ABSTRACT

Title of dissertation: ANALYSIS OF STEADY-STATE AND DYNAMICAL RADIALLY-SYMMETRIC PROBLEMS OF NONLINEAR VISCOELASTICITY

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This thesis treats radially symmetric steady states and radially symmetric motions of nonlinearly elastic and viscoelastic plates and shells subject to dead-load and hydrostatic pressures on their boundaries and with the plate subject to centrifugal force. The plates and shells are described by specializations of the exact (nonlinear) equations of three-dimensional continuum mechanics. The treatment in every case is very general and encompasses large classes of constitutive functions (characterizing the material response).

We first treat the radially symmetric steady states of plates and shells and the radially symmetric steady rotations of plates. We show that the existence, multiplicity, and qualitative behavior of solutions for problems accounting for the live loads due to hydrostatic pressure and centrifugal force depend critically on the material properties of the bodies, physically reasonable refined descriptions of which are given and examined here with great care, and on the nature of boundary conditions. The treatment here, giving new and sharp results, employs several different

mathematical tools, ranging from phase-plane analysis to the mathematically more sophisticated direct methods of the calculus of variations, fixed-point theorems, and global continuation methods, each of which has different strengths and weaknesses for handling intrinsic difficulties in the mechanics.

We then treat the initial-boundary-value problems for the radially symmetric motions of annular plates and spherical shells that consist of a nonlinearly viscoelastic material of strain-rate type. We discuss a range of physically natural constitutive equations. We first show that when the material is strong in a suitable sense relative to externally applied loads, solutions exist for all time, depend continuously on the data, and consequently are unique. We study the role of the constitutive restrictions and that of the regularity of the data in ensuring the preclusion of a total compression and of an infinite extension for finite time. We then show that when the material is not sufficiently strong then under certain conditions on the (hydrostatic) pressure terms there are globally defined unbounded solutions and there are solutions that blow up in finite time.

The practical importance of these results is that for each problem involving live loads they furnish thresholds in material response delimiting materials for which solutions are ill behaved. A mathematical or numerical study limited to a particular class of materials may dangerously indicate well-behaved solutions when there are other realistic materials for which solutions are ill behaved. Moreover this work furnishes so-called trivial solutions for the subsequent study (not given here) of bifurcation of stable equilibrium configurations from these trivial solutions.

ANALYSIS OF STEADY-STATE AND DYNAMICAL RADIALLY-SYMMETRIC PROBLEMS OF NONLINEAR VISCOELASTICITY

by

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Table of Contents

List of Symbols v

1 Introduction 1

2 Radially Symmetric Steady States of Nonlinearly Elastic Plates and Shells 6
- 2.1 Introduction . . . 6
- 2.2 Formulation of the governing equations . . . 9
- 2.3 Constitutive Assumptions . . . 12
- 2.4 A Dual Formulation . . . 21
- 2.5 Phase-Plane Methods for Homogeneous Bodies . . . 24
- 2.6 Existence and Uniqueness via Variational Methods . . . 41
- 2.7 Existence and Qualitative Behavior via Fixed-Point Methods . . . 50
- 2.8 Perturbation Methods for Nearly Homogeneous Materials . . . 63
- 2.9 Global Continuation Methods for Nonhomogeneous Materials and for Weak Materials . . . 69
- 2.10 Comparison of Analytic Methods . . . 71
- 2.11 Comments . . . 73

3 Radially Symmetric Motions of Nonlinearly Viscoelastic Plates and Shells 75
- 3.1 Introduction . . . 75
- 3.2 Notation . . . 78
- 3.3 Mechanics and Material Behavior . . . 80
- 3.4 Boundary conditions. Weak formulation . . . 82
- 3.5 Constitutive restrictions . . . 85
- 3.6 Fundamental Theorems and Plan of the Analysis . . . 96
- 3.7 Energy Estimates . . . 99
- 3.8 Preclusion of Total Compression and Infinite Extension . . . 101
- 3.9 Estimates of the Accelerations and the Strain Rates . . . 105
- 3.10 Pointwise Estimates on the Strain Rates . . . 108
- 3.11 The Role of the Bounds and the Modified Problem . . . 109

3.12 A Faedo-Bubnov-Galerkin Method 111
3.13 Approximating Ordinary Differential Equations 112
3.14 Global Existence of Solutions of the Approximating Ordinary Differential Equations 113
3.15 Estimates of Higher Derivatives 115
3.16 Convergence and Regularity 120
3.17 Continuous Dependence on the Data. Uniqueness 123
3.18 The Treatment of Other Boundary Conditions 126
3.19 Blowup of solutions under hydrostatic pressures 136
3.20 Open Problems. Comments 142

Bibliography 146

List of Symbols

The following table lists the principal symbols used, their meanings, and where they are defined (section or equation number). Some symbols have a different meaning in different parts of the thesis. This is indicated in the table.

a	Inner radius of the plate or shell. Sec. 2.2, Sec. 3.3.
B	Boundary power term. Sec. 3.7.
B^M	Galerkin approximation for B. Sec. 3.14.
C	Large positive constant depending on the data. Sec. 3.2.
c	Small positive constant depending on the data. Sec. 3.2.
c_{V}	Positive constant characterizing viscosity effects. (3.5.15).
H	Energy-like quantity. Sec. 3.9.
H^M	Galerkin approximation for H. Sec. 3.15.
K	Kinetic energy. Sec. 3.7.
K^M	Galerkin approximation for K. Sec. 3.14.
M	Dimension of Galerkin approximations. Sec. 3.12.
n	Dual variable. Sec. 2.4.
N	Normal Piola-Kirchhoff stress of the first kind acting in the radial direction. Sec. 2.2, Sec. 3.3.
$\hat{N}$	Constitutive function for N. Sec. 2.2, Sec. 3.5.
N^{D}	Dissipative part of $\hat{N}$. Sec. 3.5.
N^{E}	Equilibrium part of $\hat{N}$. Sec. 3.5.
N^M	Galerkin approximation for $\hat{N}$. Sec. 3.12.
p	Initial values of r. Sec. 3.4.
q	Initial values of r_t. Sec. 3.4.
$r(s)$	Position of material point s. Sec. 2.2.
$r(s,t)$	Position of material point s at time t. Sec. 3.3.
r^M	Galerkin approximation for r. Sec. 3.12
s	Material (Lagrangian) radial coordinate. Sec. 2.2, Sec. 3.3.
t	Time. Sec. 3.3.
t^+	Arbitrary fixed time. Sec. 3.6.
T	Normal Piola-Kirchhoff stress of the first kind acting in the azimuthal (circumferential) direction. Sec. 2.2, Sec. 3.3.
$\hat{T}$	Constitutive function for T. Sec. 2.2, Sec. 3.5.
$T^\sharp$	Dual constitutive function. (2.4.2).
T^{D}	Dissipative part of $\hat{T}$. Sec. 3.5.
T^{E}	Equilibrium part of $\hat{T}$. Sec. 3.5.
T^M	Galerkin approximation for $\hat{T}$. Sec. 3.12.
$\mathfrak{W}$	Test function space. (3.4.10).
W	Work of dissipative internal forces. Sec. 3.7.
W^M	Galerkin approximation for W. Sec. 3.14.
y_m	Shape function for Galerkin approximations. Sec. 3.12.
γ	Dimension. Sec. 2.2, Sec. 3.3.
$\mathit{\Gamma}$	(Large) positive-valued continuous function depending on the data. Sec. 3.2.

Γ^{-1}	(Small) positive-valued continuous function depending on the data. Sec. 3.2.
Δ	Dissipative term. (3.5.39).
Δ^M	Galerkin approximation for Δ. (3.14.3).
ε	Small positive constant depending on the data. Sec. 3.2.
$\lambda_1(t)$	Pressure intensity at $s = 1$. Sec. 3.4.
$\lambda_a(t)$	Pressure intensity at $s = a$. Sec. 3.4.
$\Lambda(t)$	Boundary term. (3.19.1).
μ	Mass density. Sec. 2.2.
ν	Radial strain variable. (2.2.1), (3.3.2).
$\nu^\sharp$	Dual constitutive function. (2.4.1).
$\dot{\nu}$	Symbol identifying the argument occupied by ν_t. Sec. 3.3.
ρ	Mass density. Sec. 3.3.
τ	Azimuthal (circumferential) strain variable. (2.2.1), (3.3.2).
$\dot{\tau}$	Symbol identifying the argument occupied by τ_t. Sec. 3.3.
$\varphi^{(\gamma)}$	Stored-energy density function. Sec. 2.3, Sec. 3.5.
Φ	Stored-energy. Sec. 2.2, Sec. 3.5.
Φ^M	Galerkin approximation for Φ. Sec. 3.14.
Ψ	Energy-like quantity. (3.19.1).
	Stored-energy density. Sec. 2.3.

Chapter 1: Introduction

This thesis treats radially symmetric steady states and radially symmetric motions of nonlinearly elastic and viscoelastic plates and shells. The plates and shells are described by specializations of the exact (nonlinear) equations of three-dimensional continuum mechanics [5, Ch. 13]. The treatment in every case is very general and encompasses large classes of constitutive functions (characterizing the material response). In particular our constitutive assumptions and models can be used for modeling soft biological structures [39]. (In fact our models are more comprehensive than some of the standard models discussed in [39] in that they account for the behavior at a total compression.)

This mathematical work has practical importance because it accounts for live loads (which are force systems depending on the configuration of the body). In particular, in the presence of live loads the behavior of solutions depends critically on the strength of the material in resisting large strains. E.g., for steady rotations of an annular plate due to the constant rotation of its inner boundary there is a threshold in material response above which there are angular speeds for which there are no steady solutions. And linear elastic response lies on this threshold. Consequently, a study, numerical or analytical, that does not carefully account for material response can give misleading information with possibly dangerous consequences about structural stability. There are similar thresholds for bodies subject to hydrostatic pressure. There are more spectacular effects for dynamical problems: there are thresholds in material response beyond which there are solutions of problems for bodies subject to hydrostatic pressure that blow up in finite time. These observations justify our careful study of a wide range of material behavior.

A popular constitutive equation for numerical studies of nonlinear elastic responses is that of the St. Venant-Kirchhoff material. This material has a stored-energy function of the same form as that for linear elasticity, except that in place of the linear strain tensor $\boldsymbol{E}_1$ it uses the material strain tensor $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$ (where the Cauchy-Green deformation tensor $\boldsymbol{C}$ is defined in Section 2.2). Now $\boldsymbol{E}$ is quadratic in the displacement gradient whereas $\boldsymbol{E}_1$ is linear. Thus the stored-energy function for $\boldsymbol{E}$, which is the St. Venant-Kirchhoff stored-energy function, is quartic in the displacement gradient, whereas the stored-energy function for $\boldsymbol{E}_1$ is merely quadratic. Consequently, as our analysis shows, solutions of the problems we treat for St. Venant-Kirchhoff materials behave very nicely. Any belief that such materials are representative of nonlinearly materials in general is dangerous: As we amply demonstrate, problems for elastic materials with milder growth in the strain can suffer a range of unstable behavior in the presence of live loads.

A second practical motivation for this work is that it furnishes so-called trivial solutions for the subsequent study (not given here) of bifurcation of stable equilibrium configurations from these trivial solutions. These bifurcated states include out-of-plane buckling of plates, non-spherical states of shells, and configurations in which radial symmetry is lost.

Chapter 2 treats the radially symmetric steady states of plates and shells as well as radially symmetric steady rotations of plates. Despite the apparent simplicity, the theory is very rich: The existence, multiplicity, and qualitative behavior of solutions for problems accounting for the live loads due to hydrostatic pressure and centrifugal force depend critically on the material properties of the bodies, physically reasonable refined descriptions of which are given and examined here with great care, and on the nature of boundary conditions. The treatment here, giving new and sharp results, employs several different mathematical tools, ranging from phase-plane analysis to the mathematically more sophisticated direct methods of the calculus of variations, fixed-point theorems, and global continuation methods, each of which has different strengths and weaknesses for handling intrinsic difficulties in the mechanics. Two underlying themes in these investigations are (i) the pervasive role of physically natural constitutive restrictions (including those that have not been frequently employed in the literature) in determining the existence, uniqueness, regularity and qualitative behavior of solutions, (ii) the use of the constitutive restrictions in an array of different mathematical methods that provide different kinds of information about solutions.

Chapter 3 treats the initial-boundary-value problems for radially symmetric motions of nonlinearly viscoelastic annular plates and spherical shells of strain-rate type. The governing equation is a third-order quasilinear parabolic-hyperbolic partial differential equation in one space variable. The equation is singular in the sense that the constitutive functions blow up as the strain variables approach zero.

This corresponds to a state of total compression: the Jacobian of the deformation gradient (local volume or length ratio) goes to zero. The existence theory for this problem has never been studied. One of the underlying themes in our studies is to determine thresholds in constitutive equations separating qualitatively different responses.

We discuss a range of physically natural constitutive equations. We first show that when the material is strong in a suitable sense relative to externally applied live loads, solutions exist for all time, depend continuously on the data, and consequently are unique. We study the role of the constitutive restrictions and that of the regularity of the data in ensuring the preclusion of a total compression and of an infinite extension for finite time. A priori bounds that we obtain on the strain variables and on the strain-rates allow us to replace the original singular problem with an equivalent regular problem. This we analyze by using the Faedo-Bubnov-Galerkin method. Our constitutive hypotheses support bounds and consequent compactness properties for the Galerkin approximations so strong that these approximations are shown to converge to the solution of the initial-boundary-value problem without appeal to the theory of monotone operators to handle the weak convergence of composite functions [37].

We then consider the case when the material is not sufficiently strong relative to externally applied live loads. We show that in that case under certain conditions on the (hydrostatic) pressure terms and initial conditions (i) radially symmetric motions of annular plates and spherical shells become unbounded at various rates as time approaches infinity; (ii) radially symmetric motions of spherical shells blow up in finite time. We show that although the equations for annular plates and spherical shells differ slightly, there are major qualitative differences between the nonlinear dynamical behavior of annular plates and spherical shells.

Chapter 2: Radially Symmetric Steady States of Nonlinearly Elastic Plates and Shells

2.1 Introduction

Within the linear theory of elasticity, the problems of determining equilibrium states of annular plates and spherical shells composed of homogeneous isotropic materials subject to pressures on their boundaries were solved by Lamé [34] in 1852. His solutions are presented in all the standard texts: [38, Secs. 98–100, 173], [48, Sec. 94], [53, Sec. 121], inter alios. The solutions of the same problems for radially symmetric aeolotropic materials are given by [35, 36]. (The radially symmetric equilibrium of an *intact isotropic* nonlinearly elastic (full) disk or (solid) ball is elementary: Under reasonable constitutive restrictions such an equilibrium state is unique and corresponds to a state of uniform deformation [5, 26]. If, however, the disk or ball is aeolotropic, even for a homogeneous linearly elastic material, the solution can exhibit a rich range of singular behavior at the origin [7, 10, 25, 50]. Moreover, if the center of the disk or ball is not assumed to be intact and if the material is not sufficiently strong in a suitable sense, then there exist cavitating solutions [17].)

We study boundary-value problems for the radially symmetric steady states of annular plates and spherical shells composed of radially symmetric nonlinearly elastic materials subject to pressures or displacement conditions on their boundaries. (Our equations for plates are the exact equations for radially symmetric plane-strain equilibria of cylindrical shells.) The theory is surprisingly rich. In contrast to the case of an isotropic disk or ball, a radially symmetric steady state of an annular plate or spherical shell is not generally unique and does not typically correspond to a state of uniform deformation. Two underlying themes in our work are (i) the pervasive role of physically natural constitutive restrictions that have not been frequently employed in the literature, and (ii) their use in an array of different methods for analyzing the equations, methods that provide different kinds of information about solutions. In particular, the restriction of our analysis to steady radially symmetric solutions means that the governing equations are ordinary differential equations, a fact that simplifies the analysis. Our problems, however, must account for transverse effects (nonlinear Poisson-ratio effects), which subtly permeate the analysis.

After formulating the governing equations, we give a careful discussion of reasonable constitutive restrictions (which exhibit some subtleties). We use phase-plane methods to exhibit examples of homogeneous materials for which there exist multiple radially symmetric states and examples of other such materials for which there are no radially symmetric states under certain reasonable boundary conditions. Under favorable conditions, these methods support existence and uniqueness theorems. For nonhomogeneous materials, we then use the direct methods of the Calculus of Variations to give restrictions on the boundary conditions and on the material response for which the potential-energy functional has a minimizer, which is a classical solution of the boundary-value problem. We give further restrictions that ensure uniqueness. This functional fails to have a minimizer when the material is not sufficiently strong in resisting extension for certain problems in which the boundary conditions include a hydrostatic pressure and for certain problems in which there is a centrifugal force. Such problems include not only those for which the phase-plane methods show nonexistence, but also those for which there is not a steady state for all pressures or for all rotational speeds. In this case, other functionals have extremizers, showing that there are, however, solutions for all sizes.

We next treat such problems by fixed-point methods, which complement the variational methods by providing an estimate of the range of pressures for which there are equilibrium states when there are (other) pressures for which there are none. We then treat nonhomogeneous materials by perturbation methods combined with shooting and global continuation methods. The latter technique elegantly deals with nonexistence and multiplicity in solution-parameter space. None of these methods is universally effective. We discuss the virtues and weaknesses of each in Section 2.10.

Notation. We often denote partial and ordinary derivatives by subscripts, and often denote ordinary derivatives by primes.

2.2 Formulation of the governing equations

Let the reference configuration of a nonlinearly elastic body when it is subject to zero tractions on its boundary and zero body force be either an annular plate or a spherical shell of inner radius $a \in (0, 1)$ and unit outer radius. (Such a body could be subject to residual stress provided that it is radially symmetric.) We consider only planar steady motions of the annular plate in which it rotates at constant angular velocity ω about the axis through its center perpendicular to its plane and in which the material points with reference radius s move along their material rays a distance depending only on s. We consider only spherically symmetric deformations of the spherical shell in which the material points with reference radius s move along their rays a distance depending only on s. (The restriction to such deformations precludes the study of problems for rotating spherical shells.)

Let $r(s)$ denote the radius in a deformed configuration of a typical material point with reference radius s. Then

$$\nu(s) := r'(s) \quad \text{and} \quad \tau(s) := r(s)/s \tag{2.2.1}$$

are the radial and azimuthal stretches at s. The shear strains with respect to polar coordinates are all zero by virtue of the symmetry of the deformation. For brevity we often refer to the annular plate and spherical shell as the plate and the shell.

We assume that the materials of these bodies have enough symmetry for a radially symmetric deformation to correspond to a radially symmetric stress distribution. (For a spherical shell undergoing such a deformation, this means that there is no shear stress on concentric material spheres and on material surfaces consisting of rays and that all normal stresses in any azimuthal direction for a given radius s and all normal stresses for a given radius s should respectively be equal in magnitude.)

Let $N(s)$ and $T(s)$ be the normal Piola-Kirchhoff stresses of the first kind (i.e., forces per reference length or area) at the (material) radius s in the radial and azimuthal directions. We assume that the material of the bodies is *nonlinearly elastic*, so that for the restricted classes of deformations treated here there are functions $(0,\infty)\times(0,\infty)\times(a,1)\ni(\tau,\nu,s)\mapsto\hat{T}(\tau,\nu,s),\hat{N}(\tau,\nu,s)$ such that

$$T(s)=\hat{T}(\tau(s),\nu(s),s),\qquad N(s)=\hat{N}(\tau(s),\nu(s),s). \tag{2.2.2}$$

Unless there is a statement to the contrary, we tacitly assume that the constitutive functions $\hat{T}$, $\hat{N}$, and other constitutive functions expressed in terms of these have as many continuous derivatives as are exhibited in the analysis. (See [10] for a treatment of problems where this assumption is not valid.)

The reference configuration is *natural* if the stress resultants vanish in it:

$$\hat{T}(1,1,s)=0=\hat{N}(1,1,s). \tag{2.2.3}$$

Much of our work does not require that the reference configuration be natural.

Equations of steady rotation. We derive the equations for the steady rotation of plates, leaving the equilibrium equations for shells as an exercise: Let $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ be a fixed right-handed orthonormal basis, let the origin be at the center of the plate, and let $\{\boldsymbol{i},\boldsymbol{j}\}$ span the plane of the plate. Set

$$\boldsymbol{e}_1(\phi):=\cos\phi\,\boldsymbol{i}+\sin\phi\,\boldsymbol{j},\qquad \boldsymbol{e}_2(\phi):=-\sin\phi\,\boldsymbol{i}+\cos\phi\,\boldsymbol{j}. \tag{2.2.4}$$

Let $\mu(s)$ be the mass per unit area of the plate at a material point with radial coordinate s. Consider the material that in the reference configuration has polar coordinates $(\bar{s}, \bar{\theta} + \omega t) \in [s, 1] \times [0, \theta]$. The requirement that at time t the resultant force on this material equal the time rate of change of its linear momentum is

$$\begin{aligned}
&\int_0^\theta N(1)\boldsymbol{e}_1(\omega t + \bar{\theta})\, d\bar{\theta} - \int_0^\theta N(s)\boldsymbol{e}_1(\omega t + \bar{\theta})\, s\, d\bar{\theta} \\
&+ \int_s^1 T(s)\boldsymbol{e}_2(\omega t + \theta)\, ds - \int_s^1 T(s)\boldsymbol{e}_2(\omega t)\, ds \\
&\quad = \frac{d}{dt}\int_s^1 \int_0^\theta \mu(\bar{s})\omega r(\bar{s})\boldsymbol{e}_2(\omega t + \bar{\theta})\, \bar{s}\, d\bar{s}\, d\bar{\theta} \\
&\quad \equiv -\omega^2 \int_s^1 \int_0^\theta \mu(\bar{s}) r(\bar{s})\boldsymbol{e}_1(\omega t + \bar{\theta})\, \bar{s}\, d\bar{s}\, d\bar{\theta}.
\end{aligned} \tag{2.2.5}$$

Under our favorable regularity assumptions, operate on this equation with $\partial^2/\partial s \partial \theta$ to get the equation for the plate. To combine the equation for the rotating plate with that for the shell in a compact notation, let $\gamma = 1$ for plates and $\gamma = 2$ for shells. Then the substitution of the constitutive equations (2.2.2) into these steady-state equations yields the quasilinear ordinary differential equation

$$\frac{d}{ds}[s^\gamma \hat{N}(s^{-1}r, r_s, s)] + (2 - \gamma)\omega^2 \mu(s) s r = \gamma s^{\gamma - 1} \hat{T}(s^{-1}r, r_s, s). \tag{2.2.6}$$

We shall seek classical solutions of (2.2.6) for which the strains strictly positive:

$$s^{-1}r(s) > 0, \quad r'(s) > 0 \qquad \forall\, s \in [a, 1]. \tag{2.2.7}$$

Boundary conditions. On the outer boundary $s = 1$, we prescribe either the deformed outer radius:

$$r(1) = \rho_1 > a, \tag{2.2.8}$$

or else a normal force of the form

$$N(1) = -\lambda_1 r(1)^{\delta_1} \equiv -\lambda_1 \tau(1)^{\delta_1}, \qquad \delta_1 = 0, \gamma. \tag{2.2.9}$$

If $\delta_1 = 0$, condition (2.2.9) represents a normal force per unit *reference* length or area (a dead load) of intensity λ_1. If $\delta = \gamma = 1$, condition (2.2.9) represents a hydrostatic pressure, i.e., a normal force per unit *actual* length of intensity λ_1 acting on the outer edge of the plate. If $\delta = \gamma = 2$, condition (2.2.9) represents a hydrostatic pressure, i.e., a normal force per unit *actual* area of intensity λ_1 acting on the outer surface of the shell. We allow λ_1 to have either sign. When λ_1 is positive, $N(1)$ is compressive. We adopt analogous conditions on the inner boundary $s = a$:

$$r(a) = \rho_a \quad \text{or else} \quad N(a) = -\lambda_a \left[\frac{r(a)}{a}\right]^{\delta_a} \equiv -\lambda_a \tau(a)^{\delta_a}, \qquad \delta_a = 0, \gamma. \tag{2.2.10}$$

When λ_a is positive, $N(a)$ is compressive (and therefore tends to inflate the body).

2.3 Constitutive Assumptions

We require that the pair $(\hat{T}, \hat{N})$ of constitutive functions satisfy the (specialization to our class of deformations of the) *Strong Ellipticity Condition*:

$$\hat{T}_\tau > 0, \quad \hat{N}_\nu > 0. \tag{2.3.1}$$

It is also reasonable to require that

$$\hat{T}_\nu > 0, \quad \hat{N}_\tau > 0. \tag{2.3.2}$$

Conditions (2.3.2) express what might be called the Poisson-ratio effect: If ν is fixed and τ is increased, then $\hat{N}(\tau, \nu, s)$ must increase, etc. In some of our analysis we use the more restrictive *Quasi-Monotonicity (QM) Condition*, which says that $(\hat{T}, \hat{N})$ is monotone:

$$\begin{bmatrix} \hat{T}_\tau & \hat{T}_\nu \\ \hat{N}_\tau & \hat{N}_\nu \end{bmatrix} \quad \text{is positive-definite.} \tag{2.3.3}$$

When (2.3.2) and (2.3.3) both hold, the effect of transverse deformation is dominated by direct deformation. In particular, (2.3.3) implies that

$$\frac{\hat{T}_\nu \hat{N}_\tau}{\hat{T}_\tau \hat{N}_\nu} < 1. \tag{2.3.4}$$

Mild strengthenings of this condition are

$$\frac{\hat{T}_\nu}{\hat{T}_\tau} < 1, \quad \frac{\hat{N}_\tau}{\hat{N}_\nu} < 1, \quad \frac{\hat{T}_\nu}{\hat{N}_\nu} < 1, \quad \frac{\hat{N}_\tau}{\hat{T}_\tau} < 1. \tag{2.3.5}$$

The first of these conditions says that the effect on T of ν is less than the effect on T of τ, etc. Below we discuss the subtle relationship of the QM condition with the Monotonicity Condition of 2- and 3-dimensional elasticity.

We supplement these conditions with a corresponding set of physically reasonable growth conditions:

$$\hat{T}(\tau,\nu,s) \to \begin{Bmatrix} \infty \\ -\infty \end{Bmatrix} \quad \text{as} \quad \tau \to \begin{Bmatrix} \infty \\ 0 \end{Bmatrix} \quad \text{if} \quad \nu \begin{Bmatrix} \text{has a positive lower bound} \\ \text{is bounded above} \end{Bmatrix}, \tag{2.3.6a}$$

$$\hat{N}(\tau,\nu,s) \to \begin{Bmatrix} \infty \\ -\infty \end{Bmatrix} \quad \text{as} \quad \nu \to \begin{Bmatrix} \infty \\ 0 \end{Bmatrix} \quad \text{if} \quad \tau \begin{Bmatrix} \text{has a positive lower bound} \\ \text{is bounded above} \end{Bmatrix}, \tag{2.3.6b}$$

$$\hat{T}(\tau,\nu,s) \to \begin{Bmatrix} \infty \\ -\infty \end{Bmatrix} \quad \text{as} \quad \nu \to \begin{Bmatrix} \infty \\ 0 \end{Bmatrix} \quad \text{if} \quad \tau \begin{Bmatrix} \text{is bounded above} \\ \text{has a positive lower bound} \end{Bmatrix}, \tag{2.3.6c}$$

$$\hat{N}(\tau,\nu,s) \to \left\{ \begin{matrix} \infty \\ -\infty \end{matrix} \right\} \quad \text{as} \quad \tau \to \left\{ \begin{matrix} \infty \\ 0 \end{matrix} \right\} \quad \text{if} \quad \nu \left\{ \begin{matrix} \text{is bounded above} \\ \text{has a positive lower bound} \end{matrix} \right\}. \tag{2.3.6d}$$

Conditions (2.3.6c) and (2.3.6d) correspond to (2.3.2). (By characterizing these restrictions as reasonable, we do not infer that unreasonable behavior attends the violation of these conditions. E.g., one could account for yielding by taking T and N bounded as τ or $\nu \to \infty$. The treatment of such problems would be similar to our treatments below of live loads.)

The material is *hyperelastic* (for radially symmetric deformations) if there is a real-valued *stored-energy density function* $(\tau,\nu,s) \mapsto \varphi(\tau,\nu,s)$ such that

$$\gamma\hat{T} = \varphi_\tau, \qquad \hat{N} = \varphi_\nu. \tag{2.3.7}$$

The reason for the presence of γ will be evident from the first equality in (2.3.12) below. For a hyperelastic material, the Strong Ellipticity Condition is equivalent to the Legendre-Hadamard condition of the Calculus of Variations. For a hyperelastic material, the QM Condition (2.3.3) is equivalent to the convexity of φ.

Isotropy. For brevity we treat isotropy only for hyperelastic materials. Let a 3-dimensional body be described by spherical coordinates. At a material point where there is no shear with respect to these coordinates, the stored-energy function has the form $\Psi(\tau_1,\tau_2,\nu,s)$ where ν is the stretch in the radial direction, and τ_1 and τ_2 are orthogonal stretches in azimuthal directions. (At such a point the deformation tensor $\boldsymbol{F}$ reduces to the square root $\boldsymbol{C}^{1/2}$ of the Cauchy-Green deformation tensor.) In general, the material is isotropic if the stored-energy function depends on the principal invariants of $\boldsymbol{C}^{1/2}$. At the material point in question, an isotropic material accordingly has a stored-energy density function of the form

$$\Psi(\tau_1, \tau_2, \nu, s) = \Omega(\tau_1 + \tau_2 + \nu, \tau_1\tau_2 + \tau_1\nu + \tau_2\nu, \tau_1\tau_2\nu, s). \tag{2.3.8}$$

Let the first three arguments of Ω be denoted A, B, C. For such an isotropic material, the hoop Piola-Kirchhoff stresses T_1 and T_2, and the radial stress N have constitutive equations of the form

$$\begin{aligned} T_1 &= \Psi_{\tau_1} = \Omega_A + (\tau_2 + \nu)\Omega_B + \tau_2\nu\Omega_C, \\ T_2 &= \Psi_{\tau_2} = \Omega_A + (\tau_1 + \nu)\Omega_B + \tau_1\nu\Omega_C, \\ N &= \Psi_\nu = \Omega_A + (\tau_1 + \tau_2)\Omega_B + \tau_1\tau_2\Omega_C. \end{aligned} \tag{2.3.9}$$

Denote the arguments of T_1, T_2, N as (τ_1, τ_2, ν, s). Then the isotropy ensures that

$$N(\nu, \tau_2, \tau_1, s) = T_1(\tau_1, \tau_2, \nu, s), \quad \text{etc.} \tag{2.3.10}$$

Now consider radially symmetric deformations of the isotropic material for which $\tau_1 = \tau_2 =: \tau$ and $T_1 = T_2$. Suppress the argument s and set

$$\begin{aligned} \varphi(\tau, \nu) &:= \Psi(\tau, \tau, \nu) = \Omega(2\tau + \nu, \tau^2 + 2\tau\nu, \tau^2\nu), \\ \hat{T}(\tau, \nu) &:= T_1(\tau, \tau, \nu) \equiv T_2(\tau, \tau, \nu), \qquad \hat{N}(\tau, \nu) := N(\tau, \tau, \nu). \end{aligned} \tag{2.3.11}$$

Then

$$\begin{aligned} 2\hat{T}(\tau, \nu) &= \varphi_\tau(\tau, \nu) \equiv \Psi_{\tau_1}(\tau, \tau, \nu) + \Psi_{\tau_2}(\tau, \tau, \nu) \equiv 2\Omega_A + 2(\tau + \nu)\Omega_B + 2\tau\nu\Omega_C, \\ \hat{N}(\tau, \nu) &= \varphi_\nu(\tau, \nu) \equiv \Psi_\nu(\tau, \tau, \nu) \equiv \Omega_A + 2\tau\Omega_B + \tau^2\Omega_C \end{aligned} \tag{2.3.12}$$

where the arguments of the derivatives of Ω are those of Ω in (2.3.11). Thus

$$\hat{T}(\tau, \tau) = \hat{N}(\tau, \tau). \tag{2.3.13}$$

for isotropic shells undergoing radially symmetric deformations. Beyond this, there seems to be no useful and simple consequence of isotropy that generalizes (2.3.10) and gives a relationship between the T and N of (2.3.12) that is independent of the derivatives of Ω. (There is for plates, as we soon show.)

Note that for shells isotropy transverse to the radial direction implies that

$$\Psi(\tau_1, \tau_2, \nu) = \Omega(\tau_1 + \tau_2, \tau_1\tau_2, \nu), \tag{2.3.14}$$

so that

$$\begin{aligned} T_1(\tau_1, \tau_2, \nu) &= \Psi_{\tau_1} = \Omega_A + \tau_2\Omega_B, \\ T_2(\tau_1, \tau_2, \nu) &= \Psi_{\tau_2} = \Omega_A + \tau_1\Omega_B. \end{aligned} \tag{2.3.15}$$

Thus $T_1(\tau, \tau, \nu) = T_2(\tau, \tau, \nu) = \hat{T}(\tau, \nu)$. This observation was exploited in our formulation in Section 2.2.

For isotropic plates we replace (2.3.8) with

$$\varphi(\tau, \nu, s) \equiv \Psi(\tau, \nu, s) = \Omega(\tau + \nu, \tau\nu, s). \tag{2.3.16}$$

Thus

$$\varphi(\tau, \nu, s) = \varphi(\nu, \tau, s). \tag{2.3.17}$$

These conditions immediately imply the attractive characterization of isotropy for *plates*:

$$\hat{T}(\tau, \nu, s) = \hat{N}(\nu, \tau, s), \tag{2.3.18}$$

a simple analog of (2.3.10), which, as (2.3.12) shows, does not characterize isotropy for shells.

Monotonicity and quasi-monotonicity. We pause to examine some delicate aspects of the relationship of the unacceptable Monotonicity Condition of 2- and 3-dimensional nonlinear elasticity to the QM condition (2.3.3). In 2- and 3-dimensional nonlinear elasticity the first Piola-Kirchhoff stress tensor with Cartesian components T_{ij} is given as a function $\hat{T}_{ij}$ of the (transposed) deformation gradient with Cartesian components F_{kl} and of the material point. The Strong Ellipticity Condition in summation convention is that

$$A_{ij}\frac{\partial\,\hat{T}_{ij}}{\partial\,F_{kl}}A_{kl} > 0 \tag{2.3.19}$$

for all nonzero tensors A_{ij} of rank 1. This condition ensures that the dynamical equations have the richest wave properties, that the equilibrium equations have a natural existence theory [14], and that each component $\hat{T}_{ij}$ of the first Piola-Kirchhoff stress is a strictly increasing function of the corresponding deformation gradient F_{ij} [5, Eq. (13.3.5)]. (On the other hand, much of the extensive work on coexistent phases [18] is based on the violation of the Strong Ellipticity Condition on bounded sets.) The Monotonicity Condition is that (2.3.19) holds for all nonzero tensors A_{ij}. This condition is deemed unacceptable for nonlinear elasticity because (i) it is incompatible with frame-indifference, (ii) it ensures uniqueness of equilibrium states under dead loads and thereby precludes buckling under such loads, and (iii) it prevents a corresponding convex stored-energy function from penalizing total compression by blowing up on the boundary of the nonconvex set of admissible deformation gradients, which have positive determinant [5, Sec. 13.3].

Clearly the Strong Ellipticity Condition implies (2.3.1) and the Monotonicity

Condition implies (2.3.3). In the next paragraph we show that the Monotonicity Condition implies an additional constitutive restriction that is patently inappropriate for our problems. Thus the QM Condition (2.3.3) is more general than the consequences of Monotonicity Condition for our problems. We can entertain the QM Condition as physically reasonable for our restricted axisymmetric problems because: (i) The rotational invariance of frame-indifference does not intervene. (ii) For (2.3.1) to hold and for (2.3.3) to be violated (so that an unexpected axisymmetric nonuniqueness might be allowed), off-diagonal terms in (2.3.3), which correspond to $\varphi_{\tau\nu}$ for hyperelastic materials, would have to dominate the diagonal terms. This would mean, e.g., that a stretching in one direction would produce a more severe transverse contraction (in a way unlike the Poisson-ratio effect of linear elasticity, but compatible with the behavior of some artificially constructed materials). (iii) For our problem the domain of the constitutive functions is the convex quadrant $\{(\tau,\nu) : \tau > 0, \nu > 0\}$.

An unacceptable consequence of the Monotonicity Condition. As above, let F_{ij} be the Cartesian components of the 2-dimensional (transposed) deformation gradient. An isotropic 2-dimensional hyperelastic material has a stored-energy function of the form

$$W(F_{11}, F_{12}, F_{21}, F_{22}) = \Omega(I, J),$$
$$I := \tfrac{1}{2}(F_{11}^2 + F_{21}^2 + F_{12}^2 + F_{22}^2), \qquad J := \det \boldsymbol{F} = F_{11}F_{22} - F_{21}F_{12}. \tag{2.3.20}$$

($2I$ is the trace of Cauchy-Green deformation tensor $\boldsymbol{C}$, and J^2 is its determinant.)

Then

$$
\begin{aligned}
T_{11} &= W_{F_{11}} = \mathit{\Omega}_I F_{11} + \mathit{\Omega}_J F_{22},\\
T_{22} &= W_{F_{22}} = \mathit{\Omega}_I F_{22} + \mathit{\Omega}_J F_{11},\\
T_{12} &= W_{F_{12}} = \mathit{\Omega}_I F_{12} - \mathit{\Omega}_J F_{21},\\
T_{21} &= W_{F_{21}} = \mathit{\Omega}_I F_{21} - \mathit{\Omega}_J F_{12}.
\end{aligned} \tag{2.3.21}
$$

Note the ordering of the indices here and in the next two sets of equations. If $F_{12} = 0 = F_{21}$, then the matrix of partial derivatives of the T_{ij} with respect to the F_{kl} is

$$
\frac{\partial\,(T_{11}, T_{22}, T_{12}, T_{21})}{\partial\,(F_{11}, F_{22}, F_{12}, F_{21})} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 \\ 0 & 0 & \mathit{\Omega}_I & -\mathit{\Omega}_J \\ 0 & 0 & -\mathit{\Omega}_J & \mathit{\Omega}_I \end{bmatrix} \tag{2.3.22}
$$

where

$$
\begin{aligned}
Q_{11} &:= \mathit{\Omega}_I + \mathit{\Omega}_{II} F_{11}^2 + 2\mathit{\Omega}_{IJ} F_{11} F_{22} + \mathit{\Omega}_{JJ} F_{22}^2,\\
Q_{22} &:= \mathit{\Omega}_I + \mathit{\Omega}_{II} F_{22}^2 + 2\mathit{\Omega}_{IJ} F_{11} F_{22} + \mathit{\Omega}_{JJ} F_{11}^2,\\
Q_{12} \equiv Q_{21} &:= \mathit{\Omega}_J + \mathit{\Omega}_{II} F_{11} F_{22} + 2\mathit{\Omega}_{IJ} (F_{11}^2 + F_{22}^2) + \mathit{\Omega}_{JJ} F_{11} F_{22}.
\end{aligned} \tag{2.3.23}
$$

The Monotonicity Condition (2.3.19) implies the positive-definiteness of the symmetric matrix (2.3.22), which is equivalent to

$$
\mathit{\Omega}_I > |\mathit{\Omega}_J|, \quad Q_{11} > 0, \quad Q_{22} > 0, \quad Q_{11} Q_{22} - Q_{12}^2 > 0. \tag{2.3.24}
$$

These same conditions hold for polar coordinates with 1 corresponding to the radial direction and 2 to the azimuthal direction. For the equilibrium of an isotropic disk, $F_{11} = F_{22} = k > 0$, $k = \text{const}$. The specialization of $(2.3.21)_1$ to this case implies that $N = T_{11}$ is a positive constant, which is nonsense if the disk is subjected to compression on its outer boundary.

A family of examples. Both to show that the various conditions we have imposed are not inconsistent and to have a general class of hyperelastic materials for which we can perform specific computations, we consider stored-energy functions of the form

$$\begin{aligned}(\tau, \nu) \mapsto \varphi(\tau, \nu) = &\frac{A_1 \tau^{1-a_1}}{a_1 - 1} + \frac{A_2 \nu^{1-a_2}}{a_2 - 1} + \frac{B_1 \tau^{1+b_1}}{b_1 + 1} + \frac{B_2 \nu^{1+b_2}}{b_2 + 1} \\ &+ C\tau^{-c_1}\nu^{-c_2} + D\tau^{d_1}\nu^{d_2} + E_1\tau^{-e_1}\nu^{f_1} + E_2\tau^{f_2}\nu^{-e_2}\end{aligned} \tag{2.3.25}$$

where $A_1, A_2, B_1, B_2, C, D, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2$ are positive constants and E_1, E_2 are non-negative constants. More generally, all these constants could be replaced by functions of s. The corresponding functions T and N clearly satisfy the Strong Ellipticity condition (2.3.1). The first five terms on the right-hand side of (2.3.25) define convex functions. The D term is convex if and only if $d_1 + d_2 < 1$. The E_1 term is convex if and only if $f_1 > c_1 + 1$. The E_2 term is convex if and only if $f_2 > c_2 + 1$. The hyperelastic material defined by (2.3.25) is isotropic if φ is symmetric in its arguments, i.e., $\varphi(\tau, \nu) = \varphi(\nu, \tau)$. This holds if each of the constants bearing the subscript 1 equals the corresponding constant bearing the subscript 2. The cross terms in (2.3.25) are needed to ensure (2.3.6c) and (2.3.6d).

2.4 A Dual Formulation

Conditions $(2.3.1)_2$ and (2.3.6b) imply that the function $\hat{N}(\tau, \cdot, s)$ is invertible, so that there is a function $\nu^\sharp$ such that

$$n = \hat{N}(\tau, \nu, s) \quad \Longleftrightarrow \quad \nu = \nu^\sharp(\tau, n, s). \tag{2.4.1}$$

Define

$$T^\sharp(\tau, n, s) := \hat{T}(\tau, \nu^\sharp(\tau, n, s), s). \tag{2.4.2}$$

Combining these dual constitutive equations with (2.2.6) and the equation $r'(s) = \nu(s)$ yields the semilinear system

$$\frac{d}{ds}(s\tau) = \nu^\sharp(\tau, n, s), \tag{2.4.3a}$$

$$\frac{d}{ds}(s^\gamma n) = \gamma s^{\gamma-1} T^\sharp(\tau, n, s) - (2-\gamma)\omega^2 \mu(s)\tau. \tag{2.4.3b}$$

The condition (2.2.3) that the reference configuration is natural is equivalent to

$$T^\sharp(1, 0, s) = 0, \qquad \nu^\sharp(1, 0, s) = 1. \tag{2.4.4}$$

The analogs of the growth conditions (2.3.6) are

$$T^\sharp(\tau,n,s) \to \begin{Bmatrix} +\infty \\ -\infty \end{Bmatrix} \text{ as } \tau \to \begin{Bmatrix} +\infty \\ 0 \end{Bmatrix} \text{ if } n \text{ is bounded} \begin{Bmatrix} \text{below} \\ \text{above} \end{Bmatrix}, \tag{2.4.5a}$$

$$\nu^\sharp(\tau,n,s) \to \begin{Bmatrix} +\infty \\ 0 \end{Bmatrix} \text{ as } n \to \begin{Bmatrix} +\infty \\ -\infty \end{Bmatrix} \text{ if } \tau \begin{Bmatrix} \text{has a positive lower bound} \\ \text{is bounded above} \end{Bmatrix}, \tag{2.4.5b}$$

$$T^\sharp(\tau,n,s) \to \begin{Bmatrix} +\infty \\ -\infty \end{Bmatrix} \text{ as } n \to \begin{Bmatrix} +\infty \\ -\infty \end{Bmatrix} \text{ if } \tau \begin{Bmatrix} \text{has a positive lower bound} \\ \text{is bounded above} \end{Bmatrix}, \tag{2.4.5c}$$

$$\nu^\sharp(\tau,n,s) \to \begin{Bmatrix} 0 \\ +\infty \end{Bmatrix} \text{ as } \tau \to \begin{Bmatrix} +\infty \\ 0 \end{Bmatrix} \text{ if } n \text{ is bounded} \begin{Bmatrix} \text{above} \\ \text{below} \end{Bmatrix}. \tag{2.4.5d}$$

The Strong Ellipticity Condition (2.3.1) becomes

$$T^\sharp_\tau \nu^\sharp_n - T^\sharp_n \nu^\sharp_\tau > 0, \quad \nu^\sharp_n > 0, \tag{2.4.6}$$

The conditions $(2.3.2)_2$ and (2.4.6) imply that

$$T^\sharp_n = \hat{T}_\nu \nu^\sharp_n > 0, \qquad T^\sharp_\tau = \hat{T}_\tau + (\nu^\sharp_n)^{-1} T^\sharp_n \nu^\sharp_\tau > 0. \qquad \nu^\sharp_\tau < 0. \tag{2.4.7}$$

The QM Condition (2.3.3) becomes

$$\begin{bmatrix} T^\sharp_\tau \nu^\sharp_n - T^\sharp_n \nu^\sharp_\tau & T^\sharp_n \\ -\nu^\sharp_\tau & 1 \end{bmatrix} \text{ is positive-definite.} \tag{2.4.8}$$

Condition (2.4.8) says that the symmetric part of the matrix of (2.4.8) is positive-definite. A simple argument then shows that the determinant of the matrix of (2.4.8) is positive, i.e.,

$$T^\sharp_\tau > 0, \tag{2.4.9}$$

which is $(2.4.7)_2$. By forming the quadratic form for (2.4.8) with coefficients α, β and then choosing $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (1, \nu^\sharp_\tau)$ we show that (2.4.7) implies that

$$\begin{bmatrix} \nu^\sharp_n & \nu^\sharp_\tau \\ T^\sharp_n & T^\sharp_\tau \end{bmatrix} \quad \text{is positive-definite,} \tag{2.4.10}$$

and a further calculation shows that (2.4.10) is equivalent to (2.4.8). Differentiating the identity $n = \hat{N}(\tau, \nu^\sharp(\tau, n, s), s)$ from (2.4.1) with respect to τ yields $\nu^\sharp_\tau = -\hat{N}_\tau / \hat{N}_\nu$ and differentiating (2.4.2) with respect to n yields $T^\sharp_n = \hat{T}_\nu \nu^\sharp_n = \hat{T}_\nu {\hat{N}_\tau}^{-1}$. Thus the Strong Ellipticity Condition (2.4.6) and $(2.3.5)_3$ yield complements to (2.4.7):

$$T^\sharp_n < 1, \qquad -1 < \nu^\sharp_\tau. \tag{2.4.11}$$

The inequalities $0 < T^\sharp_n < 1$ from (2.4.9) and (2.4.11) play important roles in Section 2.7. For a linear isotropic material with Poisson's ratio σ, straightforward calculations [48] show that

$$T^\sharp_n = \frac{\sigma}{1-\sigma}, \qquad T^\sharp_n = \sigma \tag{2.4.12}$$

respectively in plane strain and in plane stress. The usual range for σ, namely $(0, \frac{1}{2})$, shows that $(2.4.12)_1$ agrees with the bounds $0 < T^\sharp_n < 1$, while $(2.4.12)_2$ delivers the sharper bounds $0 < T^\sharp_n < \frac{1}{2}$.

If the material is hyperelastic, the Legendre transform of φ is the dual energy

Θ given by

$$\Theta(\tau, n, s) = n\nu^\sharp(\tau, n, s) - \varphi(\tau, \nu^\sharp(\tau, n), s), \tag{2.4.13}$$

in terms of which

$$T^\sharp(\tau, n, s) = -\Theta_\tau(\tau, n, s), \quad \nu^\sharp(\tau, n, s) = \Theta_n(\tau, n, s). \tag{2.4.14}$$

To construct examples it is convenient to have explicit representations for $\nu^\sharp$ (which deliver explicit representations for $T^\sharp$). By choosing

$$\begin{gathered} a_2 = b_2, \quad c_2 + 1 = b_2, \quad d_2 - 1 = b_2, \\ f_1 - 1 = b_2 \quad \text{when} \quad E_1 \neq 0, \quad -e_2 - 1 = b_2 \quad \text{when} \quad E_2 \neq 0, \end{gathered} \tag{2.4.15}$$

we convert $(2.2.2)_2$ to a quadratic equation for ν^{b_2}, the solution of which has the form $(2.4.14)_2$.

2.5 Phase-Plane Methods for Homogeneous Bodies

The material is *homogeneous* if the constitutive functions $\hat{T}$,$\hat{N}$ and the density μ do not depend explicitly on s, equivalently, if $\nu^\sharp, T^\sharp$ and μ do not depend explicitly on s. In this section we construct solutions to our boundary-value problems for various homogeneous materials (when there is no rotation), providing examples of problems for which there are multiple solutions and for which there are no solutions.

Let

$$s = e^{\xi - 1}, \quad \tau(s) = \tilde{\tau}(\xi), \quad n(s) = \tilde{n}(\xi). \tag{2.5.1}$$

Then for homogeneous materials, system (2.4.3) is equivalent to

$$\frac{d\tau}{d\xi} = \nu^\sharp(\tau, n) - \tau, \quad \frac{dn}{d\xi} = \gamma[T^\sharp(\tau, n) - n] - (2 - \gamma)\omega^2\mu e^{2(\xi - 1)}\tau, \quad \xi \in (1 + \ln a, 1) \tag{2.5.2}$$

where we have dropped the superposed tildes. This system is autonomous when $\omega = 0$, a condition we impose in this section. We analyze solutions of (2.5.2) with $\omega = 0$ by extensions of phase-plane methods developed for it in [7, 10].

In accordance with the discussion in Section 2, on the outer boundary $\xi = 1$, we prescribe one of the conditions:

$$n(1) = -\lambda_1 \tau^{\delta_1}(1), \quad \tau(1) = \tau_1, \tag{2.5.3}$$

and on the inner boundary $\xi = \ln a + 1$, we prescribe one of the conditions:

$$n(a) = -\lambda_a \left(a\tau(a)\right)^{\delta_a}, \quad \tau(a) = \tau_a. \tag{2.5.4}$$

Any solution of such a boundary-value problem for (2.5.2) must correspond to a phase-plane trajectory that originates on the *initial curve* $\mathcal{B}_a$ defined by one of the conditions of (2.5.4), that terminates on the *terminal curve* $\mathcal{B}_1$ defined by one of the conditions of (2.5.3), and that uses up exactly $-\ln a$ units of ξ in going from $\mathcal{B}_a$ to $\mathcal{B}_1$. The existence and uniqueness of solutions are determined exactly by the number of such trajectories.

On a trajectory of (2.5.2) that passes through $\mathcal{B}_1$ at a (terminal) point $(\tau_{\mathrm{T}}, n_{\mathrm{T}})$ we identify the *precursor point* $(\tau_{\mathrm{P}}, n_{\mathrm{P}})$ such that the trajectory from $(\tau_{\mathrm{P}}, n_{\mathrm{P}})$ to $(\tau_{\mathrm{T}}, n_{\mathrm{T}})$ uses up exactly $-\ln a$ units of ξ. The locus of all such precursor points is the *precursor curve* $\mathcal{P}$ for $\mathcal{B}_1$. Then a solution of our boundary-value problem corresponds to a trajectory of (2.5.2) starting at a point of intersection of $\mathcal{P}$ and $\mathcal{B}_a$ and terminating on $\mathcal{B}_1$. (Cf. [45].)

Isoclines. We take the τ-axis to be the abscissa of the phase portrait for (2.5.2).

In this case the horizontal isocline $\mathcal{H}$ consists of the (τ, n) satisfying

$$n = T^\sharp(\tau, n) \equiv \hat{T}(\tau, \nu^\sharp(\tau, n)) \quad \Leftrightarrow \quad \hat{T}(\tau, \nu^\sharp(\tau, n)) = \hat{N}(\tau, \nu^\sharp(\tau, n)) \tag{2.5.5}$$

(by virtue of definitions (2.4.1) and (2.4.2)), and the vertical isocline $\mathcal{V}$ consists of the (τ, n) satisfying

$$\tau = \nu^\sharp(\tau, n) \qquad \Longleftrightarrow \qquad n = \hat{N}(\tau, \tau). \tag{2.5.6}$$

The singular points of (2.5.2) are the intersections of these two isoclines, i.e., they satisfy (2.5.5) and (2.5.6). At a singular point, we can replace n by $(2.5.6)_2$ and replace $\nu^\sharp(\tau, n)$ by $(2.5.6)_1$ to get

$$\hat{N}(\tau, \tau) = \hat{T}(\tau, \tau). \tag{2.5.7}$$

By (2.3.13) and (2.3.18), this equation is an identity for isotropic plates and shells. Thus for an isotropic plate or shell, all the singular points of (2.5.2) lie on the curve $\mathcal{H} = \mathcal{V}$,

$$n = \hat{N}(\tau, \tau) \equiv \hat{T}(\tau, \tau), \tag{2.5.8}$$

so they are not isolated. In this case, we denote this curve of singular points by $\mathcal{S}$. If the reference configuration is natural, i.e., if (2.4.4) holds, then $(\tau, n) = (1, 0)$ is a singular point.

As discussed in [5, Sec. 10.2], the solution of the analog of our boundary-value problem for the isotropic disk or ball is given by such a curve of singular points and corresponds to a state of uniform deformation. A typical boundary-value problem for the annular plate or spherical shell cannot have such a solution, because such a solution occurs only when $\mathcal{B}_a$ and $\mathcal{B}_1$ intersect on $\mathcal{S}$. (Using this observation we could contrive special sets of boundary conditions for which the strains and stresses are constants throughout the body.)

If (2.3.1) and (2.3.2) hold, or if (2.3.3) holds, then $\tau \mapsto \hat{N}(\tau, \tau)$ (which gives the graph of $\mathcal{V}$; see (2.5.6)) is strictly increasing. If, furthermore, (2.3.6) holds, then $\hat{N}(\tau, \tau)$ strictly increases from $-\infty$ to ∞ as τ increases from 0 to ∞.

Equation $(2.5.5)_1$ for $\mathcal{H}$ can be solved for n as a function of τ everywhere if $1 - T^\sharp_n > 0$ everywhere. Condition (2.4.11) ensures this inequality. Conditions (2.4.5a) imply that on $\mathcal{H}$, n strictly increases from $-\infty$ to ∞ in this case, so the horizontal isocline has the same behavior as the vertical isocline.

Equation $(2.5.2)_1$ implies that τ increases along a trajectory wherever $\nu^\sharp(\tau, n) > \tau$. At such points $n \equiv \hat{N}(\tau, \nu^\sharp(\tau, n)) > \hat{N}(\tau, \tau)$. Thus τ increases along a trajectory wherever it lies above (or equivalently to the left of) $\mathcal{V}$, and likewise τ decreases along a trajectory wherever it lies below (or equivalently to the right of) $\mathcal{V}$.

Equation $(2.5.2)_2$ implies that n increases along a trajectory wherever $n < T^\sharp(\tau, n)$. If $T^\sharp_\tau(\tau, n) > 0$, as a consequence of the conditions ensuring either of the equivalent inequalities $(2.4.7)_2$ and (2.4.9), then on a line in the phase portrait with constant n, the derivative $dn/d\xi$ can change sign only when this line crosses $\mathcal{H}$. In this case, n decreases along a trajectory to the left of $\mathcal{H}$, and n increases along a trajectory to the right of $\mathcal{H}$. As shown above, condition (2.4.11) ensures that what is to the left of $\mathcal{H}$ is above it, etc. We assume that this condition holds.

Precursor curves. To find the precursor curve for $\mathcal{B}_1$ we need formulas for the amount of independent variable ξ used up on intervals of phase-plane trajectories. Suppose that $[\tau_1, \tau_2] \equiv [\tau(\xi_1), \tau(\xi_2)] \ni \tau \mapsto \tilde{n}(\tau)$ gives the graph over the τ-axis of a segment of a phase-plane trajectory. Then $(2.5.2)_1$ implies that

$$\xi_2 - \xi_1 = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\nu^\sharp(\tau, \tilde{n}(\tau)) - \tau}. \tag{2.5.9}$$

Likewise, if $[n_1, n_2] \equiv [n(\xi_1), n(\xi_2)] \ni \tau \mapsto \tilde{\tau}(n)$ gives the graph over the n-axis of a segment of a phase-plane trajectory, then $(2.5.2)_2$ implies that

$$\xi_2 - \xi_1 = \int_{n_1}^{n_2} \frac{dn}{\gamma[T^\sharp(\tilde{\tau}(n), n) - n]}. \tag{2.5.10}$$

We find the precursor curve $\mathcal{P}$, consisting of points $(\tau_{\mathrm{P}}, \nu_{\mathrm{P}})$ thus: Fix a point $(\tau_{\mathrm{T}}, n_{\mathrm{T}})$ on $\mathcal{B}_1$. If the phase-plane trajectory terminating at $(\tau_{\mathrm{T}}, n_{\mathrm{T}})$ has a graph over the τ-axis given by $\tau \mapsto \tilde{n}(\tau)$, then (2.5.9) implies that τ_{P} is the solution of

$$-\ln a = \int_{\tau_{\mathrm{P}}}^{\tau_{\mathrm{T}}} \frac{d\tau}{\nu^\sharp(\tau, \tilde{n}(\tau)) - \tau} \tag{2.5.11}$$

and $n_{\mathrm{P}} = \tilde{n}(\tau_{\mathrm{P}})$. Likewise, if the phase-plane trajectory terminating at $(\tau_{\mathrm{T}}, n_{\mathrm{T}})$ has a graph over the n-axis given by $n \mapsto \tilde{\tau}(n)$, then (2.5.10) implies that τ_{P} is the solution of

$$-\ln a = \int_{n_{\mathrm{P}}}^{n_{\mathrm{T}}} \frac{dn}{\gamma[T^\sharp(\tilde{\tau}(n), n) - n]} \tag{2.5.12}$$

and $\tau_{\mathrm{P}} = \tilde{\tau}(n_{\mathrm{P}})$. Every trajectory of (2.5.2) for an isotropic material, other than a rare constant state lying on the curve $\mathcal{S}$ of singular points, admits the representations supporting (2.5.11) and (2.5.12). In general, when the trajectory terminating at $(\tau_{\mathrm{T}}, n_{\mathrm{T}})$ lacks simple projections on the τ- or n-axes, the right-hand side of (2.5.11) or (2.5.12) can be replaced by sums of integrals from (2.5.9) and (2.5.10).

Depending on the terminal curve and the character of the phase portrait, we employ formulas (2.5.9)–(2.5.12) to compute numerically the precursor curves for various terminal curves. The MATLAB solver function `ode45` is used to compute several trajectories touching the terminal curve. Each of these trajectories is obtained as a collection of pairs $\{(\tau_i, n_i), i = 1, \ldots, K\}$ where the positive integer K depends on the trajectory. For each of the trajectories, the MATLAB function `trapz` is then used to find $i \in \{1, ..., K\}$ for which (2.5.11) or (2.5.12) with $n_{\mathrm{P}} = n_i$ holds with the best accuracy.

Examples illustrating the range of physical phenomena. In the following examples of materials of the form (2.3.25) we have chosen the parameters for mathematical simplicity. Our choices correspond to rescalings of physically natural parameters leading to phase portraits topologically equivalent to those given here. In all of our examples, we consider the case of an annular plate, i.e., we study (2.5.2) with $\gamma = 1$, we treat only hyperelastic materials, and we assume that the reference configuration is stress free.

In the phase portraits of (2.5.2) (with $\omega = 0$) for these examples we illustrate some typical boundary curves $\mathcal{B}_1$ and their precursor curves $\mathcal{P}$. The number and qualitative behavior of solutions are determined by the intersections (if any) of the boundary curve $\mathcal{B}_a$ with $\mathcal{P}$. The absence from the phase portraits of $\mathcal{B}_a$ (which for plates are merely straight lines) is intended to allow the reader to consider the range of possibilities for $\mathcal{B}_a$.

Example: An isotropic material with the convex stored-energy density function

$$\varphi(\tau,\nu) = \frac{1}{\tau} + \frac{1}{\nu} + \frac{\tau^3}{3} + \frac{\nu^3}{3} \tag{2.5.13}$$

producing constitutive functions

$$\begin{aligned} \hat{T}(\tau) &= \varphi_\tau(\tau,\nu) = -\frac{1}{\tau^2} + \tau^2, \qquad & \hat{N}(\nu) &= \varphi_\nu(\tau,\nu) = -\frac{1}{\nu^2} + \nu^2, \\ \nu^\sharp(n) &= \sqrt{\frac{n+\sqrt{n^2+4}}{2}}, \qquad & T^\sharp(\tau) &= -\frac{1}{\tau^2} + \tau^2, \end{aligned} \tag{2.5.14}$$

which satisfy the QM Condition (2.3.3) and the growth conditions (2.3.6a). Fig. 2.1 illustrates the non-uniqueness of solutions due to hydrostatic loads on the boundary. For example, the initial curve corresponding to condition $(2.5.4)_1$ with $\delta_a = 0$ has multiple points of intersection with $\mathcal{P}$ when λ_a is in a suitable range. Thus there exist multiple trajectory segments, each representing a different radially symmetric solution.

To further investigate the existence of radially symmetric states in case when condition $(2.5.4)_1$ or $(2.5.4)_2$ is prescribed on the inner boundary, we study the asymptotic behavior of a typical precursor curve in the region below the singular separatrix $\mathcal{S}$. Let $\tau = \tau^\dagger(n,\beta)$ be the trajectory which intersects the terminal line $n = -\lambda_1\tau$ at point $(\beta, -\lambda_1\beta)$ and let $(n_{\text{P}}^\beta, \tau_{\text{P}}^\beta)$ be the precursor point which lies on this trajectory. Then n_{P}^β is the solution of

$$-\ln a = \int_{n_{\text{P}}^\beta}^{-\lambda_1\beta} \frac{dn}{T^\sharp(\tau^\dagger(n,\beta)) - n} =: \Upsilon(\beta), \tag{2.5.15}$$

where $\varphi(\beta)$ is the amount of the independent variable used up on the segment of the trajectory between the point $(\beta, -\lambda_1\beta)$ and the line $n = n_{\text{P}}^\beta$. Note that $\tau^\dagger(n,\beta) \to \infty$ as $\beta \to \infty$ for any fixed n. Hence $T^\sharp(\tau^\dagger(n,\beta)) \sim (\tau^\dagger(n,\beta))^2$ as $\beta \to \infty$. We further

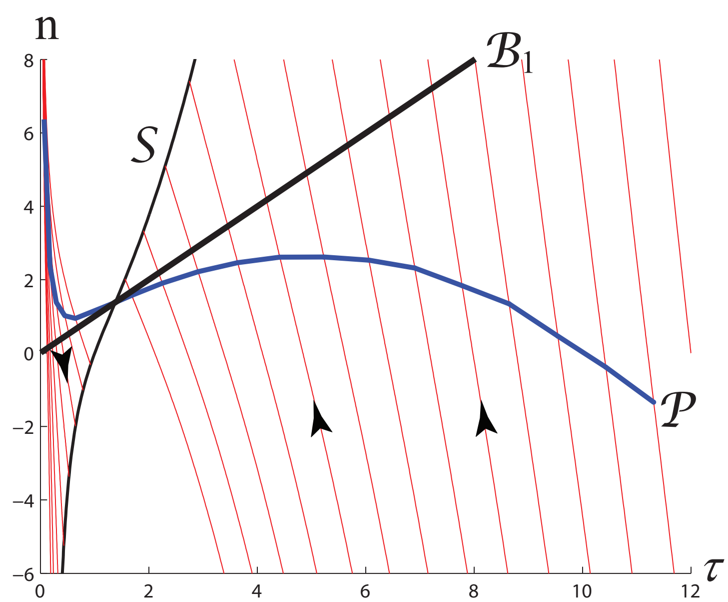

Figure 2.1: Example (2.5.13) (convex stored-energy density function). The precursor curve $\mathcal{P}$ for the terminal line $\mathcal{B}_1$ corresponds to condition $(2.5.3)_2$ with $\delta_1 = 1$ and $-\lambda_1 = 1$.

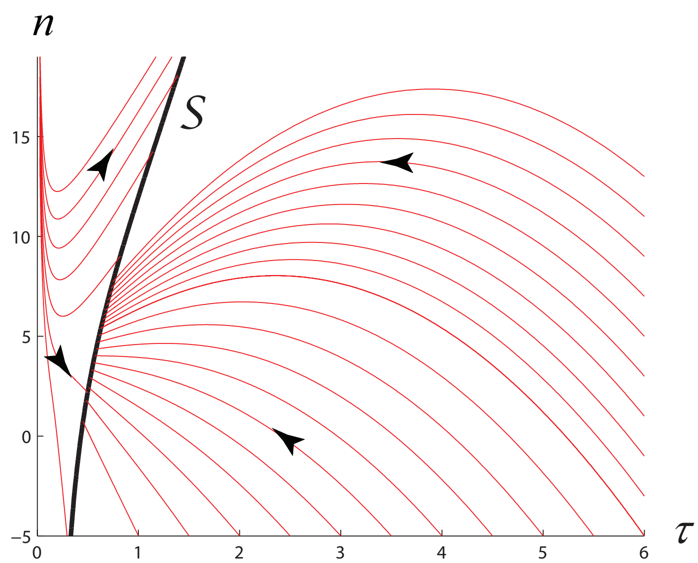

Figure 2.2: Phase portrait for example (2.5.19) (non-convex stored-energy density function) showing that n need not be monotone along trajectories.

note that for fixed n we have

$$\frac{d\tau^\dagger}{dn} = \frac{\nu^\sharp(n) - \tau^\dagger}{T^\sharp(\tau^\dagger) - n} \to 0 \quad \text{as } \beta \to \infty \tag{2.5.16}$$

and so $\tau^\dagger(n, \beta) \sim \beta$ as $\beta \to \infty$. Thus

$$-\ln a = \lim_{\beta\to\infty} \Upsilon(\beta) \sim \int_{n_{\mathrm{P}}^\beta}^{-\lambda_1\beta} \frac{dn}{\beta^2 - n} \equiv -\ln \frac{\beta^2 + \lambda_1\beta}{\beta^2 - n_{\mathrm{P}}^\beta}. \tag{2.5.17}$$

Consequently,

$$n_p^\beta \sim (1 - a^{-1})\beta^2 \quad \text{as } \beta \to \infty. \tag{2.5.18}$$

Thus the precursor curve (regarded as the graph of some function $\tilde{n} = \tilde{n}(\tau)$) decays quadratically to $-\infty$ in the region below the singular separatrix $\mathcal{S}$.

Example: An isotropic material with the non-convex stored-energy density function

$$\varphi(\tau, \nu) = \frac{1}{\tau} + \frac{1}{\nu} + \frac{\tau^3}{3} + \frac{\nu^3}{3} + 12\nu\tau - 12\tau - 12\nu. \tag{2.5.19}$$

producing constitutive functions

$$\begin{aligned}\hat{T}(\tau,\nu) &= \varphi_\tau(\tau,\nu) = -\frac{1}{\tau^2} + \tau^2 + 12\nu - 12 \\ \hat{N}(\tau,\nu) &= \varphi_\nu(\tau,\nu) = -\frac{1}{\nu^2} + \nu^2 + 12\tau - 12, \\ \nu^\sharp(\tau,n) &= \sqrt{\frac{-12\tau + n + 12 + \sqrt{(12\tau - n - 12)^2 + 4}}{2}}, \\ T^\sharp(\tau,n) &= -\frac{1}{\tau^2} + \tau^2 + 12\nu^\sharp(\tau,n) - 12,\end{aligned} \tag{2.5.20}$$

which satisfy the Strong Ellipticity Condition (2.3.1) and growth conditions (2.3.6a), (2.3.6b), but not the QM Condition (2.3.3). The curve $\mathcal{S}$ of singular points in the (τ, n)-half-space is given by

$$\tau = \nu^\sharp(\tau, n). \tag{2.5.21}$$

The phase portrait for this example is shown in Fig. 2.2. As in Example (2.5.13), τ increases along a trajectory wherever it lies in a region above $\mathcal{S}$, and likewise τ decreases along a trajectory wherever it lies in a region below $\mathcal{S}$. However, unlike Example (2.5.13), n is not generally monotone along trajectories.

Fig. 2.3, a blowup of part of Fig. 2.2, shows the precursor curve $\mathcal{P}$ constructed for the terminal line corresponding to condition $(2.5.3)_1$ with $-\lambda_1 = 14$. In this example, the behavior of the precursor curve $\mathcal{P}$ is complicated because n is not monotone along the trajectories touching the line $n = 14$. In particular, the initial curve corresponding to condition $(2.5.4)_1$ may have as many as three different points of intersection with $\mathcal{P}$ when λ_a is in a suitable range. This means that there exist three distinct radially symmetric solutions for certain boundary conditions.

Example: An aeolotropic material with the convex stored-energy density function

$$\varphi(\tau,\nu) = \frac{1}{\tau} + \frac{1}{\nu} + \frac{\tau^2}{2} + \frac{\nu^3}{3} \tag{2.5.22}$$

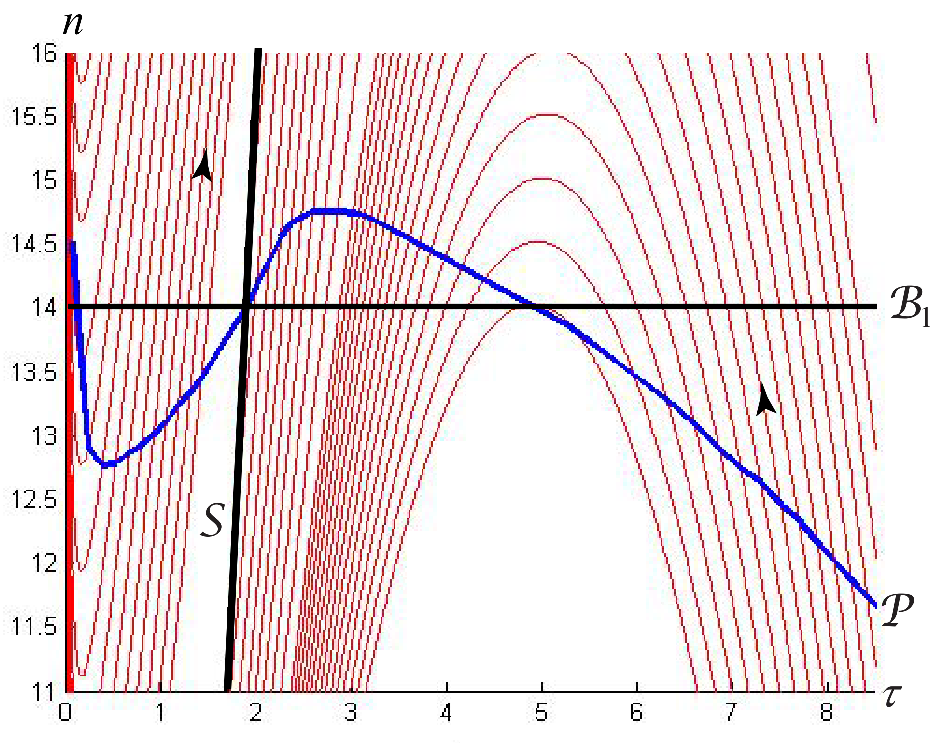

Figure 2.3: A blowup of Fig. 2.2 for the non-convex stored-energy density function (2.5.19) showing the precursor curve for the terminal line corresponding to condition $(2.5.3)_1$ with $-\lambda_1 = 14$ and $\delta_1 = 0$.

producing constitutive functions

$$\begin{gathered} T(\tau,\nu)=\varphi_\tau(\tau,\nu)=-\frac{1}{\tau^2}+\tau, \qquad N(\tau,\nu)=\varphi_\nu(\tau,\nu)=-\frac{1}{\nu^2}+\nu^2, \\ \nu^\sharp(\tau,n)=\sqrt{\frac{n+\sqrt{n^2+4}}{2}}, \qquad T^\sharp(\tau,n)=-\frac{1}{\tau^2}+\tau \end{gathered} \tag{2.5.23}$$

satisfying the QM Condition (2.3.1) and growth conditions (2.3.6a), (2.3.6b). The vertical isocline $\mathcal{V}$ is given by

$$\tau=\nu^\sharp(\tau,n) \quad \Longleftrightarrow \quad n=-\frac{1}{\tau^2}+\tau^2, \tag{2.5.24}$$

The horizontal isocline $\mathcal{H}$ is given by

$$n=-\frac{1}{\tau^2}+\tau. \tag{2.5.25}$$

The horizontal and vertical isoclines intersect at the singular point $(1,0)$. Fig. 2.4 shows the precursor curve constructed for the terminal line corresponding to condition $(2.5.3)_1$ with $-\lambda_1=-.5$ and $\delta_1=0$.

Example: An isotropic material weak in tension with the convex stored-energy density function

$$\varphi(\tau,\nu)=-2\tau^{\frac{1}{2}}-2\nu^{\frac{1}{2}}+\tfrac{2}{3}\tau^{3/2}+\tfrac{2}{3}\nu^{3/2} \tag{2.5.26}$$

producing constitutive functions

$$\begin{gathered} \hat{T}(\tau,\nu)=\varphi_\tau(\tau,\nu)=-\tau^{-1/2}+\tau^{1/2}, \quad \hat{N}(\tau,\nu)=\varphi_\nu(\tau,\nu)=-\nu^{-1/2}+\nu^{1/2} \\ \nu^\sharp(n)=\tfrac{1}{4}(n+\sqrt{n^2+4})^2, \qquad T^\sharp(\tau)=-\tau^{-1/2}+\tau^{1/2} \end{gathered} \tag{2.5.27}$$

satisfying the QM Condition (2.3.1) and growth conditions (2.3.6a), (2.3.6b). (φ does not become infinite in a total compression, although $\hat{T}$ and $\hat{N}$ respectively

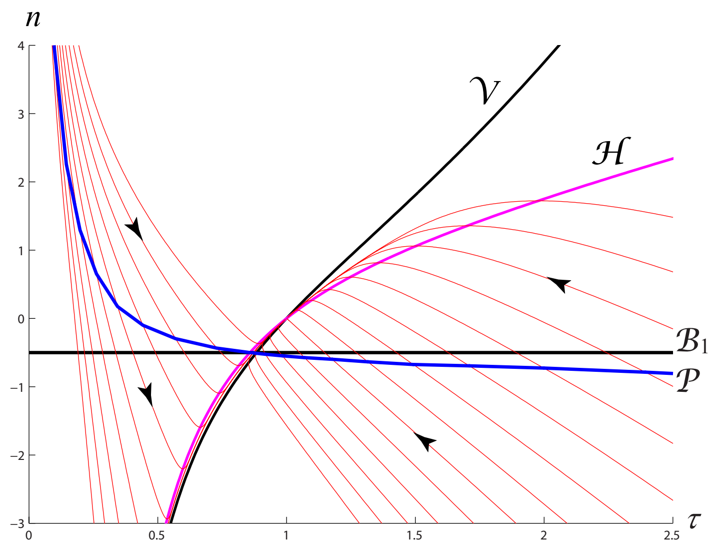

Figure 2.4: Example (2.5.22) (aeolotropic material). The precursor curve $\mathcal{P}$ is for the terminal line defined by $(2.5.3)_1$ with $\lambda_1 = 0.5$ and $\delta_1 = 0$.

approach $-\infty$ as $\tau \searrow 0$ and $\nu \searrow 0$.) The curve $\mathcal{S}$ of singular points in the (τ, n)-plane is given by

$$n = -\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}}. \tag{2.5.28}$$

Let $\tau = \tau^{\dagger}(n, \beta)$ be the equation of the trajectory that intersects the terminal line $n = -\lambda_1$ at point $(\beta, -\lambda_1)$ and let $(n_{\mathrm{P}}^{\beta}, \tau_{\mathrm{P}}^{\beta})$ be the precursor point which lies on this trajectory. Then n_{P}^{β} satisfies

$$-\ln a = \int_{n_{\mathrm{P}}^{\beta}}^{-\lambda_1} \frac{dn}{T^{\sharp}(\tau^{\dagger}(n, \beta)) - n} =: \Upsilon(\beta), \tag{2.5.29}$$

where $\Upsilon(\beta)$ is the amount of the independent variable used up on the segment of the trajectory between the point $(\beta, -\lambda_1)$ and the line $n = n_{\mathrm{P}}^{\beta}$.

By an asymptotic argument like that leading to (2.5.18), we find

$$n_p^{\beta} \sim \frac{2}{3}\left(1 - a^{-3/2}\right)\sqrt{\beta} \quad \text{as } \beta \to \infty. \tag{2.5.30}$$

Fig. 2.5 shows the precursor curve constructed for the terminal line corresponding to condition(2.5.3)$_1$ with $-\lambda_1 = 2$. The behavior of the precursor curve agrees with the *sublinear decay* predicted by the asymptotic formula (2.5.30). In particular, we see that when a hydrostatic load (2.5.3)$_2$ is prescribed on the inner boundary there are no solutions for λ_a in a suitable range.

Example: An isotropic material that is exhibiting a nonlinear Poisson-ratio effect with the following (not necessarily convex) stored-energy density function

$$\varphi(\tau, \nu) = \frac{1}{\tau} + \frac{1}{\nu} + \frac{\tau^3}{3} + \frac{\nu^3}{3} + D\nu\tau + \frac{D}{\nu\tau}. \tag{2.5.31}$$

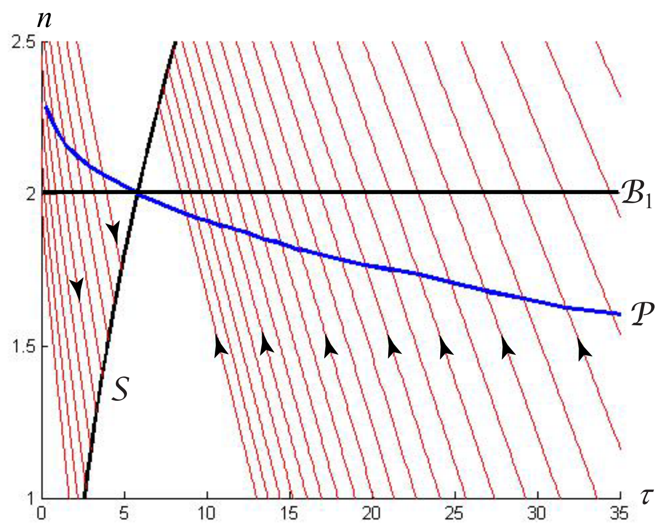

Figure 2.5: Example (2.5.26). The precursor curve $\mathcal{P}$ is for the terminal line corresponding to condition $(2.5.3)_1$ with $-\lambda_1 = 2$ and $\delta_1 = 0$.

producing constitutive functions

$$\begin{aligned}
T = \varphi_\tau = -\frac{1}{\tau^2} + \tau^2 + D\nu - \frac{D}{\nu\tau^2}, \quad N = \varphi_\nu = -\frac{1}{\nu^2} + \nu^2 + D\tau - \frac{D}{\nu^2\tau}, \\
\nu^\sharp(\tau, n) = \sqrt{\frac{n - \tau + \sqrt{(\tau - n)^2 + 4(1 + \tau^{-1})}}{2}} \\
T^\sharp(\tau, n) = -\frac{1}{\tau^2} + \tau^2 + \nu^\sharp(\tau, n) - \frac{1}{\nu^\sharp(\tau, n)\,\tau^2}.
\end{aligned} \tag{2.5.32}$$

satisfying the Strong Ellipticity Condition (2.3.1) and growth conditions (2.3.6a), (2.3.6b). We choose $D = 1$ to ensure that (T, N) satisfies the QM Condition (2.3.3). The curve $\mathcal{S}$ of singular points in the (τ, n)-half-space has the equation

$$\tau = \nu^\sharp(\tau, n). \tag{2.5.33}$$

To investigate the existence and uniqueness of solutions in case when condition $(2.5.4)_1$ or $(2.5.4)_2$ is prescribed on the inner boundary, we study the asymptotic behavior of a typical precursor curve as $\tau \to \infty$. Let $\tau = \tau^\dagger(n, \beta)$ be the trajectory which intersects the terminal line $n = -\lambda_1$ at point $(\beta, -\lambda_1)$ and let $(n_{\mathrm{P}}^\beta, \tau_{\mathrm{P}}^\beta)$ be the precursor point which lies on this trajectory. Then n_{P}^β is the solution of

$$-\ln a = \int_{n_{\mathrm{P}}^\beta}^{-\lambda_1} \frac{dn}{T^\sharp(\tau^\dagger(n, \beta), n) - n} =: \Upsilon(\beta), \tag{2.5.34}$$

where $\Upsilon(\beta)$ is the amount of the independent variable used up on the segment of the trajectory between the point $(\beta, -\lambda_1)$ and the line $n = n_{\mathrm{P}}^\beta$.

An asymptotic argument like that leading to (2.5.18) implies that that

$$n_p^\beta \sim (1 - a^{-1})\beta^2 \quad \text{as } \beta \to \infty. \tag{2.5.35}$$

Thus the precursor curve (regarded as the graph of a function $\tau \mapsto \tilde{n}(\tau)$) decays quadratically to $-\infty$ in the region below the singular separatrix $\mathcal{S}$. See Fig. 1.6.

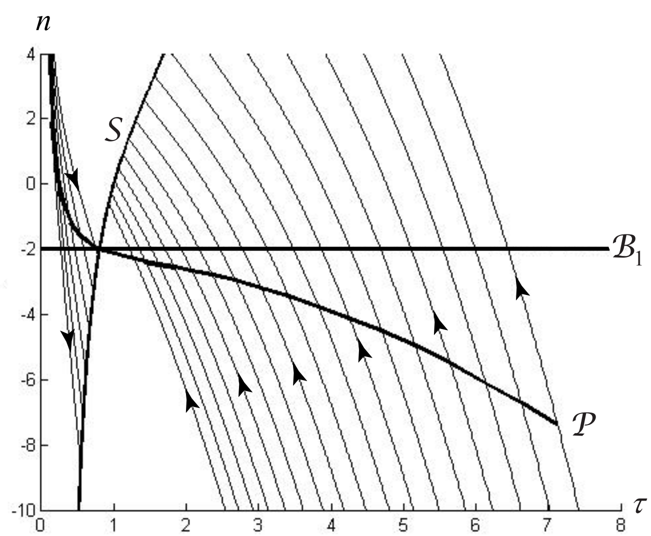

Figure 2.6: Example (2.5.31) (material with nonlinear Poisson ratio effects). The precursor curve $\mathcal{P}$ is for the terminal line corresponding to condition $(2.5.3)_1$ with $-\lambda_1 = -2$ and $\delta_1 = 0$.

2.6 Existence and Uniqueness via Variational Methods

We use the direct methods of the Calculus of Variations to establish general existence and uniqueness results for the solutions of the boundary-value problems for (2.2.6), especially for nonhomogeneous materials, for which phase-plane methods are not readily adapted. (This work extends part of [3].)

We begin by studying the minimization of the potential energy functionals for our problems, the Euler-Lagrange equations for which are the equilibrium equations for our plates and shells. Define the *potential energy functional*

$$\begin{aligned} \Pi[r] := \int_a^1 \varphi\Big(\frac{r(s)}{s}, r'(s), s\Big) s^\gamma \, ds - \tfrac{2-\gamma}{2}\omega^2 \int_a^1 \mu(s) r(s)^2 s^\gamma \, ds \\ - \varepsilon_a \lambda_a \frac{r(a)^{\delta_a+1}}{\delta_a+1} + \varepsilon_1 \lambda_1 \frac{r(1)^{\delta_1+1}}{\delta_1+1} \end{aligned} \tag{2.6.1}$$

where $\varepsilon_a = 0$ if the position condition $(2.2.10)_1$ holds, in which case the admissible r must satisfy this condition; where $\varepsilon_a = 1$ if the traction boundary condition $(2.2.10)_2$ holds; where $\varepsilon_1 = 0$ if the position condition (2.2.8) holds, in which case the admissible r must satisfy this condition; and where $\varepsilon_1 = 1$ if the traction boundary condition (2.2.9) holds.

2.6.2. Basic assumptions. Let $\mathcal{V} := (0, \infty) \times (0, \infty)$. (i) The plates and shells are hyperelastic. (ii) $(\tau, \nu, s) \mapsto \varphi(\tau, \nu, s)$ is twice continuously differentiable on $\mathcal{V} \times [a, 1]$. (iii) $\varphi_{\nu\nu} \equiv \hat{N}_\nu > 0$ everywhere (this is $(2.3.1)_2$). (iv) There are numbers $c > 0$, $C \geq 0$, $\alpha > 1$, $\beta > 1$ such that φ satisfies the *coercivity condition*:

$$\varphi(\tau, \nu, s) \geq c\Big[\tau^\beta + \nu^\alpha\Big] - C. \tag{2.6.3}$$

(This condition ensures that the material cannot yield under a dead load. Some of the ensuing analysis does not require the positivity of $\beta - 1$.) (v) The stored-energy function is infinite at a total compression:

$$\varphi(\tau, \nu, s) \to \infty \quad \text{as} \quad \tau \searrow 0 \quad \text{or} \quad \nu \searrow 0. \tag{2.6.4}$$

(vi) If position conditions are prescribed at $s = a, 1$, then $\rho_a < \rho_1$ (since we do not consider everted states [5, Sec. 14.7]).

Denote by $\mathcal{B}$ the Banach (Sobolev) space of functions r satisfying

$$\|r\| := \Big[\int_a^1 \Big|\frac{r(s)}{s}\Big|^\beta s^\gamma\, ds\Big]^{1/\beta} + \Big[\int_a^1 |r'(s)|^\alpha s^\gamma\, ds\Big]^{1/\alpha} < \infty \tag{2.6.5}$$

The positivity of $\alpha - 1$ ensures that $\mathcal{B}$ is reflexive and is compactly embedded in $C^0[a, 1]$. Our admissible class $\mathcal{A} \subset \mathcal{B}$ of functions r consists of those functions of $\mathcal{B}$ that satisfy any prescribed position boundary conditions and that satisfy the almost tautological condition that $r(s) \le \rho_1$ when $r(1)$ is prescribed to equal ρ_1. We briefly treat standard parts of the theory, for which we shall invoke the following theorem, concentrating on those aspects with physical consequences.

2.6.6. Fundamental Abstract Existence Theorem [5, 24, 56]**.** *A sequentially weakly lower semicontinuous real-valued functional on a bounded sequentially weakly closed nonempty subset of a reflexive Banach space has a minimum there.*

That our potential energy functionals, the specializations of (2.6.1), are sequentially weakly lower semicontinuous on the Sobolev space $\mathcal{B}$ follows from the $(2.3.1)_2$ by virtue of the theory in [5, 24]. A physically critical aspect of our analysis is to show that the potential energy functionals become infinite as a suitable norm of the admissible functions becomes infinite, whence it suffices to seek minimizers for admissible functions on a bounded set (as needed for Theorem 2.6.6).

We shall need some simple inequalities. Let $p > 1$ and $q = p/(p-1)$. If $|f|^p$ and $|g|^q$ are integrable with weight s^γ on $[a, 1]$, then the Hölder inequality implies that

$$\int_a^1 fgs^\gamma \, ds \leq \Big[\int_a^1 |f|^p s^\gamma \, ds \Big]^{1/p} \Big[\int_a^1 |g|^q s^\gamma \, ds \Big]^{1/q}. \tag{2.6.7}$$

In getting inequalities we use the convention that C represents any positive constant depending on data, the dependence on which is not worth exhibiting. The meaning of C may change in each appearance.

The mean value of r is

$$m[r] := \frac{1}{1-a} \int_a^1 r(s) \, ds. \tag{2.6.8}$$

Let $r \in \mathcal{B}$. The Hölder inequality (2.6.7) implies that

$$m[r] \leq C \Big[\int_a^1 r^\beta \, ds \Big]^{1/\beta} \leq C \Big[\int_a^1 (s^{-1} r)^\beta s^\gamma \, ds \Big]^{1/\beta}. \tag{2.6.9}$$

The function $r - m[r]$ vanishes at at least one point $s^* \in [a, 1]$.

If

$$\left\{ \begin{matrix} r(1) = \rho_1 \\ r(a) = \rho_a, \quad N(1) = -\lambda_1 \tau(1)^{\delta_1} \\ N(a) = -\lambda_a \tau(a)^{\delta_a}, \quad N(1) = -\lambda_1 \tau(1)^{\delta_1} \end{matrix} \right\} \equiv \left\{ \begin{matrix} \varepsilon_1 = 0 \\ \varepsilon_a = 0, \quad \varepsilon_1 = 1 \\ \varepsilon_a = 1 = \varepsilon_1 \end{matrix} \right\}, \tag{2.6.10}$$

then the Fundamental Theorem of Calculus, the Hölder inequality (2.6.7), and (2.6.9) imply that

$$r(s) = \left\{ \begin{matrix} \rho_1 - \int_s^1 r_s(\xi) \, d\xi \\ \rho_a + \int_a^s r_s(\xi) \, d\xi \\ m[r] + \int_{s^*}^s r_s(\xi) \, d\xi \end{matrix} \right\} \leq C \left\{ \begin{matrix} 1 \\ \Big[\int_a^1 {r_s}^\alpha s^\gamma \, ds \Big]^{1/\alpha} \\ \Big[\int_a^1 \tau^\beta s^\gamma \, ds \Big]^{1/\beta} + \Big[\int_a^1 {r_s}^\alpha s^\gamma \, ds \Big]^{1/\alpha} \end{matrix} \right\}, \tag{2.6.11}$$

so that (2.6.3) implies that the potential energy of the pressure terms in (2.6.1) satisfy

$$\begin{aligned}\Big|-\varepsilon_a\lambda_a\frac{r(a)^{\delta_a+1}}{\delta_a+1}+\varepsilon_1\lambda_1\frac{r(1)^{\delta_1+1}}{\delta_1+1}\Big| &= \left\{\begin{matrix}\varepsilon_a\lambda_a\frac{r(a)^{1+\delta_a}}{1+\delta_a}\\ \lambda_1\frac{r(1)^{1+\delta_1}}{1+\delta_1}\\ \big|-\lambda_a\frac{r(a)^{\delta_a+1}}{\delta_a+1}+\lambda_1\frac{r(1)^{\delta_1+1}}{\delta_1+1}\big|\end{matrix}\right\}\\ &\le C\left\{\begin{matrix}1\\ 1\\ \Upsilon(\delta_a,\beta)+\Upsilon(\delta_1,\beta)\end{matrix}\right\}+C\left\{\begin{matrix}\Upsilon(\delta_a,\alpha)\\ \Upsilon(\delta_1,\alpha)\\ \Upsilon(\delta_a,\alpha)+\Upsilon(\delta_1,\alpha)\end{matrix}\right\}\end{aligned} \tag{2.6.12}$$

where $\Upsilon(\delta,\eta) := \left[\int_a^1 \varphi s^\gamma\, ds\right]^{(1+\delta)/\eta}$. Consequently, if $\omega = 0$ and if (2.6.10) holds, then

$$\Pi[r]\to\infty \quad \text{as} \quad \|r\|\to\infty$$

$$\text{if}\quad \left\{\begin{matrix}\alpha>1,\\ \alpha>1+\delta_1\\ \alpha,\beta>1+\delta_a,1+\delta_1\end{matrix}\right\}\quad \text{or if}\quad \varepsilon_a\lambda_a\le 0 \quad\text{and}\quad \varepsilon_1\lambda_1\ge 0. \tag{2.6.13}$$

2.6.14. Minimization Theorem for Nonrotating Plates and Shells. *Let assumptions 2.6.2 hold. Let $\omega = 0$. If*

(i) $r(1)=\rho_1,\ \ \alpha>1$, *or if*

(ii) $r(a)=\rho_a,\ N(1)=-\lambda_1 r(1)^{\delta_1},\ \ \alpha>1+\delta_1$, *or if*

(iii) $N(a)=-\lambda_a[r(a)/a]^{\delta_a},\ N(1)=-\lambda_1 r(1)^{\delta_1},\ \ \alpha,\beta>1+\delta_a,1+\delta_1,$

then Π attains its minimum on $\mathcal{A}$.

Proof. $\mathcal{A}$ is clearly nonempty. The functional Π defined in (2.6.1) is sequentially weakly lower semicontinuous since $\varphi_{\nu\nu}\equiv N_\nu>0$ and $\varphi\ge 0$ everywhere [5, Thm. 7.3.26]. $\mathcal{A}$ is sequentially weakly closed because $\mathcal{B}$ is compactly embedded in $C^0([a,1])$. The coercivity of Π is given by (2.6.13). □

2.6.15. Minimization Theorem for Rotating Plates. *Let assumptions 2.6.2 hold. Let $\gamma = 1$ and $\omega \neq 0$. If condition* (i) *of Theorem 2.6.14 holds, or if condition* (ii) *holds and* $\max\{\alpha, \beta\} > 2$, *or if condition* (iii) *holds and* $\alpha, \beta > 2$, *then Π attains its minimum on $\mathcal{A}$.*

Proof. We follow the proof of Theorem 2.6.14, only needing to show that the stored energy dominates $\Omega[r] := \omega^2 \int_a^1 \mu(s) r(s)^2 s^\gamma \, ds$. This is immediate for condition (i) for which $r(1) = \rho_1$ since $r(s) \leq r(1)$. If (ii) holds, then $\Omega[r] \leq C[\int_a^1 \varphi s \, ds]^{2/\beta}$, so there is dominance if $\beta > 2$. Alternatively, the second line of (2.6.11) shows this dominance if $\alpha > 2$. Likewise, if (iii) holds, then the third line of (2.6.11) shows this dominance. □

Note the role of the resistance to stretching in the radial and azimuthal directions for these different cases in Theorems 2.6.14 and 2.6.15.

2.6.16. Uniqueness Theorem. *Let assumptions 2.6.2 hold. Let the QM Condition* (2.3.3) *hold, i.e., let $(\tau, \nu) \mapsto \varphi(\tau, \nu, s)$ be strictly convex for every $s \in [a, 1]$. Let* (i) $r(1) = \rho_1$ *or let* $N(1) = -\lambda_1 r(1)^{\delta_1}$ *with* (ii) $\delta_1 = 0$ *or* (iii) $\lambda_1 \geq 0$. *Let* (iv) $r(a) = \rho_a$ *or let* $N(a) = -\lambda_a r(a)^{\delta_a}$ *with* (v) $\delta_a = 0$ *or* (vi) $\lambda_a \leq 0$. *Then there is at most one minimizer $\bar{r}$ of Π on $\mathcal{A}$.*

Proof. This result, an example of the principle that a strictly convex functional can have at most one minimizer, has a standard proof [24, Sec. 3.4]: To be specific, assume that $N(1) = -\lambda_1 r(1)^{\delta_1}$ with $\lambda_1 \geq 0$ and $N(a) = -\lambda_a r(a)^{\delta_a}$ with $\lambda_a \leq 0$.

Then the QM Condition and our hypotheses on the signs of λ_1 and λ_a imply that Π is strictly convex. Let u and v minimize Π. The convexity of $\mathcal{B}$ implies that $\theta u + (1-\theta)v \in \mathcal{A} = \mathcal{B}$ for all $\theta \in [0,1]$. The convexity of Π implies that

$$\Pi[\theta u + (1-\theta)v] \leq \theta \Pi[u] + (1-\theta)\Pi[v] \quad \text{for all} \quad \theta \in [0,1]. \tag{2.6.17}$$

In particular, $w := \frac{1}{2}(u+v)$ is also a minimizer of Π. Thus

$$\tfrac{1}{2}\Pi[u] + \tfrac{1}{2}\Pi[v] - \Pi[\tfrac{1}{2}(u+v)] = 0. \tag{2.6.18}$$

which violates the strict convexity of Π unless $u = v$. □

We have seen in Section 2.5 that solutions need not be unique when the hypotheses of Theorem 2.6.16 do not hold.

We now study the regularity of minimizer $\bar{r}$ under the assumption (v) of (2.6.2) (stronger than (2.3.6a), (2.3.6b)) that φ becomes infinite in a total compression. We may define $\varphi(\tau, \nu, s) = \infty$ if $(\tau(s), \nu(s)) \notin \mathcal{V}$. The physically important consequence of the regularity, which is the source of difficulty, is that τ and ν are everywhere positive (cf. (2.2.7)).

2.6.19. Theorem. *Let the assumptions 2.6.2 hold and let the relevant hypotheses of the Minimization Theorems 2.6.14 and 2.6.15 hold. Let $(\tau, \nu) \mapsto \varphi(\tau, \nu, s)$ be twice continuously differentiable on $\mathcal{V}$. The minimizer $\bar{r}$ of Π on $\mathcal{A}$ is twice continuously differentiable and satisfies the Euler-Lagrange equation*

$$\frac{d}{ds}\Big[s\hat{N}\Big(\frac{\bar{r}(s)}{s}, \bar{r}'(s), s\Big)\Big] + (2-\gamma)\omega^2\mu(s)s\bar{r}(s) = \hat{T}\Big(\frac{\bar{r}(s)}{s}, \bar{r}'(s), s\Big) \tag{2.6.20}$$

everywhere with $\bar{r}'(s)$ and $\bar{r}(s)/s$ everywhere positive.

Proof. The proof of this theorem is an elaborate version of that used in [16, Théorème 2] (cf. [5, Chap. 7]). We show that Π has a directional (Gâteaux) derivative in suitable directions and then employ the bootstrap method. We focus on the case $\omega = 0$. The case $\omega \neq 0$ is treated analogously.

We construct variations that vanish where the minimizer is close to violating (2.2.7): For any positive integer n, let

$$\begin{aligned} G_k(s) :&= \sup \Big\{ \Big|\varphi_\tau\Big(\tau, \nu, s\Big)\Big| + \Big|\varphi_\nu\Big(\tau, \nu, s\Big)\Big| : \\ &\qquad (\tau, \nu) \in \mathbb{R} \times \mathbb{R}, \Big|\nu - \bar{r}'(s)\Big| + \Big|\tau - s^{-1}\bar{r}(s)\Big| < 1/k \Big\}, \qquad (2.6.21) \\ \Omega_k &:= \{ s \in [a, 1] : G_k(s) \leq k \}, \end{aligned}$$

and let χ_k be the characteristic function of Ω_k. Our hypotheses on φ ensure that the set Ω_k, on which φ_τ and φ_ν are not badly behaved, is measurable. Clearly, $\Omega_k \subset \Omega_{k+1}$ and $[a, 1] \setminus \cup_{k=1}^{\infty} \Omega_k$ has measure zero.

Note that if $s_* \in [a, 1]$ is a singular point of φ, i.e., $\tau(s_*) \leq 0$ or $\nu(s_*) \leq 0$, then $s_* \in \Omega_k^c$ for all n where $\Omega_k^c = [a, 1] \setminus \Omega_k$. Moreover, for each k there is a non-empty neighborhood $\mathcal{N}(s_*)$ of s_* such that $\mathcal{N}(s_*) \subset \Omega_k^c$ (if $s_* = a$ or $s_* = 1$ then $\mathcal{N}(s_*)$ is a half-open interval). This implies that $\chi_k(s)$ is differentiable almost everywhere in $[a, 1]$ for all k and

$$\frac{d\chi_k}{ds}(s) = 0 \quad \text{for a.e. } s \in [a, 1]. \qquad (2.6.22)$$

We first choose an arbitrary bounded function $s \mapsto v(s)$ with $\int_{\Omega_k} v(s)\, ds = 0$. For $|t|$ small define

$$r_k(s; t) := \bar{r}(s) + t\chi_k(s) \int_a^s \chi_k(\xi) v(\xi)\, d\xi. \qquad (2.6.23)$$

Then (2.6.22) implies that

$$r_k'(s;t) := \bar{r}'(s) + t\chi_k(s)v(s) \quad \text{a.e.} \tag{2.6.24}$$

Since

$$\Pi[r_k(\cdot;t)] = \Pi[\bar{r}] + \int_{\Omega_k} \Big[\varphi\Big(\frac{r_k(s;t)}{s}, \bar{r}'(s) + tv(s), s\Big) - \varphi\Big(\frac{\bar{r}(s)}{s}, \bar{r}'(s), s\Big)\Big] s\, ds, \tag{2.6.25}$$

we find that $\Pi[r_k(\cdot;t)] < \infty$ for $|t|$ sufficiently small, so that $r_k(\cdot;t) \in \mathcal{A}$ for $|t|$ sufficiently small. By dividing (2.6.25) by t and letting $t \searrow 0$, we obtain

$$\int_{\Omega_k} T\Big(\frac{\bar{r}(s)}{s}, \bar{r}'(s), s\Big)\Big(\int_a^s \chi_k(\xi)v(\xi)\, d\xi\Big)\, ds + \int_{\Omega_k} sN\Big(\frac{\bar{r}(s)}{s}, \bar{r}'(s), s\Big)v(s)\, ds = 0 \tag{2.6.26}$$

where we used the Lebesgue Dominated Convergence Theorem. An integration by parts yields

$$\int_{\Omega_k} \Big[sN\Big(\frac{\bar{r}(s)}{s}, \bar{r}'(s), s\Big) - \int_a^s \chi_k(\xi)T\Big(\frac{\bar{r}(\xi)}{\xi}, \bar{r}'(\xi), \xi\Big)\, d\xi\Big]v(s)\, ds = 0. \tag{2.6.27}$$

Since v is arbitrary, We take v to be the difference between its bracketed coefficient in integrand of (2.6.27) and the mean value of this coefficient, whence (2.6.27) yields

$$sN\Big(\frac{\bar{r}(s)}{s}, \bar{r}'(s), s\Big) - \int_a^s \chi_k(\xi)T\Big(\frac{\bar{r}(\xi)}{\xi}, \bar{r}'(\xi), \xi\Big)\, d\xi = C \tag{2.6.28}$$

for a.e. $s \in \Omega_k$. Since $\Omega_k \subset \Omega_{n+1}$, it is clear that C is independent of k. Hence (2.6.28) is valid for a.e. $s \in [a, 1]$. By differentiating (2.6.28) with respect to s, we find that (2.6.20) is valid for a.e. $s \in [a, 1]$. Thus

$$N\Big(\frac{\bar{r}(s)}{s}, \bar{r}'(s), s\Big) = Cs^{-1} + s^{-1}\int_a^s T\Big(\frac{\bar{r}(\xi)}{\xi}, \bar{r}'(\xi), \xi\Big)\, d\xi \tag{2.6.29}$$

is also valid for a.e. $s \in [a, 1]$.

We now show that $\bar{r}$ is a classical solution of the Euler-Lagrange equations by using the bootstrap method. Since $\varphi_{\nu\nu} \equiv N_\nu > 0$ everywhere, then $N(\bar{r}(s)s^{-1}, \cdot, s)$ has an inverse $\nu^\sharp(\bar{r}(s)s^{-1}, \cdot, s)$ on its range. Since we know that (2.6.29) has a solution $\bar{r}$, we know that the right-hand side of (2.6.29) is in this range. Therefore (2.6.29) is equivalent to

$$\bar{r}'(s) = \nu^\sharp\Big(\bar{r}(s)s^{-1}, s^{-1}\int_a^s T\Big(\frac{\bar{r}(\xi)}{\xi}, \bar{r}'(\xi), \xi\Big)\, d\xi + Cs^{-1}, s\Big) \tag{2.6.30}$$

The classical Implicit-Function Theorem implies that $\nu^\sharp$ is continuously differentiable. Since its arguments in (2.6.30) are continuous functions of s, it follows that $\bar{r}'$ is continuous. Consequently, the arguments of $\nu^\sharp$ in (2.6.30) are continuously differentiable functions of s. Therefore, (2.6.30) implies that $\bar{r}$ is twice continuously differentiable. Hence $\bar{r}$ is a classical solution of the Euler-Lagrange equations. $\square$

Materials weak in tension. As we have seen in Section 2.5, when there is an inflational hydrostatic pressure and when the corresponding coercivity hypotheses of Theorems 2.6.14 are not met, i.e., when Π is not coercive, or, equivalently, when the material is weak is tension, there may not be solutions for all pressures. Likewise, if $r(1)$ is not specified and the plate is rotating, there may not be solutions for all rotational speeds. (We have not yet exhibited a proof of nonexistence in this case. See Section 2.9 for comments on how the continuation method can provide such proofs.) We can nevertheless in principle solve the boundary-value problem for such materials for some pressures and for some rotational speeds: We simply replace the pressure boundary conditions with position boundary conditions, and from the solution compute the corresponding pressure(s). This process works for phase-plane methods.

This process can be efficiently carried out by variational methods to produce alternative existence theorems that illuminate the difficulty. To sketch the basic ideas, consider the boundary-value problem of (2.2.6) with $\omega = 0$ subject to

$$n(a) = -\lambda_a r(a)^\gamma, \qquad n(1) = 0 \tag{2.6.31}$$

with $\lambda_a > 0$, so that there is a hydrostatic pressure on the inner boundary $s = a$. Consider the variational problem of either minimizing or maximizing the functional $P[r] := r(a)$ or $P[r] = r(1)$ (which furnish a measure of the size of solutions) on the set

$$\mathcal{E} := \left\{ \int_a^1 \varphi(s^{-1}r, r', s)s^\gamma \, ds \leq A \right\} \subset \mathcal{B} \tag{2.6.32}$$

where A is prescribed. The sequential weak lower semi-continuity of the stored-energy functional ensures that $\mathcal{E}$ is weakly closed. It is easily shown that P is weakly continuous on $\mathcal{E}$. Therefore P attains its maximum and minimum on $\mathcal{E}$, and a simple estimate shows that these are attained on the boundary of $\mathcal{E}$. A Lagrange Multiplier Rule shows that there is a Lagrange multiplier $\lambda_a r(a)^\gamma$ such that (2.6.20) holds with $n(a) = -\lambda_a r(a)^\gamma$. For details see [3, 5].

2.7 Existence and Qualitative Behavior via Fixed-Point Methods

As Section 2.6 showed, the Existence Theorems 2.6.14 and 2.6.15 require strong resistance to extension to compensate for inflational hydrostatic pressures and centrifugal forces. Existence results of the sort discussed in the paragraph containing

(2.6.32) do not delimit the range of pressures for which there are solutions. In this section, we use a fixed-point method to exhibit a range of pressures $\lambda_a > 0$ for which the problem treated in that paragraph has solutions. We thereby analyze problems not covered by Theorems 2.6.14 and 2.6.15. (Of course, fixed-point methods, which can handle any set of boundary conditions are not restricted to hyperelastic materials. The examples treated here are illustrative of methods for treating any boundary conditions. For brevity, we take the centrifugal force to be zero.) We generalize (2.6.31) to account also for a dead-load pressure by taking boundary conditions

$$n(a) = -\lambda_a \tau(a)^{\delta_a}, \qquad n(1) = 0 \tag{2.7.1}$$

with $\lambda_a > 0$ prescribed and with $\delta_a = \gamma$ or 0. Thus there is an (inflational) hydrostatic pressure on the inner boundary $s = a$. This method gives some qualitative information about solutions. (This boundary-value problem is one of the trickiest for the fixed-point method.) In the following treatment of the boundary conditions (2.7.1) we drop the subscript a from λ_a.

The dual differential equations (2.4.3) subject to these boundary conditions yield the integral equations

$$\begin{aligned} s\tau(s) &= a\tau(a) + \int_a^s \nu^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi, \\ s^\gamma n(s) &= -\lambda a^\gamma \tau(a)^{\delta_a} + \int_a^s T^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi^\gamma, \end{aligned} \tag{2.7.2}$$

or, alternatively,

$$\begin{aligned} s\tau(s) &= \tau(1) - \int_s^1 \nu^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi, \\ s^\gamma n(s) &= -\int_s^1 T^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi^\gamma \end{aligned} \tag{2.7.3}$$

where $d\xi^\gamma = \gamma \xi^{\gamma-1} d\xi$.

$\tau(a)$ and $\tau(1)$ are not known. Evaluating $(2.7.2)_2$ at $s = 1$ subject to the boundary condition that $n(1) = 0$ or evaluating $(2.7.3)_2$ at $s = a$ subject to the boundary condition that $n(a) = -\lambda\tau(a)^{\delta_a}$ yields

$$\lambda a^\gamma \tau(a)^{\delta_a} = \int_a^1 T^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi^\gamma. \tag{2.7.4}$$

If $\delta_a = \gamma$, then this equation gives an integral representation for the unknown $\tau(a)$. We treat this case first:

Combine (2.7.2) and (2.7.4) to get

$$\begin{bmatrix} \tau(s) \\ n(s) \end{bmatrix} = \begin{bmatrix} \mathfrak{t}[\tau, n](s) \\ \mathfrak{n}[\tau, n](s) \end{bmatrix} \\ := \begin{bmatrix} \frac{1}{s}\left\{\frac{1}{\lambda}\int_a^1 T^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi^\gamma\right\}^{1/\gamma} + \frac{1}{s}\int_a^s \nu^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi \\ -\frac{1}{s^\gamma}\int_s^1 T^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi^\gamma \end{bmatrix}. \tag{2.7.5}$$

A solution of this (system of) integral equation(s) is a fixed point of $(\mathfrak{t}, \mathfrak{n})$, i.e., a pair (τ, n) that is taken by $(\mathfrak{t}, \mathfrak{n})$ to itself. To prove that (2.7.5) has a solution we use the

2.7.6. Schauder Fixed-Point Theorem. *Let $\mathcal{G}$ be a closed bounded convex subset of a Banach space. If $\mathbf{g}$ is a sequentially compact mapping from $\mathcal{G}$ to itself (i.e., if $\mathbf{g}$ takes any bounded sequence from $\mathcal{G}$ into a sequence in $\mathcal{G}$ that has a convergent subsequence), then $\mathbf{g}$ has a fixed point in $\mathcal{G}$.*

For proofs see [5, 22, 57], e.g.

Assume that (2.4.4) holds (so that the reference configuration is natural) and that the growth conditions (2.4.7) hold. Take the Banach space for (2.7.5) to be

$$\mathcal{C} := C^0[a,1] \times C^0[a,1]. \tag{2.7.7}$$

Hydrostatic pressure: $\delta_a = \gamma$. To apply the Schauder Fixed-Point Theorem to (2.7.5) we must make a judicious choice of $\mathcal{G}$. For given constants b, c, k with $1 < b < c$, $k > 0$ we propose to take $\mathcal{G}$ to be the closed bounded convex subset of $\mathcal{C}$ consisting of pairs (τ, n) satisfying the inequalities

$$b \le \tau(s) \le c, \qquad -k^\gamma \le n(s) \le 0 \quad \forall\, s \in [a,1]. \tag{2.7.8}$$

(It might seem reasonable to expect that a solution would satisfy $n(s) \ge -\lambda\tau^\gamma$, and replace k^γ in (2.7.8) with $\lambda\tau^\gamma$, but if $\gamma = 2$, then the resulting version of (2.7.8) would not define a convex set in (τ, n)-space.) To show that the hypotheses of the Schauder Fixed-Point Theorem are satisfied, we show that for suitable b, c, k, λ, the mapping $(\mathfrak{t}, \mathfrak{n})$ takes $\mathcal{G}$ into itself, i.e., $(\mathfrak{t}[\tau, n], \mathfrak{n}[\tau, n])$ satisfies the same inequalities (2.7.8) as (τ, n):

$$b \le \mathfrak{t}[\tau, n](s) \le c, \qquad -k^\gamma \le \mathfrak{n}[\tau, n](s) \le 0 \quad \forall\, s \in [a,1]. \tag{2.7.9}$$

We impose the following

Constitutive restrictions. Conditions (2.4.4), (2.4.6), and (2.4.7) hold. The material is weak in extension in the sense that

$$\tau^{-\gamma} T^\sharp(\tau, 0, \xi) \to 0 \quad \text{as} \quad \tau \to \infty. \tag{2.7.10}$$

(This is the very kind of condition that prevented the minimization of the potential energy functional in Section 2.6.) For every $q > 0$, there is a constant $\underline{B} > 0$ such that

$$T^\sharp(Bq, -q, \xi) > 0 \quad \forall\, B \geq \underline{B}. \tag{2.7.11}$$

(This condition has the flavor of (2.4.5a), $(2.4.7)_{1,2}$, $(2.4.11)_1$.)

Now we give sufficient conditions for (2.7.9) to hold. The inequalities for $\mathfrak{n}$ in (2.7.9) imply that

$$0 \leq \frac{1}{s^\gamma} \int_s^1 T^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi^\gamma \leq k^\gamma. \tag{2.7.12}$$

Our constitutive restrictions, just stated, and (2.7.8) imply that a sufficient condition for (2.7.12) is

$$0 \leq \frac{1}{s^\gamma} \int_s^1 T^\sharp(b, -k^\gamma, \xi)\, d\xi^\gamma \leq \frac{1}{s^\gamma} \int_s^1 T^\sharp(c, 0, \xi)\, d\xi^\gamma \leq k^\gamma. \tag{2.7.13}$$

Constitutive restriction (2.7.11) ensures that the first inequality holds for $b = Bk^\gamma$, $B \geq \underline{B}k^\gamma$. Let $c = Ck^\gamma > b$. Then (2.7.10) implies that the last inequality of (2.7.13) holds for large enough k.

Now study $\mathfrak{t}$. Our constitutive assumptions show that a sufficient condition for $\mathfrak{t}[\tau, n](s) \leq c$ is that

$$\frac{1}{a}\Big\{\frac{1}{\lambda} \int_a^1 T^\sharp(c, 0, \xi)\, d\xi^\gamma\Big\}^{1/\gamma} + \frac{1}{s} \int_a^s \nu^\sharp(b, 0, \xi)\, d\xi \leq c. \tag{2.7.14}$$

The second summand in (2.7.14) is < 1 (by (2.4.4) and $(2.4.7)_3$ since $b \geq 1$), so a sufficient condition for (2.7.14) is that

$$\frac{1}{\lambda} \int_a^1 T^\sharp(c, 0, \xi)\, d\xi^\gamma \leq [a(c-1)]^\gamma. \tag{2.7.15}$$

A sufficient condition for $\mathfrak{t}[\tau, n](s) \geq b$ is that

$$\frac{1}{\lambda} \int_a^1 T^\sharp(b, -k^\gamma, \xi)\, d\xi^\gamma \geq b^\gamma. \tag{2.7.16}$$

Combine these two conditions to get

$$\frac{1}{[a(c-1)]^\gamma}\int_a^1 T^\sharp(c,0,\xi)\,d\xi^\gamma \le \lambda \le \frac{1}{b^\gamma}\int_a^1 T^\sharp(b,-k^\gamma,\xi)\,d\xi^\gamma. \tag{2.7.17}$$

With the choices of b and c made above, (2.7.17) becomes

$$\frac{\int_a^1 T^\sharp(Ck^\gamma,0,\xi)\,d\xi^\gamma}{[a(Ck^\gamma-1)]^\gamma} \le \lambda \le \frac{\int_a^1 T^\sharp(Bk^\gamma,-k^\gamma,\xi)\,d\xi^\gamma}{(Bk^\gamma)^\gamma}. \tag{2.7.18}$$

The right-hand side of (2.7.18) is positive, and gives an upper bound for λ. The left-hand side is also positive, but it can be made arbitrarily small by taking C large. Under these conditions, $(\mathfrak{t},\mathfrak{n})$ maps the convex $\mathcal{G}$ into itself.

The Arzelà-Ascoli Theorem says that $(\mathfrak{t},\mathfrak{n})$ is sequentially compact if its image of $\mathcal{G}$ is uniformly bounded and equicontinuous. These properties follow immediately from the definitions of $\mathfrak{t},\mathfrak{n}$ as innocuous integral operators on a bounded subset of $\mathcal{C}$. Thus

2.7.19. Theorem. *If $\delta_a = \gamma$ and if the constitutive restrictions* (2.7.10) *and* (2.7.11) *hold, then there are numbers b, c, k for which the integral equation* (2.7.5), *has a classical solution provided that λ is sufficiently small. In this case, this integral equation is equivalent to the boundary-value problem* (2.4.3), (2.7.1).

Dead-load pressure: $\delta_a = 0$. A recurrent theme in this chapter is that the live loads of hydrostatic pressure and centrifugal force are a source of interesting mechanics requiring careful analysis. Here we treat dead-load pressures by a fixed-point method, which surprisingly is much trickier than that for hydrostatic pressures leading to Theorem 2.7.19.

Since $\delta_a = 0$ we cannot use (2.7.4) to get an explicit representation for $\tau(a)$. Instead, we substitute $(2.7.2)_1$ into (2.7.4) to get an implicit equation for $\tau(a)$ or for

$\tau(1)$:

$$\begin{aligned}\lambda a^\gamma &= \int_a^1 T^\sharp\Big(\xi^{-1}\big[a\tau(a) + \textstyle\int_a^\xi \nu^\sharp(\tau(\eta), n(\eta), \eta)\, d\eta\big], n(\xi), \xi\Big)\, d\xi^\gamma \\ &= \int_a^1 T^\sharp\Big(\xi^{-1}\big[\tau(1) - \textstyle\int_\xi^1 \nu^\sharp(\tau(\eta), n(\eta), \eta)\, d\eta\big], n(\xi), \xi\Big)\, d\xi^\gamma.\end{aligned} \tag{2.7.20}$$

Constitutive restrictions (2.4.5a) and $(2.4.7)_2$ imply that $T^\sharp(\tau, n, s)$ strictly increases from $-\infty$ to ∞ as τ increases from 0 to ∞ for fixed values of its other arguments. Thus $(2.7.20)_1$ can be uniquely solved for $\tau(a)$ and $(2.7.20)_2$ can be uniquely solved for $\tau(1)$:

$$\tau(a) = \sigma_a[\tau, n], \qquad \tau(1) = \sigma_1[\tau, n], \tag{2.7.21}$$

operator equations equivalent to $(2.7.20)_{1,2}$. We can use either version of (2.7.21). We choose the second.

We study the following operator equation for τ, n coming from (2.7.3) and $(2.7.21)_2$:

$$\begin{aligned}\tau(s) = \mathfrak{t}[\tau, n](s) :=& \frac{1}{s}\sigma_1[\tau, n] - \frac{1}{s}\int_s^1 \nu^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi, \\ n(s) = \mathfrak{n}[\tau, n](s) :=& -\frac{1}{s^\gamma}\int_s^1 T^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi^\gamma.\end{aligned} \tag{2.7.22}$$

We employ the constitutive assumptions that

$$T^\sharp_\tau > 0, \quad 0 < T^\sharp_n < 1, \quad -1 < \nu^\sharp_\tau < 0, \quad \nu^\sharp_n > 0, \tag{2.7.23}$$

which were stated in (2.4.6), (2.4.7), and (2.4.11), we assume that the reference configuration is natural, and we employ the growth conditions (2.4.5). The constitutive bounds on $T^\sharp_n$ and $\nu^\sharp_\tau$ will play a crucial role in our analysis.

We shall show that (2.7.22) has a solution by showing that $(\mathfrak{t}, \mathfrak{n})$ of (2.7.22) has a fixed point on a suitable convex subset $\mathcal{G}$ of elements (τ, n) in $C^0[a, 1] \times C^0[a, 1]$

satisfying the inequalities

$$1 \le \tau(s) \le c, \quad -\mathit{\Lambda} \le n(s) \le N \tag{2.7.24}$$

where $c, \mathit{\Lambda}, N$ are positive constants to be chosen so that $(\mathfrak{t}, \mathfrak{n})$ maps $\mathcal{G}$ into itself. (The trick in using fixed-point methods is to make a judicious choice of $\mathcal{G}$.) The mapping $(\mathfrak{t}, \mathfrak{n})$ of (2.7.22) takes $\mathcal{G}$ of (2.7.24) into itself if and only if

$$s \le \sigma_1[\tau, n] - \int_s^1 \nu^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi \le cs, \tag{2.7.25}$$

$$-Ns^\gamma \le \int_s^1 T^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi^\gamma \le \mathit{\Lambda} s^\gamma \tag{2.7.26}$$

when (τ, n) satisfies (2.7.24).

If (2.7.24) holds, then (2.7.20) imply that

$$\begin{aligned} &\int_a^1 T^\sharp\Big(\xi^{-1}\big[\sigma_1 - \textstyle\int_\xi^1 \nu^\sharp(1, N, \eta)\, d\eta\big], -\mathit{\Lambda}, \xi\Big)\, d\xi^\gamma \\ &\le \lambda a^\gamma \le \int_a^1 T^\sharp\Big(\xi^{-1}\big[\sigma_1 - \textstyle\int_\xi^1 \nu^\sharp(c, -\mathit{\Lambda}, \eta)\, d\eta\big], N, \xi\Big)\, d\xi^\gamma. \end{aligned} \tag{2.7.27}$$

For any constants b, B define

$$T^\flat(b, B) := \int_a^1 T^\sharp(b, B, \xi)\, d\xi^\gamma. \tag{2.7.28}$$

Sufficient conditions on the parameters $c, \mathit{\Lambda}, N$ for the τ of (2.7.22) to satisfy the bounds in (2.7.24) are

$$T^\flat(c, -\mathit{\Lambda}) \equiv \int_a^1 T^\sharp(c, -\mathit{\Lambda}, \xi)\, d\xi^\gamma \le \lambda a^\gamma \le \int_a^1 T^\sharp(1, N, \xi)\, d\xi^\gamma \equiv T^\flat(1, N). \tag{2.7.29}$$

Sufficient conditions on c, Λ, N for the two inequalities of (2.7.26) are

$$-Na^\gamma \le \int_a^1 T^\sharp(1, -\mathit{\Lambda}, \xi)\, d\xi^\gamma \equiv T^\flat(1, -\mathit{\Lambda}), \tag{2.7.30}$$

$$T^\flat(c, N) \equiv \int_a^1 T^\sharp(c, N, \xi)\, d\xi^\gamma \le \mathit{\Lambda} a^\gamma. \tag{2.7.31}$$

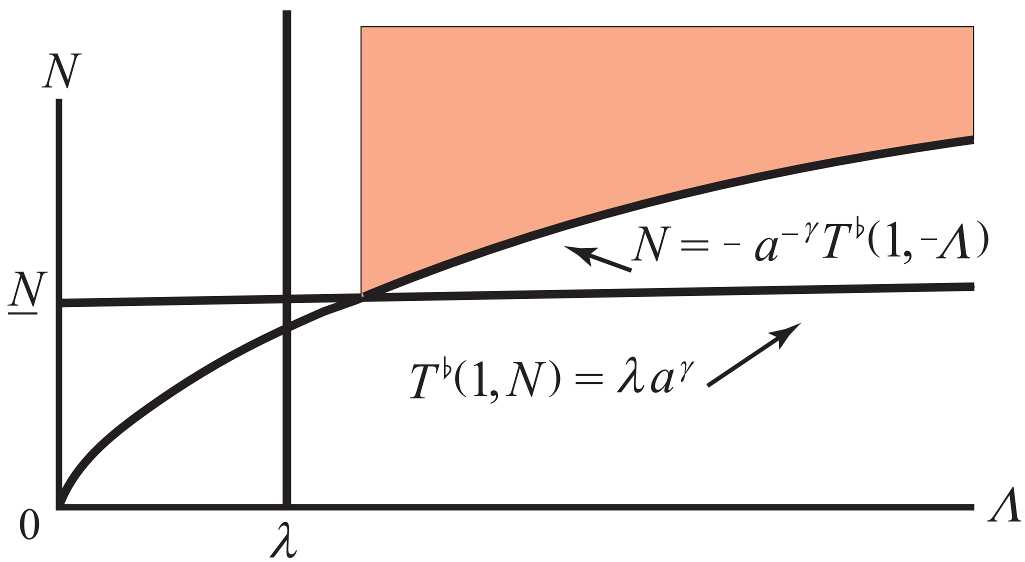

Figure 2.7: The shaded region shows the set of points (Λ, N) satisfying the first and fourth inequalities of (2.7.32) together with the inequality $\Lambda \geq \lambda$.

We combine inequalities (2.7.29)–(2.7.31) with inequalities coming from the monotonicity of $T^\flat$ with respect to its arguments:

$$-Na^\gamma \leq T^\flat(1,-\Lambda) < T^\flat(c,-\Lambda) \leq \lambda a^\gamma \leq T^\flat(1,N) < T^\flat(c,N) \leq \Lambda a^\gamma. \qquad (2.7.32)$$

(Note that the two strict inequalities are automatically satisfied.)

Now we show that there are constants c, Λ, N such that these inequalities are compatible and that they suffice to ensure that $(\mathfrak{t}, \mathfrak{n})$ maps $\mathcal{G}$ into itself. We start with the leftmost inequality of (2.7.32). Denote the corresponding *equation* by $N = \bar{N}(\Lambda) := -a^{-\gamma}T^\flat(1,-\Lambda)$. Our constitutive hypotheses imply that $\bar{N}(\Lambda)$ strictly increasing from 0 to ∞ as Λ increases from 0 to ∞. Thus the (Λ, N) satisfying the first inequality of (2.7.32) lie above the graph of $\bar{N}$ in (Λ, N)-space. See Figure 2.7.

Let $\underline{N}$ satisfy the *equation* $T^\flat(1,N) = \lambda a^\gamma$. Then the N satisfying the fourth inequality of (2.7.32) lie above the line $N = \underline{N}$ in (Λ, N)-space. See Figure 2.7. (This figure shows why we could not take $N = 0$ and shows why in general we should not take $\Lambda = \lambda$.)

Now examine the third inequality of (2.7.32). Let $\bar{c}(\Lambda)$ be the unique solution of the corresponding equation: $T^\flat(c,-\Lambda) = \lambda a^\gamma$. The function $\bar{c}$ strictly increases to ∞ as $\Lambda \to \infty$. Then any c satisfying the third inequality of (2.7.32) lies below the graph of $\bar{c}$ in the (Λ, c)-plane.

Finally examine the last inequality of (2.7.32), which says that Λ exceeds the strictly increasing function $a^{-\gamma}T^\flat(\cdot,\cdot)$. So the Λ satisfying the the last inequality of (2.7.32) lies below or to the right of the surface defined by $T^\flat(c,N) = \lambda a^\gamma$ in (Λ, c, N)-space with the N-axis vertical. See Figure 2.9.

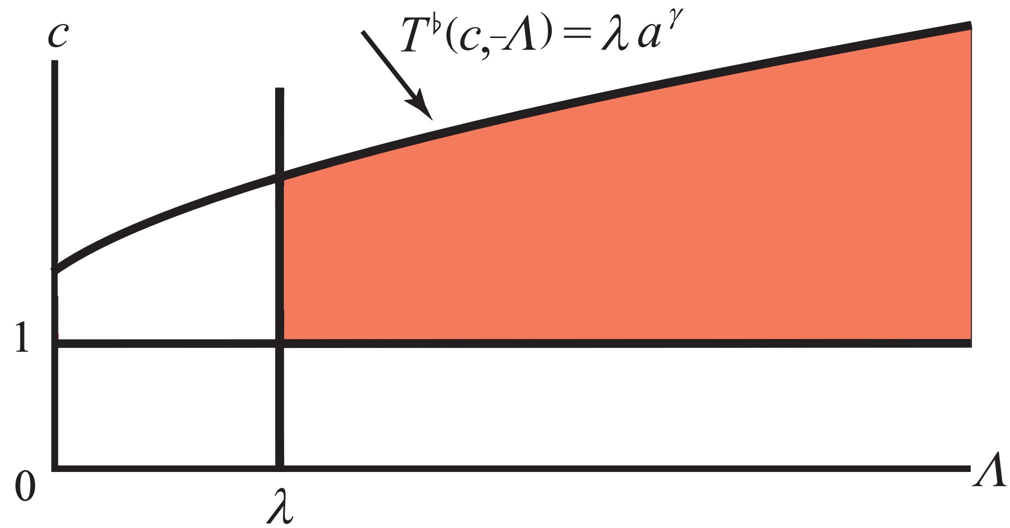

Figure 2.8: The shaded region shows the set of points (Λ, c) satisfying the third inequality of (2.7.32) together with the inequality $\Lambda \geq \lambda$.

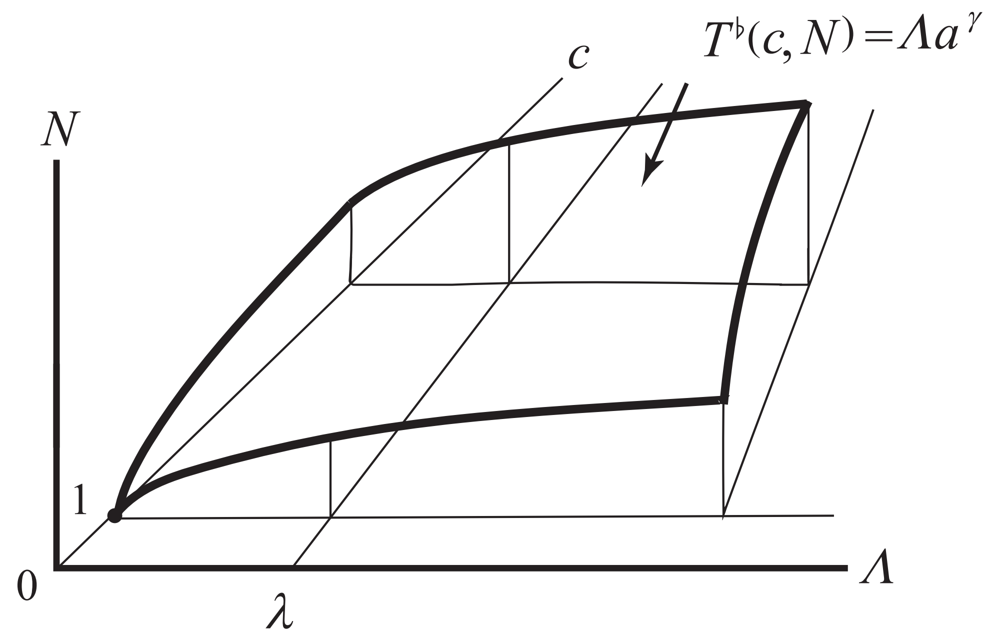

Figure 2.9: The region between the plane $N = 0$ and the surface defined by $T^\flat(c, N) = \Lambda a^\gamma$ satisfies the last inequality of (2.7.32).

Now Figure 2.7 defines a cylindrical region in (Λ, c, N)-space with generators parallel to the c-axis, and Figure 2.8 defines a cylindrical region in (Λ, c, N)-space with generators parallel to the N-axis. As a consequence of our constitutive assumptions, the intersection of these cylindrical regions with the region between the plane $N = 0$ and the surface defined by $T^\flat(c, N) = \Lambda a^\gamma$ in Figure 2.9 is not empty. Consequently for *any* (Λ, c, N) in this intersection, $(\mathfrak{t}, \mathfrak{n})$ maps $\mathcal{G}$ defined by (2.7.22) into itself. As in the treatment of the preceding problem with hydrostatic pressure, the hypotheses of the Schauder Fixed-Point Theorem are satisfied, whence there is an equilibrium state for every λ:

2.7.33. Theorem. *If* $\delta_a = 0$ *and if the constitutive restrictions* (2.7.23) *and* (2.4.5) *hold, then the integral equation* (2.7.5)*, has a classical solution for all* λ*. In this case, this integral equation is equivalent to the boundary-value problem* (2.4.3), (2.7.1) *(with* $\omega = 0$*).*

This solution satisfies

$$n(s) \leq 0 \quad \text{for all} \quad s \in [a, 1], \tag{2.7.34}$$

for suppose not. Then there would be an interval (s_1, s_2) in $(a, 1)$ with $n(s_1) = 0 = n(s_2)$ and with $n(s) > 0$ for $s_1 < s < s_2$. Then $(2.7.22)_3$ would imply that

$$\int_{s_1}^{1} T^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi^\gamma = 0 = \int_{s_2}^{1} T^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi, \tag{2.7.35}$$

so that $\int_{s_1}^{s_2} T^\sharp(\tau(\xi), n(\xi), \xi)\, d\xi^\gamma = 0$. But our constitutive assumptions and the inequality $\tau(s) \geq 1$ show that this equation *contradicts* the positivity of the integrand.

One can determine qualitative properties of solutions when various constitutive restrictions are imposed. As a cartoon of such a process, note that for the dead-load pressure,

$$n_s(a) = \gamma[T^\sharp(\tau(a), -\lambda, a) + \lambda], \qquad n_s(1) = \gamma T^\sharp(\tau(1), 0, 1). \tag{2.7.36}$$

Under the assumption that $\tau \geq 1$ from (2.7.24), which solutions of Theorem 2.7.33 satisfy, condition $(2.7.36)_2$ implies that $n_s(1) \geq 0$, but the sign of $n_s(a)$ depends on λ and the constitutive function $T^\sharp$.

If $\delta_a = \gamma$, we could regard (2.7.4) as an implicit equation for $\tau(a)$, and conceivably treat it like (2.7.20). Similar techniques handle other boundary-value problems. Treating our governing equations as a first-order system leads to particularly simple integral equations when one boundary condition is a traction condition and the other is a position condition. (Implicit function theorems like that used to treat (2.7.20) are unnecessary for such problems.)

2.8 Perturbation Methods for Nearly Homogeneous Materials

In this section we show how to treat nonhomogeneous materials by perturbation methods, limiting our attention to non-rotating annular plates subject to position boundary conditions

$$r(a) = \rho_a, \qquad r(1) = \rho_1 \tag{2.8.1}$$

or to the traction boundary conditions

$$N(a) = -\lambda_a a^{-1} r(a), \qquad N(1) = 0 \tag{2.8.2}$$

in the interesting case that $\lambda_a \geq 0$ (for which there need not be equilibrium solutions for all pressures). The treatment of other problems is similar.

Assume that the material is nearly homogeneous in the sense that the constitutive functions have the form

$$\hat{T}(\tau,\nu,s,\varepsilon) = T_0(\tau,\nu) + \varepsilon T_1(\tau,\nu,s), \quad \hat{N}(\tau,\nu,s,\varepsilon) = N_0(\tau,\nu) + \varepsilon N_1(\tau,\nu,s) \quad (2.8.3)$$

where ε is a small real (imperfection) parameter and where the constitutive functions have as many derivatives as appear in the analysis. We seek solutions of our boundary-value problems of the form

$$r(s,\varepsilon) = r_0(s) + \sum_{k=1}^{K} \frac{\varepsilon^k}{k!} r_k(s) + o(\varepsilon^K), \quad (2.8.4)$$

where r_0 (the solution of the *reduced problem*) describes a radially symmetric configuration of a homogeneous annular plate with the constitutive functions N_0 and T_0. Since (2.8.4) implies that

$$r_k(s) = \left. \frac{\partial^k r(s,\varepsilon)}{\partial \varepsilon^k} \right|_{\varepsilon=0}, \quad k = 1, 2, ..., \quad (2.8.5)$$

we find the problem formally satisfied by r_k by substituting (2.8.4) into equilibrium equation (2.2.6) and boundary conditions (2.8.1) or (2.8.2), differentiating the resulting equations k times with respect to ε, and then setting $\varepsilon = 0$. The equations of the first perturbation are

$$\frac{d}{ds}\left[s\left\{ N^0_{0\tau}\frac{r_1}{s} + N^0_{0\nu} r_1' + N^0_1 \right\}\right] = T^0_{0\tau}\frac{r_1}{s} + T^0_{0\nu} r_1' + T^0_1, \quad (2.8.6)$$

$$r_1(a) = 0 = r_1(1) \quad (2.8.7)$$

or

$$N^0_{0\tau}(a)\frac{r_1(a)}{a} + N^0_{0\nu}(a)r_1'(a) + N^0_1(a) = -\lambda_a a^{-1} r_1(a),$$
$$N^0_{0\tau}(1)r_1(1) + N^0_{0\nu}(1)r_1'(1) + N^0_1(1) = 0 \tag{2.8.8}$$

where $N^0_{0\tau}(s) := N_{0\tau}\Big(\frac{r_0(s)}{s}, r_0'(s)\Big)$, etc.

The homogenous versions of these boundary-value problems are obtained by dropping the constitutive functions bearing the subscript 1. Solutions of our nonhomogeneous boundary-value problems for (2.8.6) exist and are unique if the homogeneous boundary-value problem has only the zero solution [49, Sec. 3.2]. To find conditions ensuring the uniqueness of the zero solution, multiply the homogeneous version of (2.8.6) by r_1, integrate the resulting equation by parts from a to 1 and use the homogeneous versions of (2.8.7) or (2.8.8) to get

$$\int_a^1 s\Big[N^0_{0\nu}(r_1')^2 + \Big(N^0_{0\tau} + T^0_{0\nu}\Big)\frac{r_1}{s}r_1' + T^0_{0\tau}\frac{r_1^2}{s^2}\Big]\,ds = \eta\lambda_a r_1(a)^2 \tag{2.8.9}$$

where $\eta = 0$ if (2.8.7) holds and $\eta = 1$ if (2.8.8) holds. The QM condition (2.3.3), which we assume to hold here, implies that the integrand in (2.8.9) is positive-definite. If $\eta = 0$, then (2.8.9) implies that the solution of the homogeneous problem is 0. If $\eta = 1$, we know that the nonlinear problem need not have solutions for large λ_a, so we cannot expect good behavior for all λ_a here. To study this case, let $m[{r_1}^2]$ denote the mean value of ${r_1}^2$ (cf. (2.6.8)). Then ${r_1}^2 - m[{r_1}^2]$ vanishes at at least one point $s^* \in [a, 1]$, so that

$$\begin{aligned} r_1(a)^2 &= m[{r_1}^2] - 2\int_a^{s^*} r_1 r_1'\,ds \\ &\le \frac{1}{(1-a)}\int_a^1 {r_1}^2\,ds + \int_a^1 [r_1^2 + r_{1s}^2]\,ds \le C\int_a^1 [r_1^2 + r_{1s}^2]\,ds. \end{aligned} \tag{2.8.10}$$

In summary, it follows from the QM condition (2.3.3) that if (2.8.7) holds or if

(2.8.8) holds with λ_a small enough, then the nonhomogeneous problem has a unique solution.

We can explicitly construct solutions to (2.8.6), (2.8.7) in the special case when $r_0(s) = ks$ with $k > 0$, i.e, r_0 is a state of uniform deformation. In this case we have $\nu_0 \equiv r_0' = k$ and $\tau_0 \equiv r_0/s = k$. Under the assumption of isotropy $N_0(\tau, \nu) = T_0(\nu, \tau)$, we also have

$$N^0_{0\tau} = T^0_{0\nu} = \text{const.}, \quad N^0_{0\nu} = T^0_{0\tau} = \text{const.} \tag{2.8.11}$$

The equation (2.8.6) thus reduces to

$$Lr := s^2 r_1'' + s r_1' - r_1 = f := \frac{s}{N^0_{0\nu}}(T^0_1 - (sN^0_1)'). \tag{2.8.12}$$

The homogeneous equation $Lr = 0$ is a second-order Cauchy-Euler equation. Its general solution is given by

$$r_1(s) = C_1 s + C_2 s^{-1}. \tag{2.8.13}$$

The solution to (2.8.12), (2.8.7) is given by

$$r_1(s) = \int_a^1 G(s, \xi) f(\xi) d\xi \tag{2.8.14}$$

where Green's function G for L subject to (2.8.7) is given by

$$G(s, \xi) = \begin{cases} \frac{1}{2(1-a^2)}(s - \frac{a^2}{s})(\xi - \frac{1}{\xi}) & \text{for} \quad s < \xi, \\ \\ \frac{1}{2(1-a^2)}(\xi - \frac{a^2}{\xi})(s - \frac{1}{s}) & \text{for} \quad \xi < s. \end{cases} \tag{2.8.15}$$

Justification of perturbation methods. We use the Shooting Method [5, Sec. 20.2], based upon the Implicit-Function Theorem, to prove that the solution of the equilibrium equation (2.2.6) with $\gamma = 1$, $\omega = 0$ subject to the representative boundary condition (2.8.2) in the interesting case that $\lambda_a \geq 0$ admits a solution of the form (2.8.4) provided that λ_a is sufficiently small and the constitutive functions are sufficiently smooth:

2.8.16. Theorem. *Let $\hat{N}, \hat{T}$ be continuous functions having representations* (2.8.3) *and satisfying growth conditions* (2.3.6a), (2.3.6b) *and the QM condition in the equivalent forms* (2.3.3) *and* (2.4.10)*. Let r_0 satisfy* (2.2.6) *with $\hat{T}, \hat{N}$ replaced with T_0, N_0 subject to the boundary conditions* (2.8.2)*. Let $T_0, N_0, \hat{T}_1(\cdot, \cdot, s)$, $\hat{N}_1(\cdot, \cdot, s) \in C^{K+1}(0, \infty) \times C^{K+1}(0, \infty)$ for a positive integer K. Then for sufficiently small λ_a, there is a number η such that for $|\varepsilon| < \eta$ the boundary-value problem* (2.2.6), (2.8.2) *with $\hat{T}, \hat{N}$ given by* (2.8.3) *has a unique solution $r(\cdot, \varepsilon)$ with $r(\cdot, \varepsilon) \in C^2[a, 1]$ and $r(s, \cdot) \in C^{K+1}(-\eta, \eta)$. Thus $r(s, \varepsilon)$ admits an expansion like* (2.8.4)*.*

Proof. We replace (2.2.6) with its dual formulation with $\nu^\sharp, T^\sharp$ coming from $\hat{T}, \hat{N}$ of (2.8.3) and we replace the boundary condition $(2.8.2)_2$ with an initial condition, obtaining the semilinear initial-value problem

$$\frac{d\tau}{ds} = s^{-1}\nu^\sharp(\tau, n, s, \varepsilon) - s^{-1}\tau, \qquad \frac{dn}{ds} = s^{-1}T^\sharp(\tau, n, s, \varepsilon) - s^{-1}n, \tag{2.8.17a}$$

$$\tau(a) = b, \qquad n(a) = -\lambda_a \tau(a) \equiv -\lambda_a b \tag{2.8.17b}$$

where b is an unknown parameter. The solution r_0 of the reduced problem generates a solution (τ_0, n_0) to (2.8.17) with $\varepsilon = 0$, which satisfies the initial conditions

$$\tau_0(a) = a^{-1}r_0(a), \qquad n_0(a) = -\lambda_a a^{-1} r_0(a) \equiv -\lambda_a \tau_0(a). \tag{2.8.18}$$

The theory of ordinary differential equations implies that the initial-value problem (2.8.17) has a unique solution $(\tilde{\tau}(\cdot, b, \varepsilon), \tilde{n}(\cdot, b, \varepsilon))$ defined on the whole interval $[a, 1]$

if ε and b are close to 0 and $\tau_0(a)$. Moreover, $\big(\tilde{\tau}(s,\cdot,\cdot),\tilde{n}(s,\cdot,\cdot)\big)$ is $(K+1)$-times continuously differentiable. This solution of the initial-value problem would be a solution of our boundary-value problem if b can be chosen so that

$$\tilde{n}(1,b,\varepsilon)=0. \tag{2.8.19}$$

We know that this equation for b has the solution $\tau_0(a)$ for $\varepsilon=0$. The Implicit-Function Theorem then implies that there is a number $\eta>0$ such that (2.8.19) has a unique solution $(-\eta,\eta)\ni\varepsilon\mapsto\tilde{b}(\varepsilon)$ with $\tilde{b}\in C^{K+1}(-\eta,\eta)$ and with $\hat{b}(0)=\tau_0(a)$ provided that

$$\frac{\partial\tilde{n}}{\partial b}(1,\tau_0(a),0)\neq 0. \tag{2.8.20}$$

To demonstrate this inequality set

$$\mathsf{t}(s):=\tilde{\tau}_b(s,\tau_0(a),0),\qquad \mathsf{n}(s):=\tilde{n}_b(s,\tau_0(a),0). \tag{2.8.21}$$

The theory of ordinary differential equations implies that (t,n) satisfies the initial-value problem obtained by formally differentiating (2.8.17) with respect to b and then setting $(b,\varepsilon)=(\tau_0(a),0)$:

$$(s\mathsf{t})'=\nu_\tau^0\mathsf{t}+\nu_n^0\mathsf{n},\quad (s\mathsf{n})'=T_\tau^0\mathsf{t}+T_n^0\mathsf{n}, \tag{2.8.22a}$$

$$\mathsf{t}(a)=1,\quad \mathsf{n}(a)=-\lambda_a\mathsf{t}(a)\equiv-\lambda_a \tag{2.8.22b}$$

where $\nu_\tau^0(s):=\nu_\tau^\sharp(\tau_0,n_0,s,0)$, etc. This solution depends smoothly on λ_a. Multiply $(2.8.22\text{a})_1$ by $s\mathsf{n}$ and $(2.8.22\text{a})_2$ by $s\mathsf{t}$, and then integrate the sum of products by parts from a to 1 to obtain

$$s^2\mathsf{t}(s)\mathsf{n}(s)\Big|_a^1\equiv\mathsf{t}(1)\mathsf{n}(1)+a^2\lambda_a=2\int_a^1 s[\nu_n^0\mathsf{n}^2+(\nu_\tau^0+T_n^0)\mathsf{t}\mathsf{n}+T_\tau^0\mathsf{t}^2]\,ds. \tag{2.8.23}$$

Now the quadratic form in the integrand is positive-definite by the QM condition (2.4.10). The continuous dependence of the coefficients of this form on λ_a and the continuous dependence of the solution (t, n) on λ_a implies that the right-hand side is close to its value for $\lambda_a = 0$ for λ_a close to 0. Thus for small λ_a the right-hand side has a positive lower bound independent of λ_a. It follows that if λ_a is sufficiently small, then $\mathsf{n}(1)$ cannot vanish, for if so, (2.8.23) would imply that its right-hand side would vanish, whence $\mathsf{t} = 0 = \mathsf{n}$, in contradiction to (2.8.22b). □

2.9 Global Continuation Methods for Nonhomogeneous Materials and for Weak Materials

We now sketch out how to extend the results of the previous section to a global analysis of problems with 'large' nonhomogeneities that gives a detailed picture of the large deformation of weak materials under hydrostatic pressures, without imposing a priori restrictions on λ_a as in Theorem 2.8.16. We continue to use the traction boundary conditions (2.8.2), we adopt constitutive functions with values $\hat{T}(\tau, \nu, s, \varepsilon), \hat{N}(\tau, \nu, s, \varepsilon)$ having linear approximations in ε given by (2.8.3), and we take λ_a to be a continuously differentiable function λ of ε with $\lambda(0)$ positive and so small that the hypothesis on it given in Theorem 2.8.16 holds for $\lambda_a = \lambda(0)$. Under these conditions our boundary-value problem is governed by the integral equation (2.7.5) with λ and the constitutive functions depending on ε.

As in the development of Section 2.7, the operator $(\mathfrak{t}, \mathfrak{n})$ is compact from $C^0[a, 1] \times C^0[a, 1]$ to itself. The analysis of Section 8 shows that this integral equation has a solution (τ_0, n_0) for $\varepsilon = 0$ and that $(\mathfrak{t}, \mathfrak{n})$ is Fréchet differentiable here. That the difference between the identity and the Fréchet derivative is bijective is equivalent to the requirement that the linearized problem (2.8.6), (2.8.8) has a unique solution. These conditions ensure that the hypotheses of a one-parameter version of the Global Continuation Theorem [2] (cf. [43, 31]) are met, whence

2.9.1. Theorem. *Let T, N satisfy* (2.3.3), (2.3.6a), (2.3.6b) (*which support the formulation* (2.7.5) *and the treatment of* (2.8.9)) *and the conditions stated above. Then the boundary-value problem corresponding to* (2.7.5) *has a maximal connected set Σ of solution pairs $(\tau, n, \lambda; \varepsilon)$ satisfying at least one of the conditions:*

(i) *Σ is unbounded in $[C^0[a,1] \times C^0[a,1] \times \mathbb{R}] \times \mathbb{R}$,*

(ii) *$\Sigma \setminus \{(\tau_0, n_0, \lambda(0); 0)\}$ is connected.*

Property (ii) *does not hold if (τ_0, n_0) is the unique solution of* (2.7.5) *for $\lambda = \lambda(0)$ and $\varepsilon = 0$.*

The virtue of this result is evident when the material is weak. Suppose for simplicity of exposition that ε is fixed, with λ_a being the parameter for the boundary-value problem. In this case there are no equilibrium states for large λ_a, and alternative (i) of Theorem 2.9.1 says that there is an unbounded connected set Σ of solution pairs. If Σ is represented in a bifurcation diagram showing some norm of (τ, n) versus λ_a, then Σ is unbounded in the '(τ, n)-direction' with λ_a lying in a bounded interval. Such an analysis shows the range of λ_a for which equilibrium solutions exist and shows how the solution grows with λ. (See [5, Chap. 6] for full treatments of analogous problems.)

A generalization [2] of Rabinowitz's Theorem to a finite number of parameters (here ε and λ_a) gives an analogous conclusion without λ_a depending on ε. (Indeed, a generalization [1] of [2] to an infinite number of parameters, allows the constitutive functions themselves to be regarded as parameters; cf. [5, Sec. 3.3]). The treatment of the parameter ω^2 when there is centrifugal force is like that of λ_a. The presence of both hydrostatic pressure and centrifugal force (or of other parameters) suggests the use of multiparameter continuation theory.

2.10 Comparison of Analytic Methods

Phase-plane methods for our boundary-value problems, requiring the computation of precursor curves, deliver detailed qualitative behavior, clearly show existence or non-existence and uniqueness or non-uniqueness, and show where N and τ have extrema, information from which the extrema of ν and T can be obtained. They are limited to homogeneous materials (although some techniques studying trajectories in (τ, n, s)-space may work [47]). They are not limited to hyperelastic materials. (Mathematical formulations involving non-hyperelastic materials can come from certain approximation processes.) In contrast to our computation approach, phase-plane methods can give qualitative treatments of whole classes of materials.

Variational methods can readily handle inhomogeneous materials, but are limited to hyperelastic materials subject to conservative loads. These methods show the distinct roles of radial and azimuthal stiffness. They do not give sharp results for materials weak in tension, and cannot be used to demonstrate nonexistence. Minimizers of potential energy functionals are traditionally given a tautological stability interpretation. But rigorous dynamic stability theorems about local minimizers require delicate analyses of evolutionary partial differential equations [19, 33].

Fixed-Point methods give fairly sharp results on properties of solutions. They handle non-variational problems. They cannot handle cases of nonexistence. As our treatment of dead loads in Section 7 shows, for different classes of boundary conditions these methods may require different creative choices of regions $\mathcal{G}$ that get mapped into themselves by the operator $(\mathfrak{t}, \mathfrak{n})$.

Shooting methods for ordinary differential equations, which are used in the proof of Theorem 2.8.16 and which could be used elsewhere in our analysis, lead directly to a finite-dimensional problems, which are readily accessible to degree theory and its consequences: fixed-point theorems and continuation theorems. They are limited neither to autonomous equations nor variational problems. They can deliver results on existence, nonexistence, and multiplicity. They are the basis for numerical methods [30] (which do not carry over to partial differential equations).

Continuation methods readily handle dependence of solutions on parameters such as λ_a and ω, especially for cases in which solutions do not exist for a range of parameters or for all sizes. These methods give a global branch of solution pairs, its location respecting the bounds suggested by the variational method, the fixed-point method, and the phase-plane-method (when available). (They can often be illuminated by detailed estimates in solution-parameter space, not done here. Such estimates provide parameter ranges for which there do not exist steady-state solutions.) A virtue is that one can use numerical continuation methods [27] to construct a solution globally without worrying about whether there are solutions for all parameters. In other words, these methods do not rely on coercivity assumptions (like (2.6.3)).

2.11 Comments

The assertion of [7] that (2.3.18) characterizes isotropy for radially symmetric deformations of spherical shells is false, as shown in Section 2.3. But the only place in [7] where this condition is used is in the study of curves of singular points in the phase plane, just as in our Section 2.5, and as our analysis of this section shows, only the valid (2.3.13) is needed.

The differential equations (2.4.3) of the dual formulation can of course be given a direct variational characterization as the Euler-Lagrange equations for an action functional, just like the Hamiltonian equations of discrete mechanics.

When the material is weak in tension, so that there do not exist steady solutions for all hydrostatic pressures or for all rotational speeds, certain initial-value problems can blow up, i.e., have position fields that become infinite in finite or infinite time. This issue for corresponding initial-boundary-value problems for radially symmetric motions of nonlinearly viscoelastic plates and shells is treated in Chapter 2 and in [52].

Our work in Section 2.6 refines the variational existence theory of [3] by accounting for more delicate growth conditions to accommodate live loads.

The theory developed here is crucial for the study of buckling and dynamic instabilities, some of which result in the loss of symmetry.

Chapter 3: Radially Symmetric Motions of Nonlinearly Viscoelastic Plates and Shells

3.1 Introduction

The radially symmetric motions of cylindrical and spherical shells described by a geometrically exact Cosserat theory are studied in [6]. Within this constrained theory, the motions are governed by fourth-order systems of ordinary differential equations in time. The existence of solutions immediately follows from an energy estimate. Paper [6] gives comprehensive global treatments of the radially symmetric motions and in particular exhibits restrictions on the constitutive functions and restrictions on the forcing pressure and initial conditions for which (i) radially symmetric motions become unbounded at various rates as time approaches infinity; (ii) radially symmetric motions blow up in finite time.

The radially symmetric motions of incompressible cylindrical and spherical shells have been extensively studied, e.g. [20, 21, 54] (see also references in [6]). The constraint of incompressibility ensures that such motions of elastic shells are governed by systems of ordinary differential equations in time.

Paper [55] treats systems of partial differential equations that govern the motion of a viscoelastic material of strain-rate type on a bounded domain. However, the hypotheses imposed in [55] on the constitutive functions are physically unreasonable. In particular these hypotheses (i) are incompatible with the principle of frame indifference, (ii) preclude the physically natural requirement that compressive stress become infinite at a state of total compression.

Quasilinear parabolic-hyperbolic equations governing the motions of nonlinearly viscoelastic rods have been studied in [8, 9, 11]. The paper [8] treats purely longitudinal motions of nonlinearly viscoelastic rods, governed by a scalar quasilinear third-order parabolic-hyperbolic equation. The paper [9] represents a generalization of [8] to a much broader class of spatial motions. The governing equations in [9] form an eighteenth-order quasilinear parabolic-hyperbolic system of partial differential equations. The existence theory developed in [8, 9] is based on a version of the Faedo-Bubnov-Galerkin method [37, 57, 58].

We treat the initial-boundary-value problems for the radially symmetric motions of compressible annular plates and spherical shells that consist of nonlinearly viscoelastic materials of strain-rate type. The plates and shells are described by the specialization of exact equations of three-dimensional continuum mechanics to radially symmetric motions. The governing equation is a third-order quasilinear parabolic-hyperbolic partial differential equation in one space variable. The equation is singular in the sense that the constitutive functions blow up as some of our strain variables approach 0, which corresponds to a state of total compression. Our analysis consists in showing that some standard techniques such as those developed and used in [6, 8, 9, 11] can be carried over to our problem, while the inherent two-dimensional nature of our problem provides new technical obstacles.

We discuss a range of physically natural constitutive equations. We first show that when the material is strong in a suitable sense relative to externally applied live loads, solutions exist for all time, depend continuously on the data, and consequently are unique. We require that the dissipative mechanism be uniformly strong, which ensures that the governing equation has a parabolic-hyperbolic character. To ensure the preclusion of a total compression, we require that the viscosity effects become infinite as one of our strain variables $\nu \to 0$. We significantly refine the arguments used for ensuring the preclusion in one-dimensional problems [8, 9, 11] and study the role of the constitutive restrictions and that of the regularity of the data in the preclusion. The bounds on the strain variables and additional a priori bounds on the strain-rates allow us to replace the original singular problem with an equivalent regular problem. This we analyze by using the Faedo-Bubnov-Galerkin method [29, 37, 57, 58]. Our constitutive hypotheses support bounds and consequent compactness properties for the Galerkin approximations so strong that these approximations are shown to converge to the solution of the initial-boundary-value problem without appeal to the theory of monotone operators to handle the weak convergence of composite functions [37].

We then consider the case when the material is not sufficiently strong relative to externally applied live loads. We show that in that case under certain conditions on the the (hydrostatic) pressure terms and initial conditions (i) radially symmetric motions of annular plates and spherical shells become unbounded at various rates as time approaches infinity; (ii) radially symmetric motions of spherical shells blow up in finite time. We show that although the equations for annular plates and spherical shells differ slightly, there are major qualitative differences between the nonlinear dynamical behavior of annular plates and spherical shells.

In the course of our study we give a detailed treatment of a variety of non-homogeneous boundary conditions, not only because they are physically important, but also because they lead to significant differences in the behavior of solutions. On the other hand, we content ourselves with a reasonable set of regularity assumptions on the data, sufficient to produce solutions with regularity adequate to our needs; we make no effort to produce a scale of the sharpest results, because the emphasis is on the material behavior.

3.2 Notation

We let c, ε and C denote typical positive constants that are supplied as data or can be estimated in terms of data. Their meanings usually change with each appearance (even in the same equation or inequality. C may be regarded as increasing and c and ε as decreasing with each appearance). c is used for (small) lower bounds whereas ε is used for (small) upper bounds. Similarly, $t \mapsto \Gamma(t)$ and $t \mapsto \Gamma^{-1}(t)$ denote typical (large and small) positive-valued continuous functions depending on the data. We also use $\bar{c}$ and $\bar{C}$ in the formulation of our hypotheses. Their meaning is analogous to that of c and C.

We use without comment the Hölder inequality, Cauchy-Bunyakovskiĭ-Schwarz inequality, and Young's inequality:

$$2|ab| \leq \frac{\eta^p |a|^p}{p} + \frac{|b|^q}{q\eta^q} \tag{3.2.1}$$

for real a, b and for positive η, p, q with $p^{-1} + q^{-1} = 1$. If we take η to be small, we can replace η^p with ε, and use the convention just discussed to write the last estimate as $|ab| \leq \varepsilon|a|^p + C|b|^q$.

We use the Gronwall inequality in the form: If f and g are positive-valued functions on $[0, \infty)$, if u is real and continuous and

$$\begin{aligned} &\text{if } u(t) \leq f(t) + g(t) \int_0^t u(\bar{t})\, d\bar{t} \qquad \forall\, t \in (0, t^+), \\ &\text{then } u(t) \leq f(t) + g(t) \exp\Big[\int_0^t g(s)\, ds\Big] \int_0^t \exp\Big[-\int_0^{\bar{t}} g(s)\, ds\Big] f(\bar{t})\, d\bar{t}. \end{aligned} \tag{3.2.2}$$

for all $t \in (0, t^+)$. In light of our notational convention on the use of Γ, we might write a consequence of the Gronwall inequality as $u \leq \Gamma$ when f and g are supplied as data.

We use only real function spaces. $L^\infty(0, t^+)$ denotes the space of essentially bounded functions on $[0, t^+]$. For each nonnegative integer k, $C^k[0, t^+]$ denotes the space of functions that are k times continuously differentiable on $[0.t^+]$, $H^k(a, 1)$ denotes the Sobolev space of square-integrable functions defined on the interval $(a, 1)$ whose weak derivatives up to order k are square-integrable, and $W^{k,p}(a, 1)$ denotes the Sobolev space of L^p functions defined on the interval $(a, 1)$ whose weak derivatives up to order k belong to $L^p(a, 1)$ ($W^{k,2}(a, 1)$ is equivalent to $H^k(a, 1)$). $W^{k,p}_{\text{loc}}[0, \infty)$ denotes the set of functions that are of class $W^{k,p}$ on every on compact subset of $[0, \infty)$, etc.

We denote the norm of a Banach space $\mathcal{X}$ by $\|\cdot, \mathcal{X}\|$, but omit $\mathcal{X}$ when

$\mathcal{X} = L^2(a,1)$. If $\mathcal{X}$ is a Banach space of functions on the interval $(a,1)$ and $\mathcal{Y}$ a Banach space of real-valued functions on the interval $[0,t^+]$, then as usual $\mathcal{Y}(0,t^+;\mathcal{X})$ denotes the Banach space of mappings $[0,t^+] \ni t \mapsto w(\cdot,t) \in \mathcal{X}$ with norm $\| [t \mapsto \|w(\cdot,t),\mathcal{X}\|], \mathcal{Y}\|$. In particular, the square of a norm of w in $H^1(0,t^+,H^1(a,1))$ is

$$\begin{aligned} \|w\|^2 &= \int_0^{t^+} \{\|w(\cdot,t),H^1(a,1)\|^2 + \|w_t(\cdot,t),H^1(a,1)\|^2\}\,dt \\ &= \int_0^{t^+}\int_a^1 [w(s,t)^2 + w_s(s,t)^2 + w_t(s,t)^2 + w_{ts}(s,t)^2]\,dt. \end{aligned} \tag{3.2.3}$$

3.3 Mechanics and Material Behavior

Let the reference configuration of a body when it is subject to zero tractions on its boundary and zero body force be either an annular plate or a spherical shell of inner radius $a \in (0,1)$ and unit outer radius. The radially symmetric motion of an annular plate or spherical shell is defined here by the scalar function

$$[a,1] \times [0,\infty) \ni (s,t) \mapsto r(s,t) \in \mathbb{R}, \tag{3.3.1}$$

which is interpreted as the distance at time t from the origin of a typical material point s. Let

$$\tau(s,t) := \frac{r(s,t)}{s}, \quad \nu(s,t) := r_s(s,t) \tag{3.3.2}$$

be the stretches (or strain variables) corresponding to the azimuthal and radial directions.

Our governing equation of motion is the specialization of exact (nonlinear) equations of three-dimensional continuum mechanics to the radially symmetric motions [5, Ch. 12]:

$$\frac{\partial}{\partial s}\left(s^{\gamma}N(s,t)\right)-\gamma s^{\gamma-1}T(s,t)=s^{\gamma}\rho(s)\,r_{tt}(s,t),\qquad s\in(a,1)\tag{3.3.3}$$

where $T(s,t)$, $N(s,t)$ are normal Piola-Kirchhoff stresses of the first kind at the (material) radius s acting in the azimuthal and radial directions at time t. Here and below $\gamma=1$ corresponds to an annular plate and $\gamma=2$ to a spherical shell. (The equation (3.3.3) with $\gamma=1$ also governs the radially symmetric motions of an infinite cylindrical shell under the assumption of plane stress.) $\rho(s)$ is the given mass density per unit reference area (when $\gamma=2$) or per unit reference length (when $\gamma=1$) at s in the reference configuration. We assume that it is bounded above and that it has a positive lower bound on $[a,1]$.

We treat materials that are *viscoelastic of strain-rate type (of complexity 1)*, which have the defining property that there are constitutive functions

$$(\tau,\nu,\dot{\tau},\dot{\nu},s)\mapsto\hat{T}(\tau,\nu,\dot{\tau},\dot{\nu},s),\quad(\tau,\nu,\dot{\tau},\dot{\nu},s)\mapsto\hat{N}(\tau,\nu,\dot{\tau},\dot{\nu},s)\tag{3.3.4}$$

such that

$$T(s,t)=\hat{T}(\tau(s,t),\nu(s,t),\tau_t(s,t),\nu_t(s,t),s),\quad\text{etc.}\tag{3.3.5}$$

Throughout our exposition, superposed dots, like those over τ and ν in (3.3.4), have no operational significance; in (3.3.4) the $\dot{\tau}$, $\dot{\nu}$ merely identify the third and fourth arguments of $\hat{N}$ which are typically occupied by the time derivatives τ_t, ν_t, which are known as the strain-rates or velocity gradients.

The substitution of the constitutive equations (3.3.5) into the governing equation (3.3.3) of motion yields a third-order quasilinear partial differential equation in one space variable. We shall seek solutions of the governing equation (3.3.3) that never suffer a total compression, i.e., solutions for which the two strain variables are everywhere positive:

$$\tau(s,t) > 0, \quad \nu(s,t) > 0 \quad \forall\, s \in [a,1] \quad \forall\, t \geq 0. \tag{3.3.6}$$

We assume that the materials of the bodies have enough symmetry for a radially symmetric configuration at each time t to correspond to a radially symmetric stress distribution. (For a spherical shell undergoing a radially symmetric motion, this means that at each time t there is no shear stress on concentric material spheres and on material surfaces consisting of rays and that all normal stresses in any azimuthal direction for a given radius s and all normal stresses for a given radius s should respectively be equal in magnitude.)

3.4 Boundary conditions. Weak formulation

On the outer boundary $s = 1$, we either prescribe a time-dependent position condition:

$$r(1,\cdot) \text{ is prescribed} \tag{3.4.1}$$

or else prescribe a normal force of the form

$$N(1,t) = -\lambda_1(t) r(1,t)^{\delta_1} \equiv -\lambda_1(t)\tau(1,t)^{\delta_1}, \qquad \delta_1 = 0, \gamma. \tag{3.4.2}$$

If $\delta_1 = 0$, (3.4.1) represents a normal force per unit *reference* length or area (a dead load) of intensity $\lambda_1(t)$ at time t. If $\delta = \gamma = 1$, (3.4.1) represents a hydrostatic pressure, i.e., a normal force per unit *actual* length of intensity $\lambda_1(t)$ at time t acting on the outer edge of the plate. If $\delta = \gamma = 2$, (3.4.1) represents a hydrostatic pressure, i.e., a normal force per unit *actual* length of intensity $\lambda_1(t)$ at time t acting on the outer surface of the shell. We allow $\lambda_1(t)$ to have either sign. When $\lambda_1(t)$ is positive, $N(1,t)$ is compressive. We adopt analogous conditions on the inner boundary $s = a$:

$$r(a,t) = r_a(t) \quad \text{or else} \quad N(a,t) = -\lambda_a(t)\left[\frac{r(a,t)}{a}\right]^{\delta_a} \equiv -\lambda_a \tau(a,t)^{\delta_a},\ \delta_a = 0, \gamma. \tag{3.4.3}$$

In (3.4.3) λ_a represents the intensity of a hydrostatic load when $\delta_a = \gamma$. For brevity, in the sequel we shall impose a hydrostatic load condition of the form

$$N(a,t) = -\lambda_a(t) r(a,t)^\gamma, \tag{3.4.4}$$

in which λ_a corresponds to the intensity divided by a^γ.

In the main part of this work, we limit our attention to a single set of simple boundary conditions, discussing other possibilities later: We assume here that the radius of the inner boundary is fixed at $s = r_a$ and that the outer boundary is free and is subject to a dead load of intensity $\lambda_1(t)$:

$$r(a,t) = r_a, \quad N(1,t) = -\lambda_1(t) \tag{3.4.5}$$

with

$$\lambda_1 \in H^2_{\text{loc}}(0,\infty). \tag{3.4.6}$$

The initial conditions are

$$r(s,0) = p(s), \quad r_t(s,0) = q(s). \tag{3.4.7}$$

3.4.8. Hypothesis. *The initial value p of r lies in $C^1(a,1)$, the initial value q of r_t lies in $C^1(a,1)$, and the initial value $r_{tt}(\cdot,0)$, which is defined by the partial* *differential equation* (3.3.3)*, lies in* $L^2(a,1)$*. Initially there is no total compression, i.e.,* $\min\{p(s) : s \in [a,1]\} > 0$ *and* $\min\{p'(s) : s \in [a,1]\} > 0$.

We assume that p, q satisfy the compatibility conditions: e.g., if (3.4.5) are the prescribed boundary conditions then

$$p(a) = r_a, \quad \hat{N}\left(p(1), p'(1), q(1), q'(1), 1\right) = -\lambda_1(0). \tag{3.4.9}$$

In accord with the boundary conditions (3.4.5) we define

$$\mathfrak{W} := \{\chi \in H^1(a,1) : \chi(a) = 0\}. \tag{3.4.10}$$

A (spatially) weak version of (3.3.3) may be formally obtained by multiplying the equation by test function $\chi \in \mathfrak{W}$ depending only on s and integrating by parts. This process yields

$$\begin{gathered}\int_a^1 s^\gamma \left(\rho(s) r_{tt}\chi + N\chi_s + \frac{\gamma}{s} T\chi\right) ds - s^\gamma N(s,t)\chi(s)\Big|_{s=a}^{s=1} \equiv \\ \int_a^1 s^\gamma \left(\rho(s) r_{tt}\chi + N\chi_s + \frac{\gamma}{s} T\chi\right) ds + \lambda_1(t)\chi(1) = 0.\end{gathered} \tag{3.4.11}$$

We can replace χ in (3.4.11) with a function of both s and t in a suitable function space, because arbitrary functions in these spaces can be approximated by finite linear combinations of products of functions of s with functions of t. Likewise, we can formally take the time derivative of (3.4.11) and then replace χ with a function of s and t, obtaining

$$\begin{gathered}\int_a^1 s^\gamma \left(\rho(s) r_{ttt}\,\chi + N_t\chi_s + \frac{\gamma}{s} T_t\chi\right) ds - s^\gamma N_t(s,t)\chi(s,t)\Big|_{s=a}^{s=1} \equiv \\ \int_a^1 s^\gamma \left(\rho(s) r_{ttt}\chi + N_t\chi_s + \frac{\gamma}{s} T_t\chi\right) ds + \lambda_1'(t)\chi(1,t) = 0.\end{gathered} \tag{3.4.12}$$

3.5 Constitutive restrictions

The constitutive functions $\hat{N}$ and $\hat{T}$ admit the decomposition into *equilibrium* and *dissipative* parts:

$$\begin{aligned} \hat{T}(\tau,\nu,\dot{\tau},\dot{\nu},s) &= T^{\mathrm{E}}(\tau,\nu,s) + T^{\mathrm{D}}(\tau,\nu,\dot{\tau},\dot{\nu},s), \\ \hat{N}(\tau,\nu,\dot{\tau},\dot{\nu},s) &= N^{\mathrm{E}}(\tau,\nu,s) + N^{\mathrm{D}}(\tau,\nu,\dot{\tau},\dot{\nu},s) \end{aligned} \tag{3.5.1}$$

with

$$T^{\mathrm{D}}(\tau,\nu,0,0,s) = 0, \quad N^{\mathrm{D}}(\tau,\nu,0,0,s) = 0. \tag{3.5.2}$$

We assume that the equilibrium response is hyperelastic, i.e., there exists a stored-energy density function $\varphi^{(\gamma)} = \varphi^{(\gamma)}(\tau,\nu,s)$ such that

$$N^{\mathrm{E}} = \varphi^{(\gamma)}_{\nu}, \quad \gamma T^{\mathrm{E}} = \varphi^{(\gamma)}_{\tau} \tag{3.5.3}$$

where

$$\varphi^{(\gamma)}(\tau,\nu,s) = \begin{cases} \varphi(\tau,\nu,s), & \gamma = 1, \\ \varphi(\tau,\tau,\nu,s), & \gamma = 2. \end{cases} \tag{3.5.4}$$

We define the stored energy at time t:

$$\Phi(t) := \int_a^1 s^{\gamma}\, \varphi^{(\gamma)}(\tau(s,t),\nu(s,t),s)\, ds. \tag{3.5.5}$$

To establish global existence results we assume the following

3.5.6. Hypothesis. *(a) There are positive numbers C, c and there are numbers $\alpha_1,\alpha_2 > 1$ such that*

$$\varphi^{(\gamma)}(\tau,\nu,s) \geq c[\nu^{\alpha_1} + \tau^{\alpha_2}] - C. \tag{3.5.7}$$

(b) (3.5.7) *holds with $\alpha_1,\alpha_2 > 2$ if $\gamma = 1$ and with $\alpha_1,\alpha_2 > 4$ if $\gamma = 2$.*

Hypothesis 3.5.6a ensures that the material is strong enough not to yield under dead loads. The stronger Hypothesis 3.5.6b is used in our analysis of boundary conditions including hydrostatic loads.

The following physically reasonable hypothesis is used to establish a positive lower bound for τ as well as bounds on the strain-rates:

3.5.8. Hypothesis. *There are positive numbers C, c and there are positive numbers α_3, α_4 such that*

$$\varphi^{(\gamma)}(\tau, \nu, s) \geq c\left[\frac{1}{\nu^{\alpha_3}} + \frac{1}{\tau^{\alpha_4}}\right] - C. \tag{3.5.9}$$

We shall need a pointwise bound for $r(\cdot, t)$ in terms of Φ, which follows from (3.5.7). Let us define the mean value of r on the interval $[a, 1]$ for any fixed t:

$$m[r](t) := \frac{1}{1-a}\int_a^1 r(\bar{s}, t)\, d\bar{s}. \tag{3.5.10}$$

By the Mean-Value Theorem for Integrals, for each t there is a point $s^*(t)$ where $r(s^*(t), t) - m[r](t)$ vanishes. Hence for any $s \in [a, 1]$

$$\begin{aligned} r(s,t) &= m[r](t) + \int_{s^*(t)}^{s} r_s(\bar{s}, t)\, d\bar{s} \\ &\leq C\left(\int_a^1 \bar{s}^{\gamma}\left(\frac{r(\bar{s},t)}{\bar{s}}\right)^{\alpha_2} d\bar{s}\right)^{1/\alpha_2} + C\left(\int_a^1 \bar{s}^{\gamma} r_s(\bar{s},t)^{\alpha_1}\, d\bar{s}\right)^{1/\alpha_1}. \end{aligned} \tag{3.5.11}$$

By using the inequality $(a+b)^n \leq C(a^n + b^n)$, which is valid, in particular, for $a, b \geq 0$ and $n \in \mathbb{N}$, we get

$$\begin{aligned} r(s,t)^{\gamma+1} &\leq C\int_a^1 \bar{s}^{\gamma}\left[\tau(\bar{s},t)^{\alpha_2} + \nu(\bar{s},t)^{\alpha_1}\right] d\,\bar{s} + C \\ &\leq C\int_a^1 \bar{s}^{\gamma}\varphi(\tau(\bar{s},t), \nu(\bar{s},t), \bar{s})\, d\bar{s} + C \equiv C\Phi(t) + C. \end{aligned} \tag{3.5.12}$$

provided that $\alpha_1, \alpha_2 > \gamma + 1$.

If $r(a,\cdot)$ or $r(1,\cdot)$ is prescribed, say, in $L^\infty_{\text{loc}}[0,\infty)$, it suffices to require that $\alpha_1 > \gamma+1$ to obtain a pointwise bound similar to (3.5.12). E.g., if $r(a,t)$ is prescribed then instead of (3.5.11) we have

$$r(s,t) = r(a,t) + \int_a^s r_s(\bar{s},t)\,d\,\bar{s} \le C + C\left(\int_a^1 \bar{s}^\gamma r_s(\bar{s},t)^{\alpha_1} d\,\bar{s}\right)^{1/\alpha_1}. \tag{3.5.13}$$

whence $r(s,t)^{\gamma+1} \le C\Phi + C$ if $\alpha_1 > \gamma + 1$.

We require that the effects of internal friction grow with the strain rates:

3.5.14. Hypothesis. *There is a positive constant c_{V} such that*

$$\begin{aligned}&\left(\hat{N}(\tau,\nu,\dot{\tau}_1,\dot{\nu}_1,s) - \hat{N}(\tau,\nu,\dot{\tau}_2,\dot{\nu}_2,s)\right)(\dot{\nu}_1 - \dot{\nu}_2)\\ &\quad + \gamma\left(\hat{T}(\tau,\nu,\dot{\tau}_1,\dot{\nu}_1,s) - \hat{T}(\tau,\nu,\dot{\tau}_2,\dot{\nu}_2,s)\right)(\dot{\tau}_1 - \dot{\tau}_2)\\ &\quad \ge c_{\text{V}}\left((\dot{\nu}_1 - \dot{\nu}_2)^2 + \gamma(\dot{\tau}_1 - \dot{\tau}_2)^2\right)\end{aligned} \tag{3.5.15}$$

for all values of the variables that appear.

This monotonicity condition ensures that the dissipative mechanism is uniformly strong and that the governing equation of motion has a parabolic character. Inequality (3.5.15) is responsible for the regularity of solutions, and, in particular, for the absence of shocks, which are typically present in analogous problems for nonlinearly elastic bodies (in which the stress resultants depend only on the strains and the material point). Of course, we can replace $\hat{N}$, $\hat{T}$ in (3.5.15) with N^{D}, T^{D}. Since the constitutive functions $\hat{N}$, $\hat{T}$ are assumed to be continuously differentiable, (3.5.15) is equivalent to

$$a^2\partial_{\dot{\nu}}N^{\text{D}} + ab\left[\partial_{\dot{\tau}}N^{\text{D}} + \gamma\partial_{\dot{\nu}}T^{\text{D}}\right] + \gamma b^2\partial_{\dot{\tau}}T^{\text{D}} \ge c_{\text{V}}(a^2 + \gamma b^2) \quad \forall a,b \in \mathbb{R}. \tag{3.5.16}$$

(In particular $\partial_{\dot{\tau}}T^{\text{D}} > 0$, $\partial_{\dot{\nu}}N^{\text{D}} > 0$.)

To obtain (3.5.16) from (3.5.15) let $\dot\nu_1 = \dot\nu_2 + ah$ and $\dot\tau_1 = \dot\tau_2 + bh$. Then (3.5.15) implies that

$$\begin{aligned}
&\Big(\hat N(\tau,\nu,\dot\tau_1,\dot\nu_2+ah,s) - \hat N(\tau,\nu,\dot\tau_1,\dot\nu_2,s)\Big)\, ah \\
&\quad + \Big(\hat N(\tau,\nu,\dot\tau_2+bh,\dot\nu_2,s) - \hat N(\tau,\nu,\dot\tau_2,\dot\nu_2,s)\Big)\, ah \\
&\quad + \gamma\Big(\hat T(\tau,\nu,\dot\tau_1,\dot\nu_2+ah,s) - \hat T(\tau,\nu,\dot\tau_1,\dot\nu_2,s)\Big)\, bh \\
&\quad + \gamma\Big(\hat T(\tau,\nu,\dot\tau_2+bh,\dot\nu_2,s) - \hat T(\tau,\nu,\dot\tau_2,\dot\nu_2,s)\Big)\, bh \\
&\quad \ge c_{\mathrm{V}}\left((a^2+\gamma b^2\right) h^2.
\end{aligned} \tag{3.5.17}$$

Dividing both sides of (3.5.17) by h^2 and passing to the limit as $h \to 0$ gives (3.5.16).

The following hypothesis is an analog of Hypothesis 6.11 of [9]. It says that when the strains are suitably controlled, the "elasticity" $\partial(\hat N, \hat T)/\partial(\nu,\tau)$ is dominated by the "viscosity" $\partial(\hat N, \hat T)/\partial(\dot\nu,\dot\tau)$. For this to occur, the viscosities must depend appropriately on the strains. This dependence is the underlying theme of our constitutive restrictions.

3.5.18. Hypothesis. *Let $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\tau,\nu,\dot\tau,\dot\nu,s)$ be the positive-definite square root of the (positive-definite) symmetric part of*

$$\frac{\partial(\hat N, \gamma\hat T)}{\partial(\dot\nu,\dot\tau)} \equiv \begin{bmatrix} \hat N_{\dot\nu} & \hat N_{\dot\tau} \\ \gamma\hat T_{\dot\nu} & \gamma\hat T_{\dot\tau} \end{bmatrix}. \tag{3.5.19}$$

Let $\bar c$, $\bar C$ be positive numbers such that $\bar c \le \tau, \nu \le \bar C$. Then there is a positive number C (depending on $\bar c$, $\bar C$) such that

$$\left| \boldsymbol{\Lambda}^{-1} \cdot \frac{\partial(\hat N, \gamma\hat T)}{\partial(\nu,\tau)} \cdot \begin{bmatrix} \dot\nu \\ \dot\tau \end{bmatrix} \right|^2 \le C[1 + N^{\mathrm{D}}\dot\nu + \gamma T^{\mathrm{D}}\dot\tau + \varphi^{(\gamma)}]. \tag{3.5.20}$$

The requirement that

$$\varphi(\tau, \nu, s) \to \infty \quad \text{as } \tau \to 0 \text{ or } \nu \to 0, \tag{3.5.21}$$

which follows from (3.5.9), can be used to show that total compression cannot occur for reasonable *equilibrium (steady-state)* problems. The energy estimate (3.7.7) below immediately shows that for any fixed t, the set of s on which there is a total compression must have measure 0. However (3.5.21) has never been shown capable by itself of preventing total compression everywhere for dynamical problems. For this purpose we require that viscous effects become infinitely large in a suitable way at a total compression and at an infinite extension.

We now introduce an hypothesis that is used to preclude total compression and infinite extension with respect to the strain ν.

3.5.22. Hypothesis Related to Total Compression and Infinite Extension. *Let $\bar{c}$, $\bar{C}$ be positive numbers such that $\bar{c} \leq \tau \leq \bar{C}$.*

(a) There is a number $y_* \in (0,1)$ (depending on $\bar{c}$, $\bar{C}$) and there is a continuously differentiable function ψ on $(0, y_*)$ with

$$\psi \geq 0, \quad \psi(y) \to \infty \qquad \textit{as} \quad y \to 0 \tag{3.5.23}$$

such that

$$\hat{N}(\tau, \nu, \dot{\tau}, \dot{\nu}, s) \leq -\psi'(\nu)\dot{\nu} \tag{3.5.24}$$

for all $\dot{\tau}$, s, and $\bar{c} \leq \tau \leq \bar{C}$, $\dot{\nu} \leq 0$, $\nu \leq y_*$;

(b) There is a number $y^* \in (1, \infty)$ (depending on $\bar{c}$, $\bar{C}$) and there is a contin-

uously differentiable function ψ *on* (y^*, ∞) *with*

$$\psi \geq 0, \quad \psi(y) \to \infty \qquad as \quad y \to \infty \tag{3.5.25}$$

$$\hat{N}(\tau, \nu, \dot{\tau}, \dot{\nu}, s) \geq \psi'(\nu)\dot{\nu} \tag{3.5.26}$$

for all $\dot{\tau}$*,* s *and* $\bar{c} \leq \tau \leq \bar{C}$*,* $\dot{\nu} \geq 0$*,* $\nu \geq y^*$*.*

(3.5.24) and (3.5.26) are analogs of Hypotheses 3.8 and 3.12 of [11] in our two-dimensional setting. Hypothesis 3.7 of [8] and Hypothesis 6.10 of [9] are also analogous to (3.5.24). Hypothesis 3.5.22 is only mildly restrictive because we require that (3.5.24) hold only for nonpositive $\dot{\nu}$ and that (3.5.26) hold only for nonnegative $\dot{\nu}$. In contrast, Hypothesis 3.7 of [8] (which is analogous to (3.5.24)) is required to hold for all values of $\dot{\nu}$, which may be unreasonably restrictive. [8] shows how to relax its restrictive Hypothesis 3.7 by using a more complicated weaker hypothesis. We show that Hypothesis 3.5.22 can be entertained provided the data is sufficiently regular.

To motivate Hypothesis 3.5.22 it is instructive to look at the constitutive equation for a compressible Newtonian fluid in the material (Lagrangian) formulation in order to see how the viscosities depend on the strains. We now derive this constitutive equation in the material formulation from the corresponding constitutive equation in the spatial (Eulerian) formulation. Naturally, we limit our attention to the class of radially symmetric motions. We focus only on the two-dimensional case because the three-dimensinal case is similar.

We introduce polar coordinates $\boldsymbol{x} := (s, \phi)$ and denote the reference position of the material point with coordinates $\boldsymbol{x}$ by $\boldsymbol{z} = \tilde{\boldsymbol{z}}(\boldsymbol{x})$. Let $\{\boldsymbol{i}, \boldsymbol{j}\}$ be a fixed right-handed orthonormal basis for Euclidean 2-space $\mathbb{E}^2$ and define

$$\boldsymbol{e}_1(\phi) := \cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j}, \qquad \boldsymbol{e}_2(\phi) := -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}. \tag{3.5.27}$$

Let $\boldsymbol{p}(\boldsymbol{z},t)$ be the position of material point $\boldsymbol{z}$ at time t. Under our assumption that $\boldsymbol{p}(\cdot,t)$ is one-to-one for all t, it has an inverse $\boldsymbol{y} \mapsto \boldsymbol{q}(\boldsymbol{y},t)$ so that $\boldsymbol{y} = \boldsymbol{p}(\boldsymbol{z},t)$ if and only if $\boldsymbol{z} = \boldsymbol{q}(\boldsymbol{y},t)$. For our class of radial motions, the position is given by

$$\boldsymbol{p}(\boldsymbol{z},t) = r(s,t)\boldsymbol{e}_1. \tag{3.5.28}$$

We define the velocities in material and spatial coordinates:

$$\boldsymbol{v}(\boldsymbol{z},t) := \boldsymbol{p}_t(\boldsymbol{z},t) \equiv r_t(s,t)\boldsymbol{e}_1, \quad \breve{\boldsymbol{v}}(\boldsymbol{y},t) := \boldsymbol{p}_t(\boldsymbol{q}(\boldsymbol{y},t),t). \tag{3.5.29}$$

Thus the deformation tensor and the velocity gradient are given by

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{z},t) &= r_s(s,t)\boldsymbol{e}_1\boldsymbol{e}_1 + s^{-1}r(s,t)\boldsymbol{e}_2\boldsymbol{e}_2 \equiv \nu\boldsymbol{e}_1\boldsymbol{e}_1 + \tau\boldsymbol{e}_2\boldsymbol{e}_2, \\ \boldsymbol{v}_{\boldsymbol{z}}(\boldsymbol{z},t) &= r_{st}(s,t)\boldsymbol{e}_1\boldsymbol{e}_1 + s^{-1}r_t(s,t)\boldsymbol{e}_2\boldsymbol{e}_2 \equiv \nu_t\boldsymbol{e}_1\boldsymbol{e}_1 + \tau_t\boldsymbol{e}_2\boldsymbol{e}_2. \end{aligned} \tag{3.5.30}$$

The constitutive equation for a compressible Newtonian fluid in the spatial formulation is

$$\boldsymbol{\Sigma}(\boldsymbol{y},t) = -p\boldsymbol{I} + \mu\left(\breve{\boldsymbol{v}}_{\boldsymbol{y}}(\boldsymbol{y},t) + \breve{\boldsymbol{v}}_{\boldsymbol{y}}(\boldsymbol{y},t)^*\right) + \lambda\,\mathrm{div}\breve{\boldsymbol{v}}\,\boldsymbol{I}. \tag{3.5.31}$$

where $\boldsymbol{\Sigma}$ is the Cauchy stress tensor, p is the pressure, μ and λ are the coefficients of viscosity. Let $\boldsymbol{T}$ be the first Piola-Kirchhoff stress tensor. Using the relations $\boldsymbol{\Sigma}(\boldsymbol{y},t) = (\det\boldsymbol{F}(\boldsymbol{z},t))^{-1}\boldsymbol{T}(\boldsymbol{z},t)\cdot\boldsymbol{F}(\boldsymbol{z},t)^*$ and $\breve{\boldsymbol{v}}_{\boldsymbol{y}}(\boldsymbol{y},t) = \boldsymbol{v}_{\boldsymbol{z}}(\boldsymbol{z},t)\cdot\boldsymbol{F}^{-1}(\boldsymbol{z},t)$ (e.g., see [5, Eqs. (12.15.15) and (12.15.34))], we rewrite (3.5.46) as

$$\begin{aligned} &(\det\boldsymbol{F}(\boldsymbol{z},t))^{-1}\boldsymbol{T}(\boldsymbol{z},t)\cdot\boldsymbol{F}(\boldsymbol{z},t)^* \\ &\quad = -p\boldsymbol{I} + \mu\left(\boldsymbol{v}_{\boldsymbol{z}}(\boldsymbol{z},t)\cdot\boldsymbol{F}^{-1}(\boldsymbol{z},t) + \boldsymbol{F}^{-*}(\boldsymbol{z},t)\cdot\boldsymbol{v}_{\boldsymbol{z}}(\boldsymbol{z},t)^*\right) + \lambda\,\mathrm{div}\breve{\boldsymbol{v}}\,\boldsymbol{I}, \end{aligned} \tag{3.5.32}$$

whence we obtain the constitutive equation in the material formulation:

$$\begin{aligned}\boldsymbol{T} = -p \det \boldsymbol{F}\, \boldsymbol{F}^{-*} + \mu \det \boldsymbol{F}\, \big(\boldsymbol{v}_{\boldsymbol{z}} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-*} + (\boldsymbol{v}_{\boldsymbol{z}} \cdot \boldsymbol{F}^{-1})^{*} \cdot \boldsymbol{F}^{-*}\big) \\ + \lambda \det \boldsymbol{F}\, \operatorname{div}\breve{\boldsymbol{v}}\, \boldsymbol{F}^{-*}.\end{aligned} \tag{3.5.33}$$

Using (3.5.30) we find that

$$\begin{aligned}\boldsymbol{v}_{\boldsymbol{z}}(\boldsymbol{z},t) \cdot \boldsymbol{F}^{-1}(\boldsymbol{z},t) = \nu_t(s,t)\nu(s,t)^{-1}\boldsymbol{e}_1\boldsymbol{e}_1 + \tau_t(s,t)\tau(s,t)^{-1}\boldsymbol{e}_2\boldsymbol{e}_2, \\ \det \boldsymbol{F}(\boldsymbol{z},t) = \tau(s,t)\nu(s,t).\end{aligned} \tag{3.5.34}$$

From the formula $\operatorname{div}\boldsymbol{v} = \operatorname{tr}\, \boldsymbol{v}_{\boldsymbol{z}}$ [5, (11.2.25)] and from the invariance of the divergence operator under a coordinate transformation we find that

$$\operatorname{div}\breve{\boldsymbol{v}} = \operatorname{div}\boldsymbol{v} = \operatorname{tr}\, \boldsymbol{v}_{\boldsymbol{z}} = \tau_t + \nu_t. \tag{3.5.35}$$

Thus we obtain from (3.5.33), (3.5.34), (3.5.35) that

$$\boldsymbol{T} = -p(\tau\boldsymbol{e}_1\boldsymbol{e}_1 + \nu\boldsymbol{e}_2\boldsymbol{e}_2) + 2\mu\left(\frac{\nu_t\tau}{\nu}\boldsymbol{e}_1\boldsymbol{e}_1 + \frac{\nu\tau_t}{\tau}\boldsymbol{e}_2\boldsymbol{e}_2\right) + \lambda(\tau_t+\nu_t)(\tau\boldsymbol{e}_1\boldsymbol{e}_1 + \nu\boldsymbol{e}_2\boldsymbol{e}_2). \tag{3.5.36}$$

The dissipative part of $\boldsymbol{T}$ accordingly is

$$\left[\Big(\lambda\tau + 2\mu\frac{\tau}{\nu}\Big)\nu_t + \lambda\tau\tau_t\right]\boldsymbol{e}_1\boldsymbol{e}_1 + \left[\Big(\lambda\nu + 2\mu\frac{\nu}{\tau}\Big)\tau_t + \lambda\nu\nu_t\right]\boldsymbol{e}_2\boldsymbol{e}_2. \tag{3.5.37}$$

It is also illuminating and particularly easy to convert the constitutive equation (3.5.31) to the material (Lagrangian) coordinates in the case of one dimension. The normal Piola-Kirchhoff stress is then given by

$$N = -p + \Big(\frac{2\mu}{\nu} + \lambda\Big)\nu_t. \tag{3.5.38}$$

The dissipative parts in (3.5.36) and (3.5.38) have the property that the coefficients of the strain-rates (viscosities) blow up as the strain rates approach zero.

In our treatment of the preclusion of total compression, we find it necessary to impose further technical conditions. We shall be able to estimate space-time integrals of

$$\varDelta^{(\gamma)} := N^{\mathrm{D}}\nu_t + \gamma T^{\mathrm{D}}\tau_t \tag{3.5.39}$$

and shall need to estimate space-time integrals of T^{D} and N^{D}. ($\varDelta^{(\gamma)}$ is nonnegative under Hypothesis 3.5.14). We omit the superscript γ and write simply $\varDelta$ for visual clarity.

The following hypothesis is partially motivated by the form of the dissipative part in (3.5.36), which can be estimated by Young's inequality (3.2.1).

3.5.40. Hypothesis. *Assume $\bar{c} \leq \tau \leq \bar{C}$ for some positive numbers $\bar{c}$, $\bar{C}$. Then there is a number $\varepsilon \in (0,1)$ such that*

$$\begin{aligned} |N^{\mathrm{D}}(\tau,\nu,\dot{\tau},\dot{\nu},s)| &\leq C\left(\nu^{\alpha_1} + \frac{1}{\nu^{\alpha_3}} + \varepsilon\dot{\nu}^2 + 1\right), \\ |T^{\mathrm{D}}(\tau,\nu,\dot{\tau},\dot{\nu},s)| &\leq C\left(\nu^{\alpha_1} + \frac{1}{\nu^{\alpha_3}} + \dot{\tau}^2 + 1\right). \end{aligned} \tag{3.5.41}$$

for all ν, $\dot{\tau}$, $\dot{\nu}$, s and $\bar{c} \leq \tau \leq \bar{C}$ (α_1, α_3 are as defined in Hypotheses 3.5.6, 3.5.8).

The following hypothesis is used to get a priori bounds for $|\dot{\nu}|$. It is likewise partially motivated by the form of the dissipative parts in (3.5.36).

3.5.42. Hypothesis. *Let $\bar{c}$, $\bar{C}$ be some positive numbers such that $\bar{c} \leq \nu, \tau \leq \bar{C}$ and $|\dot{\tau}| \leq \bar{C}$. Then there exists a positive number C depending on $\bar{c}$, $\bar{C}$ such that*

$$|T^{\mathrm{D}}| \leq C, \tag{3.5.43}$$

and there exists a positive number c depending on $\bar{c}$, $\bar{C}$ such that

$$c|\dot{\nu}| \leq |N^{\mathrm{D}}| \quad \textit{if} \quad |\dot{\nu}| \geq c. \tag{3.5.44}$$

Condition (3.5.43) can be replaced with a more general condition:

$$|T^{\mathrm{D}}| \le C(\dot{\nu}^2 + 1) \text{ if } \quad |\dot{\nu}| \ge c, \quad \bar{c} \le \nu,\ \tau \le \bar{C}, \quad |\dot{\tau}| \le \bar{C}. \tag{3.5.45}$$

Our analysis with condition (3.5.45) would be very similar to that with condition (3.5.43).

In our treatment of the problem with double Dirichlet conditions we use the following mild hypothesis that says that T^{E} can be controlled by φ for $\bar{c} \le \tau \le \bar{C}$.

3.5.46. Hypothesis. *Let $\bar{c}$, $\bar{C}$ be some positive numbers such that $\bar{c} \le \tau \le \bar{C}$. Then*

$$|T^{\mathrm{E}}(\tau, \nu, s)| \le C(1 + \varphi^{(\gamma)}(\tau, \nu, s)) \tag{3.5.47}$$

for all ν, s and $\bar{c} \le \tau \le \bar{C}$.

The next (technical) hypothesis is also used in our treatment of the problem with double Dirichlet conditions and says that the dissipative behavior, as embodied in the properties of ψ, must compensate to some extent for the fact that $\varphi_\nu^{(\gamma)}$ grows at a rate faster than $\varphi^{(\gamma)}$ for ν small. This hypothesis is an analog of Hypothesis 3.18 of [9].

3.5.48. Hypothesis. *Let $\bar{C}$ be a positive number such that $\tau \le \bar{C}$. Then there are constants $\beta_1, \beta_2 \in (0, 1]$ such that*

$$|N^{\mathrm{E}}(\tau, \nu, s)| \le C\left(\varphi^{(\gamma)}(\tau, \nu, s) + \varphi^{(\gamma)}(\tau, \nu, s)^{\beta_1} \psi(\nu)^{\beta_2} + 1\right), \tag{3.5.49}$$

for all ν, s and $\tau \le \bar{C}$.

Consistency of the Hypotheses: An Example. We demonstrate that our constitutive hypotheses are compatible by producing a class of examples that satisfies them. We accordingly consider the functions of the form

$$\varphi(\tau,\nu) = \frac{A_1\tau^{1-a_1}}{a_1-1} + \frac{A_2\nu^{1-a_2}}{a_2-1} + \frac{B_1\tau^{1+b_1}}{b_1+1} + \frac{B_2\nu^{1+b_2}}{b_2+1},$$

$$\psi(y) = Q_1 y^{-q_1} + Q_2 y^{q_2+1},$$

$$N^{\mathrm{D}}(\tau,\nu,\dot\tau,\dot\nu) = (G_1\nu^{-g_1} + H_1\nu^{h_1} + G_2\tau^{-g_2} + H_2\tau^{h_2})R_1(\dot\nu) + R_3(\dot\nu),$$

$$T^{\mathrm{D}}(\tau,\nu,\dot\tau,\dot\nu) = (G_3\nu^{-g_3} + H_3\nu^{h_3} + G_4\tau^{-g_4} + H_4\tau^{h_4})R_2(\dot\tau) + R_4(\dot\tau), \qquad (3.5.50)$$

$$R_j(x) = \begin{cases} K_j x^{k_j} & \text{for } x \geq 1, \\ P_j(x) & \text{for } |x| \leq 1, \\ -L_j|x|^{l_j} & \text{for } x \leq -1 \end{cases}$$

with $j = 1,2,3,4$. Here the lower-case symbols a_1,..., l_2 represent positive real numbers and the upper-case symbols A_1,..., H_4 represent nonnegative real numbers. The functions P_j are polynomials satisfying $P_j(0) = 0$, $P_j(1) = 1$, $P_j(-1) = -1$, $P_j'(1) = K_j k_j$, $P_j'(-1) = L_j l_j$, $P'(x) > 0$ for $|x| \leq 1$.

Condition (3.5.24) is satisfied if

$$l_1 \geq 1, \quad l_3 \geq 1, \quad q_1 Q_1 \leq G_1 L_1, \quad g_1 \leq q_1 + 1, \qquad (3.5.51)$$

provided y_* is taken sufficiently small. Condition (3.5.26) is satisfied if

$$k_1 \geq 1, \quad k_3 \geq 1, \quad (q_2+1)Q_2 \leq H_1 K_1, \quad h_1 \geq q_2, \qquad (3.5.52)$$

provided y^* is taken sufficiently large. Condition (3.5.49) is satisfied if

$$A_2 \leq Q_1 q_1, \quad a_2 = \beta_1(1-a_2) + \beta_2 q_1. \qquad (3.5.53)$$

The remaining hypotheses are justified likewise.

A consistent set of inequalities is easily chosen from the list of inequalities we have just developed. E.g., a material that satisfies Hypotheses 3.5.6, 3.5.8, 3.5.14, 3.5.22, 3.5.40, 3.5.42 is described by the following functions:

$$\begin{aligned} &\varphi(\tau,\nu) = \tau^{-4} + \nu^{-4} + \tau^4 + \nu^4, \\ &N^{\mathrm{D}}(\tau,\nu,\dot\nu) = \frac{\tau\dot\nu}{\nu} + \dot\nu^3, \quad T^{\mathrm{D}}(\tau,\nu,\dot\tau) = \frac{\nu\dot\tau}{\tau} + \dot\tau^3, \quad \psi(y) = y^{-2} + y. \end{aligned} \tag{3.5.54}$$

3.6 Fundamental Theorems and Plan of the Analysis

Our *initial-boundary-value problem* for a viscoelastic annular plate or spherical shell consists of the equation (3.3.3) of motion, the constitutive relations (3.3.5), (3.5.1), the boundary conditions (3.4.5) and the initial conditions (3.4.7). In the next section we begin the analysis of our initial-boundary-value problem leading to our fundamental existence theorem.

3.6.1. Theorem. *Let t^+ be a fixed positive number. Let the initial data satisfy Hypothesis 3.4.8. Let the boundary conditions have the form* (3.4.5)*, satisfy* (3.4.6)*, and be compatible with the initial conditions. Let Hypotheses 3.5.6, 3.5.8, 3.5.14, 3.5.18, 3.5.22, 3.5.40, 3.5.42 hold. Then there is a unique solution $(s,t) \mapsto r(s,t)$ of* (3.4.11) *such that*

$$\begin{aligned} r_t &\in L^\infty(0,t^+;H^2(a,1)), \\ r_{tt} &\in L^\infty(0,t^+;H^1(a,1)), \\ r_{stt} &\in C^0([0,t^+]\times[a,1]). \end{aligned} \tag{3.6.2}$$

Moreover, r satisfies the initial conditions and boundary conditions pointwise.

The existence theory carried out in Secs. 3.12–3.16 shows that if the data are sufficiently regular, then so are the solutions as long as they exist. For the purpose of obtaining estimates, we may accordingly take the strains τ, ν and the derivatives r_t, r_{st}, r_{stt} to be continuous. We obtain bounds for these quantities in Secs. 3.12–3.15. We avoid inconsistency in the existence theory by seeking solutions that satisfy these bounds.

In Sec. 3.7 we obtain an energy estimate, based primarily on Hypothesis 3.5.14, which leads to a useful bound on the space-time integral

$$\int_0^t \int_a^1 s^\gamma \Big[\Big(\frac{r_t}{s}\Big)^2 + r_{st}^2 \Big] \, ds \, d\bar{t} \tag{3.6.3}$$

(that represents the work of dissipative internal forces) as well as on the kinetic and stored energies.

In Sec. 3.8 we use our energy estimate and Hypotheses 3.5.6, 3.5.22 to prove the preclusion of a total compression and of an infinite extension with respect to τ and ν, i.e., we prove that $s \mapsto r_s(s,t)$ and $s \mapsto s^{-1}r(s,t)$ are pointwise bounded below and above by positive-valued functions of t. Hypothesis 3.5.22 is an analog of Hypothesis 3.7 of [8] in our two-dimensional setting. Hypothesis 3.5.22 is only moderately restrictive because we require that (3.5.24) hold only for nonpositive $\dot{\nu}$ and that (3.5.26) hold only for nonnegative $\dot{\nu}$. In contrast, in [8] the condition analogous to (3.5.24) is required to hold for all values of $\dot{\nu}$, which may be unreasonably restrictive. As discussed in Sec. 5 of [8] a technical obstacle associated with relaxing the condition analogous to (3.5.24) (so that it holds only for nonpositive $\dot{\nu}$) is related to the fact that the strain variable $r_s(s,\cdot)$ may have an infinite number of oscillations on a time interval of interest. A similar obstacle is present in our analysis. In Sec. 3.8 we show however that the potential oscillatory behavior of $r_s(s, \cdot)$ can be handled if r is sufficiently regular, in particular, if r_{stt} is continuous. The required regularity of r is ensured by the regularity of the data that we impose as shown in Secs. 3.12–3.16.

In Sec. 3.9 we get integral bounds on the accelerations and the strain-rates. In Sec. 3.10 we use the estimates of Secs. 3.8, 3.9 and Hypothesis 3.5.42 to obtain pointwise bounds on the strain-rates. The technical part of our work consists in deriving these bounds and their analogs for the Galerkin approximations.

Since we have these bounds on the arguments of the constitutive functions, we do not change the solutions of our governing equations if we modify our constitutive functions where they do not obey these bounds. In Sec. 3.11 we effect such a modification that replaces $\hat{N}$, $\hat{T}$ with modified functions that satisfy the uniform monotonicity condition with respect to the strain-rates and that behave regularly at a total compression and at an infinite extension. This replacement makes our problem more accessible to a version of the Faedo-Bubnov-Galerkin method designed to accomodate the technical challenges posed by the underlying mechanics. We use this method to carry out the existence theory in Sec. 3.12–3.16. In Sec. 3.17 we show that the solution depends continuously on the data and is therefore unique.

Sec. 3.18 treats other boundary conditions. In Sec. 3.19 we exhibit alternative constitutive restrictions for which the solutions blow up or become unbounded at different rates.

3.7 Energy Estimates

We introduce the kinetic energy $K(t)$, the work $W(t)$ of the dissipative internal forces, and the boundary power term $B(t)$ at time t:

$$
\begin{aligned}
K(t) &:= \frac{1}{2}\int_a^1 s^\gamma \rho(s) r_t(s,t)^2\, ds, \\
W(t) &:= \int_0^t \int_a^1 s^\gamma \varDelta(s,\bar{t})\, ds\, d\bar{t} \equiv \int_0^t \int_a^1 s^\gamma \Big[N^{\mathrm{D}}\nu_t + \gamma T^{\mathrm{D}}\tau_t\Big]\, ds\, d\bar{t}, \qquad (3.7.1)\\
B(t) &:= N(1,t)r_t(1,t) - a^\gamma N(a,t) r_t(a,t).
\end{aligned}
$$

We formally multiply the governing equation (3.3.3) by r_t and then integrate the resulting equation over $[a,1]\times[0,t]$ to obtain the energy equation

$$
K(t) + \varPhi(t) + W(t) - \int_0^t B(\bar{t})\, d\bar{t} = K(0) + \varPhi(0). \qquad (3.7.2)
$$

($\varPhi$ was defined in (3.7.3).) Using the notation introduced in (3.4.7) we find that

$$
\begin{aligned}
\varPhi(0) &= \int_a^1 s^\gamma\, \varphi^{(\gamma)}(s^{-1}p(s), p'(s), s)\, ds, \\
K(0) &= \frac{1}{2}\int_a^1 s^\gamma \rho(s) q(s)^2\, ds.
\end{aligned} \qquad (3.7.3)
$$

Hypothesis 3.4.8 then ensures that $\varPhi(0)$ and $K(0)$ are bounded.

Since $N^{\mathrm{D}}(\tau,\nu,0,0,s)=0$, we have

$$
\begin{aligned}
N^{\mathrm{D}}(\tau,\nu,\tau_t,\nu_t,s) &= \int_0^1 \frac{d}{d\alpha} N^{\mathrm{D}}(\tau,\nu,\alpha\tau_t,\alpha\nu_t,s)\, d\alpha \\
&= \int_0^1 \left(\partial_{\dot\tau} N^{\mathrm{D}}\, \tau_t + \partial_{\dot\nu} N^{\mathrm{D}}\, \nu_t\right)\, d\alpha.
\end{aligned} \qquad (3.7.4)
$$

Obtaining a similar formula for T^{D} and then using the monotonicity condition (3.5.16), we deduce that

$$
\varDelta = \int_0^1 \Big[\gamma\partial_{\dot\tau}T^{\mathrm{D}}\, \tau_t^2 + (\gamma\partial_{\dot\nu}T^{\mathrm{D}} + \partial_{\dot\tau}N^{\mathrm{D}})\, \nu_t\tau_t + \partial_{\dot\nu}N^{\mathrm{D}}\, \nu_t^2\Big]\, d\alpha \geq c_{\mathrm{V}}\Big[\gamma\tau_t^2 + \nu_t^2\Big] \geq 0. \quad (3.7.5)
$$

3.7.6. Theorem. *Let* (3.5.14) *hold (so that* (3.5.16) *holds). Let boundary conditions* (3.4.5) *hold subject to* (3.4.6), *and let the initial conditions satisfy the Hypothesis 3.4.8. Let* t^+ *be any positive number. Then the energy estimate*

$$K(t) + \Phi(t) + W(t) \le \Gamma(t^+) \tag{3.7.7}$$

holds for $0 \le t \le t^+$ *with* $\Gamma(t^+)$ *depending only on* t^+, *the constitutive functions, and the bounds for the data. In particular,*

$$\|r_t(\cdot,t)\|, \ \int_0^t \|r_{st}(\cdot,\bar{t})\|^2 \, d\bar{t} \le \Gamma(t^+). \tag{3.7.8}$$

If the material is sufficiently strong in the sense that Hypothesis 3.5.6a holds, then

$$r(\cdot,t), \ \tau(\cdot,t) \le \Gamma(t^+). \tag{3.7.9}$$

If λ_1 *is independent of* t *and if Hypothesis 3.5.6a holds, then the bounds in* (3.7.7), (3.7.8), (3.7.9) *are independent of* t^+.

Proof. Hypothesis 3.4.8 ensures that the right-hand side of (3.7.2) is bounded. Thus we only need to obtain a tractable bound on the boundary term $\int_0^t B(\bar{t})\, d\bar{t}$. By using the condition $r_t(a,t) = 0$, which follows from $(3.4.5)_1$, and the inequality (3.7.5), we obtain

$$\begin{aligned} |r_t(1,t)| &\le \int_a^1 |r_{st}(\xi,t)|\, d\xi \le \left(\int_a^1 r_{st}^2(\xi,t)\, d\xi \right)^{1/2} \\ &\le (a^\gamma c_{\mathrm{V}})^{-1/2} \left(\int_a^1 s^\gamma \Delta(s,t)\, ds \right)^{1/2} \end{aligned} \tag{3.7.10}$$

whence

$$\int_0^t r_t^2(1,\bar{t})\, d\bar{t} \le C W(t). \tag{3.7.11}$$

By using the last estimate and our assumption $N(1,t) \equiv -\lambda_1(t) \in L^2_{\text{loc}}[0,\infty)$, we get

$$\left|\int_0^t B(\bar{t})\,d\bar{t}\right| = \left|\int_0^t -\lambda_1(\bar{t}) r_t(1,\bar{t})\,d\bar{t}\right| \le C\int_0^t \lambda_1^2(\bar{t})\,d\bar{t} + \varepsilon\int_0^t r_t^2(1,\bar{t})\,d\bar{t} \\ \le \Gamma(t^+) + \varepsilon W(t). \tag{3.7.12}$$

We substitute the last estimate into (3.7.2) and choose ε sufficiently small to get (3.7.8). The bounds (3.7.9) now follow from Hypothesis 3.5.6a and (3.5.11).

We get a slicker derivation of a bound for $\int_0^t B(\bar{t})\,d\bar{t}$ if λ_1 is independent of t:

$$\left|\int_0^t B(\bar{t})\,d\bar{t}\right| = |\lambda_1|\left|\int_0^t r_t(1,\bar{t})\,d\bar{t}\right| = Cr(1,t) + C \le \varepsilon\Phi(t) + C \tag{3.7.13}$$

where the last inequality follows from (3.5.11) provided $\alpha_1, \alpha_2 > 1$ (which holds in view of Hypothesis 3.5.6a). Substituting (3.7.13) into (3.7.2) and choosing ε sufficiently small we get (3.7.7) with Γ independent of t^+. □

By slightly modifying the argument encompassing (3.7.10), (3.7.11) we can obtain a tractable bound on $|r_t(1,t)|$ without assuming that the boundary condition $r(a,t) = r_a$ is prescribed. By an argument analogous to that encompassing (3.5.10), (3.5.11), (3.5.12) we find that for any $s \in [a,1]$

$$r_t(s,t) \le C\sqrt{\int_a^1 \bar{s}^\gamma \rho(\bar{s}) r_t^2(\bar{s},t)\,d\bar{s}} + C\sqrt{\int_a^1 \bar{s}^\gamma r_{st}^2(\bar{s},t)\,d\bar{s}}, \tag{3.7.14}$$

so that

$$\int_0^t r_t(1,\bar{t})^2\,d\bar{t} \le C\int_0^t K(\bar{t})\,d\bar{t} + CW(t). \tag{3.7.15}$$

3.8 Preclusion of Total Compression and Infinite Extension

We now establish positive lower and upper bounds on the strains τ, ν for our problem with boundary conditions (3.4.5). The preclusion of total compression and inifinite extension for one-dimensional problems was shown in [9]. Our approach uses some techniques of [9] but also exhibits novel features related to the necessity to handle the inherent higher dimensionality of the problem. In particular, the preclusion of inifinite extension and total compression related to the extreme values of τ is ensured by imposing conditions on the stored-energy density φ (Hypothesis 3.5.8), while the extreme of values of ν are precluded by virtue of Hypothesis 3.5.22 controlling the dissipative part N^{D}.

A priori bounds for τ. A consequence (3.7.9) of the energy estimate (3.7.8) ensures that $\tau(\xi,t) \leq \mathit{\Gamma}(t^+)$ for all $(\xi,t) \in [a,1] \times [0,t^+]$, which gives an upper bound for τ.

The condition $r(a,t) = r_a$ in (3.4.5) gives a positive lower bound for τ. We can also establish such a bound for τ by using Hypothesis 3.5.8. This argument can be used for other boundary conditions. We argue in analogy with (3.5.11):

$$\begin{aligned} r(s,t)^{-1} &= m[r^{-1}](t) + \int_{s^*(t)}^{s} \frac{\partial}{\partial s}\left(r(\bar{s},t)^{-1}\right) d\,\bar{s} \\ &= m[r^{-1}](t) - \int_{s^*(t)}^{s} r(\bar{s},t)^{-2} r_s(\bar{s},t) d\,\bar{s} \\ &\leq C\left(\int_a^1 r(\bar{s},t)^{-\alpha_4}\, d\bar{s}\right)^{1/\alpha_4} \\ &\quad + C\left(\int_a^1 r(\bar{s},t)^{-2\alpha_1^*} d\bar{s}\right)^{1/\alpha_1^*}\left(\int_a^1 r_s(\bar{s},t)^{\alpha_1} d\bar{s}\right)^{1/\alpha_1} \end{aligned} \tag{3.8.1}$$

where $\alpha_1^* = (1-1/\alpha_1)^{-1}$. We now require that α_4 in Hypothesis 3.5.8 is such that

$$\alpha_4 \geq 2\alpha_1^*. \tag{3.8.2}$$

(E.g., if $\alpha_1 = 2$, then $\alpha_4 \geq 4$.) Then using (3.7.7) and Hypothesis 3.5.6a we deduce from (3.8.1) that

$$r(s,t)^{-1} \leq \Gamma(t^{+}) \tag{3.8.3}$$

whence $\tau(s,t) \geq \Gamma^{-1}(t)$.

A priori bounds for ν. The existence theory of Sec. 3.16 shows that if the data are sufficiently regular then so is the solution as long as it exists. We accordingly assume that r_s, r_{st} and r_{stt} are continuous. We assume that Hypothesis 3.5.22 holds and the hypotheses of Theorem 3.7.6 hold, so that initially there is no total compression and the energy estimate (3.7.7) holds. Without loss of generality we may choose the number y_* introduced in Hypothesis 3.5.22 so that

$$\min_s r_s(s,0) \equiv \min_s p'(s) \geq y_*. \tag{3.8.4}$$

Let (ξ,t) be a point such that

$$0 < \nu(\xi,t) < y_*. \tag{3.8.5}$$

(Were there no such (ξ,t), then ν would be greater than y_* and we would be done). Since r and r_s are taken to be continuous, there is a time θ before t such that $\nu(\xi,\theta) = y_*$ and $\nu(\xi,\bar{t}) < y_*$ for $\theta < \bar{t} < t$.

We first consider the case that

$$r_{st}(\xi,\bar{t}) \leq 0 \quad \text{for} \quad \bar{t} \in (\theta,t). \tag{3.8.6}$$

(We shall later use the continuity of r_{stt} to argue that in fact it suffices to assume (3.8.6).)

From the equation (3.3.3) of motion and Hypothesis 3.5.22a we obtain

$$\begin{aligned}\int_\xi^1 s^\gamma \rho(s)\, r_t(s,\bar t)\, ds\Big|_{\bar t=\theta}^{\bar t=t} &= \int_\theta^t s^\gamma N(s,\bar t)\, d\bar t\Big|_{s=\xi}^{s=1} - \int_\theta^t \int_\xi^1 \gamma s^{\gamma-1} T(s,\bar t)\, ds\, d\bar t \\ &\geq \psi(\nu(\xi,t)) - \int_\theta^t \int_\xi^1 \gamma s^{\gamma-1} T(s,\bar t)\, ds\, d\bar t - \mathit{\Gamma}(t^+).\end{aligned} \tag{3.8.7}$$

The energy estimate (3.7.7) implies that

$$\left|\int_\xi^1 s^\gamma \rho(s)\, r_t(s,\bar t)\, ds\right| \leq CK(\bar t) \leq \mathit{\Gamma}(t^+) \quad \text{for } \bar t \in [0,t^+]. \tag{3.8.8}$$

Hypotheses 3.5.6, 3.5.40, 3.5.46a and the energy estimate (3.7.7) imply that

$$\begin{aligned}\left|\int_\theta^t \int_\xi^1 \gamma s^{\gamma-1} T(s,\bar t)\, ds\, d\bar t\right| &\leq C\int_\theta^t \int_a^1 |T(s,\bar t)|\, ds\, d\bar t \\ &\leq \mathit{\Gamma}(t^+) + C\int_\theta^t \mathit{\Phi}(\bar t)\, d\bar t + C\int_\theta^t K(\bar t)\, d\bar t \leq \mathit{\Gamma}(t^+).\end{aligned} \tag{3.8.9}$$

By combining (3.8.7)–(3.8.9) we get

$$\psi(\nu(\xi,t)) \leq \mathit{\Gamma}(t^+), \tag{3.8.10}$$

and the properties of ψ then imply that $\nu(\xi,t) \geq \mathit{\Gamma}^{-1}(t)$.

We now consider the case when (3.8.6) does not hold. Clearly if $r_{st}(\xi,\cdot)$ changes sign only a finite number of times on $[\theta,t]$ then dividing (θ,t) into the intervals on which r_{st} has the same sign and using the above argument on each interval where r_{st} is non-positive would again lead to (3.8.10). Suppose now $r_{st}(\xi,\cdot)$ changes sign infinitely many times on $[\theta,t]$. Then there exists a $\theta^* \in [\theta,t]$ in every neighborhood of which $r_{st}(\xi,\cdot)$ changes sign infinitely many times. Since $r_{stt}(\xi,\cdot)$ is taken to be continuous then $r_{stt}(\xi,\theta^*) = 0 = r_{st}(\xi,\theta^*)$. Thus the amplitude of oscillations of $r_s(\xi,t)$ becomes smaller as t approaches θ^*. This shows that a neighborhood of θ^*

is not important for our analysis and it suffices to consider a suitable subinterval on which r_{st} changes sign only a finite number of times.

We likewise obtain an upper bound for ν that depends on the data by using Hypothesis 3.5.22b. Therefore

3.8.11. Theorem. *Let the hypotheses of Theorem 3.7.6* (*namely, Hypotheses 3.5.6a, 3.5.14, 3.4.8*) *hold, and let Hypotheses 3.5.22, 3.5.40 and Hypothesis 3.5.46a hold. Let boundary conditions* (3.4.5) *hold subject to* (3.4.6). *Then any solution of the initial-boundary-value problem with requisite smoothness satisfies*

$$\varGamma^{-1}(t) \leq \nu(s,t) \leq \varGamma(t), \quad \varGamma^{-1}(t) \leq \tau(s,t) \leq \varGamma(t) \quad \forall\,(s,t) \in [a,1] \times [0,t^+] \tag{3.8.12}$$

where $\varGamma^{-1}$, $\varGamma$ *denote continuous positive-valued functions on* $[0,t^+]$.

3.9 Estimates of the Accelerations and the Strain Rates

We shall obtain an energy-like estimate for the functional

$$H(t) := \frac{1}{2}\int_a^1 s^\gamma \rho(s)\, r_{tt}^2(s,t)\, ds + \int_0^t \int_a^1 s^\gamma\, [\tau_{tt}\, \nu_{tt}] \cdot \frac{\partial(\hat{N}, \gamma\hat{T})}{\partial(\dot{\nu}, \dot{\tau})} \cdot \begin{bmatrix} \tau_{tt} \\ \nu_{tt} \end{bmatrix} ds\, d\bar{t}. \tag{3.9.1}$$

Taking $\chi = r_{tt}$ in (3.4.12) yields

$$-s^\gamma N_t r_{tt}\Big|_{s=a}^{s=1} + \int_a^1 s^\gamma \left(\hat{N}_t r_{stt} + \frac{\gamma}{s}\hat{T}_t r_{tt}\right) ds + \int_a^1 s^\gamma \rho\, r_{tt} r_{ttt}\, ds = 0. \tag{3.9.2}$$

Using the constitutive equations (3.3.5), we find that

$$N_t = \hat{N}_\tau \tau_t + \hat{N}_\nu \nu_t + N_{\dot{\tau}}^{\mathrm{D}} \tau_{tt} + N_{\dot{\nu}}^{\mathrm{D}} \nu_{tt}, \quad \text{etc.} \tag{3.9.3}$$

Integrate (3.9.2) from 0 to t and use (3.9.1), (3.9.3) to obtain the equation involving the accelerations and the strain rates:

$$H(t) - \frac{1}{2}\int_a^1 s^\gamma \rho(s)\, r_{tt}^2(s,0)\, ds + \int_0^t \int_a^1 s^\gamma [\nu_{tt}\ \tau_{tt}] \cdot \begin{bmatrix} \hat{N}_\nu, \hat{N}_\tau \\ \gamma\hat{T}_\nu, \gamma\hat{T}_\tau \end{bmatrix} \cdot \begin{bmatrix} \nu_t \\ \tau_t \end{bmatrix} ds\, d\bar{t} = \int_0^t s^\gamma N_t r_{tt}\Big|_{s=a}^{s=1} d\bar{t}. \tag{3.9.4}$$

3.9.5. Theorem. *Let the hypotheses of Theorem 3.8.11 hold and let Hypothesis 3.5.18 hold. Let boundary conditions* (3.4.5) *hold subject to* (3.4.6)*. Let t^+ be any positive number. Then*

$$H(t) \le \Gamma(t^+) \quad \textit{for} \quad 0 \le t \le t^+ \tag{3.9.6}$$

with $\Gamma(t^+)$ depending only on t^+, the constitutive functions, and the bounds for the data. In particular,

$$\|r_{tt}(\cdot,t)\| + \int_0^t \|r_{stt}(\cdot,\bar{t})\|^2\, d\bar{t} \le \Gamma(t^+) \quad \textit{for} \quad 0 \le t \le t^+. \tag{3.9.7}$$

Proof. We use Hypothesis 3.5.18, which was expressly designed to handle the integrand of the third term on the left-hand side of (3.9.4), to obtain that

$$\left| [\ddot{\nu}\ \ddot{\tau}] \cdot \frac{\partial(N,\gamma T)}{\partial(\nu,\tau)} \cdot \begin{bmatrix} \dot{\nu} \\ \dot{\tau} \end{bmatrix} \right| \equiv \left| \left(\boldsymbol{\Lambda} \cdot \begin{bmatrix} \ddot{\nu} \\ \ddot{\tau} \end{bmatrix} \right) \cdot \left(\boldsymbol{\Lambda}^{-1} \cdot \frac{\partial(N,\gamma T)}{\partial(\nu,\tau)} \cdot \begin{bmatrix} \dot{\nu} \\ \dot{\tau} \end{bmatrix} \right) \right|$$
$$\le \varepsilon [\ddot{\nu}\ \ddot{\tau}] \cdot \frac{\partial(N,\gamma T)}{\partial(\dot{\nu},\dot{\tau})} \cdot \begin{bmatrix} \ddot{\nu} \\ \ddot{\tau} \end{bmatrix} + C\left[1 + N^{\mathrm{D}}\dot{\nu} + T^{\mathrm{D}}\dot{\tau} + \varphi^{(\gamma)}\right]. \tag{3.9.8}$$

The monotonicity condition (3.5.16) (ensured by Hypothesis 3.5.14) implies that

$$\int_0^t \int_a^1 s^\gamma \left(\gamma \frac{r_{tt}^2}{s^2} + r_{stt}^2 \right) ds\, d\bar{t} \le CH(t). \tag{3.9.9}$$

By using the condition $r_{tt}(a,t)=0$, which follows from $(3.4.5)_1$, we further obtain that

$$|r_{tt}(1,t)| \le \int_a^1 |r_{stt}(\xi,t)|\,d\xi \le \left(\int_a^1 r_{stt}^2(\xi,t)\,d\xi\right)^{1/2}. \tag{3.9.10}$$

Combining (3.9.9) and (3.9.10) yields

$$\int_0^t r_{tt}^2(1,\bar{t})\,d\bar{t} \le \int_0^t \int_a^1 r_{stt}^2(\xi,\bar{t})\,d\xi\,d\bar{t} \le CH(t). \tag{3.9.11}$$

Since $N(1,t) \equiv -\lambda_1(t) \in H^1_{\mathrm{loc}}[0,\infty)$ we then get

$$\begin{aligned} \left|\int_0^t N_t(1,\bar{t}) r_{tt}(1,\bar{t})\,d\bar{t}\right| &\le C\int_0^t N_t^2(1,\bar{t})\,d\bar{t} + \varepsilon \int_0^t r_{tt}^2(1,\bar{t})\,d\bar{t} \\ &\le \mathit{\Gamma}(t) + \varepsilon H(t). \end{aligned} \tag{3.9.12}$$

We substitute the last estimate and the estimate (3.9.8) into (3.9.4) (not forgetting that (3.9.8) is to be integrated over $[a,1]\times[0,t]$), use the energy estimate (3.7.7), and choose ε sufficiently small to get (3.9.6). □

By slightly modifying the argument encompassing (3.9.10), (3.9.11) we can obtain a tractable bound on the integral $r_{tt}(1,t)^2$ without assuming that the boundary condition $r(a,t)=r_a$ is prescribed. Indeed, in analogy with (3.7.14) we find that for any $s\in[a,1]$

$$|r_{tt}(s,t)| \le C\sqrt{\int_a^1 \bar{s}^\gamma \rho(\bar{s}) r_{tt}^2(\bar{s},t)\,d\bar{s}} + C\sqrt{\int_a^1 \bar{s}^\gamma r_{stt}^2(\bar{s},t)\,d\bar{s}}, \tag{3.9.13}$$

so that

$$\int_0^t r_{tt}(1,t)^2 \le C\int_0^t H(\bar{t})\,d\bar{t}. \tag{3.9.14}$$

A priori estimate for $\|r_{st}\|$. Hypothesis 3.4.8 on the initial data and the energy estimate (3.7.8) imply that

$$\begin{aligned}\int_a^1 r_{st}^2(s,t)\,ds &= \int_a^1 r_{st}^2(s,0)\,ds + 2\int_a^1\int_0^t r_{st}r_{stt}\,d\bar{t}\,ds \\ &\le C + 2\sqrt{\int_0^t\int_a^1 r_{st}^2\,ds\,d\bar{t}}\sqrt{\int_0^t\int_a^1 r_{stt}^2\,ds\,d\bar{t}} \\ &\le \mathit{\Gamma}(1+\sqrt{H}) \le \mathit{\Gamma}(t^+)\end{aligned} \tag{3.9.15}$$

where the last inequality follows from (3.9.6).

3.10 Pointwise Estimates on the Strain Rates

In this section we use the estimates obtained in Sec.3.7-3.9 to establish pointwise bounds on the strain rates ν_t and τ_t.

A priori pointwise estimate for $|\tau_t|$. The estimates (3.7.8), (3.7.14), and (3.9.15) immediately imply that

$$|\tau_t| \le \mathit{\Gamma}(t^+). \tag{3.10.1}$$

A priori pointwise estimate for $|\nu_t|$. The estimate (3.8.12) implies that the resultants N^{E}, T^{E} are bounded for all $(s,t)\in[a,1]\times[0,t^+]$:

$$|N^{\mathrm{E}}|,\ |T^{\mathrm{E}}| \le \mathit{\Gamma}(t^+). \tag{3.10.2}$$

Integrating the governing equation (3.3.3) over $[s,1]$ and using our boundary condition $(3.4.5)_2$ with (3.4.6) yields

$$\begin{aligned}-s^\gamma N(s,t) &= \lambda_1(t) + \int_s^1 \gamma s^{\gamma-1}T\,ds + \int_s^1 s^\gamma\rho(s)r_{tt}(s,t)\,ds \\ &\le \mathit{\Gamma}(t^+) + C\int_a^1 |T|\,ds + C\left(\int_a^1 r_{tt}(s,t)^2\,ds\right)^{1/2} \\ &\le \mathit{\Gamma}(t^+)\end{aligned} \tag{3.10.3}$$

where we used (3.9.7), (3.10.2). By combining (3.10.2) and (3.10.3) and using (3.5.43) of Hypothesis 3.5.42 we obtain that

$$|N^{\mathrm{D}}| \leq \Gamma(t^+) + \int_a^1 |T^{\mathrm{D}}| \, ds \leq \Gamma(t^+). \tag{3.10.4}$$

Invoking (3.5.44) of Hypothesis 3.5.42, we conclude that

$$|\nu_t| \leq \Gamma(t^+). \tag{3.10.5}$$

Instead of using Hypothesis 3.5.42a to obtain (3.10.4) we could use the more general condition (3.5.45) and invoke (3.9.15).

3.11 The Role of the Bounds and the Modified Problem

In obtaining a priori estimates of Secs. 3.7–3.9 we have assumed that our initial-boundary-value problem has a solution (with more regularity than needed to give the weak equation (3.4.11) meaning). These estimates are used in Secs. 3.12-3.16 to prove that the weak equation (3.4.5) has a sufficiently regular solution. It may seem that this statement suggests that the proof of such existence is based on circular reasoning. A more careful description of the ways the a priori bounds are used shows that there is no logical inconsistency.

Namely the estimates (3.8.12), (3.10.1) and (3.10.4) imply that for any regular solution r and for any positive number t^+ there are positive numbers $\Gamma(t^+)$ and $\Gamma^{-1}(t^+)$ such that

$$\begin{gathered} \Gamma^{-1}(t^+) \leq \nu(\cdot, t),\, \tau(\cdot, t) \leq \Gamma(t^+), \\ |\nu_t(\cdot, t)|,\, |\tau_t(\cdot, t)| \leq \Gamma(t^+) \end{gathered} \tag{3.11.1}$$

for $t \in [0, t^+]$. Hence only the restriction of $\hat{N}(\cdot,\cdot,\cdot,\cdot,s)$, $\hat{T}(\cdot,\cdot,\cdot,\cdot,s)$ to the corresponding values of the arguments $(\tau, \nu, \dot{\tau}, \dot{\nu})$ actually intervenes in our initial-boundary-value problem for $0 \le t \le t^+$. This means that we can replace our constitutive functions, which were originally defined for all strains (τ, ν) satisfying (3.3.6) and for all strain rates (τ_t, ν_t) and which exhibit unpleasant behavior at extreme values of these variables, by nicer constitutive functions, which are well-behaved at the extreme values and which accordingly remove some of the difficulties in the existence theory. We now show how this can be done.

For our fixed $\mathit{\Gamma}^{-1}(t^+)$, $\mathit{\Gamma}(t^+)$ we introduce the cut-off function

$$[\nu] := \begin{cases} \frac{1}{2}\mathit{\Gamma}^{-1}(t^+), & \text{if} \quad \nu \le \frac{1}{2}\mathit{\Gamma}^{-1}(t^+), \\ \nu, & \text{if} \quad \frac{1}{2}\mathit{\Gamma}^{-1}(t^+) \le \nu \le 2\mathit{\Gamma}(t^+), \\ 2\mathit{\Gamma}(t^+) & \text{if} \quad 2\mathit{\Gamma}(t^+) \le \nu. \end{cases} \tag{3.11.2}$$

We likewise define $[\tau]$.

We define the modified constitutive functions $N^\sharp$, $T^\sharp$:

$$N^\sharp(\tau, \nu, \dot{\tau}, \dot{\nu}, s) := \begin{cases} \hat{N}([\tau], [\nu], \dot{\tau}, \dot{\nu}, s) & \text{if} \quad \sqrt{|\dot{\tau}|^2 + |\dot{\nu}|^2} \le 2\mathit{\Gamma}(t^+), \\ \hat{N}([\tau], [\nu], \dot{\tau}, \dot{\nu}, s) & \\ \qquad +2\mu\mathit{\Gamma}(t^+)\left(1 - \frac{1}{\sqrt{\dot{\tau}^2+\dot{\nu}^2}}\right)\dot{\nu} & \text{if} \quad \sqrt{|\dot{\tau}|^2 + |\dot{\nu}|^2} \ge 2\mathit{\Gamma}(t^+) \end{cases} \tag{3.11.3}$$

where μ is a positive number. We likewise define $T^\sharp$. The smoothness of our constitutive functions $\hat{N}$, $\hat{T}$ implies a uniform regularity for the modified constitutive functions $N^\sharp$, $T^\sharp$:

$$
\begin{aligned}
&|N^\sharp(\tau,\nu,\dot\tau,\dot\nu,s)| \le \varGamma(t^+)(1+|\dot\tau|+|\dot\nu|),\\
&|N^\sharp_\nu|,\ |N^\sharp_{\dot\nu}|,\ |N^\sharp_\tau|,\ |N^\sharp_{\dot\tau}|,\ |N^\sharp_s|, |N^\sharp_{\nu\nu}|, |N^\sharp_{\nu\dot\nu}| \le \varGamma(t^+),\ \text{etc.}
\end{aligned}
\tag{3.11.4}
$$

Moreover (3.5.16) implies that

$$
a^2\partial_{\dot\nu}N^\sharp + ab\left[\partial_{\dot\tau}N^\sharp + \gamma\partial_{\dot\nu}T^\sharp\right] + \gamma b^2\partial_{\dot\tau}T^\sharp \ge c_{\mathrm{V}}(a^2+\gamma b^2) \quad \forall a,b\in\mathbb{R} \tag{3.11.5}
$$

whenever $\sqrt{|\dot\tau|^2+|\dot\nu|^2} \le 2\varGamma(t^+)$. It was proved in [4] that the constitutive functions modified in this way have the virtue that (3.11.5) holds for all $\dot\tau$, $\dot\nu$ provided that μ is large enough.

In view of these remarks, we replace the actual problem with the modified problem. In doing so, we drop the sharp signs. This means that we are treating a problem of the same form as the original problem, but with the bounds (3.11.4), with the uniform monotonicity (3.11.5), and without the restriction (3.11.1).

3.12 A Faedo-Bubnov-Galerkin Method

We carry out the existence theory by a version of the Faedo-Bubnov-Galerkin method [28, 37, 57, 58]. This method (as a tool for proving the existence of solutions) was first developed by Sandro Faedo [29].

We seek Galerkin approximations of standard type, taking these in the form

$$
r^M(s,t) := \sum_{m=1}^{M} r_m(t)y_m(s) \tag{3.12.1}
$$

where the shape functions y_m are chosen so that span $\{y_1, y_2, ..., y_m, ...\}$ is dense in $H^1(a,1)$. The functions r_m are to be determined. (These unknown functions should also be indexed by M because the equations for them depend on M, but we suppress this index for visual clarity).

We choose y_m to be normalized eigenfunctions of the equation $-y'' = \kappa^2 y$ subject to appropriate boundary conditions. This special choice of the shape functions is used in Sec. 3.15 to derive an a priori estimate for $\|r^M_{sst}\|$. Our existence theory of Secs. 3.13–3.16 is carried out for the case of the boundary conditions (3.4.5). For these boundary conditions (3.4.5) we take

$$y_m(s) = \sqrt{2}\sin\left(\vartheta_m \frac{s-a}{1-a}\right), \quad \vartheta_m = \frac{\pi}{2}\left(2m-1\right). \tag{3.12.2}$$

so that $y_m(a) = 0$.

We define

$$\begin{aligned} N^M(s,t) &:= \hat{N}(\tau^M(s,t), \nu^M(s,t), \tau^M_t(s,t), \nu^M_t(s,t), s), \\ N^M_{\mathrm{D}}(s,t) &:= \hat{N}^{\mathrm{D}}(\tau^M(s,t), \nu^M(s,t), \tau^M_t(s,t), \nu^M_t(s,t), s), \quad \text{etc.}, \end{aligned} \tag{3.12.3}$$

where $\tau^M(s,t) := r^M(s,t)/s,\ \nu^M(s,t) := r^M_s(s,t)$, etc.

3.13 Approximating Ordinary Differential Equations

To get the approximating differential equations from (3.4.11), we replace χ with y_j:

$$\int_a^1 s^\gamma \left(\rho(s)\, r^M_{tt}\, y_j + N^M y_j' + \frac{\gamma}{s} T^M y_j\right) ds + \lambda_1(t)\, y_j(1) = 0 \tag{3.13.1}$$

where $j = 1, 2, ..., M$.

As initial conditions for r^M in (3.13.1) we take

$$\begin{aligned} r^M(s,0) &= \sum_{m=1}^{M} r_m(0)y_m(s) = \sum_{m=1}^{M} \int_a^1 p(\xi)y_m(\xi)\,d\xi\ y_m(s) =: \sum_{m=1}^{M} p_m y_m, \\ r_t^M(s,0) &= \sum_{m=1}^{M} r'_m(0)y_m(s) = \sum_{m=1}^{M} \int_a^1 q(\xi)y_m(\xi)\,d\xi\ y_m(s) =: \sum_{m=1}^{M} q_m y_m. \end{aligned} \tag{3.13.2}$$

Thus $r^M(s,0)$ and $r_t^M(s,0)$ are equal to the orthogonal projections of p and q onto span $\{y_1, ..., y_M\}$. By Bessel's inequality

$$\|r^M(\cdot,0)\| \leq \|p\|, \qquad \|r_t^M(\cdot,0)\| \leq \|q\|. \tag{3.13.3}$$

System (3.13.1), (3.13.2) is a well-defined initial-value problem for a finite-dimensional system of ordinary differential equations for the variables r_m. Equations (3.13.1) can be put into standard form in which each r''_m is expressed as a function of r_1,...,r_M and r'_1,..., r'_M because the associated mass matrix is positive-definite. Since $\hat{N}$, $\hat{T}$ are assumed to be continuously differentiable and since λ_1 is continuous in view of (3.4.6), the standard theory of ordinary differential equations [23] then implies that there is a $t^M > 0$ such that the initial-value problem (3.13.1), (3.13.2) has a unique classical solution on $[0, t^M]$. In Sec. 3.14 we get an energy estimate that will allow us to take $t^M = t^+$ for all M.

3.14 Global Existence of Solutions of the Approximating Ordinary Differential Equations

The convergence of the Faedo-Bubnov-Galerkin Method for our problem hinges on obtaining sharp estimates for (3.12.1), some of which are analogous to those obtained in Secs. 3.7 and 3.9. In all these estimates it is essential to note that the constant C and the function $t \mapsto \mathit{\Gamma}(t)$ are independent of M.

We multiply (3.13.1) by r_j' and then sum in j to get

$$\int_a^1 s^\gamma \left(\rho(s)\, r_{tt}^M r_t^M + N^M r_{st}^M + \frac{\gamma}{s} T^M r_t^M\right) ds + \lambda_1(t) r_t^M(1,t) = 0, \tag{3.14.1}$$

whence we obtain

$$\frac{d}{dt}\left[K^M + \mathit{\Phi}^M\right] + B^M + \int_a^1 \mathit{\Delta}^M \, ds = 0 \tag{3.14.2}$$

where in analogy with (3.5.51) and (3.7.1) we defined

$$\begin{aligned} \mathit{\Delta}^M &:= N_{\mathrm{D}}^M \nu_t + \gamma T_{\mathrm{D}}^M \tau_t, \\ \mathit{\Phi}^M &:= \int_a^1 s^\gamma \varphi^{(\gamma)}(\tau^M, \nu^M, s)\, ds, \\ K^M &:= \frac{1}{2}\int_a^1 \rho(s)\, s^\gamma \left(r_t^M\right)^2 \, ds, \\ B^M &:= \lambda_1(t) r_t^M(1,t). \end{aligned} \tag{3.14.3}$$

Integrating (3.7.5) from 0 to t yields the energy equation

$$K^M(t) + \mathit{\Phi}^M(t) + W^M(t) + \int_0^t B^M(\bar{t})\, d\bar{t} = K^M(0) + \mathit{\Phi}^M(0) \tag{3.14.4}$$

where

$$W^M := \int_0^t \int_a^1 s^\gamma \mathit{\Delta}^M(s,\bar{t})\, ds\, d\bar{t}. \tag{3.14.5}$$

In analogy with (3.7.5), we find

$$\mathit{\Delta}^M \ge c_{\mathrm{V}}\Big[\gamma\Big(\frac{r_t^M}{s}\Big)^2 + \left(r_{st}^M\right)^2\Big] \ge 0. \tag{3.14.6}$$

Then by using estimates analogous to (3.7.10), (3.7.11), we get

$$\left|\int_0^t B^M(\bar{t})\, d\bar{t}\right| \le \mathit{\Gamma}(t) + \varepsilon W^M(t), \tag{3.14.7}$$

which is similar to (3.7.12). From the second inequality in (3.13.3) we deduce that $K^M(0) \le C$ where C is independent of M. The uniform convergence of the sum in the first line of (3.13.2) implies that $\varPhi^M(0)$ is also bounded by a constant independent of M. Thus substituting (3.14.7) into (3.14.4) and choosing ε sufficiently small yields

$$K^M(t) + W^M(t) + \varPhi^M(t) \le \varGamma(t^+) \tag{3.14.8}$$

with $0 \le t \le t^+$. In particular,

$$\|r_t^M(\cdot,t)\|, \int_0^t \int_a^1 s^\gamma r_{st}^M(s,\bar{t})^2 \, d\bar{t} \, d\bar{t} \le \varGamma(t^+) \tag{3.14.9}$$

where we used the uniform monotonicity condition (3.11.5). Moreover, Hypothesis 3.5.6 and (3.14.8) imply that

$$|r^M(\cdot,t)| \le \varGamma(t^+). \tag{3.14.10}$$

Let us emphasize again that here and throughout this section $\varGamma$ is independent of M.

Thus $\sum_{m=1}^M r_m(t)^2$ and $\sum_{m=1}^M r_m'(t)^2$ are uniformly bounded for $t \in [0, t^+]$. The continuation theory of ordinary differential equations [23] now implies that (3.13.1), (3.13.2) has a solution defined for all $t \in [0, t^+]$ that satisfies (3.14.9).

3.15 Estimates of Higher Derivatives

A priori estimate for accelerations. Since $N^M(s,\cdot), T^M(s,\cdot)$ are continuously differentiable and since $\lambda_1 \in H^2_{\text{loc}}[0,\infty)$, we can differentiate (3.13.1) with respect to t:

$$\int_a^1 s^\gamma \left(\rho(s)\, r_{ttt}^M y_j + N_t^M y_j' + \frac{\gamma}{s} T_t^M y_j\right)\, ds + \lambda_1'(t)\, y_j(1) = 0 \tag{3.15.1}$$

where $j = 1, 2, ..., M$. We thus find that $r_{ttt}^M(s, \cdot)$ belongs to $L^2_{\text{loc}}[0, \infty)$. We multiply (3.15.1) by r_j'' and sum them in j to obtain that

$$\int_a^1 s^\gamma \left(N_t^M r_{stt}^M + \frac{\gamma}{s} T_t^M r_{tt}^M\right)\, ds + \frac{d}{dt}\left(\frac{1}{2}\int_a^1 s^\gamma \rho(s)\, \left(r_{tt}^M\right)^2\, ds\right) = -\lambda_1'(t) r_{tt}^M(1, t). \tag{3.15.2}$$

In imitation of (3.9.1) we define

$$H^M(t) := \frac{1}{2}\int_a^1 s^\gamma \rho(s)\, r_{tt}^M(s, t)^2\, ds + \int_0^t \int_a^1 s^\gamma [\tau_{tt}^M\ \nu_{tt}^M]\, \frac{\partial(N^M, \gamma T^M)}{\partial(\dot\nu^M, \dot\tau^M)} \begin{bmatrix} \tau_{tt}^M \\ \nu_{tt}^M \end{bmatrix} ds\, d\bar{t}. \tag{3.15.3}$$

By mimicking the proof of Theorem 3.9.5 we can get the estimate analogous to (3.9.6):

$$H^M(t) \le \mathit{\Gamma}(t^+) \quad \text{for} \quad 0 \le t \le t^+ \tag{3.15.4}$$

with $\mathit{\Gamma}(t^+)$ depending only on t^+, the constitutive functions, and the bounds for the data. In particular,

$$\|r_{tt}^M(\cdot, t)\| + \int_0^t \|r_{stt}^M(\cdot, \bar{t})\|^2\, d\bar{t} \le \mathit{\Gamma}(t^+) \quad \text{for} \quad 0 \le t \le t^+. \tag{3.15.5}$$

A priori estimate for $\|r_{st}^M(\cdot, t)\|$. As in Section 3.9, we obtain

$$\|r_{st}^M(\cdot, t)\| \le \mathit{\Gamma}(t^+)(1 + \sqrt{H^M}), \tag{3.15.6}$$

which, in view of (3.15.4), yields

$$\|r_{st}^M(\cdot, t)\| \le \mathit{\Gamma}(t^+). \tag{3.15.7}$$

A priori estimate for $\|r^M_{sst}\|$. To justify the convergence of our Galerkin approximations to the solution, we must obtain an estimate for $\|r^M_{sst}\|$. Toward this end, we multiply (3.13.1) by $\kappa_j^2 r_j'$ with $\kappa_j := (1-a)^{-1}\theta_j$ (θ_j were defined in (3.12.2)), use the identity $y_j'' = -\kappa_j^2 y_j$, sum the product over j from 1 to M, and integrate the sum by parts to obtain

$$\begin{aligned}\int_a^1 s^\gamma \rho(s)\, r^M_{sst} r^M_{tt}\, ds = \int_a^1 \left[\frac{d}{ds}(s^\gamma N^M)\nu^M_{st} + \gamma \frac{d}{ds}\left(s^{\gamma-1}T^M\right)(s\tau^M_t)_s\right] ds \\ - s^{\gamma-1}T^M r^M_{st}\Big|_{s=a}^{s=1}.\end{aligned} \tag{3.15.8}$$

Since

$$\frac{dN^M}{ds} = N^M_\tau \tau^M_s + N^M_\nu \nu^M_s + N^M_{\dot\tau}\tau_{st} + N^M_{\dot\nu}\nu^M_{st} + N^M_s, \text{ etc.} \tag{3.15.9}$$

then we obtain from (3.15.8)

$$\begin{aligned}&\int_a^1 s^\gamma \rho(s)\, r^M_{sst} r^M_{tt}\, ds + \int_a^1 s^\gamma [\tau^M_{st}\ \nu^M_{st}]\,\frac{\partial(N^M, \gamma T^M)}{\partial(\dot\nu^M, \dot\tau^M)}\begin{bmatrix}\tau^M_{st}\\ \\ \nu^M_{st}\end{bmatrix} ds \\ &+ \int_a^1 \gamma s^{\gamma-1}\nu_t\left(T^M_s + T^M_\nu \nu_s + T_\tau \tau_s + (\gamma-1)s^{-1}T^M\right) + \gamma\frac{d}{ds}\left(s^{\gamma-1}T^M\right)\tau_t\, ds \\ &+ \int_a^1 s^\gamma \nu^M_{st}\left(N^M_s + N^M_\nu \nu_s + N_\tau \tau_s\right) + \gamma s^{\gamma-1}N\nu_{st}\, ds - s^{\gamma-1}T^M r^M_{st}\,\Big|_{s=a}^{s=1} = 0.\end{aligned} \tag{3.15.10}$$

By (3.11.5) we have

$$c\,\|r^M_{sst}(\cdot,t)\|^2 \le \int_a^1 s^\gamma [\tau^M_{st}\ \nu^M_{st}]\,\frac{\partial(N^M, \gamma T^M)}{\partial(\dot\nu^M, \dot\tau^M)}\begin{bmatrix}\tau^M_{st}\\ \\ \nu^M_{st}\end{bmatrix} ds. \tag{3.15.11}$$

To estimate the first term on the left-hand side of (3.15.10) we use (3.15.5) and the fact that ρ is assumed to be bounded from above:

$$\int_a^1 s^\gamma \rho(s)\, r^M_{sst} r^M_{tt}\, ds \le \varepsilon\|r_{sst}(\cdot,t)\|^2 + C\|r_{tt}(\cdot,t)\|^2 \le \varepsilon\|r_{sst}(\cdot,t)\|^2 + \Gamma(t^+). \tag{3.15.12}$$

We use the energy estimate (3.14.8), the regularity estimates (3.11.4) and the pointwise bounds

$$
\begin{aligned}
|r_s^M(s,t)| &\le C + \int_0^t \|r_{sst}^M(\cdot,t)\|, \\
|r_{st}^M(s,t)| \le C + \|r_{sst}(\cdot,t)\| &\le C + \varepsilon \|r_{sst}^M(\cdot,t)\|^2, \\
\|r_{ss}^M(\cdot,t)\|^2 &\le C + \int_0^t \|r_{sst}^M(\cdot,\bar{t})\|^2 \, d\bar{t}.
\end{aligned} \tag{3.15.13}
$$

valid for all $s \in [a,1]$ and $t \ge 0$, to estimate the terms in the last two lines of (3.15.10):

$$
\begin{aligned}
\left| \int_a^1 s^\gamma \dot{\nu}_s^M N_s^M \, ds \right| \equiv \left| \int_a^1 s^\gamma r_{sst}^M N_s^M \, ds \right| &\le \varepsilon \|r_{sst}^M\|^2 + \Gamma(t^+), \\
\left| \int_a^1 \gamma s^{\gamma-1} T_s^M r_{st}^M \, ds \right| &\le \Gamma(t^+) + \varepsilon \|r_{sst}^M\|^2, \\
\left| \int_a^1 s^\gamma N_\nu^M \nu_s \dot{\nu}_s^M \right| &\le \Gamma(t^+) \|r_{sst}^M\| \sqrt{C + \int_0^t \|r_{sst}^M(\cdot,\bar{t})\|^2 \, d\bar{t}} \\
&\le \varepsilon \|r_{sst}^M\|^2 + \Gamma(t^+) \left(1 + \int_0^t \|r_{sst}(\cdot,\bar{t})\|^2 \, d\bar{t} \right).
\end{aligned} \tag{3.15.14}
$$

The remaining terms can be shown to have similar bounds. By substituting these bounds into (3.15.10) and choosing ε sufficiently small we get

$$
\|r_{sst}^M(\cdot,t)\|^2 \le \Gamma(t^+) + \Gamma(t^+) \int_0^t \|r_{sst}(\cdot,\bar{t})\|^2 \, d\bar{t}, \tag{3.15.15}
$$

so that the Gronwall inequality implies that

$$
\|r_{sst}^M(\cdot,t)\| \le \Gamma(t^+) \quad \forall\, t \in [0,t^+]. \tag{3.15.16}
$$

Further estimates on higher derivatives. We now establish further estimates that will allow us to prove higher regularity of the solution.

We first prove that the solution $(r_1, ..., r_M)$ to the system (3.13.1), (3.13.2) enjoys more regularity. Replace r^M in (3.15.1) with an unknown function of the form $\tilde{r}^M := \sum_{m=1}^M \tilde{r}_m(t) y_m(s)$ and consider the system of equations

$$\int_a^1 s^\gamma \left(\rho(s)\,\tilde{r}^M_{ttt} y_j + N^M_t y_j' + \frac{\gamma}{s} T^M_t y_j\right)\, ds + \lambda_1'(t)\, y_j(1) = 0 \qquad (3.15.17)$$

where $j = 1, 2, ..., M$. Equations (3.15.17) can be put into standard form in which each $\tilde{r}_m'''$ is expressed as a function of $\tilde{r}_1$,...,$\tilde{r}_M$, $\tilde{r}_1'$,..., $\tilde{r}_M'$, and $\tilde{r}_1''$,..., $\tilde{r}_M''$.

We supplement (3.15.17) with initial conditions that are obtained from (3.13.1) and (3.13.2):

$$\begin{aligned}
&\tilde{r}^M(s,0) \equiv \sum_{m=1}^M \tilde{r}_m(0) y_m(s) = \sum_{m=1}^M p_m y_m,\\
&\tilde{r}^M_t(s,0) \equiv \sum_{m=1}^M \tilde{r}_m'(0) y_m(s) = \sum_{m=1}^M q_m y_m,\\
&\int_a^1 s^\gamma \left(\rho(s)\,\tilde{r}^M_{tt}(s,0) y_j + \hat{N}(s^{-1}p, p', s^{-1}q, q', s) y_j'\right)\, ds\\
&+\int_a^1 \gamma s^{\gamma-1} \hat{T}(s^{-1}p, p', s^{-1}q, q', s) y_j\, ds + \lambda_1(0)\, y_j(1) = 0.
\end{aligned} \qquad (3.15.18)$$

$(3.15.18)_3$ can be put into form in which each $\tilde{r}_m''$ is expressed in terms of the data.

System (3.15.17), (3.15.18) is a well-defined initial-value problem for a finite-dimensional system of ordinary differential equations for the variables $\tilde{r}_m$. The estimates (3.14.8), (3.15.5) and the standard theory of ordinary differential equations imply that the system has a unique classical solution $(\tilde{r}_1, ..., \tilde{r}_M)$. By integrating (3.15.17) from 0 to t and taking into account $(3.15.18)_3$, we find that the functions $\tilde{r}_1, ..., \tilde{r}_M$ satisfy the equations (3.13.1). Since these functions also satisfy the initial conditions (3.13.2), we conclude that they can be identified with $r_1, ..., r_M$. We thus have established higher regularity for $(r_1, ..., r_M)$.

Since $N^M(s, \cdot), T^M(s, \cdot)$ are continuously differentiable and since $\lambda_1 \in H^2_{\text{loc}}[0, \infty)$, we can differentiate (3.15.1) with respect to t, finding that $r^M_{tttt}(s, \cdot)$ belongs to

$L^2_{\mathrm{loc}}[0,\infty)$. We multiply the resulting equations by r_j''' and sum them in j to obtain

$$\int_a^1 s^\gamma \left(N_{tt}^M r_{sttt}^M + \frac{\gamma}{s} T_{tt}^M r_{ttt}^M \right) ds + \frac{d}{dt}\Big(\frac{1}{2} \int_a^1 s^\gamma \rho(s)\, r_{ttt}^M(s,t)^2\, ds \Big) = -\lambda_1''(t) r_{ttt}^M(1,t). \tag{3.15.19}$$

In analogy with (3.15.3) we define

$$J^M(t) := \frac{1}{2} \int_a^1 s^\gamma \rho(s)\, r_{ttt}^M(s,t)^2\, ds + \int_0^t \int_a^1 s^\gamma [\tau_{ttt}^M\ \nu_{ttt}^M]\, \frac{\partial(N^M, T^M)}{\partial(\dot\nu^M, \dot\tau^M)} \begin{bmatrix} \tau_{ttt}^M \\ \\ \nu_{ttt}^M \end{bmatrix} ds\, d\bar t. \tag{3.15.20}$$

By using the ideas similar to those of Theorem 3.9.5 along with the estimate (3.15.5), pointwise estimates like $(3.15.13)_2$ and the regularity estimates (3.11.4), we get from (3.15.19) that

$$J^M(t) \le \mathit{\Gamma}(t^+) \quad \text{for} \quad 0 \le t \le t^+. \tag{3.15.21}$$

In particular,

$$\|r_{ttt}^M(\cdot,t)\| + \int_0^t \|r_{sttt}^M(\cdot,\bar t)\|^2\, d\bar t \le \mathit{\Gamma}(t^+) \quad \text{for} \quad 0 \le t \le t^+. \tag{3.15.22}$$

Finally by analogy with (3.15.7) and (3.15.16) we obtain that

$$\|r_{stt}^M(\cdot,t)\|,\ \|r_{sstt}^M(\cdot,t)\| \le \mathit{\Gamma}(t^+) \quad \forall\, t \in [0,t^+]. \tag{3.15.23}$$

3.16 Convergence and Regularity

Convergence. We now show that the bounds obtained in Sec. 3.14, 3.15 support compactness properties for the Galerkin approximations so strong that these approximations converge to the solution of the initial-boundary-value problem without appeal to the theory of monotone operators to handle the weak convergence of composite functions [37].

We invoke a generalization of a compactness lemma of Aubin [12, 44]:

3.16.1. Lemma. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Banach spaces of functions with $\mathcal{X}$ compactly embedded in $\mathcal{Y}$ and $\mathcal{Y}$ compactly embedded in $\mathcal{Z}$. Let $\mathcal{C}$ be a set of functions w for which w_t lies in a bounded subset of $L^p(0, t^+, \mathcal{Z})$ with $p > 1$ and for which w lies in a bounded subset of $L^\infty(0, t^+, \mathcal{X})$. Then $\mathcal{C}$ lies in a compact subset of $C^0(0, t^+, \mathcal{Y})$.*

Let $\mathcal{X} = H^1(a, 1)$, $\mathcal{Y} = C^0[a, 1]$, $\mathcal{Z} = L^2(a, 1)$, so that $\mathcal{X}$ is compactly embedded in $\mathcal{Y}$ and $\mathcal{Y}$ is embedded in $\mathcal{Z}$. Then the estimates (3.15.7), (3.15.16) ensure that r_{st}^M lies a bounded subset of $L^\infty(0, t^+, \mathcal{X})$. The estimate (3.15.5) implies that r_{stt}^M lies a bounded subset of $L^2(0, t^+, \mathcal{Z})$. Thus Lemma 3.16.1 implies that

$$r_{st}^M \text{ lies in a compact subset of } C^0(0, t^+, \mathcal{Y}) \equiv C^0([a, 1] \times [0, t^+]), \tag{3.16.2}$$

so that

$$\begin{aligned} &r_{st}^M \text{ has a subsequence converging uniformly} \\ &\text{to a continuous limit } (r_{st})^\infty \text{ on } [0, 1] \times [0, t^+]. \end{aligned} \tag{3.16.3}$$

Since Hypothesis 3.4.8 implies that the initial values of r_s are continuous, it follows from the Fundamental Theorem of Calculus and the Arzelà-Ascoli Theorem that

$$\begin{aligned} &r_s^M \text{ itself lies in a compact subset of } C^0([a, 1] \times [0, t^+]), \\ &\text{and converges uniformly to a continuous limit } (r_s)^\infty, \end{aligned} \tag{3.16.4}$$

whose t-derivative is $(r_{st})^\infty$. Since $r(a, \cdot)$ is prescribed in (3.4.5), it likewise follows from (3.16.3) and the Arzelà-Ascoli Theorem that

$$\begin{aligned} &r_t^M \text{ itself lies in a compact subset of } C^0([a, 1] \times [0, t^+]), \\ &\text{and converges uniformly to a continuous limit } r_t^\infty, \end{aligned} \tag{3.16.5}$$

whose s-derivative is $(r_{st})^\infty$. (If $r(a,\cdot)$ were not prescribed we could use the functional defined in (3.5.10) and the Mean-Value Theorem for Integration, and again invoke the Arzelà-Ascoli Theorem.) Similarly it follows that

$$\begin{aligned} &r^M \text{ lies in a compact subset of } C^0([a,1]\times[0,t^+]), \\ &\text{and converges uniformly to a continuous limit } r^\infty, \end{aligned} \tag{3.16.6}$$

whose s-derivative is $(r_s)^\infty$ and whose t-derivative is $(r_t)^\infty$. We then can identify $(r^\infty)_s$ with r_s^∞, etc. Since $N(\cdot,\cdot,\cdot,\cdot,s)$, $T(\cdot,\cdot,\cdot,\cdot,s)$ are continuous then N^M, T^M must converge to the continuous limits $N^\infty := \hat{N}(s^{-1}r^\infty, r_s^\infty, s^{-1}r_t^\infty, r_{st}^\infty, s)$, $T^\infty := \hat{T}(s^{-1}r^\infty, r_s^\infty, s^{-1}r_t^\infty, r_{st}^\infty, s)$ as $M \to \infty$.

From (3.15.5) we obtain that r_{tt}^M is a bounded sequence in the reflexive space $L^2((a,1)\times(0,t^+))$ and accordingly by the Banach-Alaoglu Theorem has a weakly convergent subsequence (denoted in the same way as the parent sequence):

$$r_{tt}^M \rightharpoonup (r_{tt})^\infty \quad \text{in} \quad L^2((a,1)\times(0,t^+)). \tag{3.16.7}$$

Since r_t^M converges uniformly to r_t^∞, it follows that $(r_{tt})^\infty$ is the distributional t-derivative of r_t^∞, i.e., $(r_{tt})^\infty = r_{tt}^\infty$ in the sense of distributions. We accordingly drop the parentheses in (3.16.7).

Regularity. Let Y_1, Y_2,... be arbitrary functions of t in $C^1[0,t^+]$. We multiply (3.13.1) by Y_j, sum in j from 1 to L and then integrate with respect to t to find that

$$\int_0^t \int_a^1 s^\gamma \left(\rho(s)\, r_{tt}^M Y + N^M Y_s + \frac{\gamma}{s} T^M Y\right) ds\, d\bar{t} + \lambda_1(t) Y(1,t) = 0 \tag{3.16.8}$$

for all Y having the form

$$Y(s,t) = \sum_{j=1}^{L} Y_j(t) y_j(s). \tag{3.16.9}$$

But such finite sums are dense in $L^2(0, t^+; \mathfrak{W})$ and so (3.16.8) holds for all $Y \in L^2(0, t^+; \mathfrak{W})$. ($\mathfrak{W}$ was defined in (3.4.10)).

We use (3.16.2)–(3.16.7) to take the subsequential limit of (3.16.8) as $M \to \infty$:

$$\int_0^t \int_a^1 s^\gamma \left(\rho(s)\, r_{tt}^\infty Y + N^\infty Y_s + \frac{\gamma}{s} T^\infty Y \right) ds\, d\bar{t} + \lambda_1(t) Y(1,t) = 0. \tag{3.16.10}$$

By using the arbitrariness of Y in (3.16.10) (i.e., by using a version of the Fundamental Lemma of the Calculus of Variations), we readily find that r^∞ satisfies the weak formulation (3.4.11) for almost every $t \in [0, t^+]$.

The method for showing that r^∞ satisfies the boundary and initial conditions is standard and is accordingly omitted [28, 37, 57]. We thus have proved the existence part of Theorem 3.6.1.

We then can use the estimates (3.15.22), (3.15.23) to prove that the solution actually enjoys more regularity as asserted in Theorem 3.6.1.

3.17 Continuous Dependence on the Data. Uniqueness

In this section we show that solutions depend continuously on the initial and boundary data, from which it follows that solutions are unique.

We continue to treat the boundary conditions (3.4.5). We examine the difference of solutions of two problems for the same material in which nonzero initial and boundary data are distinguished by superscripts 1 and 2. We denote the corresponding solutions and the constitutive functions evaluated at these solutions by the same superscripts: r^1 and r^2, etc. and set

$$\delta r := r^1 - r^2, \quad \delta N := N^1 - N^2, \quad \delta\tau := \tau^1 - \tau^2, \text{ etc.} \tag{3.17.1}$$

In the weak formulation (3.4.11), we replace r with δr and y with δr_t to obtain

$$\int_a^1 s^\gamma \left(\rho(s)\delta r_{tt}\delta r_t + \delta N \delta r_{ts} + \frac{\gamma}{s}\delta T \delta r_t\right) ds - s^\gamma \delta N \delta r_t\Big|_{s=a}^{s=1} = 0. \tag{3.17.2}$$

We integrate (3.17.2) from 0 to t to obtain

$$\begin{aligned}\frac{1}{2}\int_a^1 s^\gamma \rho(s)|\delta r_t|^2\, ds + \int_0^t \int_a^1 s^\gamma \left(\delta N \delta r_{ts} + \frac{\gamma}{s}\delta T \delta r_t\right) ds\, d\bar{t} \\ = \frac{1}{2}\int_a^1 s^\gamma \rho(s)|\delta q|^2\, ds - \int_0^t \delta\lambda_1 \delta r_t(1,\cdot)\, d\bar{t}.\end{aligned} \tag{3.17.3}$$

Note

$$\begin{aligned}\int_0^t \delta\lambda_1 \delta r_t(1,\cdot)\, d\bar{t} &\equiv \int_0^t \delta\lambda_1 \int_a^1 \delta r_{st}(s,\cdot)\, ds\, d\bar{t} \\ &\le \left(\int_0^t (\delta\lambda_1)^2\, d\bar{t}\right)^{1/2} \left(\int_0^t \int_a^1 (\delta r_{st})^2(s,\cdot)\, ds\, d\bar{t}\right)^{1/2} \\ &\le C\int_0^t (\delta\lambda_1)^2\, d\tau + \varepsilon \int_0^t \int_a^1 (\delta\nu_t)^2(s,\cdot)\, ds\, d\bar{t}.\end{aligned} \tag{3.17.4}$$

For α, β ranging over $1, 2$, set

$$\begin{aligned}N^{\alpha\beta} &:= N(\tau^\alpha, \nu^\alpha, \dot{\tau}^\beta, \dot{\nu}^\beta, \cdot), \quad T^{\alpha\beta} := T(\tau^\alpha, \nu^\alpha, \dot{\tau}^\beta, \dot{\nu}^\beta, \cdot), \\ N_\tau^\beta &:= \int_0^1 N_\tau(\mu\tau^1 + (1-\mu)\tau^2, \mu\nu^1 + (1-\mu)\nu^2, \dot{\tau}^\beta, \dot{\nu}^\beta, \cdot)\, d\mu, \text{ etc.}\end{aligned} \tag{3.17.5}$$

The Mean-Value Theorem and the monotonicity condition (3.5.15) imply that

$$\begin{aligned}\delta N \delta r_{ts} + \frac{\gamma}{s}\delta T \delta r_t &\equiv (N^1 - N^2)\delta\nu_t + \gamma(T^1 - T^2)\delta\tau_t \\ &= (N^{11} - N^{12} + N^{12} - N^{22})\delta\nu_t + \gamma(T^{11} - T^{12} + T^{12} - T^{22})\delta\tau_t \\ &\ge c_{\mathrm{V}}\left(|\delta\nu_t|^2 + |\delta\tau_t|^2\right) + (N_\tau^2\delta\tau + N_\nu^2\delta\nu)\delta\nu_t + \gamma(T_\tau^2\delta\tau + T_\nu^2\delta\nu)\delta\tau_t \\ &\ge (c_{\mathrm{V}} - \varepsilon)\left(|\delta\nu_t|^2 + |\delta\tau_t|^2\right) - C(|\delta\nu|^2 + |\delta\tau|^2).\end{aligned} \tag{3.17.6}$$

where to obtain the last inequality we used the estimates like

$$\left|N_\tau^2 \delta\tau \delta\nu_t\right| \leq C|\delta\tau|^2 + \varepsilon|\delta\nu_t|^2. \tag{3.17.7}$$

Thus using (3.17.3), (3.17.4) and (3.17.6) we get

$$\begin{aligned}\frac{1}{2}\int_a^1 s^\gamma \rho(s)|\delta r_t|^2\, ds + (c_{\mathrm{V}} - \varepsilon)\int_0^t \int_a^1 s^\gamma \left(|\delta\nu_t|^2 + |\delta\tau_t|^2\right)\, ds\, d\bar{t} \\ \leq \frac{1}{2}\int_a^1 s^\gamma \rho(s)|\delta q|^2\, ds + C\int_0^t \int_a^1 \left(|\delta\nu|^2 + |\delta\tau|^2\right)\, ds\, d\bar{t} + C\int_0^t (\delta\lambda_1)^2\, d\bar{t}.\end{aligned} \tag{3.17.8}$$

Choosing ε sufficiently small in (3.17.8) and using (Poincaré-type) inequalities

$$|\delta\nu(s,t)| \leq |\delta p'| + \sqrt{t\int_0^t |\delta\nu_t(s,\bar{t})|^2\, d\bar{t}}, \quad |\delta\tau(s,t)| \leq \left|\frac{\delta p(s)}{s}\right| + \sqrt{t\int_0^t |\delta\tau_t(s,\bar{t})|^2\, d\bar{t}} \tag{3.17.9}$$

we obtain

$$\begin{aligned}\int_0^t \int_a^1 \left(|\delta\nu_t|^2 + |\delta\tau_t|^2\right)\, ds\, d\bar{t} \leq \mathit{\Gamma}(t)\int_0^t \int_a^1 \int_0^{\bar{t}} \left(|\delta\nu_t(s,\zeta)|^2 + |\delta\tau_t(s,\zeta)|^2\right)\, d\zeta\, ds\, d\bar{t} \\ + C\int_a^1 \left(|\delta q|^2 + |\delta p'|^2 + |\delta p|^2\right)\, ds + C\int_0^t (\delta\lambda_1)^2\, d\bar{t}.\end{aligned} \tag{3.17.10}$$

The Gronwall inequality then imples that

$$\int_0^t \int_a^1 \left(|\delta\nu_t|^2 + |\delta\tau_t|^2\right)\, ds\, d\bar{t} \leq C\int_a^1 \left(|\delta q|^2 + |\delta p'|^2 + |\delta p|^2\right)\, ds + C\int_0^t (\delta\lambda_1)^2\, d\bar{t}. \tag{3.17.11}$$

This inequality gives the continuous dependence with respect to the norms that intervene here. Obviously, the vanishing of the left-hand side of (3.17.11) when the right-hand side vanishes ensures uniqueness.

3.18 The Treatment of Other Boundary Conditions

3.18.1 Energy Estimates

We now obtain energy estimates like those of Sec. 3.7 for other boundary conditions. We consider only several illuminating cases, for which we modify the treatment of Theorem 3.7.6 to obtain a tractable bound on the absolute value of the integral of the boundary term B defined in (3.7.1).

By the methods that we use here we can obtain energy estimates for any boundary conditions except for time-dependent double Dirichlet conditions (where position is prescribed on both boundaries). For this trickier case we shall need to obtain energy estimates simultaneously with obtaining a priori bounds on the strains. That this case require special treatment is not surprising: If the material strongly resists having its length changed, then changing its length against a large resistance might cause the generation of so much work that the boundary power term B in the energy equation (3.7.2) cannot be controlled. We postpone the treatment of such trickier cases to Sec. 3.18.2.

Dead loads both on $s = a$ and $s = 1$. Suppose that both the inner and outer boundary are subjected to prescribed forces:

$$N(a, \cdot) \text{ and } N(1, \cdot) \text{ are prescribed in } L^2_{\text{loc}}[0, \infty). \tag{3.18.1}$$

We use (3.8.1) to obtain

$$\begin{aligned}\int_0^t N(1,\bar t) r_t(1,\bar t)\,d\bar t &\le C\int_0^t N(1,\bar t)^2\,d\bar t + \varepsilon\int_0^t r_t(1,\bar t)^2\,d\bar t\\ &\le C + C\int_0^t K(\bar t)\,d\bar t + \varepsilon W(t).\end{aligned}\tag{3.18.2}$$

The boundary term $\int_0^t N(a,\bar t) r_t(a,\bar t)\,d\bar t$ can be estimated in the same way.

Hydrostatic loads both on $s = a$ and $s = 1$. Now suppose the inner surface $s = a$ is subjected to a hydrostatic pressure $\lambda_a(t)$ and the outer surface $s = 1$ is subjected to a hydrostatic pressure $\lambda_1(t)$:

$$N(a,t) = -\lambda_a(t) r^\gamma(a,t), \qquad N(1,t) = -\lambda_1(t) r^\gamma(1,t) \tag{3.18.3}$$

with $\lambda_1, \lambda_a \in L^\infty_{\text{loc}}[0,\infty)$. We focus on obtaining tractable bounds for the term $\int_0^t \lambda_1(\bar t) r^\gamma(1,\bar t) r_t(1,\bar t)\,d\bar t$. The bounds for $\int_0^t \lambda_1(\bar t) r^\gamma(a,\bar t) r_t(a,\bar t)\,d\bar t$ can be obtained in a similar way. Using that $\lambda_1 \in L^\infty_{\text{loc}}[0,\infty)$ we get

$$\begin{aligned}\left|\int_0^t \lambda_1(\bar t) r^\gamma(1,\bar t) r_t(1,\bar t)\,d\bar t\right| &\le C\left|\int_0^t r^\gamma(1,\bar t) r_t(1,\bar t)\,d\bar t\right|\\ &\le C\int_0^t r(1,\bar t)^{2\gamma}\,d\bar t + \varepsilon\int_0^t r_t(1,\bar t)^2\,d\bar t.\end{aligned}\tag{3.18.4}$$

Case $\gamma = 1$. We first consider the case $\gamma = 1$. We estimate the integral of $\Pi_1(t) := r(1,t)^2$:

$$\begin{aligned}\Pi_1(t) &= \Pi_1(0) + 2\int_0^t r(1,\bar t) r_t(1,\bar t)\,d\bar t\\ &\le C + C\sqrt{\int_0^t \Pi_1(\bar t)\,d\bar t}\sqrt{\int_0^t r_t(1,\bar t)^2\,d\bar t}\\ &\le C + C\int_0^t \Pi_1(\bar t)\,d\bar t + \varepsilon\int_0^t r_t(1,\bar t)^2\,d\bar t\\ &\le C + C\int_0^t \Pi_1(\bar t)\,d\bar t + C\int_0^t K(\bar t)\,d\bar t + \varepsilon W(t)\end{aligned}\tag{3.18.5}$$

where to obtain the last inequality we used (3.8.1). We apply the Gronwall inequality to (3.18.5), obtaining

$$\int_0^t \Pi_1(\bar{t})\,d\bar{t} \le \Gamma(t^+) + C\int_0^t K(\bar{t})\,d\bar{t} + CW(t). \tag{3.18.6}$$

Thus by combining (3.18.4) (with $\gamma = 1$) and (3.18.6), and by using again (3.8.1), we obtain a tractable bound on the boundary term at $s = 1$:

$$\left|\int_0^t \lambda_1(\bar{t}) r^\gamma(1,\bar{t}) r_t(1,\bar{t})\,d\bar{t}\right| \le \Gamma(t^+) + \Gamma(t^+)\int_0^t K(\bar{t})\,d\bar{t} + C\int_0^t W(\bar{t})\,d\bar{t}. \tag{3.18.7}$$

Case $\gamma = 2$. To handle the case $\gamma = 2$ and to get a slicker derivation of a bound for Π_1 we use Hypothesis 3.5.6b (with $\alpha_1, \alpha_2 > 2\gamma = 4$). Then (3.5.7) and its consequence, inequality (3.5.12), yield

$$\int_0^t r(1,\bar{t})^{2\gamma}\,d\bar{t} \le \int_0^t \Phi(\bar{t})\,d\bar{t} + \Gamma(t^+) \tag{3.18.8}$$

provided $\alpha_1, \alpha_2 > 2\gamma$. We then substitute (3.18.8) into (3.18.4) to obtain a tractable bound on $\int_0^t \lambda_1(\bar{t}) r^\gamma(1,\bar{t}) r_t(1,\bar{t})\,d\bar{t}$.

We can get by with $\alpha_1, \alpha_2 > \gamma + 1$ if λ_1 is independent of time. In that case

$$\begin{aligned} \left|\int_0^t \lambda_1 r^\gamma(1,\bar{t}) r_t(1,\bar{t})\,d\bar{t}\right| &\le \left|\lambda_1 \int_0^t r^\gamma(1,\bar{t}) r_t(1,t)\,d\bar{t}\right| \\ &\le C\left(r^{\gamma+1}(1,t) + 1\right). \end{aligned} \tag{3.18.9}$$

Hypothesis 3.5.6 with $\alpha_1, \alpha_2 > \gamma + 1$ then ensures that the last term in the last line of (3.18.4) is bounded by $\varepsilon\Phi(t) + C$. In this case the bounds that we obtain are independent of t^+.

These calculations lead us to the following analog of Theorem 3.7.6 for boundary conditions (3.18.3):

3.18.10. Theorem. *Let* (3.5.14) *hold (so that* (3.5.16) *holds). Let boundary conditions* (3.18.3) *hold with* $\lambda_1, \lambda_a \in L^\infty_{loc}[0,\infty)$*, and let the initial conditions satisfy the Hypothesis 3.4.8. Let the material be sufficiently strong in resisting extension in the sense that Hypothesis 3.5.6b holds. Let* t^+ *be any positive number. Then the energy estimate*

$$K(t) + \varPhi(t) + W(t) \leq \varGamma(t^+) \tag{3.18.11}$$

holds for $0 \leq t \leq t^+$ *with* $\varGamma(t^+)$ *depending only on* t^+*, the constitutive functions, and the bounds for the data. In particular,*

$$\|r_t(\cdot,t)\|, \ \int_0^t \|r_{st}(\cdot,\bar{t})\|^2 \, d\bar{t}, \ |r(\cdot,t)| \leq \varGamma(t^+). \tag{3.18.12}$$

Moreover if λ_1*,* λ_a *are independent of* t*, then then the bounds in* (3.18.11)*,* (3.18.12) *are ensured by a weaker version of Hypothesis 3.5.6b with* $\alpha_1, \alpha_2 > \gamma + 1$ *and these bounds are independent of* t^+*.*

Time-dependent position on $s = a$ and dead load on $s = 1$. Suppose that

$$r(a,\cdot) \text{ is prescribed in } W^{2,\infty}_{\text{loc}}[0,\infty) \text{ and } N(1,\cdot) \text{ is prescribed in } L^2_{\text{loc}}[0,\infty) \tag{3.18.13}$$

with

$$r(a,t) > 0 \quad \forall\, t \in [0,t^+]. \tag{3.18.14}$$

By a simple modification of the argument of Theorem 3.7.6 we find that the term $\int_0^t N(1,\bar{t}) r_t(1,\bar{t}) \, d\bar{t}$ is bounded by $\varGamma(t^+) + \varepsilon W(t)$.

To handle the term $\int_0^t N(a,\bar{t}) r_t(a,\bar{t}) \, d\bar{t}$, use the governing equation (3.3.3) to

obtain

$$
\begin{aligned}
\int_0^t N(a,\bar{t}) r_t(a,\bar{t})\, d\bar{t} &= a^{-\gamma} \int_0^t N(1,\bar{t}) r_t(a,\bar{t})\, d\bar{t} \\
&\quad - a^{-\gamma} \int_0^t \int_a^1 r_t(a,\bar{t}) \left(\gamma s^{\gamma-1} \hat{T} + s^\gamma \rho(s) r_{tt}(s,\bar{t}) \right) ds\, d\bar{t} \\
&= a^{-\gamma} \int_0^t N(1,\bar{t}) r_t(a,\bar{t})\, d\bar{t} + a^{-\gamma} \int_0^t \int_a^1 s^\gamma \rho(s) r_{tt}(a,\bar{t}) r_t(s,\bar{t})\, ds\, d\bar{t} \\
&\quad - a^{-\gamma} \int_a^1 s^\gamma \rho(s) r_t(a,\bar{t}) r_t(s,\bar{t})\, ds \Big|_{\bar{t}=0}^{\bar{t}=t} - a^{-\gamma} \int_0^t \int_a^1 r_t(a,\bar{t}) \gamma s^{\gamma-1} \hat{T}\, ds\, d\bar{t}.
\end{aligned}
\tag{3.18.15}
$$

By the techniques we have been using, we readily find that the first three terms on the extreme right-hand side of (3.18.15) are dominated by the tractable bound $\varGamma(t^+) + \varepsilon K(t) + C \int_0^t K(\bar{t})\, d\bar{t}$. To get a tractable bound for the last term on the right-hand side of (3.18.15), we use Hypotheses 3.5.6, 3.5.40, and 3.5.46a:

$$
\begin{aligned}
\left| \int_0^t \int_a^1 r_t(a,\bar{t}) \gamma s^{\gamma-1} \hat{T}\, ds\, d\bar{t} \right| &\le C \int_0^t \int_a^1 |\hat{T}|\, ds\, d\bar{t} \\
&\le \varGamma(t^+) + C \int_0^t \varPhi(\bar{t})\, d\bar{t} + C \int_0^t K(\bar{t})\, d\bar{t}.
\end{aligned}
\tag{3.18.16}
$$

We thus establish an energy estimate analogous to (3.7.8).

3.18.2 Preclusion of total compression and infinite extension

In Sec. 3.9 we established a strengthened version of (3.3.6) for the case of boundary conditions (3.4.5). We now discuss how to obtain analogs of Theorem 3.8.11 for other boundary conditions. As remarked at the beginning of Sec. 3.18.1, for the case of time-dependent double Dirichlet conditions we need to obtain energy estimates simultaneously with obtaining positive lower and upper bounds on the strain variables. In this section we focus only on this trickier case. Other cases can be treated by the methods of Section 3.8.

Position conditions both on $s = a$ and $s = 1$. We now outline the treatment of the problem with double Dirichlet boundary conditions:

$$r(a,\cdot) \text{ is prescribed in } W^{2,\infty}_{\text{loc}}[0,\infty) \text{ and } r(1,\cdot) \text{ is prescribed in } W^{2,\infty}_{\text{loc}}[0,\infty) \tag{3.18.17}$$

with

$$0 < a^{-1}r(a,t) < r(1,t) \quad \forall t. \tag{3.18.18}$$

Conditions (3.18.17) imply that τ has positive lower and upper bounds. We thus only need to obtain appropriate bounds for ν. The treatment parallels that of the one-dimensional problem of [9], pp. 154 – 159. A novelty here is that we use our elegant constitutive Hypotheses 3.5.8, 3.5.40, 3.5.42 instead of the more complicated hypotheses used in [9]. We supplement our hypotheses with the assumption that there is a function $D \in L^{\infty}_{\text{loc}}[0,\infty)$ such that

$$D(t) \geq |r_t(1,t) - r_t(a,t)|. \tag{3.18.19}$$

We shall obtain additional restrictions on D which would give an upper bound for

$$\psi^{\sharp}(t) := \max\{\psi(\nu(s,\bar{t})) : a \leq s \leq 1,\ 0 \leq \bar{t} \leq t\} \tag{3.18.20}$$

and thus provide appropriate bounds for ν.

Step 1. We need a useful integral representation for $N(s,t)$. Let us replace s in (3.3.3) with σ. We multiply (3.3.3) by $\sigma - a$ and integrate the resulting equation with respect to σ over $[a, s]$. Next, we multiply (3.3.3) by $\sigma - 1$ and integrate the resulting equation with respect to σ over $[s, 1]$. We add the resulting equations to

obtain

$$
\begin{aligned}
s^\gamma(1-a)N(s,t) = \int_a^1 \sigma^\gamma \left(N(\sigma,t) + \gamma\sigma^{\gamma-1} G(s,\sigma)\, T(\sigma,t) \right) d\sigma \\
+ \int_a^1 \sigma^\gamma G(s,\sigma)\rho(\sigma) r_{tt}(\sigma,t)\, d\sigma
\end{aligned} \tag{3.18.21}
$$

where

$$
G(s,\sigma) = \begin{cases} \sigma - a, & \text{for} \quad \sigma < s, \\ \\ \sigma - 1, & \text{for} \quad s < \sigma. \end{cases} \tag{3.18.22}
$$

Step 2. We use (3.18.21) to dominate the boundary term of (3.7.2):

$$
\begin{aligned}
\left| \int_0^t B(\bar{t})\, d\bar{t} \right| &= \left| \int_0^t s^\gamma N(s,\bar{t}) r_t(s,\bar{t}) \Big|_{s=a}^{s=1} d\bar{t} \right| \\
&\leq C \int_0^t D(\bar{t}) \int_a^1 \left(|N(s,\bar{t})| + |T(s,\bar{t})| \right) ds\, d\bar{t} \\
&\quad + C \int_0^t K(\bar{t})\, d\bar{t} + \mathit{\Gamma}(t^+).
\end{aligned} \tag{3.18.23}
$$

If $r_t(1,t) = r_t(a,t)$ for all t then $D = 0$ and we immediately get a tractable bound on $\int_0^t B(\bar{t})\, d\bar{t}$. Otherwise we use Hypotheses 3.5.6, 3.5.8, 3.5.40, 3.5.46 to estimate the right-hand side of (3.18.23). Let

$$
\mathit{\Phi}^\sharp(t) := \max\{\mathit{\Phi}(\bar{t}) : 0 \leq \bar{t} \leq t\}, \quad K^\sharp(t) := \max\{K(\bar{t}) : 0 \leq \bar{t} \leq t\}. \tag{3.18.24}
$$

By $(3.5.49)_1$ of Hypothesis 3.5.46b,

$$
\begin{aligned}
\left| \int_0^t D(\bar{t}) \int_a^1 N^{\mathrm{E}}\, ds\, d\bar{t} \right| &\leq \mathit{\Gamma}(t) + C \int_0^t D(\bar{t}) \int_a^1 \left(\varphi + \varphi^{\beta_1} \psi(\nu)^{\beta_2} \right) ds\, d\bar{t} \\
&\leq \mathit{\Gamma}(t) + C \int_0^t D(\bar{t}) \left(\mathit{\Phi}(\bar{t}) + \mathit{\Phi}(\bar{t})^{\beta_1} \psi^\sharp(\bar{t})^{\beta_2} \right) d\bar{t} \\
&\leq \mathit{\Gamma}(t) + C \int_0^t \mathit{\Phi}(\bar{t})\, d\bar{t} + C\mathit{\Phi}^\sharp(t)^{\beta_1} \int_0^t D(\bar{t})\, \psi^\sharp(\bar{t})^{\beta_2}\, d\bar{t} \\
&\leq \mathit{\Gamma}(t) + C \int_0^t \mathit{\Phi}(\bar{t})\, d\bar{t} \\
&\quad + \mathit{\Gamma}(t) \mathit{\Phi}^\sharp(t)^{\beta_1} \left(\int_0^t D(\bar{t})\, d\bar{t} \right)^{1-\beta_2} \left(\int_0^t D(\bar{t}) \psi^\sharp(\bar{t})\, d\bar{t} \right)^{\beta_2}.
\end{aligned} \tag{3.18.25}
$$

By Hypothesis 3.5.46a,

$$\left|\int_0^t \int_a^1 T^{\mathrm{E}}\, ds\, d\bar{t}\right| \le \Gamma(t) + C \int_0^t \Phi(\bar{t})\, d\bar{t}. \tag{3.18.26}$$

By Hypotheses 3.5.6, 3.5.8 and 3.5.40,

$$\left|\int_0^t \int_a^1 N^{\mathrm{D}}\, ds\, d\bar{t}\right| \le \Gamma(t) + C \int_0^t \Phi(\bar{t})\, d\bar{t} + \varepsilon W(t), \tag{3.18.27}$$

$$\left|\int_0^t \int_a^1 T^{\mathrm{D}}\, ds\, d\bar{t}\right| \le \Gamma(t) + C \int_0^t \Phi(\bar{t})\, d\bar{t} + C \int_0^t K(\bar{t})\, d\bar{t}. \tag{3.18.28}$$

The relevant version of the energy inequality coming from (3.7.2), (3.18.23), and the estimates (3.18.25)–(3.18.28) has the form

$$\begin{aligned} K(t) + \Phi^\sharp(t) + W(t) \le \Gamma(t) &+ C \int_0^t \Phi^\sharp(\bar{t})\, d\bar{t} + C \int_0^t K(\bar{t})\, d\bar{t} \\ &+ \Gamma(t) \Phi^\sharp(t)^{\beta_1} \left(\int_0^t D(\bar{t})\, d\bar{t}\right)^{1-\beta_2} \left(\int_0^t D(\bar{t}) \psi^\sharp(\bar{t})\, d\bar{t}\right)^{\beta_2}. \end{aligned} \tag{3.18.29}$$

Hence

$$\begin{aligned} K(t) + \Phi^\sharp(t) + W(t) \le \Gamma(t) &+ C \int_0^t \Phi^\sharp(\bar{t})\, d\bar{t} + C \int_0^t K(\bar{t})\, d\bar{t} \\ &+ \Gamma(t) \left(\int_0^t D(\bar{t})\, d\bar{t}\right)^{(1-\beta_2)/(1-\beta_1)} \left(\int_0^t D(\bar{t}) \psi^\sharp(\bar{t})\, d\bar{t}\right)^{\beta_2/(1-\beta_1)}. \end{aligned} \tag{3.18.30}$$

By using Gronwall's inequality we deduce that (assuming $\beta_1 \in (0,1)$)

$$K^\sharp(t) + \Phi^\sharp(t) + W(t) \le \Gamma(t) + \Gamma(t) \left(\int_0^t D(\bar{t})\, d\bar{t}\right)^{(1-\beta_2)/(1-\beta_1)} \left(\int_0^t D(\bar{t}) \psi^\sharp(\bar{t})\, d\bar{t}\right)^{\beta_2/(1-\beta_1)}. \tag{3.18.31}$$

Step 3. Let us first consider the case when

$$\psi^\sharp(t) = \psi(\nu(\xi, t)) \quad \text{with} \quad r_s(\xi, t) < y_*. \tag{3.18.32}$$

We argue as in the proof of Theorem 3.8.11 and use Hypothesis 3.5.22 and (3.18.21) to obtain that

$$\begin{aligned}\psi(\xi,t) &= \psi(y_*) + \int_\theta^t \psi'(\nu(\xi,\bar t))\nu_t(\xi,\bar t)\,d\bar t \\ &\le \Gamma(t) - \int_\theta^t N(\xi,\bar t)\,d\bar t \\ &\le \Gamma(t) - C\int_\theta^t\int_a^1 s^\gamma\left(N(s,\bar t) + \gamma s^{\gamma-1}G(\xi,s)\,T(s,\bar t)\right)\,ds\,d\bar t \\ &\quad - C\int_a^1 s^\gamma G(\xi,s)\rho(s)r_t(s,\bar t)\Big|_{\bar t=\theta}^{\bar t=t}\,ds \\ &\le \Gamma(t) + C\int_0^t\int_a^1 \left(|N(s,\bar t)| + |T(s,\bar t)|\right)\,ds\,d\bar t + K^\sharp(t).\end{aligned} \tag{3.18.33}$$

By using the estimates (3.18.25)–(3.18.28) and (3.18.31), we deduce from (3.18.33) that

$$\psi^\sharp(t) \le \Gamma(t) + \Gamma(t)\left(\int_0^t D(\bar t)\,d\bar t\right)^{d_1}\left(\int_0^t D(\bar t)\psi^\sharp(\bar t)\,d\bar t\right)^{d_2} \tag{3.18.34}$$

where d_1, d_2 are some positive numbers (that can be expressed in terms of β_1, β_2). Let $Q := \int_0^t D(\bar t)\psi^\sharp(\bar t)\,d\bar t$. Then (3.18.34) becomes

$$Q'(t) \le \Gamma(t)\left(1 + D(t)\left(\int_0^t D(\bar t)\,d\bar t\right)^{d_1} Q(t)^{d_2}\right). \tag{3.18.35}$$

We obtain

$$Q^{-d_2}Q'(t) \le \Gamma(t)D(t)\left(\int_0^t D(\bar t)\,d\bar t\right)^{d_1}. \tag{3.18.36}$$

Since $Q \mapsto \int_1^Q q^{-d_2}\,dq =: J(Q)$ is an increasing function of Q, it follows from the integral of (3.18.36) that $Q(t) \le \Gamma(t)$ if

$$J(\infty) > \int_0^\infty D(t)\left(\int_0^t D(\bar t)\,d\bar t\right)^{d_1} dt \equiv \frac{1}{d_1+1}\left(\int_0^\infty D(\bar t)\,d\bar t\right)^{d_1+1}. \tag{3.18.37}$$

In this case (3.18.34) implies that $\psi^\sharp(t) \le \Gamma(t)$, which gives a lower bound on ν that depends on the data.

We likewise consider the case when

$$\psi^{\sharp}(t) = \psi(\nu(\xi, t)) \quad \text{with} \quad r_s(\xi, t) > y^* \tag{3.18.38}$$

and obtain an upper bound on ν that depends on the data.

3.18.3 Estimates on the accelerations and the strain rates

We now show how to get some of the estimates of Sec. 3.15 for other boundary conditions. We treat only a few illuminating cases of boundary conditions. We modify the treatment of Theorem 3.9.5 beginning with (3.9.10) to estimate the boundary term of (3.9.4).

Hydrostatic loads both on $s = a$ and $s = 1$. Suppose (3.18.3) are the prescribed boundary conditions with $\lambda_1, \lambda_a \in W^{1,\infty}_{\text{loc}}[0, \infty)$. We assume the conditions required for the energy estimate to hold as discussed in Sec. 3.18.1 (including the restrictions on the exponents α_1, α_2). To estimate the term $\int_0^t N_t(1, \bar{t}) r_{tt}(1, \bar{t})\, d\bar{t}$ we compute

$$N_t(1, t) = -\lambda_1'(t) r^{\gamma}(1, t) - \lambda_1(t) \gamma r^{\gamma - 1}(1, t) r_t(1, t). \tag{3.18.39}$$

The estimate (3.5.12) and the energy estimate established in Sec. 3.18.1 imply that $r(1, t) \in L^{\infty}_{\text{loc}}[0, \infty)$. The estimate (3.7.14) and the energy estimate imply that $r_t(1, t) \in L^2_{\text{loc}}[0, \infty)$. Hence (3.18.39) implies that $N_t(1, t) \in L^2_{\text{loc}}[0, \infty)$ for $\gamma = 1, 2$. Therefore (3.9.12) again holds. Estimating the boundary term $\int_0^t N_t(a, \bar{t}) r_{tt}(a, \bar{t})\, d\bar{t}$ in a similar way, we obtain the analog of Theorem 3.9.5.

Dead loads both on $s = a$ and $s = 1$. Now suppose (3.18.1) are the prescribed boundary conditions with $N(a, \cdot)$ and $N(1, \cdot)$ in $W^{1,2}_{\text{loc}}[0, \infty)$. We use (3.9.14) to

obtain

$$\begin{aligned}\left|\int_0^t N_t(1,\bar t) r_{tt}(1,\bar t)\,d\bar t\right| &\le C\int_0^t N_t(1,\bar t)^2\,d\bar t + C\int_0^t r_{tt}(1,\bar t)^2\,d\bar t \\ &\le C + C\int_0^t H(\bar t)\,d\bar t.\end{aligned} \tag{3.18.40}$$

Estimating the boundary term $\int_0^t N_t(a,\bar t) r_{tt}(a,\bar t)\,d\bar t$ in a similar way, we obtain the analog of Theorem 3.9.5.

Position conditions both on $s = a$ and $s = 1$. Now suppose that the double Dirichlet boundary conditions (3.18.17) are prescribed. We assume that $r(1,\cdot)$, $r(a,\cdot)$ are in $W^{3,\infty}_{\mathrm{loc}}[0,\infty)$. We differentiate the integral representation (3.18.21) with respect to t and use the resulting equation to estimate the boundary term of (3.9.4):

$$\begin{aligned}(1-a)&\int_0^t s^\gamma N_t(s,\bar t) r_{tt}(s,\bar t)\Big|_{s=a}^{s=1} d\bar t \\ &= \int_0^t (r_{tt}(1,\bar t) - r_{tt}(a,\bar t)) \int_a^1 \sigma^\gamma \left(N_t(\sigma,\bar t) + \gamma\sigma^{\gamma-1} G(s,\sigma)\, T_t(\sigma,\bar t)\right)\,d\sigma\,d\bar t)\Big|_{s=a}^{s=1} \\ &\quad + \int_0^t (r_{tt}(1,\bar t) - r_{tt}(a,\bar t)) \int_a^1 \sigma^\gamma G(s,\sigma)\rho(\sigma) r_{ttt}(\sigma,t)\,d\sigma\,d\bar t)\Big|_{s=a}^{s=1}.\end{aligned} \tag{3.18.41}$$

By integrating by parts the second term on the right-hand side of (3.18.41), we find that it is bounded by $\varGamma(t^+)(\varepsilon H + 1)$. Hypothesis 3.5.18 and the relevant energy etimate ensure that the first term on the right-hand side of (3.18.41) is bounded by $\varGamma(t^+)(1+\sqrt{H(t)})$. By substituting these bounds into (3.9.4) and using a version of Gronwall's inequality, we obtain the analog of Theorem 3.9.5.

3.19 Blowup of solutions under hydrostatic pressures

We establish several results related to the blowup and unboundedness of solutions to the equation (3.3.3) subject to the boundary conditions (3.18.3) involving hydrostatic pressures. We shall employ constitutive inequalities asserting that the material is weak in resisting inflation and/or that the material is dissipatively weak and impose certain conditions on the data. Our treatment in this section uses some of the techniques developed in [6, 32]. See also [5, Sec. 18.7] and [20, 21].

We set

$$\begin{aligned} \varPsi(t) &:= \frac{1}{2}\int_a^1 s^\gamma \rho(s) r(s,t)^2\,ds, \\ \varLambda(\lambda_1,\lambda_a,r) &:= \frac{s^\gamma}{\gamma+1} N(s,t) r(s,t)\Big|_{s=a}^{s=1} \\ &\equiv \frac{1}{\gamma+1}\left[-\lambda_1(t) r^{\gamma+1}(1,t) + a^\gamma \lambda_a(t) r^{\gamma+1}(a,t)\right]. \end{aligned} \tag{3.19.1}$$

We shall seek various differential inequalities for $\varPsi$ that ensure that $\varPsi$ grows at various rates as time approaches infinity or blows up in finite time.

Energy-like estimate. We shall need an energy-like estimate which complements our energy estimate of Sec. 3.7. We easily find that

$$\begin{aligned} &\int_0^t B(\bar t)\,d\bar t + \int_0^t \varLambda(\lambda_1'(\bar t),\lambda_a'(\bar t), r(s,\bar t))\,d\bar t \\ &\quad = \varLambda(\lambda_1(t),\lambda_a(t),r(s,t)) - \varLambda(\lambda_1(0),\lambda_a(0),r(s,0)). \end{aligned} \tag{3.19.2}$$

(B was defined in (3.7.1).) We compute $\varPsi_{tt}$, replace the term involving r_{tt} with its expression obtained from the product of (3.3.3) with r, and then replace $\varLambda$ with its expression given by the energy equation (3.7.2) and formula (3.19.2) to obtain that

$$\begin{aligned} \varPsi_{tt} &= 2K - \int_a^1 s^\gamma \left(\hat N r_s + \gamma \hat T \frac{r}{s}\right)\,ds - (\gamma+1)\varLambda = \\ &= (3+\gamma)K + (\gamma+1)\varPhi - \int_a^1 s^\gamma \left(\hat N r_s + \gamma \hat T \frac{r}{s}\right)\,ds - (\gamma+1)E \\ &\quad + (\gamma+1)W(t) + (\gamma+1)\int_0^t \varLambda(\lambda_1'(\bar t),\lambda_a'(\bar t), r(s,\bar t))\,d\bar t \end{aligned} \tag{3.19.3}$$

where

$$E := K(0) + \varPhi(0) - \varLambda(\lambda_1(0),\lambda_a(0),r(s,0)). \tag{3.19.4}$$

We shall exploit inequalities following from this fundamental identity to prove blowup and unboundedness results.

Elastic plates and shells under hydrostatic pressures constant in time. We now study the dynamic behavior of solutions for elastic plates and shells under static pressures, i.e., we assume that λ_1, λ_2 are independent of time. Since we assume the material to be elastic, $T^{\mathrm{D}} = N^{\mathrm{D}} = 0$ and so $W = 0$. Since the pressures are constant, the last term in (3.19.3) vanishes. We choose the pressures to have so large an inflational effect at $t = 0$ that

$$E < 0. \tag{3.19.5}$$

We assume that the equilibrium response is weak in resisting inflation so that

$$(\gamma + 1)\varphi^{(\gamma)}(\tau, \nu, s) \geq N^{\mathrm{E}}\nu + \gamma T^{\mathrm{E}}\tau, \tag{3.19.6}$$

which in view of (3.7.3) implies that

$$(\gamma + 1)\Phi \geq \int_a^1 s^\gamma \left(N^{\mathrm{E}} r_s + \gamma T^{\mathrm{E}} \frac{r}{s}\right) ds. \tag{3.19.7}$$

(Condition (3.19.6) is incompatible with Hypothesis 3.5.6b, which was used in Sec. 3.18.1 to establish the energy estimate that supports the existence theory). In this case, invoking (3.19.3) we obtain that

$$\Psi_{tt} > (3 + \gamma)K. \tag{3.19.8}$$

Thus (3.19.8) and Hölder's inequality imply that

$$\Psi_{tt}\Psi > (3 + \gamma)K\Psi \geq \frac{3 + \gamma}{4}\Psi_t^2. \tag{3.19.9}$$

Assume that $\Psi_t(0) > 0$. Since (3.19.8) implies that $\Psi_{tt} > 0$, it follows that $\Psi_t(t) > 0$ for all $t \geq 0$, in which case (3.19.9) yields

$$\frac{\Psi_{tt}}{\Psi_t} \geq \frac{3+\gamma}{4} \frac{\Psi_t}{\Psi}. \tag{3.19.10}$$

The following theorem shows that the behavior of the spherical shell differs significantly from that of the annular plate.

3.19.11. Theorem. *Let an elastic annular plate* ($\gamma = 1$) *or spherical shell* ($\gamma = 2$) *be weak in tension in the sense that* (3.19.6) *holds. Let it be subject to static pressures that are inflational in the sense that* (3.19.5) *holds. Then there are initial conditions, namely, those for which* $\Psi_t(0) > 0$, *for which (i)* Ψ *grows exponentially fast if* $\gamma = 1$; *(ii) the solution blows up in finite time if* $\gamma = 2$.

Proof. First let $\gamma = 1$. Integrating (3.19.10) from 0 to t we obtain that

$$\ln(\Psi_t(t)/\Psi_t(0)) \geq \ln(\Psi(t)/\Psi(0)) \tag{3.19.12}$$

whence

$$\Psi_t(t) \geq \Psi(t)\frac{\Psi_t(0)}{\Psi(0)}. \tag{3.19.13}$$

Using the differential form of Gronwall's inequality we deduce from (3.19.13) that

$$\Psi(t) \geq \Psi(0) \exp\left[\frac{\Psi_t(0)}{\Psi(0)} t\right]. \tag{3.19.14}$$

Clearly if $\Psi_t(0) > 0$ then Ψ grows exponentially fast as $t \to \infty$.

Now let $\gamma = 2$. By an integration process like that yielding (3.19.14) we find that

$$\Psi(t) \geq \frac{\Psi(0)^5}{\left[\Psi(0) - \frac{1}{4}\Psi_t(0)t\right]^4}. \tag{3.19.15}$$

Clearly if $\Psi_t(0) > 0$ then the solution blows up in finite time. $\square$

Viscoelastic plates and shells under hydrostatic pressures constant in time. Assume that

$$N^{\mathrm{D}}(\tau, \nu, \dot{\tau}, \dot{\nu}, s)\nu + \gamma T^{\mathrm{D}}(\tau, \nu, \dot{\tau}, \dot{\nu}, s)\tau \leq \frac{1}{2} c_1 s^2 \rho(s) \dot{\tau}^2 + Q(\tau, \nu) \tag{3.19.16}$$

where $c_1 \in (0, 3+\gamma)$ and Q is some positive-valued function. This condition says that the material is dissipatively weak. It is restrictive because it prohibits a linear dependce of N^{D} on $\dot{\nu}$. It is incompatible with Hypothesis 3.5.22 and condition (3.5.44), which support the existence theory analogous to that carried out in Secs. 3.12 – 3.16. Note that if T^{D} and N^{D} were to blow up as $\tau \to 0$ or as $\nu \to 0$, then Q could blow up in these limits as well. Assume further that

$$(\gamma + 1)\varphi^{(\gamma)}(\tau, \nu, s) \geq N^{\mathrm{E}}\nu + \gamma T^{\mathrm{E}}\tau + Q(\tau, \nu). \tag{3.19.17}$$

This last condition says that the material is sufficiently weak in resisting tension and that any blowup in Q at a total compression is dominated by that of φ. (3.19.16) and (3.19.17) imply that

$$c_1 K + (\gamma + 1)\Phi - \int_a^1 s^\gamma \left(\hat{N} r_s + \gamma \hat{T} \frac{r}{s} \right) ds \geq 0. \tag{3.19.18}$$

When (3.19.16), (3.19.17) hold and $E < 0$, the energy-like equation (3.19.3) yields

$$\Psi_{tt} > (4 - c_1)K \quad \Rightarrow \quad \frac{\Psi_{tt}}{\Psi_t} \geq \alpha \frac{\Psi_t}{\Psi} \tag{3.19.19}$$

provided $\Psi_t(0) > 0$. Here $\alpha := (3 + \gamma - c_1)/4$. Note that $\alpha \in (0, 1)$ for $\gamma = 1$ and $\alpha \in (0, 5/4)$ for $\gamma = 2$. For $\gamma = 1, 2$ and $\alpha \in (0, 1)$, inequality (3.19.19) implies that

$$\Psi(t)^{1-\alpha} - \Psi(0)^{1-\alpha} \geq (1 - \alpha)\Psi(0)^{-\alpha}\Psi_t(0)t. \tag{3.19.20}$$

Likewise, for $\gamma = 2$ and $\alpha \in (1, 5/4)$ we get

$$\Psi(t) \geq \frac{\Psi(0)^{\alpha/(\alpha-1)}}{[\Psi(0) - (\alpha - 1)\Psi_t(0)t]^{1/(\alpha-1)}}. \tag{3.19.21}$$

Finally for $\gamma = 2$ and $\alpha = 1$ we obtain (3.19.14). Hence

3.19.22. Theorem. *Let a viscoelastic annular plate ($\gamma = 1$) or spherical shell ($\gamma = 2$) be dissipatively weak in the sense that* (3.19.16) *holds and be weak in tension in the sense that* (3.19.17) *holds. Let it be subject to pressures that are inflational in the sense that* (3.19.5) *holds. Let the initial conditions be inflational in the sense that $\Psi_t(0) > 0$. Then (i) Ψ grows like a positive power of t if $\alpha \in (0, 1)$ and $\gamma = 1, 2$; (ii) Ψ grows exponentially fast if $\alpha = 1$ and $\gamma = 2$; (iii) the solution blows up in finite time if $\alpha \in (1, 5/4)$ and $\gamma = 2$.*

Viscoelastic plates and shells under time-dependent hydrostatic pressures. For time-dependent pressures, we have to treat the last term on the extreme right-hand side of (3.19.3). The analysis of this case becomes a trivial modification of the analysis of viscoelastic plates and shells under pressures constant in time if in addition to (3.19.5), (3.19.16), (3.19.17) we assume that

$$\lambda_a'(t) \geq 0, \qquad \lambda_1'(t) \leq 0, \tag{3.19.23}$$

which ensures that $\Lambda(\lambda_1', \lambda_a', r) \geq 0$. We then immediately obtain the analog of Theorem 3.19.22.

In this section we have given conditions under which energy-like quantity Ψ blows up. However, we have not shown that the solution survives long enough for Ψ to blow up. It is conceivable that it does not [15]. This issue does not occur for ordinary differential equations.

3.20 Open Problems. Comments

Centrifugal force. In Sec. 3.19 we showed that when the material is not sufficiently strong relative to externally applied hydrostatic pressures then the solutions may blow up in finite or infinite time. It is natural to expect the blowup of solutions under other types of live loads, in particular under a centrifugal force.

Specifically one can consider planar motions of an annular plate in which it rotates at constant angular velocity ω about the axis through its center perpendicular to its plane and in which the material points with reference radius s move along their material rays a distance $r(s,t)$ depending only on s and t. (The analogous problem for a spherical shell is considerably more difficult as the shell loses its sphericity under the action of a centrifugal force.)

The governing equation is obtained by introducing into the equation (3.3.3) the appropriate acceleration term, which gives the centrifugal force:

$$\frac{\partial}{\partial s}(sN(s,t)) - T(s,t) + \omega^2\rho(s)sr = s\rho(s)\, r_{tt}(s,t), \qquad s \in (a,1). \tag{3.20.1}$$

We assume that the inner radius $s = a$ is fixed and that the outer boundary $s = 1$ is traction-free:

$$r(a,t) = r_a, \qquad N(1,t) = 0. \tag{3.20.2}$$

We formally multiply (3.20.1) by r_t and then integrate the resulting equation over $[a,1]\times[0,t]$ to obtain the energy equation

$$K(t) + \Phi(t) + W(t) - \omega^2\Psi(t) - \int_0^t B(\bar{t})\, d\bar{t} = K(0) + \Phi(0) - \omega^2\Psi(0) =: E_c \tag{3.20.3}$$

where we used the notation of (3.7.1) and (3.19.1).

Assume the material is strong in resisting extension in the sense that (3.5.7) holds with $\alpha_1 > 1$ and $\alpha_2 > 2$. Then the stored energy $\varPhi$ dominates the term $\omega^2\varPsi$, i.e., $\varepsilon\varPhi \geq \omega^2\varPsi - C$ and the energy inequality (3.7.7) again holds. In this case our methods can be used to establish the global existence theory for the equation (3.20.1).

Assume now the material is elastic and weak in resisting inflation in the sense that (3.19.7) holds with $\gamma = 1$. We compute $\varPsi_{tt}$, replace the term involving r_{tt} with its expression obtained from the product of (3.20.1) with r, and then replce $\varPsi$ with its expression given by the energy equation (3.20.3) to obtain that

$$\begin{aligned} \varPsi_{tt} &= 2K + 2\omega^2\varPsi(t) - \int_a^1 s\left(\hat{N}r_s + \hat{T}\frac{r}{s}\right) ds - aN(a,t)r_a \\ &= 4K + 2\varPhi - \int_a^1 s\left(\hat{N}r_s + \gamma\hat{T}\frac{r}{s}\right) ds - 2E_c - aN(a,t)r_a \\ &\geq 4K - 2E_c - aN(a,t)r_a. \end{aligned} \tag{3.20.4}$$

We can choose ω to be so large that $E_c < 0$. The remaining issue is to get a tractable bound on $N(a,t)$ so that we could deduce from (3.20.4) that $\varPsi_{tt} > 4K$ and prove the blowup of solutions in infinite time.

Disks and balls. The initial-boundary-value problems for disks and balls (where the inner radius $a = 0$) require a special treatment because of the presence of a singularity at the origin $s = 0$. In such cases there is a possibility that the stress becomes infinite at the origin, which may result either in cavitation [17, 40, 41, 42, 46] or in a total compression at the origin.

Let us obtain an analog of the energy estimate (3.7.7) for a disk ($\gamma = 1$) or a ball ($\gamma = 2$) subject to a dead load condition $(3.4.2)_1$ with $\delta_1 = 0$ on the boundary

$s = 1$. We assume that the center is intact in the sense that $r(0,t) = 0$ for all t. We modify the proof of Theorem 3.7.6 starting with (3.7.10). As a approaches 0 the constant on the extreme right-hand side of (3.7.10) blows up. We accordingly replace (3.7.10) with

$$\begin{aligned} r_t(1,t) &= \int_0^1 (s^\gamma r_t(s,t))_s \, ds \equiv \int_0^1 \left[s^\gamma r_{st}(s,t) + \gamma s^\gamma \frac{r_t(s,t)}{s} \right] ds \\ &\leq \sqrt{\int_0^1 s^\gamma r_{st}(s,t)^2 \, ds} + \gamma \sqrt{\int_0^1 s^\gamma \left[\frac{r_t(s,t)}{s} \right]^2 ds} \\ &\leq C \sqrt{\int_a^1 s^\gamma \Delta(s,t) \, ds}. \end{aligned} \tag{3.20.5}$$

Then (3.7.11) again holds and the rest of the proof is the same as that of Theorem 3.7.6.

We have therefore established an analog of the energy estimate (3.7.7), which implies, in particular, that $\Phi \leq \Gamma(t^+)$. Consequently $\int_{\mathcal{B}} \varphi \, dv \leq \Gamma(t^+)$ where $\mathcal{B}$ is either a unit disk or ball and v denotes the standard area or volume (Lebesgue) measure. The Sobolev embedding theorem implies that if the equilibrium response is sufficiently strong in the sense that (3.5.7) holds with $\alpha_1, \alpha_2 > \gamma + 1$, then r is continuous so that no type of fracture can occur. Otherwise there is a possibility of cavitation when the center opens into a hole. The problem of dynamic cavitation in viscoelastic materials has never been studied.

Total compression. We have exhibited various constitutive restrictions that allowed us to prove the preclusion of a total compression for our two-dimensional problems. Constitutive restrictions that can be used to preclude total compression in one-dimensional problems for nonlinearly viscoelastic rods are discussed in [8, 9, 11]. Such restrictions are therefore sufficient for the preclusion. It is now a natural question to ask whether such restrictions are necessary. In other words, can there be a total compression under weaker hypotheses? This question remains open both for one- and two-dimensional problems.

Role of the regularity of the data. In obtaining our a priori bounds in Secs. 3.12–3.15, we assumed that r is more regular than needed for the weak formulation (3.4.11) to make sense. This level of regularity was needed in our arguments for proving the preclusion of a total compression and of an infinite extension based on Hypothesis 3.5.22. To justify such regularity of r we required in Theorem 3.6.1 that the data be sufficiently regular.

As remarked in Sec. 3.3, the conditions that are used in [11] to ensure the preclusion of a total compression and of an infinite extension are analogous to our mild Hypothesis 3.5.22. However paper [11] ignores the role of the regularity of a solution in ensuring the preclusion and does not impose the appropriate level of regularity on the data.

Moreover in deriving our bounds on the accelerations and strain rates in Sec. 3.9 we used (3.9.2). For this equation to be meaningful it suffices that r_{ttt} is of class $L^\infty(0, t^+; L^2[a, 1])$. This level of regularity is supported by the bounds on higher derivatives of the Galerkin approximations obtained in Sec. 3.14. In turn such bounds are based on the data being sufficiently regular.

The existence theories for nonlinearly viscoelastic rods developed in [8, 9] make use of equations analogous to our equation (3.9.2). However, the regularity needed for such equations in [8, 9] to be meaningful is not properly justified there.

Bibliography

[1] J. C. Alexander & S. S. Antman (1983), Global behavior of solutions of nonlinear equations depending on infinite-dimensional parameters, *Indiana Univ. Math. J.*, **32** 39–62.

[2] J. C. Alexander & J. A. Yorke (1976), The implicit function theorem and global methods of cohomology, *J. Funct. Anal.* **21** 330–339.

[3] S. S. Antman (1971), Existence and nonuniqueness of axisymmetric equilibrium states of nonlinearly elastic shells, *Arch. Rational Mech. Anal.* **40** 329–371.

[4] S. S. Antman (1996), Extensions of monotone mappings, *C.R. Acad. Sci., Paris*, Sér I 323 235 – 239.

[5] S. S. Antman (2005), *Nonlinear Problems of Elasticity*, 2nd edn., Springer.

[6] S. S. Antman & W. Lacarbonara (2009), Forced radial motions of nonlinearly viscoelastic shells, *J. Elasticity* **96** 155–190.

[7] S. S. Antman & P. V. Negrón-Marrero (1987), The remarkable nature of radially symmetric equilibrium states of aeolotropic nonlinearly elastic bodies, *J. Elasticity* **18** 131–164.

[8] S. S. Antman & T. I. Seidman (1996), Quasilinear hyperbolic-parabolic equations of one-dimensional viscoelasticity, *J. Diff. Eqs*, **124** 132 –185.

[9] S. S. Antman & T. I. Seidman (2004), Parabolic-hyperbolic systems governing the spatial motion of nonlinearly viscoelastic rods, *Arch. Rational Mech. Anal.*, **175** 85–150.

[10] S. S. Antman & M. M. Shvartsman (1995), The shrink-fit problem for aeolotropic nonlinearly elastic bodies, *J. Elasticity* **37** 157–166.

[11] S. S. Antman & S. Ulusoy (2015), Global attractors for quasilinear parabolic-hyperbolic equations governing longitudinal motions of nonlinearly viscoelastic rods, *Physica D*, **291** 31 – 44.

[12] J. P. Aubin (1963), Un théorème de compacité , *C.R. Acad. Sci. Paris*, **265** 5042 – 5045.

[13] J. M. Ball (1977) , Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, *Q. J. Math.*, **28**, 473 – 486.

[14] J. M. Ball (1977), Convexity conditions and existence theory in nonlinear elasticity, *Arch. Rational Mech. Anal.* **63** 337–403.

[15] J. M. Ball (1978), Finite time blow-up in nonlinear problems. In: M.G. Crandall (ed.) Nonlinear Evolution Equations. Academic Press, 189 – 205.

[16] J. M. Ball (1981), Remarques sur l'éxistence et la régularité des solutions d'élastostatique nonlinéaire, *Recent Contributions to Nonlinear Partial Differential Equations*, H. Berestycki & H. Brezis, eds., Pitman, 50–62.

[17] J.M. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity (1982), *Phil. Trans. of the Royal Society of London.* Series A, Math. and Phys. Sci. **306** 557–611.

[18] J. M. Ball & R. D. James (1987), Fine phase mixtures as minimizers of energy, *Arch. Rational Mech. Anal.* **100** 13–52.

[19] J. M. Ball & J. E. Marsden (1984), Quasiconvexity at the boundary, positivity of the second variation and elastic stability. *Arch. Rational Mech. Anal.* **86** 251–277.

[20] M. C. Calderer (1983), The dynamic behavior of nonlinearly elastic spherical shells, *J. Elasticity*, **13**, 17–47.

[21] M. C. Calderer (1986) The dynamic behavior of viscoelastic spherical shells, *J. Diff. Eqs.*, **63**, 289–305.

[22] P. G. Ciarlet (2013), *Linear and Nonlinear Functional Analysis with Applications*, SIAM.

[23] E. A. Coddington & N. Levinson (1955), Theory of Ordinary Differential Equations, *McGraw-Hill*.

[24] B. Dacorogna (2008), *Direct Methods in the Calculus of Variations*, 2nd edn., Springer.

[25] R. C. Gauss & S. S. Antman (1984), Large thermal buckling of nonuniform beams and plates, *Intl. J. Solids Structures* **20** 979–1000.

[26] A. E. Green & W. Zerna (1968), *Theoretical Elasticity*, 2nd edn., Oxford Univ. Pr.

[27] E. Doedel (2010), *Numerical Analysis of Nonlinear Equations*, Lecture Notes, Concordia Univ.

[28] L. C. Evans (2010), *Partial Differential Equations*, 2nd Edition, Amer. Math. Soc.

[29] S. Faedo (1949), Un nuovo metodo per l'analisi esistenziale e quantitativa dei problemi di propagazione. *Ann. Sc. Norm. Pisa*, Ser. III. 1(14), 1–41.

[30] H. B. Keller (1992), *Numerical Methods for Two-Point Boundary Value Problems*, Dover.

[31] H. Kielhöfer (2012), *Bifurcation Theory*, 2nd edn., Springer.

[32] R. Knops (1973), Logarithmic convexity and other techniques applied to problems in continuum mechanics. In: R. Knops (ed.) Symposium on Non-Well-Posed Problems and Logarithmic Convexity. Lecture Notes in Mathematics, vol. 316, pp. 31–54, Springer

[33] R. Knops & E. W. Wilkes (1973), Theory of Elastic Stability, in *Handbuch der Physik*, Vol. VIa/3, Springer, 125–302.

[34] G. Lamé (1852), *Leçons sur la théorie mathématique de l'élasticité de corps solides*, Bachelier, Paris.

[35] S. G. Lekhnitskii (1981), *Theory of Elasticity of an Anisotropic Body*, Mir.

[36] S. G. Lekhnitskii, S. W. Tsai, & Th. Chéron (1987), *Anisotropic Plates*, Gordon & Breach.

[37] J. L. Lions (1969), Quelques méthodes de résolution des problèmes aux limites non linéaires, *Dunod*

[38] A. E. H. Love (1927) *A Treatise on the Mathematical Theory of Elasticity*, 4th edn., Cambridge Univ. Pr.

[39] L. A. Mihai, L. Chin, P. Janmey, A. Goriely (2015), A comparison of hyperelastic constitutive models applicable to brain and fat tissues, *J. Roy. Soc. Interface.*

[40] A. Miroshnikov & A. Tzavaras (2015), On the construction and properties of weak solutions describing dynamic cavitation, *J. Elasticity*, **118-2**, 141–185.

[41] K.A. Pericak-Spector & S.J. Spector (1987), Dynamic Cavitation with Shocks in Nonlinear Elasticity. *Proc. Royal Soc. Edinburgh* **127A**, 837–857

[42] K. A. Pericak-Spector & S. J. Spector (1988), Nonuniqueness for a hyperbolic system: cavitation in non-linear elastodynamics, *Arch. Rational Mech. Anal.* **101**, 293–317.

[43] P. H. Rabinowitz (1971), Some global results for nonlinear eigenvalue problems, *J. Functional Anal.*, **7** 487–513.

[44] T. I. Seidman (1989), The transient semiconductor problem with generating terms, II. In: *Nonlinear Semigroups, Partial Differential Equations, and Attractors.* T. E. Gill, W. W. Zachary (eds.), 1394 Springer Lecture Notes Math., pp. 185–198.

[45] J. Sivaloganathan (1986), Uniqueness of regular and singular equilibria for spherically symmetric problems of nonlinear elasticity, *Arch. Rational Mech. Anal.* **96**, 97–136.

[46] J. Sivaloganathan & S. Spector (2008), Energy minimizing properties of the radial cavitation solution in incompressible nonlinear elasticity, *J. of Elasticity*, **93**, 177–187.

[47] M. M. Shvartsman & S. S. Antman (1995), Coexistent phases in nonlinear thermoelasticity: radially symmetric equilibrium states of aeolotropic bodies, *J. Elasticity* **41**, 107–136.

[48] I. S. Sokolnikoff (1956), *The Mathematical Theory of Elasticity*, 2nd edn., McGraw-Hill.

[49] I. Stakgold (1998), *Green's Functions and Boundary Value Problems*, 2nd edn., Wiley-Interscience.

[50] C. R. Steele & R. F. Hartung (1965), Symmetric loading of orthotropic shells of revolution. *J.Appl. Mech.* **32** 337–345.

[51] A. B. Stepanov & S. S. Antman (2015), Radially symmetric steady states of nonlinearly elastic plates and shells, *J. of Elasticity*.

[52] A. B. Stepanov & S. S. Antman (2015), Radially symmetric motions of nonlinearly viscoelastic plates and shells, in preparation.

[53] S. P. Timoshenko & J. N. Goodier (1951), *Theory of Elasticity*, 2nd edn., McGraw-Hill.

[54] C. Truesdell & W. Noll (2004), *Non-linear Field Theories of Mechanics*, 3rd edition, Springer.

[55] B. Tvedt (2008), *Quasilinear Equations for Viscoelasticity of Strain-Rate Type*, Arch. Rational Mech. Anal. 189 (2008) 237–281.

[56] M. M. Vainberg (1964), *Variational Methods for the Study of Nonlinear Operators*, Holden-Day.

[57] E. Zeidler (1986), *Nonlinear Functional Analysis and it Applications I, Fixed-Point Theorems*, Springer.

[58] S. Zheng (2004), *Nonlinear Evolution Equations*, Chapman & Hall/CRC.