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Stability and Liapunov Functionals for Fractional Differential Equations

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STABILITY AND LIAPUNOV FUNCTIONALS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. This project is devoted to developing Liapunov direct method for fractional differential equations and systems. The method (constructing a system related scalar function) enables investigators to analyze the qualitative behavior of solutions of a differential equation without actually solving it. We are able to convert some fractional differential equations (semi-linear) to integral equations with singular kernels and construct Liapunov functionals for the integral equations to deduce conditions for boundedness and stability of solutions. Extending such a method to fully nonlinear equations presents a significant challenge to investigators and will be a major area of research for many years to come.

1. Motivation

Fractional Calculus, in a certain sense, is as old as classical calculus as we know it today: The origins can be traced back to the end of the seventeenth century, the time when Newton and Leibniz developed the foundations of differential and integral calculus. Leibniz introduced the symbol

$$\frac{d^n}{dx^n} f(x)$$

to denote the nth derivative of a function f. When he reported this in a letter to de L'Hospital (apparently with the implicit assumption that n is a positive integer), L'Hospital replied (1695): "What does $\frac{d^n}{dx^n} f(x)$ mean if $n = 1/2$?" (see Diethelm [9]). It is now well-known that there is no reason to restrict n to the set of positive integers. In fact, the fractional derivative of order 1/2 appears in many applications such as super-diffusion processes, stochastic fractional dynamical systems, and feedback control.

For an n-fold integral there is a well-known formula

$$\int_0^x dx \int_0^x dx \cdots \int_0^x \phi(s)ds = \frac{1}{\Gamma(n)} \int_0^x (x-s)^{n-1}\phi(s)ds$$

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for $\phi \in L^1[0,b]$ since $\Gamma(n) = (n-1)!$, where Γ is the gamma function. We then see

$$D^n \left[\frac{1}{\Gamma(n)} \int_0^x (x-s)^{n-1} \phi(s) ds \right] = \phi(x)$$

where $D^n = \frac{d^n}{dx^n}$. For $n = 1$, we have $\displaystyle \frac{d}{dx} \int_0^x \phi(s) ds = \phi(x)$.
If we write

$$D^q \left[\frac{1}{\Gamma(q)} \int_0^x (x-s)^{q-1} \phi(s) ds \right] = \phi(x)$$

for $0 < q < 1$, what is the definition of D^q ?

Abel formulated (the "tochtochrone" - problem)

$$\frac{1}{\Gamma(q)} \int_0^t \frac{\phi(s)}{(t-s)^{1-q}} ds = f(t)$$

where $0 < q < 1$, and solved the equation (for q=1/2) in 1825. The solution is

$$\phi(x) = \frac{d}{dx} \left[\frac{1}{\Gamma(1-q)} \int_0^t \frac{f(s)}{(t-s)^q} ds \right] =: (D^q f)(t).$$

We now view the right-hand side of this equation as the fractional derivative of f (of order q). If we write

$$(J^q \phi)(x) = \frac{1}{\Gamma(q)} \int_0^x (x-t)^{q-1} \phi(t) dt$$

then $D^q (J^q \phi)(x) = \phi(x)$.

2. Fractional Integrals and Derivatives

The theory starts with the generalization of $n!$ by $\Gamma(n+1)$, where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

where the integral converges for $\mathrm{Re}(z) > 0$ with $\Gamma(z+1) = z\Gamma(z)$. We assume that $f(x)$ is defined for $x \geq 0$ and define $\displaystyle (Jf)(x) = \int_0^x f(t) dt$.
Then $\displaystyle (J^2 f)(x) = \int_0^x (x-t) f(t) dt$ and

$$(J^n f)(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) dt.$$

This gives a general form of a fractional integral (with $0 < q < 1$)

$$(J^q f)(x) = \frac{1}{\Gamma(q)} \int_0^x (x-t)^{q-1} f(t) dt.$$

The fractional derivative D^q is defined so that $D^q(J^q f)(x) = f(x)$ and the definition is

$$(D^q f)(x) = \frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_0^x (x-t)^{-q} f(t)dt.$$

We see that D^q is the "inverse operator" of J^q and that

$$D^q x^n = \frac{d^q}{dx^q}\, x^n = \frac{\Gamma(n+1)}{\Gamma(n+1-q)}\, x^{n-q}$$

which may be viewed as a generalization of the Power Rule. For example,

$$D^{1/2} x^2 = \frac{d^{1/2}}{dx^{1/2}}\, x^2 = \frac{8}{3\sqrt{\pi}}\, x^{3/2}$$

Note that the fractional derivative of a constant function may not be zero

$$D^{1/2}[C] = \frac{1}{\Gamma(1-1/2)} \frac{d}{dx} \int_0^x (x-t)^{-1/2} [C] dt = \frac{[C]}{\sqrt{\pi}} x^{-1/2}.$$

Remark.
* Riemann-Liouville fractional derivative of $q > 0$:

$$(D^q f)(t) = \frac{d^n}{dt^n} \left[\frac{1}{\Gamma(n-q)} \int_a^t \frac{f(s)}{(t-s)^{q-n+1}} ds \right]$$

for $n-1 < q < n$. It follows that $D^q(J^q f)x) = f(x)$ and

$$J^q(D^q f)x) = f(x) + C(x-a)^{q-1}, \;\; 0 < q < 1$$

for a "class" of functions f.

* Caputo fractional derivative of order $q > 0$:

$$({}^c D^q f)(t) = \frac{1}{\Gamma(n-q)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{q-n+1}} ds$$

for $n-1 < q < n$. It is then true that ${}^c D^q(J^q f)(x) = f(x)$ and

$$J^q({}^c D^q f)x) = f(x) + C, \;\; 0 < q < 1$$

for a "class" of functions f. We also have ${}^c D^q([C])(t) = 0$.

Remark.
There are many other generalizations of derivatives. A function f is said to be weakly differentiable on $[0, 1]$ if the "integration by parts " formula works, i.e., if

$$\int_{[0,1]} f(x)\phi'(x)dx = -\int_{[0,1]} g(x)\phi(x)dx$$

for any $\phi \in C_0([0,1])$, then $Df(x) = g(x)$ is called the generalized derivative of f. Sobolev spaces (1930) of weakly differentiable functions arise in connection with numerous problems in partial differential equations, approximation theory, and many other areas of pure and applied mathematics (see Adams and Fournier [1]).

* What happens to

$$J^q(D^q\phi)(x) = ?$$

It is a major topic in differential equations. If $0 < q < 1$ and if $J^{m-n}f \in AC[0,b]$, then

$$J^q(D^q\phi)(x) = f(x) - \frac{x^{q-1}}{\Gamma(q)} \lim_{u \to 0^+} J^{m-n}f(u)$$

(see Diethelm [9, p.39] and Samko, Kilbas, and Marichev [13, p.45])

3. Fractional Differential Equations

We are concerned with a fractional functional differential equation

$$^cD^q x(t) = f(t, x(t)), \ \ x(0) = x_0, \ \ 0 < q < 1, \tag{1}$$

where the differentiation $^cD^q x$ is of Caputo type. We assume that $f : [0,\infty) \times R \to R$ is continuous. Under the continuity condition, one can show that (1) is equivalent to the Volterra fractional integral equation

$$\begin{aligned} x(t) &= x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))ds, \quad t \geq 0 \\ x(0) &= x_0 \end{aligned} \tag{2}$$

where Γ is the gamma function. Equation (2) has a wide range of application to real-world problems. We observe that (2) is a singular integral equation with a convex kernel.

A classical example of (1) is

$$^cD^q x = ax(t) + f(t), \quad x(0) = x_0, \quad 0 < q < 1, \tag{3}$$

where a is a constant, $f : R \to R$ is continuous. The solution of (3) is given by

$$x(t) = E_{q,1}(at^q)x_0 + \int_0^t (t-s)^{q-1} E_{q,q}[a(t-s)^q] f(s)ds.$$

where $E_{q,1}$ and $E_{q,q}$ are members of the two parameter family of Mittag-Leffler functions (the generalized exponential functions) defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \alpha, \ \beta > 0.$$

Note that the solution of the ordinary differential equation

$$x'(t) = ax(t) + f(t), \quad x(0) = x_0 \tag{4}$$

can be expressed as

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-s)} f(s)ds.$$

There is also a solution formula for (4) when a is a function, say $a(t)$. Can we derive a solution formula for (3) when a is function of t ?

4. Stability in L^p spaces

We now consider the equation

$${}^cD^q x = h(t) - kg(x(t-r)), \quad x_0 = \phi, \quad 0 < q < 1/2,$$

where k, r are positive constants, $g : R \to R$ is continuous with

$$0 < \mu \le \frac{g(x)}{x} \le 1 \text{ for } x \neq 0,$$

and h is continuous on $[0, \infty)$. We invert the equation as

$$x(t) = \phi(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[h(s) - kg(x(s-r))]ds \tag{5}$$

For $\phi(0) = 0$, we write (5) as

$$\begin{aligned} x(t) &= H(t) - \frac{k}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(x(s-r))ds \qquad (6) \\ &= H(t) - \int_0^{t-r} C(t-r-u)g(x(u))du \\ &\quad - \int_{-r}^0 C(t-r-u)g(\phi(u))du \end{aligned}$$

for $t \ge r$, where

$$H(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)ds \quad \text{for} \quad t \ge 0$$

and

$$C(t) = \frac{k}{\Gamma(q)} t^{q-1} \quad \text{for} \quad t > 0.$$

Then the resolvent R satisfies

$$R(t) = C(t) - \int_0^t C(t-s)R(s)ds.$$

This resolvent R is completely monotone on $(0, \infty)$. Moreover,

$$0 \le R(t) \le C(t), \ tR(t) \to 0 \text{ as } t \to \infty, \text{ and } \int_0^\infty R(s)ds = 1$$

(see Miller [11, p.205-204]). If $y(t)$ is a solution of

$$y(t) = H(t) - \int_0^t C(t-s)y(s)ds$$

then

$$y(t) = H(t) - \int_0^t R(t-s)H(s)ds. \tag{7}$$

If we write (6) as

$$x(t) = H(t) - \int_0^t C(t-s)[x(s) + g(x(s-r)) - x(s)]ds$$

then its solution $x(t)$ satisfies

$$\begin{aligned} x(t) &= y(t) - \int_0^t R(t-s)[g(x(s-r)) - x(s)]ds \qquad (8)\\ &= y(t) + \int_0^t R(t-s)x(s)ds - \int_0^{t-r} R(t-r-u)g(x(u))du \\ &\quad - \int_{-r}^0 R(t-r-u)g(\phi(u))du. \end{aligned}$$

For the inversion see Lakshmikantham, Leela, and Vasundhara Devi [10, p.54]. Let us assume that

$$\Big[2(1-q) + 2 + \frac{1}{q}\Big]\frac{k}{\Gamma(q)}\, r^q < 1. \tag{9}$$

We observe that (9) is equivalent to

$$2r^2|C'(r)| + 2rC(r) + \int_0^r C(u)du < 1. \tag{10}$$

Theorem 4.1. *Suppose (9) holds and $H \in L^2[0,\infty)$ with $H(t) \to 0$ as $t \to \infty$. If ϕ is any continuous initial function with $\phi(0) = 0$, then the solution of (6) satisfies $x \in L^2[0,\infty)$ and $x(t) \to 0$ as $t \to \infty$.*

The proof of the theorem is based on the construction of a Liapunov functional and application of differential-integral inequalities. The technique used here has its root in Burton ([3], [5], [4]) and Zhang [14]. We can apply the method to nonlinear equations.

We now consider the nonlinear equation

$${}^cD^q x = -a(t)x^m(t) + f(t), \quad 0 < q < 1 \tag{11}$$

where $a, f : [0, \infty) \to \Re$ are continuous, $m \geq 1$ is an odd integer, and there are positive numbers ϵ and M such that

$$0 < \epsilon \leq a(t) \leq M. \tag{12}$$

We will exchange the kernel in (10) for $R(t-s)$, but first we will reduce $a(t)$ to a function bounded by $\alpha < 1$. Define $J = \epsilon + (1/2)(M - \epsilon)$. Then there is an α with

$$J > 0, \quad 0 < \alpha < 1, \quad |a(t) - J| < \alpha J. \tag{13}$$

Note that we may choose $\alpha = (M - \epsilon)/(M + \epsilon)$. In fact, if we write $J = (M + \epsilon)/2$, then by (11) we have

$$\epsilon - J \leq a(t) - J \leq M - J.$$

This implies that

$$-\frac{1}{2}(M - \epsilon) \leq a(t) - J \leq \frac{1}{2}(M - \epsilon)$$

and so

$$|a(t) - J| \leq \frac{1}{2}(M - \epsilon) = \frac{M - \epsilon}{M + \epsilon} J =: \alpha J.$$

(12) holds. We then find $J > 0, \alpha < 1$ with $|J - a(t)| \leq \alpha J$. We invert (11) as

$$x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [a(s)x^m(s) - f(s)] ds$$

which we write

$$x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [Jx(t) - Jx(t) + a(s)x^m(s) - f(s)] ds.$$

We now decompose it into

$$z(t) = x(0) - \frac{J}{\Gamma(q)} \int_0^t (t-s)^{q-1} z(s) ds$$

with solution $z(t) = x(0)\widetilde{R}(t)$ and

$$x(t) = z(t) + \int_0^t R(t-s)[x(s) - x^m(s) + \frac{(J - a(s))}{J} x^m(s)] ds + F(t) \tag{14}$$

where $R(t)$ is defined in (4), $\widetilde{R}(t) = 1 - \int_0^t R(s) ds$, and

$$F(t) := \frac{1}{J} \int_0^t R(t-s) f(s) ds.$$

Note that $F \in L^p$ if $f \in L^p$ for $p \geq 1$ and $F(t) \to 0$ as $t \to 0$ if f is bounded.

Theorem 4.2. *Suppose (12) holds and $f \in L^p[0,\infty)$ with $p \geq 1$. Then there exists $\eta > 0$ such that if $|F(t)| < \eta$ for $t \geq 0$, then the solution of (11) with $x(0) = 0$ is in $L^{mp}[0,\infty)$ with $\|x\|_{mp} \leq L\|f\|_p^{1/m}$ for some constant $L > 0$.*

For $x(0) \neq 0$, it will require a different technique. We derive conditions to ensure that $z(t) \in L^p[0,\infty)$.

Theorem 4.3. *Suppose (12) holds and $f \in L^p[0,\infty)$ for $p > 1/q$. Then there exists $\eta > 0$ such that if $|F(t)| < \eta$ for $t \geq 0$, then the solution of (11) is in $L^{mp}[0,\infty)$ with*

$$\|x\|_{mp} \leq L\left[|x(0)|\|\widetilde{R}\|_p + \|f\|_p\right]^{1/m} \tag{15}$$

for some constant $L > 0$.

Remark: If $f \equiv 0$, then by (15), the zero solution of (11) is L^{mp}-asymptotically stable. One may also note that Theorem 4.2 and Theorem 4.3 are general results. Everything would work for

$${}^cD^q x = -a(t)g(x) + f(t), \quad 0 < q < 1$$

for an odd function $g(x)$ that resembles x^m. The technique used here has its root in Burton and Zhang [6] and [7].

5. Liapunov Stability

If we view the right-hand side of equation (14) as a mapping function on a metric space, then we can combine Liapunov method and fixed point theory (contraction mapping principle) to prove stability in BC, the Banach space of bounded continuous functions $\phi : [0,\infty) \to \Re$. The stability theory is parallel to that of Liapunov for ordinary differential equations.

Consider the equation

$${}^cD^q x = -a(t)x^m(t), \quad 0 < q < 1 \tag{16}$$

where $a : [0,\infty) \to \Re$ is continuous and $m \geq 1$ is an odd integer.

Theorem 5.1. *If (12) holds, then the zero solution of (16) is asymptotically stable.*

We define the mapping function P (see (14)) as follows.

$$(P\phi)(t) = z(t) + \int_0^t R(t-s)[\phi(s) - \phi^m(s) + \frac{(J - a(s))}{J}\phi^m(s)]ds. \tag{17}$$

That integrand will be a large contraction. The contraction constant for the first term is obtained from $y = x - x^3$ with $y' = 1 - 3x^2$, while the x derivative of $\frac{(J-a(s))}{J}x^3$ is bounded by $3\alpha x^2$, yielding the sum $1 - 3x^2 + 3\alpha x^2$. This will yield a large contraction on a certain space of bounded continuous functions. It is clear that the fixed point of P is a solution of (16). For references and detailed discussions, we refer the readers to Burton [2], Burton and Zhang [8], Zhang [15].

For the linear equation

$$^{c}D^{q}x = -a(t)x, \quad 0 < q < 1 \tag{18}$$

we can obtain global stability.

Theorem 5.2. *If (12) holds, then the zero solution of (18) is globally asymptotically stable.*

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