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Fully distributed quantized secure bipartite consensus control of nonlinear multiagent systems subject to denial-of-service attacks

Qiang Wang [a], Lorenzo Zino [b], Dayu Tan [c], Jiapeng Xu [d], Weimin Zhong [a,e,*]

[a] Key Laboratory of Smart Manufacturing in Energy Chemical Process, Ministry of Education, East China University of Science and Technology, Shanghai 200237, China
[b] Faculty of Science and Engineering, University of Groningen, Groningen 9747AG, the Netherlands
[c] Key Laboratory of Intelligent Computing and Signal Processing of Ministry of Education, Institutes of Physical Science and Information Technology, Anhui University, Hefei 230601, China
[d] Department of Electrical and Computer Engineering, University of Windsor, Windsor, ON N9B 3P4, Canada
[e] Shanghai Institute of Intelligent Science and Technology, Tongji University, Shanghai 201804, China

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ABSTRACT

This paper is intended to solve the fully distributed secure bipartite consensus problem of nonlinear multi-agent systems (MASs) with quantized information under Denial-of-Service (DoS) attacks. The attacks, which constrained on attack frequency and duration are studied. Firstly, we propose a novel secure output feedback control protocol integrated of the logarithmic quantizer and relative output measurements of neighboring agents, which can realize secure control under DoS attacks by choosing the design parameters correctly. Secondly, an adaptive control protocol that includes dynamic coupling strengths into the control law and the state observer function is developed. Contrast to the single adaptive control strategy, two adaptive couplings constructed in sensor-to-observer, and controller-to-actuator channels, respectively, which can alleviate the burden of the limited bandwidth and energy consumption more effectively. Furthermore, this control strategy with dynamic coupling gains is fully distributed, under which agents are not required to know a priori knowledge of any global information and the quantizer only needs to quantize the output state error information of agents. Then, theoretical guarantees on the effectiveness of the proposed controllers in steering the system to a secure bipartite (bounded) consensus under quantized output measurements and intermittent DoS attacks are derived. Finally, the numerical simulation inspired by a real-world physical network system is developed to verify the usefulness of the presented controllers.

1. Introduction

In recent years, the cooperative control of multi-agent systems (MASs) has attracted much attention in the scientific community, as many real-world problems require multiple agents to cooperate with each other to perform a task collectively. Therefore, cooperative control of MASs has been extensively studied by many scholars [1,2]. These theoretical advances have been widely used in several practical applications, including networked control systems [3], unmanned air vehicles [4], neural networks [5], and networked cyber-physical systems [6]. The consensus, as a basic problem, aims to propose a suitable control principle by utilizing the local state information of its and neighbor nodes so that all the agents can reach the same value [7]. Depending on the presence of further requirements and specific characteristics of the scenario considered, consensus problem has been investigated in [8]. In particular, we mention the leader–follower consensus, in which a flock of (follower) agents has to coordinate with the state of a leader by using distributed algorithms [9].

However, in many practical situations, competitive and antagonistic interactions are also proposed in [10]. For instance, in [11], it was observed that, in the industrial market, companies not only collaborate, but also compete for market resources. The coexistence of cooperative and competitive interactions has been observed as a key feature in government formation process in parliamentary democracies [12]. In a leader–follower framework, the authors of [13] investigated the cooperation and competition between employer and employees in management control systems. Signed graphs, originally proposed in [10], became a universal tool, widely used to describe and study networks with both cooperative and antagonistic relationships. The study on the

* Corresponding author at: Key Laboratory of Smart Manufacturing in Energy Chemical Process, Ministry of Education, East China University of Science and Technology, Shanghai 200237, China.
E-mail address: wmzhong@ecust.edu.cn (W. Zhong).

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dynamic behaviour of MASs over signed graphs can be traced back to the seminal work on linear systems, in which the authors investigated how the network structure determines whether the system converges to a collective agreement or a polarized scenario, termed bipartite consensus [14]. Concerning leader–follower networks, we mention leader–follower bipartite consensus under fuzzy sliding mode control [15], nonlinear dynamics [16], adaptive control [17], and observer-based control [18].

In the classical literature, authors often concentrated on the implementation for MASs and the design of control algorithms to achieve consensus in idealized scenarios, that is, ignoring important characteristics and limitations of the real environment in which the algorithms are implemented, such as secure state estimation [19], attack detection and identification [20] and the actual deception attacks [21]. These limitations hinder the possibility to apply classical consensus algorithms to achieve secure consensus against malicious cyber attacks in many real-world settings. Typically, there exist many different kinds of cyber attacks in MASs, including attacks on the dynamics of agents (for which we refer to [22,23]) and on their communication [24,25]. Within the second category, DoS attacks have received considerable attention in the past few years as it can be realized efficiently, thereby constituting a serious threat to the well-functioning of MASs [26,27]. In [28], the authors presented a structure of the distributed interval estimator over sensor networks under the aperiodic DoS attack. In [29], the authors resolved the secure control problem for observer-based dynamic event-triggered control of a networked control system with the aperiodic DoS attack. In [30], the authors developed an efficient distributed filter to practically reflect the impact from both DoS attacks and gain perturbations. Although many studies have been performed on secure consensus control of MASs, it is usually supposed that only cooperative interactions are present and that the dynamic of each agent is linear, while competitive interactions and nonlinear systems are often present in the real world, calling for the development of new tools to deal with them. These motivated us to study secure bipartite consensus for nonlinear MASs under DoS attacks.

With the development of digital communication technology, communication constraints are becoming increasingly important in many practical situations. Quantized communication is a successful strategy to deal with these problems [31,32]. Specifically, it has been shown that logarithmic quantizer can solve the consensus problem effectively in many different settings [33,34]. Nevertheless, DoS attacks are commonly encountered in practical applications and it is more significant to analyze the application environment of MASs and consider the situation that the system faces both quantitative communication environment and DoS attacks. Then, the secure consensus control problem of MASs can be resolved by combining quantized information and DoS attacks [35–37]. As mentioned in above researches, global nonzero eigenvalues of Laplacian matrix were always needed, which would consume a lot of energy to process especially for a large-scale network. Therefore, developing a fully distributed control strategy without knowing a priori knowledge of any global information is in great demand. Then to avoid the utilization of global information, several attempts have been made to investigate the fully distributed control approach based on the event-triggered control strategy [38,39]. In particular, the authors investigated the neural network-based control of unknown discrete-time nonlinear systems subject to a DoS attack and an adaptive event-triggered strategy [40]. However, the event-triggered condition could cause unnecessary triggered instants and a few of efforts have been taken to study quantized secure consensus and fully distributed control under DoS attacks. This is another motivation for us to consider the fully distributed quantized secure bipartite consensus of MASs.

In addition, the relative state information of agents can not always be obtained in practical engineering. Therefore, the output feedback control played an important role in achieving asymptotic tracking by constructing a distributed controller [41–43]. However, the above mentioned literatures only study secure consensus problem under DoS attacks without quantized communication, and the limited relative state information of agents makes it difficult to consider the secure consensus, involving how to construct the output feedback control strategy without using any state information, how to combine the nonlinear control condition, how to construct the dynamic parameter without utilizing any global information and how to deal with the effects of competitive relationship between agents. These problems are challenging for realizing quantized secure bipartite consensus under DoS attacks.

Motivated by these works, we fill in this gap by considering the fully distributed secure bipartite consensus for nonlinear MASs with quantized communication subject to DoS attacks. After having formally defined the two controllers and illustrated the theorems to set the gain matrices, we performed a theoretical analysis of the proposed approaches. Through a Lyapunov-based argument, we prove that the two controllers are able to guarantee convergence of the system to a leader–follower bipartite (bounded) consensus. Then, our theoretical findings are illustrated via a numerical simulation that based on a real-world physical network system [21]. The numerical findings show the good performances of the proposed controllers under the adaptive coupling gains, corroborating our theoretical results. The following fundamental issues are listed:

- Compared with some results using the full relative states of neighboring agents [8,31], a novel distributed bipartite consensus control law based on quantized relative state measurements is proposed, in which only the relative output information of neighboring agents is utilized. Also different from [1,17] that the dynamic of system state is linear, we consider a broad class of Lipschitz nonlinear dynamics, which are reflective of many real-world scenarios.
- Inspired by [33,36], both quantized communication and aperiodic DoS attacks are studied in the context of secure bipartite consensus over signed graph. The observer-based control strategy, based on the leader–follower framework, can guarantee that the consensus and observer errors go to zero or bounded by selecting the control parameters properly.
- Contrast to the traditional control protocols in related researches [20,22], we also develop a new control law depending on quantized output state measurements, which is fully distributed and do not need to know a priori knowledge of any global information. In particular, two adaptive couplings constructed in sensor-to-observer, and controller-to-actuator channels, respectively, which can alleviate the burden of the limited bandwidth and energy consumption more effectively. Afterwards, some criteria are presented to guarantee the secure bipartite (bounded) consensus under DoS attacks.
- The elements, including fully distributed control, logarithmic quantizer, DoS attacks, observer based control approach, and antagonistic interactions are investigated simultaneously for the first time to consider secure control. The derived results are more general.

The rest of the article is summarized as follows. In Section 2, we develop the notation and some preliminary results. In Section 3, we formulate the problem. In Section 4, we present our main findings, with proofs reported in the appendices. In Section 5, we formulate the numerical simulations. In Section 6, we conclude the paper and outline avenues for future research.

2. Notation and preliminaries

2.1. Notation

We gather here the notation used throughout this paper. Let $\mathbb{R}^n$ and $\mathbb{N}_+$ denote the n-dimensional Euclidean space and the set of strictly positive integers. The $N \times N$ identity matrix is denoted by I_N. The Euclidean norm is denoted as $\|\cdot\|$. The symbol $\otimes$ is the Kronecker product, $\mathrm{sgn}(\cdot)$ is the sign function, and $\mathrm{diag}(\cdot)$ is the diagonalization operator. Given a matrix $M, \lambda_{\min}(M)$ and $\lambda_{\max}(M)$ represent its minimum and maximum eigenvalues, respectively. We also let $\mathscr{S} = \{S = \mathrm{diag}(s_1, s_2, \cdots, s_N), s_i \in \{-1, 1\}\}$.

2.2. Graph theory

Consider a set $\mathscr{V} = \{1, \ldots, N\}$ of N agents (also referred to as *followers*) and one leader, labeled as $\{0\}$. Followers are connected through a (signed di-) graph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$. Specifically, $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ is the set of directed edges, with $(j, i) \in \mathscr{E}$ if i can access information from j; and $\mathscr{A} \in \mathbb{R}^{N\times N}$ is the (signed) weighted adjacency matrix, whose generic entry a_{ij} measures the information that i receives from $j; a_{ij} \neq 0$ if and only if $(j, i) \in \mathscr{E}, i \neq j$, and $a_{ij} = 0$ otherwise. In addition, assume $a_{ii} = 0, i = 1, 2, \cdots, N$. Given the (signed) graph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$, we define its (signed) Laplacian matrix $L = (l_{ij}) \in \mathbb{R}^{N\times N}$ entry-wise as follows:

$$l_{ij} := \begin{cases} \sum\limits_{j\in\mathscr{V}\setminus\{i\}} |a_{ij}|, & \text{if } i = j, \\ -a_{ij}, & \text{if } i \neq j. \end{cases} \tag{1}$$

Definition 1. [14] A signed graph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ is structurally balanced if there is a partition of the agent set $\mathscr{V}_1, \mathscr{V}_2$ satisfying i) $\mathscr{V}_1 \cup \mathscr{V}_2 = \mathscr{V}$, ii) $\mathscr{V}_1 \cap \mathscr{V}_2 = \varnothing$, iii) $a_{ij} \geqslant 0, \forall v_i, v_j \in \mathscr{V}_k$ $(k \in \{1, 2\})$, and iv) $a_{ij} \leqslant 0, \forall v_i \in \mathscr{V}_k, v_j \notin \mathscr{V}_k (k \in \{1, 2\})$. if not, it is said to be structurally unbalanced.

Lemma 1. [16] *For the graph $\mathscr{G}$, there is a diagonal matrix $S \in \mathscr{S}$ such that the diagonal entries of SLS are positive, and the off-diagonal entries of SLS are negative. In addition, S produces a division, i.e., $\mathscr{V}_1 = \{i|s_i > 0\}$ and $\mathscr{V}_2 = \{i|s_i < 0\}$ that satisfies properties i)–iv) in Definition 1.*

In this paper, we consider an augmented graph $\mathscr{G}_R$ formed by the set of N followers and the leader. The augmented graph has thus agent set $\mathscr{V}_R = \mathscr{V} \cup \{0\}$ and edge set $\mathscr{E}_R = \mathscr{E} \cup \{(j, 0) : j \in \mathscr{N}_0\}$, in which $\mathscr{N}_0$ is the set of followers that can access the information on the leader's state. We define a non-negative $N \times N$-dimensional diagonal matrix $R = \mathrm{diag}([a_{10}, \ldots a_{N0}])$, whose entry $a_{i0} \geqslant 0$ measures how much follower i interacts with the leader, with the understanding that $a_{i0} > 0$ if and only if $(i, 0)$ in $\mathscr{E}_R$. We can finally define $\bar{L} = SLS + R, L_R = L + R$. Based on Definition 1, one obtains $\bar{L}$is positive definite, i.e., $\bar{L} > 0$.

2.3. Cyber attack: aperiodic DoS attack model

In this paper, assume the DoS attacks could damage temporarily both the communication and the control channel. Each DoS attack occurs over a finite time window, termed *attack interval*, after which the MAS could recover to the initial communication and control channels. Hence, DoS attacks constitute a sequence of attack intervals parametrized by a positive integer $k \in \mathbb{N}_+$. Specifically, the kth attack interval is defined as $\mathrm{T}_k := [t_k, t_k + \tau_k)$, where t_k is the time instant at which the kth attack begins, and τ_k is the duration of the kth attack. Consider a generic time interval $[t_1, t_2)$, when attack exists, a sequence of time intervals can be denoted as

$$\mathrm{T}_d(t_1, t_2) = \bigcup_{k\in\mathbb{N}_+} \{\mathrm{T}_k\} \cap [t_1, t_2], \tag{2}$$

and its complement $\mathrm{T}_f(t_1, t_2) := [t_1, t_2) \setminus \mathrm{T}_d(t_1, t_2)$ denotes a sequence of time intervals that no attacks occur.

During the attach intervals, every $\& > 0$ time units, starting from the time instant in which the attach has occurred. The MAS does not resume communication immediately after the DoS attack, but only after a (successful) attempt of communication. Hence, the *effective attack interval* of the kth DoS attack may be longer than the attack interval, and it is equal to $\overline{\mathrm{T}}_k = [t_k, t_k + \bar{\tau}_k)$, where $\bar{\tau}_k = \min\{t \geqslant \tau_k : t/\& \in \mathbb{N}_+\}$. The effective DoS attack time interval set and its complement can be denoted as

$$\overline{\mathrm{T}}_d(t_1, t_2) = \bigcup_{k\in\mathbb{N}_+} \{\overline{\mathrm{T}}_k\} \cap [t_1, t_2],$$
$$\overline{\mathrm{T}}_f(t_1, t_2) = [t_1, t_2) \setminus \overline{\mathrm{T}}_d(t_1, t_2).$$

Definition 2. [35] Denote $N(t_1, t_2)$ as the number of DoS attacks in the interval $[t_1, t_2)$, and the attack frequency can be concluded as

$$\Lambda(t_1, t_2) = \frac{N(t_1, t_2)}{t_2 - t_1}.$$

Assumption 1. [22] Define $|\mathrm{T}_d(t_1, t_2)|$ as the total duration of the DoS attacks in the time interval $[t_1, t_2)$. And there exist $T_0 \geqslant 0, \Lambda_0 \geqslant 0, T_1 > 1, \Lambda_1 > 1$ such that

$$|\mathrm{T}_d(t_1, t_2)| \leqslant T_0 + \tfrac{t_2 - t_1}{T_1},$$
$$N(t_1, t_2) \leqslant \Lambda_0 + \tfrac{t_2 - t_1}{\Lambda_1}.$$

2.4. Logarithmic quantizer

The quantizer $q : \mathbb{R} \to \mathbb{R}$ is assumed to be logarithmic and can be described by

$$q(r) = \begin{cases} \mathfrak{I}_i, & \text{if } \frac{1}{1+\xi}\mathfrak{I}_i < r \leqslant \frac{1}{1-\xi}\mathfrak{I}_i, \quad r > 0, \\ 0, & \text{if } r = 0, \\ -q(-r), & \text{if } r < 0, \end{cases} \tag{3}$$

Then the accuracy constant $\xi \in (0, 1)$. The set of quantized levels can be denoted as

$$\bar{\mathfrak{I}} = \left\{\pm\mathfrak{I}_i, \mathfrak{I}_i = \left(\tfrac{1-\xi}{1+\xi}\right)^i \mathfrak{I}_0, i = \pm1, \pm2, \ldots\right\} \cup \{\pm\mathfrak{I}_0\} \cup \{0\}.$$

According to the conception of the quantizer, one has $|q(a) - a| \leqslant \xi|a|, \forall a \in \mathbb{R}$. For $\aleph = [\aleph_1, \aleph_2, \cdots \aleph_n]^{\mathrm{T}} \in \mathbb{R}^n$, and $q(\aleph) = [q(\aleph_1), q(\aleph_2), \cdots, q(\aleph_n)]^{\mathrm{T}}$, one has $q(\aleph) - \aleph = \mathrm{H}\aleph$, in which $\mathrm{H} = \mathrm{diag}\{\mathrm{H}_1, \mathrm{H}_2, \cdots, \mathrm{H}_n\}$ and $\mathrm{H}_i \in [-\xi, +\xi]$.

3. Problem formulation

In this paper, consider the MAS made of a group of N followers and a leader. Each follower $i \in \mathscr{V}$ is characterized by a *state vector* $x_i(t) \in \mathbb{R}^n$, an *input vector* $u_i(t) \in \mathbb{R}^m$, and an *output measurement vector* $z_i(t) \in \mathbb{R}^r$, and its dynamic is described by

$$\begin{aligned}\dot{x}_i(t) &= Ax_i(t) + f(x_i(t),t) + Bu_i(t),\\ z_i(t) &= Cx_i(t),\end{aligned} \tag{4}$$

in which $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{r\times n}$, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a function continuous and differentiable in t. Note that all the followers have the same dynamics. The dynamic of the leader can be instead given by the following equation

$$\dot{x}_0(t) = Ax_0(t) + f(x_0(t),t), \tag{5}$$

in which $x_0(t) \in \mathbb{R}^n$ stands for the leader's state. Note that, the state of the leader evolved as an autonomous nonlinear system, that is, $u_0(t) = 0$, while the states of the followers are influenced by the external input. In this paper, we will study whether the MAS made by the leader and the N followers converges to a bipartite consensus. Specifically, we introduce the following definitions.

Definition 3. [Leader–follower bipartite consensus] The leader–follower bipartite consensus problem of MAS (4) and (5) can be resolved for some $k \in \{1,2\}$ if

$$\lim_{t\to\infty}\|x_i(t) - x_0(t)\| = 0, \forall i \in \mathscr{V}_k,$$
$$\lim_{t\to\infty}\|x_i(t) + x_0(t)\| = 0, \forall i \in \mathscr{V}_{3-k},$$

which can be further written by

$$\lim_{t\to\infty}\|x_i(t) - s_i x_0(t)\| = 0, i = 1,2,\cdots,N.$$

Definition 4. [Leader–follower bipartite bounded consensus] The leader–follower bipartite bounded consensus for MAS (4) and (5) can be achieved if

$$\lim_{t\to\infty}\|x_i(t) - s_i x_0(t)\| = \imath, i = 1,2,\cdots,N,$$

where $\imath$ is a positive constant that yields the bound on the deviation of leader–follower bipartite consensus.

Note that, differently from other notions of consensus [8], in Definitions 3 and 4, we say that a leader–follower bipartite consensus problem can be solved if the entire system synchronizes toward a trajectory in which a set of followers has the same state of the leader, and the remaining followers have the opposite state.

Assumption 2. The pair (A,B,C) is stabilizable and detectable.

Assumption 3. There exists a non-negative constant $\rho > 0$ such that

$$\|f(a_1,t) - s_i f(a_2,t)\| \leqslant \rho\|a_1 - s_i a_2\|, \forall a_1, a_2 \in \mathbb{R}^n. \tag{6}$$

Lemma 2. [27] *Consider a MAS with dynamics from (4) and (5) that satisfy Assumption 3, with a sequence of DoS attacks. If the system-related piecewise Lyapunov function satisfies: 1) When there are no DoS attacks, that is* $t \in \overline{\mathrm{T}}_f$,

$$V(t) = \tilde{V}(t), \dot{V}(t) \leqslant -a_1\tilde{V}(t) + a_2,$$

2) When there exist DoS attacks, that is $t \in \overline{\mathrm{T}}_d$,

$$V(t) = \hat{V}(t), \dot{V}(t) \leqslant a_3\hat{V}(t) + a_4,$$

in which $a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0, T_1$ *and* Λ_1 *satisfy the conditions:*

$$\frac{1}{T_1} < \frac{a_1-\tau}{a_1+a_3},$$
$$\frac{1}{\Lambda_1} < \frac{\tau}{2\ln\mu + (a_1+a_3)\&},$$

where $0 < \tau < a_1, \& > 0, \mu \geqslant 1$, *and the following inequalities hold:*

$$\begin{cases}\mu\hat{V}((t_k+\bar{\tau}_k)^-) - \tilde{V}(t_k+\bar{\tau}_k) \geqslant 0,\\ \mu\tilde{V}(t_{k+1}^-) - \hat{V}(t_{k+1}) \geqslant 0,\end{cases}$$

in which $k \in N$. *Therefore,* $V(t)$ *is bounded.*

Remark 1. Assumption 1 is concerned with the attack strength and attack frequency for the DoS attacks and it is a standard assumption made in the literature on consensus problems of MASs subjects to DoS attacks [26,41], which bounds the maximum attack frequency. However, different from many other works in the literature [35,43], we do not rely on the more restrictive assumption that DoS attacks are periodic.

Remark 2. Assumption 3 restricts the set of nonlinear functions that our controller is able to deal with those that verifies the Lipschitz condition. From an engineering point of view, this assumption is meaningful, all linear and piece-wise linear time-invariant continuous functions satisfy this condition, Also practical systems such as mass-spring- damper systems and van der Pol oscillators satisfy this assumption, with many nonlinear functions often used in cyber-physical systems [27,40]. However, it is still an open problem to extend these theoretical findings to ensure secure bipartite consensus for MASs under DoS attacks and nonlinear dynamics which do not satisfy the Lipschitz condition. The design approaches presented in [8,9,21,26] might be useful for investigating this direction in the future research.

Remark 3. The leader in the MASs (3) can be a real or a virtual agent that provides a reference state being tracked by the followers. So the states of the followers are not only required to achieve the predefined time-varying formation but also need to track the state of the leader, especially when do not consider the control input or disturbance produced by external systems, the same dynamic of each follower is essential. Then, based on the relative output information among neighboring agents, the nonlinear MASs can achieve secure bipartite consensus under DoS attacks, whilst the control law depending on quantized output state measurements, which is fully distributed and do not need to know a priori knowledge of any global information. Examples of practical physical systems are the formation control of unmanned air vehicles [4], the cooperative control of mobile robots, the design of distributed moving neural networks [5], and so forth. Therefore, the bipartite consensus is much more general and the consensus can be recognized as a special case of it.

4. Main results

In this section, secure bipartite consensus control of nonlinear MASs subject to DoS attacks and quantized communication is solved by both the static protocol and the adaptive protocol, respectively.

4.1. Secure bipartite consensus with static protocol under DoS attacks

In this subsection, consider the bipartite consensus of the MASs in (4)- ((5) over a static control protocol. An observer-based controller based on the output measurements is developed by defining the following input functions for the followers:

$$u_i(t) = \begin{cases} cK\varphi_i(t), & \text{if } t \in \overline{\mathrm{T}}_f(t_0,t),\\ 0, & \text{if } t \in \overline{\mathrm{T}}_d(t_0,t),\end{cases} \tag{7}$$

in which $c > 0$ is a coupling strength, K is the feedback gain matrix, and the combining measurement $\varphi_i(t)$ satisfying

$$\varphi_i(t) = \left[\sum_{j=1}^{N}|a_{ij}|\left(\hat{x}_i(t) - \mathrm{sgn}(a_{ij})\hat{x}_j(t)\right) + a_{i0}(\hat{x}_i(t) - s_i x_0(t))\right], \quad (8)$$

and $\hat{x}_i(t)$ denotes the state observer. Then, one obtains

$$\dot{x}_i(t) = \begin{cases} Ax_i(t) + f(x_i(t),t) + cBK\left[\sum_{j=1}^{N}|a_{ij}|\left(\hat{x}_i(t) - \mathrm{sgn}(a_{ij})\hat{x}_j(t)\right)\right. \\ \quad \left. + a_{i0}(\hat{x}_i(t) - s_i x_0(t))\right], \quad \text{if} t \in \overline{\mathrm{T}}_f(t_0,t), \\ Ax_i(t) + f(x_i(t),t), \quad \text{if} t \in \overline{\mathrm{T}}_d(t_0,t), \end{cases}$$

and

$$\dot{\hat{x}}_i(t) = \begin{cases} A\hat{x}_i(t) + f(\hat{x}_i(t),t) + \varepsilon Fq\left[\sum_{j=1}^{N}|a_{ij}|\left(\tilde{e}_i(t) - \mathrm{sgn}(a_{ij})\tilde{e}_j(t)\right)\right. \\ \quad \left. + a_{i0}(\tilde{e}_i(t) - \mathrm{sgn}(a_{i0})\tilde{e}_0(t))\right] + Bu_i(t), \quad \text{if} t \in \overline{\mathrm{T}}_f(t_0,t), \\ A\hat{x}_i(t) + f(\hat{x}_i(t),t), \quad \text{if} t \in \overline{\mathrm{T}}_d(t_0,t), \end{cases}$$

in which $\varepsilon > 0$ and F is the feedback matrix to be designed. After that, for any follower $i \in \mathscr{V}$, define $\tilde{e}_i(t) = z_i(t) - C\hat{x}_i(t)$ as the error between the measurement output, $z_i(t)$, and the corresponding quantity computed from the state observer, $C\hat{x}_i(t)$, for all $i \in \mathscr{V}$. Since the leader acts as a reference signal generator, it is supposed that $\hat{x}_0(t) = x_0(t)$, i.e., the leader does not need to observe its own state, and it holds $\tilde{e}_0(t) = z_0(t) - C\hat{x}_0(t) = z_0(t) - Cx_0(t) = 0$. Define the following two errors:

$$\bar{e}_i(t) = x_i(t) - \hat{x}_i(t), \quad \hat{e}_i(t) = \hat{x}_i(t) - s_i x_0(t), \quad (9)$$

in which $\bar{e}_i(t)$ represents the observer error between the agent i and its observer and $\hat{e}_i(t)$ denotes consensus tracking error between the observer of agent i and the leader or its opposite side, respectively. Based on the forgoing analysis, since $s_i s_j a_{ij} \geqslant 0, i,j = 1,\cdots N$, one obtains $a_{ij}s_i = |a_{ij}|s_j$ and $|a_{ij}|s_i = a_{ij}\mathrm{sgn}(a_{ij})s_i = |a_{ij}|s_i\mathrm{sgn}(a_{ij})$. Hence, one obtains

$$\dot{\bar{e}}_i(t) = \begin{cases} A\bar{e}_i(t) + f(x_i(t),t) - f(\hat{x}_i(t),t) - \varepsilon Fq\left[\sum_{j=1}^{N}|a_{ij}|\cdot\right. \\ \quad \left.\cdot\left(\tilde{e}_i(t) - \mathrm{sgn}(a_{ij})\tilde{e}_j(t)\right) + a_{i0}\tilde{e}_i(t)\right], \quad \text{if} t \in \overline{\mathrm{T}}_f(t_0,t), \\ A\bar{e}_i(t) + f(x_i(t),t) - f(\hat{x}_i(t),t), \quad \text{if} t \in \overline{\mathrm{T}}_d(t_0,t). \end{cases}$$

Similarly, we compute

$$\dot{\hat{e}}_i(t) = \begin{cases} A\hat{e}_i(t) + f(\hat{x}_i(t),t) - s_i f(x_0(t),t) + \varepsilon Fq\left[\sum_{j=1}^{N}|a_{ij}|\cdot\right. \\ \quad \left.\cdot\left(\tilde{e}_i(t) - \mathrm{sgn}(a_{ij})\tilde{e}_j(t)\right) + a_{i0}\tilde{e}_i(t)\right] + Bu_i(t), \quad \text{if} t \in \overline{\mathrm{T}}_f(t_0,t), \\ A\hat{e}_i(t) + f(\hat{x}_i(t),t) - s_i f(x_0(t),t), \quad \text{if} t \in \overline{\mathrm{T}}_d(t_0,t). \end{cases}$$

According to the concepts of the Krasovskii solution and logarithmic quantizer [20], one choose $\mathrm{Z}_i(t) \in \mathscr{K}(\mathrm{H}_i\bar{e}_i(t))$ and by utilizing Kronecker products, one can write the equations for the errors into a compact matrix form as

$$\dot{\bar{e}}(t) = \begin{cases} [(I_N \otimes A) - \varepsilon(L_R \otimes FC)]\bar{e}(t) - \varepsilon(L_R \otimes FC)\mathrm{Z}(t) \\ \quad + I_N \otimes (f(x(t),t) - f(\hat{x}(t),t)), \quad \text{if} t \in \overline{\mathrm{T}}_f(t_0,t), \\ (I_N \otimes A)\bar{e}(t) + I_N \otimes (f(x(t),t) - f(\hat{x}(t),t)), \quad \text{if} t \in \overline{\mathrm{T}}_d(t_0,t), \end{cases}$$

and

$$\dot{\hat{e}}(t) = \begin{cases} [I_N \otimes A + c(L_R \otimes BK)]\hat{e}(t) + \varepsilon(L_R \otimes FC)\mathrm{Z}(t) + \varepsilon(L_R \otimes FC)\bar{e}(t) \\ \quad + (f(\hat{x}(t),t) - (SI_N \otimes f(x_0(t),t))), \quad \text{if} t \in \overline{\mathrm{T}}_f(t_0,t), \\ (I_N \otimes A)\hat{e}(t) + (f(\hat{x}(t),t) \\ \quad - (SI_N \otimes f(x_0(t),t))), \quad \text{if} t \in \overline{\mathrm{T}}_d(t_0,t), \end{cases}$$

in which $f(x(t),t) := \left[f^\top(x_1(t),t),\cdots,f^\top(x_N(t),t)\right]^\top$, $f(\hat{x}(t),t) := \left[f^\top(\hat{x}_1(t),t),\cdots,f^\top(\hat{x}_N(t),t)\right]^\top$, $\bar{e}(t) := \left[\bar{e}_1^\top(t),\bar{e}_2^\top(t),\cdots,\bar{e}_N^\top(t)\right]^\top$, $\hat{e}(t) := \left[\hat{e}_1^\top(t),\hat{e}_2^\top(t),\cdots,\hat{e}_N^\top(t)\right]^\top$.

Remark 4. Contrast to the related works in which the network environment is secure, the observer-based controller investigated in this paper will be blocked by DoS attacks when the attacker being active. Therefore, based on (7), the controller is available only when t belongs to a sequence of time intervals that no attacks occur, that is, $t \in \overline{\mathrm{T}}_f(t_0,t)$. For practical implement, how to determine the secure time sequence is a hot topic and some scholars have made some attempts [25,26]. Then the detection of DoS attacks will be the first priority to fight against DoS attacks. Noted that many works on detecting DoS attacks can be searched from the computer science literature, and there exist three main attack detection strategies for the cyber-phisical systems: signature-based, anomaly-based, and hybrid-based. Also according to [20], the authors designed a consistent monitor to detect (respectively, identifies) the DoS attack, the monitor is a deterministic algorithm related to continuous-time measurements and the system dynamics. Furthermore, we usually use some detection techniques to detect the DoS attacks. For example, the activity profiling, changepoint detection, and wavelet-based signal analysis-face the considerable challenge of discriminating network-based flooding attacks from sudden increases in legitimate activity or flash events. However, to completely solve the detection is still a challenging problem for limited testing of every detector. Therefore, how to combine the controller with the detectors to improve the detection efficiency will be considered in our future works and the related results in [25,41] will be helpful for us.

Theorem 1. *Assume the MAS in (4)- (5) satisfying Assumptions 1–3. Under the controller form (7), the MAS achieves leader–follower bipartite consensus (Definition 3) with feedback matrices* $K = -B^{\mathrm{T}}P, F = P^{-1}C^{\mathrm{T}}, \Gamma = PBB^{\mathrm{T}}P$, *and* $\tilde{\Gamma} = C^{\mathrm{T}}C$, *if there are two positive definite matrices* P *and* Q *and positive constants* $m_1, m_2, n_1, n_2, \rho > 0$ *and* $\tau \in (0, m_0)$ *such that the following conditions are satisfied:*

$$\begin{bmatrix} \Theta_1 & \rho P & C \\ \rho P & -I & 0 \\ * & 0 & -I \end{bmatrix} < 0, \begin{bmatrix} \Theta_2 & \rho P \\ * & -I \end{bmatrix} < 0, \quad (10)$$

$$\begin{bmatrix} \Theta_3 & \rho Q \\ * & -I \end{bmatrix} < 0, \begin{bmatrix} \Theta_4 & \rho Q \\ * & -I \end{bmatrix} < 0, \quad (11)$$

$$\mu = \max\left\{\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}, \frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)}\right\}, \quad (12)$$

$$\frac{1}{\Lambda(t_0,t)} \leqslant \frac{\tau}{2\ln\mu + (m_0+n_0)\&}, \tau \in (0, m_0), \quad (13)$$

$$\frac{1}{T_1} \leqslant \frac{m_0 - \tau}{m_0 + n_0}, \quad (14)$$

in which

$$\Theta_1 = A^T P + PA - \left(PB + B^T P\right) + I_N + m_1 P,$$
$$\Theta_2 = A^T P + PA - C^{\mathrm{T}}C + I_N + m_2 P,$$
$$\Theta_3 = A^T Q + QA + I_N - n_1 Q,$$
$$\Theta_4 = A^T Q + QA + I_N - n_2 Q,$$

with constant T_1 *defined in Assumption 1,* $m_0 = \min_{i\in\{1,2\}}\{m_i\}, n_0 = \max_{i\in\{1,2\}}\{n_i\}, c \geqslant \frac{1}{2\lambda_1}$,

$\varsigma = \varepsilon(k_1+k_2) \leqslant \frac{1}{\lambda_N}, \bar{\varsigma} = \left(2\varepsilon - \frac{\xi^2}{k_1} - \frac{1}{k_2} - k_3 - \frac{\xi^2}{k_3}\right) \geqslant \frac{1}{\lambda_1}$, *and* λ_1 *and* λ_N *are the smallest nonzero eigenvalue and largest eigenvalue of* $\bar{L}$.

Proof 1. At this stage, based on a structurally balanced communication network, we can analytically prove that the observer-based control law (7) solves the quantized secure bipartite consensus for nonlinear MAS in (4)- (5) subject to DoS attacks. The following result formally guarantees our claim. The proof, which is based on a Lyapunov argument to show convergence to 0 for the two quantities in (9), is quite cumbersome and is thus reported in Appendix A, for the sake of readability.

Remark 5. Different from the related references [6,20,35], a static observer-based control law based on the output information is developed to ensure the secure bipartite consensus of MASs subject to DoS attacks. The controller here is more meaningful because the state information of agents are not always available. That is to say, our results are not limited by the state information of agents. Furthermore, unlike [34,36], the controller involved with logarithmic quantizer has been proposed, which just need to quantize the output information of agents, and it can adjust the measurement of quantized step based on the output value. Therefore, the secure bipartite consensus problem with communication constraints can be resolved more effectively. As partly depicted in the proof of Theorem 1, the influences of non-uniform quantitative information, DoS attacks, the limitation of state information of agents, nonlinear term and competitive relationships between agents make it more challenging to achieve secure bipartite consensus.

4.2. Secure bipartite consensus with fully adaptive protocol under DoS attacks

In this subsection, consider the bipartite consensus of the MASs in (4)- (5) over a fully distributed control protocol. Then, the fully adaptive control law can be designed as follows

$$u_i(t) = \begin{cases} \hat{c}_{ij}(t)K\varphi_i(t), & \text{if } t \in \overline{\mathrm{T}}_f(t_0,t), \\ 0, & \text{if } t \in \overline{\mathrm{T}}_d(t_0,t), \end{cases} \tag{15}$$

where $\hat{c}_{ij}(t)$ denotes the adaptive coupling strength, and it satisfies

$$\dot{\hat{c}}_{ij}(t) = -\zeta\sum_{j=1}^{N} a_{ij}\hat{c}_{ij}(t) + \zeta\varphi_i^{\mathrm{T}}(t)\Gamma\varphi_i(t), \tag{16}$$

in which $\hat{c}_{ij}(0) > 0, \zeta$ is an arbitrarily chosen positive constant. Then, one has

$$\dot{x}_i(t) = \begin{cases} Ax_i(t) + f(x_i(t),t) + \hat{c}_{ij}(t)BK\left[\sum_{j=1}^{N}|a_{ij}|\left(\hat{x}_i(t) - \mathrm{sgn}(a_{ij})\hat{x}_j(t)\right)\right. \\ \quad \left. + a_{i0}(\hat{x}_i(t) - s_i x_0(t))\right], \quad \text{if } t \in \overline{\mathrm{T}}_f(t_0,t), \\ Ax_i(t) + f(x_i(t),t), \quad \text{if } t \in \overline{\mathrm{T}}_d(t_0,t), \end{cases}$$

and

$$\dot{\hat{x}}_i(t) = \begin{cases} A\hat{x}_i(t) + f(\hat{x}_i(t),t) + \hat{\varepsilon}_{ij}(t)Fq\left[\sum_{j=1}^{N}|a_{ij}|\left(\tilde{e}_i(t) - \mathrm{sgn}(a_{ij})\tilde{e}_j(t)\right)\right. \\ \quad \left. + a_{i0}(\tilde{e}_i(t) - \mathrm{sgn}(a_{i0})\tilde{e}_0(t))\right] + Bu_i(t), \quad \text{if } t \in \overline{\mathrm{T}}_f(t_0,t), \\ A\hat{x}_i(t) + f(\hat{x}_i(t),t), \quad \text{if } t \in \overline{\mathrm{T}}_d(t_0,t), \end{cases}$$

where the $\hat{\varepsilon}_{ij}(t)$ is an another adaptive coupling strength, and it satisfies

$$\dot{\hat{\varepsilon}}_{ij}(t) = -\tilde{\zeta}\sum_{j=1}^{N} a_{ij}\hat{\varepsilon}_{ij}(t) - \tilde{\zeta}\varphi_i^{\mathrm{T}}(t)\tilde{\Gamma}\varphi_i(t) + \tilde{\zeta}\tilde{\varphi}_i^{\mathrm{T}}(t)\tilde{\Gamma}\tilde{\varphi}_i(t), \tag{17}$$

in which the initial condition $\hat{\varepsilon}_{ij}(0) > 0$, the constant $\tilde{\zeta} > 0$, and

$$\tilde{\varphi}_i(t) = q\left[\sum_{j=1}^{N}|a_{ij}|\left(\tilde{e}_i(t) - \mathrm{sgn}(a_{ij})\tilde{e}_j(t)\right) + a_{i0}\tilde{e}_i(t)\right]. \tag{18}$$

According to (9), one has

$$\dot{\tilde{e}}_i(t) = \begin{cases} A\tilde{e}_i(t) + f(x_i(t),t) - f(\hat{x}_i(t),t) - \hat{\varepsilon}_{ij}(t)Fq\left[\sum_{j=1}^{N}|a_{ij}|\cdot\right. \\ \quad \left.\cdot\left(\tilde{e}_i(t) - \mathrm{sgn}(a_{ij})\tilde{e}_j(t)\right) + a_{i0}\tilde{e}_i(t)\right], \quad \text{if } t \in \overline{\mathrm{T}}_f(t_0,t), \\ A\tilde{e}_i(t) + f(x_i(t),t) - f(\hat{x}_i(t),t), \quad \text{if } t \in \overline{\mathrm{T}}_d(t_0,t), \end{cases}$$

and

$$\dot{\hat{e}}_i(t) = \begin{cases} A\hat{e}_i(t) + f(\hat{x}_i(t),t) - s_i f(x_0(t),t) + \hat{\varepsilon}_{ij}(t)Fq\left[\sum_{j=1}^{N}|a_{ij}|\cdot\right. \\ \quad \left.\cdot\left(\tilde{e}_i(t) - \mathrm{sgn}(a_{ij})\tilde{e}_j(t)\right) + a_{i0}\tilde{e}_i(t)\right] + Bu_i(t), \quad \text{if } t \in \overline{\mathrm{T}}_f(t_0,t), \\ A\hat{e}_i(t) + f(\hat{x}_i(t),t) - s_i f(x_0(t),t), \quad \text{if } t \in \overline{\mathrm{T}}_d(t_0,t). \end{cases}$$

Similarly, by utilizing Kronecker products, rewrite the above equations into a compact matrix form as

$$\dot{\tilde{e}}(t) = \begin{cases} [(I_N \otimes A) - (L_\varepsilon \otimes FC)]\bar{e}(t) - (L_\varepsilon \otimes FC)\mathrm{Z(t)} \\ \quad + I_N \otimes (f(x(t),t) - f(\hat{x}(t),t)), \quad \text{if } t \in \overline{\mathrm{T}}_f(t_0,t), \\ (I_N \otimes A)\bar{e}(t) + I_N \otimes (f(x(t),t) - f(\hat{x}(t),t)), \quad \text{if } t \in \overline{\mathrm{T}}_d(t_0,t), \end{cases}$$

and

$$\dot{\hat{e}}(t) = \begin{cases} [I_N \otimes A + (L_c \otimes BK)]\hat{e}(t) + (L_\varepsilon \otimes FC)\mathrm{Z(t)} + (L_\varepsilon \otimes FC)\bar{e}(t) \\ \quad + (f(\hat{x}(t),t) - (SI_N \otimes f(x_0(t),t))), \quad \text{if } t \in \overline{\mathrm{T}}_f(t_0,t), \\ (I_N \otimes A)\hat{e}(t) + (f(\hat{x}(t),t) \\ \quad - (SI_N \otimes f(x_0(t),t))), \quad \text{if } t \in \overline{\mathrm{T}}_d(t_0,t), \end{cases}$$

in which $L_c = \bar{L} + \bar{R}, L_\varepsilon = \tilde{L} + \tilde{R}$. $\bar{L}$ and $\tilde{L}$ are defined as $\bar{L}_{cij} = (\hat{c}_{ij}l_{ij}), j \neq i, \bar{L}_{cii} = \sum_{j=1, j\neq i}^{N}\bar{L}_{cij}, \tilde{L}_{\varepsilon ij} = (\hat{\varepsilon}_{ij}l_{ij}), j \neq i, \tilde{L}_{\varepsilon ii} = \sum_{j=1, j\neq i}^{N}\tilde{L}_{\varepsilon ij}, \bar{R} = \mathrm{diag}(\hat{c}_{1j}a_{10}, \ldots \hat{c}_{Nj}a_{N0})$, and $\tilde{R} = \mathrm{diag}(\hat{\varepsilon}_{1j}a_{10}, \ldots \hat{\varepsilon}_{Nj}a_{N0})$.

Remark 6. Note that in [5,9], the developed controllers that needed to obtain the Laplacian of the graph for designing the control gains, however, in our paper, we propose an approach for adaptively tuning the gains $\hat{c}_{ij}(t)$ and $\hat{\varepsilon}_{ij}(t)$ depending on sampled relative quantized output information, and thereby the application of global information on the basis of the Laplacian is avoided. On the other hand, according to the Finsler's lemma in [44] and bounded real lemma in [45], the Assumption 2 is a necessary condition for the feasibility of LMIs (10) and (11). However, how to relax the constraints on the system's dynamic and involved with more general practical applications will be considered in our next work.

Theorem 2. *Assume that the MAS in (4)- (5) satisfying Assumptions 1–3. Under the controller (15) with adaptive control laws (16) and (17), the MAS achieves leader–follower bipartite bounded consensus (Definition 4) and* $\hat{c}_{ij}(t), \forall (j,i) \in \mathscr{E}$ *and* $\hat{\varepsilon}_{ij}(t), \forall (j,i) \in \mathscr{E}$, *converge to some positive constants with feedback matrices* $K = -B^{\mathrm{T}}P, F = P^{-1}C^{\mathrm{T}}, \Gamma = PBB^{\mathrm{T}}P$, *and* $\tilde{\Gamma} = C^{\mathrm{T}}C$, *if there are two positive definite matrices* P, Q *and positive constants* m_1, m_2, n_1, n_2, ρ, *and* $\tau \in (0, m_0)$ *such that the following conditions are satisfied:*

$$\begin{bmatrix} \tilde{\Theta}_1 & \rho P & C \\ \rho P & -I & 0 \\ * & 0 & -I \end{bmatrix} < 0, \begin{bmatrix} \tilde{\Theta}_2 & \rho P \\ * & -I \end{bmatrix} < 0, \tag{19}$$

$$\begin{bmatrix} \tilde{\Theta}_3 & \rho Q \\ * & -I \end{bmatrix} < 0, \begin{bmatrix} \tilde{\Theta}_4 & \rho Q \\ * & -I \end{bmatrix} < 0, \tag{20}$$

$$\mu = \max\left\{\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}, \frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)}\right\}, \tag{21}$$

$$\Lambda(t_0, t) \leqslant \frac{\tau}{2\ln\mu + (m_0+n_0)\&}, \tau \in (0, m_0), \tag{22}$$

$$\frac{1}{T_1} \leqslant \frac{m_0-\tau}{m_0+n_0}, \tag{23}$$

in which

$$\begin{aligned} \tilde{\Theta}_1 &= A^T P + PA - \left(PB + B^T P\right) + I_N + m_1 P, \\ \tilde{\Theta}_2 &= A^T P + PA - C^T C + I_N + m_2 P, \\ \tilde{\Theta}_3 &= A^T Q + QA + I_N - n_1 Q, \\ \tilde{\Theta}_4 &= A^T Q + QA + I_N - n_2 Q, \end{aligned}$$

with $\hat{\varsigma} = k_4 + k_5 = \tilde{\varsigma}(1+\xi)^2, \tilde{\varsigma} = \frac{\xi^2}{k_4} + \frac{1}{k_5} + k_6 + \frac{\xi^2}{k_3}$, $\bar{c}_0 \geqslant \frac{1}{\lambda_1}, \bar{\varepsilon}_0 \leqslant \frac{1}{(1-\xi)^2\lambda_N}$, $\ell = \bar{c}_0, \tilde{\ell} = \bar{\varepsilon}_0(1-\xi)^2, m_0 = \min_{i\in\{1,2\}}\{m_i\}$, $n_0 = \max_{i\in\{1,2\}}\{n_i\}, \Delta = \frac{m_0}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\bar{c}_0^2 + \frac{m_0}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\bar{c}_0^2, \iota(z) = \max\{\Delta_1, \Delta_2\}, \bar{c} = e^{(m_0+n_0)T_0+}$ $[(m_0+n_0)\& + \ln\mu]\Lambda_0, \nu = m_0 - (m_0+n_0)\frac{1}{T_1} - \tau > 0$, *and* λ_1 *and* λ_N *are the smallest nonzero eigenvalue and largest eigenvalue of* $\bar{L}$.

Proof 2. Similar to the scenario with static protocol, we can analytically prove that the fully distributed control law (15) and observer-based control strategies with gain matrices defined via (19)- (20) can solve the leader–follower bipartite bounded consensus of nonlinear MAS in (4)- (5) under DoS attacks. The following result, whose proof is reported in Appendix B, formally guarantees our claim, under some conditions on the Lemma 2. □

Remark 7. For general nonlinear MASs, it will be challenging to design a fully distributed protocol only based on relative states of neighboring agents and quantized output information over signed networks. To render the MASs secure bipartite bounded consensus, an observer-based control rule with two independent adaptive couplings is developed to accomplish fully distributed schemes without any global information [38,39]. Contrast to the single adaptive control strategy [32], two adaptive couplings constructed in sensor-to-observer, and controller-to-actuator channels, respectively, which can alleviate the burden of the limited bandwidth and energy consumption more effectively. And it is theoretically proved in Theorem 2 that the leader–follower control errors and observer errors are bounded, which reflects that the fully distributed secure bipartite bounded consensus subject to DoS attacks and quantized communication can be realized successfully.

Remark 8. Notice that the matrix inequalities in Theorems 1 and 2 should scaled by the Yang's inequality. However, by ullizing the Yang's inequality, some new decision variables parameters will increase the complexity. To deal with this problem, we utilize the property of the Kronecker product to reduce the computational complexity. In addition, the introduction of Lipschitz nonlinear condition could combine two terms of the inequality into one, thereby reducing the computational burden. On the other hand, these new parameters will increase the flexibility of the conditions, and making it easier to find the solution of matrix inequality. Therefore, the increase in time complexity caused by the calculation of Yang's inequality is controllable and it is essential [7,17,20,22].

5. Simulation

In this section, we conclude the paper by providing a simulation example [22], which is given to corroborate the theoretical results presented in the previous section. In our example, we consider a network made by coupled Chua's circuits to verify the feasibility of the theoretical results, and it is described as follows:

$$\begin{cases} \dot{x}_1(t) = a(x_2(t) - x_1(t) - f(x_1(t))), \\ \dot{x}_2(t) = x_1(t) - x_2(t) + x_3(t), \\ \dot{x}_3(t) = -bx_2(t), \end{cases} \tag{24}$$

in which $a = 10, b = 14.87$. Then, the system matrices are designed by

$$A = \begin{bmatrix} -c & c & 0 \\ 1 & -1 & 1 \\ 0 & -d & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The description for chaotic behavior of Chua's circuit can be expressed by $x_0 = [5,5,5]^T, x_1 = [0.5, 0.7, .2]^T, x_2 = [1.5, 1.2, 1.6]^T$, $x_3 = [5,5,5]^T, x_4 = [3.2, 2.4, 2.3]^T, x_5 = [0.1, 0.1, 0.1]^T, x_6 = [0.8, 0.8, 0.8]^T$. In addition, $f(x_i(t)) = [0.333\sin(x_{i3}(t)), 0, 0]^T$ is the nonlinear function of the system. Select parameters as $\mu = 0.04, \tilde{\varsigma} = 12, \bar{c}_0 = 0.04, \hat{\varsigma} = 0.1, k_4 = 0.08, k_5 = 1$, $k_6 = 0.01, \tilde{\ell} = 0.9, \bar{\varepsilon}_0 = 2.3, \bar{c} = 0.08, \nu = 0.043, \ell = 0.071$. The logarithmic quantizer is considered with the parameters $\xi = 0.005$. Solving the LMI, one choose $t_{\min} = -0.0146, n_0 = 3, m_0 = 4$. Based on (19) and (20), we compute the following matrices P and Q

$$P = \begin{bmatrix} 1.0770 & 0.0386 & -1.0067 \\ 0.0386 & 0.4641 & -0.2941 \\ -1.0067 & -0.2941 & 6.4827 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1.3126 & 0.1631 & -2.1389 \\ 0.1631 & 0.6582 & -0.4017 \\ -2.1389 & -0.4017 & 1.5214 \end{bmatrix}.$$

Then, one can further get (See Figs. 1)

$$K = \begin{bmatrix} -1.0770 & -0.0386 & 1.0067 \\ -0.0386 & -0.4641 & 0.2941 \\ 1.0067 & 0.2941 & -6.4827 \end{bmatrix},$$

$$F = \begin{bmatrix} 2.1896 & 1.2558 & 1.2728 \\ 2.2528 & 0.1203 & 2.3391 \\ 0.4422 & 0.3547 & 0.4580 \end{bmatrix}.$$

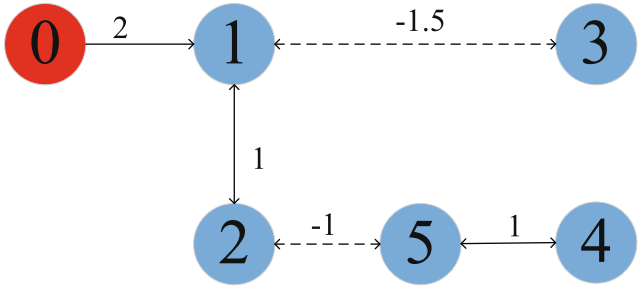


Fig. 1. The communication topology.

The simulation results are shown in Figs. 2–8. As indicated in Fig. 2, after attacks destroy the control channels, the system errors turn from convergence to divergence, and the destroyed controller can no longer guarantee the normal quantization communication. In the attack's sleeping interval, the communication has a complete recovery, the system can continue to converge, and finally the system error will be zero or bounded. Then, the sequences of DoS attacks are shown in Fig. 2. Under the controller (7), the states of five followers and the leader with antagonistic edges are shown in Fig. 3, which reflects the control law could solve the secure bipartite consensus problem for nonlinear MASs under DoS attacks. Contrast to the linear controller and observer designed in the literature [17,34], which cannot deal with nonlinear scenario, also the controller designed in [35] does not hold in this bipartite consensus model.

On the other hand, the Fig. 4 depicts the error between the (signed) leader's state and each follower agent on a logarithmic quantizer under the proposed control protocol (7), and it's observed that although the state difference diverges during the attack period, the secure bipartite consensus of the state can be achieved after the attacks. Note that different from [32,37] that the quantization parameter only has an effect on the convergence rate, but also has the effect on the state differences in our paper. That is, the state differences of $\hat{e}_i(t)$ are given under different quantization parameter ξ. With smaller ξ, the state differences become smaller. The reason why we can get this result is that the quantizer in this paper only needs to quantize the output state error information of agents, and it can adjust the measurement of quantized step based on the output value. Similarly, Fig. 5 reports the temporal evolution of the state of the agents, when the observer-based con-

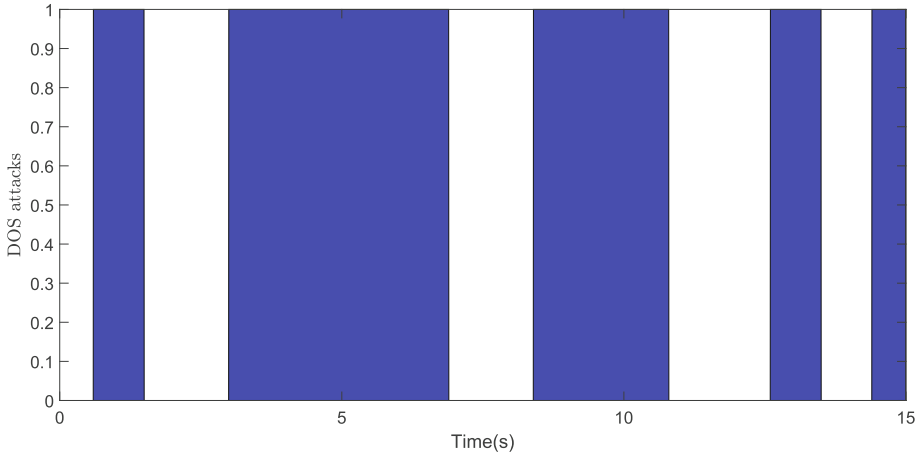
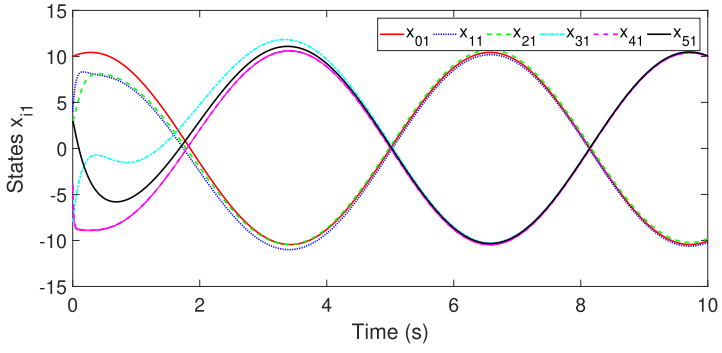
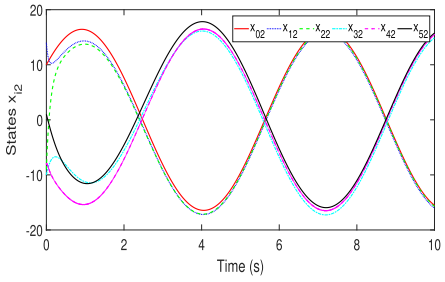


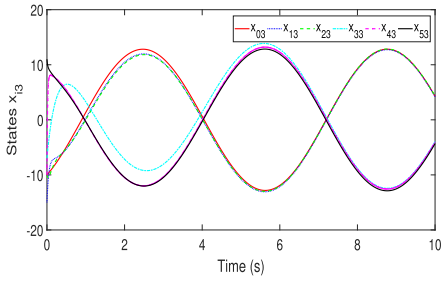
Fig. 2. Time sequences of DoS attacks.



(a) State trajectories of $x_{i1}\,(t)$ under controller (5)

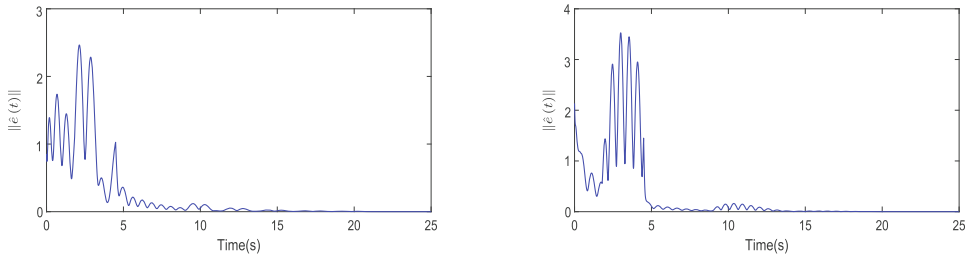


(b) State trajectories of $x_{i2}\,(t)$



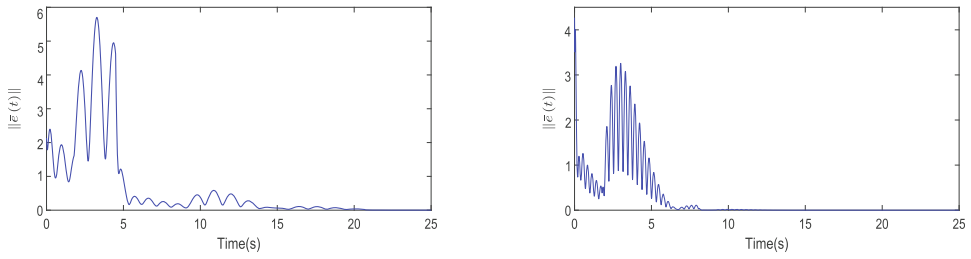
(c) State trajectories of $x_{i3}\,(t)$

Fig. 3. Temporal evolutions of state $x_{i1}(t), x_{i2}(t)$, and $x_{i3}(t)$ under the controller (7) integrated quantization and DoS attacks.



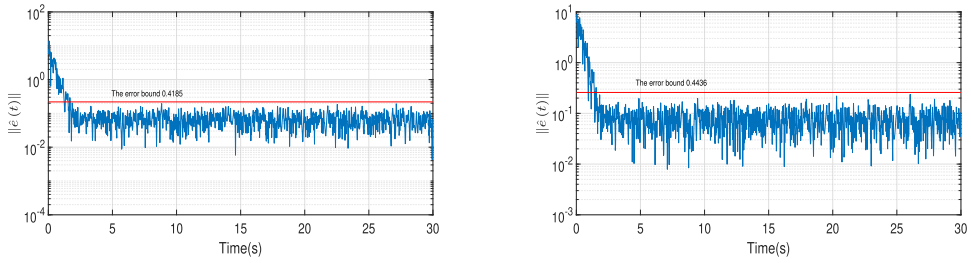
(a) Consensus error with $\xi = 0.02$ (b) Consensus error with $\xi = 0.01$

Fig. 4. Consensus tracking error under (7) with DoS attacks and different ξ.



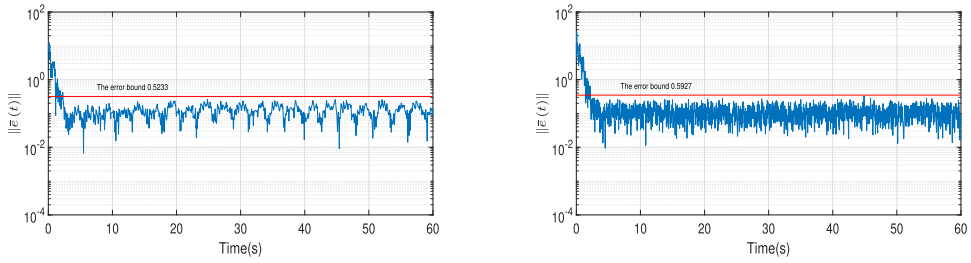
(a) Observer error with $\xi = 0.02$ (b) Observer error with $\xi = 0.01$

Fig. 5. Observer error under (7) with DoS attacks and different ξ.



(a) Consensus error with $\xi = 0.01$ (b) Consensus error with $\xi = 0.02$

Fig. 6. Consensus error under (15) with DoS attacks and different ξ.

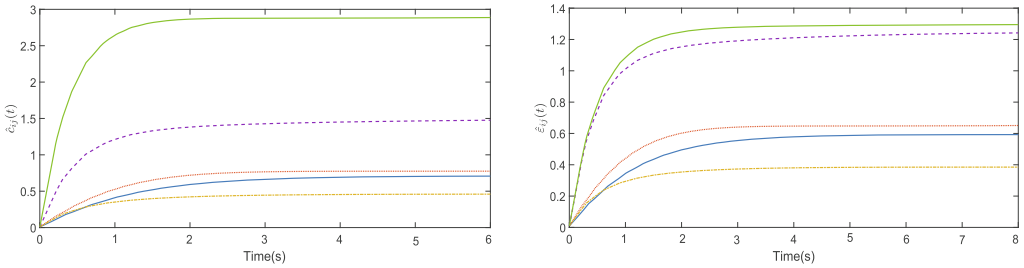


(a) Observer error with $\xi = 0.01$ (b) Observer error with $\xi = 0.02$

Fig. 7. Observer error under (15) with DoS attacks and different ξ.

troller is enacted, showing that the state of the followers converges to a leader–follower bipartite consensus despite of the existence of DoS attacks.

In Figs. 6,7, it's clear to see that secure bipartite bounded consensus of the MAS can be reached with the developed fully distributed control strategy (15) under quantized communication and DoS attacks. Note that the simulation results of (15) is based on the (7), so we don't need to give the similar explanations of (15). As a comparison, we will show that the distributed state feedback controller proposed in [16,20] cannot be applied to solve the fully distributed output secure bipartite consensus problem of MASs over DoS attacks in this paper, even though the concerned nonlinear agent dynamics are the same. The selection of simulation parameters is the same as the previous setting. However, the sim-



(a) (b)

Fig. 8. Time evolutions of $\hat{c}_{ij}(t)$ and $\hat{\varepsilon}_{ij}(t)$.

ulation result can not handle the state differences by selecting the ξ adaptively. The similar results are omitted. On the other hand, the controller (15) is fully distributed, and it is not required to know a priori knowledge of any global information. While the common controllers should include a priori information of states, so the the static controller and state feedback control design in [23] does not hold in this model simulation. Furthermore, the Fig. 8 depicted that the evolutions of adaptive parameters $\hat{c}_{ij}(t)$ and $\hat{\varepsilon}_{ij}(t)$. Under the energy constraints and the designed distributed controllers (7) and (15) based on relative quantitation information, MASs (4)- (5) can finally achieve the secure bipartite (bounded) consensus.

6. Conclusions

In this paper, we have solved the secure bipartite consensus problem for nonlinear MASs with quantized information subject to DoS attacks. Based on a connected structurally balanced signed graph, a new secure output feedback control protocol integrated of logarithmic quantizer and relative output measurements of neighboring agents is proposed to realize secure control under DoS attacks. Furthermore, we also develop a control strategy with dynamic coupling gains, which is fully distributed and agents are not required to know a priori knowledge of any global information and the quantizer only need to quantize the output state error information of agents. Then, theoretical guarantees on the effectiveness of the proposed controller in steering the system to a secure bipartite leader–follower consensus under quantized output measurements and intermittent DoS attacks are derived. Finally, numerical simulations inspired by a real-world physical MAS are provided to verify the usefulness of the presented controllers.

The promising results, supported by the example illustrated in Section 5, suggest the possible extension of our methodology to different practical scenarios. In particular, it would be interesting to extend our results to solve the filtering issues of networked systems under cyber-attacks [21,24]. In addition, following [8,28,30,40], a promising idea can be that of implementing dynamic event-triggered strategies to realize the fully distributed secure bipartite consensus for MASs under DoS attacks. The two ideas will be investigated in our future study.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

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Appendix A. Proof of Theorem 1

Construct the Lyapunov candidate $V(t)$ as follows

$$V(t)=\begin{cases}\hat{e}^{\top}(t)(L_R\otimes P)\hat{e}(t)+\bar{e}^{\top}(t)(L_R\otimes P)\bar{e}(t), t\in\overline{\mathrm{T}}_f(t_0,t),\\ \hat{e}^{\top}(t)(L_R\otimes Q)\hat{e}(t)+\bar{e}^{\top}(t)(L_R\otimes Q)\bar{e}(t), t\in\overline{\mathrm{T}}_d(t_0,t).\end{cases}$$

Without DoS attacks on the system, that is, $t\in\overline{\mathrm{T}}_f(t_0,t)$, let $V_1(t)=\hat{e}^{\top}(t)(L_R\otimes P)\hat{e}(t), V_2(t)=\bar{e}^{\top}(t)(L_R\otimes P)\bar{e}(t)$, then taking the derivative $V_1(t)$ and $V_2(t)$, one obtains

$$\begin{aligned}\dot{V}_1(t)=&2\hat{e}^{\top}(t)(L_R\otimes P)\dot{\hat{e}}(t)\\ \leqslant&\hat{e}^{\top}(t)\Big[L_R\otimes\Big(A^{\mathrm{T}}P+PA\Big)-2cL_R^2\otimes\Gamma\Big]\hat{e}(t)\\ &+2\varepsilon\hat{e}^{\top}(t)\Big(L_R^2\otimes PFC\Big)\mathrm{Z}(t)+2\varepsilon\hat{e}^{\top}(t)\Big(L_R^2\otimes PFC\Big)\bar{e}(t)\\ &+2\hat{e}^{\top}(t)(L_R\otimes P)\times(f(\hat{x}(t),t)-(SI_N\otimes f(x_0(t),t))),\\ \dot{V}_2(t)=&2\bar{e}^{\top}(t)(L_R\otimes P)\dot{\bar{e}}(t)\\ \leqslant&\bar{e}^{\top}(t)\Big[L_R\otimes\Big(A^{\mathrm{T}}P+PA\Big)-2\varepsilon L_R^2\otimes PFC\Big]\bar{e}^{\top}(t)\\ &-2\varepsilon\bar{e}^{\top}(t)\Big(L_R^2\otimes PFC\Big)\mathrm{Z}(t)\\ &+2\bar{e}^{\top}(t)(L_R\otimes P)\times(f(x(t),t)-f(\hat{x}(t),t)).\end{aligned}\tag{A.1}$$

Based on $F=P^{-1}C^{\top}, \mathrm{Z}(t)\in\mathscr{K}(\mathrm{H}\bar{e}(t)), \mathrm{H}_i\in[-\xi,+\xi]$, and according to Assumption 3 and Young's inequality, we obtain

$$\begin{aligned}&2\varepsilon\hat{e}^{\mathrm{T}}(t)\Big(L_R^2\otimes C^{\mathrm{T}}C\Big)\mathrm{Z}(t)\\ \leqslant&\varepsilon k_1\hat{e}^{\mathrm{T}}(t)\Big(L_R^2\otimes C^{\mathrm{T}}C\Big)\hat{e}(t)+\frac{1}{k_1}\mathrm{Z}^{\mathrm{T}}(t)\Big(L_R^2\otimes C^{\mathrm{T}}C\Big)\mathrm{Z}(t)\\ \leqslant&\varepsilon k_1\hat{e}^{\mathrm{T}}(t)\Big(L_R^2\otimes C^{\mathrm{T}}C\Big)\hat{e}(t)+\frac{\xi^2}{k_1}\bar{e}^{\mathrm{T}}(t)\Big(L_R^2\otimes C^{\mathrm{T}}C\Big)\bar{e}(t),\end{aligned}\tag{A.2}$$

and

$$\begin{aligned}&2\varepsilon\hat{e}^{\mathrm{T}}(t)\Big(L_R^2\otimes C^{\mathrm{T}}C\Big)\bar{e}(t)\\ \leqslant&\varepsilon k_2\hat{e}^{\mathrm{T}}(t)\Big(L_R^2\otimes C^{\mathrm{T}}C\Big)\hat{e}(t)+\frac{1}{k_2}\bar{e}^{\mathrm{T}}(t)\Big(L_R^2\otimes C^{\mathrm{T}}C\Big)\bar{e}(t),\end{aligned}$$

and

$$\begin{aligned}&2\bar{e}^{\mathrm{T}}(t)(L_R\otimes P)\times(f(x(t),t)-f(\hat{x}(t),t))\\ \leqslant&2\bar{e}^{\mathrm{T}}(t)\Big(\sqrt{L_R}\otimes I_N\Big)\Big(\sqrt{L_R}\otimes P\Big)\rho\bar{e}(t)\\ \leqslant&\bar{e}^{\mathrm{T}}(t)(L_R\otimes I_N)\bar{e}(t)+\rho^2\bar{e}^{\mathrm{T}}(t)\Big(L_R\otimes P^{\mathrm{T}}P\Big)\bar{e}(t).\end{aligned}\tag{A.4}$$

Similarly, one gets

$$
\begin{aligned}
&-2\bar{e}^{\mathrm{T}}(t)\left(L_R^2\otimes C^{\mathrm{T}}C\right)\mathrm{Z}(t)\\
&\leqslant k_3\bar{e}^{\mathrm{T}}(t)\left(L_R^2\otimes C^{\mathrm{T}}C\right)\bar{e}^{\mathrm{T}}(t)+\frac{1}{k_3}\mathrm{Z}^{\mathrm{T}}(t)\left(L_R^2\otimes C^{\mathrm{T}}C\right)\mathrm{Z}(t)\\
&\leqslant k_3\bar{e}^{\mathrm{T}}(t)\left(L_R^2\otimes C^{\mathrm{T}}C\right)\bar{e}(t)+\frac{\xi^2}{k_3}\bar{e}^{\mathrm{T}}(t)\left(L_R^2\otimes C^{\mathrm{T}}C\right)\bar{e}(t),
\end{aligned}
\tag{A.5}
$$

and

$$
\begin{aligned}
&2\hat{e}^{\mathrm{T}}(t)(L_R\otimes P)\times(f(\hat{x}(t),t)-(SI_N\otimes f(x_0(t),t)))\\
&\leqslant 2\hat{e}^{\mathrm{T}}(t)\left(\sqrt{L_R}\otimes I_N\right)\left(\sqrt{L_R}\otimes P\right)\rho\hat{e}(t)\\
&\leqslant\hat{e}^{\mathrm{T}}(t)(L_R\otimes I_N)\hat{e}(t)+\rho^2\hat{e}(t)(L_R\otimes P^{\top}P)\hat{e}(t).
\end{aligned}
\tag{A.6}
$$

Then, substituting (A.2)-(A.6) into (A.1), we obtain

$$
\begin{aligned}
\dot{V}(t)\leqslant\ &\hat{e}^{\mathrm{T}}(t)\Big[L_R\otimes\left(A^{\mathrm{T}}P+PA\right)-2cL_R^2\otimes\Gamma+\varsigma L_R^2\otimes C^{\mathrm{T}}C\\
&+L_R\otimes I_N+\rho^2L_R\otimes P^{\mathrm{T}}P\Big]\hat{e}(t)+\bar{e}^{\mathrm{T}}(t)\Big[L_R\otimes\left(A^{\mathrm{T}}P+PA\right)\\
&-\bar{\varsigma}L_R^2\otimes C^{\mathrm{T}}C+L_R\otimes I_N+\rho^2L_R\otimes P^{\mathrm{T}}P\Big]\bar{e}(t),
\end{aligned}
\tag{A.7}
$$

in which $\varsigma=\varepsilon(k_1+k_2),\bar{\varsigma}=\left(2\varepsilon-\frac{\xi^2}{k_1}-\frac{1}{k_2}-k_3-\frac{\xi^2}{k_3}\right)$. Define $\hat{\varphi}(t)=(S\otimes I_n)\hat{e}(t)$ and $\bar{\varphi}(t)=(S\otimes I_n)\bar{e}(t)$, we have

$$
\begin{aligned}
\dot{V}(t)\leqslant\ &\hat{\varphi}^{\mathrm{T}}(t)\Big[(\bar{L}\otimes I_n)\otimes\left(A^{\mathrm{T}}P+PA-2c\bar{L}\otimes\Gamma+\varsigma\bar{L}\otimes C^{\mathrm{T}}C\right.\\
&\left.+I_N+\rho^2P^{\mathrm{T}}P\right]\hat{\varphi}(t)+\bar{\varphi}^{\mathrm{T}}(t)\Big[(\bar{L}\otimes I_n)\otimes\left(A^{\mathrm{T}}P+PA\right)\\
&-\bar{\varsigma}\bar{L}\otimes C^{\mathrm{T}}C+I_N+\rho^2P^{\mathrm{T}}P\Big]\bar{\varphi}(t).
\end{aligned}
\tag{A.8}
$$

Depended on Lemma 1, one concludes $U^T\bar{L}U=\mathrm{diag}(\lambda_1,\ldots,\lambda_N)=\Delta$. Then we can get $\bar{L}=U^{\mathrm{T}}\Delta U$. Let $\tilde{\varphi}(t)=\left(U^{\mathrm{T}}\otimes I_n\right)\hat{\varphi}(t)$ and $\widehat{\varphi}(t)=\left(U^{\mathrm{T}}\otimes I_n\right)\bar{\varphi}(t)$, and it follows from the facts $c\geqslant\frac{1}{2\lambda_1},\varsigma\leqslant\frac{1}{\lambda_N},\bar{\varsigma}\geqslant\frac{1}{\lambda_1}$, one further obtains

$$
\begin{aligned}
\dot{V}(t)\leqslant\ &\tilde{\varphi}^{\mathrm{T}}(t)\Big[(\Delta\otimes I_n)\otimes\left(A^{\mathrm{T}}P+PA-PBB^{\mathrm{T}}P+C^{\mathrm{T}}C\right.\\
&\left.+I_N+\rho^2P^{\mathrm{T}}P\right)\Big]\tilde{\varphi}(t)+\widehat{\varphi}^{\mathrm{T}}(t)\Big[(\Delta\otimes I_n)\otimes\left(A^{\mathrm{T}}P+PA\right.\\
&\left.-C^{\mathrm{T}}C+I_N+\rho^2P^{\mathrm{T}}P\right)\Big]\widehat{\varphi}(t),
\end{aligned}
\tag{A.9}
$$

Based on (10), one has

$$
\begin{aligned}
\dot{V}(t)&\leqslant-m_1\tilde{\varphi}^{\mathrm{T}}(t)(\Delta\otimes P)\tilde{\varphi}(t)-m_2\widehat{\varphi}^{\mathrm{T}}(t)(\Delta\otimes P)\widehat{\varphi}(t)\\
&\leqslant-m_1\hat{e}^{\top}(t)(L_R\otimes P)\hat{e}(t)-m_2\bar{e}^{\top}(t)(L_R\otimes P)\bar{e}(t)\\
&\leqslant-m_0V(t).
\end{aligned}
\tag{A.10}
$$

With DoS attacks on the system, that is $t\in\overline{\mathrm{T}}_d(t_0,t)$, let $\widetilde{V}_1(t)=\hat{e}^{\top}(t)(L_R\otimes Q)\hat{e}(t),\widetilde{V}_2(t)=\bar{e}^{\top}(t)(L_R\otimes Q)\bar{e}(t)$, one obtains

$$V(t)=\widetilde{V}_1(t)+\widetilde{V}_2(t).$$

Then, one has

$$
\begin{aligned}
\dot{\widetilde{V}}_1(t)&=2\hat{e}^{\mathrm{T}}(t)(L_R\otimes Q)\dot{\hat{e}}(t)\\
&\leqslant\hat{e}^{\mathrm{T}}(t)\Big[L_R\otimes\left(A^{\mathrm{T}}Q+QA\right)\Big]\hat{e}(t)\\
&\quad+2\hat{e}^{\mathrm{T}}(t)(L_R\otimes Q)\times(f(\hat{x}(t),t)-(SI_N\otimes f(x_0(t),t))),\\
\dot{\widetilde{V}}_2(t)&=2\bar{e}^{\mathrm{T}}(t)(L_R\otimes Q)\dot{\bar{e}}(t)\\
&\leqslant\bar{e}^{\mathrm{T}}(t)\Big[L_R\otimes\left(A^{\mathrm{T}}Q+QA\right)\Big]\bar{e}^{\mathrm{T}}(t)\\
&\quad+2\bar{e}^{\mathrm{T}}(t)(L_R\otimes Q)\times(f(x(t),t)-f(\hat{x}(t),t)).
\end{aligned}
$$

Similar to the forgoing analysis, we have

$$
\begin{aligned}
\dot{V}(t)\ &\leqslant\hat{e}^{\mathrm{T}}(t)\Big[L_R\otimes\left(A^{\mathrm{T}}Q+QA\right)+L_R\otimes I_N+\rho^2L_R\otimes Q^{\mathrm{T}}Q\Big]\hat{e}(t)\\
&\quad+\bar{e}^{\mathrm{T}}(t)\Big[L_R\otimes\left(A^{\mathrm{T}}Q+QA\right)+L_R\otimes I_N+\rho^2L_R\otimes Q^{\mathrm{T}}Q\Big]\bar{e}^{\mathrm{T}}(t).
\end{aligned}
\tag{A.11}
$$

Then, according to (11), one obtains $\dot{V}_1(t)<n_1\bar{e}^{\mathrm{T}}(t)(L_R\otimes Q)\bar{e}(t)$, $\dot{V}_2(t)<n_2\hat{e}^{\mathrm{T}}(t)(L_R\otimes Q)\hat{e}(t)$, therefore,

$$\dot{V}(t)\leqslant\ n_0V(t),\tag{A.12}$$

where $n_0=\max_{i\in\{1,2\}}\{n_i\}$. The derivation function of Lyapunov function satisfies the following conditions

$$\dot{V}(t)\leqslant\begin{cases}-m_0V(t),t\in[t_{2k},t_{2k+1}),\\ n_0V(t),t\in[t_{2k+1},t_{2k+2}),\end{cases}$$

in which $[t_{2k},t_{2k+1})$ and $[t_{2k+1},t_{2k+2})$ denote the time sequences of $\overline{\mathrm{T}}_f(t_0,t)$ and $\overline{\mathrm{T}}_d(t_0,t)$, respectively. After that, one has

$$\dot{V}(t)\leqslant\begin{cases}-m_0V(t),t\in\overline{\mathrm{T}}_f(t_0,t),\\ n_0V(t),t\in\overline{\mathrm{T}}_d(t_0,t).\end{cases}$$

For any time interval $[t_0,t)$, when $t\in[t_{2k},t_{2k+1})$, by the mathematical induction, one obtains

$$
\begin{aligned}
V(t)&\leqslant e^{-m_0(t-t_{2k})}V(t_{2k})\\
&\leqslant\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}e^{-m_0(t-t_{2k})+n_0(t_{2k}-t_{2k-1})}V(t_{2k-1})\\
&\leqslant\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)}e^{-m_0(t-t_{2k})+n_0(t_{2k}-t_{2k-1})}\times\\
&\quad e^{-m_0(t_{2k-1}-t_{2k-2})+\cdots-m_0(t_1-t_0)}V(t_0)\\
&\leqslant\mu^{2k}e^{-m_0\left|\overline{\mathrm{T}}_f(t_0,t)\right|+n_0\left|\overline{\mathrm{T}}_d(t_0,t)\right|}V(t_0)\\
&\leqslant\mu^{N(t_0,t)}e^{-m_0\left|\overline{\mathrm{T}}_f(t_0,t)\right|+n_0\left|\overline{\mathrm{T}}_d(t_0,t)\right|}V(t_0).
\end{aligned}
\tag{A.13}
$$

When $t\in[t_{2k+1},t_{2k+2})$, there is

$$
\begin{aligned}
V(t)&\leqslant e^{-m_0\left(t-t_{2k+1}\right)}V(t_{2k+1})\\
&\leqslant\mu^{2k}e^{-m_0\left|\overline{\mathrm{T}}_f(t_0,t)\right|+n_0\left|\overline{\mathrm{T}}_d(t_0,t)\right|}V(t_0)\\
&\leqslant\mu^{N(t_0,t)}e^{-m_0\left|\overline{\mathrm{T}}_f(t_0,t)\right|+n_0\left|\overline{\mathrm{T}}_d(t_0,t)\right|}V(t_0).
\end{aligned}
\tag{A.14}
$$

Thus, for any time interval $[t_0,t)$, there is

$$V(t)\leqslant e^{\ln\mu N(t_0,t)}e^{-m_0\left|\overline{\mathrm{T}}_f(t_0,t)\right|+n_0\left|\overline{\mathrm{T}}_d(t_0,t)\right|}V(t_0).$$

Then, one has

$$
\begin{aligned}
&-m_0\left|\overline{\mathrm{T}}_f(t_0,t)\right|+n_0\left|\overline{\mathrm{T}}_d(t_0,t)\right|\\
&=-m_0\left(t-t_0-\overline{\mathrm{T}}_d(t_0,t)\right)+n_0\left|\overline{\mathrm{T}}_d(t_0,t)\right|\\
&\leqslant-m_0(t-t_0)+(m_0+n_0)\left(T_0+\frac{t-t_0}{T_1}\right)\\
&\leqslant-\tau(t-t_0)+(m_0+n_0)T_0.
\end{aligned}
\tag{A.15}
$$

According to (12), (13) and (14), there is

$$e^{\ln\mu N(t_0,t)}\leqslant e^{(2\ln\mu+(m_0+n_0)\&)N(t_0,t)}\leqslant e^{\tau(t-t_0)}.$$

In conclusion, the multi-agent systems (4) and (5) can achieve the secure bipartite consensus under the static control protocol (7). ■

Appendix B. Proof of Theorem 2

Construct a Lyapunov function candidate $V(t)$ as follows

$$V(t)=\begin{cases} V_1(t)+V_2(t)+\frac{1}{2\bar{\zeta}}\sum_{i=1}^{N}\left(\hat{\varepsilon}_{ij}(t)-\bar{\varepsilon}_0\right)^2 \\ \quad +\frac{1}{2\bar{\zeta}}\sum_{i=1}^{N}\left(\hat{c}_{ij}(t)-\bar{c}_0\right)^2, t\in\overline{\mathrm{T}}_f(t_0,t), \\ \bar{V}_1(t)+\bar{V}_2(t), t\in\overline{\mathrm{T}}_d(t_0,t), \end{cases} \tag{B.1}$$

where $V_1(t)=\hat{e}^\top(t)(L_R\otimes P)\hat{e}(t), V_2(t)=\bar{e}^\top(t)(L_R\otimes P)\bar{e}(t)$. Similar to Appendix A, when considering the case that without DoS attacks, taking the derivative $V_1(t)$ and $V_2(t)$, one obtains

$$\begin{aligned} \dot{V}_1(t)=&2\hat{e}^\top(t)(L_R\otimes P)\dot{\hat{e}}(t)\\ \leqslant&\hat{e}^\top(t)\left[L_R\otimes\left(A^{\mathrm{T}}P+PA\right)-2L_RL_c\otimes PBB^{\mathrm{T}}P\right]\hat{e}(t)\\ &+2\hat{e}^\top(t)(L_RL_\varepsilon\otimes PFC)\overline{Z}(t)+2\hat{e}^\top(t)(L_RL_\varepsilon\otimes PFC)\bar{e}(t)\\ &+2\hat{e}^\top(t)(L_R\otimes P)\times(f(\hat{x}(t),t)-(SI_N\otimes f(x_0(t),t))),\\ \dot{V}_2(t)=&2\bar{e}^\top(t)(L_R\otimes P)\dot{\bar{e}}(t)\\ \leqslant&\bar{e}^\top(t)\left[L_R\otimes\left(A^{\mathrm{T}}P+PA\right)-2L_RL_\varepsilon\otimes PFC\right]\bar{e}^\top(t)\\ &-2\bar{e}^\top(t)(L_RL_\varepsilon\otimes PFC)\overline{Z}(t)+2\bar{e}^\top(t)(L_R\otimes P)\\ &\times(f(x(t),t)-f(\hat{x}(t),t)). \end{aligned} \tag{B.2}$$

Then, one has

$$\begin{aligned} 2\quad&\hat{e}^{\mathrm{T}}(t)\left(L_RL_\varepsilon\otimes C^{\mathrm{T}}C\right)Z(t)\\ \leqslant\quad&k_1\hat{e}^{\mathrm{T}}(t)\left(L_RL_\varepsilon\otimes C^{\mathrm{T}}C\right)\hat{e}(t)+\tfrac{1}{k_1}Z^{\mathrm{T}}(t)\left(L_RL_\varepsilon\otimes C^{\mathrm{T}}C\right)Z(t)\\ \leqslant\quad&k_1\hat{e}^{\mathrm{T}}(t)\left(L_RL_\varepsilon\otimes C^{\mathrm{T}}C\right)\hat{e}(t)+\tfrac{\xi^2}{k_1}\bar{e}^{\mathrm{T}}(t)\left(L_RL_\varepsilon\otimes C^{\mathrm{T}}C\right)\bar{e}(t), \end{aligned} \tag{B.3}$$

$$\begin{aligned} 2\quad&\hat{e}^{\mathrm{T}}(t)\left(L_RL_\varepsilon\otimes C^{\mathrm{T}}C\right)\bar{e}(t)\\ \leqslant\quad&k_2\hat{e}^{\mathrm{T}}(t)\left(L_RL_\varepsilon\otimes C^{\mathrm{T}}C\right)\hat{e}(t)+\tfrac{1}{k_2}\bar{e}^{\mathrm{T}}(t)\left(L_RL_\varepsilon\otimes C^{\mathrm{T}}C\right)\bar{e}(t), \end{aligned} \tag{B.4}$$

and

$$\begin{aligned} 2\quad&\bar{e}^{\mathrm{T}}(t)(L_R\otimes P)\times(f(x(t),t)-f(\hat{x}(t),t))\\ \leqslant\quad&2\bar{e}^{\mathrm{T}}(t)\left(\sqrt{L_R}\otimes I_N\right)\left(\sqrt{L_R}\otimes P\right)\rho\bar{e}(t)\\ \leqslant\quad&\bar{e}^{\mathrm{T}}(t)(L_R\otimes I_N)\bar{e}(t)+\rho^2\bar{e}^{\mathrm{T}}(t)\left(L_R\otimes P^{\mathrm{T}}P\right)\bar{e}(t). \end{aligned} \tag{B.5}$$

Similarly, one gets

$$\begin{aligned} 2\quad&\bar{e}^{\mathrm{T}}(t)\left(L_RL_\varepsilon\otimes C^{\mathrm{T}}C\right)Z(t)\\ \leqslant\quad&k_3\bar{e}^{\mathrm{T}}(t)\left(L_RL_\varepsilon\otimes C^{\mathrm{T}}C\right)\bar{e}^{\mathrm{T}}(t)+\tfrac{1}{k_3}Z^{\mathrm{T}}(t)\left(L_RL_\varepsilon\otimes C^{\mathrm{T}}C\right)Z(t)\\ \leqslant\quad&k_3\bar{e}^{\mathrm{T}}(t)\left(L_RL_\varepsilon\otimes C^{\mathrm{T}}C\right)\bar{e}(t)+\tfrac{\xi^2}{k_3}\bar{e}^{\mathrm{T}}(t)\left(L_RL_\varepsilon\otimes C^{\mathrm{T}}C\right)\bar{e}(t), \end{aligned} \tag{B.6}$$

and

$$\begin{aligned} 2\quad&\hat{e}^{\mathrm{T}}(t)(L_R\otimes P)\times(f(\hat{x}(t),t)-(SI_N\otimes f(x_0(t),t)))\\ \leqslant\quad&2\hat{e}^{\mathrm{T}}(t)\left(\sqrt{L_R}\otimes I_N\right)\left(\sqrt{L_R}\otimes P\right)\rho\hat{e}(t)\\ \leqslant\quad&\hat{e}^{\mathrm{T}}(t)(L_R\otimes I_N)\hat{e}(t)+\rho^2\hat{e}(t)\left(L_R\otimes P^{\top}P\right)\hat{e}(t). \end{aligned} \tag{B.7}$$

Also according to $\Gamma=PBB^{\mathrm{T}}P$, one concludes

$$\begin{aligned} \dot{V}(t)\leqslant\quad&\hat{e}^{\mathrm{T}}(t)\left[L_R\otimes\left(A^{\mathrm{T}}P+PA\right)-2L_RL_c\otimes\Gamma\right.\\ &\left.+\hat{\varsigma}L_RL_\varepsilon\otimes C^{\mathrm{T}}C+L_R\otimes I_N+\rho^2L_R\otimes P^{\mathrm{T}}P\right]\hat{e}(t)\\ &+\bar{e}^{\mathrm{T}}(t)\left[L_R\otimes\left(A^{\mathrm{T}}P+PA\right)-2L_RL_\varepsilon\otimes C^{\mathrm{T}}C\right.\\ &\left.+\tilde{\varsigma}L_RL_\varepsilon\otimes C^{\mathrm{T}}C+L_R\otimes I_N+\rho^2L_R\otimes P^{\mathrm{T}}P\right]\bar{e}^{\mathrm{T}}(t)\\ &+1/\zeta\sum_{i=1}^{N}\hat{c}_{ij}(t)\dot{\hat{c}}_{ij}(t)-\bar{c}_0/\zeta\sum_{i=1}^{N}\dot{\hat{c}}_{ij}(t)\\ &+1/\tilde{\zeta}\sum_{i=1}^{N}\hat{\varepsilon}_{ij}(t)\dot{\hat{\varepsilon}}_{ij}(t)-\bar{c}_0/\tilde{\zeta}\sum_{i=1}^{N}\dot{\hat{\varepsilon}}_{ij}(t), \end{aligned} \tag{B.8}$$

where $\hat{\varsigma}=k_1+k_2, \tilde{\varsigma}=\frac{\xi^2}{k_1}+\frac{1}{k_2}+k_3+\frac{\xi^2}{k_3}$. Based on the fact $2-\tilde{\varsigma}=\hat{\varsigma}$, one has

$$\begin{aligned} \dot{V}(t)\leqslant\quad&\hat{e}^{\mathrm{T}}(t)\left[L_R\otimes\left(A^{\mathrm{T}}P+PA\right)-2L_RL_c\otimes\Gamma+\hat{\varsigma}L_RL_\varepsilon\otimes C^{\mathrm{T}}C\right.\\ &\left.+L_R\otimes I_N+\rho^2L_R\otimes P^{\mathrm{T}}P\right]\hat{e}(t)+\bar{e}^{\mathrm{T}}(t)\left[L_R\otimes\left(A^{\mathrm{T}}P+PA\right)\right.\\ &\left.-\tilde{\varsigma}L_RL_\varepsilon\otimes C^{\mathrm{T}}C+L_R\otimes I_N+\rho^2L_R\otimes P^{\mathrm{T}}P\right]\bar{e}^{\mathrm{T}}(t)+1/\zeta\sum_{i=1}^{N}\hat{c}_{ij}(t)\dot{\hat{c}}_{ij}(t)\\ &-\bar{c}_0/\zeta\sum_{i=1}^{N}\dot{\hat{c}}_{ij}(t)+1/\tilde{\zeta}\sum_{i=1}^{N}\hat{\varepsilon}_{ij}(t)\dot{\hat{\varepsilon}}_{ij}(t)-\bar{c}_0/\tilde{\zeta}\sum_{i=1}^{N}\dot{\hat{\varepsilon}}_{ij}(t). \end{aligned}$$

On the other hand, one has

$$\begin{aligned} \sum_{i=1}^{N}\hat{c}_{ij}(t)\dot{\hat{c}}_{ij}(t)=&-\zeta\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\hat{c}_{ij}^2(t)+\zeta\sum_{i=1}^{N}\hat{c}_{ij}(t)\varphi^{\mathrm{T}}(t)\Gamma\varphi(t)\\ \leqslant&-\zeta\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\hat{c}_{ij}^2(t)+\zeta\hat{e}^\top(t)(L_RL_c\otimes\Gamma)\hat{e}(t), \end{aligned} \tag{B.9}$$

and

$$\begin{aligned} -\sum_{i=1}^{N}\bar{c}_0(t)\dot{\hat{c}}_{ij}(t)=&\zeta\bar{c}_0\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\hat{c}_{ij}(t)-\zeta\sum_{i=1}^{N}\bar{c}_0\varphi^{\mathrm{T}}(t)\Gamma\varphi(t)\\ \leqslant&\zeta\bar{c}_0\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\hat{c}_{ij}(t)-\zeta\bar{c}_0\hat{e}^\top(t)\left(L_R^2\otimes\Gamma\right)\hat{e}(t), \end{aligned} \tag{B.10}$$

with

$$\begin{aligned} \sum_{i=1}^{N}\hat{\varepsilon}_{ij}(t)\dot{\hat{\varepsilon}}_{ij}(t)=&-\tilde{\zeta}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\hat{\varepsilon}_{ij}^2(t)-\tilde{\zeta}\sum_{i=1}^{N}\hat{\varepsilon}_{ij}(t)\varphi^{\mathrm{T}}(t)\tilde{\Gamma}\varphi(t)\\ &+\tilde{\zeta}\sum_{i=1}^{N}\hat{\varepsilon}_{ij}(t)\tilde{\varphi}^{\mathrm{T}}(t)\Gamma\tilde{\varphi}(t)\\ \leqslant&-\tilde{\zeta}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\hat{\varepsilon}_{ij}^2(t)-\tilde{\zeta}(1+\xi)^2\hat{e}^\top(t)\left(L_RL_\varepsilon\otimes\tilde{\Gamma}\right)\hat{e}(t)\\ &+\tilde{\zeta}(1+\xi)^2\bar{e}^\top(t)\left(L_RL_\varepsilon\otimes\tilde{\Gamma}\right)\bar{e}(t),\} \end{aligned}$$

and

$$\begin{aligned} -\sum_{i=1}^{N}\tilde{\varepsilon}_0\dot{\hat{\varepsilon}}_{ij}(t)=&\tilde{\zeta}\tilde{\varepsilon}_0\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\hat{\varepsilon}_{ij}(t)+\tilde{\zeta}\sum_{i=1}^{N}\tilde{\varepsilon}_0\varphi^{\mathrm{T}}(t)\tilde{\Gamma}\varphi(t)\\ &-\tilde{\zeta}\sum_{i=1}^{N}\tilde{\varepsilon}_0\hat{\varepsilon}_{ij}(t)\tilde{\varphi}^{\mathrm{T}}(t)\Gamma\tilde{\varphi}(t)\\ \leqslant&\tilde{\zeta}\tilde{\varepsilon}_0\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\hat{\varepsilon}_{ij}(t)+\tilde{\zeta}\tilde{\varepsilon}_0(1-\xi)^2\hat{e}^\top(t)\left(L_R^2\otimes\tilde{\Gamma}\right)\hat{e}(t)\\ &-\tilde{\zeta}\tilde{\varepsilon}_0(1-\xi)^2\bar{e}^\top(t)\left(L_R^2\otimes\tilde{\Gamma}\right)\bar{e}(t). \end{aligned}$$

According to $\tilde{\Gamma}=C^{\mathrm{T}}C$, one obtains

$$\begin{aligned} \dot{V}(t)\leqslant\quad&\hat{e}^{\mathrm{T}}(t)\left[L_R\otimes\left(A^{\mathrm{T}}P+PA\right)-\ell L_R^2\otimes\Gamma+\tilde{\ell}L_R^2\otimes\tilde{\Gamma}+L_R\otimes I_N\right.\\ &\left.+\rho^2L_R\otimes P^{\mathrm{T}}P\right]\hat{e}(t)+\bar{e}^{\mathrm{T}}(t)\left[L_R\otimes\left(A^{\mathrm{T}}P+PA\right)-\tilde{\ell}L_R^2\otimes\tilde{\Gamma}\right.\\ &\left.+L_R\otimes I_N+\rho^2L_R\otimes P^{\mathrm{T}}P\right]\bar{e}^{\mathrm{T}}(t)-2\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\hat{c}_{ij}^2(t)\\ &+2\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\bar{c}_0\hat{c}_{ij}(t)-\tilde{\zeta}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\hat{\varepsilon}_{ij}^2(t)+\tilde{\zeta}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\bar{\varepsilon}_0\hat{\varepsilon}_{ij}(t). \end{aligned}$$

Then, depended on the facts $\bar{c}_0\geqslant\frac{1}{\lambda_1}, \ell=\bar{c}_0, \varsigma=(1+\xi)^2, \tilde{\ell}=\bar{\varepsilon}_0(1-\xi)^2, \bar{\varepsilon}_0\leqslant\frac{1}{(1-\xi)^2\lambda_N}$, and similar to the proof of Appendix A, one obtains

$$\begin{aligned}\dot{V}(t) \leqslant & \tilde{\wp}^{\mathrm{T}}(t)\Big[(\Delta\otimes I_n)\otimes\Big(A^{\mathrm{T}}P+PA-PBB^{\mathrm{T}}P+C^{\mathrm{T}}C\\ &+I_N+\rho^2P^{\mathrm{T}}P\Big)\Big]\tilde{\wp}(t)+\widehat{\wp}^{\mathrm{T}}(t)\Big[(\Delta\otimes I_n)\otimes\Big(A^{\mathrm{T}}P+PA\\ &-C^{\mathrm{T}}C+I_N+\rho^2P^{\mathrm{T}}P\Big)\Big]\widehat{\wp}(t)-2\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\hat{c}_{ij}^2(t)\\ &+2\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\bar{c}_0\hat{c}_{ij}(t)-\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\hat{\varepsilon}_{ij}^2(t)+\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\bar{\varepsilon}_0\hat{\varepsilon}_{ij}(t).\end{aligned}$$

Since (19) holds, one concludes

$$\dot{V}(t)<-m_0V(t)+\Delta,$$

where $m_0=\min_{i\in\{1,2\}}\{m_i\}=\min\{2,m_2\}$, $\Delta=\frac{m_0}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\bar{c}_0^2+\frac{m_0}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\bar{\varepsilon}_0^2$. With DoS attacks on the system, similarly, we have

$$\dot{V}(t)\leqslant\ n_0V(t), \tag{B.11}$$

where $n_0=\max_{i\in\{1,2\}}\{n_i\}$. The derivation function of Lyapunov function satisfies the following conditions

$$\dot{V}(t)\leqslant\begin{cases}-m_0V(t)+\Delta, t\in\overline{\mathrm{T}}_f(t_0,t)\\ n_0V(t), t\in\overline{\mathrm{T}}_d(t_0,t)\end{cases}$$

After that, we utilize the intervals $[t_{2k},t_{2k+1})$ and $[t_{2k+1},t_{2k+2})$. And $V(t)$ can be further expressed as

$$\dot{V}(t)\leqslant\begin{cases}-m_0V(t)+\Delta, t\in[t_{2k},t_{2k+1}),\\ n_0V(t), t\in[t_{2k+1},t_{2k+2}).\end{cases}$$

After that, let

$$V(t)=\begin{cases}\tilde{V}(t), t\in[t_{2k},t_{2k+1}),\\ \hat{V}(t), t\in[t_{2k+1},t_{2k+2}),\end{cases}\quad\chi(t)=\begin{cases}-m_0, t\in[t_{2k},t_{2k+1}),\\ n_0, t\in[t_{2k+1},t_{2k+2}),\end{cases}$$

and

$$\iota(t)=\begin{cases}\Delta, t\in[t_{2k},t_{2k+1}),\\ 0, t\in[t_{2k+1},t_{2k+2}).\end{cases}$$

Then, according to Lemma 2, one obtains

$$V(t)\leqslant\begin{cases}e^{\chi(t_{2k})(t-t_{2k})}\tilde{V}(t_{2k})+\int_{t_{2k}}^{t}e^{\chi(t_{2k})(t-z)}\iota(z)dz, t\in[t_{2k},t_{2k+1}),\\ e^{\chi(t_{2k+1})(t-t_{2k+1})}\hat{V}(t_{2k+1}), t\in[t_{2k+1},t_{2k+2}),\end{cases}$$

and

$$\begin{cases}\mu\hat{V}(t_{2k}^-)-\tilde{V}(t_{2k})\geqslant 0,\\ \mu\tilde{V}(t_{2k+1}^-)-\hat{V}(t_{2k+1})\geqslant 0.\end{cases}$$

For any time interval $[t_0,t)$, when $t\in[t_{2k},t_{2k+1})$, by the mathematical induction, we conclude

$$\begin{aligned}V(t)\leqslant&\mu e^{\chi(t_{2k})(t-t_{2k})}\hat{V}(t_{2k}^-)+\int_{t_{2k}}^{t}e^{\chi(t_{2k})(t-z)}\iota(z)dz\\ &\vdots\\ \leqslant&\mu^{2k}e^{\chi(t_{2k})|\overline{\mathrm{T}}_f(t_0,t)|+\chi(t_{2k-1})|\overline{\mathrm{T}}_d(t_0,t)|}V(t_0)\\ &+\mu^{2k}\int_{t_0}^{t_1}e^{\psi_{2k}(t,2)+\chi(t_0)(t_1-z)}\iota(z)dz\\ &+\mu^{2k}\int_{t_1}^{t_2}e^{\psi_{2k}(t,3)+\chi(t_1)(t_2-z)}\iota(z)dz\\ &+\cdots+\mu^2\int_{t_{2k-2}}^{t_{2k-1}}e^{\psi_{2k}(t,2k)+\chi(t_{2k-2})(t_{2k-1}-z)}\iota(z)dz\\ &+\mu^2\int_{t_{2k-1}}^{t_{2k}}e^{\psi_{2k}(t,2k+1)+\chi(t_{2k-1})(t_{2k}-z)}\iota(z)dz\\ &+\int_{t_{2k}}^{t}e^{\chi(t_{2k})(t-z)}\iota(z)dz,\end{aligned} \tag{B.12}$$

where $\psi_{2k}(t,p)=\chi(t_{2k})(t-t_{2k})+\sum_{q=p}^{2k}\chi(t_{q-1})(t_q-t_{q-1})$. When $t\in[t_{2k+1},t_{2k+2})$, there is

$$\begin{aligned}V(t)\leqslant&\mu^{2k+2}e^{\chi(t_{2k})|\overline{\mathrm{T}}_f(t_0,t)|+\chi(t_{2k+1})|\overline{\mathrm{T}}_d(t_0,t)|}V(t_0)\\ &+\mu^{2k+2}\int_{t_0}^{t_1}e^{\psi_{2k+1}(t,2)+\chi(t_0)(t_1-z)}\iota(z)dz\\ &+\mu^{2k+2}\int_{t_1}^{t_2}e^{\psi_{2k+1}(t,3)+\chi(t_1)(t_2-z)}\iota(z)dz\\ &+\cdots+\tau^4\int_{t_{2k-1}}^{t_{2k}}e^{\psi_{2k+1}(t,2k+1)+\chi(t_{2k-1})(t_{2k}-z)}\iota(z)dz\\ &+\mu^2\int_{t_{2k}}^{t_{2k+1}}e^{\psi_{2k+1}(t,2k+2)+\chi(t_{2k})(t_{2k+1}-z)}\iota(z)dz\\ &+\mu^2\int_{t_{2k+1}}^{t}e^{\chi(t_{2k+1})(t-z)}\iota(z)dz,\end{aligned} \tag{B.13}$$

where $\psi_{2k+1}(t,p)=\chi(t_{2k+1})(t-t_{2k+1})+\sum_{q=p}^{2k+1}\chi(t_{q-1})(t_q-t_{q-1})$. According to Definition 2, it can be found that when $t\in[t_{2k},t_{2k+1})$, the number of DoS attacks is $N(t_0,t)=k$, and when $t\in[t_{2k+1},t_{2k+2})$, the number of attacks is $N(t_0,t)=k+1$. Combined with (B.13) and (B.14), we can further have

$$\begin{aligned}V(t)\leqslant&\mu^{2N(t_0,t)}e^{-m_0|\overline{\mathrm{T}}_f(t_0,t)|+\chi(t_{2k+1})|\overline{\mathrm{T}}_d(t_0,t)|}V(t_0)\\ &+\int_{t_0}^{t}\mu^{2N(z,t)}e^{-m_0|\overline{\mathrm{T}}_f(t_0,t)|+\chi(t_{2k+1})|\overline{\mathrm{T}}_d(t_0,t)|}\iota(z)dz,\end{aligned} \tag{B.14}$$

where $\iota(z)=\max\{\Delta,0\}$. Then, one has $|\overline{\mathrm{T}}_f(t_0,t)|=t-t_0-|\overline{\mathrm{T}}_d(t_0,t)|$, and $|\overline{\mathrm{T}}_d(t_0,t)|\leqslant|\mathrm{T}_d(t_0,t)|+N(t_0,t)\&$. Then, by utilizing Assumption 1 and Lemma 2, one concludes

$$V(t)\leqslant\bar{c}e^{-\nu(t-t_0)}V(t_0)+\frac{\iota\bar{c}}{\nu},$$

in which $\bar{c}=e^{(m_0+n_0)T_0+[(m_0+n_0)\&+\ln\mu]\Lambda_0}$ and $\nu=m_0-(m_0+n_0)\frac{1}{T_1}-\tau>0$. Define $0\leqslant t_f<\infty$, and it satisfies the equation $V(t_f)=V_{\max}$, in which $V_{\max}>0$. Furthermore, recall the conditions (21)- (23), one has

$$t_f=\frac{1}{\nu}\ln\frac{V_{\max}-\frac{\iota\bar{c}}{\nu}}{\bar{c}V(t_0)}+t_0,$$

and $V(t)\leqslant V_{\max}$ if $t>t_f$. Therefore, $V(t)$ is bounded, which reflects that the leader–follower control errors $\hat{e}(t)$ and the observer errors $\bar{e}(t)$ are bounded. ■

References

[1] L. Zino, A. Rizzo, M. Porfiri, Consensus over activity-driven networks, IEEE Trans. Control Network Syst. 7 (2) (2019) 866–877.

[2] D. Wang and W. Wang, Necessary and sufficient conditions for containment control of multi-agent systems with time delay, Automatica 103 (2019) 418–23.

[3] H. Sun, C. Peng, T. Yang, H. Zhang, W. He, Resilient control of networked control systems with stochastic denial of service attacks, Neurocomputing 270 (2017) 170–177.

[4] Y. Xie, L. Han, X. Dong, Q. Li, Z. Ren, Bio-inspired adaptive formation tracking control for swarm systems with application to UAV swarm systems, Neurocomputing 453 (Sep. 2021) 272–285.

[5] H. Yang, Z. Wang, Event-triggered state estimation for markovian jumping neural networks: On mode-dependent delays and uncertain transition probabilities, Neurocomputing 424 (10) (2021) 226–235.

[6] X. Qiu, Y. Wang, X. Xie, H. Zhang, Resilient model-free adaptive control for cyber-physical systems against jamming attack, Neurocomputing 413 (2020) 423–430.

[7] H. Zhang, J. Duan, Y. Wang, Z. Gao, Bipartite fixed-time output consensus of heterogeneous linear multiagent systems, IEEE Trans. Cybern. 51 (2) (2021) 548–557.

[8] D. Wang, Z. Wang, C. Wen, Distributed optimal consensus control for a class of uncertain nonlinear multiagent networks with disturbance rejection using adaptive technique, IEEE Trans. Syst. Man Cybern.: Syst. 51 (7) (2021) 4389–4399.

[9] W. He, B. Zhang, Q. Han, F. Qian, J. Kurths, J. Cao, Leader-following consensus of nonlinear multiagent systems with stochastic sampling, IEEE Trans. Cybern. 47 (2) (2017) 327–338.

[10] J. Duan, H. Zhang, Y. Liang, Y. Cai, Bipartite finite-time output consensus of heterogeneous multi-agent systems by finite-time event-triggered observer, Neurocomputing 365 (2019) 86–93.

[11] M. Bengtsson, S. Kock, Cooperation and competition in relationships between competitors in business networks, J. Business Ind. Market. 14 (3) (1999) 178–194.
[12] A. Fontan, C. Altafini, A signed network perspective on the government formation process in parliamentary democracies, Scientific Rep. 11 (1) (2021).
[13] J. Luft, Cooperation and competition among employees: Experimental evidence on the role of management control systems, Management Accounting Research, 25th Anniversary Conference 31 (2016) 75–85.
[14] C. Altafini, Consensus problems on networks with antagonistic interactions, IEEE Trans. Autom. Control 58 (4) (2013) 935–946.
[15] J. Wang, A.B. Rad, P. Chan, Indirect adaptive fuzzy sliding mode control: Part I: fuzzy switching, Fuzzy Sets Syst. 122 (1) (2001) 21–30.
[16] Q. Wang, W. Zhong, J. Xu, W. He, D. Tan, Bipartite tracking consensus control of nonlinear high-order multi-agent systems subject to exogenous disturbances, IEEE Access 7 (2019) 145910–145920.
[17] W. He, B. Xu, Q. Han, F. Qian, Adaptive consensus control of linear multiagent systems with dynamic event-triggered strategies, IEEE Trans. Cybern. 50 (7) (2020) 2996–3008.
[18] Q. Wang, W. He, D. Tan and W. Zhong, Consensus disturbance rejection of nonlinear multi-agent systems over cooperation-competition networks, 2021 China Automation Congress (CAC), pp. 510–515, 2021.
[19] D. Ding, Q. Han, X. Ge, J. Wang, Secure state estimation and control of cyber-physical systems: A survey, IEEE Trans. Syst. Man Cybern.: Syst. 51 (1) (2021) 176–190.
[20] F. Pasqualetti, F. Dörfler, F. Bullo, Attack detection and identification in cyber-physical systems, IEEE Trans. Autom. Control 58 (11) (2013) 2715–2729.
[21] H. Song, D. Ding, H. Dong and Q. Han, Distributed maximum correntropy filtering for stochastic nonlinear systems under deception attacks, IEEE Trans. Cybern. doi:10.1109/TCYB.2020.3016093.
[22] D. Liu, D. Ye, Pinning-observer-based secure synchronization control for complex dynamical networks subject to DoS attacks, IEEE Trans. Circuits Syst. I Regul. Pap. 67 (12) (2020) 5394–5404.
[23] W. Xu, Z. Wang, L. Hu and J. Kurths, State estimation under joint false data injection attacks: Dealing with constraints and insecurity, IEEE Trans. Autom. Control. doi: 10.1109/TAC.2021.3131145.
[24] H. Song, D. Ding, H. Dong, X. Yi, Distributed filtering based on Cauchy-kernel-based maximum correntropy subject to randomly occurring cyber-attacks, Automatica 135 (2022) 110004.
[25] G. Carl, G. Kesidis, R. Brooks, S. Rai, Denial-of-service attack-detection techniques, IEEE Internet Comput. 10 (1) (2006) 82–89.
[26] D. Ding, G. Wei, S. Zhang, Y. Liu, F. Alsaadi, On scheduling of deception attacks for discrete-time networked systems equipped with attack detectors, Neurocomputing 219 (2017) 99–106.
[27] X. Guo, P. Liu, J. Wang, C. Ahn, Event-triggered adaptive fault-tolerant pinning control for cluster consensus of heterogeneous nonlinear multi-agent systems under aperiodic DoS attacks, IEEE Trans. Network Sci. Eng. (2021).
[28] X. Li, G. Wei, D. Ding, Distributed resilient interval estimation for sensor networks under aperiodic denial-of-service attacks and adaptive event-triggered protocols, Appl. Math. Comput. 409 (2021) 126371.
[29] S. Hu, D. Yue, Z. Cheng, E. Tian, X. Xie, X. Chen, Co-design of dynamic event-triggered communication scheme and resilient observer-based control under aperiodic DoS attacks, IEEE Trans. Cybern. 51 (9) (2021) 4591–4601.
[30] W. Chen, D. Ding, H. Dong, G. Wei, Distributed Resilient Filtering for Power Systems Subject to Denial-of-Service Attacks, IEEE Trans. Syst. Man Cybern.: Syst. 49 (8) (2019) 1688–1697.
[31] Q. Wang, W. He, D. Tan, W. Zhong, Event-triggered Control for leader-following bipartite bounded consensus of multi-agent systems under quantized information, in: IECON 2021–47th Annual Conference of the IEEE Industrial Electronics Society, 2021, pp. 1–6.
[32] K. You, W. Su, M. Fu, L. Xie, Attainability of the minimum data rate for stabilization of linear systems via logarithmic quantization, Automatica 47 (1) (2011) 170–176.
[33] Y. Zhu, S. Li, J. Ma, Y. Zheng, Bipartite consensus in networks of agents with antagonistic interactions and quantization, IEEE Trans. Circuits Syst. II Express Briefs 65 (12) (2018) 2012–2016.
[34] Q. Wang, S. Li, W. He and W. Zhong, Fully distributed event-triggered bipartite consensus of linear multi-agent systems with quantized communication, IEEE Trans. Circuits Syst. II: Express Briefs. doi: 10.1109/TCSII.2022.3154465.
[35] Z. Xu and W. He, Quantized synchronization of master-slave systems under event-triggered control against DoS attacks, in: IECON 2020 The 46th Annual Conference of the IEEE Industrial Electronics Society, 2020, pp. 3568–3573.
[36] H. Yang, S. Ju, Y. Xia, J. Zhang, Predictive cloud control for networked multiagent systems with quantized signals under DoS attacks, IEEE Trans. Syst. Man Cybern.: Syst. (2019).
[37] X. Chen, Y. Wang, S. Hu, Event-based robust stabilization of uncertain networked control systems under quantization and denial-of-service attacks, Inf. Sci. 459 (2018) 369–386.
[38] W. Xu, W. He, D. Ho, J. Kurths, Fully distributed observer-based consensus protocol: Adaptive dynamic event-triggered schemes, Automatica 139 (2022).
[39] W. Xu, J. Kurths, G. Wen and X. Yu, Resilient event-triggered control strategies for second-order consensus, IEEE Trans. Automatic Control. doi: 10.1109/TAC.2021.3122382.
[40] X. Wang, D. Ding, X. Ge, Q. Han, Neural-network-based control for discrete–time nonlinear systems with denial-of—ervice attack: The adaptive event–triggered case, Int. J. Robust Nonlinear Control 32 (5) (2022) 2760–2779.
[41] D. Ding, Q. Han, Y. Xiang, X. Ge, X. Zhang, A survey on security control and attack detection for industrial cyber-physical systems, Neurocomputing 275 (2018) 1674–1683.
[42] Q. Wang, W. He, L. Zino, D. Tan, W. Zhong, Bipartite consensus for a class of nonlinear multi-agent systems under switching topologies: A disturbance observer-based approach, Neurocomputing 488 (2022) 130–143.
[43] Y. Zhu, W. Zheng, Observer-based control for cyber-physical systems with periodic DoS attacks via a cyclic switching strategy, IEEE Trans. Autom. Control 65 (8) (2019) 3714–3721.
[44] M. Oliveira, R. Skelton, Stability tests for constrained linear systems, Perspectives in robust control, Springer-Verlag, London, U.K, 2001, pp. 241–257.
[45] K. Zhou, J. Doyle, Essentials of Robust Control, Upper Saddle River, Prentice-Hall, NJ, USA, 1998.

Qiang Wang received the M.S. degree in automation and electrical engineering with Qingdao University, Qingdao, in 2016. He is currently pursuing the Ph.D. degree in the Key Laboratory of Advanced Control and Optimization for Chemical Processes, Ministry of Education, East China University of Science and Technology, Shanghai, China. He is a visiting scholar of Faculty of Science and Engineering, University of Groningen, Groningen, 9747AG, Netherlands. His current research interests include multi-agent systems, cooperative control, and security control of networked systems.

Lorenzo Zino is a Postdoctoral Researcher with the University of Groningen, The Netherlands, since 2019. He received his Ph.D. in Pure and Applied Mathematics (cum laude) from Politecnico di Torino and Universitá di Torino (joint doctorate program), in 2018. His research interests encompass the modeling, the analysis, and the control aspects of dynamical processes over networks, applied probability, network modeling and analysis, and game theory.

Dayu Tan received the Ph.D. degree in Key Laboratory of Advanced Control and Optimization for Chemical Processes, Ministry of Education, East China University of Science and Technology, Shanghai, China, in 2021. He was a visiting Ph.D. student of School of Engineering Practice and Technology, McMaster University, Hamilton, ON, Canada, for the period from Sep. 2019 to Oct. 2020.
He is currently a Lecturer with the Institute of Physical Science and Information Technology, Anhui University, China. His research interests include machine learning, computer vision, and data mining.

Jiapeng Xu received the Ph.D. degree in control science and engineering at the East China University of Science and Technology, Shanghai, China in 2021. From October 2019 to October 2020, he was a visiting scholar in the Department of Electrical Engineering, University of Notre Dame, South Bend, IN, USA. He is currently a postdoctoral fellow in the Department of Electrical and Computer Engineering, University of Windsor, Windsor, ON, Canada. His research interests include networked feedback control, event-triggered state estimation, distributed filtering and mean-field games.

Weimin Zhong received the B.S. degree in industry automation and Ph.D. degree in control science and engineering from Zhejiang University in 1998 and 2006, respectively. From 2006 to 2008, he was a post-doctoral research fellow at East China University of Science and Technology. From September 2013 to August 2014, he was a visiting research fellow in the department of chemical engineering at Lehigh University. He is currently a full professor in process control at East China University of Science and Technology. His research interests mainly focuses on the issues in the fields of chemical process automation, smart manufacturing, and neural networks, especially the researches on the topics of the theories and practical applications in terms of the modelling and optimization of modern industrial processes.