

A Remark on Quantum Brownian Motion⁰⁾

Joachim A. Messer¹⁾²⁾

Institut für Theoretische Physik der Justus-Liebig-Universität Giessen, Heinrich-Buff-Ring 16, D-35392 Giessen, Germany

Abstract

An implicit exact solution is found for the semiclassical limit of Roumen Tsekov's quantum Smoluchowski equation for the harmonic oscillator.

PACS numbers: 05.40.-a, 03.65.Yz, 02.30.Hq

Last year Roumen Tsekov [1] derived a solution of his quantum Smoluchowski equation for the quantum harmonic oscillator, which is a Gaussian distribution with dispersion given by Equ. (3) of [1] (see also [2]):

$$\frac{\partial}{\partial t} \sigma^2 = 2D \left(1 + \sigma^2 \int_0^\beta d\beta \frac{\hbar^2}{4m\sigma^4} - \beta m \omega_0^2 \sigma^2 \right), \quad (1)$$

where $\beta > 0$ is the inverse temperature of the bath, ω_0 the frequency of the oscillator and $D > 0$ is the Einstein diffusion constant. For small displacement $\lambda^2 = \frac{\hbar^2 \beta}{4m}$ we approximate this equation by

$$\frac{\partial}{\partial t} \sigma^2 = 2D \left(1 + \frac{\lambda^2}{\sigma^2} - \beta m \omega_0^2 \sigma^2 \right). \quad (2)$$

At first we found a stationary solution, i.e. if $\frac{\partial}{\partial t} \sigma^2 = 0$. This gives an algebraic quadratic equation with only one positive solution σ_∞ :

⁰⁾ Dedicated to the memory of Professor Dr. Carl Friedrich Freiherr von Weizsäcker

¹⁾ Privatdozent für Theoretische Physik an der Ludwig-Maximilians-Universität München

²⁾ E-Mail: Joachim.A.Messer@theo.physik.uni-giessen.de

$$\sigma_{\infty}^2 = \frac{1}{2\chi^2} \left(1 + \sqrt{4\lambda^2\chi^2 + 1} \right) \quad (3)$$

For the time-dependent solution we integrate

$$\int \frac{\sigma^2 d\sigma^2}{(\sigma^2 + \lambda^2 - \chi^2 \sigma^4)} = 2D \int dt = 2Dt + C$$

where $\chi^2 = \beta m \omega_0^2$, and thus

$$\int \frac{z dz}{(z + \lambda^2 - \chi^2 z^2)} = \int \frac{z dz}{\left[\left(\lambda^2 + \frac{1}{4\chi^2} \right) - \left(\chi z - \frac{1}{2\chi} \right)^2 \right]}$$

This integral consists of two summands. The first is, up to a constant factor given by

$$\int \frac{d\left(\chi z - \frac{1}{2\chi}\right)}{\left(\lambda^2 + \frac{1}{4\chi^2}\right) - \left(\chi z - \frac{1}{2\chi}\right)^2} = \int \frac{dy}{\psi - y^2} = -\frac{1}{2\sqrt{\psi}} \ln C \frac{y - \sqrt{\psi}}{y + \sqrt{\psi}}$$

The second integral summand is up to a constant factor

$$\int \frac{y dy}{(\psi - y^2)}; y = \chi z - \frac{1}{2\chi}$$

with $\psi = \lambda^2 + \frac{1}{4\chi^2}$.

However we know that

$$\int \frac{y dy}{(\psi - y^2)} = -\frac{1}{2} \ln C (y^2 - \psi)$$

Therefore

$$2Dt + C = \int \frac{1}{\chi^2} \frac{(\chi\sigma^2 - \frac{1}{2\chi})d(\chi\sigma^2 - \frac{1}{2\chi})}{\left[\psi - (\chi\sigma^2 - \frac{1}{2\chi})\right]} + \int \frac{1}{2\chi^3} \frac{d(\chi\sigma^2 - \frac{1}{2\chi})}{\left[\psi - (\chi\sigma^2 - \frac{1}{2\chi})\right]}$$

The implicit solution is

$$Const. \cdot e^{2\chi^2 Dt} = \frac{\left[\bar{y} + \sqrt{\psi}\right]^{a-1}}{\left[\bar{y} - \sqrt{\psi}\right]^a} \quad (4)$$

with $a = \frac{1}{2} \left(1 + \frac{1}{\sqrt{4\lambda^2 \chi^2 + 1}}\right)$

and $\bar{y} = \chi\sigma^2 - \frac{1}{2\chi}$. One sees easily that the solution,

independent of the initial values tend for $t \rightarrow \infty$ exactly to σ_∞^2 , because $\bar{y} \rightarrow \sqrt{\psi}$ necessarily.

If at time $t=0$ the inertial width is σ_0 we calculate the integration constant *Const.* to be in every case equal to

$$Const. = \frac{(\sigma_0^2 + \sigma_\infty^2 - \frac{1}{\chi^2})^{a-1}}{\chi(\sigma_0^2 - \sigma_\infty^2)^a} . \quad (5)$$

Thus the final implicit solution is

$$e^{2\chi^2 Dt} = \frac{(\sigma_0^2 - \sigma_\infty^2)^a (\sigma^2 + \sigma_\infty^2 - \frac{1}{\chi^2})^{a-1}}{(\sigma_0^2 + \sigma_\infty^2 - \frac{1}{\chi^2})^{a-1} (\sigma^2 - \sigma_\infty^2)^a} . \quad (6)$$

Most recently, this solution, which already having been presented in [3], has been quoted in [4], where also the starting differential equation (2) appears in a similar context.

References:

- [1] Roumen Tsekov: *Comment on ‘Semiclassical Klein-Kramers and Smoluchowski equations for the Brownian motion of a particle in an external potential’*, J. Phys. A: Math. Theor., **40** (2007), 10945-10947
- [2] Roumen Tsekov: *Dissipation in quantum systems*, J. Phys. A: Math. Gen., **28** (1995) L557 – L561
- [3] Joachim A. Messer: *A Remark on Nuclear Friction*, <http://www.ptjm.org> (5th March 2008)
- [4] Roumen Tsekov: *Thermo-Quantum Diffusion*, arXiv:0803.4409 (31st March 2008)

Acknowledgements:

Thanks to Professor Roumen Tsekov for interesting and helpful exchange of ideas.