot \tilde{t}^{-1}\tilde{t}^{\sigma} y t^{\sigma} = a \cdot y\tilde{t}^{-1}\tilde{t}^{\sigma} \cdot \big(N(\lambda)q(a)^{\sigma} + N(e)N(\mu)q(a) + f(a \cdot \lambda, ae \cdot \mu)^{\sigma} + s\big) \ .$$

Therefore, the first component of $\gamma\big(h_{(ax,t)}(a \cdot y, u)\big)$ is

$$\tilde{a} \cdot \phi\big(y\tilde{t}^{-1}\tilde{t}^{\sigma} \cdot \big(N(\lambda)q(a)^{\sigma} + N(e)N(\mu)q(a) + s\big)\big) + \tilde{b} \cdot \phi\big(y^{\sigma}\tilde{t}^{-\sigma}\tilde{t}f(a,a)\lambda\mu\big)^{\tilde{\sigma}} \ .$$

On the other hand, the first component of

$$\tilde{h}_{(\tilde{a}\phi(\lambda)+\tilde{b}\phi(\mu)^{\tilde{\sigma}},\phi(\tilde{t}))}\big(\tilde{a} \cdot \phi(y), \varphi_2(a \cdot y, u)\big)$$

is

$$\begin{aligned}
&\tilde{a}\phi(y)\phi(\tilde{t})^{\tilde{\sigma}} - \big(\tilde{a}\phi(\lambda) + \tilde{b}\phi(\mu)^{\tilde{\sigma}}\big) \cdot \phi(\tilde{t})^{-1}\tilde{f}\big(\tilde{a}\phi(\lambda) + \tilde{b}\phi(\mu)^{\tilde{\sigma}}, \tilde{a}\phi(y)\big)\phi(\tilde{t})^{\tilde{\sigma}} \\
&\quad = \tilde{a}\phi(y\tilde{t}^{-1}\tilde{t}^{\sigma})\phi(\tilde{t}) - \tilde{a}\phi(\lambda\tilde{t}^{-1}\lambda^{\sigma}\tilde{f}(\tilde{a},\tilde{a})y\tilde{t}^{\sigma}) - \tilde{b}\phi\big(\mu^{\sigma}\tilde{t}^{-1}\lambda^{\sigma}\phi^{-1}\big(\tilde{f}(\tilde{a},\tilde{a})\big)y\tilde{t}^{\sigma}\big) \\
&\quad = \tilde{a}\phi(y\tilde{t}^{-1}\tilde{t}^{\sigma})\phi(\tilde{t}) - \tilde{a}\phi(y\tilde{t}^{-1}\tilde{t}^{\sigma})\phi\big(N(\lambda)(\phi(q(a)) - \phi(q(a))^{\sigma})\big) - \tilde{b}\phi\big(y\tilde{t}^{-1}\tilde{t}^{\sigma}f(a,a)\lambda^{\sigma}\mu^{\sigma}\big) \\
&\quad = \tilde{a}\phi\big(y\tilde{t}^{-1}\tilde{t}^{\sigma} \cdot \big(N(x)q(a) + s - N(\lambda)q(a) + N(\lambda)q(a)^{\sigma}\big)\big) - \tilde{b}\phi\big(y^{\sigma}\tilde{t}^{-\sigma}\tilde{t}f(a,a)^{\sigma}\lambda\mu\big)^{\tilde{\sigma}} \\
&\quad = \tilde{a}\phi\big(y\tilde{t}^{-1}\tilde{t}^{\sigma} \cdot \big(N(e)N(\mu)q(a) + N(\lambda)q(a)^{\sigma} + s\big)\big) + \tilde{b}\phi\big(y^{\sigma}\tilde{t}^{-\sigma}\tilde{t}f(a,a)\lambda\mu\big)^{\tilde{\sigma}} \ .
\end{aligned}$$

(b) This follows analogously.

□

(14.17) Theorem Let Ξ and $\tilde{\Xi}$ be proper pseudo-quadratic spaces, let $(\mathbb{A}, \mathbb{F}, \sigma)$ and $(\tilde{\mathbb{A}}, \tilde{\mathbb{F}}, \tilde{\sigma})$ be quadratic of type (iv) and (iii), respectively, let

$$\dim_{\mathbb{A}} L_0 = 1 \ , \qquad\qquad \dim_{\tilde{\mathbb{A}}} \tilde{L}_0 = 2 \ ,$$

let $a \in L_0^*$ and $e \in \mathbb{E}_a^{\perp}$, and let $\phi : \mathbb{E}_a \to \tilde{\mathbb{A}}$ be an isomorphism of fields. Moreover, let $\gamma : T \to \tilde{T}$ be a map such that

$$\forall\ x = s + et \in \mathbb{A},\ u \in \mathbb{F} : \qquad \gamma\big(a \cdot x, x^{\sigma}q(a)x + u\big) = \big(\tilde{a} \cdot \phi(s) + \tilde{b} \cdot \phi(t)^{\tilde{\sigma}}, \phi\big(N(x)q(a) + u\big)\big)$$

for some $\tilde{a}, \tilde{b} \in \tilde{L}_0$ such that $\tilde{f}(\tilde{a}, \tilde{b}) = 0_{\tilde{\mathbb{A}}}$. Then γ is a Jordan isomorphism.

Proof

- Putting $x := 0_{\mathbb{A}}$ and $u := 1_{\mathbb{A}}$ yields

$$\gamma(0_{L_0}, 1_{\mathbb{A}}) = \big(0_{\tilde{L}_0}, \phi(1_{\mathbb{A}})\big) = (0_{\tilde{L}_0}, 1_{\tilde{\mathbb{A}}}) \ .$$

- By proposition (14.10), γ is an isomorphism of groups.
- Let $x, y \in T$. By lemma (14.8), there are $\hat{a} \in A$ and $\hat{b} \in B$ such that $y = \hat{a} \cdot \hat{b}$. By lemma (7.19) and proposition (14.16), we have

$$\begin{aligned}
\gamma\big(h_x(y)\big) &= \gamma\big(h_x(\hat{a} \cdot \hat{b})\big) = \gamma\big(h_x(\hat{a}) \cdot h_x(\hat{b})\big) = \gamma\big(h_x(\hat{a})\big) \cdot \gamma\big(h_x(\hat{b})\big) \\
&= \tilde{h}_{\gamma(x)}\big(\gamma(\hat{a})\big) \cdot \tilde{h}_{\gamma(x)}\big(\gamma(\hat{b})\big) = \tilde{h}_{\gamma(x)}\big(\gamma(\hat{a}) \cdot \gamma(\hat{b})\big) = \tilde{h}_{\gamma(x)}\big(\gamma(\hat{a} \cdot \hat{b})\big) = \tilde{h}_{\gamma(x)}\big(\gamma(y)\big)
\end{aligned}$$

□

Chapter 15 Exceptional Isomorphisms II

We show that a Moufang set T defined by a one-dimensional pseudo-quadratic space over a quaternion division algebra could equally defined by a 2-dimensional pseudo-quadratic space over a separable quadratic extension and vice versa.

(15.1) Lemma Let $(\mathbb{A}, \mathbb{F}, \sigma)$ be quadratic of type (iii) and assume $\dim_{L_0} \mathbb{A} = 2$. Let f be a skew-hermitian form on L_0 and let $\{a, b\}$ be an orthogonal basis of L_0. Then f is anisotropic if and only if we have

$$f(b,b)f(a,a)^{-1} \notin -N(\mathbb{A}) \ .$$

Proof
Given $s, t \in \mathbb{A}$, we have

$$\begin{aligned} f(as+bt, as+bt) = 0_{\mathbb{A}} \ &\Leftrightarrow \ N(s)f(a,a) + N(t)f(b,b) = 0_{\mathbb{A}} \\ &\Leftrightarrow \ f(b,b)f(a,a)^{-1} = -N(st^{-1}) \in -N(\mathbb{A}) \ . \end{aligned}$$

□

(15.2) Lemma Let Ξ be a pseudo-quadratic space such that dim $\dim_{\mathbb{A}} L_0 = 1$ and such that $(\mathbb{A}, \mathbb{F}, \sigma)$ is quadratic of type (iv). Let $a \in L_0^*$, $e \in \mathbb{E}_a^{\perp}$, $\beta := -N(e)$ and $b := ae$. Then the following holds:

(a) The set $\{a, b\}$ is an $\mathbb{E}_a$-basis of L_0.

(b) The skew-hermitian form $\tilde{f} : L_0 \times L_0 \to \mathbb{E}_a$ defined by

$$\tilde{f}(a,a) := f(a,a) \ , \qquad \tilde{f}(b,b) := N(e)f(a,a) \ , \qquad \tilde{f}(a,b) := 0_{\mathbb{A}}$$

is anisotropic.

(c) There is a pseudo-quadratic form $\tilde{q}$ on L_0 with respect to $\mathbb{F}$, σ and $\tilde{f}$ such that

$$\tilde{\Xi} := (\mathbb{E}_a, \mathbb{F}, \sigma, L_0, \tilde{q})$$

is a pseudo-quadratic space, satisfying $\tilde{q}(a) \equiv q(a) \mod \mathbb{F}$ and $\tilde{q}(b) \equiv N(e)q(a) \mod \mathbb{F}$.

(d) The map $\gamma : T \to \tilde{T}$ defined by

$$\forall\ x = s + et \in \mathbb{A},\ u \in \mathbb{F} : \qquad \big(a \cdot x, x^{\sigma}q(a)x + u\big) \mapsto \big(a \cdot s + b \cdot t^{\sigma}, N(x)q(a) + u\big)$$

is a Jordan isomorphism.

Proof

(a) We have $L_0 = a \cdot (\mathbb{E}_a + e\mathbb{E}_a) = a \cdot \mathbb{E}_a + b \cdot \mathbb{E}_a$.

(b) Since we have $\mathbb{A} \cong (\mathbb{E}_a/\mathbb{F}, \beta)$, we have $\tilde{f}(b,b)\tilde{f}(a,a)^{-1} = N(e) = -\beta \notin -N(\mathbb{E}_a)$.

(c) This results from (11.28) and (11.30) of [TW], respectively, and remark (7.4). Notice that $\tilde{f}$ is trace-valued with

$$\tilde{f}(a,a) = f(a,a) = q(a) + q(a)^{\sigma} \ , \quad \tilde{f}(b,b) = N(e)f(a,a) = N(e)q(a) + \big(N(e)q(a)\big)^{\sigma}$$

if we have $\operatorname{Char} \mathbb{A} = 2$, so that we choose $\tilde{\beta}_a := q(a)$, $\tilde{\beta}_b := N(e)q(a)$ in this case.

(d) By theorem (14.17), it suffices to show that γ is well-defined. Given $s, t \in \mathbb{E}_a$, we have

$$\begin{aligned} \tilde{q}(a \cdot s + b \cdot t^{\sigma}) &\equiv s^{\sigma}\tilde{q}(a)s + t^{\sigma}\tilde{q}(b)t + \tilde{f}(a \cdot s, b \cdot t^{\sigma}) \\ &\equiv N(s)q(a) + N(e)N(t)q(a) = N(x)q(a) \equiv N(x)q(a) + u \mod \mathbb{F} \ . \end{aligned}$$

□

(15.3) Lemma Let Ξ be a pseudo-quadratic space such that dim $\dim_{\mathbb{A}} L_0 = 2$ and such that $(\mathbb{A}, \mathbb{F}, \sigma)$ is quadratic of type (iii). Let $\{a, b\}$ be an orthogonal basis of L_0 (which exists by theorem (6.3) in chapter 7 of [WSch]) and let

$$\beta := -f(b,b)f(a,a)^{-1} .$$

Then the following holds:

(a) We have $\beta \in \mathbb{F} \setminus N(\mathbb{A})$. In particular, $\tilde{\mathbb{H}} := (\mathbb{A}/\mathbb{F}, \beta)$ is a quaternion division algebra.

(b) Let $e \in \mathbb{A}^{\perp}$ such that $N(e) = -\beta$. We extend the scalar multiplication on L_0 to $\tilde{\mathbb{H}}$ by $ae := b$. Then the skew-hermitian form $\tilde{f} : L_0 \times L_0 \to \tilde{\mathbb{H}}$ defined by $\tilde{f}(a,a) := f(a,a)$ is anisotropic, satisfying

$$\tilde{f}(b,b) = -N(e)f(a,a) , \qquad \tilde{f}(a,b) = -ef(a,b) .$$

(c) There is a pseudo-quadratic form $\tilde{q}$ on L_0 with respect to $\mathbb{F}$, σ and $\tilde{f}$ such that

$$\tilde{\Xi} := (\tilde{\mathbb{H}}, \mathbb{F}, \sigma, L_0, \tilde{q})$$

is a pseudo-quadratic space, satisfying $q(a) \equiv \tilde{q}(a) \mod \mathbb{F}$ and $q(b) \equiv N(e)\tilde{q}(a) \mod \mathbb{F}$.

(d) The map $\gamma : \tilde{T} \to T$ defined by

$$\forall\ x = s + et \in \tilde{\mathbb{H}},\ u \in \mathbb{F} : \qquad \big(a \cdot x, x^{\sigma}\tilde{q}(a)x + u\big) \mapsto \big(a \cdot s + b \cdot t^{\sigma}, N(x)\tilde{q}(a) + u\big)$$

is a Jordan isomorphism.

Proof
Notice that we have $\mathrm{Fix}(\sigma) = \mathbb{F}$ since $(\mathbb{A}, \mathbb{F}, \sigma)$ is quadratic of type (iii).

(a) We have

$$\beta^{\sigma} = -f(b,b)^{\sigma}f(a,a)^{-\sigma} = -f(b,b)f(a,a)^{-1} = \beta , \qquad \beta \in \mathrm{Fix}(\sigma) = \mathbb{F} .$$

By lemma (15.1), we have $\beta = -f(b,b)f(a,a)^{-1} \notin N(\mathbb{A})$.

(b) We have

$$\tilde{f}(a,a) = f(a,a)_{\mathbb{A}} \neq 0_{\mathbb{A}} , \qquad \tilde{f}(b,b) = \tilde{f}(ae,ae) = e^{\sigma}f(a,a)e = -N(e)f(a,a) .$$

(c) This results from (11.28) and (11.30) of [TW], respectively, and remark (7.4). Notice that

$$q(b) + q(b)^{\sigma} = N(e)\big(q(a) + q(a)^{\sigma}\big) , \quad q(b) + N(e)q(a) = \big(q(b) + N(e)q(a)\big)^{\sigma} \in \mathrm{Fix}(\sigma) = \mathbb{F}$$

and that $\tilde{f}$ is trace-valued with $\tilde{f}(a,a) = f(a,a) = q(a) + q(a)^{\sigma}$ if we have $\mathrm{Char}\,\mathbb{A} = 2$, so that we choose $\tilde{\beta}_a := q(a)$ in this case to obtain

$$\begin{aligned} q(b) &= f(b,b) + q(b)^{\sigma} = N(e)f(a,a) + q(b)^{\sigma} \\ &= N(e)q(a) + N(e)q(a)^{\sigma} + q(b)^{\sigma} \equiv N(e)q(a) \equiv N(e)\tilde{q}(a) \mod \mathbb{F} . \end{aligned}$$

(d) By theorem (14.17), it suffices to show that γ is well-defined. Given $s, t \in \mathbb{A}$, we have

$$\begin{aligned} q(a \cdot s + b \cdot t^{\sigma}) &\equiv s^{\sigma}q(a)s + t^{\sigma}q(b)t + f(a \cdot s, b \cdot t^{\sigma}) \\ &\equiv N(s)\tilde{q}(a) + N(e)N(t)\tilde{q}(a) = N(x)\tilde{q}(a) \equiv N(x)\tilde{q}(a) + u \mod \mathbb{F} . \end{aligned}$$

$\square$

Chapter 16 The Field $\mathbb{F}_4$

We return to the case $\mathbb{A} \cong \mathbb{F}_4$ which we excluded in chapter 11.

(16.1) Notation Throughout this chapter, let $a \in L_0^*$ and $\mathbb{A} \cong \mathbb{F}_4$, which implies $\tilde{\mathbb{A}} \cong \mathbb{F}_4$ and thus $\mathbb{F} \cong \mathbb{F}_2 \cong \tilde{\mathbb{F}}$, and let $\gamma : T \to \tilde{T}$ be a Jordan isomorphism.

(16.2) Remark By remark (11.7), the map $(\varphi_1, \phi_a) : (\langle a \rangle_{\mathbb{A}}, \mathbb{A}) \to (\langle \varphi_1(a) \rangle_{\tilde{\mathbb{A}}}, \tilde{\mathbb{A}})$ is an isomorphism of vector spaces, where $\phi_a : \mathbb{A} \to \tilde{\mathbb{A}}$ is defined by

$$\forall\, t \in \mathbb{A} : \qquad \varphi_1(a \cdot t) = \varphi_1(a) \cdot \phi_a(t) \ .$$

Moreover, we have $\tilde{X}_{\varphi_1(a)} = \{\varphi_2(a,t), \varphi_2(a,t)^{\tilde{\sigma}}\}$ and thus either $\phi_a(t) = \varphi_2(a,t)$ for each $t \in X_a$ or $\phi_a(t) = \varphi_2(a,t)^{\tilde{\sigma}}$ for each $t \in X_a$.

(16.3) Lemma Assume $\dim_{\mathbb{A}} L_0 \geq 2$. Then we have

$$\forall\, t \in X_a : \qquad \phi_a(t) = \varphi_2(a,t) \ .$$

Proof
Notice that $(\mathbb{A}, \mathbb{F}, \sigma)$ is quadratic of type (iii), hence there are no inseparable elements. Let $b \in a^\perp$ and $t \in X_a$. By identity (7.1), we have

$$\begin{aligned}\varphi_1(a) \cdot \phi_a(t^\sigma) &= \varphi_1(a \cdot t^\sigma) = \varphi_1(a \cdot t^\sigma - b \cdot t^{-1} f(a,b) t^\sigma) \\ &= \varphi_1(a) \cdot \varphi_2(a,t)^{\tilde{\sigma}} - \varphi_1(b) \cdot \varphi_2(a,t)^{-1} \tilde{f}\big(\varphi_1(a), \varphi_1(b)\big) \varphi_2(a,t)^{\tilde{\sigma}} \ .\end{aligned}$$

The linear independence of $\varphi_1(a)$ and $\varphi_1(b)$ yields $\phi_a(t)^{\tilde{\sigma}} = \phi_a(t^\sigma) = \varphi_2(a,t)^{\tilde{\sigma}}$. □

(16.4) Remark Now we can go on as in chapter 11 and we obtain that γ is induced by an isomorphism $\Phi : \Xi \to \tilde{\Xi}$ of pseudo-quadratic spaces. In the case $\dim_{\mathbb{A}} L_0 = 1$ however, each isomorphism of groups turns out to be a Jordan isomorphism. At this point, we drop the assumption that γ is a Jordan isomorphism.

(16.5) Lemma Assume $\dim_{\mathbb{A}} L_0 = 1$. Then we have $h_a = \mathrm{id}_T$ for each $a \in L_0^*$.

Proof
Let $a \in L_0^*$. Given $x, y \in \mathbb{A}$ and $s \in X_{ax},\ t \in X_{ay}$, we have

$$\begin{aligned}h_{(ax,s)}(a \cdot y, t) &= \big(a \cdot y s^\sigma - a \cdot x s^{-1} f(ax, ay) s^\sigma, s t s^\sigma\big) \\ &= \big(a \cdot y s^\sigma - a \cdot y N(x) f(a,a) s, N(s) t\big) = \big(a \cdot y(s^\sigma + s), t\big) = (a \cdot y, t) \ .\end{aligned}$$

□

(16.6) Corollary Assume $\dim_{\mathbb{A}} L_0 = 1$ and let $\gamma : T \to \tilde{T}$ be an isomorphism of groups. Then γ is a Jordan isomorphism.

Proof
Because of $|Z(\tilde{T})| = 2$, we have $\gamma(0_{L_0}, 1_{\mathbb{A}}) = (0_{\tilde{L}_0}, 1_{\tilde{\mathbb{A}}})$, and because of $|\tilde{T}| = |T| = 8$, we have $\dim_{\tilde{\mathbb{A}}} \tilde{L}_0 = 1$. By lemma (16.5), we have

$$\forall\, a \in L_0^* : \qquad h_a = \mathrm{id}_T \ , \qquad \tilde{h}_{\gamma(a)} = \mathrm{id}_{\tilde{T}}$$

and thus

$$\forall\, a \in L_0^*,\ x \in L_0 : \qquad \gamma\big(h_a(x)\big) = \gamma(x) = \tilde{h}_{\gamma(a)}\big(\gamma(x)\big) \ .$$

□

(16.7) Remark As a consequence, it is convenient to determine the isomorphism class of the group T and hence the structure of $\mathrm{Aut}(T)$.

(16.8) Lemma We have $T \cong Q_8$, where Q_8 denotes the quaternion group.

Proof
We have $|T| = 8$, and given $(a,t) \in T \setminus Z(T)$, we have

$$(a,t)^2 = \big(a + a, t + t + f(a,a)\big) = (0_{L_0}, 1_{\mathbb{A}}) \neq (0_{L_0}, 0_{\mathbb{A}}) \ .$$

□

(16.9) Remark The outer automorphisms of T are represented by the restrictions of isomorphisms of pseudo-quadratic spaces, i.e., by the group

$$\langle \gamma_\sigma, \rho_s \mid \gamma_\sigma : T \to \tilde{T},\ (a \cdot x, t) \mapsto (a \cdot x^\sigma, t^\sigma),\ \rho_s : T \to \tilde{T},\ (a \cdot x, t) \mapsto (a \cdot xs, t) \rangle \cong \Sigma_3 \ ,$$

where $s \in \mathbb{A} \setminus \mathbb{F}$. The three non-trivial inner automorphisms yield some exceptional Jordan automorphisms:

$$\begin{aligned}
\gamma_a : T \to T : &\quad (a,t) \mapsto (a,t)\ , &\quad (as,t) \mapsto (as,t^\sigma)\ , &\quad (as^\sigma,t) \mapsto (as^\sigma,t^\sigma)\ ,\\
\gamma_{as} : T \to T : &\quad (a,t) \mapsto (a,t^\sigma)\ , &\quad (as,t) \mapsto (as,t)\ , &\quad (as^\sigma,t) \mapsto (as^\sigma,t^\sigma)\ ,\\
\gamma_{as^\sigma} : T \to T : &\quad (a,t) \mapsto (a,t^\sigma)\ , &\quad (as,t) \mapsto (as,t^\sigma)\ , &\quad (as^\sigma,t) \mapsto (as^\sigma,t)\ .
\end{aligned}$$

(16.10) Lemma Suppose that $\dim_{\mathbb{A}} L_0 = 1$ and let $\gamma : T \to \tilde{T}$ be a Jordan isomorphism. Then there are an isomorphism $\Phi : \Xi \to \tilde{\Xi}$ of pseudo-quadratic spaces and an inner automorphism $\tilde{\gamma} \in \mathrm{Aut}(\tilde{T})$ such that γ is induced by $\tilde{\gamma} \circ \Phi$.

Proof
By remark (11.7), the map

$$(\varphi_1, \phi_a) : (\langle a \rangle_{\mathbb{A}}, \mathbb{A}) \to (\langle \varphi_1(a) \rangle_{\tilde{\mathbb{A}}}, \tilde{\mathbb{A}})$$

is an isomorphism of vector spaces. Let $\Phi : \Xi \to \tilde{\Xi}$ be defined by

$$\Phi := (\varphi_1, \phi_a) : (L_0, \mathbb{A}) \to (\tilde{L}_0, \tilde{\mathbb{A}}),\ (a,t) \mapsto \big(\varphi_1(a), \phi_a(t)\big) \ .$$

Then the map $\tilde{\gamma} := \gamma \circ \Phi^{-1}_{|\tilde{T}} : \tilde{T} \to \tilde{T}$ is an automorphism of $\tilde{T}$ such that $\tilde{\varphi}_1 = \mathrm{id}_{\tilde{L}_0}$, hence $\tilde{\gamma}$ is inner. □

(16.11) Theorem Assume $\mathbb{K} \cong \mathbb{F}_4$. A map $\gamma : T \to \tilde{T}$ is a Jordan isomorphism if and only if one of the following holds:

(i) There is an isomorphism $\Phi : \Xi \to \tilde{\Xi}$ that induces γ.

(ii) We have $\dim_{\mathbb{K}} L = 1$ and there are an isomorphism $\Phi : \Xi \to \tilde{\Xi}$ of pseudo-quadratic spaces and a non-trivial inner automorphism $\tilde{\gamma} \in \mathrm{Aut}(\tilde{T})$ such that γ is induced by $\tilde{\gamma} \circ \Phi$.

Proof

"$\Rightarrow$" This results from remark (16.4) and lemma (16.10).

"$\Leftarrow$" This result from theorem (14.2) and corollary (16.6).

□

Chapter 17 Conclusion

The complete description of the Jordan isomorphisms between the Moufang sets of two proper pseudo-quadratic spaces is as follows:

(17.1) Theorem (**Jordan Isomorphisms of Moufang Sets of Pseudo-Quadratic Form Type**) Let Ξ and $\tilde{\Xi}$ be proper pseudo-quadratic spaces. A map $\gamma : T \to \tilde{T}$ is a Jordan isomorphism if and only if one of the following holds:

(i) There is an isomorphism $\Phi : \Xi \to \tilde{\Xi}$ of pseudo-quadratic spaces that induces γ.

(ii) The involutory sets $(\mathbb{K}, \mathbb{K}_0, \sigma)$ and $(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ both are quadratic of type (iv), we have

$$\mathbb{K} \ncong \tilde{\mathbb{K}} \,, \qquad \dim_{\mathbb{K}} L_0 = 2 = \dim_{\tilde{\mathbb{K}}} \tilde{L}_0$$

and there are an $i \in \{2,3\}$ and an isomorphism $\Phi : \Xi \to \tilde{\Xi}_i$ of pseudo-quadratic spaces such that γ is induced by $(\mathrm{id}^i_{\tilde{T}})^{-1} \circ \Phi$, where $\mathrm{id}^i_{\tilde{T}}$ and $\tilde{\Xi} =: \tilde{\Xi}_1, \tilde{\Xi}_2, \tilde{\Xi}_3$ are as in notation (13.4).

(iii) The involutory sets $(\mathbb{K}, \mathbb{K}_0, \sigma)$ and $(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ are quadratic of type (iv) and (iii), respectively, we have $\dim_{\mathbb{K}} L_0 = 1$, $\dim_{\tilde{\mathbb{K}}} \tilde{L}_0 = 2$ and γ can be described by

$$\forall\ x = s + et \in \mathbb{K},\ u \in \mathbb{K}_0 : \qquad \gamma(ax, x^{\sigma} q(a) x + u) = \big(\tilde{a}\phi(s) + \tilde{b}\phi(t)^{\tilde{\sigma}}, \phi\big(N(x) q(a) + u\big)\big) \,,$$

where $a \in L_0^*$ is arbitrary, $\phi : \mathbb{E}_a \to \tilde{\mathbb{K}}$ is an isomorphism of fields, $e \in \mathbb{E}_a^{\perp}$, $\tilde{a} \in \tilde{L}_0$ and $\tilde{b} \in \tilde{a}^{\perp}$.

(iv) We have $\mathbb{K} \cong \mathbb{F}_4 \cong \tilde{\mathbb{K}}$, $\dim_{\mathbb{K}} L = 1$ and there are an isomorphism $\Phi : \Xi \to \tilde{\Xi}$ of pseudo-quadratic spaces and a non-trivial inner automorphism $\tilde{\gamma} \in \mathrm{Aut}(\tilde{T})$ such that γ is induced by $\tilde{\gamma} \circ \Phi$.

Proof

"$\Rightarrow$" If (i) or (iv) holds, we have $\mathbb{K} \cong \tilde{\mathbb{K}}$, thus neither (ii) nor (iii) holds. If (iv) holds, (i) can't hold since $\tilde{\gamma}$ is not induced by an isomorphism of pseudo-quadratic spaces by remark (16.9). Suppose that neither (i) nor (iv) holds. By theorem (8.1), $(\mathbb{K}, \mathbb{K}_0, \sigma)$ and $(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ are non-proper, hence they are of quadratic type by remark (9.9), and by theorem (16.11), we have $\mathbb{K} \ncong \mathbb{F}_4$. But then we have $\mathbb{K} \ncong \tilde{\mathbb{K}}$ by theorem (8.1). Finally, either (ii) or (iii) holds by theorem (13.15).

"$\Leftarrow$" This results from theorem (14.2), theorem (14.17) and theorem (16.11).

□

Part IV

Simply Laced Foundations

The subject of this part is the classification of simply laced twin buildings via foundations, which are amalgams of parametrized Moufang triangles. Given a simply laced twin building, we obtain a foundation by taking the set of rank 2 residues, which are Moufang triangles, and by parametrizing the corresponding root group sequences. These parametrizations make the glueings visible, and they turn out to be Jordan isomorphisms.

By looking at foundations of rank 3, we can deduce more information about the appearing glueings, e.g., the glueing of a foundation of type A_3 is an isomorphism of skew-fields, which is quite restrictive, e.g., concerning foundations involving octonions or residues of type D_4.

Chapter 18 Parametrized Moufang Triangles

(18.1) Definition Let $\mathbb{A}$ be an alternative division ring.

- The root group sequence
$$\mathcal{T}(\mathbb{A}) := \big(U_{[1,3]}, x_1(\mathbb{A}), x_2(\mathbb{A}), x_3(\mathbb{A})\big)$$
with commutator relations
$$\forall\ s,t \in \mathbb{A}: \qquad [x_1(s), x_3(t)] := x_2(st)$$
is the *parametrized standard triangle with respect to* $\mathbb{A}$.
- The root group sequence
$$\mathcal{T}^o(\mathbb{A}) := \big(U_{[1,3]}, x_1(\mathbb{A}), x_2(\mathbb{A}), x_3(\mathbb{A})\big)$$
with commutator relations
$$\forall\ s,t \in \mathbb{A}: \qquad [x_1(s), x_3(t)] := x_2(-st)$$
is the *parametrized opposite triangle with respect to* $\mathbb{A}$.

(18.2) Remark For reasons of brevity, we will write
$$\mathcal{T}^{(o)}(\mathbb{A}) = \big(x_1(\mathbb{A}), \ldots, x_3(\mathbb{A})\big)$$
instead of $\mathcal{T}^{(o)}(\mathbb{A}) = \big(U_{[1,3]}, x_1(\mathbb{A}), \ldots, x_3(\mathbb{A})\big)$.

(18.3) Lemma Given an alternative division ring $\mathbb{A}$, we have
$$\mathcal{T}^o(\mathbb{A}) \cong \mathcal{T}(\mathbb{A}) \ .$$

Proof
Let
$$\mathcal{T}^o(\mathbb{A}) = \big(x_1(\mathbb{A}), x_2(\mathbb{A}), x_3(\mathbb{A})\big) \ , \qquad\qquad \mathcal{T}(\mathbb{A}) = \big(\tilde{x}_1(\mathbb{A}), \tilde{x}_2(\mathbb{A}), \tilde{x}_3(\mathbb{A})\big) \ .$$
Then $\alpha = (\alpha^1, \alpha^2, \alpha^3)$ with
$$\begin{aligned} \alpha_1 &: x_1(\mathbb{A}) \to \tilde{x}_1(\mathbb{A}),\ x_1(t) \mapsto \tilde{x}_1(t) \ , \\ \alpha_2 &: x_2(\mathbb{A}) \to \tilde{x}_2(\mathbb{A}),\ x_2(t) \mapsto \tilde{x}_2(-t) \ , \\ \alpha_3 &: x_3(\mathbb{A}) \to \tilde{x}_3(\mathbb{A}),\ x_3(t) \mapsto \tilde{x}_3(t) \end{aligned}$$
preserves the commutator relations: Given $s,t \in \mathbb{A}$, we have
$$\alpha\big([x_1(s), x_3(t)]\big) = \alpha\big(x_2(-st)\big) = \tilde{x}_2(st) = [\tilde{x}_1(s), \tilde{x}_3(t)] = [\alpha\big(x_1(s)\big), \alpha\big(x_3(t)\big)] \ .$$
□

(18.4) Lemma Let $\mathcal{T}(\mathbb{A}) = \big(x_1(\mathbb{A}), x_2(\mathbb{A}), x_3(\mathbb{A})\big)$ be a parametrized standard triangle. Then we have
$$\big(x_3(\mathbb{A}^o), x_2(\mathbb{A}^o), x_1(\mathbb{A}^o)\big) = \mathcal{T}^o(\mathbb{A}^o) \ .$$

Proof
Given $s, t \in \mathbb{A}^o$, we have

$$[x_3(s), x_1(t)] = [x_1(t), x_3(s)]^{-1} = x_2(ts)^{-1} = x_2(-s \circ t) \ .$$

□

(18.5) Notation In the following, a *(parametrized) Moufang triangle* always denotes a parametrized standard Moufang triangle.

(18.6) Definition Let $\mathcal{T}(\mathbb{A})$, $\mathcal{T}(\tilde{\mathbb{A}})$ be parametrized Moufang triangles.

- An *isomorphism* $\alpha : \mathcal{T}(\mathbb{A}) \to \mathcal{T}(\tilde{\mathbb{A}})$ is a triple $(\alpha_1, \alpha_2, \alpha_3)$ such that $\alpha_1, \alpha_2, \alpha_3 : \mathbb{A} \to \tilde{\mathbb{A}}$ are isomorphisms of additive groups satisfying
$$\forall\ s, t \in \mathbb{A} : \qquad \alpha_2(st) = \alpha_1(s)\alpha_3(t) \ .$$
- A *reparametrization for* $\mathcal{T}(\mathbb{A})$ is an ordered set $\alpha = (\tilde{\mathbb{A}}, \alpha_1, \alpha_2, \alpha_3)$ such that $\tilde{\mathbb{A}}$ is an alternative division ring and $\alpha_1, \alpha_2, \alpha_3 : \tilde{\mathbb{A}} \to \mathbb{A}$ are isomorphisms of additive groups satisfying
$$\forall\ s, t \in \tilde{\mathbb{A}} : \qquad \alpha_2(st) = \alpha_1(s)\alpha_3(t) \ .$$

(18.7) Lemma Let $\mathcal{T}(\mathbb{A})$ be a parametrized Moufang triangle and let $\alpha = (\tilde{\mathbb{A}}, \alpha_1, \alpha_2, \alpha_3)$ be a reparametrization for $\mathcal{T}(\mathbb{A})$. Then we have

$$\big(\tilde{x}_1(\tilde{\mathbb{A}}), \tilde{x}_2(\tilde{\mathbb{A}}), \tilde{x}_3(\tilde{\mathbb{A}})\big) = \mathcal{T}(\tilde{\mathbb{A}}) \ , \qquad\qquad \forall\ i = 1,2,3 :\ \tilde{x}_i := x_i \circ \alpha_i \ .$$

Proof
Given $s, t \in \tilde{\mathbb{A}}$, we have

$$[\tilde{x}_1(s), \tilde{x}_3(t)] = [x_1\big(\alpha_1(s)\big), x_3\big(\alpha_3(t)\big)] = x_2\big(\alpha_1(s)\alpha_3(t)\big) = x_2\big(\alpha_2(st)\big) = \tilde{x}_2(st) \ .$$

□

(18.8) Lemma Let $\mathcal{T}(\mathbb{A})$ be a parametrized Moufang triangle and let $a, b \in \mathbb{A}^*$. Then there are reparametrizations $\alpha = (\mathbb{A}, \alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\mathbb{A}, \beta_1, \beta_2, \beta_3)$ for $\mathcal{T}(\mathbb{A})$ such that

$$x_1\big(\alpha_1(1_{\mathbb{A}})\big) = x_1(1_{\mathbb{A}}) \ , \quad x_3\big(\alpha_3(1_{\mathbb{A}})\big) = x_3(a) \ , \quad x_1\big(\beta_1(1_{\mathbb{A}})\big) = x_1(b) \ , \quad x_3\big(\beta_3(1_{\mathbb{A}})\big) = x_3(1_{\mathbb{A}}) \ .$$

Proof
Set $\alpha := (\mathbb{A}, \phi, \rho_a\phi, \rho_a\phi)$ with ϕ as in (20.25) of [TW]. For the second statement, apply the first result to $\mathcal{T}(\mathbb{A}^o)$. □

(18.9) Lemma Let $\mathcal{T}(\mathbb{A}) = \big(x_1(\mathbb{A}), x_2(\mathbb{A}), x_3(\mathbb{A})\big)$ be a parametrized Moufang triangle. Then the action of the *Hua automorphism* $h_1(s) := \mu\big(x_1(1_{\mathbb{A}})\big)^{-1}\mu\big(x_1(s)\big)$ on $x_1(\mathbb{A}) \times x_3(\mathbb{A})$ corresponds to the map

$$(t, u) \mapsto (sts, s^{-1}u) \ ,$$

and the action of the *Hua automorphism* $h_3(s) := \mu\big(x_3(1_{\mathbb{A}})\big)^{-1}\mu\big(x_3(s)\big)$ on $x_1(\mathbb{A}) \times x_3(\mathbb{A})$ corresponds to the map

$$(t, u) \mapsto (ts^{-1}, sus) \ .$$

Proof
This is (33.10) of [TW]. □

Chapter 19 Foundations

In this chapter, we introduce the objects we will mainly deal with and which turn out to be a classifying invariant for simply laced twin buildings.

§ 19.1 Definition

The definition given here combines ideas and concepts of B. Mühlherr and R. Weiss. In this part, we only consider simply laced foundations, cf. part VIII for a general definition.

(19.1) Definition

- Let M be a *simply laced* Coxeter matrix, i.e., we have $m_{ij} \in \{2,3\}$ for all $i,j \in I$. A *foundation of type M* is a set
$$\mathcal{F} := \{\mathcal{T}(\mathbb{A}_{(i,j)}), \gamma_{(i,j,k)} \mid (i,j) \in A(M), (i,j,k) \in G(M)\}$$
such that:
  - (F1) Given $(i,j) \in A(M)$, then $\mathcal{T}(\mathbb{A}_{(i,j)})$ is a Moufang triangle over $\mathbb{A}_{(i,j)}$.
  - (F2) Given $(i,j) \in A(M)$, we have $\mathbb{A}_{(i,j)} = \mathbb{A}^o_{(j,i)}$.
  - (F3) Given $(i,j,k) \in G(M)$, then $\gamma_{(i,j,k)} : \mathbb{A}_{(i,j)} \to \mathbb{A}_{(j,k)}$ is an isomorphism of additive groups satisfying
$$\gamma_{(i,j,k)}(1) = 1 \ , \qquad \gamma_{(i,j,k)} = \mathrm{id}^o \circ \gamma^{-1}_{(k,j,i)} \circ \mathrm{id}^o \ .$$
  - (F4) Given $(i,j,k), (i,j,l), (l,j,k) \in G(M)$, we have
$$\gamma_{(i,j,k)} = \gamma_{(l,j,k)} \circ \mathrm{id}^o \circ \gamma_{(i,j,l)} \ .$$
- Given a foundation $\mathcal{F}$, we denote the corresponding Coxeter Matrix by F.
- A foundation $\mathcal{F}$ is a *Moufang foundation* if each *glueing* $\gamma := \gamma_{(i,j,k)}$ is a Jordan isomorphism, i.e., we have
$$\forall\ s,t \in \mathbb{A}_{(i,j)} : \qquad \gamma(sts) = \gamma(s)\gamma(t)\gamma(s) \ .$$

(19.2) Definition Let $\mathcal{F}$ be a foundation over $I = V(F)$ and let $J \subseteq I$. The *J-residue of $\mathcal{F}$* is the foundation
$$\mathcal{F}_J := \{\mathcal{T}(\mathbb{A}_{(i,j)}), \gamma_{(i,j,k)} \mid (i,j) \in J^2 \cap A(F), (i,j,k) \in J^3 \cap G(F)\} \ .$$

(19.3) Remark Since a foundation is, in fact, an amalgam of Moufang triangles, an isomorphism of foundations is a system of isomorphism of Moufang triangles preserving the glueings.

(19.4) Definition Let $\mathcal{F}, \tilde{\mathcal{F}}$ be foundations.

- An *isomorphism* $\alpha : \mathcal{F} \to \tilde{\mathcal{F}}$ is a system $\alpha = \{\pi, \alpha_{(i,j)} \mid (i,j) \in A(F)\}$ of isomorphisms
$$\pi : F \to \tilde{F} \ , \qquad \alpha_{(i,j)} = (\alpha^i_{(i,j)}, \alpha^{ij}_{(i,j)}, \alpha^j_{(i,j)}) : \mathcal{T}(\mathbb{A}_{(i,j)}) \to \mathcal{T}(\tilde{\mathbb{A}}_{(\pi(i),\pi(j))})$$
such that
$$\forall\ (i,j,k) \in G(F) : \qquad \tilde{\gamma}_{(\pi(i),\pi(j),\pi(k))} \circ \alpha^j_{(i,j)} = \alpha^j_{(j,k)} \circ \gamma_{(i,j,k)}$$
and $\alpha_{(i,j)} = \alpha^o_{(j,i)}$ for each $(i,j) \in A(F)$.
- An isomorphism $\alpha : \mathcal{F} \to \tilde{\mathcal{F}}$ is *special* if $F = \tilde{F}$ and $\pi = \mathrm{id}_F$.
- An *automorphism of* $\mathcal{F}$ is an isomorphism $\alpha : \mathcal{F} \to \mathcal{F}$.

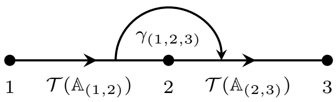
§ 19.2 Visualizing Foundations

Given a foundation $\mathcal{F}$ of type M, we can extend the corresponding Coxeter diagram Π_M in such a way that it contains all the information of the given foundation $\mathcal{F}$:

- Given an edge $\{i,j\} \in E(M)$, we label it by either $\mathcal{T}(\mathbb{A}_{(i,j)})$ or $\mathcal{T}(\mathbb{A}_{(j,i)})$ and add an arrow to indicate in which direction we have the given standard root group sequence.
- Given $(i,j,k) \in G(M)$, we choose either (i,j,k) or (k,j,i), we add a directed arc from $\{i,j\}$ to $\{j,k\}$ or vice versa, and label it by $\gamma_{(i,j,k)}$, resp. $\gamma_{(k,j,i)}$.

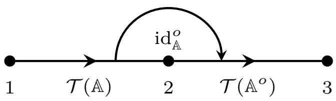
The remaining information can be deduced from the given ones. Notice that the constructed diagram is not uniquely determined by $\mathcal{F}$ as there is a choice in the directions.

(19.5) Example An arbitrary foundation of type A_3 is given by



,

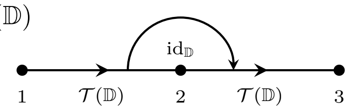
a concrete example is



,

where $\mathbb{A}$ is an arbitrary alternative division ring. We will see that an integrable foundation of type A_3 is isomorphic to the foundation

$\mathcal{A}_3(\mathbb{D})$



for some skew-field $\mathbb{D}$, i.e., the previous example is not integrable.

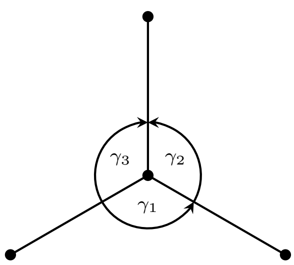
(19.6) Remark

(a) Concerning a given problem, we sometimes don't need the whole visualization to get a feeling for the crucial step in the solution. In this case, we restrict to a diagram with the essential information, e.g., we just indicate whether some glueings are iso- or anti-isomorphisms of skew-fields.

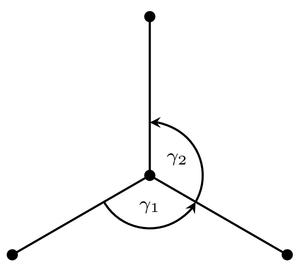
(b) Notice that in the above construction, the resulting diagram possibly carries redundant information: Given $(i,j,k),(i,j,l) \in G(M)$ (and thus $(l,j,k) \in G(M)$), we have

$$\gamma_{(i,j,k)} = \gamma_{(l,j,k)} \circ \mathrm{id}^o \circ \gamma_{(i,j,l)} \ ,$$

which means that



and



carry the same information, where $\gamma_1 = \gamma_{(i,j,l)}$, $\gamma_2 = \gamma_{(l,j,k)}$ and $\gamma_3 = \gamma_{(i,j,k)}$.

§ 19.3 Root Group Systems

The fact that a root group systems is a classifying invariant of the corresponding twin building is a fundamental result in twin building theory.

(19.7) Definition Let $\mathcal{B}$ be a simply laced twin building of type M, let Σ be a twin apartment of $\mathcal{B}$ and let $c \in \mathcal{O}_\Sigma$.

- Given $(i,j) \in A(M)$, let α_i, α_j be the simple roots with respect to (Σ, c) and let $\Theta_{(i,j)}$ be as in theorem (2.32) (d). Then
$$U_{(i,j)} := (U_{[i,j]}, U^i_{(i,j)}, U^{ij}_{(i,j)}, U^j_{(i,j)}) := \Theta_{(i,j)}$$
denotes the root group sequence of $\mathcal{B}$ from α_i to α_j, which is isomorphic to the root group sequence of $\mathcal{B}_{ij}$ from $\alpha_i \cap \mathcal{B}_{ij}$ to $\alpha_j \cap \mathcal{B}_{ij}$.
- The resulting set
$$\mathcal{U}(\mathcal{B}, M, \Sigma, c) := \{U_{(i,j)} \mid (i,j) \in A(M)\}$$
is the *root group system of* $\mathcal{B}$ *based at* (Σ, c).

(19.8) Lemma Given $(i,j,k) \in G(F)$, we have $U^j_{(i,j)} = U^j_{(j,k)}$.

Proof
This holds by definition. □

(19.9) Definition Let $\mathcal{U} := \mathcal{U}(\mathcal{B}, M, \Sigma, c)$ and $\tilde{\mathcal{U}} := \mathcal{U}(\tilde{\mathcal{B}}, \tilde{M}, \tilde{\Sigma}, \tilde{c})$ be root group systems.

- An *isomorphism* $\alpha : \mathcal{U} \to \tilde{\mathcal{U}}$ is a system
$$\alpha = \{\pi, \alpha_{(i,j)} \mid (i,j) \in A(M)\}$$
of isomorphisms
$$\pi : M \to \tilde{M}\ , \qquad \alpha_{(i,j)} : U_{(i,j)} \to \tilde{U}_{(\pi(i),\pi(j))}$$
such that
$$\forall\ (i,j,k) \in G(M):\ \alpha_{(i,j)|U^j_{(i,j)}} = \alpha_{(j,k)|U^j_{(j,k)}}\ , \qquad \forall\ (i,j) \in A(M):\ \alpha_{(i,j)} = \alpha^o_{(j,i)}\ .$$
- An isomorphism $\alpha : \mathcal{U} \to \tilde{\mathcal{U}}$ is *special* if $M = \tilde{M}$ and $\pi = \mathrm{id}_M$.
- An *automorphism of* $\mathcal{U}$ is an isomorphism $\alpha : \mathcal{U} \to \mathcal{U}$.

(19.10) Theorem Two root group systems $\mathcal{U}(\mathcal{B}, M, \Sigma, c)$ and $\mathcal{U}(\mathcal{B}, M, \tilde{\Sigma}, \tilde{c})$ of of a twin building $\mathcal{B}$ are specially isomorphic.

Proof
This is a consequence of theorem (2.22). □

(19.11) Theorem Let $\mathcal{U} := \mathcal{U}(\mathcal{B}, M, \Sigma, c)$ be a root group system of a twin building $\mathcal{B}$. Then the isomorphism class of $\mathcal{U}$ is a classifying invariant of the isomorphism class of $\mathcal{B}$.

Proof
This is a consequence of the extension theorem (2.23). □

§ 19.4 Foundations and Root Group Systems

Given a root group system, there is a natural way to attach a foundation to it.

(19.12) Definition Let $\mathcal{U}(\mathcal{B}, M, \Sigma, c)$ be a root group system.

- Given $(i,j) \in A(M)$, there is an alternative division ring $\mathbb{A}_{(i,j)}$ such that $U_{(i,j)} \cong \mathcal{T}(\mathbb{A}_{(i,j)})$. In particular, there is a system of parametrizations

$$x^*_{(i,j)} : \mathbb{A}_{(i,j)} \to U^*_{(i,j)} \ , \ t \mapsto x^*_{(i,j)}(t) \ , \qquad\qquad * \in \{i, ij, j\}$$

extending to the defining relations for $\mathcal{T}(\mathbb{A}_{(i,j)})$, i.e., we have

$$\forall\ s,t \in \mathbb{A}_{(i,j)} : \qquad [x^i_{(i,j)}(s), x^j_{(i,j)}(t)] = x^{ij}_{(i,j)}(st) \ .$$

By lemma (18.4) and lemma (18.3), such a parametrization yields an opposite system of parametrizations

$$\begin{aligned} x^j_{(j,i)} &: \mathbb{A}^o_{(i,j)} \to U^j_{(j,i)},\ t \mapsto x^j_{(i,j)}\big(\mathrm{id}^o(t)\big) \ ,\\ x^{ji}_{(j,i)} &: \mathbb{A}^o_{(i,j)} \to U^{ji}_{(j,i)},\ t \mapsto x^{ij}_{(i,j)}\big(\mathrm{id}^o(-t)\big) \ ,\\ x^i_{(j,i)} &: \mathbb{A}^o_{(i,j)} \to U^i_{(j,i)},\ t \mapsto x^i_{(i,j)}\big(\mathrm{id}^o(t)\big) \ . \end{aligned}$$

The resulting set $\Lambda := \{\mathcal{T}(\mathbb{A}_{(i,j)}) \mid (i,j) \in A(M)\}$ is a *parameter system for $\mathcal{U}$*.

- Given $(i,j,k) \in G(M)$ and parametrizations $\mathcal{T}(\mathbb{A}_{(i,j)})$ and $\mathcal{T}(\mathbb{A}_{(j,k)})$, we define the glueing $\gamma_{(i,j,k)} : \mathbb{A}_{(i,j)} \to \mathbb{A}_{(j,k)}$ by

$$x^j_{(i,j)}(t) = x^j_{(j,k)}\big(\gamma_{(i,j,k)}(t)\big)$$

which is justified by lemma (19.8). Then $\gamma_{(i,j,k)}$ is an isomorphism of additive groups satisfying $\gamma_{(i,j,k)} = \mathrm{id}^o \circ \gamma^{-1}_{(k,j,i)} \circ \mathrm{id}^o$. By lemma (18.8), we may adjust all the parametrizations such that

$$\forall\ (i,j,k) \in G(F) : \qquad \gamma_{(i,j,k)}(1) = 1 \ .$$

Notice that for this purpose we need the following fact: The adjustment of a glueing $\gamma_{(i,j,k)}$ can be realized by a reparametrization for $\mathcal{T}(\mathbb{A}_{(i,j)})$ which fixes $x^i_{(i,j)}(1)$. Thus we can make sure that we don't alter glueings which have already been adjusted before.

(19.13) Lemma Given a root group system $\mathcal{U} := \mathcal{U}(\mathcal{B}, M, \Sigma, c)$, a parameter system Λ as in definition (19.12) induces a foundation

$$\mathcal{F}(U,\Lambda) := \{\mathcal{T}(\mathbb{A}_{(i,j)}), \gamma_{(i,j,k)} \mid (i,j) \in A(M), (i,j,k) \in G(M)\} \ .$$

Proof

We emphasize that the glueings in definition (19.12) are identifications with respect to directed edges. Given $(i,j,k), (i,j,l), (l,j,k) \in G(M)$ and $t \in \mathbb{A}_{(i,j)}$, we have

$$\begin{aligned} x^j_{(j,k)}\big(\gamma_{(i,j,k)}(t)\big) &= x^j_{(i,j)}(t) = x^j_{(j,l)}\big(\gamma_{(i,j,l)}(t)\big) \\ &= x^j_{(l,j)}\big(\mathrm{id}^o \circ \gamma_{(i,j,l)}(t)\big) = x^j_{(j,k)}\big(\gamma_{(l,j,k)} \circ \mathrm{id}^o \circ \gamma_{(i,j,l)}(t)\big) \end{aligned}$$

and thus $\gamma_{(i,j,k)} = \gamma_{(l,j,k)} \circ \mathrm{id}^o \circ \gamma_{(i,j,l)}$. □

(19.14) Definition A foundation $\mathcal{F}$ is *integrable* if it is the foundation of a twin building $\mathcal{B}$, i.e., if there are a root group system $\mathcal{U} := \mathcal{U}(\mathcal{B}, M, \Sigma, c)$ and a parameter system Λ for $\mathcal{U}$ such that

$$\mathcal{F} = \mathcal{F}(\mathcal{U}, \Lambda) \ .$$

(19.15) Remark

(a) We will see in lemma (19.27) (a) that an integrable foundation is necessarily a Moufang foundation.

(b) The next step is to show that the foundation attached to a root group system is unique up to isomorphism. Moreover, we want to prove that the building corresponding to an integrable foundation is unique up to isomorphism.

(19.16) Proposition Let $\mathcal{U} := \mathcal{U}(\mathcal{B}, M, \Sigma, c)$ and $\tilde{\mathcal{U}} := \mathcal{U}(\tilde{\mathcal{B}}, \tilde{M}, \tilde{\Sigma}, \tilde{c})$ be root group systems and let Λ and $\tilde{\Lambda}$ be parameter systems for $\mathcal{U}$ and $\tilde{\mathcal{U}}$, respectively. Then the following holds:

(a) An isomorphism $\tilde{\alpha} : \mathcal{F}(\mathcal{U}, \Lambda) \to \mathcal{F}(\tilde{\mathcal{U}}, \tilde{\Lambda})$ induces an isomorphism $\alpha : \mathcal{U} \to \tilde{\mathcal{U}}$.

(b) An isomorphism $\alpha : \mathcal{U} \to \tilde{\mathcal{U}}$ induces an isomorphism $\tilde{\alpha} : \mathcal{F}(\mathcal{U}, \Lambda) \to \mathcal{F}(\tilde{\mathcal{U}}, \tilde{\Lambda})$.

Proof
Each isomorphism

$$\alpha_{(i,j)} : U_{(i,j)} \to \tilde{U}_{(\pi(i),\pi(j))}$$

induces an isomorphism

$$\tilde{\alpha}_{(i,j)} : \mathcal{T}(\mathbb{A}_{(i,j)}) \to \mathcal{T}(\tilde{\mathbb{A}}_{(\pi(i),\pi(j))})$$

and vice versa. Given $(i,j) \in A(M)$, we have

$$\alpha_{(i,j)} = \alpha^o_{(j,i)} \;\Leftrightarrow\; \tilde{\alpha}_{(i,j)} = \tilde{\alpha}^o_{(j,i)} \ .$$

(a) We show that

$$\alpha := \{\pi, \alpha_{(i,j)} \mid (i,j) \in A(M)\} : \mathcal{U} \to \tilde{\mathcal{U}}$$

is an isomorphism.
Given $(i,j,k) \in G(M)$ and $t \in \mathbb{A}_{(i,j)}$, we have

$$\begin{aligned}\big(x^j_{(i,j)}(t)\big)^{\alpha_{(i,j)}} &= \tilde{x}^{\pi(j)}_{(\pi(i),\pi(j))}\big(\tilde{\alpha}^j_{(i,j)}(t)\big) = \tilde{x}^{\pi(j)}_{(\pi(j),\pi(k))}\big(\tilde{\gamma}_{(\pi(i),\pi(j),\pi(k))} \circ \tilde{\alpha}^j_{i,j}(t)\big)\\ &= \tilde{x}^{\pi(j)}_{(\pi(j),\pi(k))}\big(\tilde{\alpha}^j_{(j,k)} \circ \gamma_{(i,j,k)}(t)\big) = \big(x^j_{(j,k)}(\gamma_{(i,j,k)}(t))\big)^{\alpha_{(j,k)}}\end{aligned}$$

and therefore

$$\alpha_{(i,j)|U^j_{(i,j)}} = \alpha_{(j,k)|U^j_{(j,k)}} \ .$$

(b) We show that

$$\tilde{\alpha} := \{\pi, \tilde{\alpha}_{(i,j)} \mid (i,j) \in A(M)\} : \mathcal{F}(\mathcal{U}, \Lambda) \to \mathcal{F}(\tilde{\mathcal{U}}, \tilde{\Lambda})$$

is an isomorphism.
Given $(i,j,k) \in G(M)$ and $t \in \mathbb{A}_{(i,j)}$, we have

$$U^j_{(i,j)} \ni x^j_{(i,j)}(t) = x^j_{(j,k)}\big(\gamma_{(i,j,k)}(t)\big) \in U^j_{(j,k)} \ .$$

As we have

$$\alpha_{(i,j)|U^j_{(i,j)}} = \alpha_{(j,k)|U^j_{(j,k)}} \ , \tag{19.1}$$

it follows that

$$\begin{aligned}\tilde{x}^{\pi(j)}_{(\pi(j),\pi(k))}\big(\tilde{\gamma}_{(\pi(i),\pi(j),\pi(k))} \circ \tilde{\alpha}^j_{i,j}(t)\big) &= \tilde{x}^{\pi(j)}_{(\pi(i),\pi(j))}\big(\tilde{\alpha}^j_{(i,j)}(t)\big)\\ &\overset{(19.1)}{=} \tilde{x}^{\pi(j)}_{(\pi(j),\pi(k))}\big(\tilde{\alpha}^j_{(j,k)} \circ \gamma_{(i,j,k)}(t)\big)\end{aligned}$$

and therefore

$$\tilde{\gamma}_{(\pi(i),\pi(j),\pi(k))} \circ \tilde{\alpha}^j_{(i,j)} = \tilde{\alpha}^j_{(j,k)} \circ \gamma_{(i,j,k)} \ .$$

□

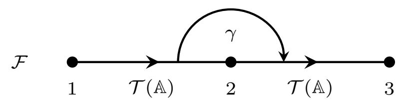
§ 19.5 Reparametrizations and Isomorphisms

The concept of reparametrizations is quite similar to that of isomorphisms. However, we deal with a single foundation and produce (in fact, all the) foundations which are isomorphic to a given one. Moreover, this concept allows us to complete the proof that a foundation is a classifying invariant of the corresponding twin building.

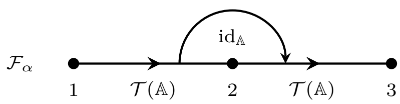
(19.17) Definition Let $\mathcal{F}$ be a foundation.

- A system of reparametrizations
$$\alpha := \{\alpha_{(i,j)} \mid (i,j) \in A(F)\}$$
satisfying $\alpha_{(i,j)} = \alpha^o_{(j,i)}$ for each $(i,j) \in A(F)$ and
$$\gamma_{(i,j,k)} \circ \alpha^j_{(i,j)}(1) = \alpha^j_{(j,k)}(1)$$
for each $(i,j,k) \in G(F)$ is a *reparametrization for $\mathcal{F}$*.
- Given a reparametrization α for $\mathcal{F}$, we set
$$\mathcal{F}_\alpha := \{\mathcal{T}(\tilde{\mathbb{A}}_{(i,j)}), \tilde{\gamma}_{(i,j,k)} \mid (i,j) \in A(F), (i,j,k) \in G(F)\}$$
with
$$\tilde{\gamma}_{(i,j,k)} := (\alpha^j_{(j,k)})^{-1} \circ \gamma_{(i,j,k)} \circ \alpha^j_{(i,j)}$$
for each $(i,j,k) \in G(F)$.

(19.18) Example Given the foundation



with $\gamma \in \mathrm{Aut}(\mathbb{A})$ and $\alpha := \{\alpha_{(1,2)} := (\mathbb{A}, \mathrm{id}_\mathbb{A}, \mathrm{id}_\mathbb{A}, \mathrm{id}_\mathbb{A}), \alpha_{(2,3)} := (\mathbb{A}, \gamma, \gamma, \gamma)\}$, we have



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(19.19) Lemma Let $\mathcal{U} := \mathcal{U}(\mathcal{B}, M, \Sigma, c)$ be a root group system, let $\mathcal{F} := \mathcal{F}(\mathcal{U}, \Lambda)$ for some parameter system Λ for $\mathcal{U}$, let α be a reparametrization for $\mathcal{F}$ and let $\tilde{\Lambda}$ be the parameter system induced by α. Then we have $\tilde{\mathcal{F}} := \mathcal{F}(\mathcal{U}, \tilde{\Lambda}) = \mathcal{F}_\alpha$.

Proof
We have
$$\begin{aligned}\tilde{x}^j_{(j,k)}\big(\tilde{\gamma}_{(i,j,k)}(t)\big) &= \tilde{x}^j_{(i,j)}(t) = x^j_{(i,j)}\big(\alpha^j_{(i,j)}(t)\big)\\ &= x^j_{(j,k)}\big(\gamma_{(i,j,k)} \circ \alpha^j_{(i,j)}(t)\big) = \tilde{x}^j_{(j,k)}\big((\alpha^j_{(j,k)})^{-1} \circ \gamma_{(i,j,k)} \circ \alpha^j_{(i,j)}(t)\big)\end{aligned}$$
for each $t \in \tilde{\mathbb{A}}_{(i,j)}$. □

(19.20) Corollary Let $\mathcal{U} := \mathcal{U}(\mathcal{B}, M, \Sigma, c)$ be a root group system, let $\mathcal{F} := \mathcal{F}(\mathcal{U}, \Lambda)$ for some parameter system Λ for $\mathcal{U}$ and let
$$\alpha = \{\pi, \alpha_{(i,j)} \mid (i,j) \in A(F)\} : \mathcal{F} \to \tilde{\mathcal{F}}$$
be an isomorphism. Then $\tilde{\mathcal{F}}$ is integrable.

Proof
Take $\big(\tilde{\mathbb{A}}_{(i,j)} := \tilde{\mathbb{A}}_{(\pi(i),\pi(j))}, (\alpha^i_{(i,j)})^{-1}, (\alpha^{ij}_{(i,j)})^{-1}, (\alpha^j_{(i,j)})^{-1}\big)$ as reparametrization for $\mathcal{T}(\mathbb{A}_{(i,j)})$, then replace $i \in I$ by $\pi(i) \in \tilde{I}$. The resulting parameter system $\tilde{\Lambda}$ satisfies

$$\mathcal{F}(\mathcal{U}, \tilde{\Lambda}) = \mathcal{F}_\alpha = \tilde{\mathcal{F}}\ .$$

□

(19.21) Theorem The isomorphism class of an integrable foundations $\mathcal{F} = \mathcal{F}(\mathcal{U}, \Lambda)$ is a classifying invariant of the isomorphism class of the corresponding building.

Proof
This results from corollary (19.20), proposition (19.16) and theorem (19.11). □

(19.22) Remark The following theorem shows that the concept of reparametrization is useful if we want to determine all the foundations isomorphic to a given foundation $\mathcal{F}$.

(19.23) Theorem Let $\mathcal{F}, \tilde{\mathcal{F}}$ be foundations with $F = \tilde{F}$. Then the following holds:

(a) Let $\tilde{\alpha} = \{\tilde{\alpha}_{(i,j)} \mid (i,j) \in A(F)\} : \mathcal{F} \to \tilde{\mathcal{F}}$ be a special isomorphism. Then there is a reparametrization α of $\mathcal{F}$ such that $\mathcal{F}_\alpha = \tilde{\mathcal{F}}$.

(b) Let $\alpha = \{\alpha_{(i,j)} \mid (i,j) \in A(F)$ be a reparametrization for $\mathcal{F}$ such that $\mathcal{F}_\alpha = \tilde{\mathcal{F}}$. Then there is a special isomorphism $\tilde{\alpha} : \mathcal{F} \to \tilde{\mathcal{F}}$.

Proof

(a) If we take $\alpha := \{\alpha_{(i,j)} \mid (i,j) \in A(F)\}$ with

$$\alpha_{(i,j)} := \{\tilde{\mathbb{A}}_{(i,j)}, (\tilde{\alpha}^i_{(i,j)})^{-1}, (\tilde{\alpha}^{ij}_{(i,j)})^{-1}, (\tilde{\alpha}^j_{(i,j)})^{-1}\}$$

as reparametrization for $\mathcal{F}$, then $\mathcal{F}_\alpha = \tilde{\mathcal{F}}$.

(b) We have

$$\alpha_{(i,j)} = (\tilde{\mathbb{A}}_{(i,j)}, \alpha^i_{(i,j)}, \alpha^{ij}_{(i,j)}, \alpha^j_{(i,j)})$$

for each $(i,j) \in A(F)$, thus $\tilde{\alpha} := \{\mathrm{id}_F, \tilde{\alpha}_{(i,j)} \mid (i,j) \in A(F)\} : \mathcal{F} \to \tilde{\mathcal{F}}$ with

$$\tilde{\alpha}_{(i,j)} := \big((\alpha^i_{(i,j)})^{-1}, (\alpha^{ij}_{(i,j)})^{-1}, (\alpha^j_{(i,j)})^{-1}\big)$$

is an isomorphism.

□

(19.24) Remark

(a) Let $\mathcal{F}$ and $\tilde{\mathcal{F}}$ be foundations and let

$$\alpha = \{\pi, \alpha_{(i,j)} \mid (i,j) \in A(F)\} : \mathcal{F} \to \tilde{\mathcal{F}}$$

be an isomorphism. As we may replace $i \in V(F)$ by $\pi(i) \in V(\tilde{F})$, we may consider α as special. Thus it suffices to determine all foundations which are specially isomorphic to $\mathcal{F}$. The remaining foundations isomorphic to $\mathcal{F}$ are obtained by relabelings of the vertex set.

(b) The theorem is useful if we want to show that two given foundations $\mathcal{F}$ and $\tilde{\mathcal{F}}$ with isomorphic residues $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are isomorphic. In this case we may replace $\mathcal{R}$ by $\tilde{\mathcal{R}}$, observing that there is a relabeling of the corresponding vertices involved.

§ 19.6 Glueings

At this point we collect some first results about glueings of integrable foundations.

(19.25) Notation Throughout the rest of this part, $\mathcal{F}$ denotes an integrable foundation.

(19.26) Theorem Let $(i,j) \in A(F)$. Then a Hua automorphism of $\mathcal{T}(\mathbb{A}_{(i,j)})$ is induced by an automorphism of $\mathcal{F}$.

Proof
This results from theorem (2.35). □

(19.27) Lemma Let $(i,j,k) \in G(F)$ and let $\mathbb{A} := \mathbb{A}_{(i,j)}$, $\tilde{\mathbb{A}} := \mathbb{A}_{(j,k)}$. Then the following holds:

(a) The glueing $\gamma := \gamma_{(i,j,k)}$ is a Jordan isomorphism. In particular, $\mathcal{F}$ is a Moufang foundation.

(b) If $\gamma_{(i,j,k)}$ is an isomorphism of alternative rings, then $\mathcal{T}(\tilde{\mathbb{A}}) \cong \mathcal{T}(\mathbb{A})$.

(c) If $\gamma_{(i,j,k)}$ is an anti-isomorphism of alternative rings, then $\mathcal{T}(\tilde{\mathbb{A}}) \cong \mathcal{T}(\mathbb{A}^o)$.

Proof

(a) If we set
$$h(t) := \mu\big(x^j_{(i,j)}(1_{\mathbb{A}})\big)^{-1}\mu\big(x^j_{(i,j)}(t)\big),\ t \in \mathbb{A}\ ,\quad \tilde{h}(\tilde{t}) := \mu\big(x^j_{(j,k)}(1_{\tilde{\mathbb{A}}})\big)^{-1}\mu\big(x^j_{(j,k)}(\tilde{t})\big),\ \tilde{t} \in \tilde{\mathbb{A}}\ ,$$
we have
$$\tilde{h}\big(\gamma(t)\big) = \mu\big(x^j_{(j,k)}(1_{\tilde{\mathbb{A}}})\big)^{-1}\mu\big(x^j_{(j,k)}(\gamma(t))\big) = \mu\big(x^j_{(i,j)}(1_{\mathbb{A}})\big)^{-1}\mu\big(x^j_{(i,j)}(t)\big) = h(t)$$
for each $t \in \mathbb{A}$. Moreover, we have
$$x^j_{(i,j)}(t)^{h(s)} = x^j_{(i,j)}(sts)\ , \qquad x^j_{(j,k)}(\tilde{t})^{\tilde{h}(\tilde{s})} = x^j_{(j,k)}(\tilde{s}\tilde{t}\tilde{s})$$
for all $s,t \in \mathbb{A}$, $\tilde{s},\tilde{t} \in \tilde{\mathbb{A}}$, cf. lemma (18.9). Combining these two facts yields
$$x^j_{(j,k)}\big(\gamma(s)\gamma(t)\gamma(s)\big) = x^j_{(j,k)}\big(\gamma(t)\big)^{\tilde{h}(\gamma(s))} = x^j_{(i,j)}(t)^{h(s)} = x^j_{(i,j)}(sts) = x^j_{(j,k)}\big(\gamma(sts)\big)$$
and therefore
$$\gamma(sts) = \gamma(s)\gamma(t)\gamma(s)$$
for all $s,t \in \mathbb{A}$, thus γ is a Jordan isomorphism.

(b) If γ is an isomorphism, then $(\mathbb{A},\gamma,\gamma,\gamma)$ is a reparametrization for $\mathcal{T}(\tilde{\mathbb{A}})$, thus $\mathcal{T}(\tilde{\mathbb{A}}) \cong \mathcal{T}(\mathbb{A})$.

(c) If γ is an anti-isomorphism, then $(\mathbb{A}^o,\gamma^o,\gamma^o,\gamma^o)$ is a reparametrization for $\mathcal{T}(\tilde{\mathbb{A}})$, thus $\mathcal{T}(\tilde{\mathbb{A}}) \cong \mathcal{T}(\mathbb{A}^o)$.

□

(19.28) Definition

- A glueing is *negative* if it is an isomorphism of alternative rings.
- A glueing is *positive* if it is an anti-isomorphism of alternative rings.
- A glueing is *exceptional* if it is neither positive nor negative.
- A foundation is *negative* if each glueing is negative.
- A foundation is *positive* if each glueing is positive.
- A foundation is *mixed* if there are both positive and negative glueings.

(19.29) Proposition A foundation

$$\mathcal{F} = \{\mathcal{T}(\mathbb{A}_{(1,2)}), \mathcal{T}(\mathbb{A}_{(2,3)}), \gamma := \gamma_{(1,2,3)}\}$$

of type A_3 is negative. Moreover, $\mathbb{A}$ (and thus $\tilde{\mathbb{A}}$) is associative.

Proof
If we set $\mathbb{A} := \mathbb{A}_{(1,2)}, \tilde{\mathbb{A}} := \mathbb{A}_{(2,3)}$,

$$x_1 := x^1_{(1,2)} \ , \qquad x_2 := x^2_{(1,2)} \ , \qquad \tilde{x}_2 := x^2_{(2,3)} \ , \qquad \tilde{x}_3 := \tilde{x}^3_{(2,3)}$$

and

$$\begin{aligned} h_1(t) &:= \mu\big(x_1(1_{\mathbb{A}})\big)^{-1}\mu\big(x_1(t^{-1})\big) \in U^1_{(1,2)}, \qquad t \in \mathbb{A} \ , \\ h_3(\tilde{t}) &:= \mu\big(\tilde{x}_3(1_{\tilde{\mathbb{A}}})\big)^{-1}\mu\big(\tilde{x}_3(\tilde{t}^{-1})\big) \in U^3_{(2,3)}, \qquad \tilde{t} \in \tilde{\mathbb{A}} \ , \end{aligned}$$

lemma (18.9) yields

$$x_2(s)^{h_1(t)} = x_2(t \cdot s) \ , \qquad \tilde{x}_2(\tilde{s})^{h_3(\tilde{t})} = \tilde{x}_2(\tilde{s} * \tilde{t})$$

for all $s, t \in \mathbb{A}$, $\tilde{s}, \tilde{t} \in \tilde{\mathbb{A}}$. Combining these two facts and observing $[U^1_{(1,2)}, U^3_{(2,3)}] = 1$ yield

$$\begin{aligned} \tilde{x}_2\big(\gamma(t) * \gamma(s)\big) &= \tilde{x}_2\big(\gamma(t)\big)^{h_3(\gamma(s))} = x_2(t)^{h_3(\gamma(s))} = x_2(1_{\mathbb{A}})^{h_1(t)h_3(\gamma(s))} \\ &= \tilde{x}_2(1_{\tilde{\mathbb{A}}})^{h_3(\gamma(s))h_1(t)} = \bar{x}_2\big(\gamma(s)\big)^{h_1(t)} = x_2(s)^{h_1(t)} = x_2(t \cdot s) = \tilde{x}_2\big(\gamma(t \cdot s)\big) \end{aligned}$$

and thus $\gamma(t \cdot s) = \gamma(t) * \gamma(s)$ for all $s, t \in \mathbb{A}$. As a consequence, we obtain

$$x_2(s \cdot t) = \tilde{x}_2\big(\gamma(s \cdot t)\big) = \tilde{x}_2\big(\gamma(s) * \gamma(t)\big) = \tilde{x}_2\big(\gamma(s)\big)^{h_3(\gamma(t))} = x_2(s)^{h_3(\gamma(t))}$$

for all $s, t \in \mathbb{A}$. This implies

$$\begin{aligned} x_2\big((s \cdot t) \cdot u\big) &= x_2(s \cdot t)^{h_3(\gamma(u))} = x_2(t)^{h_1(s)h_3(\gamma(u))} \\ &= x_2(t)^{h_3(\gamma(u))h_1(s)} = x_2(t \cdot u)^{h_1(s)} = x_2\big(s \cdot (t \cdot u)\big) \end{aligned}$$

and therefore $(s \cdot t) \cdot u = s \cdot (t \cdot u)$ for all $s, t, u \in \mathbb{A}$. $\square$

(19.30) Corollary Let $\mathcal{F}$ be a positive foundation. Then $\mathcal{G}_F$ is a complete graph.

(19.31) Theorem (**Hua's Theorem**) Let $\mathbb{A}$ and $\tilde{\mathbb{A}}$ be alternative division rings and let $\gamma : \mathbb{A} \to \tilde{\mathbb{A}}$ be a Jordan homomorphism. If $\tilde{\mathbb{A}}$ is associative, then $\gamma : \mathbb{A} \to \gamma(\mathbb{A})$ is an iso- or anti-isomorphism of alternative rings. In particular, $\mathbb{A}$ and $\gamma(\mathbb{A})$ are also skew-fields, and, if $\mathbb{A}$ or $\gamma(\mathbb{A})$ is a field, the map γ is an isomorphism of fields.

Proof

(i) We show: Given $s, t \in \mathbb{A}$, we have $\gamma(st) = \gamma(s)\gamma(t)$ or $\gamma(st) = \gamma(t)\gamma(s)$.
The assertion is clearly true for $s = 0_{\mathbb{A}}$ or $t = 0_{\mathbb{A}}$, so assume $s \neq 0_{\mathbb{A}} \neq t$. As we have

$$\gamma(u^2) = \gamma(u \cdot 1_{\mathbb{A}} \cdot u) = \gamma(u)\gamma(1_{\mathbb{A}})\gamma(u) = \gamma(u) \cdot 1_{\tilde{\mathbb{A}}} \cdot \gamma(u) = \gamma(u)^2$$

for each $u \in \mathbb{A}$, it follows that

$$\begin{aligned} \gamma\big((s+t)^2\big) &= \gamma(s^2 + st + ts + t^2) = \gamma(s)^2 + \gamma(st) + \gamma(ts) + \gamma(t)^2 \ , \\ \gamma\big((s+t)^2\big) &= \gamma(s+t)^2 = \big(\gamma(s) + \gamma(t)\big)^2 = \gamma(s)^2 + \gamma(s)\gamma(t) + \gamma(t)\gamma(s) + \gamma(t)^2 \end{aligned}$$

and thus

$$\gamma(st) + \gamma(ts) = \gamma(s)\gamma(t) + \gamma(t)\gamma(s) \ . \tag{19.2}$$

On the other hand, we have

$$s\big(t(st)^{-1}t\big)s = s([t(t^{-1}s^{-1})]t)s = [s(s^{-1}t)]s = ts$$

by the Moufang identities and the inverse properties and, by lemma (27.3), therefore,

$$\gamma(ts) = \gamma\big(s(t(st)^{-1}t)s\big) = \gamma(s)\big(\gamma(t)\gamma(st)^{-1}\gamma(t)\big)\gamma(s) \ . \tag{19.3}$$

Observing the associativity of $\tilde{\mathbb{A}}$, we obtain

$$\begin{aligned}
&[1_{\tilde{\mathbb{A}}} - \gamma(s)\gamma(t)\gamma(st)^{-1}][\gamma(st) - \gamma(t)\gamma(s)] \\
&= \gamma(st) - \gamma(s)\gamma(t) - \gamma(t)\gamma(s) + \gamma(s)\gamma(t)\gamma(st)^{-1}\gamma(t)\gamma(s) \\
&\overset{(19.3)}{=} \gamma(st) - \gamma(s)\gamma(t) - \gamma(t)\gamma(s) + \gamma(ts) \overset{(19.2)}{=} 0_{\tilde{\mathbb{A}}} \ .
\end{aligned}$$

Since $\tilde{\mathbb{A}}$ has no zero divisors, it finally follows that

$$1_{\tilde{\mathbb{A}}} - \gamma(s)\gamma(t)\gamma(st)^{-1} = 0_{\tilde{\mathbb{A}}} \qquad \vee \qquad \gamma(st) - \gamma(t)\gamma(s) = 0_{\tilde{\mathbb{A}}} \ .$$

(ii) Given $s \in \mathbb{A}$, let

$$N_s := \{t \in \mathbb{A} \mid \gamma(st) = \gamma(s)\gamma(t)\} \ , \qquad P_s := \{t \in \mathbb{A} \mid \gamma(st) = \gamma(t)\gamma(s)\} \ .$$

By step (i), the subgroups N_s and P_s of $(\mathbb{A}, +)$ satisfy $\mathbb{A} = N_s \cup P_s$. As no group is the union of two proper subgroups, we obtain

$$N_s = \mathbb{A} \qquad \vee \qquad P_s = \mathbb{A} \ .$$

(iii) If we set

$$N := \{s \in \mathbb{A} \mid N_s = \mathbb{A}\} \ , \qquad P := \{s \in \mathbb{A} \mid P_s = \mathbb{A}\} \ ,$$

step (ii) shows that N and P are subgroups of $(\mathbb{A}, +)$ satisfying $\mathbb{A} = N \cup P$, hence

$$N = \mathbb{A} \qquad \vee \qquad P = \mathbb{A} \ .$$

□

(19.32) Remark If we suppose $\mathbb{A}$ instead of $\tilde{\mathbb{A}}$ to be associative and if $\gamma(\mathbb{A})$ is an alternative division ring, the theorem remains true as we may apply it to $\gamma^{-1} : \gamma(\mathbb{A}) \to \mathbb{A}$.

(19.33) Corollary Let $\mathcal{F}$ be a foundation such that there exists an edge $(a, b) \in A(F)$ with $\mathbb{A} := \mathbb{A}_{(a,b)}$ associative. Then we have

$$\forall\ (i,j) \in A(F) : \quad \mathbb{A}_{(i,j)} \cong \mathbb{A} \ \vee \ \mathbb{A}_{(i,j)} \cong \mathbb{A}^o \ , \quad \mathcal{T}(\mathbb{A}_{(i,j)}) \cong \mathcal{T}(\mathbb{A}) \ \vee \ \mathcal{T}(\mathbb{A}_{(i,j)}) \cong \mathcal{T}(\mathbb{A}^o) \ .$$

In particular, the alternative division ring $\mathbb{A}_{(i,j)}$ is associative for each $(i, j) \in A(F)$. Moreover, the assumption is satisfied if $\mathcal{F}$ has a residue of type A_3.

Proof
This results from lemma (19.27) (a), Hua's theorem and lemma (19.27) (b), (c), using an easy induction. The final assertion results from proposition (19.29). □

(19.34) Corollary Let $\mathcal{F}$ be a foundation such that there exists an edge $(a, b) \in A(F)$ with $\mathbb{A} := \mathbb{A}_{(a,b)}$ non-associative, i.e., $\mathbb{A}$ is an octonion division algebra. Then $\mathcal{G}_F$ is a complete graph.

Proof
By proposition (19.29), $\mathcal{F}$ satisfies the assumption of corollary (19.33) if it has a residue of type A_3. □

Chapter 20 Integrability of Certain Foundations

In this chapter, we prove the integrability of certain foundations which are not negative or whose defining field is an octonion division algebra. Afterwards we will prove that these foundations are the only integrable foundations of this type, all of them defined over an octonion or quaternion division algebra. For this purpose, we need the theory of fixed point structures developed in [MHab].

The idea is as follows: Given an integrable foundation $\mathcal{F}$ and an automorphism α of the corresponding twin building $\mathcal{B}$, the foundation $\tilde{\mathcal{F}}$ corresponding to the twin building $\tilde{\mathcal{B}} := \mathrm{Fix}(\alpha)$ can be constructed out of $\mathcal{F}$ and is, in fact, the fixed point structure of the automorphism induced on the group generated by all the root groups. Conversely, all such fixed point structures arise in this way, i.e., if we can realize a given foundation $\tilde{\mathcal{F}}$ as the fixed point structure of an automorphism α of an integrable foundation $\mathcal{F}$, then $\tilde{\mathcal{F}}$ itself is integrable.

We apply the so called theory of Tits indices without introducing it in detail since this would involve a lot of technical considerations. We just follow the recipes, hopefully in a natural and intuitive way, except for the canonical triangle over an octonion division algebra, where we just indicate the main idea.

But first of all we start with some general integrability criterions. In combination with Kac-Moody theory, they allow us to handle all the foundations that are defined over skew-fields distinct from a quaternion division algebra (which includes fields).

§ 20.1 Integrability Criterions

The first one is a straight forward result as each residue of a twin building is again a twin building, the second one is taken from [MLoc], and the third relies on Kac-Moody theory.

(20.1) Theorem Let M be a Coxeter matrix over $I = V(M)$, let $\mathcal{F}$ be an integrable foundation of type M and let $J \subseteq I$ such that $|J| \geq 2$. Then $\mathcal{F}_J$ is integrable.

Proof
By assumption, we have $\mathcal{F} = \mathcal{F}(\mathcal{U}, \Lambda)$ for some root group system $\mathcal{U} = \mathcal{U}(\mathcal{B}, M, \Sigma, c)$ and some parameter system Λ. By theorem (2.14), $\mathcal{B}_J := \mathcal{B}_J(c)$ is a twin building, and by theorem (2.7) (b), $\Sigma_J := \Sigma_J(c)$ is a twin apartment of $\mathcal{B}_J$ containing c. Since we have

$$\forall\ i,j \in A(M_J): \qquad \Sigma \cap \mathcal{B}_{\{i,j\}}(c) = \Sigma_J \cap (\mathcal{B}_J)_{\{i,j\}}(c)\ ,$$

we have

$$\mathcal{U}_J := \mathcal{U}(\mathcal{B}_J, M_J, \Sigma_J, c) = \{U_{(i,j)} \mid i,j \in A(M_J)\}$$

and thus $\mathcal{F}_J = \mathcal{F}(\mathcal{U}_J, \Lambda_J)$, where Λ_J is the parameter system induced by Λ. □

(20.2) Definition Let $\mathcal{F}$ be a foundation.

- Let $(\tilde{F}, \varphi)$ be a cover of F. Then the foundation

  $$\mathcal{F}(\tilde{F}, \varphi) := \{\mathcal{T}(\tilde{\mathbb{A}}_{(i,j)}), \tilde{\gamma}_{(i,j,k)} \mid (i,j) \in A(\tilde{F}), (i,j,k) \in G(\tilde{F})\}$$

  with

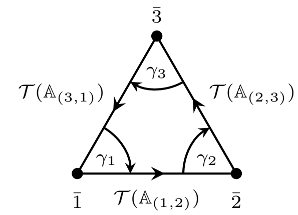
  $$\forall\ (i,j) \in A(\tilde{F}):\ \tilde{\mathbb{A}}_{(i,j)} = \mathbb{A}_{(\varphi(i),\varphi(j))}\ , \qquad \forall\ (i,j,k) \in G(\tilde{F}):\ \tilde{\gamma}_{(i,j,k)} = \gamma_{(\varphi(i),\varphi(j),\varphi(k))}$$

  is the *cover corresponding to* $(\tilde{F}, \varphi)$.

- A foundation $\tilde{\mathcal{F}}$ is a *cover of* $\mathcal{F}$ if there is a cover $(\tilde{F}, \varphi)$ of F such that

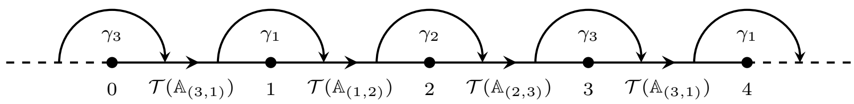
  $$\tilde{\mathcal{F}} \cong \mathcal{F}(\tilde{F}, \varphi)\ .$$

(20.3) Example Given the foundation



,

a cover is given by



,

where $\varphi : \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z},\ z \mapsto \bar{z}$ is the natural homomorphism.

(20.4) Theorem Let $\mathcal{F}$ be a foundation and let $\tilde{\mathcal{F}}$ be a cover of $\mathcal{F}$. Then $\mathcal{F}$ is integrable if $\tilde{\mathcal{F}}$ is integrable.

Proof
This is a consequence of theorem C in [MLoc]. □

(20.5) Definition A foundation $\mathcal{F}$ such that

$$\forall\ (i,j,k) \in G(F): \qquad \gamma_{(i,j,k)} = \mathrm{id}$$

is a *canonical* foundation.

(20.6) Lemma Let $\mathcal{F}$ be a negative foundation such that $\mathcal{G}_F$ is a tree. Then $\mathcal{F}$ is isomorphic to the corresponding canonical foundation.

Proof
Since $\mathcal{G}_F$ is a tree, it suffices to show the following: Given $(i,j,k) \in G(F)$, there is a reparametrization

$$\alpha_{(j,k)} = (\mathbb{A}_{(i,j)}, \alpha^{j}_{(j,k)}, \alpha^{jk}_{(j,k)}, \alpha^{k}_{(j,k)})$$

such that $(\alpha^{j}_{(j,k)})^{-1} \circ \gamma_{(i,j,k)} = \mathrm{id}_{\mathbb{A}_{(i,j)}}$. This holds for $\alpha_{(j,k)} := (\mathbb{A}_{(i,j)}, \gamma_{(i,j,k)}, \gamma_{(i,j,k)}, \gamma_{(i,j,k)})$. □

(20.7) Theorem Let $\mathcal{F}$ be a canonical foundation such that one of the following holds:

(a) The defining field $\mathbb{A}$ (cf. definition (21.7)) is a field, and F is a tree.

(b) The defining field is a non-commutative skew-field, and F is a string, a ray or a chain.

Then $\mathcal{F}$ is integrable.

Proof

(a) Kac-Moody theory provides the existence of an integrable foundation $\tilde{\mathcal{F}}$ over $\mathbb{A}$ such that $\tilde{F} = F$, cf. [T92]. Since $\tilde{\mathcal{F}}$ is negative by proposition (19.29), we have $\tilde{\mathcal{F}} \cong \mathcal{F}$ by lemma (20.6). Therefore, $\mathcal{F}$ is integrable by corollary (19.20).

(b) The corresponding twin building is the limit of a sequence of twin buildings $A_n(\mathbb{A})$.

□

§ 20.2 The Canonical Triangle over an Octonion Division Algebra

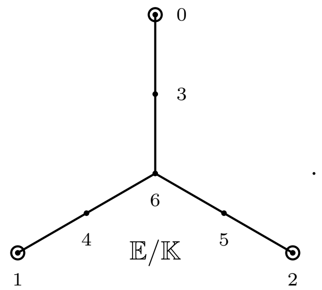
(20.8) Definition Let $\mathbb{O}$ be an octonion division algebra. The foundation

$$\tilde{A}_2(\mathbb{O}) := \{\mathcal{T}(\mathbb{A}_{(1,2)}) := \mathcal{T}(\mathbb{A}_{(2,3)}) := \mathcal{T}(\mathbb{A}_{(3,1)}) := \mathcal{T}(\mathbb{O}), \gamma_{(1,2,3)} := \gamma_{(2,3,1)} := \gamma_{(3,1,2)} := \mathrm{id}_{\mathbb{O}}\}$$

is the *canonical triangle over* $\mathbb{O}$.

(20.9) Theorem The foundation $\tilde{A}_2(\mathbb{O})$ is integrable.

Proof **(Sketch)**
Let $\mathbb{K} := Z(\mathbb{O})$ and let $\mathbb{E} \subseteq \mathbb{O}$ be such that $\mathbb{E}/\mathbb{K}$ is a quadratic separable extension. The foundation $\tilde{A}_2(\mathbb{O})$ is the fixed point structure of the Tits index



Let $\mathcal{B}$ be the twin building associated with the diagram, let $\mathcal{R} := \mathcal{R}_{\{3,4,5,6\}}$ and let τ be the triality associated with $\mathcal{R}$. Given $k \in \{0,1,2\}$, let $\bar{k} := \{0,\dots,6\} \setminus \{k\}$ and $\tilde{k} := \{0,1,2\} \setminus \{k\}$. By similar arguments as in [MGeo], there are automorphisms $\varphi_k \in \mathrm{Aut}(\mathcal{R}_{\bar{k}})$ such that

$$\mathrm{Fix}(\varphi_k) \cong \mathcal{T}(\mathbb{O}) \ , \qquad \varphi_k(R_{\tilde{k}}) = R_{\tilde{k}} \ , \qquad \varphi_{k|\mathcal{R}} \circ \tau = \tau \circ \varphi_{k|\mathcal{R}} \ ,$$

and φ_0 can be extended to an automorphism $\varphi \in \mathrm{Aut}(\mathcal{B})$ such that $\varphi_{|\mathcal{R}_{\bar{k}}} = \varphi_k$. By a generalization of the arguments in chapter 3 of [MPhD], the fixed point set of φ is a twin building of type $\tilde{A}_2$ whose foundation is $\tilde{A}_2(\mathbb{O})$. □

§ 20.3 Positive and Mixed Foundations over Quaternions

(20.10) Notation Throughout this paragraph, $\mathbb{H} := (\mathbb{E}/\mathbb{K}, \beta)$ is a quaternion division algebra with standard involution σ_s.

(20.11) Notation Let $B := \left(\begin{smallmatrix} 0 & \beta \\ 1 & 0 \end{smallmatrix}\right) \in GL_2(\mathbb{E})$ and let

$$\sigma : M_2(\mathbb{E}) \to M_2(\mathbb{E}),\ X \mapsto B\bar{X}B^{-1} \ .$$

(20.12) Lemma We have

$$\mathrm{Fix}(\sigma) = \left\{ \begin{pmatrix} s & \beta\bar{t} \\ t & \bar{s} \end{pmatrix} \mid s,t \in \mathbb{E} \right\} \cong \mathbb{H} \ .$$

Proof
Given $X := \left(\begin{smallmatrix} s & u \\ t & v \end{smallmatrix}\right) \in M_2(\mathbb{E})$, we have

$$X \in \mathrm{Fix}(\sigma) \ \Leftrightarrow \ \begin{pmatrix} \bar{v} & \beta\bar{t} \\ \beta^{-1}\bar{u} & \bar{s} \end{pmatrix} = \begin{pmatrix} s & u \\ t & v \end{pmatrix} \ \Leftrightarrow \ v = \bar{s} \ \wedge \ u = \beta\bar{t} \ \Leftrightarrow \ X \in \mathbb{H} \ .$$

□

(20.13) Corollary If we extend σ to an involution

$$\sigma : M_6(\mathbb{E}) \to M_6(\mathbb{E}),\ X \mapsto \begin{pmatrix} B & & \\ & B & \\ & & B \end{pmatrix} \bar{X} \begin{pmatrix} B^{-1} & & \\ & B^{-1} & \\ & & B^{-1} \end{pmatrix} ,$$

we have

$$\bar{U}_+ := \mathrm{Fix}(\sigma) \cap U_+ = \left\{ \begin{pmatrix} I_2 & s & t \\ & I_2 & u \\ & & I_2 \end{pmatrix} \mid s, t, u \in \mathbb{H} \right\} .$$

(20.14) Remark

(a) We translate the map σ to root groups:

On the group $\langle U_{1,2}, U_{2,3}, U_{3,4}, U_{2,1}, U_{3,2}, U_{4,3} \rangle$, the map σ is given by

$$\begin{aligned} &x_{1,2}(t) \mapsto x_{2,1}(\beta^{-1}\bar{t}) , && x_{1,3}(t) \mapsto x_{2,4}(\bar{t}) , \\ &x_{2,3}(t) \mapsto x_{1,4}(\beta\bar{t}) , && x_{2,4}(t) \mapsto x_{1,3}(\bar{t}) , \\ &x_{3,4}(t) \mapsto x_{4,3}(\beta^{-1}\bar{t}) , && x_{1,4}(t) \mapsto x_{2,3}(\beta^{-1}\bar{t}) . \end{aligned}$$

Moreover, the fixed point set in $\langle U_{1,2}, U_{2,3}, U_{3,4} \rangle$ is

$$\{ y^a(s,t) := x_{1,3}(s) x_{1,4}(\beta\bar{t}) x_{2,3}(t) x_{2,4}(\bar{s}) \mid (s,t) \in \mathbb{H} \} .$$

On the group $\langle U_{3,4}, U_{4,5}, U_{5,6}, U_{4,3}, U_{5,4}, U_{6,5} \rangle$, the map σ is given by

$$\begin{aligned} &x_{3,4}(t) \mapsto x_{4,3}(\beta^{-1}\bar{t}) , && x_{3,5}(t) \mapsto x_{4,6}(\bar{t}) , \\ &x_{4,5}(t) \mapsto x_{3,6}(\beta\bar{t}) , && x_{4,6}(t) \mapsto x_{3,5}(\bar{t}) , \\ &x_{5,6}(t) \mapsto x_{6,5}(\beta^{-1}\bar{t}) , && x_{3,6}(t) \mapsto x_{4,5}(\beta^{-1}\bar{t}) . \end{aligned}$$

Moreover, the fixed point set in $\langle U_{3,4}, U_{4,5}, U_{5,6} \rangle$ is

$$\{ y^b(s,t) := x_{3,5}(s) x_{3,6}(\beta\bar{t}) x_{4,5}(t) x_{4,6}(\bar{s}) \mid (s,t) \in \mathbb{H} \} .$$

If we set

$$\{ y^{ab}(s,t) := x_{1,5}(s) x_{1,6}(\beta\bar{t}) x_{2,5}(t) x_{2,6}(\bar{s}) \mid (s,t) \in \mathbb{H} \} \subseteq \bar{U}_+ ,$$

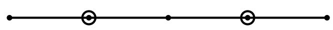
we have

$$[y^a(s_1,t_1), y^b(s_2,t_2)] = y^{ab}\big((s_1,t_1) \cdot (s_2,t_2)\big)$$

for all $(s_1,t_1), (s_2,t_2) \in \mathbb{H}$, hence

$$\big(\bar{U}_+, y^a(\mathbb{H}), y^{ab}(\mathbb{H}), y^b(\mathbb{H})\big) \cong \mathcal{T}(\mathbb{H}) .$$

(b) The corresponding Tits index is



.

$\mathbb{E}/\mathbb{K}$

(20.15) Theorem Let $\mathcal{G}$ be a complete graph over $I := \{1, \ldots, n\}$. Then there exists an integrable positive foundation $\mathcal{F} := \mathcal{P}_n^+(\mathbb{H})$ over $\mathbb{H}$ such that $\mathcal{G}_F = \mathcal{G}$.

More precisely: For each $i \in I$, let $\mathcal{B}_i$ be a a copy of $\tilde{\mathcal{B}}$, and for all $1 \leq i < j \leq n$ let $\mathcal{B}_{\{i,j\}}$ be a copy of $\mathcal{B}$. We identify $\mathcal{B}_i$ with $\mathcal{B}_{\{i,j\}}$ as in the first case and $\mathcal{B}_j$ with $\mathcal{B}_{\{i,j\}}$ as in the second case. Then the fixed point foundation $\mathcal{F} := \mathcal{P}_n^+(\mathbb{H})$ is a positive foundation with defining field $\mathbb{H}$ (cf. definition (21.7)) and $\mathcal{G}_F = \mathcal{G}$. Moreover, we have $\gamma_{(i,j,k)} \in \{\sigma_s, \mathrm{id}^o\}$ for each $(i,j,k) \in G(F)$.

Proof
Let

$$\mathcal{B} := \big(A_5(\mathbb{E}) = (U_+, U_{1,2}, U_{2,3}, U_{3,4}, U_{4,5}, U_{5,6}), \sigma\big)\ , \quad \tilde{\mathcal{B}} := \big(A_3(\mathbb{E}) = (\tilde{U}_+, \tilde{U}_{1,2}, \tilde{U}_{2,3}, \tilde{U}_{3,4}), \tilde{\sigma}\big)$$

with the usual parametrizations.

(i) We identify $\tilde{U}_+$ and $U_{1,2}U_{2,3}U_{3,4}$ via

$$\begin{array}{lll} \tilde{x}_{1,2}(t) \mapsto x_{1,2}(t)\ , & \tilde{x}_{2,3}(t) \mapsto x_{2,3}(t)\ , & \tilde{x}_{3,4}(t) \mapsto x_{3,4}(t)\ , \\ \tilde{x}_{1,3}(t) \mapsto x_{1,3}(t)\ , & \tilde{x}_{2,4}(t) \mapsto x_{2,4}(t)\ , & \tilde{x}_{1,4}(t) \mapsto x_{1,4}(t)\ . \end{array}$$

(ii) We identify $\tilde{U}_+$ and $U_{3,4}U_{4,5}U_{5,6}$ via

$$\begin{array}{lll} \tilde{x}_{1,2}(t) \mapsto x_{5,6}(t)\ , & \tilde{x}_{2,3}(t) \mapsto x_{4,5}(-t)\ , & \tilde{x}_{3,4}(t) \mapsto x_{3,4}(t)\ , \\ \tilde{x}_{1,3}(t) \mapsto x_{4,6}(t)\ , & \tilde{x}_{2,4}(t) \mapsto x_{3,5}(t)\ , & \tilde{x}_{1,4}(t) \mapsto x_{3,6}(-t)\ . \end{array}$$

We have

$$\forall\ t \in \mathbb{E}: \qquad \alpha\big(\tilde{x}_{i,j}(t)\big)^{\sigma} = \alpha\big(\tilde{x}_{i,j}(t)^{\tilde{\sigma}}\big)$$

for all $1 \leq i < j \leq 3$ and therefore

$$\alpha\big(\mathrm{Fix}(\bar{\sigma})\big) = \mathrm{Fix}_{U_a U_b U_c}(\sigma)\ .$$

Thus we may identify $\mathrm{Fix}(\bar{\sigma})$ and $\mathrm{Fix}_{U_a U_b U_c}(\sigma)$ via

(i) $\tilde{y}(s,t) := \tilde{x}_{1,3}(s)\tilde{x}_{1,4}(\beta\bar{t})\tilde{x}_{2,3}(t)\tilde{x}_{2,4}(\bar{s}) \mapsto x_{1,3}(s)x_{1,4}(\beta\bar{t})x_{2,3}(t)x_{2,4}(\bar{s}) = y^a(s,t)$.

(ii) $\tilde{y}(s,t) := \tilde{x}_{1,3}(s)\tilde{x}_{1,4}(\beta\bar{t})\tilde{x}_{2,3}(t)\tilde{x}_{2,4}(\bar{s}) \mapsto x_{4,6}(s)x_{3,6}(-\beta\bar{t})x_{4,5}(-t)x_{3,5}(\bar{s}) = y^b(\bar{s},-t)$.

So if we have two copies $\mathcal{B}_1, \mathcal{B}_2$ of $\mathcal{B}$ and if we identify $\tilde{\mathcal{B}}$ with subgroups of $\mathcal{B}_1, \mathcal{B}_2$ as above, we have:

- $y_1^a(s,t) = y_2^a(s,t)$ if both the identifications are of type (i). Moreover, we have

  $$\big(y_2^b(\mathbb{H}^o), y_2^{ab}(\mathbb{H}^o), y_2^a(\mathbb{H}^o)\big) \cong \mathcal{T}(\mathbb{H}^o)\ , \qquad \big(y_1^a(\mathbb{H}), y_1^{ab}(\mathbb{H}), y_1^b(\mathbb{H})\big) \cong \mathcal{T}(\mathbb{H})$$

  and therefore $\gamma_{(2,1)} = \mathrm{id}^o : \mathbb{H}^o \to \mathbb{H}$.

- $y_1^b(s,t) = y_2^b(s,t)$ if both the identifications are of type (ii). Moreover, we have

  $$\big(y_1^a(\mathbb{H}), y_1^{ab}(\mathbb{H}), y_1^b(\mathbb{H})\big) \cong \mathcal{T}(\mathbb{H})\ , \qquad \big(y_2^b(\mathbb{H}^o), y_2^{ab}(\mathbb{H}^o), y_2^a(\mathbb{H}^o)\big) \cong \mathcal{T}(\mathbb{H}^o)$$

  and therefore $\gamma_{(1,2)} = \mathrm{id}^o : \mathbb{H} \to \mathbb{H}^o$.

- $y_1^a(s,t) = y_2^b(\bar{s},-t)$ if the identifications are of different type. Moreover, we have

  $$\big(y_2^a(\mathbb{H}), y_2^{ab}(\mathbb{H}), y_2^b(\mathbb{H})\big) \cong \mathcal{T}(\mathbb{H}) \cong \big(y_1^a(\mathbb{H}), y_1^{ab}(\mathbb{H}), y_1^b(\mathbb{H})\big)$$

  and therefore $\gamma_{(2,1)} = \sigma_s : \mathbb{H} \to \mathbb{H}$.

$\square$

(20.16) Remark We obtain integrable mixed foundations over quaternions by glueing integrable positive foundations in a suitable way.

(20.17) Theorem Let $\mathcal{G}$ be a graph over I and let P be a collection of finite complete subgraphs such that

$$\bigcup_{\mathcal{P}\in P} \mathcal{P} = \mathcal{G} \ , \quad \forall\, \mathcal{P}_1 \neq \mathcal{P}_2 \in P : |V(\mathcal{P}_1) \cap V(\mathcal{P}_2)| \leq 1 \ , \quad \forall\, i \in I : |\{\mathcal{P} \in P \mid i \in V(\mathcal{P})\}| \leq 2$$

and such that the graph $\mathcal{G}^P$ with vertex set P and

$$\{\mathcal{P}_1, \mathcal{P}_2\} \in E(\mathcal{G}^P) \ :\Leftrightarrow \ |V(\mathcal{P}_1) \cap V(\mathcal{P}_2)| = 1$$

is a tree. Then there exists an integrable mixed foundation $\mathcal{F} := \mathcal{F}(\mathcal{G}^P, \mathbb{H})$ over $\mathbb{H}$ such that:

- $\mathcal{G}_F = \mathcal{G}$,
- The $V(\mathcal{P})$-residue is isomorphic to $\mathcal{P}^+_{|V(\mathcal{P})|}(\mathbb{H})$ for each $\mathcal{P} \in P$.
- Given $\mathcal{P}_1, \mathcal{P}_2 \in P$ such that $\mathcal{P}_1 \cap \mathcal{P}_2 = \{j\}$, we have

$$\forall\, (i,k) \in \big(V(\mathcal{P}_1) \setminus \{j\}\big) \times \big(V(\mathcal{P}_2) \setminus \{j\}\big) : \qquad \gamma_{(i,j,k)} \in \{\mathrm{id}, \sigma_s^o\} \ ,$$

  i.e., these glueings are negative.

Proof
Given a graph $\mathcal{P} \in P$, there is an integrable positive foundation

$$\mathcal{P}^+_{|V(\mathcal{P})|}(\mathbb{H})$$

which can be realized as fix foundation of a foundation $\mathcal{B}_{\mathcal{P}}$. We likewise want to realize the desired foundation $\mathcal{F}$ as a fixed point foundation. For this purpose, we will connect the foundations $\mathcal{B}_{\mathcal{P}}$ in a suitable way. Since $\mathcal{G}^P$ is a tree and $|\{\mathcal{P} \in P \mid i \in V(\mathcal{P})\}| \leq 2$ for each vertex $i \in I$, it suffices to prove the assertion for $|P| = 2$.

Let $P = \{\mathcal{P}_1, \mathcal{P}_2\}$, let $V(\mathcal{P}_1) \cap V(\mathcal{P}_2) = \{j\}$, for $\lambda = 1, 2$ let $n_\lambda := |V(\mathcal{P}_\lambda)|$, let j_λ be the copy of j in $\mathcal{P}_\lambda$ and let $\tilde{\mathcal{B}}_{j_\lambda}$ be the corresponding copy of $\tilde{\mathcal{B}}$ in the construction of $\mathcal{B}_{\mathcal{P}_\lambda}$ in theorem (20.15). We identify $\tilde{\mathcal{B}}_{j_1}$ and $\tilde{\mathcal{B}}_{j_2}$ via

$$\tilde{x}^{j_1}_{1,2}(t) \mapsto \tilde{x}^{j_2}_{3,4}(t) \ , \qquad \tilde{x}^{j_1}_{2,3}(t) \mapsto \tilde{x}^{j_2}_{2,3}(-t) \ , \qquad \tilde{x}^{j_1}_{3,4}(t) \mapsto \tilde{x}^{j_2}_{1,2}(t) \ .$$

If $\mathcal{B}_{\{i,j_1\}}$ and $\mathcal{B}_{\{j_2,k\}}$ are copies of $\mathcal{B}$ in the construction of $\mathcal{B}_{\mathcal{P}_1}$ and $\mathcal{B}_{\mathcal{P}_2}$, respectively, we have the following possibilities:

- $y^a_{j_1}(s,t) = y^a_{j_2}(\bar{s}, -t)$ if both the identifications are of type (i). Moreover, we have

$$\big(y^b_{j_1}(\mathbb{H}^o), y^{ab}_{j_1}(\mathbb{H}^o), y^a_{j_1}(\mathbb{H}^o)\big) \cong \mathcal{T}(\mathbb{H}^o) \ , \qquad \big(y^a_{j_2}(\mathbb{H}), y^{ab}_{j_2}(\mathbb{H}), y^b_{j_2}(\mathbb{H})\big) \cong \mathcal{T}(\mathbb{H})$$

  and therefore $\gamma_{(i,j,k)} = \sigma_s^o : \mathbb{H}^o \to \mathbb{H}$.

- $y^b_{j_1}(s,t) = y^b_{j_2}(\bar{s}, -t)$ if both the identifications are of type (ii). Moreover, we have

$$\big(y^a_{j_1}(\mathbb{H}), y^{ab}_{j_1}(\mathbb{H}), y^b_{j_1}(\mathbb{H})\big) \cong \mathcal{T}(\mathbb{H}) \ , \qquad \big(y^b_{j_2}(\mathbb{H}^o), y^{ab}_{j_2}(\mathbb{H}^o), y^a_{j_2}(\mathbb{H}^o)\big) \cong \mathcal{T}(\mathbb{H}^o)$$

  and therefore $\gamma_{(i,j,k)} = \sigma_s^o : \mathbb{H} \to \mathbb{H}^o$.

- $y^b_{j_1}(s,t) = y^a_{j_2}(s,t)$ if both the identifications are of different type. Moreover, we have

$$\big(y^a_{j_1}(\mathbb{H}), y^{ab}_{j_1}(\mathbb{H}), y^b_{j_1}(\mathbb{H})\big) \cong \mathcal{T}(\mathbb{H}) \cong \big(y^a_{j_2}(\mathbb{H}), y^{ab}_{j_2}(\mathbb{H}), y^b_{j_2}(\mathbb{H})\big)$$

  and therefore $\gamma_{(i,j,k)} = \mathrm{id} : \mathbb{H} \to \mathbb{H}$.

By construction, we have

$$\mathcal{G}_F = \mathcal{P}_1 \cup \mathcal{P}_2 = \mathcal{G} \ .$$

□

Chapter 21 Triangle Foundations

The smallest building bricks of foundations involving glueings are A_3- and $\tilde{A}_2$-residues. Since there are $\tilde{A}_2$-foundations which do not have an integrable cover, it is not enough to consider A_3-residues. We will show that the only $\tilde{A}_2$-residues without an integrable cover are those constructed in chapter 20.

(21.1) Notation Throughout this chapter, $\mathcal{B}$ is of type $\tilde{A}_2$ with foundation

$$\mathcal{F} = \{\mathcal{T}(\mathbb{A}_{(1,2)}), \mathcal{T}(\mathbb{A}_{(2,3)}), \mathcal{T}(\mathbb{A}_{(3,1)}), \gamma_2 := \gamma_{(1,2,3)}, \gamma_3 := \gamma_{(2,3,1)}, \gamma_1 := \gamma_{(3,1,2)}\} .$$

We denote its building at infinity by $\mathcal{B}^\infty$.

(21.2) Remark Since $\mathcal{B}$ is a Moufang twin building, it is a Bruhat-Tits building by [VV], i.e., $\mathcal{B}^\infty$ is a Moufang triangle $\mathcal{T}(\mathbb{A}^\infty)$ for some alternative division ring $\mathbb{A}^\infty$. As a consequence, we may apply the results of [W].

§ 21.1 The Defining Field

First of all we show that each A_2-residue is isomorphic to the same Moufang triangle $\mathcal{T}(\mathbb{A})$, up to opposition. Then we prove that there is an embedding $\mathbb{A} \hookrightarrow \mathbb{A}^\infty$.

(21.3) Proposition There is an automorphism $\alpha \in \mathrm{Aut}(\mathcal{B})$ inducing $1 \mapsto 2 \mapsto 3 \mapsto 1$ on $\mathcal{G}_F$.

Proof
By corollary (18.15) of [W], $\mathrm{Aut}(\mathcal{B})$ acts transitively on the set of gems, hence on the vertices and thus on I. As $|I| = 3$ we are done. □

(21.4) Corollary There is a (1 2 3)-automorphism of $\mathcal{F}$.

Proof
Let $\mathcal{U} := \mathcal{U}(\mathcal{B}, F, \Sigma, c)$ be a root group system such that $\mathcal{F} = \mathcal{F}(\mathcal{U}, \Lambda)$ for some parameter system Λ for $\mathcal{U}$ and let $\alpha \in \mathrm{Aut}(\mathcal{B})$ be as in proposition (21.3). Then α induces a (1 2 3)-isomorphism from $\mathcal{U}$ to $\mathcal{U}(\mathcal{B}, F, \alpha(\Sigma), \alpha(c))$ which is specially isomorphic to $\mathcal{U}$ by theorem (19.10). By proposition (19.16), there is a (1 2 3)-automorphism of $\mathcal{U}$ which induces a (1 2 3)-automorphism of $\mathcal{F}$. □

(21.5) Corollary We have $\mathbb{A} := \mathbb{A}_{(1,2)} \cong \mathbb{A}_{(2,3)} \cong \mathbb{A}_{(3,1)}$.

Proof
By corollary (21.4), we have

$$\mathcal{T}(\mathbb{A}_{(1,2)}) \cong \mathcal{T}(\mathbb{A}_{(2,3)}) \cong \mathcal{T}(\mathbb{A}_{(3,1)}) ,$$

thus the claim results from (35.6) of [TW]. □

(21.6) Theorem Let $\tilde{\mathcal{F}}$ be an integrable foundation. Then there is an alternative division ring $\mathbb{A}$ such that

$$\forall\ (i,j) \in A(\tilde{F}) : \qquad \tilde{\mathbb{A}}_{(i,j)} \cong \mathbb{A} \ \vee\ \tilde{\mathbb{A}}_{(i,j)} \cong \mathbb{A}^o .$$

Proof
By corollary (19.33) and corollary (21.5), this is true for each irr. rank 3 residue. The assertion results from an easy induction, starting with $\mathbb{A} := \tilde{\mathbb{A}}_{(a,b)}$ for an arbitrary edge $(a,b) \in A(\tilde{F})$. □

(21.7) Definition Given an integrable foundation $\tilde{\mathcal{F}}$, we call the alternative division ring $\tilde{\mathbb{A}}$ of theorem (21.6) the *defining field for* $\tilde{\mathcal{F}}$, and we say that $\tilde{\mathcal{F}}$ is defined over $\tilde{\mathbb{A}}$. By (35.6) of [TW], it is unique up to (anti-)isomorphism.

(21.8) Notation Throughout the rest of this part, given an arbitrary integrable foundation $\tilde{\mathcal{F}}$, the alternative division ring $\tilde{\mathbb{A}}$ always denotes its defining field.

(21.9) Theorem There is an embedding $\sigma : \mathbb{A} \hookrightarrow \mathbb{A}^\infty$.

Proof
Let

$$\mathcal{R}_0 \cong \mathcal{T}(\mathbb{A}) = \{x_1(\mathbb{A}), x_2(\mathbb{A}), x_3(\mathbb{A})\}$$

be the gem as in (18.1) of [W]. As $\mathcal{R}_0$ is a special residue, it is a subbuilding of

$$\mathcal{B}^\infty \cong \mathcal{T}(\mathbb{A}^\infty) = \{\tilde{x}_1(\mathbb{A}^\infty), \tilde{x}_2(\mathbb{A}^\infty), \tilde{x}_3(\mathbb{A}^\infty)\}$$

by theorem (6.3) of [MV]. Given $t \in \mathbb{A}$ and $i \in \{1,2,3\}$, there is a unique element $\tilde{x}_i\big(\sigma_i(t)\big)$ inducing $x_i(t)$, cf. proposition (29.61) of [W]. By lemma (18.8), we may reparametrize $\mathcal{T}(\mathbb{A}^\infty)$ such that

$$\sigma_1(1_{\mathbb{A}}) = 1_{\mathbb{A}^\infty} \ , \qquad\qquad \sigma_3(1_{\mathbb{A}}) = 1_{\mathbb{A}^\infty} \ .$$

Given $s, t \in \mathbb{A}$ and $i \in \{1,2,3\}$, we have[1]

$$\tilde{x}_i\big(\sigma_i(s+t)\big) = x_i(s+t) = x_i(s)x_i(t) = \tilde{x}_i\big(\sigma_i(s)\big)\tilde{x}_i\big(\sigma_i(t)\big) = \tilde{x}_i\big(\sigma_i(s)+\sigma_i(t)\big) \tag{21.1}$$

and

$$\tilde{x}_2\big(\sigma_2(st)\big) = x_2(st) = [x_1(s), x_3(t)] = \big[\tilde{x}_1\big(\sigma_1(s)\big), \tilde{x}_3\big(\sigma_3(t)\big)\big] = \tilde{x}_2\big(\sigma_1(s)\sigma_3(t)\big) \ . \tag{21.2}$$

Putting $s := 1_{\mathbb{A}}$ and $t := 1_{\mathbb{A}}$ in equation (21.2) shows that

$$\sigma := \sigma_3 = \sigma_2 = \sigma_1 \ ,$$

hence

$$\forall\ s, t \in \mathbb{A} : \qquad \sigma(s+t) \overset{(21.1)}{=} \sigma(s) + \sigma(t) \ , \qquad \sigma(st) \overset{(21.2)}{=} \sigma(s)\sigma(t) \ .$$

□

(21.10) Remark

(a) The root group valuation $\phi := \phi_{\mathcal{R}_0}$ with respect to $\mathcal{R}_0$ as in definition (13.8) of [W] induces a discrete valuation ν of $\mathbb{A}^\infty$. As a consequence, we have

$$\sigma(\mathbb{A}) \subseteq \{k \in \mathbb{A}^\infty \mid \nu(k) = 0\} \ .$$

Moreover, $\sigma(\mathbb{A})$ is a set of representatives for the residue field $\bar{\mathbb{A}}^\infty := \mathcal{O}/m$, where $\mathcal{O}$ is the valuation ring of $\mathbb{A}^\infty$ and m is its unique maximal ideal.

In particular, $\tilde{x}_i\big(\sigma(\mathbb{A})\big) \subseteq U_{i,0}$ is a set of representatives for $\bar{U}_{i,0}$ and, given a uniformizer $\pi \in \mathbb{A}^\infty$, $\tilde{x}_i(\sigma(\mathbb{A})\pi) \subseteq U_{i,1}$ and $\tilde{x}_i\big(\pi\sigma(\mathbb{A})\big) \subseteq U_{i,1}$ are sets of representatives for $\bar{U}_{i,1}$, where $U_{i,k}$ is defined as in definition (3.21) of [W] and $\bar{U}_{i,k}$ is defined as in (18.21) of [W].

(b) By the results of [W], we may suppose $\mathbb{A}^\infty$ to be complete with respect to the valuation ν and $\nu(\mathbb{A}^\infty) = \mathbb{Z}$. By lemma (3.39) (a), $Z(\mathbb{A}^\infty)$ is complete with respect to the induced valuation. As a consequence, we may apply the results of [P] if $\mathbb{A}^\infty$ is an octonion division algebra.

(c) For brevity, we will write $\tilde{x}_i(t)$ instead of $\tilde{x}_i\big(\sigma(t)\big)$ in the following.

[1] We replace $\tilde{x}_i(\sigma_i(t))_{|\mathcal{R}_0} = x_i(t)$ by $\tilde{x}_i(\sigma_i(t)) = x_i(t)$.

§ 21.2 Triangles over Octonions

At this point, as well as in §21.4 and §21.6, we heavily make use of the theory of affine buildings developed in [W]. The main point is the fact that a parametrization for the building at infinity induces a parametrization for a given root group system. As uniformizers play a central role, we need some results of [P].

(21.11) Notation Throughout this paragraph, $\mathbb{O} := \mathbb{A}$ is an octonion division algebra. As a consequence, $\mathbb{O}^\infty$ is an octonion division algebra.

(21.12) Proposition We have $\nu\big(Z(\mathbb{O}^\infty)\big) = \mathbb{Z}$. As a consequence, there is a uniformizer $\pi \in Z(\mathbb{O}^\infty)$.

Proof
As we have $\mathbb{O} = \bar{\mathbb{O}}^\infty$ by remark (21.10), this is a consequence of proposition 2 in [P]. □

(21.13) Theorem The foundation $\mathcal{F}$ is specially isomorphic to the foundation $\tilde{\mathcal{A}}_2(\mathbb{O})$.

Proof
Let (x_1, x_2, x_3) be a parametrization for $\mathcal{T}(\mathbb{O}^\infty)$. Then the maps

$$\begin{aligned}
x^i_{(i,j)} &: \mathbb{O} \to x_1(\mathbb{O}^\infty)\ , & t &\mapsto x_1(t) \in U_{1,0}\ ,\\
x^{ij}_{(i,j)} &: \mathbb{O} \to x_2(\mathbb{O}^\infty)\ , & t &\mapsto x_2(t) \in U_{2,0}\ ,\\
x^j_{(i,j)} &: \mathbb{O} \to x_3(\mathbb{O}^\infty)\ , & t &\mapsto x_3(t) \in U_{3,0}
\end{aligned}$$

yield a parametrization for the gem $\mathcal{R}_0 = \mathcal{R}_{\{i,j\}}$, cf. remark (21.10). Moreover, the maps

$$\begin{aligned}
x^j_{(j,k)} &: \mathbb{O} \to x_3(\mathbb{O}^\infty)\ , & t &\mapsto x_3(t) \in U_{3,0}\ ,\\
x^{jk}_{(j,k)} &: \mathbb{O} \to x_{-1}(\mathbb{O}^\infty)\ , & t &\mapsto x_{-1}(t\pi) \in U_{-1,1}\ ,\\
x^k_{(j,k)} &: \mathbb{O} \to x_{-2}(\mathbb{O}^\infty)\ , & t &\mapsto x_{-2}(t\pi) \in U_{-2,1}
\end{aligned}$$

and

$$\begin{aligned}
x^k_{(k,i)} &: \mathbb{O} \to x_{-2}(\mathbb{O}^\infty)\ , & t &\mapsto x_{-2}(\pi t) \in U_{-2,1}\ ,\\
x^{ki}_{(k,i)} &: \mathbb{O} \to x_{-3}(\mathbb{O}^\infty)\ , & t &\mapsto x_{-3}(\pi t) \in U_{-3,1}\ ,\\
x^i_{(k,i)} &: \mathbb{O} \to x_1(\mathbb{O}^\infty)\ , & t &\mapsto x_1(t) \in U_{1,0}
\end{aligned}$$

yield parametrizations for two gems $\mathcal{R}_{\{j,k\}}$ and $\mathcal{R}_{(k,i)}$ at distance one to each other and to $\mathcal{R}_{\{i,j\}}$. By construction, this parameter system Λ parametrizes a root group system $\mathcal{U} := \mathcal{U}(\mathcal{B}, F, \Sigma, c)$. As

$$\begin{aligned}
x^j_{(i,j)}(t) &= x_3(t) = x^j_{(j,k)}(t)\ ,\\
x^k_{(j,k)}(t) &= x_{-2}(t\pi) = x_{-2}(\pi t) = x^k_{(k,i)}(t)\ ,\\
x^i_{(k,i)}(t) &= x_1(t) = x^i_{(i,j)}(t)
\end{aligned}$$

it follows that

$$\gamma_{(i,j,k)} = \gamma_{(j,k,i)} = \gamma_{(k,i,j)} = \mathrm{id}_{\mathbb{O}}\ .$$

Therefore, the resulting foundation

$$\tilde{\mathcal{F}} := \mathcal{F}(\mathcal{U}, \Lambda)$$

is the canonical foundation $\tilde{\mathcal{A}}_2(\mathbb{O})$. Finally we have $\mathcal{F} \cong \tilde{\mathcal{F}} = \tilde{\mathcal{A}}_2(\mathbb{O})$ by theorem (19.10). □

§ 21.3 Non-Existence of Tetrahedrons over Octonions

Before we continue the examination of triangle foundations, we use the results of §21.2 to prove that there are no further integrable foundations over octonions. To apply them, we have to investigate the structure of $\mathrm{Aut}_J(\mathbb{O})$. In particular, we construct a certain subgroup $\Gamma \leq \mathrm{Aut}_J(\mathbb{O})$.

(21.14) Notation

- Given a quaternion subalgebra $\mathbb{H}$, $e \in \mathbb{H}^\perp$ and $w, p \in \mathbb{H}$, we set

$$\psi_{(\mathbb{H},e,w)} : \mathbb{O} \to \mathbb{O},\ x + e \cdot y \mapsto x + e \cdot w^{-1} y w \ ,$$
$$\phi_{(\mathbb{H},e,w,p)} : \mathbb{O} \to \mathbb{O},\ x + e \cdot y \mapsto w^{-1} x w + e \cdot w^{-1} y w p \ .$$

- We set

$$\Psi := \{\psi_{(\mathbb{H},e,w)} \mid \mathbb{H} \text{ a quaternion subalgebra},\ e \in \mathbb{H}^\perp,\ w \in \mathbb{H}\} \ ,$$
$$\Phi := \{\phi_{(\mathbb{H},e,w,p)} \mid \mathbb{H} \text{ a quaternion subalgebra},\ e \in \mathbb{H}^\perp,\ w, p \in \mathbb{H},\ N(p) = 1_{\mathbb{O}}\} \ .$$

- Given $\psi_{(\mathbb{H},e,w)} \in \Psi$, we set

$$\psi^o_{(\mathbb{H},e,w)} : \mathbb{O}^o \to \mathbb{O}^o,\ x + e \circ y \mapsto x + e \circ w^{-1} y w \ .$$

- We set

$$\Gamma := \{\psi\phi \mid \psi \in \Psi,\ \phi \in \mathrm{Aut}(\mathbb{O})\} \ , \qquad \Gamma^o := \{\psi^o\phi \mid \psi \in \Psi,\ \phi \in \mathrm{Aut}(\mathbb{O}^o)\} \ .$$

(21.15) Lemma Let $\psi := \psi_{(\mathbb{H},e,w)} \in \Psi$. Then the following holds:

(a) Given $s, t \in \mathbb{O}$, we have

$$\psi(st) = \big(\psi(s) \cdot \psi(t) w\big) w^{-1} \ .$$

(b) $\psi \in \mathrm{Aut}_J(\mathbb{O})$.

Proof

(a) This is (20.24) of [TW].

(b) By the Moufang identities and the inverse properties, We have

$$\begin{aligned}\psi(sts) &\overset{(a)}{=} \big(\psi(s) \cdot \psi(ts) w\big) w^{-1} \overset{(a)}{=} \big(\psi(s) \cdot \big[\big(\psi(t) \cdot \psi(s) w\big) w^{-1}\big] w\big) w^{-1} \\ &= \big[\psi(s)\big(\psi(t) \cdot \psi(s) w\big)\big] w^{-1} = \big[\big(\psi(s)\psi(t) \cdot \psi(s)\big) w\big] w^{-1} = \psi(s)\psi(t)\psi(s) \ .\end{aligned}$$

for all $s, t \in \mathbb{O}$.

□

(21.16) Lemma We have $\Phi \subseteq \mathrm{Aut}_{Z(\mathbb{O})}(\mathbb{O})$. Given an element $\tilde{\phi} \in \mathrm{Aut}_{Z(\mathbb{O})}(\mathbb{O})$ leaving a quaternion subalgebra $\mathbb{H}$ invariant, there is an element $\phi \in \Phi$ such that $\tilde{\phi} = \phi$.

Proof
This is section (2.1) of [Sp]. Notice that our point of view is that of the opposite multiplication. □

(21.17) Lemma Given $\gamma = \psi_{(\mathbb{H},e,w)}\phi \in \Gamma$ and $v \in \mathbb{O}^*$, let $\alpha_{v,\gamma} : \mathcal{T}(\mathbb{O}) \to \mathcal{T}(\mathbb{O})$,

$$x^1(t) \mapsto x^1\big(v \cdot \gamma(t)\big) \ , \qquad x^{12}(t) \mapsto x^{12}\big(v \cdot \gamma(t)w \cdot v\big) \ , \qquad x^2(t) \mapsto x^2\big(\gamma(t)w \cdot v\big)$$

and given $\gamma^o = \psi^o_{(\mathbb{H},e,w)}\phi \in \Gamma$ and $v \in \mathbb{O}^*$, let $\alpha^o_{v,\gamma^o} : \mathcal{T}(\mathbb{O}^o) \to \mathcal{T}(\mathbb{O}^o)$,

$$x^2(t) \mapsto x^2\big(v \circ \gamma^o(t)\big) \ , \qquad x^{12}(t) \mapsto x^{12}\big(v \circ \gamma^o(t)w \circ v\big) \ , \qquad x^1(t) \mapsto x^1\big(\gamma^o(t)w \circ v\big) \ .$$

Then we have

$$\{\alpha_{v,\gamma} \mid v \in \mathbb{O}^*,\ \gamma \in \Gamma\} = \mathrm{Aut}\big(\mathcal{T}(\mathbb{O})\big) = \mathrm{Aut}\big(\mathcal{T}(\mathbb{O}^o)\big) = \{\alpha^o_{v,\gamma^o} \mid v \in \mathbb{O}^*,\ \gamma^o \in \Gamma^o\} \ .$$

Proof
This holds by the proof of (37.12) in [TW]. □

(21.18) Lemma We have $\Gamma = \Gamma^o$.

Proof
Given $\psi := \psi_{(\mathbb{H},e,w)} \in \Psi$, we have

$$\begin{aligned}\psi^o(x + e \cdot y) &= \psi^o(x + \bar{y} \cdot e) = \psi^o(x + e \circ \bar{y}) \\ &= x + e \circ w^{-1}\bar{y}w = x + w\bar{y}w^{-1} \cdot e = x + e \cdot \overline{w\bar{y}w^{-1}} = x + e \cdot wyw^{-1}\end{aligned}$$

for all $x, y \in \mathbb{H}$ and therefore

$$\psi^o_{(\mathbb{H},e,w)} = \psi_{(\mathbb{H},e,w^{-1})} \ .$$

□

(21.19) Lemma The set Γ is a subgroup of $\mathrm{Aut}_J(\mathbb{O})$.

Proof
Given $\alpha_{v,\gamma} \in \mathrm{Aut}\big(\mathcal{T}(\mathbb{O})\big)$, we have

$$\alpha^1_{v,\gamma}(1_{\mathbb{O}}) = v \cdot \gamma(1_{\mathbb{O}}) = v \ . \tag{21.3}$$

- We have
$$\mathrm{id}_{\mathbb{O}} = \psi_{(\mathbb{H},e,1_{\mathbb{O}})} \circ \mathrm{id}_{\mathbb{O}} \in \Gamma \ .$$
- Let $\gamma \in \Gamma$. By equation (21.3) and lemma (21.17), there is an element $\rho \in \Gamma$ such that
$$\alpha^{-1}_{1_{\mathbb{O}},\gamma} = \alpha_{1_{\mathbb{O}},\rho} \ .$$
We obtain
$$\gamma^{-1} = (\alpha^1_{1_{\mathbb{O}},\gamma})^{-1} = \alpha^1_{1_{\mathbb{O}},\rho} = \rho \in \Gamma \ .$$
- Let $\gamma_1, \gamma_2 \in \Gamma$. By equation (21.3) and lemma (21.17), there is an element $\rho \in \Gamma$ such that
$$\alpha_{1_{\mathbb{O}},\gamma_1} \circ \alpha_{1_{\mathbb{O}},\gamma_2} = \alpha_{1_{\mathbb{O}},\rho} \ .$$
We obtain
$$\gamma_1\gamma_2 = \alpha^1_{1_{\mathbb{O}},\gamma_1}\alpha^1_{1_{\mathbb{O}},\gamma_2} = \alpha^1_{1_{\mathbb{O}},\rho} = \rho \in \Gamma \ .$$

□

(21.20) Lemma We have $\Psi \cup \Phi \subseteq \Gamma$.

Proof
Given a quaternion subalgebra $\mathbb{H}$ and $e \in \mathbb{H}^\perp$, we have

$$\mathrm{id}_{\mathbb{O}} \in \mathrm{Aut}(\mathbb{O}) \ , \qquad \mathrm{id}_{\mathbb{O}} = \psi_{(\mathbb{H},e,1_{\mathbb{O}})} \in \Psi \ .$$

□

(21.21) Lemma We have $\gamma_w \in \Gamma$ for each $w \in \mathbb{O}$.

Proof
By lemma (3.25) (c), there is a quaternion subalgebra $\mathbb{H}$ containing w. Let $e \in \mathbb{H}^\perp$. Then we have

$$\gamma_w(x + e \cdot y) = w^{-1}xw + w^{-1}(e \cdot y)w = w^{-1}xw + e \cdot \bar{w}^{-1}wy$$

for all $x, y \in \mathbb{H}$. Notice that

$$N(\bar{w}^{-1}w) = N(\bar{w})^{-1}N(w) = N(w)^{-1}N(w) = 1_{\mathbb{O}} \ ,$$

thus we obtain

$$\gamma_w = \phi_{(\mathbb{H},e,w,\bar{w}^{-1}w)} \circ \psi_{(\mathbb{H},e,w^{-2}\bar{w})} \in \Gamma \ .$$

□

(21.22) Lemma We have $\sigma_s \notin \Gamma$.

Proof
Let $\psi = \psi_{(\mathbb{H},e,w)} \in \Psi$ and $\phi \in \mathrm{Aut}(\mathbb{O})$ such that $\sigma_s = \psi\phi$. Then

$$\mathrm{id}_{\mathbb{H}} = \psi_{|\mathbb{H}} = \sigma_s \phi^{-1}_{|\mathbb{H}}$$

is both negative and positive $\lightning$. □

(21.23) Lemma Let

$$\mathcal{F} = \{\mathcal{T}(\mathbb{A}_{(1,2)}) := \mathcal{T}(\mathbb{A}_{(2,3)}) := \mathcal{T}(\mathbb{A}_{(3,1)}) := \mathcal{T}(\mathbb{O}),\ \gamma_2 := \gamma_{(1,2,3)}, \gamma_3 := \gamma_{(2,3,1)}, \gamma_1 := \gamma_{(3,1,2)}\}$$

be an integrable triangle foundation over $\mathbb{O}$. Then we have $\gamma_i \in \Gamma$ for $i = 1, 2, 3$.

Proof
We have $\mathcal{F} \cong \tilde{\mathcal{A}}_2(\mathbb{O})$ by theorem (21.13), thus theorem (19.23) (a) yields a reparametrization α for $\tilde{\mathcal{A}}_2(\mathbb{O})$ such that

$$\tilde{\mathcal{A}}_2(\mathbb{O})_\alpha = \mathcal{F} \ .$$

From definition (19.17) it follows that

$$\gamma_{(i,j,k)} = (\alpha^j_{(j,k)})^{-1} \circ \mathrm{id}_{\mathbb{O}} \circ \alpha^j_{(i,j)} \ .$$

By lemma (21.17) and lemma (21.18), there are $\gamma_{(i,j)}, \gamma_{(j,k)} \in \Gamma$ and $w \in \mathbb{O}$ such that

$$\alpha^j_{(i,j)}(t) = w \cdot \gamma_{(i,j)}(t) \ , \qquad \alpha^j_{(j,k)}(t) = \gamma_{(j,k)}(t) \cdot w$$

and thus

$$(\alpha^j_{(j,k)})^{-1}(t) = \gamma^{-1}_{(j,k)}(t \cdot w^{-1}) \ , \qquad (\alpha^j_{(j,k)})^{-1} \circ \alpha^j_{(i,j)}(t) = (\gamma^j_{(j,k)})^{-1}\big(w \cdot \gamma_{(i,j)}(t) \cdot w^{-1}\big)$$

for each $t \in \mathbb{O}$. We finally obtain

$$\gamma_{(i,j,k)} = (\gamma^j_{(j,k)})^{-1} \circ \gamma_{w^{-1}} \circ \gamma_{(i,j)} \in \Gamma$$

by lemma (21.21). □

(21.24) Proposition Let $\tilde{\mathcal{F}}$ be a foundation over $\mathbb{O}$ such that $\mathcal{G}_{\tilde{F}}$ is a tetrahedron. Then $\tilde{\mathcal{F}}$ is not integrable.

Proof
Assume that $\tilde{\mathcal{F}}$ is integrable. Then each rank 3 residue is integrable, hence specially isomorphic to $\tilde{\mathcal{A}}_2(\mathbb{O})$ by theorem (21.13). By remark (19.24) (b), we may assume that the $\{1,2,3\}$-residue is $\tilde{\mathcal{A}}_2(\mathbb{O})$. Moreover, we may assume

$$\mathcal{T}(\tilde{\mathbb{A}}_{(3,4)}) = \mathcal{T}(\tilde{\mathbb{A}}_{(4,1)}) = \mathcal{T}(\tilde{\mathbb{A}}_{(2,4)}) = \mathcal{T}(\mathbb{O}) \ .$$

Now we are in the situation of lemma (21.23), thus we have $\tilde{\gamma}_{(1,2,4)} \in \Gamma$ and

$$\tilde{\gamma}_{(4,2,3)} = \mathrm{id}^o \circ \mathrm{id}^o \circ \tilde{\gamma}^{-1}_{(1,2,4)} \circ \mathrm{id}^o = \tilde{\gamma}^{-1}_{(1,2,4)} \circ \mathrm{id}^o \ .$$

We extend the reparametrization

$$\alpha_{(4,2)} := \{\mathbb{O}, \sigma_s^o, \sigma_s^o, \sigma_s^o\}$$

to a reparametrization α for $\tilde{\mathcal{F}}$ and obtain an integrable foundation

$$\mathcal{F} := \tilde{\mathcal{F}}_\alpha$$

satisfying

$$\gamma_{(4,2,3)} = \mathrm{id}_{\mathbb{O}} \circ \tilde{\gamma}_{(4,2,3)} \circ \sigma_s^o = \tilde{\gamma}^{-1}_{(1,2,4)} \circ \sigma_s \ .$$

As we have

$$\mathcal{T}(\mathbb{A}_{(4,2)}) = \mathcal{T}(\mathbb{A}_{(2,3)}) = \mathcal{T}(\mathbb{A}_{(3,4)}) = \mathcal{T}(\mathbb{O}) \ ,$$

it follows that

$$\tilde{\gamma}^{-1}_{(1,2,4)} \circ \sigma_s = \gamma_{(4,2,3)} \in \Gamma \ , \qquad\qquad \sigma_s \in \Gamma \qquad \lightning \ .$$

□

(21.25) Theorem A foundation $\mathcal{F}$ over an octonion division algebra $\mathbb{O}$ is integrable if and only if we have $\mathcal{F} \cong \mathcal{A}_2(\mathbb{O})$ or $\mathcal{F} \cong \tilde{\mathcal{A}}_2(\mathbb{O})$.

Proof
As $\mathcal{G}_F$ is complete and as each residue is integrable, proposition (21.24) implies $|V(F)| \leq 3$. If $|V(F)| = 2$, we have $\mathcal{F} \cong \mathcal{T}(\mathbb{O})$, and if $|V(F)| = 3$, we have $\mathcal{F} \cong \tilde{\mathcal{A}}_2(\mathbb{O})$ by theorem (21.13). Finally, $\tilde{\mathcal{A}}_2(\mathbb{O})$ is integrable by theorem (20.9). □

§ 21.4 Triangles over Skew-Fields

As we are done with the octonion case, we next deal with positive foundations over skew-fields. The first step is to show that a triangle foundation $\mathcal{F}$ over a skew-field $\mathbb{A}$ is negative if $\mathbb{A}^\infty$ is a skew-field. Thus $\mathbb{A}^\infty$ is necessarily an octonion division algebra if $\mathcal{F}$ is positive. Then we prove that $\mathbb{A}$ is a quaternion division algebra, using the fact that $\mathbb{A}$ embeds into $\mathbb{A}^\infty$. At last we obtain a parametrization for a root group sequence via the building at infinity.

(21.26) Notation Throughout this paragraph, $\mathbb{A}$ is a skew-field.

(21.27) Lemma Let $\mathcal{T}(\tilde{\mathbb{A}})$ and $\mathcal{T}(\hat{\mathbb{A}})$ be isomorphic Moufang triangles over skew-fields and let $\alpha = (\alpha_1, \alpha_2, \alpha_3) : \mathcal{T}(\tilde{\mathbb{A}}) \to \mathcal{T}(\hat{\mathbb{A}})$ be an isomorphism. Then there are $a, b \in \hat{\mathbb{A}}$ and an isomorphism $\phi : \tilde{\mathbb{A}} \to \hat{\mathbb{A}}$ of skew-fields such that

$$\alpha = (\lambda_a \phi, \lambda_a \rho_b \phi, \rho_b \phi) \ .$$

Proof
Let

$$a := \alpha_1(1_{\tilde{\mathbb{A}}}) \ , \qquad\qquad b := \alpha_3(1_{\tilde{\mathbb{A}}}) \ .$$

Since $\hat{\mathbb{A}}$ is associative, the map

$$\tilde{\alpha} := \left(\lambda_{a^{-1}}\alpha_1, \lambda_{a^{-1}}\rho_{b^{-1}}\alpha_2, \rho_{b^{-1}}\right) : \mathcal{T}(\tilde{\mathbb{A}}) \to \mathcal{T}(\hat{\mathbb{A}})$$

is an isomorphism. Moreover, we have

$$\lambda_{a^{-1}}\alpha_1(1_{\tilde{\mathbb{A}}}) = a^{-1}a = 1_{\hat{\mathbb{A}}} \ , \qquad\qquad \rho_{b^{-1}}\alpha_3(1_{\tilde{\mathbb{A}}}) = bb^{-1} = 1_{\hat{\mathbb{A}}} \ .$$

By (35.23) of [TW] therefore, there is an isomorphism $\phi : \tilde{\mathbb{A}} \to \hat{\mathbb{A}}$ of skew-fields such that

$$\tilde{\alpha} = (\phi, \phi, \phi) \ .$$

Thus we have

$$\alpha = (\lambda_a\phi, \lambda_a\rho_b\phi, \rho_b\phi) \ .$$

□

(21.28) Lemma Let $\tilde{\mathcal{F}}, \hat{\mathcal{F}}$ be isomorphic foundations over a skew-field, let $\alpha : \tilde{\mathcal{F}} \to \hat{\mathcal{F}}$ be an isomorphism and let $(i,j,k) \in G(F)$. If $\tilde{\gamma}_{(i,j,k)}$ is negative, then $\hat{\gamma} := \hat{\gamma}_{(\pi(1),\pi(2),\pi(3))}$ is negative.

Proof
By lemma (21.27), there are isomorphisms $\phi_i : \tilde{\mathbb{A}}_{(i,j)} \to \hat{\mathbb{A}}_{(\pi(i),\pi(j))}$ and $\phi_k : \tilde{\mathbb{A}}_{(j,k)} \to \hat{\mathbb{A}}_{(\pi(j),\pi(k))}$ of skew-fields and elements $a_i \in \hat{\mathbb{A}}_{(\pi(i),\pi(j))}$ and $a_k \in \hat{\mathbb{A}}_{(\pi(j),\pi(k)k)}$ such that

$$\alpha^j_{(i,j)} = \rho_{a_i} \circ \phi_i \ , \qquad\qquad \alpha^j_{(j,k)} = \lambda_{a_k} \circ \phi_k \ .$$

By the definition of an isomorphism, we have

$$\hat{\gamma} = \alpha^j_{(j,k)} \circ \tilde{\gamma}_{(i,j,k)} \circ (\alpha^j_{(i,j)})^{-1} \ .$$

Combining these two facts implies that there are elements $a, b \in \hat{\mathbb{A}}_{(\pi(j),\pi(k))}$ such that

$$\hat{\gamma} = \lambda_a \circ \rho_b \circ \phi_k \circ \tilde{\gamma}_{(i,j,k)} \circ \phi_i^{-1} \ .$$

Moreover, we have

$$ab = \hat{\gamma}(1) = 1$$

and therefore $b = a^{-1}$. It finally follows that

$$\hat{\gamma} = \gamma_b \circ \phi_k \circ \tilde{\gamma}_{(i,j,k)} \circ \phi_i^{-1}$$

is negative. □

(21.29) Corollary The following holds:

(a) The foundation $\mathcal{F}$ is either negative or positive.

(b) If a foundation isomorphic to $\mathcal{F}$ is negative, then $\mathcal{F}$ is negative.

Proof

(a) This results from lemma (21.28) and corollary (21.4).

(b) This is an immediate consequence of lemma (21.28).

□

(21.30) Theorem Let $\mathbb{A}^\infty$ be a skew-field. Then $\mathcal{F}$ is negative.

Proof
Let (x_1, x_2, x_3) be a parametrization for $\mathcal{T}(\mathbb{A}^\infty)$ and let $\pi \in \mathbb{A}^\infty$ be a uniformizer. Then the maps

$$\begin{aligned} x^{i}_{(i,j)} &: \mathbb{A} \to x_1(\mathbb{A}^\infty)\ , & t &\mapsto x_1(t) \in U_{1,0}\ , \\ x^{ij}_{(i,j)} &: \mathbb{A} \to x_2(\mathbb{A}^\infty)\ , & t &\mapsto x_2(t) \in U_{2,0}\ , \\ x^{j}_{(i,j)} &: \mathbb{A} \to x_3(\mathbb{A}^\infty)\ , & t &\mapsto x_3(t) \in U_{3,0} \end{aligned}$$

yield a parametrization for the gem $\mathcal{R}_0 = \mathcal{R}_{\{i,j\}}$, cf. remark (21.10). Moreover, the maps

$$\begin{aligned} x^{j}_{(j,k)} &: \mathbb{A} \to x_3(\mathbb{A}^\infty)\ , & t &\mapsto x_3(t) \in U_{3,0}\ , \\ x^{jk}_{(j,k)} &: \mathbb{A} \to x_{-1}(\mathbb{A}^\infty)\ , & t &\mapsto x_{-1}(t\pi) \in U_{-1,1}\ , \\ x^{k}_{(j,k)} &: \mathbb{A} \to x_{-2}(\mathbb{A}^\infty)\ , & t &\mapsto x_{-2}(t\pi) \in U_{-2,1} \end{aligned}$$

yield a parametrization for a gem $\mathcal{R}_{\{j,k\}}$ at distance one to $\mathcal{R}_{\{i,j\}}$. By construction, these two parametrizations are part of a parameter system Λ for a root group system $\mathcal{U} := \mathcal{U}(\mathcal{B}, F, \Sigma, c)$. As we have

$$x^{j}_{(i,j)}(t) = x_3(t) = x^{j}_{(j,k)}(t)\ ,$$

it follows that $\gamma_{(i,j,k)} = \mathrm{id}_{\mathbb{A}}$ is negative. Therefore, the resulting foundation

$$\tilde{\mathcal{F}} := \mathcal{F}(\mathcal{U}, \Lambda)$$

is negative, cf. corollary (21.29) (a) with $\tilde{\mathcal{F}}$ in place of $\mathcal{F}$. As we have $\mathcal{F} \cong \tilde{\mathcal{F}}$ by theorem (19.10), corollary (21.29) (b) finally shows that $\mathcal{F}$ is negative. □

§ 21.5 Skew-Fields inside Octonions

As we want to show that $\mathbb{A}$ is a quaternion division algebra if $\mathcal{F}$ is positive, we prove that each non-commutative division subring of an octonion division algebra is a quaternion division algebra. Then the claim results from the fact that $\mathbb{A}$ embeds into $\mathbb{A}^\infty$.

In §21.6, we will need once again the results of [P] to find a suitable uniformizer for a parametrization. In fact, we can choose a uniformizer $\pi \in \mathbb{A}^\perp$, thus conjugating elements of $\mathbb{A}$ by π is equal to applying the standard involution.

(21.31) Lemma Let $\mathbb{O}$ be an octonion algebra and let $\mathbb{D}$ be a non-commutative division subring. Then we have

$$Z(\mathbb{D}) \subseteq Z(\mathbb{O}) =: \mathbb{K}\ .$$

Proof
The octonion division algebra $\mathbb{O}$ quadratic over $\mathbb{K}$. Then $\tilde{\mathbb{D}} := \langle \mathbb{D} \rangle_{\mathbb{K}}$ is non-commutative and quadratic over $\mathbb{K} \subseteq Z(\tilde{\mathbb{D}})$, thus we have $\mathbb{K} = Z(\tilde{\mathbb{D}})$ by proposition (3.32) and therefore

$$Z(\mathbb{D}) \subseteq Z(\tilde{\mathbb{D}}) = \mathbb{K}\ .$$

□

(21.32) Lemma Let $\mathbb{D}$ be a skew-field such that $\dim_{Z(\mathbb{D})} \mathbb{D} < \infty$. Then there is an $n \in \mathbb{N}^*$ such that

$$\dim_{Z(\mathbb{D})} \mathbb{D} = n^2\ .$$

Proof
This results from the fact that $\mathbb{D}$ is central simple over $Z(\mathbb{D})$. □

(21.33) Theorem Let $\mathbb{O}$ be an octonion division algebra, let $\mathbb{K} := Z(\mathbb{O})$ and let $\mathbb{D}$ be a non-commutative alternative division subring. Then the following holds:

(a) $\mathbb{D} \otimes_{Z(\mathbb{D})} \mathbb{K}$ is isomorphic to a division subalgebra of $\mathbb{O}$.

(b) $\mathbb{D}$ is a quaternion division algebra or an octonion division algebra.

Proof
The non-commutative division subalgebra $\tilde{\mathbb{D}} := \langle \mathbb{D} \rangle_{\mathbb{K}} \subseteq \mathbb{O}$ is quadratic over $\mathbb{K} \subseteq Z(\tilde{\mathbb{D}})$ and therefore a quaternion division algebra or an octonion division algebra by proposition (3.32).

(a) By lemma (21.31), we have $Z(\mathbb{D}) \subseteq \mathbb{K}$, hence

$$\mathbb{D} \otimes_{Z(\mathbb{D})} \mathbb{K}$$

is a central simple algebra over $\mathbb{K}$. By the universal property, there is an epimorphism

$$\pi : \mathbb{D} \otimes_{Z(\mathbb{D})} \mathbb{K} \to \tilde{\mathbb{D}}$$

which is injective due to simplicity.

(b) Part (a) yields

$$\dim_{Z(\mathbb{D})} \mathbb{D} = \dim_{\mathbb{K}}(\mathbb{D} \otimes_{Z(\mathbb{D})} \mathbb{K}) = \dim_{\mathbb{K}} \tilde{\mathbb{D}} \in \{4, 8\} \ .$$

In particular, each $Z(\mathbb{D})$-basis of $\mathbb{D}$ is a $\mathbb{K}$-basis of $\tilde{\mathbb{D}}$.

- If $\mathbb{D}$ is non-associative, then $\mathbb{D}$ is an octonion division algebra and thus $\dim_{Z(\mathbb{D})} \mathbb{D} = 8$.
- If $\mathbb{D}$ is associative, then we have

$$\dim_{Z(\mathbb{D})} \mathbb{D} \in \{4, 8\} \cap \{n^2 \mid n \in \mathbb{N}^*\} = 4$$

by lemma (21.32). Let $x \in \mathbb{D} \setminus Z(\mathbb{D})$ and let $\{1_{\mathbb{O}}, x, y, z\}$ be a $Z(\mathbb{D})$-basis of $\mathbb{D}$. Then there are $\lambda_1, \ldots, \lambda_4 \in Z(\mathbb{D}) \subseteq \mathbb{K}$ such that

$$x^{-1} = \lambda_1 \cdot 1_{\mathbb{O}} + \lambda_2 \cdot x + \lambda_3 \cdot y + \lambda_4 \cdot z \ .$$

Since $\{1_{\mathbb{O}}, x, y, z\}$ is a $\mathbb{K}$-basis of $\tilde{\mathbb{D}}$, it follows that

$$\lambda_1 = N(x)^{-1} T(x) \ , \qquad \lambda_2 = -N(x)^{-1} \ , \qquad \lambda_3 = 0_{\mathbb{O}} = \lambda_4$$

and therefore

$$N(x) = -\lambda_2^{-1} \in Z(\mathbb{D}) \ , \qquad T(x) = \lambda_1 \cdot N(x) \in Z(\mathbb{D}) \ .$$

As a consequence, $\mathbb{D}$ is quadratic over $Z(\mathbb{D})$ and therefore a quaternion division algebra by proposition (3.32).

□

§ 21.6 Positive Triangles over Skew-Fields

(21.34) Notation Throughout this paragraph, $\mathbb{A}$ is a skew-field and $\mathcal{F}$ is positive.

(21.35) Remark By theorem (21.30), $\mathbb{A}^\infty$ is an octonion division algebra.

(21.36) Lemma The defining field $\mathbb{A}$ is a quaternion division algebra.

Proof
We have $\mathbb{A} \subseteq \mathbb{A}^\infty$ by theorem (21.9), thus the claim results from theorem (21.33). □

(21.37) Notation We set $\mathbb{H} := \mathbb{A}$, $\mathbb{O} := \mathbb{A}^\infty$ and $\mathbb{K} := Z(\mathbb{O})$.

(21.38) Proposition There is a uniformizer $\pi \in \mathbb{H}^\perp$.

Proof
As we have $\mathbb{H} = \bar{\mathbb{O}}$ by remark (21.10), proposition 2 of [P] implies $\nu(\mathbb{K}) = 2\mathbb{Z}$. As a consequence, $\mathbb{O}$ is ramified and the quaternion division algebra $\mathbb{H}' := \langle\mathbb{H}\rangle_{\mathbb{K}}$ is an unramified composition algebra such that $\bar{\mathbb{H}}' = \mathbb{H}$. Theorem 2 of [P] shows that

$$\mathbb{O} \cong (\mathbb{H}', \pi')$$

for some $\nu_{|\mathbb{K}}$-uniformizer π' of $\mathbb{K}$, i.e., we have $\nu(\pi') = 2$. As a consequence, there is an element $\pi \in \mathbb{H}'^{\perp} \subseteq \mathbb{H}^{\perp}$ satisfying $N(\pi) = -\pi'$ and, therefore,

$$\nu(\pi) = \frac{\nu\big(N(\pi)\big)}{2} = \frac{\nu(-\pi')}{2} = \frac{\nu(\pi')}{2} = 1 \ .$$

□

(21.39) Theorem The foundation $\mathcal{F}$ is specially isomorphic to the standard positive foundation

$$\mathcal{P}_3^+(\mathbb{H}) := \{\mathcal{T}(\mathbb{A}^+_{(1,2)}) := \mathcal{T}(\mathbb{A}^+_{(2,3)}) := \mathcal{T}(\mathbb{A}^+_{(3,1)}) := \mathcal{T}(\mathbb{H}), \gamma_1^+ := \gamma_2^+ := \gamma_3^+ := \sigma_s\} \ .$$

Proof
By proposition (21.38), there is a uniformizer $\pi \in \mathbb{H}^{\perp}$. Let (x_1, x_2, x_3) be a parametrization for $\mathcal{T}(\mathbb{O})$. Then the maps

$$\begin{aligned}
x^i_{(i,j)} &: \mathbb{H} \to x_1(\mathbb{O}) \ , & t &\mapsto x_1(t) \in U_{1,0} \ , \\
x^{ij}_{(i,j)} &: \mathbb{H} \to x_2(\mathbb{O}) \ , & t &\mapsto x_2(t) \in U_{2,0} \ , \\
x^j_{(i,j)} &: \mathbb{H} \to x_3(\mathbb{O}) \ , & t &\mapsto x_3(t) \in U_{3,0}
\end{aligned}$$

yield a parametrization for the gem $\mathcal{R}_0 = \mathcal{R}_{\{i,j\}}$, cf. remark (21.10). Moreover, the maps

$$\begin{aligned}
x^j_{(j,k)} &: \mathbb{H} \to x_3(\mathbb{O}) \ , & t &\mapsto x_3(\bar{t}) \in U_{3,0} \ , \\
x^{jk}_{(j,k)} &: \mathbb{H} \to x_{-1}(\mathbb{O}) \ , & t &\mapsto x_{-1}(\bar{t}\pi) \in U_{-1,1} \ , \\
x^k_{(j,k)} &: \mathbb{H} \to x_{-2}(\mathbb{O}) \ , & t &\mapsto x_{-2}(\bar{t}\pi) \in U_{-2,1}
\end{aligned}$$

and

$$\begin{aligned}
x^k_{(k,i)} &: \mathbb{H} \to x_{-2}(\mathbb{O}) \ , & t &\mapsto x_{-2}(\pi\bar{t}) \in U_{-2,1} \ , \\
x^{ki}_{(k,i)} &: \mathbb{H} \to x_{-3}(\mathbb{O}) \ , & t &\mapsto x_{-3}(\pi\bar{t}) \in U_{-3,1} \ , \\
x^i_{(k,i)} &: \mathbb{H} \to x_1(\mathbb{O}) \ , & t &\mapsto x_1(\bar{t}) \in U_{1,0}
\end{aligned}$$

yield parametrizations for two gems $\mathcal{R}_{\{j,k\}}$ and $\mathcal{R}_{(k,i)}$ at distance one to each other and to $\mathcal{R}_{\{i,j\}}$, e.g., we have

$$[x^j_{(j,k)}(s), x^k_{(j,k)}(t)] = [x_3(\bar{s}), x_{-2}(\bar{t}\pi)] = x_{-1}\big(\bar{s}(\bar{t}\pi)\big) = x_{-1}\big((\bar{t}\bar{s})\pi\big) = x_{-1}\big((\overline{st})\pi\big) = x^{jk}_{(j,k)}(st)$$

for all $s, t \in \mathbb{H}$. By construction, this parameter system Λ parametrizes a root group system $\mathcal{U} := \mathcal{U}(\mathcal{B}, F, \Sigma, c)$. As we have

$$\begin{aligned}
x^j_{(i,j)}(t) &= x_3(t) = x^j_{(j,k)}(\bar{t}) \ , \\
x^k_{(j,k)}(t) &= x_{-2}(\bar{t}\pi) = x_{-2}\big(\pi(\pi^{-1}\bar{t}\pi)\big) = x_{-2}(\pi t) = x^k_{(k,i)}(\bar{t}) \ , \\
x^i_{(k,i)}(t) &= x_1(\bar{t}) = x^i_{(i,j)}(\bar{t}) \ ,
\end{aligned}$$

it follows that

$$\gamma_{(i,j,k)} = \gamma_{(j,k,i)} = \gamma_{(k,i,j)} = \sigma_s \ .$$

Therefore, the resulting foundation

$$\tilde{\mathcal{F}} := \mathcal{F}(\mathcal{U}, \Lambda)$$

is $\mathcal{P}_3^+(\mathbb{H})$. Finally $\mathcal{F} \cong \tilde{\mathcal{F}} = \mathcal{P}_3^+(\mathbb{H})$ by theorem (19.10). □

Chapter 22 Positive Foundations over Skew-Fields

Since each residue of an integrable foundation is itself integrable, a positive foundation is built up of positive triangle foundations. Their uniqueness enables us to show the uniqueness of positive foundations for a given quaternion division algebra $\mathbb{H}$ and a given value of $|I|$.

(22.1) Notation Throughout this chapter, $\mathcal{F}$ is a positive foundation over $I := \{1, \dots, n\}$. Since each rank 3 residue is a positive triangle, $\mathbb{H} := \mathbb{A}$ is a quaternion division algebra by lemma (21.36). Moreover, $\mathcal{G}_F$ is complete by corollary (19.30).

(22.2) Lemma Let $\tilde{\mathcal{F}}, \hat{\mathcal{F}}$ be isomorphic positive foundations over a skew-field, let $\alpha : \tilde{\mathcal{F}} \to \hat{\mathcal{F}}$ be an isomorphism and let $(i, j, k) \in G(F)$. By lemma (21.27), there are isomorphisms

$$\phi_i : \tilde{\mathbb{A}}_{(i,j)} \to \hat{\mathbb{A}}_{(\pi(i),\pi(j))} \ , \qquad \phi_k : \tilde{\mathbb{A}}_{(j,k)} \to \hat{\mathbb{A}}_{(\pi(j),\pi(k))}$$

of skew-fields and elements $a_i \in \hat{\mathbb{A}}_{(\pi(i),\pi(j))}$ and $a_k \in \hat{\mathbb{A}}_{(\pi(j),\pi(k)k)}$ such that

$$\alpha^j_{(i,j)} = \rho_{a_i} \circ \phi_i \ , \qquad \alpha^j_{(j,k)} = \lambda_{a_k} \circ \phi_k \ .$$

Then we have

$$\hat{\gamma} := \hat{\gamma}_{(\pi(i),\pi(j),\pi(k))} = \phi_k \circ \tilde{\gamma}_{(i,j,k)} \circ \phi_i^{-1} \ .$$

Proof
By the definition of an isomorphism, we have

$$\hat{\gamma} = \alpha^j_{(j,k)} \circ \tilde{\gamma}_{(i,j,k)} \circ (\alpha^j_{(i,j)})^{-1} = \lambda_{a_k} \circ \phi_k \circ \tilde{\gamma}_{(i,j,k)} \circ \circ \phi_i^{-1} \circ \rho_{a_i}^{-1} \ .$$

As $\tilde{\gamma}_{(i,j,k)}$ is positive, it follows that there are elements $a, b \in \hat{\mathbb{A}}_{(\pi(j),\pi(k))}$ such that

$$\hat{\gamma} = \lambda_a \circ \lambda_b \circ \phi_k \circ \tilde{\gamma}_{(i,j,k)} \circ \phi_i^{-1} \ .$$

Moreover, we have

$$ab = \hat{\gamma}(1) = 1$$

and therefore $a = b^{-1}$. It finally follows that

$$\hat{\gamma} = \phi_k \circ \tilde{\gamma}_{(i,j,k)} \circ \phi_i^{-1} \ .$$

□

(22.3) Remark The standard involution σ_s satisfies

$$\sigma_s \circ \phi = \phi \circ \sigma_s$$

for all $\phi \in \mathrm{Aut}(\mathbb{H})$, cf corollary (28.6).

(22.4) Lemma Let

$$\tilde{\mathcal{F}} := (\mathcal{T}(\tilde{\mathbb{A}}_{(1,2)}) := \mathcal{T}(\tilde{\mathbb{A}}_{(2,3)}) := \mathcal{T}(\tilde{\mathbb{A}}_{(3,1)}) := \mathcal{T}(\mathbb{H}), \tilde{\gamma}_2 := \tilde{\gamma}_{(1,2,3)}, \tilde{\gamma}_3 := \tilde{\gamma}_{(2,3,1)}, \tilde{\gamma}_1 := \tilde{\gamma}_{(3,1,2)})$$

be a positive triangle foundation over $\mathbb{H}$. Then we have

$$\tilde{\gamma}_3\tilde{\gamma}_2\tilde{\gamma}_1 = \sigma_s \ .$$

Proof
The reparametrizations

$$\tilde{\alpha}_{(1,2)} := (\mathbb{H}, \tilde{\gamma}_2^{-1}\tilde{\gamma}_3^{-1}, \tilde{\gamma}_2^{-1}\tilde{\gamma}_3^{-1}, \tilde{\gamma}_2^{-1}\tilde{\gamma}_3^{-1}) \ , \qquad \tilde{\alpha}_{(2,3)} := (\mathbb{H}, \tilde{\gamma}_3^{-1}\sigma_s, \tilde{\gamma}_3^{-1}\sigma_s, \tilde{\gamma}_3^{-1}\sigma_s)$$

show that $\tilde{\mathcal{F}}$ is isomorphic to the foundation

$$\hat{\mathcal{F}} := \{\mathcal{T}(\hat{\mathbb{A}}_{(1,2)}) := \mathcal{T}(\hat{\mathbb{A}}_{(2,3)}) := \mathcal{T}(\hat{\mathbb{A}}_{(3,1)}) := \mathcal{T}(\mathbb{H}), \hat{\gamma}_2 := \hat{\gamma}_3 := \sigma_s, \hat{\gamma}_1 := \tilde{\gamma}_3\tilde{\gamma}_2\tilde{\gamma}_1\} \ .$$

Since we have $\hat{\mathcal{F}} \cong \mathcal{P}_3^+(\mathbb{H})$ by theorem (21.39), there are isomorphisms

$$\alpha_{(i,j)} = (\lambda_{a_i}\phi_{(i,j)}, \lambda_{a_i}\rho_{b_j}\phi_{(i,j)}, \rho_{b_j}\phi_{(i,j)}) : \mathcal{T}(\tilde{\mathbb{A}}_{(i,j)}) \to \mathcal{T}(\hat{\mathbb{A}}_{(i,j)})$$

satisfying

$$\hat{\gamma}_1 = \phi_{(1,2)}\sigma_s\phi_{(3,1)}^{-1} \ , \qquad \sigma_s = \phi_{(2,3)}\sigma_s\phi_{(1,2)}^{-1} \ , \qquad \sigma_s = \phi_{(3,1)}\sigma_s\phi_{(2,3)}^{-1}$$

by the definition of an isomorphism and lemma (22.2). Now remark (22.3) implies

$$\phi_{(1,2)} = \phi_{(2,3)} = \phi_{(3,1)} \ , \qquad \tilde{\gamma}_3\tilde{\gamma}_2\tilde{\gamma}_1 = \hat{\gamma}_1 = \sigma_s \ .$$

□

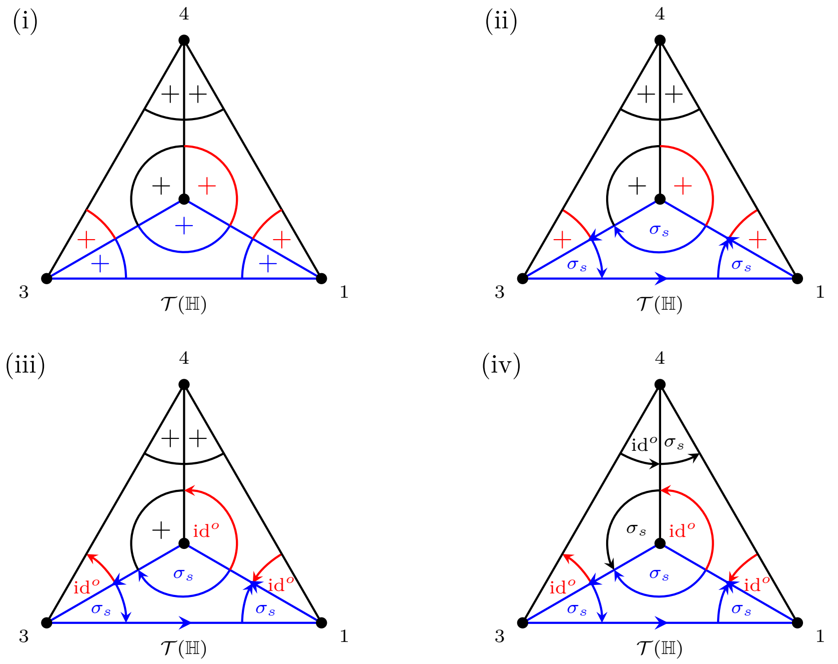
(22.5) Theorem The foundation $\mathcal{F}$ is specially isomorphic to $\tilde{\mathcal{F}} := \mathcal{P}_n^+(\mathbb{H})$.

Proof
Induction on n:

- $n = 3$: This is theorem (21.39).
- $n \to n+1$: By induction assumption, we may assume $\tilde{\mathcal{F}}_{[1,n]} = \mathcal{F}_{[1,n]}$, where $\tilde{\mathcal{F}}_{[1,n]}$ and $\mathcal{F}_{[1,n]}$ are the $[1,n]$-residues of $\tilde{\mathcal{F}}$ and $\mathcal{F}$, respectively. If we reparametrize the rank 2 residues of $\mathcal{F} \setminus \mathcal{F}_{[1,n]}$ and $\tilde{\mathcal{F}} \setminus \tilde{\mathcal{F}}_{[1,n]}$ in such a way that

  $$\tilde{\gamma}_{(j,j+1,n+1)} = \gamma_{(j,j+1,n+1)} = \mathrm{id}^o = \gamma_{(n+1,1,2)} = \tilde{\gamma}_{(n+1,1,2)} \ , \qquad j = 1, \dots, n-1 \ ,$$

  the remaining glueings are uniquely determined by lemma (22.4), thus corresponding glueings are equal.



□

Chapter 23 Mixed Foundations over Skew-Fields

In order to determine the integrable mixed foundations, we attach a graph to each of them. If this graph is a tree, it turns out that the corresponding foundation is isomorphic to one of the foundations constructed in §20.3. If it is not a tree, then the corresponding foundation is covered by a foundation of §20.3.

(23.1) Notation Throughout this chapter, $\mathcal{F}$ is a mixed foundation over a skew-field, i.e., there are positive and negative glueings.

(23.2) Remark Since $\mathcal{F}$ has positive residues, $\mathbb{H} := \mathbb{A}$ is a quaternion division algebra by lemma (21.36).

(23.3) Definition Let $\mathcal{F}$ be a foundation over $I = V(F)$. Given $i \in I$, the *set of neighbours of i* is

$$B_1(i) := \{j \in I \mid \{i, j\} \in E(F)\} .$$

(23.4) Lemma Let $\tilde{\mathcal{F}}$ be a foundation over $I = \{1, 2, 3, 4\}$ such that its defining field is a skew-field and such that $|B_1(1)| = 3$. Let

$$\tilde{\Gamma} := \{\tilde{\gamma}_{(2,1,3)}, \tilde{\gamma}_{(3,1,4)}, \tilde{\gamma}_{(2,1,4)}\} .$$

Then we have

$$n := |\{\tilde{\gamma} \in \tilde{\Gamma} \mid \tilde{\gamma} \text{ positive}\}| \in \{1, 3\} .$$

Proof
Notice that a glueing is either negative or positive.

(i) Suppose that $\tilde{\gamma}_{(2,1,3)}$ and $\tilde{\gamma}_{(3,1,4)}$ are negative. Then

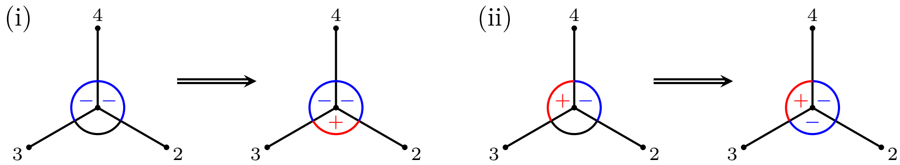
$$\tilde{\gamma}_{(2,1,4)} = \tilde{\gamma}_{(3,1,4)} \circ \mathrm{id}^o \circ \tilde{\gamma}_{(2,1,3)}$$

is positive, thus $n \geq 1$.

(ii) Suppose that $\tilde{\gamma}_{(2,1,3)}$ is positive and that $\tilde{\gamma}_{(3,1,3)}$ is negative. Then

$$\tilde{\gamma}_{(2,1,4)} = \tilde{\gamma}_{(3,1,4)} \circ \mathrm{id}^o \circ \tilde{\gamma}_{(2,1,3)}$$

is negative, thus $n \neq 2$.



Notice that there could be more edges than the drawn ones. □

(23.5) Notation Given a mixed foundation $\tilde{\mathcal{F}}$ over $\mathbb{H}$, we denote the collection of all maximal positive residues of $\tilde{\mathcal{F}}$ by $\mathcal{M}(\tilde{\mathcal{F}})$. We set

$$P(\tilde{\mathcal{F}}) := \{\mathcal{G}_{\tilde{P}} \mid \tilde{\mathcal{P}} \in \mathcal{M}(\tilde{\mathcal{F}})\} .$$

(23.6) Remark Rank 2 two residues are considered to be positive, thus

$$\bigcup_{\mathcal{P} \in \mathcal{M}(\mathcal{F})} \mathcal{G}_P = \mathcal{G}_F .$$

(23.7) Theorem The set $\mathcal{M}(\mathcal{F})$ satisfies the following conditions:

(i) Each $\mathcal{P} \in \mathcal{M}(\mathcal{F})$ is integrable, in particular, $\mathcal{G}_P$ is complete.

(ii) We have
$$\bigcup_{\mathcal{P} \in \mathcal{M}(\mathcal{F})} \mathcal{G}_P = \mathcal{G}_F \ .$$

(iii) Given $\mathcal{P}_1 \neq \mathcal{P}_2 \in \mathcal{M}(\mathcal{F})$, we have $|V(P_1) \cap V(P_2)| \leq 1$.

(iv) Given $i \in I$, we have $|\{\mathcal{P} \in \mathcal{M}(\mathcal{F}) \mid i \in V(P)\}| \leq 2$.

Proof

(i) Each $\mathcal{P} \in \mathcal{M}(\mathcal{F})$ is integrable because $\mathcal{F}$ itself is integrable.

(ii) This holds by remark (23.6).

(iii) Since the elements of $\mathcal{M}(\mathcal{F})$ are maximal and G_P is complete for each $\mathcal{P} \in \mathcal{M}(\mathcal{F})$, lemma (23.4) implies that
$$\forall\ \mathcal{P}_1 \neq \mathcal{P}_2 \in \mathcal{M}(\mathcal{F}) : \qquad |V(P_1) \cap V(P_2)| \leq 1 \ .$$
Otherwise $\mathcal{P}_1 \cup \mathcal{P}_2$ would be positive with $\mathcal{P}_1 \subsetneq \mathcal{P}_1 \cup \mathcal{P}_2$.

(iv) By step (ii), the glueings connecting two elements $\mathcal{P}_1 \neq \mathcal{P}_2 \in \mathcal{M}(\mathcal{F})$ are necessarily negative. From lemma (23.4) again it follows that
$$\forall\ i \in I : \qquad |\{\mathcal{P} \in \mathcal{M}(\mathcal{F}) \mid i \in V(P)\}| \leq 2 \ .$$

□

(23.8) Proposition Let $\tilde{\mathcal{F}}$ be a mixed foundation over $\mathbb{H}$ such that $\mathcal{M}(\tilde{\mathcal{F}})$ satisfies the conditions (i)-(iv) of theorem (23.7). If the graph $\mathcal{G} := \mathcal{G}_{\tilde{F}}^{P(\tilde{\mathcal{F}})}$ is a tree, we have $\tilde{\mathcal{F}} \cong \mathcal{F}(\mathcal{G}, \mathbb{H}) =: \hat{\mathcal{F}}$. In particular, $\tilde{\mathcal{F}}$ is integrable.

Proof

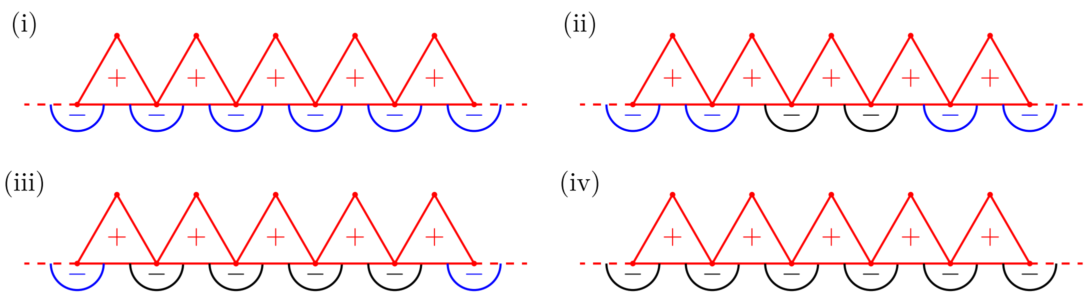
First of all we observe that $\hat{\mathcal{F}}$ is well-defined the conditions (i)-(iv). By construction, we have $P(\tilde{\mathcal{F}}) = P(\hat{\mathcal{F}})$, and by theorem (22.5), we have
$$\tilde{\mathcal{P}} \cong \mathcal{P}^+_{|V(\tilde{P})|}(\mathbb{H}) =: \hat{\mathcal{P}} \in \mathcal{M}(\hat{\mathcal{F}})$$
for each $\tilde{P} \in \mathcal{M}(\tilde{\mathcal{F}})$. Since the isomorphisms are special, we may assume $\tilde{\mathcal{P}} = \hat{\mathcal{P}}$ for each $\tilde{P} \in \mathcal{M}(\tilde{\mathcal{F}})$. It remains to adjust the glueings connecting two elements of $\mathcal{M}(\tilde{\mathcal{F}})$. Since $\mathcal{G}$ is a tree and
$$|\{\tilde{\mathcal{P}} \in \mathcal{M}(\tilde{\mathcal{F}}) \mid i \in V(\tilde{P})\}| \leq 2$$
for each vertex $i \in I$, it suffices to show the following:

Given $\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2 \in \mathcal{M}(\tilde{F})$ with $\{\mathcal{G}_{\tilde{P}_1}, \mathcal{G}_{\tilde{P}_2}\} \in E(\mathcal{G})$, there is an isomorphism $\alpha : \tilde{\mathcal{P}}_1 \cup \tilde{\mathcal{P}}_2 \to \hat{\mathcal{P}}_1 \cup \hat{\mathcal{P}}_2$ fixing $\tilde{\mathcal{P}}_1$.

Let $V(\tilde{P}_1) \cap V(\tilde{P}_2) = \{b\}$, let $a \in V(\tilde{P}_1) \setminus \{b\}$, let $c \in V(\tilde{P}_2) \setminus \{b\}$ and let $\gamma := \hat{\gamma}_{(a,b,c)} \circ \tilde{\gamma}^{-1}_{(a,b,c)}$. Then
$$\alpha := \{\mathrm{id}_F, \alpha_{(i,j)} \mid (i,j) \in A(\tilde{P}_1 \cup \tilde{P}_2)\}$$
with
$$\forall\ (i,j) \in A(\tilde{P}_1) : \alpha_{(i,j)} := (\mathrm{id}, \mathrm{id}, \mathrm{id}) \ , \qquad \forall\ (i,j) \in A(\tilde{P}_2) : \alpha_{(i,j)} = (\gamma, \gamma, \gamma)$$
satisfies the required condition.

Finally $\tilde{\mathcal{F}}$ is integrable by corollary (19.20) and theorem (20.17). □

(23.9) Example



The red triangles represent the elements of $\mathcal{M}(\tilde{\mathcal{F}})$, the blue glueings represent the corresponding glueings of $\tilde{\mathcal{F}}$, while the black ones represent those of $\hat{\mathcal{F}}$.

(23.10) Corollary If $\mathcal{G} := \mathcal{G}_F^{P(\mathcal{F})}$ is a tree, then we have

$$\mathcal{F} \cong \mathcal{F}(\mathcal{G}, \mathbb{H}) \ .$$

Proof
This results immediately from theorem (23.7) and proposition (23.8). $\square$

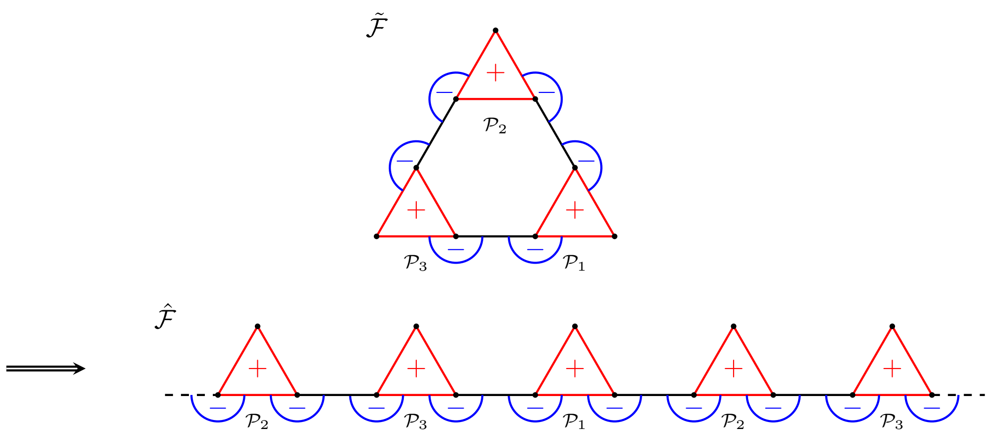
(23.11) Theorem Let $\tilde{\mathcal{F}}$ be a mixed foundation over $\mathbb{H}$ such that $\mathcal{M}(\tilde{\mathcal{F}})$ satisfies the conditions (i)-(iv) of theorem (23.7). Then $\tilde{\mathcal{F}}$ is integrable.

Proof
Let $\mathcal{U}$ be the the universal cover of $\mathcal{G}_{\tilde{F}}^{P(\tilde{\mathcal{F}})}$ and let $\hat{\mathcal{F}}$ be the cover corresponding to the induced cover of $\mathcal{G}_{\tilde{F}}$. By theorem (20.4), $\tilde{\mathcal{F}}$ is integrable if $\hat{\mathcal{F}}$ is integrable. But $\mathcal{M}(\hat{\mathcal{F}})$ equally satisfies the conditions (i)-(iii) of proposition (23.8) and, moreover,

$$\mathcal{G}_{\hat{F}}^{P(\hat{\mathcal{F}})} \cong \mathcal{U}$$

is a tree. Therefore, $\hat{\mathcal{F}}$ is integrable by proposition (23.8). $\square$

(23.12) Example



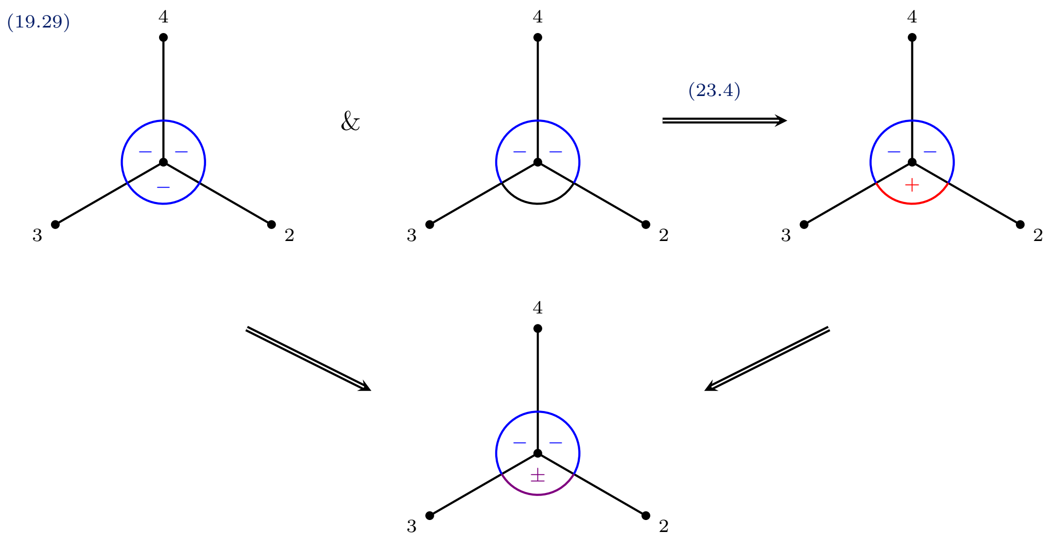
Chapter 24 Negative Foundations

Negative foundations are quite easy to handle as we may apply theorems (20.4) and (20.7). If $\mathbb{A}$ is a field, there are no restrictions; each foundation is integrable. If $\mathbb{A}$ is non-commutative, then lemma (24.2) is very restrictive; the structure of an integrable foundation is very simple.

(24.1) Notation Throughout this chapter, $\mathcal{F}$ is a negative foundation over a skew-field.

(24.2) Lemma If $\mathcal{F}$ has a residue $\mathcal{R}$ of type D_4, then $\mathbb{A}$ is a field. In particular, $\mathcal{F}$ is negative.

Proof
We label the vertices of $\mathcal{R}$ such that $\{1,2\},\{1,3\},\{1,4\} \in E(R)$. By lemma (19.29), each glueing is negative, and by lemma (23.4), at least one is positive.



□

(24.3) Theorem If $\mathbb{A}$ is a non-commutative skew-field, then $\mathcal{G}_F$ is a string, a ray, a chain or a circle.

Proof
By lemma (24.2), $\mathcal{G}_F$ has no branches. □

(24.4) Theorem Let $\tilde{\mathcal{F}}$ be a negative foundation. Then the following holds:

(a) If $\tilde{\mathbb{A}}$ is a field, then $\tilde{\mathcal{F}}$ is integrable.

(b) If $\tilde{\mathbb{A}}$ is a non-commutative skew-field and $\mathcal{G}_{\tilde{F}}$ is a string, a ray, a chain or a circle, then $\tilde{\mathcal{F}}$ is integrable.

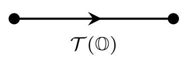
Proof
By theorem (20.4), $\tilde{\mathcal{F}}$ is integrable if its universal cover $\mathcal{U}$ is integrable. But since $\mathcal{G}_U$ is a tree, $\mathcal{U}$ is isomorphic to the corresponding canonical foundation by lemma (20.6), which is integrable by theorem (20.7). □

Chapter 25 Conclusion

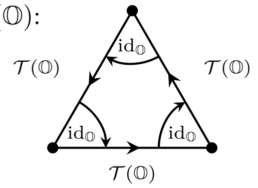
(25.1) Theorem (**Classification of Simply Laced Twin Buildings**) Let $\mathcal{F}$ be an irr. simply laced foundation. Then $\mathcal{F}$ is integrable if and only if one of the following holds:

- The defining field is an octonion division algebra $\mathbb{O}$, and $\mathcal{F}$ is isomorphic to one of the following foundations:

$\mathcal{A}_2(\mathbb{O})$:



$\tilde{\mathcal{A}}_2(\mathbb{O})$:

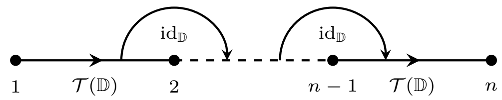


.

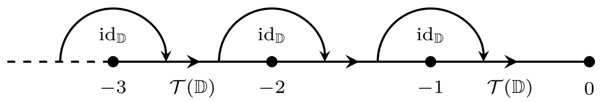
- The defining field is a quaternion division algebra $\mathbb{H}$, and $\mathcal{M}(\mathcal{F})$ satisfies the conditions (i)-(iv) of theorem (23.7).

- The defining field is a non-commutative skew-field $\mathbb{D}$ different from a quaternion algebra, and $\mathcal{F}$ is isomorphic to one of the following foundations (where $\gamma_1, \ldots, \gamma_{n+1} \in \mathrm{Aut}(\mathbb{D})$):

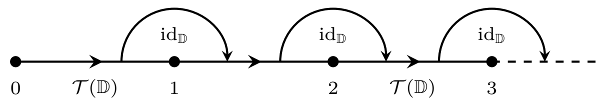
$\mathcal{A}_n(\mathbb{D})$:



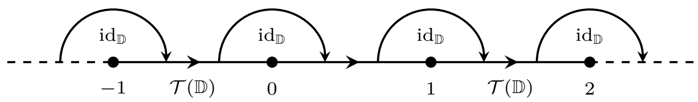
$\mathcal{A}^l_\infty(\mathbb{D})$:



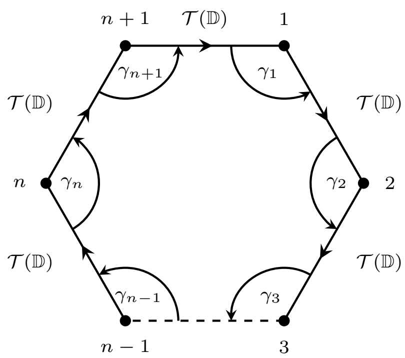
$\mathcal{A}^r_\infty(\mathbb{D})$:



$\mathcal{A}_\infty(\mathbb{D})$:



$\tilde{\mathcal{A}}_n(\mathbb{D}, \gamma_i)$:



- The defining field is a field, and there are no further restrictions on the foundation $\mathcal{F}$.

Proof
This results from theorems (21.25), (23.7), (23.11), (24.3) and (24.4). □

(25.2) Remark

(a) Given a non-commutative skew-field $\mathbb{D}$, we have

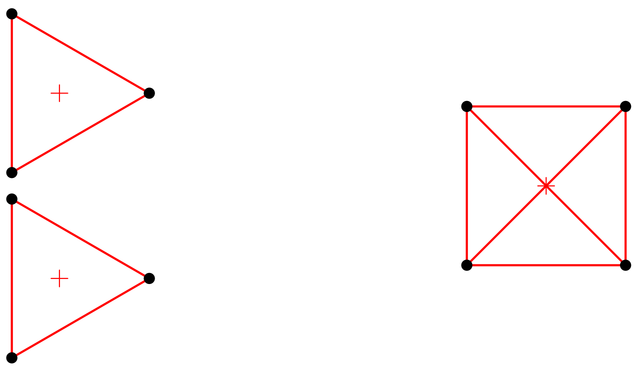
$$\mathcal{A}^l_\infty(\mathbb{D}) \cong \mathcal{A}^r_\infty(\mathbb{D}) \;\Leftrightarrow\; \mathbb{D} \cong \mathbb{D}^o \ .$$

(b) The theorem can be stated in a more precise way by using classifying invariants. For example, given a skew-field $\mathbb{D}$, the foundation $\tilde{\mathcal{A}}_n(\mathbb{D}, \gamma_1, \ldots, \gamma_{n+1})$ only depends on the coset

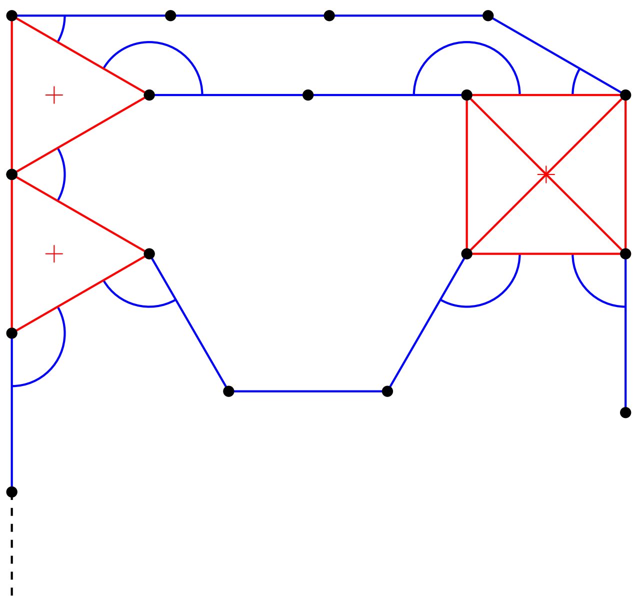
$$\left(\prod_{i=1}^{n+1} \gamma_i\right) \cdot \mathrm{Inn}(\mathbb{D}) \in \mathrm{Aut}(\mathbb{D})/\mathrm{Inn}(\mathbb{D}) \ .$$

(c) Each integrable mixed foundation over a quaternion division algebra $\mathbb{H}$ arises in the following way: Start with some integrable positive foundations, then add some negative chains or strings of arbitrary length with the rule that each vertex of a positive foundation is part of at most one negative chain or string, e.g.,

(i)



(ii)



Part V

Jordan Automorphisms of Alternative Division Rings

In this part, we determine the structure of the group $\mathrm{Aut}_J(\mathbb{A})$ of Jordan automorphisms for an alternative division ring $\mathbb{A}$. If $\mathbb{A}$ is a skew-field, we know the answer by Hua's theorem. Since a non-associative alternative division ring is an octonion division algebra by the Bruck-Kleinfeld theorem (3.33), it remains to consider the octonion case. It turns out that we just have to add a certain class of Jordan automorphisms, each of them fixing a quaternion subalgebra.

One basic tool is the possibility to extend isomorphisms between subalgebras to an automorphism of the whole algebra, see [Sp] for a detailed reference. Moreover, the crucial thing is the fact that nothing can go wrong with Jordan homomorphisms on fields, i.e., they are just monomorphisms of rings. This is not true for skew-fields, cf. §132.

Chapter 26 Composition Algebras and Norm Similarities

The theorems of this chapter provide the existence of automorphisms which we will need in §28.3.

(26.1) Definition A *composition algebra over a field* $\mathbb{K}$ is a unital algebra $\mathbb{A}$ over $\mathbb{K}$ together with a non-defective quadratic form $N : \mathbb{A} \to \mathbb{K}$ which *permits composition*, i.e., we have

$$\forall\, x, y \in \mathbb{A} : \qquad N(xy) = N(x)N(y) \ .$$

(26.2) Lemma An octonion division algebra $\mathbb{O}$ is a composition algebra over $\mathbb{K} := Z(\mathbb{O})$.

Proof
The norm N is non-defective by corollary (3.24) and multiplicative by (9.9)(iii) of [TW]. □

(26.3) Definition For $i = 1, 2$, let V_i be a vector space over $\mathbb{K}_i$ with non-defective quadratic form N_i, and let $\sigma : \mathbb{K}_1 \to \mathbb{K}_2$ be an isomorphism of fields.

- A *σ-similarity* is an isomorphism $(\varphi, \sigma) : (V_1, \mathbb{K}_1) \to (V_2, \mathbb{K}_2)$ of vector spaces such that
$$\forall\, v \in V_1 : \qquad N_2\big(\varphi(v)\big) = \rho_\varphi \cdot \sigma\big(N_1(v)\big)$$
for some element $\rho_\varphi \in \mathbb{K}_2^*$, which is the *multiplier of* φ.
- A *similarity* is a σ-similarity such that $\mathbb{K}_2 = \mathbb{K}_1$ and $\sigma = \mathrm{id}_{\mathbb{K}_1}$.
- A *σ-isometry* is a σ-similarity such that $\rho_\varphi = 1_{\mathbb{K}_2}$.
- An *isometry* is a σ-isometry such that $\mathbb{K}_2 = \mathbb{K}_1$ and $\sigma = \mathrm{id}_{\mathbb{K}_1}$.

(26.4) Lemma For $i = 1, 2$, let V_i be a vector space over $\mathbb{K}_i$ with non-defective quadratic form N_i and associated bilinear form $\langle \cdot, \cdot \rangle_i$, and let $\varphi : V_1 \to V_2$ be a σ-similarity. Then φ satisfies

$$\forall\, x, y \in V_1 : \qquad \langle \varphi(x), \varphi(y) \rangle_2 = \rho_\varphi \cdot \sigma_\varphi(\langle x, y \rangle_1) \ .$$

In particular, we have $\varphi(M^\perp) = \varphi(M)^\perp$ for each subset $M \subseteq V_1$.

Proof
Given $x \in M$ and $y \in M^\perp$, we have

$$\begin{aligned} \langle \varphi(x), \varphi(y) \rangle_2 &= N_2\big(\varphi(x) + \varphi(y)\big) - N_2\big(\varphi(x)\big) - N_2\big(\varphi(y)\big) \\ &= \rho_\varphi \cdot \sigma_\varphi\big(N_1(x+y) - N_1(x) - N_1(y)\big) = \rho_\varphi \cdot \sigma_\varphi(\langle x, y \rangle_1) = 0_{\mathbb{K}_2} \ . \end{aligned}$$

□

(26.5) Notation Let V be a vector space over $\mathbb{K}$ with non-defective quadratic form N. We denote the *group of σ-isometries of* V by

$$\mathit{\Gamma L}_N(V, \mathbb{K}) := \big\{ (\varphi, \sigma) \in \mathit{\Gamma L}(V) \mid \forall\, v \in V : \ N\big(\varphi(v)\big) = \sigma\big(N(v)\big) \big\} \ .$$

(26.6) Theorem Let $\mathbb{A}$ be a composition algebra over $\mathbb{K}$ and let $\sigma \in \mathrm{Aut}(\mathbb{K})$. Then there exists a σ-automorphism $\phi \in \mathrm{Aut}(\mathbb{A},\mathbb{K})$ if and only if there exists a σ-isometry $\varphi \in \mathit{\Gamma} L_N(\mathbb{A},\mathbb{K})$.

Proof
This is corollary (1.7.2) of [S]. □

(26.7) Theorem Let $\mathbb{A}$ be a composition algebra over $\mathbb{K}$, let $\sigma \in \mathrm{Aut}(\mathbb{K})$ and let $\mathbb{B}_1, \mathbb{B}_2$ be subalgebras of the same dimension. If there exists a σ-isometry $\varphi \in \mathit{\Gamma} L_N(\mathbb{A},\mathbb{K})$, then each σ-isomorphism $\phi : \mathbb{B}_1 \to \mathbb{B}_2$ of algebras can be extended to a σ-automorphism $\tilde{\phi} \in \mathrm{Aut}(\mathbb{A},\mathbb{K})$. In particular, each linear isomorphism $\psi : \mathbb{B}_1 \to \mathbb{B}_2$ of algebras can be extended to a linear automorphism $\tilde{\psi} \in \mathrm{Aut}_{\mathbb{K}}(\mathbb{A},\mathbb{K})$.

Proof
This is corollary (1.7.3) of [Sp]. □

Chapter 27 Jordan Homomorphisms

At this point we recall the definition of Jordan homomorphisms which play a central role in the classification of simply laced twin buildings. As the classification involves some detailed calculations concerning Jordan homomorphisms, it is natural to use this knowledge for a generalization of Hua's theorem to octonions.

§ 27.1 Basic Definitions and Basic Properties

(27.1) Definition

- Let $\mathbb{A}, \tilde{\mathbb{A}}$ be alternative division rings. A *Jordan homomorphism* is an additive monomorphism $\gamma : \mathbb{A} \to \tilde{\mathbb{A}}$ such that

$$\gamma(1_{\mathbb{A}}) = 1_{\tilde{\mathbb{A}}} \ , \qquad \forall\, x, y \in \mathbb{A} : \ \gamma(xyx) = \gamma(x)\gamma(y)\gamma(x) \ .$$

- Given an alternative division ring $\mathbb{A}$, we denote the *group of Jordan automorphisms of* $\mathbb{A}$ by

$$\mathrm{Aut}_J(\mathbb{A}) := \{\gamma : \mathbb{A} \to \mathbb{A} \mid \gamma \text{ Jordan automorphism}\} \ .$$

(27.2) Lemma Let $\mathbb{A}, \tilde{\mathbb{A}}$ be alternative division rings. A Jordan homomorphism $\gamma : \mathbb{A} \to \tilde{\mathbb{A}}$ satisfies

$$\forall\, x, y \in \mathbb{A} : \qquad \gamma(xy) + \gamma(yx) = \gamma(x)\gamma(y) + \gamma(y)\gamma(x) \ .$$

Proof
The assertion is clearly true for $x = 0_{\mathbb{A}}$ or $y = 0_{\mathbb{A}}$, so assume $x \neq 0_{\mathbb{A}} \neq y$. As we have

$$\gamma(z^2) = \gamma(z \cdot 1_{\mathbb{A}} \cdot z) = \gamma(z)\gamma(1_{\mathbb{A}})\gamma(z) = \gamma(z) \cdot 1_{\tilde{\mathbb{A}}} \cdot \gamma(z) = \gamma(z)^2$$

for each $z \in \mathbb{A}$, it follows that

$$\begin{aligned} \gamma\big((x+y)^2\big) &= \gamma(x^2 + xy + yx + y^2) = \gamma(x)^2 + \gamma(xy) + \gamma(yx) + \gamma(y)^2 \ , \\ \gamma\big((x+y)^2\big) &= \gamma(x+y)^2 = \big(\gamma(x) + \gamma(y)\big)^2 = \gamma(x)^2 + \gamma(x)\gamma(y) + \gamma(y)\gamma(x) + \gamma(y)^2 \end{aligned}$$

and thus

$$\gamma(xy) + \gamma(yx) = \gamma(x)\gamma(y) + \gamma(y)\gamma(x) \ .$$

□

(27.3) Lemma Let $\mathbb{A}, \tilde{\mathbb{A}}$ be alternative division rings. A Jordan homomorphism $\gamma : \mathbb{A} \to \tilde{\mathbb{A}}$ satisfies

$$\forall\, x \in \mathbb{A}^* : \qquad \gamma(x^{-1}) = \gamma(x)^{-1} \ .$$

Proof
Given $x \in \mathbb{A}^*$, we have

$$\gamma(x^{-1}) = \gamma(x^{-1}xx^{-1}) = \gamma(x^{-1})\gamma(x)\gamma(x^{-1})$$

and thus $\gamma(x^{-1}) = \gamma(x)^{-1}$ by lemma (3.6). □

§ 27.2 Jordan Homomorphisms on Fields

A Jordan homomorphism from a field $\mathbb{F}$ to an alternative division ring $\mathbb{A}$ is a monomorphism of rings. In particular, the image of $\mathbb{A}$ is a subfield of $\mathbb{A}$.

(27.4) Lemma Let $\mathbb{F}$ be a field, let $\mathbb{A}$ be an alternative division ring and let $\gamma : \mathbb{F} \to \mathbb{A}$ be a Jordan homomorphism. Then $\gamma : \mathbb{F} \to \gamma(\mathbb{F})$ is an isomorphism of fields.

Proof
By Hua's theorem, we may assume that $\mathbb{A}$ is an octonion division algebra. By lemma (27.2), we have

$$2\gamma(xy) = \gamma(x)\gamma(y) + \gamma(y)\gamma(x) \tag{27.1}$$

for all $x, y \in \mathbb{F}$.

- $\operatorname{Char}\mathbb{F} \neq 2$: Given $x, y \in \mathbb{F}^*$, we have

  $$\gamma(xy)\gamma(x^{-1}y^{-1}) \overset{(3.6)}{=} \gamma(xy)\gamma\big((xy)^{-1}\big) \overset{(27.3)}{=} \gamma(xy)\gamma(xy)^{-1} = 1_{\mathbb{A}}$$

  and thus by equation (27.1) and lemma (3.25) (a)

  $$\begin{aligned} 4 &= 2\gamma(xy)2\gamma(x^{-1}y^{-1}) = [\gamma(x)\gamma(y) + \gamma(y)\gamma(x)][\gamma(x^{-1})\gamma(y^{-1}) + \gamma(y^{-1})\gamma(x^{-1})] \\ &= \gamma(x)\gamma(y)\gamma(x)^{-1}\gamma(y)^{-1} + 1_{\mathbb{A}} + 1_{\mathbb{A}} + \gamma(y)\gamma(x)\gamma(y)^{-1}\gamma(x)^{-1} \ . \end{aligned}$$

  We set $z := \gamma(x)\gamma(y)\gamma(x)^{-1}\gamma(y)^{-1}$ and obtain

  $$2 = z + z^{-1} \ , \qquad\qquad (z - 1_{\mathbb{A}})^2 = z^2 - 2z + 1_{\mathbb{A}} = 0_{\mathbb{A}}$$

  and therefore

  $$\gamma(x)\gamma(y)\gamma(x)^{-1}\gamma(y)^{-1} = z = 1_{\mathbb{A}} \ , \qquad\qquad \gamma(x)\gamma(y) = \gamma(y)\gamma(x) \ .$$

  From equation (27.1) it follows that $\gamma(xy) = \gamma(x)\gamma(y)$.

- $\operatorname{Char}\mathbb{F} = 2$: In this case, equation (27.1) implies

  $$\gamma(x)\gamma(y) = \gamma(y)\gamma(x)$$

  for all $x, y \in \mathbb{F}$. As a consequence,

  $$\tilde{\mathbb{F}} := \langle \gamma(\mathbb{F}) \rangle_{Z(\mathbb{A})}$$

  is a commutative subalgebra of $\mathbb{A}$, hence a field. Since $\gamma : \mathbb{F} \to \tilde{\mathbb{F}}$ is a Jordan homomorphism, Hua's theorem implies that $\gamma : \mathbb{F} \to \gamma(\mathbb{F})$ is an iso- or anti-isomorphism of skew-fields, and thus, in fact, an isomorphism of fields.

□

Chapter 28 Jordan Automorphisms

In this chapter, we determine the structure of $\mathrm{Aut}_J(\mathbb{A})$ for an alternative division ring $\mathbb{A}$. If $\mathbb{A}$ is a skew-field, then Hua's theorem gives the answer:

(28.1) Theorem Let $\mathbb{A}$ be skew-field. Then we have

$$\mathrm{Aut}_J(\mathbb{A}) = \mathrm{Aut}(\mathbb{A}) \cup \mathrm{Aut}^o(\mathbb{A}) \ .$$

Thus it remains to consider the Jordan automorphisms of an octonion division algebra $\mathbb{O}$.

(28.2) Notation Throughout the rest of this chapter, $\mathbb{O}$ is an octonion division algebra and $\mathbb{K} := Z(\mathbb{O})$ is its center.

§ 28.1 Jordan Automorphisms on Subfields

As a consequence of the last paragraph, a Jordan automorphism restricted to a subfield is a monomorphism of rings. In particular, the image of a subfield is again a subfield.

This is not true for skew-subfields. Otherwise, a Jordan automorphism would be, in fact, an auto- or anti-automorphism by the proof of Hua's theorem. But there are indeed Jordan automorphism neither of the first nor of the second kind, cf. lemma (28.13) and remark (28.12).

(28.3) Lemma Let $\gamma \in \mathrm{Aut}_J(\mathbb{O})$ and let $\mathbb{F}$ be a subfield of $\mathbb{O}$. Then $\gamma_{|\mathbb{F}} : \mathbb{F} \to \gamma(\mathbb{F})$ is an isomorphism of fields.

Proof
This results from Lemma (27.4). □

(28.4) Corollary An element $\gamma \in \mathrm{Aut}_J(\mathbb{O})$ satisfies

$$\gamma(\lambda) \cdot \gamma(x) = \gamma(x) \cdot \gamma(\lambda)$$

for all $\lambda \in \mathbb{K}$, $x \in \mathbb{O}$.

Proof
Since each element $x \in \mathbb{O}$ is contained in a subfield $\mathbb{F}$ of $\mathbb{O}$ with $\mathbb{K} \subseteq \mathbb{F}$ by (20.9) of [TW], we may apply lemma (28.3) to obtain

$$\gamma(\lambda) \cdot \gamma(x) = \gamma(\lambda \cdot x) = \gamma(x \cdot \lambda) = \gamma(x) \cdot \gamma(\lambda)$$

for all $\lambda \in \mathbb{K}$, $x \in \mathbb{O}$. □

§ 28.2 Jordan Automorphisms and Norm Similarities

The results of §28.1 enable us to show that Jordan automorphisms are norm similarities. As a consequence, we may apply the results of chapter 26. Moreover, the results of [J] show the reverse inclusion for $\mathrm{Char}\,\mathbb{O} \neq 2$.

(28.5) Proposition Let $\gamma \in \mathrm{Aut}_J(\mathbb{O})$ and $\sigma := \gamma_{|\mathbb{K}}$. Then the following holds:

(a) $\sigma \in \mathrm{Aut}(\mathbb{K})$.

(b) $(\gamma, \sigma) \in \mathit{\Gamma L}(\mathbb{O}, \mathbb{K})$.

(c) $(\gamma, \sigma) \in \mathit{\Gamma L}_N(\mathbb{O}, \mathbb{K})$.

Proof

(a) By corollary (28.4) and lemma (3.28), we have $\gamma(\mathbb{K}) = \mathbb{K}$ and therefore $\sigma \in \mathrm{Aut}(\mathbb{K})$ by lemma (27.4).

(b) This is a consequence of (a) and corollary (28.4).

(c) By (a), the assertion is true for $x \in \mathbb{K}$. Given $x \in \mathbb{O} \setminus \mathbb{K}$, we have

$$x^2 - T(x)x + N(x) = 0_{\mathbb{O}} \ , \qquad \gamma(x)^2 - T\big(\gamma(x)\big)\gamma(x) + N\big(\gamma(x)\big) = 0_{\mathbb{O}}$$

and hence

$$\begin{aligned} 0_{\mathbb{O}} &= \gamma^{-1}\big(\gamma(x)^2 - T(\gamma(x))\gamma(x) + N(\gamma(x))\big) = \gamma^{-1}\gamma\big(x^2 - \gamma^{-1}\big(T(\gamma(x))\big)x + \gamma^{-1}(N(\gamma(x)))\big) \\ &= x^2 - \gamma^{-1}\big(T(\gamma(x))\big)x + \gamma^{-1}\big(N(\gamma(x))\big) \ . \end{aligned}$$

As the maps T and N are uniquely determined by the minimum equation, we obtain

$$\gamma^{-1}\big(N(\gamma(x))\big) = N(x) \ , \qquad N\big(\gamma(x)\big) = \gamma\big(N(x)\big) = \sigma\big(N(x)\big)$$

for all $x \in \mathbb{O} \setminus \mathbb{K}$.

□

(28.6) Corollary We have

$$\sigma_s \in Z\big(\mathrm{Aut}_J(\mathbb{O})\big) \ .$$

Proof
Given $\gamma \in \mathrm{Aut}_J(\mathbb{O})$ and $x \in \mathbb{O}$, we have

$$\gamma\sigma_s(x) = \gamma(\bar{x}) = \gamma\big(N(x) \cdot x^{-1}\big) = \sigma\big(N(x)\big) \cdot \gamma(x^{-1}) = N\big(\gamma(x)\big) \cdot \gamma(x)^{-1} = \overline{\gamma(x)} = \sigma_s\gamma(x) \ .$$

□

(28.7) Remark The results of [J] are valid for Jordan algebras with characteristic different from 2, cf. definition (1.3) of [J].

(28.8) Lemma If $\mathrm{Char}\,\mathbb{O} \neq 2$, the Jordan algebra $\mathbb{O}$ is separable

Proof
By corollary (3.24), the bilinear form $\langle\cdot,\cdot\rangle$ and thus the trace form

$$\bar{T} : \mathbb{O} \times \mathbb{O} \to \mathbb{K}, \ (x,y) \mapsto T(xy) = xy + \bar{y}\bar{x} = \langle x, \bar{y}\rangle$$

is non-degenerate. Now the assertion results from theorem (6.5) of [J]. □

(28.9) Proposition If $\mathrm{Char}\,\mathbb{O} \neq 2$, we have

$$O(\mathbb{O},\mathbb{K}) = GL_N(\mathbb{O},\mathbb{K}) \subseteq \mathrm{Aut}_J(\mathbb{O}) \ .$$

Proof
Let $\varphi = (\varphi, \mathrm{id}_{\mathbb{K}}) \in GL_N(\mathbb{O},\mathbb{K})$. As we have $\varphi(1_{\mathbb{O}}) = \varphi(1_{\mathbb{O}} \cdot 1_{\mathbb{O}}) = 1_{\mathbb{O}} \cdot \varphi(1_{\mathbb{O}})$, the isometry φ satisfies $\varphi(1_{\mathbb{O}}) = 1_{\mathbb{O}}$, and because of $|\mathbb{K}| > 2 = \deg \mathbb{O}$, the assertion results from lemma (28.8) and theorem (6.7)(a) of [J]. □

(28.10) Theorem If $\mathrm{Char}\,\mathbb{O} \neq 2$, we have

$$\mathit{\Gamma} L_N(\mathbb{O},\mathbb{K}) = \mathrm{Aut}_J(\mathbb{O}) \ .$$

Proof

"$\subseteq$" Let $(\varphi, \sigma) \in \mathit{\Gamma L}_N(\mathbb{O}, \mathbb{K})$. By theorem (26.6), there is a σ-automorphism $(\phi, \sigma) \in \mathrm{Aut}(\mathbb{O}, \mathbb{K})$. By proposition (28.9), we have

$$(\phi^{-1}, \sigma^{-1}) \cdot (\varphi, \sigma) \in GL_N(\mathbb{O}, \mathbb{K}) \subseteq \mathrm{Aut}_J(\mathbb{O})$$

and thus

$$(\varphi, \sigma) \in (\phi, \sigma) \cdot \mathrm{Aut}_J(\mathbb{O}) = \mathrm{Aut}_J(\mathbb{O}) \ .$$

"$\supseteq$" This is proposition (28.5) (c).

□

§ 28.3 The Structure of $\mathrm{Aut}_J(\mathbb{O})$

Now we are ready to tackle the main problem which we split up into three steps. First of all we construct a subgroup $\Gamma \leq \mathrm{Aut}_J(\mathbb{O})$. The second step shows that we may suppose $\gamma \in \mathrm{Aut}_J(\mathbb{O})$ to fix a quaternion subalgebra pointwise. Then we finally prove that γ is the product of elements of the given group Γ.

(28.11) Notation

- Given a quaternion subalgebra $\mathbb{H}$, $e \in \mathbb{H}^\perp$ and $w, p \in \mathbb{H}$, we set

$$\psi_{(\mathbb{H},e,w)} : \mathbb{O} \to \mathbb{O},\ x + e \cdot y \mapsto x + e \cdot w^{-1} y w \ ,$$
$$\phi_{(\mathbb{H},e,w,p)} : \mathbb{O} \to \mathbb{O},\ x + e \cdot y \mapsto w^{-1} x w + e \cdot w^{-1} y w p \ .$$

- We set

$$\Psi := \{\psi_{(\mathbb{H},e,w)} \mid \mathbb{H} \text{ a quaternion subalgebra},\ e \in \mathbb{H}^\perp,\ w \in \mathbb{H}\} \ ,$$
$$\Phi := \{\phi_{(\mathbb{H},e,w,p)} \mid \mathbb{H} \text{ a quaternion subalgebra},\ e \in \mathbb{H}^\perp,\ w, p \in \mathbb{H},\ N(p) = 1_{\mathbb{K}}\} \ .$$

- We set

$$\Gamma := \{\psi\phi \mid \psi \in \Psi,\ \phi \in \mathrm{Aut}(\mathbb{O})\} \ .$$

(28.12) Remark By remark (20.29) of [TW], a map $\psi \in \Psi$ is neither an auto- nor an anti-automorphism.

(28.13) Lemma We have

$$\Psi \cup \Phi \subseteq \Gamma \leq \mathrm{Aut}_J(\mathbb{O}) \ .$$

Proof
This results from lemma (21.20) and lemma (21.19). □

(28.14) Lemma Let $\mathbb{A}$ be a subalgebra of $\mathbb{O}$ such that $\mathbb{A}^\perp \nsubseteq \mathbb{A}$ and let $e \in \mathbb{A}^\perp \setminus \mathbb{A}$. Then we have

$$(e \cdot x)(e \cdot y)(e \cdot x) = -N(e) e \cdot x \bar{y} x$$

for all $x, y \in \mathbb{A}$.

Proof
This results from lemma (3.26) (c). □

(28.15) Remark The following lemma is helpful since we know that Jordan homomorphisms on subfields are in fact isomorphisms between subfields and that Jordan automorphisms of skew-fields are iso- or anti-isomorphisms.

(28.16) Lemma Let $\mathbb{A}$ be a subalgebra of $\mathbb{O}$ such that $\mathbb{A}^\perp \not\subseteq \mathbb{A}$, let $e \in \mathbb{A}^\perp \setminus \mathbb{A}$ and let $\gamma \in \mathrm{Aut}_J(\mathbb{O}) \cap GL_N(\mathbb{O}, \mathbb{K})$ such that $\gamma(e) = e$. Then the map $\tilde{\gamma} : \mathbb{A} \to \mathbb{O}$ defined by

$$\gamma(e \cdot x) = e \cdot \tilde{\gamma}(x)$$

is a linear Jordan homomorphism.

Proof
Notice that we have

$$\forall\, x \in \mathbb{A} : \qquad \big(e, \tilde{\gamma}(a)\big) = \big(e, -N(e)^{-1} e \cdot \gamma(e \cdot a)\big) = -\big(1_{\mathbb{K}}, \gamma(e \cdot a)\big) = 0_{\mathbb{K}}$$

by lemma (3.26) and lemma (26.4), hence $e \in \tilde{\gamma}(\mathbb{A})^\perp \setminus \tilde{\gamma}(\mathbb{A})$.

- We have
$$e \cdot \tilde{\gamma}(1_{\mathbb{O}}) = \gamma(e \cdot 1_{\mathbb{O}}) = \gamma(e) = e \cdot 1_{\mathbb{O}} \ .$$
- Given $\lambda \in \mathbb{K},\ x \in \mathbb{A}$, we have
$$e \cdot \tilde{\gamma}(\lambda x) = \gamma(e \cdot \lambda x) = \lambda \gamma(e \cdot x) = \lambda\big(e \cdot \tilde{\gamma}(x)\big) = e \cdot \lambda \tilde{\gamma}(x) \ .$$
- Given $x, y \in \mathbb{A}$, we have
$$e \cdot \tilde{\gamma}(x + y) = \gamma\big(e \cdot (x + y)\big) = \gamma(e \cdot x) + \gamma(e \cdot y) = e \cdot \big(\tilde{\gamma}(x) + \tilde{\gamma}(y)\big) \ .$$
- Given $x, y \in \mathbb{A}$, we have
$$\begin{aligned} e \cdot \tilde{\gamma}(xyx) = \gamma(e \cdot xyx) &= -N(e)^{-1} \cdot \gamma\big((e \cdot x)(e \cdot \overline{y})(e \cdot y)\big) \\ &= -N(e)^{-1} \cdot \big(e \cdot \tilde{\gamma}(x)\big)\big(e \cdot \tilde{\gamma}(\overline{y})\big)\big(e \cdot \tilde{\gamma}(x)\big) = e \cdot \tilde{\gamma}(x)\overline{\tilde{\gamma}(\overline{y})}\tilde{\gamma}(x) \end{aligned}$$
  by lemma (28.14). We set $x := 1_{\mathbb{O}}$ to obtain
$$\tilde{\gamma}(y) = \overline{\tilde{\gamma}(\overline{y})}$$
  for each $y \in \mathbb{A}$ and thus
$$\tilde{\gamma}(xyx) = \tilde{\gamma}(x)\tilde{\gamma}(y)\tilde{\gamma}(x)$$
  for all $x, y \in \mathbb{A}$.

□

(28.17) Definition Let $e_1, e_2 \in \mathbb{O}^*$ such that $\lambda := N(e_1) \neq 0_{\mathbb{K}}, \mu := N(e_2) \neq 0_{\mathbb{K}}$. Then (e_1, e_2) is a *special* (λ, μ)*-pair* if

- we have
$$\langle e_1, 1_{\mathbb{O}} \rangle = \langle e_2, 1_{\mathbb{O}} \rangle = \langle e_1, e_2 \rangle = 0_{\mathbb{K}}$$
  for $\mathrm{Char}\, \mathbb{O} \neq 2$;
- we have
$$\langle e_1, 1_{\mathbb{O}} \rangle = 1_{\mathbb{K}} \ , \qquad\qquad \langle e_2, 1_{\mathbb{O}} \rangle = \langle e_1, e_2 \rangle = 0_{\mathbb{K}}$$
  for $\mathrm{Char}\, \mathbb{O} = 2$.

(28.18) Remark We will need special (λ, μ)-pairs to extend isomorphisms between quaternion subalgebras to the whole octonion division algebra.

(28.19) Lemma Let (e_1, e_2) be a special (λ, μ)-pair and $\mathbb{E} := \langle 1_{\mathbb{O}}, e_1 \rangle_{\mathbb{K}}$. Then we have

$$\bar{e}_1 \neq e_1 \ , \qquad e_2 \in \mathbb{E}^{\perp} \setminus \mathbb{E} \ .$$

Proof

- $\operatorname{Char} \mathbb{O} \neq 2$: We have

  $$e_1 + \bar{e}_1 = \langle e_1, 1_{\mathbb{O}} \rangle = 0_{\mathbb{K}} \ , \qquad \bar{e}_1 = -e_1 \neq e_1 \ .$$

  Let $x = s + te_1 \in \mathbb{E}^*$. If $s \neq 0_{\mathbb{K}}$, we have

  $$\langle x, 1_{\mathbb{O}} \rangle = \langle s + te_1, 1_{\mathbb{O}} \rangle = s \cdot \langle e_1, e_1 \rangle = 2s \cdot N(e_1) \neq 0_{\mathbb{K}}$$

  and thus $x \notin \mathbb{E}^{\perp}$. If $s = 0_{\mathbb{K}}$ (and thus $t \neq 0_{\mathbb{K}}$), we have

  $$\langle x, e_1 \rangle = \langle te_1, e_1 \rangle = t \cdot \langle e_1, e_1 \rangle = 2t \cdot N(e_1) \neq 0_{\mathbb{K}}$$

  and thus $x \notin \mathbb{E}^{\perp}$.

- $\operatorname{Char} \mathbb{O} = 2$: We have

  $$e_1 + \bar{e}_1 = \langle e_1, 1_{\mathbb{O}} \rangle = 1_{\mathbb{K}} \ , \qquad \bar{e}_1 = e_1 + 1_{\mathbb{K}} \neq e_1 \ .$$

  Let $x = s + te_1 \in \mathbb{E}^*$. If $t \neq 0_{\mathbb{K}}$, we have

  $$\langle x, 1_{\mathbb{O}} \rangle = \langle s + te_1, 1_{\mathbb{O}} \rangle = s \cdot \langle 1_{\mathbb{O}}, 1_{\mathbb{O}} \rangle + t \cdot \langle e_1, 1_{\mathbb{O}} \rangle = 2s \cdot N(1_{\mathbb{O}}) + t = t \neq 0_{\mathbb{K}}$$

  and thus $x \notin \mathbb{E}^{\perp}$. If $t = 0_{\mathbb{K}}$ (and thus $s \neq 0_{\mathbb{K}}$), we have

  $$\langle e_1, x \rangle = \langle e_1, s \rangle = s \cdot \langle e_1, 1_{\mathbb{O}} \rangle = s \neq 0_{\mathbb{K}}$$

  and thus $x \notin \mathbb{E}^{\perp}$.

$\square$

(28.20) Remark The following lemma allows us to choose a suitable $\mathbb{K}$-basis for a quaternion division algebra containing two elements $x, y \in \mathbb{O}$.

(28.21) Lemma Let $x \in \mathbb{O} \setminus \mathbb{K}$, let $\mathbb{E} := \langle 1_{\mathbb{O}}, x \rangle_{\mathbb{K}}$ and let $y \in \mathbb{O} \setminus \mathbb{E}$. Then

$$\{1_{\mathbb{O}}, x, y, xy\}$$

is linearly independent over $\mathbb{K}$.

Proof

Notice that $\mathbb{E}$ is a field. Let $a, b, c, d \in \mathbb{K}$ such that

$$a \cdot 1_{\mathbb{O}} + b \cdot x + c \cdot y + d \cdot xy = 0_{\mathbb{O}} \ .$$

Then we have

$$(c \cdot 1_{\mathbb{O}} + d \cdot x) \cdot y = -a \cdot 1_{\mathbb{O}} - b \cdot x \ .$$

- $c \cdot 1_{\mathbb{O}} + d \cdot x \neq 0_{\mathbb{O}}$: In this case, we have

  $$y = (c \cdot 1_{\mathbb{O}} + d \cdot x)^{-1} \cdot (-a \cdot 1_{\mathbb{O}} - b \cdot x) \in \mathbb{E} \qquad ↯ \ .$$

- $c \cdot 1_{\mathbb{O}} + d \cdot x = 0_{\mathbb{O}}$: In this case, we have

  $$c \cdot 1_{\mathbb{O}} + d \cdot x = 0_{\mathbb{O}} = a \cdot 1_{\mathbb{O}} + b \cdot x \ , \qquad c = d = 0_{\mathbb{K}} = a = b \ .$$

$\square$

(28.22) Proposition Given an element $\gamma \in \operatorname{Aut}_J(\mathbb{O})$, there is an element $\tilde{\gamma} \in \langle \sigma_s, \Gamma \rangle$ such that $\tilde{\gamma}\gamma$ fixes a quaternion subalgebra $\mathbb{H}$ pointwise.

Proof

(i) Let $\sigma := \gamma_{|\mathbb{K}} \in \mathrm{Aut}(\mathbb{K})$. Then γ is a σ-isometry by proposition (28.5) (c), thus γ^{-1} is a σ^{-1}-isometry. By theorem (26.6), there is a σ^{-1}-automorphism $\bar{\gamma} \in \mathrm{Aut}(\mathbb{O}, \mathbb{K})$. As a consequence, we have $\bar{\gamma}\gamma \in \mathrm{Aut}_J(\mathbb{O}) \cap GL_N(\mathbb{O}, \mathbb{K})$.

(ii) By lemma (3.23), there is an element $e_1 \in \mathbb{O}$ such that $\bar{e}_1 \neq e_1$. Let $\mathbb{E} := \langle 1, e_1 \rangle_{\mathbb{K}}$ and $e_2 \in \mathbb{E}^{\perp} \setminus \mathbb{E}$. By lemma (3.25) (b), e_1 and e_2 are contained in a quaternion subalgebra $\tilde{\mathbb{H}}$ which contains a (λ, μ)-special pair $(\tilde{e}_1, \tilde{e}_2)$ by definition (1.7.4) of [Sp]. Thus we may assume that (e_1, e_2) is (λ, μ)-special by lemma (28.19).

(iii) As we may suppose γ to be an isometry by (i), the pairs (e_1, e_2) and $\big(\gamma(e_1), \gamma(e_2)\big)$ are (λ, μ)-special. By corollary (1.7.5) of [Sp], there is a linear automorphism $\bar{\gamma} \in \mathrm{Aut}_{\mathbb{K}}(\mathbb{O}, \mathbb{K})$ extending

$$\gamma(e_1) \mapsto e_1 \ , \qquad\qquad \gamma(e_2) \mapsto e_2 \ .$$

As a consequence, $\gamma := \bar{\gamma}\gamma$ fixes $\langle \mathbb{E}, e_2 \rangle_{\mathbb{K}}$ pointwise.

(iv) By lemma (28.16), the map $\tilde{\gamma} : \mathbb{E} \to \tilde{\gamma}(\mathbb{E})$ defined by

$$\gamma(e_2 \cdot x) = e_2 \cdot \tilde{\gamma}(x)$$

is a linear Jordan isomorphism and thus an isomorphism of fields by lemma (27.4).

(v) If $\tilde{\gamma}(\mathbb{E}) = \mathbb{E}$, we have

$$\gamma(\mathbb{E} + e_2 \cdot \mathbb{E}) = \mathbb{E} + e_2 \cdot \mathbb{E} =: \mathbb{H} \ .$$

In this case, $\gamma_{|\mathbb{H}}$ is an auto- or anti-automorphism by Hua's theorem.

- If $\gamma_{|\mathbb{H}}$ is an automorphism, we have

$$\gamma(e_2 \cdot e_1) = \gamma(e_2) \cdot \gamma(e_1) = e_2 \cdot e_1 \ , \qquad\qquad \gamma_{|\mathbb{H}} = \mathrm{id}_{\mathbb{H}} \ .$$

- If $\gamma_{|\mathbb{H}}$ is an anti-automorphism, then $\phi := \sigma_s \gamma_{|\mathbb{H}}$ is a linear automorphism. By theorem (26.7), we may extend ϕ to a linear automorphism $\tilde{\phi} \in \mathrm{Aut}_{\mathbb{K}}(\mathbb{O}, \mathbb{K})$. Then

$$\tilde{\phi}^{-1} \sigma_s \gamma$$

fixes $\mathbb{H}$ pointwise.

Thus we may assume $\tilde{\gamma}(e_1) \notin \mathbb{E}$.

(vi) By lemma (3.25) (b), e_1 and $\tilde{\gamma}(e_1)$ are contained in a quaternion subalgebra $\tilde{\mathbb{H}}$, and by (v) and lemma (28.21), we have

$$\tilde{\mathbb{H}} = \langle 1_{\mathbb{O}}, e_1, \tilde{\gamma}(e_1), e_1 \tilde{\gamma}(e_1) \rangle_{\mathbb{K}} \ ,$$

thus we may extend $\tilde{\gamma} : \mathbb{E} \to \tilde{\gamma}(\mathbb{E})$ to a linear automorphism $\phi \in \mathrm{Aut}_{\mathbb{K}}(\tilde{\mathbb{H}}, \mathbb{K})$ by theorem (26.7). By the Skolem-Noether theorem, there is an element $w \in \tilde{\mathbb{H}}$ such that

$$\phi = \gamma_w \ .$$

(vii) We have $e_2 \cdot \mathbb{E} \subseteq \mathbb{E}^{\perp}$ and therefore

$$\langle e_2, 1_{\mathbb{O}} \rangle = \langle e_2, e_1 \rangle = 0_{\mathbb{K}} \ .$$

Moreover, we have

$$e_2 \cdot \phi(\mathbb{E}) = \gamma(e_2 \cdot \mathbb{E}) \subseteq \gamma(\mathbb{E}^{\perp}) = \gamma(\mathbb{E})^{\perp} = \mathbb{E}^{\perp}$$

by corollary (26.4) and therefore

$$\langle e_2, e_1 \phi(e_1) \rangle = \langle e_2 \phi(e_1)^{-1} \phi(e_1), e_1 \phi(e_1) \rangle = \langle e_2 \phi(e_1)^{-1}, e_1 \rangle \cdot N\big(\phi(e_1)\big) = 0_{\mathbb{K}} \ ,$$
$$\langle e_2, \phi(e_1) \rangle = \langle e_2, -N(e_2)^{-1} e_2 \cdot e_2 \phi(e_1) \rangle = -N(e_2)^{-1} N(e_2) \cdot \langle 1_{\mathbb{O}}, e_2 \phi(e_1) \rangle = 0_{\mathbb{K}} \ ,$$

hence $e_2 \in \tilde{\mathbb{H}}^{\perp}$, and $\gamma_{(\tilde{\mathbb{H}}, e_2, w^{-1})} \gamma$ fixes $\mathbb{H} := \mathbb{E} + e_2 \cdot \mathbb{E}$ pointwise, cf. notation (28.11).

□

(28.23) Proposition Let $\gamma \in \mathrm{Aut}_J(\mathbb{O})$ be a Jordan automorphism fixing a quaternion subalgebra $\mathbb{H}$ pointwise. Then we have $\gamma \in \langle \sigma_s, \Gamma \rangle$.

Proof
Let $e \in \mathbb{H}^\perp \setminus \mathbb{H}$.

(i) As we have
$$\gamma(e \cdot \mathbb{H}) = \gamma(\mathbb{H}^\perp) = \gamma(\mathbb{H})^\perp = \mathbb{H}^\perp = e \cdot \mathbb{H} \ ,$$
by corollary (26.4), we may define a map $\tilde{\gamma} : \mathbb{H} \to \mathbb{H}$ via
$$\gamma(e \cdot x) = e \cdot \tilde{\gamma}(x) \ .$$

(ii) We have
$$N(e) = N\big(\gamma(e \cdot 1_{\mathbb{O}})\big) = N\big(e \cdot \tilde{\gamma}(1_{\mathbb{O}})\big) = N(e) \cdot N\big(\tilde{\gamma}(1_{\mathbb{O}})\big)$$
and thus $N\big(\tilde{\gamma}(1_{\mathbb{O}})\big) = 1_{\mathbb{K}}$. Therefore, the map
$$\phi_{(\mathbb{H}, e, 1_{\mathbb{O}}, \tilde{\gamma}(1_{\mathbb{O}})^{-1})} \gamma$$
fixes $\langle \mathbb{H}, e \rangle_{\mathbb{K}}$ pointwise.

(iii) By lemma (28.16), the map $\tilde{\gamma}$ is a linear Jordan automorphism and thus an auto- or an anti-automorphism by Hua's theorem.

(iv)
- $\tilde{\gamma}$ is an automorphism: As $\tilde{\gamma}$ is linear, the Skolem-Noether theorem yields an element $w \in \mathbb{H}$ such that $\tilde{\gamma} = \gamma_w$, hence
$$\gamma = \gamma_{(\mathbb{H}, e, w)} \in \Gamma \ .$$
- $\tilde{\gamma}$ is an anti-automorphism: Then there is an element $w \in \mathbb{H}$ such that
$$\gamma(x + e \cdot y) = x + e \cdot w^{-1} \bar{y} w$$
for all $x, y \in \mathbb{H}$, hence
$$\gamma_e \gamma(x + e \cdot y) = e^{-1} x e + w^{-1} \bar{y} w \cdot e = \bar{x} + e \cdot w^{-1} y w$$
for all $x, y \in \mathbb{H}$ and therefore
$$\sigma_s \phi_{(\mathbb{H}, e, 1, -1)} \gamma_e \gamma = \gamma_{(\mathbb{H}, e, w)} \in \Gamma \ , \gamma \in \langle \sigma_s, \Gamma \rangle \ .$$

□

§ 28.4 Conclusion

(28.24) Theorem (Jordan Automorphisms of Octonion Division Algebras) Given an octonion division algebra $\mathbb{O}$, we have
$$\mathrm{Aut}_J(\mathbb{O}) = \langle \sigma_s, \Gamma \rangle \cong \langle \sigma_s \rangle \times \Gamma \ .$$

Proof
The first equality results from proposition (28.22) and proposition (28.23), the second assertion from corollary (28.6). □

Part VI

Moufang Sets

Now we turn to the general description of the root groups of Moufang Polygons. In fact, all of them are parametrized by Moufang sets, more precisely, by their associated groups. As the glueings of integrable foundations turn out to be Jordan isomorphisms, the classification of twin buildings is closely related to the solution of the isomorphism problem for Moufang sets.

We list the examples of Moufang sets which will appear in the sequel, then we give a complete overview of the Jordan isomorphisms between these Moufang sets before we give the missing proofs.

Chapter 29 Basic Definitions

(29.1) Definition A *Moufang set* is a pair $\mathbb{M} = \big(X, \{U_x\}_{x \in X}\big)$ consisting of a set X with $|X| \geq 3$ and a set of *root groups* $\{U_x\}_{x \in X}$ satisfying the following conditions:

(M1) For each $x \in X$, the group $U_x \leq \mathrm{Sym}(X)$ fixes x and acts regularly on $X \setminus \{x\}$.

(M2) For each $x \in X$ and for each $\varphi \in \langle U_y \mid y \in X \rangle$, we have $U_x^\varphi = U_{\varphi(x)}$.

(29.2) Remark Let $(U, +)$ be a not necessarily commutative group, let $X := U \cup \{\infty\}$ be the disjoint union of U and $\{\infty\}$ and let $\tau \in \mathrm{Sym}(X)$ be a permutation interchanging 0 and ∞, which means that we have $\tau_{|U^*} \in \mathrm{Sym}(U^*)$. By theorem 2 of [DW], the pair (U, τ) gives rise to a Moufang set $\mathbb{M}(U, \tau)$ if and only if we have $h_a \in \mathrm{Aut}(U)$ for each $a \in U^*$, where h_a is the *Hua map with respect to* a as in definition 2 of [DW], more precisely, we consider h_a to be the restriction to U of that map given there. Conversely, each Moufang set $\mathbb{M}$ arises in such a way, cf page 5 of [DW], or lemma 1.3.4 of [DS] for a more precise statement,.

As both the descriptions are equivalent, we consider a Moufang set to be a pair $\mathbb{M} = (U, \tau)$ consisting of a not necessarily commutative group $U = (U, +)$ and an element $\tau \in \mathrm{Sym}(U^*)$ such that $h_a \in \mathrm{Aut}(U)$ for each $a \in U^*$.

(29.3) Definition

- A Moufang set $\mathbb{M} = (U, \tau)$ is *commutative* if the group U is commutative.
- A Moufang set $\mathbb{M} = (U, \tau)$ is *unital* if there is an element $1_{\mathbb{M}} \in U^*$ such that $h_{1_{\mathbb{M}}} = \mathrm{id}_U$.

(29.4) Remark If a Moufang set is unital, the element $1_{\mathbb{M}}$ is not necessarily uniquely determined by the defining property. However, we will distinguish a canonical element for the examples we mainly deal with, cf. the next paragraph.

(29.5) Definition Let $\mathbb{M} = (U, \tau)$, $\tilde{\mathbb{M}} = (\tilde{U}, \tilde{\tau})$ be Moufang sets.

- An *isomorphism* $\varphi : \mathbb{M} \to \tilde{\mathbb{M}}$ is an isomorphism of groups $\varphi : U \to \tilde{U}$ such that
$$\forall\ x \in U : \qquad \varphi\big(\tau(x)\big) = \tilde{\tau}\big(\varphi(x)\big)\ .$$
- An *automorphism of* $\mathbb{M}$ is an isomorphism $\varphi : \mathbb{M} \to \mathbb{M}$.
- A *Jordan isomorphism* $\gamma : \mathbb{M} \to \tilde{\mathbb{M}}$ is an isomorphism of groups $\gamma : U \to \tilde{U}$ such that
$$\forall\ x \in U,\ a \in U^* : \qquad \gamma\big(h_a(x)\big) = \tilde{h}_{\gamma(a)}\big(\gamma(x)\big)\ ,$$
and, moreover, such that $\gamma(1_{\mathbb{M}}) = 1_{\tilde{\mathbb{M}}}$ if $\mathbb{M}$ and $\tilde{\mathbb{M}}$ both are unital.
- A *Jordan automorphism of* $\mathbb{M}$ is a Jordan isomorphism $\gamma : \mathbb{M} \to \mathbb{M}$.

(29.6) Remark The list in the following chapter is not complete, we only list those Moufang sets appearing in triangles and quadrangles since we only classify foundations involving polygons of this type. Moreover, we exclude the non-commutative Moufang sets appearing in quadrangles of type E_n.

Chapter 30 Examples

(30.1) Example (Moufang Sets of Linear Type) Given an alternative division ring $\mathbb{A}$, the corresponding *Moufang set of linear type* is

$$\mathbb{M}(\mathbb{A}) := (\mathbb{A}, \tau)\ , \qquad \tau : \mathbb{A}^* \to \mathbb{A}^*,\ x \mapsto -x^{-1}\ .$$

Given $a \in \mathbb{A}^*$, the Hua map with respect to a is

$$h_a : \mathbb{A} \to \mathbb{A},\ x \mapsto axa\ .$$

As a consequence, $\mathbb{M}(\mathbb{A})$ is unital with $1_{\mathbb{M}} = 1_{\mathbb{A}}$.

(30.2) Example (Moufang Sets of Involutory Type) Given an involutory set $(\mathbb{K}, \mathbb{K}_0, \sigma)$, the corresponding *Moufang set of involutory type* is

$$\mathbb{M}(\mathbb{K}, \mathbb{K}_0, \sigma) := (\mathbb{K}_0, \tau)\ , \qquad \tau : \mathbb{K}_0^* \to \mathbb{K}_0^*,\ x \mapsto -x^{-1}\ .$$

Given $a \in \mathbb{K}_0^*$, the Hua map with respect to a is

$$h_a : \mathbb{K}_0 \to \mathbb{K}_0,\ x \mapsto axa\ .$$

As a consequence, $\mathbb{M}(\mathbb{K}, \mathbb{K}_0, \sigma)$ is unital with $1_{\mathbb{M}} = 1_{\mathbb{K}} \in \mathbb{K}_0$.

(30.3) Example (Moufang Sets of Indifferent Type) Given an indifferent set $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$, the corresponding *Moufang set of indifferent type* is

$$\mathbb{M}(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0) := (\mathbb{K}_0, \tau)\ , \qquad \tau : \mathbb{K}_0^* \to \mathbb{K}_0^*,\ x \mapsto -x^{-1}\ .$$

Given $a \in \mathbb{K}_0^*$, the Hua map with respect to a is

$$h_a : \mathbb{K}_0 \to \mathbb{K}_0,\ x \mapsto axa\ .$$

As a consequence, $\mathbb{M}(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$ is unital with $1_{\mathbb{M}} = 1_{\mathbb{K}} \in \mathbb{K}_0$.

(30.4) Example (Moufang Sets of Quadratic Form Type) Given a quadratic space $(L_0, \mathbb{K}, q)$ with basepoint ϵ, the corresponding *Moufang set of quadratic form type with basepoint ϵ* is

$$\mathbb{M}(L_0, \mathbb{K}, q) := (L_0, \tau)\ , \qquad \tau : L_0^* \to L_0^*,\ a \mapsto -a^\sigma \cdot q(a)^{-1}\ .$$

Given $a \in L_0^*$, the Hua map with respect to a is

$$h_a : L_0 \to L_0,\ v \mapsto \pi_a \pi_\epsilon(v) \cdot q(a)\ .$$

As a consequence, $\mathbb{M}(L_0, \mathbb{K}, q)$ is unital with $1_{\mathbb{M}} = \epsilon$.

(30.5) Example (Moufang Sets of Pseudo-Quadratic Form Type)
Let $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma, L_0, q)$ be a pseudo-quadratic space and let $T = T(\Xi)$ be the group as in definition (7.5) and corollary (7.9). The corresponding *Moufang set of pseudo-quadratic form type* is

$$\mathbb{M}(\Xi) := (T, \tau)\ , \qquad \tau : T^* \to T^*,\ (a, t) \mapsto (at^{-1}, -t^{-1})\ .$$

Given $(a, t) \in T^*$, the Hua map with respect to (a, t) is

$$h_{(a,t)} : T \to T,\ (b, v) \mapsto \left(bt^\sigma - at^{-1} f(a, b) t^\sigma, tvt^\sigma\right)\ .$$

As a consequence, $\mathbb{M}(\Xi)$ is unital with $1_{\mathbb{M}} = (0, 1_{\mathbb{K}})$.

Chapter 31 The Isomorphism Problem for Moufang Sets

Since the glueings appearing in a foundation are in fact Jordan isomorphisms, it is natural to solve the isomorphism problem for the appearing Moufang sets before we tackle the classification of foundations.

As mentioned, we consider Jordan isomorphisms, not isomorphisms of Moufang sets in the proper sense, which are a subset of the Jordan isomorphisms. It would be interesting in which cases both the definitions coincide, which should be correct in almost all the cases.

First of all we give an overview of the results, some of them already proved in the previous parts, then we give the missing proofs.

§ 31.1 Results

(31.1) Definition Let $\mathbb{O}$ be an octonion division algebra.

- Given a quaternion subalgebra $\mathbb{H}$, $e \in \mathbb{H}^\perp$ and $w, p \in \mathbb{H}$, we set
$$\psi_{(\mathbb{H},e,w)} : \mathbb{O} \to \mathbb{O},\ x + e \cdot y \mapsto x + e \cdot w^{-1} y w \ .$$
- We set
$$\Psi := \{\psi_{(\mathbb{H},e,w)} \mid \mathbb{H} \text{ a quaternion subalgebra},\ e \in \mathbb{H}^\perp,\ w \in \mathbb{H}\} \ .$$
and $\Gamma := \{\psi\phi \mid \psi \in \Psi,\ \phi \in \mathrm{Aut}(\mathbb{O})\}$, which is a subgroup of $\mathrm{Aut}_J(\mathbb{O})$ by lemma (21.19).

(31.2) Theorem (**Moufang Sets of Linear Type**) Let $\mathbb{M} := \mathbb{M}(\mathbb{A})$, $\tilde{\mathbb{M}} := \mathbb{M}(\tilde{\mathbb{A}})$ be Moufang sets of linear type. A map $\gamma : \mathbb{M} \to \tilde{\mathbb{M}}$ is a Jordan isomorphism such that $\gamma(1_{\mathbb{M}}) = \gamma(1_{\tilde{\mathbb{M}}})$ if and only if one of the following holds:

(i) The alternative division rings $\mathbb{A}$ and $\tilde{\mathbb{A}}$ are skew-fields and γ is an iso- or anti-isomorphism of skew-fields.

(ii) The alternative division rings $\mathbb{A}$ and $\tilde{\mathbb{A}}$ are isomorphic octonion division algebras and we have
$$\phi^{-1}\gamma \in \langle \sigma_s, \Gamma \rangle \cong \langle \sigma_s \rangle \times \Gamma \ ,$$
where $\phi : \mathbb{A} \to \tilde{\mathbb{A}}$ is an isomorphism of alternative rings, σ_s is the standard involution of $\mathbb{A}$ and Γ is the group defined in (31.1).

Proof

"⇒" If $\mathbb{A}$ is a skew-field, then (i) holds by Hua's theorem. If $\mathbb{A}$ (and thus $\tilde{\mathbb{A}}$) is an octonion division algebra, we may adapt proposition (28.5) to obtain that $\gamma : \mathbb{A} \to \tilde{\mathbb{A}}$ is a σ-isometry. By theorem (1.7.1) of [Sp], there is a σ-isomorphism $\phi : \mathbb{A} \to \tilde{\mathbb{A}}$, hence $\phi^{-1}\gamma \in \mathrm{Aut}_J(\mathbb{A})$, and we may apply theorem (28.24).

"⇐" ✓

Notice that definition (3.8) and definition (27.1) include the condition $\gamma(1_{\mathbb{A}}) = 1_{\tilde{\mathbb{A}}}$. □

(31.3) Theorem (**Moufang Sets of Involutory Type**) Let $(\mathbb{K}, \mathbb{K}_0, \sigma)$ be a proper involutory set, let $(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ be an involutory set and let $\mathbb{M} := \mathbb{M}(\mathbb{K}, \mathbb{K}_0, \sigma)$, $\tilde{\mathbb{M}} := \mathbb{M}(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ be the corresponding Moufang sets of involutory type. A map $\gamma : \mathbb{M} \to \tilde{\mathbb{M}}$ is a Jordan isomorphism such that $\gamma(1_{\mathbb{M}}) = 1_{\tilde{\mathbb{M}}}$ if and only if there is an isomorphism $\phi : (\mathbb{K}, \mathbb{K}_0, \sigma) \to (\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ of involutory sets such that $\gamma = \phi_{|\mathbb{K}_0}$.

Proof

This is theorem (5.3). Notice that the proof is not complete yet. □

(31.4) Definition Let $(\mathbb{A}, \mathbb{F}, \sigma)$ be quadratic of type (iv) and suppose that $\dim_{\mathbb{A}} L_0 = 2$. By [D], there are exactly three pseudo-quadratic spaces

$$(\mathbb{A}, \mathbb{F}, \sigma, L_0, q) = (\mathbb{A}_1, \mathbb{F}, \sigma, L_0, q_1) = \Xi_1 \ , \quad (\mathbb{A}_2, \mathbb{F}, \sigma, L_0, q_2) = \Xi_2 \ , \quad (\mathbb{A}_3, \mathbb{F}, \sigma, L_0, q_3) = \Xi_3$$

with pairwise non-isomorphic quaternion division algebras $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ which define the group T. When we switch between the parametrizing pseudo-quadratic spaces, we indicate this by the map

$$\mathrm{id}_T^i : T \to T, \ (a,t) \mapsto (a,t) \ ,$$

i.e., after applying id_T^i, we consider T to be defined by Ξ_i.

(31.5) Theorem (**Moufang Sets of Pseudo-Quadratic Form Type**) Let Ξ and $\tilde{\Xi}$ be proper pseudo-quadratic spaces and let $\mathbb{M} := \mathbb{M}(\Xi)$, $\tilde{\mathbb{M}} := \mathbb{M}(\tilde{\Xi})$ be the corresponding Moufang sets of pseudo-quadratic form type. A map $\gamma : \mathbb{M} \to \tilde{\mathbb{M}}$ is a Jordan isomorphism such that $\gamma(1_{\mathbb{M}}) = 1_{\tilde{\mathbb{M}}}$ if and only if one of the following holds:

(i) There is an isomorphism $\Phi : \Xi \to \tilde{\Xi}$ of pseudo-quadratic spaces that induces γ.

(ii) The involutory sets $(\mathbb{K}, \mathbb{K}_0, \sigma)$ and $(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ both are quadratic of type (iv), we have

$$\mathbb{K} \ncong \tilde{\mathbb{K}} \ , \qquad \dim_{\mathbb{K}} L_0 = 2 = \dim_{\tilde{\mathbb{K}}} \tilde{L}_0$$

and there are an $i \in \{2,3\}$ and an isomorphism $\Phi : \Xi \to \tilde{\Xi}_i$ of pseudo-quadratic spaces such that γ is induced by $(\mathrm{id}_{\tilde{T}}^i)^{-1} \circ \Phi$, where $\mathrm{id}_{\tilde{T}}^i$ and $\tilde{\Xi} =: \tilde{\Xi}_1, \tilde{\Xi}_2, \tilde{\Xi}_3$ are as in definition (31.4).

(iii) The involutory sets $(\mathbb{K}, \mathbb{K}_0, \sigma)$ and $(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}_0, \tilde{\sigma})$ are quadratic of type (iv) and (iii), respectively, we have $\dim_{\mathbb{K}} L_0 = 1$, $\dim_{\tilde{\mathbb{K}}} \tilde{L}_0 = 2$ and γ can be described by

$$\forall \ x = s + et \in \mathbb{K}, \ u \in \mathbb{K}_0 : \qquad \gamma(ax, x^\sigma q(a)x + u) = \big(\tilde{a}\phi(s) + \tilde{b}\phi(t)^{\tilde{\sigma}}, \phi\big(N(x)q(a) + u\big)\big) \ ,$$

where $a \in L_0^*$ is arbitrary, $\phi : \mathbb{E}_a \to \tilde{\mathbb{K}}$ is an isomorphism of fields, $e \in \mathbb{E}_a^\perp$, $\tilde{a} \in \tilde{L}_0$ and $\tilde{b} \in \tilde{a}^\perp$.

(iv) We have $\mathbb{K} \cong \mathbb{F}_4 \cong \tilde{\mathbb{K}}$, $\dim_{\mathbb{K}} L = 1$ and there are an isomorphism $\Phi : \Xi \to \tilde{\Xi}$ of pseudo-quadratic spaces and a non-trivial inner automorphism $\tilde{\gamma} \in \mathrm{Aut}(\tilde{T})$ such that γ is induced by $\tilde{\gamma} \circ \Phi$.

Proof
This is theorem (17.1). Notice that definition (7.16) includes the condition $\gamma(0, 1_{\mathbb{K}}) = (0, 1_{\tilde{\mathbb{K}}})$. □

(31.6) Theorem (**Moufang Sets of Quadratic Form Type**) Let Ξ be a quadratic space with basepoint ϵ, let $\tilde{\Xi}$ be a proper quadratic space with basepoint $\tilde{\epsilon}$ and let $\mathbb{M} := \mathbb{M}(\Xi)$, $\tilde{\mathbb{M}} := \mathbb{M}(\tilde{\Xi})$ be the corresponding Moufang sets of quadratic form type. A map $\gamma : \mathbb{M} \to \tilde{\mathbb{M}}$ is a Jordan isomorphism such that $\gamma(1_{\mathbb{M}}) = 1_{\tilde{\mathbb{M}}}$ if and only if one of the following holds:

(i) We have $\dim_{\tilde{\mathbb{K}}} \tilde{L}_0 \geq 3$ and there is an isomorphism $\phi : \mathbb{K} \to \tilde{\mathbb{K}}$ of fields such that the map

$$(\gamma, \phi) : (L_0, \mathbb{K}, q) \to (\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$$

is an isomorphism of quadratic spaces. In particular, we have $\dim_{\mathbb{K}} L_0 = \dim_{\tilde{\mathbb{K}}} \tilde{L}_0$.

(ii) We have $\dim_{\tilde{\mathbb{K}}} \tilde{L}_0 \leq 2$, the map

$$\phi : \mathbb{K} \to \hat{\mathbb{K}} := \gamma(\langle \epsilon \rangle_{\mathbb{K}}) \subseteq \hat{\mathbb{F}} := \mathbb{F}(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}), \ s \mapsto \gamma(\epsilon \cdot s)$$

is an isomorphism of fields, the field $\hat{\mathbb{F}}$ is quadratic over $\hat{\mathbb{K}}$, and the map

$$(\gamma, \phi) : (L_0, \mathbb{K}, q) \to (\hat{\mathbb{F}}, \hat{\mathbb{K}}, N_{\hat{\mathbb{K}}}^{\hat{\mathbb{F}}})$$

is an isomorphism of quadratic spaces. This is true even if $\tilde{\Xi}$ is non-proper.

Proof
This is theorem (31.41). □

(31.7) Theorem (**Moufang Sets of Quadratic Form and Linear Type**) Let $\mathbb{M} := \mathbb{M}(L_0, \mathbb{K}, q)$ be a Moufang set of quadratic form type with basepoint ϵ and let $\tilde{\mathbb{M}} := \mathbb{M}(\tilde{\mathbb{A}})$ be a Moufang set of linear type. A map $\gamma : \mathbb{M} \to \tilde{\mathbb{M}}$ is a Jordan isomorphism such that $\gamma(1_{\mathbb{M}}) = 1_{\tilde{\mathbb{M}}}$ if and only if the map

$$\phi : \mathbb{K} \to \tilde{\mathbb{K}} := \gamma(\langle \epsilon \rangle_{\mathbb{K}}) \subseteq \tilde{\mathbb{A}},\ s \mapsto \gamma(\epsilon \cdot s)$$

is an isomorphism of fields, $\tilde{\mathbb{A}}$ is quadratic over $\tilde{\mathbb{K}}$ and the map $(\gamma, \phi) : (L_0, \mathbb{K}, q) \to (\tilde{\mathbb{A}}, \tilde{\mathbb{K}}, N_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{A}}})$ is an isomorphism of quadratic spaces.

Proof
This is theorem (31.21). □

(31.8) Theorem (**Moufang Sets of Indifferent and Linear Type**) Let $\mathbb{M} := \mathbb{M}(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$ be a Moufang set of indifferent type and let $\tilde{\mathbb{M}} := \mathbb{M}(\tilde{\mathbb{A}})$ be a Moufang set of linear type. A map $\gamma : \tilde{\mathbb{M}} \to \mathbb{M}$ is a Jordan isomorphism such that $\gamma(1_{\tilde{\mathbb{M}}}) = 1_{\mathbb{M}}$ if and only if $\mathbb{K}_0 = \mathbb{K}$, $\tilde{\mathbb{A}}$ is a field and the map $\gamma : \tilde{\mathbb{A}} \to \mathbb{K}_0$ is an isomorphism of fields. In particular, the indifferent set $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$ is non-proper if we have $\mathbb{M} \cong \tilde{\mathbb{M}}$.

Proof
This is theorem (31.24). □

(31.9) Theorem (**Moufang Sets of Involutory and Linear Type**) Let $\tilde{\mathbb{M}} := \mathbb{M}(\tilde{\mathbb{A}})$ be a Moufang set of linear type and let $\mathbb{M} := \mathbb{M}(\mathbb{K}, \mathbb{K}_0, \sigma)$ be a Moufang set of involutory type. A map $\gamma : \tilde{\mathbb{M}} \to \mathbb{M}$ is a Jordan isomorphism such that $\gamma(1_{\tilde{\mathbb{M}}}) = 1_{\mathbb{M}}$ if and only if $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is of quadratic type, $\tilde{\mathbb{A}}$ and $\mathbb{K}_0$ are fields and the map $\gamma : \tilde{\mathbb{A}} \to \mathbb{K}_0$ is an isomorphism of fields. In particular, the involutory set $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is non-proper if we have $\mathbb{M} \cong \tilde{\mathbb{M}}$.

Proof
This is theorem (31.26). □

(31.10) Remark Most of the following proofs or different versions can also be found in [K]. Notice, however, that some of them only give the idea for the proof so that we had to work out some details, especially in the following paragraph.

§ 31.2 $\mathbb{M}(L_0, \mathbb{K}, q) \cong \mathbb{M}(\tilde{\mathbb{A}})$

If a Moufang set $\mathbb{M}(L_0, \mathbb{K}, q)$ of quadratic form type with basepoint ϵ is isomorphic to a Moufang set $\mathbb{M}(\tilde{\mathbb{A}})$ of linear type, then $\tilde{\mathbb{A}}$ is quadratic over the field $\tilde{\mathbb{K}} := \gamma(\langle \epsilon_{\mathbb{K}} \rangle)$ and γ is induced by an isomorphism $(\gamma, \phi) : (L_0, \mathbb{K}, q) \to (\tilde{\mathbb{A}}, \tilde{\mathbb{K}}, N_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{A}}})$ of quadratic spaces.

(31.11) Lemma Given a quadratic space $(L_0, \mathbb{K}, q)$ such that $\dim_{\mathbb{K}} L_0 \leq 2$, we have

$$\mathbb{M} := \mathbb{M}(L_0, \mathbb{K}, q) = \mathbb{M}\big(\mathbb{F}(L_0, \mathbb{K}, q)\big) =: \tilde{\mathbb{M}}\ .$$

In particular, the corresponding Hua maps coincide.

Proof
We have

$$\forall\ x \in L_0 : \qquad \tau(x) = -x^\sigma \cdot q(x)^{-1} = -x^\sigma * \epsilon \cdot q(x)^{-1} = -x^\sigma * N_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{F}}}(x)^{-1} = -x^{-1} = \tilde{\tau}(x)\ .$$

□

(31.12) Notation Until proposition (31.20), $\mathbb{M} := \mathbb{M}(L_0, \mathbb{K}, q)$ is a Moufang set of quadratic form type with basepoint ϵ, $\tilde{\mathbb{M}} := \mathbb{M}(\tilde{\mathbb{A}})$ is a Moufang set of linear type and $\gamma : \mathbb{M} \to \tilde{\mathbb{M}}$ is a Jordan isomorphism such that $\gamma(1_{\mathbb{M}}) = 1_{\tilde{\mathbb{M}}}$.

(31.13) Lemma The map

$$\phi : \mathbb{K} \to \tilde{\mathbb{K}} := \gamma(\langle \epsilon \rangle_{\mathbb{K}}) \subseteq \tilde{\mathbb{A}},\ s \mapsto \gamma(\epsilon \cdot s)$$

is an isomorphism of fields.

Proof
By lemma (4.7), we have

$$\forall\ s, t \in \mathbb{K} : \qquad \phi(sts) = \gamma(\epsilon \cdot sts) = \gamma\big(h_{\epsilon \cdot s}(\epsilon \cdot t)\big) = \tilde{h}_{\gamma(\epsilon \cdot s)}\big(\gamma(\epsilon \cdot t)\big) = \phi(s)\phi(t)\phi(s)\ .$$

As a consequence, $\phi : \mathbb{K} \to \tilde{\mathbb{A}}$ is a Jordan homomorphism, hence $\phi : \mathbb{K} \to \phi(\mathbb{K}) = \gamma(\langle \epsilon \rangle_{\mathbb{K}})$ is an isomorphism of fields by lemma (27.4). □

(31.14) Notation Given $x \in L_0 \setminus \langle \epsilon \rangle_{\mathbb{K}}$, we set $R_x := \langle \epsilon, x \rangle_{\mathbb{K}}$.

(31.15) Remark Given $x \in L_0 \setminus \langle \epsilon \rangle_{\mathbb{K}}$, the triple $(R_x, \mathbb{K}, q)$ is a quadratic space such that $\dim_{\mathbb{K}} R_x = 2$.

(31.16) Notation Given $x \in L_0 \setminus \langle \epsilon \rangle_{\mathbb{K}}$, we set $\mathbb{F}_x := \mathbb{F}(R_x, \mathbb{K}, q)$.

(31.17) Lemma Given $x \in L_0 \setminus \langle \epsilon \rangle_{\mathbb{K}}$, the map

$$\gamma_{|\mathbb{F}_x} : \mathbb{F}_x \to \gamma(\mathbb{F}_x) \subseteq \tilde{\mathbb{A}},\ y \mapsto \gamma(y)$$

is an isomorphism of fields.

Proof
By lemma (31.11), the Hua maps of $\mathbb{M}(R_x, \mathbb{K}, q)$ and $\mathbb{M}(\mathbb{F}_x)$ coincide, hence $\gamma_{|\mathbb{F}_x} : \mathbb{F}_x \to \tilde{\mathbb{A}}$ is a Jordan homomorphism so that we may apply lemma (27.4). □

(31.18) Proposition Let $x \in L_0$. Then the following holds:

(a) We have

$$\forall\ s \in \mathbb{K} : \qquad \gamma(x \cdot s) = \gamma(x) \cdot \phi(s)\ .$$

In particular, the map $(\gamma, \phi) : (L_0, \mathbb{K}) \to (\tilde{\mathbb{A}}, \tilde{\mathbb{K}})$ is an isomorphism of vector spaces.

(b) We have

$$\forall\ s \in \mathbb{K} : \qquad \gamma(x) \cdot \phi(s) = \phi(s) \cdot \gamma(x)\ .$$

In particular, we have $\tilde{\mathbb{K}} \subseteq Z(\tilde{\mathbb{A}})$.

Proof
If $x \in \langle \epsilon \rangle_{\mathbb{K}}$, the assertions result from lemma (31.13), so assume $x \in L_0 \setminus \langle \epsilon \rangle_{\mathbb{K}}$.

(a) Given $s \in \mathbb{K}$, we have

$$\gamma(x \cdot s) = \gamma\big(x * (\epsilon \cdot s)\big) = \gamma(x) \cdot \gamma(\epsilon \cdot s) = \gamma(x) \cdot \phi(s)\ .$$

(b) Given $s \in \mathbb{K}$, we have

$$\gamma(x) \cdot \phi(s) = \gamma(x \cdot s) = \gamma\big((\epsilon \cdot s) * x\big) = \gamma(\epsilon \cdot s) \cdot \gamma(x) = \phi(s) \cdot \gamma(x)\ .$$

□

(31.19) Lemma Given $x \in L_0$, we have

$$h_x(\epsilon) - x \cdot T(x) + \epsilon \cdot q(x) = 0_{L_0} \ .$$

Proof
Let $x \in L_0$. By remark (4.9) (b), we have

$$\begin{aligned} h_x(\epsilon) - x \cdot T(x) + \epsilon \cdot q(x) &= x \cdot f_q(x, \bar{\epsilon}) - \bar{\epsilon} \cdot q(x) - x \cdot T(x) + \epsilon \cdot q(x) \\ &= x \cdot T(x) - x \cdot T(x) - \epsilon \cdot q(x) + \epsilon \cdot q(x) = 0_{L_0} \ . \end{aligned}$$

□

(31.20) Proposition Given $x \in L_0^*$, we have

$$\gamma(x)^2 - \gamma(x) \cdot \phi\big(T(x)\big) + \phi\big(q(x)\big) = 0_{\tilde{\mathbb{A}}} \ .$$

In particular, the alternative division ring $\tilde{\mathbb{A}}$ is quadratic over $\tilde{\mathbb{K}}$ with norm $\tilde{N} := N_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{A}}}$, satisfying

$$\tilde{N} \circ \gamma = \phi \circ q \ .$$

Proof
Given $x \in L_0^*$, we have

$$\begin{aligned} 0_{\tilde{\mathbb{A}}} = \gamma(0_{L_0}) &= \gamma\big(h_x(\epsilon) - x \cdot T(x) + \epsilon \cdot q(x)\big) = \gamma\big(h_x(\epsilon)\big) - \gamma\big(x \cdot T(x)\big) + \gamma\big(\epsilon \cdot q(x)\big) \\ &= \tilde{h}_{\gamma(x)}\big(\gamma(\epsilon)\big) - \gamma(x) \cdot \phi\big(T(x)\big) + \phi\big(q(x)\big) = \gamma(x)^2 - \gamma(x) \cdot \phi\big(T(x)\big) + \phi\big(q(x)\big) \end{aligned}$$

by lemma (31.19) and proposition (31.18) (a), which shows that $\tilde{\mathbb{A}}$ is quadratic over $\tilde{\mathbb{K}}$. Given $x \in L_0 \setminus \langle \epsilon \rangle_{\mathbb{K}}$, we have $\gamma(x) \in \tilde{\mathbb{A}} \setminus \tilde{\mathbb{K}}$ and thus

$$\tilde{N}\big(\gamma(x)\big) = \phi\big(q(x)\big)$$

since the minimum equation is unique. Finally, given $s \in \mathbb{K}$, we have

$$\tilde{N}\big(\gamma(\epsilon \cdot s)\big) = \tilde{N}\big(\phi(s)\big) = \phi(s)^2 = \phi(s^2) = \phi\big(q(\epsilon \cdot s)\big)$$

by lemma (31.13). □

(31.21) Theorem Let $\mathbb{M} := \mathbb{M}(L_0, \mathbb{K}, q)$ be a Moufang set of quadratic form type with basepoint ϵ and let $\tilde{\mathbb{M}} := \mathbb{M}(\tilde{\mathbb{A}})$ be a Moufang set of linear type. A map $\gamma : \mathbb{M} \to \tilde{\mathbb{M}}$ is a Jordan isomorphism such that $\gamma(1_{\mathbb{M}}) = 1_{\tilde{\mathbb{M}}}$ if and only if the map

$$\phi : \mathbb{K} \to \tilde{\mathbb{K}} := \gamma(\langle \epsilon \rangle_{\mathbb{K}}) \subseteq \tilde{\mathbb{A}}, \ s \mapsto \gamma(\epsilon \cdot s)$$

is an isomorphism of fields, $\tilde{\mathbb{A}}$ is quadratic over $\tilde{\mathbb{K}}$ with norm $\tilde{N} := N_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{A}}}$ and the map

$$(\gamma, \phi) : (L_0, \mathbb{K}, q) \to (\tilde{\mathbb{A}}, \tilde{\mathbb{K}}, \tilde{N})$$

is an isomorphism of quadratic spaces.

Proof

"⇒" By lemma (31.13), the map $\phi : \mathbb{K} \to \tilde{\mathbb{K}}$ is an isomorphism of fields. By proposition (31.20) and proposition (31.18) (a), $\tilde{\mathbb{A}}$ is quadratic over $\tilde{\mathbb{K}}$, and the map $(\gamma, \phi) : (L_0, \mathbb{K}, q) \to (\tilde{\mathbb{A}}, \tilde{\mathbb{K}}, \tilde{N})$ is an isomorphism of quadratic spaces.

"$\Leftarrow$" We have $\gamma(1_{\mathbb{M}}) = \gamma(\epsilon) = \phi(\epsilon) = 1_{\tilde{\mathbb{A}}} = 1_{\tilde{\mathbb{M}}}$. By lemma (4.2), we have

$$\begin{aligned}
\gamma\big(h_a(x)\big) &= \gamma\big(a \cdot f_q(a, x^\sigma) - x^\sigma \cdot q(a)\big) = \gamma(a) \cdot \phi\big(f_q(a, x^\sigma)\big) - \gamma(x^\sigma) \cdot \phi\big(q(a)\big) \\
&= \gamma(a) \cdot \phi\big(q(a + x^\sigma) - q(a) - q(x^\sigma)\big) - \gamma(x)^{\tilde{\sigma}} \cdot \tilde{N}\big(\gamma(a)\big) \\
&= \gamma(a) \cdot \big(\tilde{N}\big(\gamma(a) + \gamma(x^\sigma)\big) - \tilde{N}\big(\gamma(a)\big) - \tilde{N}\big(\gamma(x^\sigma)\big)\big) - \gamma(x)^{\tilde{\sigma}} \cdot \tilde{N}\big(\gamma(a)\big) \\
&= \gamma(a) \cdot \big(\gamma(a)^{\tilde{\sigma}}\gamma(x^\sigma) + \gamma(x^\sigma)^{\tilde{\sigma}}\gamma(a)\big) - \gamma(x)^{\tilde{\sigma}} \cdot \tilde{N}\big(\gamma(a)\big) \\
&= \tilde{N}\big(\gamma(a)\big) \cdot \gamma(x)^{\tilde{\sigma}} + \gamma(a)(\gamma(x)^{\tilde{\sigma}})^{\tilde{\sigma}}\gamma(a) - \gamma(x)^{\tilde{\sigma}} \cdot \tilde{N}\big(\gamma(a)\big) \\
&= \gamma(a)\gamma(x)\gamma(a) = \tilde{h}_{\gamma(a)}\big(\gamma(x)\big)
\end{aligned}$$

for all $a \in L_0^*$, $x \in L_0$.

□

(31.22) Corollary Let $(\mathbb{A}, \mathbb{K}, N_{\mathbb{K}}^{\mathbb{A}})$ be a quadratic space of type (m). Then we have

$$\mathbb{M}(\mathbb{A}, \mathbb{K}, N_{\mathbb{K}}^{\mathbb{A}}) \cong \mathbb{M}(\mathbb{A}) \ .$$

Proof
Let $\gamma := \mathrm{id}_{\mathbb{A}}$. Then the map $\phi = \mathrm{id}_{\mathbb{K}} : \mathbb{K} \to \mathbb{K} = \gamma(\langle 1_{\mathbb{A}} \rangle_{\mathbb{K}})$ is an isomorphism of fields, and the map

$$(\gamma, \phi) = (\mathrm{id}_{\mathbb{A}}, \mathrm{id}_{\mathbb{K}}) : (\mathbb{A}, \mathbb{K}, N_{\mathbb{K}}^{\mathbb{A}}) \to (\mathbb{A}, \mathbb{K}, N_{\mathbb{K}}^{\mathbb{A}})$$

is an isomorphism of quadratic spaces so that we may apply theorem (31.21). □

(31.23) Lemma More precisely: Let $(\mathbb{A}, \mathbb{K}, N_{\mathbb{K}}^{\mathbb{A}})$ be a quadratic space of type (m). Then we have

$$\mathbb{M} := \mathbb{M}\big(\mathbb{A}, \mathbb{K}, N_{\mathbb{K}}^{\mathbb{A}}\big) = \mathbb{M}(\mathbb{A}) =: \tilde{\mathbb{M}} \ .$$

Proof
We have

$$\forall\ x \in \mathbb{A} : \qquad \tau(x) = -x^\sigma \cdot N_{\mathbb{K}}^{\mathbb{A}}(x)^{-1} = -x^{-1} = \tilde{\tau}(x) \ .$$

□

§ 31.3 $\mathbb{M}(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0) \cong \mathbb{M}(\tilde{\mathbb{A}})$

(31.24) Theorem Let $\mathbb{M} := \mathbb{M}(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$ be a Moufang set of indifferent type and let $\tilde{\mathbb{M}} := \mathbb{M}(\tilde{\mathbb{A}})$ be a Moufang set of linear type. A map $\gamma : \tilde{\mathbb{M}} \to \mathbb{M}$ is a Jordan isomorphism such that $\gamma(1_{\tilde{\mathbb{M}}}) = 1_{\mathbb{M}}$ if and only if $\mathbb{K}_0 = \mathbb{K}$, $\tilde{\mathbb{A}}$ is a field and the map $\gamma : \tilde{\mathbb{A}} \to \mathbb{K}_0$ is an isomorphism of fields. In particular, the indifferent set $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$ is non-proper if we have $\mathbb{M} \cong \tilde{\mathbb{M}}$.

Proof

"$\Rightarrow$" The map $\gamma : \tilde{\mathbb{A}} \to \mathbb{K}_0 \subseteq \mathbb{K}$ is a Jordan homomorphism. Since $\mathbb{K}$ is associative, Hua's theorem implies that $\mathbb{K}_0 = \gamma(\tilde{\mathbb{A}})$ is a skew-field, which is thus, in fact, a subfield of $\mathbb{K}$. By Hua's theorem again, the map γ is an isomorphism of fields. In particular, $\tilde{\mathbb{A}} = \gamma^{-1}(\mathbb{K}_0)$ is a field. Moreover, we have

$$\mathbb{K}_0 = \langle \mathbb{K}_0 \rangle = \mathbb{K} \ ,$$

hence $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0) = (\mathbb{K}, \mathbb{K}, \mathbb{L}_0)$ is non-proper.

"$\Leftarrow$" We have $\gamma(1_{\tilde{\mathbb{M}}}) = \gamma(1_{\tilde{\mathbb{A}}}) = 1_{\tilde{\mathbb{K}}_0} = 1_{\mathbb{K}} = 1_{\mathbb{M}}$. Given $a \in \tilde{\mathbb{A}}^*$ and $x \in \tilde{\mathbb{A}}$, we have

$$\gamma\big(\tilde{h}_a(x)\big) = \gamma(axa) = \gamma(a)\gamma(x)\gamma(a) = h_{\gamma(a)}\big(\gamma(x)\big) \ .$$

□

§ 31.4 $\mathbb{M}(\mathbb{K},\mathbb{K}_0,\sigma)\cong\mathbb{M}(\tilde{\mathbb{A}})$

(31.25) Lemma Let $(\mathbb{K},\mathbb{K}_0,\sigma)$ be a non-proper involutory set with the additional assumption that $\mathbb{K}_0$ is a field if $\sigma=\mathrm{id}_{\mathbb{K}}$ and $\operatorname{Char}\mathbb{K}=2$. Then $(\mathbb{K},\mathbb{K}_0,\sigma)$ is of quadratic type.

Proof
Assume that $(\mathbb{K},\mathbb{K}_0,\sigma)$ is not quadratic of type (v), i.e., $\mathbb{K}$ is a skew-field.

- $\sigma=\mathrm{id}_{\mathbb{K}}$, $\operatorname{Char}\mathbb{K}\neq 2$: Then $\mathbb{K}$ is a field and $\mathbb{K}_\sigma=\mathbb{K}_0=\operatorname{Fix}(\sigma)=\mathbb{K}$, hence $(\mathbb{K},\mathbb{K}_0,\sigma)$ is quadratic of type (ii).
- $\sigma=\mathrm{id}_{\mathbb{K}}$, $\operatorname{Char}\mathbb{K}=2$: Then $\mathbb{K}$ is a field and $\mathbb{K}^2\subseteq\mathbb{K}_0\subseteq\mathbb{K}$, hence $(\mathbb{K},\mathbb{K}_0,\sigma)$ is quadratic of type (m)$\in\{$(i), (ii)$\}$.
- $\sigma\neq\mathrm{id}_{\mathbb{K}}$: By (23.23) of [TW], $(\mathbb{K},\mathbb{K}_0,\sigma)$ is quadratic of type (m)$\in\{$(iii), (iv)$\}$.

$\square$

(31.26) Theorem Let $\tilde{\mathbb{M}}:=\mathbb{M}(\tilde{\mathbb{A}})$ be a Moufang set of linear type and let $\mathbb{M}:=\mathbb{M}(\mathbb{K},\mathbb{K}_0,\sigma)$ be a Moufang set of involutory type. A map $\gamma:\tilde{\mathbb{M}}\to\mathbb{M}$ is a Jordan isomorphism such that $\gamma(1_{\tilde{\mathbb{M}}})=1_{\mathbb{M}}$ if and only if $(\mathbb{K},\mathbb{K}_0,\sigma)$ is of quadratic type, $\tilde{\mathbb{A}}$ and $\mathbb{K}_0$ are fields and the map $\gamma:\tilde{\mathbb{A}}\to\mathbb{K}_0$ is an isomorphism of fields. In particular, the involutory set $(\mathbb{K},\mathbb{K}_0,\sigma)$ is non-proper if we have $\mathbb{M}\cong\tilde{\mathbb{M}}$.

Proof

"$\Rightarrow$" If $(\mathbb{K},\mathbb{K}_0,\sigma)$ is quadratic of type (v), $\mathbb{K}_0$ is a field and the map $\gamma:\mathbb{M}(\tilde{\mathbb{A}})\to\mathbb{M}(\mathbb{K}_0)$ is a Jordan isomorphism, hence an isomorphism of fields by theorem (31.2) since $\mathbb{K}_0$ is associative and commutative. In particual, $\tilde{\mathbb{A}}$ is a field. In the sequel we suppose $\mathbb{K}$ to be associative.

The map $\gamma:\tilde{\mathbb{A}}\to\mathbb{K}_0\subseteq\mathbb{K}$ is a Jordan homomorphism. Since $\mathbb{K}$ is associative, Hua's theorem implies that $\mathbb{K}_0=\gamma(\tilde{\mathbb{A}})$ is a skew-subfield, which is thus, in fact, a subfield of $\mathbb{K}$ since we have $\mathbb{K}_0\subseteq\operatorname{Fix}(\sigma)$ and, therefore,

$$\forall\, x,y\in\mathbb{K}_0:\qquad xy=(xy)^\sigma=y^\sigma x^\sigma=yx\ .$$

By Hua's theorem again, γ is an isomorphism of fields. In particular, $\tilde{\mathbb{A}}=\gamma^{-1}(\mathbb{K}_0)$ is a field. Moreover, $(\mathbb{K},\mathbb{K}_0,\sigma)$ is non-proper by lemma (5.2) and thus quadratic of type (m)$\in\{$(i),...,(iv)$\}$ by lemma (31.25).

"$\Leftarrow$" We have

$$\gamma(1_{\tilde{\mathbb{M}}})=\gamma(1_{\tilde{\mathbb{A}}})=1_{\tilde{\mathbb{K}}_0}=1_{\mathbb{K}}=1_{\mathbb{M}}\ .$$

Given $a\in\tilde{\mathbb{A}}^*$ and $x\in\tilde{\mathbb{A}}$, we have

$$\gamma\big(\tilde{h}_a(x)\big)=\gamma(axa)=\gamma(a)\gamma(xa)=\gamma(a)\gamma(x)\gamma(a)=h_{\gamma(a)}\big(\gamma(x)\big)\ .$$

$\square$

§ 31.5 $\mathbb{M}(L_0,\mathbb{K},q)\cong\mathbb{M}(\tilde{L}_0,\tilde{\mathbb{K}},\tilde{q})$

(31.27) Remark Cf. chapter 4.6 in [K] for another proof of the main result of this paragraph.

(31.28) Notation Until theorem (31.41), $\mathbb{M}:=\mathbb{M}(L_0,\mathbb{K},q)$ is a Moufang set of quadratic form type with basepoint ϵ, $\tilde{\mathbb{M}}:=\mathbb{M}(\tilde{L}_0,\tilde{\mathbb{K}},\tilde{q})$ is a Moufang set of proper quadratic form type with basepoint $\tilde{\epsilon}$ and $\dim_{\tilde{\mathbb{K}}}\tilde{L}_0\geq 3$, and $\gamma:\mathbb{M}\to\tilde{\mathbb{M}}$ is a Jordan isomorphism such that $\gamma(1_{\mathbb{M}})=1_{\tilde{\mathbb{M}}}$.

(31.29) Lemma If we have

$$\forall\, x \in L_0^* : \qquad \gamma(\langle x\rangle_{\mathbb{K}}) \subseteq \langle \gamma(x)\rangle_{\tilde{\mathbb{K}}} \ ,$$

we have

$$\forall\, x \in L_0^* : \qquad \gamma(\langle x\rangle_{\mathbb{K}}) = \langle \gamma(x)\rangle_{\tilde{\mathbb{K}}} \ .$$

Proof
Notice that we have $\dim_{\tilde{\mathbb{K}}} \tilde{L}_0 \geq 3$. First of all, assume $\dim_{\mathbb{K}} L \leq 2$. Then $\gamma(L_0)$ is contained in a two-dimensial $\tilde{\mathbb{K}}$-subspace of $\tilde{L}_0$ and hence in a proper subspace of $\tilde{L}_0$, which contradicts the fact that $\gamma : L_0 \to \tilde{L}_0$ is a bijection. We obtain $\dim_{\mathbb{K}} \geq 3$, ans thus, by symmetry,

$$\forall\, x \in L_0^* : \qquad \gamma^{-1}(\langle \gamma(x)\rangle_{\tilde{\mathbb{K}}}) \subseteq \langle \gamma^{-1}\big(\gamma(x)\big)\rangle_{\mathbb{K}} = \langle x\rangle_{\mathbb{K}} \ ,$$

hence

$$\forall\, x \in L_0^* : \qquad \gamma(\langle x\rangle_{\mathbb{K}}) \supseteq \langle \gamma(x)\rangle_{\tilde{\mathbb{K}}} \ .$$

□

(31.30) Notation Given $x \in L_0^*$, $y \in L_0$ and $s \in \mathbb{K}$, we set

$$\begin{aligned}
\psi_1(x,y,s) &:= \gamma(x\cdot s)\cdot f_{\tilde{q}}\big(\gamma(x\cdot s),\gamma(y)^{\tilde{\sigma}}\big) - \gamma(y)^{\tilde{\sigma}}\cdot \tilde{q}\big(\gamma(x\cdot s)\big) \ ,\\
\psi_2(x,y,s) &:= \gamma(x)\cdot f_{\tilde{q}}\big(\gamma(x),\gamma(y\cdot s^2)^{\tilde{\sigma}}\big) - \gamma(y\cdot s^2)^{\tilde{\sigma}}\cdot \tilde{q}\big(\gamma(x)\big) \ ,\\
\psi_3(x,y,s) &:= \gamma(x)\cdot f_{\tilde{q}}\big(\gamma(x\cdot s),\gamma(y)^{\tilde{\sigma}}\big) + \gamma(x\cdot s)\cdot f_{\tilde{q}}\big(\gamma(x),\gamma(y)^{\tilde{\sigma}}\big) - \gamma(y)^{\tilde{\sigma}}\cdot f_{\tilde{q}}\big(\gamma(x\cdot s),\gamma(x)\big) \ ,\\
\psi_4(x,y,s) &:= \gamma(x)\cdot f_{\tilde{q}}\big(\gamma(x),2\gamma(y\cdot s)^{\tilde{\sigma}}\big) - 2\gamma(y\cdot s)^{\tilde{\sigma}}\cdot \tilde{q}\big(\gamma(x)\big) \ .
\end{aligned}$$

(31.31) Lemma Let $x \in L_0^*$, $y \in L_0$ and $s \in \mathbb{K}$. Then the following holds:

(a) We have
$$\psi_1(x,y,s+1_{\mathbb{K}}) = \psi_1(x,y,s) + \psi_1(x,y,1_{\mathbb{K}}) + \psi_3(x,y,s) \ .$$

(b) We have
$$\psi_2(x,y,s+1_{\mathbb{K}}) = \psi_2(x,y,s) + \psi_2(x,y,1_{\mathbb{K}}) + \psi_4(x,y,s) \ .$$

(c) We have $\psi_1(x,y,s) = \psi_2(x,y,s)$.

(d) We have $\psi_3(x,y,s) = \psi_4(x,y,s)$.

Proof

(a) This is a direct calculation using the facts that γ is additive and that we have
$$\forall\, x \in L_0,\ s \in \mathbb{K} : \qquad \tilde{q}\big(\gamma(x\cdot s)+\gamma(x)\big) = \tilde{q}\big(\gamma(x\cdot s)\big) + \tilde{q}\big(\gamma(x)\big) + f_{\tilde{q}}\big(\gamma(x\cdot s),\gamma(x)\big) \ .$$

(b) This a direct calculation using the fact that γ is additive.

(c) By lemma (4.5) and lemma (4.7), we have
$$\begin{aligned}
\psi_1(x,y,s) &= \gamma(x\cdot s)\cdot f_{\tilde{q}}\big(\gamma(x\cdot s),\gamma(y)^{\tilde{\sigma}}\big) - \gamma(y)^{\tilde{\sigma}}\cdot \tilde{q}\big(\gamma(x\cdot s)\big)\\
&= \tilde{h}_{\gamma(x\cdot s)}\big(\gamma(y)\big) = \gamma\big(h_{x\cdot s}(y)\big) = \gamma\big(h_x(y\cdot s^2)\big) = \tilde{h}_{\gamma(x)}\big(\gamma(y\cdot s^2)\big)\\
&= \gamma(x)\cdot f_{\tilde{q}}\big(\gamma(x),\gamma(y\cdot s^2)^{\tilde{\sigma}}\big) - \gamma(y\cdot s^2)^{\tilde{\sigma}}\cdot \tilde{q}\big(\gamma(x)\big) = \psi_2(x,y,s) \ .
\end{aligned}$$

(d) By (a), (b) and (c), we have
$$\begin{aligned}
\gamma_1(x,y,s) + \gamma_1(x,y,1_{\mathbb{K}}) + \gamma_3(x,y,s) &= \gamma_1(x,y,s+1_{\mathbb{K}}) = \gamma_2(x,y,s+1_{\mathbb{K}})\\
&= \gamma_2(x,y,s) + \gamma_2(x,y,1_{\mathbb{K}}) + \gamma_4(x,y,s) \ ,
\end{aligned}$$
hence $\gamma_3(x,y,s) = \gamma_4(x,y,s)$ by (c) again.

□

(31.32) Lemma Assume $\operatorname{Char}\mathbb{K} = 2$. Given $x \in L_0$ such that $\gamma(x) \notin \tilde{L}_0^\perp$, we have

$$\forall\ s \in \mathbb{K}: \qquad \gamma(x \cdot s) \in \langle \gamma(x) \rangle_{\tilde{\mathbb{K}}} \ .$$

Proof
Lemma (31.31) (d) simplifies to

$$\gamma(x) \cdot f_{\tilde{q}}\big(\gamma(x \cdot s), \gamma(y)^{\tilde{\sigma}}\big) + \gamma(x \cdot s) \cdot f_{\tilde{q}}\big(\gamma(x), \gamma(y)^{\tilde{\sigma}}\big) + \gamma(y)^{\tilde{\sigma}} \cdot f_{\tilde{q}}\big(\gamma(x \cdot s), \gamma(x)\big) = 0_{\tilde{L}_0} \tag{31.1}$$

for all $x \in L_0^*$, $y \in L_0$, $s \in \mathbb{K}$. Since we have $\dim_{\tilde{\mathbb{K}}} \tilde{L}_0 \geq 3$, there is an element $y \in L_0$ such that $\gamma(y)^{\tilde{\sigma}} \in \gamma(x)^\perp \cap \gamma(x \cdot s)^\perp$. Therefore, we have

$$\forall\ s \in \mathbb{K}: \qquad f_{\tilde{q}}\big(\gamma(x \cdot s), \gamma(x)\big) = 0_{\tilde{\mathbb{K}}} \ ,$$

and equation (31.1) simplifies to

$$\gamma(x) \cdot f_{\tilde{q}}\big(\gamma(x \cdot s), \gamma(y)^{\tilde{\sigma}}\big) = \gamma(x \cdot s) \cdot f_{\tilde{q}}\big(\gamma(x), \gamma(y)^{\tilde{\sigma}}\big)$$

for all $x \in L_0^\perp$, $y \in L_0$, $s \in \mathbb{K}$. By assumption, there is an element $y \in L_0$ such that $f_{\tilde{q}}\big(\gamma(x), \gamma(y)^{\tilde{\sigma}}\big) \neq 0_{\tilde{\mathbb{K}}}$, hence

$$\gamma(x \cdot s) = \gamma(x) \cdot f_{\tilde{q}}\big(\gamma(x \cdot s), \gamma(y)^{\tilde{\sigma}}\big) f_{\tilde{q}}\big(\gamma(x), \gamma(y)^{\tilde{\sigma}}\big)^{-1} \in \langle \gamma(x) \rangle_{\tilde{\mathbb{K}}} \ .$$

□

(31.33) Notation

- Until proposition (31.37), we assume $\operatorname{Char}\mathbb{K} = 2$.
- Given $x \in L_0$ such that $\gamma(x) \notin L_0^\perp$, let $\phi_x : \mathbb{K} \to \tilde{\mathbb{K}}$ defined by

$$\forall\ s \in \mathbb{K}: \qquad \gamma(x \cdot s) = \gamma(x) \cdot \phi_x(s) \ .$$

(31.34) Lemma Let $x \in L_0$ be such that $\gamma(x) \notin \tilde{L}_0^\perp$ and let $y \in L_0$ be such that $\gamma(y) \in \tilde{L}_0^\perp$. Then we have

$$\gamma(x + y) \notin \tilde{L}_0^\perp \ .$$

Proof
Given $z \in L_0$, we have

$$f_{\tilde{q}}\big(\gamma(x + y), \gamma(z)\big) = f_{\tilde{q}}\big(\gamma(x), \gamma(z)\big) \ ,$$

hence $\gamma(x + y) \notin L_0^\perp$. □

(31.35) Corollary Let $x \in L_0$ be such that $\gamma(x) \notin \tilde{L}_0^\perp$ and let $y \in L_0$ be such that $\gamma(y) \in \tilde{L}_0^\perp$. Then we have

$$\forall\ t \in \mathbb{K}^*: \qquad \gamma\big(x \cdot t + y\big) \notin \tilde{L}_0^\perp \ .$$

Proof
By lemma (31.32), we have

$$\forall\ t \in \mathbb{K}^*: \qquad \gamma(x \cdot t) = \gamma(x) \cdot \phi_x(t) \notin \tilde{L}_0^\perp$$

so that we may apply lemma (31.34). □

(31.36) Lemma Let $x, y \in L_0$ be such that $\gamma(y) \notin \langle \gamma(x) \rangle_{\tilde{\mathbb{K}}}$ and $\gamma(x), \gamma(y) \notin \tilde{L}_0^\perp$. Then we have

$$\phi_x = \phi_y \ .$$

Proof
The assertion is clearly true for $\mathbb{K} = \mathbb{F}_2$ so that we may assume $|\mathbb{K}| \geq 4$.

(i) $\gamma(y) \notin \langle \gamma(x) \rangle_{\tilde{\mathbb{K}}}$, $\gamma(x+y) \notin L_0^{\perp}$: Given $s \in \mathbb{K}$, we have

$$\begin{aligned}\gamma(x) \cdot \phi_{x+y}(s) + \gamma(y) \cdot \phi_{x+y}(s) &= \gamma\big((x+y) \cdot s\big) = \gamma(x \cdot s) + \gamma(y \cdot s) \\ &= \gamma(x) \cdot \phi_x(s) + \gamma(y) \cdot \phi_y(s) \ ,\end{aligned}$$

hence

$$\phi_x(s) = \phi_{x+y}(s) = \phi_y(s) \ .$$

The condition in this step is always fulfilled if we have $\tilde{L}_0^{\perp} = \{0\}$ so that we may assume $\tilde{L}_0^{\perp} \neq \{0\}$ in the following.

(ii) $y \in \langle x \rangle_{\mathbb{K}}$, which means that we allow $\gamma(y) \in \langle \gamma(x) \rangle_{\tilde{\mathbb{K}}}$ in this case, cf. lemma (31.32): Since we have $|\mathbb{K}| \geq 4$, there is an element $t \in \mathbb{K}^*$ such that

$$x \cdot t \notin \{x, y\} \ .$$

Let $z \in L_0^*$ be such that $\gamma(z) \in \tilde{L}_0^{\perp}$, which implies $\gamma(z) \notin \langle \gamma(x) \rangle_{\tilde{\mathbb{K}}}$. Then we have

$$\gamma(x \cdot t + z) \notin \tilde{L}_0^{\perp} \ , \qquad \gamma(x + x \cdot t + z) \notin \tilde{L}_0^{\perp} \ , \qquad \gamma(y + x \cdot t + z) \notin \tilde{L}_0^{\perp}$$

by corollary (31.35) and thus $\phi_x = \phi_{x \cdot t + z} = \phi_y$ by (i).

(iii) $\gamma(y) \notin \langle \gamma(x) \rangle_{\tilde{\mathbb{K}}}$, $\gamma(x+y) \in L_0^{\perp}$: Let $t \in \mathbb{K} \setminus \{0_{\mathbb{K}}, 1_{\mathbb{K}}\}$. By lemma (31.32) and corollary (31.35), we have

$$\gamma(x \cdot t) \notin \tilde{L}_0^{\perp} \ , \qquad \gamma(x \cdot t + y) = \gamma\big(x \cdot (t + 1_{\mathbb{K}}) + (x+y)\big) \notin \tilde{L}_0^{\perp} \ ,$$

hence $\phi_x \overset{\text{(i)}}{=} \phi_{x \cdot t} \overset{\text{(ii)}}{=} \phi_y$ by the previous steps.

□

(31.37) Proposition If we have $\operatorname{Char} \mathbb{K} = 2$, the map $\gamma : L_0 \to \tilde{L}_0$ is an isomorphism of vector spaces.

Proof
We show that we have

$$\forall\ x \in L_0^*,\ s \in \mathbb{K} : \qquad \gamma(x \cdot s) \in \langle \gamma(x) \rangle_{\tilde{\mathbb{K}}}$$

so that we may apply the fundamental theorem of projective geometry by lemma (31.29).

Let $x \in L_0^*$. By lemma (31.32), we may assume $\gamma(x) \in L_0^{\perp}$. Since $(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ is proper, there is an element $y \in L_0$ such that $\gamma(y) \notin L_0^{\perp}$. Moreover, we have $\gamma(y) \notin \langle \gamma(x+y) \rangle_{\tilde{\mathbb{K}}}$, and, by lemma (31.34), $\gamma(x+y) \notin \tilde{L}_0$. By lemma (31.36) therefore, we have

$$\phi_y = \phi_{x+y}$$

and thus

$$\begin{aligned}\gamma(x \cdot s) + \gamma(y) \cdot \phi_y(s) &= \gamma(x \cdot s) + \gamma(y \cdot s) = \gamma\big((x+y) \cdot s\big) \\ &= \gamma(x+y) \cdot \phi_{x+y}(s) = \gamma(x) \cdot \phi_y(s) + \gamma(y) \cdot \phi_y(s)\end{aligned}$$

for each $s \in \mathbb{K}$, hence

$$\forall\ s \in \mathbb{K} : \qquad \gamma(x \cdot s) = \gamma(x) \cdot \phi_y(s) \in \langle \gamma(x) \rangle_{\tilde{\mathbb{K}}} \ .$$

□

(31.38) Remark We drop the condition $\operatorname{Char}\mathbb{K} = 2$.

(31.39) Lemma Assume $\operatorname{Char}\mathbb{K} \neq 2$. Let $y \in L_0^*$ and $s \in \mathbb{K}$. Given $x \in L_0^*$ such that

$$f_{\tilde{q}}\big(\gamma(x), \gamma(y)^{\tilde{\sigma}}\big) = 0_{\tilde{\mathbb{K}}} = f_{\tilde{q}}\big(\gamma(x), \gamma(y \cdot s)\big) \ ,$$

we have

$$f_{\tilde{q}}\big(\gamma(x \cdot s), \gamma(y)^{\tilde{\sigma}}\big) = 0_{\tilde{\mathbb{K}}} \ .$$

Proof
By lemma (31.31) (d), we have

$$\gamma(x) \cdot f_{\tilde{q}}\big(\gamma(x \cdot s), \gamma(y)^{\tilde{\sigma}}\big) = \gamma(y)^{\tilde{\sigma}} \cdot f_{\tilde{q}}\big(\gamma(x \cdot s), \gamma(x)\big) - 2\gamma(y \cdot s)^{\tilde{\sigma}} \cdot \tilde{q}\big(\gamma(x)\big) \ .$$

Assume $f_{\tilde{q}}\big(\gamma(x \cdot s), \gamma(y)^{\tilde{\sigma}}\big) \neq 0_{\tilde{\mathbb{K}}}$. Then we have

$$\gamma(x) \in \langle \gamma(y)^{\tilde{\sigma}}, \gamma(y \cdot s)^{\tilde{\sigma}} \rangle_{\tilde{\mathbb{K}}} \subseteq \gamma(x)^{\perp} \qquad \lightning \ .$$

□

(31.40) Proposition If we have $\operatorname{Char}\mathbb{K} \neq 2$, the map $\gamma : L_0 \to \tilde{L}_0$ is an isomorphism of vector spaces.

Proof
We show that we have

$$\forall\, y \in L_0^*,\ s \in \mathbb{K} : \qquad \gamma(y \cdot s) \in \langle \gamma(y) \rangle_{\tilde{\mathbb{K}}}$$

so that we may apply the fundamental theorem of projective geometry by lemma (31.29).

Let $y \in L_0^*$. By assumption, there is an element $x \in L_0^*$ such that

$$f_{\tilde{q}}\big(\gamma(x), \gamma(y)^{\tilde{\sigma}}\big) = 0_{\tilde{\mathbb{K}}} = f_{\tilde{q}}\big(\gamma(x), \gamma(y \cdot s)\big) \ .$$

By lemma (31.31) (d) and lemma (31.39), we have

$$\gamma(y)^{\tilde{\sigma}} \cdot f_{\tilde{q}}\big(\gamma(x \cdot s), \gamma(x)\big) = 2\gamma(y \cdot s)^{\tilde{\sigma}} \cdot \tilde{q}\big(\gamma(x)\big) \ ,$$

hence

$$\gamma(y \cdot s) = \frac{1}{2}\gamma(y) \cdot f_{\tilde{q}}\big(\gamma(x \cdot s), \gamma(x)\big)\tilde{q}\big(\gamma(x)\big)^{-1} \in \langle \gamma(y) \rangle_{\tilde{\mathbb{K}}} \ .$$

□

(31.41) Theorem Let Ξ be a quadratic space with basepoint ϵ, let $\tilde{\Xi}$ be a proper quadratic space with basepoint $\tilde{\epsilon}$ and let $\mathbb{M} := \mathbb{M}(\Xi)$, $\tilde{\mathbb{M}} := \mathbb{M}(\tilde{\Xi})$ be the corresponding Moufang sets of quadratic form type. A map $\gamma : \mathbb{M} \to \tilde{\mathbb{M}}$ is a Jordan isomorphism such that $\gamma(1_{\mathbb{M}}) = 1_{\tilde{\mathbb{M}}}$ if and only if one of the following holds:

(i) We have $\dim_{\tilde{\mathbb{K}}} \tilde{L}_0 \geq 3$ and there is an isomorphism $\phi : \mathbb{K} \to \tilde{\mathbb{K}}$ of fields such that the map

$$(\gamma, \phi) : (L_0, \mathbb{K}, q) \to (\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$$

is an isomorphism of quadratic spaces. In particular, we have $\dim_{\mathbb{K}} L_0 = \dim_{\tilde{\mathbb{K}}} \tilde{L}_0$.

(ii) We have $\dim_{\tilde{\mathbb{K}}} \tilde{L}_0 \leq 2$, the map

$$\phi : \mathbb{K} \to \hat{\mathbb{K}} := \gamma(\langle \epsilon \rangle_{\mathbb{K}}) \subseteq \hat{\mathbb{F}} := \mathbb{F}(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}),\ \ s \mapsto \gamma(\epsilon \cdot s)$$

is an isomorphism of fields, the field $\hat{\mathbb{F}}$ is quadratic over $\hat{\mathbb{K}}$, and the map

$$(\gamma, \phi) : (L_0, \mathbb{K}, q) \to (\hat{\mathbb{F}}, \hat{\mathbb{K}}, N_{\hat{\mathbb{K}}}^{\hat{\mathbb{F}}})$$

is an isomorphism of quadratic spaces. This is true even if $\tilde{\Xi}$ is non-proper.

Proof

"⇒"

- $\dim_{\tilde{\mathbb{K}}} \tilde{L}_0 \geq 3$: By proposition (31.37) and proposition (31.40), there is an isomorphism $\phi : \mathbb{K} \to \tilde{\mathbb{K}}$ of fields such that the map $(\gamma, \phi) : (L_0, \mathbb{K}) \to (\tilde{L}_0, \tilde{\mathbb{K}})$ is an isomorphism of vector spaces. By lemma (31.19), we have

$$\tilde{h}_{\gamma(x)}(\tilde{\epsilon}) - \gamma(x) \cdot \phi\big(T(x)\big) + \tilde{\epsilon} \cdot \phi\big(q(x)\big) = \gamma\big(h_x(\epsilon) - x \cdot T(x) + \epsilon \cdot q(x)\big) = 0_{\tilde{L}_0}$$

and

$$\tilde{h}_{\gamma(x)}(\tilde{\epsilon}) - \gamma(x) \cdot \tilde{T}\big(\gamma(x)\big) + \tilde{\epsilon} \cdot \tilde{q}\big(\gamma(x)\big) = 0_{\tilde{L}_0}$$

for each $x \in L_0^*$, hence

$$\forall\ x \in L_0^* : \qquad -\gamma(x) \cdot \phi\big(T(x)\big) + \tilde{\epsilon} \cdot \phi\big(q(x)\big) = -\gamma(x) \cdot \tilde{T}\big(\gamma(y)\big) + \tilde{\epsilon} \cdot \tilde{q}\big(\gamma(x)\big)\ ,$$

which implies

$$\forall\ x \in L_0 \setminus \langle \epsilon \rangle_{\mathbb{K}} : \qquad \tilde{q}\big(\gamma(x)\big) = \phi\big(q(x)\big)\ .$$

Given $s \in \mathbb{K}$, we have

$$\tilde{q}\big(\gamma(\epsilon \cdot s)\big) = \tilde{q}\big(\tilde{\epsilon} \cdot \phi(s)\big) = \phi(s)^2 = \phi(s^2) = \phi\big(q(\epsilon \cdot s)\big)\ .$$

- $\dim_{\tilde{\mathbb{K}}} \tilde{L}_0 \leq 2$: By lemma (31.11), we have

$$\mathbb{M}(L_0, \mathbb{K}, q) \cong \mathbb{M}(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}) = \mathbb{M}\big(\mathbb{F}(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})\big)\ .$$

Now we apply theorem (31.7).

"⇐" Let (γ, ϕ) be an isomorphism of quadratic spaces $\big($independent of the target space $(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ or $(\hat{\mathbb{F}}, \hat{\mathbb{K}}, N_{\hat{\mathbb{K}}}^{\hat{\mathbb{F}}})\big)$. By lemma (4.2), we have

$$\begin{aligned}\gamma\big(h_x(y)\big) &= \gamma\big(x \cdot f_q(x, y^\sigma) - y^\sigma \cdot q(x)\big) = \gamma(x) \cdot \phi\big(f_q(x, y^\sigma)\big) - \gamma(y^\sigma) \cdot \phi\big(q(x)\big) \\ &= \gamma(x) \cdot f_{\tilde{q}}\big(\gamma(x), \gamma(y)^{\tilde{\sigma}}\big) - \gamma(y)^{\tilde{\sigma}} \cdot \tilde{q}\big(\gamma(y)\big) = \tilde{h}_{\gamma(x)}\big(\gamma(y)\big)\end{aligned}$$

for all $x \in L_0^*$, $y \in L_0$ which completes the case $\dim_{\tilde{\mathbb{K}}} \tilde{L}_0 \geq 3$. In the case $\dim_{\tilde{\mathbb{K}}} \tilde{L}_0 \leq 2$ moreover, we have

$$\mathbb{M}(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}) = \mathbb{M}(\hat{\mathbb{F}}) = \mathbb{M}(\hat{\mathbb{K}}, \hat{\mathbb{F}}, N_{\hat{\mathbb{K}}}^{\hat{\mathbb{F}}})$$

by lemma (31.11) and lemma (31.23) so that

$$\gamma : \mathbb{M}(L_0, \mathbb{K}, q) \to \mathbb{M}(\hat{\mathbb{K}}, \hat{\mathbb{F}}, N_{\hat{\mathbb{K}}}^{\hat{\mathbb{F}}}) = \mathbb{M}(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$$

is a Jordan isomorphism.

□

Part VII

An Inventory of Moufang Polygons

As in the simply laced case, the parametrization for the appearing Moufang polygons in root group sequences of a given twin building is an important technical tool for the classification of twin buildings: We can make use of the knowledge about the parametrizing Moufang sets and Jordan isomorphisms between them, i.e., we may apply the results of chapter 31.

Chapter 32 Parametrized Moufang Polygons

Before we give the appearing examples, we have to establish the concept of standard and opposite parametrized Moufang polygons as we can read a root group sequence in two directions.

(32.1) Definition

- The symbol $\mathcal{T}$ is *standard of type 3*.
- The symbols $\mathcal{Q}_I, \mathcal{Q}_Q, \mathcal{Q}_D, \mathcal{Q}_P, \mathcal{Q}_E$ and $\mathcal{Q}_F$ are *standard of type 4*.
- The symbol $\mathcal{H}$ is *standard of type 6*.
- The symbol $\mathcal{O}$ is *standard of type 8*.

(32.2) Definition

- A *parameter system of type* $\mathcal{T}$ is an alternative division ring $\mathbb{A}$.
- A *parameter system of type* $\mathcal{Q}_I$ is a proper involutory set $(\mathbb{K}, \mathbb{K}_0, \sigma)$.
- A *parameter system of type* $\mathcal{Q}_Q$ is a quadratic space $(L_0, \mathbb{K}, q)$ with basepoint ϵ.
- A *parameter system of type* $\mathcal{Q}_D$ is a proper indifferent set $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$.
- A *parameter system of type* $\mathcal{Q}_P$ is a proper right pseudo-quadratic space $(\mathbb{K}, \mathbb{K}_0, \sigma, L_0, q)$.
- A *parameter system of type* $\mathcal{Q}_E$ is a quadratic space $(L_0, \mathbb{K}, q)$ of type E_6, E_7, E_8.
- A *parameter system of type* $\mathcal{Q}_F$ is a quadratic space $(L_0, \mathbb{K}, q)$ of type F_4.
- A *parameter system of type* $\mathcal{H}$ is a hexagonal system $(J, \mathbb{F}, \#)$.
- A *parameter system of type* $\mathcal{O}$ is an octagonal set $(\mathbb{K}, \sigma)$.

(32.3) Definition A *parametrized standard Moufang n-gon* is a standard root group sequence

$$\mathcal{X}(\Xi) = \big(U_{[1,n]}, x_1(\mathbb{M}_1), \ldots, x_n(\mathbb{M}_n)\big) \ ,$$

where $\mathcal{X}$ is a standard symbol of type $n \in \{3, 4, 6, 8\}$, Ξ is a parameter system of type $\mathcal{X}$ and $\mathbb{M}_1, \ldots, \mathbb{M}_n$ are the parameter groups with respect to the parametrizations $x_1, \ldots, x_n$ and the corresponding commutator relations, cf. chapter 16 of [TW].

(32.4) Remark

(a) Given a parametrized standard Moufang n-gon $\mathcal{X}(\Xi)$, the parameter groups $\mathbb{M}_1, \ldots, \mathbb{M}_n$ are Moufang sets.

(b) For reasons of brevity, we will write

$$\mathcal{X}(\Xi) = \big(x_1(\mathbb{M}_1), \ldots, x_n(\mathbb{M}_n)\big)$$

instead of $\mathcal{X}(\Xi) = \big(U_{[1,n]}, x_1(\mathbb{M}_1), \ldots, x_n(\mathbb{M}_n)\big)$.

(32.5) Definition Let $\mathcal{X}$ be a standard symbol of type $n \in \{3,4,6,8\}$.

- The symbol $\mathcal{X}^o$ is the corresponding *opposite symbol of type* n. We set $(\mathcal{X}^o)^o := \mathcal{X}$.
- A *parameter system of type* $\mathcal{X}^o$ is just a parameter system of type $\mathcal{X}$, except for the following standard symbols:
  - A *parameter system of type* $\mathcal{Q}_P^o$ is a proper left pseudo-quadratic space.
  - Hexagonal systems and octagonal sets are not taken into account yet.

(32.6) Notation

- In the following, a *symbol* $\mathcal{X}$ denotes either a standard or an opposite symbol of type n for some $n \in \{3,4,6,8\}$.
- In the following, a *parameter system* Ξ denotes a parameter system of type $\mathcal{X}$ for some symbol $\mathcal{X}$.

(32.7) Remark The following list is not complete since we restrict to parameter systems for n-gons with $n \in \{3,4\}$. The list can be extended easily.

(32.8) Definition Given a parameter system Ξ, there is a natural way to define an *opposite parameter system* Ξ^o:

- Given an alternative division ring $\mathbb{A}$, the corresponding opposite parameter system $\mathbb{A}^o$ is just the opposite alternative division ring $\mathbb{A}^o$ as in definition (3.7).
- Given an involutory set $(\mathbb{K}, \mathbb{K}_0, \sigma)$, the corresponding opposite parameter system is
$$(\mathbb{K}, \mathbb{K}_0, \sigma)^o := (\mathbb{K}^o, \mathbb{K}_0, \sigma) \ ,$$
which is an involutory set.
- Given an indifferent set $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$, the corresponding opposite parameter system is
$$(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)^o := (\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0) \ ,$$
which is the indifferent set itself.
- Given a right (resp. left) pseudo-quadratic space $(\mathbb{K}, \mathbb{K}_0, \sigma, L_0, q)$, the corresponding opposite parameter system is
$$(\mathbb{K}, \mathbb{K}_0, \sigma, L_0, q)^o := (\mathbb{K}^o, \mathbb{K}_0, \sigma, L_0, q) \ ,$$
which is a left (resp. right) pseudo-quadratic space.
- Given a quadratic space $(L_0, \mathbb{K}, q)$, the corresponding opposite parameter system is
$$(L_0, \mathbb{K}, q)^o := (L_0, \mathbb{K}, q) \ ,$$
which is the quadratic space itself.

(32.9) Remark Let Ξ be a parameter system. If Ξ is of type $\mathcal{X}$, then Ξ^o is of type $\mathcal{X}^o$.

(32.10) Definition An *parametrized opposite Moufang n-gon* is a root group sequence
$$\mathcal{X}^o(\Xi) = \big(x_1(\mathbb{M}_1), \dots, x_n(\mathbb{M}_n)\big)$$
such that
$$\big(x_n(\mathbb{M}_n^o), \dots, x_1(\mathbb{M}_1^o)\big) = \mathcal{X}(\Xi^o) \ ,$$
where $\mathcal{X}$ is a standard symbol of type $n \in \{3,4,6,8\}$ and Ξ is a parameter system of type $\mathcal{X}^o$.

(32.11) Remark

(a) Parametrized opposite Moufang n-gons can be obtained as follows: Take the opposite root group sequence $\big(x_n(\mathbb{M}_n), \ldots, x_1(\mathbb{M}_1)\big)$ of some parametrized standard Moufang n-gon $\mathcal{X}(\Xi) = \big(x_1(\mathbb{M}_1), \ldots, x_n(\mathbb{M}_n)\big)$, calculate the commutator relations and interpret them in the opposite parameter system Ξ^o.

(b) By definition, each parametrized opposite Moufang n-gon arises in this way.

(c) We will make this more explicit in the next chapter.

(32.12) Definition A *parametrized Moufang polygon of type* $\mathcal{X}$ is a parametrized standard or opposite Moufang n-gon $\mathcal{X}(\Xi)$ for some symbol $\mathcal{X}$ of type n and some parameter system of type $\mathcal{X}$.

(32.13) Remark Two isomorphic Moufang polygons are necessarily n-gons for the same value $n \in \{3, 4, 6, 8\}$, cf. p. 419 of [TW]. However, there are six families of Moufang quadrangles, and there are indeed quadrangles belonging to different families, cf. chapter 38 of [TW]. But the six families of parametrized Moufang quadrangles are disjoint if we use the above list of parameter systems, i.e., two isomorphic parametrized quadrangles are necessarily of the same type $\mathcal{X}$, cf. (38.9) of [TW].

(32.14) Definition Let $\mathcal{X}(\Xi)$, $\mathcal{X}(\tilde{\Xi})$ be parametrized Moufang polygons of the same type $\mathcal{X}$.

- An *isomorphism* $\alpha : \mathcal{X}(\Xi) \to \mathcal{X}(\tilde{\Xi})$ is an ordered set $(\alpha_1, \ldots, \alpha_n)$ such that $\alpha_i : \mathbb{M}_i \to \tilde{\mathbb{M}}_i$ is an isomorphism of groups for each $i \in \{1, \ldots, n\}$ and such that
$$\big(x_1\alpha_1(\mathbb{M}_1), \ldots, x_n\alpha_n(\mathbb{M}_n)\big) = \mathcal{X}(\Xi) \ .$$
- A *reparametrization for* $\mathcal{X}(\Xi)$ is an ordered set $\alpha = (\tilde{\Xi}, \alpha_1, \ldots, \alpha_n)$ such that $\tilde{\Xi}$ is a parameter system of type $\mathcal{X}$ and $\alpha_i : \tilde{\mathbb{M}}_i \to \mathbb{M}_i$ is an isomorphism of groups for each $i \in \{1, \ldots, n\}$ and such that
$$\big(x_1\alpha_1(\tilde{\mathbb{M}}_1), \ldots, x_n\alpha_n(\tilde{\mathbb{M}}_n)\big) = \mathcal{X}(\tilde{\Xi}) \ .$$

(32.15) Remark The following results enable us to define parametrizations for root group sequences such that we have $\gamma(1_{\mathbb{M}}) = 1_{\mathbb{M}}$ for each appearing glueing γ.

(32.16) Lemma Given a parametrized Moufang n-gon $\mathcal{X}(\Xi)$, we have
$$1_{\mathbb{M}_1} \in Z(\mathbb{M}_1) \ , \qquad\qquad 1_{\mathbb{M}_n} \in Z(\mathbb{M}_n) \ .$$

Proof
This results from the fact that $\mathbb{M}_1$ and $\mathbb{M}_n$ are commutative except for the symbols $\mathcal{Q}_P$, $\mathcal{Q}_E$, $\mathcal{O}$. In these cases, the assertion results from definition (10.15) and (38.10) of [TW], cf. Fig. 5 on page 354 of [TW]. □

(32.17) Lemma Let $\mathcal{X}(\Xi)$ be a parametrized Moufang n-gon and let $a \in Z(\mathbb{M}_1)$, $b \in Z(\mathbb{M}_n)$. Then the following holds:

(a) If $\mathcal{X}$ is of type $\mathcal{T}$, $\mathcal{Q}_I$, $\mathcal{Q}_D$, $\mathcal{Q}_P$, $\mathcal{H}$, $\mathcal{O}$, there is a reparametrization $\alpha = (\tilde{\Xi}, \alpha_1, \ldots, \alpha_n)$ s.t.
$$x_1\big(\alpha_1(1_{\tilde{\mathbb{M}}_1})\big) = x_1(a) \ , \qquad\qquad x_n\big(\alpha_n(1_{\tilde{\mathbb{M}}_n})\big) = x_n(b) \ .$$

(b) If $\mathcal{X}$ is of type $\mathcal{Q}_Q$, there is a reparametrization $\alpha = (\tilde{\Xi}, \alpha_1, \ldots, \alpha_{n-1}, \mathrm{id}_{\mathbb{M}_n})$ such that
$$x_1\big(\alpha_1(1_{\tilde{\mathbb{M}}_1})\big) = x_1(a) \ , \qquad\qquad \tilde{q}\big(b\big) = 1_{\tilde{\mathbb{K}}} \ .$$
In particular, we have $1_{\tilde{\mathbb{M}}_n} = b$.

Proof

(a) $\mathcal{T}$: This results from lemma (18.8).
$\mathcal{Q}_I$: This results from (35.16) and (22.39) in [TW].
$\mathcal{Q}_D$: This results from (35.18) in [TW].
$\mathcal{Q}_P$: This results from (35.19) and (25.20) in [TW].
$\mathcal{H}$: This results from (29.40) and (29.42) in [TW].
$\mathcal{O}$: This results from (31.35) in [TW].

(b) $\mathcal{Q}_Q$: This results from (35.17) and (23.25) in [TW].

□

(32.18) Lemma Given a parametrized quadrangle $\mathcal{Q}_E(L_0, \mathbb{K}, q)$ and $a \in Z(\mathbb{M}_1)$, $b \in Z(\mathbb{M}_n)$, there is a reparametrization $\alpha = (\tilde{\Xi}, \alpha_1, \dots, \alpha_{n-1}, \mathrm{id}_{\mathbb{M}_n})$ such that

$$x_1\big(\alpha_1(1_{\tilde{\mathbb{M}}_1})\big) = x_1(a) \ , \qquad\qquad \tilde{q}\big(b\big) = 1_{\tilde{\mathbb{K}}} \ .$$

In particular, we have $1_{\tilde{\mathbb{M}}_n} = b$.

Proof

By remark (21.17) of [TW], $\mathcal{Q}_E(L_0, \mathbb{K}, q)$ is an extension of $\mathcal{Q}_Q(L_0, \mathbb{K}, q)$, and by proposition (21.4) and (38.10) of [TW], the first root group is $Y_1 = Z(U_1)$. We apply lemma (32.17) (b) to the quadrangle $\mathcal{Q}_Q(L_0, \mathbb{K}, q)$ and extend the reparametrization for $\mathcal{Q}_Q(L_0, \mathbb{K}, q)$ to a parametrization for $\mathcal{Q}_E(L_0, \mathbb{K}, q)$, which is possible by theorem (21.12) of [TW], more precisely, by its proof. □

(32.19) Lemma Let $\mathcal{Q}_F(L_0, \mathbb{K}, q)$ be a parametrized quadrangle, let $(\mathbb{F}, \hat{L}_0, \hat{q})$ be as in definition (14.12) of [TW] and let $a \in \mathrm{Def}(\hat{q})^*$, $b \in \mathrm{Def(q)}^*$. Then there is a reparametrization $\alpha = (\tilde{\Xi}, \alpha_1, \dots, \alpha_{n-1}, \mathrm{id}_{\mathbb{M}_n})$ such that

$$x_1\big(\alpha_1(1_{\tilde{M}_1})\big) = x_1(a) \ , \qquad\qquad \tilde{q}\big(b\big) = 1_{\tilde{\mathbb{K}}} \ .$$

In particular, we have $1_{\tilde{\mathbb{M}}_n} = b$.

Proof

By remark (21.18) of [TW], $\mathcal{Q}_E(L_0, \mathbb{K}, q)$ is an extension of $\mathcal{Q}_Q(L_0, \mathbb{K}, q)$, and by proposition (21.4), remark (21.18) and the proof of (14.13) of [TW], the first root group is

$$Y_1 = \{x_1(0, t) \mid t \in \mathbb{K}\} = x_1\big(\mathrm{Def}(\hat{q})\big) \ . \tag{32.1}$$

We apply lemma (32.17) (b) to the quadrangle $\mathcal{Q}_Q(L_0, \mathbb{K}, q)$ and extend the reparametrization for $\mathcal{Q}_Q(L_0, \mathbb{K}, q)$ to a parametrization for $\mathcal{Q}_E(L_0, \mathbb{K}, q)$ which is possible by theorem (21.12) of [TW], more precisely, by its proof. □

(32.20) Notation Let I be an index set, let $i \neq j \in I$ and let

$$\mathcal{B}_{(i,j)} = \mathcal{X}(\Xi) = \big(U_{[1,n]}, x_1(\mathbb{M}_1), \dots, x_n(\mathbb{M}_n)\big)$$

be a parametrized Moufang n-gon. Then $\mathbb{M}^i_{(i,j)} := \mathbb{M}_1$ denotes the parameter group of the first root group and $\mathbb{M}^j_{(i,j)} := \mathbb{M}_n$ denotes the parameter group of the last root group with corresponding parametrizations $x^i_{(i,j)} := x_1$ and $x^j_{(i,j)} := x_n$. Moreover, we set

$$\mathcal{B}^o_{(i,j)} := \mathcal{X}^o(\Xi^o) \ .$$

(32.21) Remark Let $\mathcal{B}_{(1,2)} := \mathcal{Q}_F(L_0, \mathbb{K}, q)$ and $\mathcal{B}_{(2,3)} := \mathcal{Q}_F(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ be quadrangles of type F_4, and let $\gamma : \mathbb{M}^2_{(1,2)} \to \mathbb{M}^2_{(2,3)}$ be a Jordan isomorphism. Then we have $\gamma(0, \mathbb{F}) = (0, \tilde{\mathbb{K}})$, cf. theorem (31.6) and equation (32.1).

Chapter 33 Parametrized Quadrangles

§ 33.1 Quadrangles of Involutory Type

(33.1) Definition Let $(\mathbb{K}, \mathbb{K}_0, \sigma)$ be a (proper) involutory set.

- The root group sequence

$$\mathcal{Q}_I(\mathbb{K}, \mathbb{K}_0, \sigma) := \big(x_1(\mathbb{K}_0), x_2(\mathbb{K}), x_3(\mathbb{K}_0), x_4(\mathbb{K})\big)$$

with commutator relations

$$\begin{aligned} \forall\ s, t \in \mathbb{K} : \qquad & [x_2(s), x_4(t)^{-1}] := x_3(s^\sigma t + t^\sigma s)\ , \\ \forall\ s \in \mathbb{K},\ u \in \mathbb{K}_0 : \qquad & [x_1(u), x_4(s)^{-1}] := x_2(us) x_3(s^\sigma us) \end{aligned}$$

is the *parametrized standard quadrangle of involutory type with respect to* $(\mathbb{K}, \mathbb{K}_0, \sigma)$.

- The root group sequence

$$\mathcal{Q}_I^o(\mathbb{K}, \mathbb{K}_0, \sigma) := \big(x_1(\mathbb{K}), x_2(\mathbb{K}_0), x_3(\mathbb{K}), x_4(\mathbb{K}_0)\big)$$

with commutator relations

$$\begin{aligned} \forall\ s, t \in \mathbb{K} : \qquad & [x_1(s)^{-1}, x_3(t)] := x_2(-st^\sigma - ts^\sigma)\ , \\ \forall\ s \in \mathbb{K},\ u \in \mathbb{K}_0 : \qquad & [x_1(s)^{-1}, x_4(u)] := x_2(-sus^\sigma) x_3(-su) \end{aligned}$$

is the *parametrized opposite quadrangle of involutory type with respect to* $(\mathbb{K}, \mathbb{K}_0, \sigma)$.

(33.2) Lemma Let $(\mathbb{K}, \mathbb{K}_0, \sigma)$ be a (proper) involutory set and let

$$\mathcal{Q}_I^o(\mathbb{K}, \mathbb{K}_0, \sigma) = \big(x_1(\mathbb{K}), x_2(\mathbb{K}_0), x_3(\mathbb{K}), x_4(\mathbb{K}_0)\big)$$

be the corresponding opposite quadrangle. Then the action of the *Hua automorphism*

$$h_1(s) := \mu\big(x_1(1_\mathbb{K})\big)^{-1} \mu\big(x_1(s)\big)$$

on $x_1(\mathbb{K}) \times x_4(\mathbb{K}_0)$ corresponds to the map

$$(t, u) \mapsto (sts, s^{-1}us^{-\sigma})\ ,$$

and the action of the *Hua automorphism*

$$h_4(s) := \mu\big(x_4(1_\mathbb{K})\big)^{-1} \mu\big(x_4(s)\big)$$

on $x_1(\mathbb{K}) \times x_4(\mathbb{K}_0)$ corresponds to the map

$$(t, u) \mapsto (ts^{-1}, s^\sigma us)\ .$$

Proof
If we consider the quadrangle $\mathcal{Q}_I(\mathbb{K}^o, \mathbb{K}_0^o, \sigma)$, then the action of $h_1(s)$ on $x_4(\mathbb{K}_0) \times x_1(\mathbb{K})$ corresponds to the map

$$(u, t) \mapsto (s^{-\sigma} \circ u \circ s^{-1}, s \circ t \circ s) = (s^{-1}us^{-\sigma}, sts)\ ,$$

and the action of $h_4(s)$ on $x_4(\mathbb{K}_0) \times x_1(\mathbb{K})$ corresponds to the map

$$(u, t) \mapsto (s \circ u \circ s^\sigma, s^{-1} \circ t) = (s^\sigma us, ts^{-1})\ ,$$

cf. (33.13) of [TW]. □

§ 33.2 Quadrangles of Pseudo-Quadratic Form Type

(33.3) Definition

- Let $(\mathbb{K},\mathbb{K}_0,\sigma,L_0,q)$ be a (proper) right pseudo-quadratic space. Then the root group sequence
$$\mathcal{Q}_P(\mathbb{K},\mathbb{K}_0,\sigma,L_0,q) := \big(x_1(T),x_2(\mathbb{K}),x_3(T),x_4(\mathbb{K})\big)$$
with commutator relations
$$\begin{aligned}[x_1(a,t),x_4(b,u)^{-1}] &:= x_2\big(f(a,b)\big)\ ,\\ [x_2(v),x_4(w)^{-1}] &:= x_3(0,v^\sigma w+w^\sigma v)\ ,\\ [x_1(a,t),x_4(v)^{-1}] &:= x_2(tv)x_3(av,v^\sigma tv)\end{aligned}$$
for all $v,w\in\mathbb{K}$, $(a,t),(b,u)\in T$ is the *parametrized standard quadrangle of pseudo-quadratic form type with respect to* $(\mathbb{K},\mathbb{K}_0,\sigma,L_0,q)$.
- Let $(\mathbb{K},\mathbb{K}_0,\sigma,L_0,q)$ be a (proper) left pseudo-quadratic space. Then the root group sequence
$$\mathcal{Q}_P^o(\mathbb{K},\mathbb{K}_0,\sigma,L_0,q) := \big(x_1(\mathbb{K}),x_2(T),x_3(\mathbb{K}),x_4(T)\big)$$
with commutator relations
$$\begin{aligned}[x_2(b,u)^{-1},x_4(a,t)] &:= x_3\big(-f(a,b)\big)\ ,\\ [x_1(w)^{-1},x_3(v)] &:= x_2(0,-wv^\sigma-vw^\sigma)\ ,\\ [x_1(v)^{-1},x_4(a,t)] &:= x_2(-va,-v^\sigma t^\sigma v)x_3(-vt)\end{aligned}$$
for all $v,w\in\mathbb{K}$, $(a,t),(b,u)\in T$ is the *parametrized opposite quadrangle of pseudo-quadratic form type with respect to* $(\mathbb{K},\mathbb{K}_0,\sigma,L_0,q)$.

(33.4) Lemma Let $(\mathbb{K},\mathbb{K}_0,\sigma,L_0,q)$ be a (proper) left pseudo-quadratic space and let
$$\mathcal{Q}_P^o(\mathbb{K},\mathbb{K}_0,\sigma,L_0,q) = \big(x_1(\mathbb{K}),x_2(T),x_3(\mathbb{K}),x_4(T)\big)$$
be the corresponding opposite quadrangle. Then the action of the *Hua automorphism*
$$h_1(s) := \mu\big(x_1(1_\mathbb{K})\big)^{-1}\mu\big(x_1(s)\big)$$
on $x_1(\mathbb{K})\times x_4(T)$ corresponds to the map
$$\big(u,(b,v)\big)\mapsto\big(sus,(s^{-1}b,s^{-1}vs^{-\sigma})\big)\ ,$$
and the action of the *Hua automorphism*
$$h_4(a,t) := \mu\big(x_4(0,1_\mathbb{K})\big)^{-1}\mu\big(x_4(a,t)\big)$$
on $x_1(\mathbb{K})\times x_4(T)$ corresponds to the map
$$\big(u,(b,v)\big)\mapsto\big(ut^{-\sigma},(t^\sigma b-t^\sigma f(a,b)t^{-1}a,t^\sigma vt)\big)\ .$$

Proof
If we consider the quadrangle $\mathcal{Q}_P(\mathbb{K}^o,\mathbb{K}_0^o,\sigma,L_0,q)$, then the action of $h_1(s)$ on $x_4(T)\times x_1(\mathbb{K})$ corresponds to the map
$$\big((b,v),u\big)\mapsto\big((b\circ s^{-1},s^{-\sigma}\circ v\circ s^{-1}),s\circ u\circ s\big)=\big((s^{-1}b,s^{-1}vs^{-\sigma}),sus\big)\ ,$$
and the action of $h_4(a,t)$ on $x_4(T)\times x_1(\mathbb{K})$ corresponds to the map
$$\big((b,v),u\big)\mapsto\big((b\circ t^\sigma-a\circ t^{-1}\circ f(a,b)\circ t^\sigma,t\circ v\circ t^\sigma),t^{-\sigma}\circ u\big)=\big((t^\sigma b-t^\sigma f(a,b)t^{-1}a,t^\sigma vt),ut^{-\sigma}\big)\ ,$$
cf. (33.13) of [TW]. □

§ 33.3 Quadrangles of Quadratic Form Type

(33.5) Definition Let $(L_0, \mathbb{K}, q)$ be a quadratic space with basepoint ϵ.

- The root group sequence

$$\mathcal{Q}_Q(L_0, \mathbb{K}, q) := \big(x_1(\mathbb{K}), x_2(L_0), x_3(\mathbb{K}), x_4(L_0)\big)$$

with commutator relations

$$[x_2(a), x_4(b)^{-1}] := x_3\big(f(a,b)\big) \ ,$$
$$[x_1(t), x_4(a)^{-1}] := x_2(at)x_3\big(tq(a)\big)$$

for all $a, b \in L_0,\ t \in \mathbb{K}$ is the *parametrized standard quadrangle of quadratic form type with respect to* $(L_0, \mathbb{K}, q)$.

- The root group sequence

$$\mathcal{Q}_Q^o(L_0, \mathbb{K}, q) := \big(x_1(L_0), x_2(\mathbb{K}), x_3(L_0), x_4(\mathbb{K})\big)$$

with commutator relations

$$[x_1(b)^{-1}, x_1(a)] := x_2\big(-f(a,b)\big) \ ,$$
$$[x_4(a)^{-1}, x_1(t)] := x_2\big(-tq(a)\big)x_3(-at)$$

for all $a, b \in L_0,\ t \in \mathbb{K}$ is the *parametrized opposite quadrangle of quadratic form type with respect to* $(L_0, \mathbb{K}, q)$.

(33.6) Lemma Let $(L_0, \mathbb{K}, q)$ be a quadratic space with basepoint ϵ and let

$$\mathcal{Q}_Q^o(L_0, \mathbb{K}, q) = \big(x_1(L_0), x_2(\mathbb{K}), x_3(L_0), x_4(\mathbb{K})\big)$$

be the corresponding opposite quadrangle. Then the action of the *Hua automorphism*

$$h_1(a) := \mu\big(x_1(\epsilon)\big)^{-1}\mu\big(x_1(a)\big)$$

on $x_1(L_0) \times x_4(\mathbb{K})$ corresponds to the map

$$(v, u) \mapsto \big(\pi_a\pi_\epsilon(v) \cdot q(a), u/q(a)\big) \ ,$$

and the action of the *Hua automorphism*

$$h_4(s) := \mu\big(x_4(1_\mathbb{K})\big)^{-1}\mu\big(x_4(s)\big)$$

on $x_1(L_0) \times x_4(\mathbb{K})$ corresponds to the map

$$(b, u) \mapsto (b \cdot t^{-1}, t^2 u) \ .$$

Proof
If we consider the quadrangle $\mathcal{Q}_P(L_0, \mathbb{K}, q)$, then the action of $h_1(a)$ on $x_4(\mathbb{K}) \times x_1(L_0)$ corresponds to the map

$$(u, v) \mapsto \big(u/q(a), \pi_a\pi_\epsilon(v) \cdot q(a)\big) \ ,$$

and the action of $h_4(s)$ on $x_4(\mathbb{K}) \times x_1(L_0)$ corresponds to the map

$$(u, b) \mapsto (t^2 u, b \cdot t^{-1}) \ ,$$

cf. (33.11) of [TW]. □

§ 33.4 Quadrangles of Indifferent Type

(33.7) Definition Let $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$ be a (proper) indifferent set.

- The root group sequence

$$\mathcal{Q}_D(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0) := \big(x_1(\mathbb{K}_0), x_2(\mathbb{L}_0), x_3(\mathbb{K}_0), x_4(\mathbb{L}_0)\big)$$

with commutator relations

$$\forall\ t \in \mathbb{K}_0, a \in \mathbb{L}_0 : \qquad [x_1(t), x_4(a)] = x_2(t^2a)x_3(ta)$$

is the *parametrized standard quadrangle of indifferent type with respect to* $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$.

- The root group sequence

$$\mathcal{Q}_D^o(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0) := \big(x_1(\mathbb{L}_0), x_2(\mathbb{K}_0), x_3(\mathbb{L}_0), x_4(\mathbb{K}_0)\big)$$

with commutator relations

$$\forall\ t \in \mathbb{K}_0, a \in \mathbb{L}_0 : \qquad [x_1(a), x_4(t)] = x_2(-ta)x_3(-t^2a) = x_2(ta)x_3(t^2a)$$

is the *parametrized opposite quadrangle of indifferent type with respect to* $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$.

(33.8) Lemma Let $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$ be a (proper) indifferent set and let

$$\mathcal{Q}_D^o(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0) = \big(x_1(\mathbb{L}_0), x_2(\mathbb{K}_0), x_3(\mathbb{L}_0), x_4(\mathbb{K}_0)\big)$$

be the corresponding opposite quadrangle. Then the action of the *Hua automorphism*

$$h_1(a) := \mu\big(x_1(1_\mathbb{K})\big)^{-1}\mu\big(x_1(a)\big)$$

on $x_1(\mathbb{L}_0) \times x_4(\mathbb{K}_0)$ corresponds to the map

$$(b, u) \mapsto (ba^2, ua^{-1}) ,$$

and the action of the *Hua automorphism*

$$h_4(t) := \mu\big(x_4(1_\mathbb{K})\big)^{-1}\mu\big(x_4(t)\big)$$

on $x_1(\mathbb{L}_0) \times x_4(\mathbb{K}_0)$ corresponds to the map

$$(b, u) \mapsto (bt^{-2}, ut^2) .$$

Proof
If we consider the quadrangle $\mathcal{Q}_D(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$, then the action of $h_1(a)$ on $x_4(\mathbb{K}_0) \times x_1(\mathbb{L}_0)$ corresponds to the map

$$(u, b) \mapsto (ua^{-1}, ba^2) ,$$

and the action of $h_4(s)$ on $x_4(\mathbb{K}_0) \times x_1(\mathbb{L}_0)$ corresponds to the map

$$(u, b) \mapsto (ut^2, bt^{-2}) ,$$

cf. (33.12) of [TW]. □

§ 33.5 Quadrangles of Type E_n, F_4

For an overview of those quadrangles, we refer to (16.6), (33.14), (16.7) and (33.15) of [TW].

Chapter 34 The Moufang Sets of Moufang Polygons

We give an overview of the Moufang sets appearing as root groups of Moufang triangles and quadrangles.

(34.1) Remark Let $\mathcal{B}_{(1,2)} := \mathcal{X}(\Xi)$ be a parametrized standard Moufang polygon. Then one of the following holds:

(i) We have
$$\mathbb{M}^1_{(1,2)} = \mathbb{M}(\mathbb{A}) = \mathbb{M}^2_{(1,2)}$$
if $\mathcal{B}_{(1,2)} = \mathcal{T}(\mathbb{A})$ for some alternative division ring $\mathbb{A}$.

(ii) We have
$$\mathbb{M}^1_{(1,2)} = \mathbb{M}(\mathbb{K}, \mathbb{K}_0, \sigma) \ , \qquad \mathbb{M}^2_{(1,2)} = \mathbb{M}(\mathbb{K})$$
if $\mathcal{B}_{(1,2)} = \mathcal{Q}_I(\mathbb{K}, \mathbb{K}_0, \sigma)$ for some (proper) involutory set $(\mathbb{K}, \mathbb{K}_0, \sigma)$.

(iii) We have
$$\mathbb{M}^1_{(1,2)} = \mathbb{M}(\mathbb{K}, \mathbb{K}_0, \sigma, L_0, q) \ , \qquad \mathbb{M}^2_{(1,2)} = \mathbb{M}(\mathbb{K})$$
if $\mathcal{B}_{(1,2)} = \mathcal{Q}_P(\mathbb{K}, \mathbb{K}_0, \sigma, L_0, q)$ for some (proper) pseudo-quadratic space $(\mathbb{K}, \mathbb{K}_0, \sigma, L_0, q)$.

(iv) We have
$$\mathbb{M}^1_{(1,2)} = \mathbb{M}(\mathbb{K}) \ , \qquad \mathbb{M}^2_{(1,2)} = \mathbb{M}(L_0, \mathbb{K}, q)$$
if $\mathcal{B}_{(1,2)} = \mathcal{Q}_Q(L_0, \mathbb{K}, q)$ for some quadratic space $(L_0, \mathbb{K}, q)$ with basepoint ϵ.

(v) We have
$$\mathbb{M}^1_{(1,2)} = \mathbb{M}(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0) \ , \qquad \mathbb{M}^2_{(1,2)} = \mathbb{M}(\mathbb{L}, \mathbb{L}_0, \mathbb{K}_0^2)$$
if $\mathcal{B}_{(1,2)} = \mathcal{Q}_I(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$ for some (proper) indifferent set $(\mathbb{K}, \mathbb{K}_0, \mathbb{L}_0)$.

(vi) We have
$$\mathbb{M}^1_{(1,2)} = \mathbb{M}(S) \ , \qquad \mathbb{M}^2_{(1,2)} = \mathbb{M}(L_0, \mathbb{K}, q)$$
if $\mathcal{B}_{(1,2)} = \mathcal{Q}_E(L_0, \mathbb{K}, q)$ for some quadratic space $(L_0, \mathbb{K}, q)$ of type E_n.

(vii) We have
$$\mathbb{M}^1_{(1,2)} = \mathbb{M}(\mathbb{F}, \hat{L}_0, \hat{q}) \ , \qquad \mathbb{M}^2_{(1,2)} = \mathbb{M}(L_0, \mathbb{K}, q)$$
if $\mathcal{B}_{(1,2)} = \mathcal{Q}_4(L_0, \mathbb{K}, q)$ for some quadratic space $(L_0, \mathbb{K}, q)$ of type F_4.

(viii) We have
$$\mathbb{M}^1_{(1,2)} = \mathbb{M}(J, \mathbb{F}, \#) \ , \qquad \mathbb{M}^2_{(1,2)} = \mathbb{M}(\mathbb{F})$$
if $\mathcal{B}_{(1,2)} = \mathcal{H}(J, \mathbb{F}, \#)$ for some hexagonal system $(J, \mathbb{F}, \#)$.

(ix) We have
$$\mathbb{M}^1_{(1,2)} = \mathbb{M}(\mathbb{K}) \ , \qquad \mathbb{M}^2_{(1,2)} = \mathbb{M}(\mathbb{K}, \sigma)$$
if $\mathcal{B}_{(1,2)} = \mathcal{O}(\mathbb{K}, \sigma)$ for some octagonal system $(\mathbb{K}, \sigma)$.

(34.2) Remark The Hua automorphisms of lemma (18.9) and chapter 33, which can be defined for each polygon, cf. chapter 33 of [TW], induce the Hua maps on the corresponding Moufang sets. Each Hua automorphism of a polygon which is part of an integrable foundation is induced by an automorphism of the whole building, cf. theorem (2.35).

Part VIII

Foundations

We generalize the definitions and results of chapter 19, i.e., we give the definition of a foundation involving arbitrary Moufang polygons and show that we can attach a foundation to each twin building. Once again, this foundation turns out to be a classifying invariant of the corresponding twin building, and the integrability criterions of chapter 19 hold as well.

Chapter 35 Definition

(35.1) Definition

- Let M be a Coxeter matrix. A *foundation of type M* is a set
$$\mathcal{F} := \{\mathcal{B}_{(i,j)}, \gamma_{(i,j,k)} \mid (i,j) \in A(M), (i,j,k) \in G(M)\}$$
such that:
  - (F1) Given $(i,j) \in A(M)$, then $\mathcal{B}_{(i,j)} = \mathcal{X}_{(i,j)}(\Xi_{(i,j)})$ for some symbol $\mathcal{X}_{(i,j)}$ of type m_{ij} as in notation (32.6) and some parameter system $\Xi_{(i,j)}$ of type $\mathcal{X}_{(i,j)}$.
  - (F2) Given $(i,j) \in A(M)$, we have $\mathcal{B}_{(i,j)} = \mathcal{B}^o_{(j,i)}$.
  - (F3) Given $(i,j,k) \in G(M)$, then $\gamma_{(i,j,k)} : \mathbb{M}^j_{(i,j)} \to \mathbb{M}^j_{(j,k)}$ is an isomorphism of groups satisfying
$$\gamma_{(i,j,k)}(1_{\mathbb{M}}) = 1_{\mathbb{M}} \ , \qquad \gamma_{(i,j,k)} = \mathrm{id}^o \circ \gamma^{-1}_{(k,j,i)} \circ \mathrm{id}^o \ .$$
  - (F4) Given $(i,j,k), (i,j,l), (l,j,k) \in G(M)$, we have
$$\gamma_{(i,j,k)} = \gamma_{(l,j,k)} \circ \mathrm{id}^o \circ \gamma_{(i,j,l)} \ .$$
- Given a foundation $\mathcal{F}$, we denote the corresponding Coxeter Matrix by F.
- A foundation $\mathcal{F}$ is a *Moufang foundation* if each *glueing* $\gamma := \gamma_{(i,j,k)}$ is a Jordan isomorphism, i.e., we have
$$\forall\ a \in \mathbb{M}^*_{(i,j)},\ x \in \mathbb{M}_{(i,j)} : \qquad \gamma\big(h_a(x)\big) = h_{\gamma(a)}\big(\gamma(x)\big) \ .$$

(35.2) Definition Let $\mathcal{F}$ be a foundation over $I = V(F)$ and let $J \subseteq I$. The *J-residue of $\mathcal{F}$* is the foundation
$$\mathcal{F}_J := \{\mathcal{B}_{(i,j)}, \gamma_{(i,j,k)} \mid (i,j) \in J^2 \cap A(F), (i,j,k) \in J^3 \cap G(F)\} \ .$$

(35.3) Remark Since a foundation is, in fact, an amalgam of Moufang polygons, an isomorphism of foundations is a system of isomorphism of Moufang polygons preserving the glueings.

(35.4) Definition Let $\mathcal{F}, \tilde{\mathcal{F}}$ be foundations.

- An *isomorphism* $\alpha : \mathcal{F} \to \tilde{\mathcal{F}}$ is a system $\alpha = \{\pi, \alpha_{(i,j)} \mid (i,j) \in A(F)\}$ of isomorphisms
$$\pi : F \to \tilde{F} \ , \qquad \alpha_{(i,j)} = (\alpha^i_{(i,j)}, \dots, \alpha^j_{(i,j)}) : \mathcal{B}_{(i,j)} \to \tilde{\mathcal{B}}_{(\pi(i),\pi(j))}$$
such that
$$\forall\ (i,j,k) \in G(F) : \qquad \tilde{\gamma}_{(\pi(i),\pi(j),\pi(k))} \circ \alpha^j_{(i,j)} = \alpha^j_{(j,k)} \circ \gamma_{(i,j,k)}$$
and
$$\forall\ (i,j) \in A(F) : \qquad \alpha_{(i,j)} = \alpha^o_{(j,i)} \ .$$
- An isomorphism $\alpha : \mathcal{F} \to \tilde{\mathcal{F}}$ is *special* if $F = \tilde{F}$ and $\pi = \mathrm{id}_F$.
- An *automorphism of $\mathcal{F}$* is an isomorphism $\alpha : \mathcal{F} \to \mathcal{F}$.

Chapter 36 Root Group Systems

The fact that a root group systems is a classifying invariant of the corresponding twin building is a fundamental result in twin building theory.

(36.1) Definition Let $\mathcal{B}$ be a twin building of type M, let Σ be a twin apartment of $\mathcal{B}$ and let $c \in \mathcal{O}_\Sigma$.

- Given $(i,j) \in A(M)$, let α_i, α_j be the simple roots with respect to (Σ, c) and let $\Theta_{(i,j)}$ be as in theorem (2.32) (d). Then
$$U_{(i,j)} := (U_{[i,j]}, U^i_{(i,j)}, \ldots, U^j_{(i,j)}) := \Theta_{(i,j)}$$
denotes the root group sequence of $\mathcal{B}$ from α_i to α_j, which is isomorphic to the root group sequence of $\mathcal{B}_{ij}$ from $\alpha_i \cap \mathcal{B}_{ij}$ to $\alpha_j \cap \mathcal{B}_{ij}$.
- The resulting set
$$\mathcal{U}(\mathcal{B}, M, \Sigma, c) := \{U_{(i,j)} \mid (i,j) \in A(M)\}$$
is the *root group system of $\mathcal{B}$ based at (Σ, c)*.

(36.2) Lemma Given $(i,j,k) \in G(F)$, we have $U^j_{(i,j)} = U^j_{(j,k)}$.

Proof
This holds by definition. $\square$

(36.3) Definition Let $\mathcal{U} := \mathcal{U}(\mathcal{B}, M, \Sigma, c)$ and $\tilde{\mathcal{U}} := \mathcal{U}(\tilde{\mathcal{B}}, \tilde{M}, \tilde{\Sigma}, \tilde{c})$ be root group systems.

- An *isomorphism* $\alpha : \mathcal{U} \to \tilde{\mathcal{U}}$ is a system
$$\alpha = \{\pi, \alpha_{(i,j)} \mid (i,j) \in A(M)\}$$
of isomorphisms
$$\pi : M \to \tilde{M} \;, \qquad\qquad \alpha_{(i,j)} : U_{(i,j)} \to \tilde{U}_{(\pi(i),\pi(j))}$$
of root group sequences such that
$$\forall\ (i,j,k) \in G(M) :\ \alpha_{(i,j)|U^j_{(i,j)}} = \alpha_{(j,k)|U^j_{(j,k)}} \;, \qquad \forall\ (i,j) \in A(M) :\ \alpha_{(i,j)} = \alpha^o_{(j,i)} \;.$$
- An isomorphism $\alpha : \mathcal{U} \to \tilde{\mathcal{U}}$ is *special* if $M = \tilde{M}$ and $\pi = \mathrm{id}_M$.
- An *automorphism of $\mathcal{U}$* is an isomorphism $\alpha : \mathcal{U} \to \mathcal{U}$.

(36.4) Theorem Two root group systems $\mathcal{U}(\mathcal{B}, M, \Sigma, c)$ and $\mathcal{U}(\mathcal{B}, M, \tilde{\Sigma}, \tilde{c})$ of a twin building $\mathcal{B}$ are specially isomorphic.

Proof
This is a consequence of theorem (2.22). $\square$

(36.5) Theorem Let $\mathcal{U} := \mathcal{U}(\mathcal{B}, M, \Sigma, c)$ be a root group system of a twin building $\mathcal{B}$ which satisfies condition (CO). Then the isomorphism class of $\mathcal{U}$ is a classifying invariant of the isomorphism class of $\mathcal{B}$.

Proof
This is a consequence of the extension theorem (2.23). $\square$

Chapter 37 Foundations and Root Group Systems

Given a root group system, there is a natural way to attach a foundation to it.

(37.1) Definition Let $\mathcal{U}(\mathcal{B}, M, \Sigma, c)$ be a root group system.

- Given $(i,j) \in A(M)$, there is a symbol $\mathcal{X}_{(i,j)}$ and a parameter system $\Xi_{(i,j)}$ such that $U_{(i,j)} \cong \mathcal{X}_{(i,j)}(\Xi_{(i,j)})$. In particular, there is a system of parametrizations
$$x^*_{(i,j)} : \mathbb{M}^*_{(i,j)} \to U^*_{(i,j)} \ , \ t \mapsto x^*_{(i,j)}(t) \ , \qquad\qquad * \in \{i,j\}$$
extending to the defining relations for $\mathcal{X}_{(i,j)}(\Xi_{(i,j)})$. Such a parametrization yields an opposite system of parametrizations
$$x^*_{(j,i)} : (\mathbb{M}^*_{(i,j)})^o \to U^*_{(j,i)} \ , \ t \mapsto x^*_{(i,j)}\big(\mathrm{id}^o(t)\big) \ , \qquad\qquad * \in \{j,i\} \ .$$
The resulting set
$$\Lambda := \{\mathcal{X}_{(i,j)}(\Xi_{(i,j)}) \mid (i,j) \in A(M)\}$$
is a *parameter system for* $\mathcal{U}$.
- Given $(i,j,k) \in G(M)$ and parametrizations $\mathcal{X}_{(i,j)}(\Xi_{(i,j)})$ and $\mathcal{X}_{(j,k)}(\Xi_{(j,k)})$, we define the glueing $\gamma_{(i,j,k)} : \mathbb{M}^j_{(i,j)} \to \mathbb{M}^j_{(j,k)}$ by
$$x^j_{(i,j)}(t) = x^j_{(j,k)}\big(\gamma_{(i,j,k)}(t)\big)$$
which is justified by lemma (36.2). Then $\gamma_{(i,j,k)}$ is an isomorphism of groups satisfying $\gamma_{(i,j,k)} = \mathrm{id}^o \circ \gamma^{-1}_{(k,j,i)} \circ \mathrm{id}^o$. By lemma (32.16), (32.17), (32.18) and (32.19), we may adjust all the parametrizations such that
$$\forall \ (i,j,k) \in G(F) : \qquad \gamma_{(i,j,k)}(1_{\mathbb{M}}) = 1_{\mathbb{M}} \ .$$
In the first instance, we have to adjust the glueings connecting quadrangles of type F_4 (for which we need remark (32.21)) since we have the least flexibility in this case: The element $1_{\mathbb{M}}$ is an element in the corresponding defect.

(37.2) Lemma Given a root group system $\mathcal{U} := \mathcal{U}(\mathcal{B}, M, \Sigma, c)$, a parameter system Λ as in definition (37.1) induces a foundation
$$\mathcal{F}(U,\Lambda) = \{\mathcal{X}_{(i,j)}(\Xi_{(i,j)}), \gamma_{(i,j,k)} \mid (i,j) \in A(M), (i,j,k) \in G(M)\} \ .$$

Proof
We emphasize that the glueings in definition (37.1) are identifications with respect to directed edges. Given $(i,j,k), (i,j,l), (l,j,k) \in G(M)$ and $t \in \mathbb{M}^j_{(i,j)}$, we have
$$\begin{aligned} x^j_{(j,k)}\big(\gamma_{(i,j,k)}(t)\big) &= x^j_{(i,j)}(t) = x^j_{(j,l)}\big(\gamma_{(i,j,l)}(t)\big) \\ &= x^j_{(l,j)}\big(\mathrm{id}^o \circ \gamma_{(i,j,l)}(t)\big) = x^j_{(j,k)}\big(\gamma_{(l,j,k)} \circ \mathrm{id}^o \circ \gamma_{(i,j,l)}(t)\big) \end{aligned}$$
and thus $\gamma_{(i,j,k)} = \gamma_{(l,j,k)} \circ \mathrm{id}^o \circ \gamma_{(i,j,l)}$. □

(37.3) Definition A foundation $\mathcal{F}$ is *integrable* if it is the foundation of a twin building $\mathcal{B}$, i.e., if there are a root group system $\mathcal{U} := \mathcal{U}(\mathcal{B}, M, \Sigma, c)$ and a parameter system Λ for $\mathcal{U}$ such that $\mathcal{F} = \mathcal{F}(\mathcal{U}, \Lambda)$.

(37.4) Theorem Let $\mathcal{F}$ be an integrable foundation. Then $\mathcal{F}$ is a Moufang foundation.

Proof
Let $(i,j,k) \in G(F)$ and $\gamma := \gamma_{(i,j,k)}$. If we set

$$h(a) := \mu\big(x^j_{(i,j)}(1_{\mathbb{M}})\big)^{-1}\mu\big(x^j_{(i,j)}(a)\big) \ , \qquad a \in (M^j_{(i,j)})^* \ ,$$
$$\tilde{h}(a) := \mu\big(x^j_{(j,k)}(1_{\mathbb{M}})\big)^{-1}\mu\big(x^j_{(j,k)}(a)\big) \ , \qquad a \in (M^j_{(j,k)})^* \ ,$$

we have

$$\tilde{h}\big(\gamma(a)\big) = \mu\big(x^j_{(j,k)}(1_{\mathbb{M}})\big)^{-1}\mu\big(x^j_{(j,k)}(\gamma(a))\big) = \mu\big(x^j_{(i,j)}(1_{\mathbb{M}})\big)^{-1}\mu\big(x^j_{(i,j)}(a)\big) = h(a)$$

for each $a \in \mathbb{M}^j_{(i,j)}$. Moreover, we have

$$x^j_{(i,j)}(x)^{h(a)} = x^j_{(i,j)}\big(h_a(x)\big) \ , \qquad x^j_{(j,k)}(y)^{\tilde{h}(b)} = x^j_{(j,k)}\big(\tilde{h}_b(y)\big)$$

for all $a \in (\mathbb{M}^j_{(i,j)})^*$, $x \in \mathbb{M}^j_{(i,j)}$, $b \in (\mathbb{M}^j_{(j,k)})^*$, $y \in \mathbb{M}^j_{(j,k)}$. Combining these two facts yields

$$x^j_{(j,k)}\big(\gamma(h_a(x))\big) = x^j_{(i,j)}\big(h_a(x)\big) = x^j_{(i,j)}(x)^{h(a)} = x^j_{(j,k)}\big(\gamma(x)\big)^{\tilde{h}(\gamma(x))} = x^j_{(j,k)}\big(\tilde{h}_{\gamma(a)}(\gamma(x))\big)$$

and thus $\gamma\big(h_a(x)\big) = \tilde{h}_{\gamma(a)}\big(\gamma(x)\big)$ for all $a \in (\mathbb{M}^j_{(i,j)})^*$, $x \in \mathbb{M}^j_{(i,j)}$. □

(37.5) Remark The following result provides an integrability criterion.

(37.6) Definition Let $\mathcal{F}$ be a foundation.

- Let $(\tilde{F},\varphi)$ be a cover of F. Then the foundation

$$\mathcal{F}(\tilde{F},\varphi) := \{\tilde{\mathcal{B}}_{(i,j)}, \tilde{\gamma}_{(i,j,k)} \mid (i,j) \in A(\tilde{F}), (i,j,k) \in G(\tilde{F})\}$$

  with

$$\forall\ (i,j) \in A(\tilde{F}):\ \tilde{\mathcal{B}}_{(i,j)} = \mathcal{B}_{(\varphi(i),\varphi(j))} \ , \qquad \forall\ (i,j,k) \in G(\tilde{F}):\ \tilde{\gamma}_{(i,j,k)} = \gamma_{(\varphi(i),\varphi(j),\varphi(k))}$$

  is the *cover corresponding to* $(\tilde{F},\varphi)$.
- A foundation $\tilde{\mathcal{F}}$ is a *cover of* $\mathcal{F}$ if there is a cover $(\tilde{F},\varphi)$ of F such that

$$\tilde{\mathcal{F}} \cong \mathcal{F}(\tilde{F},\varphi) \ .$$

(37.7) Theorem Let $\mathcal{F}$ be a foundation and let $\tilde{\mathcal{F}}$ be a cover of $\mathcal{F}$. Then $\mathcal{F}$ is integrable if $\tilde{\mathcal{F}}$ is integrable.

Proof
This is a consequence of theorem C in [MLoc]. □

(37.8) Remark The next step is to show that the foundation attached to a root group system is unique up to isomorphism. Moreover, we want to prove that the building corresponding to an integrable foundation is unique up to isomorphism.

(37.9) Proposition Let $\mathcal{U} := \mathcal{U}(\mathcal{B}, M, \Sigma, c)$ and $\tilde{\mathcal{U}} := \mathcal{U}(\tilde{\mathcal{B}}, \tilde{M}, \tilde{\Sigma}, \tilde{c})$ be root group systems and let Λ and $\tilde{\Lambda}$ be parameter systems for $\mathcal{U}$ and $\tilde{\mathcal{U}}$, respectively. Then the following holds:

(a) An isomorphism $\tilde{\alpha} : \mathcal{F}(\mathcal{U},\Lambda) \to \mathcal{F}(\tilde{\mathcal{U}},\tilde{\Lambda})$ induces an isomorphism $\alpha : \mathcal{U} \to \tilde{\mathcal{U}}$.

(b) An isomorphism $\alpha : \mathcal{U} \to \tilde{\mathcal{U}}$ induces an isomorphism $\tilde{\alpha} : \mathcal{F}(\mathcal{U},\Lambda) \to \mathcal{F}(\tilde{\mathcal{U}},\tilde{\Lambda})$.

Proof
Each isomorphism

$$\alpha_{(i,j)} : U_{(i,j)} \to \tilde{U}_{(\pi(i),\pi(j))}$$

induces an isomorphism

$$\tilde{\alpha}_{(i,j)} : \mathcal{X}_{(i,j)}(\Xi_{(i,j)}) \to \mathcal{X}_{(i,j)}(\tilde{\Xi}_{(\pi(i),\pi(j))})$$

and vice versa. Given $(i,j) \in A(M)$, we have

$$\alpha_{(i,j)} = \alpha^o_{(j,i)} \;\Leftrightarrow\; \tilde{\alpha}_{(i,j)} = \tilde{\alpha}^o_{(j,i)} \;.$$

Now we may go on as in the proof of proposition (19.16). □

Chapter 38 Reparametrizations and Isomorphisms

The concept of reparametrizations is quite similar to that of isomorphisms. However, we deal with a single foundation and produce (in fact, all the) foundations which are isomorphic to a given one. Moreover, this concept allows us to complete the proof that a foundation is a classifying invariant of the corresponding twin building.

(38.1) Definition Let $\mathcal{F}$ be a foundation.

- A system of reparametrizations

  $$\alpha := \{\alpha_{(i,j)} \mid (i,j) \in A(F)\}$$

  satisfying $\alpha_{(i,j)} = \alpha^o_{(j,i)}$ for each $(i,j) \in A(F)$ and

  $$\gamma_{(i,j,k)} \circ \alpha^j_{(i,j)}(1) = \alpha^j_{(j,k)}(1)$$

  for each $(i,j,k) \in G(F)$ is a *reparametrization for $\mathcal{F}$*.

- Given a reparametrization α for $\mathcal{F}$, we set

  $$\mathcal{F}_\alpha := \{\mathcal{X}_{(i,j)}(\tilde{\Xi}_{(i,j)}), \tilde{\gamma}_{(i,j,k)} \mid (i,j) \in A(F), (i,j,k) \in G(F)\}$$

  with

  $$\tilde{\gamma}_{(i,j,k)} := (\alpha^j_{(j,k)})^{-1} \circ \gamma_{(i,j,k)} \circ \alpha^j_{(i,j)}$$

  for each $(i,j,k) \in G(F)$.

(38.2) Lemma Let $\mathcal{U} := \mathcal{U}(\mathcal{B}, M, \Sigma, c)$ be a root group system, let $\mathcal{F} := \mathcal{F}(\mathcal{U}, \Lambda)$ for some parameter system Λ for $\mathcal{U}$, let α be a reparametrization for $\mathcal{F}$ and let $\tilde{\Lambda}$ be the parameter system induced by α. Then we have $\tilde{\mathcal{F}} := \mathcal{F}(\mathcal{U}, \tilde{\Lambda}) = \mathcal{F}_\alpha$.

Proof
We have

$$\begin{aligned}\tilde{x}^j_{(j,k)}\big(\tilde{\gamma}_{(i,j,k)}(t)\big) &= \tilde{x}^j_{(i,j)}(t) = x^j_{(i,j)}\big(\alpha^j_{(i,j)}(t)\big)\\ &= x^j_{(j,k)}\big(\gamma_{(i,j,k)} \circ \alpha^j_{(i,j)}(t)\big) = \tilde{x}^j_{(j,k)}\big((\alpha^j_{(j,k)})^{-1} \circ \gamma_{(i,j,k)} \circ \alpha^j_{(i,j)}(t)\big)\end{aligned}$$

for each $t \in \tilde{\mathbb{M}}^j_{(i,j)}$. □

(38.3) Corollary Let $\mathcal{U} := \mathcal{U}(\mathcal{B}, M, \Sigma, c)$ be a root group system, let $\mathcal{F} := \mathcal{F}(\mathcal{U}, \Lambda)$ for some parameter system Λ for $\mathcal{U}$ and let

$$\alpha = \{\pi, \alpha_{(i,j)} \mid (i,j) \in A(F)\} : \mathcal{F} \to \tilde{\mathcal{F}}$$

be an isomorphism. Then $\tilde{\mathcal{F}}$ is integrable.

Proof
Take $\big(\tilde{\Xi}_{(i,j)} := \tilde{\Xi}_{(\pi(i),\pi(j))}, (\alpha^i_{(i,j)})^{-1}, \ldots, (\alpha^j_{(i,j)})^{-1}\big)$ as reparametrization for $\mathcal{X}_{(i,j)}(\Xi_{(i,j)})$, then replace $i \in I$ by $\pi(i) \in \tilde{I}$. The resulting parameter system $\tilde{\Lambda}$ satisfies

$$\mathcal{F}(\mathcal{U}, \tilde{\Lambda}) = \mathcal{F}_\alpha = \tilde{\mathcal{F}} \ .$$

□

(38.4) Theorem The isomorphism class of an integrable foundations $\mathcal{F} = \mathcal{F}(\mathcal{U}, \Lambda)$ is a classifying invariant of the isomorphism class of the corresponding building.

Proof
This results from corollary (38.3), proposition (37.9) and theorem (36.5). □

(38.5) Remark The following theorem shows that the concept of reparametrization is useful if we want to determine all the foundations isomorphic to a given foundation $\mathcal{F}$.

(38.6) Theorem Let $\mathcal{F}, \tilde{\mathcal{F}}$ be foundations with $F = \tilde{F}$. Then the following holds:

(a) Let $\tilde{\alpha} = \{\tilde{\alpha}_{(i,j)} \mid (i,j) \in A(F)\} : \mathcal{F} \to \tilde{\mathcal{F}}$ be a special isomorphism. Then there is a reparametrization α of $\mathcal{F}$ such that $\mathcal{F}_\alpha = \tilde{\mathcal{F}}$.

(b) Let $\alpha = \{\alpha_{(i,j)} \mid (i,j) \in A(F)$ be a reparametrization for $\mathcal{F}$ such that $\mathcal{F}_\alpha = \tilde{\mathcal{F}}$. Then there is a special isomorphism $\tilde{\alpha} : \mathcal{F} \to \tilde{\mathcal{F}}$.

Proof

(a) If we take $\alpha := \{\alpha_{(i,j)} \mid (i,j) \in A(F)\}$ with

$$\alpha_{(i,j)} := \{\tilde{\Xi}_{(i,j)}, (\tilde{\alpha}^i_{(i,j)})^{-1}, \ldots, (\tilde{\alpha}^j_{(i,j)})^{-1}\}$$

as reparametrization for $\mathcal{F}$, then $\mathcal{F}_\alpha = \tilde{\mathcal{F}}$.

(b) We have

$$\alpha_{(i,j)} = (\tilde{\Xi}_{(i,j)}, \alpha^i_{(i,j)}, \ldots, \alpha^j_{(i,j)})$$

for each $(i,j) \in A(F)$, thus $\tilde{\alpha} := \{\mathrm{id}_F, \tilde{\alpha}_{(i,j)} \mid (i,j) \in A(F)\} : \mathcal{F} \to \tilde{\mathcal{F}}$ with

$$\tilde{\alpha}_{(i,j)} := \big((\alpha^i_{(i,j)})^{-1}, \ldots, (\alpha^j_{(i,j)})^{-1}\big)$$

is an isomorphism.

□

(38.7) Remark

(a) Let $\mathcal{F}$ and $\tilde{\mathcal{F}}$ be foundations and let

$$\alpha = \{\pi, \alpha_{(i,j)} \mid (i,j) \in A(F)\} : \mathcal{F} \to \tilde{\mathcal{F}}$$

be an isomorphism. As we may replace $i \in V(F)$ by $\pi(i) \in V(\tilde{F})$, we may consider α as special. Thus it suffices to determine all foundations which are specially isomorphic to $\mathcal{F}$. The remaining foundations isomorphic to $\mathcal{F}$ are obtained by relabelings of the vertex set.

(b) The theorem is useful if we want to show that two given foundations $\mathcal{F}$ and $\tilde{\mathcal{F}}$ with isomorphic residues $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are isomorphic. In this case, we may replace $\mathcal{R}$ by $\tilde{\mathcal{R}}$, observing that there is a relabeling of the corresponding vertices involved.

Part IX

443-Foundations

Now we are ready to turn to the classification of integrable 443-foundations, whose Moufang polygons are two quadrangles and one triangle and whose Coxeter diagrams are complete graphs.

The first step is to exclude quadrangles of type E_n, of type F_4 and of indifferent type. Then we turn to unitary quadrangles, i.e., quadrangles of pseudo-quadratic form or involutory type. As we restrict to proper parameter systems, there are not many possibilities to glue these polygons together.

The final class is that of quadrangles of quadratic form type, which is rich in integrable foundations. In order to avoid characteristic 2 trouble, there is one point where we restrict to proper quadratic spaces although a small gap is the consequence.

Chapter 39 Definition

(39.1) Definition A *443-foundation* is a foundation

$$\mathcal{F} := \{\mathcal{B}_{(1,2)}, \mathcal{B}_{(2,3)}, \mathcal{B}_{(3,1)}, \gamma_{(1,2,3)}, \gamma_{(2,3,1)}, \gamma_{(3,1,2)}\}$$

such that $\mathcal{B}_{(1,2)}$ and $\mathcal{B}_{(2,3)}$ are quadrangles and $\mathcal{B}_{(3,1)}$ is a triangle.

(39.2) Notation

- Given a 443-foundation $\mathcal{F}$, we set

$$\gamma_1 := \gamma_{(3,1,2)} \ , \qquad \gamma_2 := \gamma_{(1,2,3)} \ , \qquad \gamma_3 := \gamma_{(2,3,1)} \ .$$

- Throughout the rest of this part, $\mathcal{F}$ is an integrable 443-foundation.

Chapter 40 The Quadrangles Are Not of Type E_n

(40.1) Lemma Let $\mathcal{B} = \big(x_1(\mathbb{M}_1), \ldots, x_4(\mathbb{M}_4)\big)$ be a parametrized standard quadrangle of type E_n. Then the following holds:

(a) The Moufang set $\mathbb{M}_1$ is non-commutative.

(b) We have $\mathbb{M}_4 = \mathbb{M}(L_0, \mathbb{K}, q)$ for some quadratic space $(L_0, \mathbb{K}, q)$ of type E_n.

Proof

(a) This results from (38.10) of [TW].

(b) This holds by definition, cf. remark (34.1) (vi) or example (16.6) of [TW].

□

(40.2) Theorem The quadrangles are not of type E_n.

Proof

Suppose that $\mathcal{B}_{(1,2)}$ or $\mathcal{B}_{(2,1)}$ is a parametrized standard quadrangle of type E_n. The glueing $\gamma_1 : \mathbb{M}^1_{(3,1)} \to \mathbb{M}^1_{(1,2)}$ is a Jordan isomorphism. We have $\mathbb{M}^1_{(3,1)} = \mathbb{M}(\mathbb{A})$ for some alternative division ring $\mathbb{A}$, thus $\mathbb{M}^1_{(3,1)}$ is commutative. Hence we have $\mathbb{M}^1_{(1,2)} = \mathbb{M}(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ for some quadratic space $(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ of type E_n by lemma (40.1).

Notice that $(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ is proper by remark (4.27). By theorem (31.7), the alternative division ring $\mathbb{A}$ is quadratic over a subfield $\mathbb{F}$ of its center $\mathbb{K} := Z(\mathbb{A})$ with $N := N^{\mathbb{A}}_{\mathbb{F}}$, and $(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ and $(\mathbb{A}, \mathbb{F}, N)$ are isomorphic as quadratic spaces. In particular, we have $\tilde{\mathbb{K}} \cong \mathbb{F}$ and

$$\dim_{\tilde{\mathbb{K}}} \tilde{L}_0 = \dim_{\mathbb{F}} \mathbb{A} \in \{1, 2, 4, 8\} \cap \{6, 8, 12\} = \{8\} \ .$$

As a consequence, $(\mathbb{O}, \mathbb{K}, N) = (\mathbb{A}, \mathbb{F}, N)$ is of type (v) and $(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ is of type E_7. But by corollary (4.29) (c), we have $(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}) \ncong (\mathbb{O}, \mathbb{K}, N)$ as quadratic spaces $\lightning$. □

Chapter 41 The Quadrangles Are Not of Type F_4

(41.1) Remark Let $\mathcal{Q}_F(L_0,\mathbb{K},q) = \big(x_1(\mathbb{M}_1), x_2(\mathbb{M}_2), x_3(\mathbb{M}_3), x_4(\mathbb{M}_4)\big)$ be a quadrangle of type F_4, where $(L_0,\mathbb{K},q)$ is a quadratic space of type F_4.

(a) By remark (21.18) of [TW], the quadrangle $\mathcal{Q}_F(L_0,\mathbb{K},q)$ is an extension of the quadrangle $\mathcal{Q}_Q(L_0,\mathbb{K},q)$, thus we have $\mathbb{M}_4 = \mathbb{M}(L_0,\mathbb{K},q)$.

(b) By remark (21.18) of [TW] again, the quadrangle $\mathcal{Q}_F^o(L_0,\mathbb{K},q)^o$ is an extension of the quadrangle $\mathcal{Q}_Q(\hat{L}_0,\mathbb{F},\hat{q})$, thus we have $\mathbb{M}_1 = \mathbb{M}(\hat{L}_0,\mathbb{F},\hat{q})$, where $(\hat{L}_0,\mathbb{F},\hat{q})$ is the quadratic space of type F_4 as defined in (14.12) of [TW], cf. (14.13) of [TW].

(41.2) Theorem The quadrangles are not of type F_4.

Proof
Suppose that $\mathcal{B}_{(1,2)}$ or $\mathcal{B}_{(2,1)}$ is a parametrized standard quadrangle of type F_4. The glueing $\gamma_1 : \mathbb{M}^1_{(3,1)} \to \mathbb{M}^1_{(1,2)}$ is a Jordan isomorphism. We have $\mathbb{M}^1_{(3,1)} = \mathbb{M}(\tilde{\mathbb{A}})$ for some alternative division ring $\tilde{\mathbb{A}}$. By remark (41.1), we have

$$\mathbb{M}^1_{(1,2)} = \mathbb{M}(L_0,\mathbb{K},q)$$

for some quadratic space $(L_0,\mathbb{K},q)$ of type F_4. By theorem (31.7), the alternative division ring $\tilde{\mathbb{A}}$ is quadratic over a subfield $\tilde{\mathbb{F}}$ of its center $\tilde{\mathbb{K}} := Z(\tilde{\mathbb{A}})$, and $(L_0,\mathbb{K},q)$ and $(\tilde{\mathbb{A}},\tilde{\mathbb{F}},N^{\tilde{\mathbb{A}}}_{\tilde{\mathbb{F}}})$ are isomorphic as quadratic spaces, which contradicts lemma (4.36). $\square$

Chapter 42 The Quadrangles Are Not of Indifferent Type

(42.1) Remark Let $\mathcal{B} := \mathcal{Q}_D(\mathbb{K},\mathbb{K}_0,\mathbb{L}_0) = \big(x_1(\mathbb{M}_1), x_2(\mathbb{M}_2), x_3(\mathbb{M}_3), x_4(\mathbb{M}_4)\big)$ be a quadrangle of indifferent type, where $(\mathbb{K},\mathbb{K}_0,\mathbb{L}_0)$ is a proper indifferent set.

(a) By definition, the Moufang set $\mathbb{M}_1 = \mathbb{M}(\mathbb{K},\mathbb{K}_0,\mathbb{L}_0)$ is of indifferent type.

(b) By remark (35.9) of [TW], we have $\mathcal{B}^o = \mathcal{Q}_D(\mathbb{L},\mathbb{L}_0,\mathbb{K}_0^2)$, thus we have $\mathbb{M}_4 = \mathbb{M}(\mathbb{L},\mathbb{L}_0,\mathbb{K}_0^2)$, where $(\mathbb{L},\mathbb{L}_0,\mathbb{K}_0^2)$ is the opposite of $(\mathbb{K},\mathbb{K}_0,\mathbb{L}_0)$, which is proper by lemma (6.4).

(42.2) Theorem The quadrangles are not of indifferent type.

Proof
Suppose that $\mathcal{B}_{(1,2)}$ or $\mathcal{B}_{(2,1)}$ is a parametrized standard quadrangle of indifferent type. By remark (42.1), we have

$$\mathbb{M}^1_{(1,2)} = \mathbb{M}(\mathbb{K},\mathbb{K}_0,\mathbb{L}_0)$$

for some proper indifferent set $(\mathbb{K},\mathbb{K}_0,\mathbb{L}_0)$, we have $\mathbb{M}^1_{(3,1)} = \mathbb{M}(\mathbb{A})$ for some alternative division ring $\mathbb{A}$, and the glueing $\gamma_1 : \mathbb{M}^1_{(3,1)} \to \mathbb{M}^1_{(1,2)}$ is a Jordan isomorphism, which contradicts theorem (31.24). $\square$

(42.3) Remark Now we are done with the exclusion of certain families of quadrangles. Next we pass to unitary 443-foundations, which can not be obtained as fixed point foundations of covers. As in the $\tilde{A}_2$-case with positive glueings, the parametrizing structures are quaternion division algebras, and the existence can be shown via Tits indices.

Then we finally come to 443-foundations involving quadrangles of quadratic form type, which can be constructed as fixed point foundations of covers.

In both cases however, we leave off the existence proofs which require different kinds of techniques.

Chapter 43 Unitary Quadrangles

As quadrangles of pseudo-quadratic form type are extensions of quadrangles of involutory type (which are not necessarily of purely involutory type) and as quadrangles of involutory type can be considered as quadrangles of pseudo-quadratic form type of a non-proper pseudo-quadratic space, it is natural to treat them in a common setup. As a consequence, we sometimes omit the assumption of a proper parameter system to obtain a general statement for both the families.

§ 43.1 Definitions

(43.1) Definition

- A foundation

$$\mathcal{F} := \{\mathcal{B}_{(1,2)} = \mathcal{Q}_I^o(\hat{\Xi}), \mathcal{B}_{(2,3)} = \mathcal{Q}_I(\Xi), \mathcal{B}_{(3,1)} = \mathcal{T}(\tilde{\mathbb{K}}), \gamma_{(1,2,3)}, \gamma_{(2,3,1)}, \gamma_{(3,1,2)}\}$$

  for some proper involutory sets Ξ and $\hat{\Xi}$ is a *443-foundation of involutory type*.

- A foundation

$$\mathcal{F} := \{\mathcal{B}_{(1,2)} = \mathcal{Q}_P^o(\hat{\Xi}), \mathcal{B}_{(2,3)} = \mathcal{Q}_P(\Xi), \mathcal{B}_{(3,1)} = \mathcal{T}(\tilde{\mathbb{K}}), \gamma_{(1,2,3)}, \gamma_{(2,3,1)}, \gamma_{(3,1,2)}\}$$

  for some proper pseudo-quadratic spaces Ξ and $\hat{\Xi}$ is a *443-foundation of pseudo-quadratic form type*.

- A 443-foundation is of *unitary type* if it is either of involutory type or of pseudo-quadratic form type.

(43.2) Notation Given a 443-foundation $\mathcal{F}$, we set

$$\gamma_1 := \gamma_{(3,1,2)} \ , \qquad \gamma_2 := \gamma_{(1,2,3)} \ , \qquad \gamma_3 := \gamma_{(2,3,1)} \ .$$

(43.3) Lemma Let $\mathcal{F}$ be an integrable 443-foundation of unitary type. Then $\tilde{\mathbb{K}}$ is associative.

Proof
The glueings $\gamma_3 = \gamma_{(1,3,2)}$ and $\gamma_1 = \gamma_{(3,1,2)}$ are positive or negative by Hua's theorem. In particular, $\tilde{\mathbb{K}}$ is associative. □

(43.4) Lemma Let $\mathbb{K}$ be a skew-field, let $M \subseteq \mathbb{K}$ and let $a, b, c \in \mathbb{K}$ such that

$$1_{\mathbb{K}} \in M \ , \qquad \forall\, x \in M : \quad axb = cx \ .$$

Then we have

$$b \in C_{\mathbb{K}}(M) \ , \qquad M \subseteq C_{\mathbb{K}}(b) \ .$$

Proof
We have

$$c = c \cdot 1_{\mathbb{K}} = a \cdot 1_{\mathbb{K}} \cdot b = ab$$

and therefore

$$\forall\, x \in M : \quad xb = a^{-1}(axb) = a^{-1}(abx) = bx \ .$$

□

(43.5) Theorem Let $\mathcal{F}$ be an integrable 443-foundation of pseudo-quadratic form type with $\hat{\Xi} = \Xi^o$ and $\gamma_2 = \mathrm{id}_T^o$ and let

$$\pi_{\mathbb{K}} : T \to \mathbb{K},\ (a,t) \mapsto t\ .$$

If one of the glueings is negative, we have

$$\pi_{\mathbb{K}}(T) \subseteq Z(\mathbb{K})\ .$$

In particular, we have $\mathbb{K}_0 \subseteq Z(\mathbb{K})$.

Proof
Assume that $\gamma_3 = \gamma_{(2,3,1)}$ is negative (otherwise, we consider the opposite buildings and glueings). By taking $(\mathbb{K}, \gamma_3, \gamma_3, \gamma_3)$ as reparametrization for $\mathcal{T}(\tilde{\mathbb{K}})$, we may assume

$$\tilde{\mathbb{K}} = \mathbb{K}\ , \qquad\qquad \gamma_3 = \mathrm{id}_{\mathbb{K}}\ .$$

If we set

$$\begin{aligned} h(s) &:= \mu\big(x^1_{(3,1)}(1_{\mathbb{K}})\big)^{-1} \mu\big(x^1_{(3,1)}(s^{-1})\big)\ , \qquad s \in \mathbb{K}^*\ , \\ \tilde{h}(s) &:= \mu\big(x^1_{(1,2)}(1_{\mathbb{K}})\big)^{-1} \mu\big(x^1_{(1,2)}(s^{-1})\big)\ , \qquad s \in \mathbb{K}^*\ , \end{aligned}$$

we have

$$h(s) = \mu\big(x^1_{(3,1)}(1_{\mathbb{K}})\big)^{-1} \mu\big(x^1_{(3,1)}(s^{-1})\big) = \mu\big(x^1_{(1,2)}(1_{\mathbb{K}})\big)^{-1} \mu\big(x^1_{(1,2)}(\gamma_1(s)^{-1})\big) = \tilde{h}\big(\gamma_1(s)\big)$$

for each $s \in \mathbb{K}^*$,

$$x^3_{(2,3)}(t)^{h(s)} = x^3_{(3,1)}(t)^{h(s)} = x^3_{(3,1)}(ts) = x^3_{(2,3)}(ts)$$

for all $s \in \mathbb{K}^*,\ t \in \mathbb{K}$ by lemma (18.9) and

$$\begin{aligned} x^2_{(2,3)}(a,t)^{h(s)} &= x^2_{(1,2)}(a,t)^{\tilde{h}(\gamma_1(s))} \\ &= x^2_{(1,2)}\big(\gamma_1(s) \circ a, \gamma_1(s) \circ t \circ \gamma_1(s)^\sigma\big) = x^2_{(2,3)}\big(a \cdot \gamma(s), \gamma_1(s)^\sigma \cdot t \cdot \gamma_1(s)\big) \end{aligned}$$

for all $s \in \mathbb{K}^*,\ (a,t) \in T$ by lemma (33.4). Given $s \in \mathbb{K}^*$, the Hua automorphism $h\big(\gamma_1^{-1}(s)\big)$ induces an automorphism $\alpha_s \in \mathrm{Aut}(\mathcal{B}_{(2,3)})$ which satisfies

$$x^2_{(2,3)}(a,t) \mapsto x^2_{(2,3)}(a \cdot s, s^\sigma \cdot t \cdot s)\ , \qquad\qquad x^3_{(2,3)}(t) \mapsto x^3_{(2,3)}(t \cdot \gamma_1^{-1}(s))\ .$$

If we set $\tilde{\alpha} := \alpha_{(\gamma_1^{-1}(s), 1_{\mathbb{K}}, (\mathrm{id}_{L_0}, \mathrm{id}_{\mathbb{K}}))}$ as in (37.33) of [TW], then $\tilde{\alpha}\alpha_s$ satisfies

$$x^2_{(2,3)}(a,t) \mapsto x^2_{(2,3)}(a \cdot s, s^\sigma \cdot t \cdot s)\ , \qquad\qquad x^3_{(2,3)}(t) \mapsto x^3_{(2,3)}(t)\ .$$

By (37.33) of [TW], there is an element $c \in \mathbb{K}_0^*$ such that

$$\forall\ t \in \pi_{\mathbb{K}}(T) : \qquad s^\sigma t s = ct\ .$$

Lemma (43.4) implies that we have

$$\pi_{\mathbb{K}}(T) \subseteq C_{\mathbb{K}}(s)\ .$$

Since $s \in \mathbb{K}^*$ is arbitrary, it follows that

$$\pi_{\mathbb{K}}(T) \subseteq Z(\mathbb{K})\ .$$

□

(43.6) Remark Notice that we don't need the fact that Ξ is proper by definition of our parameter systems, i.e., if we allow Ξ to be non-proper, the theorem remains true. As a consequence, we get a similar result for 443-foundations of involutory type which can be considered as 443-foundations of pseudo-quadratic form type for some non-proper pseudo-quadratic spaces, cf. lemma (43.19).

§ 43.2 Quadrangles of Pseudo-Quadratic Form Type

(43.7) Notation Throughout this paragraph, $\mathcal{F}$ is an integrable 443-foundation such that at least one quadrangle is of pseudo-quadratic form type.

(43.8) Lemma The foundation $\mathcal{F}$ is of pseudo-quadratic form type.

Proof
We may assume that $\mathcal{B}_{(1,2)}$ or $\mathcal{B}_{(2,1)}$ is a standard quadrangle $\mathcal{Q}_P(\hat{\mathbb{K}}, \hat{\mathbb{K}}_0, \hat{\sigma}, \hat{L}_0, \hat{q})$ of pseudo-quadratic form type and thus $\mathbb{M}^1_{(1,2)} = \mathbb{M}(\hat{\mathbb{K}})$ or $\mathbb{M}^1_{(1,2)} = \mathbb{M}(\hat{\mathbb{K}}, \hat{\mathbb{K}}_0, \hat{\sigma}, \hat{L}_0, \hat{q})$. But since the map $\gamma_1 : \mathbb{M}(\tilde{\mathbb{K}}) = \mathbb{M}^1_{(3,1)} \to \mathbb{M}^1_{(1,2)}$ is a Jordan isomorphism and Moufang sets of linear type are commutative while Moufang sets of pseudo-quadratic form type are not, we obtain that

$$\mathcal{B}_{(1,2)} = \mathcal{Q}_P^o\big((\hat{\mathbb{K}}, \hat{\mathbb{K}}_0, \hat{\sigma}, \hat{L}_0, \hat{q})^o\big) \ .$$

Now $\mathcal{B}_{(2,3)}$ is a Moufang quadrangle such that $\mathbb{M}^2_{(2,3)}$ is non-commutative, and since we excluded quadrangles of type E_n in chapter 40, we have

$$\mathcal{B}_{(2,3)} = \mathcal{Q}_P(\Xi)$$

for some proper pseudo-quadratic space Ξ. □

(43.9) Notation Until proposition (43.12), $\mathcal{F}$ is an integrable 443-foundation of pseudo-quadratic form type such that $\mathbb{K}$ is non-commutative.

(43.10) Remark By theorem (8.1), the Jordan isomorphism $\gamma_2 = \gamma_{(1,2,3)} : \hat{T} \to T$ is induced by an isomorphism $\Phi : \hat{\Xi}^o \to \Xi$ of pseudo-quadratic spaces. By taking $(\Xi^o, \phi^{-1}, \Phi^{-1}, \phi^{-1}, \Phi^{-1})$ as reparametrization for $\mathcal{Q}_P^o(\hat{\Xi})$, we may assume

$$\hat{\Xi} = \Xi^o \ , \qquad\qquad \gamma_2 = \mathrm{id}_T^o \ .$$

(43.11) Lemma Both the glueings γ_1 and γ_3 are positive.

Proof
If one of the glueings is negative, then we have

$$\mathbb{K}_0 \subseteq \pi_{\mathbb{K}}(T) \subseteq Z(\mathbb{K})$$

by theorem (43.5), thus $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is non-proper by lemma (5.2). By remark (9.9), we have

$$(\mathbb{K}, \mathbb{K}_0, \sigma) = (\mathbb{H}, Z(\mathbb{H}), \sigma_s)$$

for some quaternion division algebra $\mathbb{H}$. Given $a \in L_0$, we have

$$q(a) \in \pi_{\mathbb{H}}(T) \subseteq Z(\mathbb{H}) \ , \qquad\qquad a = 0_{L_0}$$

and thus $L_0 = \{0_{L_0}\}$. But then Ξ is non-proper $\lightning$. □

(43.12) Proposition The skew-field $\mathbb{K}$ is a quaternion division algebra and $\mathcal{F}$ is isomorphic to the foundation

$$\mathcal{F}_{443}(\Xi) := \{\tilde{\mathcal{B}}_{(1,2)} = \mathcal{Q}_P^o(\Xi^o), \tilde{\mathcal{B}}_{(2,3)} = \mathcal{Q}_P(\Xi), \tilde{\mathcal{B}}_{(3,1)} = \mathcal{T}(\mathbb{K}^o), \tilde{\gamma}_1 = \sigma_s, \tilde{\gamma}_2 = \mathrm{id}_T^o, \tilde{\gamma}_3 = \mathrm{id}_{\mathbb{K}}^o\} \ .$$

Proof
As $\gamma_3 = \gamma_{(2,3,1)}$ is positive, we may take $(\mathbb{K}^o, \gamma_3^o, \gamma_3^o, \gamma_3^o)$ as reparametrization for $\mathcal{T}(\tilde{\mathbb{K}})$. Therefore, we may assume

$$\tilde{\mathbb{K}} = \mathbb{K}^o \ , \qquad\qquad \gamma_3 = \mathrm{id}_{\mathbb{K}}^o \ .$$

Let $s \in \mathbb{K}^o$. As in the proof of proposition (43.20), we obtain an automorphism $\alpha_s \in \mathrm{Aut}(\mathcal{B}_{(2,3)})$ satisfying

$$x^2_{(2,3)}(a,t) \mapsto x^2_{(2,3)}\big(a \cdot \gamma_1^o(s) \cdot \mathrm{id}_{\mathbb{K}}^o(s), \mathrm{id}_{\mathbb{K}}^o(s)^\sigma \cdot \gamma_1^o(s)^\sigma \cdot t \cdot \gamma_1^o(s) \cdot \mathrm{id}_{\mathbb{K}}^o(s)\big) \ , \quad x^3_{(2,3)}(t) \mapsto x^3_{(2,3)}(t) \ .$$

By (37.33) of [TW], the map

$$\varphi_1 : L_0 \to L_0, \ a \mapsto a \cdot \gamma_1^o(s) \cdot \mathrm{id}_{\mathbb{K}}^o(s)$$

is an isomorphism of vector spaces satisfying

$$\forall \ a \in L_0, \ t \in \mathbb{K} : \qquad \varphi_1(a \cdot t) = \varphi_1(a) \cdot t.$$

As Ξ is proper, we have $L_0 \neq \{0_{L_0}\}$ and thus

$$\forall \ t \in \mathbb{K} : \qquad t \cdot \gamma_1^o(s) \cdot \mathrm{id}_{\mathbb{K}}^o(s) = \gamma_1^o(s) \cdot \mathrm{id}_{\mathbb{K}}^o(s) \cdot t \ , \qquad\qquad \gamma_1^o(s) \cdot \mathrm{id}_{\mathbb{K}}^o(s) \in Z(\mathbb{K}) \ .$$

Since $\mathbb{K}$ is non-commutative by assumption, lemma (5.6) (b) shows that $\mathbb{K}$ is a quaternion division algebra and that we have

$$\gamma_1 = \sigma_s : \mathbb{K}^o \to \mathbb{K}^o \ .$$

□

(43.13) Remark Let $\mathcal{F}$ be an integrable 443-foundation of pseudo-quadratic form type such that $\mathbb{K}$ is a field, but $\mathbb{K} \ncong \mathbb{F}_4$ if $\dim_{\mathbb{K}} L_0 = 1$. Then $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is non-proper by lemma (5.2) and thus quadratic of type (iii) by remark (9.9). Moreover, we may reparametrize as in the non-commutative case, but we cannot get more information concerning the glueing γ_1, i.e., we have

$$\mathcal{F} \cong \mathcal{F}_{443}(\Xi, \gamma) := \{\tilde{\mathcal{B}}_{(1,2)} = \mathcal{Q}_P^o(\Xi^o), \tilde{\mathcal{B}}_{(2,3)} = \mathcal{Q}_P(\Xi), \tilde{\mathcal{B}}_{(3,1)} = \mathcal{T}(\mathbb{K}), \tilde{\gamma}_1 = \gamma, \tilde{\gamma}_2 = \mathrm{id}_T^o, \tilde{\gamma}_3 = \mathrm{id}_{\mathbb{K}}\}$$

for some pseudo-quadratic space Ξ such that $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is quadratic of type (iii) and for some $\gamma \in \mathrm{Aut}(\mathbb{K})$.

(43.14) Theorem Let $\mathcal{F}$ be an integrable 443-foundation such that at least one quadrangle is of pseudo-quadratic form type and such that $\tilde{\mathbb{K}} \ncong \mathbb{F}_4$ if $\dim_{\mathbb{K}} L_0 = 1$, where $\mathcal{B}_{(3,1)} = \mathcal{T}(\tilde{\mathbb{K}})$. Then one of the following holds:

(i) We have

$$\mathcal{F} \cong \mathcal{F}_{443}(\Xi) = \{\tilde{\mathcal{B}}_{(2,1)} = \tilde{\mathcal{B}}_{(2,3)} = \mathcal{Q}_P(\Xi), \tilde{\mathcal{B}}_{(3,1)} = \mathcal{T}(\mathbb{K}^o), \tilde{\gamma}_1 = \sigma_s, \tilde{\gamma}_2 = \mathrm{id}_T^o, \tilde{\gamma}_3 = \mathrm{id}_{\mathbb{K}}^o\}$$

for some proper pseudo-quadratic space $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma, L_0, q)$ such that $\mathbb{K}$ is a quaternion division algebra and σ_s is its standard involution.

(ii) We have

$$\mathcal{F} \cong \mathcal{F}_{443}(\Xi, \gamma) = \{\tilde{\mathcal{B}}_{(2,1)} = \tilde{\mathcal{B}}_{(2,3)} = \mathcal{Q}_P(\Xi), \tilde{\mathcal{B}}_{(3,1)} = \mathcal{T}(\mathbb{K}), \tilde{\gamma}_1 = \gamma, \tilde{\gamma}_2 = \mathrm{id}_T^o, \tilde{\gamma}_3 = \mathrm{id}_{\mathbb{K}}\}$$

for some proper pseudo-quadratic space $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma, L_0, q)$ such that $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is quadratic of type (iii) and for some $\gamma \in \mathrm{Aut}(\mathbb{K})$.

Proof
This results from lemma (43.8), proposition (43.12) and remark (43.13). □

§ 43.3 Quadrangles of Involutory Type

(43.15) Notation Throughout this paragraph, $\mathcal{F}$ is an integrable 443-foundation such that at least one quadrangle is of involutory type.

(43.16) Lemma The foundation $\mathcal{F}$ is of involutory type.

Proof
We may assume that $\mathcal{B}_{(1,2)}$ or $\mathcal{B}_{(2,1)}$ is a standard quadrangle $\mathcal{Q}_I(\hat{\mathbb{K}}, \hat{\mathbb{K}}_0, \hat{\sigma})$ of involutory type and thus $\mathbb{M}^1_{(1,2)} = \mathbb{M}(\hat{\mathbb{K}})$ or $\mathbb{M}^1_{(1,2)} = \mathbb{M}(\hat{\mathbb{K}}, \hat{\mathbb{K}}_0, \hat{\sigma})$. Since $(\hat{\mathbb{K}}, \hat{\mathbb{K}}_0, \hat{\sigma})$ is proper, we have

$$\mathbb{M}(\tilde{\mathbb{K}}) = \mathbb{M}^1_{(3,1)} \cong \mathbb{M}^1_{(1,2)} \neq \mathbb{M}(\hat{\mathbb{K}}, \hat{\mathbb{K}}_0, \hat{\sigma})$$

by theorem (31.26) and thus

$$\mathcal{B}_{(1,2)} = \mathcal{Q}^o_I\big((\hat{\mathbb{K}}, \hat{\mathbb{K}}_0, \hat{\sigma})^o\big) \ .$$

Since we excluded quadrangles of type E_n, F_4, of indifferent type and of pseudo-quadratic form type in the previous paragraphs, the quadrangle $\mathcal{B}_{(2,3)}$ is either of quadratic form type or of involutory type.

Assume that $\mathcal{B}_{(2,3)}$ is of quadratic form type. Then we have $\mathbb{M}^2_{(2,3)} = \mathbb{M}(\mathbb{K})$ or $\mathbb{M}^3_{(2,3)} = \mathbb{M}(\mathbb{K})$ for some field $\mathbb{K}$. But we have

$$\mathbb{M}^3_{(2,3)} \cong \mathbb{M}^3_{(3,1)} = \mathbb{M}(\tilde{\mathbb{K}}) = \mathbb{M}^1_{(3,1)} \cong \mathbb{M}^1_{(1,2)} = \mathbb{M}(\hat{\mathbb{K}}) \ ,$$

where $\hat{\mathbb{K}}$ is a non-commutative skew-field since $(\hat{\mathbb{K}}, \hat{\mathbb{K}}_0, \hat{q})$ is proper, and thus $\mathbb{M}^3_{(2,3)} \ncong \mathbb{M}(\mathbb{K})$. Moreover, we have

$$\mathbb{M}^2_{(2,3)} \cong \mathbb{M}^2_{(1,2)} = \mathbb{M}(\hat{\mathbb{K}}, \hat{\mathbb{K}}_0, \hat{\sigma})$$

and thus $\mathbb{M}^2_{(2,3)} \neq \mathbb{M}(\mathbb{K})$ by theorem (31.26) $\lightning$.

Therefore, the quadrangle $\mathcal{B}_{(2,3)}$ = is of involutory type, i.e., $\mathcal{B}_{(2,3)}$ or $\mathcal{B}_{(3,2)}$ is a standard quadrangle $\mathcal{Q}_I(\mathbb{K}, \mathbb{K}_0, \sigma)$. Since we have

$$\mathbb{M}^3_{(2,3)} \cong \mathbb{M}^3_{(3,1)} = \mathbb{M}(\tilde{\mathbb{K}})$$

and $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is proper, we have $\mathbb{M}^3_{(2,3)} \ncong \mathbb{M}(\mathbb{K}, \mathbb{K}_0, \sigma)$ by theorem (31.26) again and thus

$$\mathcal{B}_{(2,3)} = \mathcal{Q}_I(\mathbb{K}, \mathbb{K}_0, \sigma) \ .$$

□

(43.17) Notation Throughout the rest of this paragraph, $\mathcal{F}$ is an integrable 443-foundation of involutory type.

(43.18) Remark
By theorem (5.3), the Jordan isomorphism

$$\gamma_2 = \gamma_{(1,2,3)} : \hat{\mathbb{K}}_0 \to \mathbb{K}_0$$

is induced by an isomorphism

$$\phi : (\hat{\mathbb{K}}, \hat{\mathbb{K}}_0, \hat{\sigma}) \to (\mathbb{K}, \mathbb{K}_0, \sigma) =: \Xi$$

of involutory sets. As a consequence, the map

$$\tilde{\phi} := \phi \circ \sigma^o : (\hat{\mathbb{K}}^o, \hat{\mathbb{K}}^o_0, \hat{\sigma}) \to (\mathbb{K}, \mathbb{K}_0, \sigma)$$

is an isomorphism of involutory sets that induces γ_2 as well. By taking $(\Xi^o, \tilde{\phi}^{-1}, \tilde{\phi}^{-1}, \tilde{\phi}^{-1}, \tilde{\phi}^{-1})$ as reparametrization for $\mathcal{Q}^o_I(\hat{\mathbb{K}}, \hat{\mathbb{K}}_0, \hat{\sigma})$, we may assume

$$(\hat{\mathbb{K}}, \hat{\mathbb{K}}_0, \hat{\sigma}) = (\mathbb{K}^o, \mathbb{K}^o_0, \sigma) \ , \qquad \gamma_2 = \mathrm{id}^o_{\mathbb{K}_0} \ .$$

(43.19) Lemma Both the glueings γ_1 and γ_3 are positive.

Proof
Notice that we have
$$\mathcal{Q}_I(\mathbb{K},\mathbb{K}_0,\sigma)=\mathcal{Q}_P(\mathbb{K},\mathbb{K}_0,\sigma,L_0,q)$$
for $L_0:=\{0\}$, $q:=0$. If one of the glueings is negative, we have
$$\mathbb{K}_0=\pi_{\mathbb{K}}(T)\subseteq Z(\mathbb{K})$$
by theorem (43.5), where we did not use the fact that $(\mathbb{K},\mathbb{K}_0,\sigma,L_0,q)$ is proper in the given setup. But then $(\mathbb{K},\mathbb{K}_0,\sigma)$ is non-proper by lemma (5.2) $\lightning$. $\square$

(43.20) Proposition The skew-field $\mathbb{K}$ is a quaternion division algebra and $\mathcal{F}$ is isomorphic to the foundation
$$\mathcal{F}_{443}(\Xi):=\{\tilde{\mathcal{B}}_{(1,2)}=\mathcal{Q}_I^o(\Xi^o),\tilde{\mathcal{B}}_{(2,3)}=\mathcal{Q}_I(\Xi),\tilde{\mathcal{B}}_{(3,1)}=\mathcal{T}(\mathbb{K}^o),\tilde{\gamma}_1=\sigma_s,\tilde{\gamma}_2=\mathrm{id}^o_{\mathbb{K}_0},\tilde{\gamma}_3=\mathrm{id}^o_{\mathbb{K}}\}\ .$$

Proof
As $\gamma_3=\gamma_{(2,3,1)}$ is positive, we may take $(\mathbb{K}^o,\gamma_3^o,\gamma_3^o,\gamma_3^o)$ as reparametrization for $\mathcal{T}(\tilde{\mathbb{K}})$. Therefore, we may assume
$$\tilde{\mathbb{K}}=\mathbb{K}^o\ ,\qquad\qquad \gamma_3=\mathrm{id}^o_{\mathbb{K}}\ .$$

- If we set
$$h_1(s):=\mu\big(x^1_{(3,1)}(1_{\mathbb{K}^o})\big)^{-1}\mu\big(x^1_{(3,1)}(s^{-1})\big)\ ,\qquad s\in\mathbb{K}^o\ ,$$
$$\tilde{h}_1(s):=\mu\big(x^1_{(1,2)}(1^o_{\mathbb{K}})\big)^{-1}\mu\big(x^1_{(1,2)}(s^{-1})\big)\ ,\qquad s\in\mathbb{K}^o\ ,$$
we have
$$\begin{aligned}h_1(s)&=\mu\big(x^1_{(3,1)}(1_{\mathbb{K}^o})\big)^{-1}\mu\big(x^1_{(3,1)}(s^{-1})\big)\\&=\mu\big(x^1_{(1,2)}(1^o_{\mathbb{K}})\big)^{-1}\mu\big(x^1_{(1,2)}(\gamma_1(s)^{-1})\big)=\tilde{h}_1\big(\gamma_1(s)\big)\end{aligned}$$
for each $s\in\mathbb{K}^o$,
$$x^3_{(2,3)}(t)^{h_1(s)}=x^3_{(3,1)}\big(\mathrm{id}^o_{\mathbb{K}}(t)\big)^{h_1(s)}=x^3_{(3,1)}\big(\mathrm{id}^o_{\mathbb{K}}(t)\circ s\big)=x^3_{(2,3)}\big(\mathrm{id}^o_{\mathbb{K}}(s)\cdot t\big)$$
for all $s\in\mathbb{K}^o,\ t\in\mathbb{K}$ by lemma (18.9) and
$$\begin{aligned}x^2_{(2,3)}(t)^{h_1(s)}&=x^2_{(1,2)}\big(\mathrm{id}^o_{\mathbb{K}_0}(t)\big)^{\tilde{h}_1(\gamma_1(s))}\\&=x^2_{(1,2)}\big(\gamma_1(s)\circ\mathrm{id}^o_{\mathbb{K}_0}(t)\circ\gamma_1(s)^\sigma\big)=x^2_{(2,3)}\big(\gamma_1^o(s)^\sigma\cdot t\cdot\gamma_1^o(s)\big)\end{aligned}$$
for all $s\in\mathbb{K}^o,\ t\in\mathbb{K}_0$ by lemma (33.2).

- If we set
$$h_3(s):=\mu\big(x^3_{(3,1)}(1_{\mathbb{K}^o})\big)^{-1}\mu\big(x^3_{(3,1)}(s)\big)\ ,\qquad s\in\mathbb{K}^o\ ,$$
$$\tilde{h}_3(s):=\mu\big(x^3_{(2,3)}(1_{\mathbb{K}})\big)^{-1}\mu\big(x^3_{(2,3)}(s)\big)\ ,\qquad s\in\mathbb{K}\ ,$$
we have
$$h_3(s)=\mu\big(x^1_{(3,1)}(1_{\mathbb{K}^o})\big)^{-1}\mu\big(x^1_{(3,1)}(s^{-1})\big)=\mu\big(x^1_{(1,2)}(1_{\mathbb{K}})\big)^{-1}\mu\big(x^1_{(1,2)}(\mathrm{id}^o_{\mathbb{K}}(s^{-1}))\big)=\tilde{h}_3\big(\mathrm{id}^o_{\mathbb{K}}(s)\big)$$
for each $s\in\mathbb{K}^o$,
$$\begin{aligned}x^3_{(2,3)}(t)^{h_3(s)}&=x^3_{(3,1)}\big(\mathrm{id}^o_{\mathbb{K}}(t)\big)^{h_3(s)}\\&=x^3_{(3,1)}\big(s^{-1}\circ\mathrm{id}^o_{\mathbb{K}}(t)\circ s^{-1}\big)=x^3_{(2,3)}\big(\mathrm{id}^o_{\mathbb{K}}(s)^{-1}\cdot t\cdot\mathrm{id}^o_{\mathbb{K}}(s)^{-1}\big)\end{aligned}$$
for all $s\in\mathbb{K}^o,\ t\in\mathbb{K}$ by lemma (18.9) and
$$x^2_{(2,3)}(t)^{h_3(s)}=x^2_{(2,3)}(t)^{\tilde{h}_3(\mathrm{id}^o_{\mathbb{K}}(s))}=x^2_{(2,3)}\big(\mathrm{id}^o_{\mathbb{K}}(s)^\sigma\cdot t\cdot\mathrm{id}^o_{\mathbb{K}}(s)\big)$$
for all $s\in\mathbb{K}^o,\ t\in\mathbb{K}_0$ by (33.13) of [TW].

Therefore, given $s \in \mathbb{K}^o$, $h_3(s)h_1(s)$ induces an automorphism $\alpha_s \in \mathrm{Aut}(\mathcal{B}_{(2,3)})$ which satisfies

$$x^2_{(2,3)}(t) \mapsto x^2_{(2,3)}\big(\mathrm{id}^o_{\mathbb{K}}(s)^\sigma \cdot \gamma^o_1(s)^\sigma \cdot t \cdot \gamma^o_1(s) \cdot \mathrm{id}^o_{\mathbb{K}}(s)\big) \ , \qquad x^3_{(2,3)}(t) \mapsto x^3_{(2,3)}\big(t \cdot \mathrm{id}^o_{\mathbb{K}}(s)^{-1}\big) \ .$$

If we set $\tilde{\alpha} := \alpha_{(\mathrm{id}^o_{\mathbb{K}}(s), 1_{\mathbb{K}}, (\mathrm{id}_{L_0}, \mathrm{id}_{\mathbb{K}}))}$ as in (37.33) of [TW], then $\tilde{\alpha}\alpha_s \in \mathrm{Aut}(\mathcal{B}_{(2,3)})$ satisfies

$$x^2_{(2,3)}(t) \mapsto x^2_{(2,3)}\big(\mathrm{id}^o_{\mathbb{K}}(s)^\sigma \cdot \gamma^o_1(s)^\sigma \cdot t \cdot \gamma^o_1(s) \cdot \mathrm{id}^o_{\mathbb{K}}(s)\big) \ , \qquad x^3_{(2,3)}(t) \mapsto x^3_{(2,3)}(t) \ .$$

By (37.33) of [TW], there is an element $c \in \mathbb{K}^*_0$ such that

$$\forall\, t \in \mathbb{K}_0 : \qquad \mathrm{id}^o_{\mathbb{K}}(s)^\sigma \cdot \gamma^o_1(s)^\sigma \cdot t \cdot \gamma^o_1(s) \cdot \mathrm{id}^o_{\mathbb{K}}(s) = c \cdot t \ .$$

Lemma (43.4) implies that we have

$$\gamma^o_1(s) \cdot \mathrm{id}^o_{\mathbb{K}}(s) \in C_{\mathbb{K}}(\mathbb{K}_0) = C_{\mathbb{K}}(\langle \mathbb{K}_0 \rangle) = Z(\mathbb{K}) \ .$$

Since $\mathbb{K}$ is non-commutative by lemma (5.2), lemma (5.6) (b) shows that $\mathbb{K}$ is a quaternion division algebra and that we have

$$\gamma_1 = \sigma_s : \mathbb{K}^o \to \mathbb{K}^o \ .$$

□

(43.21) Theorem Let $\mathcal{F}$ be an integrable 443-foundation such that at least one quadrangle is of involutory type. Then we have

$$\mathcal{F} \cong \mathcal{F}_{443}(\Xi) = \{\tilde{\mathcal{B}}_{(1,2)} = \mathcal{Q}^o_I(\Xi^o), \tilde{\mathcal{B}}_{(2,3)} = \mathcal{Q}_I(\Xi), \tilde{\mathcal{B}}_{(3,1)} = \mathcal{T}(\mathbb{K}^o), \tilde{\gamma}_1 = \sigma_s, \tilde{\gamma}_2 = \mathrm{id}^o_{\mathbb{K}_0}, \tilde{\gamma}_3 = \mathrm{id}^o_{\mathbb{K}}\}$$

for some proper involutory set $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma)$ such that $\mathbb{K}$ is a quaternion division algebra.

Proof
This results from lemma (43.16) and proposition (43.20). □

(43.22) Remark

(a) By remark (11.2) of [TW], the pair $(\mathbb{K}, \sigma)$ uniquely determines $\mathbb{K}_0$ if we have $\mathrm{Char}\,\mathbb{K} \neq 2$.

(b) By lemma (3.1.6) of [K] and (35.7) of [TW], we may assume $\sigma = \gamma_s$ if we have $\mathrm{Char}\,\mathbb{K} = 2$ and σ is an involution of the first kind.

Chapter 44 Quadrangles of Quadratic Form Type

By the previous chapters, there is only one case left: Both the quadrangles are of quadratic form type.

(44.1) Notation

- Throughout this chapter, $\mathcal{F}$ is an integrable 443-foundation such that both the quadrangles are of quadratic form type.
- Given a foundation, if neither $\gamma_{(i,j,k)}$ nor $\gamma_{(k,j,i)}$ is specified, these glueings are supposed to be the identity map.

(44.2) Proposition If we have

$$\mathcal{F} = \{\mathcal{B}_{(2,1)} = \mathcal{Q}_Q(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}), \mathcal{B}_{(2,3)} = \mathcal{Q}_Q(\hat{L}_0, \hat{\mathbb{K}}, \hat{q}), \mathcal{B}_{(3,1)} = \mathcal{T}(\mathbb{A}), \gamma_1, \gamma_2, \gamma_3\} \ ,$$

then $\mathbb{A}$ is quadratic over subfields $\mathbb{F}_1, \mathbb{F}_2$ of its center, and $\mathcal{F}$ is isomorphic to the foundation

$$\mathcal{F}_{443}\big(\mathbb{A}, (\mathbb{F}_1, \mathbb{F}_2), \gamma_2\big) := \{\mathcal{B}_{(2,1)} = \mathcal{Q}_Q(\mathbb{A}, \mathbb{F}_1, N^{\mathbb{A}}_{\mathbb{F}_1}), \mathcal{B}_{(2,3)} = \mathcal{Q}_Q(\mathbb{A}, \mathbb{F}_2, N^{\mathbb{A}}_{\mathbb{F}_2}), \mathcal{B}_{(3,1)} = \mathcal{T}(\mathbb{A}), \gamma_2\}$$

for some isomorphism $\gamma_2 : \mathbb{F}_1 \to \mathbb{F}_2$ of fields.

Proof
We have

$$\mathbb{M}(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}) = \mathbb{M}^1_{(2,1)} \cong \mathbb{M}^1_{(3,1)} = \mathbb{M}(\mathbb{A}) = \mathbb{M}^3_{(3,1)} \cong \mathbb{M}^3_{(2,3)} = \mathbb{M}(\hat{L}_0, \hat{\mathbb{K}}, \hat{q}) \ .$$

By theorem (31.7), the alternative division ring $\mathbb{A}$ is quadratic over subfields $\mathbb{F}_1, \mathbb{F}_2$ of its center, and the maps

$$\gamma_1 : \mathbb{A} \to \tilde{L}_0 \ , \qquad \gamma_3^{-1} : \mathbb{A} \to \hat{L}_0$$

are induced by isomorphisms

$$(\gamma_1, \phi_1) : (\mathbb{A}, \mathbb{F}_1, N^{\mathbb{A}}_{\mathbb{F}_1}) \to (\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}) \ , \qquad (\gamma_3^{-1}, \phi_3^{-1}) : (\mathbb{A}, \mathbb{F}_2, N^{\mathbb{A}}_{\mathbb{F}_2}) \to (\hat{L}_0, \hat{\mathbb{K}}, \hat{q})$$

of quadratic spaces. Now we take

$$\alpha_{(2,1)} := (\Xi_1, \phi_1, \gamma_1, \phi_1, \gamma_1) \ , \qquad \alpha_{(2,3)} := (\Xi_2, \phi_3^{-1}, \gamma_3^{-1}, \phi_3^{-1}, \gamma_3^{-1})$$

as reparametrizations for $\mathcal{B}_{(2,1)}$ and $\mathcal{B}_{(2,3)}$, respectively, where $\Xi_i := (\mathbb{A}, \mathbb{F}_i, N^{\mathbb{A}}_{\mathbb{F}_i})$. □

(44.3) Proposition If we have

$$\mathcal{F} = \{\mathcal{B}_{(1,2)} = \mathcal{Q}_Q(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}), \mathcal{B}_{(2,3)} = \mathcal{Q}_Q(\hat{L}_0, \hat{\mathbb{K}}, \hat{q}), \mathcal{B}_{(3,1)} = \mathcal{T}(\mathbb{K}), \gamma_1, \gamma_2, \gamma_3\} \ ,$$

then $\mathbb{K}$ is a field which is quadratic over some subfield $\mathbb{F}$ of its center, $\mathbb{F}$ is quadratic over some subfield $\mathbb{E}$ of its center, and we have

$$\mathcal{F} \cong \mathcal{F}_{443}(\mathbb{K}, \mathbb{F}, \mathbb{E}, \gamma_1) := \{\mathcal{B}_{(1,2)} = \mathcal{Q}_Q(\mathbb{F}, \mathbb{E}, N^{\mathbb{F}}_{\mathbb{E}}), \mathcal{B}_{(2,3)} = \mathcal{Q}_Q(\mathbb{K}, \mathbb{F}, N^{\mathbb{K}}_{\mathbb{F}}), \mathcal{B}_{(3,1)} = \mathcal{T}(\mathbb{K}), \gamma_1\}$$

for some isomorphism $\gamma_1 : \mathbb{K} \to \mathbb{E}$ of fields.

Proof
Since we have

$$\mathbb{M}(\mathbb{K}) = \mathbb{M}^1_{(3,1)} \cong \mathbb{M}^1_{(1,2)} = \mathbb{M}(\tilde{\mathbb{K}}) \ ,$$

the alternative division ring $\mathbb{K} \cong \tilde{\mathbb{K}}$ is a field by Hua's theorem. We have

$$\mathbb{M}(\hat{L}_0, \hat{\mathbb{K}}, \hat{q}) = \mathbb{M}^3_{(2,3)} \cong \mathbb{M}^3_{(3,1)} = \mathbb{M}(\mathbb{K}) \ ,$$

therefore, $\mathbb{K}$ is quadratic over some subfield $\mathbb{F}$ of its center by theorem (31.7), and the map $\gamma_3^{-1} : \mathbb{K} \to \hat{L}_0$ is induced by an isomorphism $(\gamma_3^{-1}, \phi_3^{-1}) : (\mathbb{K}, \mathbb{F}, N^{\mathbb{K}}_{\mathbb{F}}) \to (\hat{L}_0, \hat{\mathbb{K}}, \hat{q})$ of quadratic spaces. By taking

$$\alpha_{(2,3)} := \big((\mathbb{K}, \mathbb{F}, N^{\mathbb{K}}_{\mathbb{F}}), \phi_3^{-1}, \gamma_3^{-1}, \phi_3^{-1}, \gamma_3^{-1}\big)$$

as reparametrization for $\mathcal{B}_{(2,3)}$, we may assume $\mathcal{B}_{(2,3)} = \mathcal{Q}_Q(\mathbb{K}, \mathbb{F}, N^{\mathbb{K}}_{\mathbb{F}})$ and $\gamma_3 = \mathrm{id}_{\mathbb{K}}$. Moreover, we have

$$\mathbb{M}(\mathbb{F}) = \mathbb{M}^2_{(2,3)} \cong \mathbb{M}^2_{(1,2)} = \mathbb{M}(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}) \ ,$$

therefore, $\mathbb{F}$ is quadratic over some subfield $\mathbb{E}$ of its center by theorem (31.7) again, and the map $\gamma_2^{-1} : \mathbb{F} \to \tilde{L}_0$ is induced by an isomorphism $(\gamma_2^{-1}, \phi_2^{-1}) : (\mathbb{F}, \mathbb{E}, N^{\mathbb{F}}_{\mathbb{E}}) \to (\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ of quadratic spaces. By taking

$$\alpha_{(1,2)} := \big((\mathbb{F}, \mathbb{E}, N^{\mathbb{F}}_{\mathbb{E}}), \phi_2^{-1}, \gamma_2^{-1}, \phi_2^{-1}, \gamma_2^{-1}\big)$$

as reparametrization for $\mathcal{B}_{(1,2)}$, we may assume $\mathcal{B}_{(1,2)} = \mathcal{Q}_Q(\mathbb{F}, \mathbb{E}, N^{\mathbb{F}}_{\mathbb{E}})$ and $\gamma_2 = \mathrm{id}_{\mathbb{F}}$. Finally, the map $\gamma_1 : \mathbb{K} \to \mathbb{E}$ is an isomorphism of fields by Hua's theorem. □

(44.4) Proposition If we have

$$\mathcal{F} = \{\mathcal{B}_{(1,2)} = \mathcal{Q}_Q(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}), \mathcal{B}_{(3,2)} = \mathcal{Q}_Q(\hat{L}_0, \hat{\mathbb{K}}, \hat{q}), \mathcal{B}_{(3,1)} = \mathcal{T}(\mathbb{A}), \gamma_1, \gamma_2, \gamma_3\} \ ,$$

such that $(\hat{L}_0, \hat{\mathbb{K}}, \hat{q})$ is proper with $\dim_{\hat{\mathbb{K}}} \hat{L}_0 \geq 3$, we have

$$\mathcal{F} \cong \mathcal{F}_{443}\big((\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}), \gamma_3\big) := \{\mathcal{B}_{(1,2)} = \mathcal{B}_{(3,2)} = \mathcal{Q}_Q(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}), \mathcal{B}_{(3,1)} = \mathcal{T}(\tilde{\mathbb{K}}), \gamma_3\}$$

for some $\gamma_3 \in \mathrm{Aut}(\tilde{\mathbb{K}})$.

Proof
By Hua's theorem, the map $\gamma_1^{-1} : \tilde{\mathbb{K}} \to \mathbb{A}$ is an isomorphism of fields. By taking

$$\alpha_{(3,1)} := \big(\tilde{\mathbb{K}}, \gamma_1^{-1}, \gamma_1^{-1}, \gamma_1^{-1}, \big)$$

as reparametrization for $\mathcal{B}_{(3,1)}$, we may assume $\mathcal{B}_{(3,1)} = \mathcal{T}(\tilde{\mathbb{K}})$ and $\gamma_3 = \mathrm{id}_{\tilde{\mathbb{K}}}$. Moreover, by theorem (31.6), the map $\gamma_2 : \tilde{L}_0 \to \hat{L}_0$ is induced by an isomorphism $(\gamma_2, \phi_2) : (\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}) \to (\hat{L}_0, \hat{\mathbb{K}}, \hat{q})$ of quadratic spaces. By taking

$$\alpha_{(3,2)} := \big((\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}), \phi_2, \gamma_2, \phi_2, \gamma_2\big)$$

as reparametrization for $\mathcal{B}_{(3,2)}$, we may assume $\mathcal{B}_{(3,2)} = \mathcal{Q}_Q(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q})$ and $\gamma_2 = \mathrm{id}_{\tilde{L}_0}$. Now the map $\gamma_3 : \tilde{\mathbb{K}} \to \tilde{\mathbb{K}}$ is an isomorphism of fields by Hua's theorem. □

(44.5) Proposition If we have

$$\mathcal{F} = \{\mathcal{B}_{(1,2)} = \mathcal{Q}_Q(\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}), \mathcal{B}_{(3,2)} = \mathcal{Q}_Q(\hat{L}_0, \hat{\mathbb{K}}, \hat{q}), \mathcal{B}_{(3,1)} = \mathcal{T}(\mathbb{A}), \gamma_1, \gamma_2, \gamma_3\}\ ,$$

such that $\dim_{\hat{\mathbb{K}}} \hat{L}_0 \leq 2$, we have

$$\mathcal{F} \cong \mathcal{F}_{443}(\tilde{\Xi}, \hat{\Xi}, \gamma_2) := \{\mathcal{B}_{(1,2)} = \mathcal{Q}_Q(\mathbb{K}, \hat{L}_0, q), \mathcal{B}_{(3,2)} = \mathcal{Q}_Q(\hat{L}_0, \hat{\mathbb{K}}, \hat{q}), \mathcal{B}_{(3,1)} = \mathcal{T}(\mathbb{K}), \gamma_3\}\ ,$$

for the quadratic space $\hat{\Xi} := (\hat{L}_0, \hat{\mathbb{K}}, \hat{q})$ with $\dim_{\hat{\mathbb{K}}} \hat{L}_0 \leq 2$ (and thus of type (m)∈ {(ii),(iii)}), some quadratic space $\tilde{\Xi} := (\hat{L}_0, \mathbb{K}, q)$ of type (m)∈ {(i),(ii),(iii)} and some isomorphism $\gamma_3 : \hat{\mathbb{K}} \to \mathbb{K}$ of fields.

Proof
By theorem (31.6), there is a quadratic space $(\hat{L}_0, \mathbb{K}, q)$ of type (m)∈ {(i),(ii),(iii)} such that the map $\gamma_2 : \tilde{L}_0 \to \hat{L}_0$ is induced by an isomorphism $(\gamma_2, \phi_2) : (\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}) \to (\hat{L}_0, \mathbb{K}, q)$ of quadratic spaces. Notice that we don't need $(\hat{L}_0, \hat{\mathbb{K}}, \hat{q})$ to be proper to establish case (ii) of theorem (31.6). By taking

$$\alpha_{(1,2)} := \big((\hat{L}_0, \mathbb{K}, q), \phi_2^{-1}, \gamma_2^{-1}, \phi_2^{-1}, \gamma_2^{-1}\big)$$

as reparametrization for $\mathcal{B}_{(1,2)}$, we may assume $\mathcal{B}_{(1,2)} = \mathcal{Q}_Q(\hat{L}_0, \mathbb{K}, q)$ and $\gamma_2 = \mathrm{id}_{\hat{L}_0}$. By Hua's theorem, the map $\gamma_1^{-1} : \mathbb{K} \to \mathbb{A}$ is an isomorphism of fields. By taking

$$\alpha_{(3,1)} := \big(\mathbb{K}, \gamma_1^{-1}, \gamma_1^{-1}, \gamma_1^{-1}, \big)$$

as reparametrization for $\mathcal{B}_{(3,1)}$, we may assume $\mathcal{B}_{(3,1)} = \mathcal{T}(\mathbb{K})$ and $\gamma_1 = \mathrm{id}_{\mathbb{K}}$. Now the map $\gamma_3 : \hat{\mathbb{K}} \to \mathbb{K}$ is an isomorphism of fields by Hua's theorem. □

(44.6) Theorem Let $\mathcal{F}$ be an integrable 443-foundation such that both the quadrangles are of quadratic form type and such that $(\hat{L}_0, \hat{\mathbb{K}}, \hat{q})$ is proper. Then $\mathcal{F}$ is isomorphic to one of the following foundations:

(i) $\mathcal{F}_{443}\big(\mathbb{A}, (\mathbb{F}_1, \mathbb{F}_2), \gamma_2\big)$ as in proposition (44.2)

(ii) $\mathcal{F}_{443}(\mathbb{K}, \mathbb{F}, \mathbb{E}, \gamma_1)$ as in proposition (44.3)

(iii) $\mathcal{F}_{443}\big((\tilde{L}_0, \tilde{\mathbb{K}}, \tilde{q}), \gamma_3\big)$ as in proposition (44.4)

(iv) $\mathcal{F}_{443}(\tilde{\Xi}, \hat{\Xi}, \gamma_3)$ as in proposition (44.5)

Proof
This results from propositions (44.2), (44.3), (44.4), and (44.5). □

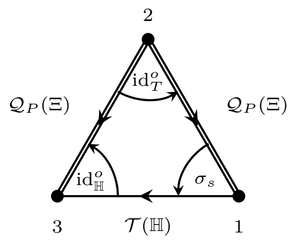
(44.7) Remark Notice that we supposed $(\hat{L}_0, \hat{\mathbb{K}}, \hat{q})$ to be proper only in proposition (44.4). The remaining results are valid even if both the parametrizing quadratic spaces are non-proper. In particular, case (ii) of theorem (31.6) doesn't require $(\hat{L}_0, \hat{\mathbb{K}}, \hat{q})$ to be proper.

Chapter 45 Conclusion

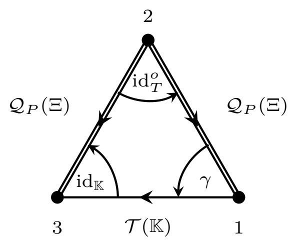
We summarize the previous results to have a complete list of integrable 443-Foundations. By remark (44.7), the theorem can be extended to non-proper quadratic spaces by adding quadratic spaces of type (i) except for case (vi). However, we don't give the existence proofs.

(45.1) Theorem (**Classification of 443 Twin Buildings**) An integrable 443-foundation $\mathcal{F}$ with proper parameter systems is isomorphic to one of the following foundations:

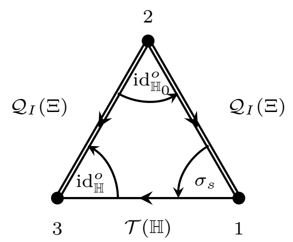
(i) $\mathcal{F}_{443}(\Xi)$ for some proper pseudo-quadratic space $\Xi = (\mathbb{H}, \mathbb{H}_0, \sigma, L_0, q)$ such that $\mathbb{H}$ is a quaternion division algebra:



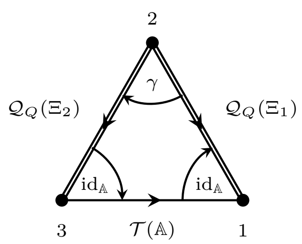
(ii) $\mathcal{F}_{443}(\Xi, \gamma)$ for some proper pseudo-quadratic space $\Xi = (\mathbb{K}, \mathbb{K}_0, \sigma, L_0, q)$ such that $(\mathbb{K}, \mathbb{K}_0, \sigma)$ is quadratic of type (iii) and for some automorphism $\gamma \in \mathrm{Aut}(\mathbb{K})$:



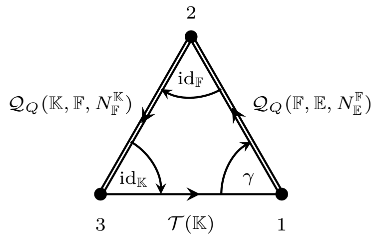
(iii) $\mathcal{F}_{443}(\Xi)$ for some proper involutory set $\Xi = (\mathbb{H}, \mathbb{H}_0, \sigma)$ such that $\mathbb{H}$ is a quaternion division algebra:



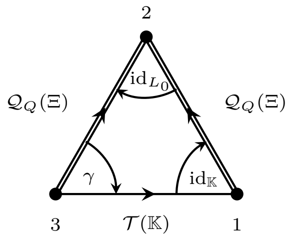
(iv) $\mathcal{F}_{443}(\mathbb{A}, (\mathbb{F}_1, \mathbb{F}_2), \gamma)$ for some proper quadratic spaces $\Xi_i := (\mathbb{A}, \mathbb{F}_i, N_{\mathbb{F}_i}^{\mathbb{A}})$, $i = 1, 2$, of type (ii)-(v) and some isomorphism $\gamma : \mathbb{F}_1 \to \mathbb{F}_2$ of fields:



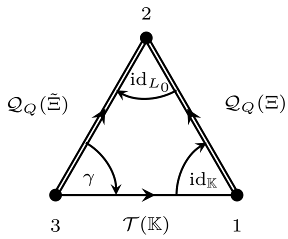
(v) $\mathcal{F}_{443}(\mathbb{K}, \mathbb{F}, \mathbb{E}, \gamma)$ for some proper quadratic spaces $(\mathbb{K}, \mathbb{F}, N_{\mathbb{F}}^{\mathbb{K}})$, $(\mathbb{F}, \mathbb{E}, N_{\mathbb{E}}^{\mathbb{F}})$ of type (ii)-(iii) and some isomorphism $\gamma : \mathbb{K} \to \mathbb{E}$ of fields:



(vi) $\mathcal{F}_{443}(\Xi, \gamma)$ for some proper quadratic space $\Xi = (L_0, \mathbb{K}, q)$ such that $\dim_{\mathbb{K}} L_0 \geq 3$ and some automorphism $\gamma \in \mathrm{Aut}(\mathbb{K})$:



(vii) $\mathcal{F}_{443}(\Xi, \tilde{\Xi}, \gamma)$ for some proper quadratic space $\Xi = (L_0, \mathbb{K}, q)$ such that $\dim_{\mathbb{K}} L_0 \leq 2$, some proper quadratic space $\tilde{\Xi} = (L_0, \tilde{\mathbb{K}}, \tilde{q})$ of type (ii)-(iii) and some isomorphism $\gamma : \tilde{\mathbb{K}} \to \mathbb{K}$ of fields:



Proof
This holds by theorems (40.2), (41.2), (42.2), (43.14), (43.21) and (44.6). □

(45.2) Remark

(a) Notice the restriction in theorem (43.14) which we did not mention in the above formulation.

(b) Let $(\mathbb{A}, \mathbb{K}, N_{\mathbb{K}}^{\mathbb{A}})$ be a quadratic space of type (m). Then we have

$$\mathbb{M} := \mathbb{M}\big(\mathbb{A}, \mathbb{K}, N_{\mathbb{K}}^{\mathbb{A}}\big) = \mathbb{M}(\mathbb{A}) =: \tilde{\mathbb{M}}$$

by lemma (31.23).

(45.3) Corollary If we have $\dim_{\mathbb{F}_1} \mathbb{A} \geq 3$ in case (iv) of theorem (45.1), we have

$$\Xi_2 = \Xi_1 \ .$$

Proof
This results from theorem (31.6) as we have $\mathbb{M}(\Xi_1) = M(\mathbb{A}) = \mathbb{M}(\Xi_2)$. □

Appendix

Chapter 46 $\tilde{A}_2$-Buildings Revisited

We prove theorem (21.39) without using the building at infinity.

(46.1) Proposition Let

$$\mathcal{F} := \{\mathcal{B}_{(1,2)} = \mathcal{T}(\hat{\mathbb{A}}), \mathcal{B}_{(2,3)} = \mathcal{T}(\mathbb{A}), \mathcal{B}_{(3,1)} = \mathcal{T}(\tilde{\mathbb{A}}), \gamma_{(1,2,3)}, \gamma_{(2,3,1)}, \gamma_{(3,1,2)}\}$$

be an integrable foundation of type $\tilde{A}_2$ such that the defining field is a non-commutative skew-field and such that at least one glueing is positive. Then each glueing is positive.

Proof
Assume that $\gamma_2 = \gamma_{(1,2,3)}$ is positive. Without loss of generality we may assume that $\gamma_3 = \gamma_{(2,3,1)}$ is negative (otherwise, we consider the opposite buildings and glueings). By taking $(\mathbb{A}, \gamma_3, \gamma_3, \gamma_3)$ as reparametrization for $\mathcal{T}(\tilde{\mathbb{A}})$, we may assume

$$\tilde{\mathbb{A}} = \mathbb{A} \ , \qquad\qquad \gamma_3 = \mathrm{id}_{\mathbb{A}} \ ,$$

and by taking $(\mathbb{A}^o, \gamma_2^{-1}, \gamma_2^{-1}, \gamma_2^{-1})$ as reparametrization for $\mathcal{T}(\hat{\mathbb{A}})$, we may assume

$$\hat{\mathbb{A}} = \mathbb{A}^o \ , \qquad\qquad \gamma_2 = \mathrm{id}^o_{\mathbb{A}} \ .$$

If we set

$$\begin{aligned} h(s) &:= \mu\big(x^1_{(3,1)}(1_{\mathbb{K}})\big)^{-1}\mu\big(x^1_{(3,1)}(s^{-1})\big) \ , \qquad s \in \mathbb{A}^* \ , \\ \tilde{h}(s) &:= \mu\big(x^1_{(1,2)}(1_{\mathbb{K}})\big)^{-1}\mu\big(x^1_{(1,2)}(s^{-1})\big) \ , \qquad s \in \mathbb{A}^* \ , \end{aligned}$$

we have

$$h(s) = \mu\big(x^1_{(3,1)}(1_{\mathbb{A}})\big)^{-1}\mu\big(x^1_{(3,1)}(s^{-1})\big) = \mu\big(x^1_{(1,2)}(1_{\mathbb{A}})\big)^{-1}\mu\big(x^1_{(1,2)}(\gamma_1(s)^{-1})\big) = \tilde{h}\big(\gamma_1(s)\big)$$

for each $s \in \mathbb{A}^*$,

$$x^3_{(2,3)}(t)^{h(s)} = x^3_{(3,1)}(t)^{h(s)} = x^3_{(3,1)}(t) = x^3_{(2,3)}(t \cdot s)$$

and

$$\begin{aligned} x^2_{(2,3)}(t)^{h(s)} &= x^2_{(1,2)}(t)^{\tilde{h}(\gamma_1(s))} \\ &= x^2_{(1,2)}\big(\gamma_1(s) \circ t \circ \gamma_1(s)\big) = x^2_{(2,3)}\big(\gamma_1(s) \cdot t \cdot \gamma_1(s)\big) \end{aligned}$$

for all $s \in \mathbb{A}^*,\ t \in \mathbb{A}$ by lemma (18.9). Given $s \in \mathbb{K}$, the Hua automorphism $h\big(\gamma_1^{-1}(s)\big)$ induces an automorphism $\alpha_s \in \mathrm{Aut}(\mathcal{B}_{(2,3)})$ which satisfies

$$x^2_{(2,3)}(t) \mapsto x^2_{(2,3)}(s \cdot t \cdot s) \ , \qquad\qquad x^3_{(2,3)}(t) \mapsto x^3_{(2,3)}\big(t \cdot \gamma_1^{-1}(s)\big) \ .$$

If we set $\tilde{\alpha} := (\mathrm{id}_{\mathbb{A}}, \rho_{\gamma_1(s)}, \rho_{\gamma_1(s)})$, then $\tilde{\alpha}\alpha_s$ satisfies

$$x^2_{(2,3)}(t) \mapsto x^2_{(2,3)}(s \cdot t \cdot s) \ , \qquad\qquad x^3_{(2,3)}(t) \mapsto x^3_{(2,3)}(t) \ .$$

By lemma (21.27), there is an element $c \in \mathbb{A}^*$ such that

$$\forall\, t \in \mathbb{A} : \qquad sts = ct \ .$$

Lemma (43.4) implies that we have

$$s \in Z(\mathbb{A}) \ .$$

Since $s \in \mathbb{A}$ is arbitrary, it follows that

$$\mathbb{A} \subseteq Z(\mathbb{A}) \qquad \lightning \ .$$

$\square$

(46.2) Proposition Let

$$\mathcal{F} := \{\mathcal{B}_{(1,2)} = \mathcal{T}(\hat{\mathbb{A}}), \mathcal{B}_{(2,3)} = \mathcal{T}(\mathbb{A}), \mathcal{B}_{(3,1)} = \mathcal{T}(\tilde{\mathbb{A}}), \gamma_{(1,2,3)}, \gamma_{(2,3,1)}, \gamma_{(3,1,2)}\}$$

be an integrable foundation of type $\tilde{A}_2$ such that the defining field is a non-commutative skew-field and such that each glueing is positive. Then the skew-field $\mathbb{A}$ is a quaternion division algebra and $\mathcal{F}$ is isomorphic to the foundation

$$\mathcal{P}_3^+(\mathbb{A}) := \{\tilde{\mathcal{B}}_{(1,2)} = \mathcal{T}(\mathbb{A}^o), \tilde{\mathcal{B}}_{(2,3)} = \mathcal{T}(\mathbb{A}), \tilde{\mathcal{B}}_{(3,1)} = \mathcal{T}(\mathbb{A}^o), \tilde{\gamma}_1 = \sigma_s, \tilde{\gamma}_2 = \mathrm{id}^o_{\mathbb{A}}, \tilde{\gamma}_3 = \mathrm{id}^o_{\mathbb{A}}\} \ .$$

Proof
As $\gamma_2 = \gamma_{(2,3,1)}$ is positive, we may take $(\mathbb{A}^o, \gamma_2^{-1}, \gamma_2^{-1}, \gamma_2^{-1})$ as reparametrization for $\mathcal{T}(\hat{\mathbb{A}})$. Therefore, we may assume

$$\hat{\mathbb{A}} = \mathbb{A}^o \ , \qquad\qquad \gamma_2 = \mathrm{id}^o_{\mathbb{A}_0} \ .$$

As $\gamma_3 = \gamma_{(2,3,1)}$ is positive, we may take $(\mathbb{A}^o, \gamma_3^o, \gamma_3^o, \gamma_3^o)$ as reparametrization for $\mathcal{T}(\tilde{\mathbb{A}})$. Therefore, we may assume

$$\tilde{\mathbb{A}} = \mathbb{A}^o \ , \qquad\qquad \gamma_3 = \mathrm{id}^o_{\mathbb{A}} \ .$$

- If we set

$$\begin{aligned} h_1(s) &:= \mu\big(x^1_{(3,1)}(1_{\mathbb{K}})\big)^{-1}\mu\big(x^1_{(3,1)}(s^{-1})\big) \ , && s \in \mathbb{A}^o \ , \\ \tilde{h}_1(s) &:= \mu\big(x^1_{(1,2)}(1_{\mathbb{K}})\big)^{-1}\mu\big(x^1_{(1,2)}(s^{-1})\big) \ , && s \in \mathbb{A}^o \ , \end{aligned}$$

  we have

$$\begin{aligned} h_1(s) &= \mu\big(x^1_{(3,1)}(1_{\mathbb{A}^o})\big)^{-1}\mu\big(x^1_{(3,1)}(s^{-1})\big) \\ &= \mu\big(x^1_{(1,2)}(1_{\mathbb{A}})\big)^{-1}\mu\big(x^1_{(1,2)}(\gamma_1(s)^{-1})\big) = \tilde{h}_1\big(\gamma_1(s)\big) \end{aligned}$$

  for each $s \in \mathbb{A}^o$,

$$x^3_{(2,3)}(t)^{h_1(s)} = x^3_{(3,1)}\big(\mathrm{id}^o_{\mathbb{A}}(t)\big)^{h_1(s)} = x^3_{(3,1)}\big(\mathrm{id}^o_{\mathbb{A}}(t) \circ s\big) = x^3_{(2,3)}\big(\mathrm{id}^o_{\mathbb{A}}(s) \cdot t\big)$$

  for all $s \in \mathbb{A}^o, \ t \in \mathbb{A}$ by lemma (18.9) and

$$\begin{aligned} x^2_{(2,3)}(t)^{h_1(s)} &= x^2_{(1,2)}\big(\mathrm{id}^o_{\mathbb{A}}(t)\big)^{\tilde{h}_1(\gamma_1(s))} \\ &= x^2_{(1,2)}\big(\gamma_1(s) \circ \mathrm{id}^o_{\mathbb{A}}(t) \circ \gamma_1(s)\big) = x^2_{(2,3)}\big(\gamma^o_1(s) \cdot t \cdot \gamma^o_1(s)\big) \end{aligned}$$

  for all $s \in \mathbb{A}^o, \ t \in \mathbb{A}$ by lemma (18.9).

- If we set

$$\begin{aligned} h_3(s) &:= \mu\big(x^3_{(3,1)}(1_{\mathbb{A}})\big)^{-1}\mu\big(x^3_{(3,1)}(s)\big) \ , && s \in \mathbb{A}^o \ , \\ \tilde{h}_3(s) &:= \mu\big(x^3_{(2,3)}(1_{\mathbb{A}})\big)^{-1}\mu\big(x^3_{(2,3)}(s)\big) \ , && s \in \mathbb{A} \ , \end{aligned}$$

  we have

$$h_3(s) = \mu\big(x^1_{(3,1)}(1_{\mathbb{A}^o})\big)^{-1}\mu\big(x^1_{(3,1)}(s^{-1})\big) = \mu\big(x^1_{(1,2)}(1_{\mathbb{A}})\big)^{-1}\mu\big(x^1_{(1,2)}(\mathrm{id}^o_{\mathbb{A}}(s^{-1}))\big) = \tilde{h}_3\big(\mathrm{id}^o_{\mathbb{A}}(s)\big)$$

  for each $s \in \mathbb{A}^o$,

$$\begin{aligned} x^3_{(2,3)}(t)^{h_3(s)} &= x^3_{(3,1)}\big(\mathrm{id}^o_{\mathbb{A}}(t)\big)^{h_3(s)} \\ &= x^3_{(3,1)}\big(s^{-1} \circ \mathrm{id}^o_{\mathbb{A}}(t) \circ s^{-1}\big) = x^3_{(2,3)}\big(\mathrm{id}^o_{\mathbb{A}}(s)^{-1} \cdot t \cdot \mathrm{id}^o_{\mathbb{A}}(s)^{-1}\big) \end{aligned}$$

  for all $s \in \mathbb{A}^o, \ t \in \mathbb{A}$ by lemma (18.9) and

$$x^2_{(2,3)}(t)^{h_3(s)} = x^2_{(2,3)}(t)^{\tilde{h}_3(\mathrm{id}^o_{\mathbb{A}}(s))} = x^2_{(2,3)}\big(\mathrm{id}^o_{\mathbb{A}}(s)^\sigma \cdot t \cdot \mathrm{id}^o_{\mathbb{A}}(s)\big)$$

  for all $s \in \mathbb{A}^o, \ t \in \mathbb{A}_0$ by lemma (18.9).

Given $s \in \mathbb{A}^o$, then $h_3(s)h_1(s)$ induces an automorphism $\alpha_s \in \mathrm{Aut}(\mathcal{B}_{(2,3)})$ which satisfies

$$x^2_{(2,3)}(t) \mapsto x^2_{(2,3)}(\mathrm{id}^o_{\mathbb{A}}(s) \cdot \gamma^o_1(s) \cdot t \cdot \gamma^o_1(s) \cdot \mathrm{id}^o_{\mathbb{A}}(s)) \ , \qquad x^3_{(2,3)}(t) \mapsto x^3_{(2,3)}(t \cdot \mathrm{id}^o_{\mathbb{A}}(s)^{-1}) \ .$$

If we set $\tilde{\alpha} := (\mathrm{id}_{\mathbb{A}^o}, \rho_{\mathrm{id}_{\mathbb{A}^o}(s)}, \rho_{\mathrm{id}_{\mathbb{A}^o}(s)})$, then $\tilde{\alpha}\alpha_s \in \mathrm{Aut}(\mathcal{B}_{(2,3)})$ satisfies

$$x^2_{(2,3)}(t) \mapsto x^2_{(2,3)}(\mathrm{id}^o_{\mathbb{A}}(s) \cdot \gamma^o_1(s) \cdot t \cdot \gamma^o_1(s) \cdot \mathrm{id}^o_{\mathbb{A}}(s)) \ , \qquad x^3_{(2,3)}(t) \mapsto x^3_{(2,3)}(t) \ .$$

By lemma (21.27), there is an element $c \in \mathbb{A}^*$ such that

$$\forall\, t \in \mathbb{A} : \qquad \mathrm{id}^o_{\mathbb{A}}(s) \cdot \gamma^o_1(s) \cdot t \cdot \gamma^o_1(s) \cdot \mathrm{id}^o_{\mathbb{A}}(s) = c \cdot t \ .$$

Lemma (43.4) implies that we have

$$\gamma^o_1(s) \cdot \mathrm{id}^o_{\mathbb{A}}(s) \in C_{\mathbb{A}}(\mathbb{A}) = Z(\mathbb{A}) \ .$$

Since $\mathbb{A}$ is non-commutative, lemma (5.6) (b) shows that $\mathbb{A}$ is a quaternion division algebra and that we have

$$\gamma_1 = \sigma_s : \mathbb{A}^o \to \mathbb{A}^o \ .$$

□

(46.3) Theorem Let $\mathcal{F}$ be an integrable foundation of type $\tilde{A}_2$ such that the defining field is a non-commutative skew-field and such that at least one glueing is positive. Then there is is a quaternion $\mathbb{H}$ such that

$$\mathcal{F} \cong \mathcal{P}^+_3(\mathbb{H}) = \{\tilde{\mathcal{B}}_{(1,2)} = \mathcal{T}(\mathbb{H}^o), \tilde{\mathcal{B}}_{(2,3)} = \mathcal{T}(\mathbb{H}), \tilde{\mathcal{B}}_{(3,1)} = \mathcal{T}(\mathbb{H}^o), \tilde{\gamma}_1 = \sigma_s, \tilde{\gamma}_2 = \mathrm{id}^o_{\mathbb{H}}, \tilde{\gamma}_3 = \mathrm{id}^o_{\mathbb{H}}\} \ .$$

Proof
This results from proposition (46.1) and proposition (46.2). □

(46.4) Remark As we did not use the fact that the residues embed into the building at infinity, which is a result in twin building theory, the above theorem is true for any foundation of an arbitrary affine building of type $\tilde{A}_2$ in which each Hua automorphism is induced by an automorphism of the whole building.

Chapter 47 Jordan Automorphisms of Octonion Division Algebras Revisited

We give a direct proof for proposition (28.9), independent of the characteristic.

(47.1) Theorem Let $\mathbb{O}$ be an octonion division algebra with center $\mathbb{K} := Z(\mathbb{O})$ and norm $N := N^{\mathbb{O}}_{\mathbb{K}}$. Then we have

$$\mathrm{Aut}_J(\mathbb{O}) = \Gamma L_N(\mathbb{O}, \mathbb{K}) \ .$$

Proof
Let $(\varphi, \sigma) \in \Gamma L_N(\mathbb{O}, \mathbb{K})$. Then we have

$$\varphi(1_{\mathbb{O}}) = \varphi(1_{\mathbb{O}} \cdot 1_{\mathbb{O}}) = \sigma(1_{\mathbb{O}}) \cdot \varphi(1_{\mathbb{O}}) = 1_{\mathbb{O}} \cdot \varphi(1_{\mathbb{O}})$$

and thus $\varphi(1_{\mathbb{O}}) = 1_{\mathbb{O}}$. By definition, lemma (31.23) and theorem (31.41), we have

$$\begin{aligned} \mathrm{Aut}_J(\mathbb{O}) &= \mathrm{Aut}_J\big(\mathbb{M}(\mathbb{O})\big) = \mathrm{Aut}_J\big(\mathbb{M}(\mathbb{O}, \mathbb{K}, N)\big) \\ &= \{(\varphi, \sigma) \in \Gamma L_N(\mathbb{O}, \mathbb{K}) \mid \varphi(1_{\mathbb{O}}) = 1_{\mathbb{O}}\} = \Gamma L_N(\mathbb{O}, \mathbb{K}) \ . \end{aligned}$$

□

Chapter 48 The Defining Field Revisited

With theorem (47.1), we can prove theorem (21.6) without using proposition (21.3).

(48.1) Lemma Let $\mathcal{F}$ be a foundation such that there exists an edge $(a,b) \in A(F)$ with $\mathbb{A} := \mathbb{A}_{(a,b)}$ an octonion division algebra. Then we have

$$\forall\ (i,j) \in A(F): \quad \mathbb{A}_{(i,j)} \cong \mathbb{A} \ \vee\ \mathbb{A}_{(i,j)} \cong \mathbb{A}^o\ , \quad \mathcal{T}(\mathbb{A}_{(i,j)}) \cong \mathcal{T}(\mathbb{A}) \ \vee\ \mathcal{T}(\mathbb{A}_{(i,j)}) \cong \mathcal{T}(\mathbb{A}^o)\ .$$

Proof
By theorem (47.1), each glueing $\gamma_{(i,j,k)}$ is a norm similarity. Therefore, we have

$$\forall\ (i,j,k) \in G(F): \qquad \mathbb{A}_{(i,j)} \cong \mathbb{A}_{(j,k)}$$

by theorem (1.7.1) of [Sp]. □

(48.2) Theorem Let $\tilde{\mathcal{F}}$ be an integrable foundation. Then there is an alternative division ring $\mathbb{A}$ such that

$$\forall\ (i,j) \in A(\tilde{F}): \qquad \tilde{\mathbb{A}}_{(i,j)} \cong \mathbb{A} \ \vee\ \tilde{\mathbb{A}}_{(i,j)} \cong \mathbb{A}^o\ .$$

Proof
This is an immediate consequence of corollary (19.33) and lemma (48.1). □

Chapter 49 Introduction in German

Historischer und theoretischer Hintergrund

Bei der folgenden Darstellung orientieren wir uns stark an den Ausführungen in [MLoc] und [AB].

Zwillingsgebäude

Gebäude wurden von J. Tits eingeführt, um halbeinfache algebraische Gruppen von einem geometrischen Standpunkt aus zu untersuchen. Eines der wichtigsten Resultate in der Gebäude-Theorie ist die Klassifikation der irreduziblen sphärischen Gebäuden vom Rang mindestens 3 in [T74]. Mittlerweile gibt es einen vereinfachten Beweis in [TW], der auf der Klassifikation der Moufang-Polygone beruht.

Vor über 25 Jahren definierten M. Ronan und J. Tits die Klasse der Zwillingsgebäude, eine natürliche Verallgemeinerung der sphärischen Gebäude. Motiviert wurde diese Definition durch die Theorie der Kac-Moody-Gruppen. Wir verweisen an diesem Punkt auf [T92] für allgemeine, weitergehende Informationen über Zwillingsgebäude.

Zwillingsgebäude verallgemeinern sphärische Gebäude in folgender Hinsicht: Bei sphärischen Gebäuden gibt es eine natürliche *Oppositions-Relation* auf der Menge der Kammern, die die Struktur des Gebäudes wesentlich einschränkt. Die oben erwähnte Klassifikation der irreduziblen sphärischen Gebäuden vom Rang mindestens 3 basiert letzten Endes genau auf dieser Oppositions-Relation. Ein Zwillingsgebäude besteht nun aus zwei verschieden Gebäuden gleichen Typs, auf deren Kammern eine symmetrische Relation eingeführt wird, die ähnliche Eigenschaften wie die Oppositions-Relation von sphärischen Gebäuden besitzt. Ein Zwillingsgebäude ist also ein Tripel bestehend aus zwei Gebäuden gleichen Typs und einer Oppositions-Relation auf den zwei „Hälften“ des Zwillingsgebäudes.

Das Klassifikations-Programm für 2-sphärische Zwillingsgebäude

Im Hinblick auf die Klassifikation der sphärischen Gebäude ergibt sich ganz natürlich die Frage, ob es auch möglich ist, Zwillingsgebäude höheren Ranges zu klassifizieren. Ein großer Teil von [T92] beschäftigt sich mit genau dieser Problemstellung. Zunächst stellt sich heraus, dass eine solche Klassifikation nur unter der zusätzlichen Annahme möglich ist, dass die Einträge der zugehörigen Coxeter-Matrizen endlich sind. Wir nennen diese Gebäude *2-sphärisch*. Das in [T92] beschriebene Klassifikations-Programm basiert auf der Vermutung, dass es für jede 2-sphärische Coxeter-Matrix vom Typ M eine Bijektion zwischen Zwillingsgebäuden vom Typ M und bestimmten Moufang-Fundamenten vom Typ M gibt.

Fundamente wurden von M. Ronan und J. Tits in [RT] eingeführt, um Kammer-Systeme zu beschreiben, die Kandidaten für die lokale Struktur eines Gebäudes sind. Grob gesagt sind Fundamente Amalgame von Rang-2-Gebäuden, die entlang bestimmter Rang-1-Residuen verklebt sind. Ist c eine Kammer eines Gebäudes $\mathcal{B}$ vom Typ M, so bildet die Vereinigung $E_2(c)$ der Rang-2-Residuen, die diese Kammer c enthalten, ein Fundament vom Typ M, das *Fundament von $\mathcal{B}$ in c*. Der Ausdruck „lokale Struktur“ ist also als eine Art 2-Umgebung einer Kammer c des Gebäudes zu verstehen.

Es ist eine (nicht triviale) Tatsache, dass die Fundamente zweier Kammern in derselben Hälfte eines Zwillingsgebäudes isomorph sind. Darüber hinaus ist der Isomorphie-Typ des Fundamentes der einen Hälfte durch den Isomorphie-Typ des Fundaments der anderen Hälfte eindeutig bestimmt. Umgekehrt besagt eine Verallgemeinerung von Tits’ Erweiterungs-Satz durch B. Mühlherr und M. Ronan in [MR], dass ein Zwillingsgebäude in fast allen Fällen durch das Fundament einer seiner Hälften eindeutig bestimmt ist, siehe (5.10), (*5.11), (*9.11) und (*9.12) in [AB] für eine Zusammenfassung. Somit ist das Fundament in einer Kammer eine klassifizierende Invariante des zugehörigen Zwillingsgebäudes, falls die folgende Bedingung erfüllt ist:

(CO) Kein Rang-2-Residuum ist isomorph zu einem Gebäude, das zu einer der Gruppen $B_2(2)$, $G_2(2)$, $G_2(3)$ und ${}^2F_4(2)$ gehört.

Diese Bedingung garantiert, dass die Menge c^o der einer Kammer $c \in \mathcal{B}_\epsilon$ ($\epsilon \in \{\pm\}$) gegenüber liegenden Kammern stets eine Galerie-zusammenhängende Teilmenge von $\mathcal{B}_{-\epsilon}$ ist.

In Anbetracht der bisherigen Überlegungen reduziert sich die Klassifikation der 2-sphärischen Zwillingsgebäude also auf die Klassifikation aller Fundamente, die als lokale Struktur eines Zwillingsgebäudes realisiert werden können. Wir nennen ein solches Fundament *integrierbar*. Bei der Bestimmung der integrierbaren Fundamente verfährt man in zwei Schritten.

Schritt 1: Herausfiltern der nicht integrierbaren Fundamente

In [T92] wird bewiesen, dass ein integrierbares Fundament Moufang ist, die Rang-2-Gebäude also Moufang-Polygone sind, deren Verklebungen mit den induzierten Moufang-Mengen auf den Rang-1-Residuen kompatibel sind. Eine erste notwendige Bedingung für die Integrierbarkeit eines Fundaments ist also die Moufang-Eigenschaft.

Als Folge dessen sind die Klassifikation der Moufang-Polygone in [TW] und die Lösung des Isomorphie-Problems für Moufang-Mengen grundlegend bei der Untersuchung, welche Moufang-Polygone zu einem Fundament zusammengefügt werden können. Zudem kann man die Liste der möglicherweise integrierbaren Fundamente weiter einschränken, indem man bestimmte Automorphismen des Zwillingsgebäudes betrachtet, die sogenannten *Hua-Automorphismen*, die in einem engen Zusammenhang mit den Doppel-μ-Maps der auftauchenden Moufang-Mengen stehen.

Schritt 2: Existenz- / Integrierbarkeits-Beweis

Schließlich muss man beweisen, dass jeder der verbleibenden Kandidaten auch wirklich integrierbar ist. Das zugehörige Zwillingsgebäude ist dann bis auf Isomorphie eindeutig. In [MLoc] und seiner Habilitationsschrift [MHab] entwickelte B. Mühlherr Techniken, die bestimmte Zwillingsgebäude als Fixpunktmenge in zu Kac-Moody-Gruppen gehörenden Zwillingsgebäuden realisieren. Er, H. Petersson und R. Weiss bereiten momentan ein Buch vor, das weitergehende, substanzielle Hintergründe liefert.

Ziele und Ergebnisse

Diese Arbeit widmet sich der Erstellung vollständiger Listen integrierbarer Fundamente für bestimmte Diagramm-Typen. Wir folgen hierbei dem Ansatz für die Klassifikation sphärischer Gebäude in [TW], wobei wir jedoch die dort verwendeten Techniken verfeinern müssen, da Fundamente im Allgemeinen nicht nur vom zugehörigen Diagramm und dem definierenden (Alternativ-) Körper abhängen. Zum Beispiel gibt es für einen festen Schiefkörper $\mathbb{A}$ in der Regel mehrere nicht isomorphe Fundamente vom Typ $\tilde{A}_n$: Automorphismen von $\mathbb{A}$ spielen ebenfalls eine Rolle, was der Tatsache Rechnung trägt, dass es mehrere Möglichkeiten gibt, Moufang Polygone entlang eines Rang-1-Residuums zu verkleben.

Ein wesentlicher Aspekt ist die passende Parametrisierung von Sequenzen von Moufang-Polygonen bzw. deren Wurzelgruppen-Sequenzen mit den zugehörigen Kommutator-Relationen, um die verschiedenen Verklebungen sichtbar zu machen. Die entscheidende Subtilität ist die folgende: Jedes Moufang-Polygon wird zweimal parametrisiert, einmal für jede Richtung, in der die zugehörige Wurzelgruppen-Sequenz gelesen werden kann. Folglich erhalten wir Verklebungen zwischen gerichteten Moufang-Polygonen, und es macht einen Unterschied, ob wir $\mathrm{id}_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$ oder $\mathrm{id}_{\mathbb{A}}^o : \mathbb{A} \to \mathbb{A}^o$ betrachten, wobei $\mathbb{A}^o$ der zu $\mathbb{A}$ entgegengesetzte Schiefkörper ist: Ersteres ist ein Isomorphismus, während Letzteres ein Anti-Isomorphismus von Schiefkörpern ist.

Wie bereits erwähnt, ist das Herausfiltern nicht integrierbarer Fundamente eng verknüpft mit der Betrachtung von Moufang-Mengen und deren Isomorphismen. Deshalb beschäftigt sich ein großer Teil dieser Arbeit mit der Einführung der zugrundeliegenden Parameter-Systeme und der Lösung des Isomorphie-Problems für Moufang-Mengen. Viele Probleme wurden bereits gelöst, siehe [K], aber wir müssen die existierenden Ergebnisse für unsere Anforderungen sowohl verfeinern als auch erweitern und übersetzen deren Beweise in unser Setup.

Fundamente mit einfachen Kanten (Simply Laced Foundations)

Das Hauptergebnis dieser Arbeit ist die vollständige Klassifikation der Zwillingsgebäude mit einfachen Kanten via ihrer Fundamente. Natürlich ist die Hauptvoraussetzung für ein integrierbares Fundament die Moufang-Eigenschaft: Die Verklebungen sind Jordan-Isomorphismen, d.h., sie sind mit dem Jordan-Produkt xyx verträglich.

Ein mächtiges Werkzeug ist der Satz von Hua, siehe [H] für eine Referenz, der das Isomorphie-Problem für Moufang-Mengen von Schiefkörpern löst: Jeder Jordan-Isomorphismus ist letztlich ein Iso- oder Anti-Isomorphismus von Schiefkörpern. Leider beinhaltet die Klasse von Parameter-Systemen für Moufang-Dreiecke zusätzlich Oktaven-Divisionsalgebren, was wegen der fehlenden Assoziativität zu einem gewissen Mehraufwand führt. Ein Nebenprodukt ist die Existenz von Jordan-Isomorphismen, die weder Iso- noch Anti-Isomorphismen von alternativen Ringen sind. Der aufwändigste Teil handelt von den Ausnahmefällen, in denen Oktaven auftauchen.

Wir geben an dieser Stelle einen Überblick über den Klassifikations-Prozess und streichen die Hauptideen heraus. Die folgenden Beobachtungen liefern die erste Einschränkung an Möglichkeiten:

(1) Jedes Moufang-Dreieck ist über demselben alternativen Divisionsring $\mathbb{A}$ definiert.

(2) Ein integrierbares Fundament vom Typ A_3 ist notwendigerweise über einem Schiefkörper definiert, und die Verklebung ist notwendigerweise ein Isomorphismus von Schiefkörpern.

Folglich ist der entscheidende Schritt die Klassifikation der integrierbaren Fundamente vom Typ $\tilde{A}_2$, da dies die kleinsten sind, bei denen „Nicht-Standard"-Phänomene auftreten können. Die Theorie affiner Gebäude, von Bruhat-Tits-Gebäuden sowie die Theorie von Kompositions-Algebren, die bzgl. einer diskreten Bewertung komplett sind, ermöglichen uns weitere Einschränkungen:

(3) Zu einer Oktaven-Divisionsalgebra $\mathbb{O}$ gibt es genau ein Zwillingsgebäude vom Typ $\tilde{A}_2$.

(4) Ein integrierbares Fundament vom Typ $\tilde{A}_2$, dessen Verklebungen Anti-Isomorphismen sind, ist notwendigerweise über einer Quaternionen-Divisionsalgebra definiert, und zu einem Quaternionen-Schiefkörper $\mathbb{H}$ gibt es genau ein solches „positives" Zwillingsgebäude vom Typ $\tilde{A}_2$.

Eine genauere Betrachtung der Gruppe der Jordan-Automorphismen einer Oktaven-Divisionsalgebra hilft dabei, die Klassifikation der über Oktaven definierten Zwillingsgebäude abzuschließen:

(5) Es gibt keine integrierbaren Fundamente über Oktaven, deren zugehöriger Graph ein Tetraeder ist. Insbesondere sind $\mathcal{A}_2(\mathbb{O}) = \mathcal{T}(\mathbb{O})$ und $\tilde{\mathcal{A}}_2(\mathbb{O})$ die einzigen integrierbaren Fundamente über einer Oktaven-Divisionsalgebra $\mathbb{O}$.

Schließlich liefert die folgende Beobachtung in Verbindung mit (4) eine wesentliche Einschränkung der Liste integrierbarer Fundamente über echten Schiefkörpern, die keine Quaternionen-Schiefkörper sind:

(6) Ein integrierbares Fundament vom Typ D_4 ist notwendigerweise über einem Körper definiert.

Als Konsequenz sind über echten Schiefkörpern, die keine Quaternionen-Schiefkörper sind, nur Fundamente integrierbar, deren Diagramm ein Kreis, ein String, ein Strahl oder eine Kette ist und deren Verklebungen Isomorphismen von Schiefkörpern sind.

Schließlich stellt die Kac-Moody-Theorie die Integrierbarkeit sicher, solange das zugehörige Coxeter-Diagramm ein Baum ist. Die restlichen Integrierbarkeits-Beweise basieren auf Techniken, die von B. Mühlherr entwickelt wurden.

Jordan-Automorphismen von alternativen Divisionsringen

In Hinblick auf den Satz von Hua, nach dem

$$\mathrm{Aut}_J(\mathbb{D}) = \mathrm{Aut}(\mathbb{D}) \cup \mathrm{Aut}^o(\mathbb{D})$$

für jeden Schiefkörper $\mathbb{D}$, seine Gruppe $\mathrm{Aut}_J(\mathbb{D})$ von Jordan-Automorphismen, seine Untergruppe $\mathrm{Aut}(\mathbb{D})$ von Automorphismen und seine Menge $\mathrm{Aut}^o(\mathbb{D})$ von Anti-Automorphismen gilt, stellt sich die Frage nach einem ähnlichen Ergebnis für Oktaven-Divisionsalgebren.

Im Beweis, dass integrierbare Tetraeder-Fundamente über Oktaven nicht existieren, definieren wir eine Teilmenge $\Gamma \subseteq \mathrm{Aut}_J(\mathbb{O})$, für die sich herausstellt, dass sie nicht die Standard-Involution σ_s enthält. Die Elemente von Γ sind Automorphismen von $\mathbb{O}$, die zusätzlich mit einem der in [TW] definierten „Ausnahme"-Jordan-Automorphismen multipliziert werden. Diese fixieren eine Quaternionen-Unteralgebra $\mathbb{H}$ punktweise und wirken auf dem orthogonalen Komplement von $\mathbb{H}$ als Konjugation.

Dass Γ eine Untergruppe von $\mathrm{Aut}_J(\mathbb{O})$ ist, kann man aus der Kenntnis der Automorphismen-Gruppe des zugehörigen Moufang-Dreiecks $\mathcal{T}(\mathbb{O})$ ableiten. Diese Untergruppe Γ entspricht der Untergruppe $\mathrm{Aut}(\mathbb{D})$ im Satz von Hua, d.h., wir erhalten

$$\mathrm{Aut}_J(\mathbb{O}) = \langle \sigma_s, \Gamma \rangle = \Gamma \cup \sigma_s \Gamma\ .$$

Die Strategie beim Beweis ist wie folgt:

(1) Jordan-Automorphismen eingeschränkt auf Unterkörper sind Ring-Monomorphismen, d.h., das Bild eines Unterkörpers ist wieder ein Unterkörper.

(2) Als unmittelbare Konsequenz ergibt sich, dass Jordan-Automorphismen von Oktaven Norm-Ähnlichkeiten sind.

(3) Die Ergebnisse aus [Sp] erlauben uns, die Problemstellung auf Isometrien zurückzuführen, die eine Quaternionen-Unteralgebra punktweise fixieren.

(4) Der Satz von Hua und das Skolem-Noether-Theorem erlauben uns zu zeigen, dass jeder Jordan-Automorphismus wirklich ein Produkt in $\langle \sigma_s, \Gamma \rangle$ ist.

443-Fundamente

Das zweite Ergebnis im Zusammenhang mit der Klassifikation der Zwillingsgebäude ist die Durchführung von Schritt 1 für 443-Fundamente. Dies sind Fundamente, deren Diagramm ein Dreieck ist und deren Moufang-Polygone zwei Vierecke und ein Dreieck sind. Obwohl wir uns in diesem Fall mit nur einem einzigen Diagramm beschäftigen, gibt es dennoch eine Vielzahl verschiedener integrierbarer 443-Fundamente, da es sechs Familien von Moufang-Vierecken gibt, die in dieser Konfiguration zudem auch noch oft zusammenpassen. Allerdings treten keine Vierecke vom Typ E_n, vom Typ F_4 und vom indifferenten Typ auf, weil ihre Moufang-Mengen nicht vom linearen Typ, also keine projektiven Geraden sind.

Das Gleiche gilt zwar für Moufang-Mengen vom pseudo-quadratischen und involutorischen Typ, allerdings ist das zweite Panel des zugehörigen *unitären* Vierecks vom linearen Typ, sodass es genau eine Möglichkeit für die Orientierung des Vierecks gibt. Die Lösung des Isomorphie-Problems für die auftauchenden Moufang-Mengen und die Kenntnis der Automorphismen-Gruppe eines unitären Vierecks führen zu folgendem Ergebnis:

(1) Die auftauchenden pseudo-quadratischen Räume sind über einem Quaternionen-Schiefkörper $\mathbb{H}$ oder einer separablen quadratischen Erweiterung $\mathbb{E}$ definiert. Im ersten Fall gibt es genau ein integrierbares 443-Fundament über einem solchen pseudo-quadratischen Raum Ξ; im zweiten Fall hängt die Isomorphie-Klasse eines integrierbaren Fundaments zusätzlich von einem Automorphismus $\gamma \in \mathrm{Aut}(\mathbb{E})$ ab.

(2) Die auftauchenden involutorischen Mengen sind über einem Quaternionen-Schiefkörper $\mathbb{H}$ definiert, und es gibt genau ein integrierbares 443-Fundament über einer solchen involutorischen Menge Ξ.

Schließlich bilden Vierecke vom quadratischen Typ die flexibelste Familie, da es Moufang-Mengen gibt, die sowohl vom quadratischen als auch vom linearen Typ sind, sodass diese Vierecke in jeder Orientierung verklebbar sind. Des Weiteren gibt es eine Stelle, an der wir uns auf echte quadratische Räume als parametrisierende Strukturen beschränken müssen, um für eine zufriedenstellende Darstellung Charakteristik-2-Phänomene ausschließen zu können.

Im Gegensatz zur Klassifikation der integrierbaren Fundamente mit einfachen Kanten verzichten wir im Rahmen dieser Arbeit jedoch auf Schritt 2 des Klassifikations-Programms, da die Integrierbarkeits-Beweise andersartige, von B. Mühlherr, H. Petersson und R. Weiss eingeführte Techniken verwenden. Wie zuvor gibt es zwei Möglichkeiten, die Integrierbarkeit eines gegebenen Fundamentes zu beweisen: Entweder ist die universelle Überlagerung isomorph zu einem kanonischen Fundament, also einem Fundament, dessen Verklebungen alle die Identität sind und das integrierbar ist, falls das zugehörige Diagramm ein Baum ist, oder das Fundament kann als Fixpunkt-Struktur eines Tits-Index realisiert werden. Die erste Methode funktioniert bei 443-Fundamenten mit Vierecken vom quadratischen Typ, während die zweite bei 443-Fundamenten mit unitären Vierecken angewendet wird.

Jordan-Isomorphismen von pseudo-quadratischen Räumen

Genauso, wie der Satz von Hua bei der Klassifikation der integrierbaren Fundamente mit einfachen Kanten entscheidend ist, ist die Lösung des Isomorphie-Problems für die auftauchenden Moufang-Mengen ein wesentlicher Bestandteil bei der Klassifikation der integrierbaren 443-Fundamente. Wie bereits erwähnt, wurden viele Fälle von R. Knop in seiner Dissertation [K] abgehandelt. Allerdings beschäftigt er sich nur mit kommutativen Moufang-Mengen, sodass wir die entsprechenden Resultate für Moufang-Mengen vom pseudo-quadratischen Typ ergänzen müssen.

Wir erhalten, dass Jordan-Isomorphismen zwischen zwei Moufang-Mengen vom pseudo-quadratischen Typ in der Regel von Isomorphismen zwischen den zugehörigen pseudo-quadratischen Räumen induziert werden. Genauer ist dies immer der Fall, wenn die Dimension mindestens 3 ist oder die beteiligte involutorische Menge echt ist. Folglich tauchen Ausnahmen nur bei pseudo-quadratischen Räumen kleiner Dimension auf, die über einem Quaternionen-Schiefkörper oder einer separablen quadratischen Erweiterung definiert sind. Glücklicherweise tauchen diese Ausnahme-Fälle nicht bei der Klassifikation integrierbarer 443-Fundamente auf, sodass beide Vierecke über demselben pseudo-quadratischen Raum Ξ definiert sind.

Ausblick und offene Probleme

Jordan-Isomorphismen

In der Theorie der Moufang-Mengen spielen die μ-Maps und die Hua-Maps eine entscheidende Rolle, da sie viele Informationen enthalten. Folglich stehen Jordan-Isomorphismen – dies sind additive Isomorphismen, die mit den Hua-Maps verträglich sind – in engem Zusammenhang mit Isomorphismen von Moufang-Mengen. Da die Hua-Maps in Summen und der Permutation τ dargestellt werden können, ist jeder Isomorphismus von Moufang-Mengen letztlich auch ein Jordan-Isomorphismus.

In diesem Zusammenhang stellt sich ganz natürlich die Frage, ob jeder Jordan-Isomorphismus auch ein Isomorphismus von Moufang-Mengen ist. Die Hua-Maps von scharf 2-fach transitiven Moufang-Mengen sind trivial, sodass die Frage in diesem Fall natürlich negativ beantwortet werden muss. Experten auf diesem Gebiet wie R. Weiss und T. De Medts gehen aber davon aus, dass beide Definitionen äquivalent sind, solange man sich auf „echte“ Moufang-Mengen beschränkt.

Das Klassifikations-Programm

Die Hauptvermutung im Zusammenhang mit dem Klassifikations-Programm ist die folgende, siehe Seite 5 in [MHab]:

> Ein Moufang-Fundament vom 2-sphärischen Typ ist genau dann integrierbar, falls jedes Rang-3-Residuum integrierbar ist.

In seiner Habilitationsschrift [MHab] deutet B. Mühlherr an, dass die Vermutung unter der zusätzlichen Annahme beweisbar wäre, dass alle Rang-3-Residuen sphärisch sind, was natürlich eine starke Einschränkung ist. Allerdings gibt es bislang noch keinen veröffentlichten Beweis.

Sobald die Vermutung bewiesen ist, reduziert sich das Klassifikations-Programm auf die Klassifikation der integrierbaren Moufang-Fundamente vom Rang 3. Die meisten Fälle können mit den in [MHab] und [MLoc] eingeführten Methoden abgehandelt werden. Allerdings gibt es ein paar Ausnahmen, von denen die kompliziertesten vom Typ $\tilde{C}_2$, $\tilde{A}_2$ und 443-Fundamente sind. Der $\tilde{A}_2$- und der 443-Fall werden in dieser Arbeit gelöst, während es für den $\tilde{C}_2$-Fall (unveröffentlichte) Teilergebnisse von T. De Medts, B. Mühlherr, H. Van Maldeghem und R. Weiss gibt.

Die Klassifikation der Zwillingsgebäude mit einfachen Kanten

Obwohl die Klassifikation der integrierbaren Fundamente mit einfachen Kanten abgeschlossen ist, machen wir keine Aussage darüber, ob zwei Fundamente unserer Liste isomorph sind. Verwendet man klassifizierende Invarianten und führt passende Parameter ein, kann man eine Liste paarweise nicht isomorpher Fundamente erstellen.

Ist das zugehörige Coxeter-Diagramm $\mathcal{G}_F$ ein Baum, so hängt das Fundament $\mathcal{F}$ nur vom definierenden Körper ab. Kreise im Diagramm sorgen für eine zusätzliche Abhängigkeit von „Twists“, also von Automorphismen des definierenden Körpers $\mathbb{A}$. Genauer:

- Ist $\mathbb{A}$ ein Körper, so ist ein integrierbares Fundament $\mathcal{F}$ durch $\mathcal{G}_F$ und einen Homomorphismus $\varphi : \Pi_1(\mathcal{G}_F) \to \mathrm{Aut}(\mathbb{A})/\mathrm{Inn}(\mathbb{A}) \cong \mathrm{Aut}(\mathbb{A})$ eindeutig bestimmt, wobei $\Pi_1(\mathcal{G}_F)$ die Fundamentalgruppe von $\mathcal{G}_F$ ist.

- Ist $\mathbb{A}$ ein Schiefkörper, aber kein Quaternionen-Schiefkörper, und $\mathcal{F}$ ein integrierbares Fundament vom Typ $\tilde{A}_n$, so ist das Fundament durch n und ein Element von $\mathrm{Aut}(\mathbb{A})/\mathrm{Inn}(\mathbb{A})$ eindeutig bestimmt.

- Ist der definierende Körper ein Quaternionen-Schiefkörper, so gilt ein ähnliches Resultat wie für einen Körper.

Zudem könnte man die Integrierbarkeits-Beweise in einigen Punkten überarbeiten, sobald die angewandte Theorie von B. Mühlherr, H. Petersson und R. Weiss vollständig entwickelt wurde.

Endliche Moufang-Fundamente

Die eingeführte Terminologie und die Methoden von [MHab] können verwendet werden, um zu zeigen, dass jedes lokal endliche Zwillingsgebäude vom 2-sphärischen Typ das Fixpunkt-Gebäude einer Galois-Wirkung im Sinne von B. Rémy ist, was bedeutet, dass es algebraischen Ursprungs ist.

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Zunächst möchte ich meinem Betreuer Bernhard Mühlherr meinen ganz besonderen Dank dafür aussprechen, dass er meine Aufmerksamkeit auf das interessante Gebiet der Moufang-Fundamente und deren Moufang-Mengen gelenkt hat. Es war mir eine Freude, meinen Teil zum Klassifikations-Programm für Zwillingsgebäude beisteuern zu können. Viele ergiebige Diskussionen zeigten mir die richtigen Ansätze, also die möglicherweise zu beweisenden Aussagen und die zugehörigen Beweisideen. Seine Intuition ist beeindruckend.

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Ralf Köhl erweiterte die Arbeitsgruppe um viele nette Leute und gab mir die Möglichkeit, an einem Forschungsprojekt über kompakte Untergruppen von Kac-Moody-Gruppen teilzunehmen. Seine Begeisterung und seine Hingabe sind bewundernswert und bilden die Grundlage für unsere gedeihende Arbeitsgruppe.

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Selbständigkeitserklärung

Ich erkläre: Ich habe die vorgelegte Dissertation selbständig und ohne unerlaubte fremde Hilfe und nur mit den Hilfen angefertigt, die ich in der Dissertation angegeben habe.

Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen sind, und alle Angaben, die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht.

Bei den von mir durchgeführten und in der Dissertation erwähnten Untersuchungen habe ich die Grundsätze guter wissenschaftlicher Praxis, wie sie in der "'Satzung der Justus-Liebig-Universität Gießen zur Sicherung guter wissenschaftlicher Praxis"' niedergelegt sind, eingehalten.

Nidderau, im September 2013