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Symmetry reductions related to specific nonlinear models

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Abstract

The purpose of this work is to explore possible side conditions involving high order differential invariants with the aim of reducing the original mathematical model. It is shown that, in the case of the Tzitzeica curve equation, a suitable side condition that leads to exact solutions is a system consisting of two third order differential invariants involving the arbitrary functions and a third order differential invariant involving the dependent variable of the equation. In this situation, the equation can be reduced to a linear equation for the equation's constant. Similarly, in the case of a proposed generalization of the Tzitzeica curve equation, the above side condition also leads to a reduced model.

1 Introduction

There is no general theory for solving nonlinear differential equations, and, therefore, this is one of the most difficult tasks. Symmetry analysis is one of the most powerful theories that may be applied to determine particular exact solutions to a mathematical model represented by a nonlinear differential equation. Introduced by Sophus Lie [9], the symmetry analysis theory is based on continuous groups of transformations (called today *Lie groups of transformations*) that leave the differential structure of the equation invariant and may be used to reduce the original model. More exactly, if the original model is represented by an ordinary differential equation (ODE)

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that is invariant under a one-parameter Lie group of transformations, then the order of the equation can be lowered by one or if the model is given by a partial differential equation (PDE), then this can be reduced to a dimensionless model. In this context, the corresponding Lie groups of transformations are named *symmetry reductions*. Once these are determined, they can be used to determine particular exact solutions of the model. Another way of finding exact solutions of the model is to consider *side conditions*. These are differential equations augmented to the original model that would allow the reduction of the its equation. The choice of the side conditions it may be randomly but this is not a good technique because the selected auxiliary equation may not reduce the original model.

In this paper, we propose a technique for finding compatible side conditions, namely, differential equations involving higher order differential invariants. We will focus on nonlinear models involving arbitrary functions. This idea is based on the fact that, in the symmetry analysis theory, a PDE is augmented with an auxiliary first order linear PDE (called *invariant surface condition*) which is, in fact, a first order linear PDE involving the coefficients of the infinitesimal generator related to the corresponding symmetry reduction. The invariant surface condition is invariant under the symmetry reduction itself, and, therefore, it is a first order invariant. Therefore, the main question is "How one can use higher order differential invariants to construct side conditions for a nonlinear model?". To answer this question, we will consider models described by differential equations involving arbitrary functions and will look for suitable side conditions that will lead to the reduction of the order or of the dimension of the model.

In a recent work [2], it has been shown that particular solutions of the Tzitzeica curve equation may be found if this equation is augmented with a system of three third order homogeneous linear ODEs with constant coefficients. In this paper, we show that this particular side condition is suitable for the Tzitzeica curve equation because the original equation is invariant with respect to translations in the t-space and the side condition itself consists of third order differential invariants related to this group of transformations. In addition, we will introduce a generalization of the Tzitzeica curve equation and show that the above side condition is also compatible with our proposed generalization.

The paper is structured as follows. In Section 2, we discuss the Tzitzeica curve equation and its generalization. Specific suitable side conditions are discussed in Section 3. The last section is reserved for conclusions.

2 The Tzitzeica curve equation

Gheorghe Tzitzeica (1873-1939), the founder of the Romanian school of differential geometry, has introduced a special type of curves that carry today his name. A *Tzitzeica curve* is a spatial curve for which the ratio of its torsion τ and the square of the distance d from the origin to the osculating plane at any arbitrary point of the curve is constant, i.e.,

$$\frac{\tau}{d^2} = \alpha, \tag{1}$$

where $\alpha \neq 0$ is a real constant. In this paper, we propose a generalization of the Tzitzeica curve equation, namely,

$$\frac{\tau}{d^{2n}} = \alpha ||\mathbf{r}'(t) \times \mathbf{r}''(t)||^{2n-2} \tag{2}$$

where $n \geq 1$ is a natural number. We will discuss the possibility of augmenting the original equation associated with (2) with a side condition involving third order differential invariants.

Let us consider a curve defined parametrically by

$$\mathbf{r}(t) = (x(t), y(t), z(t)), \tag{3}$$

where $t \in I \subset \mathbf{R}$ is the curve parameter. Assume that (3) has nonzero curvature k. The torsion of the curve is defined as

$$\tau(t) = \frac{(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t))}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||^2}$$

where here the primes denote the derivatives with respect to t, the vector $\mathbf{r}' \times \mathbf{r}''$ is the cross product of the tangent vector $\mathbf{r}'$ and the acceleration vector $\mathbf{r}''$, $||\mathbf{r}'(t) \times \mathbf{r}''(t)||$ is the magnitude of $\mathbf{r}' \times \mathbf{r}''$, and

$$(\mathbf{r}'(t), \mathbf{r}''(t), \mathbf{r}'''(t)) = \begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix}$$

is the mixed product of vectors $\mathbf{r}'$, $\mathbf{r}''$, and $\mathbf{r}'''$ (see, for instance, [11], page 48). We assume that (3) has nonzero torsion at each point on the curve. Next, we consider the equation of the osculating plane

$$\begin{vmatrix} x - x(t) & y - y(t) & z - z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = 0.$$

The osculating plane is generated by the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$ at each point of the curve or, equivalently, by the tangent vector $\mathbf{r}'(t)$ and the acceleration vector $\mathbf{r}''(t)$. Next, the distance from the origin to the osculating plane of the curve is

$$d^2 = \frac{1}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||^2} \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2 .$$

The substitution of τ and d^2 into the condition (1) yields the following equation

$$\begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix} = \alpha \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2 , \tag{4}$$

which may also be written as

$$az''' - a'z'' + bz' = \alpha \left(cz'' - c'z' + az\right)^2 , \tag{5}$$

where

$$a = x'y'' - x''y', \quad b = x''y''' - x'''y'', \quad \text{and} \quad c = xy' - x'y$$

are functions of the curve parameter t. The equation defined by (5) will be called the *Tzitzeica curve equation.*

Proposition 1. *The space curve (3) is a Tzitzeica curve if and only if the functions x, y, and z are solutions to the nonlinear ODE (5).*

Next, assume that a space curve (3) has nonzero curvature and torsion satisfies the condition (2). Similarly, it may be shown that the functions x, y, and z satisfy the following equation

$$\begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix} = \alpha \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^{2n} , \tag{6}$$

or, equivalently,

$$az''' - a'z'' + bz' = \alpha \left(cz'' - c'z' + az\right)^{2n} , \tag{7}$$

where

$$a = x'y'' - x''y', \quad b = x''y''' - x'''y'', \quad \text{and} \quad c = xy' - x'y$$

depend on the curve parameter t. Throughout this paper, the equation defined in (7) will be called the *generalization of the Tzitzeica curve equation.*

3 Side conditions involving higher order differential invariants

In this section, we will consider specific side conditions that will lead to reduced order models. It can be shown [2] that the equation (5) may be written in terms of Wronskians as follows

$$W(x', y', z')(t) = \alpha \left[W(x, y, z)(t)\right]^2, \tag{8}$$

where

$$W(x, y, z)(t) = \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}.$$

Since a determinant is invariant under a cyclic permutation of its rows, the left-hand side of (5) becomes

$$\begin{vmatrix} x'''(t) & y'''(t) & z'''(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = \alpha \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2. \tag{9}$$

Assume that the functions x, y, and z satisfy the auxiliary equations

$$\begin{aligned} x''' + \beta x'' + \gamma x' + \delta x = 0, \\ y''' + \beta y'' + \gamma y' + \delta y = 0, \\ z''' + \beta z'' + \gamma z' + \delta z = 0, \end{aligned} \tag{10}$$

where β, γ, and $\delta \neq 0$ are real numbers. Then the two determinants in (9) are equal (here we consider $\delta \neq 0$ because the curve's torsion should be nonzero). The ODEs (10) are equivalent to the condition that x, y, and z satisfy

$$u''' + \beta u'' + \gamma u' + \delta u = 0, \tag{11}$$

which is a third order linear homogeneous ODE with constant coefficients in the unknown function $u = u(t)$, with β, γ, and δ real numbers such that $\delta \neq 0$. Here the solutions x, y, and z need to be chosen linearly independent because the curve's torsion is nonzero. After substituting (10) into (9), we get

$$-\delta \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = \alpha \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}^2,$$

which may be written as

$$-\delta W(t) = \alpha[W(t)]^2,$$

for any $t \in I$, where we denote $W(t) = W(x, y, z)(t)$. Since $W(t)$ is nonzero on the interval I (the functions x, y, and z are assumed to be linearly independent), the above equation turns into

$$W(t) = -\frac{\delta}{\alpha}. \tag{12}$$

In conclusion, the Tzitzeica curve equation has been reduced to the ODEs (11) and (12). By applying the Abel's differential equation identity, we have

$$W'(t) = -\beta W(t). \tag{13}$$

Replacing (12) into (13) (and taking into account that $W(t)$ is nonzero) implies $\beta = 0$. Hence, the equation (11) becomes

$$u''' + \gamma u' + \delta u = 0. \tag{14}$$

In [2], the equation (14) is solved and its solutions are discussed in detail. It is shown that, in this case, new solutions for the Tzitzeica curve equation may be obtained. Observe that the equation (12) to which the Tzitzeica curve equation is reduced, is, in fact, a linear equation for α, i.e.,

$$\alpha = -\frac{\delta}{W}. \tag{15}$$

Theorem 1. [2] *Any linearly independent solutions of the third order linear homogeneous ODE with constant coefficients (14) define a Tzitzeica curve.*

On the other hand, the Tzitzeica curve equation (5) is invariant under t-translations $t \mapsto \tilde{t} = t + \varepsilon$, where ε is a real number. Indeed, the equation does not depend explicitly on the variable t. This result may be shown by applying the classical method for finding the symmetry group related to a differential equation (see, for instance, Olver's book [10]). In this case, the differential invariants related to the Lie group of transformations that defines the t-translations are the following

$$\begin{aligned}
&I_1 = x, \quad I_2 = x', \quad I_3 = x'', \quad I_4 = x''', \ldots, \quad I_n = x^{(n-1)}, \\
&J_1 = x, \quad J_2 = y', \quad J_3 = y'', \quad J_4 = y''', \ldots, \quad J_n = y^{(n-1)} \\
&K_1 = x, \quad K_2 = z', \quad K_3 = z'', \quad K_4 = z''', \ldots, \quad K_n = z^{(n-1)}.
\end{aligned} \tag{16}$$

The linear ODEs (10) may be rewritten in terms of the above invariants as

$$\begin{aligned} &I_4+\beta I_3+\gamma I_2+\delta I_1=0,\\ &J_4+\beta J_3+\gamma J_2+\delta J_1=0,\\ &K_4+\beta K_3+\gamma K_2+\delta K_1=0. \end{aligned} \tag{17}$$

The condition $\beta = 0$ results from the compatibility of the side conditions (10) with the original equation (5). It follows

Proposition 2. *The side conditions*

$$\begin{aligned} x'''+\gamma x'+\delta x=0,\\ y'''+\gamma y'+\delta y=0,\\ z'''+\gamma z'+\delta z=0, \end{aligned} \tag{18}$$

represent third order differential invariants related to the Lie group of transformations that defines the translations in the t-space under which the original equation (5) is invariant.

Next, we are going to discuss the case of the generalization of the Tzitzeica curve equation (7) and the possibility of augmenting side conditions of the form (18) to it. Notice that, in the Wronskian form, the equation (7) may be written as

$$W(x',y',z')(t)=\alpha\,[W(x,y,z)(t)]^{2n}\,, \tag{19}$$

and, in the determinant form, the equation (7) is equivalent to

$$\begin{vmatrix} x'''(t) & y'''(t) & z'''(t)\\ x'(t) & y'(t) & z'(t)\\ x''(t) & y''(t) & z''(t) \end{vmatrix} = \alpha \begin{vmatrix} x(t) & y(t) & z(t)\\ x'(t) & y'(t) & z'(t)\\ x''(t) & y''(t) & z''(t) \end{vmatrix}^{2n}. \tag{20}$$

Observe that the generalization of the Tzitzeica curve equation (7) is also invariant under translations in the t-space. Indeed, the equation does not depend explicitly on the variable t. Therefore, a side condition of the form (18) means an auxiliary equation that will be attached to the original equation (7) that is expressed in terms of the differential I_4, I_3, I_2, J_4, J_3, J_2, K_4, K_3, and K_2. In this case, the ODE (7) is reduced to the linear equation

$$-\delta W=\alpha W^{2n} \tag{21}$$

which can be rewritten as a linear equation for the constant α, i.e.,

$$\alpha=-\frac{\delta}{W^{2n-1}}. \tag{22}$$

Proposition 3. *The side conditions (18) represent third order differential invariants related to the group of the translations in the t-space under which the original equation (7) is invariant.*

4 Conclusion

In this paper, suitable side conditions involving high order differential invariants have been analyzed for particular nonlinear models, namely, the Tzitzeica curve equation (5) and its generalization (7). It has been shown that, to obtain a reduced model, for nonlinear models involving arbitrary functions, the original equation may be augmented with higher order differential equations involving differential invariants of the arbitrary functions. An intriguing side condition given by a third order linear ODE with constant coefficients is, in fact, a third order differential invariant. Since the equation is invariant with respect to t-translations, this particular side condition leads indeed to reduced models. It is well-known that not any side condition leads to a reduced model. Finding those specific side conditions that are compatible with the model will always be a challenge. In a future work, we will be interested in exploring other particular intriguing side conditions for nonlinear models involving arbitrary functions and to generalize these results by introducing an efficient method based on side conditions involving higher order differential invariants that would allow the reduction of the model.

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