ABSTRACT

Title of dissertation: ABSOLUTELY CONTINUOUS SPECTRUM FOR PARABOLIC FLOWS/MAPS

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This work is devoted to creating an abstract framework for the study of certain spectral properties of parabolic systems. Specifically, we determine under which general conditions to expect the presence of absolutely continuous spectral measures. We use these general conditions to derive results for spectral properties of time-changes of unipotent flows on homogeneous spaces of semisimple groups regarding absolutely continuous spectrum as well as maximal spectral type; the time-changes of the horocycle flow are special cases of this general category of flows. In addition we use the general conditions to derive spectral results for twisted horocycle flows and to rederive spectral results for skew products over translations and Furstenberg transformations.

ABSOLUTELY CONTINUOUS SPECTRUM FOR PARABOLIC FLOWS/MAPS

by

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Dedication

To my parents, Brenda and Italo, my North Stars.

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Chapter 1: Introduction

1.1 Motivation

Spectral theory of dynamical systems has long been studied [16]; of particular interest, is the notion of when to expect the presence of absolutely continuous spectral measures. Since absolutely continuous spectrum implies mixing, this property can be thought of as an indicator of how chaotic, or how far from orderly, a system is. In the hyperbolic setting, systems are characterized as having a correlation decay that is exponential. As a result, techniques derived from the existence of a spectral gap as well as probabilistic tools are available for the study of spectral properties, and therefore, it is in the hyperbolic setting where the existence of absolutely continuous spectrum predominantly occurs. Interestingly, certain parabolic systems also share this property despite having at most polynomial decay of correlations. This slower decay of correlations precludes the use of the tools available in the spectral study of hyperbolic systems, and consequently, spectral theory of smooth parabolic flows and smooth perturbations of well known parabolic flows has been much less studied. This work is devoted to creating an abstract framework for the study of certain spectral properties of parabolic systems. Specifically, we attempt to answer the question: under what general conditions can we expect the existence of absolutely continuous spectral measures?

First we will provide a bit of background on operators and spectral theory. Then we will describe two methods that have been used in the spectral study of parabolic systems and discuss their applications to both a simple example and a more complex example. We use this as motivation to develop general conditions under which we expect a system to have absolutely continuous measures. We use these general conditions to derive results for spectral properties of time-changes of unipotent flows on homogeneous spaces of semisimple groups regarding absolute continuity of the spectrum as well as maximal spectral type; the time-changes of the horocycle flow are special cases of this general category of flows. In addition we use the general conditions to derive spectral results for twisted horocycle flows and to rederive spectral results for skew products over translations and Furstenberg transformations [29].

1.2 Background [12], [21], [22], [30]

Let $\mathcal{H}$ be a Hilbert Space and let A, V be bounded, densely defined operators acting on $\mathcal{H}$.

If for $f, g \in \mathcal{H}$,

$$\langle Af, g\rangle_{\mathcal{H}} = \langle f, Ag\rangle_{\mathcal{H}}$$

then A is **symmetric**, and if

$$\langle Af, g\rangle_{\mathcal{H}} = -\langle f, Ag\rangle_{\mathcal{H}}$$

then A is **skew-symmetric**.

The adjoint, A^*, of an operator A, is defined on all $g \in \mathcal{H}$ such that

$$\langle Af, g \rangle_{\mathcal{H}}$$

is a continuous linear functional of f. Since $\overline{Dom(A)} = \mathcal{H}$, there is a unique A^*g such that

$$\langle Af, g \rangle_{\mathcal{H}} = \langle f, A^*g \rangle_{\mathcal{H}}$$

for all $f \in Dom(A)$.

A is **self-adjoint** if

$$A = A^*$$

and A is **skew-adjoint** if

$$A = -A^*.$$

A is **essentially self-adjoint** if

$$\overline{A} = A^*$$

and A is **essentially skew-adjoint** if

$$\overline{A} = -A^*.$$

An operator V is **unitary** if $V^* = V^{-1}$.

The **operator norm** is defined as

$$\| A \|_{op} = \sup\{ \| Af \|_{\mathcal{H}} : \| f \|_{\mathcal{H}} \leqslant 1\}.$$

The **spectrum** $\sigma(A)$ of a self-adjoint operator A, is given by the the collection of $z \in \mathbb{C}$ for which

$$A - zI$$

does not have a bounded inverse.

A **spectral projector** E is set function that maps Borel subsets of $\mathbb{R}$ into projections on $\mathcal{H}$.

For $S \subset \mathbb{R}$, the **spectral measure** is given by

$$\mu_f(S) = \int_{\mathbb{R}} \chi_S(x) d\mu_f(x) = \int_{\mathbb{R}} \chi_S(x) d\langle E(S)f, f\rangle.$$

Consider a flow, ϕ_t (map ϕ_n), generated by a skew-adjoint operator A:

$$f \circ \phi_t^A = e^{tA} f \qquad\qquad (f \circ \phi_n^A = e^{nA} f).$$

The Spectral Theorem gives an expression for $\hat{\mu}_f$.

For $t \in \mathbb{R}$:

$$\begin{aligned}\langle f \circ \phi_t^A, f\rangle_{\mathcal{H}} &= \int_{\mathbb{R}} e^{it\xi} d\langle E(\xi)f, f\rangle \\ &= \int_{\mathbb{R}} e^{it\xi} d\mu_f(\xi) = \hat{\mu}_f(t)\end{aligned}$$

For $n \in \mathbb{Z}$:

$$\begin{aligned}\langle f \circ \phi_n^A, f\rangle_{\mathcal{H}} &= \int_{-\pi}^{\pi} e^{in\xi} d\langle E(\xi)f, f\rangle \\ &= \int_{-\pi}^{\pi} e^{in\xi} d\mu_f(\xi) = \hat{\mu}_f(n).\end{aligned}$$

The Hilbert Space $\mathcal{H}$ has the following orthogonal decomposition:

$$\mathcal{H} = \mathcal{H}_{ac} \bigoplus \mathcal{H}_{pp} \bigoplus \mathcal{H}_{sc}$$

where

$\mathcal{H}_{ac}$ is the subspace of vectors $f \in \mathcal{H}$ for which μ_f is absolutely continuous with respect to the Lebesgue measure m, i.e., for any $S \in \mathbb{R}$ such that $m(S) = 0$, $\mu_f(S) = 0$.

$\mathcal{H}_{pp}$ is the subspace of vectors $f \in \mathcal{H}$ for which μ_f is discrete with respect to Lebesgue, i.e., μ_f is supported on at most a countable set.

$\mathcal{H}_{sc}$ is the subspace of vectors $f \in \mathcal{H}$ for which μ_f is singularly continuous with respect to Lebesgue, i.e., it is continuous but supported on a set of Lebesgue measure 0.

The **maximal spectral type** μ_A of the operator A is a positive measure (defined up to equivalence) such that for every $f \in \mathcal{H}$, μ_f is absolutely continuous with respect to μ_A and no measure absolutely continuous with respect to μ_A but not equivalent to μ_A has the same property. If $\mu_A = m$, the Lebesgue measure, then the maximal spectral type is said to be Lebesgue.

The main question we investigate is:

Under which conditions do we expect the existence of a subspace of $\mathcal{H}$ on which the associated spectral measures are absolutely continuous with respect to the Lebesgue measure?

Additionally, in our results for time-changes of unipotent flows, we determine that the maximal spectral type is Lebesgue following the method in [8]. In the applications to maps, the maximal spectral type is implied by the purity law in [14].

1.3 Methods Used in the Parabolic Setting

The following two methods can be applied to certain parabolic systems to show the existence of absolutely continuous measures. We provide brief descriptions of the methods followed by examples of their applications.

1.3.1 Method 1. Limiting Estimates for the Resolvent

On the level of the generator, spectral properties can be derived from limiting properties involving the resolvent,

$$R(z) = (A - zI)^{-1}$$

for A self-adjoint, $z = \lambda + i\mu$, $\lambda \in \sigma(A)$. Since, $\| R(z) \| = |\mu|^{-1}$, $R(z)$ does not have a bounded limit as $\mu \to +0$. If, however, there exists a dense subset of vectors in $\mathcal{H}$ for which the $\lim F(\lambda + i\mu) = \langle f, R(\lambda + i\mu) f \rangle$ exists as $\mu \to +0$, then we can obtain results on the spectral properties of H.

Theorem 1. *[2] If $S \subset \mathbb{R}$ is an open set and $|\langle f, \mathcal{I}m R(z) f \rangle| \leqslant C(f) < \infty$ for all $\lambda \in S$ and $\mu > 0$, then f is A-absolutely continuous on S.*

1.3.2 Method 2. Regularity of the Spectral Measure

Another method useful in answering the first question is by directly showing that $\hat{\mu}_f(t) \in L^2(\mathbb{R})$ (or $\hat{\mu}_f(n) \in \ell^2(\mathbb{Z})$ in the discrete case) as this implies that μ_f is absolutely continuous with respect to the Lebesgue measure in the following way (we include the proof for the continuous case):

Suppose $\hat{\mu}_f \in L^2(\mathbb{R})$. Let a set $S \in \mathbb{R}$ be such that $m(S) = 0$ for m the Lebesgue measure. Let χ_S be the indicator function of S. Since χ_S can be approximated by smooth functions with compact support, the following is well-defined:

$$\mu_f(S) = \int_{\mathbb{R}} \chi_S(t)\, d\mu_f(t).$$

χ_S can be expressed as the inverse Fourier Transform of $\hat{\chi_S}$,

$$\mu_f(S) = \int_{\mathbb{R}} \chi_S(t)\, d\mu_f(t) = \int_{\mathbb{R}} (\int_{\mathbb{R}} \hat{\chi_S}(s) \cdot e^{ist} ds)\, d\mu_f(t)$$

$$= \int_{\mathbb{R}} \hat{\chi_S}(t)(\int_{\mathbb{R}} e^{ist}\, d\mu_f(s))\, dt = \int_{\mathbb{R}} \hat{\chi_S}(t) \cdot \hat{\mu}_f(t)\, dt$$

and from Hölder's Inequality,

$$|\mu_f(S)| \leqslant \| \hat{\chi_S}(t) \|_{L^2(\mathbb{R})} \cdot \| \hat{\mu}_f(t) \|_{L^2(\mathbb{R})} \leqslant \sqrt{m(S)} \cdot \| \hat{\mu}_f(t) \|_{L^2(\mathbb{R})} = 0.$$

To show that the assumption $\hat{\mu}_f \in L^2$ holds, one can show that the growth of $|\langle f \circ \phi_t^A, f\rangle_{\mathcal{H}}| = \mathrm{O}(\frac{1}{t^\beta})$ for $\beta > \frac{1}{2}$. Polynomial decay of correlations of general smooth functions does not guarantee a fast enough rate to achieve this bound, even in the simplest examples.

Note that both methods rely on a particular choice of subspace of $\mathcal{H}$. If the methods are applicable on a dense subspace of $\mathcal{H}$ then the spectrum of A is purely absolutely continuous since $\mathcal{H}_{ac}$ is closed.

1.4 A Simple Case - The Horocycle Flow

For a simple example, we begin with the classical horocycle flow. The horocycle flow on compact, hyperbolic surfaces, is minimal [13], uniquely ergodic [10], strongly mixing [24], and has zero entropy [11].

On $M = \Gamma \backslash PSL(2, \mathbb{R})$, where M is either compact or of finite area, we consider the basis

$$\left\{ U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, X = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \right\}$$

of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, where U and V are the generators of the positive and negative horocycle flows, $\{h_t^U\}$ and $\{h_t^V\}$ respectively, and X is the generator of the geodesic flow, $\{\phi_s^X\}$. It follows from [1] that $U$and V are essentially skew-adjoint. Our Hilbert Space is $L^2(M, vol)$ for vol the h_t^U-invariant volume form.

The key to the simplicity of this example lies in the commutation relations:

$$[X, U] = \boxed{\text{U}}$$

and

$$e^{-tU}[X, e^{tU}] = (\textstyle\int_0^t 1 \circ h_\tau^U(x)\, d\tau)\, U = \boxed{\text{tU}} \qquad \text{(See 6.2 for the calculation.)}$$

1.4.1 Method 1. Applied to the Horocycle Flow

For cases involving such a simple commutator, it is possible to prove that $\lim_{\mu\to+0} F(\lambda+i\mu)$ exists by showing that

$$\int_0^1 |\frac{d}{d\mu}F(\lambda+i\mu)|d\mu < \infty.$$

This is achieved by expressing a bound for $\frac{d}{d\mu}F(\lambda+i\mu)$ in terms of $F(\lambda+i\mu)$. In many situations this requires considering a new function $F(\lambda+i\mu+i\epsilon)$ and then applying the Gronwall Inequality. However, in the case of the horocycle flow, the simplicity of the commutator enables us to calculate the estimate directly. In the following calculations we will use the operator iU as it is essentially self-adjoint. We include more details in 6.1; what follows are the main steps. This proof is motivated by an overview of conjugate operator methods in [2] (Chapter 7).

Let $\lambda \in \sigma(iU)$ and $z = \lambda + i\mu$ for $\mu > 0$. The resolvent of iU is given by,

$$R(z) = (iU - z)^{-1}.$$

We will use the following important identities,

$$\frac{d}{dz}R(z) = R(z)^2,$$

$$[R(z), X] = R(z)[X, iU]R(z),$$

and since $[X, iU] = iU$,

$$z\frac{d}{dz}R(z) = [R(z), X] - R(z). \tag{1.1}$$

For $f \in Dom(X)$,

$$z\frac{d}{dz}F(z) = -F(z) - \langle R(z)f, Xf\rangle + \langle Xf, R(\overline{z})f\rangle.$$

Let $\| \cdot \| = \| \cdot \|_{L^2(M,vol)}$. Given our choice of z, the following equality holds,

$$\| R(z)f \| = \| R(\overline{z})f \| = \mu^{-\frac{1}{2}} \cdot |\mathcal{I}mF(z)|^{\frac{1}{2}}.$$

Using the above bound, we have the following inequality,

$$|\frac{d}{d\mu}F(\lambda + i\mu)| \leqslant |\lambda|^{-1}(\| f \| + 2 \| Xf \|)\mu^{-\frac{1}{2}}|F(\lambda + i\mu)|^{\frac{1}{2}}. \tag{1.2}$$

We divide both sides by $|F(\lambda + i\mu)|^{\frac{1}{2}}$, and since $|F(\lambda + i\mu)|^{\frac{1}{2}} = |F(\lambda + i\mu)^{\frac{1}{2}}|$, we obtain, after integrating with respect to μ for $0 < \mu < 1$,

$$|F(\lambda + i\mu)|^{\frac{1}{2}} \leqslant |F(\lambda + i)|^{\frac{1}{2}} + 2|\lambda|^{-1}(\| f \| + 2 \| Xf \|).$$

We use this to bound the right hand side of (1.2). Since λ is bounded away from 0, there exists a constant M such that,

$$|\frac{d}{d\mu}F(\lambda + i\mu)| \leqslant \frac{M}{\sqrt{\mu}}(\| f \|^2 + 2 \| Xf \|^2).$$

The above inequality enables us to show the following is finite,

$$\int_0^1 |\frac{d}{d\mu}F(\lambda + i\mu)|\, d\mu \leqslant \int_0^1 \frac{M}{\sqrt{\mu}}(\| f \|^2 + 2 \| Xf \|^2)\, d\mu$$

$$= 2M(\| f \|^2 + 2 \| Xf \|^2) < \infty,$$

which implies the existence of $\lim_{\mu \to 0+} F(\lambda + i\mu)$.

1.4.2 Method 2. Applied to the Horocycle Flow

For this method we show that

$$\langle f \circ h_t^U, f \rangle_{L^2(M,vol)} \in L^2(\mathbb{R}, dt).$$

Because the geodesic flow is volume preserving,

$$\langle f \circ h_t^U, f \rangle_{L^2(M,vol)} = \langle f \circ h_t^U \circ \phi_s^X, f \circ \phi_s^X \rangle_{L^2(M,vol)}.$$

We integrate both sides from 0 to σ with respect to s,

$$\langle f \circ h_t^U, f \rangle_{L^2(M,vol)} = \frac{1}{\sigma} \int_0^\sigma \langle f \circ h_t^U \circ \phi_s^X, f \circ \phi_s^X \rangle_{L^2(M,vol)} ds.$$

and then integrate by parts,

$$\begin{array}{cc} f \circ \phi_s^X & Xf \circ \phi_s^X \\ \int_0^\sigma f \circ h_t^U \circ \phi_s^X ds & f \circ h_t^U \circ \phi_s^X \end{array}$$

$$\frac{1}{\sigma} \int_0^\sigma \langle f \circ h_t^U \circ \phi_s^X, f \circ \phi_s^X \rangle_{L^2(M,vol)} ds$$
$$= \frac{1}{\sigma} \langle \int_0^\sigma f \circ h_t^U \circ \phi_s^X ds, f \circ \phi_s^X \rangle_{L^2(M,vol)} - \frac{1}{\sigma} \int_0^\sigma \langle \int_0^S f \circ h_t^U \circ \phi_s^X ds, Xf \circ \phi_s^X \rangle_{L^2(M,vol)} dS.$$

The only term that is not clearly bounded is

$$\int_0^\sigma f \circ h_t^U \circ \phi_s^X ds.$$

We consider functions of the form

$$f = Ug$$

for $g \in L^2(M, vol)$. Also, we use that

$$\frac{d}{ds}(g \circ h_t^U \circ \phi_s^X) = tUg \circ h_t^U \circ \phi_s^X + Xg \circ h_t^U \circ \phi_s^X,$$

and thus,

$$tUg \circ h_t^U \circ \phi_s^X = \frac{d}{ds}(g \circ h_t^U \circ \phi_s^X) - Xg \circ h_t^U \circ \phi_s^X.$$

So,

$$\begin{aligned}\int_0^\sigma f \circ h_t^U \circ \phi_s^X ds &= \int_0^\sigma Ug \circ h_t^U \circ \phi_s^X ds = \frac{1}{t}\int_0^\sigma tUg \circ h_t^U \circ \phi_s^X ds \\ &= \frac{1}{t}\int_0^\sigma \frac{d}{ds}(g \circ h_t^U \circ \phi_s^X)ds - \frac{1}{t}\int_0^\sigma Xg \circ h_t^U \circ \phi_s^X ds,\end{aligned}$$

and hence,

$$\| \int_0^\sigma f \circ h_t^U \circ \phi_s^X ds \|_{L^2(M,vol)}) \leqslant \frac{1}{t} \cdot (\| g \|_{L^2(M,vol)} + \| Xg \|_{L^2(M,vol)}).$$

This shows that

$$| < f \circ h_t^U(x), f >_{L^2(M,vol)} | = O(\frac{1}{t}).$$

It is an important observation that

$$\int_0^\sigma f \circ h_t^U \circ \phi_s^X ds$$

is approximately (for small σ and large t) an ergodic average for the horocycle flow, i.e.,

$$\lim_{t\to\infty} \int_0^\sigma f \circ h_t^U \circ \phi_s^X \, ds = \int_M f \, d\mu.$$

This geometric property was initially used by Marcus in his proof of mixing of the horocycle flow [18]; above we have a quantitative version along coboundaries. This technique was a key tool in [8] to prove spectral results for the time-changes of the horocycle flow. The following figure, Figure 1., from [17] illustrates that the image of a small geodesic segment γ_1, under the horocycle flow $h_t^U(\gamma_1)$ for large t, is approximately a horocycle segment, γ_2.

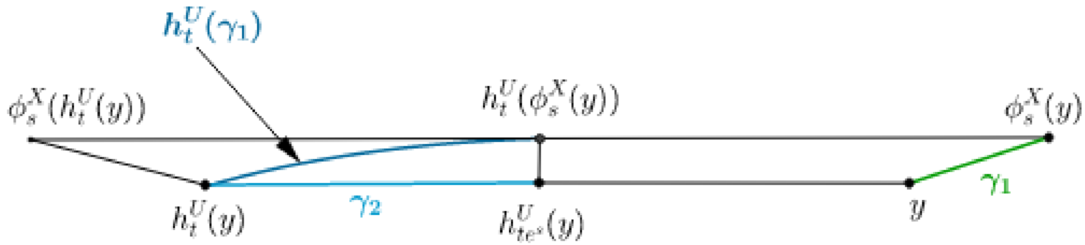

Figure 1.

1.4.3 Relationship with the Geodesic Flow

In fact, the commutator relations are so nice that the spectral properties can be derived from a direct analysis of these relations. The commutation relation between the geodesic and horocycle flows can be written as

$$\phi_s^X \circ h_t^U \circ \phi_{-s}^X = h_{te^s}^U.$$

It also has the following geometric interpretation.

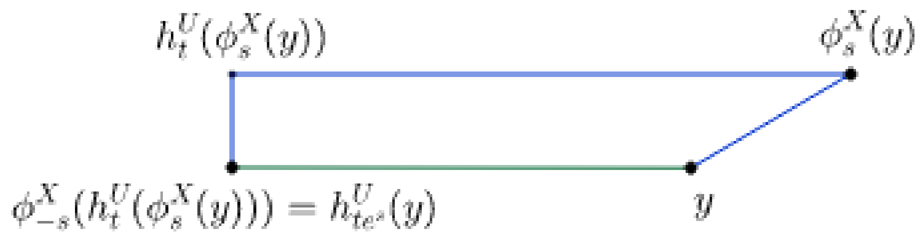

Figure 2.

If you travel along a geodesic for a given time s, and then you travel along a horocycle for time t before traveling back along a geodesic for time $-s$, it is the same as having traveled along a horocycle arc for a rescaled time, te^s. This shows that e^{tU} is unitarily equivalent to the renormalized one-parameter group e^{te^sU}. Consequently, the generators U and e^sU are also unitarily equivalent, and thus, spectrally isomorphic. Since this means that the spectral measure is invariant under multiplication by e^s, the spectrum must be Lebesgue [12] (p 664).

1.5 Time-Changes of the Horocycle Flow (Compact Case)

Even a small perturbation can disturb the simplicity of these relations, and thus, create obstacles in the study of spectral properties. We will use the example of adding a time-change to the horocycle flow to briefly describe the methods used in [28], [29], [8] to show that the spectrum remains absolutely continuous under this reparametrization when M is compact. Through these methods we will try to understand, on the level of the generators, to what extent we can increase the complexity in the commutation relations and still maintain absolute continuity of the spectrum.

Let $\tau : M \times \mathbb{R} \to \mathbb{R}$ such that

$$\tau(x, t + t') = \tau(x, t) + \tau(h_t^U(x), t').$$

Let $\alpha : M \to \mathbb{R}^+$, be the infinitesimal generator of τ, such that $\alpha \in C^\infty(M)$ and

$$\int_M \alpha vol = \int_M vol_\alpha = 1$$

where vol is the h_t^U-invariant volume form and vol_α is the $h_t^{U_\alpha}$-invariant volume form.

Now we consider a time-change on the positive horocycle flow, $\{h_t^{U_\alpha}\}$, generated by

$$U_\alpha =: U/\alpha.$$

The commutation relations are now as follows,

$$[X, U_\alpha] = \boxed{(\frac{X\alpha}{\alpha} - 1)U_\alpha} = G(\alpha)U_\alpha$$

and

$$e^{-tU_\alpha}[X, e^{tU_\alpha}] = \boxed{(\int_0^t (\frac{X\alpha}{\alpha} - 1) \circ h_\tau^{U_\alpha}(x)\, d\tau) U_\alpha} = G(\alpha, t) U_\alpha.$$

The fact that applying the aforementioned spectral methods is not so straightforward reflects these more complicated relations.

1.5.1 Method 1. Applied to the Time-Changes of the Horocycle Flow (Compact Case) [28], [29]

The more basic techniques for proving the existence of such limits require a simple expression for the commutator $[X, iU_\alpha]$ as seen in 1.4.1; for example, in this case we cannot verify the identity in equation (1.1) and thus, cannot proceed with the calculations as before. To extend these methods to more general and complicated cases, Mourre [20] derived the operator (we describe his estimate in the time-change setting)

$$E(S)[X, iU_\alpha]E(S)$$

for S a Borel set in $\mathbb{R}$, $f \in L_0^2(M, vol_\alpha)$ (zero average functions in $L^2(M, vol_\alpha)$), and μ_f the spectral measure associated to U_α as described in 1.2. For each bounded Borel set $S \in \mathbb{R}$, $E(S)[X, iU_\alpha]E(S)$ is bounded and self-adjoint.

Definition 1. *If there exists a number $a > 0$ such that*

$$E(S)[X, iU_\alpha]\mu_f(S) \geqslant aE(S)$$

*then the "**Mourre estimate**" is satisfied.*

The importance of this estimate for us lies in the following theorem:

Theorem 2. *[29] Let U_α and X be skew-adjoint operators in a Hilbert space $\mathcal{H}$. Suppose that U_α is of class $C^2(X)$ (i.e. $e^{-tX}U_\alpha e^{tX}$ is of class $C^2(X)$) and satisfies a Mourre Estimate on a bounded Borel set $S \subset \mathbb{R}$. Then U_α has no singular spectrum in S.*

The idea is to show that $[X, iU_\alpha]$ has a definite sign when localized in a neighborhood of λ, for $\lambda \in \sigma(iU_\alpha)$. Showing this positivity condition in the time-change case does not follow immediately from the natural commutator $[X, iU_\alpha]$. Instead the author in [28] relies on the following,

$$H_1 = -i\mathscr{L}_U,\ H_2 = -i\mathscr{L}_X$$

$$H = \alpha^{\frac{1}{2}}U\alpha^{\frac{1}{2}},\ H^2 = (\alpha^{\frac{1}{2}}U\alpha^{\frac{1}{2}})^2$$

and

$$[iH^2, H_2] = H^2 g + 2HgH + gH^2$$

for $g = -\frac{1}{2}G(\alpha)$.

$[iH^2, H_2]$ satisfies the Mourre Estimate under the added assumption that $g > 0$, equivalent to the Kushnirenko Condition [15]. Since the spectral properties of H can be derived from those of H^2, and since H is spectrally equivalent to U_α, the author concludes that U_α has purely absolutely continuous spectrum except on $\mathbb{C}^\perp$.

In a subsequent paper [29], the author modifies the commutator differently and replaces the Kushnirenko Condition by exploiting the unique ergodicity present in the compact case.

$$A_L = \frac{1}{L}\int_0^L e^{itH} H_2 e^{-itH}\, dt$$
$$g_L = \frac{1}{L}\int_0^L g \circ h_t^{U_\alpha}(x)\, dt$$

Using unique ergodicity, the author shows that

$$\lim_{L\to\infty} g_L = \frac{1}{2}$$

and thus, for large L,

$$g_L > 0.$$

He then proves the Mourre Estimate for $[H^2, A_L]$ on $S \in (0, \infty)$ with

$$a := 2\inf(S) \inf_{x\in M} g_L(x).$$

Again, since the spectral properties of H can be derived from those of H^2, and since H is spectrally equivalent to U_α, the author concludes that U_α has purely absolutely continuous spectrum except on $\mathbb{C}^\perp$.

1.5.2 Method 2. Applied to the Time-Changes of the Horocycle Flow (Compact Case) [8]

As mentioned previously, a direct approach to proving absolute continuity of the spectral measure involves deriving square mean bounds on the Fourier transform of the measure and requires that the decay of correlations of a general smooth function under the flow be square-integrable (this is not even satisfied in the classical horocycle case [25]). In the time-change case, this condition is not satisfied either; to circumvent this problem, the authors in [8] derive square integrable decay of correlations for smooth coboundaries.

Definition 2. *A function f on M is called a coboundary for the flow $\{h_t^{U_\alpha}\}$ if there exists a function g on M, called a transfer fuction, such that $U_\alpha g = f$.*

In contrast to the horocycle case, it is much more difficult to bound

$$\int_0^\sigma f \circ h_t^{U_\alpha} \circ \phi_s^X ds.$$

If we try to continue as in 1.4.2,

$$\int_0^\sigma f \circ h_t^{U_\alpha} \circ \phi_s^X ds = \int_0^\sigma U_\alpha g \circ h_t^{U_\alpha} \circ \phi_s^X ds = \frac{1}{t}\int_0^\sigma tU_\alpha g \circ h_t^{U_\alpha} \circ \phi_s^X ds,$$

but tU_α is not equivalent to the commutator $e^{-tU_\alpha}[X, e^{tU_\alpha}]$.

Instead the authors introduce the commutator in the following way,

$$\int_0^\sigma (U_\alpha - \frac{G(\alpha,t)}{t}U_\alpha + \frac{G(\alpha,t)}{t}U_\alpha) g \circ h_t^{U_\alpha} \circ \phi_s^X ds$$

$$= \int_0^\sigma (U_\alpha - \frac{G(\alpha,t)}{t}U_\alpha) g \circ h_t^{U_\alpha} \circ \phi_s^X ds + \int_0^\sigma \frac{G(\alpha,t)}{t}U_\alpha g \circ h_t^{U_\alpha} \circ \phi_s^X ds.$$

Bounds for the second integral follow from 1.4.2, however, bounding the first integral is difficult. The required bounds are ultimately achieved by deriving bounds on integrals along the push-forward of geodesic arcs - the technique described at the end of 1.4.2, followed by a bootstrap of the estimates. The authors also rely on the unique ergodicity of the flow in the compact setting as the uniform convergence of

$$\lim_{t\to\infty} \frac{G(\alpha, t)}{t} = -1$$

is very important. The reason for this becomes apparent in the following section.

In an effort to find general conditions applicable to parabolic flows, we take from Method 1. the idea of imposing conditions on the level of the generators and thus eliminate any reliance on geometric behavior and interactions. We express these conditions in terms of restrictions on the growth with respect to t of the relevant commutators in order to achieve the estimate in Method 2. To do this, we mimic the proof from [8] using general operators in order to identify exactly which terms must be controlled. An advantage to this method is that it provides a segue to determining maximal spectral type.

Chapter 2: General Results

2.1 Conditions

Let U be a skew-adjoint operator in a Hilbert space $\mathcal{H}$ with norm $\|\cdot\|_{\mathcal{H}}$. We define the spectral measure, μ_f by its Fourier Transform,

$$\hat{\mu}_f(t) = \langle e^{tU} f, f\rangle_{\mathcal{H}} = \int_{\mathbb{R}} e^{it\xi} d\mu_f(\xi)$$

for $f \in \mathcal{H}$.

In the discrete case we have

$$\hat{\mu}_f(n) = \langle e^{nU} f, f\rangle_{\mathcal{H}} = \int_{-\pi}^{\pi} e^{in\xi} d\mu_f(\xi)$$

We find conditions under which μ_f is absolutely continuous with respect to the Lebesgue measure.

Preliminary Assumptions

Suppose that for some operator X on $\mathcal{H}$, on a subspace $D \subset Dom(X)$ dense in $\mathcal{H}$, $e^{tU}(D) \subset D$, and the commutator

$$H(t) = e^{-tU}[X, e^{tU}]$$

is defined on D.

For $u \in D$, let

$$\frac{H(t)}{t^\beta} u \xrightarrow[t\to\infty]{\mathcal{H}} Hu.$$

Suppose also that $\{e^{sX}\}$ is a group of bounded operators for which

$$\sup_{s\in[0,\sigma]} \| e^{sX} \|_{op} < +\infty$$

and $\lim_{s\to 0} \frac{e^{sX}-I}{s} = X$ on D.

For B_1, B_2 bounded operators on $\mathcal{H}$ such that

$B_2 : D \to D$, let

$$\| < e^{tU} f, f >_{\mathcal{H}} \|_{L^2(\mathbb{R})} \leqslant \| \frac{1}{\sigma} \int_0^\sigma < e^{sX} e^{tU} f, B_1 e^{sX} B_2 f >_{\mathcal{H}} ds \ \|_{L^2(\mathbb{R})} .$$

Note: In the discrete case, instead of the continuous parameter t and norm $\| \cdot \|_{L^2(\mathbb{R}}$ we use the discrete parameter n and norm $\| \cdot \|_{\ell^2(\mathbb{Z})}$.

Theorem 3. *If for $\beta > \frac{1}{2}$, $H(t)$ and H satisfy:*

(i) $\frac{H(t)}{t^\beta} H^{-1} : D \to D$, *is defined on Ran$(H)$, extends to a bounded operator with uniformly bounded $\| \cdot \|_{op}$ norm in t, and satisfies on Ran(H),*

$$\limsup_{t\to\infty} \| I - \frac{H(t)H^{-1}}{t^\beta} \|_{op} < 1$$

(ii) $[X, \frac{H(t)}{t^\beta} H^{-1}]$ *is defined on Ran(H) and extends to a bounded operator with uniformly bounded $\| \cdot \|_{op}$ norm in t*

(iii) $[H(t), H]H^{-1}$ *is defined on Ran(H) and extends to a bounded operator with uniformly bounded $\| \cdot \|_{op}$ norm in t*

then for $f \in$ Ran$(H) \cap D$, μ_f is absolutely continuous.

Furthermore, if

(iv) $\overline{Ran(H) \cap D} = \mathcal{H}$,

then the spectrum of H *is purely absolutely continuous.*

Remark: It is never true in ergodic theory that $\overline{Ran(H) \cap D} = \mathcal{H}$. However, in many cases, $\overline{Ran(H) \cap D} = \mathcal{F}$ for $\mathcal{F}$ a subspace of $\mathcal{H}$; for example, $\mathcal{F} = L^2_0(M)$ the space of zero-average functions in $\mathcal{H} = L^2(M)$. While this doesn't give a result for purely absolutely continuous spectrum it implies the existence of an absolutely continuous component.

Proof. Let $f \in Ran(H) \cap D$.

$$\| \hat{\mu}_f(t) \|_{L^2(\mathbb{R})} = \| < e^{tU} f, f >_{\mathcal{H}} \|_{L^2(\mathbb{R})} \leqslant \| \frac{1}{\sigma} \int_0^\sigma < e^{sX} e^{tU} f, B_1 e^{sX} B_2 f >_{\mathcal{H}} \; ds \|_{L^2(\mathbb{R})} .$$

For $s \in [0, \sigma]$, we integrate by parts:

$$B_1 e^{sX} B_2 f \qquad\qquad \frac{d}{ds}(B_1 e^{sX} B_2 f) = B_1 e^{sX} \mathscr{L}_X(B_2 f)$$

$$e^{sX} e^{tU} f \qquad\qquad \int_0^\sigma e^{sX} e^{tU} f ds.$$

$$\frac{1}{\sigma} \int_0^\sigma < e^{sX} e^{tU} f, B_1 e^{sX} B_2 f >_{\mathcal{H}} \; ds$$

$$= \frac{1}{\sigma} < \int_0^\sigma e^{sX} e^{tU} f ds, B_1 e^{\sigma X} B_2 f >_{\mathcal{H}}$$

$$-\frac{1}{\sigma} \int_0^\sigma < \int_0^S e^{sX} e^{tU} f ds, B_1 e^{sX} \mathscr{L}_X(B_2 f) >_{\mathcal{H}} dS$$

From our assumptions, both $B_1 e^{sX} B_2 f$ and $B_1 e^{sX} \mathscr{L}_X(B_2 f)$ are bounded in $\mathcal{H}$. Thus, in order to show that $\hat{\mu}_f(t) = O(\frac{1}{t^\beta})$, we need a bound (in t) for

$$\| \int_0^\sigma e^{sX} e^{tU} f ds \|_{\mathcal{H}} .$$

Suppose that conditions (i), (ii), and (iii) hold, and let f be a coboundary of the form $f = Hg$,

for $g \in Dom(H) \cap D$.

$$\int_0^\sigma e^{sX} e^{tU} f ds = \int_0^\sigma e^{sX} e^{tU} Hg ds = \underbrace{\int_0^\sigma e^{sX} e^{tU} (H - \frac{H(t)}{t^\beta}) g ds}_{\text{I.}} + \underbrace{\int_0^\sigma e^{sX} e^{tU} \frac{H(t)}{t^\beta} g ds}_{\text{II.}}$$

I. Let

$$\tilde{H}(s,t) = e^{sX} e^{tU} (I - \frac{H(t) H^{-1}}{t^\beta}) e^{-tU} e^{-sX}.$$

(H may not be invertible in $\mathcal{H}$, however it is on coboundaries of the form $f = Hg$.)

It follows from the assumption $\limsup_{t \to \infty} \| I - \frac{H(t)H^{-1}}{t^\beta} \|_{op} < 1$, that for large t,

$$\| \tilde{H}(s,t) \|_{\mathcal{H}} < C_1 < 1.$$

Now we can rewrite

$$\int_0^\sigma e^{sX} e^{tU} (H - \frac{H(t)}{t^\beta}) g ds = \int_0^\sigma e^{sX} e^{tU} (I - \frac{H(t)H^{-1}}{t^\beta}) Hg ds = \int_0^\sigma \tilde{H}(s,t) e^{sX} e^{tU} f ds,$$

and integration by parts gives

$$\int_0^\sigma \tilde{H}(s,t) e^{sX} e^{tU} f ds = \tilde{H}(\sigma,t) \int_0^\sigma e^{sX} e^{tU} f ds - \int_0^\sigma \frac{\partial \tilde{H}(S,t)}{\partial S} [\int_0^S e^{sX} e^{tU} f ds] dS.$$

So we must bound $\frac{\partial \tilde{H}(s,t)}{\partial s}$:

$$\frac{\partial \tilde{H}(s,t)}{\partial s} = \frac{\partial}{\partial s}[\frac{1}{t^\beta} e^{sX} e^{tU} H(t) H^{-1} e^{-tU} e^{-sX}]$$

$$= \frac{1}{t^\beta}[e^{sX} X e^{tU} H(t) H^{-1} e^{-tU} e^{-sX} - e^{sX} e^{tU} H(t) H^{-1} e^{-tU} X e^{-sX}]$$

Since e^{sX} and e^{-sX} are bounded, we can factor e^{sX} from the left and e^{-sX} from the right and now consider bounding the term

$$\frac{1}{t^\beta}([X, e^{tU} H(t) H^{-1} e^{-tU}])$$

$$= \frac{1}{t^\beta}([X, e^{tU}] H(t) H^{-1} e^{-tU} + e^{tU}[X, H(t)H^{-1}] e^{-tU} + e^{tU} H(t) H^{-1}[X, e^{-tU}]$$

Using the identity $e^{-tU}[X, e^{tU}] = -[X, e^{-tU}]e^{tU}$ we can simplify and combine terms:

$$\frac{1}{t^\beta} e^{tU}(H(t)^2 H^{-1} + [X, H(t)H^{-1}] - H(t) H^{-1} H(t)) e^{-tU}$$

$$= e^{tU}([X, \frac{H(t)H^{-1}}{t^\beta}] - \frac{H(t)H^{-1}}{t^\beta}[H(t), H] H^{-1}) e^{-tU}$$

Conditions (i), (ii), and (iii) imply

$$\| \frac{\partial \tilde{H}(s,t)}{\partial s} \|_{op} \leqslant C(\| \frac{H(t)}{t^\beta} H^{-1} \|_{op} \| [H(t), H] H^{-1} \|_{op} + \| [X, \frac{H(t)}{t^\beta} H^{-1}]) \|_{op} \leqslant C_2$$

for some constants C and C_2.

II. For $g \in Dom(H) \cap D$,

$$\frac{1}{t^\beta}\int_0^\sigma e^{sX}e^{tU}H(t)gds = \frac{1}{t^\beta}\int_0^\sigma e^{sX}e^{tU}Xgds - \frac{1}{t^\beta}\int_0^\sigma \frac{d}{ds}e^{sX}e^{tU}gds$$

$$\Longrightarrow \| \frac{1}{t^\beta}\int_0^\sigma e^{sX}e^{tU}H(t)gds \|_{\mathcal{H}} \leqslant \frac{C_3}{t^\beta} \| Xg \|_{\mathcal{H}} + \frac{C_4}{t^\beta} \| g \|_{\mathcal{H}} .$$

Finally, from I. and II.,

$$\sup_{s\in[0,\sigma]} \| \int_0^s e^{sX}e^{tU}fds \|_{\mathcal{H}}$$

$$\leqslant \sup_{s\in[0,\sigma]} (\| \tilde{H}(s,t) \|_{op} \cdot \| \int_0^s e^{sX}e^{tU}fds \|_{\mathcal{H}})$$

$$+ \sigma \cdot \sup_{s\in[0,\sigma]} (\| \frac{\partial \tilde{H}(s,t)}{\partial s} \|_{op} \cdot \| \int_0^s e^{sX}e^{tU}fds \|_{\mathcal{H}}) + \frac{C_3 \| Xg \|_{\mathcal{H}} + C_4 \| g \|_{\mathcal{H}}}{t^\beta}.$$

So for $\sigma > 0$, chosen such that $0 < C_1 + \sigma C_2 < 1$, for all t sufficiently large,

$$\sup_{s\in[0,\sigma]} \| \int_0^s e^{sX}e^{tU}fds \|_{\mathcal{H}} \leqslant \frac{C_3 \| Xg \|_{\mathcal{H}} + C_4 \| g \|_{\mathcal{H}}}{t^\beta} \frac{1}{(1 - C_1 - \sigma C_2)} = O(\frac{1}{t^\beta}).$$

Thus, since $\hat{\mu}_f(t) \in L^2(\mathbb{R})$, μ_f is absolutely continuous for $f \in \overline{Ran(H) \cap D}$. Furthermore, if $Ran(H) \cap D$ is dense, then μ_f is absolutely continuous for a dense subspace of functions in $\mathcal{H}$, and thus, the spectrum of U is purely absolutely continuous.

Note: In the discrete case, the conclusion is that

$$\| < e^{nU}f, f >_{\mathcal{H}} \|_{\ell^2(\mathbb{Z})} = O(\frac{1}{n^\beta})$$

for $\beta > \frac{1}{2}$, *and thus,* $\mu_f(n) \in \ell^2(\mathbb{Z})$.

□

Chapter 3: Applications to Flows

3.1 Time-Changes of Unipotent Flows on Homogeneous Spaces of Semisimple Groups

As a direct consequence of *Theorem* 1, we derive a result for a specific category of generating operators.

Let G be a semisimple Lie group and let the manifold $M = \Gamma\backslash G$ for some lattice Γ in G such that M has finite area.

By the Jacobson-Morozov Theorem, any nilpotent element U of the semisimple Lie algebra of G is contained in a subalgebra isomorphic to $\mathfrak{sl}_2$. This means that this subalgebra contains an element X, such that $[U, X] = U$. Let e^{tU} be a unitary operator of the Hilbert space $L^2(M, vol)$. Thus, if the unipotent flow generated by U, $f \circ \phi_t^U = e^{tU} f$, $f \in L^2(M, vol)$, is ergodic, then from Lemma 5.1 in [19], it has purely absolutely continuous spectrum on

$$L_0^2(M, vol) = \{f \in L^2(M) \mid \int_M f \, vol = 0\}.$$

Let $\tau : M \times \mathbb{R} \to \mathbb{R}$ such that

$$\tau(x, t + t') = \tau(x, t) + \tau(\phi_t^U(x), t').$$

Let $\alpha : M \to \mathbb{R}^+$, be the infinitesimal generator of τ, such that $\alpha \in C^\infty(M)$ and

$$\int_M \alpha vol = \int_M vol_\alpha = 1$$

where vol is the ϕ_t^U-invariant volume form and vol_α is the $\phi_t^{U_\alpha}$-invariant volume form. Now we consider a time-changed flow, $\{\phi_t^{U_\alpha}\}$, generated by

$$U_\alpha =: U/\alpha.$$

Let e^{tU_α} be a unitary operator on the Hilbert space $L^2(M, vol_\alpha)$, and let $D = C^\infty(M)$.

$$[X, U_\alpha] = G(\alpha)U_\alpha = (\frac{X\alpha}{\alpha} - 1)U_\alpha = H$$

$$e^{-tU_\alpha}[X, e^{tU_\alpha}] = G(\alpha, t)U_\alpha = \boxed{(\int_0^t (\frac{X\alpha}{\alpha} - 1) \circ \phi_\tau^{U_\alpha}(x) d\tau) U_\alpha = H(t).}$$

The ergodicity of $\phi_t^{U_\alpha}$ gives us the following pointwise limit,

$$\lim_{t\to\infty} \frac{G(\alpha, t)}{t} = \lim_{t\to\infty} \frac{1}{t} \int_0^t (\frac{X\alpha}{\alpha} - 1) \circ \phi_\tau^{U_\alpha}(x) d\tau$$

$$= \lim_{t\to\infty} (\frac{1}{t} \int_0^t \frac{X\alpha}{\alpha} \circ \phi_\tau^{U_\alpha}(x) d\tau + \frac{1}{t} \int_0^t -1 \circ \phi_\tau^{U_\alpha}(x) d\tau)$$

$$= \lim_{t\to\infty} (\frac{1}{t} \int_0^t \frac{X\alpha}{\alpha} \circ \phi_\tau^{U_\alpha}(x) d\tau - 1)$$

$$= \int_M \frac{X\alpha}{\alpha} dvol_\alpha - 1 = -1$$

and thus, for $u \in C^\infty(M)$,

$$\lim_{t\to\infty} \frac{G(\alpha, t)}{t} U_\alpha u = \boxed{-U_\alpha u = Hu.}$$

Lastly,

$$\int_M e^{tU_\alpha} f \ \cdot \overline{f} \, dvol_\alpha = \int_M e^{tU_\alpha} f \ \cdot \overline{\alpha f} \, dvol$$

$$= \int_M e^{sX} e^{tU_\alpha} f \ \cdot \overline{e^{sX} \alpha f} \, dvol = \int_M e^{sX} e^{tU_\alpha} f \ \cdot \overline{\frac{1}{\alpha} e^{sX} \alpha f} \, dvol_\alpha.$$

So if we integrate both sides of

$$< e^{tU_\alpha} f, f >_{L^2(M, vol_\alpha)} = < e^{sX} e^{tU_\alpha} f, \frac{1}{\alpha} e^{sX} \alpha f >_{L^2(M, vol_\alpha)}$$

with respect to s, we obtain the following equality

$$\| < e^{tU_\alpha} f, f >_{L^2(M, vol_\alpha)} \|_{L^2(\mathbb{R}, dt)} = \| \frac{1}{\sigma} \int_0^\sigma < e^{sX} e^{tU_\alpha} f, \frac{1}{\alpha} e^{sX} \alpha f >_{L^2(M, vol_\alpha)} ds \|_{L^2(\mathbb{R}, dt)} .$$

Thus, the preliminary assumptions are satisfied with $B_1 = \frac{1}{\alpha} I$ and $B_2 = \alpha I$.

Theorem 4. ***a.*** *Any smooth time-change of an ergodic flow on M generated by a non-central nilpotent element of a semisimple Lie algebra has absolutely continuous spectrum on $L^2_0(M, vol_\alpha)$ if $\| \frac{X\alpha}{\alpha} \|_\infty < 1$.*

b. *Any smooth time-change of a uniquely ergodic flow on M generated by a non-central nilpotent element of a semisimple Lie algebra has absolutely continuous spectrum on $L^2_0(M, vol_\alpha)$.*

Proof. *(a)*

(i) Let $f = U_\alpha g$ for $g \in C^\infty(M)$.

$$\| \frac{H(t)}{t} H^{-1} f \|_{L^2(M, vol_\alpha)} = \| \frac{G(\alpha, t)}{t} U_\alpha(-U_\alpha^{-1} f) \|_{L^2(M, vol_\alpha)}$$

$$= \| \frac{G(\alpha, t)}{t} f \|_{L^2(M, vol_\alpha)} \leqslant 2 \| G(\alpha) \|_\infty \cdot \| f \|_{L^2(M, vol_\alpha)} \leqslant 2 \| f \|_{L^2(M, vol_\alpha)}$$

Since the above holds for $f \in Ran(U_\alpha)$, $\frac{H(t)}{t}H^{-1}$ extends to a bounded operator on $\overline{Ran(U_\alpha)} = L^2_0(M, vol_\alpha)$ with uniformly bounded norm in t,

$$\| \frac{H(t)}{t}H^{-1} \|_{op} \leqslant 2.$$

Also,

$$\| (I - \frac{H(t)}{t}H^{-1})f \|_{L^2(M,vol_\alpha)} = \| (1 + \frac{G(\alpha,t)}{t})f \|_{L^2(M,vol_\alpha)}$$

$$= \| (1 + (\frac{1}{t}\int_0^t (\frac{X\alpha}{\alpha} - 1) \circ \phi_\tau^{U_\alpha}(x)d\tau))f \|_{L^2(M,vol_\alpha)}$$

$$= \| (\frac{1}{t}\int_0^t \frac{X\alpha}{\alpha} \circ \phi_\tau^{U_\alpha}(x)d\tau)f \|_{L^2(M,vol_\alpha)} \leqslant \| (\frac{1}{t}\int_0^t \frac{X\alpha}{\alpha} \circ \phi_\tau^{U_\alpha}(x)d\tau) \|_\infty \cdot \| f \|_{L^2(M,vol_\alpha)}$$

$$\leqslant \| \frac{X\alpha}{\alpha} \|_\infty \cdot \| f \|_{L^2(M,vol_\alpha)} \leqslant \| f \|_{L^2(M,vol_\alpha)} .$$

Since the above holds on $Ran(U_\alpha)$ the following is true on $\overline{Ran(U_\alpha)} = L^2_0(M, vol_\alpha)$,

$$\limsup_{t\to\infty} \| I - \frac{H(t)}{t}H^{-1}I \|_{op} = \limsup_{t\to\infty} \| I + \frac{G(\alpha,t)}{t}I \|_{op} < 1.$$

(ii) In the following calculation we use that

$$Dh_t^{U_\alpha}(X) = G(\alpha,t)U_\alpha \circ h_t^{U_\alpha} + X \circ h_t^{U_\alpha}.$$

(Please see 6.3 for this computation.)

$$[X, \frac{H(t)}{t}H^{-1}] = X(\frac{1}{t}\int_0^t (\frac{X\alpha}{\alpha} - 1) \circ \phi_\tau^{U_\alpha})d\tau) - (\frac{1}{t}\int_0^t (\frac{X\alpha}{\alpha} - 1) \circ \phi_\tau^{U_\alpha})d\tau)X$$

$$= \frac{1}{t}\int_0^t (D\phi_\tau^{U_\alpha}(X) \circ \phi_{-\tau}^{U_\alpha})(\frac{X\alpha}{\alpha}) \circ \phi_\tau^{U_\alpha} d\tau = \frac{1}{t}\int_0^t (G(\alpha,t)U_\alpha + X)(\frac{X\alpha}{\alpha}) \circ \phi_\tau^{U_\alpha} d\tau$$

$$= \frac{1}{t}\int_0^t G(\alpha,t)U_\alpha(\frac{X\alpha}{\alpha}) \circ \phi_\tau^{U_\alpha}(x)d\tau + \frac{1}{t}\int_0^t X(\frac{X\alpha}{\alpha}) \circ \phi_\tau^{U_\alpha}(x)d\tau$$

$$= \frac{1}{t}\int_0^t G(\alpha,t)\frac{d}{d\tau}(\frac{X\alpha}{\alpha})\circ\phi_\tau^{U_\alpha}(x)d\tau + \frac{1}{t}\int_0^t X(\frac{X\alpha}{\alpha})\circ\phi_\tau^{U_\alpha}(x)d\tau$$

We integrate

$$\frac{1}{t}\int_0^t G(\alpha,t)\frac{d}{d\tau}(\frac{X\alpha}{\alpha})\circ\phi_\tau^{U_\alpha}(x)d\tau$$

by parts,

$$\frac{1}{t}\int_0^t G(\alpha,t)\frac{d}{d\tau}(\frac{X\alpha}{\alpha})\circ\phi_\tau^{U_\alpha}(x)d\tau$$

$$= \frac{G(\alpha,t)}{t}(\frac{X\alpha}{\alpha})\circ\phi_t^{U_\alpha} - \frac{1}{t}\int_0^t(\frac{X\alpha}{\alpha}-1)(\frac{X\alpha}{\alpha})\circ\phi_\tau^{U_\alpha}(x)d\tau,$$

and obtain the bound,

$$\| \frac{1}{t}\int_0^t(G(\alpha,t)U_\alpha + X)(\frac{X\alpha}{\alpha})\circ\phi_\tau^{U_\alpha}d\tau \|_\infty$$

$$\leqslant \| \frac{G(\alpha,t)}{t}(\frac{X\alpha}{\alpha}) \|_\infty + \| \frac{1}{t}\int_0^t(\frac{X\alpha}{\alpha}-1)(\frac{X\alpha}{\alpha})\circ\phi_\tau^{U_\alpha}(x)d\tau \|_\infty$$

$$+ \| \frac{1}{t}\int_0^t X(\frac{X\alpha}{\alpha})\circ\phi_\tau^{U_\alpha}(x)d\tau \|_\infty$$

$$\leqslant 2\cdot \| \frac{X\alpha}{\alpha} - 1 \|_\infty \cdot \| (\frac{X\alpha}{\alpha}) \|_\infty + \| X(\frac{X\alpha}{\alpha}) \|_\infty$$

$$\leqslant 2(2) + C_\alpha$$

where C_α depends on the second derivative of α.

Since

$$[X, \frac{H(t)}{t}H^{-1}]$$

is the multiplication operator given by

$$(\frac{1}{t}\int_0^t(G(\alpha,t)U_\alpha + X)(\frac{X\alpha}{\alpha})\circ\phi_\tau^{U_\alpha}d\tau)\cdot I,$$

we obtain the following bound,

$$\| [X, \frac{H(t)}{t}H^{-1}]f \|_{L^2(M,vol_\alpha)} \leqslant \| \frac{1}{t}\int_0^t (G(\alpha,t)U_\alpha + X)(\frac{X\alpha}{\alpha}) \circ \phi_\tau^{U_\alpha} d\tau \|_\infty \cdot \| f \|_{L^2(M,vol_\alpha)}$$

$$\leqslant (4 + C_\alpha) \cdot \| f \|_{L^2(M,vol_\alpha)} .$$

Thus, $[X, \frac{H(t)}{t}H^{-1}]$ extends to a bounded operator on $\overline{Ran(U_\alpha)}$ with operator norm uniformly bounded in t:

$$\| [X, \frac{H(t)}{t}H^{-1}] \|_{op} \leqslant 4 + C_\alpha$$

(iii)

$$\| [H(t), H]H^{-1} f \|_{L^2(M,vol_\alpha)} = \| [G(\alpha,t)U_\alpha, -U_\alpha](-U_\alpha^{-1} f) \|_{L^2(M,vol_\alpha)} X$$

$$\leqslant 2 \| G(\alpha) \|_\infty \cdot \| f \|_{L^2(M,vol_\alpha)} \leqslant 2(2) \cdot \| f \|_{L^2(M,vol_\alpha)}$$

$$= \| [U_\alpha(\int_0^t (\frac{X\alpha}{\alpha} - 1) \circ \phi_\tau^{U_\alpha} d\tau)] \cdot f \|_{L^2(M,vol_\alpha)} = \| [\int_0^t \frac{d}{d\tau}(\frac{X\alpha}{\alpha} - 1) \circ \phi_\tau^{U_\alpha} d\tau)] \cdot f \|_{L^2(M,vol_\alpha)}$$

$$\leqslant \| G(\alpha) \circ \phi_t^{U_\alpha} - G(\alpha) \|_\infty \cdot \| f \|_{L^2(M,vol_\alpha)}$$

$$\leqslant 2 \| G(\alpha) \|_\infty \cdot \| f \|_{L^2(M,vol_\alpha)} \leqslant 2(2) \cdot \| f \|_{L^2(M,vol_\alpha)}$$

The above holds on coboundaries of the form $f = U_\alpha g$, so on $\overline{Ran(U_\alpha)} = L_0^2(M, vol_\alpha)$,

$$\| [H(t), H]H^{-1} \|_{op} < 4.$$

Since conditions $(i) - (iii)$ of Theorem 3. are satisfied on $Ran(U_\alpha)$, the time-changed flow, $\{\phi_t^{U_\alpha}\}$, has purely absolutely continuous spectrum on $\overline{Ran(U_\alpha)} = L_0^2(M, vol_\alpha)$.

This concludes the proof of part **a**.

b. Now we assume that the flow $\{\phi_t^U\}$, and hence $\{\phi_t^{U_\alpha}\}$, are uniquely ergodic.

$$\| (I - \frac{H(t)}{t} H^{-1}) f \|_{L^2(M,vol_\alpha)} = \| (1 + \frac{G(\alpha,t)}{t}) f \|_{L^2(M,vol_\alpha)}$$

$$= \| (1 + (\frac{1}{t} \int_0^t (\frac{X\alpha}{\alpha} - 1) \circ \phi_\tau^{U_\alpha}(x) d\tau)) f \|_{L^2(M,vol_\alpha)}$$

$$= \| (\frac{1}{t} \int_0^t \frac{X\alpha}{\alpha} \circ \phi_\tau^{U_\alpha}(x) d\tau) f \|_{L^2(M,vol_\alpha)} \leqslant \| (\frac{1}{t} \int_0^t \frac{X\alpha}{\alpha} \circ \phi_\tau^{U_\alpha}(x) d\tau) \|_\infty \cdot \| f \|_{L^2(M,vol_\alpha)} .$$

If $\{\phi_t^{U_\alpha}\}$ is uniquely ergodic, then the following converges uniformly,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{X\alpha}{\alpha} \circ \phi_\tau^{U_\alpha} d\tau = \int_M \frac{X\alpha}{\alpha} \, dvol_\alpha = 0,$$

and thus,

$$\limsup_{t\to\infty} \| I + \frac{H(t)}{t} H^{-1} \|_{op} \leqslant \limsup_{t\to\infty} \| (\frac{1}{t} \int_0^t \frac{X\alpha}{\alpha} \circ \phi_\tau^{U_\alpha}(x) d\tau) \|_\infty$$
$$= \| \int_M \frac{X\alpha}{\alpha} \, dvol_\alpha \|_\infty = 0.$$

Hence,

$$\limsup_{t\to\infty} \| I + \frac{H(t)}{t} H^{-1} \|_{op} < 1$$

is satisfied on $\overline{Ran(U_\alpha)} = L_0^2(M, vol_\alpha)$ without imposing any further conditions on $\frac{X\alpha}{\alpha}$. The remainder of the proof is the same as in **a** except that

$$\| \frac{X\alpha}{\alpha} \|_\infty = M$$

where M is finite but not necessary equal to 1.

□

Theorem 5 (Maximal Spectral Type)**.** *The maximal spectral type of the uniqely ergodic flow* $\{\phi_t^{U_\alpha}\}$ *is Lebesgue on the subspace* $\overline{Ran(U_\alpha)}$.

Proof. We follow the method in [8].

Lemma 1. *[8] Suppose that the maximal spectral type of* $\{\phi_t^{U_\alpha}\}$ *is not Lebesgue. Then there exists a smooth non-zero function* $\omega \in L^2(\mathbb{R}, dt)$ *such that for all functions* $g \in C^\infty(M)$ *the following holds:*

$$\int_{\mathbb{R}} \omega(t) \int_0^\sigma e^{sX} e^{tU_\alpha} U_\alpha g ds dt = 0$$

Proof. Since the maximal spectral type is not Lebesgue, then there exists a compact set $A \subset \mathbb{R}$ such that A has positive Lebesgue measure but measure 0 with respect to the maximal spectral type. So we let $\omega \in L^2(\mathbb{R})$ be the complex conjugate of the Fourier transform of the characteristic function χ_A of the set $A \subset \mathbb{R}$. For $f, h \in RanU_\alpha$, let $\mu_{f,h}$ denote the joint spectral measure (which we know is absolutely continuous with respect to Lebesgue since $f, h \in RanU_\alpha$. Thus,

$$\int_{\mathbb{R}} \omega(t) < e^{tU_\alpha} f, h >_{L^2(M,vol)} dt = \int_{\mathbb{R}} \chi_A(\xi) d\mu_{f,h}(\xi) = 0.$$

In particular, when $f = U_\alpha g$ we have

$$\int_0^\sigma \int_{\mathbb{R}} \omega(t) < e^{sX} e^{tU_\alpha} U_\alpha g, h >_{L^2(M,vol)} \; dt \; ds = 0$$

$$=< \int_{\mathbb{R}} \omega(t) \int_0^\sigma e^{sX} e^{tU_\alpha} U_\alpha g \; ds \; dt, h >_{L^2(M,vol)} = 0.$$

Recall that satisfying conditions (i) and (ii) and (iii) in Theorem 3. results in the bound

$$\sup_{s\in[0,\sigma]} \| \int_0^{\sigma} e^{sX}e^{tU_\alpha}U_\alpha g ds \|_{L^2(\mathbb{R},dt)} \leqslant \frac{C_\sigma(\alpha)}{t^\beta} \max\{\| g \|_{L^2(M)}, \| Xg \|_{L^2(M)}, \| U_\alpha g \|_{L^2(M)}\}$$

where $\beta = 1$ and $C_\sigma(\alpha)$ is a constant that depends on the time-change function α and parameter $\sigma > 0$.

Because

$$\int_0^{\sigma} e^{sX}e^{tU_\alpha}U_\alpha g\, ds$$

is bounded on M, it follows that

$$\int_{\mathbb{R}} \omega(t) \int_0^{\sigma} e^{sX}e^{tU_\alpha}U_\alpha g\, ds\, dt$$

vanishes.

□

Lemma 2. *[8] For $\omega \in L^2(\mathbb{R}, dt)$, for some $x \in M$, and for all $g \in C^\infty(M)$,*

$$\int_{\mathbb{R}} \omega(t) \int_0^{\sigma} e^{sX}e^{tU_\alpha}U_\alpha g ds dt = 0.$$

Thus, ω vanishes identically.

Proof. Fix $x \in M$ and $\sigma > 0$. For any $T > 0$, $\rho > 0$, and $\frac{1}{2} > \gamma > 0$, let $E^T_{\rho,\sigma}$ be the flow-box for the the flow $\{\phi^{U_\alpha}_t\}$ defined as follows:

$$E^T_{\rho,\sigma} = (\phi^{U_\alpha}_t \circ \phi^X_s \circ \phi^V_r)(x), \text{ for all } (r, s, t) \in (-\gamma, \gamma) \times (-\rho, \rho) \times (-\sigma, \sigma).$$

For any $\chi \in C^\infty_0(-1, 1)$ and any $\psi \in C^\infty_0(-T, T)$, let

$$\tilde{g}(r,s,t) := \chi(\frac{r}{\rho})\chi(\frac{s}{\sigma})\psi(t).$$

Let $g \in C^\infty(M)$ such that $g = 0$ on $M \backslash Im(E^T_{\rho,\sigma})$ and

$$g \circ E^T_{\rho,\sigma} = \begin{cases} 0 & \text{on } M\backslash Im(E^T_{\rho,\sigma}) \\ \tilde{g}(r,s,t) & \text{on } Im(E^T_{\rho,\sigma}) \end{cases}$$

Let $T_{\rho,\sigma} > 0$ be defined as:

$$T_{\rho,\sigma} := \min\{|t| > T : \cup_{s\in[-\sigma,\sigma]}(\phi^{U_\alpha}_t \circ \phi^X_s)(x) \cap Im(E^T_{\rho,\sigma}) \neq \emptyset\}$$

From unique ergodicity,

$$lim_{\rho\to 0^+} T_{\rho,\sigma} = +\infty.$$

The composition of the flow box with $U_\alpha g$ and Xg follow from the commutation relations:

$$(U_\alpha g) \circ E^T_{\rho,\sigma} := \chi(\frac{r}{\rho})\chi(\frac{s}{\sigma})\frac{d\psi(t)}{dt}(t)$$

and

$$(Xg) \circ E^T_{\rho,\sigma}$$

$$:= \frac{1}{\sigma}\chi(\frac{r}{\rho})\frac{d\chi}{ds}(\frac{s}{\sigma})\psi(t) - (\int_0^t (\frac{X\alpha}{\alpha} - 1) \circ \phi^{U_\alpha}_\tau \circ \phi^X_s \circ \phi^V_r(x)d\tau)\chi(\frac{r}{\rho})\chi(\frac{s}{\sigma})\frac{d\psi}{dt}(t). \quad (3.1)$$

From the assumptions of Lemma 1 and by integrating (3.1), we have

$$\chi(0)\,(\int_0^\sigma \chi(\frac{s}{\sigma})\,ds)(\int_{-T}^T \omega(t)\frac{d\psi(t)}{dt}\,dt)$$

$$+\int_{\mathbb{R}\setminus[-T_{\rho,\sigma},T_{\rho,\sigma}]}\omega(t)\int_0^\sigma e^{sX}e^{tU_\alpha}(U_\alpha g)\,ds\,dt=0. \tag{3.2}$$

The bound $C_\sigma(\alpha)$ of

$$\int_0^\sigma e^{sX}e^{tU_\alpha}U_\alpha g ds$$

derived for the spectral results, combined with (3.1) and (3.2), give us the following L^2 bound,

$$\| \int_0^\sigma e^{sX}e^{tU_\alpha}U_\alpha g ds \|_{L^2(\mathbb{R},dt)}\leqslant \frac{C_\sigma(\alpha)}{t}\max\{\| g \|_\infty, \| Xg \|_\infty, \| U_\alpha g \|_\infty\}$$

$$\leqslant \frac{C_\sigma(\alpha)}{t}\max\{1,T\}\times \max{}^2\{\| \chi \|_{L^\infty(\mathbb{R})}, \| \chi' \|_{L^\infty(\mathbb{R})}, \| \psi \|_{L^\infty(\mathbb{R})}, \| \psi' \|_{L^\infty(\mathbb{R})}\}$$

Since the above bound is uniform with respect to ρ, we can conclude that the following limit holds,

$$\lim_{\rho\to 0^+}\int_{\mathbb{R}\setminus[-T_{\rho,\sigma},T_{\rho,\sigma}]}\omega(t)\int_0^\sigma e^{sX}e^{tU_\alpha}(U_\alpha g)dsdt=0. \tag{3.3}$$

Combining equation (3.2) with the limit result in (3.3) implies that

$$\int_{\mathbb{R}}\omega(t)\frac{d\psi(t)}{dt}\,dt=0$$

and thus, $\omega\equiv 0$. □

□

3.2 Time Changes of the Horocycle Flow - Compact and Finite Area

From the description in 1.5, it follows that time-changes of the horocycle flow are special cases of Theorem 4, and thus, Theorem 5. When M is of finite volume, $\{h_t^{U_\alpha}\}$ is ergodic, and when M is compact, $\{h_t^{U_\alpha}\}$ is uniquely ergodic. We state the spectral results in the following Corollary.

Corollary 1. ***a.*** *Any smooth time-change* $\{h_t^{U_\alpha}\}$ *of the horocycle flow on* M *(finite volume) has absolutely continuous spectrum on* $L_0^2(M, vol_\alpha)$ *if* $\| \frac{X\alpha}{\alpha} \|_\infty < 1$.

b. *Any smooth time-change* $\{h_t^{U_\alpha}\}$ *of the horocycle flow on* M *(compact) has Lebesgue spectrum on* $L_0^2(M, vol_\alpha)$.

3.3 Twisted Horocycle Flows

We would like to examine the conditions under which the spectral properties persist or do not persist after we combine the horocycle time-change with a circle rotation. Our new space is $\hat{M} = (\Gamma \backslash PSL(2,\mathbb{R})) \times S^1$ for Γ a cocompact lattice. We define the following operators:

$\hat{X} = (X, 0)$ where X is the generator of the geodesic flow.

$\hat{V} = (V, 0)$ where V is the generator of the negative horocycle flow.

$\hat{\frac{d}{d\theta}} = (0, \frac{d}{d\theta})$ where $\frac{d}{d\theta}$ is a rotation on S^1.

$W = (U, 0) + (0, \alpha\frac{d}{d\theta})$ where U is the generator of the positive horocycle flow and $\alpha = \alpha(x)$, $x \in \Gamma \backslash PSL(2,\mathbb{R})$, is the time change function as in 1.5.

Proposition 1. *The flow $\{\phi_t^W\}$ is uniquely ergodic.*

Proof. Consider the time-change $\{\phi_t^{W_\alpha}\} = \frac{1}{\alpha}W = \hat{U}_\alpha \times \hat{\frac{d}{d\theta}}$. Since $\{h_t^{U_\alpha}\}$ is mixing [17], then it is weakly mixing, and thus $\{\phi_t^{W_\alpha}\}$ is ergodic [5]. This implies the ergodicity of $\{\phi_t^W\}$. Since $\{\phi_t^W\}$ is ergodic and $\{h_t^U\}$ is uniquely ergodic [10], then from [9] (applied to flows), $\{\phi_t^W\}$ is uniquely ergodic. $\square$

We are interested in the spectrum of the flow $\{\phi_t^W\}$, so we compute the commutation relations with $\hat{X}$, (for details of the calculation, please see 6.4).

$$e^{-tW}[\hat{X}, e^{tW}] = \boxed{tW + (\int_0^t (\frac{\hat{X}\alpha}{\alpha} - 1) \circ \phi_\tau^W(x)\, d\tau)\frac{\hat{d}}{d\theta} = H(t)}$$

For $u \in C^\infty(\hat{M})$,

$$\lim_{t\to\infty}\frac{H(t)}{t}u = \boxed{(W - \hat{\frac{d}{d\theta}})\, u = Hu}$$

Since,

$$\| \ \langle e^{tW}f, f\rangle \ \|_{L^2(\mathbb{R},dt)} = \| \ \frac{1}{\sigma}\int_0^\sigma \langle e^{s\hat{X}}e^{tW}f, e^{s\hat{X}}f\rangle \, ds \, \|_{L^2(\mathbb{R},dt)},$$

the preliminary assumptions are satisfied with $B_1 = B_2 = I$ and $D = C^\infty(\hat{M})$.

However, when we proceed with verifying the conditions for the functions in the range of H, we are unable to extend pointwise bounds in L^2 to uniform bounds in the operator norm. For example,

$$\begin{aligned}
\frac{H(t)}{t}H^{-1} &= (W + (\frac{1}{t}\int_0^t (\frac{\hat{X}\alpha}{\alpha} - 1)\circ \phi_\tau^W(x)\, d\tau)\frac{\hat{d}}{d\theta})H^{-1} \\
&= (H + \frac{\hat{d}}{d\theta} + (\frac{1}{t}\int_0^t (\frac{\hat{X}\alpha}{\alpha} - 1)\circ \phi_\tau^W(x)\, d\tau)\frac{\hat{d}}{d\theta})H^{-1} \\
&= (H + (\frac{1}{t}\int_0^t \frac{\hat{X}\alpha}{\alpha}\circ \phi_\tau^W(x)\, d\tau)\frac{\hat{d}}{d\theta})H^{-1} \\
&= I + (\frac{1}{t}\int_0^t \frac{\hat{X}\alpha}{\alpha}\circ \phi_\tau^W(x)\, d\tau)\frac{\hat{d}}{d\theta}H^{-1}
\end{aligned}$$

Since $\frac{\hat{d}}{d\theta}$ commutes with everything, we can restrict to the subspace

$$\mathbb{E}_n = \{\frac{\hat{d}}{d\theta}u = inu\}$$

since it is invariant under all of the operators.

For $f = Hg$, $f, g \in C^\infty(M)$,

$$\| \ (I + (\frac{1}{t}\int_0^t \frac{\hat{X}\alpha}{\alpha}\circ \phi_\tau^W(x)\, d\tau) inH^{-1})g \ \|_{L^2(\hat{M})}$$

$$=\| g + (\frac{1}{t}\int_0^t \frac{\hat{X}\alpha}{\alpha} \circ \phi_\tau^W(x)\, d\tau) inH^{-1}g \|_{L^2(\hat{M})}$$

$$=\| g + (\frac{1}{t}\int_0^t \frac{\hat{X}\alpha}{\alpha} \circ \phi_\tau^W(x)\, d\tau) inf \|_{L^2(\hat{M})}$$

$$\leqslant \| g \|_{L^2(\hat{M})} + \| (\frac{1}{t}\int_0^t \frac{\hat{X}\alpha}{\alpha} \circ \phi_\tau^W(x)\, d\tau) inf \|_{L^2(\hat{M})}$$

$$\leqslant \| g \|_{L^2(\hat{M})} + nC_\alpha \cdot \| f \|_{L^2(\hat{M})}$$

since in the compact setting

$$C_\alpha = \| \frac{\hat{X}\alpha}{\alpha} \|_\infty < \infty.$$

Because we have an L^2 bound in terms of both $\| g \|_{L^2(\hat{M})}$ and $\| f \|_{L^2(\hat{M})} = \| Hg \|_{L^2(\hat{M})}$, we are unable to extend this to a bound in the operator norm.

Instead we modify our operators by introducing an operator P, defined in such a way that it not only acts as a projection operator but also preserves regularity.

Let $\chi \in C_0^\infty(\mathbb{R}\backslash\{0\})$ such that the support of χ is compact subset of the spectrum of H away from 0. For $f, g \in L^2(\hat{M})$,

$$\langle Pf, g\rangle_{L^2(\hat{M})} = \int_{\mathbb{R}} \chi(x) d\mu_{f,g}(x)$$

$$= \int_{\mathbb{R}} \hat{\chi}(t)\hat{\mu_{f,g}}(t)\, dt = \int_{\mathbb{R}} \hat{\chi}(t)\langle e^{tH} f, g\rangle_{L^2(\hat{M})}\, dt$$

since H is a vector field, and thus,

$$Pf = \int_{\mathbb{R}} \hat{\chi}(t) e^{tH} f\, dt.$$

The decay of $e^{tH}f = f \circ \phi_t^H$ is at most polynomial in t, however, since $\chi \in C_0^\infty(\hat{M})$, $\hat{\chi} \in \mathcal{S}(\mathbb{R})$, and thus, must decay faster than any power of $\frac{1}{t}$. In this way, we guarantee that

$$P : C^\infty \to C^\infty.$$

Now we introduce our modified operators.

$$\hat{X}_p = P\hat{X}P$$

$$e^{-tW}[X_P, e^{tW}] = Pe^{-tW}[X, e^{tW}]P = PH(t)P = H_P(t)$$

For $u \in C^\infty(\hat{M})$,

$$\lim_{t\to\infty} \frac{H_P(t)}{t} u = PHPu = HP^2u = H_Pu.$$

Note that now H_P is a bounded, invertible operator. Let

$$C_{H_P} = \| H_P \|_{op}$$

$$C_{H_P}^{-1} = \| H_P^{-1} \|_{op}$$

$$C_P^k = \| P^k \|_{op}$$

$$C_H^P = \| HP \|_{op}$$

and

$$C_\alpha' = \| \alpha \|_\infty .$$

Theorem 6. *The flow $\{\phi_t^W\}$ has absolutely continuous spectrum on $\overline{Ran(H)}$.*

Proof. We will verify the conditions of Theorem 3. on each subspace

$$\mathbb{E}_n = \{\frac{\hat{d}}{d\theta}u = inu\}.$$

(*i*)

$$\begin{aligned}
\frac{H(t)}{t} &= W + (\frac{1}{t}\int_0^t (\frac{\hat{X}\alpha}{\alpha} - 1) \circ \phi_\tau^W(x)\, d\tau) in \\
&= H + in + (\frac{1}{t}\int_0^t \frac{\hat{X}\alpha}{\alpha} \circ \phi_\tau^W(x)\, d\tau) in + (\frac{1}{t}\int_0^t -1 \circ \phi_\tau^W(x)\, d\tau) in \\
&= H + in + (\frac{1}{t}\int_0^t \frac{\hat{X}\alpha}{\alpha} \circ \phi_\tau^W(x)\, d\tau) in - in \\
&= H + (\frac{1}{t}\int_0^t \frac{\hat{X}\alpha}{\alpha} \circ \phi_\tau^W(x)\, d\tau) in = H + \frac{L(t)}{t} in
\end{aligned}$$

Let $f = H_p g$,

$$\begin{aligned}
&\| \frac{H_p(t)}{t} H_p^{-1} f \|_{L^2(\hat{M})} = \| \frac{H_p(t)}{t} g \|_{L^2(\hat{M})} \\
&= \| H_p g + inP\frac{L(t)}{t} Pg \|_{L^2(\hat{M})} \leqslant \| H_p g \|_{L^2(\hat{M})} + \| inP\frac{L(t)}{t} Pg \|_{L^2(\hat{M})} \\
&\leqslant C_{H_p} \| g \|_{L^2(\hat{M})} + n(C_P^1)^2 \| \frac{\hat{X}\alpha}{\alpha} \|_\infty \| g \|_{L^2(\hat{M})} \\
&= (C_{H_P} + n(C_P^1)^2 C_\alpha) \| g \|_{L^2(\hat{M})} .
\end{aligned}$$

So,

$$\| \frac{H_p(t)}{t} H_p^{-1} \|_{op} \leqslant (C_{H_P} + n(C_P^1)^2 C_\alpha).$$

Also,

$$\begin{aligned}
&\| (I - \frac{H_p(t)}{t} H_p^{-1}) f \|_{L^2(\hat{M})} = \| (I - I - inP\frac{L(t)}{t} P) f \|_{L^2(\hat{M})} \\
&= \| inP\frac{L(t)}{t} Pf \|_{L^2(\hat{M})} \leqslant n(C_P^1)^2 \| \frac{L(t)}{t} \|_\infty \cdot \| f \|_{L^2(\hat{M})}
\end{aligned}$$

Since $\{\phi_t^W\}$ is uniquely ergodic, the following converges uniformly,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\hat{X}\alpha}{\alpha} \circ \phi_\tau^W(x)\, d\tau = \lim_{t\to\infty} \frac{L(t)}{t} = 0.$$

So,

$$\limsup_{t\to\infty} \parallel I - \frac{H_p(t)}{t} H_p^{-1} \parallel_{op} \leqslant \limsup_{t\to\infty} n(C_P^1)^2 \parallel \frac{L(t)}{t} \parallel_\infty = 0,$$

and hence,

$$\limsup_{t\to\infty} \parallel I - \frac{H_p(t)}{t} H_p^{-1} \parallel_{op} < 1.$$

(ii)

$$[\hat{X}_P, \frac{H_p(t)}{t}(H_P)^{-1}]$$

$$= \underbrace{P[\hat{X}, P]P\frac{H(t)}{t}PH_P^{-1}}_{a} + \underbrace{P^2[\hat{X}, P]\frac{H(t)}{t}PH_P^{-1}}_{b} + \underbrace{P^3[\hat{X}, \frac{H(t)}{t}]PH_P^{-1}}_{c}$$

$$+ \underbrace{P^3\frac{H(t)}{t}[\hat{X}, P]H_P^{-1}}_{d} + \underbrace{P^3\frac{H(t)}{t}P[\hat{X}, H_P^{-1}]}_{e}$$

$$[\hat{X}, P]f = \int_{\mathbb{R}} \hat{\chi}(t)[\hat{X}, e^{tH}]f\, dt$$

$$= \int_{\mathbb{R}} \hat{\chi}(t)e^{tW}e^{-tW}[\hat{X}, e^{tH+tin-tin}]f\, dt$$

$$= \int_{\mathbb{R}} \hat{\chi}(t)e^{tW}e^{-tW}[\hat{X}, e^{tW}]e^{-itn}f\, dt$$

$$= \int_{\mathbb{R}} \hat{\chi}(t)e^{tW}H(t)e^{-itn}f\, dt$$

$$= \int_{\mathbb{R}} \hat{\chi}(t)e^{tW}tWe^{-itn}f\, dt + \int_{\mathbb{R}} \hat{\chi}(t)e^{tW}(L(t)-t)e^{-itn}f\, dt$$

$$= \int_{\mathbb{R}} \hat{\chi}(t)e^{-itn}t\frac{d}{dt}(e^{tW}f)\, dt + \int_{\mathbb{R}} \hat{\chi}(t)e^{tW}(L(t)-t)e^{-itn}f\, dt.$$

The first term we integrate by parts:

$$\hat{\chi}(t)e^{-itn}t \qquad \hat{\chi}'(t)e^{-itn}t + \hat{\chi}(t)e^{-itn} - in\hat{\chi}(t)e^{-itn}t$$

$$e^{tW}f \qquad \frac{d}{dt}(e^{tW}f)$$

$$\int_{\mathbb{R}} \hat{\chi}(t)e^{-itn}t\frac{d}{dt}(e^{tW}f)\, dt$$

$$= \hat{\chi}(t)e^{-itn}te^{tW}f\Big|_{\infty}^{-\infty} + \int_{\mathbb{R}}(\hat{\chi}^{'}(t)e^{-itn}t + \hat{\chi}(t)e^{-itn} - in\hat{\chi}(t)e^{-itn}t)e^{tW}f\,dt$$

$$= \int_{\mathbb{R}}(\hat{\chi}^{'}(t)e^{-itn}t + \hat{\chi}(t)e^{-itn} - in\hat{\chi}(t)e^{-itn}t)e^{tW}f\,dt$$

So,

$$\| \int_{\mathbb{R}}(\hat{\chi}^{'}(t)e^{-itn}t + \hat{\chi}(t)e^{-itn} - in\hat{\chi}(t)e^{-itn}t)e^{tW}f\,dt \|_{L^2(\hat{M})}$$

$$\leqslant (\int_{\mathbb{R}}|\hat{\chi}^{'}(t)t|\,dt + \int_{\mathbb{R}}|\hat{\chi}(t)|\,dt + \int_{\mathbb{R}}|n\hat{\chi}(t)|\,dt) \parallel f \parallel_{L^2(\hat{M})} \leqslant C_1 \parallel f \parallel_{L^2(\hat{M})}.$$

The boundedness of the second term follows immediately,

$$\| \int_{\mathbb{R}}\hat{\chi}(t)e^{tW}(L(t)-t)e^{-itn}f\,dt \|_{L^2(\hat{M})}$$

$$\leqslant \int_{\mathbb{R}}|\hat{\chi}(t)(L(t)-t)|\,dt \parallel f \parallel_{L^2(\hat{M})} \leqslant C_2 \parallel f \parallel_{L^2(\hat{M})}.$$

Thus,

$$\| [\hat{X},P]f \|_{L^2(\hat{M})} \leqslant (C_1 + C_2) \parallel f \parallel_{L^2(\hat{M})} = C \parallel f \parallel_{L^2(\hat{M})},$$

and hence,

$$\| [\hat{X},P] \|_{op} \leqslant C.$$

Also,

$$[\hat{X},\frac{H(t)}{t}]P = [\hat{X},W]P + [\hat{X},\int_0^t(\frac{\hat{X}\alpha}{\alpha}-1)\circ\phi_\tau^W(x)\,d\tau\,in]P$$

$$= (\hat{U} + \hat{X}\alpha\,in)P + [\hat{X},\int_0^t(\frac{\hat{X}\alpha}{\alpha}-1)\circ\phi_\tau^W(x)\,d\tau\,in]P$$

$$= (\hat{U} + (\alpha-1)in - (\alpha-1)in + \hat{X}\alpha\,in)P + [\hat{X},\int_0^t(\frac{\hat{X}\alpha}{\alpha}-1)\circ\phi_\tau^W(x)\,d\tau\,in]P$$

$$= HP + (\hat{X}\alpha - \alpha + 1)\,inP + [\hat{X},\int_0^t(\frac{\hat{X}\alpha}{\alpha}-1)\circ\phi_\tau^W(x)\,d\tau\,in]P$$

In order to bound the following term,

$$[\hat{X},\int_0^t(\frac{\hat{X}\alpha}{\alpha}-1)\circ\phi_\tau^W(x)\,d\tau\,in]$$

we need the following,

$$D\phi_t^W(X) = tW \circ \phi_t^W + (\int_0^t (\frac{\hat{X}\alpha}{\alpha} - 1) \circ \phi_\tau^W(x)\, d\tau)\; in \circ \phi_t^W$$

(Please see 6.4 for this derivation.) Let

$$G(\alpha, t) = \int_0^t (\frac{\hat{X}\alpha}{\alpha} - 1) \circ \phi_\tau^W(x)\, d\tau.$$

$$[\hat{X}, \int_0^t (\frac{\hat{X}\alpha}{\alpha} - 1) \circ \phi_\tau^W(x)\, d\tau\; in]$$

$$= \hat{X}(\frac{1}{t}\int_0^t (\frac{\hat{X}\alpha}{\alpha} - 1) \circ \phi_\tau^W)(x)\, d\tau)in - (\frac{1}{t}\int_0^t (\frac{\hat{X}\alpha}{\alpha} - 1)\phi_\tau^W)(x)\, d\tau)\hat{X}in$$

$$= \frac{1}{t}\int_0^t (D\phi_\tau^W(\hat{X}) \circ \phi_{-\tau}^W)(\frac{\hat{X}\alpha}{\alpha}) \circ \phi_\tau^W(x) d\tau in = \frac{1}{t}\int_0^t (tW + G(\alpha, t)in)(\frac{\hat{X}\alpha}{\alpha}) \circ \phi_\tau^W(x) d\tau in$$

$$= (\frac{\hat{X}\alpha}{\alpha} \circ \phi_t^W - \frac{\hat{X}\alpha}{\alpha})in + (\frac{1}{t}\int_0^t G(\alpha, t)(\frac{\hat{X}\alpha}{\alpha}) \circ \phi_\tau^W(x)\, d\tau\; in)in$$

So,

$$\| \frac{1}{t}\int_0^t (tW + G(\alpha, t)in)(\frac{\hat{X}\alpha}{\alpha}) \circ \phi_\tau^W d\tau \|_\infty$$

$$\leqslant 2n \cdot \| \frac{\hat{X}\alpha}{\alpha} \|_\infty + n^2 \cdot \| \frac{G(\alpha, t)}{t} \cdot \frac{\hat{X}\alpha}{\alpha} \|_\infty$$

$$\leqslant 2n \cdot \| \frac{\hat{X}\alpha}{\alpha} \|_\infty + n^2 \cdot \| (\frac{\hat{X}\alpha}{\alpha} - 1) \cdot \frac{\hat{X}\alpha}{\alpha} \|_\infty$$

$$\leqslant 2n \cdot \| \frac{\hat{X}\alpha}{\alpha} \|_\infty + n^2 \cdot \| (\frac{\hat{X}\alpha}{\alpha} - 1) \|_\infty \| \frac{\hat{X}\alpha}{\alpha} \|_\infty$$

$$\leqslant 2nC_\alpha + n^2(C_\alpha + 1)C_\alpha.$$

Thus, $[\hat{X}, \frac{H(t)}{t}]P$ extends to a bounded operator on $\overline{Ran(H_p)}$ with operator norm uniformly bounded in t:

$$\| [\hat{X}, \frac{H(t)}{t}]P \|_{op} \leqslant (C_H^P + n(C_\alpha' C_\alpha + C_\alpha' + 1)C_P^1) + (2nC_\alpha + n^2(C_\alpha + 1)C_\alpha)C_P^1.$$

a:

$$\| P[\hat{X}, P]P\frac{H(t)}{t}PH_P^{-1} \|_{op} \leqslant C_P^1 \cdot C \cdot \| \frac{H_P(t)}{t}H_P^{-1} \|_{op}$$

$$\leqslant C_P^1 \cdot C \cdot (C_{H_P} + n(C_P^1)^2 C_\alpha).$$

$b:$

$$\| P^2[\hat{X}, P]\frac{H(t)}{t}PH_P^{-1} \|_{op} = \| P^2[\hat{X}, P](H + \frac{L(t)}{t}in)PH_P^{-1} \|_{op}$$

$$\leqslant C_P^2 \cdot C \cdot (\| HP \|_{op} + n \| \frac{\hat{X}\alpha}{\alpha} \|_\infty C_P^1) \cdot \| H_P^{-1} \|_{op}$$

$$= C_P^2 \cdot C \cdot (C_H^P + nC_\alpha C_P^1)C_{H_P}^{-1}$$

$c:$

$$\| P^3[\hat{X}, \frac{H(t)}{t}]PH_P^{-1} \|_{op}$$

$$\leqslant C_P^3((C_H^P + n(C_\alpha' C_\alpha + C_\alpha' + 1)C_P^1) + (2nC_\alpha + n^2(C_\alpha + 1)C_\alpha)C_P^1)C_{H_P}^{-1}.$$

$d:$

$$\| P^3\frac{H(t)}{t}[\hat{X}, P]H_P^{-1} \|_{op} \leqslant C_P^2 \cdot C \cdot (C_H^P + nC_P^1 C_\alpha)C_{H_P}^{-1}$$

$e:$

$$[\hat{X}, H_P^{-1}] = H_P^{-1}[H_P, \hat{X}]H_P^{-1}$$

$$= H_P^{-1}[P, \hat{X}]PHH_P^{-1} + H_P^{-1}P[P, \hat{X}]HH_P^{-1} + H_P^{-1}P^2[\hat{X}, H]H_P^{-1}.$$

$$\| [\hat{X}, H_P^{-1}] \|_{op}$$

$$\leqslant C_{H_P}^{-1} \cdot C \cdot C_H^P C_{H_P}^{-1} + C_{H_P}^{-1} C_P^1 \cdot C \cdot (C_P^2)^{-1} + C_{H_P}^{-1}(C_{H_P} + C_P^2 n(C_\alpha' C_\alpha + C_\alpha' + 1))C_{H_P}^{-1}.$$

$$\| P^3\frac{H(t)}{t}P[\hat{X}, H_P^{-1}] \|_{op}$$

$$\leqslant C_P^2(C_{H_P} + n(C_P^1)^2 C_\alpha)(C_{H_P}^{-1} \cdot C \cdot C_H^P C_{H_P}^{-1} + C_{H_P}^{-1} C_P^1 \cdot C \cdot (C_P^2)^{-1}$$

$$+ C_{H_P}^{-1}(C_{H_P} + C_P^2 n(C_\alpha' C_\alpha + C_\alpha' + 1))C_{H_P}^{-1}).$$

(iii).

$$[H_p(t), H_P]H_P^{-1} = [tH_p + PL(t)Pin, H_P]H_p^{-1} = [PL(t)P, H_P]H_P^{-1}.$$

$$= P^3 HL(t) Pin H_P^{-1} = P^3 WL(t) Pin H_P^{-1} - P^3 in Pin H_P^{-1}$$

$$= P^3 (\frac{\hat{X}\alpha}{\alpha} \circ \phi_t^W - \frac{\hat{X}\alpha}{\alpha}) in H_P^{-1} - P^3 in Pin H_P^{-1}.$$

$$\| [H_p(t), H_P] H_P^{-1} \|_{op}$$

$$\leqslant 2n C_P^3 C_\alpha C_{H_P}^{-1} + n^2 C_P^4 C_{H_P}^{-1}.$$

We have shown that the conditions of Theorem 3 are satisfied on $Ran(H_P)$. We would like to extend this to $Ran(H)$. Recall that H_P depends upon a choice of $\chi \in C_0^\infty(\mathbb{R}\backslash\{0\})$. For $f \in Dom(H)$, we can express the following in terms of integrals involving the spectral projector as

$$Hf = \int_{\mathbb{R}} x \, dE(x) f,$$

$$H_P f = \int_{\mathbb{R}} x\chi(x) \, dE(x) f,$$

and since $f \in Dom(H)$,

$$\int_{\mathbb{R}} x^2 \, dE(x) f < +\infty.$$

Let χ be such that

$$\chi(x) = 1 \qquad \text{for } x \in (-K, -\epsilon) \cup (\epsilon, K) = I_{\epsilon,K}$$

and $supp(\chi)$ vanishes outside of $I_{\epsilon,K}$. Since on $I_{\epsilon,K}$,

$$H_P f = Hf,$$

we consider

$$H_P f - Hf$$

on $\mathbb{R}\backslash I_{\epsilon,K}$, i.e.,

$$\int_{\mathbb{R}\backslash I_{\epsilon,K}} x(\chi(x) - 1) \, dE(x) f.$$

For $|x| \leqslant \epsilon$,

$$\lim_{\epsilon \to 0} \| \int_{|x| \leqslant \epsilon} x(\chi(x) - 1) \, dE(x) f \|^2_{L^2(\mathbb{R})} = \lim_{\epsilon \to 0} \int_{|x| \leqslant \epsilon} |x(\chi(x) - 1)|^2 \, dE(x) f$$

$$\leqslant \lim_{\epsilon \to 0} 4\epsilon^2 \int_{|x| \leqslant \epsilon} dE(x) f \leqslant \lim_{\epsilon \to 0} 4\epsilon^2 \| f \|_{L^2(\hat{M})} = 0.$$

For $|x| \geqslant K$,

$$\lim_{K \to \infty} \| \int_{|x| \geqslant K} x(\chi(x) - 1) \, dE(x) f \|^2_{L^2(\mathbb{R})} = \lim_{K \to \infty} \int_{|x| \geqslant K} |x(\chi(x) - 1)|^2 \, dE(x) f$$

$$\leqslant \lim_{K \to \infty} \int_{|x| \geqslant K} 4x^2 \, dE(x) f = 0$$

since

$$\int_{\mathbb{R}} x^2 \, dE(x) f < +\infty.$$

Thus,

$$\inf_{\chi \in C_0^\infty(\mathbb{R} \setminus \{0\})} \| H_P f - H f \|_{L^2(\hat{M})} = 0.$$

So, for any Hf, there exists a sequence $\{H_{P_n}\}$ such that

$$H_{P_n} \to Hf,$$

and thus,

$$\overline{Ran(H)} = \overline{\{\bigcup_P Ran(H_P)\}}.$$

Consequently, for every $f \in \overline{Ran(H)}$, μ_f is absolutely continuous. $\square$

Chapter 4: Applications to Maps

The author in [29] uses the Mourre Estimate technique (described in 1.5.1) to prove the following spectral results. Here we rederive these results by showing that the conditions of the Theorem 3. are satisfied.

4.1 Skew Products over Translations

Let X be a compact metric abelian Lie group with normalized Haar measure μ. Let $\{F_t\}$ be a uniquely ergodic [9] translation flow (we assume that F_1 is ergodic),

$$F_t = y_t x \text{ with vector field } Y.$$

The associated operators $\{V_t\}$ are given by

$$V_t\psi = \psi \circ F_t \text{ with generator } P = -i\mathscr{L}_Y.$$

Let G be a compact metric abelian group. Let $\phi : X \to G$ such that ϕ can be written as $\phi = \xi\eta$ where ξ is a group homomorphism and η satisfies

$$\sup_{t>0} \| \frac{\mathscr{L}_Y(\chi \circ \eta) \circ F_t - \mathscr{L}_Y(\chi \circ \eta)}{t}) \|_\infty < \infty$$

and

$$\chi \circ \eta = e^{i\tilde{\eta}_\chi}$$

for $\tilde{\eta}_\chi \in Dom(P)$ a real-valued function determined by χ and η.

The skew product, $T : X \times G \to X \times G$, is defined by

$$T(x, z) = (y_1 x, \phi(x) z)$$

with corresponding unitary operator

$$W\Psi = \Psi \circ T.$$

Let $\hat{G}$ be the character group of G. The decomposition $L^2(X \times G) = \bigoplus_{\chi \in \hat{G}} L_\chi$ and the restriction of W to the subspaces L_χ allow us to study the spectrum of convenient, unitarily equivalent operators to $W|_{L_\chi}$, namely,

$$U_\chi \psi = (\chi \circ \psi) V_1 \psi \text{ (here } U_\chi \text{ takes the place of } e^U \text{ as given in the conditions)}$$

for $\chi \circ \xi \not\equiv 1$.

We will choose to take the commutator with P; from [1], P is essentially self-adjoint. Let $D = C^\infty(X)$.

Remark: Describing these systems in full generality inevitably leads to cumbersome notation. In an effort to simplify the reading, we provide the following example:

Let $X, G = S^1$. For $x \in X$ and $z \in G$ and $\alpha \in S^1$,

$$T(x, z) = (\alpha x, \phi(x) z).$$

$$\phi = e^{i\tilde{\phi}}, \text{ and } \eta = e^{i\tilde{\eta}}, \text{ for } \tilde{\phi} \text{ and } \tilde{\eta} \text{ real valued functions.}$$

$$\chi(y) = y^k \text{ for } k \in \mathbb{Z} \backslash \{0\}, \text{ and thus, } \chi \circ \phi = e^{ik\tilde{\phi}} \text{ and } \chi \circ \eta = e^{ik\tilde{\eta}}.$$

$$U_\chi \psi = e^{ik\tilde{\phi}(x)}\psi(\alpha x)$$

$$P = -i\frac{d}{dx}.$$

Now we compute the commutators using the general notation.

$$[P, U_\chi] = [P, (\chi \circ \phi)V_1] = [P, (\chi \circ \phi)I]V_1$$

$$= -i\mathscr{L}_Y(\chi \circ \phi)V_1 = -i[\mathscr{L}_Y(\chi \circ \xi)(\chi \circ \eta) + (\chi \circ \xi)\mathscr{L}_Y(\chi \circ \eta)]V_1$$

$$= -i[\xi_0 + \tfrac{\mathscr{L}_Y(\chi \circ \eta)}{(\chi \circ \eta)}](\chi \circ \phi)V_1 = -i[\xi_0 + \tfrac{\mathscr{L}_Y(\chi \circ \eta)}{(\chi \circ \eta)}]U_\chi$$

where $\xi_0 = \frac{d}{dt}(\chi \circ \xi)(y_t)|_{t=0} \in i\mathbb{R}\backslash\{0\}$.

So,

$$[P, U_\chi] = (-i\xi_0 - \frac{i\mathscr{L}_Y(\chi \circ \eta)}{(\chi \circ \eta)})U_\chi = GU_\chi.$$

Thus,

$$U_\chi^{-n}[P, U_\chi^n] = \textstyle\sum_{k=1}^n U_\chi^{-k} G U_\chi^k =$$
$$\boxed{\sum_{k=1}^n (-i\xi_0 - \frac{i\mathscr{L}_Y(\chi \circ \eta)}{(\chi \circ \eta)}) \circ F_{-k} = \sum_{k=1}^n G \circ F_{-k} = H(n).}$$

Note that

$$\frac{-i\mathscr{L}_Y(\chi \circ \eta)}{(\chi \circ \eta)} = \frac{-i\mathscr{L}_Y(e^{i\tilde{\eta}_\chi})}{e^{i\tilde{\eta}_\chi}} = \frac{-ie^{i\tilde{\eta}_\chi} \cdot \mathscr{L}_Y(\tilde{\eta}_\chi)}{e^{i\tilde{\eta}_\chi}} = -i\mathscr{L}_Y(\tilde{\eta}_\chi) = P\tilde{\eta}_\chi$$

From unique ergodicity we get the following convergence

$$\lim_{n\to\infty} \frac{H(n)}{n}u = (-i\xi_0 + \int_X P\tilde{\eta}_\chi \, d\mu)u = \boxed{-i\xi_0 u = Hu}$$

uniformly in n for $u \in L_\chi$.

Since

$$\| < U_\chi^n f, f >_{L_\chi} \|_{\ell^2(\mathbb{Z})} = \| \frac{1}{\sigma}\int_0^\sigma < e^{sP} U_\chi^n f, e^{sP} f >_{L_\chi} ds \ \|_{\ell^2(\mathbb{Z})} .$$

the preliminary assumptions are satisfied with $B_1 = B_2 = I$.

(i) It is unnecessary to consider coboundaries since both $H = -i\xi_0 I$ and $H^{-1} = \frac{i}{\xi_0} I$ are constants. Instead we take any $f \in L_\chi$.

$$\frac{H(n)}{n} H^{-1} f = (\frac{1}{n}\sum_{k=1}^{n}(-i\xi_0 - \frac{i\mathscr{L}_Y(\chi\circ\eta)}{(\chi\circ\eta)}) \circ F_{-k}) \cdot \frac{i}{\xi_0} f = f + (\frac{1}{n}\sum_{k=1}^{n} P\tilde{\eta}_\chi \circ F_{-k}) \cdot \frac{i}{\xi_0} f.$$

So,

$$\| \frac{H(n)}{n} H^{-1} f \|_{L_\chi} \leqslant (1 + \frac{\| P\tilde{\eta}_\chi \|_{L_\chi}}{|\xi_0|}) \cdot \| f \|_{L_\chi} .$$

Since $\tilde{\eta}_\chi \in Dom(P)$,

$$\| P\tilde{\eta}_\chi \|_{L_\chi} \leqslant C_1.$$

Thus, $\frac{H(n)}{n} H^{-1}$ is a bounded operator with uniformly bounded norm in n,

$$\| \frac{H(n)H^{-1}}{n} \|_{op} \leqslant 1 + \frac{C_1}{|\xi_0|}.$$

Also,

$$\| (I - \frac{H(n)}{n} H^{-1}) f \|_{L_\chi} = \| (1 - (1 + (\frac{1}{n}\sum_{k=1}^{n} P\tilde{\eta}_\chi \circ F_{-k}) \cdot \frac{i}{\xi_0}) f \|_{L_\chi}$$

$$= \| (\frac{1}{n}\sum_{k=1}^{n} P\tilde{\eta}_\chi \circ F_{-k} \cdot \frac{i}{\xi_0}) f \|_{L_\chi} \leqslant \| (\frac{1}{n}\sum_{k=1}^{n} P\tilde{\eta}_\chi \circ F_{-k}) \cdot \frac{i}{\xi_0} \|_\infty \cdot \| f \|_{L_\chi} .$$

As a result of unique ergodicity, the following converges uniformly,

$$\lim_{n\to\infty} (\frac{1}{n}\sum_{k=1}^{n} P\tilde{\eta}_\chi \circ F_{-k}) \cdot \frac{i}{\xi_0} = \frac{i}{\xi_0}\int_X P\tilde{\eta}_\chi \, d\mu = 0,$$

and thus,

$$\limsup_{n\to\infty} \| I - \frac{H(n)}{n} H^{-1} \|_{op} \leqslant \limsup_{n\to\infty} \| (\frac{1}{n} \sum_{k=1}^{n} P\tilde{\eta}_{\chi} \circ F_{-k}) \cdot \frac{i}{\xi_0} \|_{\infty}$$

$$= \| \frac{i}{\xi_0} \int_X P\tilde{\eta}_{\chi} \, d\mu \|_{\infty} = 0.$$

Hence,

$$\limsup_{n\to\infty} \| I + \frac{H(n)}{n} H^{-1} \|_{op} < 1.$$

(ii)

$$[P, \frac{H(n)}{n} H^{-1}] = [P, I + (\frac{1}{n} \sum_{k=1}^{n} P\tilde{\eta}_{\chi} \circ F_{-k}) \cdot \frac{i}{\xi_0} I] = [P, (\frac{1}{n} \sum_{k=1}^{n} P\tilde{\eta}_{\chi} \circ F_{-k}) \cdot \frac{i}{\xi_0} I]$$

$$= (\frac{1}{n} \sum_{k=1}^{n} P(P\tilde{\eta}_{\chi}) \circ F_{-k}) \cdot \frac{i}{\xi_0} I.$$

Since $\sup_{t>0} \| \frac{\mathscr{L}_Y(\chi\circ\eta)\circ F_t - \mathscr{L}_Y(\chi\circ\eta)}{t}) \|_{\infty} < \infty$, $P(P\tilde{\eta}_{\chi})$ is bounded in L_{χ} and

$$\| [P, \frac{H(n)}{n} H^{-1}] f \|_{L_{\chi}} \leqslant \frac{\| P(P\tilde{\eta}_{\chi}) \|_{L_{\chi}}}{|\xi_0|} \cdot \| f \|_{L_{\chi}} \leqslant \frac{C_2}{|\xi_0|} \| f \|_{L_{\chi}} .$$

Thus, $[P, \frac{H(n)}{n} H^{-1}]$ extends to a bounded operator on L_{χ} with uniformly bounded norm in n,

$$\| [P, \frac{H(n)}{n} H^{-1}] \|_{op} \leqslant \frac{C_2}{|\xi_0|}.$$

.

(iii) Since the operator H is just multiplication by the constant $-i\xi_0$,

$$[H(n), H] H^{-1} = 0.$$

Thus, condition (iii) is immediately satisfied.

Since conditions (i), (ii), and (iii) of Theorem 3 are satisfied on each L_{χ}, we have shown that the operator U_{χ} has purely absolutely continuous spectrum on

L_χ. Thus, W has purely absolutely continuous spectrum when restricted to the subspace $\bigoplus_{\chi\in\hat{G},\chi\circ\xi\not\equiv 1} L_\chi$.

In addition, from the purity law in [14] extended to translations, the maximal spectral type is either purely Lebesgue, purely singularly continuous, or purely discrete with respect to μ (the Haar measure). Since we know that the spectrum is absolutely continuous from above, we have rederived the following result from [29],

Theorem 7. *The operator U_χ has Lebesgue spectrum on L_χ. Thus, W has countable Lebesgue spectrum when restricted to the subspace $\bigoplus_{\chi\in\hat{G},\chi\circ\xi\not\equiv 1} L_\chi$.*

4.2 Furstenberg Transformations

Let μ_n be the normalized Haar measure on $\mathbb{T}^n \simeq \mathbb{R}^n/\mathbb{Z}^n$ and $\mathcal{H}_n = L^2(\mathbb{T}^n, \mu_n)$. Let $T_d : \mathbb{T}^d \to \mathbb{T}^d$, $d \geqslant 2$, be the uniquely ergodic map [9]

$$T_d(x_1, x_2, ..., x_d) =$$

$$(x_1 + y, x_2 + b_{2,1}x_1 + h_1(x_1), ..., x_d + b_{d,1}x_1 + \cdots + b_{d,d-1}x_{d-1} + h_{d-1}(x_1, x_2, ..., x_{d-1}))$$

$$(mod\ \mathbb{Z}^d)$$

for $y \in \mathbb{R}\backslash\mathbb{Q}$, $b_{j,k} \in \mathbb{Z}$, $b_{l,l-1} \neq 0$, and $l \in \{2, ..., d\}$. (For $n = 2$, we get the skew product in 4.1). Let each $h_j : \mathbb{T}^j \to \mathbb{R}$ satisfy a uniform Lipschitz condition in x_j and be in $C^2(\mathbb{T}^j)$. What follows is very similar to the case of the skew products over translations. We begin by considering the operator

$$W_d : \mathcal{H}_d \to \mathcal{H}_d.$$

The space $\mathcal{H}_d$ can be decomposed into

$$\mathcal{H}_d = \mathcal{H}_1 \bigoplus_{j\in\{2,...d\},\, k\in\mathbb{Z}\setminus\{0\}} \mathcal{H}_{j,k}$$

for $\mathcal{H}_{j,k} = \overline{Span\{\eta \bigotimes \chi_k | \eta \in \mathcal{H}_{j-1}\}}$ and $\chi_k(x_j) = e^{2\pi i k x_j} \in \hat{\mathbb{T}}$.

The restriction of W_d, $W_d|_{\mathcal{H}_{j,k}}$ is unitarily equivalent to the operator

$$U_{j,k}\eta = e^{2\pi i k \phi_j} W_{j-1}\eta$$

for $\eta \in \mathcal{H}_{j-1}$ and $\phi_j(x_1, x_2, ..., x_{j-1}) = b_{j,1}x_1 + \cdots + b_{j,j-1}x_{j-1} + h_{j-1}(x_1, x_2, ..., x_{j-1})$.

We will choose to take the commutator with $P_{j-1} = -i\partial_{j-1}$, the essentially self-adjoint [1] generator of the translation group $\{V_{t,j-1}\}_{t\in\mathbb{R}}$ in $\mathcal{H}_{j-1}$. Let $D = C^\infty(\mathbb{T}^{j-1})$.

$$[P_{j-1}, U_{j,k}] = [P_{j-1}, e^{2\pi i k \phi_j} I]W_{j-1}$$

$$= -i\partial_{j-1}(e^{2\pi i k \phi_j})W_{j-1} = (2\pi k b_{j,j-1} + 2\pi k \partial_{j-1}h_{j-1})e^{2\pi i k \phi_j} W_{j-1}.$$

So,

$$[P_{j-1}, U_{j,k}] = (2\pi k b_{j,j-1} + 2\pi k \partial_{j-1}h_{j-1})U_{j,k} = GU_{j,k}.$$

Thus,

$$U_{j,k}^{-n}[P_{j,k}, U_{j,k}^n] = \boxed{(\sum_{l=1}^{n} U_{j,k}^{-l} G U_{j,k}^l) = (\sum_{l=1}^{n} G \circ T_{j-1}^{-l}) = H(n).}$$

From unique ergodicity we get the following convergence

$$\lim_{n\to\infty} \frac{H(n)}{n} u = 2\pi k b_{j,j-1} + 2\pi k \int_{\mathbb{T}^{j-1}} \partial_{j-1}h_{j-1}\, d\mu = \boxed{2\pi k b_{j,j-1} u = Hu}$$

uniformly in n for $u \in D = \mathcal{H}_{j-1}$.

Since

$$\| < U_{j,k}^n f, f >_{\mathcal{H}_{j-1}} \|_{\ell^2(\mathbb{Z})} = \| \frac{1}{\sigma} \int_0^\sigma < e^{sP_{j-1}} U_{j,k}^n f, e^{sP_{j-1}} f >_{\mathcal{H}_{j-1}} ds \ \|_{\ell^2(\mathbb{Z})} .$$

the preliminary assumptions are satisfied with $B_1 = B_2 = I$.

(i) It is unnecessary to consider coboundaries since both $H = 2\pi k b_{j,j-1} I$ and $H^{-1} = \frac{1}{2\pi k b_{j,j-1}} I$ are constants. Instead we take any $f \in \mathcal{H}_{j-1}$.

$$\frac{H(n)}{n} H^{-1} f = (\frac{1}{n}(\sum_{l=1}^{n}(2\pi k b_{j,j-1} + 2\pi k \partial_{j-1} h_{j-1}) \circ T_{j-1}^{-l})) \cdot \frac{1}{2\pi k b_{j,j-1}} f$$

$$= f + (\frac{1}{n}\sum_{l=1}^{n}(2\pi k \partial_{j-1} h_{j-1}) \circ T_{j-1}^{-l}) \frac{1}{2\pi k b_{j,j-1}} f.$$

Hence,

$$\| \frac{H(n)}{n} H^{-1} f \|_{\mathcal{H}_{j-1}} \leqslant (1 + \frac{\| 2\pi k \partial_{j-1} h_{j-1} \|_{\mathcal{H}_{j-1}}}{|2\pi k b_{j,j-1}|}) \| f \|_{\mathcal{H}_{j-1}} .$$

Since h_{j-1} satisfies a uniform Lipschitz condition in x_{j-1},

$$\| \partial_{j-1} h_{j-1} \|_{\mathcal{H}_{j-1}} \leqslant C_1.$$

So $\frac{H(n)}{n} H^{-1}$ extends to a bounded operator on $\mathcal{H}_{j,k}$ with uniformly bounded norm in n,

$$\| \frac{H(n)}{n} H^{-1} \|_{op} \leqslant 1 + \frac{|2\pi k| \ \| \partial_{j-1} h_{j-1} \|_{\mathcal{H}_{j-1}}}{|2\pi k b_{j,j-1}|} \leqslant \frac{C_1}{|b_{j,j-1}|}.$$

Also,

$$\| (I - \frac{H(n)}{n} H^{-1}) f \|_{\mathcal{H}_{j-1}} = \| (1 - (1 + (\frac{1}{n}\sum_{l=1}^{n}(2\pi k \partial_{j-1} h_{j-1}) \circ T_{j-1}^{-l}) \frac{1}{2\pi k b_{j,j-1}}) f \|_{\mathcal{H}_{j-1}}$$

$$= \| (\frac{1}{n}\sum_{l=1}^{n}(2\pi k \partial_{j-1} h_{j-1}) \circ T_{j-1}^{-l}) \frac{1}{2\pi k b_{j,j-1}} f \|_{\mathcal{H}_{j-1}}$$

$$\leqslant \| \left(\frac{1}{n}\sum_{l=1}^{n}(2\pi k\partial_{j-1}h_{j-1})\circ T_{j-1}^{-l}\right)\frac{1}{2\pi k b_{j,j-1}} \|_\infty \cdot \| f \|_{\mathcal{H}_{j-1}} .$$

As a result of unique ergodicity, the following converges uniformly,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{l=1}^{n}(2\pi k\partial_{j-1}h_{j-1})\circ T_{j-1}^{-l}\right)\frac{1}{2\pi k b_{j,j-1}} = \frac{1}{b_{j,j-1}}\int_{\mathbb{T}^{j-1}}\partial_{j-1}h_{j-1}\,d\mu = 0,$$

and thus,

$$\limsup_{n\to\infty} \| I - \frac{H(n)}{n}H^{-1} \|_{op} \leqslant \limsup_{n\to\infty} \| \left(\frac{1}{n}\sum_{l=1}^{n}(2\pi k\partial_{j-1}h_{j-1})\circ T_{j-1}^{-l}\right)\frac{1}{2\pi k b_{j,j-1}} \|_\infty$$

$$= \| \frac{1}{b_{j,j-1}}\int_{\mathbb{T}^{j-1}}\partial_{j-1}h_{j-1}\,d\mu \|_\infty = 0$$

Hence,

$$\limsup_{n\to\infty} \| I + \frac{H(n)}{n}H^{-1} \|_{op} < 1.$$

(ii)

$$[P_{j-1}, \frac{H(n)}{n}H^{-1}] = [P_{j-1}, I + \left(\frac{1}{n}\sum_{k=1}^{n}(2\pi k\partial_{j-1}h_{j-1})\circ T_{j-1}^{-l}\right)\cdot\frac{1}{2\pi k b_{j,j-1}}I]$$

$$= [P_{j-1}, \left(\frac{1}{n}\sum_{k=1}^{n}(2\pi k\partial_{j-1}h_{j-1})\circ T_{j-1}^{-l}\right)\cdot\frac{1}{2\pi k b_{j,j-1}}I]$$

$$= \left(\frac{1}{n}\sum_{k=1}^{n}(2\pi k\partial_{j-1}(\partial_{j-1}h_{j-1}))\circ T_{j-1}^{-l}\right)\cdot\frac{1}{2\pi k b_{j,j-1}}I.$$

Since $h_{j-1}\in C^2(\mathbb{T}^{j-1})$, $\partial_{j-1}(\partial_{j-1}h_{j-1})$ is bounded in $\mathcal{H}_{j-1}$,

$$\| [P, \frac{H(n)}{n}H^{-1}]f \|_{\mathcal{H}_{j-1}}$$

$$\leqslant \frac{|2\pi k| \| \partial_{j-1}(\partial_{j-1}h_{j-1}) \|_{\mathcal{H}_{j-1}}}{|2\pi k b_{j,j-1}|}\cdot \| f \|_{\mathcal{H}_{j-1}} \leqslant \frac{C_2}{|b_{j,j-1}|} \| f \|_{\mathcal{H}_{j-1}} .$$

Thus, $[P, \frac{H(n)}{n}H^{-1}]$ extends to a bounded operator on $\mathcal{H}_{j-1}$ with uniformly bounded norm in n,

$$\| [P, \frac{H(n)}{n}H^{-1}] \|_{op} \leqslant \frac{C_2}{|b_{j,j-1}|}.$$

(iii) Since the operator H is just multiplication by $2\pi k b_{j,j-1}$,

$$[H(n), H]H^{-1} = 0.$$

Thus, condition (iii) is immediately satisfied.

Since conditions (i), (ii), and (iii) of Theorem 1. are satisfied, we obtain the following result. The operator $U_{j,k}$ has purely absolutely continuous spectrum on each $\mathcal{H}_{j,k}$. Thus, W_d has purely absolutely continuous spectrum on the orthocomplement of $\mathcal{H}_1$.

Again from the purity law in [14], we rederive the following result from [29].

Theorem 8. *W_d has countable Lebesgue spectrum on the orthocomplement of $\mathcal{H}_1$.*

Chapter 5: Open Questions

5.1 Further Study of the Twisted Horocycle Flow

The properties of the subspace $\overline{Ran(H)}$ are linked to the properties of the cocyle

$$a = \int_0^t (\alpha - 1) \circ h_s^U(x) ds$$

since

$$\phi_t^H(x, \theta) = (h_t^U(x), \theta + \int_0^t (\alpha - 1) \circ h_s^U(x) ds).$$

To help understand the properties of a, it may be useful to consider both Anzai's Theorem [3] and results from the theory of Essential Values [26].

Also, we would like to determine the maximal spectral type on $\overline{Ran(H)}$.

5.2 Time-Changes of Nilflows

The 3-dimensional Heisenberg Group is a connected, simply connected, Lie Group whose Lie Alebra is generated by two elements X, Y that satisfy the following commutation relations:

$$[X, Y] = Z \text{ and } [X, Z] = [Y, Z] = 0.$$

In [4], the authors show that any nontrivial time-change of a uniquely ergodic Heisenberg nilflow is mixing. They mention that it is still unknown whether the spectrum of mixing time-changes is singularly continuous, absolutely continuous, or possibly mixed.

This is an example of when the best possible upper bound when satisfying the conditions of Theorem 3 is achieved with $\beta = \frac{1}{2}$, for which we cannot determine square-integrability of the spectral measure. It is of interest to develop tools to include such borderline cases.

Appendix 6: Appendix

6.1 Detailed Computation for Resolvent Estimates for the Horocycle Flow

We will use the following important identities,

$$\frac{d}{dz}R(z) = \frac{d}{dz}(iU - z)^{-1} = -(iU - z)^{-2} \cdot -1 = (iU - z)^{-2} = R(z)^2,$$

$$[R(z), X] = R(z)X - XR(z)$$

$$= R(z)X(iU - z)R(z) - R(z)(iU - z)XR(z)$$

$$= R(z)XiUR(z) - R(z)iUXR(z) = R(z)[X, iU]R(z)$$

and since $[X, iU] = iU$,

$$z\frac{d}{dz}R(z) = [R(z), X] - R(z).$$

For $f \in Dom(X)$,

$$z\frac{d}{dz}F(z) = -F(z) - < R(z)f, Xf > + < Xf, R(\overline{z})f > .$$

Let $\| \cdot \| = \| \cdot \|_{L^2(M,vol)}$. Given our choice of z, the following equality holds,

$$\| R(z)f \| = \| R(\overline{z})f \| = \mu^{-\frac{1}{2}} \cdot |\mathcal{I}mF(z)|^{\frac{1}{2}}.$$

Using the above bound, we have the following inequality,

$$|\frac{d}{d\mu}F(\lambda+i\mu)| \leqslant |z|^{-1}(\parallel R(z)f \parallel \cdot \parallel f \parallel + \parallel R(z)f \parallel \cdot \parallel Xf \parallel + \parallel R(\overline{z})f \parallel \cdot \parallel f \parallel)$$

$$= |z|^{-1}(\parallel f \parallel +2 \parallel Xf \parallel) \parallel R(z)f \parallel$$

$$\leqslant |\lambda|^{-1}(\parallel f \parallel +2 \parallel Xf \parallel) \cdot \mu^{-\frac{1}{2}}|F(\lambda+i\mu)|^{\frac{1}{2}}. \tag{6.1}$$

We divide both sides by $|F(\lambda+i\mu)|^{\frac{1}{2}}$, and since $|F(\lambda+i\mu)|^{\frac{1}{2}} = |F(\lambda+i\mu)^{\frac{1}{2}}|$, we obtain, after integrating with respect to μ for $0 < \mu < 1$,

$$|F(\lambda+i\mu)|^{\frac{1}{2}} - |F(\lambda+ix)|^{\frac{1}{2}} \leqslant \int_x^\mu |\frac{d}{d\mu}F(\lambda+i\mu)|^{\frac{1}{2}}d\mu \leqslant 2|\lambda|^{-1}(\parallel f \parallel +2 \parallel Xf \parallel)\mu^{\frac{1}{2}}.$$

Since the right hand side is maximized when $\mu = 1$,

$$|F(\lambda+i\mu)|^{\frac{1}{2}} \leqslant |F(\lambda+i)|^{\frac{1}{2}} + 2|\lambda|^{-1}(\parallel f \parallel +2 \parallel Xf \parallel).$$

We use this to bound the right hand side of (1).

$$|\lambda|^{-1}(\parallel f \parallel +2 \parallel Xf \parallel)\mu^{-\frac{1}{2}}|F(\lambda+i\mu)|^{\frac{1}{2}}$$

$$\leqslant |\lambda|^{-1}(\parallel f \parallel +2 \parallel Xf \parallel)\mu^{-\frac{1}{2}}(|F(\lambda+i)|^{\frac{1}{2}} + 2|\lambda|^{-1}(\parallel f \parallel +2 \parallel Xf \parallel))$$

$$= (\frac{\parallel f \parallel}{|\lambda|^{-1}} + 2\frac{\parallel Xf \parallel}{|\lambda|^{-1}})\mu^{-\frac{1}{2}}(|F(\lambda+i)|^{\frac{1}{2}} + |(2\frac{\parallel f \parallel}{|\lambda|^{-1}} + 4\frac{\parallel Xf \parallel)}{|\lambda|^{-1}}).$$

Let

$$a = \frac{\parallel f \parallel}{|\lambda|^{-1}}$$

$$b = \frac{\parallel Xf \parallel}{|\lambda|^{-1}}$$

$$c = |F(\lambda + i)|^{\frac{1}{2}}$$

Now we have

$$\mu^{-\frac{1}{2}}(a+2b)(c+2a+4b) = \mu^{-\frac{1}{2}}(ac+2a^2+4ab+2bc+4ab+8b^2)$$

$$= \mu^{-\frac{1}{2}}(\frac{a^2c}{a} + 2a^2 + \frac{8a^2b}{a} + \frac{2b^2c}{b} + 8b^2)$$

$$= \mu^{-\frac{1}{2}}(a^2(\frac{c}{a} + 2 + \frac{8b}{a}) + b^2(\frac{2c}{b} + 8)).$$

Let

$$M = \max\{(a^2(\frac{c}{a} + 2 + \frac{8b}{a}), (\frac{2c}{b} + 8)\}.$$

Since λ is bounded away from 0, the following inequality holds,

$$|\frac{d}{d\mu}F(\lambda + i\mu)| \leqslant \frac{M}{\sqrt{\mu}}(\| f \|^2 + 2 \| Xf \|^2).$$

The above inequality enables us to show the following is finite,

$$\int_0^1 |\frac{d}{d\mu}F(\lambda + i\mu)|\, d\mu \leqslant \int_0^1 \frac{M}{\sqrt{\mu}}(\| f \|^2 + 2 \| Xf \|^2)\, d\mu$$

$$= 2M(\| f \|^2 + 2 \| Xf \|^2) < \infty,$$

which implies the existence of $\lim_{\mu \to 0+} F(\lambda + i\mu)$.

6.2 Commutator Calculation for the Horocycle Flow

$$(h_t^U)_*(S) = a_tU + b_tV + c_tX$$

for a general vector field S on M.

$$\frac{d}{dt}(h_t^U)_*(S) = \frac{da_t}{dt}U + \frac{db_t}{dt}V + \frac{dc_t}{dt}X.$$

Also,

$$(h^U_{s+t})_*(S) = (a_t \circ h^U_{-s})(h^U_s)_*(U) + (b_t \circ h^U_{-s})(h^U_s)_*(V) + (c_t \circ h^U_{-s})(h_s)_*(X).$$

If we take the derivative with respect to s at $s = 0$, we have

$$\frac{d}{dt}(h^U_t)_*(S) = -(Ua_t)U + (b_t[U,V] - Ub_t)U + c_t([U,X] - Uc_t)X,$$

and thus,

$$\frac{da_t}{dt}U + \frac{db_t}{dt}V + \frac{dc_t}{dt}X = -(Ua_t)U + (b_t[U,V] - Ub_t)U + (c_t[U.X] - Uc_t)X.$$

We now have

$$\frac{d}{dt}(a_t \circ h^U_t) = \frac{da_t}{dt} \circ h^U_t + (Ua_t) \circ h^U_t$$
$$\frac{d}{dt}(b_t \circ h^U_t) = \frac{db_t}{dt} \circ h^U_t + (Ub_t) \circ h^U_t$$
$$\frac{d}{dt}(c_t \circ h^U_t) = \frac{dc_t}{dt} \circ h^U_t - (Uc_t) \circ h^U_t$$

which gives

$$[\frac{d}{dt}(a \circ h^U_t)]U \circ h^U_t + [\frac{d}{dt}(b \circ h^U_t)]V \circ h^U_t + [\frac{d}{dt}(c \circ h^U_t)]X \circ h^U_t$$

$$= (b_t \circ h^U_t)[U,V] \circ h^U_t + (c_t \circ h^U_t)[U,X] \circ h^U_t$$

$$= (b_t \circ h^U_t)X \circ h^U_t + (c_t \circ h^U_t)U \circ h^U_t$$

$\Longrightarrow$

$\frac{d}{dt}(a_t \circ h^U_t) = c_t \circ h^U_t$

$\frac{d}{dt}(b_t \circ h^U_t) \equiv 0$

$\frac{d}{dt}(c_t \circ h^U_t) = b_t \circ h^U_t$

Let $\tilde{a}_t = a_t \circ h_t^U$, $\tilde{b}_t = b_t \circ h_t^U$, and $\tilde{c}_t = c_t \circ h_t^U$.

When $S = X$, the initial condition is $(a_0, b_0, c_0) = (0, 0, 1)$.

$\tilde{a}_t = t$

$\tilde{b}_t = 0$

$\tilde{c}_t = 1$

$\Longrightarrow$

$a_t = t \circ h_{-t}^U = t$

$b_t = 0$

$c_t = 1 \circ h_{-t}^U = 1$

$\Longrightarrow$

$$Dh_t^U(X) = \boxed{tU \circ h_t^U + X \circ h_t^U}$$

and

$$e^{-tU}[X, e^{tU}] = \boxed{tU}$$

6.3 Commutator Calculation for the Time-Changes of the Horocycle Flow

From [8].

$$(h_t^{U_\alpha})_*(S) = a_t U_\alpha + b_t V + c_t X$$

for a general vector field S on M. If we follow the calculation in 4.2, we get

$$[\frac{d}{dt}(a_t \circ h_t^{U_\alpha})]U_\alpha \circ h_t^{U_\alpha} + [\frac{d}{dt}(b_t \circ h_t^{U_\alpha})]V + [\frac{d}{dt}(c_t \circ h_t^{U_\alpha})]X \circ h_t^{U_\alpha}$$

$$= (b_t \circ h_t^{U_\alpha})[U_\alpha, V] \circ h_t^{U_\alpha} + (c_t \circ h_t^{U_\alpha})[U_\alpha, X] \circ h_t^{U_\alpha}$$

$$= (b_t \circ h_t^{U_\alpha})(X/\alpha + \frac{V\alpha}{\alpha}U_\alpha) \circ h_t^{U_\alpha} + (c_t \circ h_t^{U_\alpha})(\frac{X\alpha}{\alpha} - 1)U_\alpha \circ h_t^{U_\alpha}$$

$$= (b_t\frac{1}{\alpha} \circ h_t^{U_\alpha})X \circ h_t^{U_\alpha} + (b_t\frac{V\alpha}{\alpha} \circ h_t^{U_\alpha})U_\alpha \circ h_t^{U_\alpha} + (c_t(\frac{X\alpha}{\alpha} - 1) \circ h_t^{U_\alpha})U_\alpha \circ h_t^{U_\alpha}$$

$\Longrightarrow$

$\frac{d}{dt}(a_t \circ h_t^{U_\alpha}) = b_t\frac{V\alpha}{\alpha} \circ h_t^{U_\alpha} + c_t(\frac{X\alpha}{\alpha} - 1) \circ h_t^{U_\alpha}$

$\frac{d}{dt}(b_t \circ h_t^{U_\alpha}) \equiv 0$

$\frac{d}{dt}(c_t \circ h_t^{U_\alpha} = b_t\frac{1}{\alpha} \circ h_t^{U_\alpha}$

When $S = X$, the initial condition is $(a_0, b_0, c_0) = (0, 0, 1)$.

$a_t = \int_0^t (\frac{X\alpha}{\alpha} - 1) \circ h_{\tau-t}^{U_\alpha}(x)d\tau$

$b_t = 0$

$c_t = 1$

$\Longrightarrow$

$$Dh_t^{U_\alpha}(X) = \boxed{(\int_0^t (\frac{X\alpha}{\alpha} - 1) \circ h_\tau^{U_\alpha}(x)d\tau)U_\alpha \circ h_t^{U_\alpha} + X \circ h_t^{U_\alpha}}$$

and

$$e^{-tU_\alpha}[X, e^{tU_\alpha}] = \boxed{(\int_0^t (\frac{X\alpha}{\alpha} - 1) \circ h_\tau^{U_\alpha}(x)d\tau)U_\alpha}$$

6.4 Commutator Calculation for the Twisted Horocycle Flow

$$(\phi_t^W)_*(S) = a_tW + b_tV + c_tX + d_t\frac{d}{d\theta}$$

for a general vector field S on M. Again if we follow 4.2 we obtain

$$[\frac{d}{dt}(a_t \circ \phi_t^W)]W \circ \phi_t^W + [\frac{d}{dt}(b_t \circ \phi_t^W)]V \circ \phi_t^W + [\frac{d}{dt}(c_t \circ \phi_t^W)]X \circ \phi_t^W + [\frac{d}{dt}(d_t \circ \phi_t^W)]\frac{d}{d\theta} \circ \phi_t^W$$

$$= (b_t \circ \phi_t^W)[W, V] \circ \phi_t^W + (c_t \circ \phi_t^W)[W, X] \circ \phi_t^W$$

$$= (b_t \circ \phi_t^W)(X + V\alpha\frac{d}{d\theta}) \circ \phi_t^W + (c_t \circ \phi_t^W)(U + X\alpha\frac{d}{d\theta}) \circ \phi_t^W$$

$$= (b_t \circ \phi_t^W)X \circ \phi_t^W + (b_t V\alpha \circ \phi_t^W)\frac{d}{d\theta} \circ \phi_t^W + (c_t \circ \phi_t^W)U \circ \phi_t^W$$

$$+(c_t X\alpha \circ \phi_t^W)\frac{d}{d\theta} \circ \phi_t^W + (c_t \circ \phi_t^W)\alpha\frac{d}{d\theta} \circ \phi_t^W - (c_t \circ \phi_t^W)\alpha\frac{d}{d\theta} \circ \phi_t^W$$

$$= (b_t \circ \phi_t^W)X \circ \phi_t^W + (c_t \circ \phi_t^W)W \circ \phi_t^W + (b_t V\alpha \circ \phi_t^W + c_t(X\alpha - \alpha) \circ \phi_t^W)\frac{d}{d\theta} \circ \phi_t^W$$

$\Longrightarrow$

$\frac{d}{dt}(a_t \circ \phi_t^W) = c_t \circ \phi_t^W$

$\frac{d}{dt}(b_t \circ \phi_t^W) \equiv 0$

$\frac{d}{dt}(c_t \circ \phi_t^W) = b_t \circ \phi_t^W$

$\frac{d}{dt}(d_t \circ \phi_t^W) = b_t V\alpha \circ \phi_t^W + c_t(X\alpha - \alpha) \circ \phi_t^W$

When $S = X$, the initial condition is $(a_0, b_0, c_0, d_0) = (0, 0, 1, 0)$.

$a_t = t$

$b_t = 0$

$c_t = 0$

$d_t = \int_0^t (X\alpha - \alpha) \circ \phi_{\tau - t}^W \, d\tau$

$\Longrightarrow$

$$e^{-tW}[X, e^{tW}] = \left(\int_0^t 1 \circ \phi_\tau^W d\tau\right)W + \left(\int_0^t (X\alpha - \alpha) \circ \phi_\tau^W d\tau\right)\frac{d}{d\theta}$$

$$= \boxed{tW + \left(\int_0^t (X\alpha - \alpha) \circ \phi_\tau^W d\tau\right)\frac{d}{d\theta}}$$

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