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Towards Observer-Based Fault Detection and Isolation for Branched Water Distribution Networks without Cycles

Froukje Veldman - de Roo, Arturo Tejada, Henk van Waarde, and Harry L. Trentelman

Abstract— **This paper addresses the observer-based, diagonal, fault detection and isolation (FDI) problem for branched water distribution networks without cycles. Specifically, it provides a linear, time-invariant, state-space model for water contamination in such networks based on one-dimensional mass balance and a necessary and sufficient solvability condition for the aforementioned FDI problem. The latter is based on a graph-theoretic characterization of the rank of the model's transfer matrix, which allows one to check for solvability just by analyzing the fault-output paths in the directed graph associated to the model.**

I. Introduction

Water companies are required by law to monitor and safeguard the 'quality' of the water they produce and distribute. Water quality is generally defined in toxicological terms, that is, as maximum concentration levels of particular contaminants that can be present in the water without harming consumers [1]. While in transit, water can be contaminated due to resuspension of sediments present in the water distribution network (WDN), contact with the environment, or regrowth of micro-organisms such as bacteria [2]. Traditionally, water quality is monitored through chemical and biological laboratory analyses that are performed periodically (every few weeks or days) on water samples obtained at a few locations in the WDN. However, there is currently great interest in developing automated systems that can perform such monitoring in real time (e.g., every few minutes) and at a large number of locations (see, e.g., [3]–[5]).

An ideal water quality monitoring systems should be able to ascertain whether the water quality remains within acceptable levels and, if not, to determine the locations of the contamination source(s). Thus, these systems could be studied from the perspective of model-based fault detection and isolation (FDI) theory. Although such approach has been extensively studied by the systems and control community in the past forty years (see, e.g., [6]–[11]), it is less common in the water community (see [3], [12]), where an iterative combination of simulation and optimization procedures is usually preferred [2], [4], [13]. FDI generally involves the design of residual generators that generate a vector of residuals, each of which reflects the occurrence of exactly one specific fault. The existence and design of such generators has been extensively studied both for general systems [14], [15] and for structured systems, an approach introduced by Lin [16] and further developed by Commault and others [17]–[20].

WDNs display the same interconnection structure among their components (i.e., pumps, pipes, valves, etc.) for a wide range of operating conditions. Thus, under some circumstances (see below) they can be considered to be linear, time-invariant (LTI) structured systems and studied as such. Unfortunately, most of the FDI results for LTI systems are generic. That is, they hold everywhere except for a small set of operating conditions with measure zero [20]. Thus, although these results are intuitive and easy to use, since they are based on the system graphs associated to the structure of the system in a natural way, they are difficult to apply in practice because one must first check whether a particular operating condition is not in the set of measure zero.

In this paper, we remove this difficulty. We limit our analysis to branched WDN without cycles and provide a structured LTI state-space model that describes the dynamics of contaminants dissolved in the water. In this context, faults are the unwanted injection of a given contaminant in the water stream. For such faults, we derive a necessary and sufficient solvability condition for the observer-based, diagonal, FDI problem using a graph-theoretical characterization of the rank of the model's transfer matrix [18], [20], which holds for *all* branched WDN without cycles. It is important to remark that our results apply not only to WDN but to all practical systems that satisfy the framework of the LTI models in this paper.

After the notation section below, the rest of the paper is organized as follows: Section II introduces the water contamination model. The graph-theoretical characterization of the model's transfer matrix, the precise FDI problem formulation, and its solvability condition are presented in Section III. Finally, Section IV gives our conclusions.

A. Notation

In the sequel, $\mathbb{N}^+$ ($\mathbb{R}^+$) denotes the strictly positive integer (real) numbers, $\mathbb{N} := \mathbb{N}^+ \cup \{0\}$, $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$, $\mathbb{R}^n$ is the n-dimensional Euclidean space with standard basis vectors e_i, $i = 1, \ldots, n$, and $\mathbb{R}^{n \times m}$ is the space of real $n \times m$ matrices. Also, I_n denotes the $n \times n$ identity matrix, $0_{n,r}$ is the $n \times r$ matrix of zeros, and $^\top$ denotes transposition.

II. A Model for Water Contamination in WDNs

A. Preliminaries

Practical drinking water networks are generally equipped with several pumps or water tanks that establish enough pressure differential across the pipe network to allow water

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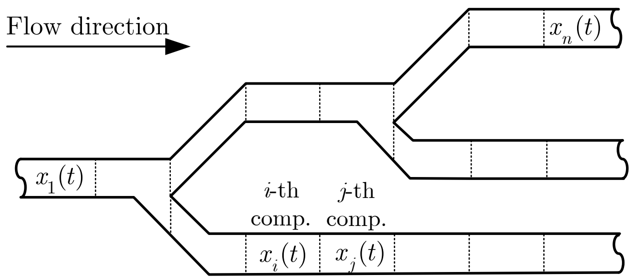

Fig. 1. WDN simplified schematics.

to flow. The pressure differentials usually vary slowly during a typical day and generally do not change signs. More over, their time constants are much slower that those of the water flow. This means that water flow in drinking water networks can be considered directed and quasi-static. These considerations inform the following assumptions, under which our contamination dynamics model holds:

The WDN is branched, has no cycles, and provides constant water flow of 'sufficient' speed[i].

In addition to this hydraulic assumptions, the following will be assumed about the possible water contaminants.

There is a single contaminant in the water that does not react with its environment, precipitates, nor re-suspends (i.e., its total mass is conserved).

Situations where the contaminants decay (or increase) over time can be handled by including reaction/decay terms in the equations below a shown in [21]. These cases, however, will not be considered here.

B. Contaminant dynamics

4 Consider the WDN shown in Figure 1and let it be subdivided into $n \in \mathbb{N}^+$ finite-volume compartments with constant volume $V_i \in \mathbb{R}^+$ (in m^3), $i = 1, 2, \ldots, n$ (in Figure 1, $n = 7$). By mass conservation, the (water + contaminant) mass entering the i-th compartment must either accumulate within it or leave it by flowing into the compartment downstream (compartment l in Figure 1). Hence, the propagation of contaminant in the i-th compartment can be described by

$$V_i x_i(t + \Delta t) = V_i x_i(t) - q_i \Delta t x_i(t) + u_i^{\Delta t}, \qquad (1)$$

where $x_i(t) \in \mathbb{R}_0^+$ is the average contaminant concentration in the i-th compartment i at time t (in g/m^3), $\Delta t \in \mathbb{R}^+$ is the time span over which the propagation process is considered (in s), $q_i \in \mathbb{R}^+$ is the average water flow (in m^3/s, constant by assumption), and $u_i^{\Delta t} \in \mathbb{R}_0^+$ is the amount of contaminant that enters the i-th compartment i over time span Δt (in g). To characterize $u_i^{\Delta t}$, the following definitions are needed.

Definition 2.1: Let $i, j \in \{1, 2, \ldots, n\}$ such that $i \neq j$. Compartment i is said to *succeed* compartment j, if the water from compartment j flows directly into compartment i (i.e., without flowing through another compartment first). The set of compartments succeeded by compartment i is denoted by $N^-(i) := \{j \in \{1, 2, \ldots, n\} \mid j \neq i, i \,\text{succeeds}\, j\}$. Similarly, compartment j is said to *precede* compartment i, if i succeeds j, and the set of compartments preceded by compartment j is denoted by $N^+(j) := \{i \in \{1, 2, \ldots, n\} \mid i \neq j, j \,\text{precedes}\, i\}$.

Definition 2.1 implies that $j \in N^-(i)$ if and only if $i \in N^+(j)$ for some $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$ and that, for any $j = 1, \ldots, n$, $q_j = \sum_{k \in N^+(j)} q_{jk}$. That is, compartment j contributes q_{jk} to the inflow of each of its succeeding compartments $k \in N^+(j)$. Hence, $u_i^{\Delta t}$ is given by:

$$u_i^{\Delta t} = \sum_{j \in N^-(i)} q_{ji} \Delta t x_j(t). \qquad (2)$$

In this context, faults are defined as the unwanted injection of contaminants in a subset $M := \{m_1, m_2, \ldots, m_r\} \subseteq \{1, 2, \ldots, n\}$ of the WDN's n compartments. Faults are modeled as arbitrary, unknown functions, $f_l(t) \in \mathbb{R}_0^+$, $l = 1, 2, \ldots, r$, that denote the mass rate of the injected contaminant (g/s). Each fault, $f_l(t)$, occurs in only one compartment, m_l, and since it is also assumed that only one fault per compartment can occur, the dynamics of the m_l-th compartment are given by

$$\begin{aligned} V_{m_l} x_{m_l}(t + \Delta t) &= V_{m_l} x_{m_l}(t) - q_{m_l} \Delta t x_{m_l}(t) + u_{m_l}^{\Delta t} \\ &\quad + \Delta t f_l(t). \end{aligned} \qquad (3)$$

Substituting (2) into (1) and (3), rearranging the terms, dividing by Δt, and taking the limit as $\Delta t \to 0$, yields

$$\dot{x}_i(t) = -\frac{q_i}{V_i} x_i(t) + \sum_{j \in N^-(i)} \frac{q_{ji}}{V_i} x_j(t), \qquad (4a)$$

if $i \notin M$, and

$$\dot{x}_i(t) = -\frac{q_i}{V_i} x_i(t) + \sum_{j \in N^-(i)} \frac{q_{ji}}{V_i} x_j(t) + \frac{f_l(t)}{V_i} \qquad (4b)$$

if $i = m_l \in M$, for $i = 1, \ldots, n$. Equations 4a,b are known as the propagation dynamics and describe the change in average contamination concentration per compartment.

C. State-space model

The propagation dynamics (4) for the whole WDN and an expression for sensor measurements are given by

$$\Sigma : \left\{ \begin{array}{rcl} \dot{x}(t) &=& Ax(t) + Ff(t) \\ y(t) &=& Cx(t) \end{array} \right. , \qquad (5)$$

where $x(t) := \begin{pmatrix} x_1(t) & x_2(t) & \ldots & x_n(t) \end{pmatrix}^\top \in \mathbb{R}^n$ is the state vector; $f(t) = \begin{pmatrix} f_1(t) & \ldots & f_r(t) \end{pmatrix}^\top \in \mathbb{R}^r$ is the fault vector; $F \in \mathbb{R}^{n \times r}$ has entries of value $1/V_{m_l}$ at entries (m_l, l), $l = 1, \ldots, r$, and zero elsewhere, and $A \in \mathbb{R}^{n \times n}$ is the adjacency matrix of the weighted directed graph $G(V, E)$. The vertex set in $G(V, E)$ is given by $V = \{x_1, x_2, \ldots, x_n\}$ and the edge set by $E = \{(x_i, x_i) \mid i = 1, 2, \ldots, n\} \cup \{(x_j, x_i) \mid \text{compartment } i \text{ succeeds compartment } j, i, j \in \{1, 2, \ldots, n\}, j \neq i\}$ where the edges $(x_i, x_i) \in E$ are weighted with negative weights $-q_i/V_i$ and the edges $(x_j, x_i) \in E$, $j \neq i$, are weighted with positive weights q_{ji}/V_i. In addition, $y(t) \in \mathbb{R}^p$, $p \leq n$ gives measurements of the average contaminant concentration from p different compartments in the network. That is, if $K :=$

[i]For sufficiently fast flows, contaminants are mainly transported (and not diffused) by the water [21].

$\{k_1, k_2, \ldots, k_p\} \subseteq \{1, 2, \ldots, n\}$ denotes the set of compartment where measurements are taken, then the output matrix $C \in \mathbb{R}^{p\times n}$ has entries of value 1 at entries (i, k_i), $i = 1, \ldots, p$, and zero elsewhere.

Remark 2.2: The results in the sequel are derived for system Σ in (5), our main subject of study. Note, however, that these results are valid for any other application whose LTI dynamics can be associated to a directed graph $G(V, E)$ with no cycles in the vertex set V (with the exception of self-loops).

Remark 2.3: For a given WDN topology, labeling a compartments with a higher index label than their predecessors yields a lower triangular matrix A with nonzero diagonal entries (since $q_i, V_i > 0$ by definition, for all $i \in \{1, 2, \ldots, n\}$).

III. Observer-Based Diagonal FDI

This section provides a necessary and sufficient condition for the existence of observer-based residual generators with diagonal transfer matrix from faults to residuals. Such "diagonal" residual generators allow for simultaneous fault detection and isolation, since only one fault influences one residual at a time. The precise problem formulation and its solvability condition are presented in Section III-B and Theorem 3.1, respectively. The latter relies on a graph-theoretic characterization of the rank of the transfer matrix of system Σ in (5). Hence some basic facts on rational matrices and graph theory are presented first.

A. Preliminaries

On proper rational matrices

Recall that a rational function, $p(s)/r(s)$, is called *proper* if the degree of polynomial $p(s)$ is smaller than or equal to the degree of polynomial $r(s)$, and a matrix $T(s)$ of rational functions is called proper if all its entries are proper. If in addition $T(s)$ has a proper inverse, it is called *bicausal*. Finally, $\operatorname{rank}(T)$ is defined as the maximum rank of $T(s)$ over all s. Proper rational matrices factorize into a bicausal matrix and a Hermite form, as shown next in the context of system Σ.

Lemma 3.1: Consider system $\Sigma = (A, F, C, 0)$ in (5) with $r = p$ and consider its $r\times r$ transfer matrix $T_{fy}(s) = C(sI - A)^{-1}F$. For a given $a \in \mathbb{R}$, $T_{fy}(s)$ can be expressed as $T_{fy}(s) = Z^{-1}(s)H(s)$, where $Z(s)$ is an $r \times r$ bicausal matrix, and $H(s)$ is an $r \times r$ upper triangular matrix with diagonal entries $h_{ii}(s) = (s+a)^{-n_i}$ and off-diagonal entries $h_{ij} = \gamma_{ij}(s)/\pi^{n_{ij}}(s)$, where $n_i, n_{ij} \in \mathbb{N}^+$, $n_{ij} < n_i$, and $\gamma_{ij}(s)$ is a polynomial with real coefficients and with order less than or equal to n_{ij}, $i, j = 1, 2, \ldots, r, i < j$.

Proof: See [22, Theorem 2]. ∎

Note that $H(s)$ is uniquely determined by $T_{fy}(s)$ and a. Next, consider the following observer-based residual generator for system Σ:

$$\Omega : \left\{ \begin{array}{rcl} \dot{\hat{x}}(t) & = & A\hat{x}(t) + K(y(t) - C\hat{x}(t)) \\ r(t) & = & Q(y(t) - C\hat{x}(t)) \end{array} \right. , \qquad (6)$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the state estimate, $r(t) \in \mathbb{R}^r$ is the generator's output (i.e., the residual), $K \in \mathbb{R}^{n\times p}$, and $Q \in \mathbb{R}^{p\times p}$ and nonsingular. Note that $T_{fr}(s)$, the transfer function of the series interconnection of Σ and Ω, can be expressed as the product of a bicausal matrix and $T_{fy}(s)$ since

$$\begin{aligned} T_{fr}(s) &= Q(I_n - C(sI_n - A + KC)^{-1}K)T_{fy}(s) \\ &= Z(s)T_{fy}(s) \end{aligned}$$

(see [22]). The converse also holds as shown next.

Lemma 3.2: Consider the transfer matrix $T_{fy}(s) = C(sI - A)^{-1}F$ of system $\Sigma = (A, F, C, 0)$ (5) and let $Z(s)$ be a $p \times p$ proper, rational matrix. There exists a matrix K and a non-singular matrix Q such that

$$Z(s)T_{fy}(s) = T_{fr}(s)$$

if and only if $Z(s)$ is bicausal and for every polynomial row vector $v(s)$ such that $v(s)C(sI - A)^{-1}$ is polynomial, the vector $v(s)Z^{-1}(s)$ is polynomial as well.

Proof: See [22, Theorem 4]. ∎

On Graph theory

The directed system graph $G_\Sigma(V_\Sigma, E_\Sigma)$ associated with the system $\Sigma = (A, F, C, 0)$ in (5) consists of a vertex set $V_\Sigma = X \cup F \cup Y$, where $X = \{x_1, x_2, \ldots, x_n\}$ denotes the set of state vertices, $F = \{f_1, f_2, \ldots, f_r\}$ denotes the set of fault vertices and $Y = \{y_1, y_2, \ldots, y_p\}$ denotes the set of output vertices. The edge set E_Σ is defined as $E_\Sigma = \{(x_j, x_i) \mid a_{ij} \neq 0\} \cup \{(f_j, x_i) \mid f_{ij} \neq 0\} \cup \{(x_j, y_i) \mid c_{ij} \neq 0\}$, where a_{ij}, f_{ij}, c_{ij} denote the (i, j)-th entries of, respectively, matrices A, F, C for appropriate indices i, j.

Let $P \in \mathbb{R}^{n\times n}$ with entries p_{ij}, $i, j = 1, 2, \ldots, n$. The directed Coates graph $G_P(V_P, E_P)$ associated with P has vertex set $V_P = \{v_1, v_2, \ldots, v_n\}$ and edge set $E_P := \{(v_j, v_i) \mid p_{ij} \neq 0\}$. Each edge $(v_j, v_i) \in E_P$ is weighted with the nonzero weight p_{ij}, for $i, j = 1, 2, \ldots, n$ [20], [23].

The system graph associated with Σ can be related to a Coates graph as follows: Let the number of faults be equal to the number of outputs, that is, let $r = p$ in Σ. The first order $(n+r)\times(n+r)$ polynomial matrix associated with system Σ, that is, its system matrix $P_\Sigma(s)$ (see [24]), is given by:

$$P_\Sigma(s) = \begin{pmatrix} sI - A & -F \\ C & 0_{r,r} \end{pmatrix}. \qquad (7)$$

The Coates graph $G_{P_\Sigma(\bar{s})}$ associated with $P_\Sigma(\bar{s})$ for a fixed $\bar{s} \in \mathbb{R}$, can be obtained from G_Σ by defining vertices $v_i = x_i$, for $i = 1, 2, \ldots, n$, and by identifying the vertex v_{n+j} with the pair (f_j, y_j), for $j = 1, 2, \ldots, r$. In other words, v_{j+n} is the vertex obtained by merging vertices f_j and y_j, $j = 1, 2, \ldots, r$. The edge set is as defined above.

Next, recall the following concepts [23]: a *path* in a directed graph $G(V, E)$ is an edge sequence $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$, with $k \geq 2$, in which all the edges $(i_j, i_{j+1}) \in E$, $j = 1, 2, \ldots, k-1$, and all the nodes $i_1, i_2, \ldots, i_k \in V$ are distinct. Paths are called vertex disjoint if they have no vertex in common. A *circuit* is a path $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$, $k \geq 2$, with $i_1 = i_k$. A *circuit family* is a set of vertex disjoint circuits which together cover the entire vertex set V. A circuit family is said to span the graph $G(V, E)$. A *subgraph* of a directed graph $G(V, E)$ is a directed graph $G_s(V_s, E_s)$ with V_s a subset of V and E_s a subset of E.

[23, Theorem 3.1] shows that the determinant of a real square matrix P can be computed in terms of the weights of the edges in its associated Coates graph G_P. Specifically, $\det(P) \neq 0$ if and only if there exists at least one circuit family in G_P. This is the basis for the following lemma, which relates the existence of a circuit family in $G_{P_\Sigma(\bar{s})}$ with the number of vertex-disjoint paths in G_Σ that connect the set of faults F to the set of outputs Y (see [20] for a proof).

Lemma 3.3: Consider the system $\Sigma = (A, F, C, 0)$ in (5) with $r = p$, the system graph G_Σ, and the Coates graph $G_{P_\Sigma(\bar{s})}$ associated with $P_\Sigma(\bar{s})$, for an arbitrary $\bar{s} \in \mathbb{R}$ (see (7)). If there exists a circuit family in $G_{P_\Sigma(\bar{s})}$, then there exists an r-tuple of vertex disjoint paths in G_Σ from the set of fault vertices F to the set of output vertices Y.

The following lemma provides a graph theoretic characterization of the rank of $T_{fy}(s)$.

Lemma 3.4: Consider system $\Sigma = (A, F, C, 0)$ in (5) with $r = p$ and its associated system graph G_Σ. The maximum number of vertex disjoint paths in G_Σ from the set of fault vertices F to the set of output vertices Y is equal to the rank the transfer matrix $T_{fy}(s) = C(sI - A)^{-1}F$.

Proof : Recall the associated system matrix $P_\Sigma(s)$ in (7) and note from [24, Lemma 8.9] that $\text{rank}(T_{fy}) = \text{rank}(P_\Sigma) - n$. Also recall that for any complex matrix P, $\text{rank}(P) = p$ implies that p is the size of the largest square submatrix of P that has nonzero determinant.

If $\det(P_\Sigma) \neq 0$, it follows that $\text{rank}(P_\Sigma) = n + r$, so $\text{rank}(T_{fy}) = r$. By [23, Theorem 3.1] it then follows that there is at least one circuit family in the Coates graph $G_{P_\Sigma(\bar{s})}$ associated with Σ, where $\bar{s} \in \mathbb{R} \setminus \text{Re}(\sigma(A))$ and $\text{Re}(\sigma(A)) \subset \mathbb{R}$ is the set of real eigenvalues of matrix A. Since there is at least one circuit family in $G_{P_\Sigma(\bar{s})}$, it follows from Lemma 3.3 that there exists an r-tuple of vertex disjoint paths in the system graph G_Σ from F to Y. Since F and Y consist both of r components, it follows directly that $\text{rank}(T_{fy})$ is equal to the maximum number of vertex disjoint paths in G_Σ from F to Y.

If $\det(P_\Sigma) = 0$, then $\text{rank}(P_\Sigma) < n + r$. Suppose that $\text{rank}(P_\Sigma) = l$ for some $l \in \mathbb{N}^+$, $n \leq l < n + r$, so $\text{rank}(T_{fy}) = l - n$. Denote by L the $l \times l$ submatrix of P_Σ which has non-zero determinant, i.e. $\text{rank}(L) = l$. Without loss of generality, we may assume that L is given by:

$$L(s) = \begin{pmatrix} sI - A & -F_L \\ C_L & 0_{l-n,l-n} \end{pmatrix}, \tag{8}$$

where F_L is composed of $l - n$ columns of F and C_L is composed of $l - n$ rows of C. By the same line of reasoning as applied to the case $\det(P_\Sigma) \neq 0$, but now applied to the case $\det(L) \neq 0$, and by remarking that the Coates graph $G_{L_{\Sigma_L}(\bar{s})}$ associated with $\Sigma_L = (A, F_L, C_L, 0)$ is a subgraph of the Coates graph $G_{P_\Sigma(\bar{s})}$ associated with Σ, $\bar{s} \in \mathbb{R} \setminus \text{Re}(\sigma(A))$, it follows that there exist at least $l - n$ vertex disjoint paths in the system graph G_Σ from F to Y.

That the are precisely $l - n$ vertex disjoint paths in G_Σ can be shown by contradiction: Suppose that next to the aforementioned $l-n$ vertex disjoint paths in G_Σ, there exists an additional path in G_Σ from F to Y which is vertex disjoint with these $l - n$ paths, while $\text{rank}(P_\Sigma) = l$. If there exist $l - n + 1$ vertex disjoint paths in G_Σ from F to Y, then there exists a $(l+1) \times (l+1)$ submatrix $\bar{L}$ of P_Σ, which is, without loss of generality, assumed to be given by:

$$\bar{L}(s) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{L(s)} & \begin{matrix} -f \\ 0_{l-n,1} \end{matrix} \\ \hline c & 0_{1,l-n} & 0_{1,1} \end{array}\right), \tag{9}$$

where f is a column vector of matrix F which is not used to compose F_L and c is a row vector of matrix C which is not used to compose C_L. Column vector f and row vector c relate how the fault vertex and the output vertex of the additionally considered $(l - n + 1)$-th path of G_Σ are connected (by edge sequences). By (9), we have $\text{rank}(\bar{L}) = \text{rank}(L) + \text{rank}(c_{\bar{L}} L^{-1} f_{\bar{L}})$ where $c_{\bar{L}} = \begin{pmatrix} c & 0_{1,l-n} \end{pmatrix}$ and $f_{\bar{L}} = \begin{pmatrix} f^{\mathrm{T}} & 0^{\mathrm{T}}_{l-n,1} \end{pmatrix}^{\mathrm{T}}$. Since $\text{rank}(L) = l$, $L^{-1}(s)$ exists and is given by:

$$L^{-1}(s) = \left[\begin{pmatrix} I_n & 0_{n,l-n} \\ C_L S(s) & I_{l-n} \end{pmatrix} \cdot \begin{pmatrix} S^{-1}(s) & 0_{n,l-n} \\ 0_{l-n,n} & C_L S(s) F_L \end{pmatrix} \cdot \begin{pmatrix} I_n & S(s)F_L \\ 0_{l-n,n} & I_{l-n} \end{pmatrix}\right]^{-1},$$

where $S(s) := (sI - A)^{-1}$. Hence, it follows that

$$c_{\bar{L}} L^{-1}(s) f_{\bar{L}} = cS(s)f + cS(s)F_L T_L^{-1}(s) C_L S(s) f$$

where $T_L(s) = C_L S(s) F_L$ is the $(l - n) \times (l - n)$ transfer matrix of system Σ_L, whose inverse exists since $\text{rank}(L) = n + \text{rank}(T_L)$ (see 8), i.e. $\text{rank}(T_L) = l - n$. In general, F_L, C_L, f and c are given by:

$$\begin{aligned} F_L &= \begin{pmatrix} \frac{e_{m_1}}{V_{m_1}} & \frac{e_{m_2}}{V_{m_2}} & \cdots & \frac{e_{m_{l-n}}}{V_{m_{l-n}}} \end{pmatrix} \\ C_L &= \begin{pmatrix} e_{k_1} & e_{k_2} & \cdots & e_{k_{l-n}} \end{pmatrix}^{\mathrm{T}} \\ f &= \frac{e_{m_{l-n+1}}}{V_{m_{l-n+1}}} \\ c &= e^{\mathrm{T}}_{k_{l-n+1}} \end{aligned}$$

and we assume, without loss of generality, that the indices $m_i, k_i \in \{1, 2, \ldots, n\}$[ii], for $i = 1, 2, \ldots, l - n + 1$, satisfy:

$$m_1 \leq k_1 < m_2 \leq k_2 < \ldots < m_{l-n+1} \leq k_{l-n+1}. \tag{10}$$

By (10) and by assuming, without loss of generality, that $S(s)$ is a lower triangular matrix (i.e. assuming A is lower triangular, see Remark 2.3), we have $C_L S(s) f = 0$ for all $s \in \mathbb{R}$. Furthermore, since A is a lower triangular matrix and there exists a path from fault vertex f_{l-n+1} to output y_{l-n+1} in G_Σ, it can be seen that $cS(s)f \neq 0$ for almost all $s \in \mathbb{R}$. Hence, by the above, for almost all $s \in \mathbb{R}$ we have $c_{\bar{L}} L^{-1}(s) f_{\bar{L}} \neq 0$ and consequently $\text{rank}(c_{\bar{L}} L^{-1} f_{\bar{L}}) = 1$. Hence, the $(l + 1) \times (l + 1)$ submatrix $\bar{L}$ of P_Σ satisfies $\text{rank}(\bar{L}) > \text{rank}(L)$ and consequently $\text{rank}(P_\Sigma) > l$, which

[ii]These indices are defined in Section II. See equations (3)-(5).

is in contradiction with the assumption $\mathrm{rank}(P_\Sigma) = l$. Hence, there exist exactly $l - n$ vertex disjoint paths in the system graph G_Σ from F to Y if $\mathrm{rank}(P_\Sigma) = l$, i.e. if $\mathrm{rank}(T_{fy}) = l - n$. ■

B. FDI problem definition and solvability condition

Consider the LTI state-space system $\Sigma = (A, F, C, 0)$ in (5) with equal number of faults and outputs, i.e., $r = p$. Denote by Σ^i, for $i = 1, 2, \dots, r$, the system obtained from Σ as follows:

$$\Sigma^i : \begin{cases} \dot{x}(t) = Ax(t) + F^{-i} f_{-i}(t) + F^i f_i(t) \\ y(t) = Cx(t) \end{cases}, \qquad (11)$$

where $f_i(t)$ is the i-th component in $f(t)$, $f_{-i}(t) = \begin{pmatrix} f_1(t) & f_2(t) & \dots & f_{i-1}(t) & f_{i+1}(t) & \dots & f_r(t) \end{pmatrix}^{\mathrm{T}}$, F^i is the i-th column of matrix F, and F^{-i} is obtained from F by deleting its i-th column. Note that

$$\begin{aligned} \begin{pmatrix} f_{-i}(t) \\ f_i(t) \end{pmatrix} &= T^{\mathrm{T}} f(t) \\ \begin{pmatrix} F^{-i} & F^i \end{pmatrix} &= FT \end{aligned} \qquad (12)$$

where $T = \begin{pmatrix} e_1 & \dots & e_{i-1} & e_{i+1} & \dots & e_r & e_i \end{pmatrix}$ is a $r \times r$ permutation matrix. The bank of observers scheme, which underlies the observer-based, diagonal FDI problem, is given by r observers (one per system Σ^i, $i = 1, 2, \dots, r$) and r residuals, which are designed using the output estimation error derived from the observers. For $i = 1, 2, \dots, r$, a resulting residual generator Ω^i for system Σ^i is given by:

$$\Omega^i : \begin{cases} \dot{\hat{x}}^i(t) = & A\hat{x}^i(t) + K^i(y(t) - C\hat{x}^i(t)) \\ r_i(t) = & Q^i(y(t) - C\hat{x}^i(t)) \end{cases}, \qquad (13)$$

where $\hat{x}^i(t) \in \mathbb{R}^n$ is the state estimate, $r_i(t) \in \mathbb{R}$ is the residual, K^i is the $n \times r$ matrix to be designed such that $\hat{x}^i(t)$ converges to $x(t)$ when no faults $f^{-i}(t), f^i(t)$ are considered, and Q^i is an $1 \times r$ row matrix to be designed such that the observer-based, diagonal, FDI problem of Definition 3.5 below is solved.

Interconnecting (11) and (13), the dynamics of the estimation error $e^i(t) := x(t) - \hat{x}^i(t)$ of system Σ^i and the residual $r_i(t)$ are respectively given by:

$$\dot{e}^i(t) = (A - K^i C)e^i(t) + F^{-i} f_{-i}(t) + F^i f_i(t)$$
$$r_i(t) = Q^i C e^i(t).$$

Consequently, the transfer matrices from the faults $\begin{pmatrix} f_{-i}^{\mathrm{T}}(s) & f_i(s) \end{pmatrix}^{\mathrm{T}}$ to the residual $r_i(t)$ are given by:

$$T_{fr}^i(s) = Q^i C(sI - A + K^i C)^{-1} \begin{pmatrix} F^{-i} & F^i \end{pmatrix}.$$

The observer-based, diagonal, FDI problem is defined as follows (see also [18]):

Definition 3.5: Consider system $\Sigma = (A, F, C, 0)$ in (5) with $r = p$, systems Σ^i in (11), and residual generators Ω^i in (13), $i = 1, 2, \dots, r$. The observer-based, diagonal, FDI problem is to find $n \times r$ matrices K^i and $1 \times r$ matrices Q^i such that, for $i = 1, 2, \dots, r$, $A - K^i C$ is stable, transfer matrix $T_{fr}^{-i}(s) = 0_{1,r-1}$ and transfer function $T_{fr}^i(s)$ is a nonzero, proper, rational function.

The observer-based, diagonal, FDI problem is solvable if matrices K^i, Q^i, $i = 1, 2, \dots, r$, exist such that the conditions of Definition 3.5 are satisfied. In such case, the series interconnection of system Σ and the residual generators have a diagonal transfer matrix $T_{fr}(s)$. That is,

$$\begin{aligned} T_{fr}(s) &= \begin{pmatrix} Q^1 C(sI - A + K^1 C)^{-1} F \\ Q^2 C(sI - A + K^2 C)^{-1} F \\ \vdots \\ Q^r C(sI - A + K^r C)^{-1} F \end{pmatrix} \\ &= \mathrm{diag}(t_{11}(s), t_{22}(s), \dots, t_{rr}(s)) \end{aligned} \qquad (14)$$

with $t_{ii}(s) \neq 0$ and proper for $i = 1, 2, \dots, r$.

The following theorem is the main result of this paper.

Theorem 3.1: Consider the linear, time-invariant system $\Sigma = (A, F, C, 0)$ of (5), with $r = p$, and its associated system graph G_Σ. The observer-based diagonal fault detection and isolation problem of Definition 3.5 is solvable if and only if the maximum number of vertex disjoint paths in G_Σ from the set of fault vertices F to the set of output vertices Y is equal to r.

Proof: Necessity: Suppose that the observer-based diagonal fault detection and isolation problem is solvable. Since $t_{ii}(s) \neq 0$ for $i = 1, 2, \dots, r$, $\det(T_{fr}(s)) = t_{11}(s) \cdot t_{22}(s) \cdot \dots \cdot t_{rr}(s) \neq 0$ (see (14)), and consequently $\mathrm{rank}(T_{fr}) = r$. By $r \times r$ transfer matrices $T_{fr}(s)$ (i.e. from the faults to the residuals), $T_{yr}(s)$ (i.e. from the outputs to the residuals) and $T_{fy}(s)$ (i.e. from the faults to the outputs) satisfying $T_{fr}(s) = T_{yr}(s) \cdot T_{fy}(s)$ and by $r = \mathrm{rank}(T_{fr}) \leq \min(T_{yr}, T_{fy})$, we have $\mathrm{rank}(T_{fy}) = r$. Hence, by Lemma 3.4 we have that the maximum number of vertex disjoint paths in G_Σ from F to Y is equal to r.

Sufficiency: Suppose that the maximum number of vertex disjoint paths in G_Σ from F to Y is equal to r, i.e. $\mathrm{rank}(T_{fy}) = r$ (Lemma 3.4). Let $i \in \{1, 2, \dots, r\}$. Now, since

$$T_{fy}^i(s) = C(sI - A)^{-1} \begin{pmatrix} F^{-i} & F^i \end{pmatrix} = T_{fy}(s)T$$

(see (12)) and $r = \mathrm{rank}(T_{fy}) \leq \min(T_{fy}^i, T)$, we have $\mathrm{rank}(T_{fy}^i) = r$, i.e. the maximum number of vertex disjoint paths in G_{Σ^i} from the set of fault vertices $\{f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_r\} \cup \{f_i\}$ to the set of output vertices Y is equal to r (Lemma 3.4). Now, let $a \in \mathbb{R}$. Then, by Lemma 3.1 there exists an $r \times r$ bicausal matrix $Z(s)$ such that $Z(s)T_{fy}^i(s) = H(s)$ where $H(s)$ is an $r \times r$ proper, rational, upper triangular matrix with nonzero diagonal elements. Next, consider an arbitrary polynomial row vector $v(s)$ such that $v(s)C(sI - A)^{-1}$ is polynomial. Consequently,

$$\begin{aligned} v(s)C(sI - A)^{-1} \begin{pmatrix} F^{-i} & F^i \end{pmatrix} &= v(s)T_{fy}^i(s) \\ &= v(s)Z^{-1}(s)H(s) \end{aligned}$$

where $Z^{-1}(s)$ exists and is proper since $Z(s)$ is bicausal. Since $v(s)C(sI - A)^{-1}$ is polynomial, it follows that $v(s)T_{fy}^i(s)$ is polynomial and consequently $v(s)Z^{-1}(s)H(s)$ is polynomial. Furthermore, since $H(s)$

is a proper, non-singular rational matrix by definition, for any $a \in \mathbb{R}^+$, it follows that $H^{-1}(s)$ is an upper triangular, polynomial matrix with real coefficients. Consequently, $v(s)Z^{-1}(s)$ is polynomial. Hence, by Lemma 3.2 there exists an $n \times r$ constant matrix $\bar{K}^i$ and $r \times r$ non-singular matrix $\bar{Q}^i$ such that

$$Z(s)T^i_{fy}(s) = T^i_{fr}(s)$$

and by $Z(s)T^i_{fy}(s) = H(s)$ it follows directly that $T^i_{fr}(s) = H(s)$ is an upper triangular, proper, rational matrix with nonzero diagonal elements. Hence, the last row of $T^i_{fr}(s)$ satisfies:

$$e_m^{\mathrm{T}} T^i_{fr}(s) = \begin{pmatrix} 0 & \dots & 0 & t_{rr}(s) \end{pmatrix}$$

with $t_{rr}(s) \neq 0$ and proper. So, by taking $K^i = \bar{K}^i$ and $Q^i = e_m^{\mathrm{T}}\bar{Q}^i$, for $i = 1, 2, \dots, r$, there exists $n \times r$ matrix K^i and $1 \times r$ nonsingular matrix Q^i such that

$$r_i(s) = \begin{pmatrix} 0 & \dots & 0 & t_{rr}(s) \end{pmatrix} \begin{pmatrix} f_{-i}(s) \\ f_i(s) \end{pmatrix}.$$

Repeating this methodology for $i = 1, 2, \dots, r$, a bank of r residual generators Ω^i (13) can be designed such that the observer-based, diagonal, FDI problem is solved. ∎

Remark 3.6: In general, it is necessary to design r residual generators Ω^i (13) for system Σ in (5) in order to obtain a vector of r residuals, such that the transfer matrix from faults to residuals is diagonal with nonzero, proper, rational entries. However, for specific practical systems Σ, the same results might be attained with fewer residual generators.

IV. Conclusions

This paper addressed the problem of detection and isolation of contamination faults in water distribution networks. A linear, time-invariant, state-space model for the propagation of a chemical contaminant in water was introduced first, based on the one-dimensional mass balance principle. Then, for this model, a necessary and sufficient condition was presented for the existence of a bank of residual generators capable of performing diagonal fault detection and isolation. This condition was given in terms of the number of vertex disjoint fault-residual paths in the system graph associated to the model. This condition is intuitive and easy to use, and can be applied to any combination of parameter values in the model. It can also be applied to any other practical system that satisfies the framework of the model in this paper.

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