ABSTRACT

Title of dissertation: ADVENTURES ON NETWORKS:
                                     DEGREES AND GAMES

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A network consists of a set of nodes and edges with the edges representing pairwise connections between nodes. Examples of real-world networks include the Internet, the World Wide Web, social networks and transportation networks often modeled as random graphs. In the first half of this thesis, we explore the degree distributions of such random graphs. In homogeneous networks or graphs, the behavior of the (generic) degree of a single node is often thought to reflect the degree distribution of the graph defined as the usual fractions of nodes with given degree. To study this preconceived notion, we introduce a general framework to discuss the conditions under which these two degree distributions coincide asymptotically in large random networks. Although Erdős-Rényi graphs along with other well known random graph models satisfy the aforementioned conditions, we show that there might be homogeneous random graphs for which such a conclusion may fail to hold. A counterexample to this common notion is found in the class of random threshold graphs. An implication of this finding is that random threshold graphs cannot be

used as a substitute to the Barabási-Albert model for scale-free network modeling, as proposed in some works.

Since the Barabási-Albert model was proposed, other network growth models were introduced that were shown to generate scale-free networks. We study one such basic network growth model, called the *fitness model*, which captures the inherent attributes of individual nodes through fitness values (drawn from a fitness distribution) that influence network growth. We characterize the tail of the network-wide degree distribution through the fitness distribution and demonstrate that the fitness model is indeed richer than the Barabási-Albert model, in that it is capable of producing power-law degree distributions with varying parameters along with other non-Poisson degree distributions.

In the second half of the thesis, we look at the interactions between nodes in a game-theoretic setting. As an example, these nodes could represent interacting agents making decisions over time while the edges represent the dependence of their payoffs on the decisions taken by other nodes. We study learning rules that could be adopted by the agents so that the entire system of agents reaches a desired operating point in various scenarios motivated by practical concerns facing engineering systems. For our analysis, we abstract out the network and represent the problem in the strategic-form repeated game setting.

We consider two classes of learning rules – a class of better-reply rules and a new class of rules, which we call, the class of monitoring rules. Motivated by practical concerns, we first consider a scenario in which agents revise their actions asynchronously based on delayed payoff information. We prove that, under the

better-reply rules (when certain mild assumptions hold), the action profiles played by the agents converge almost surely to a pure-strategy Nash equilibrium (PSNE) with finite expected convergence time in a large class of games called generalized weakly acyclic games (GWAGs). A similar result is shown to hold for the monitoring rules in GWAGs and also in games satisfying a payoff interdependency structure. Secondly, we investigate a scenario in which the payoff information is unreliable, causing agents to make erroneous decisions occasionally. When the agents follow the better-reply rules and the payoff information becomes more accurate over time, we demonstrate the agents will play a PSNE with probability tending to one in GWAGs. Under a similar setting, when the agents follow the monitoring rule, we show that the action profile weakly converges to certain characterizable PSNE(s). Finally, we study a scenario where an agent might erroneously execute an intended action from time to time. Under such a setting, we show that the monitoring rules ensure that the system reaches PSNE(s) which are resilient to deviations by potentially multiple agents.

Adventures on Networks:
Degrees and Games

by

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Part I

Study of Degrees on Networks

Chapter 1:   Networks & Degrees

In the past three decades considerable efforts have been devoted to understanding the rich structure and functions of complex networks, be they technologically engineered, found in nature or generated through social interactions. A popular research direction has been the design of "good" random graph models in the sense of exhibiting key properties found in observed networks – Historically attention was given to the simplest of network properties, namely the degrees of nodes and their distributions.

The discussion invariably starts with the work of Erdős and Rényi [17]: With n nodes and link probability p, the (binomial) Erdős-Rényi graph $\mathbb{G}(n;p)$ postulates that the $\frac{n(n-1)}{2}$ potential undirected links between these n nodes are each created with probability p, independently of each other. The degree distribution of a node in Erdős-Rényi graphs is then announced to be "Poisson"-like, the justification going roughly as follows: (i) With $D_{n,k}(p)$ denoting the degree random variable (rv) of node k in $\mathbb{G}(n;p)$, the rvs $D_{n,1}(p), \ldots, D_{n,n}(p)$ are identically distributed, each distributed according to a binomial rv $\mathrm{Bin}(n-1;p)$; (ii) If the link probability scales with n as $p_n \sim \frac{\lambda}{n}$ for some $\lambda > 0$, then Poisson Convergence ensures the distributional

convergence

$$D_{n,1}(p_n) \Longrightarrow_n D \tag{1.1}$$

where the rv D is a Poisson rv with parameter λ.

A rich asymptotic theory has been developed for Erdős-Rényi graphs in the many node regime; see the monographs [7, 13, 27]. However, as more networks have come under investigation, in many cases the data suggest that the degree distribution is not Poisson but displays instead a *power-law* behavior in the following sense: If the network comprises a large number n of nodes and $N_n(d)$ is the number of nodes with degree d, then the data reveals a behavior of the form

$$\frac{N_n(d)}{n} \simeq Cd^{-\alpha} \tag{1.2}$$

for some α in the range $[2, 3]$ and $C > 0$ – Such statements are usually left somewhat vague as the range of d is never carefully specified in relation to n; networks where (1.2) was observed are often called *scale-free* networks. On the basis of this observation, it was concluded that Erdős-Rényi graphs cannot model scale-free networks, and new random graph models were sought. The Barabási-Albert model came to prominence as the first random graph model to formally "explain" via the mechanism of preferential attachment the possibility of power law degree distributions in large networks [2].

The statement (1.2) concerns a degree distribution which is computed *network-wide*, whereas the convergence (1.1) addresses the behavior of the (generic) degree of a *single* node, its distribution being identical across nodes. A natural question is then whether this discrepancy can be resolved, at least asymptotically, in large

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networks and if so, under what conditions.

Our first contribution lies in exploring this issue in some detail. First we introduce a general framework to investigate this discrepancy in Chapter 2 through a sequence of random graphs $\{\mathbb{G}_n, \ n = 1, 2, \ldots\}$ with increasingly large and unbounded vertex sets. The discussion is carried out under a set of three assumptions, namely

(i) Weak Homogeneity: For each $n = 1, 2, \ldots$, the degree rvs in $\mathbb{G}_n$ are pairwise equidistributed across nodes – Let D_n denote the generic degree rv in $\mathbb{G}_n$;

(ii) Existence of an asymptotic (nodal) degree distribution: There exists an $\mathbb{N}$-valued rv D such that

$$D_n \Longrightarrow_n D. \tag{1.3}$$

Let $\boldsymbol{p} = (p(d), \ d = 0, 1, \ldots)$ denote the pmf of D; and

(iii) Asymptotic uncorrelatedness: The degree rvs display a weak form of asymptotic "pairwise independence."

Under the aforementioned assumptions, we show the following result: If $(p_n(d), \ d = 0, 1, \ldots)$ is the empirical degree distribution in $\mathbb{G}_n$ (with $p_n(d)$ denoting the fraction of nodes in $\mathbb{G}_n$ with degree d), then

$$p_n(d) \xrightarrow{P}_n p(d), \quad d = 0, 1, \ldots \tag{1.4}$$

where the pmf $\boldsymbol{p} = (p(d), \ d = 0, 1, \ldots)$ on $\mathbb{N}$ is as in (ii) above. Essentially this result gives us a set of necessary conditions for the (empirical) degree distribution to converge in the usual sense (1.4). As we discuss the underlying assumptions

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in Chapter 2, we see that Erdős-Rényi graphs (under the scaling yielding (1.1)) are readily subsumed in this framework, as are several well-known homogeneous networks of interest in applications. This resolves the discrepancy mentioned earlier in that the appropriate version of (1.4) does hold for both Erdős-Rényi graphs and the Barabási-Albert model.

Next we turn our attention to the belief, mostly unsubstantiated, that in homogeneous graphs the convergence (1.3) of the generic degree distribution might *automatically* imply the convergence (1.4) of the empirical degree distribution. In Chapter 2 we show in homogeneous graphs that weak asymptotic uncorrelatedness (assumption (iii)) is necessary to ensure convergence of the empirical degree distribution in the usual sense even when the (nodal) degree distribution converges according to (1.3). This brings us to the question – *Are there counterexamples in the class of homogeneous random graphs of any significant interest for which convergence (1.3) of the generic degree distribution take place while the usual convergence of the empirical degree distribution (1.4) does not hold ?* A counterexample described in Chapter 3 is found in the class of random threshold graphs.

We motivate the counterexample by going back to the Barabási-Albert network model. After the Barabási-Albert network model was proposed, researchers began to wonder whether *there were mechanisms other than preferential attachment which could also lead to power law degree distribution.* Caldarelli et al. [9] proposed a homogeneous random graph model called the random threshold model based on the "good-get-richer" mechanism. They argued that in many scenarios, the degree information about every node might not be readily available (as assumed in the

Barabási-Albert network model). Instead, two nodes might form a connection if they are *mutually beneficial* to each other depending on their intrinsic properties (maybe friendship, interaction strength, attractiveness, etc.). The intrisic property of a node is modeled as a *fitness value* drawn from a fitness distribution – A connection is said to form between any two nodes if the sum of their individual fitnesses exceeds a certain threshold. With an exponential fitness distribution, when the threshold is scaled appropriately, they argued that the empirical degree distribution is power-law in the limit of large graph size.

For the setting just described, although (1.3) is known to take place [22] with the power tail behavior

$$p(d) \sim d^{-2} \quad (d \to \infty),$$

we show that (1.4) fails to hold [Proposition 3.3]. One implication of this finding is that random threshold graphs cannot be used as an alternative scale-free model to the Barabási-Albert model (see below) as claimed by the authors [9,37]. Indeed, only the convergence (1.4) has meaning in the preferential attachment model while (1.3) is meaningless, with the situation being reversed for random threshold graphs. In other words, the two models cannot be compared in terms of their degree distributions! This also highlights the fact that *even* in homogeneous graphs, there are noteworthy situations when the network-wide degree distribution and the nodal degree distribution may capture vastly different information. We take the discussion one step further by showing in Chapter 4 that the empirical degree distribution actually converges in a weaker sense.

Echoing the modeling concern in the Barabási-Albert network model posed by Caldarelli et al. [9], Ghadge et al. [23] argued that often degree information might not be readily available, and that the inherent quality of each node captured through a fitness variable (drawn from a fitness distribution in an i.i.d. fashion) should instead be the primary driver of network growth. Through simulations, the authors were able to show a wide range of achievable degree distributions including power-law for fitness lognormally distributed. In Chapter 5, we study this model, often called the *fitness model*, in great detail. We investigate convergence of the empirical degree distribution in the following expected sense

$$\mathbb{E}\left[\frac{N_n(d)}{n}\right] \longrightarrow_n p(d), \ d = 0, 1, \dots \tag{1.5}$$

where the pmf $(p(d), \ d = 0, 1, \dots)$ on $\mathbb{N}$ is the empirical degree distribution in the limit of large graph size. The convergence (1.5) is a weaker form of convergence than (1.4). However, if the convergence (1.4) were to hold, then (1.5) must also hold with the same limiting pmf. Our results indicate that if the fitness distribution is bounded, the tail of the (limiting) empirical degree distribution shows *geometric* decay, i.e., roughly speaking

$$p(d + 1) \simeq Ce^{-\beta d}, \ d = 0, 1, \dots \tag{1.6}$$

for some $\beta > 0$ and $C > 0$. Thus, if the fitness distribution is bounded, we cannot have a power-law behavior. Conversely, for fitness distribution with infinite support, we prove that the tail of the (limiting) empirical degree distribution cannot have a geometric decay. We consider two special cases – (i) When fitness is pareto distributed we show that the asymptotic empirical degree distribution $\{p(d), \ d = 0, 1, \dots\}$ is

indeed power-law with a parameter depending on the fitness distribution. By appropriately choosing the fitness distribution, any power-law degree distribution of parameter greater than two is shown to be achievable. (ii) On the other hand, with exponentially distributed fitness we show that a power-law behavior does not emerge, implying that unbounded fitness distribution would not necessarily lead to a power-law behavior. The fitness model is therefore richer than the Barabási-Albert model in the sense that it is capable of producing a variety of tail behavior along with power-law distributions of different parameters.

This portion of the thesis is organized as follows: In Chapter 2 we introduce a general framework for studying the degree distributions of networks modeled as random graphs. We give necessary and sufficient conditions for homogeneous graphs under which the empirical degree distribution will converge and be identical to the asymptotic nodal degree distribution. We also consider well known examples of random graph models that satisfy these conditions. In Chapter 3 we investigate a specific counterexample in the class of random threshold graphs, where the convergence (1.4) of the empirical degree distribution does not hold. In Chapter 4 we study the degree distribution of random threshold graphs in further detail and show that the empirical degree distribution actually converges in a weaker sense. Finally in Chapter 5 we investigate the degree distribution of the fitness model and study its tail behavior.

Chapter 2: Degree Distribution of Networks

In this chapter we first introduce a general framework for studying various degree distributions of networks modeled as random graphs, namely the degree distribution of a particular node and the (empirical) degree distribution of the graph defined as the fraction of nodes with given degree. We obtain necessary and sufficient conditions in the large graph regime under which the network-wide degree distribution exists and is identical to the nodal degree distribution. In the later part of this chapter, we consider well known examples of random graph models that satisfy these conditions.

2.1 A general framework

We are given a sequence of random graphs $\{\mathbb{G}_n, \ n = 2, 3, \ldots\}$ defined on the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ with the following structure: Fix $n = 2, 3, \ldots$. With V_n a finite and non-empty set, the random graph $\mathbb{G}_n$ is an ordered pair $(V_n, \mathbb{E}_n)$ defined on the set of nodes V_n with random edge set $\mathbb{E}_n \subseteq V_n \times V_n$. The edge set $\mathbb{E}_n$ is equivalently determined by a set of $\{0, 1\}$-valued edge variables $\{\chi_n(k, \ell), \ k, \ell \in V_n\}$ – Thus, $\chi_n(k, \ell) = 1$ (resp. $\chi_n(k, \ell) = 0$) if there is an edge (resp. no edge) from

node k to node ℓ, so that

$$\mathbb{E}_n = \{(k, \ell) \in V_n \times V_n : \chi_n(k, \ell) = 1\}.$$

There is no loss in generality in taking $V_n = \{1, \ldots, k_n\}$ for some positive integer k_n. In most cases of interest $V_n = \{1, \ldots, n\}$ so that $k_n = n$.

We assume $\mathbb{G}_n$ to be an *undirected* graph with *no self-loops*. This amounts to

$$\chi_n(k, k) = 0 \quad \text{and} \quad \chi_n(k, \ell) = \chi_n(\ell, k), \qquad k, \ell \in V_n.$$

Under these conditions the edge set $\mathbb{E}_n$ is a symmetric subset of $V_n \times V_n$ because there is an edge from node k to node ℓ (i.e., $\chi_n(k, \ell) = 1$) if and only if there is an edge from node ℓ to node k (i.e., $\chi_n(\ell, k) = 1$).

For each k in V_n, the degree of node k in $\mathbb{G}_n$ is the rv $D_{n,k}$ given by

$$D_{n,k} = \sum_{\ell \in V_n} \chi_n(k, \ell). \tag{2.1}$$

For each $d = 0, 1, \ldots$, the rv $N_n(d)$ defined by

$$N_n(d) = \sum_{k \in V_n} \mathbf{1}\left[D_{n,k} = d\right]$$

counts the number of nodes in V_n which have degree d in $\mathbb{G}_n$. The fraction of nodes in V_n with degree d in $\mathbb{G}_n$ is then given by

$$\frac{N_n(d)}{|V_n|} = \frac{1}{|V_n|} \sum_{k \in V_n} \mathbf{1}\left[D_{n,k} = d\right].$$

This defines the pmf-valued rv

$$\left(\frac{N_n(d)}{|V_n|}, \; d = 0, 1, \ldots\right)$$

which takes its values in the space of pmfs on $\mathbb{N}$ with support contained in $V_n \cup \{0\}$.

As we focus on limiting results for n large, we assume the sets $\{V_n, \ n = 1, 2, \ldots\}$ to grow unboundedly large with n, namely $\lim_{n\to\infty} |V_n| = \infty$. Concretely, we assume the sequence $n \to k_n$ to be monotone increasing with $\lim_{n\to\infty} k_n = \infty$.

2.2 The main result

With $\lim_{n\to\infty} |V_n| = \infty$, we seek sufficient conditions to ensure the convergence

$$\frac{N_n(d)}{|V_n|} \xrightarrow{P}_n p(d), \quad d = 0, 1, \ldots \tag{2.2}$$

for some pmf $\boldsymbol{p} = (p(d), \ d = 0, 1, \ldots)$ on $\mathbb{N}$. A set of assumptions to that effect is presented next.

Assumption 2.1. *(Weak Homogeneity) For each $n = 2, 3, \ldots$, the degree rvs in $\mathbb{G}_n$ are pairwise equidistributed in the sense that*

$$(D_{n,k}, D_{n,\ell}) =_{st} (D_{n,1}, D_{n,2}) \quad \begin{array}{c} k \neq \ell \\ \\ k, \ell \in V_n. \end{array} \tag{2.3}$$

Note that (2.3) necessarily implies that the degree rvs in $\mathbb{G}_n$ are equidistributed with

$$D_{n,k} =_{st} D_{n,1}, \quad k \in V_n. \tag{2.4}$$

Assumption 2.2. *(Existence of an asymptotic (nodal) degree distribution) Under Assumption 2.1, there exists an $\mathbb{N}$-valued rv D such that*

$$D_{n,1} \Longrightarrow_n D. \tag{2.5}$$

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Let $\boldsymbol{p} = (p(d),\ d = 0, 1, \ldots)$ denote the pmf of the limiting rv D.

Assumption 2.2 can be rephrased as

$$\lim_{n\to\infty} \mathbb{P}\left[D_{n,1} = d\right] = p(d), \quad d = 0, 1, \ldots \tag{2.6}$$

Assumption 2.3. *(Asymptotic uncorrelatedness) Under Assumption 2.1, for each $d = 0, 1, \ldots$, the rvs $\mathbf{1}\left[D_{n,1} = d\right]$ and $\mathbf{1}\left[D_{n,2} = d\right]$ are asymptotically uncorrelated in the sense that*

$$\lim_{n\to\infty} \mathrm{Cov}[\mathbf{1}\left[D_{n,1} = d\right], \mathbf{1}\left[D_{n,2} = d\right]] = 0. \tag{2.7}$$

Assumption 2.3 amounts to the convergence statement

$$\lim_{n\to\infty} \left(\mathbb{P}\left[D_{n,1} = d, D_{n,2} = d\right] - \mathbb{P}\left[D_{n,1} = d\right]\mathbb{P}\left[D_{n,2} = d\right]\right) = 0 \tag{2.8}$$

for each $d = 0, 1, \ldots$. As will become apparent shortly, Assumption 2.2 and Assumption 2.3 will always be used in combination with Assumption 2.1. The main result of this chapter can now be given.

Proposition 2.1. *Under Assumptions 2.1-2.3, we have*

$$\frac{N_n(d)}{|V_n|} \xrightarrow{P}_n p(d), \quad d = 0, 1, \ldots \tag{2.9}$$

where the pmf $\boldsymbol{p} = (p(d),\ d = 0, 1, \ldots)$ is postulated in Assumption 2.2.

The proof of Proposition 2.1 mimics the classical proof of the Weak Law of Large Numbers, and is provided in Section 2.8. A careful inspection of the arguments given there shows that the following partial converse also holds; see Section 2.8 for details.

Proposition 2.2. *Assume Assumptions 2.1-2.2 to hold. If for some $d = 0, 1, \ldots,$*

we have

$$\frac{N_n(d)}{|V_n|} \xrightarrow{P}_n L(d) \tag{2.10}$$

for some constant $L(d)$ in $\mathbb{R}$, then we necessarily have $L(d) = p(d)$ where the pmf

$\boldsymbol{p} = (p(d),\ d = 0, 1, \ldots)$ is the one postulated in Assumption 2.2, and the limit

$$C(d) \equiv \lim_{n \to \infty} \mathrm{Cov}\left[\mathbf{1}\left[D_{n,1} = d\right], \mathbf{1}\left[D_{n,2} = d\right]\right] \tag{2.11}$$

must exist with $C(d) = 0$.

This converse has the following consequence to be used later: Under Assumptions 2.1-2.2, whenever we have

$$\lim_{n \to \infty} \mathrm{Cov}\left[\mathbf{1}\left[D_{n,1} = d\right], \mathbf{1}\left[D_{n,2} = d\right]\right] > 0, \tag{2.12}$$

then the conclusion (2.9) cannot hold.

Finally, under Assumption 2.1, the convergence

$$\frac{N_n(d)}{|V_n|} \xrightarrow{P}_n L(d)$$

for some $d = 0, 1, \ldots$ with some constant $L(d)$ in $\mathbb{R}$ necessarily implies

$$\lim_{n \to \infty} \mathbb{P}\left[D_{n,1} = d\right] = L(d)$$

by bounded convergence. This shows the necessity of Assumption 2.2 for (2.9) to

hold.

2.3 Concerning Assumption 2.3

Assumption 2.3 is implied by the following assumption which is easier to check

in some cases; see Section 2.5 for some examples in a commonly occurring setting.

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Assumption 2.4. *(Pairwise asymptotic independence) Under Assumptions 2.1 and 2.2, the degree rvs $D_{n,1}$ and $D_{n,2}$ are asymptotically independent in the sense that*

$$(D_{n,1}, D_{n,2}) \Longrightarrow_n (D_1, D_2) \tag{2.13}$$

where D_1 and D_2 are independent $\mathbb{N}$-valued rvs, each distributed according to the pmf $\boldsymbol{p}$ postulated in Assumption 2.2.

Assumption 2.4 can be rephrased as

$$\lim_{n \to \infty} \mathbb{P}\left[D_{n,1} = d, D_{n,2} = d'\right] = p(d)p(d'), \quad d, d' = 0, 1, \ldots \tag{2.14}$$

Assumption 2.3 does not require the joint convergence (2.13) to hold. However, if (2.13) is known to hold (with *no further* characterization of the limit), then under Assumption 2.2 it is easy to check that (2.8) is equivalent to the independence of the *binary* rvs $\mathbf{1}[D_1 = d]$ and $\mathbf{1}[D_2 = d]$ for each $d = 0, 1, \ldots$: Indeed, the existence of the limit (2.13) implies

$$\lim_{n \to \infty} \mathbb{P}\left[D_{n,1} = d, D_{n,2} = d\right] = \mathbb{P}\left[D_1 = d, D_2 = d\right]$$

and

$$\lim_{n \to \infty} \mathbb{P}\left[D_{n,j} = d\right] = \mathbb{P}\left[D_j = d\right], \quad j = 1, 2.$$

The condition (2.8) is now equivalent to

$$\mathbb{P}\left[D_1 = d, D_2 = d\right] = p(d)p(d) = \mathbb{P}\left[D_1 = d\right]\mathbb{P}\left[D_2 = d\right], \tag{2.15}$$

and states the independence of the *binary* rvs $\mathbf{1}[D_1 = d]$ and $\mathbf{1}[D_2 = d]$. It should be pointed out that the lack of independence of the rvs D_1 and D_2 does not preclude

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the possibility that the rvs $\mathbf{1}\left[D_1 = d\right]$ and $\mathbf{1}\left[D_2 = d\right]$ are independent – Indeed it is possible for (2.15) to hold even for *all* $d = 0, 1, \ldots$ without the rvs D_1 and D_2 being independent.

2.4 Erdős-Rényi Graphs

We first consider the popular Erdős-Rényi random graph model, and examine whether Assumptions 2.1-2.3 are satisfied.

Consider an undirected graph of n nodes with link probability p – The *Erdős-Rényi graph* $\mathbb{G}(n; p)$ on the vertex set $V_n := \{1, 2, \ldots, n\}$ postulates that the $\frac{n(n-1)}{2}$ potential undirected links between these n nodes are each created with probability p, independently of each other.

The links can be modelled by a collection of i.i.d. Bernoulli rvs $\{B_{ij}(p),\ i, j = 1, 2, \ldots, n\}$ with parameter p; with the restriction that $B_{ij}(p) = B_{ji}(p)$ for distinct $i, j = 1, 2, \ldots, n$ and $B_{ii}(p) = 0$ for $i = 1, 2, \ldots, n$. For $i = 1, 2, \ldots, n$ and $0 < p < 1$, let $D_{n,i}(p)$ denote the degree of node i in $\mathbb{G}(n; p)$. Clearly, we have

$$D_{n,i}(p) = \sum_{j=1, j \neq i}^{n} B_{ij}(p),$$

and Assumption 2.1 is satisfied given the homogeneity of the model. It is well known that when p is scaled according to $p^* : \mathbb{N}_0 \to [0, 1]$ given by

$$p_n^* \sim \frac{\lambda}{n}, \tag{2.16}$$

the degree rvs converge to the Poisson pmf $\boldsymbol{p}_\lambda = (p_\lambda(d),\ d = 0, 1, \ldots)$ with parameter λ given by

$$p_\lambda(d) = \frac{\lambda^d}{d!} e^{-\lambda},\ d = 0, 1, \ldots. \tag{2.17}$$

Hence Assumption 2.2 is also satisfied with $\boldsymbol{p} = \boldsymbol{p}_\lambda$. It is also relatively straightforward to show that the degree rvs become pairwise asymptotically independent. For fixed $n = 1, 2, \ldots, \ell, k = 0, 1, \ldots, n - 1$ and $0 < p < 1$, we have

$$
\mathbb{P}\left[D_{n,1}(p) = k, D_{n,2}(p) = \ell\right]
$$
$$
= \mathbb{P}\left[\sum_{j=2}^{n} B_{1j}(p) = k, \sum_{j=1, j \neq 2}^{n} B_{2j}(p) = \ell\right]
$$
$$
= \mathbb{P}\left[\sum_{j=3}^{n} B_{1j}(p) = k, \sum_{j=3}^{n} B_{2j}(p) = \ell, B_{12}(p) = 0\right]
$$
$$
+ \mathbb{P}\left[\sum_{j=3}^{n} B_{1j}(p) = k - 1, \sum_{j=3}^{n} B_{2j}(p) = \ell - 1, B_{12}(p) = 1\right]. \qquad (2.18)
$$

For the second term in (2.18), we have the bound

$$
\mathbb{P}\left[\sum_{j=3}^{n} B_{1j}(p) = k - 1, \sum_{j=3}^{n} B_{2j}(p) = \ell - 1, B_{12}(p) = 1\right] \leqslant \mathbb{P}\left[B_{12}(p) = 1\right].
$$

However under the scaling (2.16), we have

$$
\lim_{n \to \infty} \mathbb{P}\left[B_{12}(p_n^*) = 1\right] = 0, \qquad (2.19)
$$

which leads to

$$
\lim_{n \to \infty} \mathbb{P}\left[\sum_{j=3}^{n} B_{1j}(p_n^*) = k - 1, \sum_{j=3}^{n} B_{2j}(p_n^*) = \ell - 1, B_{12}(p_n^*) = 1\right] = 0. \qquad (2.20)
$$

For each $n = 2, 3, \ldots$ and $0 < p \leqslant 1$, set

$$
D_{n,i}^*(p) = \sum_{j=3}^{n} B_{ij}(p), \ \ i = 1, 2.
$$

For the first term in (2.18), using the fact that the Bernoulli rvs are assumed

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to be i.i.d., we obtain

$$\mathbb{P}\left[\sum_{j=3}^{n} B_{1j}(p) = k, \sum_{j=3}^{n} B_{2j}(p) = \ell, B_{12}(p) = 0\right]$$

$$= \mathbb{P}\left[\sum_{j=3}^{n} B_{1j}(p) = k\right] \mathbb{P}\left[\sum_{j=3}^{n} B_{2j}(p) = \ell\right] \mathbb{P}\left[B_{12}(p) = 0\right]$$

$$= \mathbb{P}\left[D_{n,1}^*(p) = k\right] \mathbb{P}\left[D_{n,2}^*(p) = \ell\right] \mathbb{P}\left[B_{12}(p) = 0\right]. \tag{2.21}$$

The following reduction step will simplify calculations.

Lemma 2.3. *Fix* $n = 2, 3, \ldots$ *and* $0 \leqslant p \leqslant 1$. *For each* $d = 0, 1, \ldots, n - 1$, *we have*

$$\left|\mathbb{P}\left[D_{n,i}(p) = d\right] - \mathbb{P}\left[D_{n,i}^*(p) = d\right]\right| \leqslant 2\mathbb{P}\left[B_{12}(p) = 1\right], \quad i = 1, 2. \tag{2.22}$$

Proof. Fix $n = 2, 3, \ldots$ and $0 \leqslant p \leqslant 1$. For $i = 1, 2$, we observe that

$$\mathbb{P}\left[D_{n,i}(p) = 0\right] = \mathbb{P}\left[D_{n,i}^*(p) + B_{12}(p) = 0\right]$$

$$= \mathbb{P}\left[D_{n,i}^*(p) = 0, B_{12}(p) = 0\right]$$

$$= \mathbb{P}\left[D_{n,i}^*(p) = 0\right] - \mathbb{P}\left[D_{n,i}^*(p) = 0, B_{12}(p) = 1\right] \tag{2.23}$$

and the bound (2.22) follows for $d = 0$.

For each $d = 1, 2, \ldots, n - 1$, and $i = 1, 2$, we observe that

$$\mathbb{P}\left[D_{n,i}(p) = d\right]$$

$$= \mathbb{P}\left[D_{n,i}^*(p) + B_{12}(p) = d\right]$$

$$= \mathbb{P}\left[D_{n,i}^*(p) = d, B_{12}(p) = 0\right] + \mathbb{P}\left[D_{n,i}^*(p) = d - 1, B_{12}(p) = 1\right]$$

$$= \mathbb{P}\left[D_{n,i}^*(p) = d\right] - \mathbb{P}\left[D_{n,i}^*(p) = d, B_{12}(p) = 1\right] + \mathbb{P}\left[D_{n,i}^*(p) = d - 1, B_{12}(p) = 1\right]$$

and the bound (2.22) follows. ∎

Under the scaling (2.16), using Lemma 2.3 and (2.19), we conclude that

$$D_{n,j}^*(p_n^*) \Longrightarrow_n \mathrm{Poi}(\lambda), \ j = 1, 2.$$

In other words

$$\lim_{n \to \infty} \mathbb{P}\left[D_{n,j}^*(p_n^*) = d\right] = \frac{\lambda^d}{d!}e^{-\lambda}, \quad \begin{array}{l} d = 0, 1, \ldots \\ \\ j = 1, 2. \end{array} \tag{2.24}$$

Returning to (2.21), we conclude from (2.24) (along with (2.19)) that

$$\lim_{n \to \infty} \mathbb{P}\left[\sum_{j=3}^n B_{1j}(p_n^*) = k, \sum_{j=3}^n B_{2j}(p_n^*) = \ell, B_{12}(p_n^*) = 0\right]$$

$$= \lim_{n \to \infty} \mathbb{P}\left[D_{n,1}^*(p) = k\right] \mathbb{P}\left[D_{n,2}^*(p) = \ell\right] \mathbb{P}\left[B_{12}(p) = 0\right]$$

$$= p_\lambda(k)p_\lambda(\ell) \tag{2.25}$$

where $\boldsymbol{p}_\lambda = (p_\lambda(d), \ d = 0, 1, \ldots)$ is the Poisson pmf with parameter λ as defined in (2.17). Putting together (2.20) and (2.25) in (2.18), we obtain the desired result

$$\lim_{n \to \infty} \mathbb{P}\left[D_{n,1}(p_n^*) = k, D_{n,2}(p_n^*) = \ell\right] = p_\lambda(k)p_\lambda(\ell).$$

This shows that the stronger Assumption 2.4 holds. Thus, we have a setting where the Assumptions described in Section 2.2 hold and the empirical degree distribution converges as announced in Proposition 2.1.

2.5   The generic setting

In many situations of interest the sequence of random graphs $\{\mathbb{G}_n, \ n = 1, 2, \ldots\}$ arises in the following natural manner: Given is an underlying parametric family of random graphs, say

$$\{\mathbb{G}(n; \alpha), \ n = 2, 3, \ldots\}, \quad \alpha \in A \subseteq \mathbb{R}^p \tag{2.26}$$

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where A is some parameter set and p is the dimension of the parameter space. With α in A, for each $n = 2, 3, \ldots$, the random graph $\mathbb{G}(n; \alpha)$ is a random graph on V_n whose statistics depend on the parameter α.

For α in A, we define a set of $\{0, 1\}$-valued edge variables $\{\chi_n(k, \ell; \alpha), \ k, \ell \in V_n\}$ corresponding to the graph $\mathbb{G}(n; \alpha)$. Thus, $\chi_n(k, \ell; \alpha) = 1$ (resp. $\chi_n(k, \ell; \alpha) = 0$) if there is an edge (resp. no edge) between node k and node ℓ. Since the graph $\mathbb{G}(n; \alpha)$ is an undirected graph with no self-loops, we must have

$$\chi_n(k, k; \alpha) = 0 \quad \text{and} \quad \chi_n(k, \ell; \alpha) = \chi_n(\ell, k; \alpha), \qquad k, \ell \in V_n.$$

For each k in V_n, let $D_{n,k}(\alpha)$ denote the degree of node k in $\mathbb{G}(n; \alpha)$. For each $d = 0, 1, \ldots$, the rv $N_n(d; \alpha)$ defined by

$$N_n(d; \alpha) = \sum_{k=1}^{n} \mathbf{1}\left[D_{n,k}(\alpha) = d\right]$$

counts the number of nodes in $\{1, \ldots, n\}$ which have degree d in $\mathbb{G}(n; \alpha)$. The fraction of nodes in $\{1, \ldots, n\}$ with degree d in $\mathbb{G}(n; \alpha)$ is then given by

$$p_n(d; \alpha) = \frac{N_n(d; \alpha)}{n}.$$

The next assumption imposes a probabilistic structure on the edge rvs.

Property 2.1. *For all α in A and $n = 2, 3, \ldots$, the set of rvs $\left\{ \chi_n(k, \ell; \alpha), \begin{array}{l} k \neq \ell \\ k, \ell \in V_n \end{array} \right\}$ constitutes an exchangeable family.*

Property 2.1 implies that the rvs $\{D_{n,k}(\alpha), \ k \in V_n\}$ also constitute an exchangeable family. Therefore in $\mathbb{G}(n; \alpha)$ there is no ambiguity as to what is the (nodal) degree distribution because *all* nodes have the same degree distribution,

namely that of the rv $D_{n,1}(\alpha)$. Further, Property 2.1 guarantees that for each α in A, and all k, ℓ in V_n with $k \neq \ell$, it holds that $(D_{n,k}(\alpha), D_{n,\ell}(\alpha)) =_{st} (D_{n,1}(\alpha), D_{n,2}(\alpha))$.

We construct the collection $\{\mathbb{G}_n, \ n = 2, 3, \ldots\}$ by setting

$$\mathbb{G}_n = \mathbb{G}(n; \alpha_n^*), \quad n = 2, 3, \ldots \tag{2.27}$$

where the scaling $\alpha^* : \mathbb{N}_0 \to A$ is the (usually unique) scaling which ensures the convergence

$$D_{n,1}(\alpha_n^*) \Longrightarrow_n D \tag{2.28}$$

for some non-degenerate $\mathbb{N}$-valued rv D. This scaling is often the critical scaling associated with the emergence of a maximal component. Under these circumstances, Assumptions 2.1 and 2.2 are automatically satisfied, and only Assumption 2.3 needs to be verified.

The example of Erdős-Rényi graphs was given earlier. In the following sections we consider additional examples of random graph models routinely discussed in the literature. The setting outlined above is used as it applies to these examples : With $\lambda > 0$,

1. Random key graphs $\mathbb{K}(n; K, P)$ ($K < P$ in $\mathbb{N}_0$) with scalings $K^*, P^* : \mathbb{N}_0 \to \mathbb{N}_0$ given by $\frac{(K_n^*)^2}{P_n^*} \sim \frac{\lambda}{n}$ [38]; and

2. Geometric random graphs $\mathbb{G}(n; \alpha)$ on a unit square ($\alpha > 0$) with scaling $\alpha^* : \mathbb{N}_0 \to \mathbb{R}_+$ given by $\pi(\alpha_n^*)^2 \sim \frac{\lambda}{n}$ [34].

In each case, it is a simple matter to check that D in (2.28) is a Poisson rv with parameter λ. We show that the stronger Assumption 2.4 actually holds in both

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cases, as was the case for Erdős-Rényi graphs.

We continue the discussion by imposing additional structure on the generic setting in order to find conditions under which the stronger Assumption 2.4 might hold. Fix integers $k, \ell = 0, 1, \ldots$ and α in A. For each $n = 2, 3, \ldots$, such that $\max(k, \ell) + 2 \leqslant n$, we have the decomposition

$$
\mathbb{P}\left[D_{n,1}(\alpha) = k, D_{n,2}(\alpha) = \ell\right] = \mathbb{P}\left[D_{n,1}(\alpha) = k, D_{n,2}(\alpha) = \ell, \chi_n(1, 2; \alpha) = 0\right]
$$
$$
+ \mathbb{P}\left[D_{n,1}(\alpha) = k, D_{n,2}(\alpha) = \ell, \chi_n(1, 2; \alpha) = 1\right].
$$

$$(2.29)$$

The second term in (2.29) satisfies the following bound

$$
\mathbb{P}\left[D_{n,1}(\alpha) = k, D_{n,2}(\alpha) = \ell, \chi_n(1, 2; \alpha) = 1\right] \leqslant \mathbb{P}\left[\chi_n(1, 2; \alpha) = 1\right]. \qquad (2.30)
$$

Together (2.29) and (2.30) suggest the following assumption, namely

Property 2.2. *Under the scaling* $\alpha^* : \mathbb{N}_0 \to A$ *satisfying* (2.9)*, it holds that*

$$
\lim_{n \to \infty} \mathbb{P}\left[\chi_n(1, 2; \alpha_n^*) = 1\right] = 0.
$$

If the above-mentioned property is satisfied, then the second term in (2.29) can be disregarded in the limit of large n under the appropriate scaling. Let $\mathcal{N}_{n,i}(\alpha)$ denote the neighbor set of node i in V_n in the graph $\mathbb{G}(n; \alpha)$, i.e.,

$$
\mathcal{N}_{n,i}(\alpha) = \{j \in V_n : \chi_n(i, j; \alpha) = 1\}, \ i \in V_n.
$$

We can further decompose the first term in (2.29) as

$$\mathbb{P}\left[D_{n,1}(\alpha) = k, D_{n,2}(\alpha) = \ell, \chi_n(1,2;\alpha) = 0\right]$$

$$= \mathbb{P}\left[D_{n,1}(\alpha) = k, D_{n,2}(\alpha) = \ell, \chi_n(1,2;\alpha) = 0, \mathcal{N}_{n,1}(\alpha) \cap \mathcal{N}_{n,2}(\alpha) = \varnothing\right]$$

$$+ \mathbb{P}\left[D_{n,1}(\alpha) = k, D_{n,2}(\alpha) = \ell, \chi_n(1,2;\alpha) = 0, \mathcal{N}_{n,1}(\alpha) \cap \mathcal{N}_{n,2}(\alpha) \neq \varnothing\right]. \quad (2.31)$$

Under Property 2.1, we bound the second term in (2.31) as

$$\mathbb{P}\left[D_{n,1}(\alpha) = k, D_{n,2}(\alpha) = \ell, \chi_n(1,2;\alpha) = 0, \mathcal{N}_{n,1}(\alpha) \cap \mathcal{N}_{n,2}(\alpha) \neq \varnothing\right]$$

$$\leqslant \mathbb{P}\left[\mathcal{N}_{n,1}(\alpha) \cap \mathcal{N}_{n,2}(\alpha) \neq \varnothing\right]$$

$$= \mathbb{P}\left[\chi_n(1,j;\alpha) = 1, \chi_n(2,j;\alpha) = 1 \text{ for some } j = 3, \ldots, n\right]$$

$$\leqslant \sum_{j=3}^{n} \mathbb{P}\left[\chi_n(1,j;\alpha) = 1, \chi_n(2,j;\alpha) = 1\right]$$

$$= (n-2)\mathbb{P}\left[\chi_n(1,3;\alpha) = 1, \chi_n(2,3;\alpha) = 1\right] \quad (2.32)$$

with the help of a simple union bound. This leads naturally to the next assumption, namely

Property 2.3. *Under the scaling $\alpha^* : \mathbb{N}_0 \rightarrow A$ satisfying (2.9), it holds that*

$$\lim_{n\to\infty} n\mathbb{P}\left[\chi_n(1,3;\alpha_n^*) = 1, \chi_n(2,3;\alpha_n^*) = 1\right] = 0.$$

Therefore, if Properties 2.1-2.3 are satisfied, then for each $k, \ell = 0, 1, \ldots$, we have

$$\lim_{n\to\infty} \mathbb{P}\left[D_{n,1}(\alpha_n^*) = k, D_{n,2}(\alpha_n^*) = \ell\right] \quad (2.33)$$

$$= \lim_{n\to\infty} \mathbb{P}\left[D_{n,1}(\alpha_n^*) = k, D_{n,2}(\alpha_n^*) = \ell, \chi_n(1,2;\alpha_n^*) = 0, \mathcal{N}_{n,1}(\alpha_n^*) \cap \mathcal{N}_{n,2}(\alpha_n^*) \neq \varnothing\right]$$

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with the understanding that if one of the limits exists, so does the other, and the limiting values coincide.

As we will see, the presence of the events $[\chi_n(1, 2; \alpha) = 0]$ and $[\mathcal{N}_{n,1}(\alpha) \cap \mathcal{N}_{n,2}(\alpha) \neq \varnothing]$ simplifies the calculations that follow. With integers $k, \ell = 0, 1, \ldots$ pick $n = 2, 3, \ldots$ such that $k + \ell + 2 \leqslant n$, and define the set

$$\Upsilon_n(\ell, k) = \left\{ (S, T) \mid S, T \subseteq V_n \backslash \{1, 2\}, S \cap T = \varnothing, |S| = k, |T| = \ell \right\}.$$

Under Property 2.1, for each α in A, we have

$$\mathbb{P}\left[D_{n,1}(\alpha) = k, D_{n,2}(\alpha) = \ell, \chi_n(1, 2; \alpha) = 0, \mathcal{N}_{n,1}(\alpha) \cap \mathcal{N}_{n,2}(\alpha) \neq \varnothing \right]$$

$$= \sum_{(S,T) \in \Upsilon_n(\ell,k)} \mathbb{P}\left[\mathcal{N}_{n,1}(\alpha) = S, \mathcal{N}_{n,2}(\alpha) = T \right]$$

$$= \binom{n-2}{\ell+k} \binom{\ell+k}{k} \mathbb{P}\left[\mathcal{N}_{n,1}(\alpha) = S_0, \mathcal{N}_{n,2}(\alpha) = T_0 \right] \tag{2.34}$$

where we have set $S_0 = \{3, \ldots, k + 2\}$ and $T_0 = \{k + 3, \ldots, k + \ell + 2\}$.

Note that

$$\mathbb{P}\left[\mathcal{N}_{n,1}(\alpha) = S_0, \mathcal{N}_{n,2}(\alpha) = T_0 \right]$$

$$= \mathbb{P}\left[\begin{array}{l} \chi_n(1, 2; \alpha) = 0 \\[2mm] \chi_n(1, s; \alpha) = 1, \chi_n(2, s; \alpha) = 0, s \in S_0 \\[2mm] \chi_n(1, t; \alpha) = 0, \chi_n(2, t; \alpha) = 1, t \in T_0 \\[2mm] \chi_n(1, r; \alpha) = 0, \chi_n(2, r; \alpha) = 0, r \in (S_0 \cup T_0 \cup \{1, 2\})^c \end{array} \right]. \tag{2.35}$$

It is easy to see that we cannot proceed further without imposing additional structure on the underlying random graph model $\mathbb{G}(n; \alpha)$. Therefore, we shall instead consider specific examples of random graph models.

2.6 Random Key Graphs

In this section we introduce a class of random graphs that belong to the class of random intersection graphs; it is naturally associated with the random key pre-distribution scheme of Eschenauer and Gligor [19] in the context of wireless sensor networks. These random graphs have also appeared recently in application areas like clustering analysis [24, 25] and collaborative filtering in recommender systems [29].

Consider n nodes and a pool of P keys. Each node is assigned a set of K distinct keys which are selected at random from the pool of P keys (with $K < P$). Two nodes form a connection if their key rings have at least one key in common. The resulting notion of adjacency defines the *random key graph* $\mathbb{K}(n; \alpha)$ on the vertex set $V_n := \{1, 2, \ldots, n\}$, with the parameter $\alpha \equiv (K, P)$. Clearly the parameter space A is here given by

$$A \equiv \{(K, P) : K, P \in \mathbb{N}_0, K < P\}.$$

Conditions on n, K and P have been sought under which the random key graph $\mathbb{K}(n; \alpha)$ exhibits a non-trivial degree distribution. In [38], the authors show that if the parameters $K^*, P^* : \mathbb{N}_0 \to \mathbb{N}_0$ are scaled such that

$$\frac{(K_n^*)^2}{P_n^*} \sim \frac{\lambda}{n}, \tag{2.36}$$

the asymptotic nodal degree distribution is the Poisson pmf $\boldsymbol{p}_\lambda = \{p_\lambda(d), \ d = 0, 1, \ldots\}$ with parameter λ. Under this scaling we shall show that weak asymptotic independence in the sense of (2.8) holds for the random key graphs. In fact, we shall prove something stronger – the pairwise asymptotic independence of the degree rvs

in the usual sense.

Throughout it is convenient to assume that the keys are labeled $1, \ldots, P$. For each node $i = 1, \ldots, n$, let $K_{n,i}(\alpha)$ denote the random set of K distinct keys assigned to node i. Let $\mathcal{P}_K$ denote the collection of all subsets of $\{1, \ldots, P\}$ which contain exactly K elements. The rvs $K_{n,1}(\alpha), \ldots, K_{n,n}(\alpha)$ are assumed to be i.i.d. rvs, each of which is distributed uniformly over $\mathcal{P}_K$ according to

$$\mathbb{P}\left[K_{n,i}(\alpha) = S\right] = \binom{P}{K}^{-1}, \quad S \in \mathcal{P}_K.$$

With this notation, distinct nodes $i, j = 1, 2, \ldots, n$ are seen to be adjacent if,

$$K_{n,i}(\alpha) \cap K_{n,j}(\alpha) \neq \varnothing.$$

It is easy to check that

$$\mathbb{P}\left[K_{n,i}(\alpha) \cap K_{n,j}(\alpha) = \varnothing\right] = q(\alpha)$$

with

$$q(\alpha) = \begin{cases} 0 & \text{if } P < 2K \\ \dfrac{\binom{P-K}{K}}{\binom{P}{K}} & \text{if } 2K \leqslant P. \end{cases} \tag{2.37}$$

Expression (2.37) is a simple consequence of the fact that

$$\mathbb{P}\left[S \cap K_{n,i}(\alpha) = \varnothing\right] = \begin{cases} 0 & \text{if } |S| > P - K \\ \dfrac{\binom{P-|S|}{K}}{\binom{P}{K}} & \text{if } |S| \leqslant P - K \end{cases}$$

for any subset S of $\{1, \ldots, P\}$. Thus, with

$$\chi_n(i, j; \alpha) = \mathbf{1}\left[K_{n,i}(\alpha) \cap K_{n,j}(\alpha) \neq \varnothing\right]$$

under the generic setting, the probability of edge occurrence in $\mathbb{K}(n; \alpha)$ is given by

$$\mathbb{P}\left[\chi_n(i, j; \alpha) = 1\right] = 1 - q(\alpha). \tag{2.38}$$

For each i in V_n, it is plain that

$$D_{n,i}(\alpha) = \sum_{j \in V_n, j \neq i} \mathbf{1}\left[K_{n,i}(\alpha) \cap K_{n,j}(\alpha) \neq \varnothing\right].$$

Now we consider a sequence of random key graphs $\{\mathbb{K}(n; \alpha_n^*),\ n = 2, 3, \ldots\}$ with $\alpha_n^* \equiv (K_n^*, P_n^*)$ such that (2.36) is satisfied.

Theorem 2.4. *For the class of random key graphs, under the scaling $\alpha^* : \mathbb{N}_0 \to A$ satisfying (2.36), we have*

$$\lim_{n \to \infty} \mathbb{P}\left[D_{n,1}(\alpha_n^*) = k, D_{n,2}(\alpha_n^*) = \ell\right] = p_\lambda(k)p_\lambda(\ell), \quad k, \ell = 0, 1, \ldots \tag{2.39}$$

where the pmf $\boldsymbol{p}_\lambda = (p_\lambda(d),\ d = 0, 1, \ldots)$ is the Poisson pmf on $\mathbb{N}$ with parameter λ.

Theorem 2.4 is established in Section 2.9, and yields a stronger form of independence compared to what is required for Assumption 2.3. Assumptions 2.1-2.3 therefore hold for the random key graphs under the aforementioned scaling. This leads to the following corollary.

Corollary 2.5. *For the class of random key graphs, under the scaling $\alpha^* : \mathbb{N}_0 \to A$ satisfying (2.36), we have*

$$\frac{N_n(d; \alpha_n^*)}{|V_n|} \xrightarrow{P}_n p_\lambda(d), \quad d = 0, 1, \ldots \tag{2.40}$$

where the pmf $\boldsymbol{p}_\lambda = (p_\lambda(d),\ d = 0, 1, \ldots)$ is the Poisson pmf on $\mathbb{N}$ with parameter λ.

2.7 Random Geometric Graphs

Next, we introduce a class of random graphs which is often considered to be a relevant model for (ad-hoc) wireless sensor networks [20, 35]: Consider n nodes which are assumed to be placed uniformly at random on the square $[0, 1]^2$. With set of nodes $V_n = \{1, \ldots, n\}$, let the position of node i in V_n be denoted as $\boldsymbol{Z}_i = (X_i, Y_i)$. Distinct nodes $i, j = 1, 2, \ldots, n$ are said to be adjacent if the distance between them is less than some $\alpha > 0$. If $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ denotes the (Euclidean) distance on $\mathbb{R}^2$, distinct nodes i and j are then adjacent if

$$d(\boldsymbol{Z}_i, \boldsymbol{Z}_j) \leqslant \alpha.$$

According to the notation developed in Section 2.5, with distinct nodes $i, j = 1, \ldots, n$, we have

$$\chi_n(i, j; \alpha) = \mathbf{1}\left[d(\boldsymbol{Z}_i, \boldsymbol{Z}_j) \leqslant \alpha\right].$$

This defines the *random geometric graph* $\mathbb{G}(n; \alpha)$ on the vertex set V_n. This time, for each i in V_n, we have

$$D_{n,i}(\alpha) = \sum_{j \in V_n, j \neq i} \mathbf{1}\left[d(\boldsymbol{Z}_i, \boldsymbol{Z}_j) \leqslant \alpha\right].$$

Under the scaling $\alpha^* : \mathbb{N}_0 \to \mathbb{R}_+$ such that

$$\pi(\alpha_n^*)^2 \sim \frac{\lambda}{n}, \tag{2.41}$$

the asymptotic nodal degree distribution can be shown to be the Poisson pmf $\boldsymbol{p}_\lambda = \{p_\lambda(d), \ d = 0, 1, \ldots\}$ on $\mathbb{N}$ with parameter λ. Under this scaling we will show that weak asymptotic independence holds in the sense of (2.8). As was done for random

key graphs, we prove the stronger pairwise asymptotic independence of the degree rvs in the usual sense.

Theorem 2.6. *For the class of random geometric graphs, under the scaling* $\alpha^* :$ $\mathbb{N}_0 \to \mathbb{R}_+$ *satisfying* (2.41), *we have*

$$\lim_{n\to\infty} \mathbb{P}\left[D_{n,1}(\alpha_n^*) = k, D_{n,2}(\alpha_n^*) = \ell\right] = p_\lambda(k)p_\lambda(\ell), \quad k, \ell \in \mathbb{N}, \tag{2.42}$$

where the pmf $\boldsymbol{p}_\lambda = (p_\lambda(d), \ d = 0, 1, \ldots)$ *is the Poisson pmf on* $\mathbb{N}$ *with parameter* λ.

We prove Theorem 2.6 in Section 2.10. This result is analogous to Theorem 2.4 for random key graphs, and here as well, implies convergence of the empirical degree distribution.

Corollary 2.7. *For the class of random geometric graphs, under the scaling* $\alpha^* :$ $\mathbb{N}_0 \to \mathbb{R}_+$ *satisfying* (2.41), *we have*

$$\frac{N_n(d; \alpha_n^*)}{|V_n|} \xrightarrow{P}_n p_\lambda(d), \quad d = 0, 1, \ldots \tag{2.43}$$

where the pmf $\boldsymbol{p}_\lambda = (p_\lambda(d), \ d = 0, 1, \ldots)$ *is the Poisson pmf on* $\mathbb{N}$ *with parameter* λ.

2.8 Proofs of Propositions 2.1 and 2.2

We begin with a preliminary technical lemma.

Lemma 2.8. *If Assumptions 2.1-2.2 hold, then we have*

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\frac{N_n(d)}{|V_n|} - p(d)\right|^2\right] = \lim_{n\to\infty} \mathrm{Cov}\left[\mathbf{1}\left[D_{n,1} = d\right], \mathbf{1}\left[D_{n,2} = d\right]\right] \tag{2.44}$$

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for each $d = 0, 1, \ldots$, with the understanding that if one of the limits exists, so does

the other and the limiting values coincide.

Proof. Fix $n = 2, 3, \ldots$ and $d = 0, 1, \ldots$. By standard properties of the variance, we note that

$$
\mathbb{E}\left[\left|\frac{N_n(d)}{|V_n|} - p(d)\right|^2\right] = \mathrm{Var}\left[\frac{N_n(d)}{|V_n|}\right] + \left(\mathbb{E}\left[\frac{N_n(d)}{|V_n|}\right] - p(d)\right)^2
$$
$$
= \frac{\mathrm{Var}\left[N_n(d)\right]}{|V_n|^2} + \left(\mathbb{E}\left[\frac{N_n(d)}{|V_n|}\right] - p(d)\right)^2. \qquad (2.45)
$$

Proceeding in the usual manner, we use the definition of the rv $N_n(d)$ to obtain the expressions

$$
\mathbb{E}\left[N_n(d)\right] = \sum_{k \in V_n} \mathbb{P}\left[D_{n,k} = d\right]
$$

and

$$
\mathrm{Var}\left[N_n(d)\right] = \sum_{k \in V_n} \mathrm{Var}\left[\mathbf{1}\left[D_{n,k} = d\right]\right] + \sum_{k, \ell \in V_n:\ k \neq \ell} \mathrm{Cov}\left[\mathbf{1}\left[D_{n,k} = d\right], \mathbf{1}\left[D_{n,\ell} = d\right]\right]
$$

by the binary nature of the involved rvs. Under Assumption 2.1, these expressions imply

$$
\mathbb{E}\left[\frac{N_n(d)}{|V_n|}\right] = \frac{\mathbb{E}\left[N_n(d)\right]}{|V_n|} = \mathbb{P}\left[D_{n,1} = d\right] \qquad (2.46)
$$

and

$$
\mathrm{Var}\left[N_n(d)\right] = |V_n|\mathrm{Var}\left[\mathbf{1}\left[D_{n,1} = d\right]\right] + |V_n|(|V_n| - 1) \cdot \mathrm{Cov}\left[\mathbf{1}\left[D_{n,1} = d\right], \mathbf{1}\left[D_{n,2} = d\right]\right],
$$

respectively. Collecting terms we then conclude that

$$
\mathbb{E}\left[\left|\frac{N_n(d)}{|V_n|} - p(d)\right|^2\right] = (\mathbb{P}\left[D_{n,1} = d\right] - p(d))^2 + \frac{\mathrm{Var}\left[\mathbf{1}\left[D_{n,1} = d\right]\right]}{|V_n|}
$$
$$
+ \frac{|V_n| - 1}{|V_n|} \cdot \mathrm{Cov}\left[\mathbf{1}\left[D_{n,1} = d\right], \mathbf{1}\left[D_{n,2} = d\right]\right].(2.47)
$$

Let n go to infinity in (2.47): Assumption 2.2 implies

$$\lim_{n\to\infty} \mathbb{P}\left[D_{n,1} = d\right] = p(d), \tag{2.48}$$

whereas

$$\lim_{n\to\infty} \frac{\text{Var}\left[\mathbf{1}\left[D_{n,1} = d\right]\right]}{|V_n|} = 0$$

always holds, and the equivalence of the limits at (2.44) immediately follows.   ∎

It is worth pointing out that under Assumptions 2.1-2.2, the arguments just given also show that

$$\liminf_{n\to\infty} \text{Cov}\left[\mathbf{1}\left[D_{n,1} = d\right], \mathbf{1}\left[D_{n,2} = d\right]\right] \geqslant 0. \tag{2.49}$$

2.8.1   A proof of Proposition 2.1

Fix $d = 0, 1, \ldots$. With $\varepsilon > 0$, Tchebychev's inequality gives

$$\mathbb{P}\left[\left|\frac{N_n(d)}{|V_n|} - p(d)\right| > \varepsilon\right] \leqslant \frac{1}{\varepsilon^2} \cdot \mathbb{E}\left[\left|\frac{N_n(d)}{|V_n|} - p(d)\right|^2\right], \quad n = 2, 3, \ldots \tag{2.50}$$

and the convergence (2.43) will be established if we show

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\frac{N_n(d)}{|V_n|} - p(d)\right|^2\right] = 0.$$

By virtue of Lemma 2.8 this is equivalent to having

$$\lim_{n\to\infty} \text{Cov}\left[\mathbf{1}\left[D_{n,1} = d\right], \mathbf{1}\left[D_{n,2} = d\right]\right] = 0,$$

and the proof of Proposition 2.1 is now complete since this limiting condition coincides with the enforced Assumption 2.3.   ∎

2.8.2   A proof of Proposition 2.2

Fix $d = 0, 1, \ldots$, and assume that the convergence

$$\frac{N_n(d)}{|V_n|} \xrightarrow[n]{P} L(d) \tag{2.51}$$

indeed takes place for some constant $L(d)$ in $\mathbb{R}$. Then, it is also the case that

$$\lim_{n \to \infty} \mathbb{E}\left[\frac{N_n(d)}{|V_n|}\right] = L(d) \tag{2.52}$$

by the Bounded Convergence Theorem. The relation (2.46) (seen earlier to hold under Assumption 2.1) and Assumption 2.2 together imply

$$\lim_{n \to \infty} \mathbb{E}\left[\frac{N_n(d)}{|V_n|}\right] = \lim_{n \to \infty} \mathbb{P}\left[D_{n,1} = d\right] = p(d). \tag{2.53}$$

Comparing (2.52) and (2.53) yields $L(d) = p(d)$.

Note that (2.51) (necessarily with $L(d) = p(d)$ under the assumed conditions) occurs if and only if

$$\frac{N_n(d)}{|V_n|} \xrightarrow[n]{L^2} p(d). \tag{2.54}$$

This is because convergence in probability and L^2-convergence are equivalent for uniformly bounded rvs. The latter being equivalent to

$$\lim_{n \to \infty} \mathbb{E}\left[\left|\frac{N_n(d)}{|V_n|} - p(d)\right|^2\right] = 0,$$

we get

$$\lim_{n \to \infty} \mathrm{Cov}\left[\mathbf{1}\left[D_{n,1} = d\right], \mathbf{1}\left[D_{n,2} = d\right]\right] = 0$$

by a final appeal to Lemma 2.8. This completes the proof of Proposition 2.2.   ∎

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2.9   A proof of Theorem 2.4

Before commencing the proof we state a few preliminary results.

2.9.1   Some well known results

We state two lemmas which present simple bounds. The details of the first lemma can be found in [38].

Lemma 2.9. *For positive integers K, L and P such that $K + L \leqslant P$, we have*

$$\left(1 - \frac{L}{P-K}\right)^K \leqslant \frac{\binom{P-L}{K}}{\binom{P}{K}} \leqslant \left(1 - \frac{L}{P}\right)^K. \tag{2.55}$$

This lemma directly leads to the following bounds.

Lemma 2.10. *With positive integers K, L and P such that $K + L \leqslant P$, we have*

$$1 - e^{-\frac{LK}{P}} \leqslant 1 - \frac{\binom{P-L}{K}}{\binom{P}{K}} \leqslant \frac{LK}{P-K}. \tag{2.56}$$

Proof. Lemma 2.9 yields the bounds

$$1 - e^{-\frac{LK}{P}} \leqslant 1 - \frac{\binom{P-L}{K}}{\binom{P}{K}} \leqslant 1 - \left(1 - \frac{L}{P-K}\right)^K.$$

The upper bound in (2.56) follows by noting that

$$1 - \left(1 - \frac{L}{P-K}\right)^K = \int_{1-\frac{L}{P-K}}^1 Kt^{K-1}dt \leqslant \frac{LK}{P-K}.$$

■

It is worth mentioning that the proof given above is based on a proof given in [38, p. 2988].

Corollary 2.11. *For any scaling $P, K : \mathbb{N}_0 \to \mathbb{N}_0$ and any sequence $L : \mathbb{N}_0 \to \mathbb{N}_0$ such that $K_n + L_n < P_n$ for all $n = 1, 2, \ldots$, it holds that*

$$1 - \frac{\binom{P_n - L_n}{K_n}}{\binom{P_n}{K_n}} \sim \frac{L_n K_n}{P_n} \tag{2.57}$$

if and only if

$$\lim_{n \to \infty} \frac{L_n K_n}{P_n} = 0. \tag{2.58}$$

Proof. From Lemma 2.10, we obtain

$$1 - e^{-\frac{L_n K_n}{P_n}} \leqslant 1 - \frac{\binom{P_n - L_n}{K_n}}{\binom{P_n}{K_n}} \leqslant \frac{L_n K_n}{P_n - K_n}, \quad n = 1, 2, \ldots \tag{2.59}$$

provided $K_n + L_n > P_n$. Multiply (2.59) by $\frac{P_n}{L_n K_n}$ and let n go to infinity in the resulting set of inequalities. Under (2.58), we get

$$\lim_{n \to \infty} \frac{P_n}{L_n K_n} \cdot \left(1 - e^{-\frac{L_n K_n}{P_n}}\right) = 1$$

from the elementary fact $\lim_{t \to 0+} \frac{1 - e^{-t}}{t} = 1$, while

$$\lim_{n \to \infty} \frac{P_n}{L_n K_n} \cdot \frac{L_n K_n}{P_n - K_n} = \lim_{n \to \infty} \frac{P_n}{P_n - K_n} = 1$$

by virtue of (2.58) (as it implies $\lim_{n \to \infty} \frac{K_n}{P_n} = 0$). The asymptotic equivalence (2.57) follows.

∎

Fix a positive integer c. Under the scaling (2.36), there exists $n_0(c)$ such that,

$$(c+1)K_n^* < P_n^*$$

for all $n \geqslant n_0(c)$ and Corollary 2.11 yields

$$1 - \frac{\binom{P_n^* - cK_n^*}{K_n^*}}{\binom{P_n^*}{K_n^*}} \sim \frac{c(K_n^*)^2}{P_n^*}. \tag{2.60}$$

2.9.2 The proof

Previously in Section 2.5, we examined a generic method for establishing the asymptotic independence of degree rvs required by Assumption 2.4. Before we can proceed further, we first need to show that Properties 2.1-2.3 are satisfied for random key graphs.

It is clear that Property 2.1 holds for the model. For $n = 2, 3, \ldots$ and $P, K > 0$ such that $2K \leqslant P$, we have from (2.38),

$$\mathbb{P}\left[\chi_n(1, 2; \alpha) = 1\right] = 1 - q(\alpha) = 1 - \binom{P - K}{K} \tag{2.61}$$

where $\alpha \equiv (K, P)$. Under the scaling $\alpha^* : \mathbb{N}_0 \to A$ satisfying (2.36), we have

$$\lim_{n \to \infty} \mathbb{P}\left[\chi_n(1, 2; \alpha_n^*) = 1\right] = \lim_{n \to \infty} \mathbb{P}\left[K_{n,1}(\alpha_n^*) \cap K_{n,2}(\alpha_n^*) \neq \varnothing\right] = 0 \tag{2.62}$$

as we use (2.60) (with $c = 1$). This implies that Property 2.2 is satisfied under this scaling. Next, we show that Property 2.3 also holds under the aforementioned

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scaling. For $\alpha \equiv (K, P)$ in A such that $2K \leqslant P$, and $n = 3, 4, \ldots$, we find

$$n\mathbb{P}\left[\chi_n(1, 3; \alpha) = 1, \chi_n(2, 3; \alpha) = 1\right]$$

$$= n\mathbb{P}\left[K_{n,1}(\alpha) \cap K_{n,3}(\alpha) \neq \varnothing, K_{n,2}(\alpha) \cap K_{n,3}(\alpha) \neq \varnothing\right]$$

$$= n\mathbb{E}\left[\left(\mathbb{P}\left[K_{n,1}(\alpha) \cap S \neq \varnothing\right]^2\right)_{S=K_{n,3}(\alpha)}\right] = n\left(1 - \frac{\binom{P-K}{K}}{\binom{P}{K}}\right)^2.$$

Under the scaling satisfying (2.36), by setting $c = 1$ in (2.60) we conclude that Property 2.3 is indeed satisfied.

Since Properties 2.1-2.3 hold for random key graphs, the equivalence (2.33) holds. Therefore, continuing from (2.35), for α in A, all $k, \ell = 0, 1, \ldots$ and $n = 2, 3, \ldots$ such that $n > k + \ell + 2$, we find

$$\mathbb{P}\left[\mathcal{N}_{n,1}(\alpha) = S_0, \mathcal{N}_{n,2}(\alpha) = T_0\right]$$

$$= \mathbb{P}\left[\begin{array}{l} \chi_n(1, 2; \alpha) = 0 \\[2mm] \chi_n(1, s; \alpha) = 1, \chi_n(2, s; \alpha) = 0, s \in S_0 \\[2mm] \chi_n(1, t; \alpha) = 0, \chi_n(2, t; \alpha) = 1, t \in T_0 \\[2mm] \chi_n(1, r; \alpha) = 0, \chi_n(2, r; \alpha) = 0, r \in (S_0 \cup T_0 \cup \{1, 2\})^c \end{array}\right]$$

$$= \mathbb{P}\left[\begin{array}{l} K_{n,1}(\alpha) \cap K_{n,2}(\alpha) = \varnothing \\[2mm] K_{n,1}(\alpha) \cap K_{n,s}(\alpha) \neq \varnothing, K_{n,2}(\alpha) \cap K_{n,s}(\alpha) = \varnothing, \; s \in S_0 \\[2mm] K_{n,1}(\alpha) \cap K_{n,t}(\alpha) = \varnothing, K_{n,2}(\alpha) \cap K_{n,t}(\alpha) \neq \varnothing, \; t \in T_0 \\[2mm] K_{n,1}(\alpha) \cap K_{n,r}(\alpha) = \varnothing, K_{n,2}(\alpha) \cap K_{n,r}(\alpha) = \varnothing, \; r \in (S_0 \cup T_0 \cup \{1, 2\})^c \end{array}\right]$$

$$(2.63)$$

with $S_0 = \{3, \ldots, k + 2\}$ and $T_0 = \{k + 3, \ldots, k + \ell + 2\}$.

To help deal with (2.63), with R and S given in $\mathcal{P}_K$, we define the events

$$E_{1,n}(R,S;\alpha) = [R \cap K_{n,s}(\alpha) \neq \varnothing, S \cap K_{n,s}(\alpha) = \varnothing, \ s \in S_0],$$

$$E_{2,n}(R,S;\alpha) = [R \cap K_{n,t}(\alpha) = \varnothing, S \cap K_{n,t}(\alpha) \neq \varnothing, \ t \in T_0],$$

and

$$E_{3,n}(R,S;\alpha) = [R \cap K_{n,r}(\alpha) = \varnothing, S \cap K_{n,r}(\alpha) = \varnothing, \ r \in V_n \backslash (\{1,2\} \cup S_0 \cup T_0)].$$

Expressing (2.63) in terms of the events defined above, we see that

$$\mathbb{P}\left[\mathcal{N}_{n,1}(\alpha) = S_0, \mathcal{N}_{n,2}(\alpha) = T_0\right]$$

$$= \mathbb{E}\Big[\mathbf{1}\left[K_{n,1}(\alpha) \cap K_{n,2}(\alpha) = \varnothing\right]$$

$$\times \mathbb{P}\left[E_{1,n}(R,S;\alpha) \cap E_{2,n}(R,S;\alpha) \cap E_{3,n}(R,S;\alpha)\big|K_{n,1}(\alpha), K_{n,2}(\alpha)\right]_{K_{n,1}(\alpha)=R, K_{n,2}(\alpha)=S}\Big]$$

$$= \mathbb{E}\Big[\mathbf{1}\left[K_{n,1}(\alpha) \cap K_{n,2}(\alpha) = \varnothing\right]$$

$$\times \left(\mathbb{P}\left[E_{1,n}(R,S;\alpha)\right]\mathbb{P}\left[E_{2,n}(R,S;\alpha)\right]\mathbb{P}\left[E_{3,n}(R,S;\alpha)\right]\right)_{R=K_{n,1}(\alpha), S=K_{n,2}(\alpha)}\Big].$$

$$(2.64)$$

For R and S in $\mathcal{P}_K$ such that $R \cap S = \varnothing$, and $\alpha \equiv (K,P)$ in A such that $3K \leqslant P$, we obtain

$$\mathbb{P}\left[E_{1,n}(R,S;\alpha)\right] = \mathbb{P}\left[R \cap K_{n,j}(\alpha) \neq \varnothing, S \cap K_{n,j}(\alpha) = \varnothing, \ j \in S_0\right]$$

$$= \mathbb{P}\left[R \cap K_{n,3}(\alpha) \neq \varnothing, S \cap K_{n,3}(\alpha) = \varnothing\right]^k$$

$$= \left(\mathbb{P}\left[S \cap K_{n,3}(\alpha) = \varnothing\right] - \mathbb{P}\left[R \cap K_{n,3}(\alpha) = \varnothing, S \cap K_{n,3}(\alpha) = \varnothing\right]\right)^k$$

$$= \left(\frac{\binom{P-K}{K}}{\binom{P}{K}} - \frac{\binom{P-2K}{K}}{\binom{P}{K}}\right)^k$$

$$= \left[\frac{\binom{P-K}{K}}{\binom{P}{K}}\left(1 - \frac{\binom{P-2K}{K}}{\binom{P-K}{K}}\right)\right]^k, \qquad (2.65)$$

and

$$\mathbb{P}\left[E_{2,n}(R,S;\alpha)\right] = \left[\frac{\binom{P-K}{K}}{\binom{P}{K}}\left(1 - \frac{\binom{P-2K}{K}}{\binom{P-K}{K}}\right)\right]^{\ell}. \tag{2.66}$$

Similarly, we obtain

$$\mathbb{P}\left[E_{3,n}(R,S;\alpha)\right] = \mathbb{P}\left[R \cap K_{n,3}(\alpha) = \varnothing, S \cap K_{n,3}(\alpha) = \varnothing\right]^{n-2-k-\ell}$$

$$= \left(\frac{\binom{P-2K}{K}}{\binom{P}{K}}\right)^{n-2-k-\ell}. \tag{2.67}$$

Substituting (2.65), (2.66) and (2.67) into (2.64) yields

$$\mathbb{P}\left[\mathcal{N}_{n,1}(\alpha) = S_0, \mathcal{N}_{n,2}(\alpha) = T_0\right]$$

$$= \left[\frac{\binom{P-K}{K}}{\binom{P}{K}}\left(1 - \frac{\binom{P-2K}{K}}{\binom{P-K}{K}}\right)\right]^{k+\ell}\left(\frac{\binom{P-2K}{K}}{\binom{P}{K}}\right)^{n-2-k-\ell} \mathbb{P}\left[K_{n,1}(\alpha) \cap K_{n,2}(\alpha) = \varnothing\right]. \tag{2.68}$$

Next, we study the asymptotics of the individual terms in (2.68) under the scaling satisfying (2.36): Investigating the first term in (2.68), we observe

$$\frac{\binom{P_n^* - K_n^*}{K_n^*}}{\binom{P_n^*}{K_n^*}}\left(1 - \frac{\binom{P_n^* - 2K_n^*}{K_n^*}}{\binom{P_n^* - K_n^*}{K_n^*}}\right)$$

$$= \left(1 - \frac{\binom{P_n^* - 2K_n^*}{K_n^*}}{\binom{P_n^* - K_n^*}{K_n^*}}\right) - \left(1 - \frac{\binom{P_n^* - K_n^*}{K_n^*}}{\binom{P_n^*}{K_n^*}}\right)\left(1 - \frac{\binom{P_n^* - 2K_n^*}{K_n^*}}{\binom{P_n^* - K_n^*}{K_n^*}}\right). \tag{2.69}$$

From equation (2.60) (with $c = 1$), we obtain

$$\lim_{n\to\infty} n\left(1 - \frac{\binom{P_n^* - K_n^*}{K_n^*}}{\binom{P_n^*}{K_n^*}}\right) = \lambda \tag{2.70}$$

and

$$\lim_{n\to\infty} n\left(1 - \frac{\binom{P_n^* - 2K_n^*}{K_n^*}}{\binom{P_n^* - K_n^*}{K_n^*}}\right) = \lambda \tag{2.71}$$

where the latter follows by substituting $P_n = P_n^* - K_n^*$, $K_n = K_n^*$ in (2.60) and using the fact that $\lim_{n\to\infty} \frac{K_n^*}{P_n^*} = 0$. Expression (2.70) and (2.71) when substituted

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in (2.69) yields

$$\lim_{n\to\infty} n \left[\frac{\binom{P_n^* - K_n^*}{K_n^*}}{\binom{P_n^*}{K_n^*}} \left(1 - \frac{\binom{P_n^* - 2K_n^*}{K_n^*}}{\binom{P_n^* - K_n^*}{K_n^*}} \right) \right] = \lambda. \tag{2.72}$$

Next, we state a simple result to be used later.

Lemma 2.12. *If for a sequence $a : \mathbb{N} \to \mathbb{R}$ there exists a^* in $\mathbb{R}$ such that*

$$\lim_{n\to\infty} n a_n = a^*,$$

then

$$\lim_{n\to\infty} (1 + a_n)^{n+b} = e^{a^*}, \ \ b \in \mathbb{R}.$$

From (2.60) (with $c = 2$), under the scaling satisfying (2.36), we obtain

$$\lim_{n\to\infty} n \left(1 - \frac{\binom{P_n^* - 2K_n^*}{K_n^*}}{\binom{P_n^*}{K_n^*}} \right) = 2\lambda. \tag{2.73}$$

Returning to (2.67), we can write

$$\left(\frac{\binom{P_n^* - 2K_n^*}{K_n^*}}{\binom{P_n^*}{K_n^*}} \right)^{n-2-k-\ell} = \left[1 - \left(1 - \frac{\binom{P_n^* - 2K_n^*}{K_n^*}}{\binom{P_n^*}{K_n^*}} \right) \right]^{n-2-k-\ell},$$

and Lemma 2.12 now yields

$$\lim_{n\to\infty} \left(\frac{\binom{P_n^* - 2K_n^*}{K_n^*}}{\binom{P_n^*}{K_n^*}} \right)^{n-2-k-\ell} = e^{-2\lambda} \tag{2.74}$$

with the help of (2.73). Applying the asymptotic results (2.62), (2.72) and (2.74), we get from (2.68) that

$$\lim_{n\to\infty} n^{k+\ell} \mathbb{P} \left[\mathcal{N}_{n,1}(\alpha_n^*) = S_0, \mathcal{N}_{n,2}(\alpha_n^*) = T_0 \right] = \lambda^{k+\ell} e^{-2\lambda}.$$

This last relation readily leads to

$$\lim_{n \to \infty} \mathbb{P}\left[D_{n,1}(\alpha_n^*) = k, D_{n,2}(\alpha_n^*) = \ell\right] = \left(\frac{\lambda^k}{k!}e^{-\lambda}\right)\left(\frac{\lambda^\ell}{\ell!}e^{-\lambda}\right)$$

$$= p_\lambda(k)p_\lambda(\ell).$$

upon using (2.33) and (2.34).   ∎

2.10 A proof of Theorem 2.6

First, we define a number of events to be used in the proof of Theorem 2.6, and compute their asymptotic probabilities under the appropriate scaling α^*.

2.10.1 Preliminaries

Fix $n = 2, 3, \ldots$ and $0 < \alpha < \frac{1}{2}$. We find it useful to define the set of points

$$A_n(\alpha) = \{(X, Y) \mid \alpha < X < 1 - \alpha, \ \alpha < Y < 1 - \alpha\}.$$

For each i in V_n, we find it useful to define the event

$$E_{n,i}(\alpha) = \left[(X_i, Y_i) \in A_n(\alpha)\right].$$

It is easy to see that

$$\mathbb{P}\left[E_{n,i}(\alpha)\right] = (1 - 2\alpha)^2. \tag{2.75}$$

On this event a particular node is at least distance α away from the borders of the square $[0, 1]^2$. Next we define the event where a pair of nodes (say nodes 1 and 2) are at least distance 2α apart from each other, namely

$$\tilde{E}_n(\alpha) = \left[d(\mathbf{Z}_1, \mathbf{Z}_2) > 2\alpha\right].$$

Using simple arguments we obtain the following lower bound

$$\mathbb{P}\left[\tilde{E}_n(\alpha)\right] = \mathbb{P}\left[d(\mathbf{Z}_1, \mathbf{Z}_2) > 2\alpha\right]$$

$$= \mathbb{E}\left[\mathbb{P}\left[d(\mathbf{Z}_1, \mathbf{Z}_2) > 2\alpha \mid \mathbf{Z}_1\right]\right]$$

$$\geqslant 1 - 4\pi\alpha^2. \tag{2.76}$$

For $0 < \alpha < \frac{1}{2}$, define the event $E_n(\alpha) = E_{n,1}(\alpha) \cap E_{n,2}(\alpha) \cap \tilde{E}_n(\alpha)$. Under the scaling $\alpha^* : \mathbb{N}_0 \to \mathbb{R}_+$ satisfying (2.41), we obtain

$$\lim_{n\to\infty} \mathbb{P}\left[E_n(\alpha_n^*)\right] = 1 \tag{2.77}$$

since

$$\lim_{n\to\infty} \mathbb{P}\left[E_{n,j}(\alpha_n^*)\right] = 1, \ j = 1, 2, \quad \text{and}$$

$$\lim_{n\to\infty} \mathbb{P}\left[\tilde{E}_n(\alpha_n^*)\right] = 1.$$

2.10.2 The proof

First, we note that random geometric graphs satisfy Property 2.1. For $0 < \alpha < \frac{1}{2}$ and $n = 2, 3, \ldots$, it is plain that

$$\mathbb{P}\left[\chi_n(1, 2; \alpha) = 1\right] = \mathbb{P}\left[|\mathbf{Z}_1 - \mathbf{Z}_2| \leqslant \alpha\right] \leqslant \pi\alpha^2$$

and Property 2.2 is therefore satisfied under the scaling specified in (2.41). Again, for α in A and $n = 2, 3, \ldots$, we have

$$n\mathbb{P}\left[\chi_n(1, 3; \alpha) = 1, \chi_n(2, 3; \alpha) = 1\right] = n\mathbb{P}\left[|\boldsymbol{Z}_1 - \boldsymbol{Z}_3| \leqslant \alpha, |\boldsymbol{Z}_2 - \boldsymbol{Z}_3| \leqslant \alpha\right]$$

$$= n\mathbb{E}\left[\left(\mathbb{P}\left[|\boldsymbol{Z}_1 - \mathbf{x}| \leqslant \alpha, |\boldsymbol{Z}_2 - \mathbf{x}| \leqslant \alpha\right]\right)_{\mathbf{x}=\boldsymbol{Z}_3}\right]$$

$$= n\mathbb{E}\left[\left(\mathbb{P}\left[|\boldsymbol{Z}_1 - \mathbf{x}| \leqslant \alpha\right]^2\right)_{\mathbf{x}=\boldsymbol{Z}_3}\right]$$

$$\leqslant n \cdot (\pi\alpha^2)^2$$

and Property 2.3 is satisfied under the scaling α^*. Properties 2.1-2.3 being satisfied, the equivalence (2.33) is valid for random geometric graphs. Continuing from (2.35), for $0 < \alpha < \frac{1}{2}$, all $k, \ell = 0, 1, \ldots$ and $n = 2, 3, \ldots$ such that $n > k + \ell + 2$, we get

$$\mathbb{P}\left[\mathcal{N}_{n,1}(\alpha) = S_0, \mathcal{N}_{n,2}(\alpha) = T_0\right]$$

$$= \mathbb{P}\left[\begin{array}{l} \chi_n(1, 2; \alpha) = 0 \\[6pt] \chi_n(1, s; \alpha) = 1, \chi_n(2, s; \alpha) = 0, s \in S_0 \\[6pt] \chi_n(1, t; \alpha) = 0, \chi_n(2, t; \alpha) = 1, t \in T_0 \\[6pt] \chi_n(1, r; \alpha) = 0, \chi_n(2, r; \alpha) = 0, r \in (S_0 \cup T_0 \cup \{1, 2\})^c \end{array}\right]$$

$$= \mathbb{P}\left[\begin{array}{l} |\boldsymbol{Z}_1 - \boldsymbol{Z}_2| > \alpha \\[6pt] |\boldsymbol{Z}_1 - \boldsymbol{Z}_s| \leqslant \alpha, |\boldsymbol{Z}_2 - \boldsymbol{Z}_s| > \alpha, \; s \in S_0 \\[6pt] |\boldsymbol{Z}_1 - \boldsymbol{Z}_t| > \alpha, |\boldsymbol{Z}_2 - \boldsymbol{Z}_t| \leqslant \alpha, \; t \in T_0 \\[6pt] |\boldsymbol{Z}_1 - \boldsymbol{Z}_r| > \alpha, |\boldsymbol{Z}_2 - \boldsymbol{Z}_r| > \alpha, \; r \in (S_0 \cup T_0 \cup \{1, 2\})^c \end{array}\right] \tag{2.78}$$

with $S_0 = \{3, \ldots, k + 2\}$ and $T_0 = \{k + 3, \ldots, k + \ell + 2\}$ as before.

To help deal with (2.78), we define the events

$$E'_{1,n}(\mathbf{x}, \mathbf{y}; \alpha) = \left[|\mathbf{x} - \mathbf{Z}_s| \leqslant \alpha, |\mathbf{y} - \mathbf{Z}_s| > \alpha, \ s \in S_0 \right],$$

$$E'_{2,n}(\mathbf{x}, \mathbf{y}; \alpha) = \left[|\mathbf{x} - \mathbf{Z}_t| > \alpha, |\mathbf{y} - \mathbf{Z}_t| \leqslant \alpha, \ t \in T_0 \right],$$

and

$$E'_{3,n}(\mathbf{x}, \mathbf{y}; \alpha) = \left[|\mathbf{y} - \mathbf{Z}_r| > \alpha, |\mathbf{y} - \mathbf{Z}_r| > \alpha, \ r \in (S_0 \cup T_0 \cup \{1,2\})^c \right] \qquad (2.79)$$

for $\mathbf{x}, \mathbf{y}$ in $(0,1)^2$. We decompose (2.78) further by writing

$$\mathbb{P}\left[\mathcal{N}_{n,1}(\alpha) = S_0, \mathcal{N}_{n,2}(\alpha) = T_0 \right] \qquad (2.80)$$

$$= \mathbb{P}\left[\mathcal{N}_{n,1}(\alpha) = S_0, \mathcal{N}_{n,2}(\alpha) = T_0, E_n(\alpha) \right] + \mathbb{P}\left[\mathcal{N}_{n,1}(\alpha) = S_0, \mathcal{N}_{n,2}(\alpha) = T_0, E_n(\alpha)^c \right].$$

Writing the first term in (2.80) in terms of the events defined above, we obtain

$$\mathbb{P}\left[\mathcal{N}_{n,1}(\alpha) = S_0, \mathcal{N}_{n,2}(\alpha) = T_0, E_n(\alpha) \right]$$

$$= \mathbb{E}\left[\mathbf{1}\left[E_n(\alpha) \right] \left(\mathbb{P}\left[E'_{1,n}(\mathbf{x}, \mathbf{y}; \alpha) \right] \mathbb{P}\left[E'_{2,n}(\mathbf{x}, \mathbf{y}; \alpha) \right] \mathbb{P}\left[E'_{3,n}(\mathbf{x}, \mathbf{y}; \alpha) \right] \right)_{\mathbf{x} = \mathbf{Z}_1, \mathbf{y} = \mathbf{Z}_2} \right].$$

$$(2.81)$$

Observe that under the event $E_{n,1}(\alpha)$, we can draw a circle of radius α centered at node 1 which is completely contained within the square $[0,1]^2$, implying that the probability of an edge forming between node 1 and any node s in S_0 is $\pi\alpha^2$. Also, under the event $\tilde{E}_n(\alpha)$ any node connected to node 1 cannot be connected to node 2. Thus, for $\mathbf{x}, \mathbf{y}$ in $A_n(\alpha)$ such that $d(\mathbf{x}, \mathbf{y}) > 2\alpha$, we have

$$\mathbb{P}\left[E'_{1,n}(\mathbf{x}, \mathbf{y}; \alpha) \right] = \mathbb{P}\left[|\mathbf{x} - \mathbf{Z}_s| \leqslant \alpha, |\mathbf{y} - \mathbf{Z}_s| > \alpha, \ s \in S_0 \right]$$

$$= \mathbb{P}\left[|\mathbf{x} - \mathbf{Z}_3| \leqslant \alpha, |\mathbf{y} - \mathbf{Z}_3| > \alpha \right]^k$$

$$= \left(\pi\alpha^2 \right)^k \qquad (2.82)$$

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and

$$\mathbb{P}\left[E_{2,n}'(\mathbf{x}, \mathbf{y}; \alpha)\right] = \left(\pi\alpha^2\right)^\ell. \tag{2.83}$$

Under the event $E_n(\alpha)$, circles of radius α centred at $\mathbf{x}$ and $\mathbf{y}$ do not intersect. Therefore the probability that a particular node does not connect to either node 1 or node 2 is given by $(1 - 2\pi\alpha^2)$. Hence, for $\mathbf{x}, \mathbf{y}$ in $A_n(\alpha)$ such that $d(\mathbf{x}, \mathbf{y}) > 2\alpha$, we have

$$\begin{aligned}
\mathbb{P}\left[E_{3,n}'(\mathbf{x}, \mathbf{y}; \alpha)\right] &= \mathbb{P}\left[|\mathbf{y} - \boldsymbol{Z}_r| > \alpha, |\mathbf{y} - \boldsymbol{Z}_r| > \alpha, \ r \in (S_0 \cup T_0 \cup \{1, 2\})^c\right] \\
&= \mathbb{P}\left[|\mathbf{y} - \boldsymbol{Z}_{k+\ell+3}| > \alpha, |\mathbf{y} - \boldsymbol{Z}_{k+\ell+3}| > \alpha\right]^{n-2-k-\ell} \\
&= (1 - 2\pi\alpha^2)^{n-2-k-\ell}. \tag{2.84}
\end{aligned}$$

Furthermore, for all $\mathbf{x}, \mathbf{y}$ in $[0,1]^2$, we have the following upper bound

$$\begin{aligned}
\mathbb{P}\left[E_{1,n}'(\mathbf{x}, \mathbf{y}; \alpha)\right] &= \mathbb{P}\left[|\mathbf{x} - \boldsymbol{Z}_s| \leqslant \alpha, |\mathbf{y} - \boldsymbol{Z}_s| > \alpha, \ s \in S_0\right] \\
&\leqslant \mathbb{P}\left[|\mathbf{x} - \boldsymbol{Z}_s| \leqslant \alpha, \ s \in S_0\right] \\
&\leqslant \left(\pi\alpha^2\right)^k \tag{2.85}
\end{aligned}$$

and

$$\mathbb{P}\left[E_{2,n}'(\mathbf{x}, \mathbf{y}; \alpha)\right] \leqslant \left(\pi\alpha^2\right)^\ell. \tag{2.86}$$

Using (2.82),(2.83) and (2.84) on (2.81), we obtain

$$\mathbb{P}\left[\mathcal{N}_{n,1}(\alpha) = S_0, \mathcal{N}_{n,2}(\alpha) = T_0, E_n(\alpha)\right] = (\pi\alpha^2)^{k+\ell}(1 - 2\pi\alpha^2)^{n-2-k-\ell}\mathbb{P}\left[E_n(\alpha)\right]. \tag{2.87}$$

Under the scaling α^* satisfying (2.41), we obtain

$$\lim_{n\to\infty} n^{k+\ell}\mathbb{P}\left[\mathcal{N}_{n,1}(\alpha_n^*) = S_0, \mathcal{N}_{n,2}(\alpha_n^*) = T_0, E_n(\alpha_n^*)\right] = \lambda^{k+\ell}e^{-2\lambda} \tag{2.88}$$

upon using Lemma 2.12 and (2.77). The bounds (2.85) and (2.86) allow us to bound the second term in (2.80) as

$$\mathbb{P}\left[\mathcal{N}_{n,1}(\alpha) = S_0, \mathcal{N}_{n,2}(\alpha) = T_0, E_n(\alpha)^c\right]$$

$$= \mathbb{E}\left[\mathbf{1}\left[E_n(\alpha)^c\right]\left(\mathbb{P}\left[E'_{1,n}(\mathbf{x}, \mathbf{y}; \alpha)\right]\mathbb{P}\left[E'_{2,n}(\mathbf{x}, \mathbf{y}; \alpha)\right]\mathbb{P}\left[E'_{3,n}(\mathbf{x}, \mathbf{y}; \alpha)\right]\right)_{\mathbf{x}=\mathbf{Z}_1, \mathbf{y}=\mathbf{Z}_2}\right]$$

$$\leqslant \mathbb{E}\left[\mathbf{1}\left[E_n(\alpha)^c\right]\left(\mathbb{P}\left[E'_{1,n}(\mathbf{x}, \mathbf{y}; \alpha)\right]\mathbb{P}\left[E'_{2,n}(\mathbf{x}, \mathbf{y}; \alpha)\right]\right)_{\mathbf{x}=\mathbf{Z}_1, \mathbf{y}=\mathbf{Z}_2}\right]$$

$$\leqslant (\pi\alpha^2)^{k+\ell}\mathbb{P}\left[E_n(\alpha)^c\right],$$

whence

$$\lim_{n\to\infty} n^{k+\ell}\mathbb{P}\left[\mathcal{N}_{n,1}(\alpha_n^*) = S_0, \mathcal{N}_{n,2}(\alpha_n^*) = T_0, E_n(\alpha_n^*)^c\right] = 0 \qquad (2.89)$$

by virtue of (2.77). Returning to (2.80) and using (2.88) and (2.89), we obtain

$$\lim_{n\to\infty} n^{k+\ell}\mathbb{P}\left[\mathcal{N}_{n,1}(\alpha_n^*) = S_0, \mathcal{N}_{n,2}(\alpha_n^*) = T_0\right] = \lambda^{k+\ell}e^{-2\lambda}. \qquad (2.90)$$

Substituting into (2.34) and using the equivalence step (2.33), we obtain

$$\lim_{n\to\infty} \mathbb{P}\left[D_{n,1}(\alpha_n^*) = k, D_{n,2}(\alpha_n^*) = \ell\right] = \left(\frac{\lambda^k}{k!}e^{-\lambda}\right)\left(\frac{\lambda^\ell}{\ell!}e^{-\lambda}\right)$$

$$= p_\lambda(k)p_\lambda(\ell).$$

∎

Chapter 3:   A Counterexample: Random Threshold Graphs

In the previous chapter, we introduced a generic framework for studying the degree distributions of random graphs. We identified a set of necessary and sufficient conditions under which the network-wide (empirical) degree distribution coincides with the nodal degree distribution in the large graph limit. We also demonstrated three instances of homogeneous graphs where these conditions are satisfied, namely Erdős-Rényi graphs, random key graphs and random geometric graphs.

In the present chapter, we give a counterexample to show that *even* in homogeneous graphs, the empirical degree distribution and the nodal degree distribution may capture vastly different information. This counterexample is found in the class of random threshold graphs, where the empirical degree distribution does not converge in the usual sense even though the asymptotic nodal degree distribution exists.

3.1   The model

As discussed earlier in Chapter 1, Caldarelli et al. [9] proposed the class of random threshold graphs as capable of achieving scale-free degree distribution without the notion of preferential attachment [2]. It belongs to the broader class of *hidden variable* models where connections are formed on the basis of *fitness variables*

associated with individual nodes. The random threshold graph model is based on the notion that connections between nodes are driven by mutual benefit based on intrinsic attributes, and is realized as follows – Two nodes form a connection if the sum of their fitness variables exceeds a certain threshold.

We now formally introduce the model: Let $\{\xi, \xi_k, \ k = 1, 2, \ldots\}$ denote a collection of i.i.d. $\mathbb{R}_+$-valued rvs defined on the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, each distributed according to a given (probability) distribution function $F : \mathbb{R} \to [0, 1]$. [1] With ξ acting as a generic representative for this sequence of i.i.d. rvs, we have

$$\mathbb{P}\left[\xi \leqslant x\right] = F(x), \quad x \in \mathbb{R}.$$

At minimum we assume that F is a continuous function on $\mathbb{R}$ with

$$F(x) = 0, \quad x \leqslant 0.$$

Once F is specified, random thresholds graphs are characterized by two parameters, namely a positive integer n and a threshold value $\alpha > 0$. Specifically, the network comprises n nodes, labelled $k = 1, \ldots, n$, and to each node k we assign a *fitness* variable (or weight) ξ_k which measures its importance or rank. For distinct $i, j = 1, \ldots, n$, nodes i and j are declared to be adjacent if

$$\xi_i + \xi_j > \alpha, \tag{3.1}$$

i.e.,

$$\chi_n(i, j; \alpha) = \mathbf{1}\left[\xi_i + \xi_j > \alpha\right]$$

---

[1]What we call here a probability distribution function is also called a cumulative distribution function in other literatures.

according to the notation developed in Section 2.1. The adjacency notion (3.1) defines the *random threshold* graph $\mathbb{T}(n; \alpha)$ on the set of vertices $V_n = \{1, \ldots, n\}$.

For each $n = 1, 2, \ldots$, and each $\alpha > 0$, we set

$$D_{n,k}(\alpha) = \sum_{\ell=1, \ \ell \neq k}^{n} \mathbf{1}\left[\xi_k + \xi_\ell > \alpha\right], \quad k = 1, \ldots, n$$

so that $D_{n,k}(\alpha)$ is the degree of node k in $\mathbb{T}(n; \alpha)$. Under the enforced assumptions, the rvs $D_{n,1}(\alpha), \ldots, D_{n,n}(\alpha)$ are exchangeable, thus equidistributed. Furthermore, the edge rvs $\left\{ \chi_n(i, j; \alpha), \ \begin{matrix} i \neq j \\ i, j \in V_n \end{matrix} \right\}$ constitute an exchangeable family, i.e., Property 2.1 is satisfied.

3.1.1 Applying Proposition 2.1 under exponential fitness

From now on we focus on the special case when ξ is exponentially distributed with parameter $\lambda > 0$, written $\xi \sim \mathrm{Exp}(\lambda)$, that is

$$\mathbb{P}\left[\xi \leqslant x\right] = 1 - e^{-\lambda x^+}, \quad x \in \mathbb{R}. \tag{3.2}$$

Here we use the standard notation $x^+ = \max(x, 0)$ for x in $\mathbb{R}$. Other distributions could be considered to develop counterexamples to Proposition 2.1. However, the exponential distribution was selected for two main reasons: This case was considered in [9, 22, 37] to show that scale-free networks can be generated through the fitness-based mechanism used in random threshold graphs; more on that later. Moreover, calculations are greatly simplified in the exponential case.

Fujihara et al. [22, Example 1, p. 366] showed that under the scaling $\alpha^\star$:

$\mathbb{N}_0 \to \mathbb{R}_+$ given by

$$\alpha_n^\star = \lambda^{-1} \log n, \quad n = 2, 3, \dots. \tag{3.3}$$

with the understanding that

$$\mathbb{G}_n = \mathbb{T}(n; \alpha_n^*), \quad n = 2, 3, \dots$$

we have the following distributional convergence $D_{n,1}(\alpha_n^*) \Longrightarrow_n D$ where the $\mathbb{N}$-valued rv D is a conditionally Poisson rv with pmf $\boldsymbol{p}_{\mathrm{Fuj}} = (p_{\mathrm{Fuj}}(d), \ d = 0, 1, \dots)$ given by

$$p_{\mathrm{Fuj}}(d) = \mathbb{P}\left[D = d\right] = \mathbb{E}\left[\frac{(e^{\lambda \xi})^d}{d!} e^{-e^{\lambda \xi}}\right], \quad d = 0, 1, \dots \tag{3.4}$$

Therefore, Assumption 2.2 holds with

$$\lim_{n \to \infty} \mathbb{P}\left[D_{n,1}(\alpha_n^*) = d\right] = p_{\mathrm{Fuj}}(d), \quad d = 0, 1, \dots \tag{3.5}$$

Hence, we are in the generic setting of Section 2.5. One way to prove Assumption 2.4 would be to first show that Properties 2.1-2.3 hold. While we have already argued that Property 2.1 is satisfied, the following result states that in the present context Property 2.2 holds whereas Property 2.3 does not.

Proposition 3.1. *Consider the class of random threshold graphs with $\xi \sim Exp(\lambda)$ for some $\lambda > 0$. Under the scaling $\alpha^* : \mathbb{N}_0 \to \mathbb{R}_+$ given by (3.3), Property 2.2 holds whereas Property 2.3 does not.*

Proof. For a fixed $\alpha > 0$, and $n = 2, 3, \ldots$, we get

$$\mathbb{P}\left[\xi_1 + \xi_2 > \alpha\right] = \mathbb{E}\left[\left(\mathbb{P}\left[\xi_2 > \alpha - t\right]\right)_{t=\xi_1}\right]$$

$$= \mathbb{E}\left[\mathbf{1}\left[\xi_1 > \alpha\right]\right] + \mathbb{E}\left[\mathbf{1}\left[\xi_1 \leqslant \alpha\right]\left(\mathbb{P}\left[\xi_2 > \alpha - t\right]\right)_{t=\xi_1}\right]$$

$$= \mathbb{P}\left[\xi_1 > \alpha\right] + \mathbb{E}\left[\mathbf{1}\left[\xi_1 \leqslant \alpha\right]e^{-\lambda(\alpha - \xi_1)}\right]$$

$$= e^{-\lambda\alpha}\left[1 + \mathbb{E}\left[\mathbf{1}\left[\xi_1 \leqslant \alpha\right]e^{\lambda\xi_1}\right]\right]$$

$$= e^{-\lambda\alpha}\left(1 + \alpha\lambda\right)$$

by easy calculation, and Property 2.2 clearly holds under the scaling α^* satisfying (3.3).

For a fixed $\alpha > 0$, and $n = 3, 4, \ldots$, we now see that

$$\mathbb{P}\left[\xi_1 + \xi_3 > \alpha, \xi_2 + \xi_3 > \alpha\right]$$

$$= \mathbb{E}\left[\mathbf{1}\left[\xi_3 > \alpha\right]\right] + \mathbb{E}\left[\mathbf{1}\left[\xi_3 \leqslant \alpha\right]\left(\mathbb{P}\left[\xi_1 > \alpha - t\right]^2\right)_{t=\xi_3}\right]$$

$$= \mathbb{P}\left[\xi_3 > \alpha\right] + \mathbb{E}\left[\mathbf{1}\left[\xi_3 \leqslant \alpha\right]e^{-2\lambda(\alpha - \xi_3)}\right]$$

$$= e^{-\lambda\alpha} + e^{-2\lambda\alpha}\cdot\mathbb{E}\left[\mathbf{1}\left[\xi_3 \leqslant \alpha\right]e^{2\lambda\xi_3}\right]$$

$$= e^{-\lambda\alpha} + e^{-2\lambda\alpha}\int_0^\alpha \lambda e^{-\lambda y}e^{2\lambda y}dy$$

$$= e^{-\lambda\alpha}\left(2 - e^{-\lambda\alpha}\right).$$

Under the scaling α^* given by (3.3), we conclude

$$\lim_{n\to\infty} n\mathbb{P}\left[\xi_1 + \xi_3 > \alpha_n^*, \xi_2 + \xi_3 > \alpha_n^*\right] = 2,$$

so that Property 2.3 does not hold. ∎

Therefore the approach formulated in Section 2.5 using the step (2.33) is not applicable here. Hence, we are left with no choice but to directly test the validity

of the weaker Assumption 2.3. The remainder of the chapter is devoted to showing that the weaker Assumption 2.3 actually fails.

Proposition 3.2. *Consider the class of random threshold graphs with $\xi \sim \mathrm{Exp}(\lambda)$ for some $\lambda > 0$. Under the scaling $\alpha^* : \mathbb{N}_0 \to \mathbb{R}_+$ given by (3.3), the limit*

$$C(d) \equiv \lim_{n \to \infty} \mathrm{Cov}[\mathbf{1}\left[D_{n,1}(\alpha_n^\star) = d\right], \mathbf{1}\left[D_{n,2}(\alpha_n^\star) = d\right]] \tag{3.6}$$

exists with $C(d) > 0$ for each $d = 0, 1, \ldots$.

The specific values for (3.6) are omitted here but are computed during the proof of Proposition 3.2 given from Section 3.2 to Section 3.4. For instance, we show that

$$C(0) = \mathbb{E}\left[e^{-\max(e^{\lambda \xi_1}, e^{\lambda \xi_2})}\right] - \mathbb{E}\left[e^{-(e^{\lambda \xi_1} + e^{\lambda \xi_2})}\right] > 0.$$

The expression of $C(d)$ for arbitrary $d \neq 0$ is rather cumbersome and is not shown at this time. However, the fact that $C(d) > 0$ on the entire range suffices to establish the desired counterexample via the observation following Proposition 2.2.

Proposition 3.3. *Consider the class of random threshold graphs with $\xi \sim \mathrm{Exp}(\lambda)$ for some $\lambda > 0$. Under the scaling $\theta^* : \mathbb{N}_0 \to \mathbb{R}_+$ given by (3.3), each of the sequence of rvs*

$$\left\{\frac{1}{n} \sum_{k=1}^{n} \mathbf{1}\left[D_{n,k}(\theta_n^\star) = d\right], \ n = 2, 3, \ldots\right\}, \ d = 0, 1, \ldots \tag{3.7}$$

does not converge in probability to a constant.

The fact that the convergence (2.43) fails to occur in the context of random threshold graphs is significant for the following reason: Caldarelli et al. [9, 37] have

proposed this subclass of hidden variable models as an alternative scale-free model to the preferential attachment model of Barabási and Albert [2]. The evidence behind this proposal lies in the provable power-law behavior [22, Example 1, p. 366]

$$p_{\text{Fuj}}(d) \sim d^{-2} \quad (d \to \infty). \tag{3.8}$$

However, a meaningful comparison between the two models would have required at minimum the validity of the convergence

$$\frac{1}{n} \sum_{k=1}^{n} \mathbf{1}\left[D_{n,k}(\alpha_n^{\star}) = d\right] \xrightarrow{P}_{n} p_{\text{Fuj}}(d), \quad d = 0, 1, \dots$$

As we now know through Proposition 3.3, this fails to happen, and the two models cannot be meaningfully compared as already explained in the introductory section.

3.2   A proof of Proposition 3.2 – Part I

We begin with an easy observation.

3.2.1   A reduction step

For every $n = 2, 3, \dots$ and $\alpha > 0$, note the decomposition

$$D_{n,j}(\alpha) = \mathbf{1}\left[\xi_1 + \xi_2 > \alpha\right] + D_{n,j}^{\star}(\alpha), \quad j = 1, 2 \tag{3.9}$$

where we have set

$$D_{n,j}^{\star}(\alpha) = \sum_{k=3}^{n} \mathbf{1}\left[\xi_j + \xi_k > \alpha\right].$$

Evaluating the limit (3.6) can be simplified through an easy reduction step which we now develop.

Lemma 3.4. *Fix $n = 2, 3, \ldots$ and $\alpha > 0$. For each $d = 0, 1, \ldots, n - 2$, we have*

$$\left| \mathbb{P}\left[D_{n,j}(\alpha) = d \right] - \mathbb{P}\left[D_{n,j}^\star(\alpha) = d \right] \right| \leqslant 2\mathbb{P}\left[\xi_1 + \xi_2 > \alpha \right], \quad j = 1, 2 \qquad (3.10)$$

and

$$\left| \mathbb{P}\left[D_{n,1}(\alpha) = d, D_{n,2}(\alpha) = d \right] - \mathbb{P}\left[D_{n,1}^\star(\alpha) = d, D_{n,2}^\star(\alpha) = d \right] \right| \leqslant 2\mathbb{P}\left[\xi_1 + \xi_2 > \alpha \right].$$

$$(3.11)$$

Proof. The first part of the Lemma can be proved similarly to that of Lemma 2.3. To prove the second part, fix $n = 2, 3, \ldots$ and $\alpha > 0$. Firstly for $d = 0$, using the decomposition (3.9) we find

$$\mathbb{P}\left[D_{n,1}(\alpha) = 0, D_{n,2}(\alpha) = 0 \right]$$

$$= \mathbb{P}\left[D_{n,1}^*(\alpha) = 0, D_{n,2}^*(\alpha) = 0, \xi_1 + \xi_2 \leqslant \alpha \right]$$

$$= \mathbb{P}\left[D_{n,1}^*(\alpha) = 0, D_{n,2}^*(\alpha) = 0 \right] - \mathbb{P}\left[D_{n,1}^*(\alpha) = 0, D_{n,2}^*(\alpha) = 0, \xi_1 + \xi_2 > \alpha \right] \quad (3.12)$$

and the bound (3.11) holds. For each $d = 1, 2, \ldots, n-2$, we have

$$\mathbb{P}\left[D_{n,1}(\alpha) = d, D_{n,2}(\alpha) = d\right]$$

$$= \mathbb{P}\left[\mathbf{1}\left[\xi_1 + \xi_2 > \alpha\right] + D_{n,1}^{\star}(\alpha) = d, \mathbf{1}\left[\xi_1 + \xi_2 > \alpha\right] + D_{n,2}^{\star}(\alpha) = d\right]$$

$$= \mathbb{P}\left[\xi_1 + \xi_2 \leqslant \alpha, D_{n,1}^{\star}(\alpha) = d, D_{n,2}^{\star}(\alpha) = d\right]$$

$$\quad + \mathbb{P}\left[\xi_1 + \xi_2 > \alpha, D_{n,1}^{\star}(\alpha) = d-1, D_{n,2}^{\star}(\alpha) = d-1\right]$$

$$= \mathbb{P}\left[\xi_1 + \xi_2 > \alpha, D_{n,1}^{\star}(\alpha) = d-1, D_{n,2}^{\star}(\alpha) = d-1\right]$$

$$\quad - \mathbb{P}\left[\xi_1 + \xi_2 > \alpha, D_{n,1}^{\star}(\alpha) = d, D_{n,2}^{\star}(\alpha) = d\right] + \mathbb{P}\left[D_{n,1}^{\star}(\alpha) = d, D_{n,2}^{\star}(\alpha) = d\right]$$

$$\tag{3.13}$$

and the bound (3.11) follows by combining (3.12) and (3.13). ∎

This simple fact leads to the following reduction step when evaluating the limit at (3.6): With $d = 0, 1, \ldots$ held fixed, for each $n = 2, 3, \ldots$ we substitute α by $\alpha_n^{\star}$ in the bound (3.11) according to (3.3), and let n go to infinity in the resulting inequality. Since $\lim_{n\to\infty} \alpha_n^{\star} = \infty$, we conclude that

$$\lim_{n\to\infty}\left|\mathbb{P}\left[D_{n,1}(\alpha_n^{\star}) = d, D_{n,2}(\alpha_n^{\star}) = d\right] - \mathbb{P}\left[D_{n,1}^{\star}(\alpha_n^{\star}) = d, D_{n,2}^{\star}(\alpha_n^{\star}) = d\right]\right| = 0,$$

whence

$$\lim_{n\to\infty}\mathbb{P}\left[D_{n,1}(\alpha_n^{\star}) = d, D_{n,2}(\alpha_n^{\star}) = d\right] = \lim_{n\to\infty}\mathbb{P}\left[D_{n,1}^{\star}(\alpha_n^{\star}) = d, D_{n,2}^{\star}(\alpha_n^{\star}) = d\right] \tag{3.14}$$

provided either limit exists. The same argument applied to the bounds (3.10) readily yields

$$\lim_{n\to\infty}\mathbb{P}\left[D_{n,j}(\alpha_n^{\star}) = d\right] = \lim_{n\to\infty}\mathbb{P}\left[D_{n,j}^{\star}(\alpha_n^{\star}) = d\right] = p_{\mathrm{Fuj}}(d), \quad j = 1, 2 \tag{3.15}$$

in light of (3.5). It follows from (3.15) and (3.14) that

$$C(d) = \lim_{n \to \infty} \mathrm{Cov}[\mathbf{1}\left[D_{n,1}^{\star}(\alpha_n^{\star}) = d\right], \mathbf{1}\left[D_{n,2}^{\star}(\alpha_n^{\star}) = d\right]] \qquad (3.16)$$

provided the limits at (3.14) exist.

3.2.2 Order statistics to the rescue

When evaluating (3.16), it will be convenient to introduce a second collection of $\mathbb{R}_+$-valued rvs $\{\eta_\ell, \ \ell = 1, 2, \ldots\}$. We assume that the rvs $\{\eta_\ell, \ \ell = 1, 2, \ldots\}$ are also i.i.d. rvs, each of which is exponentially distributed with parameter $\lambda > 0$. The two collections $\{\xi, \xi_k, \ k = 1, 2, \ldots\}$ and $\{\eta_\ell, \ \ell = 1, 2, \ldots\}$ are assumed to be mutually independent. For each integer $p = 2, 3, \ldots$, let $\eta_{p,1}, \ldots, \eta_{p,p}$ denote the values of the rvs $\eta_1, \ldots, \eta_p$ arranged in increasing order, namely $\eta_{p,1} \leqslant \ldots \leqslant \eta_{p,p}$, with a lexicographic tiebreaker when needed. The rvs $\eta_{p,1}, \ldots, \eta_{p,p}$ are known as the *order statistics* associated with the collection $\eta_1, \ldots, \eta_p$, and the rvs $\eta_{p,1}$ and $\eta_{p,p}$ are the minimum and maximum of the rvs $\eta_1, \ldots, \eta_p$, respectively [11, 16].

To evaluate (3.16), we start with the following observation: Fix $d = 0, 1, \ldots$ and take $n = 2, 3, \ldots$ such that $d < n - 2$. Under the enforced i.i.d. assumptions, for each $\alpha > 0$ we get

$$
\begin{aligned}
\left(D_{n,1}^{\star}(\alpha), D_{n,2}^{\star}(\alpha)\right) &= \left(\sum_{k=3}^{n} \mathbf{1}\left[\xi_1 + \xi_k > \alpha\right], \sum_{k=3}^{n} \mathbf{1}\left[\xi_2 + \xi_k > \alpha\right]\right) \\
&=_{st} \left(\sum_{\ell=1}^{n-2} \mathbf{1}\left[\xi_1 + \eta_\ell > \alpha\right], \sum_{\ell=1}^{n-2} \mathbf{1}\left[\xi_2 + \eta_\ell > \alpha\right]\right) \\
&= \left(\sum_{\ell=1}^{n-2} \mathbf{1}\left[\xi_1 + \eta_{n-2,\ell} > \alpha\right], \sum_{\ell=1}^{n-2} \mathbf{1}\left[\xi_2 + \eta_{n-2,\ell} > \alpha\right]\right). \quad (3.17)
\end{aligned}
$$

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With this notation the probabilities of interest can now be expressed as

$$\mathbb{P}\left[D_{n,1}^{\star}(\alpha) = d\right] = \mathbb{P}\left[\sum_{\ell=1}^{n-2} \mathbf{1}\left[\xi_1 + \eta_{n-2,\ell} > \alpha\right] = d\right] \tag{3.18}$$

and

$$\mathbb{P}\left[D_{n,1}^{\star}(\alpha) = d, D_{n,2}^{\star}(\alpha) = d\right]$$

$$= \mathbb{P}\left[\sum_{\ell=1}^{n-2} \mathbf{1}\left[\xi_1 + \eta_{n-2,\ell} > \alpha\right] = d, \sum_{\ell=1}^{n-2} \mathbf{1}\left[\xi_2 + \eta_{n-2,\ell} > \alpha\right] = d\right]. \tag{3.19}$$

Two different cases arise when applying these facts: With $d = 0$ we find

$$\mathbb{P}\left[D_{n,1}^{\star}(\alpha) = 0\right] = \mathbb{P}\left[\sum_{\ell=1}^{n-2} \mathbf{1}\left[\xi_1 + \eta_{n-2,\ell} > \alpha\right] = 0\right]$$

$$= \mathbb{P}\left[\xi_1 + \eta_\ell \leqslant \alpha, \ \ell = 1, \ldots, n-2\right]$$

$$= \mathbb{P}\left[\xi_1 + \eta_{n-2,n-2} \leqslant \alpha\right], \tag{3.20}$$

and

$$\mathbb{P}\left[D_{n,1}^{\star}(\alpha) = 0, D_{n,2}^{\star}(\alpha) = 0\right]$$

$$= \mathbb{P}\left[\sum_{\ell=1}^{n-2} \mathbf{1}\left[\xi_1 + \eta_{n-2,\ell} > \alpha\right] = 0, \sum_{\ell=1}^{n-2} \mathbf{1}\left[\xi_2 + \eta_{n-2,\ell} > \alpha\right] = 0\right]$$

$$= \mathbb{P}\left[\xi_1 + \eta_\ell \leqslant \alpha, \xi_2 + \eta_\ell \leqslant \alpha, \ \ell = 1, \ldots, n-2\right]$$

$$= \mathbb{P}\left[\xi_1 + \eta_{n-2,n-2} \leqslant \alpha, \xi_2 + \eta_{n-2,n-2} \leqslant \alpha\right]$$

$$= \mathbb{P}\left[\max(\xi_1, \xi_2) + \eta_{n-2,n-2} \leqslant \alpha\right]. \tag{3.21}$$

For the case $d = 1, 2, \ldots$, we introduce the index

$$t_n(d) = n - 2 - d. \tag{3.22}$$

Under the enforced independence assumptions we now have

$$
\mathbb{P}\left[D_{n,1}^{\star}(\alpha) = d\right] = \mathbb{P}\left[\sum_{\ell=1}^{n-2} \mathbf{1}\left[\xi_1 + \eta_{n-2,\ell} > \alpha\right] = d\right]
$$

$$
= \mathbb{P}\left[\xi_1 + \eta_{n-2,t_n(d)} \leqslant \alpha < \xi_1 + \eta_{n-2,t_n(d)+1}\right]
$$

$$
= \mathbb{P}\left[\alpha - \eta_{n-2,t_n(d)+1} < \xi_1 \leqslant \alpha - \eta_{n-2,t_n(d)}\right]
$$

$$
= \mathbb{E}\left[e^{-\lambda(\alpha - \eta_{n-2,t_n(d)+1})^+} - e^{-\lambda(\alpha - \eta_{n-2,t_n(d)})^+}\right], \tag{3.23}
$$

and

$$
\mathbb{P}\left[D_{n,1}^{\star}(\alpha) = d, D_{n,2}^{\star}(\alpha) = d\right]
$$

$$
= \mathbb{P}\left[\sum_{\ell=1}^{n-2} \mathbf{1}\left[\xi_1 + \eta_{n-2,\ell} > \alpha\right] = d, \sum_{\ell=1}^{n-2} \mathbf{1}\left[\xi_2 + \eta_{n-2,\ell} > \alpha\right] = d\right]
$$

$$
= \mathbb{P}\left[\xi_j + \eta_{n-2,t_n(d)} \leqslant \alpha < \xi_j + \eta_{n-2,t_n(d)+1}, \; j = 1, 2\right]
$$

$$
= \mathbb{P}\left[\alpha - \eta_{n-2,t_n(d)+1} < \xi_j \leqslant \alpha - \eta_{n-2,t_n(d)}, \; j = 1, 2\right]
$$

$$
= \mathbb{E}\left[\left(e^{-\lambda(\alpha - \eta_{n-2,t_n(d)+1})^+} - e^{-\lambda(\alpha - \eta_{n-2,t_n(d)})^+}\right)^2\right]. \tag{3.24}
$$

In particular we note that

$$
\mathrm{Cov}[\mathbf{1}\left[D_{n,1}^{\star}(\alpha) = d\right], \mathbf{1}\left[D_{n,2}^{\star}(\alpha) = d\right]]
$$

$$
= \mathbb{P}\left[D_{n,1}^{\star}(\alpha) = d, D_{n,2}^{\star}(\alpha) = d\right] - \mathbb{P}\left[D_{n,1}^{\star}(\alpha) = d\right]\mathbb{P}\left[D_{n,2}^{\star}(\alpha) = d\right]
$$

$$
= \mathrm{Var}\left[e^{-\lambda(\alpha - \eta_{n-2,t_n(d)+1})^+} - e^{-\lambda(\alpha - \eta_{n-2,t_n(d)})^+}\right] \geqslant 0. \tag{3.25}
$$

We need to evaluate the quantities (3.20), (3.21), (3.23) and (3.24), and explore their asymptotic behavior for large n when α is replaced by $\alpha_n^{\star}$ in these expressions.

3.3 Asymptotic results for order statistics

In order to carry the program outlined above we need to develop some simple facts concerning the asymptotic theory of order statistics. The notation and definitions are the ones introduced in Section 3.2.2. In carrying out this asymptotic analysis, we shall rely on the following facts: Recall that for each $s = 0, 1, \ldots$, we have

$$\binom{p}{s} \sim \frac{p^s}{s!} \quad (p \to \infty), \tag{3.26}$$

and for any sequence $a : \mathbb{N}_0 \to \mathbb{R}_+$, it holds that

$$\lim_{p \to \infty} \left(1 - \frac{u}{p}\right)^{a_p} = e^{-au}, \quad u > 0 \tag{3.27}$$

whenever

$$\lim_{p \to \infty} \frac{a_p}{p} = a > 0. \tag{3.28}$$

3.3.1 A result in one dimension

Fix $p = 1, 2, \ldots$ and $t = 1, \ldots, p-1$. It is well known [11, p. 9] that the probability distribution function of the rv $\eta_{p,t}$ is given by

$$\begin{aligned}
\mathbb{P}\left[\eta_{p,t} \leqslant x\right] &= \sum_{r=t}^{p} \binom{p}{r} F(x)^r (1 - F(x))^{p-r} \\
&= \sum_{r=t}^{p} \binom{p}{r} \left(1 - e^{-\lambda x}\right)^r e^{-\lambda(p-r)x}, \quad x \geqslant 0.
\end{aligned} \tag{3.29}$$

For each $s = 0, 1, \ldots$, we introduce the mapping $G_s : \mathbb{R} \to \mathbb{R}_+$ defined by

$$G_s(x_s) = \left(\sum_{m=0}^{s} \frac{e^{-mx_s}}{m!}\right) G_0(x_s), \quad x_s \in \mathbb{R} \tag{3.30}$$

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where $G_0 : \mathbb{R} \to \mathbb{R}_+$ is the well-known *Gumbel* distribution given by

$$G_0(x_0) = e^{-e^{-x_0}}, \quad x_0 \in \mathbb{R}. \tag{3.31}$$

Using well-known stochastic monotonicity properties of Poisson rvs (with respect to their mean parameter), it is easy to check that $G_s : \mathbb{R} \to \mathbb{R}_+$ is indeed a probability distribution function. Let Λ_s denote any $\mathbb{R}$-valued rv which is distributed according to G_s, i.e.,

$$\mathbb{P}[\Lambda_s \leqslant x_s] = G_s(x_s), \quad x_s \in \mathbb{R}. \tag{3.32}$$

In fact one could interpret (3.30) as

$$\mathbb{P}[\Lambda_s \leqslant x_s] = \mathbb{P}\left[\mathrm{Poi}(e^{-x_s}) \leqslant s\right], \quad x_s \in \mathbb{R}$$

where $\mathrm{Poi}(\lambda)$ is a generic Poisson rv with parameter $\lambda > 0$.

Lemma 3.5. *For each $s = 0, 1, \ldots$ we have*

$$\lambda\left(\eta_{p,p-s} - \alpha_p^\star\right) \Longrightarrow_p \Lambda_s. \tag{3.33}$$

Some comments before giving a proof:

With $s = 0$, Lemma 3.5 gives the distributional convergence $\lambda(\eta_{p,p} - \alpha_p^\star) \Longrightarrow_p \Lambda_0$ which expresses the well-known membership of exponential distributions in the maximal domain of attraction of the Gumbel distribution [16]. For future use note that (3.33) is equivalent to

$$\lim_{p \to \infty} \mathbb{P}\left[\lambda\left(\eta_{p,p-s} - \alpha_p^\star\right) \leqslant x_s\right] = G_s(x_s), \quad x_s \in \mathbb{R} \tag{3.34}$$

since every point in $\mathbb{R}$ is a point of continuity for G_s.

Proof. Pick $s = 0, 1, \ldots$ and x_s arbitrary in $\mathbb{R}$. With $p > s$ and $t = p - s$ such that $x_s + \log p > 0$, we obtain from (3.29) that

$$
\begin{aligned}
\mathbb{P}\left[\lambda\left(\eta_{p,p-s} - \alpha_p^\star\right) \leq x_s\right] &= \mathbb{P}\left[\eta_{p,p-s} \leq \lambda^{-1}\left(x_s + \log p\right)\right] \\
&= \sum_{r=p-s}^{p} \binom{p}{r} \left(1 - \frac{e^{-x_s}}{p}\right)^r \frac{e^{-(p-r)x_s}}{p^{p-r}} \\
&= \sum_{q=0}^{s} \binom{p}{p-s+q} \left(1 - \frac{e^{-x_s}}{p}\right)^{p-s+q} \frac{e^{-(s-q)x_s}}{p^{s-q}} \\
&= \sum_{m=0}^{s} \binom{p}{p-m} \left(1 - \frac{e^{-x_s}}{p}\right)^{p-m} \frac{e^{-mx_s}}{p^m} \\
&= \sum_{m=0}^{s} \frac{\binom{p}{m}}{p^m} \left(1 - \frac{e^{-x_s}}{p}\right)^{p-m} e^{-mx_s}. \quad (3.35)
\end{aligned}
$$

Let p go to infinity in this last expression: For each $m = 0, 1, \ldots$, we get

$$
\frac{\binom{p}{m}}{p^m} \sim \frac{1}{m!} \quad (p \to \infty) \quad \text{and} \quad \lim_{p \to \infty} \left(1 - \frac{e^{-x_s}}{p}\right)^{p-m} = e^{-e^{-x_s}} = G_0(x_s), \quad x_s \in \mathbb{R}
$$

as we make use of (3.26) and (3.27)-(3.28), respectively. Therefore, (3.34) holds, and this completes the proof. ∎

It is easy to see that $G_s \leq_{st} G_{s-1}$, or equivalently, $\Lambda_s \leq_{st} \Lambda_{s-1}$ for all $s = 1, 2, \ldots$ where $\leq_{st}$ denotes the usual stochastic ordering [36, Chap. 8] – It suffices to note that for all $s = 1, 2, \ldots$, we have $\eta_{p,p-s} \leq_{st} \eta_{p,p-(s-1)}$ for all $p > s$. This can also be checked analytically through the expression (3.30).

3.3.2   A result in two dimensions

To state the key asymptotic result we introduce some additional notation: For all $t = 0, 1, \ldots$, let the mapping $H_t : \mathbb{R}_+ \to \mathbb{R}_+$ be given by

$$
H_t(a) = \int_a^\infty v^t e^{-v} dv, \quad a \geq 0.
$$

Then, for each $s = 1, 2, \ldots$, we define the mapping $J_s : \mathbb{R}^2 \to \mathbb{R}_+$ by

$$J_s(x_s, x_{s-1}) = \frac{1}{s!} \left(\int_{e^{-\min(x_s, x_{s-1})}}^{\infty} v^s e^{-v} dv - e^{-sx_{s-1}} \cdot e^{-e^{-\min(x_s, x_{s-1})}} \right)$$

$$= \frac{1}{s!} \left(H_s(e^{-\min(x_s, x_{s-1})}) - e^{-sx_{s-1}} \cdot e^{-e^{-\min(x_s, x_{s-1})}} \right), \quad x_s, x_{s-1} \in \mathbb{R}.$$

(3.36)

Proposition 3.6. *For each $s = 1, 2, \ldots$, we have*

$$\lim_{p \to \infty} \mathbb{P} \left[\lambda(\eta_{p,p-s} - \alpha_p^\star) \leqslant x_s, \lambda(\eta_{p,p-(s-1)} - \alpha_p^\star) \leqslant x_{s-1} \right]$$

$$= J_s(x_s, x_{s-1}), \quad x_s, x_{s-1} \in \mathbb{R}$$

(3.37)

where the mapping $J_s : \mathbb{R}^2 \to \mathbb{R}_+$ is given by (3.36).

Some comments concerning this result before giving a proof in Section 3.5: First, given the convergence (3.37) it is easy to check that $\mathbb{R}^2 \to [0,1] : (x_s, x_{s-1}) \to J_s(x_s, x_{s-1})$ is a *bona fide* probability distribution on $\mathbb{R}^2$ with $\lim_{x_s, x_{s-1} \to \infty} J_s(x_s, x_{s-1}) = 1$,

$\lim_{x_s \to -\infty} J_s(x_s, x_{s-1}) = 0$ (with x_{s-1} held fixed) and $\lim_{x_{s-1} \to -\infty} J_s(x_s, x_{s-1}) = 0$ (with x_s held fixed).

Next, we turn to extracting the marginal distributions from (3.37): Upon setting $x_{s-1} = \infty$ (resp. $x_s = \infty$) in (3.37) with x_s (resp. x_{s-1}) arbitrary but held fixed, we get

$$\lim_{p \to \infty} \mathbb{P} \left[\lambda(\eta_{p,p-s} - \alpha_p^\star) \leqslant x_s \right] = \frac{1}{s!} H_s(e^{-x_s}), \quad x_s \in \mathbb{R}$$

(3.38)

and

$$\lim_{p \to \infty} \mathbb{P} \left[\lambda(\eta_{p,p-(s-1)} - \alpha_p^\star) \leqslant x_{s-1} \right]$$

$$= \frac{1}{s!} \left(H_s(e^{-x_{s-1}}) - e^{-sx_{s-1}} \cdot e^{-e^{-x_{s-1}}} \right), \quad x_{s-1} \in \mathbb{R}.$$

(3.39)

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As we return to Lemma 3.5, e.g., see (3.34), it is natural to wonder whether the marginalization of the two-dimensional result can be reconciled analytically with the one-dimensional convergence obtained earlier. In other words, is it indeed the case that the relations

$$G_s(x_s) = \frac{1}{s!} H_s(e^{-x_s}), \quad x_s \in \mathbb{R} \tag{3.40}$$

and

$$G_{s-1}(x_{s-1}) = \frac{1}{s!} \left(H_s(e^{-x_{s-1}}) - e^{-sx_{s-1}} \cdot e^{-e^{-x_{s-1}}} \right), \quad x_{s-1} \in \mathbb{R} \tag{3.41}$$

hold.

We argue as follows: For $t = 1, 2, \ldots$, integration by parts yields

$$H_t(a) = a^t e^{-a} + t H_{t-1}(a), \quad a \geqslant 0 \tag{3.42}$$

with boundary value $H_0(a) = e^{-a}$. Iterating on this relation, we readily check that

$$H_t(a) = t! \left(\sum_{r=0}^{t} \frac{a^r}{r!} \right) e^{-a}, \quad \begin{matrix} a \geqslant 0 \\ t = 0, 1, \ldots \end{matrix} \tag{3.43}$$

Using (3.43) it is plain that

$$\frac{1}{s!} H_s(e^{-x_s}) = G_s(x_s), \quad x_s \in \mathbb{R}$$

as we recall (3.30), and this shows that (3.40) indeed holds. Next, as we turn to (3.41), use (3.42) (with $t = s$ and $a = e^{-x_{s-1}}$) and note that

$$\frac{1}{s!} \left(H_s(e^{-x_{s-1}}) - e^{-sx_{s-1}} \cdot e^{-e^{-x_{s-1}}} \right)$$
$$= \frac{1}{s!} \left(e^{-sx_{s-1}} \cdot e^{-e^{-x_{s-1}}} + s H_{s-1}(e^{-x_{s-1}}) - e^{-sx_{s-1}} \cdot e^{-e^{-x_{s-1}}} \right)$$
$$= \frac{1}{(s-1)!} H_{s-1}(e^{-x_{s-1}}) = G_{s-1}(x_{s-1}), \quad x_{s-1} \in \mathbb{R} \tag{3.44}$$

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where we again relied on (3.43). This shows that (3.41) holds.

Therefore, as we combine Lemma 3.5 and Proposition 3.6, the following conclusion emerges.

Corollary 3.7. *For any given $s = 1, 2, \ldots$, there exist a pair $(\Lambda_s, \Lambda_{s-1})$ of $\mathbb{R}$-valued rvs such that*

$$\left(\lambda \left(\eta_{p,p-s} - \alpha_p^\star \right), \lambda \left(\eta_{p,p-(s-1)} - \alpha_p^\star \right) \right) \Longrightarrow_p (\Lambda_s, \Lambda_{s-1}) \tag{3.45}$$

with $(\Lambda_s, \Lambda_{s-1})$ jointly distributed according to J_s, and Λ_s and Λ_{s-1}, each distributed according to G_s and G_{s-1}, respectively.

In other words,

$$\mathbb{P}\left[\Lambda_s \leqslant x_s, \Lambda_{s-1} \leqslant x_{s-1} \right] = J_s(x_s, x_{s-1}), \quad x_s, x_{s-1} \in \mathbb{R}$$

with marginals

$$\mathbb{P}\left[\Lambda_s \leqslant x_s \right] = G_s(x_s) \quad \text{and} \quad \mathbb{P}\left[\Lambda_{s-1} \leqslant x_{s-1} \right] = G_{s-1}(x_{s-1}), \quad x_s, x_{s-1} \in \mathbb{R}.$$

3.4 A proof of Proposition 3.2 – Part II

The notation is that of Section 3.2.2 as we return to the expressions (3.20), (3.21), (3.23) and (3.24) obtained there. With $d = 0, 1, \ldots$ held fixed, for each $n = 2, 3, \ldots$ we substitute α by $\alpha_n^\star$ in these expressions according to (3.3), and let n go to infinity in the resulting inequality.

3.4.1 The case $d = 0$

For each $n = 3, 4, \ldots$, with the aforementioned substitution, we rewrite (3.20) and (3.21) as

$$\mathbb{P}\left[D_{n,1}^{\star}(\alpha_n^{\star}) = 0\right] = \mathbb{P}\left[\lambda(\eta_{n-2,n-2} - \alpha_n^{\star}) \leqslant -\lambda\xi_1\right]$$

and

$$\mathbb{P}\left[D_{n,1}^{\star}(\alpha_n^{\star}) = 0, D_{n,2}^{\star}(\alpha_n^{\star}) = 0\right] = \mathbb{P}\left[\max(\xi_1, \xi_2) + \eta_{n-2,n-2} \leqslant \alpha_n^{\star}\right]$$

$$= \mathbb{P}\left[\lambda(\eta_{n-2,n-2} - \alpha_n^{\star}) \leqslant -\lambda\max(\xi_1, \xi_2)\right].$$

By Lemma 3.5 (and comments following it), it is now plain that

$$p_{\text{Fuj}}(0) = \lim_{n\to\infty} \mathbb{P}\left[D_{n,1}^{\star}(\alpha_n^{\star}) = 0\right] = \mathbb{P}\left[\Lambda_0 \leqslant -\lambda\xi_1\right] = \mathbb{E}\left[e^{-e^{\lambda\xi_1}}\right]$$

and

$$\lim_{n\to\infty} \mathbb{P}\left[D_{n,1}^{\star}(\alpha_n^{\star}) = 0, D_{n,2}^{\star}(\alpha_n^{\star}) = 0\right] = \mathbb{P}\left[\Lambda_0 \leqslant -\lambda\max(\xi_1, \xi_2)\right] = \mathbb{E}\left[e^{-e^{\lambda\max(\xi_1,\xi_2)}}\right],$$

respectively, where Λ_0 is any rv which is distributed according to the Gumbel distribution (3.31). Collecting these facts and using the reduction step discussed in Section 3.2, we find

$$C(0) = \lim_{n\to\infty} \text{Cov}[\mathbf{1}\left[D_{n,1}(\alpha_n^{\star}) = 0\right], \mathbf{1}\left[D_{n,2}(\alpha_n^{\star}) = 0\right]]$$

$$= \lim_{n\to\infty} \text{Cov}[\mathbf{1}\left[D_{n,1}^{\star}(\alpha_n^{\star}) = 0\right], \mathbf{1}\left[D_{n,2}^{\star}(\alpha_n^{\star}) = 0\right]]$$

$$= \mathbb{E}\left[e^{-e^{\lambda\max(\xi_1,\xi_2)}}\right] - \mathbb{E}\left[e^{-e^{\lambda\xi_1}}\right]\mathbb{E}\left[e^{-e^{\lambda\xi_2}}\right]$$

$$= \mathbb{E}\left[e^{-e^{\lambda\max(\xi_1,\xi_2)}}\right] - \mathbb{E}\left[e^{-e^{\lambda\xi_1}} \cdot e^{-e^{\lambda\xi_2}}\right].$$

Note that

$$C(0) = \mathbb{E}\left[e^{-\max(e^{\lambda\xi_1}, e^{\lambda\xi_2})}\right] - \mathbb{E}\left[e^{-(e^{\lambda\xi_1}+e^{\lambda\xi_2})}\right] > 0$$

since $\max(e^{\lambda\xi_1}, e^{\lambda\xi_2}) < e^{\lambda\xi_1} + e^{\lambda\xi_2}$ a.s. ∎

3.4.2 The case $d = 1, 2, \ldots$

Pick $n = 3, 4, \ldots$ such that $d < n - 2$. Under the aforementioned substitution, we can rewrite (3.23) and (3.24) as

$$\mathbb{P}\left[D_{n,1}^{\star}(\alpha_n^{\star}) = d\right] = \mathbb{E}\left[e^{-\lambda(\alpha_n^{\star} - \eta_{n-2,t_n(d)+1})^+} - e^{-\lambda(\alpha_n^{\star} - \eta_{n-2,t_n(d)})^+}\right] \tag{3.46}$$

and

$$\mathbb{P}\left[D_{n,1}^{\star}(\alpha_n^{\star}) = d, D_{n,2}^{\star}(\alpha_n^{\star}) = d\right] = \mathbb{E}\left[\left(e^{-\lambda(\alpha_n^{\star} - \eta_{n-2,t_n(d)+1})^+} - e^{-\lambda(\alpha_n^{\star} - \eta_{n-2,t_n(d)})^+}\right)^2\right] \tag{3.47}$$

Applying Corollary 3.7 (with $p = n - 2$ and $s = d$) we conclude that

$$\left(\lambda\left(\eta_{n-2,t_n(d)} - \alpha_{n-2}^{\star}\right), \lambda\left(\eta_{n-2,t_n(d)+1} - \alpha_{n-2}^{\star}\right)\right) \Longrightarrow_n (\Lambda_d, \Lambda_{d-1}), \tag{3.48}$$

whence

$$\left(\lambda\left(\eta_{n-2,t_n(d)} - \alpha_n^{\star}\right), \lambda\left(\eta_{n-2,t_n(d)+1} - \alpha_n^{\star}\right)\right) \Longrightarrow_n (\Lambda_d, \Lambda_{d-1}) \tag{3.49}$$

by standard facts concerning weak convergence since $\lim_{n\to\infty}\left(\alpha_n^{\star} - \alpha_{n-2}^{\star}\right) = 0$. It immediately follows by the Continuous Mapping Theorem for weak convergence that

$$\left(\lambda\left(\alpha_{n-2}^{\star} - \eta_{n-2,t_n(d)}\right)^+, \lambda\left(\alpha_{n-2}^{\star} - \eta_{n-2,t_n(d)+1}\right)^+\right) \Longrightarrow_n \left((-\Lambda_d)^+, (-\Lambda_{d-1})^+\right). \tag{3.50}$$

Applying the Continuous Mapping Theorem once more we find that

$$e^{-\lambda(\alpha_n^\star - \eta_{n-2,t_n(d)+1})^+} - e^{-\lambda(\alpha_n^\star - \eta_{n-2,t_n(d)})^+} \Longrightarrow_n e^{-(-\Lambda_{d-1})^+} - e^{-(-\Lambda_d)^+}. \tag{3.51}$$

Let n go to infinity in (3.46) and (3.47): Making use of the convergence (3.51) we conclude by the Bounded Convergence Theorem that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(e^{-\lambda(\alpha_n^\star - \eta_{n-2,t_n(d)+1})^+} - e^{-\lambda(\alpha_n^\star - \eta_{n-2,t_n(d)})^+}\right)^a\right]$$

$$= \mathbb{E}\left[\left(e^{-(-\Lambda_{d-1})^+} - e^{-(-\Lambda_d)^+}\right)^a\right], \quad a = 1, 2. \tag{3.52}$$

This is made possible with the help of the obvious bounds

$$\left| e^{-\lambda(\alpha_n^\star - \eta_{n-2,t_n(d)+1})^+} - e^{-\lambda(\alpha_n^\star - \eta_{n-2,t_n(d)})^+} \right| \leqslant 1, \quad n = 3, 4, \ldots$$

From (3.25) we conclude that

$$\lim_{n\to\infty} \mathrm{Cov}[\mathbf{1}\left[D_{n,1}^\star(\alpha_n^\star) = d\right], \mathbf{1}\left[D_{n,2}^\star(\alpha_n^\star) = d\right]]$$

$$= \lim_{n\to\infty} \mathrm{Var}\left[e^{-\lambda(\alpha_n^\star - \eta_{n-2,t_n(d)+1})^+} - e^{-\lambda(\alpha_n^\star - \eta_{n-2,t_n(d)})^+}\right]$$

$$= \mathrm{Var}\left[e^{-(-\Lambda_{d-1})^+} - e^{-(-\Lambda_d)^+}\right], \tag{3.53}$$

and the reduction step via (3.16) finally leads to

$$C(d) = \lim_{n\to\infty} \mathrm{Cov}[\mathbf{1}\left[D_{n,1}(\alpha_n^\star) = d\right], \mathbf{1}\left[D_{n,2}(\alpha_n^\star) = d\right]]$$

$$= \mathrm{Var}\left[e^{-(-\Lambda_{d-1})^+} - e^{-(-\Lambda_d)^+}\right]. \tag{3.54}$$

Note that $C(d) > 0$ as the variance of the non-degenerate rv $e^{-(-\Lambda_{d-1})^+} - e^{-(-\Lambda_d)^+}$.

∎

Now, a sanity check: Let n go to infinity in (3.46). The arguments given earlier also yield

$$\lim_{n\to\infty} \mathbb{P}\left[D_{n,1}^\star(\alpha_n^\star) = d\right] = \mathbb{E}\left[e^{-(-\Lambda_{d-1})^+} - e^{-(-\Lambda_d)^+}\right].$$

Standard calculations give

$$
\mathbb{E}\left[e^{-(-\Lambda_d)^+}\right] = \int_0^\infty \mathbb{P}\left[e^{-(-\Lambda_d)^+} > t\right] dt
$$

$$
= \int_0^\infty \mathbb{P}\left[(-\Lambda_d)^+ < -\log t\right] dt
$$

$$
= \int_0^1 \mathbb{P}\left[(-\Lambda_d)^+ < -\log t\right] dt
$$

$$
= \int_0^1 \mathbb{P}\left[\Lambda_d > 0, 0 < -\log t\right] dt + \int_0^1 \mathbb{P}\left[\Lambda_d \leqslant 0, -\Lambda_d < -\log t\right] dt.
$$

$$
= \int_0^1 \mathbb{P}\left[\log t < \Lambda_d\right] dt, \tag{3.55}
$$

and

$$
\mathbb{E}\left[e^{-(-\Lambda_{d-1})^+}\right] = \int_0^1 \mathbb{P}\left[\log t < \Lambda_{d-1}\right] dt \tag{3.56}
$$

by similar arguments. Therefore,

$$
\mathbb{E}\left[e^{-(-\Lambda_{d-1})^+}\right] - \mathbb{E}\left[e^{-(-\Lambda_d)^+}\right] = \int_0^1 \left(\mathbb{P}\left[\log t < \Lambda_{d-1}\right] - \mathbb{P}\left[\log t < \Lambda_d\right]\right) dt
$$

$$
= \int_0^1 \left(\mathbb{P}\left[\Lambda_d \leqslant \log t\right] - \mathbb{P}\left[\Lambda_{d-1} \leqslant \log t\right]\right) dt
$$

$$
= \int_0^1 \left(G_d(\log t) - G_{d-1}(\log t)\right) dt
$$

$$
= \int_0^\infty \left(G_d(-x) - G_{d-1}(-x)\right) e^{-x} dx \quad [t = e^{-x}]
$$

$$
= \int_0^\infty \frac{(e^x)^d}{d!} e^{-e^x} e^{-x} dx
$$

$$
= p_{\mathrm{Fuj}}(d) \tag{3.57}
$$

as it should be in view of (3.15) when combined with (3.5).

As a by product it follows that the pmf $\boldsymbol{p}_{\mathrm{Fuj}}$ admits the multiple representations

$$
p_{\mathrm{Fuj}}(d) = \mathbb{E}\left[e^{-(-\Lambda_{d-1})^+} - e^{-(-\Lambda_d)^+}\right] = \mathbb{E}\left[\frac{(e^{\lambda\xi})^d}{d!} e^{-e^{\lambda\xi}}\right], \quad d = 0, 1, \ldots
$$

3.5 A proof of Proposition 3.6

3.5.1 Preliminaries

Fix $p = 2, 3, \ldots$ and $t = 1, \ldots, p-1$. It is also well known [11, p. 11] that the joint probability distribution function of the pair $(\eta_{p,t}, \eta_{p,t+1})$ admits a probability density function $f_{p,t,t+1} : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$f_{p,t,t+1}(x_t, x_{t+1})$$

$$= \begin{cases} p(p-1)\binom{p-2}{t-1}\left(1 - e^{-\lambda x_t}\right)^{t-1} e^{-\lambda(p-(t+1))x_{t+1}}\lambda e^{-\lambda x_t}\lambda e^{-\lambda x_{t+1}} & \text{if } 0 \leqslant x_t \leqslant x_{t+1} \\ \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, since $\eta_{p,t} \leqslant \eta_{p,t+1}$, with arbitrary x_t and x_{t+1} in $\mathbb{R}_+$, elementary calculations yield

$$\mathbb{P}\left[\eta_{p,t} \leqslant x_t, \eta_{p,t+1} \leqslant x_{t+1}\right]$$

$$= \mathbb{P}\left[\eta_{p,t} \leqslant \min(x_t, x_{t+1}), \eta_{p,t+1} \leqslant x_{t+1}\right]$$

$$= \int_0^{\min(x_t, x_{t+1})} \left(\int_{y_t}^{x_{t+1}} f_{p,t,t+1}(y_t, y_{t+1})dy_{t+1}\right) dy_t$$

$$= p(p-1)\binom{p-2}{t-1}\int_0^{\min(x_t, x_{t+1})}\left(\int_{y_t}^{x_{t+1}} \lambda e^{-\lambda(p-t)y_{t+1}}dy_{t+1}\right)\left(1 - e^{-\lambda y_t}\right)^{t-1}\lambda e^{-\lambda y_t}dy_t$$

$$= \frac{p(p-1)}{p-t}\binom{p-2}{t-1}\int_0^{\min(x_t, x_{t+1})}\left(1 - e^{-\lambda y_t}\right)^{t-1}\lambda e^{-\lambda y_t}\left(e^{-\lambda(p-t)y_t} - e^{-\lambda(p-t)x_{t+1}}\right)dy_t$$

$$= t\binom{p}{t}\int_0^{\lambda\min(x_t, x_{t+1})}\left(1 - e^{-y_t}\right)^{t-1}e^{-y_t}\left(e^{-(p-t)y_t} - e^{-\lambda(p-t)x_{t+1}}\right)dy_t$$

$$= t\binom{p}{t}\left(I_t(\lambda\min(x_t, x_{t+1}); p-t) - e^{-\lambda(p-t)x_{t+1}} \cdot I_t(\lambda\min(x_t, x_{t+1}); 0)\right) \qquad (3.58)$$

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as we use the notation

$$a \geqslant 0$$

$$I_t(a;r) = \int_0^a \left(1 - e^{-y}\right)^{t-1} e^{-(r+1)y} dy, \quad r = 0, 1, \ldots$$

$$t = 1, 2, \ldots$$

We begin with an intermediary result that will help us in investigating the asymptotics of the relevant pair of order statistics.

Lemma 3.8. *Fix* $s = 0, 1, \ldots$ *and* x *arbitrary in* $\mathbb{R}$. *For each* $r = 0, 1, \ldots$, *it holds that*

$$\lim_{p \to \infty} p^{r+1} I_{p-s}\left((x + \log p)^+; r\right) = \int_{e^{-x}}^\infty v^r e^{-v} dv. \tag{3.59}$$

Proof. Throughout the proof the integer $r = 0, 1, \ldots$ is held fixed. For $a \geqslant 0$ and positive integer t, elementary calculus gives

$$I_t(a;r) = \int_0^a \left(1 - e^{-y}\right)^{t-1} e^{-(r+1)y} dy$$

$$= \int_0^a \left(1 - \frac{e^z}{e^a}\right)^{t-1} \left(\frac{e^z}{e^a}\right)^{r+1} dz \quad [z = a - y]$$

$$= e^{-a} \int_1^{e^a} \left(1 - \frac{u}{e^a}\right)^{t-1} \left(\frac{u}{e^a}\right)^r du \quad [u = e^z]. \tag{3.60}$$

Now pick $s = 0, 1, \ldots$ and x arbitrary in $\mathbb{R}$, and take p sufficiently large so that $p > s$ and $x + \log p > 0$. Then, with $t = p - s$ and $a = x + \log p$, the last relation (3.60) becomes

$$I_{p-s}(x + \log p; r) = \frac{e^{-x}}{p} \int_1^{pe^x} \left(1 - \frac{u}{pe^x}\right)^{p-s-1} \left(\frac{u}{pe^x}\right)^r du$$

$$= \frac{1}{p^{r+1}} \int_{e^{-x}}^p \left(1 - \frac{v}{p}\right)^{p-s-1} v^r dv \quad [v = e^{-x}u]. \tag{3.61}$$

On the range $0 \leqslant v \leqslant p$, the bounds

$$0 \leqslant \left(1 - \frac{v}{p}\right)^{p-s-1} \leqslant e^{-\frac{p-s-1}{p}v} \leqslant 1$$

hold uniformly in p, while

$$\lim_{p \to \infty} \left(1 - \frac{v}{p}\right)^{p-s-1} = e^{-v}, \quad v \geqslant 0$$

as we again make use of (3.27)-(3.28). Using the Bounded Convergence Theorem,
we readily conclude by standard arguments that

$$\lim_{p \to \infty} \int_{e^{-x}}^{p} \left(1 - \frac{v}{p}\right)^{p-s-1} v^r \, dv = \int_{e^{-x}}^{\infty} v^r e^{-v} \, dv \tag{3.62}$$

since the rth moment of $\mathrm{Exp}(1)$ is finite and given by $\int_0^\infty v^r e^{-v} \, dv = r!$. This completes
the proof of (3.59). ∎

3.5.2 Proving Proposition 3.6

Fix positive integers p and t such that $t < p$, and pick x_t and x_{t+1} arbitrary
in $\mathbb{R}$. Take p large enough so that such $\min(x_t, x_{t+1}) + \log p > 0$. Under such
conditions, using (3.58) we find that

$$\mathbb{P}\left[\lambda(\eta_{p,t} - \alpha_p^\star) \leqslant x_t, \lambda(\eta_{p,t+1} - \alpha_p^\star) \leqslant x_{t+1}\right]$$

$$= \mathbb{P}\left[\eta_{p,t} \leqslant \frac{x_t + \log p}{\lambda}, \eta_{p,t+1} \leqslant \frac{x_{t+1} + \log p}{\lambda}\right]$$

$$= t\binom{p}{t}\left(I_t(\min(x_t, x_{t+1}) + \log p; p - t) - e^{-(p-t)(x_{t+1} + \log p)} \cdot I_t(\min(x_t, x_{t+1}) + \log p; 0)\right)$$

$$= t\binom{p}{t}\left(I_t(\min(x_t, x_{t+1}) + \log p; p - t) - \frac{e^{-(p-t)x_{t+1}}}{p^{p-t}} \cdot I_t(\min(x_t, x_{t+1}) + \log p; 0)\right).$$

Now pick $s = 1, 2, \ldots$, and x_s and x_{s-1} arbitrary in $\mathbb{R}$. With positive integer p
sufficiently large so that $p > s$ and $\min(x_s, x_{s-1}) + \log p > 0$, we use the last relation

with $t = p - s$, $x_t = x_s$ and $x_{t+1} = x_{s-1}$. This gives

$$\mathbb{P}\left[\lambda(\eta_{p,p-s} - \alpha_p^\star) \leqslant x_s, \lambda(\eta_{p,p-s+1} - \alpha_p^\star) \leqslant x_{s-1}\right]$$

$$= (p-s)\binom{p}{s}\left(I_{p-s}(\min(x_s, x_{s-1}) + \log p; s) - \frac{e^{-sx_{s-1}}}{p^s} \cdot I_{p-s}(\min(x_s, x_{s-1}) + \log p; 0)\right).$$

Let p go to infinity in this last expression: Lemma 3.8 yields

$$\lim_{p\to\infty} p^{s+1} I_{p-s}\left((\min(x_s, x_{s-1}) + \log p)^+; s\right) = \int_{e^{-\min(x_s, x_{s-1})}}^{\infty} v^s e^{-v} dv \tag{3.63}$$

and

$$\lim_{p\to\infty} p I_{p-s}\left((\min(x_s, x_{s-1}) + \log p)^+; 0\right) = \int_{e^{-\min(x_s, x_{s-1})}}^{\infty} e^{-v} dv = e^{-e^{-\min(x_s, x_{s-1})}}.$$

$$\tag{3.64}$$

Next, noting

$$(p-s)\binom{p}{s} \sim \frac{p^{s+1}}{s!} \quad (p\to\infty)$$

by virtue of (3.26), we easily conclude from (3.63) that

$$\lim_{p\to\infty} (p-s)\binom{p}{s} I_{p-s}(\min(x_s, x_{s-1}) + \log p; s)$$

$$= \lim_{p\to\infty} \frac{(p-s)\binom{p}{s}}{p^{s+1}} \cdot p^{s+1} I_{p-s}(\min(x_s, x_{s-1}) + \log p; s)$$

$$= \frac{1}{s!} \int_{e^{-\min(x_s, x_{s-1})}}^{\infty} v^s e^{-v} dv, \tag{3.65}$$

while (3.64) gives

$$\lim_{p\to\infty} (p-s)\binom{p}{s} \frac{e^{-sx_{s-1}}}{p^s} \cdot I_{p-s}(\min(x_s, x_{s-1}) + \log p; 0)$$

$$= \lim_{p\to\infty} \frac{(p-s)\binom{p}{s}}{p^{s+1}} e^{-sx_{s-1}} \cdot p I_{p-s}(\min(x_s, x_{s-1}) + \log p; 0)$$

$$= \frac{1}{s!} e^{-sx_{s-1}} \cdot e^{-e^{-\min(x_s, x_{s-1})}}. \tag{3.66}$$

Collecting these last two convergence statements we readily obtain the convergence

(3.37) with (3.36). ∎

Chapter 4:   More on Random Threshold Graphs: Weak Convergence

In Chapter 3, we considered a specific counterexample in the class of homogeneous random graphs: Let $\{\xi, \xi_k,\ k = 1, 2, \ldots\}$ represent an i.i.d. collection of $\mathbb{R}_+$-valued rvs with ξ_k being a fitness rv associated with node k denoting its importance or rank. With n nodes and threshold α, the random threshold graph $\mathbb{T}(n, \alpha)$ postulates that two distinct nodes i and j form a connection iff $\xi_i + \xi_j > \alpha$. With ξ exponentially distributed and the threshold α scaled as

$$\alpha_n^* = \frac{1}{\lambda} \log n, \ n = 1, 2, \ldots \tag{4.1}$$

the following distributional convergence in the sequence of graphs $\{\mathbb{T}(n; \alpha_n^*),\ n = 1, 2, \ldots\}$ [22] is known to take place

$$D_{n,1}(\alpha_n^*) \Longrightarrow_n D, \tag{4.2}$$

where the rv D has a power-tail. However, in Chapter 3 we concluded that when ξ is exponentially distributed, the usual convergence (2.9) of the empirical degree distribution fails to take place. Following the discussions in the previous chapter 3, two questions naturally emerge:

1. While in the previous chapter we considered the special case of ξ exponentially distributed, there are other regimes under which interesting asymptotic

behaviour of the degree distribution occurs in the sense of (4.2). An obvious question is whether the assertions made in the previous chapter can be extended to these more general fitness distributions.

2. Under the regime where the empirical degree distribution does not converge in the usual sense (which we now know includes the case of ξ exponentially distributed), are there weaker forms of convergence that could be shown to hold?

In what follows, we show that under certain conditions the empirical degree distribution actually converges weakly: For each $d = 0, 1, \ldots$, there exists a non-degenerate $[0, 1]$-valued rv $\Pi(d)$ such that

$$\frac{N_n(d; \alpha_n^{\star})}{n} \Longrightarrow_n \Pi(d) \tag{4.3}$$

where the scaling $\alpha^{\star} : \mathbb{N}_0 \to \mathbb{R}_+$ is the one yielding a non-trivial degree distribution in the sense of (4.2), and $\frac{N_n(d; \alpha_n^{\star})}{n}$ is the fraction of nodes with degree d in $\mathbb{T}(n; \alpha_n^{\star})$. The non-degeneracy of the rv $\Pi(d)$ for each $d = 0, 1, \ldots$, ascertains that the empirical degree distribution cannot converge in the sense of (2.9).

4.1   Degree distribution – Generic fitness distribution

The setting is that of Section 3.1: Let $\{\xi, \xi_k, \ k = 1, 2, \ldots\}$ denote a collection of i.i.d. $\mathbb{R}_+$-valued rvs defined on the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, each distributed according to a given (probability) distribution function $F : \mathbb{R} \to [0, 1]$. With ξ

acting as a generic representative for this sequence of i.i.d. rvs, we have

$$\mathbb{P}\left[\xi \leqslant x\right] = F(x), \quad x \in \mathbb{R}.$$

As mentioned in the previous chapter, we assume that F is a continuous function on $\mathbb{R}$ with

$$F(x) = 0, \quad x \leqslant 0. \tag{4.4}$$

Recall that with n number of nodes and threshold $\alpha > 0$, distinct nodes $i, j = 1, 2, \ldots, n$ are said to be adjacent if $\xi_i + \xi_j > \alpha$. For each $k = 1, 2, \ldots, n$, and threshold $\alpha > 0$ the degree of node k in $\mathbb{T}(n; \alpha)$ is the rv $D_{n,k}(\alpha)$ given by

$$D_{n,k}(\alpha) := \sum_{\ell=1, \ \ell \neq k}^{n} \mathbf{1}\left[\xi_k + \xi_\ell > \alpha\right].$$

Under the enforced assumptions on the rvs $\xi_1, \ldots, \xi_n$, the rv $D_{n,k}(\alpha)$ is a Binomial rv $\mathrm{Bin}(n-1; 1 - F(\alpha - \xi_k))$ conditioned on ξ_k. Throughout we make the following assumption.

Assumption 4.1. *There exists a scaling $\alpha^\star : \mathbb{N}_0 \to \mathbb{R}_+$ with the property*

$$\lim_{n \to \infty} \alpha_n^\star = \infty, \tag{4.5}$$

such that

$$\lim_{n \to \infty} n\left(1 - F(\alpha_n^\star - x)\right) = \lambda(x), \quad x \geqslant 0 \tag{4.6}$$

for some non-identically zero mapping $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$.

The mapping $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is necessarily non-decreasing. The following result overlaps with a similar result by Fujihara et al. [22, Thm. 2, p. 362]:

Proposition 4.1. *Under Assumption 4.1, there exists an $\mathbb{N}$-valued rv D such that*

$$D_n(\alpha_n^\star) \Longrightarrow_n D. \tag{4.7}$$

The rv D is conditionally Poisson with pmf given by

$$\mathbb{P}[D = d] = \mathbb{E}\left[\frac{\lambda(\xi)^d}{d!} e^{-\lambda(\xi)}\right], \quad d = 0, 1, \dots \tag{4.8}$$

Proof. Fix $n = 2, 3, \dots$, $\alpha > 0$ and z in $\mathbb{R}$. Standard pre-conditioning arguments

yield

$$\mathbb{E}\left[z^{D_n(\alpha)}\right] = \mathbb{E}\left[\prod_{\ell=2}^{n} z^{\mathbf{1}[\xi_1 + \xi_\ell > \alpha]}\right] = \mathbb{E}\left[\left(\mathbb{E}\left[z^{\mathbf{1}[x + \xi > \alpha]}\right]_{x = \xi_1}\right)^{n-1}\right]$$

where

$$
\begin{aligned}
\mathbb{E}\left[z^{\mathbf{1}[x + \xi > \alpha]}\right] &= \mathbb{P}[x + \xi \leqslant \alpha] + \mathbb{P}[x + \xi > \alpha]\, z \\
&= 1 - (1 - z)\mathbb{P}[x + \xi > \alpha] \\
&= 1 - (1 - z)(1 - F(\alpha - x)), \quad x \in \mathbb{R}. \tag{4.9}
\end{aligned}
$$

Consequently, as we use (4.4), we get

$$
\begin{aligned}
\mathbb{E}\left[z^{D_n(\alpha)}\right] &= \mathbb{E}\left[(1 - (1 - z)(1 - F(\alpha - \xi)))^{n-1}\right] \\
&= \mathbb{P}[\xi > \alpha]\, z^{n-1} + \mathbb{E}\left[\mathbf{1}[\xi \leqslant \alpha](1 - (1 - z)(1 - F(\alpha - \xi)))^{n-1}\right] \tag{4.10}
\end{aligned}
$$

For each $n = 1, 2, \dots$, replace α by $\alpha_n^\star$ in (4.10) according to the scaling

$\alpha^\star : \mathbb{N}_0 \to \mathbb{R}_+$. Let n go to infinity in the resulting equality when $|z| \leqslant 1$: It is plain

that $\lim_{n\to\infty} \mathbb{P}[\xi > \alpha_n^\star]\, z^{n-1} = 0$, while

$$\lim_{n\to\infty} (1 - (1 - z)(1 - F(\alpha_n^\star - \xi)))^{n-1} = e^{-(1-z)\lambda(\xi)}$$

by standard arguments as we note that

$$\lim_{n\to\infty} n(1-z)\left(1 - F(\alpha_n^\star - \xi)\right) = (1-z)\lambda(\xi)$$

under Assumption 4.1. Invoking the Bounded Convergence Theorem we conclude that

$$\lim_{n\to\infty} \mathbb{E}\left[z^{D_n(\alpha_n^\star)}\right] = \mathbb{E}\left[e^{-(1-z)\lambda(\xi)}\right], \quad |z| \leq 1$$

and the desired conclusion follows upon noting that the right-hand side is the pgf of the pmf (4.8). ∎

In what follows we use the standard notation $x^+ = \max(x,0)$ for x in $\mathbb{R}$. Assumption 4.1 holds in a number of interesting cases: Consider the case discussed in the previous chapter where ξ is exponentially distributed with parameter $\lambda > 0$, i.e.,

$$\mathbb{P}\left[\xi > x\right] = e^{-\lambda x^+}, \quad x \in \mathbb{R}.$$

Assumption 4.1 holds with

$$\lambda(x) = e^{\lambda x}, \quad x \geq 0$$

upon taking

$$\alpha_n^\star = \lambda^{-1}\log n, \quad n = 1, 2, \ldots$$

The rv ξ is said to be a Pareto rv with parameters $\nu > 0$ and $a > 0$, if

$$\mathbb{P}\left[\xi > x\right] = \left(\frac{a}{a + x^+}\right)^\nu, \quad x \in \mathbb{R}.$$

Assumption 4.1 holds if we take

$$\alpha_n^\star = a n^{\frac{1}{\nu}}, \quad n = 1, 2, \ldots$$

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in which case

$$\lambda(x) = 1, \quad x \geqslant 0.$$

4.2 The main result

Fix $n = 2, 3, \ldots$ and $\alpha > 0$. For each $d = 0, 1, \ldots$, the rv $N_n(d; \alpha)$ defined by

$$N_n(d; \alpha) = \sum_{k=1}^{n} \mathbf{1}\left[D_{n,k}(\alpha) = d\right]$$

counts the number of nodes in $\{1, \ldots, n\}$ which have degree d in $\mathbb{T}(n; \alpha)$. The fraction of nodes in $\{1, \ldots, n\}$ with degree d in $\mathbb{T}(n; \alpha)$ is then given by

$$p_n(d; \alpha) = \frac{N_n(d; \alpha)}{n}.$$

We now state the main result of the chapter.

Theorem 4.2. *Assume Assumption 4.1 to hold. Then, for each $d = 0, 1, \ldots$, there exists a non-degenerate $[0, 1]$-valued rv $\Pi(d)$ such that*

$$\frac{N_n(d; \alpha_n^\star)}{n} \Longrightarrow_n \Pi(d) \tag{4.11}$$

where the scaling $\alpha^\star : \mathbb{N}_0 \to \mathbb{R}_+$ is the one postulated in Assumption 4.1

In the process of proving Theorem 4.2 we will show that for each $d = 0, 1, \ldots$,

$$\mathbb{E}\left[\Pi(d)\right] = \mathbb{P}\left[D = d\right] \quad \text{with} \quad \mathrm{Var}[\Pi(d)] > 0. \tag{4.12}$$

In other words, the rv $\Pi(d)$ is never a degenerate rv with the following consequence.

Corollary 4.3. *Assume Assumption 4.1 to hold. Then, for each $d = 0, 1, \ldots$, the sequence*

$$\left\{ \frac{N_n(d; \alpha_n^\star)}{n}, \ n = 1, 2, \ldots \right\}$$

cannot converge in probability, i.e., there exists no constant $L(d)$ such that

$$\frac{N_n(d; \alpha_n^\star)}{n} \xrightarrow{P}_n L(d). \tag{4.13}$$

Proposition 3.3 is a special case of the corollary stated above.

4.3 A roadmap to a proof of Theorem 4.2

The remainder of the chapter is concerned with a proof of Theorem 4.2. The technical approach is rooted in the method of moments, and articulated through several intermediary results, the first one being a multi-dimensional version of Proposition 4.1.

Proposition 4.4. *Assume Assumption 4.1 to hold. For each $r = 1, 2, \ldots$, there exists an $\mathbb{N}^r$-valued rv $(D_1, \ldots, D_r)$ such that*

$$(D_{n,1}(\alpha_n^\star), \ldots, D_{n,r}(\alpha_n^\star)) \Longrightarrow_n (D_1, \ldots, D_r). \tag{4.14}$$

The limiting rvs $D_1, \ldots, D_r$ are exchangeable, but not independent, each being distributed according to the limiting rv D whose existence is established in Proposition 4.1.

Proposition 4.4 is a simple consequence of Proposition 4.5 discussed next, but first some notation: For each $r = 1, 2, \ldots$, let $\xi_{r,1}, \ldots, \xi_{r,r}$ denote the values of the fitness rvs $\xi_1, \ldots, \xi_r$ arranged in increasing order, namely $\xi_{r,1} \leqslant \ldots \leqslant \xi_{r,r}$, with a lexicographic tiebreaker when needed. Thus, the rvs $\xi_{r,1}, \ldots, \xi_{r,r}$ are the *order statistics* associated with the collection $\xi_1, \ldots, \xi_r$; the rvs $\xi_{r,1}$ and $\xi_{r,r}$ are simply the minimum and maximum of the rvs $\xi_1, \ldots, \xi_r$, respectively [11]. In what follows,

the random permutation $\beta : \{1, \ldots, r\} \to \{1, \ldots, r\}$ arranges the rvs $\xi_1, \ldots, \xi_r$ in increasing order, i.e.,

$$\xi_{r,s} = \xi_{\beta(s)}, \quad s = 1, \ldots, r$$

(under the lexicographic tiebreaker) – Note that β is determined by the rvs $\xi_1, \ldots, \xi_r$ and is uniformly distributed over the group of permutations of $\{1, \ldots, r\}$. Finally, with the notation introduced so far, write

$$G_r(z_1, \ldots, z_r) = \mathbb{E}\left[e^{-\sum_{t=1}^{r}(1 - z_{\beta(t)})\left(\prod_{s=t+1}^{r} z_{\beta(s)}\right)\lambda(\xi_{r,t})} \right], \quad \begin{array}{c} 0 \leqslant z_s \leqslant 1, \\[1mm] s = 1, \ldots, r. \end{array} \tag{4.15}$$

Proposition 4.5. *Assume Assumption 4.1 to hold. For each $r = 1, 2, \ldots$, we have*

$$\lim_{n \to \infty} \mathbb{E}\left[\prod_{s=1}^{r} z_s^{D_{n,s}(\alpha_n^\star)} \right] = G_r(z_1, \ldots, z_r) \tag{4.16}$$

for all $z_1, \ldots, z_r$ in $\mathbb{R}$ satisfying

$$0 \leqslant z_s \leqslant 1, \quad s = 1, \ldots, r. \tag{4.17}$$

This result is established in several steps which are presented across Section 4.5, Section 4.6 and Section 4.7. However, Proposition 4.5 does imply Proposition 4.4 by the usual arguments: Indeed, by the Bounded Convergence Theorem we get

$$\lim_{z_s \uparrow 1, \ s=1,\ldots,r} G_r(z_1, \ldots, z_r) = 1$$

where the convergence is taken from inside $[0, 1)^r$, and the mapping $G_r : [0, 1]^r \to \mathbb{R}$ is therefore continuous at the point $(1, \ldots, 1)$. This fact, coupled with the convergence (4.16), suffices to reach the conclusion that G_r is an r-dimensional pgf. Thus,

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there exists an $\mathbb{N}^r$-valued rv, denoted $(D_1, \ldots, D_r)$, such that

$$\mathbb{E}\left[\prod_{s=1}^{r} z_s^{D_s}\right] = G_r(z_1, \ldots, z_r), \qquad \begin{array}{l} 0 \leqslant z_s \leqslant 1, \\[6pt] s = 1, \ldots, r. \end{array} \tag{4.18}$$

and the convergence (4.14) follows; see details in [21, p. 431] . This completes the proof of Proposition 4.4. It is also plain that the rvs $D_1, \ldots, D_r$ are not independent; see below.

The next step establishes the requisite convergence of the moments; its proof is available in Section 4.8.

Proposition 4.6. *Assume Assumption 4.1 to hold. For each $r = 1, 2, \ldots$, we have*

$$\lim_{n \to \infty} \mathbb{E}\left[\left(\frac{N_n(d; \alpha_n^\star)}{n}\right)^r\right] = \mathbb{P}\left[D_1 = d, \ldots, D_r = d\right], \quad d = 0, 1, \ldots \tag{4.19}$$

where the $\mathbb{N}^r$-valued rv $(D_1, \ldots, D_r)$ is the limiting rv shown to exist in Proposition 4.4.

The following information is easily obtained by combining (4.15) and (4.18): As expected, we get back Proposition 4.1 by looking at the case $r = 1$. For $r = 2$, we read

$$\mathbb{E}\left[z_1^{D_1} z_2^{D_2}\right] = \mathbb{E}\left[e^{-(1-z_{\beta(1)})z_{\beta(2)}\lambda(\xi_{2,1}) - (1-z_{\beta(2)})\lambda(\xi_{2,2})}\right], \quad 0 \leqslant z_1, z_2 \leqslant 1. \tag{4.20}$$

By bounded convergence it follows from Proposition 4.6 that the first half of (4.12) holds with

$$\mathrm{Var}[\Pi(d)] = \mathbb{P}\left[D_1 = d, D_2 = d\right] - \mathbb{P}\left[D_1 = d\right]\mathbb{P}\left[D_2 = d\right].$$

It is now straightforward to argue that $\mathrm{Var}[\Pi(d)] > 0$. Rewriting (4.20), we obtain

$$
\begin{aligned}
\mathbb{E}\left[z_1^{D_1} z_2^{D_2}\right] &= \mathbb{E}\left[e^{-(1-z_{\beta(1)})z_{\beta(2)}\lambda(\xi_{2,1})-(1-z_{\beta(2)})\lambda(\xi_{2,2})}\right] \\[6pt]
&= \mathbb{E}\left[e^{-(1-z_{\beta(1)})\lambda(\xi_{2,1})-(1-z_{\beta(2)})\lambda(\xi_{2,2})+(1-z_{\beta(1)})(1-z_{\beta(2)})\lambda(\xi_{2,1})}\right] \\[6pt]
&= \mathbb{E}\left[e^{-(1-z_{\beta(1)})\lambda(\xi_{\beta(1)})-(1-z_{\beta(2)})\lambda(\xi_{\beta(2)})+(1-z_{\beta(1)})(1-z_{\beta(2)})\lambda(\xi_{2,1})}\right] \\[6pt]
&= \mathbb{E}\left[e^{-(1-z_1)\lambda(\xi_1)-(1-z_2)\lambda(\xi_2)+(1-z_1)(1-z_2)\lambda(\xi_{2,1})}\right] \qquad (4.21)
\end{aligned}
$$

for $0 \leqslant z_1, z_2 \leqslant 1$. Collecting a single term corresponding to $z_1^d z_2^d$ from (4.21), we obtain the lower bound

$$
\mathbb{P}\left[D_1 = d, D_2 = d\right] \geqslant \mathbb{E}\left[\frac{\lambda(\xi_1)^d}{d!}\frac{\lambda(\xi_2)^d}{d!}e^{-\lambda(\xi_1)-\lambda(\xi_2)}e^{\lambda(\min(\xi_1,\xi_2))}\right] \qquad (4.22)
$$

for each $d = 0, 1, \ldots$. Therefore for each $d = 0, 1, \ldots$, using (4.22) and (4.8), we obtain the desired lower bound

$$
\begin{aligned}
\mathrm{Var}[\Pi(d)] &= \mathbb{P}\left[D_1 = d, D_2 = d\right] - \mathbb{P}\left[D_1 = d\right]\mathbb{P}\left[D_2 = d\right] \\[6pt]
&\geqslant \mathbb{E}\left[\frac{\lambda(\xi_1)^d}{d!}\frac{\lambda(\xi_2)^d}{d!}e^{-\lambda(\xi_1)-\lambda(\xi_2)}\left(e^{\lambda(\min(\xi_1,\xi_2))}-1\right)\right] \\[6pt]
&> 0,
\end{aligned}
$$

where the last step follows from Assumption 4.1.

4.4   A proof of Theorem 4.2

Equipped with Propositions 4.4 and 4.6 we can now provide a proof of Theorem 4.2: Fix $d = 0, 1, \ldots$. Proposition 4.6 suggests that we consider the mapping $\phi_d : \mathbb{R} \to \mathbb{C}$ given by

$$
\phi_d(t) = 1 + \sum_{r=1}^{\infty} \frac{(it)^r}{r!}\mathbb{P}\left[D_1 = d, \ldots, D_r = d\right], \quad t \in \mathbb{R}.
$$

This definition is well posed with $\phi_d(t)$ always an element of $\mathbb{C}$ since

$$1 + \sum_{r=1}^{\infty} \frac{|t|^r}{r!} \mathbb{P}\left[D_1 = d, \ldots, D_r = d\right] \leqslant 1 + \sum_{r=1}^{\infty} \frac{|t|^r}{r!} = e^{|t|}, \quad t \in \mathbb{R}.$$

In particular, the mapping $\phi_d : \mathbb{R} \to \mathbb{C}$ is analytic on $\mathbb{R}$, hence continuous at $t = 0$.

Now, for each $n = 2, 3, \ldots$, let $\phi_{d,n} : \mathbb{R} \to \mathbb{C}$ denote the characteristic function of the rv $\frac{N_n(d;\alpha_n^\star)}{n}$, i.e.,

$$\phi_{d,n}(t) = \mathbb{E}\left[e^{it \frac{N_n(d;\alpha_n^\star)}{n}} \right], \quad t \in \mathbb{R}.$$

The obvious bounds

$$0 \leqslant \frac{N_n(d; \alpha_n^\star)}{n} \leqslant 1, \quad n = 2, 3, \ldots \tag{4.23}$$

can be used to validate the series expansion

$$\phi_{d,n}(t) = 1 + \sum_{r=1}^{\infty} \frac{(it)^r}{r!} \mathbb{E}\left[\left(\frac{N_n(d; \alpha_n^\star)}{n} \right)^r \right], \quad t \in \mathbb{R}$$

since here as well we have

$$1 + \sum_{r=1}^{\infty} \frac{|t|^r}{r!} \mathbb{E}\left[\left(\frac{N_n(d; \alpha_n^\star)}{n} \right)^r \right] \leqslant 1 + \sum_{r=1}^{\infty} \frac{|t|^r}{r!} = e^{|t|}, \quad t \in \mathbb{R}.$$

In view of these remarks, for each $n = 2, 3, \ldots$ and t in $\mathbb{R}$, we can write

$$\phi_{d,n}(t) - \phi_d(t) = \sum_{r=1}^{\infty} \frac{(it)^r}{r!} \left(\mathbb{E}\left[\left(\frac{N_n(d; \alpha_n^\star)}{n} \right)^r \right] - \mathbb{P}\left[D_1 = d, \ldots, D_r = d\right] \right).$$

Picking a positive integer R, we get

$$|\phi_{d,n}(t) - \phi_d(t)|$$

$$= \sum_{r=1}^{R} \frac{|t|^r}{r!} \left| \mathbb{E}\left[\left(\frac{N_n(d; \alpha_n^\star)}{n} \right)^r \right] - \mathbb{P}\left[D_1 = d, \ldots, D_r = d\right] \right| + 2 \sum_{r=R+1}^{\infty} \frac{|t|^r}{r!}.$$

With arbitrary $\varepsilon > 0$ there exists a positive integer $R^{\star}(\varepsilon, t)$ such that

$$\sum_{r=R+1}^{\infty} \frac{|t|^r}{r!} \leqslant \varepsilon, \quad R \geqslant R^{\star}(\varepsilon, t),$$

and on that range we obtain

$$
\begin{aligned}
&\limsup_{n \to \infty} |\phi_{d,n}(t) - \phi_d(t)| \\
&= \lim_{n \to \infty} \sum_{r=1}^{R} \frac{|t|^r}{r!} \left| \mathbb{E}\left[\left(\frac{N_n(d; \alpha_n^{\star})}{n}\right)^r\right] - \mathbb{P}\left[D_1 = d, \ldots, D_r = d\right] \right| + 2\varepsilon \\
&= 2\varepsilon
\end{aligned}
\tag{4.24}
$$

upon invoking Proposition 4.6. Since $\varepsilon > 0$ is arbitrary, we conclude that $\lim_{n \to \infty} \phi_{d,n}(t) = \phi_d(t)$.

The mapping $\phi_d : \mathbb{R} \to \mathbb{C}$ being continuous at $t = 0$, it follows by a standard result on weak convergence due to Lévy [21, p. 431] that the distributional convergence (4.11) takes place. The distribution of the limiting rv $\Pi(d)$ is determined in terms of its characteristic function through

$$\phi_d(t) = \mathbb{E}\left[e^{it\Pi(d)}\right], \quad t \in \mathbb{R}.$$

■

4.5   A proof of Proposition 4.5 – A reduction step

Throughout this section the integer $r = 1, 2, \ldots$ and the parameter $\alpha > 0$ are being held fixed. Pick $n > r$. Thus, for each $k = 1, \ldots, r$, we find

$$D_{n,k}(\alpha) = \sum_{\ell=1, \; \ell \neq k}^{r} \mathbf{1}\left[\xi_k + \xi_\ell > \alpha\right] + \tilde{D}_{n,k}^{(r)}(\alpha)$$

with

$$\tilde{D}_{n,k}^{(r)}(\alpha) = \sum_{\ell=r+1}^{n} \mathbf{1}\left[\xi_k + \xi_\ell > \alpha\right].$$

Because the scaling $\alpha^\star : \mathbb{N}_0 \to \mathbb{R}_+$ satisfies $\lim_{n\to\infty} \alpha_n^\star = \infty$, it is plain that

$$\lim_{n\to\infty} \max\left(\sum_{\ell=1,\ \ell\neq k}^{r} \mathbf{1}\left[\xi_k + \xi_\ell > \alpha_n^\star\right],\ k = 1,\dots,r\right) = 0 \quad a.s.$$

and (4.14) takes place if and only if

$$(\tilde{D}_{n,1}^{(r)}(\alpha_n^\star),\dots,\tilde{D}_{n,r}^{(r)}(\alpha_n^\star)) \Longrightarrow_n (D_1,\dots,D_r). \tag{4.25}$$

The remainder of the proof consists in establishing that (4.25) holds. This will be done by showing that the joint pgfs converge to the joint pgf of an $\mathbb{N}^r$-valued rv over a suitable range.

To do so, our first step is to evaluate the pgfs. Pick $z_1,\dots,z_r$ in $\mathbb{R}$. Under the enforced independence assumptions, it is plain that

$$\mathbb{E}\left[\prod_{s=1}^{r} z_s^{\tilde{D}_{n,s}^{(r)}(\alpha)}\right] = \mathbb{E}\left[\prod_{s=1}^{r} z_s^{\sum_{\ell=r+1}^{n} \mathbf{1}[\xi_s+\xi_\ell>\alpha]}\right]$$

$$= \mathbb{E}\left[\prod_{s=1}^{r} \prod_{\ell=r+1}^{n} z_s^{\mathbf{1}[\xi_s+\xi_\ell>\alpha]}\right]$$

$$= \mathbb{E}\left[\prod_{\ell=r+1}^{n} \prod_{s=1}^{r} z_s^{\mathbf{1}[\xi_s+\xi_\ell>\alpha]}\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[\prod_{\ell=r+1}^{n} \prod_{s=1}^{r} z_s^{\mathbf{1}[\xi_s+\xi_\ell>\alpha]}\middle|\xi_1,\dots,\xi_r\right]\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[\prod_{\ell=r+1}^{n} \prod_{s=1}^{r} z_s^{\mathbf{1}[x_s+\xi_\ell>\alpha]}\right]_{x_1=\xi_1,\dots,x_r=\xi_r}\right]. \tag{4.26}$$

With arbitrary $x_1,\dots,x_r$ in $\mathbb{R}_+$, we get

$$\mathbb{E}\left[\prod_{\ell=r+1}^{n} \prod_{s=1}^{r} z_s^{\mathbf{1}[x_s+\xi_\ell>\alpha]}\right] = \prod_{\ell=r+1}^{n} \mathbb{E}\left[\prod_{s=1}^{r} z_s^{\mathbf{1}[x_s+\xi_\ell>\alpha]}\right]$$

$$= F_r\big(\alpha; z_1,\dots,z_r; x_1,\dots,x_r\big)^{n-r}$$

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where we have set

$$F_r(\alpha; z_1, \ldots, z_r; x_1, \ldots, x_r) = \mathbb{E}\left[\prod_{s=1}^{r} z_s^{\mathbf{1}[x_s+\xi>\alpha]}\right]$$

$$= \mathbb{E}\left[\prod_{s=1}^{r} (\mathbf{1}[x_s + \xi > \alpha]\, z_s + \mathbf{1}[x_s + \xi \leqslant \alpha])\right] \quad (4.27)$$

In short, we have

$$\mathbb{E}\left[\prod_{s=1}^{r} z_s^{\tilde{D}_{n,s}^{(r)}(\alpha)}\right] = \mathbb{E}\left[F_r(\alpha; z_1, \ldots, z_r; \xi_1, \ldots, \xi_r)^{n-r}\right], \quad (4.28)$$

and the desired result (4.16) will hold if we show that

$$\lim_{n\to\infty} \mathbb{E}\left[F_r(\alpha; z_1, \ldots, z_r; \xi_1, \ldots, \xi_r)^{n-r}\right] = G_r(z_1, \ldots, z_r) \quad (4.29)$$

for all $z_1, \ldots, z_r$ in $\mathbb{R}$ which satisfy (4.17).

4.6   A proof of Proposition 4.5 – A decomposition

To further analyze this last expression, with $x_1, \ldots, x_r$ in $\mathbb{R}_+$, we introduce the index set

$$\mathcal{S}(\alpha; x_1, \ldots, x_r) = \{s = 1, \ldots, r : \ x_s > \alpha\}.$$

There are two possibilities which we now explore in turn: Either $\mathcal{S}(\alpha; x_1, \ldots, x_r)$ is empty or it is not, leading to a natural decomposition expressed through Lemmas 4.7 and 4.8.

Lemma 4.7. *With $x_1, \ldots, x_r$ in $\mathbb{R}$, whenever $\mathcal{S}(\alpha; x_1, \ldots, x_r)$ is non-empty, we have*

$$F_r(\alpha; z_1, \ldots, z_r; x_1, \ldots, x_r)$$
$$= \left(\prod_{s\in\mathcal{S}(\alpha;x_1,\ldots,x_r)} z_s\right) \cdot \mathbb{E}\left[\prod_{s\notin\mathcal{S}(\alpha;x_1,\ldots,x_r)}^{r} (1 - (1 - z_s)\, \mathbf{1}[x_s + \xi > \alpha])\right] \quad (4.30)$$

for all $z_1, \ldots, z_r$ in $\mathbb{R}$.

Proof. Pick arbitrary $x_1, \ldots, x_r$ in $\mathbb{R}_+$ with non-empty $\mathcal{S}(\alpha; x_1, \ldots, x_r)$. For all $z_1, \ldots, z_r$ in $\mathbb{R}$, it is easy to check by direct inspection from the expression (4.27) that (4.30) holds since

$$1 - (1 - z_s) \, \mathbf{1} \left[x_s + \xi > \alpha \right] = z_s$$

whenever s belongs to $\mathcal{S}(\alpha; x_1, \ldots, x_r)$. ∎

As an immediate consequence of (4.30) we have the inequality

$$x_1, \ldots, x_r \in \mathbb{R}_+$$

$$0 \leqslant F_r(\alpha; z_1, \ldots, z_r; x_1, \ldots, x_r) \leqslant 1, \qquad \text{with} \qquad (4.31)$$

$$|\mathcal{S}(\alpha; x_1, \ldots, x_r)| > 0$$

for all $z_1, \ldots, z_r$ in $\mathbb{R}$ in the range

$$|z_s| \leqslant 1, \quad s = 1, \ldots, r. \qquad (4.32)$$

This is because, it is always the case that

$$\left| 1 - (1 - z_s) \, \mathbf{1} \left[x_s + \xi > \alpha \right] \right| \leqslant 1 \quad \text{if } |z_s| \leqslant 1.$$

We now turn to the case when the index set $\mathcal{S}(\alpha; x_1, \ldots, x_r)$ is empty, a fact characterized by the conditions

$$x_s \leqslant \alpha, \quad s = 1, \ldots, r. \qquad (4.33)$$

It will be convenient to arrange the values $x_1, \ldots, x_r$ in increasing order, say $x_{(1)} \leqslant x_{(2)} \leqslant \ldots \leqslant x_{(r)}$, say with a lexicographic tiebreaker. Let β be any permutation of $\{1, \ldots, r\}$ such that $x_{(s)} = x_{\beta(s)}$ for all $s = 1, \ldots, r$ – Obviously this permutation depends on $x_1, \ldots, x_r$. In what follows we shall use the convention $x_{(0)} = -\infty$ and $x_{(r+1)} = \infty$.

Lemma 4.8. *With $x_1, \ldots, x_r$ in $\mathbb{R}$, whenever $\mathcal{S}(\alpha; x_1, \ldots, x_r)$ is empty, we have*

$$F_r(\alpha; z_1, \ldots, z_r; x_1, \ldots, x_r) = \sum_{t=0}^{r} \left(\prod_{s=t+1}^{r} z_{\beta(s)} \right) \cdot \left(F(\alpha - x_{(t)}) - F(\alpha - x_{(t+1)}) \right) \quad (4.34)$$

for all $z_1, \ldots, z_r$ in $\mathbb{R}$. In this expression the product of an empty number of factors is set to unity by convention.

Proof. In what follows, the values $z_1, \ldots, z_r$ in $\mathbb{R}$ are held fixed. Fix $x_1, \ldots, x_r$ in $\mathbb{R}$ and $\alpha > 0$. We define the events

$$A_{r|t}(x_1, \ldots, x_r; \alpha) = \left[x_{(t)} + \xi \leqslant \alpha < x_{(t+1)} + \xi \right], \quad t = 0, 1, , \ldots, r.$$

Under the enforced conventions, we have

$$A_{r|0}(x_1, \ldots, x_r; \alpha) = \left[\alpha < x_{(1)} + \xi \right] \quad \text{and} \quad A_{r|r}(x_1, \ldots, x_r; \alpha) = \left[x_{(r)} + \xi \leqslant \alpha \right].$$

When $\mathcal{S}(\alpha; x_1, \ldots, x_r)$ is empty, these $r+1$ events $A_{r|0}(x_1, \ldots, x_r; \alpha), \ldots, A_{r|r}(x_1, \ldots, x_r; \alpha)$ are mutually exclusive and form a partition of the sample space, so that

$$F_r(\alpha; z_1, \ldots, z_r; x_1, \ldots, x_r)$$
$$= \sum_{t=0}^{r} \mathbb{E}\left[\mathbf{1}\left[A_{r|t}(x_1, \ldots, x_r; \alpha) \right] \prod_{s=1}^{r} \left(\mathbf{1}\left[x_s + \xi > \alpha \right] z_s + \mathbf{1}\left[x_s + \xi \leqslant \alpha \right] \right) \right] \quad (4.35)$$

(i) On the event $A_{r|0}(x_1, \ldots, x_r; \alpha)$, we have $\alpha < x_{(1)} + \xi$, thus $\alpha < x_s + \xi$ for all $s = 1, \ldots, r$, so that

$$\prod_{s=1}^{r} \left(1 - (1 - z_s) \mathbf{1}\left[x_s + \xi > \alpha \right] \right) = \prod_{s=1}^{r} z_s.$$

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It then follows that

$$\mathbb{E}\left[\mathbf{1}\left[A_{r|0}(x_1,\ldots,x_r;\alpha)\right]\prod_{s=1}^{r}(1-(1-z_s)\mathbf{1}\left[x_s+\xi>\alpha\right])\right]$$

$$=\left(\prod_{s=1}^{r}z_s\right)\cdot\mathbb{P}\left[\alpha<x_{(1)}+\xi\right]$$

$$=\left(\prod_{s=1}^{r}z_s\right)\cdot\left(1-F(\alpha-x_{(1)})\right). \tag{4.36}$$

(ii) With $t=1,\ldots,r-1$, on the event $A_{r|t}(x_1,\ldots,x_r;\alpha)$ it holds that $x_{(1)}+\xi\leqslant$
$\alpha,\ldots,x_{(t)}+\xi\leqslant\alpha$ and $\alpha<x_{(t+1)}+\xi,\ldots,\alpha<x_{(r)}+\xi$. Therefore,

$$\prod_{s=1}^{r}(1-(1-z_s)\mathbf{1}\left[x_s+\xi>\alpha\right])=\left(\prod_{s=t+1}^{r}z_{\beta(s)}\right),$$

and we readily conclude to

$$\mathbb{E}\left[\mathbf{1}\left[A_{r|t}(x_1,\ldots,x_r;\alpha)\right]\prod_{s=1}^{r}(1-(1-z_s)\mathbf{1}\left[x_s+\xi>\alpha\right])\right]$$

$$=\left(\prod_{s=t+1}^{r}z_{\beta(s)}\right)\cdot\mathbb{P}\left[x_{(t)}+\xi\leqslant\alpha<x_{(t+1)}+\xi\right]$$

$$=\left(\prod_{s=t+1}^{r}z_{\beta(s)}\right)\cdot\left(F(\alpha-x_{(t)})-F(\alpha-x_{(t+1)})\right). \tag{4.37}$$

(iii) Finally, on the event $A_{r|r}(x_1,\ldots,x_r;\alpha)$, $x_{(r)}+\xi\leqslant\alpha$, thus $x_s+\xi\leqslant\alpha$ for
all $s=1,\ldots,r$, so that

$$\prod_{s=1}^{r}(1-(1-z_s)\mathbf{1}\left[x_s+\xi>\alpha\right])=1,$$

whence

$$\mathbb{E}\left[\mathbf{1}\left[A_{r|r}(x_1,\ldots,x_r;\alpha)\right]\prod_{s=1}^{r}(1-(1-z_s)\mathbf{1}\left[x_s+\xi>\alpha\right])\right]=\mathbb{P}\left[x_{(r)}+\xi\leqslant\alpha\right]$$

$$=F(\alpha-x_{(r)}).\tag{4.38}$$

To complete the proof we substitute (4.36), (4.37) and (4.38) into (4.35), and
recall that $F(\alpha-x_{(0)})=1$ and $F(\alpha-x_{(r+1)})=0$ under the enforced conventions. ∎

4.7 A proof of Proposition 4.5 – Taking the limit

In order to establish the convergence (4.25) we return to the expression (4.28) for the joint pgf of the relevant rvs. Fix $n = 2, 3, \ldots$ with $r < n$, and replace α by $\alpha_n^\star$ according to the scaling $\alpha^\star : \mathbb{N}_0 \to \mathbb{R}_+$ appearing in Assumption 4.1.

4.7.1 A useful intermediary fact

For arbitrary $\alpha > 0$, consider $x_1, \ldots, x_r$ in $\mathbb{R}$ and $z_1, \ldots, z_r$ in $\mathbb{R}$. In what follows it will be convenient to define

$$\Lambda_r(\alpha; z_1, \ldots, z_r; x_1, \ldots, x_r) = 1 - F_r(\alpha; z_1, \ldots, z_r; x_1, \ldots, x_r)$$

so that

$$F_r(\alpha; z_1, \ldots, z_r; x_1, \ldots, x_r) \;=\; 1 - \Lambda_r(\alpha; z_1, \ldots, z_r; x_1, \ldots, x_r). \tag{4.39}$$

Whenever $\mathcal{S}(\alpha; x_1, \ldots, x_r)$ is empty, Lemma 4.8 gives

$$\Lambda_r(\alpha; z_1, \ldots, z_r; x_1, \ldots, x_r)$$

$$= \; 1 - \sum_{t=0}^{r} \left(\prod_{s=t+1}^{r} z_{\beta(s)} \right) \cdot \left(F(\alpha - x_{(t)}) - F(\alpha - x_{(t+1)}) \right)$$

$$= \; -\sum_{t=0}^{r-1} \left(\prod_{s=t+1}^{r} z_{\beta(s)} \right) \cdot \left(F(\alpha - x_{(t)}) - F(\alpha - x_{(t+1)}) + \left(1 - F(\alpha - x_{(r)}) \right) \right) \tag{4.40}$$

Now fix $n = 2, 3, \ldots$ with $r < n$, and replace α by $\alpha_n^\star$ in (4.40) according to the scaling $\alpha^\star : \mathbb{N}_0 \to \mathbb{R}_+$ appearing in Assumption 4.1: We get

$$\lim_{n \to \infty} n \left(1 - F(\alpha_n^\star - x_{(r)}) \right) = \lambda(x_{(r)})$$

and

$$\lim_{n\to\infty} n\left(F(\alpha_n^\star - x_{(t)}) - F(\alpha_n^\star - x_{(t+1)})\right)$$

$$= \lim_{n\to\infty} n\left(\left(1 - F(\alpha_n^\star - x_{(t+1)})\right) - \left(1 - F(\alpha_n^\star - x_{(t)})\right)\right)$$

$$= \begin{cases} \lambda(x_{(1)}) & \text{if } t = 0 \\ \\ \lambda(x_{(t+1)}) - \lambda(x_{(t)}) & \text{if } t = 1, \dots, r-1. \end{cases} \tag{4.41}$$

As a result, with $\mathcal{S}(\alpha; x_1, \dots, x_r)$ empty, we have

$$\lim_{n\to\infty} n\Lambda_r(\alpha_n^\star; z_1, \dots, z_r; x_1, \dots, x_r)$$

$$= -\lambda(x_{(1)}) \left(\prod_{s=1}^{r} z_s\right) - \sum_{t=1}^{r-1} \left(\prod_{s=t+1}^{r} z_{\beta(s)}\right) \left(\lambda(x_{(t+1)}) - \lambda(x_{(t)})\right) + \lambda(x_{(r)})$$

$$= -\sum_{t=1}^{r} \lambda(x_{(t)}) \left(\prod_{s=t}^{r} z_{\beta(s)} - \prod_{s=t+1}^{r} z_{\beta(s)}\right)$$

$$= \sum_{t=1}^{r} \lambda(x_{(t)})(1 - z_{\beta(t)}) \prod_{s=t+1}^{r} z_{\beta(s)}, \tag{4.42}$$

and the conclusion

$$\lim_{n\to\infty} F_r(\alpha_n^\star; z_1, \dots, z_r; x_1, \dots, x_r)^{n-r}$$

$$= \lim_{n\to\infty} \left(1 - \Lambda_r(\alpha_n^\star; z_1, \dots, z_r; x_1, \dots, x_r)\right)^{n-r}$$

$$= e^{-\sum_{t=1}^{r} \lambda(x_{(t)})(1 - z_{\beta(t)}) \prod_{s=t+1}^{r} z_{\beta(s)}} \tag{4.43}$$

follows by standard arguments.

4.7.2 In the limit

Pick $z_1, \ldots, z_r$ in $\mathbb{R}$ such that (4.17) holds. For each $n = 2, 3, \ldots$ with $r < n$, the decomposition

$$\mathbb{E}\left[F_r(\alpha_n^\star; z_1, \ldots, z_r; \xi_1, \ldots, \xi_r)^{n-r}\right]$$

$$= \mathbb{E}\left[\mathbf{1}\left[|\mathcal{S}(\alpha_n^\star; \xi_1, \ldots, \xi_r)| > 0\right] F_r(\alpha_n^\star; z_1, \ldots, z_r; \xi_1, \ldots, \xi_r)^{n-r}\right]$$

$$+ \mathbb{E}\left[\mathbf{1}\left[|\mathcal{S}(\alpha_n^\star; \xi_1, \ldots, \xi_r)| = 0\right] F_r(\alpha_n^\star; z_1, \ldots, z_r; \xi_1, \ldots, \xi_r)^{n-r}\right] \quad (4.44)$$

holds. Because $\lim_{n\to\infty} \alpha_n^\star = \infty$, it follows that

$$\lim_{n\to\infty} \mathbb{P}\left[|\mathcal{S}(\alpha_n^\star; \xi_1, \ldots, \xi_r)| = 0\right] = \lim_{n\to\infty} \mathbb{P}\left[\xi_1 \leqslant \alpha_n^\star, \ldots, \xi_r \leqslant \alpha_n^\star\right] = 1,$$

and the inequality (4.31) readily yields

$$\lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}\left[|\mathcal{S}(\alpha_n^\star; \xi_1, \ldots, \xi_r)| > 0\right] F_r(\alpha_n^\star; z_1, \ldots, z_r; \xi_1, \ldots, \xi_r)^{n-r}\right] = 0 \qquad (4.45)$$

since the condition (4.17) is more restrictive than (4.32).

Next,

$$\mathbb{E}\left[\mathbf{1}\left[|\mathcal{S}(\alpha_n^\star; \xi_1, \ldots, \xi_r)| = 0\right] F_r(\alpha_n^\star; z_1, \ldots, z_r; \xi_1, \ldots, \xi_r)^{n-r}\right]$$

$$= \mathbb{E}\left[\mathbf{1}\left[|\mathcal{S}(\alpha_n^\star; \xi_1, \ldots, \xi_r)| = 0\right] (1 - \Lambda_r(\alpha_n^\star; z_1, \ldots, z_r; \xi_1, \ldots, \xi_r))^{n-r}\right] \quad (4.46)$$

with (4.40) yielding

$$\Lambda_r(\alpha_n^\star; z_1, \ldots, z_r; \xi_1, \ldots, \xi_r)$$

$$= 1 - F(\alpha_n^\star - \xi_{r,r}) - \sum_{t=0}^{r-1} \left(\prod_{s=t+1}^{r} z_{\beta(s)}\right) \cdot (F(\alpha_n^\star - \xi_{r,t}) - F(\alpha_n^\star - \xi_{r,t+1})) \quad (4.47)$$

Here the order statistics $\xi_{r,1},\ldots,\xi_{r,r}$ associated with $\xi_1,\ldots,\xi_r$ were introduced in the statement of Proposition 4.5, together with the random permutation $\beta:\{1,\ldots,r\}\to\{1,\ldots,r\}$.

It is plain that

$$0\leqslant\left(\prod_{s=t+1}^{r}z_{\beta(s)}\right)\leqslant 1,\quad t=0,\ldots,r-1$$

under the condition (4.17), and direct inspection gives

$$0\leqslant\Lambda_r(\alpha_n^\star;z_1,\ldots,z_r;\xi_1,\ldots,\xi_r)\leqslant 1$$

so that

$$\left|F_r(\alpha_n^\star;z_1,\ldots,z_r;\xi_1,\ldots,\xi_r)^{n-r}\right|\leqslant 1.$$

Therefore, with the fact $\lim_{n\to\infty}\mathbf{1}\left[|\mathcal{S}(\alpha_n^\star;\xi_1,\ldots,\xi_r)|=0\right]=1$ noted earlier, we see that the convergence

$$\lim_{n\to\infty}\mathbf{1}\left[|\mathcal{S}(\alpha_n^\star;\xi_1,\ldots,\xi_r)|=0\right]F_r(\alpha_n^\star;z_1,\ldots,z_r;\xi_1,\ldots,\xi_r)^{n-r}$$

$$=\quad e^{-\sum_{t=1}^{r}\lambda(\xi_{r,t})(1-z_{\beta(t)})\prod_{s=t+1}^{r}z_{\beta(s)}} \tag{4.48}$$

takes place boundedly. By the Bounded Convergence Theorem it follows that

$$\lim_{n\to\infty}\mathbb{E}\left[\mathbf{1}\left[|\mathcal{S}(\alpha_n^\star;\xi_1,\ldots,\xi_r)|=0\right]F_r(\alpha_n^\star;z_1,\ldots,z_r;\xi_1,\ldots,\xi_r)^{n-r}\right]$$

$$=\quad \mathbb{E}\left[e^{-\sum_{t=1}^{r}\lambda(\xi_{r,t})(1-z_{\beta(t)})\prod_{s=t+1}^{r}z_{\beta(s)}}\right]$$

$$=\quad G_r(z_1,\ldots,z_r) \tag{4.49}$$

Collecting (4.45) and (4.49), and using (4.44) we conclude that (4.29) holds on the range (4.17). $\blacksquare$

4.8 A proof of Proposition 4.6

Fix $\alpha > 0$, $d = 0, 1, \ldots$ and $r = 1, 2, \ldots$. With $n = r, r+1, \ldots$, let $\mathcal{P}_{n,r}$ denote the collection of all *ordered* arrangements of r distinct elements drawn from the set $\{1, \ldots, n\}$. Any such arrangement can be viewed as a one-to-one mapping $\pi : \{1, \ldots, r\} \rightarrow \{1, \ldots, n\}$.

We begin by noting that we can always write

$$
\begin{aligned}
N_n(d; \alpha)^r &= \left(\sum_{k=1}^{n} \mathbf{1}\left[D_{n,k}(\alpha) = d \right] \right)^r \\
&= \sum_{\pi \in \mathcal{P}_{n,r}} \mathbf{1}\left[D_{n,\pi(1)}(\alpha) = d \right] \ldots \mathbf{1}\left[D_{n,\pi(r)}(\alpha) = d \right] + R_{n,r}(d; \alpha) \quad (4.50)
\end{aligned}
$$

where the correction term $R_{n,r}(d; \alpha)$ is a sum comprising $\{0, 1\}$-valued rvs. Since the correction term $R_{n,r}(d; \alpha)$ is a sum of exactly $n^r - |\mathcal{P}_{n,r}|$ terms, with each term bounded by 1, we have the upper bound

$$
R_{n,r}(d; \alpha) \leqslant n^r - |\mathcal{P}_{n,r}|. \quad (4.51)
$$

Therefore, taking expectations, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[N_n(d; \alpha)^r \right] \\
&\quad = \sum_{\pi \in \mathcal{P}_{n,r}} \mathbb{E}\left[\mathbf{1}\left[D_{n,\pi(1)}(\alpha) = d \right] \ldots \mathbf{1}\left[D_{n,\pi(r)}(\alpha) = d \right] \right] + \mathbb{E}\left[R_{n,r}(d; \alpha) \right] \\
&\quad = |\mathcal{P}_{n,r}| \cdot \mathbb{P}\left[D_{n,1}(\alpha) = d, \ldots, D_{n,r}(\alpha) = d \right] + \mathbb{E}\left[R_{n,r}(d; \alpha) \right] \quad (4.52)
\end{aligned}
$$

as we make use of the fact that the rvs $D_{n,1}(\alpha), \ldots, D_{n,n}(\alpha)$ are exchangeable. As a result,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\frac{N_n(d; \alpha)}{n} \right)^r \right] \\
&\quad = \frac{|\mathcal{P}_{n,r}|}{n^r} \cdot \mathbb{P}\left[D_{n,1}(\alpha) = d, \ldots, D_{n,r}(\alpha) = d \right] + \mathbb{E}\left[\frac{R_{n,r}(d; \alpha)}{n^r} \right], \quad (4.53)
\end{aligned}
$$

and the bound (4.51) implies

$$\mathbb{E}\left[\frac{R_{n,r}(d;\alpha)}{n^r}\right] \leqslant 1 - \frac{|\mathcal{P}_{n,r}|}{n^r} \tag{4.54}$$

with

$$|\mathcal{P}_{n,r}| = n(n-1)\ldots(n-r+1).$$

Now consider the scaling $\alpha^\star : \mathbb{N}_0 \to \mathbb{R}_+$ whose existence is assumed in Assumption 4.1. For each $n = r, r+1, \ldots$ replace α by $\alpha_n^\star$ in (4.53) according to this scaling and let n go to infinity in the resulting relation: First, we note that

$$\lim_{n\to\infty} \frac{|\mathcal{P}_{n,r}|}{n^r} = 1$$

by direct inspection, so that

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{R_{n,r}(d;\alpha_n^\star)}{n^r}\right] = 0$$

as we make use of (4.54). It follows that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\frac{N_n(d;\alpha_n^\star)}{n}\right)^r\right] = \lim_{n\to\infty} \mathbb{P}\left[D_{n,1}(\alpha_n^\star) = d, \ldots, D_{n,r}(\alpha_n^\star) = d\right]$$

with the understanding that if one of the limits exists, so does the other and their value coincide. The latter exists since by Proposition 4.4 we have

$$\lim_{n\to\infty} \mathbb{P}\left[D_{n,1}(\alpha_n^\star) = d, \ldots, D_{n,r}(\alpha_n^\star) = d\right] = \mathbb{P}\left[D_1 = d, \ldots, D_r = d\right]$$

where the rvs $D_1, \ldots, D_r$ are the limiting rvs appearing in the convergence (4.14). It is now plain that (4.19) holds and the proof of Proposition 4.6 is now complete.

∎

Chapter 5:   Degree Distribution in Growth Models

As indicated in Chapter 1, we are interested in growth models that can explain power-law behavior in real-world networks. There we mentioned a number of existing models that implement preferential attachment on the basis of degree and fitness information combined in various ways. Existing works consider special cases where power-law behavior is observed under these models, but do not give a satisfactory account of what conditions are required (e.g., on the fitness distribution) to obtain power-law behavior. We consider arguably the simplest of these models, called the *fitness model* and analyze its degree distribution. Our main motivation is to explore how the fitness distribution affects the empirical degree distribution, if at all, and under what conditions power-law behavior could be obtained.

5.1   The model

A word on notation: Here we consider the sequence of graphs indexed by t instead of n (as was done in the previous chapters). This notation is natural when considering growth models where there is a notion of adding nodes and edges over time.

The fitness-based random graph model is defined by means of two collections

of rvs, namely $\{U, U_t,\ t = 0, 1, \ldots\}$ and $\{\xi, \xi_t,\ t = 0, 1, \ldots\}$, defined on some probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ – All probabilistic statements are made with respect to the probability measure $\mathbb{P}$, whose expectation operator is denoted by $\mathbb{E}$.

The discussion is carried out under the following set of assumptions:

(i) The rvs $\{U, U_t,\ t = 0, 1, \ldots\}$ and $\{\xi, \xi_t,\ t = 0, 1, \ldots\}$ are mutually independent;

(ii) The rvs $\{U, U_t,\ t = 0, 1, \ldots\}$ are i.i.d. rvs, each of which is uniformly distributed on the interval $(0, 1)$; and

(iii) The rvs $\{\xi, \xi_t,\ t = 0, 1, \ldots\}$ are i.i.d. $\mathbb{R}_+$-valued rvs. Throughout we assume the non-degeneracy condition

$$\mathbb{P}\left[\xi = 0\right] = 0, \tag{5.1}$$

as well as the finite mean condition

$$0 < \mathbb{E}\left[\xi\right] < \infty. \tag{5.2}$$

Under these assumptions we can always select (as we do from now on) the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, and the rvs $\{U, U_t,\ t = 0, 1, \ldots\}$ and $\{\xi, \xi_t,\ t = 0, 1, \ldots\}$ defined on it as mappings $\Omega \to \mathbb{R}$ which simultaneously satisfy the conditions

$$\xi > 0, \xi_t > 0 \quad \text{and} \quad 0 < U, U_t < 1, \quad t = 0, 1, \ldots$$

We shall find it convenient to write

$$\Xi_{-1} \equiv 0, \ \Xi_t = \sum_{s=0}^{t} \xi_s, \quad t = 0, 1, \ldots \tag{5.3}$$

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and

$$\xi^t = (\xi_0, \xi_1, \ldots, \xi_t), \quad t = 0, 1, \ldots \tag{5.4}$$

We shall also make use of the filtration $\{\mathcal{F}_t, \ t = 0, 1, \ldots\}$ on Ω given by

$$\mathcal{F}_t = \sigma \left(U_s, \xi_s, \ s = 0, 1, \ldots, t \right), \quad t = 0, 1, \ldots.$$

We now formally define the sequence of (undirected) random graphs $\{\mathbb{G}_t, \ t = 0, 1, \ldots\}$ studied in this chapter: For each $t = 0, 1, \ldots$, the random graph $\mathbb{G}_t$ has vertex set $V_t = \{0, 1, \ldots, t\}$ and random edge set $\mathbb{E}_t \subseteq V_t \times V_t$. As this is a growth model, imagine there being an initial node, labelled node 0, present in the system at time $t = 0$, with new nodes, labelled $t = 1, 2, \ldots$, arriving one at a time, say at times $t = 1, 2, \ldots$. The definition is a recursive one, starting with the initial random graph $\mathbb{G}_0 = (V_0, \mathbb{E}_0)$ where $V_0 = \{0\}$ and $\mathbb{E}_0 = \varnothing$. With $t = 0, 1, \ldots$, once the random graphs $\mathbb{G}_0, \ldots, \mathbb{G}_t$ have been defined, we can generate $\mathbb{G}_{t+1}$ from $\mathbb{G}_t$ by introducing a new vertex not in V_t (which we label $t + 1$), and connecting it to the node S_{t+1} randomly selected in V_t according to

$$S_{t+1} = s \quad \text{if} \quad \frac{\Xi_{s-1}}{\Xi_t} < U_{t+1} \leqslant \frac{\Xi_s}{\Xi_t}, \quad s \in V_t. \tag{5.5}$$

Only the fitness levels of the nodes already present in V_t matter in determining the likelihood of the node to which node $t + 1$ will attach. The newly created link between nodes $t + 1$ and S_{t+1} is interpreted as an *undirected* link, so that

$$\mathbb{E}_{t+1} = \mathbb{E}_t \cup \{(t + 1, S_{t+1}), (S_{t+1}, t + 1)\}.$$

Although the rvs $S_1, \ldots, S_t$ are *not* mutually independent, they are condition-

ally mutually independent *given* ξ^t. It is a simple matter to check that

$$\mathbb{P}\left[S_1 = r_1, \ldots, S_t = r_t, S_{t+1} = r_{t+1} | \mathcal{F}_t\right] = \prod_{\tau=1}^{t+1} \mathbb{P}\left[S_\tau = r_\tau | \mathcal{F}_t\right] \tag{5.6}$$

with arbitrary r_1 in V_1, ..., r_{t+1} in V_{t+1}, where

$$\mathbb{P}\left[S_\tau = r_\tau | \mathcal{F}_t\right] = \frac{\xi_{r_\tau}}{\Xi_{\tau-1}}, \quad \tau = 1, \ldots, t+1 \tag{5.7}$$

Fix $t = 1, 2, \ldots$. For each $s = 0, 1, \ldots, t$, let the rv $D_t(s)$ denote the degree of node s in $\mathbb{G}_t$. By construction the rv $D_t(s)$ can be expressed as

$$D_t(s) = 1 + \sum_{r=s+1}^{t} \mathbf{1}\left[S_r = s\right] \tag{5.8}$$

with possible values $1, \ldots, t - s + 1$. With $d = 1, 2, \ldots$, the number $N_t(d)$ of nodes in $\mathbb{G}_t$ with degree d is then the rv given by

$$N_t(d) = \sum_{s=0}^{t} \mathbf{1}\left[D_t(s) = d\right], \tag{5.9}$$

and

$$\frac{N_t(d)}{t+1} = \frac{1}{t+1} \sum_{s=0}^{t} \mathbf{1}\left[D_t(s) = d\right]$$

is the fraction of nodes in $\mathbb{G}_t$ whose degree is d.

5.2   The convergence results

Before stating the main result of the chapter, we introduce the pmf $\boldsymbol{p}_\xi = (p_\xi(d), \ d = 1, 2, \ldots)$ on $\mathbb{N}_0$ defined by

$$p_\xi(d+1) = \mathbb{E}\left[\frac{\Lambda(\xi)^d}{d!} e^{-\Lambda(\xi)}\right], \quad d = 0, 1, \ldots \tag{5.10}$$

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where the rv $\Lambda(\xi)$ is given by

$$\Lambda(\xi) = \frac{\xi}{\mathbb{E}[\xi]} \log\left(\frac{1}{U}\right). \tag{5.11}$$

The pmf $\boldsymbol{p}_\xi$ can be interpreted as the pmf of the rv $1 + Z(\xi)$ where the rv $Z(\xi)$ is a conditionally Poisson rv with random parameter $\Lambda(\xi)$, namely

$$\mathbb{P}[Z(\xi) = d] = \mathbb{E}\left[\frac{\Lambda(\xi)^d}{d!} e^{-\Lambda(\xi)}\right], \quad d = 0, 1, \ldots$$

5.2.1 The main result

With this notation we have the following result.

Theorem 5.1. *Under the foregoing assumptions, we have the convergence*

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{N_t(d)}{t+1}\right] = p_\xi(d), \quad d = 1, 2, \ldots \tag{5.12}$$

A proof of Theorem 5.1 is given in Sections 5.5-5.9. The pmf $\boldsymbol{p}_\xi$ depends on ξ only through normalization to its mean, namely $\frac{\xi}{\mathbb{E}[\xi]}$, since

$$\Lambda(\xi) = \Lambda\left(\frac{\xi}{\mathbb{E}[\xi]}\right).$$

This implies

$$p_\xi(d) = p_{\frac{\xi}{\mathbb{E}[\xi]}}(d), \quad d = 1, 2, \ldots$$

as should be expected from the form (7.4) of the link creation probabilities.

For each $t = 0, 1, \ldots$, let ν_t denote a rv which is uniformly distributed over the edge set V_t and independent of $\mathcal{F}_t$, thus of $\mathbb{G}_t$. It is plain that the representation

$$\mathbb{E}\left[\frac{N_t(d)}{t+1}\right] = \mathbb{P}[D_t(\nu_t) = d], \quad d = 1, 2, \ldots \tag{5.13}$$

holds, and Theorem 5.1 can be given the following probabilistic form.

Theorem 5.2. *Under the foregoing assumptions, we have the convergence*

$$\lim_{t\to\infty} \mathbb{P}\left[D_t(\nu_t) = d\right] = p_\xi(d), \quad d = 1, 2, \ldots \tag{5.14}$$

More compactly,

$$D_t(\nu_t) \Longrightarrow_t D(\xi) \tag{5.15}$$

where $D(\xi)$ is an $\mathbb{N}_0$-valued rv distributed according to the pmf $\boldsymbol{p}_\xi$.

5.2.2   An alternate expression for the limiting pmf $\boldsymbol{p}_\xi$

The pmf $\boldsymbol{p}_\xi$ admits an alternate expression which will yield insights into its tail behavior.

Proposition 5.3. *It holds that*

$$p_\xi(d) = \mathbb{E}\left[\frac{\mathbb{E}\left[\xi\right]}{\mathbb{E}\left[\xi\right] + \xi} \cdot \left(\frac{\xi}{\mathbb{E}\left[\xi\right] + \xi}\right)^{d-1}\right], \quad d = 1, 2, \ldots \tag{5.16}$$

In other words, the pmf $\boldsymbol{p}_\xi$ can also be viewed as the pmf of a conditionally geometric rv on $\mathbb{N}_0$ with random parameter $R(\xi)$ given by

$$R(\xi) = \frac{\xi}{\mathbb{E}\left[\xi\right] + \xi} \tag{5.17}$$

as we note from (5.16)-(5.17) that

$$p_\xi(d) = \mathbb{E}\left[(1 - R(\xi)) R(\xi)^{d-1}\right], \quad d = 1, 2, \ldots \tag{5.18}$$

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Proof. In view of the expressions (5.10)-(5.11) and (5.16)-(5.17), there is no loss of generality in assuming $\mathbb{E}[\xi] = 1$ (as we do from now on in this proof) – Just replace ξ by $\frac{\xi}{\mathbb{E}[\xi]}$, in which case the quantities $\Lambda(\xi)$ and $R(\xi)$ become

$$\Lambda(\xi) = \xi \log\left(\frac{1}{U}\right) \quad \text{and} \quad R(\xi) = \frac{\xi}{1+\xi},$$

respectively. Thus, $e^{-\Lambda(\xi)} = U^{\xi}$ and for each $d = 0, 1, \ldots$, we use (5.10) to obtain

$$
\begin{aligned}
p_{\xi}(d+1) &= \mathbb{E}\left[\frac{\xi^d}{d!}\left(\log\left(\frac{1}{U}\right)\right)^d U^{\xi}\right] \\
&= (-1)^d \cdot \mathbb{E}\left[\frac{\xi^d}{d!} \cdot \mathbb{E}\left[U^{\xi}(\log U)^d|\xi\right]\right] \\
&= (-1)^d \cdot \mathbb{E}\left[\frac{\xi^d}{d!} \cdot \mathbb{E}\left[U^t(\log U)^d\right]_{t=\xi}\right] \quad (5.19)
\end{aligned}
$$

by the independence of the rvs ξ and U, where we note that

$$\mathbb{E}\left[U^t(\log U)^d\right] = \int_0^1 x^t(\log x)^d dx = (-1)^d \frac{d!}{(1+t)^{d+1}}, \quad t > 0.$$

This last fact follows by repeated integration by parts; details are left to the interested reader. Substituting into (5.19) we get

$$p_{\xi}(d+1) = (-1)^d \cdot \mathbb{E}\left[\frac{\xi^d}{d!} \cdot (-1)^d \frac{d!}{(1+\xi)^{d+1}}\right] = \mathbb{E}\left[\frac{\xi^d}{(1+\xi)^{d+1}}\right]$$

as desired. ∎

5.3   Tail behavior

The tail behavior of the pmf $\boldsymbol{p}_{\xi}$ depends on the distributional properties of the rv ξ, with a key role being played by the quantity $\xi^{\star}$ given by

$$\xi^{\star} = \inf\left(x \geqslant 0 : \; \mathbb{P}[\xi \leqslant x] = 1\right) \quad (5.20)$$

with the customary understanding that $\xi^\star = \infty$ if the defining set in (5.20) is empty. Under (5.2) we necessarily have $\xi^\star > 0$ (possibly infinite).

The next result provides some important information regarding the tail behavior of the pmf $\boldsymbol{p}_\xi$. Of particular interest in the discussion is the bounded mapping $g_\xi : [0, \infty] \to \mathbb{R}$ given by

$$
g_\xi(x) = \begin{cases} \frac{x}{\mathbb{E}[\xi]+x} & \text{if } x \in \mathbb{R}_+ \\ \\ 1 & \text{if } x = \infty, \end{cases}
$$

the value at $x = \infty$ being determined by continuity.

Lemma 5.4. *Under the foregoing assumptions, it is always the case that*

$$
\lim_{d \to \infty} p_\xi(d)^{\frac{1}{d}} = g_\xi(\xi^\star) \tag{5.21}
$$

with

$$
g_\xi(\xi^\star) = \begin{cases} \frac{\xi^\star}{\mathbb{E}[\xi]+\xi^\star} & \text{if } 0 < \xi^\star < \infty \\ \\ 1 & \text{if } \xi^\star = \infty. \end{cases}
$$

The proof of Lemma 5.4 is given in Section 5.10. When $\xi^\star$ is finite, then $g_\xi(\xi^\star) < 1$ so that

$$
\lim_{d \to \infty} \frac{1}{d} \log p_\xi(d) = \log\left(\frac{\xi^\star}{\mathbb{E}[\xi] + \xi^\star}\right) < 0. \tag{5.22}
$$

This indicates a *geometric* decay for $\boldsymbol{p}_\xi$ in the following sense: For ϵ in $(0,1)$ sufficiently small so that $(1 + \delta)g_\xi(\xi^\star) < 1$ where we have set $\epsilon = \log(1 + \delta)$, the

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convergence (5.22) implies

$$-\epsilon < \frac{1}{d} \log p_\xi(d) - \log g(\xi^*) < \epsilon \tag{5.23}$$

or

$$\left(\frac{g_\xi(\xi^\star)}{(1+\delta)}\right)^d < p_\xi(d) < ((1+\delta)g_\xi(\xi^\star))^d$$

for all $d = 1, 2, \ldots$ sufficiently large (and determined by δ). In fact, direct inspection

of (5.17) yields

$$
\begin{aligned}
p_\xi(d) &= \mathbb{E}\left[(1 - R(\xi)) R(\xi)^{d-1}\right] \\
&\leqslant \mathbb{E}\left[(1 - R(\xi)) R(\xi^\star)^{d-1}\right] \\
&= (1 - \mathbb{E}[R(\xi)]) \cdot g_\xi(\xi^\star)^{d-1} \\
&= \left(\frac{1 - \mathbb{E}[R(\xi)]}{g_\xi(\xi^\star)}\right) \cdot g_\xi(\xi^\star)^d \tag{5.24}
\end{aligned}
$$

for all $d = 1, 2, \ldots$.

5.4   Special cases when $\xi^\star = \infty$

When $\xi^\star$ is infinite, geometric decay is not possible anymore, and many types

of tail behavior are possible for the pmf $\boldsymbol{p}_\xi$ as we now illustrate with two special cases.

The probability distribution function of the rv ξ is assumed to admit a probability

density function $f_\xi : \mathbb{R}_+ \to \mathbb{R}_+$. For each $d = 0, 1, \ldots$, the expression (5.16) becomes

$$
\begin{aligned}
p(d+1) &= \mathbb{E}\left[\left(1 - \frac{1}{1+\xi}\right)^d \frac{1}{1+\xi}\right] \\
&= \int_0^\infty \left(1 - \frac{1}{1+t}\right)^d \frac{1}{1+t} \cdot f_\xi(t)dt \\
&= \int_0^1 \frac{(1-s)^d}{s} \cdot f_\xi\left(\frac{1-s}{s}\right) ds \quad [s = \tfrac{1}{1+t} \text{ and } t = \tfrac{1-s}{s}]. \tag{5.25}
\end{aligned}
$$

5.4.1 Pareto distribution

We say that the rv ξ has a power law if

$$\mathbb{P}\left[\xi \leqslant t\right] = 1 - \left(\frac{a}{a+\alpha}\right)^{\alpha}, \qquad t \geqslant 0 \tag{5.26}$$

for some $a > 0$ and $\alpha > 0$. Its probability density function is given by

$$f_\xi(t) = \alpha a^\alpha (a+t)^{-(\alpha+1)}, \qquad t \geqslant 0 \tag{5.27}$$

and the first moment is easily computed to be

$$\mathbb{E}\left[\xi\right] = \frac{a}{(\alpha-1)^+}.$$

The requirement that $\mathbb{E}\left[\xi\right]$ be finite is equivalent to $\alpha > 1$, in which case $\mathbb{E}\left[\xi\right] = a(\alpha-1)^{-1}$. Thus, $\mathbb{E}\left[\xi\right] = 1$ amounts to $\alpha > 1$ *and* $a = \alpha - 1$.

Fix $d = 0, 1, \ldots$. If we insert (5.27) into (5.25) we obtain by elementary calculations that

$$
\begin{aligned}
p_\xi(d+1) &= \alpha a^\alpha \int_0^1 (1-s)^d s^\alpha \left(1 + (a-1)s\right)^{-(\alpha+1)} ds \\
&= \alpha a^\alpha d^{-(\alpha+1)} \int_0^d \left(1 - \frac{y}{d}\right)^d y^\alpha \left(1 + (a-1)\frac{y}{d}\right)^{-(\alpha+1)} dy \quad [s = \tfrac{y}{d}] \tag{5.28}
\end{aligned}
$$

By the Bounded Convergence Theorem it is now plain that

$$\lim_{d\to\infty} \frac{p_\xi(d+1)}{\alpha a^\alpha d^{-(\alpha+1)}} = \int_0^\infty y^\alpha e^{-y} dy,$$

and the following result follows.

Lemma 5.5. *When ξ is distributed according to (5.26) with $\alpha > 1$ and $a = \alpha - 1$, then the asymptotic equivalence*

$$p_\xi(d+1) \sim C(\alpha)\alpha a^\alpha d^{-(\alpha+1)} \quad \text{with} \quad C(\alpha) = \int_0^\infty y^\alpha e^{-y} dy$$

holds.

Note that $C(\alpha)$ is finite for all $\alpha \geqslant 1$.

5.4.2 Exponential distribution

The rv ξ is an exponentially distributed rv with unit mean if

$$\mathbb{P}\left[\xi \leqslant t\right] = 1 - e^{-t}, \quad t \geqslant 0, \tag{5.29}$$

in which case its probability density function is given by

$$f_\xi(t) = e^{-t}, \quad t \geqslant 0.$$

For each $d = 0, 1, \ldots$, with (5.25) as starting point, elementary calculations now give

$$
\begin{aligned}
p_\xi(d + 1) &= \int_0^1 (1 - s)^d \frac{e^{-\frac{1-s}{s}}}{s} ds \\
&= e \int_0^1 (1 - s)^d \frac{e^{-\frac{1}{s}}}{s} ds.
\end{aligned}
\tag{5.30}
$$

Lemma 5.6. *When ξ is distributed according to (5.29), then the asymptotic equivalence*

$$p_\xi(d + 1) \sim \sqrt{\frac{\pi}{e^2}} \frac{e^{-2\sqrt{d}}}{\sqrt[4]{d}}$$

holds.

A discussion of Lemma 5.6 can be found in Section 5.11.

5.5 A proof of Theorem 5.1 – Preliminaries

Pick $d = 0, 1, \dots$. Theorem 5.1 is concerned with the existence and value of the limit

$$\lim_{t \to \infty} \frac{1}{t+1} \sum_{s=0}^{t} \mathbb{P}\left[D_t(s) = d + 1\right].$$

Fix $t = 1, 2, \dots$ and $s = 0, \dots, t$. As we recall the definition (5.8) of the degree rv $D_t(s)$, we start by asking how can the rv $D_t(s)$ achieve the value $d + 1$. To avoid trivial situations of limited interest we assume that $t - s + 1 > d + 1$ or equivalently, $t - s > d$. Note that

$$
\begin{aligned}
\mathbb{P}\left[D_t(s) = d + 1\right] &= \mathbb{P}\left[1 + \sum_{r=s+1}^{t} \mathbf{1}\left[S_r = s\right] = d + 1\right] \\
&= \mathbb{P}\left[\sum_{r=s+1}^{t} \mathbf{1}\left[S_r = s\right] = d\right]. \qquad (5.31)
\end{aligned}
$$

Thus, the event $D_t(s) = d + 1$ corresponds to the following situation: Amongst the $t - s$ nodes arriving at time $s + 1, \dots, t$, exactly d arrivals attach themselves to node s, while the remaining $t - s - d$ arrivals attach themselves to a node other than node s. This observation naturally leads to considering the set $\mathcal{P}_d[s + 1, t]$ of partitions of $\{s + 1, \dots, t\}$ into two sets of size d and $t - s - d$, respectively. Thus, $\mathcal{P}_d[s + 1, t]$ is given by

$$
\mathcal{P}_d[s + 1, t] = \left\{ (A, B) : \ A, B \subseteq \{s + 1, \dots, t\}, \ \begin{array}{c} |A| = d, |B| = t - s - d, \\ A \cap B = \varnothing \end{array} \right\}
$$

With this notation we get

$$\mathbb{P}\left[\sum_{r=s+1}^{t} \mathbf{1}\left[S_r = s\right] = d\right]$$

$$= \sum_{(A,B)\in\mathcal{P}_d[s+1,t]} \mathbb{P}\left[S_r = s,\ s \in A,\ S_r \neq s,\ s \in B\right]. \tag{5.32}$$

For a given pair (A, B) in $\mathcal{P}_d[s + 1, t]$, a standard preconditioning argument

yields

$$\mathbb{P}\left[S_r = s,\ s \in A,\ S_r \neq s,\ s \in B\right]$$

$$= \mathbb{E}\left[\mathbb{P}\left[S_r = s,\ s \in A,\ S_r \neq s,\ s \in B|\xi^t\right]\right]$$

$$= \mathbb{E}\left[\prod_{r\in A}\left(\frac{\xi_s}{\Xi_{r-1}}\right) \cdot \prod_{r\in B}\left(1 - \frac{\xi_s}{\Xi_{r-1}}\right)\right] \tag{5.33}$$

$$= \mathbb{E}\left[\prod_{r\in A}\left(\frac{\xi}{\xi + \Xi_{r-2}}\right) \cdot \prod_{r\in B}\left(1 - \frac{\xi}{\xi + \Xi_{r-2}}\right)\right] \tag{5.34}$$

with the following justifications: The relation (5.33) is a consequence of the fact that

the rvs $S_1, \ldots, S_t$ are conditionally mutually independent given ξ^t. The equality

(5.34) takes advantage of the fact that

$$(\xi_s, \Xi_{r-1},\ r = s + 1, \ldots, t) =_{st} (\xi, \xi + \Xi_{r-2},\ r = s + 1, \ldots, t)$$

under the enforced i.i.d. assumptions on the fitness variables.

Next, elementary calculations show that

$$\prod_{r\in A}\left(\frac{\xi}{\xi + \Xi_{r-2}}\right) \cdot \prod_{r\in B}\left(1 - \frac{\xi}{\xi + \Xi_{r-2}}\right)$$

$$= \prod_{r=s+1}^{t}\left(1 - \frac{\xi}{\xi + \Xi_{r-2}}\right) \cdot \prod_{r\in A}\left(\frac{\frac{\xi}{\xi+\Xi_{r-2}}}{1 - \frac{\xi}{\xi+\Xi_{r-2}}}\right)$$

$$= \prod_{r\in A}\left(\frac{\xi}{\Xi_{r-2}}\right) \cdot \prod_{r=s+1}^{t}\left(1 - \frac{\xi}{\xi + \Xi_{r-2}}\right)$$

$$= \xi^d \prod_{r=s+1}^{t}\left(1 - \frac{\xi}{\xi + \Xi_{r-2}}\right) \cdot \prod_{r\in A}\left(\frac{1}{\Xi_{r-2}}\right). \tag{5.35}$$

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since $|A| = d$.

Therefore, we can write

$$\mathbb{P}\left[\sum_{r=s+1}^{t} \mathbf{1}\left[S_r = s\right] = d\right]$$

$$= \sum_{(A,B)\in\mathcal{P}_d[s+1,t]} \mathbb{E}\left[\xi^d \prod_{r=s+1}^{t}\left(1 - \frac{\xi}{\xi + \Xi_{r-2}}\right) \cdot \left(\prod_{r\in A}\frac{1}{\Xi_{r-2}}\right)\right]$$

$$= \mathbb{E}\left[\xi^d \prod_{r=s+1}^{t}\left(1 - \frac{\xi}{\xi + \Xi_{r-2}}\right) \cdot \left(\sum_{(A,B)\in\mathcal{P}_d[s+1,t]}\left(\prod_{r\in A}\frac{1}{\Xi_{r-2}}\right)\right)\right]. \quad (5.36)$$

In the next section we present some useful preliminary results that will be used in the main proof.

5.6   Useful technical facts

Before giving a proof of Theorem 5.1 we present three useful technical facts to be used in the course of the discussion.

5.6.1   A consequence of the Strong Law of Large Numbers

We begin with an easy consequence of the Strong Law of Large Numbers.

Lemma 5.7. *Assume* $\mathbb{E}\left[\xi\right] = 1$. *For arbitrary* α *and* ε *in the unit interval* $(0,1)$ *we have*

$$\lim_{t\to\infty} \mathbb{P}\left[B_t(\alpha,\varepsilon)\right] = 1 \quad (5.37)$$

where

$$B_t(\alpha,\varepsilon) = \cap_{r=\lceil\alpha t\rceil}^{t}\left[\left|\frac{\Xi_r}{r+1} - 1\right| \leqslant \varepsilon\right], \quad t = 0, 1, \ldots \quad (5.38)$$

Proof. Consider the event

$$C = \left[\lim_{t\to\infty} \frac{\Xi_t}{t+1} = 1\right],$$

and for every ε in $(0,1)$ note the inclusion

$$C \subset \cup_{t=0}^{\infty} \cap_{r=t}^{\infty} \left[\left|\frac{\Xi_r}{r+1} - 1\right| \le \varepsilon\right].$$

Therefore,

$$\mathbb{P}\left[C\right] \le \mathbb{P}\left[\cup_{t=0}^{\infty} \cap_{r=t}^{\infty}\left[\left|\frac{\Xi_r}{r+1} - 1\right| \le \varepsilon\right]\right] = \lim_{t\to\infty} \mathbb{P}\left[\cap_{r=t}^{\infty}\left[\left|\frac{\Xi_r}{r+1} - 1\right| \le \varepsilon\right]\right]$$

by the usual monotonicity argument. By the Strong Law of Large Numbers we have

$\mathbb{P}\left[C\right] = 1$, whence

$$\lim_{t\to\infty} \mathbb{P}\left[\cap_{r=t}^{\infty}\left[\left|\frac{\Xi_r}{r+1} - 1\right| \le \varepsilon\right]\right] = 1.$$

Applying this fact along the subsequence $t \to \lceil\alpha t\rceil$, we get

$$\lim_{t\to\infty} \mathbb{P}\left[\cap_{r=\lceil\alpha t\rceil}^{\infty}\left[\left|\frac{\Xi_r}{r+1} - 1\right| \le \varepsilon\right]\right] = 1$$

and the desired result follows by virtue of the inclusion

$$\cap_{r=\lceil\alpha t\rceil}^{\infty}\left[\left|\frac{\Xi_r}{r+1} - 1\right| \le \varepsilon\right] \subseteq B_t(\alpha, \varepsilon), \quad t = 0, 1, \ldots$$

■

5.6.2 Uniform selection

Consider α in the unit interval $(0,1)$. For each $t = 1, 2, \ldots$, let $\nu_{\alpha,t}$ denote a rv which is uniformly distributed over the set $\{\lceil\alpha t\rceil, \lceil\alpha t\rceil + 1, \ldots, t\}$. The following fact is elementary and given here for easy reference.

Lemma 5.8. *For each α in $(0,1)$, it holds*

$$\frac{\nu_{\alpha,t}}{t} \Longrightarrow_t U_\alpha \tag{5.39}$$

where the rv U_α is uniformly distributed over the interval $(\alpha, 1)$.

Proof. Fix α in $(0,1)$. With $b > 0$, elementary facts concerning geometric series yield

$$\mathbb{E}\left[e^{-b\frac{\nu_{\alpha,t}}{t}}\right] = \frac{1}{t - \lceil \alpha t \rceil + 1} \sum_{r=\lceil \alpha t \rceil}^{t} e^{-\frac{br}{t}} = \frac{1}{t - \lceil \alpha t \rceil + 1} \cdot \frac{e^{-\frac{b\lceil \alpha t \rceil}{t}} - e^{-\frac{b(t+1)}{t}}}{1 - e^{-\frac{b}{t}}}.$$

Therefore, since

$$\lim_{t \to \infty} \left(1 - e^{-\frac{b}{t}}\right) \cdot \frac{t}{b} = 1 \quad \text{and} \quad \lim_{t \to \infty} \frac{t}{t - \lceil \alpha t \rceil + 1} = (1 - \alpha)^{-1},$$

it is easy to check that

$$\lim_{t \to \infty} \mathbb{E}\left[e^{-b\frac{\nu_{\alpha,t}}{t}}\right] = \frac{e^{-b\alpha} - e^{-b}}{b(1 - \alpha)}. \tag{5.40}$$

The desired conclusion (5.39) follows by standard arguments upon noting that the right handside of (5.40) is the Laplace transform of U_α. ∎

5.6.3 Limits of certain expectations

Pick α in the unit interval $(0,1)$, and take $\lambda, \gamma > 0$. To simplify the presentation here and elsewhere, we write

$$J(d; \alpha, \lambda, \gamma) = \frac{\xi^d}{d!} \cdot U_\alpha^{\lambda \xi} \left(\log\left(\frac{1}{U_\alpha}\right)\right)^d \mathbf{1}\left[\xi \leqslant \gamma\right], \quad d = 0, 1, \ldots$$

with the rv U_α being uniformly distributed on the open interval $(\alpha, 1)$ and independent of ξ. The expected values

$$T(d; \alpha, \lambda, \gamma) = \mathbb{E}\left[J(d; \alpha, \lambda, \gamma)\right], \quad d = 0, 1, \ldots \tag{5.41}$$

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will play a crucial role in the proof of Theorem 5.1.

Fix $d = 0, 1, \ldots$. Note that $J(d; \alpha, \lambda, \gamma) \geqslant 0$ so that the expected value at (5.41) is always well defined, possibly infinite. However, the bounds $\alpha < U_\alpha < 1$ imply

$$0 \leqslant U_\alpha^{\lambda \xi} \left(\log \left(\frac{1}{U_\alpha} \right) \right)^d \leqslant \left(\log \alpha^{-1} \right)^d . \tag{5.42}$$

while it is plain that

$$0 \leqslant \frac{\xi^d}{d!} \mathbf{1} \left[\xi \leqslant \gamma \right] \leqslant \frac{\gamma^d}{d!} \mathbf{1} \left[\xi \leqslant \gamma \right] . \tag{5.43}$$

Combining these bounds we conclude that

$$0 \leqslant J(d; \alpha, \lambda, \gamma) \leqslant \frac{\gamma^d}{d!} \left(\log \alpha^{-1} \right)^d , \tag{5.44}$$

and $T(d; \alpha, \lambda, \gamma)$ is in fact finite with

$$0 \leqslant T(d; \alpha, \lambda, \gamma) \leqslant \frac{\gamma^d}{d!} \left(\log \alpha^{-1} \right)^d . \tag{5.45}$$

Lemma 5.9. *Fix $d = 0, 1, \ldots$. With α in the unit interval $(0, 1)$, and $\lambda, \gamma > 0$, it holds that*

$$\lim_{\lambda \to 1} T(d; \alpha, \lambda, \gamma) = T(d; \alpha, 1, \gamma), \tag{5.46}$$

$$\lim_{\gamma \to \infty} T(d; \alpha, 1, \gamma) = \mathbb{E} \left[\frac{\xi^d}{d!} \cdot U_\alpha^\xi \left(\log \left(\frac{1}{U_\alpha} \right) \right)^d \right] \tag{5.47}$$

and

$$\lim_{\alpha \downarrow 0} \mathbb{E} \left[\frac{\xi^d}{d!} \cdot U_\alpha^\xi \left(\log \left(\frac{1}{U_\alpha} \right) \right)^d \right] = p_\xi(d + 1). \tag{5.48}$$

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Proof. Fix $d = 0, 1, \ldots$. The deterministic bound (5.44) being uniform in λ, the conclusion (5.46) is immediate by the Bounded Convergence Theorem. For each α in $(0, 1)$ the validity of (5.47) follows by the Monotone Convergence Theorem because $T(d; \alpha, \lambda, \gamma)$ is non-negative for each $\gamma \geqslant 0$ and is non-decreasing as $\gamma \to \infty$.

We finally turn to (5.48): For each α in $(0, 1)$ note that

$$\frac{\xi^d}{d!} \cdot U_\alpha^\xi \left(\log \left(\frac{1}{U_\alpha} \right) \right)^d = \frac{1}{d!} \left(\xi \log \left(\frac{1}{U_\alpha} \right) \right)^d e^{-\xi \log \left(\frac{1}{U_\alpha} \right)} \leqslant 1 \qquad (5.49)$$

since this term corresponds to a Poisson pmf with parameter $\xi \log \left(\frac{1}{U_\alpha} \right)$ evaluated at d. Obviously, we have $U_\alpha \Longrightarrow_\alpha U$ (when α is driven to zero), and the Continuous Mapping Theorem yields

$$\frac{\xi^d}{d!} \cdot U_\alpha^\xi \left(\log \left(\frac{1}{U_\alpha} \right) \right)^d \Longrightarrow_\alpha \frac{\xi^d}{d!} \cdot U^\xi \left(\log \left(\frac{1}{U} \right) \right)^d.$$

The convergence (5.48) is now a consequence of the Bounded Convergence Theorem by virtue of the bound (5.49) (which is uniform in α). ∎

In particular, it follows that

$$\lim_{\alpha \downarrow 0} \left(\lim_{\gamma \to \infty} \left(\lim_{\lambda \to 1} T(d; \alpha, \lambda, \gamma) \right) \right)$$
$$= \lim_{\alpha \downarrow 0} \left(\lim_{\gamma \to \infty} T(d; \alpha, 1, \gamma) \right)$$
$$= \lim_{\alpha \downarrow 0} \left(\mathbb{E} \left[\frac{\xi^d}{d!} \cdot U_\alpha^\xi \left(\log \left(\frac{1}{U_\alpha} \right) \right)^d \right] \right)$$
$$= p_\xi(d + 1), \quad d = 0, 1, \ldots \qquad (5.50)$$

5.7   A proof of Theorem 5.1

We shall establish Theorem 5.1 in the equivalent form given in Theorem 5.2. To do so, fix $d = 0, 1, \ldots$. Under the assumptions of Theorem 5.1 we shall now show

that the convergence statements

$$\limsup_{t \to \infty} \mathbb{P}\left[D_t(\nu_t) = d + 1\right] \leqslant p_\xi(d + 1) \tag{5.51}$$

and

$$p_\xi(d + 1) \leqslant \liminf_{t \to \infty} \mathbb{P}\left[D_t(\nu_t) = d + 1\right] \tag{5.52}$$

hold where for each $t = 0, 1, \ldots$, the rv ν_t is uniformly distributed over the edge set

V_t and independent of $\mathcal{F}_t$, thus of $\mathbb{G}_t$.

Pick α and ε in the unit interval $(0, 1)$, and $\gamma > 0$. With $t = 1, 2, \ldots$ sufficiently

large so that $t - \lceil \alpha t \rceil \geqslant d$, consider the event

$$B_t(\alpha, \varepsilon; \gamma) = B_t(\alpha, \varepsilon) \cap [\xi \leqslant \gamma].$$

with $B_t(\alpha, \varepsilon)$ defined at (5.38).

Keeping (5.13) in mind, consider the decomposition

$$
\begin{aligned}
\mathbb{E}\left[N_t(d + 1)\right] &= \mathbb{E}\left[\sum_{s=0}^{t} \mathbf{1}\left[D_t(s) = d + 1\right]\right] \\
&= \sum_{s=0}^{\lceil \alpha t \rceil - 1} \mathbb{P}\left[D_t(s) = d + 1\right] + \sum_{s=\lceil \alpha t \rceil}^{t} \mathbb{P}\left[D_t(s) = d + 1\right] \\
&= \sum_{s=0}^{\lceil \alpha t \rceil - 1} \mathbb{P}\left[D_t(s) = d + 1\right] + T_t(d; \alpha, \varepsilon, \gamma) \\
&\quad + \sum_{s=\lceil \alpha t \rceil}^{t} \mathbb{P}\left[[D_t(s) = d + 1] \cap B_t(\alpha, \varepsilon; \gamma)^c\right]
\end{aligned}
\tag{5.53}
$$

where we have set

$$T_t(d; \alpha, \varepsilon, \gamma) = \sum_{s=\lceil \alpha t \rceil}^{t} \mathbb{P}\left[[D_t(s) = d + 1] \cap B_t(\alpha, \varepsilon; \gamma)\right] \tag{5.54}$$

for notational convenience

5.7.1 Establishing the upper bound (5.52)

The first and third terms in (5.53) are easily upper bounded. Specifically, we get

$$\sum_{s=0}^{\lceil \alpha t \rceil - 1} \mathbb{P}\left[D_t(s) = d + 1\right] \leqslant \lceil \alpha t \rceil \tag{5.55}$$

and

$$
\begin{aligned}
\sum_{s=\lceil \alpha t \rceil}^{t} \mathbb{P}\left[[D_t(s) = d + 1] \cap B_t(\alpha, \varepsilon; \gamma)^c\right] &\leqslant (t - \lceil \alpha t \rceil + 1)\, \mathbb{P}\left[B_t(\alpha, \varepsilon; \gamma)^c\right] \\
&\leqslant (t - \lceil \alpha t \rceil + 1)\left(\mathbb{P}\left[B_t(\alpha, \varepsilon)^c\right] + \mathbb{P}\left[\gamma < \xi\right]\right) \\
&\leqslant (t + 1)\left(\mathbb{P}\left[B_t(\alpha, \varepsilon)^c\right] + \mathbb{P}\left[\gamma < \xi\right]\right). \tag{5.56}
\end{aligned}
$$

Therefore,

$$\limsup_{t \to \infty} \left(\frac{1}{t + 1} \sum_{s=0}^{\lceil \alpha t \rceil - 1} \mathbb{P}\left[D_t(s) = d + 1\right]\right) \leqslant \alpha$$

and

$$
\begin{aligned}
\limsup_{t \to \infty} &\left(\frac{1}{t + 1} \sum_{s=\lceil \alpha t \rceil}^{t} \mathbb{P}\left[[D_t(s) = d + 1] \cap B_t(\alpha, \varepsilon; \gamma)^c\right]\right) \\
&\leqslant \mathbb{P}\left[\gamma < \xi\right] + \limsup_{t \to \infty} \mathbb{P}\left[B_t(\alpha, \varepsilon)^c\right] \\
&= \mathbb{P}\left[\gamma < \xi\right] \tag{5.57}
\end{aligned}
$$

as we invoke Lemma 5.7. We readily conclude from (5.13) that

$$\limsup_{t \to \infty} \mathbb{P}\left[D_t(\nu_t) = d + 1\right] \leqslant \alpha + \limsup_{t \to \infty}\left(\frac{T_t(d; \alpha, \varepsilon, \gamma)}{t + 1}\right) + \mathbb{P}\left[\gamma < \xi\right]. \tag{5.58}$$

Most of the technical work that remains consists in identifying the limiting term in this last inequality; a proof can be found in Section 5.8.

Proposition 5.10. *Under the assumptions of Theorem 5.1, we have*

$$\limsup_{t\to\infty} \left(\frac{T_t(d;\alpha,\varepsilon,\gamma)}{t+1-\lceil \alpha t \rceil} \right) \leqslant (1-\varepsilon)^{-d} \cdot T\left(d;\alpha,(1+\varepsilon)^{-1},\gamma\right) \qquad (5.59)$$

Collecting (5.58) and (5.59) we find that

$$\limsup_{t\to\infty} \mathbb{P}\left[D_t(\nu_t) = d+1\right]$$

$$\leqslant \alpha + (1-\alpha)(1-\varepsilon)^{-d} \cdot T\left(d;\alpha,(1+\varepsilon)^{-1},\gamma\right) + \mathbb{P}\left[\gamma < \xi\right] \qquad (5.60)$$

The left handside does not depend on either of the parameters α, ε or γ. Therefore, in (5.60) let ε go to zero, γ go to infinity and α go to zero in that order, and Lemma 5.9 leads to (5.51). ∎

5.7.2 Establishing the lower bound (5.51)

This time, neglecting the first and last terms in (5.53) (which are non-negative), we get

$$\liminf_{t\to\infty} \left(\frac{T_t(d;\alpha,\varepsilon,\gamma)}{t+1} \right) \leqslant \liminf_{t\to\infty} \mathbb{P}\left[D_t(\nu_t) = d+1\right] \qquad (5.61)$$

by arguments similar to those used earlier for deriving the upper bound. This time we need to show the following analog of Proposition 5.10; a proof is available in Section 5.9.

Proposition 5.11. *Under the assumptions of Theorem 5.1, we have*

$$\liminf_{t\to\infty} \left(\frac{T_t(d;\alpha,\varepsilon,\gamma)}{t+1-\lceil \alpha t \rceil} \right) \geqslant (1+\varepsilon)^{-d} \cdot T\left(d;\alpha,(1-\varepsilon)^{-1},\gamma\right) \qquad (5.62)$$

It now follows from (5.61) and (5.62) that

$$(1 - \alpha)(1 + \varepsilon)^{-d} \cdot T\left(d; \alpha, (1 - \varepsilon)^{-1}, \gamma\right) \leqslant \liminf_{t \to \infty} \mathbb{P}\left[D_t(\nu_t) = d + 1\right] \qquad (5.63)$$

Here as well the left handside does not depend on either of the parameters α, ε or γ. Therefore, in (5.63) let ε go to zero, γ go to infinity and α go to zero in that order. The validity of (5.52) is now a straightforward consequence of Lemma 5.9. ∎

5.8   A proof of Proposition 5.10

Fix $d = 0, 1, \ldots$. Fix α and ϵ in the unit interval $(0, 1)$ and $\gamma > 0$. For ease of exposition we recollect some notation: For each $d = 0, 1, \ldots$, $t = 1, 2, \ldots$, α and ϵ in $(0, 1)$ and $\gamma > 0$, recall that we had set

$$T_t(d; \alpha, \varepsilon, \gamma) = \sum_{s=\lceil \alpha t \rceil}^{t} \mathbb{P}\left[[D_t(s) = d + 1] \cap B_t(\alpha, \varepsilon; \gamma)\right]$$

where the event

$$B_t(\alpha, \epsilon; \gamma) = B_t(\alpha, \epsilon) \cap [\xi \leqslant \gamma]$$

with

$$B_t(\alpha, \varepsilon) = \cap_{r=\lceil \alpha t \rceil}^{t} \left[\left|\frac{\Xi_r}{r + 1} - 1\right| \leqslant \varepsilon\right].$$

Fix $t = 1, 2, \ldots$ and $s = \lceil \alpha t \rceil, \ldots, t$. From (5.36), we have

$$\mathbb{P}\left[D_t(s) = d + 1\right] = \mathbb{E}\left[\xi^d \prod_{r=s+1}^{t} \left(1 - \frac{\xi}{\xi + \Xi_{r-2}}\right) \cdot \left(\sum_{(A,B) \in \mathcal{P}_d[s+1,t]} \left(\prod_{r \in A} \frac{1}{\Xi_{r-2}}\right)\right)\right].$$

$$(5.64)$$

Under the event $B_t(\alpha, \epsilon; \gamma)$, we have the following upper bound

$$\prod_{r=s+1}^{t} \left(1 - \frac{\xi}{\xi + \Xi_{r-2}} \right) \leqslant \prod_{r=s+1}^{t} \left(1 - \frac{\xi}{\gamma + (r-2)(1+\epsilon)} \right)$$

$$\leqslant \prod_{r=s+1}^{t} e^{-\frac{\xi}{\gamma+(r-2)(1+\epsilon)}}$$

$$= e^{-\frac{\xi}{1+\epsilon} \sum_{r=s}^{t-1} \frac{1}{\frac{\gamma}{1+\epsilon} + (r-1)}}. \tag{5.65}$$

By observing that

$$\sum_{r=s}^{t-1} \frac{1}{\frac{\gamma}{1+\epsilon} + (r-1)} \geqslant \int_{s}^{t} \frac{1}{\frac{\gamma}{1+\epsilon} + (p-1)} dp = \log \left(\frac{t - 1 + \frac{\gamma}{1+\epsilon}}{s - 1 + \frac{\gamma}{1+\epsilon}} \right)$$

we refine the bound (5.65) as follows

$$\prod_{r=s+1}^{t} \left(1 - \frac{\xi}{\xi + \Xi_{r-2}} \right) \leqslant e^{-\frac{\xi}{(1+\epsilon)} \log \left(\frac{t-1+\frac{\gamma}{1+\epsilon}}{s-1+\frac{\gamma}{1+\epsilon}} \right)}$$

$$= \left(\frac{s - 1 + \frac{\gamma}{1+\epsilon}}{t - 1 + \frac{\gamma}{1+\epsilon}} \right)^{\frac{\xi}{(1+\epsilon)}}. \tag{5.66}$$

Upper bounding the final term in (5.64) on the set $B_t(\alpha, \epsilon; \gamma)$, we get

$$\sum_{(A,B)\in\mathcal{P}_d[s+1,t]} \left(\prod_{r\in A} \frac{1}{\Xi_{r-2}} \right) \leqslant \sum_{(A,B)\in\mathcal{P}_d[s+1,t]} \left(\prod_{r\in A} \frac{1}{(1-\epsilon)(r-2)} \right)$$

$$= \frac{1}{(1-\epsilon)^d} \sum_{(A,B)\in\mathcal{P}_d[s,t-1]} \left(\prod_{r\in A} \frac{1}{r-1} \right)$$

$$\leqslant \frac{1}{(1-\epsilon)^d} \cdot \frac{1}{d!} \left(\sum_{r=s}^{t-1} \frac{1}{r-1} \right)^{d}$$

$$\leqslant \frac{1}{(1-\epsilon)^d} \cdot \frac{1}{d!} \left(\int_{s}^{t} \frac{1}{p-1} dp \right)^{d}$$

$$= \frac{1}{(1-\epsilon)^d} \cdot \frac{1}{d!} \left(\log \left(\frac{t-1}{s-1} \right) \right)^{d}. \tag{5.67}$$

Substituting the upper bounds (5.66) and (5.67) in (5.64), we obtain

$$\mathbb{P}\left[[D_t(s) = d+1] \cap B_t(\alpha, \epsilon; \gamma)\right]$$

$$\leqslant \frac{1}{(1-\epsilon)^d} \mathbb{E}\left[\frac{\xi^d}{d!}\left(\frac{s-1+\frac{\gamma}{1+\epsilon}}{t-1+\frac{\gamma}{1+\epsilon}}\right)^{\frac{\xi}{(1+\epsilon)}}\left(\log\left(\frac{t-1}{s-1}\right)\right)^d \mathbf{1}\left[\xi < \gamma\right]\right]. \qquad (5.68)$$

For $c_0 > 1$, $t > \frac{c_0}{\alpha}$ and $s = \lceil \alpha t \rceil, \ldots, t$, we have the upper bound

$$\frac{\xi^d}{d!}\left(\frac{s-1+\frac{\gamma}{1+\epsilon}}{t-1+\frac{\gamma}{1+\epsilon}}\right)^{\frac{\xi}{(1+\epsilon)}}\left(\log\left(\frac{t-1}{s-1}\right)\right)^d \mathbf{1}\left[\xi < \gamma\right] \leqslant \frac{\gamma^d}{d!}\left(\log\left(\frac{t-1}{\alpha t -1}\right)\right)^d$$

$$= \frac{\gamma^d}{d!}\left(\log\left(\frac{1-\frac{1}{t}}{\alpha - \frac{1}{t}}\right)\right)^d$$

$$\leqslant \frac{\gamma^d}{d!}\left(\log\left(\frac{1-\frac{\alpha}{c_0}}{\alpha - \frac{\alpha}{c_0}}\right)\right)^d. \qquad (5.69)$$

Substituting (5.68) in the expression for $T_t(d; \alpha, \epsilon, \gamma)$, we obtain

$$T_t(d; \alpha, \epsilon, \gamma) \leqslant \frac{1}{(1-\epsilon)^d}\mathbb{E}\left[\frac{\xi^d}{d!}\left(\frac{\nu_{\alpha,t}-1+\frac{\gamma}{1+\epsilon}}{t-1+\frac{\gamma}{1+\epsilon}}\right)^{\frac{\xi}{(1+\epsilon)}}\left(\log\left(\frac{t-1}{\nu_{\alpha,t}-1}\right)\right)^d \mathbf{1}\left[\xi < \gamma\right]\right].$$

$$(5.70)$$

Allowing t to go to infinity in (5.70), we get the desired result using the Bounded

Convergence theorem (by virtue of the upper bound (5.69)) and Lemma 5.8.

∎

5.9   A proof of Proposition 5.11

Fix $d = 0, 1, \ldots$. Fix α and ϵ in the unit interval $(0, 1)$ and $\gamma > 0$. For each

$t = 1, 2, \ldots$ and $s = \lceil \alpha t \rceil, \ldots, t$, we have

$$\mathbb{P}\left[D_t(s) = d+1\right] = \mathbb{E}\left[\xi^d \prod_{r=s+1}^t \left(1 - \frac{\xi}{\xi + \Xi_{r-2}}\right) \cdot \left(\sum_{(A,B)\in\mathcal{P}_d[s+1,t]}\left(\prod_{r\in A}\frac{1}{\Xi_{r-2}}\right)\right)\right].$$

$$(5.71)$$

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Under the event $B_t(\alpha, \epsilon; \gamma)$, we have the lower bound

$$\prod_{r=s+1}^{t}\left(1 - \frac{\xi}{\xi + \Xi_{r-2}}\right) \geqslant \prod_{r=s}^{t-1}\left(1 - \frac{\xi}{\xi + (1-\epsilon)(r-1)}\right)$$

$$= \prod_{r=s}^{t-1} e^{-\frac{\xi}{\xi+(1-\epsilon)(r-1)} - \Psi\left(\frac{\xi}{\xi+(1-\epsilon)(r-1)}\right)}$$

$$= e^{-\sum_{r=s}^{t-1}\frac{\xi}{\xi+(1-\epsilon)(r-1)} - \sum_{r=s}^{t-1}\Psi\left(\frac{\xi}{\xi+(1-\epsilon)(r-1)}\right)}$$

$$\geqslant e^{-\frac{\xi}{1-\epsilon}\sum_{r=s}^{t-1}\frac{1}{r-1} - \sum_{r=s}^{t-1}\Psi\left(\frac{\xi}{\xi+(1-\epsilon)(r-1)}\right)} \tag{5.72}$$

where we have set

$$\Psi(x) = \int_0^x \frac{t}{1-t}dt, \ 0 \leqslant x < 1.$$

The first term in (5.72) can be bounded as follows

$$\sum_{r=s}^{t-1}\frac{1}{r-1} \leqslant \int_s^{t-1}\frac{1}{r-2}$$

$$= \log\left(\frac{t-2}{s-2}\right). \tag{5.73}$$

Before bounding the second term in (5.72) we note that for $0 \leqslant x \leqslant y < 1$,

$$\frac{1}{2} \leqslant \frac{\Psi(x)}{x^2} \leqslant \frac{\Psi(y)}{y^2} < \infty. \tag{5.74}$$

In view of the above fact for sufficiently large t we would like to uniformly bound

the sequence of rvs

$$\left\{\frac{\Psi\left(\frac{\xi}{\xi+(1-\epsilon)(r-1)}\right)}{\left(\frac{\xi}{\xi+(1-\epsilon)(r-1)}\right)^2}, \ r = \lceil \alpha t \rceil, \dots, t\right\}.$$

Since $\frac{\xi}{\xi+(1-\epsilon)(r-1)}$ is monotonically increasing in ξ and decreasing in r, on the set

$[\xi < \gamma]$ it is sufficient to ensure $\frac{\gamma}{\gamma+(1-\epsilon)(\alpha t-1)}$ is strictly less than one for all t being

considered. Under the condition $t > \frac{\gamma}{(1-\epsilon)\alpha} + 1$,

$$\frac{\gamma}{\gamma + (1-\epsilon)(\alpha t - 1)} < \frac{1}{2}$$

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which implies the uniform bound

$$\frac{\Psi\left(\frac{\xi}{\xi+(1-\epsilon)(r-1)}\right)}{\left(\frac{\xi}{\xi+(1-\epsilon)(r-1)}\right)^2} < \frac{\Psi\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^2}, \qquad \begin{array}{l} r = \lceil \alpha t \rceil, \dots, t \\[1.5ex] t > \frac{\gamma}{(1-\epsilon)\alpha} \end{array} \tag{5.75}$$

by virtue of the fact (5.74). Using the bound (5.75), on the set $[\xi < \gamma]$ we have the

upper bound

$$\sum_{r=s}^{t-1} \Psi\left(\frac{\xi}{\xi+(1-\epsilon)(r-1)}\right) = \sum_{r=s}^{t-1} \frac{\Psi\left(\frac{\xi}{\xi+(1-\epsilon)r}\right)}{\left(\frac{\xi}{\xi+(1-\epsilon)(r-1)}\right)^2} \left(\frac{\xi}{\xi+(1-\epsilon)(r-1)}\right)^2$$

$$\leqslant \frac{\Psi\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^2} \sum_{r=s}^{t-1} \left(\frac{\xi}{\xi+(1-\epsilon)(r-1)}\right)^2$$

$$\leqslant 4\Psi\left(\frac{1}{2}\right) \frac{\gamma^2}{(1-\epsilon)^2} \sum_{r=s}^{t-1} \frac{1}{(r-1)^2}$$

$$\leqslant 4\Psi\left(\frac{1}{2}\right) \frac{\gamma^2}{(1-\epsilon)^2} \int_s^{t-1} \frac{1}{p^2} dp$$

$$\leqslant 4\Psi\left(\frac{1}{2}\right) \frac{\gamma^2}{(1-\epsilon)^2} \frac{1-\alpha}{\alpha t}, \tag{5.76}$$

where the last step follows by noting that $s \geqslant \alpha t$. The bounds (5.73) and (5.76)

when substituted in (5.72) yields the lower bound

$$\prod_{r=s+1}^{t} \left(1 - \frac{\xi}{\xi+\Xi_{r-2}}\right) \geqslant e^{-4\Psi\left(\frac{1}{2}\right)\frac{\gamma^2}{(1-\epsilon)^2}\frac{1-\alpha}{\alpha t}} \mathbb{E}\left[\left(\frac{s-2}{t-2}\right)^{\frac{\xi}{(1-\epsilon)}}\right] \tag{5.77}$$

on the set $B_t(\alpha,\epsilon;\gamma)$. The final term inside the expectation in (5.71) can be lower

bounded as

$$\sum_{(A,B)\in\mathcal{P}_d[s+1,t]} \left(\prod_{r\in A} \frac{1}{\Xi_{r-2}}\right) \geqslant \sum_{(A,B)\in\mathcal{P}_d[s+1,t]} \left(\prod_{r\in A} \frac{1}{(1+\epsilon)(r-2)}\right)$$

$$= \frac{1}{(1+\epsilon)^d} \left(\sum_{(A,B)\in\mathcal{P}_d[s,t-1]} \left(\prod_{r\in A} \frac{1}{r-1}\right)\right) \tag{5.78}$$

on the set $B_t(\alpha,\epsilon;\gamma)$. Before we proceed further, we note that

$$\left(\sum_{r=s}^{t-1} \frac{1}{r-1}\right)^d = d! \left(\sum_{(A,B)\in\mathcal{P}_d[s,t-1]} \left(\prod_{r\in A} \frac{1}{r-1}\right)\right) + R(s,t) \tag{5.79}$$

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where $R(s,t)$ is an error term that we would like to bound. Observe that the number of terms in $R(s,t)$ is $\left((t-1-s)^d - \binom{t-1-s}{d}d!\right)$ where each term is at most $\frac{1}{s^d}$. Therefore for $s = \lceil \alpha t \rceil, \ldots, t$, we have the upper bound

$$
\begin{aligned}
R(s,t) &\leqslant \left[(t-1-s)^d - d!\binom{t-1-s}{d} \right] \frac{1}{(\alpha t)^d} \\
&\leqslant \frac{1}{\alpha^d} \left(\frac{(t-1-s)^d - d!\binom{t-1-s}{d}}{t^d} \right).
\end{aligned}
\tag{5.80}
$$

Also, we have the lower bound

$$
\left(\sum_{r=s}^{t-1} \frac{1}{r-1} \right)^d \geqslant \left(\int_s^t \frac{1}{p-1} dp \right)^d
$$

$$
\geqslant \left(\log\left(\frac{t-1}{s-1} \right) \right)^d
\tag{5.81}
$$

Substituting the bounds (5.80) and (5.81) in (5.78), we obtain

$$
\sum_{(A,B)\in\mathcal{P}_d[s+1,t]} \left(\prod_{r\in A} \frac{1}{\Xi_{r-2}} \right)
$$

$$
\geqslant \frac{1}{(1+\epsilon)^d} \cdot \frac{1}{d!} \left[\left(\log\left(\frac{t-1}{s-1} \right) \right)^d - \frac{1}{\alpha^d} \left(\frac{(t-1-s)^d - d!\binom{t-1-s}{d}}{t^d} \right) \right]
\tag{5.82}
$$

using the expression (5.79) on the set $B_t(\alpha, \epsilon; \gamma)$.

The lower bounds (5.77) and (5.82) when substituted in (5.71) yields

$$
\mathbb{P}\left[[D_t(s) = d+1] \cap B_t(\alpha, \epsilon; \gamma) \right] \geqslant \frac{1}{(1+\epsilon)^d} e^{-4\Psi\left(\frac{1}{2}\right) \frac{\gamma^2}{(1-\epsilon)^2} \frac{1-\alpha}{\alpha t}}
$$

$$
\times \mathbb{E}\left[\frac{\xi^d}{d!} \left(\frac{s-2}{t-2} \right)^{\frac{\xi}{(1-\epsilon)}} \left[\left(\log\left(\frac{t-1}{s-1} \right) \right)^d - \frac{1}{\alpha^d} \left(\frac{(t-1-s)^d - d!\binom{t-1-s}{d}}{t^d} \right) \right] \mathbf{1}\left[\xi < \gamma \right] \right]
\tag{5.83}
$$

Substituting the bound (5.83) in the expression of $T_t(d; \alpha, \epsilon, \gamma)$ and using Lemma 5.8 and the following limit

$$
\left(\frac{(t-1-\nu_{\alpha,t})^d - d!\binom{t-1-\nu_{\alpha,t}}{d}}{t^d} \right) \longrightarrow_t 0, \ d = 0, 1, \ldots.
$$

we get the desired result by virtue of the Bounded Convergence theorem. ■

5.10   A proof of Lemma 5.4

Let μ denote the probability measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ induced by the rv ξ, i.e.,

$$\mu(B) = \mathbb{P}\left[\xi \in B\right], \quad B \in \mathcal{B}(\mathbb{R}_+).$$

With μ we associate another measure ν on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ which is absolutely contin-

uous with respect to μ, its Radon-Nikodym derivative being given by

$$\frac{d\nu}{d\mu}(x) = \frac{\mathbb{E}\left[\xi\right]}{\mathbb{E}\left[\xi\right] + x}, \quad x \geqslant 0.$$

The measure ν is finite with

$$0 < \nu(\mathbb{R}_+) = \int_0^\infty \left(\frac{\mathbb{E}\left[\xi\right]}{\mathbb{E}\left[\xi\right] + x}\right) d\mu(x) \leqslant 1.$$

Thus, while μ is a probability measure, the positive measure ν will be a sub-

probability measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. The measures μ and ν are mutually absolutely

continuous, so that with B in $\mathcal{B}(\mathbb{R}_+)$ we have $\mu(B) = 0$ if and only if $\nu(B) = 0$.

Fix $d = 0, 1, \ldots$. We note that

$$
\begin{aligned}
p_\xi(d+1) &= \mathbb{E}\left[(1 - R(\xi)) R(\xi)^d\right] \\
&= \int_0^\infty \frac{\mathbb{E}\left[\xi\right]}{\mathbb{E}\left[\xi\right] + x} \cdot \left(\frac{x}{\mathbb{E}\left[\xi\right] + x}\right)^d d\mu(x) \\
&= \int_0^\infty g_\xi(x)^d d\nu(x)
\end{aligned}
\tag{5.84}
$$

We can now rewrite (5.84) more compactly as

$$\left(p_\xi(d+1)\right)^{\frac{1}{d}} = \|g_\xi\|_{L^d(\mathbb{R}_+;\nu)} \tag{5.85}$$

where $\| \cdot \|_{L^d(\mathbb{R}_+;\nu)}$ denotes the usual prenorm on the linear space $L^d(\nu, \mathbb{R}_+)$ of all

Borel measurable functions $\mathbb{R}_+ \to \mathbb{R}$ which are d-integrable with respect to ν.

Standard arguments based on Hölder's inequality can be used to show the convexity of the mapping

$$\mathbb{R}_+ \to \mathbb{R} : a \to \log \left(\frac{\int_0^\infty |g_\xi(x)|^a d\nu(x)}{\nu(\mathbb{R}_+)} \right),$$

and the mapping

$$(0, \infty) \to \mathbb{R} : a \to \frac{1}{a} \log \left(\frac{\int_0^\infty |g_\xi(x)|^a d\nu(x)}{\nu(\mathbb{R}_+)} \right)$$

is therefore non-decreasing. In particular for $0 < a < b$ we have

$$\left(\frac{\int_0^\infty |g_\xi(x)|^a d\nu(x)}{\nu(\mathbb{R}_+)} \right)^{\frac{1}{a}} \leqslant \left(\frac{\int_0^\infty |g_\xi(x)|^b d\nu(x)}{\nu(\mathbb{R}_+)} \right)^{\frac{1}{b}},$$

or equivalently,

$$\frac{\left(\int_0^\infty |g_\xi(x)|^a d\nu(x) \right)^{\frac{1}{a}}}{\left(\int_0^\infty |g_\xi(x)|^b d\nu(x) \right)^{\frac{1}{b}}} \leqslant \nu(\mathbb{R}_+)^{\frac{1}{a} - \frac{1}{b}}.$$

Because $\nu(\mathbb{R}_+) \leqslant 1$, the mapping

$$(0, \infty) \to \mathbb{R}_+ : a \to \left(\int_0^\infty |g_\xi(x)|^a d\nu(x) \right)^{\frac{1}{a}}$$

is also non-decreasing, and the limit

$$\lim_{a \to \infty} \left(\int_0^\infty |g_\xi(x)|^a d\nu(x) \right)^{\frac{1}{a}}$$

therefore exists. It is easy to see that

$$\lim_{a \to \infty} \left(\int_0^\infty |g_\xi(x)|^a d\nu(x) \right)^{\frac{1}{a}} = \|g_\xi\|_{L^\infty(\mathbb{R}_+;\nu)}$$

with

$$
\begin{aligned}
\|g_\xi\|_{L^\infty(\mathbb{R}_+;\nu)} &= \nu - \text{Ess sup} \left(|g_\xi(x)| : \ x \geqslant 0 \right) \\
&= \inf \left(a \in \mathbb{R}_+ : \ \nu\{x \geqslant 0 : \ |g_\xi(x)| > a\} = 0 \right) \\
&= \inf \left(a \in \mathbb{R}_+ : \ \mu\{x \geqslant 0 : \ |g_\xi(x)| > a\} = 0 \right) \quad (5.86)
\end{aligned}
$$

where the last step follows from the fact noted earlier that the measures μ and ν are mutually absolutely continuous (in which case $\nu\{x \geqslant 0 : |g_\xi(x)| > a\} = 0$ if and only if $\mu\{x \geqslant 0 : |g_\xi(x)| > a\} = 0$. From the definition of g_ξ it is plain that

$$
\begin{aligned}
\{x \geqslant 0 : |g_\xi(x)| > a\} &= \left\{x \geqslant 0 : \frac{x}{\mathbb{E}\left[\xi\right] + x} > a\right\} \\
&= \{x \geqslant 0 : (1-a)x > a\mathbb{E}\left[\xi\right]\} \\
&= \begin{cases} \left(\frac{a}{1-a} \cdot \mathbb{E}\left[\xi\right], \infty\right) & \text{if } 0 \leqslant a < 1 \\[2em] \varnothing & \text{if } 1 \leqslant a, \end{cases}
\end{aligned}
\tag{5.87}
$$

whence

$$
\mu\{x \geqslant 0 : |g_\xi(x)| > a\} = \begin{cases} \mathbb{P}\left[\xi > \frac{a}{1-a} \cdot \mathbb{E}\left[\xi\right]\right] & \text{if } 0 \leqslant a < 1 \\[2em] 0 & \text{if } 1 \leqslant a. \end{cases}
\tag{5.88}
$$

Next, note that the mapping $g_\xi : [0, \infty] \to [0, 1]$ admits an inverse mapping $g_\xi^{-1} : [0, 1] \to [0, \infty]$ given by

$$
g_\xi^{-1}(a) = \begin{cases} \frac{a}{1-a}\mathbb{E}\left[\xi\right] & \text{if } 0 \leqslant a < 1 \\[2em] \infty & \text{if } a = 1. \end{cases}
$$

Therefore, using the obvious convention

$$
\mathbb{P}\left[\xi > \frac{a}{1-a} \cdot \mathbb{E}\left[\xi\right]\right] = 0, \quad \text{if } a = 1,
$$

we readily conclude

$$\inf\left(a \in \mathbb{R}_+ : \ \mu\{x \geqslant 0 : \ |g_\xi(x)| > a\} = 0\right)$$

$$= \ \inf\left(a \in [0,1] : \ \mathbb{P}\left[\xi > \frac{a}{1-a} \cdot \mathbb{E}\left[\xi\right]\right] = 0\right)$$

$$= \ \inf\left(a \in [0,1] : \ \mathbb{P}\left[\xi > g_\xi^{-1}(a)\right] = 0\right)$$

$$= \ g_\xi\left(\inf\left(x \in [0,\infty] : \ \mathbb{P}\left[\xi > x\right] = 0\right)\right) = g_\xi(\xi^\star) \tag{5.89}$$

as we make use of the definition (5.20) of $\xi^\star$. The desired conclusion (5.21) readily

follows. ■

5.11   A proof of Lemma 5.6

Fix $d = 0, 1, \ldots$. From (5.30) we have

$$p_\xi(d+1) = e \int_0^1 (1-s)^d \frac{e^{-\frac{1}{s}}}{s} ds. \tag{5.90}$$

We shall find it helpful to write

$$I_d(s) = (1-s)^d \frac{e^{-\frac{1}{s}}}{s}, \quad s \geqslant 0$$

with the understanding that $I_d(0) = 0$ by the usual continuity argument since $\lim_{s \downarrow 0} e^{-\frac{1}{s}} = 0$ much faster than s. With

$$J_d(s) = \log I_d(s) = d \log(1-s) - \log s - \frac{1}{s}, \quad 0 < s < 1,$$

we have

$$I_d(s) = e^{J_d(s)}, \quad 0 < s < 1.$$

Note that

$$J_d(s)' = -d(1-s)^{-1} - s^{-1} + s^{-2} = -\frac{d}{1-s} + \frac{1-s}{s^2}, \quad 0 < s < 1.$$

Therefore, $J_d(s)' = 0$ is equivalent to

$$\frac{1-s}{s^2} = \frac{d}{1-s},$$

an equation with a unique solution $s^\star(d)$ in the interval $(0,1)$ given by

$$s^\star(d) = \frac{1}{1+\sqrt{d}}.$$

It is easy to check that $s \to J_d(s)$ is increasing on the interval $(0, s^\star(d))$ and decreasing on the interval $(s^\star(d), 1)$.

For future reference we note that

$$1 - s^\star(d) = 1 - \frac{1}{1+\sqrt{d}} = \frac{\sqrt{d}}{1+\sqrt{d}}$$

and

$$\frac{1}{1-s^\star(d)} = \frac{1+\sqrt{d}}{\sqrt{d}}.$$

It is plain that $s^\star(d)$ maximizes $J_d(s)$, hence $I_d(s)$, on the interval $[0,1]$, namely

$$I_d(s) \leqslant I_d(s^\star(d)), \quad 0 \leqslant s \leqslant 1$$

so that

$$\int_0^1 I_d(s)ds \leqslant I_d(s^\star(d)).$$

We have

$$
\begin{aligned}
J_d(s^\star(d)) &= d\log(1-s^\star(d)) - \log s^\star(d) - \frac{1}{s^\star(d)} \\
&= d\log\left(\frac{\sqrt{d}}{1+\sqrt{d}}\right) - \log\left(\frac{1}{1+\sqrt{d}}\right) - \left(1+\sqrt{d}\right) \\
&= d\log\left(\frac{\sqrt{d}}{1+\sqrt{d}}\right) + \log\left(1+\sqrt{d}\right) - \left(1+\sqrt{d}\right), \quad (5.91)
\end{aligned}
$$

so that

$$
\begin{aligned}
eI_d(s^\star(d)) &= ee^{J_d(s^\star(d))} \\
&= e\left(\frac{\sqrt{d}}{1+\sqrt{d}}\right)^d \left(1+\sqrt{d}\right) \cdot e^{-\left(1+\sqrt{d}\right)} \\
&= \left(\frac{\sqrt{d}}{1+\sqrt{d}}\right)^d \left(1+\sqrt{d}\right) \cdot e^{-\sqrt{d}}.
\end{aligned}
\tag{5.92}
$$

An easy induction argument shows the following fact: For all $k = 1, 2, \ldots$, we have

$$
\frac{d^k}{ds^k} J_d(s) = -(k-1)!d(1-s)^{-k} + (-1)^k(k-1)!s^{-k} - (-1)^k k! s^{-(k+1)}, \quad 0 < s < 1.
$$

Now fix $k = 1, 2, \ldots$. It is easy to check that

$$
\begin{aligned}
&\left(\frac{d^k}{ds^k} J_d(s)\right)_{s=s^\star(d)} \\
&= -(k-1)!d(1-s^\star(d))^{-k} + (-1)^k(k-1)!s^\star(d)^{-k} - (-1)^k k! s^\star(d)^{-(k+1)} \\
&= -(k-1)!d\left(\frac{1+\sqrt{d}}{\sqrt{d}}\right)^k + (-1)^k(k-1)!\left(1+\sqrt{d}\right)^k - (-1)^k k!\left(1+\sqrt{d}\right)^{k+1}.
\end{aligned}
$$

If we set

$$
a_k(d) = \frac{1}{k!} \cdot \left(\frac{d^k}{ds^k} J_d(s)\right)_{s=s^\star(d)},
$$

then

$$
\begin{aligned}
a_k(d) &= -\frac{d}{k}\left(\frac{1+\sqrt{d}}{\sqrt{d}}\right)^k + (-1)^k\frac{1}{k}\left(1+\sqrt{d}\right)^k - (-1)^k\left(1+\sqrt{d}\right)^{k+1} \\
&= \left(-\frac{d^{1-\frac{k}{2}}}{k} + \frac{(-1)^k}{k} - (-1)^k\left(1+\sqrt{d}\right)\right)\left(1+\sqrt{d}\right)^k.
\end{aligned}
\tag{5.93}
$$

We note that $a_k(d) < 0$ if and only if

$$
-\frac{d^{1-\frac{k}{2}}}{k} + \frac{(-1)^k}{k} - (-1)^k\left(1+\sqrt{d}\right) < 0,
$$

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a condition equivalent to

$$-d^{1-\frac{k}{2}} + (-1)^k < (-1)^k k \left(1 + \sqrt{d}\right). \tag{5.94}$$

When k is even, say $k = 2p$ for $p = 1, 2, \ldots$, this condition becomes

$$1 - d^{1-p} < 2p \left(1 + \sqrt{d}\right),$$

and is clearly satisfied.

When k is odd, say $k = 2p + 1$ for $p = 1, 2, \ldots$, then condition (5.94) becomes

$$(2p + 1) \left(1 + \sqrt{d}\right) < d^{-p+\frac{1}{2}} + 1,$$

and is never satisfied. The discussion can be summarized as follows.

Fact 5.12. *For $k \geqslant 2$, we always have $(-1)^k a_k(d) < 0$.*

We now turn to the question as to whether the power series

$$\sum_{k=2} a_k(d)(s - s^\star(d))^k$$

is convergent on the interval $(0, 1)$.

To answer this question, fix $k = 2, 3, \ldots$ and consider

$$\frac{a_{k+1}(d)}{a_k(d)} = \frac{\left(-\frac{d^{1-\frac{k+1}{2}}}{k+1} + \frac{(-1)^{k+1}}{k+1} - (-1)^{k+1}\left(1 + \sqrt{d}\right)\right)\left(1 + \sqrt{d}\right)^{k+1}}{\left(-\frac{d^{1-\frac{k}{2}}}{k} + \frac{(-1)^k}{k} - (-1)^k\left(1 + \sqrt{d}\right)\right)\left(1 + \sqrt{d}\right)^{k}}$$

$$= \frac{\left(-\frac{d^{1-\frac{k+1}{2}}}{k+1} + \frac{(-1)^{k+1}}{k+1} - (-1)^{k+1}\left(1 + \sqrt{d}\right)\right)}{\left(-\frac{d^{1-\frac{k}{2}}}{k} + \frac{(-1)^k}{k} - (-1)^k\left(1 + \sqrt{d}\right)\right)} \cdot \left(1 + \sqrt{d}\right)$$

$$= \frac{\left(-\frac{d^{-\frac{k-1}{2}}}{k+1} + \frac{(-1)^{k+1}}{k+1} - (-1)^{k+1}\left(1 + \sqrt{d}\right)\right)}{\left(-\frac{d^{-\frac{k-2}{2}}}{k} + \frac{(-1)^k}{k} - (-1)^k\left(1 + \sqrt{d}\right)\right)} \cdot \left(1 + \sqrt{d}\right)$$

$$= \frac{k}{k+1} \frac{\left(-d^{-\frac{k-1}{2}} + (-1)^{k+1} - (-1)^{k+1}(k + 1)\left(1 + \sqrt{d}\right)\right)}{\left(-d^{-\frac{k-2}{2}} + (-1)^k - (-1)^k k\left(1 + \sqrt{d}\right)\right)} \cdot \left(1 + \sqrt{d}\right)$$

$$= \frac{k}{k+1} \frac{\left(-(-1)^k d^{-\frac{k-1}{2}} - 1 + (k + 1)\left(1 + \sqrt{d}\right)\right)}{\left(-(-1)^k d^{-\frac{k-2}{2}} + 1 - k\left(1 + \sqrt{d}\right)\right)} \cdot \left(1 + \sqrt{d}\right). \qquad (5.95)$$

It is now plain that

$$\lim_{k \to \infty} \left|\frac{a_{k+1}(d)}{a_k(d)}\right| = 1 + \sqrt{d}.$$

Therefore, with s in $(0, 1)$ we have

$$\lim_{k \to \infty} \left|\frac{a_{k+1}(d)(s - s^\star(d))^{k+1}}{a_k(d)(s - s^\star(d))^k}\right| = \left(\lim_{k \to \infty} \left|\frac{a_{k+1}(d)}{a_k(d)}\right|\right) \cdot |s - s^\star(d)|$$

$$= |s - s^\star(d)| \cdot \left(1 + \sqrt{d}\right). \qquad (5.96)$$

Is it possible for some s in $(0, 1)$ to have

$$|s - s^\star(d)| \cdot \left(1 + \sqrt{d}\right) < 1?$$

This last condition is equivalent to having

$$-\frac{1}{1 + \sqrt{d}} < s - s^\star(d) < \frac{1}{1 + \sqrt{d}},$$

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or equivalently,

$$-s^\star(d) < s - s^\star(d) < s^\star(d).$$

In other words, the condition becomes

$$0 < s < 2s^\star(d) = \frac{2}{1 + \sqrt{d}}.$$

In particular, the interval $(0, 2s^\star(d))$ where $J_d(s)$ admits a Taylor series representation shrinks with increasing d.

Pick λ in the unit interval $[0, 1]$. Note that

$$(1 \pm \lambda)s^\star(d) = \frac{1 \pm \lambda}{1 + \sqrt{d}}$$

so that

$$1 - (1 \pm \lambda)s^\star(d) = \frac{\mp\lambda + \sqrt{d}}{1 + \sqrt{d}}.$$

Therefore,

$$
\begin{aligned}
J_d((1 \pm \lambda)s^\star(d)) &= d\log(1 - (1 \pm \lambda)s^\star(d)) - \log((1 \pm \lambda)s^\star(d)) - \frac{1}{(1 \pm \lambda)s^\star(d)} \\
&= d\log\left(\frac{\mp\lambda + \sqrt{d}}{1 + \sqrt{d}}\right) - \log\left(\frac{1 \pm \lambda}{1 + \sqrt{d}}\right) - \frac{1 + \sqrt{d}}{1 \pm \lambda}, \quad (5.97)
\end{aligned}
$$

whence

$$I_d((1 \pm \lambda)s^\star(d)) = \left(\frac{\mp\lambda + \sqrt{d}}{1 + \sqrt{d}}\right)^d \cdot \left(\frac{1 + \sqrt{d}}{1 \pm \lambda}\right) \cdot e^{-\frac{1 + \sqrt{d}}{1 \pm \lambda}}. \quad (5.98)$$

Next, with α in $[0, 1)$ and β in $[0, 1]$, we conclude that

$$
\begin{aligned}
\frac{I_d((1 + \beta)s^\star(d))}{I_d((1 - \alpha)s^\star(d))} &= \frac{\left(\frac{-\beta + \sqrt{d}}{1 + \sqrt{d}}\right)^d \cdot \left(\frac{1 + \sqrt{d}}{1 + \beta}\right) \cdot e^{-\frac{1 + \sqrt{d}}{1 + \beta}}}{\left(\frac{\alpha + \sqrt{d}}{1 + \sqrt{d}}\right)^d \cdot \left(\frac{1 + \sqrt{d}}{1 - \alpha}\right) \cdot e^{-\frac{1 + \sqrt{d}}{1 - \alpha}}} \\
&= \left(\frac{-\beta + \sqrt{d}}{\alpha + \sqrt{d}}\right)^d \cdot \frac{1 - \alpha}{1 + \beta} \cdot e^{-\frac{1 + \sqrt{d}}{1 + \beta} + \frac{1 + \sqrt{d}}{1 - \alpha}}.
\end{aligned}
$$

Note that

$$\frac{1 + \sqrt{d}}{1 + \beta} - \frac{1 + \sqrt{d}}{1 - \alpha} = -C(\alpha, \beta) \cdot \left(1 + \sqrt{d}\right)$$

where we have set

$$C(\alpha, \beta) = \frac{\alpha + \beta}{(1 - \alpha)(1 + \beta)},$$

whence

$$\frac{I_d((1 + \beta)s^\star(d))}{I_d((1 - \alpha)s^\star(d))} = \left(\frac{-\beta + \sqrt{d}}{\alpha + \sqrt{d}}\right)^d \cdot \frac{1 - \alpha}{1 + \beta} \cdot e^{C(\alpha,\beta)\cdot\left(1+\sqrt{d}\right)}.$$

Next, write

$$
\begin{aligned}
\left(\frac{-\beta + \sqrt{d}}{\alpha + \sqrt{d}}\right)^d &= \left(1 - \frac{\alpha + \beta}{\alpha + \sqrt{d}}\right)^d \\
&= e^{-d\left(\frac{\alpha+\beta}{\alpha+\sqrt{d}} + \Psi\left(\frac{\alpha+\beta}{\alpha+\sqrt{d}}\right)\right)}
\end{aligned}
\tag{5.99}
$$

where we have set

$$\Psi(x) = \int_0^x \frac{t}{1 - t} dt, \ 0 \leqslant x < 1.$$

.

$$d\Psi\left(\frac{\alpha + \beta}{\alpha + \sqrt{d}}\right) = d\left(\frac{\alpha + \beta}{\alpha + \sqrt{d}}\right)^2 r(d) \tag{5.100}$$

as we set

$$r(d) = \frac{\Psi\left(\frac{\alpha+\beta}{\alpha+\sqrt{d}}\right)}{\left(\frac{\alpha+\beta}{\alpha+\sqrt{d}}\right)^2} = \frac{1}{2}\left(1 + o(1)\right).$$

It is plain that

$$\lim_{d\to\infty} d\left(\frac{\alpha + \beta}{\alpha + \sqrt{d}}\right)^2 = (\alpha + \beta)^2,$$

whence

$$\lim_{d\to\infty} d\Psi\left(\frac{\alpha + \beta}{\alpha + \sqrt{d}}\right) = \frac{(\alpha + \beta)^2}{2}.$$

Thus,

$$d\left(\frac{\alpha+\beta}{\alpha+\sqrt{d}} + \Psi\left(\frac{\alpha+\beta}{\alpha+\sqrt{d}}\right)\right)$$
$$= \frac{(\alpha+\beta)d}{\alpha+\sqrt{d}} + \frac{(\alpha+\beta)^2}{2}\left(1+o(1)\right) \qquad (5.101)$$

and collecting we conclude that

$$\frac{I_d((1+\beta)s^\star(d))}{I_d((1-\alpha)s^\star(d))}$$
$$= e^{-\frac{(\alpha+\beta)d}{\alpha+\sqrt{d}}} \cdot e^{-\frac{(\alpha+\beta)^2}{2}(1+o(1))} \cdot \frac{1-\alpha}{1+\beta} \cdot e^{C(\alpha,\beta)\left(1+\sqrt{d}\right)}$$
$$= e^{-\frac{(\alpha+\beta)d}{\alpha+\sqrt{d}}} \cdot e^{C(\alpha,\beta)\sqrt{d}} \cdot \frac{1-\alpha}{1+\beta} e^{-\left(-C(\alpha,\beta)+\frac{(\alpha+\beta)^2}{2}(1+o(1))\right)} \qquad (5.102)$$

However,

$$\lim_{d\to\infty} \frac{1}{\sqrt{d}}\left(\frac{(\alpha+\beta)d}{\alpha+\sqrt{d}} - C(\alpha,\beta)\sqrt{d}\right) = \alpha + \beta - C(\alpha,\beta)$$
$$= \left((1-\alpha)(1+\beta)-1\right)C(\alpha,\beta)$$
$$= \left(\beta(1-\alpha)-\alpha\right)C(\alpha,\beta). \qquad (5.103)$$

In other words,

$$\frac{(\alpha+\beta)d}{\alpha+\sqrt{d}} - C(\alpha,\beta)\sqrt{d} \sim \left(\beta(1-\alpha)-\alpha\right)C(\alpha,\beta)\sqrt{d},$$

and the following conclusion follows.

Lemma 5.13. *With α and β in $(0,1]$ such that*

$$\frac{\alpha}{1-\alpha} < \beta,$$

we have

$$I_d((1+\beta)s^\star(d)) = o\left(I_d((1-\alpha)s^\star(d))\right)$$

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in the sense that

$$\lim_{d\to\infty} \frac{I_d((1+\beta)s^\star(d))}{I_d((1-\alpha)s^\star(d))} \cdot e^{(\beta(1-\alpha)-\alpha+o(1))C(\alpha,\beta)\sqrt{d}} = K(\alpha,\beta)$$

where

$$K(\alpha,\beta) = \frac{1-\alpha}{1+\beta} \cdot e^{-\left(\frac{(\alpha+\beta)^2}{2}-C(\alpha,\beta)\right)}.$$

In other words

$$\frac{I_d((1+\beta)s^\star(d))}{I_d((1-\alpha)s^\star(d))} \sim K(\alpha,\beta) \cdot e^{-(\beta(1-\alpha)-\alpha+o(1))C(\alpha,\beta)\sqrt{d}}$$

We exploit this fact as follows: With α and β in $(0,1]$ such that $\alpha < (1-\alpha)\beta$,

Lemma 5.13 ensures that for all d sufficiently large, the inequality

$$5I_d((1+\beta)s^\star(d)) < I_d((1-\alpha)s^\star(d))$$

holds. However, recall that the mapping $s \to I_d(s)$ is monotone increasing on

$(0, s^\star(d))$ and monotone decreasing on $(s^\star(d), 1)$, with a maximum at $s = s^\star(d)$.

Therefore, with d sufficiently large, there exists $t(d)$ to the right of $s^\star(d)$ in the

interval $(s^\star(d), (1+\beta)s^\star(d))$ which depends on α, β and d such that

$$I_d(t(d)) = I_d((1-\alpha)s^\star(d)).$$

It is now plain that

$$\int_{(1+\beta)s^\star(d)}^{1} I_d(s)ds \leq (1-(1+\beta)s^\star(d)) \cdot I_d((1+\beta)s^\star(d))$$

while

$$(t(d) - (1-\alpha)s^\star(d)) \cdot I_d((1-\alpha)s^\star(d)) \leq \int_{0}^{(1+\beta)s^\star(d)} I_d(s)ds.$$

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Therefore,

$$\frac{\int_{(1+\beta)s^\star(d)}^{1} I_d(s)ds}{\int_0^{(1+\beta)s^\star(d)} I_d(s)ds} \leqslant \frac{(1 - (1+\beta)s^\star(d))}{(t(d) - (1-\alpha)s^\star(d))} \cdot \frac{I_d((1+\beta)s^\star(d))}{I_d((1-\alpha)s^\star(d))}.$$

As we note that

$$t(d) - (1-\alpha)s^\star(d) > s^\star(d) - (1-\alpha)s^\star(d) = \alpha s^\star(d)$$

by construction, it follows that

$$\frac{\int_{(1+\beta)s^\star(d)}^{1} I_d(s)ds}{\int_0^{(1+\beta)s^\star(d)} I_d(s)ds} \leqslant \frac{1}{\alpha s^\star(d)} \cdot \frac{I_d((1+\beta)s^\star(d)}{I_d((1-\alpha)s^\star(d))}$$

$$= \frac{1 + \sqrt{d}}{\alpha} \cdot \frac{I_d((1+\beta)s^\star(d)}{I_d((1-\alpha)s^\star(d))} \qquad (5.104)$$

Lemma 5.14. *For every β in $[0, 1]$, we have*

$$\lim_{d \to \infty} \frac{\int_{(1+\beta)s^\star(d)}^{1} I_d(s)ds}{\int_0^{(1+\beta)s^\star(d)} I_d(s)ds} = 0$$

and

$$\int_0^1 I_d(s)ds \sim \int_0^{(1+\beta)s^\star(d)} I_d(s)ds.$$

Pick $d = 1, 2, \ldots$. Pick α and β in $(0, 1)$. Recall that for s in the interval

$((1 - \alpha)s^\star(d), (1 + \beta)s^\star(d))$, we have

$$J_d(s) - J_d(s^\star(d)) = \sum_{k=2}^{\infty} a_k(d) (s - s^\star)^k$$

$$= -\left(1 + \sqrt{d}\right)^3 (s - s^\star(d))^2 + \sum_{k=3}^{\infty} a_k(d) (s - s^\star(d))^k$$

$$= -\left(1 + \sqrt{d}\right)^3 (s - s^\star(d))^2 \left(1 - \sum_{k=3}^{\infty} \frac{a_k(d)}{\left(1 + \sqrt{d}\right)^3} (s - s^\star(d))^{k-2}\right)$$

$$= -\left(1 + \sqrt{d}\right)^3 (s - s^\star(d))^2 (1 - h_d(s - s^\star(d))) \qquad (5.105)$$

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where we have set

$$h_d(s) = \sum_{k=3}^{\infty} \frac{a_k(d)}{\left(1 + \sqrt{d}\right)^3} \cdot s^{k-2}, \quad |s| < s^\star(d)$$

Therefore,

$$\int_{(1-\alpha)s^\star(d)}^{(1+\beta)s^\star(d)} I_d(s)ds \;=\; e^{J_d(s^\star(d))} \int_{(1-\alpha)s^\star(d)}^{(1+\beta)s^\star(d)} e^{J_d(s)-J_d(s^\star(d))}ds \tag{5.106}$$

$$= I_d(s^\star(d)) \int_{(1-\alpha)s^\star(d)}^{(1+\beta)s^\star(d)} e^{-\left(1+\sqrt{d}\right)^3 (s-s^\star(d))^2 (1-h_d(s-s^\star(d)))} ds$$

$$= I_d(s^\star(d)) \int_{-\alpha s^\star(d)}^{\beta s^\star(d)} e^{-\left(1+\sqrt{d}\right)^3 y^2 (1-h_d(y))} dy \quad [y = s - s^\star(d)]$$

$$= s^\star(d) I_d(s^\star(d)) \int_{-\alpha}^{\beta} e^{-\left(1+\sqrt{d}\right) x^2 (1-h_d(s^\star(d)x))} dx \quad [y = s^\star(d) x] \tag{5.107}$$

Thus, with x such that $|x| < 1$, we obtain

$$|h_d(s^\star(d)x)| \;=\; \left| \sum_{k=3}^{\infty} \frac{a_k(d)}{\left(1 + \sqrt{d}\right)^3} (s^\star(d)x)^{k-2} \right|$$

$$\leqslant \sum_{k=3}^{\infty} \frac{|a_k(d)|}{\left(1 + \sqrt{d}\right)^3} s^\star(d)^{k-2} |x|^{k-2}$$

$$= \sum_{k=3}^{\infty} |a_k(d)| s^\star(d)^{k+1} |x|^{k-2} \tag{5.108}$$

with

$$|a_k(d)| s^\star(d)^{k+1} \;=\; \left| \left(-\frac{d^{1-\frac{k}{2}}}{k} + \frac{(-1)^k}{k} - (-1)^k \left(1 + \sqrt{d}\right) \right) \left(1 + \sqrt{d}\right)^k \right| s^\star(d)^{k+1}$$

$$\leqslant \; 1 + \frac{1 + d^{1-\frac{k}{2}}}{k} \cdot s^\star(d)$$

$$\leqslant \; 1 + \frac{1 + \sqrt{d}}{k\sqrt{d}} \cdot s^\star(d)$$

$$= \; 1 + \frac{1}{k\sqrt{d}}, \quad k = 3, 4, \ldots \tag{5.109}$$

As a result,

$$
\begin{aligned}
|h_d(s^\star(d)x)| &\leqslant \sum_{k=3}^{\infty}\left(1+\frac{1}{k\sqrt{d}}\right)|x|^{k-2} \\
&\leqslant 2\sum_{k=3}^{\infty}|x|^{k-2} \\
&= \frac{2|x|}{1-|x|}, \quad d=1,2,\ldots \qquad (5.110)
\end{aligned}
$$

Hence, with $\gamma = \max(\alpha,\beta)$, we get

$$
\sup\left(|h_d(s^\star(d)x)|,\ -\alpha\leqslant x\leqslant \beta\right)\leqslant \frac{2\gamma}{1-\gamma} \qquad (5.111)
$$

uniformly in d. It follows then for each λ in $(0,1)$,

$$
\sup\left(|h_d(s^\star(d)x)|,\ -\alpha\leqslant x\leqslant \beta\right)\leqslant \lambda
$$

uniformly in d whenever

$$
\gamma < \frac{\lambda}{2+\lambda} := \gamma^\star(\lambda)
$$

with $\gamma^\star(\lambda)$ in the unit interval $(0,1)$.

Thus, fix λ in $(0,1)$. With γ in $(0,\gamma^\star(\lambda))$, whenever $\gamma = \max(\alpha,\beta)$, we have

$$
\int_{-\alpha}^{\beta} e^{-\left(1+\sqrt{d}\right)x^2\left(1-h_d(s^\star(d)x)\right)}dx \leqslant \int_{-\alpha}^{\beta} e^{-\left(1+\sqrt{d}\right)(1-\lambda)x^2}dx
$$

and

$$
\int_{-\alpha}^{\beta} e^{-\left(1+\sqrt{d}\right)(1+\lambda)x^2}dx \leqslant \int_{-\alpha}^{\beta} e^{-\left(1+\sqrt{d}\right)x^2\left(1-h_d(s^\star(d)x)\right)}dx
$$

uniformly in d.

We will make use of these bounds together with the following asymptotics.

Lemma 5.15. *For arbitrary $\alpha > 0$ and $\beta > 0$ it holds that*

$$
\int_{-\alpha}^{\beta} e^{-\theta x^2}dx \sim \sqrt{\frac{\pi}{\theta}}
$$

as θ goes to infinity.

Proof. Fix $\theta > 0$. We get

$$
\begin{aligned}
\int_{-\alpha}^{\beta} e^{-\theta x^2} dx &= \frac{1}{\sqrt{2\theta}} \int_{-\alpha\sqrt{2\theta}}^{\beta\sqrt{2\theta}} e^{-\frac{y^2}{2}} dy \quad [y = \sqrt{2\theta}x] \\
&= \sqrt{\frac{\pi}{\theta}} \left(\Phi(\beta\sqrt{2\theta}) - \Phi(-\alpha\sqrt{2\theta}) \right) \\
&= \sqrt{\frac{\pi}{\theta}} \left(\Phi(\beta\sqrt{2\theta}) + \Phi(\alpha\sqrt{2\theta}) - 1 \right) \quad\quad (5.112)
\end{aligned}
$$

and the conclusion follows. ∎

Consequently,

$$
\int_{-\alpha}^{\beta} e^{-\left(1+\sqrt{d}\right)(1\pm\lambda)x^2} dx \sim \sqrt{\frac{\pi}{\left(1 + \sqrt{d}\right)(1 \pm \lambda)}}
$$

Assume from the time being that

$$
\int_0^1 I_d(s)ds \sim \int_{(1-\alpha)s^\star(d)}^1 I_d(s)ds, \quad \alpha \in (0,1)
$$

so that

$$
\begin{aligned}
p(d+1) &\sim \int_{(1-\alpha)s^\star(d)}^1 I_d(s)ds \\
&\sim \int_{(1-\alpha)s^\star(d)}^{(1+\beta)s^\star(d)} I_d(s)ds \\
&= s^\star(d)I_d(s^\star(d)) \int_{-\alpha}^{\beta} e^{-\left(1+\sqrt{d}\right)x^2(1-h_d(s^\star(d)x))} dx \\
&= s^\star(d)I_d(s^\star(d)) \cdot \sqrt{\frac{\pi}{1 + \sqrt{d}}} \cdot \frac{\int_{-\alpha}^{\beta} e^{-\left(1+\sqrt{d}\right)x^2(1-h_d(s^\star(d)x))} dx}{\sqrt{\frac{\pi}{1+\sqrt{d}}}} \quad (5.113)
\end{aligned}
$$

It follows that

$$
\frac{p(d+1)}{s^\star(d)I_d(s^\star(d)) \cdot \sqrt{\frac{\pi}{1+\sqrt{d}}}} \sim \frac{\int_{-\alpha}^{\beta} e^{-\left(1+\sqrt{d}\right)x^2(1-h_d(s^\star(d)x))} dx}{\sqrt{\frac{\pi}{1+\sqrt{d}}}} \quad (5.114)
$$

From earlier inequalities we conclude that

$$\limsup_{d\to\infty} \frac{\int_{-\alpha}^{\beta} e^{-\left(1+\sqrt{d}\right)x^2(1-h_d(s^\star(d)x))}\,dx}{\sqrt{\frac{\pi}{1+\sqrt{d}}}}$$

$$\leqslant \lim_{d\to\infty} \frac{\int_{-\alpha}^{\beta} e^{-\left(1+\sqrt{d}\right)(1-\lambda)x^2}\,dx}{\sqrt{\frac{\pi}{1+\sqrt{d}}}} = \frac{1}{\sqrt{1-\lambda}} \qquad (5.115)$$

and

$$\frac{1}{\sqrt{1+\lambda}} = \lim_{d\to\infty} \frac{\int_{-\alpha}^{\beta} e^{-\left(1+\sqrt{d}\right)(1+\lambda)x^2}\,dx}{\sqrt{\frac{\pi}{1+\sqrt{d}}}}$$

$$\leqslant \liminf_{d\to\infty} \frac{\int_{-\alpha}^{\beta} e^{-\left(1+\sqrt{d}\right)x^2(1-h_d(s^\star(d)x))}\,dx}{\sqrt{\frac{\pi}{1+\sqrt{d}}}} \qquad (5.116)$$

Finally,

$$\limsup_{d\to\infty} \frac{p(d+1)}{s^\star(d)I_d(s^\star(d))\cdot\sqrt{\frac{\pi}{1+\sqrt{d}}}} \leqslant \frac{1}{\sqrt{1-\lambda}}$$

and

$$\frac{1}{\sqrt{1+\lambda}} \leqslant \liminf_{d\to\infty} \frac{p(d+1)}{s^\star(d)I_d(s^\star(d))\cdot\sqrt{\frac{\pi}{1+\sqrt{d}}}}$$

Since λ is arbitrary in $(0,1)$ we conclude that

$$\lim_{d\to\infty} \frac{p_\xi(d+1)}{s^\star(d)I_d(s^\star(d))\cdot\sqrt{\frac{\pi}{1+\sqrt{d}}}} = 1,$$

or equivalently,

$$p_\xi(d+1) \sim s^\star(d)I_d(s^\star(d))\cdot\sqrt{\frac{\pi}{1+\sqrt{d}}}.$$

Substituting we find

$$p_\xi(d+1) \sim \sqrt{\frac{\pi}{e^2}}\frac{e^{-2\sqrt{d}}}{\sqrt{1+\sqrt{d}}} \sim \sqrt{\frac{\pi}{e^2}}\frac{e^{-2\sqrt{d}}}{\sqrt[4]{d}}$$

■

Part II

Games on Networks

Chapter 6:   Learning in Games

There has been a growing interest in game theory as a tool to solve complex engineering problems that involve interactions among (many) distributed agents or subsystems. As these agents are expected to interact many times over time, researchers proposed a variety of learning rules with desired properties, that helped the agents to determine their future strategies on the basis of the past information available to them. One example of such a desired property is convergence to some form of equilibrium that might be thought of as an approximation of a desired operating point for the entire system. Many researchers have worked on this interesting problem, and as a result many different types of learning rules have been already proposed in the literature, e.g., [74, 75, 78, 78, 79].

However, when casting an engineering problem in a game-theoretic framework there are a few important practical considerations that need to be taken into account. For instance, controllers in engineering systems are not necessarily synchronized; instead, they are often *event driven* and update their (control) actions as new measurements or observations become available. For this reason, guaranteeing acceptable performance under *asynchronous* operations is crucial in many engineering systems. Moreover, in settings where the system consists of many individual

subsystems with their own individual controllers, the measurements available to a particular subsystem or a controller could reflect the *past* decisions taken by the individual controllers. For example, consider the setting of a wind farm composed of a number of wind turbines [1]. The sensor data available at a particular wind turbine could reflect past decisions taken by other wind turbines because the actions taken by the other wind turbines could affect the former with varying time delays.

Unfortunately, most of the existing studies on learning in games do not consider either delays experienced by payoff information or asynchronous updates of strategies by players. In our work we propose two classes of learning rules which could be thought as a first step towards bridging this gap between the current literature and the sound design of engineering systems on a game-theoretic framework. First, we consider the better-reply rule [85] in which the players aim to improve their immediate payoffs. This learning rule has been studied extensively due to its simplicity [73, 74, 95]. More precisely, the proposed rule requires the following: (i) if there is no strictly better reply (SBR), the agent stays with the same action, and (ii) if there exists at least one SBR, it switches to each SBR with positive probability and also sticks with the previous actions with positive probability (often called *inertia*). This inertia may, for instance, model the scenario where an agent waits at least one more period and tries the same action before switching/commiting to an SBR. Clearly, the learning rule is computationally inexpensive in that only a number of simple comparisons need to be made.

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[1] A wind turbine can control how much power it draws from the wind by varying its axial induction factor [77]

In Chapter 7 we introduce a general framework for modelling asynchronous updates of strategies by the players, possibly based on delayed or even outdated payoff information. Assuming that the payoff information available to the players is accurate, we prove that if all the agents update in accordance to the better-reply rule, under a set of mild technical conditions, the action profiles [2] played by the agents converges to a pure strategy Nash equilibrium (PSNE) almost surely [58] in a class of games, which we call *generalized weakly acyclic games* (GWAGs). We note that this almost sure convergence of action profiles takes place even when the players update their strategies in an asynchronous manner on the basis of delayed payoff information. Finally, we demonstrate that if the game is *not* generalized weakly acyclic, the better-reply rule in general cannot guarantee the almost sure convergence of action profiles even when the strategies are updated synchronously using current payoff information.

In many practical scenarios, erroneous decision-making by the individual agents (possibly controllers in an engineering system) is also a major concern along with delayed payoff information and asynchronous operations. Such erroneous decision-making could stem from faulty available payoff information and that is what we consider in Section 7.6. We show in fact that as the payoff information becomes more reliable, the probability that the agents play a PSNE under the better-reply rule tends to one over time. When the probability of error is sufficiently small, the set of action profiles that are played by the system most of the time are called the *stochastically stable states* [95]. Thus for the GBRR rules the set of stochastically

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[2]An action profile specifies the actions played by all the agents in the system

stable states are a subset of the set of PSNE(s). However there are two shortcomings of the better reply rules – (i) a somewhat undesirable property is that under the exact payoff information setting the system converges almost surely to a PSNE depending on the system's initial conditions, and (ii) under the faulty payoff information setting the set of stochastically stable states are not easily characterisable. This gives us the reason to devise a learning rule which is not only robust with respect to delayed payoff information and asynchrony but also addresses the above-mentioned shortcomings. This leads us to our next piece of work where we have proposed a new class of rules, which we call the class of monitoring rules.

In Chapter 8, we discuss a simplified representative version in the class of monitoring rules called the *reduced simple experimentation with monitoring* (RSEM) rule [86]. Under the RSEM rule, the agents can either be in an *explore* state or in a *converged* state – where the state dictates how actions are chosen at a particular time. While at the explore state the agent tries out every action with a positive probability in an effort to play the action which cannot be improved any further. Once such an action is played the agent switches to the converged state with a positive probability. At the converged state, the agent simply keeps playing the previous action as long as 'playing conditions remain unchanged', i.e., the payoff remains the same and no better reply exists. As promised earlier, in Chapter 8, we show convergence of the action profile to a PSNE under asynchronous updates and payoff information delays, firstly, in the class of GWAGs, and also in games satisfying a 'payoff interdependence assumption' [79, 86, 87, 96]. When the game satisfies the 'payoff interdependence assumption', the RSEM rule ensures almost sure

convergence to the set of PSNE(s) independent of the initial conditions. Also, under the setting of erroneous determination of the SBRs, i.e., erroneous estimation of the set of better replies, the set of stochastically stable states are well characterisable. In particular, they are those PSNE(s) such that when it is in effect in the system, it is least likely that any of the agents see a better reply. It is worth pointing out that the results for the RSEM rule have been shown for erroneous determination of the SBRs but an accurate received payoff for the played action. Future work is in order to consider the situation of erroneous payoff for the played action as well.

Until now, we argued erroneous decision-making by the agents to be a result of faulty available payoff information. However there might be practical situations where the agents receive correct payoff information yet implement non-equilibrium actions either by mistake or maybe because its system has been compromised. Consider the model described earlier where the agents correspond to controllers driving their own (sub)-systems which in turn are a part of a much bigger system. There is a possibility that the controller decides on a certain action but its own (sub)-system cannot implement it either due to mistakes or maybe because its system has been compromised and the controller has lost command on its (sub)-system.

Such a scenario could be potentially detrimental to a system which is already at an equilibrium. A moment of thought suggests that the algorithm should allow brief changes in the received payoff information and instead be responsive to long term changes. A simple implementation of this idea leads to a generalization of the RSEM rule where the agents at the converged state allow at most a fixed number of successive changes in its payoff information before permanently moving away from

an equilibrium action. This rule is formally introduced in Chapter 9 as the *simple experimentation with monitoring* (SEM) rule. We show that when the agents follow the SEM rule, the action profiles selectively converge to resilient PSNE(s) that can tolerate deviations by potentially multiple agents. By controlling the number of successive changes in the payoff information that can be allowed by the agents as a tunable parameter, either (i) PSNEs with certain desired resilience or (ii) the most resilient PSNE(s) could be reached.

This portion of the thesis is organized as follows: In Chapter 7, we first introduce the framework for considering asynchronous updates and delayed payoff information, and also present the results under better-reply rules. In the same chapter we also introduce a setting for considering faulty payoff information and investigate convergence under the better-reply rules. In the next chapter, we introduce the RSEM rule and present convergence results under delays and asynchrony. We also distinguish our findings from that obtained under the better-reply rules under faulty payoff information. In Chapter 9, we generalize the RSEM rule to ensure selective convergence to *resilient* PSNEs under erroneous decision-making due to faulty implementation of intended actions. But first, in the following section we present an outline of our main contributions before doing a brief literature review.

6.1   Summary of Contributions

Our main contributions can be summarized as follows:

1. We propose a general framework for considering delayed payoff information

and asynchronous updates by agents. To the best of our knowledge, this work is the first one that introduces a general game-theoretic framework for modeling more realistic engineering systems with delays and asynchrony.

2. We demonstrate that a simple and intuitive rule such as the better-reply rule can ensure almost sure convergence to the set of PSNEs even under asynchronous updates and delays in a class of games, which we call the generalized weakly acyclic games. Not only that, we also show the probability that the players have not converged to a PSNE *decays geometrically* with time, thereby proving that the expected convergence time is finite.

3. Motivated by practical scenarios, we model erroneous decision-making by agents due to faulty payoff information. Under the better-reply rule, we show that as the payoff information becomes more reliable, the probability that the players play a PSNE tends to one over time.

4. For the better reply rules, under the setting where there are no errors, the system converges to a PSNE which could depend on its initial conditions. Also, under the aforementioned setting of faulty available payoff information, we are unable to characterise the set of stochastically stable equilibria. To resolve these issues, we propose the RSEM rule:

    (a) Firstly, the RSEM rule is shown to ensure almost sure convergence of the action profiles to a PSNE under delays and asynchrony for GWAGs and also in games that satisfy a 'payoff interdependence assumption'.

Moreover, under both the settings we show that the probability that the agents have not converged to a PSNE decays geometrically with time. When the game satisfies the payoff interdependence assumption the system converges to any PSNE independently of the initial conditions.

(b) Under the setting of faulty determination of SBRs, the stochastically stable states under the RSEM rule are those PSNE(s) which make it least likely for any of the agents to see a better reply when it is in effect.

5. While erroneous decision-making could occur due to faulty available payoff information, it could also be a result of faulty implementation of an intended action. Firstly, we consider a model to account for such a scenario. Next, we show that the RSEM rule can be generalized to ensure that the system reaches PSNE(s) which are resilient to deviations by potentially multiple agents.

6.2   Related Literature

There is already a large volume of literature on learning in games with findings of varying nature, e.g., $[39, 43, 52, 56, 63, 69, 95, 96]$. For this reason, we only provide a very brief summary of a limited number of studies, and refer an interested reader to the references therein for additional studies. First, a popular early learning procedure is fictitious play $[45, 55, 56, 82, 83]$; players form beliefs regarding the opponents' players, for instance, based on the empirical frequencies of their plays, and pick optimal strategies with respect to their beliefs. The convergence of strategy profile to equilibria is proven only in somewhat restrictive settings, such as zero-sum

games and potential games.

Bayesian learning has drawn attention of many researchers for quite some time: In [66], Jordan studied the Bayesian learning processes for normal form games and convergence to a Nash equilibrium under mild assumptions on the prior distribution over payoff functions. In [67], he also studied far-sighted Bayesian learning and its convergence to a Nash equilibrium in repeated game settings. Another related work by Kalai and Lehrer [68] showed that, if players in a repeated game start with subjective beliefs about their opponents' strategies and their beliefs are "compatible" with their true strategies, i.e., the players' prior beliefs assign positive probability to all strategy profiles that will be chosen with positive probability according to the true strategies, Bayesian learning leads to accurate prediction of future play of the game. Moreover, they illustrated that if the players know their payoff functions, the convergence to a Nash equilibrium occurs.

Researchers also considered non-Bayesian learning. For instance, Foster and Vohra [50] studied the problem of forecasting and pointed out the benefits of randomized forecasting against an oblivious or adaptive adversary. In their follow-up study [49], they demonstrated that when the players in a normal form game use a learning rule with calibrated forecast of the other players' plays and each player plays myopically with respect to the forecast distribution, the limit points of the sequence of plays are correlated equilibria. This was put forth as an alternative to Aumann's proof that the common prior assumption and rationality imply a correlated equilibrium. Interestingly, they also showed the converse of the finding also holds in the sense that, for every correlated equilibrium, there exists some calibrated

learning rule that would result in the equilibrium being played in the limit.

Another class of learning rules is based on *regrets*. Regrets capture the additional payoff a player could have received by playing a different action. There are several regret-based learning rules, e.g., [57,59,62], which guarantee convergence to Nash equilibria, correlated equilibria or Hannan set in an appropriate sense. Hart and Mas-Colell in [59] introduced a simple adaptive procedure called the *regret matching* where players switch from their current plays to others with probabilities proportional to their regrets. They showed that the empirical distribution of plays converges to a correlated equilibrium if all the players follow the procedure. In a slightly different informational setting, Hart and Mas-Colell proposed a reinforcement learning-based technique to estimate regrets, which they called the *modified regret matching*, with a similar convergence property. A summary of existing convergence results for generalized regret matching can be found in [61]. *Regret testing* is another regret-based uncoupled learning rule, which was first introduced by Foster and Young [51]. Germano and Lugosi [57] showed that when all players adopt this scheme, the (mixed) strategy profile of the players converges to a Nash equilibrium of the stage game almost surely.

Oftentimes, real world problems exhibit special structures which could be leveraged while designing learning rules. In this context many of the existing learning rules target games with special structures, such as identical interest games, potential games (PGs), and weakly acyclic games (WAGs). Although an arbitrary game is not guaranteed to possess a PSNE, one exists for PGs because a maximizer of the potential function is a PSNE [88]. This led to further research with problems

being formulated as PGs. For example, Arslan et al. [41] model the autonomous vehicle-target assignment problem as a PG. They present two learning rules and demonstrate their convergence to PSNEs. Marden et al. [76] study large-scale games with many players with large strategy spaces. They generalize the notion of PGs and propose a learning rule that guarantees convergence to a PSNE in the class of games they consider, with applications to congestion games. The WAGs were first studied in a systemic manner in [95]. Since then, there has been considerable interest in WAGs. For instance, Marden et al. [73] establish the relations between cooperative control problems (e.g., consensus problem) and game theoretic models. In addition, they propose the better reply with inertia dynamics and apply it to a class of games, which they call *sometimes weakly acyclic games*, to address time-varying objective functions and action sets. In another related study [74], Marden et al. proposed regret-based dynamics that achieve almost sure convergence to a strict Nash equilibrium in weakly acyclic games.

More recently, researchers aimed to design so-called payoff-based learning rules (also known as 'completely uncoupled dynamics') with provable convergence to efficient Nash equilibria (in an appropriate sense) in potential games [78] and weakly acyclic games [75]. Pradelski and Young extended the result to general games satisfying what they called an *interdependence* assumption, a weaker form of which has been actually used in our work. Related to these studies, Marden et al. [79] proposed learning rules that seek Pareto optimal strategy profile, i.e., a maximizer of aggregate payoff. This result was sharpened by Menon and Baras [80] to show that, under some conditions, the probability that the strategy profile lies in the set of

aggregate payoff maximizers converges to one over time. Marden and Shamma [78] studied the effects of asynchrony in log-linear learning using independent revision processes and showed that, in potential games with sufficiently small probability of players revising their strategies at each time $t \in \mathbb{N}$, only the maximizers of potential functions can be stochastically stable. We also point out an interesting study by Hart and Mansour [64] on the communication complexity of uncoupled equilibrium procedures. Not surprisingly, they showed that any pure Nash equilibrium procedure has communication complexity that grows exponentially with the number of players.

Chapter 7: Class of Better reply rules

In this chapter, we consider a simple and intuitive class of learning rules called the better-reply rules. Along with identifying a new class of games – generalized weakly acyclic games which contains well known classes of games such as weakly acyclic games, potential games and identical interest games (see Section 6.2 for practical significance of these classes of games), we also show convergence results of the action profiles under the better-reply rules in the context of these games. We draw connections between the better-reply rules and the GWAGs, and argue that the GWAGs is a class of games naturally associated with the better-reply rules. Keeping in mind our original motivation to better model practical engineering concerns in a game-theoretic setting, we outline a framework for considering asynchronous updates of strategies by the agents based on payoff information with potentially time-varying delays. We demonstrate that when all agents update their strategies according to the better-reply rules, the action profile converges to a pure-strategy Nash equilibrium with probability 1 (or almost surely) if the game is a GWAG. As discussed in Chapter 6, we take our attempt at modelling practical scenarios one step further, and consider the setting of erroneous decision-making by the agents due to faulty payoff information.

7.1   Stage and Repeated Games

In this section, we first describe the strategic-form stage game and the infinitely repeated game we adopt for analysis. A word on notation: An inequality between two action profiles is an element-wise inequality.

Finite stage game: Let $\mathcal{P} := \{1, 2, \ldots, n\}$ be the finite set of agents. The pure action or strategy space of agent $i \in \mathcal{P}$ and the joint strategy space of all agents are denoted by $\mathcal{A}_i = \{1, 2, \ldots, A_i\}$ and $\mathcal{A} := \prod_{i \in \mathcal{P}} \mathcal{A}_i$, respectively. We assume that the strategy spaces $\mathcal{A}_i$ are finite for all $i \in \mathcal{P}$. The payoff function of agent i is given by $U_i : \mathcal{A} \to \mathbb{R} := (-\infty, \infty)$. Hence, the strategic-form finite (stage) game is given by $\mathbb{G} := (\mathcal{P}, \{\mathcal{A}_i, i \in \mathcal{P}\}, \{U_i, i \in \mathcal{P}\})$.

A *mixed* strategy of agent $i \in \mathcal{P}$ is a probability distribution $\mathbf{p}_i = (p_i(a_i), a_i \in \mathcal{A}_i) \in \Delta(\mathcal{A}_i)$, where $\Delta(\mathcal{A}_i)$ denotes the probability simplex over $\mathcal{A}_i$; agent $i \in \mathcal{P}$ chooses action $a_i \in \mathcal{A}_i$ with probability $p_i(a_i)$. A *pure* strategy is a special case where the probability distribution is concentrated on a single action.

A *strategy profile* is a collection of strategies, one strategy for each agent. Throughout the remainder of the thesis, a strategy profile refers to a pure strategy profile unless stated otherwise. Furthermore, we find it convenient to differentiate the (mixed) strategy profile from the actions played by the agents (according to the mixed strategy profile). For this reason, we refer to the set of actions played by the agents as an *action profile*.

Given a strategy profile $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathcal{A}$, $\mathbf{a}_{-i}$ denotes the strategy

profile of all the agents other than agent i, i.e., $\mathbf{a}_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$. Similarly, given a subset of agents $I \subsetneq \mathcal{P}$, $\mathbf{a}_I$ is the strategy profile comprising strategies picked by the agents in I. We say that $\mathbf{a}^\star \in \mathcal{A}$ is a pure-strategy Nash equilibrium if, for every agent $i \in \mathcal{P}$,

$$U_i(a_i^\star, \mathbf{a}_{-i}^\star) \geqslant U_i(a_i, \mathbf{a}_{-i}^\star) \text{ for all } a_i \in \mathcal{A}_i \backslash \{a_i^\star\}. \tag{7.1}$$

The pure-strategy Nash equilibrium is *strict* if the inequality in (7.1) is strict for all agents $i \in \mathcal{P}$. Hereafter a Nash equilibrium refers to a pure-strategy Nash equilibrium. We denote the set of Nash equilibria of $\mathbb{G}$ by $\mathcal{A}_{NE}$.

Repeated game: We employ a repeated game setting to model the repeated interactions among agents. In an (infinitely) repeated game, the above stage game $\mathbb{G}$ is repeated at each time $t \in \mathbb{N} := \{1, 2, \ldots\}$. At time t, agent $i \in \mathcal{P}$ chooses its action $a_i(t)$ according to some mixed strategy $\mathbf{p}_i(t) \in \Delta(\mathcal{A}_i)$. We denote the action profile played by the agents at time t by $\mathbf{a}(t) = (a_i(t), i \in \mathcal{P})$. However, unlike in a traditional repeated game, we do not necessarily assume that the payoffs agents receive at time t depend on $\mathbf{a}(t)$.

The agents are allowed to revise their strategies based on (the history of) payoffs in a repeated game. In this chapter we describe two simple better-reply rules (that were first studied in [85]) which ensures almost sure convergence under asynchronous updates of strategies by agents based on their delayed payoff information in a class of games we call generalized weakly acyclic games described in the following section.

7.2 Generalized weakly acyclic games

In order to define the GWAGs [85], we first need to introduce the notion of *generalized better reply paths* (GBRPs). A GBRP is a sequence of action profiles $(\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^L)$ such that, for every $1 \leqslant \ell \leqslant L-1$, there exists a set of agents $I_\ell \subseteq \mathcal{P}$ that satisfies

i. $a_i^\ell \neq a_i^{\ell+1}$ and $U_i(\mathbf{a}^\ell) < U_i(a_i^{\ell+1}, \mathbf{a}_{-i}^\ell)$ for all $i \in I_\ell$, and

ii. $a_i^\ell = a_i^{\ell+1}$ for all $i \in \mathcal{P}\backslash I_\ell$.

These conditions mean that a GBRP consists of *transitions* from one action profile to another action profile, in each of which a set of agents that can achieve a higher payoff via unilateral deviation switch their actions *simultaneously*, while the remaining agents stay with their previous actions. A better reply path (BRP) used to define WAGs [73, 74] is a special case of GBRPs with $|I_\ell| = 1$ for all $\ell = 1, 2, \ldots, L - 1$.

Definition 7.1. *A game is a GWAG if (i) the set of PSNEs is nonempty and (ii) for every action profile $\mathbf{a}^* \in \mathcal{A}\backslash\mathcal{A}_{NE}$, there exists a GBRP $(\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^L)$ such that $\mathbf{a}^1 = \mathbf{a}^*$ and $\mathbf{a}^L \in \mathcal{A}_{NE}$.*

It is clear from the definition that WAGs are special cases of GWAGs where only BRPs are allowed, i.e., only a single agent is allowed to switch or deviate at a time. Due to this constraint, we suspect that there is a large class of games that are GWAGs, but not WAGs as illustrated by the following simple game.
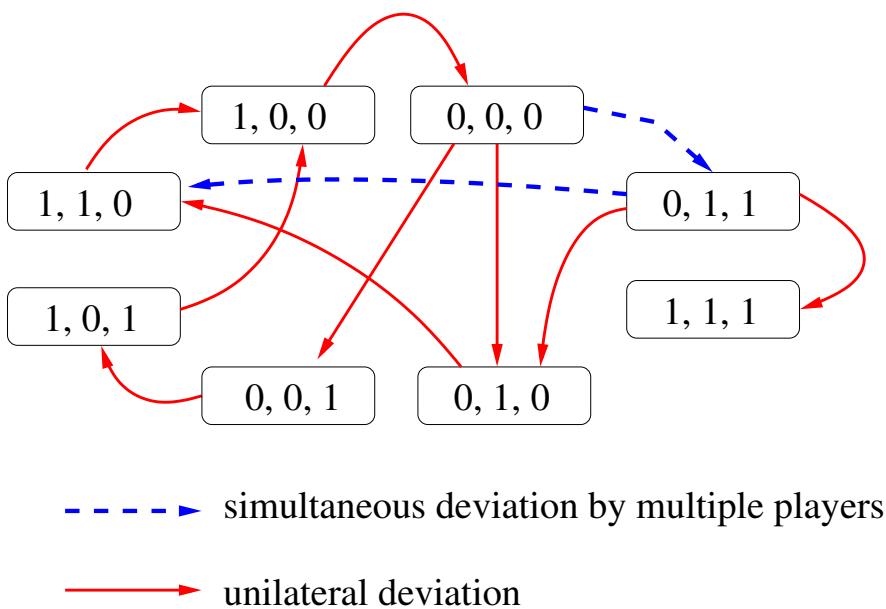
Figure 7.1: An example of a GWAG that is not a WAG. (a) Game in normal form, (b) all possible unilateral deviations and simultaneous deviations

◇ **Example of a GWAG that is not a WAG:** Consider a three-agents game shown in Fig 7.1 in normal form. All three agents have the identical action space $\{0,1\}$. The unique PSNE of the game is $\mathbf{a}^{NE} = (1,1,1)$. The solid red arrows in Fig. 7.1(b) indicate all possible *unilateral* deviations that would improve the payoff of the agent that deviates. From the figure, we can verify that there does not exist a BRP from any action profile that is not the PSNE or $\mathbf{a}^{\dagger} = (0,1,1)$; the PSNE is reachable only from $\mathbf{a}^{\dagger}$, and it is not possible to reach either the PSNE or $\mathbf{a}^{\dagger}$ from any other action profile. For this reason, this game is *not* a WAG.

On the other hand, this game is a GWAG. To see this, note that both agents 2 and 3 have an incentive to deviate at $\mathbf{a}^* = (0,0,0)$. Therefore, from action profile $\mathbf{a}^*$, we can find a GBRP given by $(\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3) = (\mathbf{a}^*, \mathbf{a}^{\dagger}, \mathbf{a}^{NE})$ with $I_1 = \{2,3\}$ and $I_2 = \{1\}$. Since it is possible to reach $\mathbf{a}^*$ from other remaining action profiles, we can construct a GBRP from them as well. Therefore, this example illustrates that the GWAGs give rise to a strictly larger class of games than WAGs.

In Section 7.4, we draw a relationship between a class of better reply algorithms and the class of GWAGs. However, next we state the following lemma which provides an alternate definition of a GWAG.

Lemma 7.1. *A game $\mathbb{G}$ is a GWAG if and only if there exists a potential function $\phi : \mathcal{A} \to \mathbb{R}$ such that, for every $\mathbf{a} \notin \mathcal{A}_{NE}$, there exists a subset of agents $I(\mathbf{a}) \subseteq \mathcal{P}$ and $a_i' \neq a_i$ for all $i \in I(\mathbf{a})$ such that $U_i(a_i', \mathbf{a}_{-i}) > U_i(\mathbf{a})$ and $\phi(\mathbf{a}_{I(\mathbf{a})}', \mathbf{a}_{-I(\mathbf{a})}) > \phi(\mathbf{a})$.*

Proof. *Sufficiency ('if'):* Pick $\mathbf{a}^0 \notin \mathcal{A}_{NE}$. Then, there exists some subset $I(\mathbf{a}^0) \subseteq \mathcal{P}$ with action profile $\mathbf{a}'_{I(\mathbf{a}^0)} \neq \mathbf{a}^0_{I(\mathbf{a}^0)}{}^1$ such that $U_i(a'_i, \mathbf{a}^0_{-i}) > U_i(\mathbf{a}^0)$ for all $i \in I(\mathbf{a}^0)$ and $\phi(\mathbf{a}^1) > \phi(\mathbf{a}^0)$, where $\mathbf{a}^1 = (\mathbf{a}'_{I(\mathbf{a}^0)}, \mathbf{a}^0_{-I(\mathbf{a}^0)})$.

If $\mathbf{a}^1 \notin \mathcal{A}_{NE}$, we can repeat this process and construct a sequence $\mathbf{a}^0, \mathbf{a}^1, \ldots$, until we have an action profile that belongs to $\mathcal{A}_{NE}$. Since $\phi(\mathbf{a}^\ell) < \phi(\mathbf{a}^{\ell+1})$ for all $\ell = 0, 1, \ldots$ and there are finitely many action profiles, this process will terminate after a finite number of iterations and the final action profile, say $\mathbf{a}^M$, must be a PSNE. It is an easy exercise to verify that $(\mathbf{a}^0, \mathbf{a}^1, \ldots, \mathbf{a}^M)$ is a GBRP from its construction. Since we can find a GBRP that leads to a PSNE for any non-PSNE action profile $\mathbf{a}^0$, the game is a GWAG.

Necessity ('only if'): Define the length of a GBRP to be the number of action profiles in the sequence. For each $\mathbf{a} \notin \mathcal{A}_{NE}$, find a shortest GBRP to a PSNE. If we represent all PSNEs in $\mathcal{A}_{NE}$ using a single node $\mathbf{a}^{NE}$, then these shortest GBRPs from all non-PSNE action profiles to $\mathbf{a}^{NE}$ give rise to a spanning tree rooted at $\mathbf{a}^{NE}$, which we denote by $\mathcal{T}$.

Suppose that we assign a value ϕ_0 to all PSNEs (represented by $\mathbf{a}^{NE}$ in $\mathcal{T}$). Let $\phi : \mathcal{A} \to \mathbb{R}$ be a potential function, where $\phi(\mathbf{a}) = \phi_0 - d(\mathbf{a})$ and $d(\mathbf{a})$ is the hop distance of $\mathbf{a}$ to $\mathbf{a}^{NE}$ in $\mathcal{T}$. Then, from the construction of the spanning tree $\mathcal{T}$, it is clear that, for any action profile $\mathbf{a} \notin \mathcal{A}_{NE}$, there exists a subset $I \subseteq \mathcal{P}$ and $\mathbf{a}'_I \neq \mathbf{a}_I$ such that $U_i(a'_i, \mathbf{a}_{-i}) > U_i(\mathbf{a})$ for all $i \in I$ and $\phi(\mathbf{a}'_I, \mathbf{a}_{-I}) > \phi(\mathbf{a})$. ∎

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[1]The inequality between two action profiles is an element-wise inequality throughout the thesis.

7.3 Proposed Update Rules - Generalized Better Reply Rules

Payoff information for updates – Let $\mathcal{S} = \{B, I, W\}$ and, for each agent $i \in \mathcal{P}$, define its *classification* mapping $\mathcal{C}_i : \mathcal{A}_i \times \mathcal{A} \to \mathcal{S}$, where

$$
\mathcal{C}_i(a_i', \mathbf{a}) = \begin{cases} B & \text{if } U_i(a_i', \mathbf{a}_{-i}) > U_i(\mathbf{a}), \\[2mm] I & \text{if } U_i(a_i', \mathbf{a}_{-i}) = U_i(\mathbf{a}), \\[2mm] W & \text{otherwise.} \end{cases}
$$

The payoff information on which agent i bases its decision lies in $\mathcal{I}_i = \mathcal{S}^{A_i}$. The interpretation is that, if the strategy profile generating the payoff information is $\mathbf{a} \in \mathcal{A}$, the payoff information agent i has available is $(\mathcal{C}_i(a_i', \mathbf{a}), a_i' \in \mathcal{A}_i)$. To simplify notation, for fixed $\mathbf{a} \in \mathcal{A}$, we denote the payoff information vector $(\mathcal{C}_i(a_i', \mathbf{a}), a_i' \in \mathcal{A}_i)$ by $\mathcal{C}_i(\mathbf{a})$.

Remark: Clearly, the payoff information we assume is less restrictive than the payoff vector $(U_i(a_i'; \mathbf{a}_{-i}), a_i' \in \mathcal{A}_i)$ assumed in [57, 59], but is more stringent than that of the completely uncoupled payoff-based learning rules, e.g., [79, 80, 87]. We argue that in some scenarios, even though agents cannot determine the exact payoffs for the strategies not played, they might be able to determine which strategies could have yielded higher (or smaller) payoffs than the previously chosen strategy.

For example, consider an interdependent security game in which each agent chooses a combination of security measures from a set of available security measures for its own protection [72], e.g., cybersecurity measures including incoming-traffic monitoring and intrusion detection systems. In this case, based on the number of successful attacks a agent suffers, the resulting losses, as well as the costs of various

security measures, it might be able to determine if another combination of security measures would have achieved a lower overall cost.

We find it convenient to define following mappings we use in the subsequent sections: For each agent $i \in \mathcal{P}$, let $BR_i : \mathcal{A} \to 2^{\mathcal{A}_i}$ and $IR_i : \mathcal{A} \to 2^{\mathcal{A}_i}$, where

$$BR_i(\mathbf{a}) = \{a_i' \in \mathcal{A}_i \mid \mathcal{C}_i(a_i', \mathbf{a}) = B\} \text{ and}$$

$$IR_i(\mathbf{a}) = \{a_i' \in \mathcal{A}_i \mid \mathcal{C}_i(a_i', \mathbf{a}) = I\}.$$

Clearly, $BR_i(\mathbf{a})$ is the set of better replies for agent i given the strategy profile of the other agents $\mathbf{a}_{-i}$.

In Sections 7.4 and 7.5, we assume that the payoff information available to the agents, i.e., $\mathcal{C}_i(\tilde{\mathbf{a}}^i(t))$, is accurate. In other words, the agents can correctly determine $BR_i(\tilde{\mathbf{a}}^i(t))$ and $IR_i(\tilde{\mathbf{a}}^i(t))$. In Section 7.6, we relax this assumption and consider the scenario where agents cannot perfectly determine these sets and, as a result, make mistakes.

In this section, we describe the better-reply rules we study. Recall that $\mathcal{T}^i = \{T_k^i, \ k \in \mathbb{N}\} \subseteq \mathbb{N}$ is the update time sequence of agent i.

At time $t = 1$, the agents choose their initial action profile $\mathbf{a}(1)$ according to some distribution $\mathcal{G}$ over $\mathcal{A}$. Subsequently, agents revise their strategies according to the following rule: Fix $\underline{\epsilon} > 0$ and $\boldsymbol{\beta}_i : \mathcal{S}^{A_i} \to \Delta(\mathcal{A}_i)$, $i \in \mathcal{P}$. The mappings $\boldsymbol{\beta}_i(\mathbf{c}_i) = (\beta_i(a_i; \mathbf{c}_i), a_i \in \mathcal{A}_i)$ are used to determine the mixed strategy to be employed when there is a better reply.

GBRR-I Rule

- if $BR_i(\tilde{\mathbf{a}}^i(t)) = \varnothing$

    - choose $a_i(t) = a_i(t-1)$;

- else (i.e., $BR_i(\tilde{\mathbf{a}}^i(t)) \neq \varnothing$)

    - choose $a_i(t) = a_i$ with probability $\beta_i(a_i; \mathcal{C}_i(\tilde{\mathbf{a}}^i(t))) \geqslant \underline{\epsilon}$ for each $a_i \in BR_i(\tilde{\mathbf{a}}^i(t))$, and $a_i(t) = a_i(t-1)$ with the remaining probability;

Note that when there is no better reply, i.e., $BR_i(\tilde{\mathbf{a}}^i(t)) = \varnothing$, agent i plays the same action $a_i(t-1)$ at time t. On the other hand, if there exists at least one better reply, agent i (a) picks each of the better replies with probability at least $\underline{\epsilon}$ or (b) continues to play the same action $a_i(t-1)$ with probability $1 - \sum_{a_i \in BR_i(\tilde{\mathbf{a}}^i(t))} \beta(a_i; \mathcal{C}_i(\tilde{\mathbf{a}}^i(t)))$. Thus, it plays a mixed strategy.

Throughout the chapter, we assume

$$\max_{i \in \mathcal{P}} \left(\max_{\mathbf{a} \in \mathcal{A}} \sum_{a'_i \in BR_i(\mathbf{a})} \beta_i(a'_i; \mathcal{C}_i(\mathbf{a})) \right) =: \mu < 1.$$

This guarantees that, even when the set of better replies is nonempty, the agent continues to play the same action $a_i(t-1)$ with strictly positive probability.

In the GBRR-I rule, the same action $a_i(t-1)$ is chosen at time t whenever $BR_i(\tilde{\mathbf{a}}^i(t)) = \varnothing$. In the following slightly modified rule, which we call the GBRR-II rule, we relax this constraint and allow additional exploration among best responses. Assume $\varepsilon^\star > 0$ and let $\boldsymbol{\vartheta}_i : \mathcal{S}^{A_i} \to \Delta(\mathcal{A}_i)$, $i \in \mathcal{P}$. The role of the mappings $\boldsymbol{\vartheta}_i(\mathbf{c}_i) = (\vartheta_i(a_i; \mathbf{c}_i), a_i \in \mathcal{A}_i), i \in \mathcal{P}$, is similar to that of $\boldsymbol{\beta}_i, i \in \mathcal{P}$, and they determine the mixed strategies to be played by the agents when there are multiple

best responses.

GBRR-II Rule: Set $\theta_i = $ TRUE at time $t = 1$.

- if $BR_i(\tilde{\mathbf{a}}^i(t)) = \varnothing$

    - if $|IR_i(\tilde{\mathbf{a}}^i(t))| = 1$ (i.e., $IR_i(\tilde{\mathbf{a}}^i(t)) = \{a_i(t-1)\}$)

        * choose $a_i(t) = a_i(t-1)$;

    - else (i.e., $|IR_i(\tilde{\mathbf{a}}^i(t))| > 1$)

        * if $\theta_i = $ TRUE

            ◇ choose each action $a_i \in IR_i(\tilde{\mathbf{a}}^i(t))$ with probability $\vartheta(a_i; \mathcal{C}_i(\tilde{\mathbf{a}}^i(t))) \geqslant$

            $\varepsilon^\star$ (with $\sum_{a_i \in IR_i(\tilde{\mathbf{a}}^i(t))} \vartheta(a_i; \mathcal{C}_i(\tilde{\mathbf{a}}^i(t))) = 1$);   (*)

            ◇ set $\theta_i = $ FALSE;

        * else (i.e., $\theta_i = $ FALSE)

            ◇ choose $a_i(t) = a_i(t-1)$;

- else (i.e., $BR_i(\tilde{\mathbf{a}}^i(t)) \neq \varnothing$)

    - choose $a_i(t) = a_i$ with probability $\beta(a_i; \mathcal{C}_i(\tilde{\mathbf{a}}^i(t))) \geqslant \underline{\epsilon}$ for each $a_i \in$ $BR_i(\tilde{\mathbf{a}}^i(t))$, and $a_i(t) = a_i(t-1)$ with the remaining probability;

    - set $\theta_i = $ TRUE;

The key difference between the two update rules is that GBRR-II permits the agent to explore among the best responses in some cases even when there is no better reply (case (*)).

These rules are quite intuitive; an agent picks every better reply with positive probability (without knowing the exact payoffs achieved by those better replies). But, if no better reply is found, the agent either keeps playing the same action (GBRR-I) or selects one of the best responses (GBRR-II).

Modeling mixed strategies in GBRRs – For the completeness of exposition and analysis, we describe how we model mixed strategies exercised by the agents in GBRRs: For every $i \in \mathcal{P}$, define $\mathbf{u}^i := (u_t^i, t \in \mathbb{N}^+)$ and $\mathbf{v}^i := (v_t^i, t \in \mathbb{N}^+)$ to be two sequences of independent uniform random variables over $(0, 1]$. We assume that $(\mathbf{u}^i, \mathbf{v}^i), i \in \mathcal{P}$, are mutually independent.

Suppose that agent i revises its strategy at time $t \in \mathcal{T}^i$ and $BR_i(\tilde{\mathbf{a}}^i(t)) \neq \varnothing$. agent i first orders the better replies, for example, by increasing index. We denote the *ordered* set of better replies by $OBR_i(\tilde{\mathbf{a}}^i(t)) = \{a_i^{(1)}, a_i^{(2)}, \ldots, a_i^{(m^*)}\}$, where m^* is the number of better replies. It then chooses

$$
a_i(t) = \begin{cases} a_i^{(l)} & \text{if } u_t^i \in (\Lambda(l-1), \Lambda(l)), \ l = 1, 2, \ldots, m^*, \\ a_i(t-1) & \text{otherwise,} \end{cases}
$$

where $\Lambda(l) := \sum_{l'=1}^{l} \beta\big(a_i^{(l')}; \mathcal{C}_i(\tilde{\mathbf{a}}^i(t))\big), \ l = 1, \ldots, m^*$.

Under GBRR-II, agent i's mixed strategy in case (*) is handled in an analogous manner. agent i first orders the best responses in $IR_i(\tilde{\mathbf{a}}^i(t))$, which we denote by $OIR_i(\tilde{\mathbf{a}}^i(t)) = \{a_i^{(1)}, a_i^{(2)}, \ldots, a_i^{(m^\dagger)}\}$, where $m^\dagger$ is the number of best responses. Then, it picks $a_i(t) = a_i^{(l)}, \ l = 1, 2, \ldots, m^\dagger$, if $v_t^i \in (\Upsilon(l-1), \Upsilon(l)]$, where

$$
\Upsilon(l) := \sum_{l'=1}^{l} \vartheta\big(a_i^{(l')}; \mathcal{C}_i(\tilde{\mathbf{a}}^i(t))\big), \ l = 1, \ldots, m^\dagger.
$$

We would like to point out one observation that we make use of in our analy-

sis. Because mixed strategies are implemented using mutually independent uniform random variables, given a mixed strategy selected by agent i at some time t, the action picked by the agent is conditionally independent of the past history.

Convergent sequences of strategy profiles – For each $t \in \mathbb{N}$, let $\mathcal{A}^t = \prod_{t'=1}^{t} \mathcal{A}$. Similarly, $\mathcal{A}^\infty = \prod_{t'=1}^{\infty} \mathcal{A}$. When the agents revise their strategies according to fixed update rules described above, together with the revision processes $\mathbb{N}_i$, $i \in \mathcal{P}$, and (the distribution of) initial action profile $\mathbf{a}(1)$, the update rules $\mathbf{S} = (S_i, i \in \mathcal{P})$ induces a distribution over $\mathcal{A}^\infty$ (with a suitable σ-field $\tilde{\mathcal{F}}$ on $\mathcal{A}^\infty$); let $\vec{\mathbf{a}} = (\mathbf{a}(t), t \in \mathbb{N})$ be the sequence of action profiles played by the agents.

We first define some subsets of $\mathcal{A}^\infty$, which are of interest to us. For each $t \in \mathbb{N}$, let

$$\mathcal{A}_t^\infty := \{\bar{\mathbf{a}} \in \mathcal{A}^\infty \mid \bar{\mathbf{a}}(t') = \mathbf{a}^\star \ \forall \ t' \geqslant t \ \text{ for some } \mathbf{a}^\star \in \mathcal{A}_{NE}\}.$$

Note that these sets are increasing in t, i.e., $\mathcal{A}_t^\infty \subsetneq \mathcal{A}_{t+1}^\infty$ for all $t \in \mathbb{N}$. In addition, define

$$\mathcal{A}_\star^\infty := \bigcup_{t \in \mathbb{N}} \mathcal{A}_t^\infty$$
$$= \{\bar{\mathbf{a}} \in \mathcal{A}^\infty \mid \bar{\mathbf{a}}(t') = \mathbf{a}^\star \ \forall \ t' \geqslant T^* \ \text{ for some } \mathbf{a}^\star \in \mathcal{A}_{NE} \text{ and } T^* < \infty\}.$$

Simply put, $\mathcal{A}_t^\infty$ (resp. $\mathcal{A}_\star^\infty$) is the set of strategy profile sequences that converge to a Nash equilibrium by time t (resp. at some finite time).

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7.4 Convergence under GBRR – Synchronous Update Case

In this section, we start with a simpler scenario in which every agent receives the payoff information $\mathcal{C}_i(\mathbf{a}(t-1))$ at each $t \in \mathbb{N}^+$, and updates its strategy according to a GBRR rule. In other words, $\mathcal{T}^i = \mathbb{N}$ for all $i \in \mathcal{P}$ and $\tilde{\mathbf{a}}^i(t) = \mathbf{a}(t-1)$ for all $i \in \mathcal{P}$ and $t \in \mathbb{N}^+$. We will consider more general settings in the following section.

Let $\vec{\mathbf{a}} = (\mathbf{a}(t), t \in \mathbb{N})$ be the sequence of action profiles selected by the agents using a GBRR rule (either GBRR-I or GBRR-II). The following theorem guarantees the *almost sure convergence* (or *convergence with probability 1*) of the played action profiles $\mathbf{a}(t)$, $t \in \mathbb{N}$, to a Nash equilibrium as $t \to \infty$ [58]. Its proof can be found in [85] and is omitted here. Instead we will prove the convergence results under the asynchronous update scenario.

Theorem 7.2. *Suppose that the game $\mathbb{G}$ is generalized weakly acyclic. Then, for any fixed initial distribution $\boldsymbol{\nu} \in \Delta(\mathcal{A})$, we have $\mathbb{P}\left[\vec{\mathbf{a}} \in \mathcal{A}_\star^\infty\right] = 1$.*

In addition to the almost sure convergence of action profiles, the following theorem states that the probability that the action profile has not converged to a Nash equilibrium by time $t \in \mathbb{N}$ decays geometrically with t.

Theorem 7.3. *Suppose that the game $\mathbb{G}$ is generalized weakly acyclic. Then, there exist $C < \infty$ and $\zeta \in (0,1)$ such that, regardless of the initial distribution $\boldsymbol{\nu} \in \Delta(\mathcal{A})$, $\mathbb{P}\left[\vec{\mathbf{a}} \notin \mathcal{A}_t^\infty\right] \leqslant \min(1, C \cdot \zeta^t)$ for all $t \in \mathbb{N}$.*

Theorem 7.3 implies that the expected convergence time $\mathbb{E}\left[N_{conv}\right]$ is finite, where $N_{conv} = \inf\{t \in \mathbb{N} \mid \vec{\mathbf{a}} \in \mathcal{A}_t^\infty\}$ is the time it takes for the agents to reach a Nash

equilibrium and remain there afterward.

Before we proceed, let us comment on the convergence rate, which is captured by ζ. Recall from Section 7.1 that, for $\mathbf{a} \notin \mathcal{A}_{NE}$, the length of a shortest generalized better reply path $h^*(\mathbf{a})$ to a Nash equilibrium is $L^\star(\mathbf{a})$. The parameter ζ in Theorem 7.3 is shaped by $\max_{\mathbf{a} \in \mathcal{A} \setminus \mathcal{A}_{NE}} L^\star(\mathbf{a})$. A possible interpretation of this observation is as follows: One can view the length of a generalized better reply path from strategy profile $\mathbf{a}$ to a Nash equilibrium as a measure of how "difficult" it is to reach the Nash equilibrium from the action profile following the generalized better reply path, and a shortest generalized better reply path offers the "easiest" path to a Nash equilibrium. Hence, the lower bound on convergence rate captured by ζ is influenced by the "most difficult" strategy profile from which the agents need to find the "easiest" path to a Nash equilibrium.

Theorem 7.4. *Suppose that the game $\mathbb{G}$ is not generalized weakly acyclic. Then, there exists at least one strategy profile $\mathbf{a}' \notin \mathcal{A}_{NE}$ such that, under the GBRR-I rule,*

$$\mathbb{P}\left[\mathbf{a}(t) \notin \mathcal{A}_{NE} \text{ for all } t \in \mathbb{N} \mid \mathbf{a}(1) = \mathbf{a}'\right] = 1.$$

Proof. Since the game is assumed to be not generalized weakly acyclic, there exists a strategy profile $\mathbf{a}^+ \notin \mathcal{A}_{NE}$ with no generalized better reply path to a Nash equilibrium. From the description of the GBRR-I rule, it is clear that any admissible transition from a strategy profile $\mathbf{a}^1$ to another strategy profile $\mathbf{a}^2$ ($\mathbf{a}^1 \neq \mathbf{a}^2$) constitutes a generalized better reply path $(\mathbf{a}^1, \mathbf{a}^2)$. Therefore, if $\mathbf{a}(1) = \mathbf{a}^+$, it is not possible to reach any Nash equilibrium under the GBRR-I rule. ∎

Note that

$$\mathbb{P}\left[\vec{\mathbf{a}} \notin \mathcal{A}_\star^\infty \mid \mathbf{a}(1) = \mathbf{a}'\right] \geqslant \mathbb{P}\left[\mathbf{a}(t) \notin \mathcal{A}_{NE} \ \forall \ t \in \mathbb{N} \mid \mathbf{a}(1) = \mathbf{a}'\right]$$

and, hence,

$$\mathbb{P}\left[\vec{\mathbf{a}} \notin \mathcal{A}_\star^\infty \mid \mathbf{a}(1) = \mathbf{a}'\right] = 1.$$

To rephrase it, conditional on $\{\mathbf{a}(1) = \mathbf{a}'\}$, the probability that $\mathbf{a}(t), t \in \mathbb{N}$, converge to a Nash equilibrium is equal to zero.

Theorems 7.2 and 7.4 together state that, if $\boldsymbol{\nu}(\mathbf{a}') > 0$ for all $\mathbf{a}' \in \mathcal{A}$, i.e., every strategy profile is selected with positive probability at time $t = 1$, the GBRR-I rule guarantees the almost sure convergence of action profiles to a Nash equilibrium *if and only if* the game is generalized weakly acyclic.

A similar, but somewhat weaker result holds for the GBRR-II rule: We call a sequence of strategy profiles $(\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^L)$ a *weak* generalized better reply path if, for every $\ell \in \{1, \ldots, L-1\}$, there exists a subset of agents $I'_\ell \subseteq \mathcal{P}$ such that

i. $a_i^\ell \neq a_i^{\ell+1}$ and $U_i(\mathbf{a}^\ell) \leqslant U_i(a_i^{\ell+1}, \mathbf{a}_{-i}^\ell)$ for all $i \in I'_\ell$, and

ii. $a_i^\ell = a_i^{\ell+1}$ for all $i \in \mathcal{P} \backslash I'_\ell$.

A game is said to be generalized weakly acyclic$^+$ if, for every $\mathbf{a}' \notin \mathcal{A}_{NE}$, there exists a weak generalized better reply path $(\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^L)$ with $\mathbf{a}^1 = \mathbf{a}'$ and $\mathbf{a}^L \in \mathcal{A}_{NE}$. One can show that if the game is not generalized weakly acyclic$^+$, there exists at least one strategy profile $\mathbf{a}^\dagger \notin \mathcal{A}_{NE}$ such that $\mathbb{P}\left[\mathbf{a}(t) \notin \mathcal{A}_{NE} \ \forall \ t \in \mathbb{N} \mid \mathbf{a}(1) = \mathbf{a}^\dagger\right] = 1$ when the agents employ the GBRR-II rule for revising strategies.

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Remark: The class of generalized weakly acyclic$^+$ games is strictly larger than that of generalized weakly acyclic games and there are generalized weakly acyclic$^+$ games that are not generalized weakly acyclic, for which GBRR-II guarantees the almost sure convergence of action profiles to a Nash equilibrium. In this sense, GBRR-II possesses a somewhat stronger convergence property than GBRR-I.

7.5 Convergence under GBRR – Asynchronous Update Case

In the previous section, we assumed that all agents update their strategies at every $t \in \mathbb{N}^+$ based on the action profile played at time $t - 1$. As already indicated in Chapter 6, in some settings this assumption might not hold. In this section, we extend the model in two directions: We allow (i) *asynchronous* updates of strategies by the agents and (ii) *time-varying delays* experienced by payoff information available to the agents. Note that the latter implies that the agents might base their decisions on *outdated* payoff information at times. We mention that there are some related studies that examine the effects of delays in evolutionary games, e.g., [94] and asynchronous distributed computation of average values, e.g., [48,93].

7.5.1 Model with asynchronous updates and delays

There are many different ways in which one can capture and model asynchronous operation and delays in payoff information. In this subsection, we describe the model we assume for our study. For concreteness, we explain it using an example.
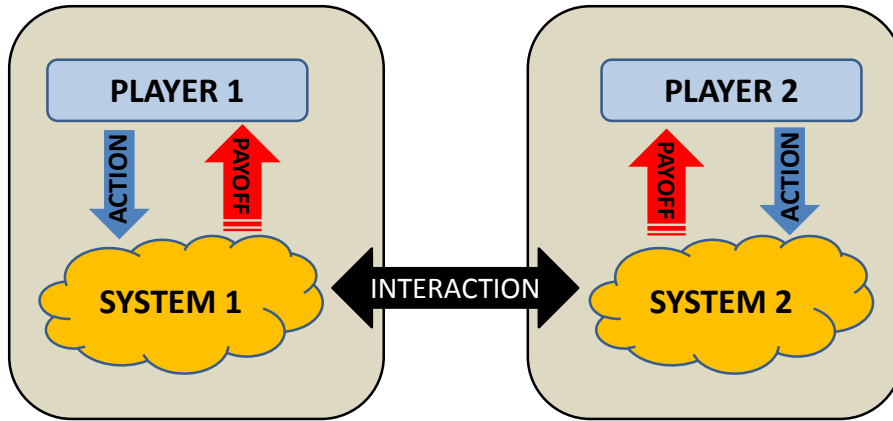
Figure 7.2: Example – two interacting systems controlled by separate agents.

Consider the system in Figure 7.2. There are two *interacting* systems, each of which is controlled by a agent (controller); for each agent i, $i = 1, 2$, a corresponding system i generates payoffs for agent i. Obviously, the case in which a single system provides payoffs to all agents is a special case of this more general setting.

In this scenario, depending on the structure of the system, new strategies adopted by agents might experience delays before taking effect at the systems. In addition, the payoffs to agents (e.g., the current states of the systems) may not be observable to the agents instantaneously. In other words, they can only observe delayed past payoffs.

To model this scenario, we introduce two types of delays – (i) *forward* delays and (ii) *feedback* delays. A forward delay refers to the amount of time that elapses before a agent's strategy goes into effect at a system after it is adopted. On the other hand, a feedback delay is the amount of time it takes for payoff information to become available to a agent after it is generated. These are illustrated in Figure 7.3 for the example in Figure 7.2. In the figure, after agent i, $i = 1, 2$, updates its
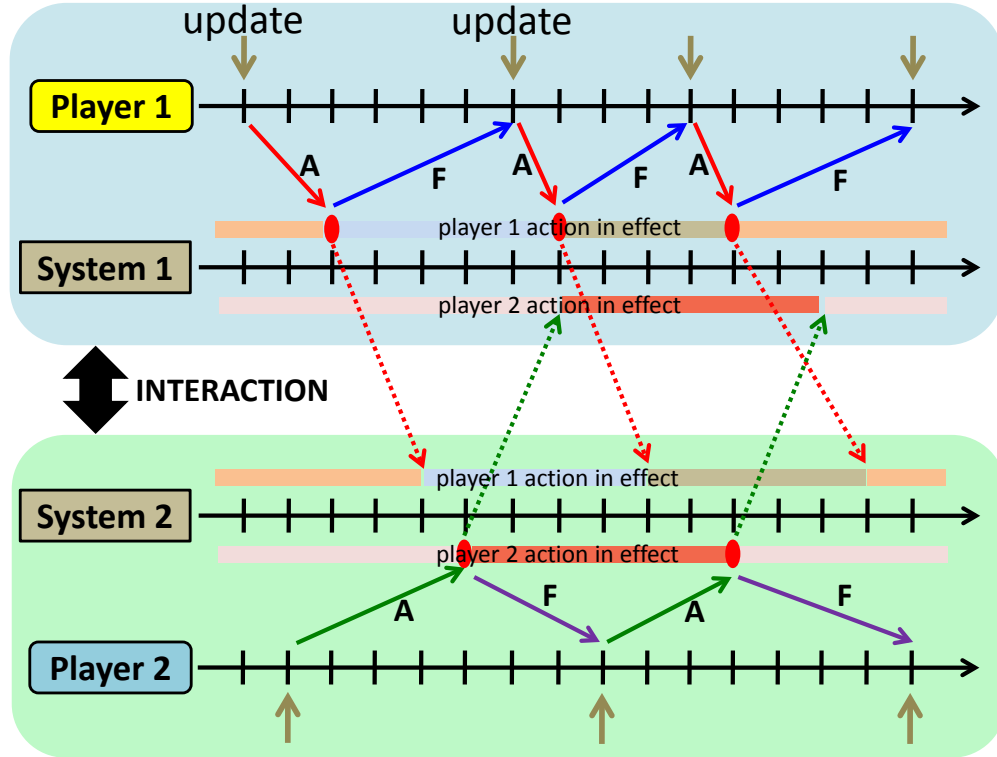
Figure 7.3: Forward and feedback delays in the system.

strategy, the forward delay experienced before system i sees the new strategy is shown as a (red or green) solid arrow (with label 'A'), whereas the *additional* forward delay to the other system appears as a dotted arrow. The feedback delay experienced by payoff information generated by a system before the corresponding agent observes it is shown as a (blue or purple) solid arrow (with label 'F').

We shall model these delays using random variables as follows. First, recall that the agents choose their initial action profile $\mathbf{a}(1) = (a_i(1), i \in \mathcal{P})$ according to some distribution $\boldsymbol{\nu}$ and the sequence of agent i's update times, namely $\mathcal{T}^i$, is determined by a discrete-time counting process. Denote the inter-update times of agent i by $U_k^i := T_{k+1}^i - T_k^i$, $k \in \mathbb{N}$.

Each inter-update time U_k^i is given by a sum of two random variables – X_k^i and Y_k^i. The random variable X_k^i models the forward delay experienced by the kth action of agent i chosen at time T_k^i, i.e., $a_i(T_k^i)$; we assume that $a(T_k^i)$ goes into effect at system i at time $T_k^i + X_k^i =: R_k^i$ after incurring a forward delay of X_k^i, at which point system i generates payoff information for agent i. This payoff information undergoes a delay of Y_k^i and is observed by agent i at the next update time $T_{k+1}^i (= R_k^i + Y_k^i)$. Throughout this section, we assume that $X_k^i \geqslant 0$ and $Y_k^i \geqslant 1$ for all $i \in \mathcal{P}$ and $k \in \mathbb{N}$. For notational simplicity, we denote the pair (X_k^i, Y_k^i) by Z_k^i and the sequences $\{Z_k^i, k \in \mathbb{N}\}$ and $\{R_k^i, \ k \in \mathbb{N}\}$ by $\mathcal{Z}^i$ and $\mathcal{R}^i$, respectively.

Similarly, the actions chosen by agent i at time T_k^i might not be seen by another system j, $j \neq i$, immediately; instead, it takes effect at system j at time $T_k^i + V_k^{i,j} =: R_k^{i,j}$, where $V_k^{i,j}$ is the forward delay that action $a_i(T_k^i)$ experiences before influencing system j. In the example shown in Figure 7.2, once the new action a_i^{new} of agent i starts affecting system i, through the interaction between the systems, the other system will also be affected by a_i^{new} possibly after some additional delay. In this case, we will have $X_k^i \leqslant V_k^{i,j}$. For every $i, j \in \mathcal{P}, i \neq j$, let $\mathcal{V}^{i,j} := \{V_k^{i,j}, k \in \mathbb{N}\}$ and $\mathcal{R}^{i,j} := \{R_k^{i,j}, k \in \mathbb{N}\}$.

In order to complete the model, we need to take care of initial conditions. Because the actions chosen by the agents at time $t = 1$ might not take effect at the systems right away, we impose the following initial conditions: For each system i, we assume that there is some strategy profile $\bar{\mathbf{a}}^i = (\bar{a}_j^i, j \in \mathcal{P}) \in \mathcal{A}$ that is in place at the system. In other words, for each $i \in \mathcal{P}$, system i sees action $\bar{a}_i^i$ from agent i till $R_1^i = 1 + X_1^i$ and $\bar{a}_j^i$ from agent j, $j \neq i$, till $R_1^{j,i} = 1 + V_1^{j,i}$. Although we can

easily handle scenarios in which these strategy profiles are random, here we assume that they are deterministic for the ease of exposition.

Remark: The settings studied in the previous section can be viewed as a special case of the general settings described in this section with $X_k^i = V_k^{i,j} = 0$ and $Y_k^i = 1$ (so that $U_k^i = X_k^i + Y_k^i = 1$) for all $i, j \in \mathcal{P}$ $(i \neq j)$ and $k \in \mathbb{N}$. To rephrase it, not only do the agents update strategies synchronously, there is no forward delay (and the strategies take effect immediately) and the feedback delay is always one.

In order to make progress, we assume that $\mathbb{N}_i$, $\mathcal{Z}^i$, and $\mathcal{V}^{i,j}$ satisfy the following assumptions.

A1 The random variables $\{\mathcal{Z}^i, \mathcal{V}^{i,j}, j \neq i\}$ for different agents $i \in \mathcal{P}$ are mutually independent.

A2. For every $i \in \mathcal{P}$, $\mathbb{P}\left[U_k^i < \infty, V_k^{i,j} < \infty \text{ for all } j \neq i \text{ and } k \in \mathbb{N}\right] = 1$.

A3. $\mathbb{P}\left[R_k^{i,j} < R_{k+1}^{i,j} \text{ for all } k \in \mathbb{N}\right] = 1$ for all $i, j \in \mathcal{P}$, $j \neq i$.

A4. Let $\mathbf{H}_k^i := \left((Z_\ell^i, V_\ell^{i,j}, j \neq i), 1 \leq \ell \leq k\right)$. There exist $\eta > 0$ and $\Delta_\eta < \infty$ such that, for all $i \in \mathcal{P}$, $k \in \mathbb{N}$ and $\varphi \in \mathbb{Z}_+$,

$$\mathbb{P}\left[U_{k+1}^i \leq \Delta_\eta + \varphi, V_{k+1}^{i,j} \leq \Delta_\eta + \varphi \text{ for all } j \neq i \mid U_{k+1}^i \geq \varphi, \mathbf{H}_k^i\right] \geq \eta.$$

A5. There exist $\delta > 0$ and $\Delta_\delta < \Delta_\eta$ such that, for all $i \in \mathcal{P}$ and $k \in \mathbb{N}$,

$$\mathbb{P}\left[X_{k+1}^i \leq \Delta_\delta \mid \mathbf{H}_k^i\right] \geq \delta \qquad (7.2)$$

and

$$\mathbb{P}\left[V_{k+1}^{i,j} \geq \Delta_\delta \text{ for all } j \neq i \mid \mathbf{H}_k^i\right] \geq \delta. \qquad (7.3)$$

A6. For all $0 \leqslant u \leqslant u' < \infty$,

$$\mathbb{P}\left[X_{k+1}^i \leqslant x \mid U_{k+1}^i \leqslant u, \mathbf{H}_k^i\right] \geqslant \mathbb{P}\left[X_{k+1}^i \leqslant x \mid U_{k+1}^i \leqslant u', \mathbf{H}_k^i\right]$$

$$\text{for all } x \in \mathbb{Z}_+,$$

i.e., the conditional distribution of X_{k+1}^i is stochastically larger given $\{U_{k+1}^i \leqslant u'\}$ than $\{U_{k+1}^i \leqslant u\}$ [91].

Assumption A2 ensures that every agent updates infinitely many times with probability 1. Assumption A3 guarantees that agents see the effects of new actions in the same order they were adopted. Assumption A4 essentially implies that the distributions of random variables $\{X_k^i, V_k^{i,j}, j \neq i\}$, $k \in \mathbb{N}$, do not have a heavy tail. Assumption A5 means that, with positive probability, agent i's action affects its own payoff no later than those of other agents.

Although these assumptions are technical in nature, we feel that they are not restrictive and are likely to hold in many cases of practical interest. For example, when the strategy update times are given by suitable delayed renewal processes with constant forward delays (with $X_k^i \leqslant V_k^{i,j}$ for all $j \neq i$), the above assumptions hold.

7.5.2 Convergence of action profiles with asynchronous updates and delays

Let $\vec{\mathbf{a}} = (\mathbf{a}(t), t \in \mathbb{N})$ be the sequence of action profiles played by the agents using a GBRR rule. The following two theorems suggest that when the game is generalized weakly acyclic and a GBRR rule is employed for updating strategies, neither

payoff information delays nor asynchronous updates of strategies among the agents prevent their action profiles from reaching a Nash equilibrium with probability 1, under Assumptions A1 through A6.

Theorem 7.5. *Suppose that the game $\mathbb{G}$ is generalized weakly acyclic and that Assumptions A1 through A6 hold. Then, for any initial distribution $\boldsymbol{\nu} \in \Delta(\mathcal{A})$, $\mathbb{P}\left[\vec{\mathbf{a}} \in \mathcal{A}_\star^\infty\right] = 1$.*

Proof. A proof of the theorem is provided in Section 7.7. ∎

Theorem 7.6. *Suppose that the game $\mathbb{G}$ is generalized weakly acyclic and that Assumptions A1 through A6 are in place. Then, there exist $C' < \infty$ and $\bar{\zeta} \in (0, 1)$ such that, for any initial distribution $\boldsymbol{\nu} \in \Delta(\mathcal{A})$, $\mathbb{P}\left[\vec{\mathbf{a}} \notin \mathcal{A}_t^\infty\right] \leqslant \min(1, C' \cdot \bar{\zeta}^t)$ for all $t \in \mathbb{N}$.*

Theorem 7.6 follows directly from Corollary 2 in the proof of Theorem 7.5 as explained in Section 7.7.

Remark: Theorem 7.5 and 7.6 are established under Assumptions A1 through A6 for generalized weakly acyclic games. However, as we explain in Section 7.7, for weakly acyclic games, they hold under Assumptions A1 through A4.

7.6 Case with Erroneous Payoff Information

In the previous sections, we assumed that accurate payoff information $\mathcal{C}_i(\tilde{\mathbf{a}}^i(t))$ is available to the agents for updates. Although this is a reasonable assumption in some cases, there are other scenarios, in which agents might not be able to reliably

determine better replies. To address this partially, we replace it with a somewhat weaker assumption that the agents are able to make more reliable determinations over time, for instance, by observing its own payoff history.

Clearly, when the agents make mistakes with the determinations of $\mathcal{C}_i(\tilde{\mathbf{a}}^i(t))$, it is unlikely that the strategy profile will converge to a Nash equilibrium and remain there forever with probability 1. Instead, we will show that, under some conditions, the probability that the agents play a Nash equilibrium at time $t \in \mathbb{N}$ goes to one over time. Although we suspect that our result can be generalized to settings with asynchronous updates studied in the previous section under suitable assumptions, here we only consider settings with synchronous updates studied in Section 7.4.

7.6.1 Preliminaries

Modeling unreliable payoff information $-$ We assume that, for all $t-1 \in \mathbb{N}$, given the action profile $\mathbf{a}(t-1)$, (a) the events of having false payoff information at time t are conditionally independent of the past and (b) their probabilities depend only on the action profile $\mathbf{a}(t-1)$. This is modeled as follows: First, the probabilities of erroneous classification of actions are given by mappings $\mathbf{q}_t^i : \mathcal{A} \times \mathcal{A}_i \to \Delta(\mathcal{S})$, $t \in \mathbb{N}^+$. The interpretation is that (i) $\mathbf{q}_t^i(\mathbf{a}, a_i') := (q_t^i(\varsigma; \mathbf{a}, a_i'), \varsigma \in \mathcal{S})$ is a probability distribution over $\mathcal{S}$ and (ii) conditional on the event $\{\mathbf{a}(t-1) = \mathbf{a}\}$, the probability of action $a_i' \in \mathcal{A}_i$ being classified as $\varsigma \in \mathcal{S}$ by agent i at time t is equal to $q_t^i(\varsigma; \mathbf{a}, a_i')$.

Second, for each agent $i \in \mathcal{P}$, we define an array of independent uniform(0,1] random variables $\mathbf{W}^i = \{\mathbf{w}_t^i, t \in \mathbb{N}^+\}$, where $\mathbf{w}_t^i = (w_{t,a}^i, a \in \mathcal{A}_i)$ with $w_{t,a}^i \sim$

uniform(0,1]. To facilitate the exposition, we order the elements in $\mathcal{S}$ as follows: $\varsigma_1 = B, \varsigma_2 = I$ and $\varsigma_3 = W$. Then, action $a_i' \in \mathcal{A}_i$ is classified as $\varsigma_l, l = 1, 2, 3$, by agent i at time $t \in \mathbb{N}^+$ if

$$w_{t,a_i'}^i \in \left(\sum_{l'=1}^{l-1} q_t^i(\varsigma_{l'}; \mathbf{a}(t-1), a_i'), \sum_{l'=1}^{l} q_t^i(\varsigma_{l'}; \mathbf{a}(t-1), a_i') \right].$$

We note that, because the uniform random variables are mutually independent, given the action profile $\mathbf{a}(t-1)$, each action is incorrectly classified by a agent independently of each other at time t and also of the past history. This allows us to model the evolution of the action profile as a Markov process in our analysis.

Assumption 7.1. *For every $i \in \mathcal{P}$ and $t \in \mathbb{N}^+$, $\mathbf{q}_t^i(\mathbf{a}, a_i') > \mathbf{0} := [0 \; 0 \; 0]^T$ for all* $\mathbf{a} \in \mathcal{A}$ *and* $a_i' \in \mathcal{A}_i$.

 Nonhomogeneous Markov chain – Before presenting our main results, we first introduce some terminologies we borrow from [53]. Once the mappings $\mathbf{q}_t^i$, $i \in \mathcal{P}$ and $t \in \mathbb{N}^+$, and $\boldsymbol{\beta}_i$ and $\boldsymbol{\vartheta}_i$, $i \in \mathcal{P}$, are fixed, the evolution of action profile $\{\mathbf{a}(t), t \in \mathbb{N}\}$ can be modeled as a *nonhomogeneous* (discrete-time) Markov chain [44], where the transition matrix at time $t \in \mathbb{N}$, denoted by $\mathbf{P}(t)$, is determined by the mappings $\boldsymbol{\beta}_i$ and $\boldsymbol{\vartheta}_i$, $i \in \mathcal{P}$, and $\mathbf{q}_t^i$, $i \in \mathcal{P}$ and $t \in \mathbb{N}^+$. Because we assume $\mathbf{q}_t^i(\mathbf{a}, a_i') > \mathbf{0}$, every action a_i' of agent i will be classified as a better reply with positive probability at every $t \in \mathbb{N}^+$ and, consequently, we have $P_{\mathbf{a}^1, \mathbf{a}^2}(t) > 0$ for all $\mathbf{a}^1, \mathbf{a}^2 \in \mathcal{A}$.

 In addition to the nonhomogeneous Markov chain $\{\mathbf{a}(t), t \in \mathbb{N}\}$, we define, for each $t \in \mathbb{N}$, a time homogeneous (discrete-time) Markov chain $\mathbb{X}^t = \{\mathbf{x}^t(n), n \in \mathbb{N}\}$ with a common state space $\mathcal{A}$ and a transition matrix $\mathbf{P}^t = \mathbf{P}(t)$. Since $P_{\mathbf{a}^1, \mathbf{a}^2}^t > 0$

for all $\mathbf{a}^1, \mathbf{a}^2 \in \mathcal{A}$ as mentioned above, the Markov chain $\mathbb{X}^t$ is ergodic with a unique stationary distribution [58, 89], which we denote by $\boldsymbol{\mu}^t = (\mu^t(\mathbf{a}), \ \mathbf{a} \in \mathcal{A})$. We call the Markov chain in which the agents make no mistake (i.e., the model assumed in Section 7.4) an *unperturbed* Markov chain. We denote the transition matrix of the unperturbed Markov chain by $\mathbf{P}^0$.

Assumption 7.2. *There exists a decreasing, positive sequence $(\epsilon_t, t \in \mathbb{N})$ such that (i) $\lim_{t \to \infty} \epsilon_t = 0$ and (ii) for every $i \in \mathcal{P}$, $\mathbf{a} \in \mathcal{A}$, $a_i' \in \mathcal{A}_i \backslash \{a_i\}$, and $\varsigma \neq \mathcal{C}_i(a_i', \mathbf{a})$, there are constants $c_i(\mathbf{a}, a_i', \varsigma) > 0$ and $\gamma_i(\mathbf{a}, a_i', \varsigma) > 0$ satisfying*

$$\lim_{t \to \infty} \frac{q_t^i(\varsigma; \mathbf{a}, a_i')}{\epsilon_t^{\gamma_i(\mathbf{a}, a_i', \varsigma)}} = c_i(\mathbf{a}, a_i', \varsigma),$$

i.e., $\mathbf{q}_t^i(\varsigma; \mathbf{a}, a_i') \sim c_i(\mathbf{a}, a_i', \varsigma) \cdot \epsilon_t^{\gamma_i(\mathbf{a}, a_i', \varsigma)}$.

First, Assumption 7.2 implies that the probability of agent i making an erroneous determination goes to zero as $t \to \infty$, i.e.,

$$\limsup_{t \to \infty} \left(\max_{i \in \mathcal{P}} \left(\max_{a_i' \in \mathcal{A}_i, \mathbf{a} \in \mathcal{A}} \left(\sum_{\varsigma \neq \mathcal{C}_i(a_i', \mathbf{a})} q_t^i(\varsigma; \mathbf{a}, a_i') \right) \right) \right) = 0.$$

Second, it states that, for all sufficiently large t, the transition probabilities $P_{\mathbf{a}^1, \mathbf{a}^2}(t)$ can be well approximated using finite sums of power functions of ϵ_t. In other words, $P_{\mathbf{a}^1, \mathbf{a}^2}(t) \sim \sum_{k=1}^{K} c_k \cdot (\epsilon_t)^{r_k}$, where $c_k \in \mathbb{R}$ and $r_k \geq 0$. As a result, for all $\mathbf{a}^1, \mathbf{a}^2 \in \mathcal{A}$, $\mathbf{a}^1 \neq \mathbf{a}^2$, there is $r(\mathbf{a}^1, \mathbf{a}^2) \geq 0$ such that

$$0 < \lim_{t \to \infty} \frac{P_{\mathbf{a}^1, \mathbf{a}^2}(t)}{\epsilon_t^{r(\mathbf{a}^1, \mathbf{a}^2)}} < \infty,$$

and we can find $0 < \underline{\xi} < \overline{\xi} < \infty$ such that, for all sufficiently large t,

$$\underline{\xi} \cdot \epsilon_t^{r(\mathbf{a}^1, \mathbf{a}^2)} < P_{\mathbf{a}^1, \mathbf{a}^2}(t) < \overline{\xi} \cdot \epsilon_t^{r(\mathbf{a}^1, \mathbf{a}^2)}. \tag{7.4}$$

Resistance of paths – A *path* from a strategy profile $\mathbf{a}^1$ to another strategy profile $\mathbf{a}^2$ is a sequence of strategy profiles $(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(M)})$ with $\mathbf{a}^{(1)} = \mathbf{a}^1$, $\mathbf{a}^{(M)} = \mathbf{a}^2$ and $\mathbf{a}^{(\ell)} \neq \mathbf{a}^{(\ell+1)}$ for all $\ell = 1, 2, \ldots, M-1$.

Definition 7.1. *The resistance of a path* $h(\mathbf{a}^1 \to \mathbf{a}^2) := (\mathbf{a}^{(1)} = \mathbf{a}^1, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(M)} = \mathbf{a}^2)$ *from* $\mathbf{a}^1$ *to* $\mathbf{a}^2$ *is equal to* $r_p(h(\mathbf{a}^1 \to \mathbf{a}^2)) = \sum_{\ell=1}^{M-1} r(\mathbf{a}^{(\ell)}, \mathbf{a}^{(\ell+1)})$. *The resistance from* $\mathbf{a}^1$ *to* $\mathbf{a}^2$*, denoted by* $\rho(\mathbf{a}^1, \mathbf{a}^2)$*, is defined to be the smallest resistance among the paths from* $\mathbf{a}^1$ *to* $\mathbf{a}^2$*, i.e.,* $\rho(\mathbf{a}^1, \mathbf{a}^2) := \inf\{r_p(h(\mathbf{a}^1 \to \mathbf{a}^2)) \mid h(\mathbf{a}^1 \to \mathbf{a}^2) \text{ is a path from } \mathbf{a}^1 \text{ to } \mathbf{a}^2\}$.

Definition 7.2. *Given a subset* $\tilde{\mathcal{A}} \subset \mathcal{A}$*, its co-radius is given by* $\tau(\tilde{\mathcal{A}}) = \max_{\mathbf{a}' \in \mathcal{A} \setminus \tilde{\mathcal{A}}} \left(\min_{\mathbf{a} \in \tilde{\mathcal{A}}} \rho(\mathbf{a}', \mathbf{a}) \right)$.

The co-radius $\tau(\tilde{\mathcal{A}})$ is the maximum resistance that must be overcome to reach some strategy profile in $\tilde{\mathcal{A}}$ from a strategy profile outside $\tilde{\mathcal{A}}$. Thus, it measures how "easy" it is for the agents to reach a strategy profile in $\tilde{\mathcal{A}}$ starting from any strategy profile outside $\tilde{\mathcal{A}}$.

Define $\kappa := \min_{\mathbf{a}^* \in \mathcal{A}_{NE}} \tau(\{\mathbf{a}^*\})$, i.e., the smallest co-radius among all Nash equilibria. Based on the above observation, κ indicates how "easily" the agents can reach a Nash equilibrium starting from *any* strategy profile.

Minimum resistance W-tree – Construct a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \mathcal{A}$ and the (directed) edge set $\mathcal{E}$ contains all possible one-step transitions in the Markov chain $\mathbb{X}^t$, i.e., $(\mathbf{a}_1, \mathbf{a}_2) \in \mathcal{A} \times \mathcal{A}$ with $P^t_{\mathbf{a}_1, \mathbf{a}_2} > 0$. Recall that, from Assumption 7.1, we have $P_{\mathbf{a}^1, \mathbf{a}^2}(t) > 0$ for all $\mathbf{a}^1, \mathbf{a}^2 \in \mathcal{A}$. Hence, the edge set $\mathcal{E} = \mathcal{A} \times \mathcal{A}$.

Given a subgraph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ of $\mathcal{G}$, where $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$, its *resistance* is defined as the sum of the resistances of all edges in the subgraph [53], and we denote it by $\pi(\mathcal{G}')$. In other words, $\pi(\mathcal{G}') = \sum_{e \in \mathcal{E}'} r(e)$, where $r(e)$ is the resistance of (directed) edge e.

For every $\mathbf{a} \in \mathcal{A}$, let $\mathcal{G}_T(\mathbf{a})$ be the set of *trees* in $\mathcal{G}$ rooted at $\mathbf{a}$, such that there exists a path from every $\mathbf{a}' \in \mathcal{A}$ to $\mathbf{a}$. The trees in $\mathcal{G}_T(\mathbf{a})$ are called W-trees rooted at $\mathbf{a}$. A minimum resistance W-tree in $\mathcal{G}_T(\mathbf{a})$ is denoted by $\Gamma^*(\mathbf{a})$, i.e., $\Gamma^*(\mathbf{a}) \in \arg\min_{\Gamma(\mathbf{a}) \in \mathcal{G}_T(\mathbf{a})} \pi(\Gamma(\mathbf{a}))$. Let $\pi_{\min}(\mathbf{a}) := \pi(\Gamma^*(\mathbf{a})) = \min_{\Gamma(\mathbf{a}) \in \mathcal{G}_T(\mathbf{a})} \pi(\Gamma(\mathbf{a}))$.

7.6.2 Main results

First, we state an auxiliary result that follows directly from Theorem 7.2 (which implies that Nash equilibria are absorbing states of the unperturbed Markov chain) and [95, Theorem 4].

Theorem 7.7. *As $t \to \infty$, $\boldsymbol{\mu}^t \to \boldsymbol{\mu}^0$, where $\boldsymbol{\mu}^0$ is a stationary distribution of the unperturbed Markov chain. In addition, $\mu^0(\mathbf{a}^\star) > 0$ if and only if (i) $\mathbf{a}^\star \in \mathcal{A}_{NE}$ and (ii) $\pi_{\min}(\mathbf{a}^\star) = \min_{\mathbf{a}' \in \mathcal{A}_{NE}} \pi_{\min}(\mathbf{a}')$.*

A Nash equilibrium $\mathbf{a}^\star$ with $\mu^0(\mathbf{a}^\star) > 0$ is said to be *stochastically stable* [53]. We denote the set of stochastically stable Nash equilibria by $\mathcal{A}_{SS} \subseteq \mathcal{A}_{NE}$.

We now prove that, under some conditions, the nonhomogeneous Markov chain $\{\mathbf{a}(t), t \in \mathbb{N}\}$ is strongly ergodic [44]; the strong ergodicity of the nonhomogeneous Markov chain means that its distribution converges to $\boldsymbol{\mu}^0$ as $t \to \infty$. Thus, it tells us $\mathbb{P}[\mathbf{a}(t) \in \mathcal{A}_{SS}] \to 1$ as $t \to \infty$, which in turn implies $\lim_{t \to \infty} \mathbb{P}[\mathbf{a}(t) \in \mathcal{A}_{NE}] = 1$

because $\mathcal{A}_{SS} \subseteq \mathcal{A}_{NE}$.

Before we proceed, we introduce the assumption under which we establish the strong ergodicity of the nonhomogeneous Markov chain.

Assumption 7.3. *For every $i \in \mathcal{P}$, $\mathbf{a} \in \mathcal{A}$, $a_i' \in \mathcal{A}_i \backslash \{a_i\}$, and $\varsigma \neq \mathcal{C}_i(a_i', \mathbf{a})$, there exist some function $f \in C^\infty$ and $\epsilon^\star > 0$ such that (i) $q_t^i(\varsigma; \mathbf{a}, a_i') = f(\epsilon_t)$ if $0 < \epsilon_t < \epsilon^\star$ and (ii) $f(\epsilon) \sim \alpha \cdot \epsilon^\beta$ for some $\alpha, \beta > 0$. In addition, $f'(\epsilon) \sim \alpha' \cdot \epsilon^{\beta'}$ for some $\alpha' > 0$ and $\beta' \in \mathbb{R}$.*

The first part of Assumption 7.3 is simply rehashing of Assumption 7.2. The assumption essentially states that both the probabilities of false classifications and their derivatives with respect to ϵ_t asymptotically behave like power functions of ϵ_t. Thus, for sufficiently large t, both the transition probabilities $P_{\mathbf{a}^1, \mathbf{a}^2}(t)$ and their derivatives can be well approximated using *finite* sums of power functions of ϵ_t. In addition, the assumption implies that the probability of correct classification of actions tends to one over time.

Theorem 7.8. *Suppose that Assumptions 7.2 and 7.3 hold with $\sum_{t \in \mathbb{N}} \epsilon_t^\kappa = \infty$. Then, the nonhomogeneous Markov chain $\{\mathbf{a}(t), t \in \mathbb{N}\}$ is strongly ergodic with limiting distribution $\boldsymbol{\mu}^0$. Consequently, it satisfies $\lim_{t \to \infty} \mathbb{P}[\mathbf{a}(t) \in \mathcal{A}_{SS}] = 1$.*

Proof. A proof of Theorem 7.8 is provided in Section 7.9. ∎

The condition $\sum_{t \in \mathbb{N}} \epsilon_t^\kappa = \infty$ in Theorem 7.8 reveals the following interesting observation. When κ is larger, it is more difficult for agents to reach a Nash equilibrium in the worst case. This demands that the erroneous classification probabilities

decrease *slower*. The intuition behind this is that greater erroneous classification probabilities make it "easier" for the played action profile to transition from one strategy profile to another, thereby facilitating the exploration of new strategy profiles, including Nash equilibria.

Remark: Suppose that Assumption 7.2 does not hold and agents continue to make mistakes with probability that does not vanish. However, if the probability of making a mistake is sufficiently small, our results can be modified to demonstrate that, for large $t \in \mathbb{N}$, the probability that the chosen action profile $\mathbf{a}(t)$ is a Nash equilibrium is close to one. Hence, even though the probability of playing a Nash equilibrium does not reach one as $t \to \infty$ in this case, the probability will be close to one.

A question that arises is whether anything more can be said about the set of stochastically stable Nash equilibria $\mathcal{A}_{SS}$. For any $\mathbf{a}^* \in \mathcal{A}_{NE}$, let the states from which we can reach $\mathbf{a}^*$ through zero resistance paths (GBRPs) be denoted as $D_{\mathbf{a}*} \subseteq \mathcal{A}$, i.e.,

$$D_{\mathbf{a}*} = \{\mathbf{a} \in \mathcal{A} \mid \rho(\mathbf{a}, \mathbf{a}^*) = 0\}$$

This represents the domain of attraction of a particular PSNE. By the GWAG assumption, $\cup_{\mathbf{a}*\in\mathcal{A}_{NE}} D_{\mathbf{a}*} = \mathcal{A}$, i.e., we can reach at least one PSNE from any non-equilibrium action profile by following a GBRP. Choose any $\mathbf{a}' \in \mathcal{A}_{NE}$. The minimum resistance W-tree $\Gamma^*(\mathbf{a}')$ contains resistance paths from other PSNE(s) and non-PSNE action profiles. For any action profile in $D_{\mathbf{a}'}$, by definition we can construct zero resistance paths to $\mathbf{a}'$. However for reaching $\mathbf{a}'$ from any action profile

outside of $D_{\mathbf{a}'}$, perturbations due to the error model is required. Therefore, nothing more can be inferred about $\mathcal{A}_{SS}$ than already stated. It would in general depend both on the error model and the structure of the game itself. This is a major reason for considering the class of RSEM rules which seem to exhibit more favourable properties in similar settings, in particular, the domain of attraction of each PSNE is better characterisable.

7.7 Proof of Theorem 7.5

Before we present the proof, we introduce the following notation. Consider a mapping $\mathcal{L} : \mathcal{A}^* \rightarrow \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, where $\mathcal{A}^*$ denotes the Kleene star on $\mathcal{A}$. Given a sequence of L action profiles, say $(\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^L)$, $\mathcal{L}((\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^L)) = L - 1$ gives us its length, i.e., the number of transitions in it.

Since the game $\mathbb{G}$ is generalized weakly acyclic, for every strategy profile $\mathbf{a} \notin \mathcal{A}_{NE}$, there exists at least one generalized better reply path that starts with $\mathbf{a}$ and leads to a Nash equilibrium. For each strategy profile $\mathbf{a} \in \mathcal{A}$, we choose a generalized better reply path with the shortest length (according to the mapping $\mathcal{L}$) and denote it by $\mathbf{p}(\mathbf{a})$. Clearly, for a Nash equilibrium $\mathbf{a}^* \in \mathcal{A}_{NE}$, we have $\mathbf{p}(\mathbf{a}^*) = (\mathbf{a}^*)$ and $\mathcal{L}(\mathbf{p}(\mathbf{a}^*)) = 0$.

Although it is not necessary, in order to facilitate the exposition, we assume that the shortest generalized better reply paths $\{\mathbf{p}(\mathbf{a}), \mathbf{a} \in \mathcal{A}\}$ satisfy the following consistency condition: Suppose that a non-Nash equilibrium strategy profile $\mathbf{a}^1$ appears in the generalized better reply path, $\mathbf{p}(\mathbf{a}^2)$, of another action profile $\mathbf{a}^2$.

Then, the subsequence in $\mathbf{p}(\mathbf{a}^2)$ that starts with $\mathbf{a}^1$ is identical to $\mathbf{p}(\mathbf{a}^1)$. Such generalized better reply paths can be constructed easily in a manner similar to Dijkstra's algorithm, starting with $\mathcal{A}_{NE}$ [42].

We introduce two mappings: $\Phi : \mathcal{A} \backslash \mathcal{A}_{NE} \to \mathcal{A}$ and $I : \mathcal{A} \backslash \mathcal{A}_{NE} \to 2^{\mathcal{P}}$, where $\Phi(\mathbf{a})$ denotes the second strategy profile in $\mathbf{p}(\mathbf{a})$ following $\mathbf{a}$ and $I(\mathbf{a}) = \{i \in \mathcal{P} \mid a_i \neq \Phi_i(\mathbf{a})\}$, i.e., the set of agents that change their strategies going from strategy profile $\mathbf{a}$ to $\Phi(\mathbf{a})$.

We prove Theorem 7.5 with the help of several lemmas, whose proofs are deferred to the following section. The first lemma states that, if the action profile at time t is not a Nash equilibrium, the action profile will reach $\Phi(\mathbf{a}(t))$ within $3\Delta_\eta$ periods with positive probability, where Δ_η is the constant in Assumption A4.

Lemma 7.9. *Under the assumptions in Theorem 7.5, there exists $q_1 > 0$ such that, for every $\mathbf{a} \notin \mathcal{A}_{NE}$ and $t \in \mathbb{N}$,*

$$\mathbb{P}\left[\mathbf{a}(t + 3\Delta_\eta) = \Phi(\mathbf{a}(t)) \mid \mathbf{a}(t) = \mathbf{a}\right] \geq q_1. \tag{7.5}$$

Repeating Lemma 7.9, we readily obtain the following corollary that tells us that, starting at $\mathbf{a}(t)$ at time t, we reach a Nash equilibrium within a finite number of periods with positive probability.

Corollary 1. *Suppose that the assumptions in Theorem 7.5 hold. Then, for any $\mathbf{a} \notin \mathcal{A}_{NE}$ and every $t \in \mathbb{N}$, we have*

$$\mathbb{P}\left[\mathbf{a}\left(t + 3\Delta_\eta \mathcal{L}(\mathbf{p}(\mathbf{a}))\right) \in \mathcal{A}_{NE} \mid \mathbf{a}(t) = \mathbf{a}\right] \geq q_1^{\mathcal{L}(\mathbf{p}(\mathbf{a}))}. \tag{7.6}$$

The next lemma states that, once the agents reach a Nash equilibrium $\mathbf{a}^*$,

there is positive probability that they will remain at the Nash equilibrium $\mathbf{a}^*$ for good.

Lemma 7.10. *Under the assumptions in Theorem 7.5, there exists $q_2 > 0$ such that, for all $\mathbf{a}^* \in \mathcal{A}_{NE}$ and $t \in \mathbb{N}$, we have*

$$\mathbb{P}\left[\mathbf{a}(t') = \mathbf{a}^* \text{ for all } t' \geqslant t + 1 \mid \mathbf{a}(t) = \mathbf{a}^*\right] \geqslant q_2. \tag{7.7}$$

Lemmas 7.9 and 7.10 yield the following corollary. Let $L_{\max} := \max_{\mathbf{a} \in \mathcal{A} \setminus \mathcal{A}_{NE}} \mathcal{L}(\mathbf{p}(\mathbf{a}))$.

Corollary 2. *Suppose that the assumptions in Theorem 7.5 are true. Then, for any $\mathbf{a} \in \mathcal{A}$, we have*

$$\mathbb{P}\left[\mathbf{a}(t') \in \mathcal{A}_{NE} \text{ for all } t' \geqslant t + 3\Delta_\eta L_{\max} \mid \mathbf{a}(t) = \mathbf{a}\right] \geqslant q_1^{L_{\max}} \cdot q_2. \tag{7.8}$$

Define $\bar{L} := 3\Delta_\eta L_{\max}$ and $\bar{q} := q_1^{L_{\max}} \cdot q_2$. Corollary 2 implies that, for all $n \in \mathbb{N}$, $\mathbb{P}\left[\vec{\mathbf{a}} \notin \mathcal{A}_{n\bar{L}+1}^\infty\right] \leqslant (1-\bar{q})^n$. Recall that $\mathcal{A}_t^\infty$ increases with t. Consequently, $\mathbb{P}\left[\vec{\mathbf{a}} \notin \mathcal{A}_t^\infty\right]$ decreases with t and $\mathbb{P}\left[\vec{\mathbf{a}} \notin \mathcal{A}_{t+1}^\infty\right] \leqslant (1-\bar{q})^{\lfloor t/\bar{L} \rfloor} \leqslant \tilde{C} \cdot \tilde{\zeta}^t$, where $\tilde{C} = (1-\bar{q})^{-1}$ and $\tilde{\zeta} = (1-\bar{q})^{1/\bar{L}} < 1$. Because $\mathcal{A}_\star^\infty = \cup_{t \in \mathbb{N}} \mathcal{A}_t^\infty$, we have $\mathbb{P}\left[\vec{\mathbf{a}} \in \mathcal{A}_\star^\infty\right] \geqslant \mathbb{P}\left[\vec{\mathbf{a}} \in \mathcal{A}_t^\infty\right]$ for all $t \in \mathbb{N}$. Finally, $\lim_{t \to \infty} \mathbb{P}\left[\vec{\mathbf{a}} \in \mathcal{A}_t^\infty\right] = 1 - \lim_{t \to \infty} \mathbb{P}\left[\vec{\mathbf{a}} \notin \mathcal{A}_t^\infty\right] = 1$, yielding the desired result $\mathbb{P}\left[\vec{\mathbf{a}} \in \mathcal{A}_\star^\infty\right] = 1$.

Remark: As mentioned in Section 7.5, Theorems 7.5 and 7.6 hold under less restrictive assumptions for weakly acyclic games. This is because Lemma 7.9 is true under Assumptions A1 through A4 for weakly acyclic games and the proof of Lemma 7.10 presented in Section 7.8.2 requires only Assumptions A1 through A4.

7.8 Proofs of Lemmas introduced in Section 7.7

7.8.1 Proof of Lemma 7.9

For each $i \in \mathcal{P}$, we define $k^i_{\max} : \mathbb{N} \to \mathbb{N}$, where $k^i_{\max}(t) = \max\{k \in \mathbb{N} \mid T^i_k \leqslant t\}$. In addition, for every $i \in \mathcal{P}$ and $t \in \mathbb{N}$, $d^F_i(t) = X^i_{k^i_{\max}(t)}$ and $d^R_i(t) = Y^i_{k^i_{\max}(t)}$. Similarly, for all $i, j \in \mathcal{P}$, $i \neq j$, and $t \in \mathbb{N}$, $d^-_{i,j}(t) = V^{i,j}_{k^i_{\max}(t)}$ and $d^+_{i,j}(t) = V^{i,j}_{k^i_{\max}(t)+1}$. For notational convenience, we denote the interval $\{t + (\ell - 1)\Delta_\eta + 1, \ldots, t + \ell \cdot \Delta_\eta\}$ by $\overline{T}_\ell(t)$, $\ell \in \mathbb{N}$, where Δ_η is the constant in Assumption A4 in Section 7.5.

Next, we define following events. Note that $T^i_{k^i_{\max}(t)}$ (resp. $T^i_{k^i_{\max}(t)+1}$) denotes the last time by time t (resp. the first time after time t) at which agent i updates its strategy.

$\mathcal{E}_0 = \{T^i_{k^i_{\max}(t)+1} \leqslant t + \Delta_\eta$ for all $i \in \mathcal{P}$

$\qquad\qquad$ and $T^i_{k^i_{\max}(t)} + d^-_{i,j}(t) \leqslant t + \Delta_\eta$ for all $i, j \in \mathcal{P}, i \neq j\}$

$\mathcal{E}_1 = \{\mathcal{T}^i \cap \overline{T}_2(t) \neq \varnothing$ for all $i \in \mathcal{P}\}$, i.e., all agents update at least once during

$\qquad\qquad$ the interval $\overline{T}_2(t)$

$\mathcal{E}_2 = \{a_i(t') = a_i(t), t' = t + 1, \ldots, T^i_{k^i_{\max}(t+2\Delta_\eta)+1} - 1$ for all $i \in \mathcal{P}\}$

$\mathcal{E}_3 = \{d^F_i(t + 2\Delta_\eta) \leqslant t + 2\Delta_\eta + \Delta_\delta - T^i_{k^i_{\max}(t+2\Delta_\eta)}$ for all $i \in \mathcal{P}\}$

$\mathcal{E}_4 = \{\mathcal{T}^i \cap \overline{T}_3(t) \neq \varnothing$ for all $i \in \mathcal{P}\}$

$\mathcal{E}_5 = \{a_i(T^i_{k^i_{\max}(t+2\Delta_\eta)+1}) = \Phi_i(\mathbf{a}(t))$ for all $i \in \mathcal{P}\}$

$\mathcal{E}_6 = \{d^+_{i,j}(t + 2\Delta_\eta) \geqslant \Delta_\delta$ for all $i, j \in \mathcal{P}, i \neq j\}$

$\mathcal{E}_7 = \{a_i(t') = \Phi_i(\mathbf{a}(t))$ for all $i \in \mathcal{P}$ and $t' = T^i_{k^i_{\max}(t+2\Delta_\eta)+1} + 1, \ldots, t + 3\Delta_\eta\}$

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Event $\mathcal{E}_0$ ensures that (a) all agents update at least once during the interval $\overline{T}_1(t)$ and (b) action profile $\mathbf{a}(t)$ starts affecting the payoffs of all agents by time $t + \Delta_\eta$. The second event $\mathcal{E}_1$ demands that all the agents revise their strategies at least once during the interval $\overline{T}_2(t)$. Events $\mathcal{E}_2$ and $\mathcal{E}_5$ together require every agent $i \in \mathcal{P}$ to continue playing the same action $a_i(t)$ till $T^i_{k^i_{\max}(t+2\Delta_\eta)+1} - 1$ and then switch to $\Phi_i(\mathbf{a}(t))$ at time $T^i_{k^i_{\max}(t+2\Delta_\eta)+1}$. In order to make sure that event $\mathcal{E}_5$ is feasible, events $\mathcal{E}_3$ and $\mathcal{E}_6$ ensure that, at the first time agents update their strategies during the interval $\overline{T}_3(t)$, they will see the payoff information in response to $\mathbf{a}(t)$ and, as a result, choose $\Phi_i(\mathbf{a}(t))$ with positive probability. Finally, event $\mathcal{E}_7$ demands that the agents continue to play strategy $\Phi_i(\mathbf{a}(t))$ till $t + 3\Delta_\eta$.

We can lower bound the conditional probability in (7.5) in Lemma 7.9 as follows. First,

$$\mathbb{P}\left[\mathbf{a}(t + 3\Delta_\eta) = \Phi(\mathbf{a}(t)) \mid \mathbf{a}(t) = \mathbf{a}\right]$$

$$\geqslant \mathbb{P}\left[\mathbf{a}(t + 3\Delta_\eta) = \Phi(\mathbf{a}(t)), \mathcal{E}_i, i = 0, 1, \ldots, 7 \mid \mathbf{a}(t) = \mathbf{a}\right] \tag{7.9}$$

We rewrite (7.9) as a product of conditional probabilities.

$$(7.9) = \mathbb{P}\left[\mathbf{a}(t + 3\Delta_\eta) = \Phi(\mathbf{a}(t)) \mid \mathbf{a}(t) = \mathbf{a}, \mathcal{E}_i, i = 0, 1, \ldots, 7\right] \tag{7.10}$$

$$\times \mathbb{P}\left[\mathcal{E}_7 \mid \mathbf{a}(t) = \mathbf{a}, \mathcal{E}_i, i = 0, 1, \ldots, 6\right] \tag{7.11}$$

$$\times \mathbb{P}\left[\mathcal{E}_5 \cap \mathcal{E}_6 \mid \mathbf{a}(t) = \mathbf{a}, \mathcal{E}_i, i = 0, \ldots, 4\right] \tag{7.12}$$

$$\times \mathbb{P}\left[\mathcal{E}_3 \cap \mathcal{E}_4 \mid \mathbf{a}(t) = \mathbf{a}, \mathcal{E}_i, i = 0, 1, 2\right] \tag{7.13}$$

$$\times \mathbb{P}\left[\mathcal{E}_2 \mid \mathbf{a}(t) = \mathbf{a}, \mathcal{E}_i, i = 0, 1\right] \tag{7.14}$$

$$\times \mathbb{P}\left[\mathcal{E}_0 \cap \mathcal{E}_1 \mid \mathbf{a}(t) = \mathbf{a}\right] \tag{7.15}$$

We now lower bound the terms in (7.10) through (7.15). First, it is clear from the definitions of the events $\mathcal{E}_i, i = 0, 1, \ldots, 7$, that (7.10) is equal to one. Since $1 - \mu$ is a lower bound to the probability with which a agent with a better reply plays the same action played at the previous time, $(7.11) \geqslant (1 - \mu)^{n(\Delta_\eta - 1)}$ and $(7.14) \geqslant (1 - \mu)^{2\Delta_\eta n}$. From Assumption A5 (more precisely, (7.3)), we have $(7.12) \geqslant (\underline{\epsilon} \cdot (1 - \mu) \cdot \delta)^n$, where $\underline{\epsilon}$ is a positive lower bound to the probability with which a agent chooses a better reply. By Assumptions A4 through A6, $(7.13) \geqslant (\eta \cdot \delta)^n$. Finally, Assumption A4 implies $(7.15) \geqslant \eta^{2n}$. From these lower bounds, the inequality in (7.5) holds with $q_1 := \left((1 - \mu)^{3\Delta_\eta} \cdot \underline{\epsilon} \cdot \delta^2 \cdot \eta^3 \right)^n$.

7.8.2 Proof of Lemma 7.10

We first define the following three events.

$$\mathcal{E}_0' = \{ T_{k_{\max}^i(t)+1}^i \leqslant t + \Delta_\eta \text{ for all } i \in \mathcal{P}$$

$$\text{and } T_{k_{\max}^i(t)}^i + d_{i,j}^-(t) \leqslant t + \Delta_\eta \text{ for all } i, j \in \mathcal{P}, i \neq j \}$$

$$\mathcal{E}_1' = \{ \mathcal{T}^i \cap \overline{T}_2(t) \neq \varnothing \text{ for all } i \in \mathcal{P} \}$$

$$\mathcal{E}_2' = \{ \mathbf{a}(t') = \mathbf{a}^* \text{ for all } t' = t + 1, \ldots, t + 2\Delta_\eta \}$$

First, events $\mathcal{E}_0'$ and $\mathcal{E}_2'$ ensure that the payoffs of all agents are determined by strategy profile $\mathbf{a}^*$ by time $t + \Delta_\eta$. Note that it also implies that every agent updates its strategy at least once during the interval $\overline{T}_1(t)$ because $T_{k_{\max}^i(t)+1}^i \leqslant t + \Delta_\eta$. Second, event $\mathcal{E}_1'$ demands the agents to revise their strategies at least once during the interval $\overline{T}_2(t)$ as well. Finally, event $\mathcal{E}_2'$ requires that the agents play the equilibrium strategies till $t + 2\Delta_\eta$.

Note that, if the events $\mathcal{E}_0'$ through $\mathcal{E}_2'$ occur, all payoff information will be produced on the basis of strategy profile $\mathbf{a}^*$ after time $t + \Delta_\eta$ and, as a result, no agent will change its action after time $t + 2\Delta_\eta$. This is because, for every agent $i \in \mathcal{P}$, we have $\tilde{\mathbf{a}}_{-i}^i(T_{k_{\max}^i(t+2\Delta)+1}^i) = \mathbf{a}_{-i}^*$. Once this happens, no agent will ever see a better reply after $t + 2\Delta_\eta$ and, consequently, will not deviate from its equilibrium action, and the strategy profile will remain $\mathbf{a}^*$ for good. Thus,

$$\mathbb{P}\left[\mathbf{a}(t') = \mathbf{a}^* \text{ for all } t' \geqslant t + 1 \mid \mathbf{a}(t) = \mathbf{a}^*, \mathcal{E}_i', i = 0, 1, 2\right] = 1.$$

Putting things together, we get

$$\mathbb{P}\left[\mathbf{a}(t') = \mathbf{a}^* \text{ for all } t' \geqslant t + 1 \mid \mathbf{a}(t) = \mathbf{a}^*\right]$$

$$\geqslant \mathbb{P}\left[\mathbf{a}(t') = \mathbf{a}^* \text{ for all } t' \geqslant t + 1 \mid \mathbf{a}(t) = \mathbf{a}^*, \mathcal{E}_i', i = 0, 1, 2\right]$$

$$\times \mathbb{P}\left[\mathcal{E}_0' \cap \mathcal{E}_1' \cap \mathcal{E}_2' \mid \mathbf{a}(t) = \mathbf{a}^*\right]$$

$$= \mathbb{P}\left[\mathcal{E}_0' \cap \mathcal{E}_1' \cap \mathcal{E}_2' \mid \mathbf{a}(t) = \mathbf{a}^*\right]$$

From Assumption A4, we get the following lower bound.

$$\mathbb{P}\left[\mathcal{E}_0' \cap \mathcal{E}_1' \cap \mathcal{E}_2' \mid \mathbf{a}(t) = \mathbf{a}^*\right] \geqslant \eta^n \cdot \eta^n (1 - \mu)^{2n\Delta_\eta} = \left(\eta \cdot (1 - \mu)^{\Delta_\eta}\right)^{2n} > 0.$$

Hence, the lemma follows with $q_2 := \left(\eta \cdot (1 - \mu)^{\Delta_\eta}\right)^{2n}$.

7.9   Proof of Theorem 7.8

From [53, Lemma 3.1, p. 177], we know that the stationary distribution $\boldsymbol{\mu}^t = (\mu^t(\mathbf{a}), \mathbf{a} \in \mathcal{A})$ is given by

$$\mu^t(\mathbf{a}) = \frac{Q_{\mathbf{a}}(t)}{\sum_{\mathbf{a}' \in \mathcal{A}} Q_{\mathbf{a}'}(t)}, \quad \mathbf{a} \in \mathcal{A}, \tag{7.16}$$

where $Q_{\mathbf{a}'}(t) = \sum_{T \in \mathcal{T}(\mathbf{a}')} \left(\prod_{(\mathbf{a}^1, \mathbf{a}^2) \in T} P_{\mathbf{a}^1, \mathbf{a}^2}(t) \right)$. Consider the derivative of $\mu^t(\mathbf{a})$ with respect to ϵ_t.

$$\left(\mu^t(\mathbf{a}) \right)' = \frac{Q'_{\mathbf{a}}(t) \sum_{\mathbf{a}' \in \mathcal{A}} Q_{\mathbf{a}'}(t) - Q_{\mathbf{a}}(t) \sum_{\mathbf{a}' \in \mathcal{A}} Q'_{\mathbf{a}'}(t)}{\left(\sum_{\mathbf{a}' \in \mathcal{A}} Q_{\mathbf{a}'}(t) \right)^2} \tag{7.17}$$

By Assumption 7.3 and the comment following it, for all sufficiently large t, the numerator of (7.17) behaves like a finite sum of power functions of ϵ_t. Therefore, there exists finite T^* such that, for all $t \geqslant T^*$ and all $\mathbf{a} \in \mathcal{A}$, the sign of $(\mu^t(\mathbf{a}))'$ remains the same. Based on this observation, we can partition the set $\mathcal{A}$ into $\mathcal{A}^-$ and $\mathcal{A}^+ = \mathcal{A} \backslash \mathcal{A}^-$, where $\mathcal{A}^- = \{\mathbf{a} \in \mathcal{A} \mid (\boldsymbol{\mu}^t(\mathbf{a}))' \leqslant 0 \text{ for all } t \geqslant T^*\}$.

Theorem 8.3 of [44, p. 242] tells us that the nonhomogeneous Markov chain $\{\mathbf{a}(t), t \in \mathbb{N}\}$ is strongly ergodic if it is weakly ergodic and the following inequality holds.

$$\sum_{t \in \mathbb{N}} \left\| \boldsymbol{\mu}^t - \boldsymbol{\mu}^{t+1} \right\|_1 < \infty \tag{7.18}$$

First, the following lemma tells us that, under the conditions in Theorem 7.8, the Markov chain is weakly ergodic. Its proof can be found in Section 7.10.

Lemma 7.11. *Suppose that Assumption 7.2 holds with $\sum_{t \in \mathbb{N}} \epsilon_t^\kappa = \infty$. Then, the nonhomogeneous Markov chain $\{\mathbf{a}(t), t \in \mathbb{N}\}$ is weakly ergodic.*

Second, we can show (7.18) as follows. We first break the summation into two summations.

$$\sum_{t \in \mathbb{N}} \left\| \boldsymbol{\mu}^t - \boldsymbol{\mu}^{t+1} \right\|_1 = \sum_{t=1}^{T^*} \left\| \boldsymbol{\mu}^t - \boldsymbol{\mu}^{t+1} \right\|_1 + \sum_{t > T^*} \left\| \boldsymbol{\mu}^t - \boldsymbol{\mu}^{t+1} \right\|_1 \tag{7.19}$$

Applying the inequality $\|\boldsymbol{\mu}^t - \boldsymbol{\mu}^{t+1}\|_1 \leqslant 2$ to the first term in (7.19) and the definition of $\|\cdot\|_1$ norm for the summands in the second term, we obtain

$$
\begin{aligned}
(7.19) &\leqslant 2T^* + \sum_{t>T^*} \left(\sum_{\mathbf{a} \in \mathcal{A}} |\mu^t(\mathbf{a}) - \mu^{t+1}(\mathbf{a})| \right) \\
&= 2T^* + \sum_{t>T^*} \left(\sum_{\mathbf{a} \in \mathcal{A}^-} \left(\mu^{t+1}(\mathbf{a}) - \mu^t(\mathbf{a}) \right) \right) \\
&\quad + \sum_{t>T^*} \left(\sum_{\mathbf{a} \in \mathcal{A}^+} \left(\mu^t(\mathbf{a}) - \mu^{t+1}(\mathbf{a}) \right) \right) \\
&= 2T^* + \sum_{\mathbf{a} \in \mathcal{A}^-} \left(\mu^0(\mathbf{a}) - \mu^{T^*+1}(\mathbf{a}) \right) \\
&\quad + \sum_{\mathbf{a} \in \mathcal{A}^+} \left(\mu^{T^*+1}(\mathbf{a}) - \mu^0(\mathbf{a}) \right) \\
&\leqslant 2T^* + 2 < \infty,
\end{aligned} \tag{7.20}
$$

where the second equality in (7.20) follows from Theorem 7.7, i.e., $\boldsymbol{\mu}^t \to \boldsymbol{\mu}^0$ as $t \to \infty$.

7.10   Proof of Lemma 7.11

Let us first introduce some notation and terminology we use in the proof. For notational ease, we denote the product $\mathbf{P}(m) \times \mathbf{P}(m+1) \times \cdots \times \mathbf{P}(n-1)$ of transition matrices by $\mathbf{P}(m, n)$. Given a stochastic matrix $\mathbf{P}$, its Dobrushin's ergodic coefficient [44, p. 235] is given by

$$
\delta(\mathbf{P}) = 1 - \min_{\mathbf{a}^1, \mathbf{a}^2 \in \mathcal{A}} \left(\sum_{\mathbf{a} \in \mathcal{A}} \min \left(P_{\mathbf{a}^1, \mathbf{a}}, P_{\mathbf{a}^2, \mathbf{a}} \right) \right). \tag{7.21}
$$

Theorem 8.2 of [44, p. 241] states that the nonhomogeneous Markov chain $\{\mathbf{a}(t), t \in \mathbb{N}\}$ is weakly ergodic if

$$
\sum_{n \in \mathbb{N}} \left(1 - \delta(\mathbf{P}(t_n, t_{n+1})) \right) = \infty. \tag{7.22}
$$

For each $\mathbf{a}^* \in \mathcal{A}_{NE}$ and $\mathbf{a} \in \mathcal{A}$, let $\ell(\mathbf{a}, \mathbf{a}^*) = \min\{\mathcal{L}(h(\mathbf{a} \to \mathbf{a}^*)) \mid r_p(h(\mathbf{a} \to \mathbf{a}^*)) = \rho(\mathbf{a}, \mathbf{a}^*)\}$, i.e., the shortest length of the paths $h(\mathbf{a} \to \mathbf{a}^*)$ with the least resistance $r_p(h) = \rho(\mathbf{a}, \mathbf{a}^*)$, and define $\ell_{\max}(\mathbf{a}^*) := \max_{\mathbf{a} \in \mathcal{A}} \ell(\mathbf{a}, \mathbf{a}^*)$. Then, the bounds in eq. (7.4) tell us that, for all $\mathbf{a} \in \mathcal{A}$ and all sufficiently large t,

$$P_{\mathbf{a},\mathbf{a}^*}^m(t) > \underline{\xi}^m \cdot \epsilon_{t+m}^{\tau(\{\mathbf{a}^*\})} \quad \text{for all } m \geqslant \ell_{\max}(\mathbf{a}^*). \tag{7.23}$$

Let $\ell_{\max} = \max_{\mathbf{a}^* \in \mathcal{A}_{NE}} \ell_{\max}(\mathbf{a}^*)$, and consider the sequence $(t_k, k \in \mathbb{N})$, where $t_k = k \cdot \ell_{\max}$. First, from (7.23), for all distinct $\mathbf{a}^1, \mathbf{a}^2 \in \mathcal{A}$ and all sufficiently large $k \in \mathbb{N}$, we have the following bound.

$$\begin{aligned}
\sum_{\mathbf{a} \in \mathcal{A}} &\min\left(P_{\mathbf{a}^1,\mathbf{a}}(t_k, t_{k+1}), P_{\mathbf{a}^2,\mathbf{a}}(t_k, t_{k+1})\right) \\
&\geqslant \sum_{\mathbf{a}^* \in \mathcal{A}_{NE}} \min\left(P_{\mathbf{a}^1,\mathbf{a}^*}(t_k, t_{k+1}), P_{\mathbf{a}^2,\mathbf{a}^*}(t_k, t_{k+1})\right) \\
&\geqslant \underline{\xi}^{\ell_{\max}} \sum_{\mathbf{a}^* \in \mathcal{A}_{NE}} \epsilon_{t_{k+1}}^{\tau(\{\mathbf{a}^*\})},
\end{aligned} \tag{7.24}$$

where the second inequality follows from (7.23).

Using the expression for Dobrushin's coefficient in (7.21), we obtain

$$\begin{aligned}
\sum_{n \in \mathbb{N}} &\left(1 - \delta(\mathbf{P}(t_n, t_{n+1}))\right) \\
&= \sum_{n \in \mathbb{N}} \min_{\mathbf{a}^1, \mathbf{a}^2 \in \mathcal{A}} \left(\sum_{\mathbf{a} \in \mathcal{A}} \min\left(P_{\mathbf{a}^1,\mathbf{a}}(t_n, t_{n+1}), P_{\mathbf{a}^2,\mathbf{a}}(t_n, t_{n+1})\right)\right).
\end{aligned} \tag{7.25}$$

In light of (7.24), it is clear that (7.25) diverges if $\sum_{n \in \mathbb{N}} \left(\sum_{\mathbf{a}^* \in \mathcal{A}_{NE}} \epsilon_{t_{n+1}}^{\tau(\mathbf{a}^*)}\right)$ diverges.

We state the following lemma without a proof, which is straightforward.

Lemma 7.12. *Suppose that $(\alpha(n), n \in \mathbb{N})$ is a decreasing, positive sequence with $\sum_{n \in \mathbb{N}} \alpha(n) = \infty$. Then, for any $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}_+$, we have $\sum_{n \in \mathbb{N}} \alpha(k \cdot n + \ell) = \infty$.*

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First, because $\kappa = \min_{\mathbf{a}^* \in \mathcal{A}_{NE}} \tau(\{\mathbf{a}^*\})$

$$\sum_{n \in \mathbb{N}} \left(\sum_{\mathbf{a}^* \in \mathcal{A}_{NE}} \epsilon_{t_{n+1}}^{\tau(\{\mathbf{a}^*\})} \right) \geq \sum_{n \in \mathbb{N}} \epsilon_{t_{n+1}}^{\kappa}. \tag{7.26}$$

Second, recall that $\sum_{t \in \mathbb{N}} \epsilon_t^{\kappa} = \infty$ by the assumption in Lemma 7.11. Thus, by Lemma 7.12, we get $\sum_{n \in \mathbb{N}} \epsilon_{t_{n+1}}^{\kappa} = \infty$. Together with (7.26), this implies that (7.25) diverges, completing the proof of (7.22).

Chapter 8:   Class of Monitoring Rules

In the previous chapter we considered the class of better reply rules which ensures convergence to a pure strategy Nash equilibrium in a large class of games, identified as the generalized weakly acyclic games, under asynchronous updates and payoff information delays. Moreover, we also showed convergence to a PSNE under faulty available payoff information, causing players to make erroneous decisions occasionally. Under the latter setting of erroneous payoff information, when the better reply rules are followed by the agents, the PSNE(s) that are stable in the *long run* or are *stochastically stable* would in general depend both on the error model and the structure of the game itself. In this chapter we consider another class of learning rules that are also robust to delays and asynchrony like the better reply rules. However, under a (partial) erroneous payoff information setting where the SBRs are estimated with errors, the stochastically stable states are better characterisable.

The learning rule in its simplest form is described in the first section, followed by the convergence results under the learning rule for the asynchronous update setting introduced in Section 7.5. Later, we consider the faulty payoff information setting introduced in Section 7.6 and contrast the convergence results from that obtained for the better-reply rules. It is worth mentioning that we adopt the

strategic-form repeated game setting described in Section 7.1.

8.1 Proposed Update Rule

In this section, we present our proposed learning rule, called the Reduced Simple Experimentation with Monitoring (RSEM). Under the RSEM update rule, at each time $t \in \mathbb{N}$, every agent is at one of two possible states – Explore (E) or Converged (C). We denote the state of agent $i \in \mathcal{P}$ at time t by $s_i(t) \in \Psi_0 := \{E, C\}$. Let $\Psi := \Psi_0^n$, and the state vector $\mathbf{s}(t) := (s_i(t), \ i \in \mathcal{P}) \in \Psi$. The rule governing the update of an agent's state will be explained shortly.

Recall that the mappings $\mathcal{C}_i$ and BR_i introduced in the previous chapter are the classification and the better reply mappings for agent i respectively. Also, recall that the update time sequence for player i is denoted as $\mathcal{T}^i = \{T_k^i, \ k \in \mathbb{N}\} \subseteq \mathbb{N}$. In the synchronous update case with no delays we have $\mathcal{T}^i = \mathbb{N}$ for all $i \in \mathcal{P}$ and $\tilde{\mathbf{a}}^i(t) = \mathbf{a}(t-1)$ for all $i \in \mathcal{P}$ and $t \in \mathbb{N}^+$, where $\tilde{\mathbf{a}}^i(t) \in \mathcal{A}$ is the strategy profile responsible for generating the payoff information observed by player i at time t.

Payoff Information for updates – We assume that the payoff information available to agent i at time T_k^i, $k \geqslant 2$, is of the form $((\mathcal{C}_i(a_i, \tilde{\mathbf{a}}^i(T_k^i)), \ a_i \in \mathcal{A}_i) \ ; \ U_i(\tilde{\mathbf{a}}^i(T_k^i)))$, compactly denoted as $(\mathcal{C}_i(\tilde{\mathbf{a}}^i(T_k^i)); U_i(\tilde{\mathbf{a}}^i(T_k^i)))$. In other words, as assumed in the previous chapter, agent i knows the actions that would lead to higher payoff with respect to the played action given the action profile of the other agents in effect (at the agent i's system) at the time the payoff feedback is generated, i.e., R_{k-1}^i.

8.1.1 Action updates

In this subsection, we first describe how the agents choose their actions according to the payoff feedback they receive at the times of updates.

Initially, at time $T_1^i = 1$, the agents choose their actions according to some (joint) distribution $\mathcal{G}$ over $\mathcal{A}$. For $k \geqslant 2$, agent $i \in \mathcal{P}$ updates its action at time T_k^i according to the rule provided below: We assume that agent i continues to play action $a_i(T_k^i)$ between T_k^i and $T_{k+1}^i - 1$, i.e., $a_i(t) = a_i(T_k^i)$ for all $t \in \{T_k^i, \ldots, T_{k+1}^i - 1\}$. Fix $\delta \in (0, 1/(\max_{i \in \mathcal{P}} |\mathcal{A}_i|))$ and $\boldsymbol{\beta}_i : \mathcal{S}^{\mathcal{A}_i} \to \Delta(\mathcal{A}_i)$, $i \in \mathcal{P}$. The mappings $\boldsymbol{\beta}_i(\mathbf{c}_i) = (\beta_i(a_i; \mathbf{c}_i), a_i \in \mathcal{A}_i)$ are used to determine the mixed strategy to be employed given the payoff information.

Action Selection Rule:

For $k = 2, 3, \ldots$,

- if $s_i(T_k^i) = E$

  – choose $a_i(T_k^i) = a_i$ with probability $\beta_i(a_i; \mathcal{C}_i(\tilde{\mathbf{a}}^i(T_k^i))) \geqslant \delta$ for all $a_i \in \mathcal{A}_i$

- else (i.e., $s_i(T_k^i) = C$)

  – set $a_i(T_k^i) = a_i(T_k^i - 1)$;

It is clear that, under RSEM, an agent may choose a new action only if it is at state E. Otherwise, it continues to play the same action employed at the previous time.

Modeling mixed strategies in RSEM rule – As was done for the GBRR rules, we describe how we model mixed strategies exercised by the players under the RSEM rule: For every $i \in \mathcal{P}$, define $\mathbf{u}^i := (u_t^i, \ t \in \mathbb{N}^+)$ to be a sequence of independent uniform random variables over $(0, 1]$.

Suppose that player i revises its strategy at time $t \in \mathcal{T}^i$ and $s_i(t) = E$. It then chooses

$$a_i(t) = \ell, \text{ if } u_t^i \in (\Lambda(\ell - 1), \Lambda(\ell)], \ \ell = 1, 2, \ldots, A_i \tag{8.1}$$

where $\Lambda(\ell) := \sum_{\ell'=1}^{\ell} \beta(a_i^{(\ell')}; \mathcal{C}_i(\tilde{\mathbf{a}}^i(t))), \ \ell = 1, \ldots, A_i$.

8.1.2 State dynamics

As explained in the previous subsection, under the RSEM rule, the state of an agent plays a key role in its action selection. Hence, the dynamics of $\mathbf{s}(t)$, $t \in \mathbb{N}$, play a major role in the algorithm. In this subsection, we explain how the agents update their states based on the received payoff feedback.

At time $t = 1$, we assume that all agents are at state E, i.e., $\mathbf{s}(1) = (E, E, \ldots, E)$. Agent i first updates its state right after it receives new payoff information at T_k^i, $k \in \mathbb{N}$, following which it chooses an action.

The state of agent i at time T_k^i depends on (i) the payoff information vector $\mathcal{C}_i(\tilde{\mathbf{a}}^i(T_k^i))$ if $s_i(T_{k-1}^i) = E$ and (ii) the payoffs obtained at time T_{k-1}^i and T_k^i if $s_i(T_{k-1}^i) = C$. Note that at time T_2^i only case S1 is applicable because at time $T_1^i = 1$ all the agents are in the Explore state. Therefore case S2 in the state update rule can be effective only for $k \geqslant 3$.

State Update Rule: Fix $p \in (0, 1)$.

For $k = 2, 3, \ldots$

S1. if $s_i(T_{k-1}^i) = E$

- if $BR_i(\tilde{\mathbf{a}}^i(T_k^i)) \neq \varnothing$, then $s_i(T_k^i) = E$

- else $\left(\text{i.e., } BR_i(\tilde{\mathbf{a}}^i(T_k^i)) = \varnothing \right)$

$$s_i(T_k^i) = \begin{cases} E & \text{with probability } p \\ \\ C & \text{with probability } 1 - p \end{cases}$$

S2. else (i.e., $s_i(T_{k-1}^i) = C$)

- if $BR_i(\tilde{\mathbf{a}}^i(T_k^i)) \neq \varnothing$, then $s_i(T_k^i) = E$

- else $\left(\text{i.e., } BR_i(\tilde{\mathbf{a}}^i(T_k^i)) = \varnothing \right)$

  - if $U_i(\tilde{\mathbf{a}}^i(T_k^i)) \neq U_i(\tilde{\mathbf{a}}^i(T_{k-1}^i))$, then $s_i(T_k^i) = E$

  - else $\left(\text{i.e., } U_i(\tilde{\mathbf{a}}^i(T_k^i)) = U_i(\tilde{\mathbf{a}}^i(T_{k-1}^i)) \right)$, then $s_i(T_k^i) = C$

In a nutshell, agent i transitions to or remains at state E if either $BR_i(\tilde{\mathbf{a}}^i(T_k^i)) \neq \varnothing$ or $U_i(\tilde{\mathbf{a}}^i(T_k^i)) \neq U_i(\tilde{\mathbf{a}}^i(T_{k-1}^i))$. Note that when an agent is at the converged state, seeing a better reply is equivalent to a change in the payoff information vector $\mathcal{C}_i(\tilde{\mathbf{a}}^i(\cdot))$. The second condition means that although there are no better replies, the received payoff has changed from the last time. In such a scenario, the agent prefers to transition to the Explore state.

The RSEM rule and the GBRR rules– In the RSEM rule when the agents are in the explore state, all the actions are chosen with positive probability. In the converged state, they play their previous action as long as the received payoff remains the same and there are no better replies. While the GBRR-I rule only allows agents to experiment among their better replies and, if there are no better replies, then the previous action is played. It is easy to see that the class of RSEM rules are more exploratory since in the RSEM rule the agent is allowed to choose actions that are not among the better replies. Hence, if we have a sequence of action profiles generated by the GBRR rule $\vec{\mathbf{a}} = (\mathbf{a}(t), \ t \in \mathbb{N})$, it can also be generated under the RSEM rule.

8.2 Convergence under RSEM

Let $\vec{\mathbf{a}} = (\mathbf{a}(t), \ t \in \mathbb{N})$ be the sequence of action profiles played by the players using the RSEM rule. First, we state the convergence results under the RSEM rule for the asynchronous update setting described in Section 7.5.1 when the game is generalized weakly acyclic.

Theorem 8.1. *Suppose that the game $\mathbb{G}$ is generalized weakly acyclic and that Assumptions A1 through A6 hold. Then, for any initial distribution $\mathcal{G} \in \Delta(\mathcal{A})$, under the RSEM rule,*

$$\mathbb{P}\left[\vec{\mathbf{a}} \in \mathcal{A}_\star^\infty\right] = 1.$$

Proof. From the discussion in Section 8.1, we conclude that the agents under the RSEM rule can also follow the GBRPs. Thus, the proof follows from arguments

similar to that for the GBRR rule in Section 7.7. ∎

Theorem 8.2. *Suppose that the game* $\mathbb{G}$ *is generalized weakly acyclic and that Assumptions A1 through A6 are in place. Then, there exist* $C_{rsem} < \infty$ *and* $\zeta_{rsem} \in (0,1)$ *such that, for any initial distribution* $\mathcal{G} \in \Delta(\mathcal{A})$ *under the RSEM rule*

$$\mathbb{P}\left[\vec{\mathbf{a}} \notin \mathcal{A}_t^{\infty}\right] \leqslant \min(1, C_{rsem} \cdot \zeta_{rsem}^t)$$

for all $t \in \mathbb{N}$.

The above-mentioned results not only hold for games that are generalized weakly acyclic, but also for games that satisfy the following interdependence assumption.

Assumption 8.1. *(Interdependence Assumption) For every* $\mathbf{a} = (a_i, \ i \in \mathcal{P}) \in \mathcal{A}$ *and* $J \subsetneq \mathcal{P}$, *there exist an agent* $i \notin J$ *and* $\mathbf{a}_J^* \in \prod_{j \in J} \mathcal{A}_j$ *such that* $U_i(\mathbf{a}) \neq U_i(\mathbf{a}_J^*, \mathbf{a}_{-J})$.

Assumption 8.1 simply states that, given any action profile and a strict subset J of the agents, we can find another agent $i \notin J$ whose payoff would change if the agents in J changed their actions to $\mathbf{a}_J^*$. Put differently, it implies that it is not possible to partition the set of agents into two subsets that do not interact with each other. The interdependence assumption has been used in the literature, for instance, to prove the convergence of action profile to efficient equilibria or Pareto optimal point [79, 87, 96]. Also, it is worth noting that there is no clear relationship between the class of GWAGs and the class of games satisfying the interdependence assumption. In general, there could be games that are GWAGs but do not satisfy the interdependence assumption and vice versa. Under this interdependence assumption, we can show that, if all agents update their actions according to the RSEM

rule, the action profile converges almost surely to a PSNE. However, we can have the desired convergence result under a weaker form of interdependence assumption.

Assumption 8.2. *(Weak Interdependence Assumption) For every $\mathbf{a} = (a_i, \ i \in \mathcal{P}) \in \mathcal{A}$ and $J \subsetneq \mathcal{P}$ such that $BR_i(\mathbf{a}) = \varnothing$ for all $i \notin J$, then there exist an agent $i^* \notin J$ and $\mathbf{a}_J^* \in \prod_{j \in J} \mathcal{A}_j$ such that either $U_{i*}(\mathbf{a}) \neq U_{i*}(\mathbf{a}_J^*, \mathbf{a}_{-J})$ or $BR_{i*}(\mathbf{a}_J^*, \mathbf{a}_{-J}) \neq \varnothing$.*

Assumption 8.2 implies that for any action profile $\mathbf{a} \in \mathcal{A}$ and a strict subset $J \subsetneq \mathcal{P}$ where all the agents outside the set J see no better reply, there exists an action profile $\mathbf{a}_J^*$ for the agents in the set J such that when the action profile $(\mathbf{a}_J^*, \mathbf{a}_{-J})$ is adopted at least one agent outside the set J either sees a better reply or a change in its received payoff.

The following two theorems suggest that when the game satisfies the weak interdependence assumption and has a non-empty set of PSNE(s), and a RSEM rule is employed for updating strategies, the action profile reaches a Nash equilibrium under payoff information delays and asynchronous updates with probability 1, under Assumptions A1 through A4.

Theorem 8.3. *Suppose that the game $\mathbb{G}$ satisfies Assumption 8.2 and has a nonempty set of PSNE(s) denoted by $\mathcal{A}_{NE}$ and that Assumptions A1 through A4 hold. Then, for any initial distribution $\mathcal{G} \in \Delta(\mathcal{A})$,*

$$\mathbb{P}\left[\vec{\mathbf{a}} \in \mathcal{A}_\star^\infty\right] = 1.$$

Proof. A proof of the theorem is provided in Section 8.3. ∎

In addition to the almost sure convergence of the action profile, we can establish that the probability that the action profile has not converged to a PSNE decays

geometrically with time t.

Theorem 8.4. *Suppose that the game $\mathbb{G}$ satisfies Assumption 8.2 and has a nonempty set of PSNE(s) denoted by $\mathcal{A}_{NE}$ and that Assumptions A1 through A4 are in place. Then, there exist $\tilde{C}_{rsem} < \infty$ and $\tilde{\zeta}_{rsem} \in (0,1)$ such that, for any initial distribution $\mathcal{G} \in \Delta(\mathcal{A})$,*

$$\mathbb{P}\left[\vec{\mathbf{a}} \notin \mathcal{A}_t^\infty\right] \leqslant \min(1, \tilde{C}_{rsem} \cdot \tilde{\zeta}_{rsem}^t).$$

Theorem 8.4 follows directly from Corollary 4 in the proof of Theorem 8.3 in Section 8.3.

Remark: While both Theorem 8.1 and 8.3 ensure convergence to the set of PSNE almost surely, it is important to make the distinction that starting from any initial action profile $\mathbf{a}(1)$ under the interdependence assumption the agents can converge to any PSNE, which is not always true for the case of GWAGs. This fact will become essential under various erroneous settings when we characterise the set of stochastically stable equilibria in this chapter and also in the next chapter.

8.3   Proof of Theorem 8.3

First, we define some notation: Let $\mathcal{Z} := \mathcal{S} \times \mathcal{A}$ and denote the pair $(\mathbf{s}(t), \mathbf{a}(t)) \in \mathcal{Z}$ by $\mathbf{z}(t)$, $t \in \mathbb{N}$. Define $\mathbf{s}^\star \in \mathcal{S}$ to be the state vector in which every agent is at state C, i.e, $\mathbf{s}^\star = (C, \ldots, C)$, and $\mathcal{Z}_{NE} = \{(\mathbf{s}^\star, \mathbf{a}^*) \in \mathcal{Z} \mid \mathbf{a}^* \in \mathcal{A}_{NE}\}$.

The theorem will be proved with the help of several lemmata we introduce. Their proofs are provided in Sections 8.4.1 through 8.4.3. For any $\mathbf{s} \in \mathcal{S}$,

$$C(\mathbf{s}) = \{i \in \mathcal{P} \mid s_i = C\} \quad \text{and} \quad E(\mathbf{s}) = \{i \in \mathcal{P} \mid s_i = E\}.$$

The first lemma states that, if the action profile at time t is not a PSNE, then even if all agents are at state C at time t, there is positive probability that at least one agent will transition to state E after $3\Delta_\eta$ periods (where Δ_η is the constant introduced in Assumption A4 in Section 7.5.1).

Lemma 8.5. *For every* $\mathbf{a} \notin \mathcal{A}_{NE}$ *and* $t \in \mathbb{N}$,

$$\mathbb{P}\left[E(\mathbf{s}(t + 3\Delta_\eta)) \neq \varnothing \mid \mathbf{z}(t) = (\mathbf{s}^\star, \mathbf{a})\right] \geqslant \zeta_0 > 0. \tag{8.2}$$

The second lemma shows that, if there is at least one agent at state E at time $t \in \mathbb{N}$, there is positive probability that the number of agents at state E will increase after a finite number of periods.

Lemma 8.6. *For every* $r \in \{1, 2, \ldots, n - 1\}$, *there exists* $0 < D_1 \leqslant 4\Delta_\eta$ *such that, for every* $t \in \mathbb{N}$ *and* $\mathbf{z} = (\mathbf{s}, \mathbf{a}) \in \mathcal{Z}$ *with* $|E(\mathbf{s})| = r$, *we have*

$$\mathbb{P}\left[|E(\mathbf{s}(t + D_1))| \geqslant r + 1 \mid \mathbf{z}(t) = \mathbf{z}\right] \geqslant \varsigma_r > 0. \tag{8.3}$$

The following corollary now follows from Lemmas 8.5 and 8.6, by repeatedly applying Lemma 8.6 until all agents switch to state E.

Corollary 3. *There exists* $0 < D \leqslant 4n\Delta_\eta$ *such that, for all* $\mathbf{z} \in \mathcal{Z}\backslash\mathcal{Z}_{NE}$ *and* $t \in \mathbb{N}$,

$$\mathbb{P}\left[|E(\mathbf{s}(t + D)| = n \mid \mathbf{z}(t) = \mathbf{z}\right] \geqslant \mu > 0. \tag{8.4}$$

The final lemma has two parts; first, it states that, if all agents are at state E at some time t, then for any $\mathbf{z}^\star \in \mathcal{Z}_{NE}$, there is positive probability that they will reach $\mathbf{z}^\star$ within $4\Delta_\eta$ periods. Second, if the agents are at some $\mathbf{z}^\star \in \mathcal{Z}_{NE}$ at time t, with positive probability they will remain at $\mathbf{z}^\star$ for good.

Lemma 8.7. *(i) Suppose that* $\mathbf{z}(t) = \mathbf{z} = (\mathbf{s}, \mathbf{a})$ *where* $|E(\mathbf{s})| = n$, *i.e., all agents are at state E. Then, for all* $\mathbf{z}^{\star} = (\mathbf{s}^{\star}, \mathbf{a}^{\star}) \in \mathcal{Z}_{NE}$, *we have*

$$\mathbb{P}\left[\mathbf{z}(t + 4\Delta_{\eta}) = \mathbf{z}^{\star} \mid \mathbf{z}(t) = \mathbf{z}\right] \geqslant \rho_1 > 0.$$

(ii) For every $\mathbf{z}^{\star} \in \mathcal{Z}_{NE}$,

$$\mathbb{P}\left[\mathbf{z}(t') = \mathbf{z}^{\star} \text{ for all } t' \geqslant t \mid \mathbf{z}(t) = \mathbf{z}^{\star}\right] \geqslant \rho_2 > 0.$$

The following corollary is a consequence of the above lemmas.

Corollary 4. *There exist* $0 < \tilde{D} \leqslant 4(n + 1)\Delta_{\eta}$ *such that, for all* $\mathbf{z} \in \mathcal{Z} \backslash \mathcal{Z}_{NE}$, $\mathbf{z}^{\star} \in \mathcal{Z}_{NE}$ *and* $t \in \mathbb{N}$, *we have*

$$\mathbb{P}\left[\mathbf{z}(t') = \mathbf{z}^{\star} \text{ for all } t' \geqslant t + \tilde{D} \mid \mathbf{z}(t) = \mathbf{z}\right] \geqslant \tilde{\mu} > 0. \tag{8.5}$$

Comparing Corollary 4 with Corollary 2 from the previous chapter, we observe that unlike the better reply rules the RSEM allows convergence to any PSNE starting from any initial action profile.

We now proceed with the proof of Theorem 8.3. Lemma 8.5 shows that, if the action profile is not a PSNE at time t, then after a finite number of periods, at least one agent will be at state E. Lemma 8.6 then claims that, whenever there is at least one agent at state E, after finitely many periods, all agents will be state E (Corollary 3). Once all agents are at state E, Lemma 8.7 asserts that they can reach any $\mathbf{z}^{\star} \in \mathcal{Z}_{NE}$ with positive probability after a finite number of periods and stay there forever. Finally, Corollary 4 implies that, for all $n \in \mathbb{N}$, $\mathbb{P}\left[\vec{\mathbf{a}} \notin \mathcal{A}_{n\tilde{D}+1}^{\infty}\right] \leqslant (1 - \tilde{\mu})^n$. The rest of the proof follows using arguments similar to that of the proof for Theorem 7.5.

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8.4 Proof of Lemmata for Theorem 8.3

8.4.1 Proof of Lemma 8.5

Because $\mathbf{a} \notin \mathcal{A}_{NE}$, when the agents adopt $\mathbf{a}$, there is at least one agent, say agent i^*, with an incentive to deviate from a_{i*} when the action profile $\mathbf{a}$ is in effect at the systems of all agents. We will prove that the state of agent i will transition to E with positive probability after $3\Delta_\eta$ periods.

We recall some of the notation introduced in the previous chapter: For each $i \in \mathcal{P}$, we define $k_{\max}^i : \mathbb{N} \to \mathbb{N}$, where $k_{\max}^i(t) = \max\{k \in \mathbb{N} \mid T_k^i \leqslant t\}$. Similarly, for all $i, j \in \mathcal{P}$, $j \neq i$, and $t \in \mathbb{N}$, $d_{i,j}^-(t) = V_{k_{\max}^i(t)}^{i,j}$ and $d_{i,j}^+(t) = V_{k_{\max}^i(t)+1}^{i,j}$, where $V_k^{i,j}$, as defined in Section 7.5, is the forward delay that action $a_i(T_k^i)$ experiences before influencing system j. For notational convenience, we denote the interval $\{t + (\ell - 1)\Delta_\eta + 1, \ldots, t + \ell \cdot \Delta_\eta\}$ by $\overline{T}_\ell(t)$, $\ell \in \mathbb{N}$, where Δ_η is the constant in Assumption A4 in Section 7.5.

Next, we define the following events. Note that $T_{k_{\max}^i(t)}^i$ (resp. $T_{k_{\max}^i(t)+1}^i$) denotes the last time by time t (resp. the first time after time t) at which player i updates its strategy.

$$\mathcal{E}_1 = \{T_{k_{\max}^i(t)}^i + d_{i,j}^-(t) \leqslant t + \Delta_\eta \text{ for all } i, j \in \mathcal{P}, j \neq i\}$$

$$\mathcal{E}_2 = \{\mathcal{T}^{i^*} \cap \overline{T}_\ell(t) \neq \varnothing, \ \ell = 2, 3\}$$

$$\mathcal{E}_3 = \{a_i(t') = a_i \text{ for all } i \in E(\mathbf{s}(t')), \ t' = t + 1, \ldots, t + 3\Delta_\eta\}$$

$$\mathcal{E}_4 = \{i^* \in E(\mathbf{s}(t' + 1)) \text{ if } i^* \in E(\mathbf{s}(t')) \text{ for all } t' = t, \ldots, t + 3\Delta_\eta - 1\}.$$

The event $\mathcal{E}_1$ implies that the action profile at time t is seen by all systems by time $t + \Delta_\eta$. The second event $\mathcal{E}_2$ simply states that agent i^* updates its action at least once in each of the intervals $\overline{T}_2$ and $\overline{T}_3$. Event $\mathcal{E}_3$ requires any agent at state E between $t + 1$ and $t + 3\Delta_\eta$ to choose the same action it did at time t (which will happen with strictly positive probability). Finally, the fourth event $\mathcal{E}_4$ demands that the agent i^*, once it switches to state E (which will happen by time $t + 3\Delta_\eta$ if the events $\mathcal{E}_1$ through $\mathcal{E}_3$ take place because $\mathbf{a} \notin \mathcal{A}_{NE}$), remain at state E till time $t + 3\Delta_\eta$.

We use the following lower bound to complete the proof.

$$\mathbb{P}\left[E(\mathbf{a}(t + 3\Delta_\eta)) \neq \varnothing \mid \mathbf{z}(t) = (\mathbf{s}^\star, \mathbf{a})\right]$$

$$\geqslant \mathbb{P}\left[E(\mathbf{a}(t + 3\Delta_\eta)) \neq \varnothing, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4 \mid \mathbf{z}(t) = (\mathbf{s}^\star, \mathbf{a})\right]$$

$$= \mathbb{P}\left[E(\mathbf{a}(t + 3\Delta_\eta)) \neq \varnothing \mid \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathbf{z}(t) = (\mathbf{s}^\star, \mathbf{a})\right]$$

$$\times \mathbb{P}\left[\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4 \mid \mathbf{z}(t) = (\mathbf{s}^\star, \mathbf{a})\right] \tag{8.6}$$

From the explanations of the events $\mathcal{E}_1$ through $\mathcal{E}_4$ above, it is clear that for every player $i \in \mathcal{P}$, we have $\tilde{\mathbf{a}}_{-i}^i\left(T_{k_{\max}^i(t+2\Delta_\eta)+1}^i\right) = \mathbf{a}_{-i}$. When this happens it is clear that at least one agent, namely agent i^* will see a better reply after $t + 2\Delta_\eta$ and move to state E. Hence, the first conditional probability in (8.6) is one. Hence, if we can show that the second conditional probability is also positive, the lemma is proved.

First, we rewrite $\mathbb{P}\left[\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4 \mid \mathbf{z}(t) = (\mathbf{s}^\star, \mathbf{a})\right]$ as follows.

$$\mathbb{P}\left[\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4 \mid \mathbf{z}(t) = (\mathbf{s}^\star, \mathbf{a})\right]$$

$$= \mathbb{P}\left[\mathcal{E}_1, \mathcal{E}_2 \mid \mathbf{z}(t) = (\mathbf{s}^\star, \mathbf{a})\right] \cdot \mathbb{P}\left[\mathcal{E}_3, \mathcal{E}_4 \mid \mathbf{z}(t) = (\mathbf{s}^\star, \mathbf{a}), \mathcal{E}_1, \mathcal{E}_2\right]. \tag{8.7}$$

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The first term in (8.7) is lower bounded by η^{n+2} from Assumption A4. Bounding the second term in (8.7),

$$\mathbb{P}\left[\mathcal{E}_3, \mathcal{E}_4 \mid \mathbf{z}(t) = (\mathbf{s}^\star, \mathbf{a}), \mathcal{E}_1, \mathcal{E}_2\right]$$

$$= \mathbb{P}\left[\mathcal{E}_4 \mid \mathbf{z}(t) = (\mathbf{s}^\star, \mathbf{a}), \mathcal{E}_1, \mathcal{E}_2\right] \cdot \mathbb{P}\left[\mathcal{E}_3 \mid \mathbf{z}(t) = (\mathbf{s}^\star, \mathbf{a}), \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_4\right]$$

$$\geq p^{3\Delta_\eta} \cdot \prod_{\ell=t+1}^{t+3\Delta_\eta} \mathbb{P}\left[a_i(\ell) = a_i \text{ for all } i \in E(s(\ell)) \mid a_i(m) = a_i \text{ for all } i \in E(s(m))\right.$$

$$\left. \text{for } m = t+1, \ldots, t+\ell-1, \mathbf{z}(t) = (\mathbf{s}^\star, \mathbf{a}), \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_4\right]$$

$$\geq p^{3\Delta_\eta} \delta^{3n\Delta_\eta} \tag{8.8}$$

where the last step follows by noting that agents in the Explore state experiment with every action with a probability of at least δ, while the step before that is a result of the fact that under the RSEM rule every agent remains at the explore state with a probability of at least p (recall that when an agent sees a better reply it continues at the explore state and when it sees no better replies it moves to the converged state with probability $1 - p$). Putting together, we have the following lower bound

$$\mathbb{P}\left[E(\mathbf{a}(t + 3\Delta_\eta)) \neq \varnothing \mid \mathbf{z}(t) = (\mathbf{s}^\star, \mathbf{a})\right] \geq (\delta^n \cdot p)^{3\Delta_\eta} \cdot \eta^{n+2} =: \zeta_0 > 0.$$

8.4.2   Proof of Lemma 8.6

We consider two cases.

c1. There exists at least one agent $i^+ \in C(\mathbf{s}(t))$ such that $U_{i^+}(\tilde{\mathbf{a}}^{i^+}(T_k^{i^+})) \neq U_{i^+}(\mathbf{a}(t))$ or $BR_{i^+}(\mathbf{a}(t)) \neq \varnothing$. In other words, either the payoff received by agent i^+ at the last time of update is different from what it would receive if its system

generated the payoff in response to the current action profile at time t, or the present action profile allows a better response for agent i^+.

c2. There is no such agent, i.e., for all $i \in C(\mathbf{s}(t))$, we have $U_i(\tilde{\mathbf{a}}^i(T_k^i)) = U_i(\mathbf{a}(t))$ and $BR_i(\mathbf{a}(t)) = \emptyset$.

Case c1: First, we define the following events.

$$\mathcal{E}_1' = \{T_{k_{\max}^i(t)}^i + d_{i,j}^-(t) \leqslant t + \Delta_\eta \text{ for all } i, j \in \mathcal{P}, i \neq j\}$$

$$\mathcal{E}_2' = \{\mathcal{T}^{i^+} \cap \overline{T}_\ell(t) \neq \emptyset, \ \ell = 2, 3\}$$

$$\mathcal{E}_3' = \{a_i(t') = a_i \text{ for all } i \in E(\mathbf{s}(t')), \ t' = t + 1, \ldots, t + 3\Delta_\eta\}$$

$$\mathcal{E}_4' = \{E(\mathbf{s}(t')) \subseteq E(\mathbf{s}(t' + 1)) \text{ for all } t' = t, \ldots, t + 3\Delta_\eta - 1\}$$

Using a similar argument used in Section 8.4.1, we obtain

$$\mathbb{P}\left[|E(\mathbf{s}(t + 3\Delta_\eta))| \geqslant r + 1 \mid \mathbf{z}(k) = \mathbf{z}\right]$$

$$\geqslant \mathbb{P}\left[|E(\mathbf{s}(t + 3\Delta_\eta))| \geqslant r + 1, \mathcal{E}_1', \mathcal{E}_2', \mathcal{E}_3', \mathcal{E}_4' \mid \mathbf{z}(k) = \mathbf{z}\right]$$

$$= \mathbb{P}\left[|E(\mathbf{s}(t + 3\Delta_\eta))| \geqslant r + 1 \mid \mathcal{E}_1', \mathcal{E}_2', \mathcal{E}_3', \mathcal{E}_4', \mathbf{z}(t) = \mathbf{z}\right] \cdot \mathbb{P}\left[\mathcal{E}_1', \mathcal{E}_2', \mathcal{E}_3', \mathcal{E}_4' \mid \mathbf{z}(t) = \mathbf{z}\right] \quad (8.9)$$

Event $\mathcal{E}_4'$ ensures that the set of exploring agents is non-decreasing, i.e., no exploring agents goes to the Converged state in the interval $(t, t + 3\Delta_\eta]$. Also the event $\mathcal{E}_3'$ does not allow any agent to change its action. Therefore, if events $\mathcal{E}_1'$ through $\mathcal{E}_4'$ take place, by time $t + 3\Delta_\eta$ agent i^+ would have received the payoff information corresponding to the action profile $\mathbf{a}(t)$ and switched its state to E, leading to an increase in the number of exploring agents. Thus, the first conditional

probability in (8.9) is equal to one. Decomposing the second term in (8.9), we obtain

$$\mathbb{P}\left[\mathcal{E}_1', \mathcal{E}_2', \mathcal{E}_3', \mathcal{E}_4' \mid \mathbf{z}(t) = \mathbf{z}\right]$$

$$= \mathbb{P}\left[\mathcal{E}_1', \mathcal{E}_2' \mid \mathbf{z}(t) = \mathbf{z}\right] \cdot \mathbb{P}\left[\mathcal{E}_3' \mid \mathbf{z}(t) = \mathbf{z}, \mathcal{E}_1', \mathcal{E}_2'\right] \cdot \mathbb{P}\left[\mathcal{E}_4' \mid \mathbf{z}(t) = \mathbf{z}, \mathcal{E}_1', \mathcal{E}_2', \mathcal{E}_3'\right]. \quad (8.10)$$

From Assumption A4, the first term in (8.10) can be lower bounded by η^{n+2}. The second term can be lower bounded by $\delta^{3n\Delta_\eta}$ following similar arguments used in the proof of Lemma 8.5. Lower bounding the third term in (8.10),

$$\mathbb{P}\left[\mathcal{E}_4' \mid \mathbf{z}(t) = \mathbf{z}, \mathcal{E}_1', \mathcal{E}_2', \mathcal{E}_3'\right] = \prod_{\ell=t}^{t+3\Delta_\eta-1} \mathbb{P}\Big[E(s(\ell)) \subseteq E(s(\ell+1)) \Big| E(s(m)) \subseteq E(s(m+1))$$

$$\text{for all } m = t, \ldots, \ell-1, \mathbf{z}(t) = \mathbf{z}, \mathcal{E}_1', \mathcal{E}_2', \mathcal{E}_3'\Big]$$

$$\geqslant p^{3n\Delta_\eta} \quad (8.11)$$

Returning to (8.9), the second conditional probability is lower bounded by $(\delta \cdot p)^{3n\Delta_\eta} \cdot \eta^{n+2} > 0$.

Case c2: Recall that, in this case, for every agent $i \in C(\mathbf{s}(t))$, we have $U_i(\tilde{\mathbf{a}}^i(T_k^i)) = U_i(\mathbf{a}(t))$ and $BR_i(\mathbf{a}(t)) = \varnothing$. Now, Assumption 8.2 implies that there exists $i' \notin E(\mathbf{s}(t)) := J_1$ and $\mathbf{a}_{J_1}^*$ such that if the agents in J_1 adopt the actions in $\mathbf{a}_{J_1}^*$ while the other agents choose the same action stipulated by $\mathbf{a}(t)$, then either agent i''s payoff changes or it sees a better reply. To move an agent in the Converged state to the Explore state, we will be using the weak interdependence assumption and argue that the exploring agents can change their actions such that at least one of the agents in the Converged state see a change in their payoff or a better reply and start exploring. As we will prove, this implies that there is positive probability that agent i' will switch its state to E after $4\Delta_\eta$ periods.

To this end, we define the following events

$$\mathcal{E}_1^+ = \{T^i_{k^i_{\max}(t+\Delta_\eta)} + d^-_{i,j}(t+\Delta_\eta) \leqslant t + 2\Delta_\eta \ \text{ for all } i,j \in \mathcal{P}, i \neq j\}$$

$$\mathcal{E}_2^+ = \{\mathcal{T}^{i'} \cap \overline{T}_\ell(t) \neq \varnothing, \ \ell = 3, 4\}$$

$$\mathcal{E}_3^+ = \{E(\mathbf{s}(t')) \subseteq E(\mathbf{s}(t'+1)) \text{ for all } t' = t, \ldots, t + 4\Delta_\eta - 1\}$$

$$\mathcal{E}_4^+ = \{\mathcal{T}^i \cap \overline{T}_1(t) \neq \varnothing \text{ for all } i \in J_1\}$$

$$\mathcal{E}_5^+ = \{a_i(t') = \tilde{a}_i \text{ for all } i \in \mathcal{P} \text{ and } t' = T^i_{k^i_{\max}(t)+1}, \ldots, t + 4\Delta_\eta\}$$

where

$$\tilde{a}_i = \begin{cases} a_i(t) & \text{if } i \notin J_1, \\[2mm] a_i^* & \text{if } i \in J_1. \end{cases}$$

The rest of the proof follows from a similar argument.

$$\mathbb{P}\left[|E(\mathbf{s}(t + 4\Delta_\eta))| \geqslant r + 1 \mid \mathbf{z}(t) = \mathbf{z}\right]$$

$$\geqslant \mathbb{P}\left[|E(\mathbf{s}(t + 4\Delta_\eta))| \geqslant r + 1, \mathcal{E}_1^+, \mathcal{E}_2^+, \mathcal{E}_3^+, \mathcal{E}_4^+, \mathcal{E}_5^+ \mid \mathbf{z}(t) = \mathbf{z}\right]$$

$$= \mathbb{P}\left[|E(\mathbf{s}(t + 3\Delta_\eta))| \geqslant r + 1 \mid \mathcal{E}_1^+, \mathcal{E}_2^+, \mathcal{E}_3^+, \mathcal{E}_4^+, \mathcal{E}_5^+, \mathbf{z}(t) = \mathbf{z}\right]$$

$$\times \mathbb{P}\left[\mathcal{E}_1^+, \mathcal{E}_2^+, \mathcal{E}_3^+, \mathcal{E}_4^+, \mathcal{E}_5^+ \mid \mathbf{z}(t) = \mathbf{z}\right]. \tag{8.12}$$

Event $\mathcal{E}_4^+$ ensures that all the agents in the Explore state update during the interval $\overline{T}_1(t)$, while event $\mathcal{E}_5^+$ ensures that the exploring agents update their actions according to the action profile $(\mathbf{a}_{-J_1}, \mathbf{a}^*_{J_1})$ and keep it fixed. Event $\mathcal{E}_1^+$ enforces that, under events $\mathcal{E}_4^+$ and $\mathcal{E}_5^+$, the current action profile at time $t + \Delta_\eta - (\mathbf{a}_{-J_1}, \mathbf{a}^*_{J_1})$ takes effect in the system of agent i' by time $t + 2\Delta_\eta$. Finally, events $\mathcal{E}_2^+$ and $\mathcal{E}_3^+$ ensure that agent i^+ reaches the Explore state by time $t + 4\Delta_\eta$ and stays there. From the definitions of the above events, the first conditional probability is one because agent

i' will have switched its state to E (from C) by time $t + 4\Delta_\eta$ and the exploring agents would still be exploring if the events $\mathcal{E}_1^+$ through $\mathcal{E}_5^+$ take place.

Lower bounding the second term in (8.12),

$$\mathbb{P}\left[\mathcal{E}_1^+, \mathcal{E}_2^+, \mathcal{E}_3^+, \mathcal{E}_4^+, \mathcal{E}_5^+ \mid \mathbf{z}(t) = \mathbf{z}\right]$$

$$= \mathbb{P}\left[\mathcal{E}_4^+ \mid \mathbf{z}(t) = \mathbf{z}\right] \cdot \mathbb{P}\left[\mathcal{E}_1^+, \mathcal{E}_2^+ \mid \mathcal{E}_4^+, \mathbf{z}(t) = \mathbf{z}\right] \tag{8.13}$$

$$\times \mathbb{P}\left[\mathcal{E}_3^+ \mid \mathcal{E}_1^+, \mathcal{E}_2^+, \mathcal{E}_4^+, \mathbf{z}(t) = \mathbf{z}\right] \cdot \mathbb{P}\left[\mathcal{E}_5^+ \mid \mathcal{E}_1^+, \mathcal{E}_2^+, \mathcal{E}_3^+, \mathcal{E}_4^+, \mathbf{z}(t) = \mathbf{z}\right] \tag{8.14}$$

Using Assumption A4, we lower bound the first and second term in (8.13) by η^r and η^{n+2} respectively. Since the event $\mathcal{E}_4'$ defined in the previous case is similar to event $\mathcal{E}_3^+$, we can use a similar argument to lower bound the first term in (8.14) by $p^{4n\Delta_\eta}$. A lower bound of $\delta^{4n\Delta_\eta}$ is obtained for the final term in (8.14) by following similar steps as in the proof for Lemma 8.5 (see (8.8)).

Therefore, the second conditional probability in (8.12) is lower bounded by –
$\eta^{n+2+r} (\delta \cdot p)^{4n\Delta_\eta} := \varsigma_r$, which is smaller than the lower bound obtained in case c1 –

$$\left((\delta \cdot p)^{3n\Delta_\eta} \cdot \eta^{n+2}\right).$$

8.4.3 Proof of Lemma 8.7

We first prove Lemma 8.7(i). First, define the following events.

$$\mathcal{E}_1^{\#} = \{\mathcal{T}^i \cap \overline{T}_1(t) \neq \varnothing \text{ for all } i \in \mathcal{P}\}$$

$$\mathcal{E}_2^{\#} = \{T_{k_{\max}^i(t+\Delta_\eta)}^i + d_{i,j}^-(t) \leqslant t + 2\Delta_\eta \text{ for all } i, j \in \mathcal{P}, j \neq i\}$$

$$\mathcal{E}_3^{\#} = \{a_i(t') = a_i^{\star} \text{ for all } i \in \mathcal{P} \text{ and } t' = T_{k_{\max}^i(t)+1}^i, \ldots, t + 4\Delta_\eta\}$$

$$\mathcal{E}_4^{\#} = \{\mathcal{T}^i \cap \overline{T}_\ell(t) \neq \varnothing \text{ for all } i \in \mathcal{P} \text{ and } \ell = 3, 4\}$$

$$\mathcal{E}_5^{\#} = \{s_i(T_{k_{\max}^i(t+3\Delta_\eta)+1}^i) = C \text{ for all } i \in E(\mathbf{s}(t + 3\Delta_\eta))\}$$

Events $\mathcal{E}_1^{\#}, \mathcal{E}_2^{\#}, \mathcal{E}_3^{\#}$ together imply that all agents update their actions to $a_i^{\star}$ during $\overline{T}_1(t)$ and their actions go into effect by $t + 2\Delta_\eta$. Events $\mathcal{E}_4^{\#}$ and $\mathcal{E}_5^{\#}$ state that all agents update at least once during the intervals $\overline{T}_3(t)$ and $\overline{T}_4(t)$, and switch their state to C. Hence, together these events mean that all agents are at state C and adopt the PSNE $\mathbf{a}^{\star}$ at time $t + 4\Delta_\eta$. Therefore,

$$\mathbb{P}\left[\mathbf{z}(t + 4\Delta_\eta) = \mathbf{z}^{\star} \mid \mathbf{z}(t) = \mathbf{z}\right]$$

$$\geqslant \mathbb{P}\left[\mathbf{z}(t + 4\Delta_\eta) = \mathbf{z}^{\star} \mid \mathcal{E}_1^{\#}, \mathcal{E}_2^{\#}, \mathcal{E}_3^{\#}, \mathcal{E}_4^{\#}, \mathcal{E}_5^{\#}, \mathbf{z}(t) = \mathbf{z}\right]$$

$$\times \mathbb{P}\left[\mathcal{E}_1^{\#}, \mathcal{E}_2^{\#}, \mathcal{E}_3^{\#}, \mathcal{E}_4^{\#}, \mathcal{E}_5^{\#} \mid \mathbf{z}(t) = \mathbf{z}\right]. \tag{8.15}$$

As argued above, the first conditional probability in (8.15) is one. Lower bounding the second conditional probability,

$$\mathbb{P}\left[\mathcal{E}_1^{\#}, \mathcal{E}_2^{\#}, \mathcal{E}_3^{\#}, \mathcal{E}_4^{\#}, \mathcal{E}_5^{\#} \mid \mathbf{z}(t) = \mathbf{z}\right]$$

$$= \mathbb{P}\left[\mathcal{E}_1^{\#}, \mathcal{E}_2^{\#}, \mathcal{E}_4^{\#} \mid \mathbf{z}(t) = \mathbf{z}\right] \cdot \mathbb{P}\left[\mathcal{E}_3^{\#} \mid \mathbf{z}(t) = \mathbf{z}, \mathcal{E}_1^{\#}, \mathcal{E}_2^{\#}, \mathcal{E}_4^{\#}\right] \tag{8.16}$$

$$\times \mathbb{P}\left[\mathcal{E}_5^{\#} \mid \mathbf{z}(t) = \mathbf{z}, \mathcal{E}_1^{\#}, \mathcal{E}_2^{\#}, \mathcal{E}_3^{\#}, \mathcal{E}_4^{\#}\right]. \tag{8.17}$$

From previous arguments the first two terms in (8.16) are lower bounded by η^{4n} and $\delta^{4n\Delta_\eta}$ respectively. To lower bound (8.17), observe that conditioned on the events $\{\mathcal{E}_i^\#, \ i = 1, 2, 3, 4\}$, the agents in the Explore state at time $t + 3\Delta_\eta$ all see an empty better reply set whenever they update in the interval $\overline{T}_4(t)$. Therefore, under the RSEM rule, (8.17) is lower bounded by $(1 - p)^n$. Collecting the lower bounds, the second conditional probability in (8.15) can be lower bounded by $\eta^{4n} \cdot \delta^{4n\Delta_\eta} \cdot (1 - p)^n$. This completes the proof of Lemma 8.7(i).

Before we prove Lemma 8.7(ii), note that even though the agents are at $\mathbf{z}^\star \in \mathcal{Z}_{NE}$ at time t, it is still possible for some agents to transition to state E due to the delays in the system. Hence, the conditional probability in the lemma is strictly positive as we will show, but is in general less than one.

First, we define the following events.

$$\mathcal{E}_1^- = \{\mathcal{T}^i \cap \overline{T}_\ell(t) \neq \varnothing \text{ for all } i \in \mathcal{P} \text{ and } \ell = 2, 3\}$$

$$\mathcal{E}_2^- = \{a_i(t') = a_i^\star \text{ for all } i \in \mathcal{P} \text{ and } t' = T_{k_{\max}^i(t)+1}^i, \ldots, t + 3\Delta_\eta\}$$

$$\mathcal{E}_3^- = \{T_{k_{\max}^i(t)}^i + d_{i,j}^-(t) \leqslant t + \Delta_\eta \text{ for all } i, j \in \mathcal{P}, j \neq i\}$$

$$\mathcal{E}_4^- = \{s_i(T_{k_{\max}^i(t+2\Delta_\eta)+1}^i) = C \text{ for all } i \in E(\mathbf{s}(t + \Delta_\eta))\}$$

Note that events $\mathcal{E}_1^-$ through $\mathcal{E}_4^-$ mean that all agents continue to play the action profile $\mathbf{a}^\star$ between time t and $t + 3\Delta_\eta$ (and afterwards). All agents see $\mathbf{a}^\star$ after time $t + \Delta_\eta$ (event $\mathcal{E}_3^-$) and all agents update during the interval $\overline{T}_2(t)$ (event $\mathcal{E}_1^-$). Thus, when the agents update during the interval $\overline{T}_3(t)$, conditional on event $\mathcal{E}_2^-$, all agents will see the payoff in response to $\mathbf{a}^\star$, i.e., $\tilde{\mathbf{a}}^i \left(T_{k_{\max}^i(t+2\Delta_\eta)+1}^i \right) = \mathbf{a}^*$. Finally, the agents' states will have changed to C by time $t + 3\Delta_\eta$ (event $\mathcal{E}_4^-$) and, as a

result, they will keep playing $\mathbf{a}^\star$ after time $t + 3\Delta_\eta$.

Following essentially the same argument, we have

$$\mathbb{P}\left[\mathbf{z}(t') = \mathbf{z}^\star \text{ for all } t' \geqslant t \mid \mathbf{z}(t) = \mathbf{z}\right]$$

$$\geqslant \mathbb{P}\left[\mathbf{z}(t') = \mathbf{z}^\star \text{ for all } t' \geqslant t \mid \mathbf{z}(t) = \mathbf{z}, \mathcal{E}_1^-, \mathcal{E}_2^-, \mathcal{E}_3^-, \mathcal{E}_4^-\right]$$

$$\times \mathbb{P}\left[\mathcal{E}_1^-, \mathcal{E}_2^-, \mathcal{E}_3^-, \mathcal{E}_4^-, \mid \mathbf{z}(t) = \mathbf{z}\right], \tag{8.18}$$

where the first conditional probability in (8.18) is one. The second conditional probability can be lower bounded by $\eta^{2n} \cdot \delta^{3n\Delta_\eta}(1-p)^n$.

8.5   Case with Erroneous Payoff Information

Until now in this chapter, we assumed that accurate payoff information $(\mathcal{C}_i(\tilde{\mathbf{a}}^i(t)); U_i(\tilde{\mathbf{a}}^i(t)))$ is available to the players for updates. For reasons already discussed in Section 7.6, it would be helpful to relax the above setting by assuming that the agents make mistakes while determining the payoff information $\mathcal{C}_i(\tilde{\mathbf{a}}^i(t))$. Once again for ease of exposition, we consider the setting of the synchronous update case, which we believe can be easily extended to the asynchronous update scenario.

8.5.1   Preliminaries

The setting is that of Section 7.6. Recall that the probabilities of erroneous classification of actions are given by mappings $\mathbf{q}_t^i : \mathcal{A} \times \mathcal{A}_i \to \Delta(\mathcal{S})$, where $\mathbf{q}_t^i(\mathbf{a}, a_i') :=$ $(q_t^i(\varsigma; \mathbf{a}, a_i'), \varsigma \in \mathcal{S})$ is a probability distribution over $\mathcal{S}$. We assume that the mappings $\{\mathbf{q}_t^i, \; i \in \mathcal{P}, \; t \in \mathbb{N}^+\}$ satisfy Assumptions 7.1 and 7.2 stated in the previous chapter.

As we did in Section 7.6, we can model the evolution of joint states $\{(\mathbf{s}(t), \mathbf{a}(t)), t \in$ $\mathbb{N}\}$ as a nonhomogeneous (discrete-time) Markov chain, where the transition matrix at time $t \in \mathbb{N}$, denoted by $\mathbf{P}(t)$, is determined by: (i) $\boldsymbol{\beta}_i$, $i \in \mathcal{P}$, and the parameters of the RSEM rule p and δ, and (ii) the mapping $\mathbf{q}_t^i$, $i \in \mathcal{P}$ and $t \in \mathbb{N}^+$. For each $t \in \mathbb{N}$, a time homogeneous (discrete-time) Markov chain $\mathbb{X}^t$ can be defined with a common state space $\mathcal{C} \subset \mathcal{Z}$ [1] and transition matrix $\mathbf{P}^t = \mathbf{P}(t)$. For each $t \in \mathbb{N}$ the Markov chain $\mathbb{X}^t$ will be ergodic, which follows from the fact that, given any initial state with $\mathbf{z}(1) = ((E, \ldots, E), \mathbf{a})$ with $\mathbf{a} \in \mathcal{A}$, we will revisit the same initial state $\mathbf{z}(1)$ w.p. 1. The MC $\mathbb{X}^t$ will thus have an unique stationary distribution denoted by $\boldsymbol{\mu}^t = (\mu^t(\mathbf{z}), \ \mathbf{z} \in \mathcal{C})$. In this context, the MC in which the players make correct payoff observation is called an unperturbed Markov chain.

As in Section 7.6 we can construct a minimum resistance W-tree rooted at any $\mathbf{z} \in \mathcal{C}$ denoted by $\Gamma^*(\mathbf{z})$ with resistance $\pi_{\min}(\mathbf{z}) := \pi(\Gamma^*(\mathbf{z}))$. From Theorem 7.7 we know that, a state will be stochastically stable if its corresponding W-tree has the least resistance among all the minimum resistance W-trees. Next, we present results that characterize the set of stochastically stable states $\mathcal{C}_{SS} \subset \mathcal{C}$.

8.5.2 Main result

Theorem 8.8. *As $t \to \infty$, $\boldsymbol{\mu}^t \to \boldsymbol{\mu}^0$, where $\boldsymbol{\mu}^0$ is a stationary distribution of the unperturbed Markov chain. In addition, $\mu^0(\mathbf{z}^\star) > 0$ if and only if (i) $\mathbf{z}^\star \in \mathcal{Z}_{NE}$ and (ii) $\psi(\mathbf{a}^\star) = \max_{\mathbf{a}' \in \mathcal{Z}_{NE}} \psi(\mathbf{a}')$, where $\psi(\mathbf{a}) = \min_{i \in \mathcal{P}} \min_{a_i' \in \mathcal{A}_i \setminus \{a_i\}} \gamma_i(\mathbf{a}, a_i', B)$ for*

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[1]There may be some states in $\mathcal{Z}$ which are unreachable from the initial state $\mathbf{z}(1)$, and we do not include those unreachable states in the state space $\mathcal{C}$.

$\mathbf{a} \in \mathcal{A}_{NE}$.

Proof. First we prove the first statement. For any $\mathbf{z}' \in \mathcal{C}\backslash\mathcal{Z}_{NE}$, from Corollary 4 we know that there exists a zero resistance path from $\mathbf{z}'$ to any joint state in $\mathcal{Z}_{NE}$. In other words according to the terminology introduced in Section 7.6, the domain of attraction of the joint state $\mathbf{z}'$ is $\mathcal{Z}\backslash\mathcal{Z}_{NE}$. Also, it is clear that the minimum resistance path from any state in $\mathcal{Z}_{NE}$ to any state in $\mathcal{C}\backslash\mathcal{Z}_{NE}$ is strictly positive. Therefore, for all $\mathbf{z}' \in \mathcal{C}\backslash\mathcal{Z}_{NE}$,

$$\pi_{\min}(\mathbf{z}') > \pi_{\min}(\mathbf{z}^*), \ \mathbf{z}^* \in \mathcal{Z}_{NE}$$

which is sufficient to prove the first statement.

To prove the second statement we once again use Corollary 4 to conclude that any state $\mathbf{z}' \in \mathcal{C}\backslash\mathcal{Z}_{NE}$ will contribute zero resistance to the resistance tree $\Gamma^*(\mathbf{z}^*)$ for any $\mathbf{z}^* \in \mathcal{Z}_{NE}$. Therefore, for $\mathbf{z}^* \in \mathcal{Z}_{NE}$

$$\pi_{\min}(\mathbf{z}^*) = \sum_{\mathbf{z} \in \mathcal{Z}_{NE}\backslash\{\mathbf{z}^*\}} \rho(\mathbf{z}, \mathbf{z}^*). \tag{8.19}$$

Choose $\mathbf{z}'' = (\mathbf{s}^*, \mathbf{a}''), \mathbf{z}^* = (\mathbf{s}^*, \mathbf{a}^*) \in \mathcal{Z}_{NE}$. The minimum resistance transition from the state $\mathbf{z}''$ to a state where at least one agent is exploring, i.e., a joint state in $\mathcal{C}\backslash\mathcal{Z}_{NE}$, has a resistance of $\psi(\mathbf{a}'')$. Once a state in $\mathcal{C}\backslash\mathcal{Z}_{NE}$ is reached, we know that a zero resistance path to $\mathbf{z}^*$ exists, which implies that $\rho(\mathbf{z}'', \mathbf{z})$ is simply $\psi(\mathbf{a}'')$. Returning to (8.19), we obtain

$$\pi_{\min}(\mathbf{z}^*) = \sum_{\mathbf{z}=(\mathbf{s}^*,\mathbf{a})\in\mathcal{Z}_{NE}\backslash\{\mathbf{z}^*\}} \psi(\mathbf{a}) = \sum_{\mathbf{z}=(\mathbf{s}^*,\mathbf{a})\in\mathcal{Z}_{NE}} \psi(\mathbf{a}) - \psi(\mathbf{a}^*) \tag{8.20}$$

which implies that

$$\mathcal{C}_{SS} = \arg\min_{\mathbf{z}^*\in\mathcal{Z}_{NE}} \pi_{\min}(\mathbf{z}^*) = \arg\max_{\mathbf{z}^*\in\mathcal{Z}_{NE}} \psi(\mathbf{a}^*). \tag{8.21}$$

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For $\mathbf{z} = (\mathbf{s}^*, \mathbf{a}) \in \mathcal{Z}_{NE}$, by definition, $\psi(\mathbf{a})$ is the resistance of the minimum resistance transition from the PSNE $\mathbf{z}$ to any other state. For all time $t \in \mathbb{N}^+$, the probability of deviation from a PSNE joint state $(\mathbf{s}^*, \mathbf{a})$ is given by $\sum_{\mathbf{z}' \in \mathcal{C} \setminus \mathcal{Z}_{NE}} P_{\mathbf{z}, \mathbf{z}'}(t)$. We know for each $\mathbf{z} = (\mathbf{s}^*, \mathbf{a}) \in \mathcal{Z}_{NE}$ there exists a constant $c(\mathbf{a}) > 0$ such that $\sum_{\mathbf{z}' \in \mathcal{C} \setminus \mathcal{Z}_{NE}} P_{\mathbf{z}, \mathbf{z}'}(t) \sim c(\mathbf{a}) \epsilon_t^{\psi(\mathbf{a})}$. Therefore for large t, Theorem 8.8 ensures stochastic stability of those PSNE(s) which under the error model has among the least probability to get destabilised.

From the expression of C_{SS} in (8.21), it is worth noting that the RSEM rule is such that the set of stochastically stable states is completely characterised by the error model $\{\mathbf{q}_t^i, \ i \in \mathcal{P}, t \in \mathbb{N}^+\}$. This is certainly not the case for the GBRR rules, where the set of stochastically stable states depend both on the structure of the game and the error model.

Chapter 9:   Monitoring Rules under Erroneous Execution

In the previous chapter we introduced the class of monitoring rules by looking at a simple version called the Reduced Simple Experimentation with Monitoring (RSEM) rule. We considered asynchronous updates by agents under delayed payoff information and errors due to faulty payoff information, more particularly, errors due erroneous estimation of SBRs. In this chapter we specifically consider errors due to faulty execution of intended actions. This requires a generalization of the simple RSEM by endowing an agent with several monitoring or alert states such that the agent/controller becomes resilient to occasional deviations from expected situations but is instead responsive to long term changes.

9.1   Resilience of PSNEs

As mentioned earlier, we are interested in designing a distributed learning rule that will allow the agents to target (more) resilient equilibria. To this end, we first define the *resilience* of a PSNE as follows.

Denote the set of PSNEs by $\mathcal{A}_{NE} \subset \mathcal{A}$. Let $d : \mathcal{A} \times \mathcal{A} \to \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ be a mapping that measures the distance between two action profiles, where

$$d(\mathbf{a}^1, \mathbf{a}^2) = \sum_{i \in \mathcal{P}} \mathbf{1}\left[a_i^1 \neq a_i^2\right], \quad \mathbf{a}^1, \mathbf{a}^2 \in \mathcal{A}. \tag{9.1}$$

For each $\tau \in \mathbb{Z}_+$, let $\mathcal{N}_\tau : \mathcal{A} \to 2^{\mathcal{A}}$, where $\mathcal{N}_\tau(\mathbf{a}) = \{\mathbf{a}' \in \mathcal{A} \mid d(\mathbf{a}, \mathbf{a}') \leqslant \tau\}$, $\mathbf{a} \in \mathcal{A}$. In other words, $\mathcal{N}_\tau(\mathbf{a})$ is the set of action profiles whose distance from $\mathbf{a}$ is at most τ according to the distance measure in (9.1).

Recall from Chapter 7 that $\mathcal{C}_i$ and BR_i are the classification and better reply mappings respectively. The resilience of a PSNE is given by a mapping $R : \mathcal{A}_{NE} \to \mathbb{Z}_+$, where for $\mathbf{a}^\star \in \mathcal{A}_{NE}$,

$$R(\mathbf{a}^\star) = \max\{\tau \geqslant 0 \mid BR_i(a_i^\star, \mathbf{a}'_{-i}) = \varnothing \text{ for all } i \in \mathcal{P} \text{ and } \mathbf{a}' \in \mathcal{N}_\tau(\mathbf{a}^\star)\}.$$

It is clear from the definition that the resilience of a PSNE is the largest number of deviating agents the PSNE can tolerate before at least one agent finds an incentive to switch its action. When $R(\mathbf{a}^\star) = K$, we say that $\mathbf{a}^\star$ is K-resilient. Define $R_{\max}^\star := \max_{\mathbf{a}^\star \in \mathcal{A}_{NE}} R(\mathbf{a}^\star)$ to be the maximum resilience among all PSNEs.

9.2   Proposed Update Rule

In this section, we present our proposed algorithm, called the Simple Experimentation with Monitoring (SEM), for seeking resilient PSNEs. This is an extension of the RSEM algorithm discussed in the previous chapter. In order to describe the algorithm, we first introduce some notation. At each time $t \in \mathbb{N}$, the state of agent $i \in \mathcal{P}$ is denoted by $s_i(t) \in \Psi'_0$. The set Ψ'_0 consists of (i) Converged (C), (ii) Explore (E), and (iii) Alert $(L_1, \ldots, L_T)$, where T is the number of alert states. We let $\Psi' = \prod_{i \in \mathcal{P}} \Psi'_0$, and the state vector at time $t \in \mathbb{N}$ is given by $\mathbf{s}(t) = (s_i(t),\ i \in \mathcal{P}) \in \Psi'$. The rule governing the update of an agent's state will be explained shortly.

- **Action profile *selected* by agents vs. action profile *adopted* by agents:** Since we are interested in scenarios where the agents intermittently make mistakes and execute incorrect actions, we distinguish (i) the action that is *selected* by agent i at time $t \in \mathbb{N}$, namely $a_i(t)$, according to the action update rule (described in Section 9.2.1) and (ii) the action that is *adopted* at time $t \in \mathbb{N}$, denoted by $\tilde{a}_i(t)$ (Section 9.2.2). We denote the action profile *selected* by the agents at time $t \in \mathbb{N}$ by $\mathbf{a}(t)$, and the action profile *adopted* at time t by $\tilde{\mathbf{a}}(t)$. Note that $\mathbf{a}(t)$ and $\tilde{\mathbf{a}}(t)$ may not be identical unless no agent makes a mistake at time t.

Payoff Information for updates – First, let us denote $\hat{\mathbf{a}}_i(t) = (a_i(t), \tilde{\mathbf{a}}_{-i}(t))$. Clearly, $\hat{\mathbf{a}}_i(t)$ is the action profile that would be in effect at time t assuming agent i did not make any erroneous implementation of its selected action. We assume that the payoff information available to agent i at time $t \geqslant 2$ is of the form – $\big((\mathcal{C}_i(a_i, \hat{\mathbf{a}}_i(t-1)),\ a_i \in \mathcal{A}_i)\ ;\ U_i(\tilde{\mathbf{a}}(t-1))\big)$. Therefore, the payoff classification information evaluates which strategies would have yielded a higher or a lower payoff than the previously *selected* strategy $a_i(t-1)$. Clearly, it is sensible to use the payoff classification information with respect to the intended action $a_i(t-1)$ irrespective of whether that action was actually implemented or not. We also assume that an agent knows whether or not an action was correctly implemented in the previous time instant. This makes sense in many practical scenarios where an agent can monitor if the intended action was actually implemented. For instance, returning to the example of distributed traffic routing considered in Chapter 6, it is reasonable to assume that a driver can comprehend in hindsight if it correctly followed the suggested route.

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Now, in case there were an error in implementation by a particular agent i, i.e., $\tilde{a}_i(t-1) \neq a_i(t-1)$, there might be scenarios where obtaining the aforementioned payoff classification information might be difficult for that particular agent. Although we do not consider such a setting here, we would however like to comment that the algorithm can be easily modified to account for such scenarios.

9.2.1 Action updates with no errors – $\mathbf{a}(t), t \in \mathbb{N}$

In this subsection, we first describe how the agents choose their actions according to the payoff feedback. At time $t = 1$, the agents choose their action profile $\mathbf{a}(1)$ according to some joint distribution $\mathcal{G}$. Starting with $t = 2$, at time $t \in \mathbb{N}$, every agent updates its action using some update rule, which will be explained shortly. Fix $\delta \in (0, 1/(\max_{i \in \mathcal{P}} |\mathcal{A}_i|))$ and $\boldsymbol{\beta}_i : \mathcal{S}^{A_i} \rightarrow \Delta(\mathcal{A}_i),\ i \in \mathcal{P}$.

Action Selection Rule:

For $t = 2, 3, \ldots$,

- if $s_i(t) = E$

  – choose $a_i(t) = a_i$ with probability $\beta_i(a_i; \mathcal{C}_i(a_i, \hat{\mathbf{a}}_i(t-1))) \geqslant \delta$ for all $a_i \in \mathcal{A}_i$

- else (i.e., $s_i(t) = C$ or L_ℓ, $\ell \in \mathcal{L} := \{1, 2, \ldots, T\}$)

  – set $a_i(t) = a_i(t-1)$

Note that, under SEM, an agent may choose a new action only if it is at state E. Otherwise, it continues to play the same action employed at the previous time.

9.2.2 Adopted action profiles – $\tilde{\mathbf{a}}(t), t \in \mathbb{N}$

As mentioned earlier, we are interested in designing a distributed learning rule that will allow the agents to seek out resilient PSNEs under erroneous decision-making by the agents. Here, we describe one way to model these erroneous actions with the help of mutually independent Bernoulli processes[1]: For each agent $i \in \mathcal{P}$, let $\mathcal{B}_i = \{B_i(t), t \in \mathbb{N}\}$ be a Bernoulli process with $\mathbb{P}[B_i(t) = 1] = \epsilon$ for some $\epsilon > 0$. The random variable $B_i(t)$, $i \in \mathcal{P}$ and $t \in \mathbb{N}$, indicates whether or not system of agent i makes a mistake and executes an incorrect action at time t as follows. For each $i \in \mathcal{P}$ and $t \in \mathbb{N}_+$,

E1. if $B_i(t) = 0$, $\tilde{a}_i(t) = a_i(t)$;

E2. else (i.e., $B_i(t) = 1$), $\mathbb{P}[\tilde{a}_i(t) = a_i] = g_i(a_i(t), a_i)$ independently of the past, where for each $a_i \in \mathcal{A}_i$, $g_i(a_i, \cdot)$ is some arbitrary distribution over $\mathcal{A}_i \backslash \{a_i\}$ with $g_i(a_i, a_i') > 0$ for all $a_i' \in \mathcal{A}_i \backslash \{a_i\}$.

According to this setup, each agent adopts an erroneous action with probability ϵ at each time $t \in \mathbb{N}_+$, independently of the past and other agents. As previously

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[1] While we use a specific model for introducing perturbations in the system as a reasonable approximation, identifying the correct or accurate faulty behavior is not the focus here and other perturbation models may be used instead.

mentioned, for each $i \in \mathcal{P}$ and $t = 2, 3, \ldots$, we assume at time t agent i can observes $B_i(t-1)$. In other words, at a particular time t, agent i knows whether an action was correctly implemented or not in the previous time instant $t-1$.

9.2.3 State dynamics

As explained in the previous subsections, under SEM, agents' selections of actions depend on their states. Hence, the dynamics of the agents' states play a key role in the algorithm. In this subsection, we explain how the agents update their states based on the received payoff information.

For each $i \in \mathcal{P}$ and $t \in \mathbb{N}_+$ such that $s_i(t) \in \{L_\ell, \ \ell \in \mathcal{L}\}$, let

$$\nu_t^i = \max\{t' < t : s_i(t') = C\} \tag{9.2}$$

and $U_i^*(t) = U_i(\tilde{\mathbf{a}}(\nu_t^i))$. Note that the payoff information vector is determined by the actions *adopted* by other agents, not those *selected* by them. Hence, even when the agents select a PSNE at time t, i.e., $\mathbf{a}(t) \in \mathcal{A}_{NE}$, the adopted action profile $\tilde{\mathbf{a}}(t)$ and, hence, the payoff information vectors seen by the agents may differ from those of the selected PSNE.

At time $t = 1$, all agents are initially at state E, i.e., $\mathbf{s}(1) = (E, \ldots, E)$. The state of agent i at time $t \geqslant 2$ depends on its state at time $t-1$ and the payoff information available at time t. Note that at time instant 2 only case S1 is applicable because at time instant 1 all the agents are in the Explore state. Therefore case S2 in the state update rule can be effective only for $t \geqslant 3$. Similarly, case S3 is applicable only for $t \geqslant 4$.

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Fix $p \in (0, 1)$.

State Update Rule:

For $t = 2, 3, \ldots$,

S1. if $s_i(t - 1) = E$

- if $BR_i(\hat{\mathbf{a}}_i(t - 1)) \neq \varnothing$: $s_i(t) = E$

- else (i.e., $BR_i(\hat{\mathbf{a}}_i(t - 1)) = \varnothing$):

$$
s_i(t) = \begin{cases} E & \text{with probability } p \\[2ex] C & \text{with probability } 1 - p \end{cases}
$$

S2. if $s_i(t - 1) = C$

- if $BR_i(\hat{\mathbf{a}}_i(t - 1)) \neq \varnothing$: $s_i(t) = E$

- else

  - if $B_i(t - 1) = 0$ and $U_i(\tilde{\mathbf{a}}(t - 1)) \neq U_i^*(t - 1)$: $s_i(t) = L_1$

  - else $\Big($i.e., (i) $B_i(t - 1) = 1$, or (ii) $B_i(t - 1) = 0$ and $U_i(\tilde{\mathbf{a}}(t - 1)) = U_i^*(t - 1)\Big)$: $s_i(t) = C$

S3. if $s_i(t - 1) = L_\ell, \ \ell = 1, 2, \ldots, T - 1$

- if $BR_i(\hat{\mathbf{a}}_i(t - 1)) \neq \varnothing$: $s_i(t) = E$

- else

  - if $B_i(t - 1) = 1$: $s_i(t) = L_\ell$

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– else if $B_i(t-1) = 0$ and $U_i(\tilde{\mathbf{a}}(t-1)) \neq U_i^*(t-1)$: $s_i(t) = L_{\ell+1}$

– else $\Big($i.e., $B_i(t-1) = 0$ and $U_i(\tilde{\mathbf{a}}(t-1)) = U_i^*(t-1)\Big)$: $s_i(t) = C$

S4. if $s_i(t-1) = L_T$

- if $BR_i(\hat{\mathbf{a}}_i(t-1)) \neq \varnothing$: $s_i(t) = E$

- else

  – if $B_i(t-1) = 1$: $s_i(t) = L_T$

  – else if $B_i(t-1) = 0$ and $U_i(\tilde{\mathbf{a}}(t-1)) \neq U_i^*(t-1)$: $s_i(t) = E$

  – else $\Big($i.e., $B_i(t-1) = 0$ and $U_i(\tilde{\mathbf{a}}(t-1)) = U_i^*(t-1)\Big)$: $s_i(t) = C$

In essence, if the SBR set is nonempty, the agent always moves to state E. Also, when the agent is at an alert state, L_ℓ, $\ell \in \mathcal{L}$, if there were erroneous implementation of the selected action then the agent stays in the same state, otherwise if the selected action was carried out and the received payoff goes back to $U_i^*(t-1)$, i.e., the payoff it was expecting the last time it was at state C, it immediately returns to state C. Otherwise, even when the SBR set is empty, it moves to the next alert state. The state transition of an agent is summarized in Figure 9.1.

9.3   Main Result

We redefine some notation from the previous chapter: Let $\mathcal{Z} := \Psi' \times \mathcal{A}$ and denote the pair $(\mathbf{s}(t), \mathbf{a}(t)) \in \mathcal{Z}$ by $\mathbf{z}(t)$, $k \in \mathbb{N}$. Define $\mathbf{s}^\star \in \Psi'$ to be the state vector
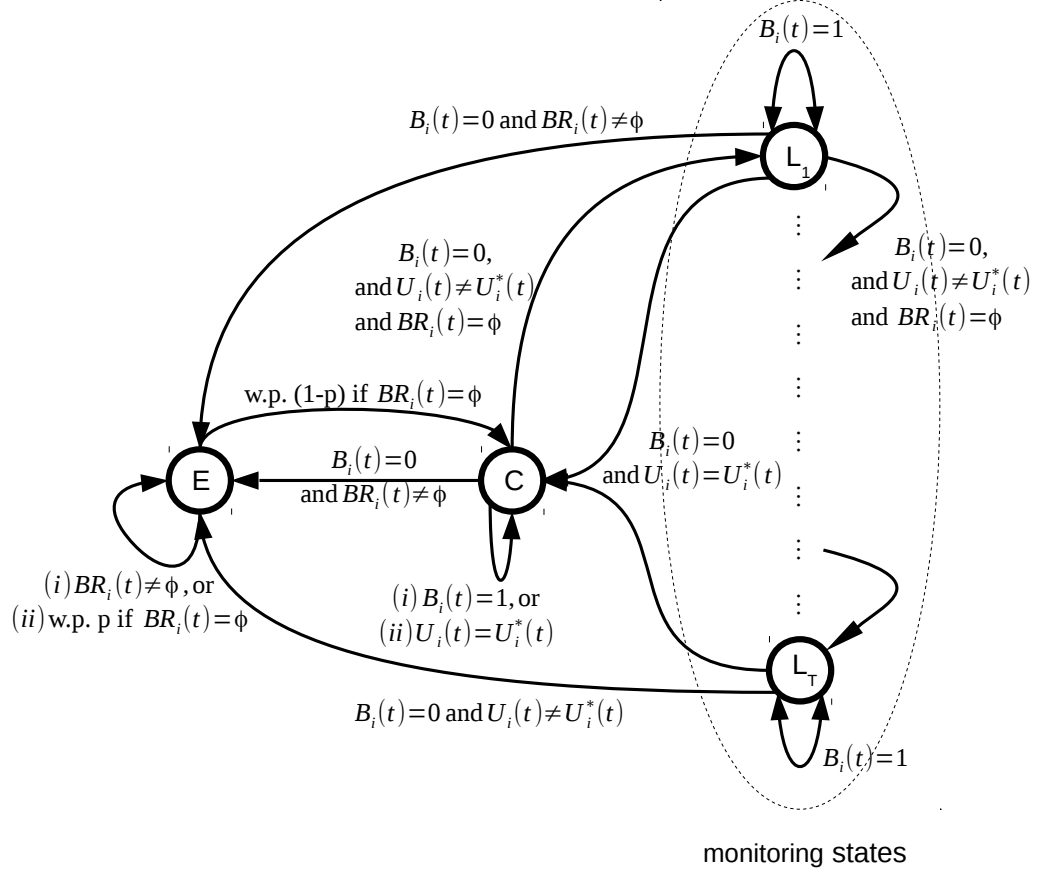
Figure 9.1: State transition diagram in SEM.

in which every agent is at state C, i.e, $\mathbf{s}^\star = (C,\ldots,C)$, and $\mathcal{Z}_{NE} = \{(\mathbf{s}^\star, \mathbf{a}^*) \in \mathcal{Z} \mid \mathbf{a}^* \in \mathcal{A}_{NE}\}$.

We assume the game satisfies the weak payoff interdependence assumption (Assumption 8.2) introduced in the previous chapter, which essentially states that, for every action profile $\mathbf{a} \in \mathcal{A}$ and subset of agents $J \subsetneq \mathcal{P}$ such that all the agents outside the set J see no better reply, there exists an agent $i^* \notin J$ and a choice of action profile $\mathbf{a}_J^* \in \prod_{j \in J} \mathcal{A}_j$ such that under the action profile $(\mathbf{a}_J^*, \mathbf{a}_{-J})$, agent i^* either sees a better reply $(BR_{i*}(\mathbf{a}_J^*, \mathbf{a}_{-J}) \neq \varnothing)$ or a change in its payoff $(U_{i*}(\mathbf{a}) \neq U_{i*}(\mathbf{a}_J^*, \mathbf{a}_{-J}))$
.

Assumption 9.1. *There exists a decreasing, positive sequence $(\epsilon_t,\ t \in \mathbb{N})$ such that* $\lim_{t\to\infty} \epsilon_t = 0$.

As in the previous chapter, we can model the evolution of joint states $\{(\mathbf{s}(t), \mathbf{a}(t)), t \in \mathbb{N}\}$ as a nonhomogeneous (discrete-time) Markov chain, where the transition matrix at time $t \in \mathbb{N}$ is denoted by $\mathbf{P}(t)$. For each $t \in \mathbb{N}$, a time homogeneous (discrete-time) Markov chain $\mathbb{X}^t$ can be defined with a common state space $\mathcal{C} \subset \mathcal{Z}^2$ and transition matrix $\mathbf{P}^t = \mathbf{P}(t)$. For reasons explained in the previous chapter, under Assumption 9.1, for every $t \in \mathbb{N}$ the present Markov chain $\mathbb{X}^t$ will also be ergodic with an unique stationary distribution denoted by $\boldsymbol{\mu}^t = (\mu^t(\mathbf{z}),\ \mathbf{z} \in \mathcal{C})$.

The following main result tells us that, if all agents adopt the proposed SEM learning rule, one of the two statements holds. The proof of the theorem is provided in Section 9.4.

Theorem 9.1. *Suppose that the game $\mathbb{G}$ satisfies Assumption 8.2 and $\mathcal{A}_{NE}$ is nonempty. Under Assumption 9.1, as $t \to \infty$, $\boldsymbol{\mu}^t \to \boldsymbol{\mu}^0$ where $\boldsymbol{\mu}^0$ is a stationary distribution of the unperturbed Markov chain.*

(a) *If $R^*_{\max} < T$, then $\mu^0(\mathbf{z}^*) > 0$ if and only if (i) $\mathbf{z}^* \in \mathcal{Z}_{NE}$, and (ii) $R(\mathbf{a}^*) = R^*_{\max}$.*

(b) *If $R^*_{\max} \geqslant T$, then $\mu^0(\mathbf{z}^*) > 0$ if and only if (i) $\mathbf{z}^* \in \mathcal{Z}_{NE}$, and (ii) $R(\mathbf{a}^*) \geqslant T$.*

The findings in Theorem 9.1 can be interpreted as follows. First, note that, when $\epsilon > 0$, due to the occasional erroneous actions, it is not possible to obtain

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[2]As we noted in the previous chapter, there may be some states in $\mathcal{Z}$ which are unreachable from the initial state $\mathbf{z}(1)$, and we do not include those unreachable states in the state space $\mathcal{C}$.

almost sure convergence to any PSNE. However, our result demonstrates that it is possible to *guide* the agents to PSNEs with certain resilience properties. If the parameter T, i.e., the number of alert states, is strictly larger than the maximum resilience among all PSNEs, when all agents adopt the SEM rule, for small error probability ϵ, the agents will spend a majority of time at the most resilient PSNE(s). On the other hand, if there are PSNEs whose resilience is at least T, then the agents will spend most of their time at the PSNEs that are at least T-resilient. Hence, the SEM rule offers a *tunable* knob, namely T, with which we can choose the desired resilience of PSNEs at which the agents will spend most of the time.

Elementary Markov chain theory allows us to assert that when $\epsilon = 0$, $\mathbf{a}(t)$ converges to a PSNE in $\mathcal{A}_{NE}$ almost surely under Assumption 8.2. Theorem 9.1 goes one step further; it states that, even when the agents are not perfect and make sporadic mistakes in executing the selected actions, under the SEM rule, the agents will stay at the resilient PSNEs most of the time, and the subset of PSNEs at which they spend most time can be chosen via the tunable parameter T in the rule.

9.4 Proof of Theorem 9.1

We first state a lemma that will be used to prove the theorem. Recall that given a state vector $\mathbf{s} \in \Psi'$, the set of agents in the Explore state is denoted by $E(\mathbf{s})$.

Lemma 9.2. *Suppose that the game $\mathbb{G}$ satisfies Assumption 8.2 and $\mathcal{A}_{NE} \neq \varnothing$. When $\epsilon = 0$, under SEM, the following hold.*

i. For all $\mathbf{z}^* \in \mathcal{Z}_{NE}$ and $\mathbf{z} = (\mathbf{s}, \mathbf{a}) \in \mathcal{C}$ with $\mathbf{s} \in \Psi'$ such that $E(\mathbf{s}) \neq \varnothing$, there exist $0 < D_1 < \infty$ and $0 < \rho_1 < 1$ such that

$$\mathbb{P}\left[\mathbf{z}(t + D_1) = \mathbf{z}^* \mid \mathbf{z}(t) = \mathbf{z}\right)] \geqslant \rho_1.$$

ii. For all $\mathbf{z} = (\mathbf{s}, \mathbf{a}^*) \in \mathcal{C}$ with $\mathbf{a}^* \in \mathcal{A}_{NE}$ and $\mathbf{s} \in \Psi'\backslash\{\mathbf{s}^\star\}$, there exist $0 < D_2 < \infty$ and $0 < \rho_2 < 1$ such that

$$\mathbb{P}\left[\mathbf{z}(t + D_2) = (\mathbf{s}^\star, \mathbf{a}^*) \mid \mathbf{z}(t) = \mathbf{z}\right] \geqslant \rho_2.$$

iii. For all $\mathbf{z}^* \in \mathcal{Z}_{NE}$ and $\mathbf{z} = (\mathbf{s}, \mathbf{a}) \in \mathcal{C}$ with $\mathbf{a} \notin \mathcal{A}_{NE}$, there exist $0 < D_3 < \infty$ and $0 < \rho_3 < 1$ such that

$$\mathbb{P}\left[\mathbf{z}(t + D_3) = \mathbf{z}^* \mid \mathbf{z}(t) = \mathbf{z}\right] \geqslant \rho_3.$$

Lemma 9.2 also implies that $\mathcal{Z}_{NE} \subset \mathcal{C}$. It also leads to the following corollary.

Corollary 9.3. *Suppose that the game* $\mathbb{G}$ *satisfies Assumption 8.2 and* $\mathcal{A}_{NE} \neq \varnothing$. *Then, when* $\epsilon = 0$, *under the SEM rule, only the states in* $\mathcal{Z}_{NE}$ *are absorbing states.*

We now proceed with the proof of the theorem. Recall that $\mathbb{X}^t$ denotes the MC with state space $\mathcal{C} \subset \mathcal{Z}$, in which each agent makes mistake with probability ϵ_t, independently of each other. Let $\mathbf{P}^t = \left[P^t_{\mathbf{z}_1, \mathbf{z}_2}; \mathbf{z}_1, \mathbf{z}_2 \in \mathcal{C}\right]$ denote the corresponding (one-step) transition matrix, where $P^t_{\mathbf{z}_1, \mathbf{z}_2}$ is the transition probability from $\mathbf{z}_1$ to $\mathbf{z}_2$.

Under Assumption 9.1, if for some $t \in \mathbb{N}$, $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{C}$, $\mathbf{z}_1 \neq \mathbf{z}_2$, and $\epsilon_t > 0$, $P_{\mathbf{z}_1, \mathbf{z}_2}(t) > 0$, then there is $r(\mathbf{z}_1, \mathbf{z}_2) \geqslant 0$ such that

$$0 < \lim_{t \to \infty} \frac{P_{\mathbf{z}_1, \mathbf{z}_2}(t)}{\epsilon_t^{r(\mathbf{z}_1, \mathbf{z}_2)}} < \infty.$$

This defines the resistance between two joint states in $\mathcal{C}$. Since the Markov chain $\mathbb{X}^t$ is ergodic for all $t \in \mathbb{N}$, we know there exists at least one sequence of transitions between any two joint states in $\mathcal{C}$. Therefore, we can define path resistances and resistance of minimum resistance paths between any two joint states as in Section 7.6. Therefore we can proceed as in Section 8.5, and construct minimum resistance W-tree $\Gamma^*(\mathbf{z})$ rooted at any $\mathbf{z} \in \mathcal{C}$, to characterize the set of stochastically stable states.

First, we can show that any state $\mathbf{z} \in \mathcal{C} \backslash \mathcal{Z}_{NE}$ is not SS. In other words, only the absorbing states of the unperturbed MC, which belong to $\mathcal{Z}_{NE}$ by Corollary 9.3, are SS from the definition of stochastic stability.

The theorem can be proved with the help of the following three claims, which we will prove shortly.

C1. For any $\mathbf{z}^* \in \mathcal{Z}_{NE}$, the resistance of the minimum resistance $\mathbf{z}^*$-tree, $\Gamma^*(\mathbf{z}^*)$, is equal to that of the Steiner tree in $\Gamma^*(\mathbf{z}^*)$ that connects only the states in $\mathcal{Z}_{NE}$ to $\mathbf{z}^*$. In order to prove this, we argue that the edges not in the Steiner tree have zero resistance.

C2. Fix any $\mathbf{z}' = (\mathbf{s}^\star, \mathbf{a}') \in \mathcal{Z}_{NE} \backslash \{\mathbf{z}^*\}$ and let $\mathbf{z}^\dagger$ be the first state in $\mathcal{Z}_{NE}$ visited along the (directed) path from $\mathbf{z}'$ to $\mathbf{z}^*$. We call $\mathbf{z}^\dagger$ the *parent state* of $\mathbf{z}'$ in

$\Gamma^*(\mathbf{z}^*)$ and denote it by $\eta(\mathbf{z}', \Gamma^*(\mathbf{z}^*))$. Then, the resistance of the subpath from $\mathbf{z}'$ to $\mathbf{z}^\dagger$ is equal to $\min\{R(\mathbf{a}'), T\} + 1$.

C3. Suppose that $\mathbf{z}'$ and $\mathbf{z}''$ ($\mathbf{z}', \mathbf{z}'' \in \mathcal{Z}_{NE}\backslash\{\mathbf{z}^*\}$) have the same parent state, say $\mathbf{z}^\dagger$. Then, the intersection of the subpaths from $\mathbf{z}'$ and $\mathbf{z}''$ to $\mathbf{z}^\dagger$ has zero resistance.

Together, these claims imply that

$$\pi(\Gamma^*(\mathbf{z}^*)) = \sum_{(\mathbf{s}^\star, \mathbf{a}') \in \mathcal{Z}_{NE}\backslash\{\mathbf{z}^*\}} \left(\min\{R(\mathbf{a}'), T\} + 1 \right). \tag{9.3}$$

The theorem then follows directly from (9.3).

Proof of C1 –   First, define

$$\mathcal{C}_A = \{(\mathbf{s}, \mathbf{a}) \in \mathcal{C} \mid \mathbf{a} \notin \mathcal{A}_{NE}\} \text{ and}$$

$$\mathcal{C}_B = \{(\mathbf{s}, \mathbf{a}^*) \in \mathcal{C} \mid \mathbf{s} \neq \mathbf{s}^\star, \ \mathbf{a}^* \in \mathcal{A}_{NE}\}.$$

Then, $\{\mathcal{Z}_{NE}, \mathcal{C}_A, \mathcal{C}_B\}$ forms a partition of $\mathcal{C}$.

Fix $\mathbf{z}^\star = (\mathbf{s}^\star, \mathbf{a}^\star) \in \mathcal{Z}_{NE}$, and consider any minimum resistance $\mathbf{z}^\star$-tree, $\Gamma^*(\mathbf{z}^\star) = (\mathcal{C}, \mathcal{E}^\star)$. Let $\mathbf{S}(\Gamma^*(\mathbf{z}^\star)) = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ denote the Steiner tree in $\Gamma^*(\mathbf{z}^\star)$ which connects the states in $\mathcal{Z}_{NE}\backslash\{\mathbf{z}^\star\}$ to $\mathbf{z}^\star$. An example of this is shown in Fig. 9.2, where the states in $\mathcal{Z}_{NE}$ appear as shaded circles, and the edges in the Steiner tree are shown as solid arrows.

We now argue that

$$\pi(\Gamma^*(\mathbf{z}^\star)) = \pi(\mathbf{S}(\Gamma^*(\mathbf{z}^\star))). \tag{9.4}$$
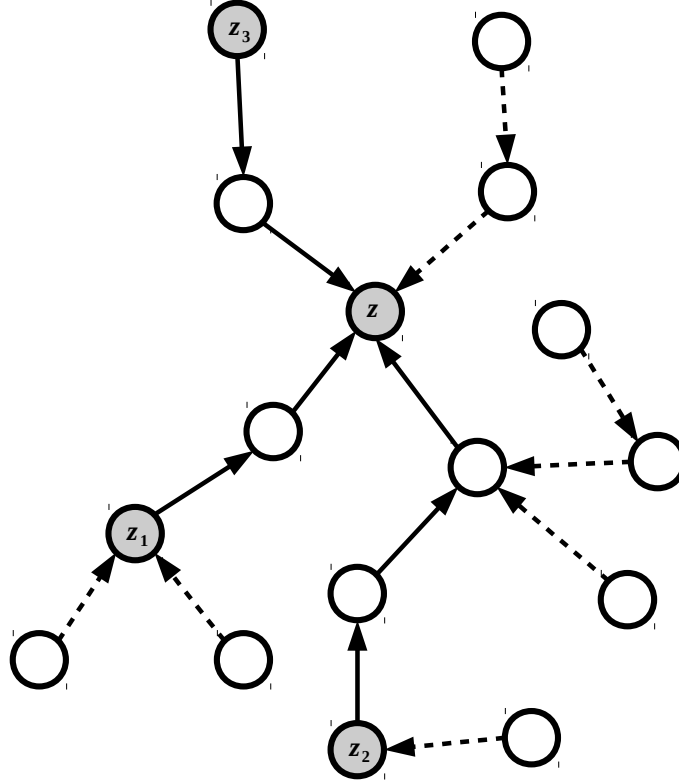
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Figure 9.2: An example of a minimum resistance $\mathbf{z}^\star$-tree, $\Gamma^*(\mathbf{z}^\star)$, and the corresponding Steiner tree $\mathbf{S}(\Gamma^*(\mathbf{z}^\star))$.

Let $\mathcal{V}^c = \mathcal{C} \backslash \tilde{\mathcal{V}}$ and $\mathcal{E}^c = \mathcal{E}^\star \backslash \tilde{\mathcal{E}}$, i.e., the set of edges in $\Gamma^*(\mathbf{z}^\star)$ which do not belong to the Steiner tree.

Since $\{\mathcal{Z}_{NE}, \mathcal{C}_A, \mathcal{C}_B\}$ is a partition of $\mathcal{C}$ and $\mathcal{Z}_{NE} \subset \tilde{\mathcal{V}}$, we have $\mathcal{V}^c \subset \mathcal{C}_A \cup \mathcal{C}_B$. Therefore, for any state $\mathbf{z} \in \mathcal{V}^c$, by Lemma 9.2-ii and -iii, there is a zero resistance path from $\mathbf{z}$ to some $\mathbf{z}^\dagger \in \mathcal{Z}_{NE}$. Therefore, since $\Gamma^*(\mathbf{z}^\star)$ is assumed to be a minimum resistance $\mathbf{z}^\star$-tree, the subpath from any state $\mathbf{z} \in \mathcal{V}^c$ to the Steiner tree $\mathbf{S}(\Gamma^*(\mathbf{z}^\star))$ should have zero resistance; otherwise, we can find another subpath to the Steiner tree with zero resistance, which contradicts the assumption that $\Gamma^*(\mathbf{z}^\star)$ is a minimum resistance $\mathbf{z}^\star$-tree. This observation means that only the edges in $\tilde{\mathcal{E}}$ can have positive

resistance and the equality in (9.4) follows.

Proof of C2 and C3 –   Define

$$\mathcal{C}_C = \{(\mathbf{s}, \mathbf{a}) \in \mathcal{C} \mid E(\mathbf{s}) \neq \varnothing\}.$$

We first introduce a lemma that will be used to prove the claims. For each $\mathbf{z}' \in$ $\mathcal{Z}_{NE}\backslash\{\mathbf{z}^\star\}$, we denote the first state in $\mathcal{C}_C$ along the path from $\mathbf{z}'$ to $\mathbf{z}^\star$ in $\Gamma^*(\mathbf{z}^\star)$ by $\zeta(\mathbf{z}', \Gamma^*(\mathbf{z}^\star))$.

Lemma 9.4. *For every $\mathbf{z} \in \mathcal{Z}_{NE}$ and its minimum resistance tree $\Gamma^*(\mathbf{z})$, the following hold.*

  i. *The (directed) path from any $\mathbf{z}' \in \mathcal{Z}_{NE}\backslash\{\mathbf{z}\}$ to $\mathbf{z}$ in $\Gamma^*(\mathbf{z})$ includes a state in $\mathcal{C}_C$.*

  ii. *For any distinct $\mathbf{z}'$ and $\mathbf{z}''$ in $\mathcal{Z}_{NE}\backslash\{\mathbf{z}\}$, the directed subpath from $\mathbf{z}'$ to $\zeta(\mathbf{z}', \Gamma^*(\mathbf{z}))$ and that from $\mathbf{z}''$ to $\zeta(\mathbf{z}'', \Gamma^*(\mathbf{z}))$ are edge disjoint, i.e., the subpaths do not share any edge.*

  iii. *The resistance of the subpath from $\mathbf{z}' = (\mathbf{s}^\star, \mathbf{a}') \in \mathcal{Z}_{NE}\backslash\{\mathbf{z}\}$ to $\zeta(\mathbf{z}', \Gamma^*(\mathbf{z}))$ is equal to $\min\{R(\mathbf{a}'), T\} + 1$.*

Proof. Lemma 9.4-i follows from the observation that a transition from $\mathbf{z}' \in \mathcal{Z}_{NE}$ to another $\mathbf{z}'' \in \mathcal{Z}_{NE}$ requires some agents to change their *selected* actions. From the description of our algorithm, this is only possible when such agents first change

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their states from C to E. Lemma 9.4-ii a consequence of the observation that, while transitioning from some $\mathbf{z}' = (\mathbf{s}^*, \mathbf{a}')$ to $\zeta(\mathbf{z}', \Gamma^*(\mathbf{z}))$, the *selected* action profile is fixed at $\mathbf{a}'$ because the agents can change their *selected* actions only when they are at state E.

Recall that there are only two possible ways for an agent to reach the state E, starting at state C: i) It can move directly to E when it finds an SBR, or ii) it can travel through the sequence of alert states $(L_\ell, \ell \in \mathcal{L})$ when it sees a change in observed payoff information. In the first case, from the definition of the resilience of a PSNE, for every $\mathbf{z}' = (\mathbf{s}^\star, \mathbf{a}') \in \mathcal{Z}_{NE}$, there exists (i) $J \subset \mathcal{P}$ with $|J| = R(\mathbf{a}') + 1$ and $\mathbf{a}_J^+ \in \prod_{i \in J} \mathcal{A}_i$ such that, if the agents in J switch their actions to $\mathbf{a}_J^+$, at least one agent sees an SBR. By definition of the resilience of a PSNE, it is clear that the resistance of such a transition is $R(\mathbf{a}') + 1$.

In the second case, using Assumption 8.2, we can find (a) an agent, say i, and action $a_i^+ \in \mathcal{A}_i$ and (b) another agent $j \neq i$ such that if agent i switches its action to a_i^+ (from a_i'), then agent j sees different payoff information. Therefore, if agent i repeats this $T + 1$ times (by mistake), then agent j will switch its state to E (through the chain of alert states). The total resistance along this sequence of transitions is equal to $T + 1$. From these two cases, the resistance of the (directed) path from $\mathbf{z}'$ to $\zeta(\mathbf{z}', \Gamma^*(\mathbf{z}))$ is equal to $\min\{R(\mathbf{a}'), T\} + 1$. ∎

Consider a state $\mathbf{z}' = (\mathbf{s}^\star, \mathbf{a}') \in \mathcal{Z}_{NE} \backslash \{\mathbf{z}^\star\}$. Lemma 9.2-i tells us that there exists a path from $\zeta(\mathbf{z}', \Gamma^*(\mathbf{z}^\star))$ to $\mathbf{z}^\star$ with zero resistance. This observation and Lemma 9.4-ii prove the claim C3. In addition, the (directed) subpath from $\mathbf{z}'$ to

$\eta(\mathbf{z}', \Gamma^*(\mathbf{z}^\star))$ has all of its resistance in the subpath from $\mathbf{z}'$ to $\zeta(\mathbf{z}', \Gamma^*(\mathbf{z}^\star))$. Together with Lemma 9.4-iii, this observation proves the claim C2.

9.5   Proof of Lemma 9.2

Before proving Lemma 9.2, we prove an intermediate result which claims that starting from any initial state with $E(\mathbf{s})$ non-empty, we can reach a state where all the agents are exploring (within a finite number of steps and with positive probability). Note that this is similar to Corollary 3 from the previous chapter. While we had proven a similar statement for the RSEM rule, we now prove the statement for the more general SEM rule.

Lemma 9.5. *Suppose that the game $\mathbb{G}$ satisfies Assumption 8.2. Then, under the SEM rule with $\epsilon = 0$, there exist $0 < D_e < \infty$ and $0 < \rho_e < 1$ such that for all $t > 0$, $\mathbf{z} = (\mathbf{s}, \mathbf{a}) \in \mathcal{C}$ with $E(\mathbf{s}) \neq \varnothing$,*

$$\mathbb{P}\left[s_i(t + D_e) = E \text{ for all } i \in \mathcal{P} \mid \mathbf{z}(t) = \mathbf{z} \right] \geqslant \rho_e. \tag{9.5}$$

Proof. First we introduce the following notation: Given a state vector $\mathbf{s} \in \mathcal{S}$,

$$L(\mathbf{s}) = \{i \in \mathcal{P} \mid s_i = L_\ell \text{ for some } \ell \in \mathcal{L}\},$$

$$C(\mathbf{s}) = \{i \in \mathcal{P} \mid s_i = C\}.$$

By definition, $L(\mathbf{s}) \cup E(\mathbf{s}) \cup C(\mathbf{s}) = \mathcal{P}$. The proof is a two step process. First we show that with positive probability and in finite time $t^\#$ we reach a state

$\mathbf{z}(t + t^\#) = \mathbf{z}^\# = (\mathbf{s}^\#, \mathbf{a}^\#)$, such that $E(\mathbf{s}^\#) \neq \varnothing$, $L(\mathbf{s}^\#) = \varnothing$ and $L(\mathbf{s}(t + t^\# + 1)) = \varnothing$. Then, we show that the rest of the agents in the converged state can be driven to the explore state in finite time with positive probability.

Let us define some events: For any time $\kappa > 0$, $K > 0$, $J \subseteq \mathcal{P}$ and $\bar{\mathbf{a}} \in \mathcal{A}$,

$$\mathcal{E}_1(\kappa, K) = \{E(\mathbf{s}(\ell)) \supseteq E(\mathbf{s}(\ell - 1)) \text{ for all } \ell = \kappa + 1, \ldots, \kappa + K\},$$

$$\mathcal{E}_2(\bar{\mathbf{a}}, \kappa, K) = \{a_i(\ell) = \bar{a}_i \text{ for } i \in E(\mathbf{s}(\ell)), \text{ for all } \ell = \kappa, \ldots, \kappa + K\},$$

$$\mathcal{E}_3(J, \kappa) = \{s_i(\kappa) = E, \ i \in J\}.$$

Fix $\mathbf{z} = (\mathbf{s}, \mathbf{a}) \in \mathcal{C}$ satisfying $E(\mathbf{s}) \neq \varnothing$. Given the state and action configuration at time t, $L(\mathbf{s}(t + 1))$ is deterministic. Consider the case where $L(\mathbf{s}(t + 1)) \neq \varnothing$. For $i \in L(\mathbf{s}(t + 1))$ it is clear that $U_i(\tilde{\mathbf{a}}(t)) \neq U_i^*(t)$. Therefore if the agents who are in the explore state, i.e., $E(\mathbf{s}(t + 1))$ continue playing their previous actions, agent i would reach state E at least within the next T periods. If agent i reached state E before T periods, we require it to continue playing its previous action for the rest of the time so that the remaining agents in $L(\mathbf{s}(t + 1))$ (if any) also reach state E. Therefore, we have

$$\mathbb{P}\left[E(\mathbf{s}(t + T + 1)) = E(\mathbf{s}) \cup L(\mathbf{s}(t + 1)) \big| \mathbf{z}(t) = \mathbf{z}\right]$$

$$\geqslant \mathbb{P}\left[\mathcal{E}_1(t, T + 1) \cap \mathcal{E}_2(\mathbf{a}, t, T + 1) \big| \mathbf{z}(t) = \mathbf{z}\right]$$

$$= \mathbb{P}\left[\mathcal{E}_1(t, T + 1) \big| \mathbf{z}(t) = \mathbf{z}\right] \cdot \mathbb{P}\left[\mathcal{E}_2(\mathbf{a}, t, T + 1) \big| \mathcal{E}_1(t, T + 1), \mathbf{z}(t) = \mathbf{z}\right]. \quad (9.6)$$

Lower bounding the first term in (9.6),

$$\mathbb{P}\left[\mathcal{E}_1(t, T+1)\big|\mathbf{z}(t) = \mathbf{z}\right]$$

$$= \prod_{\ell=t+1}^{t+T+1} \mathbb{P}\bigg[E(\mathbf{s}(\ell)) \supseteq E(\mathbf{s}(\ell-1))\big|E(\mathbf{s}(m)) \supseteq E(\mathbf{s}(m-1))$$

$$\text{for all } m = t+1, \dots, \ell-1, \mathbf{z}(t) = \mathbf{z}\bigg]$$

$$\geq \delta^{\left|E(\mathbf{s})\cup L(\mathbf{s}(t+1))\right|(T+1)} \tag{9.7}$$

where the final step follows by noting that in the Explore state each agent plays a particular (desired) action with probability at least δ. Moving to the second term in (9.6)

$$\mathbb{P}\left[\mathcal{E}_2(\mathbf{a}, t, T+1)\big|\mathbf{z}(t) = \mathbf{z}, \mathcal{E}_1(t, T+1)\right]$$

$$= \prod_{\ell=t}^{t+T+1} \mathbb{P}\bigg[a_i(\ell) = a_i \text{ for } i \in E(\mathbf{s}(\ell))\big|a_i(m) = a_i \text{ for } i \in E(\mathbf{s}(m)),$$

$$m = t, \dots, \ell-1, \ \mathbf{z}(t) = \mathbf{z}\bigg]$$

$$\geq p^{\left|E(\mathbf{s})\cup L(\mathbf{s}(t+1))\right|(T+1)}, \tag{9.8}$$

where the final bound follows from the fact that each agent at state E will remain in the same state with probability at least p. Collecting the lower bounds and returning to (9.6),

$$\mathbb{P}\left[E(\mathbf{s}(t+T+1)) = E(\mathbf{s}) \cup L(\mathbf{s}(t+1))\big|\mathbf{z}(t) = \mathbf{z}\right] \geq (\delta p)^{\left|E(\mathbf{s})\cup L(\mathbf{s}(t+1))\right|(T+1)}. \tag{9.9}$$

Define the set $J_1 = E(\mathbf{s}) \cup L(\mathbf{s}(t+1))$. By time $t_1 = t+T+1$ we have all the agents either in the explore state or the converged state. Next, we look at events that will force the agents in the converged state to the explore state. By Assumption 8.2,

there exists at least one player $i_1 \notin J_1$ such that with a choice of action $\mathbf{a}'_{J_1} \in \mathcal{A}_{J_1}$ either

$$U_{i_1}(\mathbf{a}'_{J_1}, \mathbf{a}_{-J_1}) \neq U_{i_1}(\mathbf{a}_{J_1}, \mathbf{a}_{-J_1})$$

or $BR_{i_1}(\mathbf{a}'_{J_1}, \mathbf{a}_{-J_1}) \neq \varnothing$. In case there are multiple possible action configurations then choose one using some convention, e.g., lexical.

Denote the set of agents who see a better reply due to the agents in J_1 changing their actions to $\mathbf{a}'_{J_1}$ by B_1^{BR}; while B_1^{PI} be the agents observing a change in payoff without seeing a better reply. From Assumption 8.2 we know that $B_1 \equiv B_1^{BR} \cup B_1^{PI} \neq \varnothing$.

Agents in B_1^{BR} will transition to the explore state in one time step, whereas the agents in B_1^{PI} needs $T+1$ time periods to transition through the chain of alert states before it can explore. For simplicity, we ensure that the action configuration $\mathbf{a}_1^{\dagger} = (\mathbf{a}'_{J_1}, \mathbf{a}_{-J_1})$ is continued for $T+1$ consecutive time steps. We have the following bound

$$\mathbb{P}\left[\mathcal{E}_3 \left(J_1 \cup B_1, t_1 + T + 1 \right) \middle| \mathbf{z}(t) = \mathbf{z}, \mathcal{E}_3(J_1, t_1) \right]$$

$$\geqslant \mathbb{P}\left[\mathcal{E}_1(t_1, T+1) \cap \mathcal{E}_2(\mathbf{a}_1^{\dagger}, t_1, T+1) \middle| \mathbf{z}(t) = \mathbf{z}, \mathcal{E}_3(J_1, t_1) \right]$$

$$\geqslant (\delta p)^{|J_1 \cap B_1|(T+1)}, \tag{9.10}$$

where the final step follows from similar arguments used to obtain the bound (9.9). Set $J_2 = J_1 \cap B_1 \supseteq J_1 \cup \{i_1\}$; and repeat the same argument noting that $L(\mathbf{s}(t_1 + T + 1)) = \varnothing$ if the events described above occur.

Suppose the following sequence of sets are defined $\{J_1, \ldots, J_\alpha\}$ such that $J_\alpha = $

$\mathcal{P}$ and the following sequence of times are defined $\{t_1, t_2, \ldots, t_\alpha\}$, where

$$t_j = t + j(T+1), \quad j = 1, \ldots, \alpha.$$

The time t_j denotes the time by which all the agents in the set J_j are in state E if the proper sequence of events is followed. The worst case scenario would be when $\alpha = n$. Note that the sequence of sets $\{J_1, \ldots, J_{\alpha-1}\}$ are not random sets but depends only on the initial state $\mathbf{z}(t) = (\mathbf{s}, \mathbf{a})$. Next we move on to show the lower bound as proposed in (9.5), using (9.9) and (9.10). Observing that the event $\mathcal{E}_3(\mathcal{P}, t_\alpha)$ implies that all the agents are in the Explore state by time t_α, we obtain

$$\mathbb{P}\left[\mathcal{E}_3(\mathcal{P}, t_\alpha) \middle| \mathbf{z}(k) = \mathbf{z}\right]$$

$$\geqslant \mathbb{P}\left[\mathcal{E}_3(J_\ell, t_\ell), \quad \ell = 1, \ldots, \alpha \middle| \mathbf{z}(t) = \mathbf{z}\right]$$

$$= \prod_{\ell=1}^{\alpha} \mathbb{P}\left[\mathcal{E}_3(J_\ell, t_\ell) \middle| \mathbf{z}(t) = \mathbf{z}, \mathcal{E}_3(J_m, k_m) \text{ for } m = 1, \ldots, \ell - 1\right]$$

$$\geqslant (\delta p)^{(|J_1| + \ldots + |J_\alpha|)(T+1)} \geqslant (\delta p)^{\frac{n(n+1)}{2}(T+1)}. \tag{9.11}$$

Therefore, $D_e = n(T+1)$ and set ρ_e as $(\delta p)^{\frac{n(n+1)}{2}(T+1)}$.

∎

Having proved the intermediate result Lemma 9.5, we show the proof for Lemma 9.2. Throughout the proof, for any $\mathbf{s} \in \Psi'$, the set of agents in the Explore, Converged and the Alert states are denoted as $E(\mathbf{s})$, $C(\mathbf{s})$ and $L(\mathbf{s})$ respectively.

Proof of Lemma 9.2(i) – Fix $\mathbf{z} = (\mathbf{s}, \mathbf{a}) \in \mathcal{C}$ with $E(\mathbf{s}) \neq \emptyset$. By virtue of Lemma 9.5, there exists $0 < D_e < \infty$ and $0 < \rho_e < 1$ such that

$$\mathbb{P}\left[s_i(t + D_e) = E, \text{ for all } i \in \mathcal{P}\middle|\mathbf{z}(t) = \mathbf{z}\right] \geq \rho_e. \tag{9.12}$$

Fix $\mathbf{a}^* \in \mathcal{A}_{NE}$ and let $t_e = t + D_e$. Define the following events

$$\mathcal{E}_1 = \{s_i(t_e) = E \text{ for all } i \in \mathcal{P}\}$$

$$\mathcal{E}_2 = \{a_i(t_e) = a_i^* \text{ for all } i \in \mathcal{P}\}$$

Let $D_1 = D_e + 1$. Then, we have

$$\mathbb{P}\left[\mathbf{z}(t + D_1) = \mathbf{z}^* \mid \mathbf{z}(t) = \mathbf{z}\right] = \mathbb{P}\left[\mathbf{z}(t + D_1) = \mathbf{z}^*, \mathcal{E}_1, \mathcal{E}_2 \mid \mathbf{z}(k) = \mathbf{z}\right]$$

$$= \mathbb{P}\left[\mathbf{z}(t + D_1) = \mathbf{z}^* \mid \mathcal{E}_1, \mathcal{E}_2, \mathbf{z}(k) = \mathbf{z}\right] \tag{9.13}$$

$$\times \mathbb{P}\left[\mathcal{E}_1 \mid \mathbf{z}(k) = \mathbf{z}\right] \cdot \mathbb{P}\left[\mathcal{E}_2 \mid \mathcal{E}_1, \mathbf{z}(k) = \mathbf{z}\right] \tag{9.14}$$

The term (9.13) corresponds to the probability that all the agents go to the Converged state conditioned on the events that at time t_e all the agents reached the Explore state and played the equilibrium action corresponding to $\mathbf{a}^*$. Hence, at time D_1 none of the agents will see an SBR, and as a result the conditional probability in (9.13) can be lower bounded by $(1 - p)^n$. From (9.12), the first term in (9.14) is lower bounded by $\rho_e > 0$. Further, the second conditional probability in (9.14) which corresponds to a particular action profile being played by agents in the Explore state, is lower bounded by δ^n from the description of the SEM algorithm in Section 9.2. Putting the bounds together, we obtain

$$\mathbb{P}\left[\mathbf{z}(t + D_1) = \mathbf{z}^* \mid \mathbf{z}(t) = \mathbf{z}\right] . \geq \rho_e[\delta(1 - p)]^n. \tag{9.15}$$

Proof of Lemma 9.2(ii) – Fix $\mathbf{z} = (\mathbf{s}, \mathbf{a}^*) \in \mathcal{C}$ with $\mathbf{a}^* \in \mathcal{A}_{NE}$ and $\mathbf{s} \in \Psi' \backslash \{\mathbf{s}^*\}$.

Within one time interval, the agents in $E(\mathbf{s})$ could move (with positive probability) to the converged state because they see no better reply, which yields

$$\mathbb{P}\big[s_i(t+1) = C \text{ for all } i \in E(\mathbf{s}) \mid \mathbf{z}(t) = \mathbf{z}\big] \geqslant (1-p)^{|E(\mathbf{s})|}. \qquad (9.16)$$

For those agents in $L(\mathbf{s})$ who still continue to see a change in payoff information due to the action configuration at time t, they move to the next alert state or the explore state if they were at the final alert state. Otherwise they move to the converged state. Therefore, we now look at events which ensure that agents in $E(\mathbf{s}(t+1))$ play their equilibrium action and move to the converged state; and agents in $L(\mathbf{s}(t+1))$ traverse the chain of alert states – move to the explore state – play their equilibrium action and finally end up in the converged state. Let $D_2 = T + 1$. Therefore,

$$\mathbb{P}\big[\mathbf{z}(t + D_2) = \mathbf{z}^* \mid s_i(t+1) = C \text{ for all } i \in E(\mathbf{s}), \mathbf{z}(t) = \mathbf{z}\big]$$
$$\geqslant [\delta(1-p)]^{n-|E(\mathbf{s})|}, \qquad (9.17)$$

where we assume all the agents in $L(\mathbf{s}(t+1))$ are in mood L_1 as a worst case scenario, and we use the fact that under the conditioning event, $C(\mathbf{s}(t+1)) = E(\mathbf{s})$. Combining (9.16) and (9.17) we obtain

$$\mathbb{P}\big[\mathbf{z}(t + D_2) = \mathbf{z}^* \mid \mathbf{z}(t) = \mathbf{z}\big] \geqslant [\delta(1-p)]^n.$$

Proof of Lemma 9.2(iii) – Fix $\mathbf{z} = (\mathbf{s}, \mathbf{a}) \in \mathcal{C}$ such that $\mathbf{a} \notin \mathcal{A}_{NE}$, then

$$\mathbb{P}\big[E(\mathbf{s}(t+1)) \neq \varnothing \mid \mathbf{z}(t) = \mathbf{z}\big] = 1.$$

Once, the number of agents in state E becomes positive we are back to Lemma 9.2(i).

Chapter 10:   Future work

In this part of the thesis we study two classes of learning rules that ensure convergence to PSNE(s) under delayed payoff information and asynchronous decision-making.

For the class of better reply rules, we consider faulty payoff information in that the agents cannot estimate the better reply set reliably. We show that the set of stochastically stable states, i.e., the states that are played most of the time as the payoff information becomes more reliable, are a subset of the set of PSNE(s). However due to difficulty in characterising the set of stochastically stable states we consider the class of monitoring rules. We first look at the setting where the erroneous decision-making of the agents are due to occasional misclassification of better replies while the payoff for the played action is assumed to be accurate. For this scenario, we are able to characterise the set of stochastically stable states as the PSNE(s) which when in effect would make it less likely for any of the agents to see a better reply. As a future work, we would like to formulate a more realistic setting where not only the payoff classification information is spurious but the payoff for the played action is also not accurate. Under such a setting, we would like to explore if the set of stochastically stable states can be characterised when the agents follow

the monitoring rules.

Since the existence of PSNE is not always guaranteed in strategic-form games, we would like to explore other learning rules that might yield convergence (in an appropriate sense) to possibly other kinds of equilibria (mixed Nash equilibria, correlated equilibria) under delayed payoff information and asynchronous settings.

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