Generalized renewal measures

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INTRODUCTION

1.1 Introduction and summary.

Renewal theory in its early stages concerned itself with a
probabilistic model for the failure and replacement of com-
ponents, such as electric light bulbs. For a general survey
of the theory, and its origins, we refer to Smith ([19][*)]),
for proofs of the basic results see, e.g., Feller ([6]).

Consider a population of components, where each component is
characterized by a nonnegative random variable $\underline{x}$, called its
lifetime. We suppose these random variables $\underline{x}_1, \underline{x}_2, \ldots$ to be
independent and identically distributed, and not to be degen-
erated at zero.

The first component is installed at the initial instant $t = 0$,
say, and is replaced, instantaneously, at the time $\underline{x}_1$. The
second component is, in turn, replaced at time $\underline{x}_1 + \underline{x}_2$, and
so on.

Thus, the time at which the nth renewal takes place is the
n^{th} partial sum of the $\{\underline{x}_i\}$, and we will denote it by

$$\underline{S}_n = \underline{x}_1 + \underline{x}_2 + \ldots + \underline{x}_n \qquad (n = 1, 2, \ldots).$$

Let us write $\underline{N}_t$ for the largest n such that $\underline{S}_n < t$, so that
$\underline{N}_t$ denotes the number of renewals in $[0, t)$.

A central role in renewal theory is played by the renewal
function $H(t)$, which is defined by

$$H(t) = E\{\underline{N}_t\},$$

so that $H(t)$ denotes the expected number of renewals in $[0, t)$.
The simplest and oldest general result about $H(t)$ is the

so-called elementary renewal theorem, which says that

$$H(t)/t \to \mu^{-1} \qquad as \ t \to \infty,$$

where μ denotes the expected lifetime $E\{\underline{x}_i\}$, and where μ^{-1} is
interpreted as zero if $\mu = \infty$.

A much more fundamental – and theoretically very important –
result is Blackwell's theorem, which we formulate for non-
lattice distributed lifetimes. A random variable $\underline{x}$ is called
nonlattice, if there exists no $d > 0$ such that the distribu-
tion of $\underline{x}$ is concentrated on the set $\{nd, \ n \ integer\}$.
Blackwell's theorem then says that, for any fixed $h > 0$,

$$H(t+h)-H(t) \to h\mu^{-1} \qquad as \ t \to \infty.$$

Again, the limit is interpreted as zero if $\mu = \infty$.

Soon it became clear that the mathematically most interesting
problems in the field of renewal theory – those which are
concerned with the asymptotic behaviour of the renewal func-
tion – are of independent interest and they continued to be
studied as such, the only trace of their origin being the
fact that the resulting theorems are generally called renewal
theorems.

Now the $\{\underline{x}_i\}$ no longer necessarily represent the lifetimes in
some renewal process, and the assumption that the $\{\underline{x}_i\}$ are
nonnegative is dropped.

The starting point is now a sequence $\underline{x}_1, \underline{x}_2, \ldots$ of independent
and identically distributed random variables, with common
distribution function F. We further define

$$\underline{S}_0 = 0; \qquad \underline{S}_n = \underline{x}_1 + \underline{x}_2 + \ldots + \underline{x}_n \qquad (n = 1, 2, \ldots).$$

Such a process $\{\underline{S}_n, \ n = 0, 1, \ldots\}$ is commonly called a random
walk. The random variable $\underline{S}_n$ is interpreted as the position
at time n of a particle moving with random jumps, $\underline{x}_n$ being
the jump at time n. The distribution function of $\underline{S}_n$ is given

by F^n, the n-fold convolution of F.

For a Borel set B and for $n \geq 1$ the event $\{\underline{S}_n \in B\}$ is called a visit to B at time n. The expectation of the number of visits of $\{\underline{S}_n\}$ to the set B is called the renewal measure of B, and is denoted by $U(B)$. We have

$$U(B) = \sum_{m=1}^{\infty} F^m(B),$$

where $F^m(B) = P\{\underline{S}_m \in B\}$.

In our work we will always assume that $\int |x| F(dx) < \infty$, and that $\mu = \int xF(dx)$ is positive. This implies that $U(I) < \infty$ for any bounded interval I.

If the distribution function F has positive, possibly infinite, first moment μ then again Blackwell's theorem is valid for the corresponding renewal measure U. Supposing F to be nonlattice we have, for any fixed $h > 0$,

$$U[t, t+h) \to h\mu^{-1} \qquad as \ t \to \infty.$$

In this thesis we want to study measures U_θ, which we will call generalized renewal measures, and which are defined, for any real θ, by

$$U_\theta(B) = \sum_{m=1}^{\infty} m^\theta F^m(B)$$

for all Borel sets B.

What we are interested in is the asymptotic behaviour of these generalized renewal measures, e.g. the behaviour of $U_\theta[a+t, b+t)$ as $t \to \infty$.

Our investigation of the asymptotics of U_θ was motivated by the role which such generalized renewal measures and their asymptotic behaviour play in r-dimensional renewal theory (cf. Stam, [23]).

However, the results are perhaps also interesting from a

purely analytical point of view.

Finally, some new properties of the renewal measure U itself
are derived, and a few known results are obtained by new
techniques.

Related problems were studied in a paper by Smith ([21]),
although his primary concerns are different from ours. Smith
is mainly interested in the behaviour of $U_\theta(-\infty, t)$ for $t \to \infty$,
while we concentrate our attention on local asymptotic prop-
erties of U_θ, namely on the behaviour of $U_\theta[a+t, b+t)$ for
$t \to \infty$. Some of Smith's theorems (e.g. his corollary 5.2) are
comparable with ours, but they are derived by quite different
methods and in several respects they are weaker than our
results.

The first four chapters of this thesis contain preparatory
material. In the first chapter several, mostly well-known,
renewal theorems are listed, and generalized renewal measures
are introduced. We show that U_k, with $k = 1, 2, \ldots,$ and the
$(k+1)$-fold convolution of U are closely connected. In chapter
II we formulate a theorem which shows that if two probability
distribution functions F and G have suitably many moments in
common, then their generalized renewal measures have the same
asymptotic behaviour, up to a certain order. In chapter III
the "comparison" theorem of the second chapter is used to
prove some theorems on the asymptotics of renewal densities,
and further some of the renewal theorems of the first chapter
are sharpened. In chapter IV we consider convolutions of a
certain class of signed measures; the results can be applied
to investigate convolutions of the renewal measure U.

Our main results are derived in chapters V and VI.
In chapter V we are interested in the dominant term of
$U_\theta[a+t, b+t)$, and we prove that, under appropriate conditions
for F,

$$U_\theta[a+t,b+t) = t^\theta \mu^{-\theta-1}(b-a) + o(t^\theta), \qquad as \ t \to \infty.$$

In chapter VI we go farther. There we show that, if F is absolutely continuous and has finite moment of order $\rho+1$, and if F satisfies some further conditions, then we can obtain an asymptotic expansion for the density u_θ of U_θ of the form

$$u_\theta(t) = \sum_{i=0}^{[\rho]} c_i t^{\theta-i} + o(t^{\theta-\rho}), \qquad as \ t \to \infty.$$

Expressions for the coefficients c_i, $i = 0,1,\ldots,k$, can be obtained, and we will see that c_i depends on i, θ, a and b, and further on the first $i+1$ moments of F.

An important element in our derivations is the introduction of a certain family $\{\underline{x}_t, \ t \geq t_o\}$ of integer-valued random variables. Supposing the distribution function F to be non-lattice we define, for some bounded interval I and for t_o sufficiently large,

$$P\{\underline{x}_t = m\} = F^m(I+t)/U(I+t) \qquad (m = 1,2,\ldots).$$

Assuming that $\int_{-\infty}^{o} |x|^3 F(dx) < \infty$ we find from our expressions for the dominant term of U_k, with k integer, that for $t \to \infty$

$$E\{\underline{x}_t\} = t\mu^{-1} + o(t); \qquad E\{\underline{x}_t^2\} = t^2\mu^{-2} + o(t^2).$$

It follows that $\underline{x}_t/t\mu^{-1}$ converges in quadratic mean, and therefore in probability, to 1. Hence we see that the distribution of $(\underline{x}_t/t\mu^{-1})^\theta$ converges completely to a distribution which is degenerated at the point 1. (In our work a stronger form of this weak law of large numbers will be proved, for which the assumption $\int_{-\infty}^{o} |x|^3 F(dx) < \infty$ will be seen to be superfluous.)

Now

$$U_\theta(I+t) = E\{\underline{x}_t^\theta\}U(I+t),$$

and if we may interchange limit and expectation then we find
that for $t \to \infty$, denoting the length of I by $|I|$,

$$U_\theta(I+t) = t^\theta \mu^{-\theta-1} |I| + o(t^\theta).$$

In this way in chapter V an expression for the dominant term
of U_θ, with θ real, is obtained.

In chapter VI we first show that if F has exponentially
vanishing tails then - by the results of chapter IV - an
asymptotic expansion for $U_k(I+t)$, with $k = 1,2,\ldots$, can be
derived, although in this expansion the coefficients are
rather unmanageable.

However, if F has a density with exponentially vanishing
tails, then the existence of such an asymptotic expansion,
together with Laplace inversion techniques, provides us with
other, more satisfactory, expressions for the coefficients in
this expansion, viz. in terms of moments of F.

Now, once more, the random variables $\{\underline{x}_t\}$ appear on the scene.
Assuming F to have finite second moment it turns out that,
curiously enough, the $\{\underline{x}_t\}$ for $t \to \infty$ satisfy a central limit
theorem. A satisfactory stochastic interpretation of this
property has not yet been found.

If F has a density with exponentially vanishing tails, then
it is even possible to show that the moment generating func-
tion of $\underline{y}_t = (\underline{x}_t - t/\mu)/\sqrt{t/\mu}$ converges to the moment generating
function of a normal law.

Furthermore

$$u_\theta(t) = E\{\underline{x}_t^\theta\}u(t)$$
$$= E\{(\underline{y}_t \sqrt{t/\mu} + t/\mu)^\theta\}u(t),$$

and by expanding $(\underline{y}_t \sqrt{t/\mu} + t/\mu)^\theta$ in terms of powers of $\underline{y}_t$,
and using the above-mentioned results for the moment gener-
ating function of $\underline{y}_t$, we then get an asymptotic expansion

for $u_\theta(t)$, with θ real, for the case where F has a density
with exponentially vanishing tails.

Finally the comparison theorem of chapter II is used to show
that the finite asymptotic expansions we just obtained are
valid for a more general class of probability distribution
functions F.

1.2 Notations and conventions.

With $\tilde{B}$, $\tilde{B}_b$, $\tilde{B}_i$ we denote, respectively, the classes of *Borel
sets*, *bounded Borel sets* and *bounded intervals* of $(-\infty, \infty)$. The
indicator function of a set A is written as χ_A or $\chi\{A\}$. The
set $A+t$, for real t and $A \in \tilde{B}$, denotes the *translation* of A
to the right over a distance t, so that $A+t = \{y: y-t \in A\}$.

A *signed measure* S will, as usual, be defined as an extended
real-valued, countably additive set function on the class of
all measurable sets of a measurable space $(X, \tilde{C})$, such that
$S(\emptyset) = 0$, and such that S assumes at most one of the values
$+\infty$ and $-\infty$. Every signed measure S can be written unambiguously
as the difference $S^+ - S^-$ of two measures S^+ and S^- (of which
at least one is finite) with $S^+ \perp S^-$, i.e. there exists a set
A such that $S^+(A) = 0$, $S^-(A^c) = 0$. The measures S^+ and S^- are
called the *positive* and *negative variation* of S. The sum
$S^+ + S^-$ is called the *variation* of S, and is denoted by $|S|$.
The (signed) measures we meet in the sequel are considered to
be defined on $((-\infty, \infty), \tilde{B})$. A measure R is called *interval-
finite* if $R(I) < \infty$ for all $I \in \tilde{B}_i$.

The assumption that a signed measure S cannot take both the
values $+\infty$ and $-\infty$ is needed to guarantee that S is defined on
$((-\infty, \infty), \tilde{B})$. However, our work leads us to consider differ-
ences $R_1 - R_2$, where R_1 and R_2 are interval-finite measures