ABSTRACT

Title of dissertation: ESSAYS ON MATCHING MARKETS AND THEIR EQUILIBRIA

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Matching theory and matching markets are a core component of modern economic theory and market design. This dissertation presents three original contributions to this area.

The first essay constructs a matching mechanism in an incomplete information matching market in which the positive assortative match is the unique efficient and unique stable match. The mechanism asks each agent in the matching market to reveal her privately known type. Through its novel payment rule, truthful revelation forms an ex post Nash equilibrium in this setting. This mechanism works in one-, two- and many-sided matching markets, thus offering the first mechanism to unify these matching markets under a single mechanism design framework.

The second essay confronts a problem of matching in an environment in which no efficient and incentive compatible matching mechanism exists due to matching externalities. I develop a two-stage matching game in which a contracting stage facilitates subsequent conditionally efficient and incentive compatible Vickrey auction stage. Infinite repetition of this two-stage matching game enforces the contract in

every period. This mechanism produces inequitably distributed social improvement: parties to the contract receive all of the gains and then some.

The final essay demonstrates the existence of prices which stably and efficiently partition a single set of agents into firms and workers, and match those two sets to each other. This pricing system extends Kelso and Crawford's [14] general equilibrium results in a labor market matching model and links one- and two-sided matching markets as well.

ESSAYS ON MATCHING MARKETS AND THEIR EQUILIBRIA

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2016

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Dedication

With thanks to my dad, Paul Everett Utgoff (1951 - 2008), and to my mom, Karen Lauter Utgoff.

Acknowledgments

Words cannot express my thanks to those who made this dissertation possible.

Professor Lawrence Ausubel has been a rock - an advisor who gave me guidance I needed, perspective I lacked, space to succeed, and (perhaps most important of all) space to fail so that I might try again. He encouraged me to choose the University of Maryland and I am forever grateful. This thesis would not exist without him.

Professor Daniel Vincent offered a uniquely geometric approach to economic theory; this approach coupled with his incisive insight about what economic questions really matter has been and continues to be a source of challenge, inspiration, and immeasurable improvement to my work. Professor Erkut Ozbay's excitement is infectious.

Professors Emel Filiz-Ozbay, and Peter Cramton asked difficult questions that pushed me to new results. Professors John Ham, John Wallis, and Judith Hellerstein have offered support, advice, and an invaluable non-theory perspective on economic theory. Despite many claims on their time, they have generously shared their thoughts.

Vicki Fletcher and Terry Davis have always cheered me on.

Dorothea Brosius's LaTeXtemplates (available to all University of Maryland students) were invaluable.

My family is a source of love and strength. Mom, Dad, Emily, Ariel - thank you. My husband Aryeh gave me the ultimate gift of research time by doing more loads of laundry and washing more dishes than I had any right to expect. My son

Ethan learned how pens work by making his own scribbled contributions to several pages of notes.

My father passed away as I began graduate school applications in 2008. He would have been proud and supportive of all the steps along the way, culminating in this dissertation.

The University of Maryland, and in particular the Economics Department provided financial support.

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Chapter 1: Introduction

1.1 Introduction

My dissertation consists of three essays on matching. Gale and Shapley's seminal paper [6] introduced matching theory to economics, spawning several major sub-literatures. The first, to which Gale and Shapley's paper belongs, studies matching algorithms. It assumes complete information about each agent's preference relation over other agents; thus the agents do not and cannot strategic behavior during the execution of the algorithm. The second is the closely related study of matching mechanisms. It assumes various types of incomplete information and allows strategic misreporting by one or more agents. Work in this area designs a procedure (usually an algorithm or auction-like mechanism) that mitigates agents' incentives to strategically misreport their information. The third is the study of matching in the tradition of general equilibrium. Traditional tools of general equilibrium only apply when the market for each good is reasonably thick. In a matching environment, each person who seeks a match is a unique and indivisible good. This literature develops particular tools which expand ideas of general equilibrium to these matching environments in which traditional tools are insufficient.

The first and second essays of this dissertation contribute to the second sub-literature. The first essay introduces an auction-like mechanism that implements the positive assortative match in a wide variety of incomplete information matching environments in which the positive assortative match in the unique efficient stable match. The second essay develops a second-best hybrid contracts and auction mechanism in a repeated college admissions environment that does not admit a fully efficient and incentive compatible mechanism. The last essay contributes to the third sub-literature: it extends ideas of general equilibrium from Kelso and Crawford [14] to a labor market in which the partition between firms and workers is endogenously determined.

The first essay provides a unifying framework for matching markets with incomplete information, when the positive assortative match is the unique efficient stable match. I construct a Vickrey-Clarke-Groves-like mechanism which implements assortative matching as an ex post Nash equilibrium. It achieves this result using a payment rule that distinguishes between an agent deprived of any match and an agent who merely receives a reduced match surplus. The constructed mechanism recognizes only opportunity costs arising from the former, and not the latter, effect. I also generalize the stronger condition of envy freeness to these incomplete information environments and show that the constructed equilibrium is envy free.

The second essay introduces a dynamic matching mechanism in which a persistent contracting relationship - an “old boys’ club” - occurs in a distribution-independent perfect Bayesian equilibrium when the high school in the club is sufficiently patient. Matching occurs in two stages: first, contracting between the college and a high school; second, running a Vickrey auction in the simplified post-contracting admissions market. The mechanism provides the second best total surplus among several mechanisms in a repeated college admissions market in which externalities preclude solutions using standard mechanism and market design techniques. An "old boys' club" emerges between one college and a sufficiently patient single high school as a consequence of contract enforcement rather than *ex ante* bias on the part of the college. Members of the club benefit at the expense of the non-contracted high school.

The third essay considers a modified version of Kelso and Crawford's model of labor market matching with transferrable utility in which the partition between firms and workers is endogenously determined rather than an assumption of the model. I show that there exist prices in this market which stable support the socially optimal partition of agents into firms and workers, and the consequent socially optimal match of those firms and workers.

Chapter 2: Implementation of Assortative Matching Under Incomplete Information

2.1 Introduction

Matching markets in which people have incomplete information about the qualities of potential partner(s) occur frequently and there are a variety of practical approaches that (try to) elicit this information prior to match formation to facilitate more productive or more stable matches. The interview process facilitates job matching; the dating process facilitates marriage. This paper considers matching markets in which all matches are formed through a single central clearinghouse, and in which the positive assortative match is the unique efficient and unique stable match. I model this clearinghouse as a direct revelation mechanism to which each agent reports her privately known partner quality; the mechanism implements the positive assortative match given the agents' reports. Truthful reporting in this mechanism forms an ex post Nash equilibrium in all such incomplete information matching environments (roommate, marriage, supply chain, both one-to-one and many-to-one versions).

The main contribution of this mechanism is its payment rule. It distinguishes between the opportunity cost an agent imposes on other agents whom her report deprives of membership in a match, and the opportunity cost she imposes when her report merely alters the value of other agents' respective match surpluses. When each agent's payment contains the former but not the latter opportunity costs, truthful reporting forms an ex post Nash equilibrium. The simplest example is a roommate market with three agents and a single two-person room. Each person reports her privately known roommate quality to the mechanism. The people with the two higher reported qualities each receive one bed and each of these two roommates receives a match surplus that is increasing and supermodular in her own and her roommate's qualities. The person who reports the lowest quality receives neither bed, deprives no one else of a bed, and pays zero. Each winning roommate has two effects: first, she deprives the person who submits the lowest report of a bed; second, she affects her roommate's match surplus. Her payment is the opportunity cost she imposes on the person who did not receive a bed, but does not include her effect on her roommate. This payment rule is the correct generalization of the payment rule in the VCG mechanism (with independent private values) to the interdependent value environment of these matching markets. Under independent private values, a person imposes opportunity costs on others only when her report deprives others of objects. The payment rule of the VCG-like matching mechanism of this paper recognizes that the only opportunity costs that should contribute to an agent's payment are those that arise due to depriving other agents of objects, even when other externalities are present due to interdependence.

Another way to view the VCG-like mechanism is as a position auction which implements positive assortative matching in this paper's class of matching markets. Matching theory and auction theory began contemporaneously but independently. Vickrey's seminal auction paper [21] describes a generalization of the second price auction to multiple identical objects and shows that there is no mechanism that elicits both true supply curves and demand curves. His work lays out the primary concerns addressed in subsequent auction papers: elicitation of buyers' private valuations, efficiency, and revenue. On the other hand, algorithmic design and match stability are the primary concerns in the (early) matching literature. Gale and Shapley's seminal matching paper [6] introduces their deferred acceptance algorithm. They show that it always finds a stable match in a marriage market in which preferences are public information, and that a stable match always exists in a marriage market.

Kelso and Crawford [14] were the first authors to link the matching and auction theory literatures. They exhibit a discrete wage adjustment algorithm similar to a clock auction that identifies and implements a stable match between firms and workers in a complete information setting. The intersection of these fields has since emerged as a fruitful area with numerous interesting problems and results. However, Roth [18] demonstrates that when matching theory considers the problem of preference elicitation (akin to the auction-theoretic concern of eliciting valuations) truthful revelation of preferences can be a dominant strategy for only one side of a marriage market in an efficient matching mechanism. The impossibility arises due to preference heterogeneity across agents.

Current work on matching with incomplete information assumes homogeneous, known preferences and that each agent privately observes her single-dimensional quality as a partner. These markets assume that agents' payoff functions are such that each agent strictly prefers a higher quality to a lower quality partner. It is well known that in the full information analog of this environment, positive assortative matching is both the unique efficient and unique stable match. Having eliminated preference heterogeneity, there are several interesting results concerning mechanisms which implement positive assortative matching in marriage markets. The two main papers considering these models are by Hoppe, Moldovanu, and Sela [11] and Johnson [13]. Hoppe, Moldovanu, and Sela [11] show that a fully separating signaling equilibrium exists in a marriage market with privately informed agents, but that random matching may be socially superior to a signaling equilibrium due to the signaling costs incurred by agents. Their paper adopts Spence's [20] convention that signaling costs are wasted; their equilibrium is logically equivalent to an all pay auction. Johnson [13] observes that signaling costs in these environments are often not wasted; many matching markets are run by a self-interested matchmaker serving as an information broker who receives the signaling costs as her payment. Johnson constructs revenue-maximizing auctions that implement truncated assortative matching. Both [11] and [13] implement their solutions as Bayesian Nash equilibria; while they achieve positive assortative matching (or in the case of [13], a truncation thereof) agents experience ex post regret due to the payment structure in Bayesian implementation. Bayesian implementation also assumes common knowledge of the distribution of agents' respective qualities.

Ex post Nash equilibrium is an attractive implementation choice as it is distribution independent and regret free. This paper constructs an efficient, ex post incentive compatible mechanism for any incomplete information matching market in which the match surplus functions are increasing and supermodular in each agent's one-dimensional partner quality, including the marriage markets of [11] and [13]. I describe a natural generalization of the marriage markets of [11] and [13] to a large class of matching markets in which positive assortative matching is the unique efficient stable match. I then introduce the VCG-like mechanism that unifies these markets in a single framework and prove that this mechanism always implements positive assortative matching as an ex post Nash equilibrium; in particular, this mechanism works in both one-, two-, and many-sided matching markets. This equilibrium is slightly stronger than ex post Nash equilibrium; I show that the idea of locally envy free equilibrium introduced by Edelman, Ostrovsky, and Schwarz [5] generalizes completely to this class of incomplete information environments.

2.2 Examples

The following examples of roommate and marriage matching give the flavor of the mechanism and intuition for its payment rule. In both examples, the reservation payoff is 0.

2.2.1 Roommates

There are three agents and one room with two identical beds. Each agent n has privately known roommate quality $x_n \in [0, 1]$. If agents n and n' each receive one of the beds, n receives $s(x_n, x_{n'})$ and n' receives $s(x_{n'}, x_n)$ where $s(\cdot,\cdot)$ is increasing in both arguments and supermodular. These requirements on $s(\cdot,\cdot)$ are sufficient to guarantee that positive assortative matching is the unique efficient stable match. Without loss of generality assume that $x_1 \geq x_2 \geq x_3$. Each agent reports her quality to the mechanism which implements positive assortative matching: the agents who submit the two highest reports receive one bed each. The agent who submits the lowest report receives no bed and the reservation payoff 0. Denote agent n's report $\widehat{x}_n$ to distinguish her reported quality from her actual quality. When each agent reports her quality truthfully, agents 1 and 2 each receive a bed in the positive assortative match. The positive assortative match yields the match surpluses shown in the table.

Agent	Roommate	Surplus	Payment
1	2	$s(x_1, x_2)$	$s(\widehat{x}_3, \widehat{x}_2)$
2	1	$s(x_2, x_1)$	$s(\widehat{x}_3, \widehat{x}_1)$
3	-	0	0

Truthful reporting forms an ex post Nash equilibrium under the payments shown in the table: when any two agents report their qualities truthfully, the third cannot profitably deviate from reporting her own quality truthfully. For example, suppose agents 2 and 3 report their qualities truthfully: if agent 1 reports $\widehat{x}_1 \geq x_3$, she receives one of the beds, pays $s(x_3, x_2)$, and receives payoff $s(x_1, x_2) - s(x_3, x_2) \geq 0$ because $s(\cdot,\cdot)$ is increasing in its first argument. If agent 1 reports $\widehat{x}_1 < x_3$, she receives no bed and the reservation payoff 0. Since $x_1 \geq x_3$, agent 1 is at least as well off reporting her quality truthfully as she would selecting any other report.

Agent 1's report has two effects: her report (1) deprives agent 3 of a bed and (2) affects the value of agent 2's match surplus. The intuition for agent 1's payment is that she should pay the opportunity cost she imposes on agent 3 by depriving her of a bed, but should not pay for her effect on agent 2, whom she deprived of nothing. Similarly, agent 2's payment is the opportunity cost she imposes on agent 3, but her effect on agent 1, whom she deprived of nothing. Agent 3, who deprives neither agent 1 nor 2 of a bed, pays 0. Another way to describe the payment rule is that for each agent n, the mechanism identifies the set of other agents with whom n is effectively in competition: in the above example agents 1 and 2 are not ex post in competition for a bed.

2.2.2 Marriage

The marriage market's structure differs significantly from the roommate market: the list of possible partners for a man (woman) is restricted to women (men), whereas in the roommate market any agent may match with any other. However, despite this difference, all of the observations from the roommate example apply to this marriage example.

Suppose that there are two men and two women. Each man i has privately known quality $m_i \in [0, 1]$ and each woman j has privately known quality $w_j \in [0, 1]$. If man i and woman j are matched, the man receives $s_M(m_i, w_j)$ and the woman receives $s_W(w_j, m_i)$ where both $s_M(\cdot, \cdot)$ and $s_W(\cdot, \cdot)$ are increasing in both arguments and supermodular; these conditions on $s_M(\cdot, \cdot)$ and $s_W(\cdot, \cdot)$ are sufficient to guarantee that positive assortative matching is the unique efficient stable match. Suppose without loss of generality that $m_1 \geq m_2$ and $w_1 \geq w_2$. Each agent reports his (her) type to the mechanism which implements positive assortative matching: the man with the higher report (of the two men) is matched with the woman with the higher report (of the two women), and the man with the lower report is matched with the woman with the lower report. Denote man i's (woman j's) report $\widehat{m}_i$ ($\widehat{w}_j$) to distinguish his (her) report from his (her) quality. When each agent reports his (her) quality truthfully the positive assortative match pairs man 1 with woman 1 and man 2 with woman 2. The positive assortative match yields the match surpluses shown in the table.

Agent	Spouse	Surplus	Payment
M_1	W_1	$s_M(m_1, w_1)$	$s_M(\widehat{m}_2, \widehat{w}_1) - s_M(\widehat{m}_2, \widehat{w}_2)$
M_2	W_2	$s_M(m_2, w_2)$	0
W_1	M_1	$s_W(w_1, m_1)$	$s_W(\widehat{w}_2, \widehat{m}_1) - s_W(\widehat{w}_2, \widehat{m}_2)$
W_2	M_2	$s_W(w_2, m_2)$	0

Truthful reporting again forms an ex post Nash equilibrium under the payments shown in the table: when any three agents report their qualities truthfully, the remaining agent cannot profitably deviate from reporting truthfully him/herself. The intuition behind the payment rule in the marriage market is identical to the intuition behind the payment rule in the roommate market: an agent's payment should be the total opportunity cost imposed on other agents whom his (her) report displaced from a marriage. The two-sided structure of the marriage market means that a man (woman) can only displace another man (woman) from a match; a man cannot displace a woman and a woman cannot displace a man. In this example, M_1 displaces M_2 from marriage with W_1. Therefore, his payment is the opportunity cost he imposes on M_2 by preventing him from marrying W_1. However, M_1's payment does not include the opportunity cost he imposes on either woman because he does not (and cannot) displace either woman from a marriage.

2.3 Model

There is a set of agents $\mathcal{N} = \{1, \cdots, N\}$, a set of bins $\mathcal{B} = \{1, \cdots, B\}$, and a set of roles $\mathcal{R} = \{1, \cdots, R\}$. A bin b is a set of open slots, each of which can be occupied by at most one agent acting in a role r specified by the bin. Each bin b has a publicly known number of slots $c_{b,r} \geq 0$ for each role r; assume that $c_{i,r} \geq c_{j,r}$ whenever $i < j$, so that bin i is at least as big as bin j for all $1 \leq i \leq j \leq B$. Let $c_b = \sum_{r=1}^{R} c_{b,r}$.

Each agent n has privately known type $x_n : \mathcal{R} \to [0, 1]$ where $x_n(r)$ denotes n's quality if she occupies a role r slot. Assume that each agent n has one productive role $r_n \in \mathcal{R}$; for all $r \neq r_n$, $x_n(r) = 0$. There are some matching markets (such as marriage markets) in which an agent's productive role is public information and others (such as labor markets) in which it may be private. The results hold even when every agent's productive role is private information, so assume that it is private. Assume also that the x_n are independently distributed. Let $\mathbf{x} = [x_1 \; x_2 \; \cdots \; x_N]$, $\mathbf{x}_{-n} = [x_1 \; \cdots \; x_{n-1} \; x_{n+1} \; \cdots \; x_N]$, and $(\mathbf{x}_{-n}, z) = [x_1 \; \cdots \; x_{n-1} \; z \; x_{n+1} \; \cdots \; x_N]$.

An assignment is a function mapping agents to slots in bins. Formally, it is a function $\mu : \mathcal{N} \to \mathcal{B} \times \mathcal{R} \cup (0, 0)$ which satisfies

1. $\mu(n) = (0, 0)$ if and only if n is unmatched, i.e. μ assigns n to no slot in any

bin

2. $\mu(n) = (b, r)$ if and only if agent n occupies a slot in role r in bin b

3. All capacity constraints are satisfied, i.e. $|\mu^{-1}(b, r)| \leq c_{b,r}$ for all (b, r)

Let A denote the set of all functions μ satisfying items 1 - 3 above, and let $\phi : \mathcal{B} \times \mathcal{R} \to \mathcal{B}$ and $\rho : \mathcal{B} \times \mathcal{R} \to \mathcal{R}$ denote the canonical projections. Let $\overline{\mu} = \rho \circ \mu$. For all agents n, denote the set of n's partners in role r by

$$W_\mu(n, r) = \begin{cases} \{i \in \mathcal{N} \backslash \{n\} \,|\, \phi \circ \mu(i) = \phi \circ \mu(n) \text{ and } \rho \circ \mu(i) = r\} & \text{if} \quad \phi \circ \mu(n) \neq 0 \\ \emptyset & \text{if} \quad \phi \circ \mu(n) = 0 \end{cases}$$

Let

$$y_n(\mu, r) = \begin{cases} (x_{k_1}(r), x_{k_2}(r), \cdots, x_{k_{|W_\mu(n,r)|}}(r), \underbrace{0, \cdots, 0}_{c_{b,r} - |W_\mu(n,r)|}) & \text{if} \quad \overline{\mu}(n) \neq r \\ (x_{k_1}(r), x_{k_2}(r), \cdots, x_{k_{|W_\mu(n,r)|}}(r), \underbrace{0, \cdots, 0}_{c_{b,r} - 1 - |W_\mu(n,r)|}) & \text{if} \quad \overline{\mu}(n) = r \end{cases}$$

where $k_1 < k_2 < \cdots k_{|W_\mu(n,r)|} \in W_\mu(n, r)$. Note that $y_n(\mu, r)$ is an ordered $c_{b,r}$-tuple when $\overline{\mu}(n) \neq r$ and an ordered $(c_{b,r} - 1)$-tuple when $\overline{\mu}(n) \neq r$. Finally, let

$$y_n(\mu) = (y_n(\mu, 1), y_n(\mu, 2), \cdots, y_n(\mu, R))$$

An agent who occupies a slot in her productive role derives private match surplus from her interactions with other agents in the same bin. An agent who is unmatched or occupies a slot not in her productive role receives a match surplus of zero. For each bin b and role r, there is a known private surplus function $s_{b,r} : [0,1] \times [0,1]^{c_b - 1} \to \mathbb{R}$ that is weakly increasing in all arguments and weakly supermodular. Agent n who occupies a slot in bin b in role r receives $s_{b,r}(x_n(r), y_n(\mu))$. Assume that $s_{b,r}(0, y_n) = 0$ for all b, r and y_n; intuitively, an agent whose quality in role r is 0 and who works in role r receives match surplus 0.

The conditions on the $s_{b,r}$ together with the structure of the bins' capacity constraints imply that positive assortative matching is the unique efficient stable match in this model.[1] In this model, the positive assortative match is the result of implementing the following process in the full information analog of the model environment.

1. Sort agents by productive role; within each role r, rank them from highest to lowest by quality.

2. Among agents whose productive role is 1, assign the $c_{1,1}$ highest quality agents to bin 1, the next $c_{2,1}$ highest quality agents to bin 2, and so on until there are no more agents whose productive role is 1 or until all role 1 jobs at all B bins are filled.

3. Repeat Step 2 for roles $2, \cdots, R$.

Let $L \subseteq \{1, \cdots, N\}$ be the set of agents in a bin in their respective productive roles. Agent n's match surplus is $\sum_{i \in L \setminus \{n\}} s_{r_n}(x_n(r_n), x_i(r_i))$. Agents' utilities are

[1]Shapley and Shubik [19] were the first to observe that in a two-sided one-to-one matching market, supermodularity implies that positive assortative matching is the unique efficient stable match.

quasilinear: agent n pays t_n for a slot in bin m in role r, she receives

$$\sum_{i \in L \setminus \{n\}} s_{r_n}(x_n(r_n), x_i(r_i)) - t_n$$

Under the assignment μ, agent n receives match surplus

$$s_{b,r}(x_n(\overline{\mu}(n)), y_n(\mu))$$

If $\mu(n) = (0, 0)$, agent n's match surplus is necessarily 0 since $W_\mu(n) = \emptyset$. The social surplus under μ is

$$\sum_{n \in \mathcal{N}} s_{b,r}(x_n(\overline{\mu}(n)), y_n(\mu))$$

An unmatched agent or an agent not matched in her productive role receives match surplus zero and contributes zero to the social surplus.

2.4 Mechanism Design

A mechanism is a pair of functions $\{q, t\}$ that take as their arguments messages from the N agents, and respectively map those messages into a lottery over assignments and a payment vector. A direct mechanism is one in which each agent's message space exactly coincides with her type space. By the revelation principle [15], restriction to direct mechanisms proceeds without loss of generality. Let $\widehat{x}_n$ denote agent n's report, $\widehat{\mathbf{x}} = [\widehat{x}_1 \ \widehat{x}_2 \ \cdots \ \widehat{x}_N]$, $\widehat{\mathbf{x}}_{-n} = [\widehat{x}_1 \ \cdots \ \widehat{x}_{n-1} \ \widehat{x}_{n+1} \ \cdots \ \widehat{x}_N]$, and $(\widehat{\mathbf{x}}_{-n}, z) = [\widehat{x}_1 \ \cdots \ \widehat{x}_{n-1} \ z \ \widehat{x}_{n+1} \ \cdots \ \widehat{x}_N]$. The report $\widehat{x}_n = 0$ denotes the zero function on $\mathcal{R}$. Let Δ_A denote the lottery space over A and let $\Theta = \{x : \mathcal{R} \to [0,1] \,|\, x \text{ is a function}\}$. A direct mechanism in this model is a pair of functions $\{q, t\}$ where $q : \Theta^N \to \Delta_A$ and $t : \Theta^N \to \mathbb{R}^N$. Let $q^\mu(\widehat{\mathbf{x}})$ be the probability that μ occurs under the lottery $q(\widehat{\mathbf{x}})$. The nth coordinate $t_n(\widehat{\mathbf{x}})$ of $t(\widehat{\mathbf{x}})$ denotes agent n's payment. A mechanism is feasible if and only if

$$\sum_{\mu \in A} q^\mu(\widehat{\mathbf{x}}) \leq 1 \text{ for all } \widehat{\mathbf{x}} \in \Theta^N \tag{2.1}$$

and

$$q^\mu(\widehat{\mathbf{x}}) \geq 0 \text{ for all } \widehat{\mathbf{x}} \in \Theta^N \text{ and for all } \mu \in A \tag{2.2}$$

Agent n's expected surplus under the lottery $q(\widehat{\mathbf{x}}_{-n}, \widehat{x}_n)$ is

$$H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, x_n) = \sum_{\mu \in A} \left[q^\mu(\widehat{\mathbf{x}}_{-n}, \widehat{x}_n) \sum_{i \in W_\mu(n)} s_{\overline{\mu}(n)}(x_n(\overline{\mu}(n)), x_i(\overline{\mu}(i))) \right]$$

Agent n's payoff when representing quality $\widehat{x}_n$ and the other agents represent $\widehat{\mathbf{x}}_{-n}$ is

$$U_n(q, t, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, x_n) = H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, x_n) - t_n(\widehat{\mathbf{x}}_{-n}, \widehat{x}_n) \tag{2.3}$$

A mechanism is ex post individually rational if and only if

$$U_n(q, t, \mathbf{x}_{-n}, x_n, \mathbf{x}_{-n}, x_n) \geq 0 \text{ for all } n \in \mathcal{N} \tag{2.4}$$

In other words, in equilibrium each agent must be at least as well off participating as not.

In order for agents to reveal their respective types truthfully to the mechanism, an incentive compatibility constraint must hold.

Definition 2.4.1. *A mechanism is* ex post incentive compatible *if for all $x_n, \widehat{x}_n \in \Theta$ and $n \in \mathcal{N}$*

$$U_n(q, t, \mathbf{x}_{-n}, x_n, \mathbf{x}_{-n}, x_n) \geq U_n(q, t, \mathbf{x}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, x_n) \tag{2.5}$$

Ex post incentive compatibility requires that for each agent n, when all other agents' types are revealed, n cannot profitably deviate from truthful reporting. Alternatively, truthful revelation forms an ex post Nash equilibrium.

2.5 VCG-like Mechanism: $R = 1$

This section restricts attention to $R = 1$ and suppresses role-specific notation and language; there is one role which by default is the productive role for each agent. This restriction makes it possible to illustrate the payment rule with minimal technical and notational complications. Agent n's type is simply her quality $x_n \in [0, 1]$. An assignment is a function $\mu : \mathcal{N} \to \mathcal{B} \cup \{0\}$ such that

1. $\mu(n) = 0$ if and only if n is unmatched, i.e. μ assigns n no slot in any bin

2. $\mu(n) = b$ if and only if agent n occupies a slot in bin b

3. All capacity constraints are satisfied, i.e. $|\mu^{-1}(b)| \leq c_b$ for all b

2.5.1 Existence and Structure of the VCG-like Mechanism

The mechanism of this section is the natural and correct generalization of the well-known VCG mechanism to a one-sided matching environment in which positive assortative matching is the unique efficient stable match. Define

$$\mu^*_{\widehat{\mathbf{x}}} = \underset{\mu \in A}{\operatorname{argmax}} \sum_{n \in \mathcal{N}} \left[\sum_{i \in W_\mu(n)} s(\widehat{x}_n, \widehat{x}_i) \right]$$

In other words, $\mu^*_{\widehat{\mathbf{x}}}$ is efficient given the report profile $\widehat{\mathbf{x}}$. Since the model structure guarantees that the positive assortative match is the unique efficient assignment, $\mu^*_{\widehat{\mathbf{x}}}$ must be the positive assortative match given $\widehat{\mathbf{x}}$. The key insight of this extension is that an agent's payment the opportunity cost she imposes on agent(s) whom she displaces from a *bin*, rather than every agent on whom she has an effect.

The set of agents displaced from a bin by agent n is

$$D_n(\widehat{\mathbf{x}}) = \{k \in \mathcal{N} \backslash \{n\} \mid \mu^*_{\widehat{\mathbf{x}}}(k) \neq \mu^*_{\widehat{\mathbf{x}}_{-n},0}(k)\}$$

Also define

$$v_n(\widehat{\mathbf{x}}) = \sum_{i \in W_{\mu^*_{\widehat{\mathbf{x}}}}(n)} s(x_n, x_i)$$

$$\alpha_n(\widehat{\mathbf{x}}) = \sum_{k \in D_n(\widehat{\mathbf{x}})} \left[\sum_{i \in W_{\mu^*_{\widehat{\mathbf{x}}_{-n},0}}(k)} s(\widehat{x}_k, \widehat{x}_i) \right]$$

$$\beta_n(\widehat{\mathbf{x}}) = \sum_{k \in D_n(\widehat{\mathbf{x}})} \left[\sum_{i \in W_{\mu^*_{\widehat{\mathbf{x}}}}(k)} s(\widehat{x}_k, \widehat{x}_i) \right]$$

Agent n's payment is $t_n(\widehat{\mathbf{x}}) = \alpha_n(\widehat{\mathbf{x}}) - \beta_n(\widehat{\mathbf{x}})$.

The VCG-like mechanism operates as follows. Agents simultaneously and independently report their respective types to the mechanism, which selects the socially optimal assignment $\mu^*_{\widehat{\mathbf{x}}}$. If there are multiple socially optimal assignments due to identical reports by one or more agents, the mechanism chooses a socially optimal assignment at random. Each agent receives $v_n(\widehat{\mathbf{x}})$ and pays $t_n(\widehat{\mathbf{x}})$.

The VCG-like flavor of this mechanism is now apparent: the mechanism selects a socially optimal assignment given $\widehat{\mathbf{x}}$, and charges each agent n the opportunity cost she imposes on agents in $D_n(\widehat{\mathbf{x}})$. In an independent private values setting, $D_n(\widehat{\mathbf{x}})$ is perforce the set of all agents on whom n imposes any opportunity cost. In the interdependent value setting of this matching market, the payment rule needs to be more circumspect about which opportunity costs appear in an agent's payment and which do not in order to satisfy incentive compatibility.

Theorem 2.5.1. *Truthful reporting is an ex post Nash equilibrium in the VCG-like mechanism.*

Proof. See Appendix A.1. □

Ex post incentive compatibility is the strongest achievable implementation choice; Williams and Radner [22] showed that interdependent values preclude implementation in dominant strategies.

Theorem 2.5.2. *(Williams and Radner) Positive assortative matching in the interdependent value environment of this paper is never implementable in dominant strategies.*

Theorem 2.5.2 follows from a lemma that generalizes Myerson's [15] well-known auction implementability result.

Lemma 2.5.3. *Truthful reporting in a feasible, individually rational direct mechanism is*

1. *a dominant strategy if and only if for all n and $\widehat{\mathbf{x}}_{-n}$, $H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, x_n)$ is supermodular in $(\widehat{x}_n, x_n)$ and*

$$U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n, \widehat{\mathbf{x}}_{-n}, x_n) = U_n(q, t, \widehat{\mathbf{x}}_{-n}, 0, \widehat{\mathbf{x}}_{-n}, 0) + \int_0^{x_n} \left[\frac{\partial}{\partial z} H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \widehat{\mathbf{x}}_{-n}, z)\right]\bigg|_{\widehat{x}_n = z} dz \tag{2.6}$$

2. *ex post incentive compatible if and only if for all n and $\mathbf{x}_{-n}$, $H_n(q, \mathbf{x}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, x_n)$*

is supermodular in $(\widehat{x}_n, x_n)$ *and*

$$U_n(q,t,\mathbf{x}_{-n},x_n,\mathbf{x}_{-n},x_n) = U_n(q,t,\mathbf{x}_{-n},0,\mathbf{x}_{-n},0) + \int_0^{x_n} \left[\frac{\partial}{\partial z} H_n(q,\mathbf{x}_{-n},\widehat{x}_n,\mathbf{x}_{-n},z)\right]\Bigg|_{\widehat{x}_n=z} dz \tag{2.7}$$

Proof. See Appendix A.1. □

Lemma 2.5.3 now implies Theorem 2.5.2.

Proof. Suppose that there is a dominant strategy incentive compatible mechanism that implements positive assortative matching. Suppose that $x_n = 0$ for $n = 2, \cdots, N$ and $\widehat{x}_n = 1 - \epsilon$ for $n = 2, \cdots, N$. Suppose also that $x_1 = 1$. If agent 1 reports truthfully $\widehat{x}_1 = 1$, she wins a slot in bin 1 with $c_1 - 1$ randomly assigned bin-mates. She receives $(c_1-1)s(1,0) = 0$. According to the payment rule in Lemma 2.5.3

$$\begin{aligned} t_1(\widehat{\mathbf{x}}_{-1},1) &= H_n(q,\widehat{\mathbf{x}}_{-1},1,\widehat{\mathbf{x}}_{-1},1) - \int_0^1 \left[\frac{\partial}{\partial z} H_n(q,\widehat{\mathbf{x}}_{-1},\widehat{x}_1,\widehat{\mathbf{x}}_{-1},z)\right]\Bigg|_{\widehat{x}_1=z} dz \\ &= (c_1-1)s(1-\epsilon,1-\epsilon) > 0. \end{aligned}$$

Agent 1's payoff is $-(c_1-1)s(1-\epsilon,1-\epsilon) < 0$. By reporting $\widehat{x}_1 = 0$ she would have received non-negative surplus and paid 0. Agents $2, \cdots, N$ have types and strategies that make truth-telling strictly worse for agent 1 than some other option (in this example, reporting 0). This statement is exactly that truthful reporting is not weakly dominant, a contradiction. □

Consider the following naive extension of the VCG mechanism to this matching environment, which assigns the positive assortative match given reports $\widehat{\mathbf{x}}$ and in which agent n's payment is

$$t_n^{\text{naive}}(\widehat{\mathbf{x}}) = \sum_{k \neq n} \left[\sum_{i \in W_{\mu^*_{\widehat{\mathbf{x}}_{-n},0}}(k)} s(\widehat{x}_k, \widehat{x}_i) \right] - \sum_{k \neq n} \left[\sum_{i \in W_{\mu^*_{\widehat{\mathbf{x}}}}(k)} s(\widehat{x}_k, \widehat{x}_i) \right]$$

In particular, Theorem 2.5.2 implies that truthful reporting is not, in this environment, a dominant strategy in the naive extension of the VCG mechanism. Due to the interdependence, an agent's naive VCG payment $t_n^{\text{naive}}(\widehat{\mathbf{x}})$ depends on her own report due to the externalities agent n exerts on other agents in the same bin. Since an agent's payment depends on her own report, the naive VCG mechanism loses its desirable dominant strategy property. Indeed, in this matching model, truthful reporting does not even form an ex post Nash equilibrium in the naive VCG mechanism.

Theorem 2.5.4. *Truthful reporting does not form an ex post Nash equilibrium in the naive VCG mechanism.*

Proof. Without loss of generality, assume that $x_1 \geq x_2 \geq \cdots \geq x_N$ and all agents other than n report truthfully. Observe that for all n' such that $\mu(n') \neq \mu(n)$

$$\left[\sum_{i \in W_{\mu^*_{\widehat{\mathbf{x}}_{-n},0}}(n')} s(\widehat{x}_{n'}, \widehat{x}_i) \right] - \left[\sum_{i \in W_{\mu^*_{\widehat{\mathbf{x}}}}(n')} s(\widehat{x}_{n'}, \widehat{x}_i) \right]$$

is independent of $\widehat{x}_n$. For all n' such that $\mu(n') = \mu(n)$

$$\left[\sum_{i \in W_{\mu^*_{\widehat{\mathbf{x}}_{-n},0}}(n')} s(\widehat{x}_{n'}, \widehat{x}_i)\right] - \left[\sum_{i \in W_{\mu^*_{\widehat{\mathbf{x}}}}(n')} s(\widehat{x}_{n'}, \widehat{x}_i)\right]$$

is decreasing in $\widehat{x}_n$. Thus when all agents other than n report truthfully, n's utility-maximizing report is $\widehat{x}_n = 1$ regardless of her true type. $\square$

Truthful reporting in this paper's mechanism is actually slightly stronger than ex post Nash equilibrium: Edelman, Ostrovsky, and Schwarz's [5] locally envy free equilibrium extends to this incomplete information environment.

2.5.2 Envy-Free Equilibria

Definition 2.5.1. *An ex post Nash equilibrium of a mechanism* $\{q, t\}$ *is* envy free *if each agent* n *receives identical payoffs under all* μ *that receive positive weight under* $q(\mathbf{x})$.

Envy freeness is a stronger condition than ex post incentive compatibility. Ex post incentive compatibility merely requires that agents be indifferent across tie-breaking in expectation, while envy freeness requires that agents be ex post indifferent across tie-breaking.

Theorem 2.5.5. *Truthful reporting in the VCG-like mechanism is an envy free equilibrium.*

Proof. See Appendix A.1. $\square$

2.6 Multiple Roles

This section examines the mechanism in the fully general model in which $R \geq 1$. With some minor notational modifications and one observation, the results from Section 2.5 generalize completely. Define

$$\mu^*_{\widehat{\mathbf{x}}} = \underset{\mu \in A}{\text{argmax}} \sum_{n \in \mathcal{N}} \left[\sum_{i \in W_\mu(n)} s_{\overline{\mu}(n)}(\widehat{x}_n(\overline{\mu}(n)), \widehat{x}_i(\overline{\mu}(i))) \right]$$

The set of agents displaced from a bin by agent n is

$$D_n(\widehat{\mathbf{x}}) = \{k \in \mathcal{N} \backslash \{n\} \,|\, \mu^*_{\widehat{\mathbf{x}}}(k) \neq \mu^*_{\widehat{\mathbf{x}}_{-n},0}(k)\}$$

Also define

$$v_n(\widehat{\mathbf{x}}) = \sum_{i \in W_{\mu^*_{\widehat{\mathbf{x}}}}(n)} s_{\overline{\mu}^*_{\widehat{\mathbf{x}}}(n)}(x_n(\overline{\mu}^*_{\widehat{\mathbf{x}}}(n)), x_i(\overline{\mu}^*_{\widehat{\mathbf{x}}}(i)))$$

$$\alpha_n(\widehat{\mathbf{x}}) = \sum_{k \in D_n(\widehat{\mathbf{x}})} \left[\sum_{i \in W_{\mu^*_{\widehat{\mathbf{x}}_{-n},0}}(k)} s_{\overline{\mu}^*_{\widehat{\mathbf{x}}}(k)}(\widehat{x}_k(\overline{\mu}^*_{\widehat{\mathbf{x}}}(k)), \widehat{x}_i(\overline{\mu}^*_{\widehat{\mathbf{x}}}(i))) \right]$$

$$\beta_n(\widehat{\mathbf{x}}) = \sum_{k \in D_n(\widehat{\mathbf{x}})} \left[\sum_{i \in W_{\mu^*_{\widehat{\mathbf{x}}}}(k)} s_{\overline{\mu}^*_{\widehat{\mathbf{x}}}(k)}(\widehat{x}_k(\overline{\mu}^*_{\widehat{\mathbf{x}}}(k)), \widehat{x}_i(\overline{\mu}^*_{\widehat{\mathbf{x}}}(i))) \right]$$

Agent n's payment is $t_n(\widehat{\mathbf{x}}) = \alpha_n(\widehat{\mathbf{x}}) - \beta_n(\widehat{\mathbf{x}})$. While notationally more complex than the payment defined in Section 2.5, the intuition is the same: agent n who wins a slot in bin b in role r pays the opportunity cost she imposes on agents whom her report displaces from a bin. An agent's report can displace another agent from a bin if and only if both agents report the same productive role.

The VCG-like mechanism operates exactly as in Section 2.5. Agents submit their reports to the mechanism, which selects the socially optimal assignment $\mu^*_{\widehat{\mathbf{x}}}$. If there are multiple socially optimal assignments due to identical reports by one or more agents, the mechanism chooses a socially optimal assignment at random. Each agent receives $v_n(\widehat{\mathbf{x}})$ and pays $t_n(\widehat{\mathbf{x}})$.

Fix some $r \in \mathcal{R}$. If the types of agents whose productive role is not r were public information, then the remaining mechanism design problem concerning agents whose productive role is r is identical to the mechanism design problem in which $R = 1$ and in which the types of agents whose productive role is not r enter the match surplus function $s_r(\cdot, \cdot)$ as parameters. The generalization of Section 2.5's results relies on the this observation.

Theorem 2.6.1. *Truthful reporting is an ex post Nash equilibrium is the VCG-like mechanism ($R \geq 1$).*

Proof. Fix an arbitrary $r \in \mathcal{R}$ and suppose that all agents whose productive role is not r report their types truthfully. Then set of all agents whose productive role is r report to a mechanism identical to the mechanism of Section 2.5 in which the types of the agents whose productive role is not r enter as parameters. Theorem 2.5.1 now implies the result. □

Since $R \geq 1$ includes the case $R = 1$, [22]'s result that there is no dominant strategy incentive compatible mechanism still holds.

Theorem 2.6.2. *Truthful reporting in the VCG-like mechanism is an envy free equilibrium ($R \geq 1$).*

Proof. Fix an arbitrary $r \in \mathcal{R}$ and suppose that all agents whose productive role is not r report their types truthfully. The set of all agents whose productive role is r report to a mechanism identical to the mechanism of Section 2.5 in which the types of the agents whose productive role is not r enter as parameters. Within each role, truthful reporting is envy free. Thus when all agents report truthfully, truthful reporting is envy free across all roles. $\square$

Last, using a generalization of Lemma 2.5.3 to R roles, we show that up to a constant, the VCG-like mechanism is revenue equivalent to all other (Bayesian) incentive compatible mechanisms that implement positive assortative matching.

Lemma 2.6.3. *Truthful reporting in a feasible, individually rational direct mechanism is*

1. *a dominant strategy if and only if for all n and $\widehat{\mathbf{x}}_{-n}$, $H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, x_n)$ is supermodular in $(\widehat{x}_n, x_n)$ and*

$$U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n, \widehat{\mathbf{x}}_{-n}, x_n) = U_n(q, t, \widehat{\mathbf{x}}_{-n}, 0, \widehat{\mathbf{x}}_{-n}, 0) + \int_0^{x_n} \left[\frac{\partial}{\partial z} H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \widehat{\mathbf{x}}_{-n}, z) \right] \bigg|_{\widehat{x}_n = z} dz \tag{2.8}$$

2. *ex post incentive compatible if and only if for all n and $\mathbf{x}_{-n}$, $H_n(q, \mathbf{x}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, x_n)$*

is supermodular in $(\widehat{x}_n, x_n)$ *and*

$$U_n(q, t, \mathbf{x}_{-n}, x_n, \mathbf{x}_{-n}, x_n) = U_n(q, t, \mathbf{x}_{-n}, 0, \mathbf{x}_{-n}, 0) \qquad (2.9)$$
$$+ \int_0^{x_n} \left[\frac{\partial}{\partial z} H_n(q, \mathbf{x}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, z)\right]\bigg|_{\widehat{x}_n = z} dz$$

3. *Bayesian incentive compatible if and only if for all* n *and* $\widehat{\mathbf{x}}_{-n}$*,*

$$\mathbb{E}_{\mathbf{x}_{-n}}[H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \widehat{\mathbf{x}}_{-n}, x_n)]$$

is supermodular in $\widehat{x}_n, x_n$ *and*

$$\mathbb{E}_{\mathbf{x}_{-n}}[U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n, \widehat{\mathbf{x}}_{-n}, x_n)] = \mathbb{E}_{\mathbf{x}_{-n}}[U_n(q, t, \widehat{\mathbf{x}}_{-n}, 0, \widehat{\mathbf{x}}_{-n}, 0)] \qquad (2.10)$$
$$+ \int_0^{x_n} \mathbb{E}_{\mathbf{x}_{-n}} \left[\frac{\partial}{\partial z} H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \widehat{\mathbf{x}}_{-n}, z)\right]\bigg|_{\widehat{x}_n = z} dz$$

Proof. Fix an arbitrary $r \in \mathcal{R}$ and suppose that all agents whose productive role is not r report their types truthfully. The set of all agents whose productive role is r report to a mechanism identical to the mechanism of Section 2.5 in which the types of the agents whose productive role is not r enter as parameters. The proof of Lemma 2.5.3 then applies to each role. □

Corollary 2.6.4. *A matching mechanism's expected revenue is completely determined by* q *and the (expected) payoff of to an agent whose quality is* 0 *in her productive role.*

Proof. Recall that Bayesian incentive compatibility implies ex post incentive compatibility, so that proving the revenue equivalence of Bayesian incentive compatible mechanisms includes revenue equivalence of ex post incentive compatible mechanisms. Equation (2.10) says that the expected payment of an agent n depends exclusively on q and $\mathbb{E}_{\mathbf{x}_{-n}}[U_n(q, t, \widehat{\mathbf{x}}_{-n}, 0, \widehat{\mathbf{x}}_{-n}, 0)]$. The mechanism's expected revenue is the sum of expected payments, hence expected revenue depends solely on q and $\mathbb{E}_{\mathbf{x}_{-n}}[U_n(q, t, \widehat{\mathbf{x}}_{-n}, 0, \widehat{\mathbf{x}}_{-n}, 0)]$. $\square$

2.6.1 Marriage Markets and Internet Advertising

A marriage market is a matching market in which the matched unit is a household, the two productive roles are man and woman, the N agents consist of K men and L women. The number of households is not a binding constraint; $B \geq \min\{K, L\}$. Each household b has capacity constraints $c_{b,\text{man}} = c_{b,\text{woman}} = 1$. Each man reports his quality and each woman reports her quality to the mechanism, and the man and woman reporting the kth highest qualities in their respective roles are matched.

The complete information version of this marriage market encapsulates the internet advertising market of [5]. Edelman, Ostrovsky and Schwarz study the generalized second price (GSP) auction run by internet search engines for sponsored search slots. The search engine side of the market pays nothing; the qualities of the sponsored search slots (measured as the probability of a click) are known to all advertisers, with higher-ranked slots having higher click-through rates. The advertisers bid for slots; the kth highest bidder wins the kth best sponsored search slot and pays the $k+1$st highest bid. The $k+1$st highest bid in the GSP auction exactly coincides with the kth highest bidder's payment in the VCG-like mechanism.

2.6.2 Many-to-One Matching

A labor market is a matching market in which the matched unit is a firm, and each firm hires agents into one or more productive roles. Classical models consider two roles, manager and worker. This model and mechanism are capable of handling any finite number of roles. A university, for example, might hire a one president, several provosts, many deans, full, associate, and assistant professors and staff. The N agents consist of all candidates eligible to fill whatever roles are available.

This example details the mechanism in a matching market between hospitals and doctors. This market has two productive roles, hospital h and doctor d. There are two empty bins. The capacity constraints are $c_{1h} = c_{2h} = 1$, $c_{1d} = 2$, and $c_{2d} = 1$. Hospitals and doctors report their respective qualities to the mechanism which matches the best hospital with the best doctors (until c_{1d} is satisfied or until the supply of doctors is exhausted) and the second best hospital with the next best doctors. Suppose that there are six agents $\{1, 2, 3, 4, 5, 6\}$ and two bins $\{1, 2\}$. Agent 1's and 2's productive roles are h, and agent 3's, 4's, 5's, and 6's productive roles are d. Without loss of generality, suppose that $x_1(h) \geq x_2(h)$ and $x_3(d) \geq x_4(d) \geq x_5(d) \geq x_6(d)$. The positive assortative match gives the following assignment $\mu^*_{\mathbf{x}}$, $D_n(\mathbf{x})$, and $\mu^*_{\mathbf{x}_{-n},0}$.

Agent	$W_{\mu^*_{\mathbf{x}}}(n)$	$D_n(\mathbf{x})$	$W_{\mu^*_{\mathbf{x}_{-n},0}}(k)$
1	$\{3,4\}$	$\{2\}$	$W_{\mu^*_{\mathbf{x}_{-1},0}}(2) = \{3,4\}$
2	$\{5\}$	$\emptyset$	$\emptyset$
3	$\{1,4\}$	$\{5,6\}$	$W_{\mu^*_{\mathbf{x}_{-3},0}}(5) = \{1,4\}$
			$W_{\mu^*_{\mathbf{x}_{-3},0}}(6) = \{2\}$
4	$\{1,3\}$	$\{5,6\}$	$W_{\mu^*_{\mathbf{x}_{-4},0}}(5) = \{1,3\}$
			$W_{\mu^*_{\mathbf{x}_{-4},0}}(6) = \{2\}$
5	$\{2\}$	$\{6\}$	$W_{\mu^*_{\mathbf{x}_{-5},0}}(6) = \{2\}$
6	$\emptyset$	$\emptyset$	$\emptyset$

The VCG-like mechanism results in the following assignment, surpluses, and payments:

Agent	$W_{\mu^*_{\mathbf{x}}}(n)$	Surplus	Payment
1	$\{3,4\}$	$\sum_{k \in W_{\mu^*_{\mathbf{x}}}(1)} s_H(x_1(h), x_k(\overline{\mu}^*_{\mathbf{x}}(k)))$	$\sum_{k \in W_{\mu^*_{\mathbf{x}_{-1},0}}(2)} s_H(x_2(h), x_k(\overline{\mu}^*_{\mathbf{x}}(k)))$
			$- \sum_{k \in W_{\mu^*_{\mathbf{x}}}(2)} s_H(x_2(h), x_k(\overline{\mu}^*_{\mathbf{x}}(k)))$
2	$\{5\}$	$\sum_{k \in W_{\mu^*_{\mathbf{x}}}(2)} s_H(x_2(h), x_k(\overline{\mu}^*_{\mathbf{x}}(k)))$	0
3	$\{1,4\}$	$\sum_{k \in W_{\mu^*_{\mathbf{x}}}(3)} s_D(x_3(d), x_k(\overline{\mu}^*_{\mathbf{x}}(k)))$	$\sum_{k \in W_{\mu^*_{\mathbf{x}_{-3},0}}(5)} s_D(x_5(d), x_k(\overline{\mu}^*_{\mathbf{x}}(k)))$
			$- \sum_{x \in W_{\mu^*_{\mathbf{x}}}(5)} s_D(x_5(d), x_k(\overline{\mu}^*_{\mathbf{x}}(k)))$
			$+ \sum_{k \in W_{\mu^*_{\mathbf{x}_{-3},0}}(6)} s_D(x_6(d), x_k(\overline{\mu}^*_{\mathbf{x}}(k)))$
4	$\{1,3\}$	$\sum_{k \in W_{\mu^*_{\mathbf{x}}}(4)} s_D(x_4(d), x_k(\overline{\mu}^*_{\mathbf{x}}(k)))$	$\sum_{k \in W_{\mu^*_{\mathbf{x}_{-4},0}}(5)} s_D(x_5(d), x_k(\overline{\mu}^*_{\mathbf{x}}(k)))$
			$- \sum_{x \in W_{\mu^*_{\mathbf{x}}}(5)} s_D(x_5(d), x_k(\overline{\mu}^*_{\mathbf{x}}(k)))$
			$+ \sum_{k \in W_{\mu^*_{\mathbf{x}_{-4},0}}(6)} s_D(x_6(d), x_k(\overline{\mu}^*_{\mathbf{x}}(k)))$
5	$\{2\}$	$\sum_{k \in W_{\mu^*_{\mathbf{x}}}(5)} s_D(x_5(d), x_k(\overline{\mu}^*_{\mathbf{x}}(k)))$	$\sum_{k \in W_{\mu^*_{\mathbf{x}_{-5},0}}(6)} s_D(x_6(d), x_k(\overline{\mu}^*_{\mathbf{x}}(k)))$
6	$\emptyset$	0	0

Truthful reporting is an envy free equilibrium: if two agents with the same productive role have the same quality, their payoffs are equal. For example, if

$x_1(h) = x_2(h)$, hospitals 1 and 2 each receive

$$\sum_{k \in W_{\mu^*_{\mathbf{x}}}(2)} s_H(x_2(h), x_k(\overline{\mu}^*_{\mathbf{x}}(k))) = \sum_{k \in W_{\mu^*_{\mathbf{x}}}(1)} s_H(x_2(h), x_k(\overline{\mu}^*_{\mathbf{x}}(k))) - \left[\sum_{k \in W_{\mu^*_{\mathbf{x}_{-1},0}}(2)} s_H(x_2(h), x_k(\overline{\mu}^*_{\mathbf{x}}(k))) - \sum_{x \in W_{\mu^*_{\mathbf{x}}}(2)} s_H(x_2(h), x_k(\overline{\mu}^*_{\mathbf{x}}(k))) \right]$$

Since $W_{\mu^*_{\mathbf{x}_{-1},0}}(2) = W_{\mu^*_{\mathbf{x}}}(1)$, these expressions are equal when $x_1(h) = x_2(h)$. Similarly, if $x_5(d) = x_6(d)$, doctors 5 and 6 each receive 0. This suggests a natural interpretation of payoffs as wages. If agents in a given role have approximately the same qualities, wages tend to compress. In the limiting case where $x_1(h) = x_2(h)$ the two hospitals each receive payoff 0. A similar observation obtains for doctors when $x_3(d) = x_4(d) = x_5(d) = x_6(d)$. If agents' qualities in some role r are more homogeneous, there is less to be gained by offering higher wages for higher quality agents.

2.7 Conclusion

This paper shows that all incomplete information matching markets in which positive assortative matching on agent quality is the unique efficient stable match can be unified under a single mechanism design framework. The resulting VCG-like mechanism is new to the literature and always implements positive assortative matching as ex post Nash equilibrium. Indeed, this equilibrium satisfies the slightly stronger condition of envy freeness introduced by [5] and generalizes locally envy freeness to the incomplete information matching environments considered here. The main contribution of this work is the correct generalization of the VCG mechanism to the interdependent values setting of these matching markets: the payment rule distinguishes between the opportunity cost an agent imposes on another by depriving that other agent of an object (a slot in a given bin) and the opportunity cost an agent imposes on another by changing the value of that other agent's won object (due to interdependence). In order for truthful reporting to form an ex post Nash equilibrium, each agent's payment must be based on the former but exclude the latter. One drawback to this mechanism is that in a matching market (unlike a pure auction environment) agents have an incentive to try to match outside the mechanism to avoid making payments. While this problem disappears under the assumption that the mechanism can prevent agents from matching outside, a more elegant approach would be a revenue-neutral extension of the mechanism.

Of particular note is the unification of one-sided and two-sided matching markets. These types of markets typically display very different behaviors; this paper is the only one of which this author is aware that links these two types of markets; further work on such relationships would prove interesting albeit difficult.

Thus far, progress on incomplete information matching problems has been limited to markets in which positive assortative matching is the unique stable efficient match. Interesting avenues for future work include relaxing the restrictions on surplus functions that force us to restrict our attention to positive assortative matching. Another interesting topic is the auction design implications of the VCG-like mechanism.

Chapter 3: Mitigating Matching Externalities Via The "Old Boys' Club"

3.1 Introduction

This paper shows how a persistent contracting relationship emerges in a repeated college admissions market from a randomly initiated one-shot contract offered by an unbiased expected utility maximizing college. Faced with an admissions problem that cannot be addressed by an efficient, incentive compatible market design, a college offers a contract to a randomly selected high school; they agree that the college will admit some number of the high school's most promising students immediately. The high school agrees to this contract because it is happy to guarantee admission for as many of its students as possible in advance of the general admissions market. The college offers this contract for two reasons: first, it too is happy to guarantee itself sufficiently good students in advance of the general admissions market; second, contracting away some its slots reduces the complexity of its general admissions market design problem. The contracted high school will not renege on the contract in the sense of sending subpar students to fill the contracted slots because the college learns from the general admissions market whether the contracted high school reneged. The college credibly promises to reward the contracted high school's compliance with a future contract and credibly threatens to punish the contracted high school's noncompliance by contracting with some other high school in one or more subsequent rounds of admissions.

An obvious objection grounded in a human sense of equity arises: the high school holding the contract is at a distinct advantage since students admitted via the contracted slots avoid the need to compete in the general admissions market. Moreover, the method of contract enforcement guarantees that if the initially contracted high school is sufficiently patient, it will honor the contract in every round of admissions, guaranteeing that its contracting relationship will persist *ad infinitum.* This paper assumes that the high schools are ex ante identical and the college is ex ante unbiased in the sense that it offers the initial contract at random. Nevertheless, the lifetime welfare loss to a non-contracted high school is large. In the context of college admissions in the United States, intertwined with historical cultural biases that favor of certain demographics over others, it leaves the uncomfortable conclusion that merely mandating equal consideration of all applicants is insufficient to dismantle these long standing relationships, hereinafter referred to as "old boys' clubs."

Matching markets in which there is no mechanism that is both efficient and incentive compatible abound; indeed this type of market is the rule rather than the exception. The main results in this vein are due to Vickrey [21], who showed that there is no budget-balanced auction in which both buyers and sellers reveal their private values truthfully; Roth [18], who showed that there is no matching mechanism in which participants on both sides of the market reveal their preferences truthfully; and Jehiel and Moldovanu [12], who showed that in multi-unit auctions with multidimensional signals there is in general no efficient, incentive compatible auction mechanism. The lack of a mechanism that is both efficient and incentive compatible does not mitigate the need of participants in these markets to form matches by some means, however imperfect the method or outcome may be. A college faced with this problem nonetheless needs to admit a freshman class every year: educating students is its *raison d'être*. It is therefore natural to study inefficient but incentive compatible mechanisms to understand both how these mechanisms perform and how their inefficiencies are distributed. Both the initial contract and its persistence are completely understandable from the classical economic standpoint of selfish, expected utility maximizing agents. However, despite ex ante equal expected lifetime payoffs for the high schools, as soon as the initial contract is signed, one high school benefits from being a member of the "old boys' club" in every subsequent round while the other is left out forever. The college benefits from the "old boys' club" regardless of which high school belongs to the club; however, once established, the college needs to maintain the contracting relationship in order to keep the good students coming from the contracted high school.

There are two main lenses through which to view these results. One option is in the tradition of Chatterjee and Samuelson's work on bilateral trade [3]. That work characterizes how frequently a natural trading mechanism is ex post inefficient, and how costly that inefficiency is to the trading partners. This paper considers a matching mechanism that is ex post inefficient and shows that it outperforms other candidate mechanisms. A second point of view sees this work as the generalization of Ausubel et. al.'s work on demand reduction in auctions [2] to matching environments. The college knows that it wants to admit a valedictorian, but it does not know which one. In expectation, it can admit one valedictorian at no cost to itself. From the high schools' perspective, if the college were auctioning both slots, the high school whose valedictorian was admitted should reduce demand and have her valedictorian admitted for free. The contracting stage of this paper's admissions game is analogous to the contracted high school reducing demand; the subsequent Vickrey auction is analogous to the auction for the remaining unit. Unlike a pure auction environment, the matching game must repeat since demand reduction is accomplished by contract rather than by the auction itself.

3.2 Model

There is one college C and two high schools H_1 and H_2. In each round of admissions, C has two slots for incoming freshmen and each high school has two graduating seniors. Each period, the college faces the problem of choosing two of the four graduating seniors to fill its freshman class. The college and two high schools repeat this admissions process infinitely many times, with common discount factor $\delta \in (0, 1)$ between periods. The college seeks to maximize the total lifetime payoff of its matriculated students, while each high school seeks to maximize the total lifetime payoff of its graduates.

Students at the high schools have types independently drawn from a publicly known, common density $f(\cdot)$ with support $[0, a] \subseteq [0, \infty)$ and corresponding distribution $F(\cdot)$. Assume that $f(\cdot)$ is sufficiently well behaved that every integral of interest in this paper converges. Each high school knows the type of each of its two students but not the types of the students at the other high school; the college does not know the types of any students. If students of types θ_1 and θ_2 matriculate to C in some period, student θ_1 receives payoff $U(\theta_1, \theta_2)$ and student θ_2 receives payoff $U(\theta_2, \theta_1)$, where $U(\cdot, \cdot)$ is non-negative increasing in both arguments, strictly supermodular, bounded and integrable on $[0, a]^4$. A student who does not go to college receives outside option 0. Herein lies C's market design impossibility: strict supermodularity in $U(\cdot, \cdot)$ implies sufficient payoff interdependence to preclude a mechanism that is both efficient and incentive compatible [12].

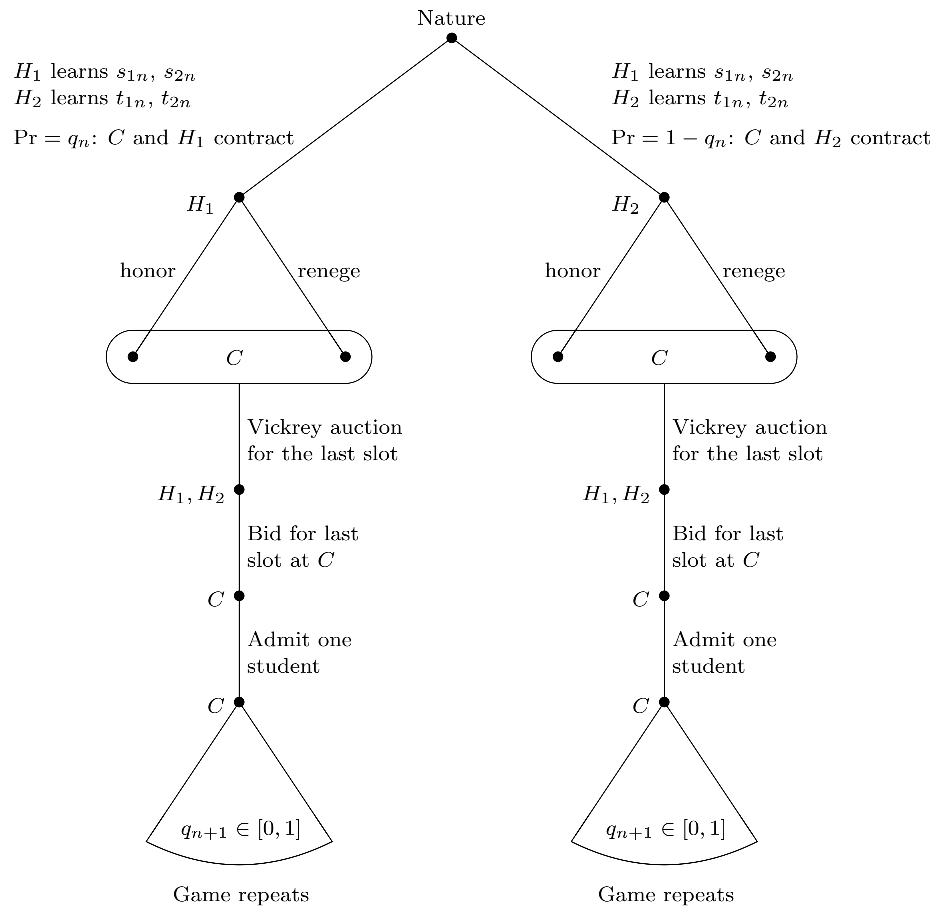
The college fully internalizes its matriculated students' payoffs; each high school fully internalizes its graduates' payoffs.

Let $s_{1n} \geq s_{2n}$ denote the types of H_1's period n good and bad students respectively; let $t_{1n} \geq t_{2n}$ denote the types of H_2's period n good and bad students respectively. Thus by standard combinatorial reasoning, $(s_{1n}, s_{2n}, t_{1n}, t_{2n})$ has joint density $4f(s_{1n})f(s_{2n})f(t_{1n})f(t_{2n})$ and its support is $\{(x, y, z, w) \in [0, a]^4 | x \geq y \text{ and } z \geq w\}$.

Abusing notation, I use a student's type as her identifier. Let X_{in} denote the ith best student and her type (across both high schools) in period n.

3.3 Contracting Mechanism

This section considers the novel combined contracting and auction admissions mechanism of this paper. In the contracting regime, C offers a randomly chosen high school a contract for its good student and runs a Vickrey auction to assign its remaining slot. If C offers H_i the contract in period n, H_i always accepts since it prefers to guarantee a berth for at least one of its students. The subsequent Vickrey auction allows C to identify and admit the best of the remaining three students as well as to discern before period $n + 1$ whether the contracted high school honored the period n contract (sent its good student on the contract) or reneged (sent its bad student). Let q_n be the probability that C contracts with H_1 in period n. Period n of this game has the extensive form shown in Figure 1.



I show that there is a distribution-independent perfect Bayesian equilibrium in this game in which C and a sufficiently patient high school contract in every period, and the contracted high school honors the contract every time.

Theorem 3.3.1. *Consider the following strategies in the contracting game:*

1. *C chooses probability $q_0 \in (0,1)$ with which it offers H_1 the contract in period 0*

2. *C offers H_1 the contract in period n with probability q_n*

3. *If offered the contract in period n, H_i accepts and honors the contract by sending C its good student immediately to fill the contracted slot*

4. *C runs a Vickrey auction to assign the remaining slot and admits the student with the higher report in the auction regardless of high school of origin*

5. *In the Vickrey auction, each high school honestly reports the type of its better (and possibly only) remaining student*

6. *C discerns from the reports in the Vickrey auction whether the contracted high school played "honor" or "renege" and chooses q_{n+1} according to*

$$q_{n+1} = \begin{cases} 1 & \text{if} \quad H_1 \text{ played "honor" in period } n \text{ or } H_2 \text{ played "renege" in period } n \\ 0 & \text{if} \quad H_1 \text{ played "renege" in period } n \text{ or } H_2 \text{ played "honor" in period } n \end{cases}$$

These strategies form a perfect Bayesian equilibrium in the contracting game whenever

$$\delta \geq \frac{S}{1+S}$$

where

$$S = \sup\left\{ \frac{A(s_{20}, s_{10}) - A(s_{10}, s_{20})}{E[A(s_{1k}, s_{2k})] - B} \middle| (s_{10}, s_{20}) \in [0,a] \times [0,a] \text{ and } s_{10} \geq s_{20} \right\}$$

$$A(s_{10}, s_{20}) = \Pr(s_{20} \geq t_{10})[U(s_{10}, s_{20}) + U(s_{20}, s_{10})]$$

$$B = \frac{5}{6} E[U(s_{1k}, t_{1k}) | s_{1k} \geq t_{2k}]$$

Proof. See Appendix A.2. □

This perfect Bayesian equilibrium is distribution-independent insofar as the players do not condition their strategies on the students' type distribution. However, I still allow $\overline{\delta}$ to depend on the distribution of students' types and the surplus function $U(\cdot, \cdot)$. The college's updating rule for q_{n+1} serves to enforce the contract and improves total welfare, as well as making the college and the contracted high school individually better off.

This enforcement and social improvement comes at the expense of the high school that is not offered the contract in period 0. The college C and the period 0 contracted high school thus form an "old boys' club" in the sense that contracting in period 0 ensures that they contract forever; the high school with no contract in period 0 is shut out from the benefits of this relationship. In each period the contracting game is ex post efficient with probability 5/6 since it selects the ex post efficient match unless the top two overall students are from the non-contracted high school. Assuming without loss off generality that H_1 wins the period 0 contract, H_1 receives $U(s_{1n}, s_{2n}) + U(s_{2n}, s_{1n})$ when its students are the two best period n students overall, $U(s_{1n}, t_{1n})$ when both valedictorians are admitted, and $U(s_{1n}, t_{1n})$ when H_2's students are the two best period n students overall. The respective probabilities of these events are 1/6, 2/3, and 1/6.

Again assuming without lost off generality that H_1 wins the period 0 contract, expected payoffs are:

$$\begin{aligned}
EC^{\text{contracting}} &= \sum_{n=0}^{\infty} \delta^n \left\{ \frac{5}{6} E[U(X_{1n}, X_{2n}) + U(X_{2n}, X_{1n})] \right. \\
&\quad \left. + \frac{1}{6} E[U(X_{1n}, X_{3n}) + U(X_{3n}, X_{1n})] \right\} \\
EH_1^{\text{contracting}} &= \sum_{n=0}^{\infty} \delta^n \left\{ \frac{1}{2} E[U(X_{1n}, X_{2n}) + U(X_{2n}, X_{1n})] \right. \\
&\quad \left. + \frac{1}{6} E[U(X_{3n}, X_{1n})] \right\} \\
EH_2^{\text{contracting}} &= \sum_{n=0}^{\infty} \delta^n \left\{ \frac{1}{3} E[U(X_{1n}, X_{2n}) + U(X_{2n}, X_{1n})] \right. \\
&\quad \left. + \frac{1}{6} E[U(X_{1n}, X_{3n})] \right\}
\end{aligned}$$

Since $U(\cdot, \cdot)$ increasing in both arguments, this result follows immediately.

Theorem 3.3.2. *The high school which wins the period* 0 *contract receives strictly higher payoff in the perfect Bayesian equilibrium described in Theorem 3.3.1.*

3.4 Performance

Having identified a distribution-independent perfect Bayesian equilibrium in the contracting game, I now compare its performance to two other incentive compatible matching mechanisms and to the socially optimal outcome under full information. This paper will consider the following options in addition to the contracting game of Section 3.3.

1. random admission in every period

2. C admits the valedictorians from H_1 and H_2 respectively

After describing each method, agents' equilibrium behavior, and expected surplus, this section will show that contracting is most likely to be ex post efficient, followed by admitting the valedictorians, and that random matching fares worst. Finally, I will consider several examples with explicit closed forms for $f(\cdot)$ and $U(\cdot,\cdot)$ to give some idea of the potential attractiveness of contracting relative to the other options.

3.4.1 Benchmark: Socially Optimal Matching

Under the unattainable full information benchmark, positive assortative matching is the unique socially optimal and $C-$optimal outcome [19]. C admits the top two of the four students, regardless of their high school(s) of origin. Lifetime ex ante expected payoffs are

$$\begin{aligned}
EC^{\text{opt}} &= \sum_{n=0}^{\infty} \delta^n E[U(X_{1n}, X_{2n}) + U(X_{2n}, X_{1n})] \\
EH_1^{\text{opt}} &= \frac{1}{2} \sum_{n=0}^{\infty} \delta^n E[U(X_{1n}, X_{2n}) + U(X_{2n}, X_{1n})] \\
EH_2^{\text{opt}} &= \frac{1}{2} \sum_{n=0}^{\infty} \delta^n E[U(X_{1n}, X_{2n}) + U(X_{2n}, X_{1n})]
\end{aligned}$$

The high schools' respective ex ante expected payoffs are equal since the probabilities that a given high school has zero, one, or two of the best two students are identical across high schools. Full efficiency always occurs under full information.

3.4.2 Alternative Matching Mechanisms

3.4.2.1 Random Matching

If C abandons all efforts at extracting students' information it avoids any incentive problems at considerable loss of efficiency. Let w and z denote the randomly chosen students. Ex ante expected payoffs are

$$
\begin{aligned}
EC^{\text{random}} &= \sum_{n=0}^{\infty} \delta^n E[U(w,z)+U(z,w)] \\
EH_1^{\text{random}} &= \frac{1}{2}\sum_{n=0}^{\infty} \delta^n E[U(w,z)+U(z,w)] \\
EH_2^{\text{random}} &= \frac{1}{2}\sum_{n=0}^{\infty} \delta^n E[U(w,z)+U(z,w)]
\end{aligned}
$$

This method is ex post efficient in period n when C randomly chooses the top two students overall in that period; this event occurs with probability 1/6. The high schools' respective ex ante expected payoffs are equal since the probabilities that C chooses zero, one, or two of a high school's students are equal across high schools.

3.4.2.2 Admit The Valedictorians

An intermediate option for C is to ask each high school to identify its good student in each period, and admit the good student from each high school. With probability 1, it is strictly dominant for each high school to reveal its valedictorian truthfully since the high school's payoff increases in its admitted student's type.[1]

[1] If H_i's students are identical, truthful revelation is only weakly dominant, but this event occurs with probability 0.

Ex ante xpected payoffs under this regime are

$$EC^{\text{Val}} = \sum_{n=0}^{\infty} \delta^n E[U(X_{1n}, X_{2n}) + U(X_{2n}, X_{1n})]$$
$$EH_1^{\text{Val}} = \frac{1}{2}\sum_{n=0}^{\infty} \delta^n E[U(X_{1n}, X_{2n}) + U(X_{2n}, X_{1n})]$$
$$EH_2^{\text{Val}} = \frac{1}{2}\sum_{n=0}^{\infty} \delta^n E[U(X_{1n}, X_{2n}) + U(X_{2n}, X_{1n})]$$

This method is ex post efficient in period n so long as the two valedictorians are the top two students overall in that period; this event occurs with probability 2/3.

3.4.3 Performance

It follows immediately from the payoffs above that the contracting game described in Section 3.3 is a social improvement on admitting the valedictorians which is a social improvement on random matching. The following tables compare performance and one period expected surplus of the three matching mechanisms to each other and to the socially optimal outcome under full information. In the case of the contracting game, assume that H_1 randomly wins the period 0 contract; its one period payoff in each of the tables is the expected payoff after making the contract but before learning students' types.

Mechanism	Pr ex post efficient	EU_C	EU_{H_1}	EU_{H_2}
Social Optimum	1	1	$\frac{1}{2}$	$\frac{1}{2}$
Contracting	5/6	$\frac{98}{108}$	$\frac{58}{108}$	$\frac{40}{108}$
Valedictorians	2/3	$\frac{96}{108}$	$\frac{48}{108}$	$\frac{48}{108}$
Random Matching	1/2	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Types i.i.d. $U[0,1]$; $U(x,y) = xy$. EU is one period ex ante expectation.

Mechanism	Pr ex post efficient	EU_C	EU_{H_1}	EU_{H_2}
Social Optimum	1	$\frac{386}{72\lambda^2}$	$\frac{193}{72\lambda^2}$	$\frac{193}{72\lambda^2}$
Contracting	5/6	$\frac{1003}{216\lambda^2}$	$\frac{598}{216\lambda^2}$	$\frac{405}{216\lambda^2}$
Valedictorians	2/3	$\frac{18}{4\lambda^2}$	$\frac{9}{4\lambda^2}$	$\frac{9}{4\lambda^2}$
Random Matching	1/2	$\frac{2}{\lambda^2}$	$\frac{1}{\lambda^2}$	$\frac{1}{\lambda^2}$

Types i.i.d. $f(x) = \lambda e^{-\lambda x}$; $U(x, y) = xy$. EU is one period ex ante expectation.

The contracting mechanism is ex post efficient with higher probability than the candidate static matching mechanisms but cannot and does not attain full efficiency. The size of the college's increased payoff and the inequity of its division between the two high schools is an artifact of the choice of surplus function and distribution of students' types.

3.5 Conclusion

Existing impossibility results show that there are many mechanism design problems lacking an efficient solution. Consequentially, there is relatively little development of mechanisms designed for such situations, despite their common occurrence. When such mechanisms do exist, they rely on assumptions (often reasonable) that successful deceit is difficult and therefore rare; therefore, one may simply ignore the possibility of strategic misrepresentation. I consider an alternative, in which a relationship is used as a partial substitute for direct information revelation.

This paper introduces a mechanism in a repeated college admissions game which uses a contracting stage to admit a good student and simplify the information elicitation problem in the post-contract admissions market. The college enforces the contract in each period with the promise of maintaining the contracting relationship in the next period. The mechanism is ex post efficient more often than other plausible incentive compatible mechanisms and thus improves total surplus. However, the gains of this matching approach are distributed inequitably, and repeatedly so. The college and the persistently contracted high school both receive higher surplus in the contracting game than in the "admit the valedictorians" mechanism; however, the non-contracted high school's surplus drops significantly. The persistent contracting relationship shows that even when a relationship is initiated at random, once established an "old boys' club" consistently exploits non-members for its own benefit. The college and contracted high school enrich themselves at considerable cost to the non-contracted high school. Future work will generalize the model to include N high schools and characterize the mechanism's performance with arbitrary type distributions and strictly supermodular payoffs.

Chapter 4: Matching with Endogenous Firm Creation

4.1 Introduction

Labor market matching studies how firms and employees agree to mutually beneficial terms while largely ignoring firms' etiology. This paper studies the labor market matching outcomes when the dichotomy between firms and workers is not determined *ex ante*. Each person in the market has two options for earning money. One is to own and manage a firm, employing herself and possibly others, and receive the proceeds of that firm after paying salaries to all other employees if any. The other is to work at a someone else's firm and receive a salary in exchange for doing so. People also receive utility from their enjoyment (or lack thereof) of their employment conditions. I show that there exist salaries that support the socially optimal division of people into sets of firms and workers as well as the socially optimal match between those two sets.

Gale and Shapley [6] introduced matching theory as a fruitful and important literature which addresses the class of problems concerned with the pairing of members of one set with members of another. The major arm of this literature in which this paper situates itself studies the intersection of matching theory and general equilibrium. Arrow-Debreu equilibrium works (well) in situations when markets for each good are reasonably thick. Workers are at best imperfect substitutes for employers and jobs are at best imperfect substitutes to workers; as such it makes sense to view each job and each worker is a unique good. In this environment, classical welfare theorems function minimally if at all.

Kelso and Crawford [14] developed an elegant a price-adjustment algorithm resembling an English auction which identifies a match and prices which stably support it. Assuming the workers are weak gross substitutes, their algorithm converges to a coalition-proof match. Gul and Stacchetti [8], [9] developed a closely related theory of labor market matching with money which more fully explored the relationship between auctions and matching. Gul and Stacchetti further showed that the set of prices supporting coalition-proof matches form a lattice, successfully relating the structure of stable matches in a labor market with prices to the lattice structure seen in models of Gale and Shapley [6] and Hatfield and Milgrom [10].

As is standard in this literature, these papers assume a preexisting division of agents into the two respective sides of the matching market. In many matching settings, this assumption is entirely reasonable: a student does not open a college and a college does not get a bachelor's degree; a resident does not open her own hospital and a hospital does not complete a residency of its own. This assumption is less tenable in the labor market: firms exist because individuals decide to undertake opening them. Work lacking this assumption has been treated almost completely separately by the coalition formation and roommate matching literatures. This paper develops a two-sided matching market in which the partition between firms and workers is endogenously determined. An agent's socially optimal role is determined by the technology she uses if she owns and manages a firm, her taste for entrepreneurship, her taste for working for other employers available, and the salaries available from these various options. Salaries exist that support both the socially optimal partition of agents into firms and workers, and stably supports the socially optimal match between them.

4.2 Model

There is a set of agents $\Omega = \{1, 2, \cdots, n\}$. In this model, owning and managing are synonymous; a firm not managed by its owner generates no utility for any agent. An agent can own and manage a firm or work for another agent, but not both. The rest of this paper refers to agents who own and manage firms as firms. It calls all other agents workers. Let F denote the set of firms and W denote the set of agents who work at a firm in F. Thus $F \cup W = \Omega$ and $F \cap W = \emptyset$.

Agent i has a publicly known utility function $u^i : \Omega \times \mathbb{R} \to \mathbb{R}$ which denotes her utility of working for agent j at salary s_{ij}. Agent i also has a publicly known production function $y^i : 2^{\Omega} \to \mathbb{R}$ which denotes the surplus generated when agent i opens a firm and employs $C \subseteq \Omega$.

Each agent is risk neutral. If agent i works for agent j, she receives $u^i(j, s_{ij}) = a_{ij} + s_{ij}$ where a_{ij} is a constant reflecting i's exogenous utility for employment with j and she earns salary s_{ij}. If agent j employs $C \subseteq \Omega$, she receives $u^j(j, s_{jj}) = a_{jj} + y^j(C) - \sum_{i \in C \setminus \{j\}} s_{ij}$ where a_{jj} is a constant reflecting j's exogenous utility managing a firm and $s_{jj} = y^j(C) - \sum_{i \in C \setminus \{j\}} s_{ij}$ is the surplus after paying her employees' salaries. For all pairs $i, j \in \Omega$, let σ_{ij} solve $u^i(j, \sigma_{ij}) = u^i(i, y^i(\{i\}))$; σ_{ij} is the salary such that i indifferent between working for j and working solely for herself.

Several assumptions are in order. First, I formalize the assumption that firm ownership and management are synonymous. A firm owner must work at her own firm in order for that firm to produce anything, i.e.

$$y^j(C) = 0 \text{ if } j \notin C \tag{4.1}$$

In particular, $y^j(\emptyset) = 0$, which is exactly Kelso and Crawford's "no free lunch" assumption.

As in Kelso and Crawford, assume that for all $C \subseteq \Omega$ containing j and for all $i \notin C$,

$$y^j(C \cup \{i\}) - y^j(C) \geq \sigma_{ij} \tag{4.2}$$

There is no requirement that $y^j(C \cup \{i\}) - y^j(C) \geq 0$; I allow the possibility that an agent i may be destructive for firm j. For example, a firm might hire an intern who is fundamentally unproductive but finds it worth it to her to pay the firm for the experience.

Finally, assume that agents, when in W, are gross substitutes per [14]. Suppose firm j faces fixed salaries $s = \{s_{1j}, \cdots, s_{j\,j-1}, s_{j\,j+1}, \cdots, s_{nj}\}$ paid by j to $i \neq j$ and suppose also that

$$C^* \in \operatorname*{argmax}_{C \subseteq \Omega} y^j(C) - \sum_{i \in C \setminus \{j\}} s_{ij}$$

Then for any other set of salaries s' paid by j to $i \neq j$ such that $s'_{ij} \geq s_{ij}$ for all i, there exists

$$C^{**} \in \operatorname*{argmax}_{C \subseteq \Omega} y^j(C) - \sum_{i \in C \setminus \{j\}} s'_{ij}$$

such that $\{i \in C^* \,|\, s_{ij} = s'_{ij}\} \subseteq C^{**}$. There are three intuitive ways to understand the gross substitutes condition. First, the gross substitutes condition requires that if a worker's salary does not increase, the firm employing that worker can continue to maximize profits without firing that worker. Alternatively, when firm j increases s_{ij}, it does not affect firm j's decision about whether to hire agent $k \neq i$. Finally, worker i' marginal product is independent of the identities of i's coworkers (although worker i's marginal product may depend on the number of coworkers she has).

A matching correspondence $\mu : F \to \Omega$ where $\mu(j)$ is the set of workers employed by firm j, and $\mu^{-1}(i)$ is agent i's employer. By Equation 4.1, firm j has non-negative output if and only if μ satisifies $j \in \mu(j)$. Say that (F, μ) satisfies the "no free management" assumption if and only if $j \in \mu(j)$ for all $j \in F$.

Definition 4.2.1. *A partition of Ω is a collection of subsets $C_1, \cdots, C_m$ of Ω such that $\bigcup_{j=1}^m C_j = \Omega$ and $C_i \cap C_j = \emptyset$ for all $i \neq j$.*

The pair (F, μ) partitions Ω if

- $j = k$ if and only if $\mu(j) \cap \mu(k) \neq \emptyset$

- $\bigcup_{j \in F} \mu(j) = \Omega$

Let $\mathbf{s} = \{s_1, \cdots, s_n\}$ denote some salary schedule in which $i \in \Omega$ receives salary s_i. Call $(F, \mu, \mathbf{s})$ an allocation whenever (F, μ) partitions Ω, (F, μ) satisfies the "no free management" assumption, and $i \in \mu(j)$ receives from $j \in F$ salary s_i specified by the schedule $\mathbf{s}$. In particular, for all $j \in F$, firm j's salary according to the schedule $\mathbf{s}$ should be j's profit after paying her workers' salaries, i.e.

$$s_j = y^j(\mu(j)) - \sum_{i \in \mu(j) \setminus \{j\}} s_i$$

Definition 4.2.2. *An allocation $(F, \mu, \mathbf{s})$ is individually rational if*

- *For all $i \in W$*

$$a_{i\mu^{-1}(i)} + s_{i\mu^{-1}(i)} \geq a_{ii} + y^i(\{i\})$$

- *For all $j \in F$, for all $j \in A \subseteq \mu(j)$*

$$y^j(\mu(j)) - \sum_{i \in \mu(j) \setminus j} s_{i\mu^{-1}(i)} \geq y^j(A) - \sum_{i \in A \setminus j} s_{i\mu^{-1}(i)}$$

The first requirement is a "no quit" condition on each worker. The second requirement is a "no terminations" condition on each firm. A termination is a unilateral deviation because it is an action taken by a single agent $j \in F$ despite that fact that it may affect more than one of j's employees.

Definition 4.2.3. *The allocation $(F, \mu, \mathbf{s})$ is a core allocation if there is no $A \subseteq \Omega$ for some $j \in A$ and salaries s_{ij} for all $i \in A\backslash\{j\}$ such that for all $i \in A$.*

$$a_{ij} + s_{ij} \geq a_{i\mu^{-1}(i)} + s_i$$

The pair (F, μ) is a strict core allocation if at least one of these inequalities is strict.

Definition 4.2.3 encapsulates pairwise stability since it necessarily includes any coalitions of two agents. The next section shows that individual rationality is sufficient for an allocation to be in the core and that given the socially optimal match (F, μ) there exists a salary schedule $\mathbf{s}(F, \mu)$ that supports it as a core allocation.

4.3 Results

I now show that there exist salaries that support a socially optimal partition of Ω into firms and workers, and stably support a socially optimal match between the two resulting sets.

Theorem 4.3.1. *Suppose that (F^*, μ^*) is socially optimal, i.e.*

$$(F^*, \mu^*) \in \underset{(F,\mu)}{\operatorname{argmax}} \sum_{j\in F} \left[y^j(\mu(j)) + \sum_{i\in\mu(j)} a_{ij} \right]$$

Then there exists a salary schedule $\mathbf{s}^* = \{s_1^*, s_2^*, \cdots, s_n^*\}$ such that $(F^*, \mu^*, \mathbf{s}^*)$ is a strict core allocation.

Proof. See Appendix A.3. □

This proof appears as a series of lemmas. The first uses the gross substitutes condition to establish an inequality concerning the marginal product of an arbitrary worker. The next four lemmas construct an $\mathbf{s}^*$ such that $(F^*, \mu^*, \mathbf{s}^*)$ is individually rational. Then, a final lemma shows that an individually rational allocation must be in the core.

There are several ways to interpret this result. One is that it shows that as in the standard Arrow-Debreu model, prices determine not just the quantity of goods transacted, but also which agents are sellers and which are buyers. An environment in which firms and workers are determined ex ante is standard for a matching model, but it is an unnecessary constraint in general equilibrium. The second interpretation is that salaries can successfully mediate both technological selection and employer / employee preferences simultaneously. A final interpretation says that prices can transform a one-sided matching problem into a corresponding two-sided market.

4.4 Conclusion

This paper shows that prices provide a means of matching firms and workers and establishing a socially optimal allocation of agents to these two sides of the market. The existence of these prices establishes that Kelso and Crawford's extension of general equilibrium also correctly partitions agents into two distinct groups, one of firms - the buyers of labor - and the other of workers - the sellers of labor. Arrow-Debreu equilibrium correctly partitions agents into buyers and sellers of the good(s) in question as well as determining prices. I have shown that Kelso and Crawford's labor market setting, there is no need to constrain agents to one side of the market; even when markets are thin as in this case, prices serve to correctly identify firms and workers. A possible further generalization is the development of a comparable price-adjustment algorithm which arrives at a socially optimal match.

Chapter 5: Conclusion

5.1 Conclusion

This dissertation contributes three essays of novel results in matching theory. The first essay contributes a VCG-like matching mechanism applicable to all matching environments in which the positive assortative match is the unique efficient stable match. Its payment rule recognizes which opportunity costs should be included and which opportunity costs should be excluded in order to achieve ex post incentive compatibility. In an environment with no interdependence, this new mechanism exactly coincides with the standard VCG mechanism. The second essay develops a hybrid contract and auction mechanism in a college admissions problem where no efficient incentive compatible mechanism exists. This hybrid mechanism improves total surplus by establishing a contracting relationship between the college and a single high school which mitigates some of the inefficiency. However, this social improvement occurs at the expense of a high school not party to the established contracting relationship. Finally, I show that in a one-sided labor market, prices support an efficient division of that single side into firms and workers, as well as stably supporting a socially optimal match between those sets.

Appendix A: Appendix

A.1 Appendix to Chapter 2

A.1.1 Proof of Theorem 2.5.1

Proof. The proof is similar to the proof that truthful revelation is a weakly dominant strategy in the VCG mechanism.

$$
\begin{aligned}
v_n(\mathbf{x}) - t_n(\mathbf{x}) &= v_n(\mathbf{x}) + \beta_n(\mathbf{x}) - \alpha_n(\mathbf{x}) \\
&\quad + \sum_{k \in \{1,\cdots,N\} \setminus [D_n(\mathbf{x}) \cup \{n\}]} \left[\sum_{i \in W_{\mu^*_{\mathbf{x}}}(k)} s(x_k, x_i) \right] - \sum_{k \in \{1,\cdots,N\} \setminus [D_n(\mathbf{x}) \cup \{n\}]} \left[\sum_{i \in W_{\mu^*_{\mathbf{x}}}(k)} s(x_k, x_i) \right] \\
&= \sum_{k \in \{1,\cdots,N\}} \left[\sum_{i \in W_{\mu^*_{\mathbf{x}}}(k)} s(x_k, x_i) \right] - \alpha_n(\mathbf{x}) - \sum_{k \in \{1,\cdots,N\} \setminus [D_n(\mathbf{x}) \cup \{n\}]} \left[\sum_{i \in W_{\mu^*_{\mathbf{x}}}(k)} s(x_k, x_i) \right]
\end{aligned}
$$

By the definition of $\mu^*_{\mathbf{x}}$,

$$
\begin{aligned}
v_n(\mathbf{x}) - t_n(\mathbf{x}) \geq \sum_{k \in \{1,\cdots,N\}} &\left[\sum_{i \in W_{\mu^*_{\mathbf{x}_{-n},z}}(k)} s(x_k, x_i) \right] \\
&- \alpha_n(\mathbf{x}) - \sum_{k \in \{1,\cdots,N\} \setminus [D_n(\mathbf{x}) \cup \{n\}]} \left[\sum_{i \in W_{\mu^*_{\mathbf{x}}}(k)} s(x_k, x_i) \right]
\end{aligned}
$$

Rearranging:

$$\begin{aligned} v_n(\mathbf{x}) - t_n(\mathbf{x}) \geq\ & v_n(\mathbf{x}_{-n}, z) - t_n(\mathbf{x}_{-n}, z) + \alpha_n(\mathbf{x}_{-n}, z) - \alpha_n(\mathbf{x}) \\ & - \sum_{k \in \{1,\cdots,N\} \setminus [D_n(\mathbf{x}) \cup \{n\}]} \left[\sum_{i \in W_{\mu^*_{\mathbf{x}}}(k)} s(x_k, x_i) \right] \\ & + \sum_{k \in \{1,\cdots,N\} \setminus [D_n(\mathbf{x}_{-n}, z) \cup \{n\}]} \left[\sum_{i \in W_{\mu^*_{\mathbf{x}_{-n},z}}(k)} s(x_k, x_i) \right] \qquad \text{(A.1)} \end{aligned}$$

If $z < x_n$, then $D_n(\mathbf{x}) \supseteq D_n(\mathbf{x}_{-n}, z)$, and Equation (A.1) consequently becomes

$$\begin{aligned} v_n(\mathbf{x}) - t_n(\mathbf{x}) \geq v_n(\mathbf{x}_{-n}, z) - t_n(\mathbf{x}_{-n}, z) - \sum_{k \in D_n(\mathbf{x}) \setminus D_n(\mathbf{x}_{-n}, z)} \left[\sum_{i \in W_{\mu^*_{\mathbf{x}_{-n},0}}(k)} s(x_k, x_i) \right] \\ + \sum_{k \in D_n(\mathbf{x}) \setminus D_n(\mathbf{x}_{-n}, z)} \left[\sum_{i \in W_{\mu^*_{\mathbf{x}_{-n},z}}(k)} s(x_k, x_i) \right] \end{aligned}$$

Agents in $D_n(\mathbf{x}) \setminus D_n(\mathbf{x}_{-n}, z)$ are at least as well off when n reports z as when n reports 0, thus

$$v_n(\mathbf{x}) - t_n(\mathbf{x}) \geq v_n(\mathbf{x}_{-n}, z) - t_n(\mathbf{x}_{-n}, z)$$

Alternatively, if $z > x_n$, then $D_n(\mathbf{x}) \subseteq D_n(\mathbf{x}_{-n}, z)$, and Equation (A.1) consequently becomes

$$\begin{aligned} v_n(\mathbf{x}) - t_n(\mathbf{x}) \geq\ & v_n(\mathbf{x}_{-n}, z) - t_n(\mathbf{x}_{-n}, z) \\ & - \sum_{k \in D_n(\mathbf{x}_{-n}, z) \backslash D_n(\mathbf{x})} \left[\sum_{i \in W_{\mu^*_{\mathbf{x}}}(k)} s(x_k, x_i) \right] \\ & + \sum_{k \in D_n(\mathbf{x}_{-n}, z) \backslash D_n(\mathbf{x})} \left[\sum_{i \in W_{\mu^*_{\mathbf{x}_{-n},0}}(k)} s(x_k, x_i) \right] \end{aligned}$$

Agents in $D_n(\mathbf{x}_{-n}, z) \backslash D_n(\mathbf{x})$ are at least as well off when n reports 0 as when n reports x_n, thus

$$v_n(\mathbf{x}) - t_n(\mathbf{x}) \geq v_n(\mathbf{x}_{-n}, z) - t_n(\mathbf{x}_{-n}, z)$$

In particular, $v_n(\mathbf{x}) - t_n(\mathbf{x}) \geq v_n(\mathbf{x}_{-n}, 0) - t_n(\mathbf{x}_{-n}, 0) = 0$, so it is ex post individually rational for agent n to report truthfully. □

A.1.2 Proof of Lemma 2.5.3

Proof. This is the proof of part 1 of Lemma 2.5.3; the proof of part 2 follows analogously.

A mechanism $\{q, t\}$ is individually rational if for all $n \in \{1, \cdots, N\}$

$$U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n, \mathbf{x}_{-n}, x_n) \geq 0 \tag{A.2}$$

A mechanism is dominant strategy incentive compatible if for all $n \in \{1, \cdots, N\}$ and for all $x_n, \widehat{x}_n \in [0,1]$

$$U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n, \mathbf{x}_{-n}, x_n) \geq U_n(q, t, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, x_n) \tag{A.3}$$

First, show that if $\{q, t\}$ is feasible, individually rational, and dominant strategy incentive compatible, it must satisfy the conditions of part 1 of Lemma 2.5.3. Rewrite the right hand side of Equation (A.3) as

$$U_n(q, t, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, \widehat{x}_n) + \sum_{\mu \in A} \left[q^{\mu}(\widehat{\mathbf{x}}_{-n}, \widehat{x}_n) \sum_{i \in W_{\mu}(n)} [s(x_n, x_i) - s(\widehat{x}_n, x_i)] \right] \tag{A.4}$$

Replace (A.4) into the right hand side of Equation (A.3). Thus the incentive compatibility constraint is equivalent to

$$\begin{aligned} U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n, \mathbf{x}_{-n}, x_n) \geq{} & U_n(q, t, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, \widehat{x}_n) \\ & + \sum_{\mu \in A} \left[q^{\mu}(\widehat{\mathbf{x}}_{-n}, \widehat{x}_n) \sum_{i \in W_{\mu}(n)} [s(x_n, x_i) - s(\widehat{x}_n, x_i)] \right] \end{aligned} \tag{A.5}$$

Transpose x_n and $\widehat{x}_n$ in Equation (A.5):

$$\begin{aligned} U_n(q, t, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, \widehat{x}_n) \geq{} & U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n, \mathbf{x}_{-n}, x_n) \\ & + \sum_{\mu \in A} \left[q^{\mu}(\widehat{\mathbf{x}}_{-n}, x_n) \sum_{i \in W_{\mu}(n)} [s(\widehat{x}_n, x_i) - s(x_n, x_i)] \right] \end{aligned} \tag{A.6}$$

Combine Equations (A.5) and (A.6), and rearrange; we see that incentive compatibility is equivalent to the supermodularity of $H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, x_n)$ in $\widehat{x}_n, x_n$:

$$\begin{aligned} &H_n(q, \widehat{\mathbf{x}}_{-n}, x_n, \mathbf{x}_{-n}, x_n) + H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, \widehat{x}_n) \\ &\qquad \geq H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, x_n) + H_n(q, \widehat{\mathbf{x}}_{-n}, x_n, \mathbf{x}_{-n}, \widehat{x}_n) \end{aligned} \tag{A.7}$$

The payment rule, Equation (2.6), follows from the standard method of equating the direct and indirect utility functions. Evaluate the incentive compatibility constraint at $\mathbf{x}_{-n} = \widehat{\mathbf{x}}_{-n}$. The resulting inequality combined with the envelope theorem implies that

$$\frac{d}{dx_n} U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n, \widehat{\mathbf{x}}_{-n}, x_n) = \left[\frac{\partial}{\partial x_n} U_n(q, t, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \widehat{\mathbf{x}}_{-n}, x_n) \right] \Bigg|_{\widehat{x}_n = x_n} \tag{A.8}$$

Integrating Equation (A.8)

$$\begin{aligned} U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n, \widehat{\mathbf{x}}_{-n}, x_n) &= U_n(q, t, \widehat{\mathbf{x}}_{-n}, 0, \widehat{\mathbf{x}}_{-n}, 0) \\ &\quad + \int_0^{x_n} \left[\frac{\partial}{\partial z} U_n(q, t, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \widehat{\mathbf{x}}_{-n}, z) \right] \Bigg|_{\widehat{x}_n = z} dz \end{aligned} \tag{A.9}$$

Since $t_n(\widehat{\mathbf{x}}_{-n}, \widehat{x}_n)$ is independent of x_n, Equation (A.9) is equivalent to

$$\begin{aligned} U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n, \widehat{\mathbf{x}}_{-n}, x_n) &= U_n(q, t, \widehat{\mathbf{x}}_{-n}, 0, \widehat{\mathbf{x}}_{-n}, 0) \\ &\quad + \int_0^{x_n} \left[\frac{\partial}{\partial z} H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \widehat{\mathbf{x}}_{-n}, z) \right] \Bigg|_{\widehat{x}_n = z} dz \end{aligned} \tag{A.10}$$

Now show that if $\{q, t\}$ is feasible, individually rational, and satisfies the conditions of part 1 of Lemma 2.5.3, it must be dominant strategy incentive compatible. Evaluating the second to last arguments of U_n and H_n in Equation (A.10) at $\widehat{\mathbf{x}}_{-n} = \mathbf{x}_{-n}$

$$U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n, \mathbf{x}_{-n}, x_n) = U_n(q, t, \widehat{\mathbf{x}}_{-n}, 0, \mathbf{x}_{-n}, 0) + \int_0^{x_n} \left[\frac{\partial}{\partial z} H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, z) \right] \Bigg|_{\widehat{x}_n = z} dz \tag{A.11}$$

By Equation (A.11)

$$U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n, \mathbf{x}_{-n}, x_n) = U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n', \mathbf{x}_{-n}, x_n') + \int_{x_n'}^{x_n} \left[\frac{\partial}{\partial z} H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, z) \right] \Bigg|_{\widehat{x}_n = z} dz \tag{A.12}$$

By the supermodularity of $H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, x_n)$ in $\widehat{x}_n, x_n$, we conclude that the integrand is increasing in its third argument.[1] Suppose that $x_n \geq x_n'$. Then

$$U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n, \mathbf{x}_{-n}, x_n) \geq U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n', \mathbf{x}_{-n}, x_n') + \int_{x_n'}^{x_n} \left[\frac{\partial}{\partial z} H_n(q, \widehat{\mathbf{x}}_{-n}, x_n', \mathbf{x}_{-n}, z) \right] \Bigg|_{\widehat{x}_n = z} dz \tag{A.13}$$

Integrating

$$U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n, \mathbf{x}_{-n}, x_n) \geq U_n(q, t, \widehat{\mathbf{x}}_{-n}, x_n', \mathbf{x}_{-n}, x_n') + \sum_{\mu \in A} \left[q^{\mu}(\widehat{\mathbf{x}}_{-n}, x_n') \sum_{i \in W_{\mu}(n)} [s(x_n, x_i) - s(x_n', x_i)] \right] \tag{A.14}$$

[1]Since $H_n(q, \widehat{\mathbf{x}}_{-n}, \widehat{x}_n, \mathbf{x}_{-n}, x_n)$ is supermodular in $\widehat{x}_n, x_n$, the cross derivative is non-negative. Therefore, $\frac{\partial H_n}{\partial x_n}$ must be increasing in $\widehat{x}_n$.

Equation (A.14) is exactly Equation (A.5) with $\widehat{x}_n$ relabeled as x'_n. Thus part 1 of Lemma 2.5.3 describes sufficient conditions for a feasible, individually rational mechanism to be dominant strategy incentive compatible. □

A.1.3 Proof of Lemma 2.5.5

Proof. Suppose that two agents, n and n' submit identical (truthful) reports. It is sufficient to show that $v_n(\mathbf{x}) - t_n(\mathbf{x}) = v_{n'}(\mathbf{x}) - t_{n'}(\mathbf{x})$ for all socially optimal μ. If $\mu(n) = \mu(n')$ for all socially optimal μ, then $D_n(\mathbf{x}) = D_{n'}(\mathbf{x})$ and the theorem follows. Now suppose that there is a socially optimal μ such that $\mu(n) \neq \mu(n')$ for some agents n, n' with identical types.

Suppose that $x_n = x_{n'}$, $\mu^*_{\mathbf{x}}(n) = b$, and $\mu^*_{\mathbf{x}}(n') = b'$. Without loss of generality, assume that $b' = b + 1$.[2] Then

$$v_n(\mathbf{x}) - t_n(\mathbf{x}) = \sum_{i \in W_{\mu^*_{\mathbf{x}}}(n)} s(x_n, x_i) + \sum_{k \in D_n(\mathbf{x})} \left[\sum_{i \in W_{\mu^*_{\mathbf{x}}}(k)} s(x_k, x_i) - \sum_{i \in W_{\mu^*_{\mathbf{x}_{-n},0}}(k)} s(x_k, x_i) \right]$$

Since $\mu(n) = b$ and $\mu(n') = b + 1$, $D_n(\mathbf{x}) = \{n'\} \cup D_{n'}(\mathbf{x})$. Therefore

$$\begin{aligned} v_n(\mathbf{x}) - t_n(\mathbf{x}) = & \sum_{i \in W_{\mu^*_{\mathbf{x}}}(n)} s(x_n, x_i) + \sum_{k \in D_{n'}(\mathbf{x})} \left[\sum_{i \in W_{\mu^*_{\mathbf{x}}}(k)} s(x_k, x_i) - \sum_{i \in W_{\mu^*_{\mathbf{x}_{-n},0}}(k)} s(x_k, x_i) \right] \\ & + \left[\sum_{i \in W_{\mu^*_{\mathbf{x}}}(n')} s(x_{n'}, x_i) - \sum_{i \in W_{\mu^*_{\mathbf{x}_{-n},0}}(n')} s(x_{n'}, x_i) \right] \end{aligned}$$

[2]If $b' > b + 1$, then there must be some other agent n'' whose quality $x_{n''} = x_n = x_{n'}$ and $\mu^*_{\mathbf{x}}(n'') = b + 1$. Transitively, it is sufficient to consider n'' and n.

Since $\mu^*_{\mathbf{x}_{-n},0}(n') = b$ and $x_n = x_{n'}$, $\sum_{i \in W_{\mu^*_{\mathbf{x}}}(n)} s(x_n, x_i) = \sum_{i \in W_{\mu^*_{\mathbf{x}_{-n},0}}(n')} s(x_{n'}, x_i)$. Thus

$$\begin{aligned} v_n(\mathbf{x}) - t_n(\mathbf{x}) = \sum_{i \in W_{\mu^*_{\mathbf{x}}}(n')} s(x_{n'}, x_i) \\ + \sum_{k \in D_{n'}(\mathbf{x})} \left[\sum_{i \in W_{\mu^*_{\mathbf{x}}}(k)} s(x_k, x_i) - \sum_{i \in W_{\mu^*_{\mathbf{x}_{-n},0}}(k)} s(x_k, x_i) \right] \end{aligned}$$

Finally, for all $k \in D_{n'}(\mathbf{x})$, $\mu^*_{\mathbf{x}_{-n},0}(k) = \mu^*_{\mathbf{x}_{-n'},0}(k)$. Therefore

$$\begin{aligned} v_n(\mathbf{x}) - t_n(\mathbf{x}) &= \sum_{i \in W_{\mu^*_{\mathbf{x}}}(n')} s(x_{n'}, x_i) \\ &\quad + \sum_{k \in D_{n'}(\mathbf{x})} \left[\sum_{i \in W_{\mu^*_{\mathbf{x}}}(k)} s(x_k, x_i) - \sum_{i \in W_{\mu^*_{\mathbf{x}_{-n'},0}}(k)} s(x_k, x_i) \right] \\ &= v_{n'}(\mathbf{x}) - t_{n'}(\mathbf{x}) \end{aligned}$$

$\square$

A.2 Appendix to Chapter 3

A.2.1 Proof of Theorem 3.3.1

Proof. Without loss of generality, suppose that H_1 wins the period 0 contract. Given s_{10} and s_{20}, the lifetime expected payoff of honoring the period 0 contract is

$$\begin{aligned}
&\Pr(s_{20} \geq t_{10})[U(s_{10}, s_{20}) + U(s_{20}, s_{10})] + \Pr(t_{10} \geq s_{20})E[U(s_{10}, t_{10})|t_{10} \geq s_{20}] \\
&\quad + \sum_{k=1}^{\infty} \delta^k \Bigg\{ \Pr(s_{2k} \geq t_{1k})E[U(s_{1k}, s_{2k}) + U(s_{2k}, s_{1k})|s_{2k} \geq t_{1k}] \\
&\quad + \Pr(t_{1k} \geq s_{2k})E[U(s_{1k}, t_{1k})|t_{1k} \geq s_{2k}] \Bigg\}
\end{aligned}$$

Given s_{10} and s_{20}, the lifetime expected payoff of reneging on the period 0 contract is

$$\begin{aligned}
&\Pr(s_{10} \geq t_{10})[U(s_{10}, s_{20}) + U(s_{20}, s_{10})] + \Pr(t_{10} \geq s_{10})E[U(s_{20}, t_{10})|t_{10} \geq s_{10}] \\
&+ \sum_{k=1}^{\infty} \delta^k \Bigg\{ \Pr(s_{1k} \geq t_{2k})E[U(s_{1k}, t_{1k})|s_{1k} \geq t_{2k}] \Bigg\}
\end{aligned}$$

The strategies form a distribution independent perfect Bayesian equilibrium if for all realizations of s_{10} and s_{20}

$$\begin{aligned}
&\Pr(s_{20} \geq t_{10})[U(s_{10}, s_{20}) + U(s_{20}, s_{10})] + \Pr(t_{10} \geq s_{20})E[U(s_{10}, t_{10})|t_{10} \geq s_{20}] \\
&\quad + \sum_{k=1}^{\infty} \delta^k \Bigg\{ \Pr(s_{2k} \geq t_{1k})E[U(s_{1k}, s_{2k}) + U(s_{2k}, s_{1k})|s_{2k} \geq t_{1k}] \\
&\quad + \Pr(t_{1k} \geq s_{2k})E[U(s_{1k}, t_{1k})|t_{1k} \geq s_{2k}] \Bigg\} \\
&\geq \Pr(s_{10} \geq t_{10})[U(s_{10}, s_{20}) + U(s_{20}, s_{10})] + \Pr(t_{10} \geq s_{10})E[U(s_{20}, t_{10})|t_{10} \geq s_{10}] \\
&\quad + \sum_{k=1}^{\infty} \delta^k \Bigg\{ \Pr(s_{1k} \geq t_{2k})E[U(s_{1k}, t_{1k})|s_{1k} \geq t_{2k}] \Bigg\}
\end{aligned}$$

Since student types are independently and identically distributed this inequality simplifies to:

$$\begin{aligned}
&F(s_{20})^2[U(s_{10}, s_{20}) + U(s_{20}, s_{10})] + (1 - F(s_{20})^2)E[U(s_{10}, t_{10})|t_{10} \geq s_{20}] \\
&+ \sum_{k=1}^{\infty} \delta^k \Bigg\{ \frac{1}{6}E[U(s_{1k}, s_{2k}) + U(s_{2k}, s_{1k})|s_{2k} \geq t_{1k}] + \frac{5}{6}E[U(s_{1k}, t_{1k})|t_{1k} \geq s_{2k}] \Bigg\} \\
&\geq F(s_{10})^2[U(s_{10}, s_{20}) + U(s_{20}, s_{10})] + (1 - F(s_{10})^2)E[U(s_{20}, t_{10})|t_{10} \geq s_{10}] \\
&+ \sum_{k=1}^{\infty} \delta^k \Bigg\{ \frac{5}{6}E[U(s_{1k}, t_{1k})|s_{1k} \geq t_{2k}] \Bigg\}
\end{aligned}$$

Let

$$A(s_{10}, s_{20}) = F(s_{20})^2[U(s_{10}, s_{20}) + U(s_{20}, s_{10})] + (1 - F(s_{20})^2)E[U(s_{10}, t_{10})|t_{10} \geq s_{20}]$$

and

$$B = \frac{5}{6} E[U(s_{1k}, t_{1k}) | s_{1k} \geq t_{2k}]$$

Note that B is a constant, and $E[A(s_{1k}, s_{2k})] > B$. Therefore

$$\frac{\delta}{1-\delta} \geq \frac{A(s_{20}, s_{10}) - A(s_{10}, s_{20})}{E[A(s_{1k}, s_{2k})] - B}$$

The assumptions about the existence of all moments of interest and the boundedness of $U(\cdot, \cdot)$ imply that

$$0 \leq \sup \left\{ \frac{A(s_{20}, s_{10}) - A(s_{10}, s_{20})}{E[A(s_{1k}, s_{2k})] - B} \middle| (s_{10}, s_{20}) \in [0, a] \times [0, a] \text{ and } s_{10} \geq s_{20} \right\} < \infty$$

Let

$$S = \sup \left\{ \frac{A(s_{20}, s_{10}) - A(s_{10}, s_{20})}{E[A(s_{1k}, s_{2k})] - B} \middle| (s_{10}, s_{20}) \in [0, a] \times [0, a] \text{ and } s_{10} \geq s_{20} \right\}$$

The critical $\overline{\delta}$ such that the strategies form a distribution independent perfect Bayesian equilibrium is therefore

$$\overline{\delta} = \frac{S}{1+S}$$

It is clear that H_1 cannot profitably deviate by reneging and the reporting $\hat{s}_{1n} > s_{2n}$ since doing so is never better than reneging and reporting $\hat{s}_{1n} \leq s_{2n}$.

Using a weak dominance argument, I now show that H_1 cannot profitably deviate by reneging on the contract and then reporting $\hat{s}_{1n} \leq s_{2n}$ in any period. Suppose that in period n, H_1 reneges and reports $\hat{s}_{1n} = s_{2n}$.[3] If H_2's good student wins the period n Vickrey auction, then H_1 is worse off for her deviation, since it receives $U(s_{2n}, t_{1n})$ in period n; H_1's payoff in subsequent periods is unchanged since C does not learn that H_1 reneged in period n. If H_1 wins the period n Vickrey auction, then H_1 receives $U(s_{1n}, s_{2n}) + U(s_{2n}, s_{1n})$ in period n. Since s_{2n} is a winning report in the period n Vickrey auction, H_1's period n payoff is the same regardless of whether it reneges and reports $\hat{s}_{1n} = s_{2n}$ or it follows the strategy specified in the theorem. However, since s_{2n} was a winning report in the period n Vickrey auction, C learns that H_1 reneged in period n and therefore sets $q_{n+1} = 0$, making H_1 strictly worse off in period $n+1$ and no better off in periods $n+2, n+3, \cdots$. □

A.3 Appendix to Chapter 4

A.3.1 Proof of Theorem 4.3.1

This proof appears as a series of lemmas. The first uses the gross substitutes condition to establish an inequality concerning the marginal product of an arbitrary worker. The next four lemmas construct an $\mathbf{s}^*$ such that $(F^*, \mu^*, \mathbf{s}^*)$ is individually rational. Then, a final lemma shows that an individually rational allocation must be in the core.

Lemma A.3.1. *Let j be any agent in Ω and let A be any subset of Ω containing j.*

[3] Conditional on reneging, the report $\hat{s}_{1n} = s_{2n}$ weakly dominates any report $\hat{s}_{1n} < s_{2n}$.

Then for any $A^{'} \subseteq A$ such that $j \in A^{'}$ and any $k \in A^{'}$

$$y^j(A) - y^j(A \backslash \{k\}) \leq y^j(A^{'}) - y^j(A^{'} \backslash \{k\})$$

Proof. Define $s = \{s_{1j}, \cdots, s_{j-1j}, s_{j+1j}, \cdots, s_{nj}\}$ by $s_{ij} = \sigma_{ij}$ if $i \in A \backslash \{k\}$, $s_{kj} = y^j(A) - y^j(A \backslash \{k\})$, and $s_{ij} = \infty$ otherwise. Then

$$A \backslash \{k\} \in \operatorname*{argmax}_{C \subseteq \Omega} y^j(C) - \sum_{i \in C \backslash \{j\}} s_{ij}$$

and

$$\begin{aligned} y^j(A \backslash \{k\}) - \sum_{i \in A \backslash \{j,k\}} s_{ij} &= y^j(A \backslash \{k\}) - \sum_{i \in A \backslash \{j,k\}} s_{ij} - s_{kj} + y^j(A) - y^j(A \backslash \{k\}) \\ &= y^j(A) - \sum_{i \in A \backslash \{j\}} s_{ij} \end{aligned}$$

ergo

$$A \in \operatorname*{argmax}_{C \subseteq \Omega} y^j(C) - \sum_{i \in C \backslash \{j\}} s_{ij}$$

Define $s^{'} = \{s^{'}_{1j}, \cdots, s^{'}_{j-1j}, s^{'}_{j+1j}, \cdots, s^{'}_{nj}\}$ by $s^{'}_{ij} = s_{ij}$ if $i \in A^{'}$ and $s_{ij} = \infty$ otherwise. By the gross substitutes condition

$$A' \in \operatorname*{argmax}_{C \subseteq \Omega} y^j(C) - \sum_{i \in C \backslash \{j\}} s^{'}_{ij}$$

Thus

$$y^j(A^{'}) - \sum_{i\in A^{'}\setminus\{j\}} s^{'}_{ij} \geq y^j(A^{'}\setminus\{k\}) - \sum_{i\in A^{'}\setminus\{j,k\}} s^{'}_{ij}$$

$$y^j(A^{'}) - y^j(A^{'}\setminus\{k\}) \geq s^{'}_{kj} = y^j(A) - y^j(A\setminus\{k\})$$

□

Lemma A.3.2. *Suppose that (F^*, μ^*) is socially optimal. Then there exists $s^j = \{s_{1j}, s_{2j}, \cdots, s_{nj}\}$ such that*

$$\mu^*(j) \in \underset{C\subseteq\Omega}{\operatorname{argmax}}\, y^j(C) - \sum_{i\in C\setminus\{j\}} s_{ij}$$

Proof. The proof constructs an appropriate s^j. Define $s^j = \{s_{1j}, s_{2j}, \cdots, s_{nj}\}$ as follows.

1. If $i \notin \mu^*(j)$, let $s_{ij} = \max_{A\subseteq\Omega}\{y^j(A\cup\{i\}) - y^j(A), a_{i\mu^{*-1}(i)} + s_{i\mu^{*-1}(i)} - a_{ij}\}$

2. If $i \in \mu^*(j)\setminus\{j\}$, let s_{ij} be any number in the closed interval

$$s_{ij} \in \left[\max_{j^{'}\in F^*}\{\sigma_{ij}, a_{ij'} + y^{j^{'}}(\mu^*(j^{'})\cup\{i\}) - y^{j^{'}}(\mu^*(j^{'})) - a_{ij}\}, \min_{A\subseteq\mu^*(j)} y^j(A) - y^j(A\setminus\{i\})\right]$$

3. Let $s_{jj} = y^j(\mu^*(j)) - \sum_{i\in\mu^*(j)\setminus\{j\}} s_{ij}$.

First, show that such an s_{ij} exists for all i, i.e. show that the interval

$$\left[\max_{j^{'}\in F^*}\{\sigma_{ij}, a_{ij'} + y^{j^{'}}(\mu^*(j^{'})\cup\{i\}) - y^{j^{'}}(\mu^*(j^{'})) - a_{ij}\}, \min_{A\subseteq\mu^*(j)} y^j(A) - y^j(A\setminus\{i\})\right]$$

is not empty. By Lemma A.3.1,

$$\min_{A \subseteq \mu^*(j)} y^j(A) - y^j(A \backslash \{i\}) = y^j(\mu^*(j)) - y^j(\mu^*(j) \backslash \{i\})$$

By Equation 4.2

$$\sigma_{ij} \leq y^j(\mu^*(j)) - y^j(\mu^*(j) \backslash \{i\})$$

Since (F^*, μ^*) is socially optimal, for all $j^{'} \in F^* \backslash \{j\}$

$$a_{ij} + y^j(\mu^*(j)) - y^j(\mu^*(j) \backslash \{i\}) \geq a_{ij'} + y^{j^{'}}(\mu^*(j^{'}) \cup \{i\}) - y^{j^{'}}(\mu^*(j^{'}))$$

ergo

$$y^j(\mu^*(j)) - y^j(\mu^*(j) \backslash \{i\}) \geq a_{ij'} + y^{j^{'}}(\mu^*(j^{'}) \cup \{i\}) - y^{j^{'}}(\mu^*(j^{'})) - a_{ij}$$

Therefore, the interval is not empty and the constructed s^j exists.

Now show that $\mu^*(j) \in \text{argmax}_{C \subseteq \Omega} y^j(C) - \sum_{i \in C \backslash \{j\}} s_{ij}$. By construction of s^j, any element of $\text{argmax}_{C \subseteq \Omega} y^j(C) - \sum_{i \in C \backslash \{j\}} s_{ij}$ is a subset of $\mu^*(j)$. Let A denote any proper subset of $\mu^*(j)$. Then for any $k \in \mu^*(j) \backslash A$

$$y^j(A \cup \{k\}) - y^j(A) \geq s_{kj}$$

Thus

$$y^j(A \cup \{k\}) - s_{kj} - \sum_{i \in A\backslash\{j\}} \geq y^j(A) - \sum_{i \in A\backslash\{j\}}$$

Iterating this process, $\mu^*(j) \in \operatorname{argmax}_{C \subseteq \Omega} y^j(C) - \sum_{i \in C\backslash\{j\}} s_{ij}$. □

To construct $\mathbf{s}^*$ from the s^j, let $s_i^* = s_{ij}$ whenever $i \in \mu^*(j)$. Now show that $(F^*, \mu^*, \mathbf{s}^*)$ is individually rational.

Lemma A.3.3. *Suppose that (F^*, μ^*) is socially optimal. At $\mathbf{s}^*$, it is not rational for any $i \in \mu^*(j)$ to quit.*

Proof. By construction, $s_i^* \geq \sigma_{ij}$. Therefore

$$a_{ij} + s_i^* \geq a_{ij} + \sigma_{ij} = a_{ii} + y^i(\{i\})$$

which is exactly the condition that it is not rational for i to quit. □

Lemma A.3.4. *Suppose that (F^*, μ^*) is socially optimal. At $\mathbf{s}^*$, it is not rational for j to fire any set of employees $C \subseteq \mu^*(j)\backslash\{j\}$.*

Proof. Let C be any subset of $\mu^*(j)\backslash\{j\}$. Denote the agents in C by $i_1, i_2, \cdots, i_{|C|}$.

Then

$$
\begin{aligned}
y^j(\mu^*(j)) - y^j(\mu^*(j)\backslash C) = {} & y^j(\mu^*(j)) - y^j(\mu^*(j)\backslash\{i_1\}) \\
& + y^j(\mu^*(j)\backslash\{i_1\}) - y^j(\mu^*(j)\backslash\{i_1, i_2\}) \\
& + \cdots \\
& + y^j(\mu^*(j)\backslash\{i_1, \cdots, i_{m-1}\}) - y^j(\mu^*(j)\backslash\{i_1, \cdots, i_{|C|}\})
\end{aligned}
$$

Since $s_i^* \leq y^j(A) - y^j(A\backslash\{i\})$ for all $A \subseteq \mu^*(j)$ such that A contains i and j :

$$y^j(\mu^*(j)) - y^j(\mu^*(j)\backslash C) \geq \sum_{i \in C} s_i^*$$

which is exactly the condition that it is not rational for j to fire any subset of its employees. □

Lemma A.3.5. *Suppose that (F^*, μ^*) is socially optimal and (F^*, μ^*, s^*) is an individually rational allocation. Then (F^*, μ^*, s^*) is in the core.*

Proof. Suppose that (F^*, μ^*, s^*) is not in the core, i.e. there exists $A \subseteq \Omega$, $f \in A$, and salaries s_{if} for all $i \in A$ such that $s_{ff} = y^f(A) - \sum_{i \in A\backslash\{f\}} s_{if}$ and

$$a_{if} + s_{if} \geq a_{i\mu^{*-1}(i)} + s_i^*$$

with at least one of these inequalities holding strictly. Thus A blocks (F^*, μ^*, s^*)

implies that

$$\sum_{i \in A}(a_{i\mu^{*-1}(i)} + s_i^*) < y^f(A) + \sum_{i \in A} a_{if}$$

Let $\overline{F} = \{j \in F^* \backslash A | \mu^*(j) \backslash \{j\} \cap A \neq \emptyset\}$ and let $\overline{W} = \{i \in W^* \backslash A | \mu^{*-1}(i) \in A\}$. These sets respectively represent the set of firms not in A who lose one or more employees to A and the set of workers not in A who lose their employer to A. Therefore, the set $A \cup \overline{F} \cup \overline{W}$ is the set of all agents whose welfare is affected when A breaks away. Since (F^*, μ^*) is socially optimal, the total welfare received by agents in $A \cup \overline{F} \cup \overline{W}$ under (F^*, μ^*) must equal or exceed the total welfare received by those agents under (F^*, μ^*) after A breaks away.

$$\sum_{i \in A}(a_{i\mu^{*-1}(i)} + s_i^*) + \sum_{i \in \overline{W}}(a_{i\mu^{*-1}(i)} + s_i^*) + \sum_{j \in \overline{F}} \left(a_{jj} + y^j(\mu^*(j)) - \sum_{i \in \mu^*(j) \backslash \{j\}} s_i^* \right)$$
$$\geq y^f(A) + \sum_{i \in A} a_{if} + \sum_{i \in \overline{W}} \left(a_{ii} + y^i(\{i\}) \right) + \sum_{j \in \overline{F}} \left(a_{jj} + y^j(\mu^*(j) \backslash A) - \sum_{i \in \mu^*(j) \backslash (A \cup \{j\})} s_i^* \right)$$

Equivalently

$$\sum_{i \in A}(a_{i\mu^{*-1}(i)} + s_i^*) + \sum_{i \in \overline{W}}(s_i^* - \sigma_{i\mu^{*-1}(i)}) + \sum_{j \in \overline{F}} \left(y^j(\mu^*(j)) - y^j(\mu^*(j) \backslash A) - \sum_{i \in \mu^*(j) \cap A} s_i^* \right)$$
$$\geq y^f(A) + \sum_{i \in A} a_{if}$$

Lemma A.3.3 says that $\sum_{i \in \overline{W}}(s_i^* - \sigma_{i\mu^{*-1}(i)})$ is non-negative. Lemma A.3.4 says that

$\sum_{j\in\overline{F}}\left(y^j(\mu^*(j)) - y^j(\mu^*(j)\backslash A) - \sum_{i\in\mu^*(j)\cap A} s_i^*\right)$ is non-negative. Therefore

$$\sum_{i\in A}(a_{i\mu^{*-1}(i)} + s_i^*) \geq y^f(A) + \sum_{i\in A} a_{if}$$

i.e. A is not a blocking coalition. $\square$

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