AN HISTORICAL AND CRITICAL DEVELOPMENT OF THE THEORY OF LEGENDRE POLYNOMIALS BEFORE 1900

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FOREWORD

Legendre Polynomials appear for the first time in the work of Legendre in 1784 in relation to problems of potential and of celestial mechanics. A more systematic exposition of their most elementary properties appears in Legendre's treatise on calculus. The polynomials may be defined by means of a generating function, an explicit form, a differential equation, an orthogonality property, an nth differential coefficient, a recurrence relation, several definite integrals, or a determinant.

In this paper, the historical approach is taken and the generating function is the point of departure. All the other definitions are made to depend on the original one. Some elementary properties of the polynomials are first derived, and the fundamental bases of the application of Legendre Polynomials to mechanical quadrature given. The significance of the Legendre differential equation and its relation to the general theory of linear differential equations of the second order is briefly considered. Some attention is given to the various definite integral representations of the polynomials.

One of the more important phases of this paper is an exposition of most of the significant properties of the Legendre Polynomials. This includes a discussion of some definite integrals which involve the polynomials in the integrand. An indication is given of some applications of the polynomials to combinatory analysis and algebraic theory. Various algebraic properties of the polynomials are displayed. A number theoretic property of their coefficients is given, and the discriminant and its properties exhibited. The zeros are considered in some detail. Bounds are given for each zero, and the properties of linear combinations of the polynomials discussed. Considerable attention is devoted to various asymptotic representations of the polynomials, and the behavior of the polynomials in the neighborhood of 1 and -1 for n large is treated also. Through the latter, a relation is derived between Legendre Polynomials and Bessel functions.

The asymptotic expansions play an important part in the discussion of Legendre Series. The prime question of sufficient conditions for convergence of the series is answered completely in one case and other results are indicated. Interpolation by finite Legendre sums is also considered and Gauss quadrature is discussed in less elementary fashion. Important continued fraction relations form the basis of the extension to general mechanical quadrature.

Finally, an attempt has been made to present developments historically without creating disjunction in a logical and systematic exposition. The bibliography is of independent value and represents some of the material the author has collected in his work as research assistant to a committee of the National Research Council, forming a bibliography on Orthogonal Polynomials.

SECTION I

HISTORICAL INTRODUCTION

A considerable part of the elementary theory of celestial mechanics was propounded in the years between the birth of Newton and the death of Läplace. One of the important phases of the basic theory was a consideration of the figures of the planets. Sir Isaac Newton was among the first to consider a mathematical theory of the physical causes of the figures of the planets. In the course of his investigations, he was forced to make several restrictive assumptions. He supposed that at "creation" the earth and the planets were in a fluid state, and that they now preserve the figure they were then given. By this hypothesis, the problem of the figure of the planets was reduced to determining the figure necessary for the equilibrium of a fluid mass.

Few mathematicians of the 17th and 18th century who followed Newton went beyond his hypotheses, and the problem in a limited sense was eventually solved. For, they succeeded in showing that a mass revolving about an axis and consisting of fluids of one or more densities will preserve its figure only if it has the form of an elliptical spheroid of revolution oblate at the poles.

However, Newton's hypotheses are not realistic; and D'Alembert attempted to generalize the problem to the consideration of attractions of non-elliptical spheroids. Because the theory of the figure of the planets was closely associated with the theory of attractions of bodies, he tried to investigate the attraction of a body of any proposed figure, and of strata varying in their densities according to any given law.

His results, complex and limited, still left the problem virtually unsolved. And it was in this form that the problem was handed on to the mathematicians of the latter part of the eighteenth century.

The principal object, then, was to investigate the figure, which a fluid, consisting of portions varying in density according to any given law, would assume, when every particle is acted upon by the attraction of every other particle and by a centrifugal force arising from a rotary motion. To what extent this may have been the original condition of the earth was not at issue, but it was made the foundation of most mathematical calculations. It was in these calculations that the Legendre Polynomials and the so-called Laplace Coefficients were first introduced.

While there is considerable question concerning the priority of Legendre or Laplace in the introduction of these functions, and even perhaps concerning the priority in the solution of the problem of attractions, it is of some interest to examine first the major difficulties facing the mathematicians of 1780 investigating the theory of the figure of the planets.

Whatever the permanent configuration of a fluid covering a solid body, that configuration will depend on the gravity at the surface. At the same time, the form of the surface determines the gravity, which is the combined effect of the attractions of all the particles of the body. That the figure of the surface is in a sense both a *datum* and *quaesitum* of the problem constitutes a difficulty. An expression must be found for the intensity of the attractive force which will be related to the form of the attracting body and shall yet be sufficiently simple.

In Laplace's "Traité de Mécanique Céleste" appears a revision of his solution of the problem. He concerns himself with the attractions of spheroids in general and especially those which differ only slightly from spheres. He succeeds in deducing a relation between the radius of the spheroid and a series expression for the attractive force on a particle without, on, or within the surface. This derivation is comparatively simple, when the complexities of the problem are considered.

Laplace begins with a potential function, from which the attractive force in any desired direction can be obtained by differentiation. The potential function is the sum of all the particle-masses of the attracting body, divided by their respective distances from the attracted particle. He expands this function into a series of descending or ascending powers of the distance of the attracted particle from the center of the spheroid, according as the particle is without or within the surface. His next step is to determine the coefficients of the terms of this expansion.

First of all, Laplace proves that every one of the coefficients satisfies a partial differential equation, which was first given by him and which is the key to his work on the theory of attractions and the figure of the planets. This equation is not integrated, but, by the use of its properties, the problem of attraction is simplified. Laplace next states a theorem he declares true at the surface of all nearly spherical spheroids. This theorem is stated without proof; and it holds, as later writers showed, only for a restricted class of spheroids. The theorem declares that the radii of all nearly spherical spheroids can be developed into series, every term of which would satisfy the Laplace differential equation. On the basis of this, he deduces the value of an expression, which is the sum, $\sum_i A_i P_i$, where the A's are known and the P's are the desired coefficients. This value is found to be proportional to the difference between that radius of the spheroid drawn through the attracted particle and the radius of the sphere nearly equal to the spheroid. Thus, he expresses the radius of the spheroid and the series for the attractive force by means of the same functions. For, in order to find the coefficients sought, we have only to develop the difference between that radius of the spheroid drawn through the attracted particle and the radius of the sphere nearly equal to the spheroid. Thus, he expresses the radius of the sphere in a series of terms, every one of which will satisfy the partial differential equation. Laplace not only gives a method for computing these terms, but he also attempts to prove that the series is unique.

Thus, the functions which serve as coefficients of the expansion play an extremely important role in the work of Laplace. These functions are the Laplace Coefficients; and it is now a question of observing in greater detail the manner in which they are used in the theory of attractions, particularly with regard to the assignment of priority in their introduction.

It is essential to discuss in some length the chronology of several memoirs of Legendre and Laplace.[1] The order in which the content of these memoirs became the common knowledge of Legendre and Laplace can easily be shown to be the following:[2]

a. Legendre - Recherches sur l'attraction des sphéroïdes homogènes (1784).

b. Laplace - Théorie du mouvement et de la figure elliptique des planètes (1784).

1.) See bibliography for detailed references to these works.

2.) Only the pertinent memoirs of these men during this period are listed. There were others which did not involve Legendre Polynomials or Laplace Coefficients.

c. Legendre - Recherches sur la figure des planètes (1784)

d. Laplace - Théorie des attractions des sphéroïdes et de la figure des planètes (1782)

e. Laplace - Mémoire sur la figure de la terre (1783)

f. Laplace - Mémoire sur la théorie de l'anneau de Saturne (1787)

g. Legendre - Suite des recherches sur la figure des planètes (1789)

h. Laplace. - Sur quelques points du système du monde (1789).

There are some peculiar circumstances in connection with these papers which cast reflections on the character of Laplace. First, it is to be remarked that papers c, d, e, f, g, h appeared in the "Mémoires de l'Académie royale des Sciences de Paris". Legendre's first paper, a. above, appeared in the "Mémoires de Mathematiques et de Physique, presentées à l'Académie royale des Sciences pars divers Savans". Laplace's first paper, b. above, appeared as a tract privately printed. Now, when a volume of the "Paris Mémoires" is of a specified year, it is usual that it contain memoirs written before, during and even after the specified year. It will appear, therefore, that Laplace was familiar with the work of Legendre and made use of it. Yet, in almost every instance, he pointedly failed to acknowledge the priority of or his indebtedness to Legendre. Legendre was sufficiently disturbed by this to devote sections of his memoirs to reminding the reader of Laplace's obligations. Thus, Laplace, who later gained for himself a reputation for remarkable political opportunism in currying favor for himself with revolutionists and royalty, was as early as 1784 guilty of intellectual dishonesty.

Oddly enough, Legendre's first paper contains an acknowledgement, indicating that Laplace had suggested to him that he approach the problem of attraction through the expansion of the potential function in series.[3] In his memoir he seeks the attraction of an ellipsoid on a particle on the prolongation of one of the axes. He succeeds in establishing that if the attraction of a solid of revolution is known for every external point which is on the prolongation of an axis, it is known for every external point. His demonstration involves the use of series of hitherto unknown functions.

Let (r,θ) and (r', θ') be the polar coordinates of the attracted particle and the element of the attracting body respectively. Let $\phi = \omega - \omega'$ be the difference of longitudes of the attracted particle and the attracting element. With the density homogeneously unity, the element of attracting mass is $r'^2 \sin\theta' d\theta' d\phi dr'$. Thus the potential function, the idea of Laplace, will be

$$(1.1) \qquad V = \int_0^{\pi}\int_0^{2\pi}\int_{-s}^{s} \frac{r'^2 \sin\theta' d\theta' d\phi dr'}{\sqrt{r'^2 - 2rr'\cos\psi + r^2}} \, ,$$

where $\cos\psi = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\phi$. Since Legendre treats the case of an ellipsoid symmetrical with respect to its equator, the limits of integration for r' are $-s$ and s, where s is the radius vector of the solid corresponding to a colatitude θ'. The reciprocal of the denominator of the integrand, today called the generating function of the Laplace Coefficients, is now expanded in ascending powers of r'/r, obtaining

$$(1.2) \qquad V = \int_0^{\pi}\int_0^{2\pi}\int_{-s}^{s} \frac{r'^2}{r}\left\{1 + X_1 \frac{r'}{r} + X_2 \frac{r'^2}{r^2} + \cdots\right\} \sin\theta' d\theta' d\phi \, dr' \, ,$$

and the coefficients $X_1, X_2, \ldots,$ are the quantities in which we are

3.) a. above P. 421

essentially interested.[4]

Legendre proceeds to perform the indicated integration obtaining an expression for V, in which the first term is mass/r. First, however, the X's are discussed in some detail as functions of θ, θ', φ. Here, for the first time, Legendre displays the remarkable property that if these functions are integrated with respect to φ from 0 to 2π, the resulting function depends only on θ and θ', which variables are separable, i.e., the function can be written as the product of a function depending only on θ and a function depending only on θ'. The proof of this important property is by an induction which is of no special interest. However, it is interesting to note that, after exhibiting as a special case of the "Laplace Coefficients" the functions which we today know as the Legendre Polynomials of even degree, Legendre becomes interested in these polynomials of themselves, presenting several properties.[5] After exhibiting a few polynomials in explicit representation, he shows

(1.3) $X_n(x)=1$ for $x=1$, and

(1.4) $\displaystyle\int_0^1 \frac{X_n(x)\,dx}{(1+Kx^2)^{n+3/2}} = \frac{(-K)^{n/2}}{(2n+1)(1+K)^{n+1/2}}$

where $X_n(x)$ represents the Legendre polynomial of degree 2n in x.

4.) Neither Legendre nor Laplace were the first to use an expression similar to the reciprocal of the distance between two points as a generating function of a series. It is unlikely that M. le Chevalier de Louville, who expanded $(p^2-2qx-x^2)^{-\frac{1}{2}}$ in ascending powers of x in his "Sur une difficulté de Statique", Mem. de l'Ac. des Sc. de Paris (1722)128-142(p.132), was really the first. One scarcely dares, as does N. Nielsen in "Sur l'introduction des fonctions sphériques dans l'analyse", Det Kgl. Danske Videnskabernes Selskab, Math.-Fys. Meddelelser 10#5(1929)9pp., attribute to de Louville the introduction of the Legendre Polynomials, for de Louville had no concern with the coefficients in his expansion and appeared in no way conscious of the fact that hidden behind his computation was a new function in analysis.

5.) If Yn satisfies $\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dY_n}{d\mu}\right\}+\frac{1}{1-\mu^2}\frac{d^2Y_n}{d\omega^2}+n(n+1)Y_n=0$, where $\mu=\cos\theta$, $\varphi=\omega-\omega'$ Y_n is the Laplace Coefficient of nth order. Any other function of θ and ω satisfying the differential equation will be a Laplace function of nth order. If Y_n is a function of the single variable μ, we have the Legendre Polynomial of degree n in μ.

In confirmation of the contention that this paper of Legendre precedes the work of Laplace is the reference in Laplace's tract of 1784 (b. above) to the researches of Legendre.[6] Furthermore, Laplace uses the potential function expressed in the usual rectangular coordinate system, and is led in his expansion of the generating function in series to laborious computation. The complex form of his result does not prevent him from recognizing some properties of the coefficients in the series, but he speaks of the results as known from the work of Legendre.[7]

On July 7, 1784, Legendre read his next memoir (c. above) to the Paris Academy. The journal bearing the memoir did not appear until 1787. In the meantime, a work by Laplace (d. above), written after Legendre's lecture, appeared in the volume of 1782, actually published in 1785. Laplace made no mention of Legendre; Legendre, peeved, had the editors insert a footnote to his paper.[8] Here he points out that his paper was read before Laplace's was submitted, that Laplace used his results to develop further the theory of attractions. Writing of the coefficients obtained from the generating function, he says: "J'ai recours aux propriétés d'une espèce particulière de fonctions rationelles <u>qui ne se sont point encore presentées aux Analystes</u>".[9] (emphasis mine). Again, he refers to his earlier work (a.above) as representing the first presentation of these functions. And so, it seems definitely established that Legendre introduced the "Laplace Coefficients" and the Legendre Polynomials.

Another remarkable feature of this paper is the evidence of Legendre's increasing interest in the Laplace Coefficient of one variable, i.e. the Legendre Polynomial. He considers the expansion of $\frac{1}{2}(1+2xz+z^2)^{-\frac{1}{2}}+\frac{1}{2}(1-2xz+z^2)^{-\frac{1}{2}}$

6., 7.) b. above, p.96.

8.) c. above, p.370.

9.) ibid-p. 371.

in ascending powers of z, obtaining the explicit representation for the coefficients, the Legendre polynomial of even degree in x, and giving seven of their properties. Of these, the following are not found in his earlier paper:[10]

(1.5) $\int_0^1 x^n X_i \, dx = \frac{n(n-2)\cdots(n-r+2)}{(n+1)(n+3)\cdots(n+r+1)}$, for integral i and n.

(1.6) $\int_0^1 X_\mu X_\nu \, dx = 0$, $\int_0^1 X_\mu^2 \, dx = \frac{1}{4\mu+1}$, $(\mu \neq \nu)$

(1.7) $X_\mu = (x^2-\alpha_1^2)(x^2-\alpha_2^2)\cdots(x^2-\alpha_\mu^2)$, α's all real, unequal, and in $(0,1)$.

(1.8) $X_\mu(x) < 1$ for $0 < x < 1$

In addition, in treating a particular case, that of obtaining the equation of the meridian of the ellipsoid in terms of series of the polynomials, he states the theorem known today: A given function can be expanded in a series of Laplace functions in only one way. It is interesting to remark that neither Legendre nor Laplace were concerned with the convergence properties of the series they obtained, a difficulty which played a significant part in the history of the development of the theory of Legendre Polynomials.

Now, the memoir of Laplace which treated Legendre so unfairly was of itself epoch-making. In it is contained the basis for the second and fifth volumes of that tour de force, "Traité de Mécanique Céleste". First of all, Laplace completes the theory of attractions of spheroids terminated by surfaces of second order, which was the problem of Legendre. The work in this part is new in its approach and simplicity. Secondly, he considers the attractions of any spheroids whatever. He makes them depend on a partial differential equation of the second order, which is the foundation of all of his researches on the figure of the planets and of the "Mécanique Céleste". This equation leads him to some general results on the expression in series of the attraction of spheroids. Assuming the spheroids approach

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10.) We use $\{X_n(x)\}$ for the Legendre Polynomials of even degree, $X_i(x)$ being of degree 2i. We use $\{P_n(x)\}$ for the usual Legendre Polynomials.

spheres and combining these results with the partial differential equation which now holds on their surfaces, he arrives at an expression in series of attractions of nearly spherical spheroids of any kind. Ordinarily, very complicated integration is necessary to arrive at an expression of this kind, but by the method of Laplace the expression is arrived at without any integration and by a single differentiation. Finally, the results are applied to obtaining all the theory of the figure of the planets and the gravitational laws on their surfaces. The equilibrium of a homogeneous planet is shown to be possible only if it takes a definite shape. Thus, for example, the earth is an ellipsoid of revolution. Because he succeeds in obtaining all the known results by use of the method, he argues that there is no loss of generality in the use of series. In the latter, his intuition misled him.

As in his tract of 1784, Laplace works, at first at least, in rectangular coordinates, a poor choice as it will appear. Let (a,b,c) and (x,y,z) be the coordinates of the attracted particle and of the particle of mass dM of the spheroid respectively. Then the potential function is given by

(1.9) $$V=\sum_i \frac{dM_i}{\sqrt{(x_i-a)^2+(y_i-b)^2+(z_i-c)^2}},$$

and interpreted as

(1.10) $$V=\int_{vol.} \frac{dM}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}.$$

From this the components of attraction parallel to any given direction can be obtained. Now, if V is expanded in series

(1.11) $$V=\int \frac{dM}{\sqrt{a^2+b^2+c^2}}\left\{1+\frac{1}{2}\,\frac{2ax+2by+2cz-x^2-y^2-z^2}{a^2+b^2+c^2}+\frac{3}{8}\,\frac{(2ax+2by+2cz-x^2-y^2-z^2)^2}{(a^2+b^2+c^2)^2}+\cdots\right\},$$

and if the center of attraction is far away, we can consider only the first term; thus $V \doteq M/(a^2+b^2+c^2)^{\frac{1}{2}}$, where M is the mass of the spheroid. This formula is even more exact if the origin of coordinates is taken to be the center of gravity of the spheroid.

In space polar coordinates, with

$$x = R\cos\theta' \qquad a = r\cos\theta \qquad dM = \rho\, dV = \rho R^2 dR\, d\theta' d\omega' \sin\theta'$$
$$y = R\sin\theta'\cos\omega' \qquad b = r\sin\theta\cos\omega \qquad \rho = \text{density of particle of spheroid}$$
$$z = R\sin\theta'\sin\omega' \qquad c = r\sin\theta\sin\omega$$
$$R = \sqrt{x^2+y^2+z^2} \qquad r = \sqrt{a^2+b^2+c^2}$$

V becomes $\displaystyle\int_0^{\pi}\int_0^{2\pi}\int_0^{\bar{R}} \frac{\rho R^2 dR\, d\theta' d\omega' \sin\theta'}{(r^2 - 2rR[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\omega-\omega')] + R^2)^{1/2}}$.

(1.12)

Letting $\mu = \cos\theta$ and making use of some observations on integrals of linear partial differential equations of second order contained in a paper of 1779, he derives the now famous Laplace differential equation

(1.13) $$\frac{\partial\left[(1-\mu^2)\frac{\partial V}{\partial\mu}\right]}{\partial\mu} + \frac{\frac{\partial^2 V}{\partial\omega^2}}{1-\mu^2} + r^2\frac{\partial^2(rV)}{\partial r^2} = 0.$$

Assuming the center of attraction to be outside the sphere, he writes[11]

(1.14) $$V = \frac{U_0}{r} + \frac{U_1}{r^2} + \frac{U_2}{r^3} + \cdots + \frac{U_n}{r^{n+1}} + \cdots,$$

and shows that the Laplace Coefficient U_i satisfies

(1.15) $$\frac{\partial\left[(1-\mu^2)\frac{\partial U_i}{\partial\mu}\right]}{\partial\mu} + \frac{\frac{\partial^2 U_i}{\partial\omega^2}}{1-\mu^2} + i(i+1)U_i = 0.$$

By comparing U_i with the integral expression for V, he concludes that U_i is a rational integral function of μ, $\sqrt{1-\mu^2}\sin\omega$, and $\sqrt{1-\mu^2}\cos\omega$, depending on the nature of the spheroid.

In order to determine the values of the U's, he considers again the generating function

(1.16) $$T = \{r^2 - 2rR[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\omega-\omega')] + R^2\}^{-1/2},$$

showing that T satisfies (1.13). Expanding T in series of ascending powers of R/r,

(1.17) $$T = \frac{Q_0}{r} + Q_1\frac{R}{r^2} + Q_2\frac{R^2}{r^3} + \cdots,$$

where the Q's satisfy (1.15) and are rational functions of μ and $\sqrt{1-\mu^2}\cos(\omega-\omega')$.

If the Q's are known, we can determine U_i by

(1.18) $$U_i = \iiint R^{i+2} dR\, Q_i\, d\omega' d\theta' \sin\theta'$$

The object then becomes to transform (1.16) in such a way that the values

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11.) The invalidity of this expansion in the general case will be treated later.

of the Q's can be easily obtained. There is a slight error in the original, corrected by Legendre[12] in 1789 and in the "Mécanique Céleste", but the chief argument is unimpaired. Essential to the determination of the Q's are the facts that they satisfy (1.15), that the variables on which they depend (θ and ω) are separable, and that the part which depends on ω can be expressed in series of cosines of multiples of $\omega - \omega'$. An expression for a general Q is finally given. Thus the U's are determined from (1.18) and V is given in series of Laplace Coefficients by (1.14).

In applying the results to the determination of the figure of the planets, Laplace for the first time introduces the orthogonality property of the Laplace Coefficients, which Legendre had previously given (1.16) for functions of a single variable. In the next papers (e. and f. above), the radii of the earth and Saturn are given in the form of series of Laplace's functions. The key differential equation (1.13) is given in rectangular coordinates as

$$(1.19) \qquad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

and the particular form of the equation determined for solids of revolution and the sphere. The results are in error, and the correction, that the right member of (1.19) should be $-4\pi\rho$ or zero according as the particle is within or without the sphere, was made by Poisson[13]. As for Saturn, Laplace shows that if the ring were circular and perfectly alike in all its parts, its equilibrium would be unstable. The demonstration involves the following property of Legendre Polynomials.

$$(1.20) \qquad \int_0^\pi P_n(\cos\theta)\,d\theta = \begin{cases} 0 & (n \text{ odd}) \\ \pi\left\{\dfrac{1\cdot 3\cdots(n-1)}{2\cdot 4\cdots n}\right\}^2 & (n \text{ even}) \end{cases}$$

which is essentially a generalization of (1.5) given by Legendre.

12.) g. above p. 432.

13.) Poisson, S.D.--Bull. de la Société Philomathique 3(1812)388, Mem. de l'Ac. roy. des Sc. de Paris (1823)463

In his memoir of 1789 (g. above), Legendre indicates the roles played by Laplace and himself in developing the knowledge of Laplace Coefficients.[14] He finds it necessary to reemphasize the prior date of his earlier memoir to the work of Laplace (a. and b. above). He points out that Laplace was the first to treat the Laplace Coefficients in this memoir published in 1785 (d. above), but these functions were generalizations of those developed in Legendre's 1784 memoir (a. above). The extension affected by Laplace consisted in treating as functions of two independent variables the functions which Legendre had formerly treated as functions of one variable.

Several new properties of the functions are derived in the course of this memoir. Requiring the value of the potential for any point within the mass or on its surface, Legendre sets up (1.12) and expands (1.16) in powers of R/r or r/R, taking precautions to insure convergence of the series so obtained. Then he lists the first eight coefficients in this series and proves again that the variables are separable. In fact, he gives the first correct and convenient expression for Laplace's nth coefficient:

(1.21) $$Y_n = P_n(\mu)P_n(\mu') + \frac{2}{n(n+1)}\frac{dP_n(\mu)}{d\mu}\frac{dP_n(\mu')}{d\mu'}\sin\theta\sin\theta'\cos(\omega-\omega') + \frac{2}{(n-1)n(n+1)(n+2)}\frac{d^2P_n(\mu)}{d\mu^2}\frac{d^2P_n(\mu')}{d\mu'^2}\sin^2\theta\sin^2\theta'\cos 2(\omega-\omega') + \frac{2}{(n-2)(n-1)\cdots(n+3)}\frac{d^3P_n(\mu)}{d\mu^3}\frac{d^3P_n(\mu')}{d\mu'^3}\sin^3\theta\sin^3\theta'\cos 3(\omega-\omega') + \cdots,$$

where P_n is the Legendre Polynomial of degree n. This served to correct an error in an earlier memoir of Laplace (d. above), in which the terms in $\cos m(\omega-\omega')$ were omitted when m+n was odd. Legendre also demonstrates

(1.22) $$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \qquad (m \neq n),$$

(1.23) $$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}.$$

14.) g. above p. 432.

Further, he presents the Legendre differential equation as a special case of the Laplace equation.[15]

Laplace in his paper of the same year (h. above) could make no addition to the work of Legendre. In fact, Lagrange's "Mécanique Analytique", which appeared about the same time, went back to the earliest work of Legendre for its approach to the problem of ellipsoidal potentials. This is even more marked in a later memoir of Lagrange.[16] In this latter, after reaffirming (1.19) and (1.10), letting $a=\rho\sin\lambda\cos\mu$, $b=\rho\sin\lambda\sin\mu$, ~~sin~~ , $c=\rho\cos\lambda$ in (1.10) and expanding the radical in increasing powers of $1/\rho$, he states that the nth coefficient is a homogeneous function of x,y, and z of degree n. This fact is already used in a fundamental way in Legendre's first paper (a. above).

There were several other papers presented during the last decade of the eighteenth century by Laplace, Legendre, Lagrange, and others. From the standpoint of celestial mechanics in general and of the problem of attractions in particular, they are of considerable importance; but one finds in them no distinct contributions to the theory of Legendre Polynomials or of Laplace Coefficients, and it is work of the latter kind with which we shall be primarily concerned hereafter.

15.) Legendre also gives the values of $\int_0^1 P_m P_n\,dx$ for $m=n$ and $m\neq n$. A simple presentation of these values is given by Todhunter, I., Note on the Value of a Certain Definite Integral, Proc. Roy. Soc. Lond., 23(1875)300-1. A more complex treatment of these integrals is given by J.W. Strutt (Lord Rayleigh), On the values of the integrals $\int_0^1 Q_n Q_{n'}\,d\mu$, $Q_n, Q_{n'}$ being Laplace's Coefficients of the orders n, n' with an application to the Theory of Radiation. Phil. Trans. 160(1870)579-590.

16.) J. Lagrange--Mémoire sur les spheroides elliptiques, Nouv. Mem. de l'Ac. roy. des Sc. et Belles-Lettres de Berlin (1792-3); Oeuvres 5(1870) 645-660

The first volume of the memoirs of the Berlin Academy of Sciences to appear in the nineteenth century contains a paper read by one J. Trembley.[17] The author takes the explicit expression for $P_n(x)$ and from it derives the Legendre differential equation:

(1.24) $\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)P_n = 0.$

Then he aims to demonstrate that $P_n(1)=1$ as follows. In (1.24) set $x=1$, then (i) $2x\frac{dP_n}{dx} = n(n+1)P_n$ or

(ii) $\frac{2}{P_n}\frac{dP_n}{dx} = \frac{n(n+1)}{x}$, from which

(iii) $\log P_n^2 = \log Kx^{n(n+1)}$ (k constant), and hence

(iv) $P_n^2(x) = Kx^{n(n+1)}$ and

(v) $P_n(1) = \sqrt{K}$.

Now k is independent of n, and since $P_1(1)=\sqrt{K}=1$, then $P_n(1)=1$. The author was apparently oblivious to the fact that he was letting x be variable and equal to one simultaneously. Thus (i) and (ii) hold only for $x=1$, and (ii) is not even a differential equation.

Other results given in Trembley's work are likewise neither new nor presented more satisfactorily than in the work of Legendre (a., c. above). Integrating (1.24) from o to x and setting $x=0$ to determine the constant of integration, Trembley obtains (1.25) $(1-x^2)P_n' + n(n+1)\int_0^x P_n dx = P_n'\Big]_{x=0}$; Let $x=1$,

(1.26) $n(n+1)\int_0^1 P_n dx = \frac{dP_n}{dx}\Big]_{x=0}$.

Now, if n is even, $P_n'(x)$ has a factor x and then the right member of (1.25) will be zero; if n is odd, Trembley gives

(1.27) $\frac{dP_n}{dx}\Big]_{x=0} = (-1)^{\frac{n-1}{2}} \frac{3\cdot 5\cdot 7\cdots n}{2\cdot 4\cdot 6\cdots(n-1)}$.

17.) J. Trembley-Observations sur l'attraction et l'équilibre des sphéroides. Mem. de L'Ac. des Sc. et B.-L. de Berlin (1799-1800) 68-109.

In general, if (1.24) is multiplied by x^m and integrated from o to x, he obtains

$$(1.28) \quad \int_0^x x^m \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)\int_0^x x^m P_n(x)\,dx = 0.$$

Integrating the first term by parts, and taking x=1 for the upper limit of integration, he presents us with a known formula of Legendre (1.5), with a superfluous $\pm$ sign before the right side of the equation. There is no evidence that Trembley misled anyone into similar error.

About the same time there were published the definitive, early volumes of the "Mécanique Céleste", in which was repeated the work of Laplace's memoir of 1782 (d. above) in a more elegant form. In addition to establishing Legendre's observation that a function of μ and θ could be expanded in series of Laplace Coefficients (1.21) in only one way, the minor errors of Laplace's previous work are corrected, virtually all of which were remarked by Legendre. For several years after, Laplace's treatise was considered an infallible source book, and little original work was forthcoming. Yet it would be false to say that interest in the problem of attractions waned, for mathematicians were evidently thoroughly digesting Laplace's work.

To end this period of quiescence came a paper by Ivory reviewing the solutions of the problems of ellipsoidal attractions as given by Legendre and Laplace.[18] Shortly afterward, Ivory presented a paper in which he very hesitantly, for Laplace's treatise had come to be looked on with reverence, indicated a fundamental error in the work of Laplace.[19] The same results had been published by Lagrange a little earlier; but Lagrange's work did not arrive in England until Ivory's paper had been read to the Royal

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18.) J. Ivory--On the Attractions of Homogeneous Ellipsoids, Phil Trans (1809) 345-372.

19.) J. Ivory--On the grounds of the Method ~~of~~ which Laplace has given in the second chapter of the third book of his Mécanique Céleste for computing the Attractions of Spheroids of every description, Phil. Trans. 102(1812) 1-45.

Society of London.[20] Laplace's error consisted in asserting that <u>any</u> function could be expanded in a series of Laplace Coefficients. Ivory and Lagrange quite correctly maintained that Laplace proved the theorem only for rational integral functions of μ', $\sqrt{1-\mu'^2}\cos\omega'$, $\sqrt{1-\mu'^2}\sin\omega'$ [μ', ω' as in (1.12)] and consequently, Laplace's work on the attraction of spheroids held only for a restricted class of spheroids in which the radii can be expressed as such rational integral functions.

The usefulness of the Laplace Coefficients was thus restricted; and when Legendre introduced elliptic integrals into the problem of the attraction of a homogeneous ellipsoid, the application of Laplace Coefficients was even less necessary.[21] However, a year later, Legendre's "Exercise du Calcul Intégral" appeared, and its presentation of the Legendre Polynomial was so elegant as to attract considerable attention.

20.) J. Lagrange--Éclaircissement d'une difficulté singulière qui se rencontre dans le Calcul de l'Attraction des Sphéroides très-peu différens de la Sphère, Jour de l'Ec. Polytech. 15(1809)57-67

21.) A.M. Legendre--Mémoire sur l'attraction des ellipsoides homogènes, Mem. de l'Institut de France (1810)#2, pp.155-183.

SECTION II

DEVELOPMENT OF ELEMENTARY FORMAL PROPERTIES

In the first edition of Legendre's "Exercise du Calcul Intégral" was published the first significant advances in the theory of Legendre Polynomials since the Legendre memoir of 1784. Beginning with the simplified generating function,[22]

(2.1) $T=(1-2xz+z^2)^{-1/2}=\sum_{n=0}^{\infty} z^n P_n(x), \quad |x|\leqq 1, \ |z|<1,$

Legendre presents the general formula for the polynomials

(2.2) $P_n(x)=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}x^n-\frac{1\cdot 3\cdot 5\cdots(2n-3)}{(n-2)!}\frac{x^{n-2}}{2}+\frac{1\cdot 3\cdot 5\cdots(2n-5)}{(n-4)!}\frac{x^{n-4}}{2\cdot 4}-\cdots,$

and exhibits them up to the ninth degree.[23] In a series of theorems, he establishes other fundamental properties of the polynomials. Considering $T=1/(1-z)$ for $x=1$ and $T=[1-2x(-z)+(-z)^2]^{-1/2}=\sum_{n=0}^{\infty}(-1)^n z^n P_n(x)$ for x replaced by $-x$, obviously

(2.3) $P_n(1)=1$, $P_n(-x)=(-1)^n P_n(x)$.

Further, let $x=\cos\varphi$, $\alpha=e^{i\varphi}$, $\beta=e^{-i\varphi}$. Then $T=(1-\alpha z)^{-1/2}(1-\beta z)^{-1/2}$ or

(2.4) $T=\left[1+\frac{1}{2}\alpha z+\frac{1\cdot 3}{2\cdot 4}\alpha^2 z^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\alpha^3 z^3+\cdots\right]\left[1+\frac{1}{2}\beta z+\frac{1\cdot 3}{2\cdot 4}\beta^2 z^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\beta^3 z^3+\cdots\right].$

The coefficient of z^n is then $P_n(\cos\varphi)=\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\alpha^n+\frac{1\cdot 3\cdots(2n-3)}{2\cdot 4\cdots(2n-2)}\frac{\alpha^{n-1}\beta}{2}+\frac{1\cdot 3\cdots(2n-5)}{2\cdot 4\cdots(2n-4)}\frac{1\cdot 3}{2\cdot 4}\alpha^{n-2}\beta^2+\cdots+\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\beta^n;$

or since $\alpha\beta=1$ and the imaginary part is zero,

(2.5) $P_n(\cos\varphi)=\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\left[\cos n\varphi+\frac{1\cdot n}{1(2n-1)}\cos(n-2)\varphi+\frac{1\cdot 3\cdot n(n-1)}{1\cdot 2(2n-1)(2n-3)}\cos(n-4)\varphi+\cdots+\cos(n-2n)\varphi\right].$

The coefficients on the right in (2.5) are all positive; thus $P_n(x)$ takes its maximum in $[-1,1]$ for $x=1$, and its minimum is greater than or equal to $-P_n(1)$.

22.) Here we present Legendre's work (vol. 2, pp. 247-263) in a more extended and more modern guise. That will be our plan in this and succeeding sections, to present historically theoretical developments without losing sight of the desirability of an organized logical structure with a consistent notation.

23.) (2.2) may be intérpreted as defining the Legendre Polynomial for all x.

Hence

(2.6) $|P_n(x)| \leqq 1 \quad \text{for } -1 \leqq x \leqq 1.$

Now Legendre applied Laplace's differential equation to T and obtained

$$(1-x^2)\frac{\partial^2 T}{\partial x^2} - 2x\frac{\partial T}{\partial x} + z\frac{\partial^2 (zT)}{\partial z^2} = 0.$$

If in this the coefficients of z^n are equated, we have the Legendre differential equation

(2.7) $(1-x^2)\dfrac{d^2 P_n}{dx^2} - 2x\dfrac{dP_n}{dx} + n(n+1)P_n = 0.$

((for $x = \cos\theta$, this yields

(2.7a) $\dfrac{1}{\sin\theta}\dfrac{d}{d\theta}\left\{\sin\theta\dfrac{dP_n}{d\theta}\right\} + n(n+1)P_n = 0.$))

Consider, as did Laplace and Legendre, the equation (2.7) for P_n to be multiplied by P_m, and integrate with respect to x between -1 and 1, i. e.,

$$n(n+1)\int_{-1}^{1} P_m P_n\,dx = -\int_{-1}^{1} P_m\left[(1-x^2)\frac{d^2P_n}{dx^2} - 2x\frac{dP_n}{dx}\right]dx = -\int_{-1}^{1} P_m\frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right]dx$$

$$= -(1-x^2)P_m\frac{dP_n}{dx}\Big]_{-1}^{1} + \int_{-1}^{1}(1-x^2)\frac{dP_m}{dx}\frac{dP_n}{dx}dx = \int_{-1}^{1}(1-x^2)\frac{dP_m}{dx}\frac{dP_n}{dx}\,dx.$$

Similarly, $m(m+1)\int_{-1}^{1} P_m P_n\,dx = \int_{-1}^{1}(1-x^2)\dfrac{dP_m}{dx}\dfrac{dP_n}{dx}dx.$

Thus $[n(n+1)-m(m+1)]\int_{-1}^{1} P_m P_n\,dx = 0$ holds for all integral m and n; and if $m \neq n$, $\int_{-1}^{1} P_m P_n\,dx = 0$. This property is called the orthogonality property of the sequence of Legendre polynomials, if the integral has a non-zero value for m=n. This can be verified and the orthogonality property obtained at the same time. Consider

$$\int_{-1}^{1}\frac{dx}{\sqrt{1-2xu+u^2}\sqrt{1-2xv+v^2}} = \frac{1}{\sqrt{uv}}\log\frac{1+\sqrt{uv}}{1-\sqrt{uv}} = 2\left[1+\frac{uv}{3}+\frac{u^2v^2}{5}+\cdots\right] = \int_{-1}^{1}(P_0+uP_1+u^2P_2+\cdots)(P_0+vP_1+v^2P_2+\cdots)\,dx.$$

Then $\int_{-1}^{1} P_m P_n\,dx$ is the coefficient of $u^m v^n$ in the above. Hence[24]

(2.8) $$\int_{-1}^{1} P_m P_n\,dx = \begin{cases} 0 & (m \neq n) \\ \dfrac{2}{2n+1} & (m = n). \end{cases}$$

24.) A proof similar to this appears in J.W.L. Glaisher's "Notes on Laplace's Coefficients", Proc. Lond. Math. Soc. 6(1875)126-136.

Returning to (2.2), we see that x^n can be represented by a series involving Legendre Polynomials, as follows:

$$x^n = a_n P_n + a_{n-2} P_{n-2} + \cdots$$

in which the coefficients can be determined by means of (2.8).[25] For,

$$(2.9) \quad a_m = \frac{2m+1}{2}\int_{-1}^{1} x^n P_m \, dx \qquad (n \geqq m).$$

Obviously

$$(2.10) \quad \int_{-1}^{1} x^n P_m \, dx = 0 \qquad (n = 0, 1, 2, \cdots, m-1) \qquad \text{or}$$

$$(2.11) \quad \int_{-1}^{1} G_{m-1}(x) P_m \, dx = 0,$$

where $G_{m-1}(x)$ is an arbitrary polynomial of degree $\leqq m-1$ in x.

Thus, Legendre obtains

$$(2.12) \quad x^n = \frac{(n!)^2 2^n}{(2n)!}\left[P_n + \frac{2n-3}{2} P_{n-2} + \frac{(2n-7)(2n-1)}{2 \cdot 4} P_{n-4} + \cdots\right].$$

Of considerable interest are the zeros of the polynomials. Legendre shows that they are real, distinct, in $(-1,1)$ and distributed symmetrically with respect to the origin. For, since $\int_{-1}^{1} P_n dx = 0$, with P_n not identically zero, there exists at least one zero, say a_1, of P_n in $(-1,1)$. But $P_n(a_1) = (-1)^n P_n(-a_1)$ and $x^2 - a_1^2$ is a factor of P_n. Since $\int_{-1}^{1} (x^2-a_1^2) P_n \, dx = \int_{-1}^{1} (x^2-a_1^2) F_{n-2} \, dx = 0$, there exists a zero of F_{n-2}, say a_2, in $(-1,1)$ and $x^2-a_2^2$ is a factor of P_n. In a similar manner for $n = 2m$, $\int_{-1}^{1} (x^2-a_1^2)(x^2-a_2^2)\cdots(x^2-a_{m-1}^2) P_{2m} \, dx = 0$; and consequently $P_{2m} = (x^2-a_1^2)(x^2-a_2^2)\cdots(x^2-a_m^2)$.

From (2.2) it appears that x is a factor of P_{2m+1}. Hence, in this case

$\int_{-1}^{1} x(x^2-a_1^2)(x^2-a_2^2)\cdots(x^2-a_{m-1}^2) P_{2m+1} \, dx = 0$, from which $P_{2m+1} = x(x^2-a_1^2)(x^2-a_2^2)\cdots(x^2-a_m^2)$.

From (2.7) and the result of differentiating (2.7) $n-2$ times with respect to x, P_n is identically zero if there is a multiple root of P_n in $(-1,1)$.

25.) Legendre actually gives

$$\int_{-1}^{1} P_m x^n \, dx = \begin{cases} 0 & (m > n) \\ \dfrac{2^{n+1}(n!)^2}{(2n+1)!} & (m = n) \\ 0 & (m < n,\ m+n \text{ odd}) \\ 2^{m+1}\dfrac{n!}{(n+m+1)!}\dfrac{[\frac{1}{2}(n+m)]!}{[\frac{1}{2}(n-m)]!} & (m < n,\ m+n \text{ even}) \end{cases}$$

For derivations of these and related integrals, see A.R. Forsyth: On the integrals $\int_{-1}^{1} P_i \mu^m d\mu$, $\int_{-1}^{1} P_i \cos m\theta \, d\mu$, $\int_{-1}^{1} P_i \nu^m d\mu$, $\int_{-1}^{1} P_i \sin m\theta \, d\mu$,

Quart. Jour. of Math. 17(1880)37-46.

Repeated differentiation of (2.7) leads to some results interesting in themselves. Already (p.19) ~~we~~ have

(2.13) $n(n+1)\int_{-1}^{1} P_m P_n dx = \int_{-1}^{1}(1-x^2)\frac{dP_m}{dx}\frac{dP_n}{dx}dx = \begin{cases} 0 & (m \neq n) \\ \frac{2n(n+1)}{2n+1} & (m=n). \end{cases}$

Differentiating (2.7), $(1-x^2)\frac{d^3P_n}{dx^3} - 4x\frac{d^2P_n}{dx^2} + (n+2)(n-1)\frac{dP_n}{dx} = 0.$

Multiplying by $1-x^2$, $\frac{d}{dx}\left\{(1-x^2)^2\frac{d^2P_n}{dx^2}\right\} = -(n+2)(n-1)\frac{dP_n}{dx}.$

But $-\int_{-1}^{1}\frac{dP_m}{dx}\frac{d}{dx}\left\{(1-x^2)^2\frac{d^2P_n}{dx^2}\right\}dx = (n+2)(n-1)\int_{-1}^{1}(1-x^2)\frac{dP_n}{dx}\frac{dP_m}{dx}dx,$

$$-(1-x^2)^2\frac{dP_m}{dx}\frac{d^2P_n}{dx^2}\Big]_{-1}^{1} + \int_{-1}^{1}(1-x^2)^2\frac{d^2P_n}{dx^2}\frac{d^2P_m}{dx^2}dx = \begin{cases} 0 & ,(m\neq n) \\ \frac{2}{2n+1}(n+2)(n+1)n(n-1), & (m=n). \end{cases}$$

Hence,

(2.14) $\int_{-1}^{1}(1-x^2)^2\frac{d^2P_m}{dx^2}\frac{d^2P_n}{dx^2}dx = \begin{cases} 0 & (m\neq n) \\ \frac{2}{2n+1}\frac{(n+2)!}{(n-2)!} & (m=n). \end{cases}$

These formulas admit the generalization

(2.15) $\int_{-1}^{1}(1-x^2)^k\frac{d^kP_m}{dx^k}\frac{d^kP_n}{dx^k}dx = \begin{cases} 0 & (m\neq n) \\ \frac{2}{2n+1}\frac{(n+k)!}{(n-k)!} & (m=n), \end{cases}$

and $\int_{-1}^{1}(1-x^2)^k G_{n-k-1}(x)\frac{d^kP_n}{dx^k}dx = 0,$ where G is defined as above-(p.20).

These appear in Legendre's work and again in a paper by Ivory.[26] In the latter, Ivory gives the generating function and differential equation of the Legendre Polynomials and ~~the~~ differentiates repeatedly the differential equation, obtaining

(2.16) $(n-k)(n+k+1)(1-x^2)^k\frac{d^kP_n}{dx^k} + \frac{d}{dx}\left\{(1-x^2)^{k+1}\frac{d^{k+1}P_n}{dx^{k+1}}\right\} = 0,$

and the orthogonality property of the derivatives as in (2.15).[27] He gives also the explicit expression

(2.17) $\frac{d^kP_n}{dx^k} = \frac{(2n)!}{(n-k)!\,n!\,2^n}\left\{x^{n-k} - \frac{(n-k)(n-k-1)}{2(2n-1)}x^{n-k-2} + \frac{(n-k)(n-k-1)(n-k-2)(n-k-3)}{2\cdot4\cdot(2n-1)(2n-3)}x^{n-k-4}\dots\right.$

Ivory's interest is in using a series of differentiated Legendre Polynomials to express the Laplace Coefficients. His expression was already known to Legendre and Laplace in a more complex form (1.21).

26.) J. Ivory, On the Attractions of an extensive Class of Spheroids, Phil. Trans. 102(1812)46-82.

27.) A very simple proof of (2.15) is given by D.D. Heath, On Laplace's Coefficients and Functions, Quart. Jour. of Math. 7(1866)23-36.

The most remarkable simplification of the expression appeared in the 1813 thesis of Rodrigues.[27] There for the first time is demonstrated the most compact representation of the Legendre Polynomials

$$(2.18) \quad P_n(x) = \frac{1}{n!\,2^n} \frac{d^n (x^2-1)^n}{dx^n}.$$

This expression has been variously attributed to Ivory, Jacobi, and Murphy, but it is clear that it appeared first in Rodrigues' work in the little known "Correspondance sur l'École Polytechnique" published for only a period of three years. It is also evident that Ivory arrived independently at the same formula in 1824.[28] Further, Jacobi's work in Crelle's Journal für Mathematik for 1827 indicates unfamiliarity with both preceding papers.[29] Although it is claimed by overenthusiastic Englishmen that Murphy in his treatise on electricity (1833) deduced the same results independently, this is not so evident in his work.[30]

In Rodrigues' dissertation appears another important formula,

$$(2.19) \quad \frac{(m+n)!}{(m-n)!} \frac{d^{m-n}(x^2-1)^m}{dx^{m-n}} = (x^2-1)^n \frac{d^{m+n}(x^2-1)^m}{dx^{m+n}},$$

which Ivory unquestionably would have derived from (2.17) had he been aware of (2.18).[31]

27.) O. Rodrigues - Mémoire sur l'attraction des sphéroides (Thesis) Correspondance sur l'École royale polytechnique 3(1816)361-85.

28.) J. Ivory - On the figure requisite to maintain the equilibrium of a homogeneous fluid mass that revolves upon an axis, Phil. Trans. 114(1824)85-150.

29.) C.G.J. Jacobi - Ueber eine besondere Gattung algebraisches Functionen, die aus der Entwichlung der Function $(1-2xz+z^2)^{-\frac{1}{2}}$ entstehen, Jour. für Math. 2(1827)223-6, Werke 6(1891)21-5.

30.) R. Murphy - Elementary Principles of the theories of electricity, heat, and molecular actions, Cambridge 1833, p.7.

31.)For proof of (2.19)see - J.W.L. Glaisher, On Rodrigues' Theorem, Mess. of Math. (2)9(1879)155-160; E.W.Hobson, Proof of Rodrigues' Theorem, Mess. of Math. (2)9(1879)53-4; W.Walton, Two Demonstrations of a Theorem due to Rodrigues, Quart. Jour. of Math. (1)15(1878)335-7; W.H.Hudson, On a Theorem due to Rodrigues, Mess. of Math. (2)7(1877)117.

Rodrigues arrived at his results with some difficulty, but both Ivory and Jacobi resort to the Lagrange expansion of the generating expansion. Assume that $1-\gamma z=\sqrt{1-2xz+z^2}$ then $\gamma = x+\frac{1}{2}z(\gamma^2-1)$. Consider γ as a function of x and apply Lagrange's method, then $\gamma = x+\frac{1}{2}z(x^2-1)+\frac{1}{2!}\frac{1}{2^2}z^2\frac{d(x^2-1)^2}{dx}+\cdots$,

and $\sqrt{1-2xz+z^2} = 1-xz-\frac{1}{2}z^2(x^2-1)-\frac{1}{2!}\frac{1}{2^2}z^3\frac{d(x^2-1)^2}{dx}-\cdots$.

Differentiating with respect to x and dividing by -z,

$$T=\frac{1}{\sqrt{1-2xz+z^2}}=1+\frac{1}{2}z\frac{d(x^2-1)}{dx}+\frac{1}{2!}\frac{1}{2^2}z^2\frac{d^2(x^2-1)^2}{dx^2}+\cdots,$$

from which the remarkable form of the Legendre Polynomials as nth differential coefficients follows.

Another exceedingly curious representation of the Legendre Polynomials was made known by Laplace in the "Paris Mémoirs" of 1817; and it appears with a considerably revised proof in the final volumes of "Mécanique Céleste"(1825).[32] In the original memoir no formal proof is really presented, but a vague reference is made to previous work on probability integrals. This situation is only slightly remedied in the "Mécanique Céleste". However, adequate proof can be easily provided. A well known integral is

$$\int_0^{2\pi}\frac{d\varphi}{u-v\cos\varphi}=\frac{2\pi}{\sqrt{u^2-v^2}},$$

where v/u is real and $|u-\sqrt{u^2-v^2}|<|v|$. Thus

$$\frac{\pi}{\sqrt{1-2xz+z^2}}=\int_0^{\pi}\frac{d\varphi}{1-xz-z\cos\varphi\sqrt{x^2-1}}$$, and comparing coefficients of z^n, we have

(2.20) $$P_n=\frac{1}{\pi}\int_0^{\pi}(x-\cos\varphi\sqrt{x^2-1})^n d\varphi,$$

which is the famous Laplace Integral Formula. Since

$$P_n=\frac{1}{\pi}\int_0^{\pi}[x^n-\binom{n}{1}x^{n-1}\sqrt{x^2-1}\cos\varphi+\binom{n}{2}x^{n-2}(x^2-1)\cos^2\varphi-\cdots]d\varphi,$$

$$P_n=\frac{1}{\pi}\int_0^{\pi}[x^n+\binom{n}{2}x^{n-2}(x^2-1)\cos^2\varphi+\binom{n}{4}x^{n-4}(x^2-1)^2\cos^4\varphi+\cdots]d\varphi,$$

as $\int_0^{\pi}\cos^{2n-1}\varphi\, d\varphi=0$. Thus, equally well

(2.20a) $$P_n=\frac{1}{\pi}\int_0^{\pi}(x+\cos\varphi\sqrt{x^2-1})^n d\varphi.$$

32.) Laplace, P.S. - Mémoires sur la figure de la Terre. Mem. de l'Ac. roy. des. Sc. de Paris (2)2(1817). Oeuvres complètes 12(1898)415-459.

The Laplace memoir is noteworthy also for a discussion of the behavior of P_n for large values of n, and its conclusion

(2.21) $$P_n(\cos\theta) \sim \sqrt{\frac{2}{n\pi\sin\theta}}\cos\left[(n+\tfrac{1}{2})\theta-\tfrac{\pi}{4}\right].$$

No estimate of the error is made, and some later writers, Todhunter notably,[33] accepted the result with reluctance because it does not hold for $\theta=0$ and is unsatisfactory when θ is small, which Laplace did not observe apparently. However, Laplace's result was verified by Cauchy in 1829 in a general treatment of definite integrals related to certain types of Lagrange expansions of which the Laplace integral is an example.[34] Todhunter's major objection is that $P_n(\cos\theta)$ should approximate unity for $\theta=0$. This difficulty was resolved by Sharpe, who showed for what values of θ (2.21) gave a good approximation and the nature of $P_n(\cos\theta)$ when θ does not have these values.[35]

We give an exceedingly simple derivation of (2.21). The differential equation (2.7a) is sometimes written as

$$\frac{d^2P_n}{d\theta^2}+\cot\theta\frac{dP_n}{d\theta}+n(n+1)P_n=0.$$

Assuming P_n of the form $u\cos a\theta + u'\sin a\theta$, where u and u' are functions of θ and $a=\sqrt{n(n+1)}$, substituting in the differential equation, and equating coefficients of $\sin a\theta$ and $\cos a\theta$, $2\frac{du}{d\theta}+u\cot\theta=\frac{1}{a}\left(\frac{d^2u'}{d\theta^2}+\frac{du'}{d\theta}\cot\theta\right)$,

$$2\frac{du'}{d\theta}+u'\cot\theta=-\frac{1}{a}\left(\frac{d^2u}{d\theta^2}+\frac{du}{d\theta}\cot\theta\right).$$

Since $1/a$ is so small for very large n, the terms on the right are neglected. A first approximation is thus $u=\frac{H}{\sqrt{\sin\theta}}$, $u'=\frac{H'}{\sqrt{\sin\theta}}$; i.e.,

$$P_n(\cos\theta)\sim\frac{1}{\sqrt{\sin\theta}}(H\cos a\theta+H'\sin a\theta)=\frac{c}{\sqrt{\sin\theta}}\cos(a\theta+\gamma).$$

33.) Todhunter, I.- An Elementary Treatise on Laplace's Functions, Lamé's Functions, and Bessel's Functions. London, Mac Millan & Co., (1875) Chap VII

34.) A. Cauchy - Mémoire sur divers points d'Analyse, Mem. de l'Ac. roy. des Sc. de l'inst de France (2)8(1829)101-129 (p.120).

35.) H.J. Sharpe - Note on Legendre's Coefficients, Quart. Jour. of Math. 24(1890)383-6.

Now $a=\sqrt{n(n+1)} \sim n+\frac{1}{2}$, and $P_n \sim \frac{c}{\sqrt{\sin\theta}}\cos(n\theta+\frac{\theta}{2}+\gamma)$.
If n is odd, $P_n=0$ when $\theta=\pi/2$, so $\gamma=-\frac{\pi}{4}$. Further, if n is even and $\theta=\pi/2$,

$$P_{2m}=\frac{(2m)!}{(m!)^2 2^{2m}}(-1)^m = c \sim \frac{1}{\sqrt{m\pi}}=\sqrt{\frac{2}{n\pi}},$$

and (2.21) is established.

The method here given can be extended to more precise approximations as does Heine,[36] who writes

$$(2.21a)\quad P_n(\cos\theta)\sim\sqrt{\frac{2}{n\pi\sin\theta}}\left[\left(1-\frac{1}{4n}\right)\cos\left\{\left(n+\tfrac{1}{2}\right)\theta-\frac{\pi}{4}\right\}+\frac{1}{8n}\cot\theta\sin\left\{\left(n+\tfrac{1}{2}\right)\theta-\frac{\pi}{4}\right\}\right].$$

Laplace does not concern himself with the asymptotic behavior of P_n for values of x outside of (-1,1), but such an extension is easily made. Let $2x=\xi+\frac{1}{\xi}>2$, and $2\sqrt{x^2-1}=\xi-\frac{1}{\xi}$, hence $\xi=x+\sqrt{x^2-1}$ and $\frac{1}{\xi}=x-\sqrt{x^2-1}$. Using $T=(1-\xi z)^{-\frac{1}{2}}\left(1-\frac{z}{\xi}\right)^{-\frac{1}{2}}$ and proceeding as in (2.4), $P_n(x)=\frac{1\cdot3\cdots(2n-1)}{n!\,2^n}\left[\xi^n+\frac{1\cdot n}{1\cdot(2n-1)}\xi^{n-2}+\frac{1\cdot3\cdot n(n-1)}{1\cdot2(2n-1)(2n-3)}\xi^{n-4}+\cdots+\xi^{-n}\right]$

$$(2.22)\qquad =\frac{(2n)!}{(n!)^2\,2^{2n}}\xi^n\left[1+\frac{1\cdot n}{1\cdot(2n-1)}\xi^{-2}+\cdots\right]\sim\frac{1}{\sqrt{n\pi}}\xi^n\left[1+\frac{1}{2}\xi^{-2}+\cdots\right]$$

$$(2.23)\qquad \sim\frac{1}{\sqrt{n\pi}}\xi^n(1-\xi^{-2})^{-\frac{1}{2}}.$$

For $x<-1$, we can use the relation (2.3).

One evident application of the Laplace result is to discussion of the behavior of terms in a series of Legendre Polynomials. Such series, and more generally series of Laplace Coefficients, were considered carefully by Ivory[37] in 1822. He repeats his earlier results[38] concerning the expansion of an arbitrary function of the sines and cosines of two angles θ and φ. Let $\mu=\cos\theta$, then the given function may be developed in a series

$$f(\theta,\varphi)=U_0+U_1+U_2+\cdots$$

in which all the U's satisfy the Laplace differential equation (1.15), without difficulty if $f(\theta,\varphi)$ is a rational integral function of μ, $\sin\varphi\sqrt{1-\mu^2}$ and $\cos\varphi\sqrt{1-\mu^2}$. This is his and Lagrange's previous correction

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36.) Heine, E. - Handbuch der Kugelfunktionen.

37.) J. Ivory - On the expansion in series of the attraction of a Spheroid Phil. Trans. 112(1822)99-112.

38.) See P. 16.

of Laplace's theorem.[39]

Poisson is concerned with the same problem in a paper extending some results on Fourier Series to series of Laplace Coefficients.[40] He proved again the result of Laplace, that the Laplace Coefficient is a rational integral function of μ, $\sqrt{1-\mu^2}\sin\varphi$, and $\sqrt{1-\mu^2}\cos\varphi$ which satisfies the Laplace differential equation. Can any arbitrary function be expressed in series of such coefficients? Poisson believed with Laplace that the question should be answered in the affirmative. However, very obviously his demonstration implies that $f(\theta, \varphi)$ be of class C'. In Poisson's paper the problem is also linked to the Laplace differential equation, and he raises the question of the determination of the general solution of the equation, which had received only slight attention before him. Also, in order to discuss the behavior of terms in series of Legendre Polynomials, particularly in the neighborhood of their extreme values, he adapts Laplace's result (2.21):

$$(2.24)\quad P_n\left(\cos\frac{\theta}{n}\right) \sim \sqrt{\frac{2}{\pi}}\left[\frac{\cos(\theta-\frac{\pi}{4})}{\sqrt{\theta}} + \frac{1^2}{8}\frac{\cos(\theta-\frac{3\pi}{4})}{\sqrt{\theta^3}} + \frac{1^2\cdot 3^2}{8\cdot 16}\frac{\cos(\theta-\frac{5\pi}{4})}{\sqrt{\theta^5}} + \cdots\right].$$

Ivory's 1824 paper takes cognizance of the work of Poisson in attempting to prove Laplace's theorem, but that is not its most noted feature. It is chiefly known for its independent derivation of (2.18). This is also the case with Jacobi's article in the second volume of Crelle's Journal. Jacobi gives an interesting consequence of Rodrigues' formula (2.19). Since $\underbrace{\int_{-1}^{x}\cdots\int_{-1}^{x}}_{(m)} P_m\,dx\cdots dx = \frac{(x^2-1)^m}{m!\,2^m}$, an application of (2.19) yields

$$\underbrace{\int_{-1}^{x}\cdots\int_{-1}^{x}}_{(n)} P_m\,dx\cdots dx = \frac{(m-n)!}{(m+n)!}(x^2-1)^n\frac{d^nP_m}{dx^n}.$$

39.) See p.16–17.

40.) Poisson, S.D. - Addition au Mémoire précédent, et au Mémoire sur la manière d'exprimer les Fonctions par des Séries de quantités périodiques Jour. de l'Ec. Polytechnique 19(1823)145-162.

Furthermore, the polynomial associated with $\frac{d^n[(x-a)^n(x-b)^n]}{dx^n}$ has the same properties as the Legendre Polynomial, only the interval of integration is now (a,b) instead of (-1,1). In particular, the Legendre Polynomials are exactly those functions whose zeros are used by Gauss in his famous paper on evaluating definite integrals. This last observation of Jacobi and his paper of the preceding year (1826) served to stimulate new interest in the whole question of Gaussian (so called, " mechanical") quadrature.

SECTION III

EARLY DEVELOPMENTS IN GAUSS QUADRATURE

The problem of quadrature is essentially to approximate an integral by replacing the integrand curve with an approximating polygon in some convenient way as, for example, Simpson's rule. Thus, to evaluate $\int_a^b f(x)dx$ when $f(x)$ is known for $\alpha_1, \alpha_2, \dots, \alpha_n$, n values of x in (a,b), Newton employs a polynomial of degree n-1 which coincides with f(x) for the given abscissas. Of course, the polynomial should approximate f(x) more closely with increasing n. The abscissas which Newton chose for various degrees were given by him in a table. Gauss obtained a closer approximation using a polynomial of the same degree and as many abscissas as did Newton. The abscissas in Gauss quadrature are the zeros of a polynomial of the nth degree. If (a,b)=(-1,1), the polynomial is the Legendre Polynomial and the general case can be correlated with the special case, for

(3.1) $x = \frac{b+a}{2} + \bar{x}\frac{b-a}{2}, \quad \int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b+a}{2} + \bar{x}\frac{b-a}{2}\right)d\bar{x}$

transforms (a,b) into (-1,1).

The first problem is to construct a polynomial which will coincide with the given function f(x) for n given values of x. Of all the functions which will coincide with the given function at (α_1, β_1), $(\alpha_2, \beta_2), \dots,$ (α_n, β_n) there is only one integral function of degree n-1. If $\varphi(x) = (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$, we use the Lagrange approximating polynomial

(3.2) $F(x) = \varphi(x)\left[\frac{\beta_1}{(x-\alpha_1)\varphi'(\alpha_1)} + \frac{\beta_2}{(x-\alpha_2)\varphi'(\alpha_2)} + \dots + \frac{\beta_n}{(x-\alpha_n)\varphi'(\alpha_n)}\right].$

Certainly this function satisfies the conditions. If there were another such function, G(x), of degree n-1, F(x) would be identical with G(x) for the n values $\alpha_1, \alpha_2, \dots, \alpha_n$; and so, F(x) is identical with G(x). It is evident that F(x) coincides with f(x) at the n given points, so we can write

(3.3) $$f(x) \sim \varphi(x)\sum_{i=1}^{n}\frac{F(\alpha_i)}{(x-\alpha_i)\varphi'(\alpha_i)}.$$

If $f(x)$ is integrable and the conditions of approximation are fulfilled, i.e. $\epsilon > 0 \ \exists \ N_\epsilon \ni |f(x)-F(x)| < \epsilon$ for $n > N_\epsilon$ and the abscissas such that $|\alpha_i - \alpha_{i-1}| < 1/n$, then

(3.4) $\int_{-1}^{1} f(x)dx \sim \int_{-1}^{1} F(x)dx$.

Let (3.5) $A_i = \frac{1}{\varphi'(\alpha_i)} \int_{-1}^{1} \frac{\varphi(x)}{x-\alpha_i}dx$.

Whereupon $\int_{-1}^{1} f(x)dx \sim \sum_{i=1}^{n} A_i F(\alpha_i)$.

All this restricts the manner of choosing the α's only in that

$|\alpha_i - \alpha_{i-1}| \longrightarrow 0$ as $n \longrightarrow \infty$.

In $(-1,1)$, Cotes[41] takes the α's in an arithmetic progression, $\alpha_i = \frac{n-2i+1}{n-1}$, and calculates the A's up to the case for n=11. Gauss quadrature likewise uses α's symmetrically distributed with respect to the origin. Hence $\alpha_{n-m+1} = -\alpha_m$, $\alpha_1 > \alpha_2 > \cdots > \alpha_n$. If n is odd, one α is zero. The A's are similarly symmetrical, for $\varphi(x) = (-1)^n \varphi(-x)$, $\varphi'(x) = (-1)^{n-1}\varphi'(-x)$, and

$$A_{n-m+1} = \frac{1}{\varphi'(\alpha_{n-m+1})} \int_{-1}^{1} \frac{\varphi(x)}{x-\alpha_{n-m+1}}dx = \frac{(-1)^{n-1}}{\varphi'(\alpha_m)} \int_{-1}^{1} \frac{\varphi(x)}{x+\alpha_m}dx = \frac{-1}{\varphi'(\alpha_m)} \int_{-1}^{1} \frac{\varphi(-x)}{x+\alpha_m}dx$$

$$= \frac{1}{\varphi'(\alpha_m)} \int_{-1}^{1} \frac{\varphi(x)}{x-\alpha_m}dx = A_m.$$

In any event, the degree of approximation depends on n, the nature of $f(x)$, and the choice of the α's. Furthermore,

(3.6) $\int_{-1}^{1} f(x)dx = A_1 F(\alpha_1) + A_2 F(\alpha_2) + \cdots + A_n F(\alpha_n) = \sum_{i=1}^{n} A_i F(\alpha_i)$

gives the true value of $\int_{-1}^{1} f(x)dx$ only when $f(x)$ is of degree $\leqq n-1$.

Suppose we were to seek the error. Set

(3.7) $E(f) = \int_{-1}^{1} f(x)dx - \sum_{i=1}^{n} A_i F(\alpha_i)$.

Then $E(f_1+f_2) = E(f_1) + E(f_2)$, and $E(Kf) = KE(f)$. In particular, if $f(x) = K_0 + K_1x + K_2x^2 + \cdots$, we have

$$E(f) = K_n E(x^n) + K_{n+1} E(x^{n+1}) + \cdots .$$

More specially

(3.8) $E(x^m) = \int_{-1}^{1} x^m dx - A_1\alpha_1^m - A_2\alpha_2^m - \cdots - A_n\alpha_n^m$.

Equating (3.8) to zero for $m=0,1,2,\ldots,n-1$ gives n linear equations for

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41.) Harmonia Mensurarum (1722).

determining the A's which are likewise n in number.

The achievement of Gauss was in showing that the degree of approximation could be made 2n by a proper choice of the abscissas, still in a symmetrical way. In this case, the error would be zero if the function approximated was a polynomial of degree 2n-1, i.e. $E(f) = K_{2n}E(x^{2n}) + K_{2n+1}E(x^{2n+1}) + \cdots$ The α's are none other than the zeros of the Legendre Polynomial of degree n. While this last fact was not immediately evident to Gauss, one can demonstrate it in a straightforward fashion.

Approximating $f(x) = K_0 + K_1x + K_2x^2 + \cdots + K_{2n-1}x^{2n-1}$ by a Lagrange approximation polynomial $F_{n-1}(x)$ of degree n-1 built from n values of the α's,

$$\frac{f(x) - F_{n-1}(x)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)} = \frac{f(x) - F_{n-1}(x)}{\varphi(x)} = q_0 + q_1x + \cdots + q_{n-1}x^{n-1} = G_{n-1}(x).$$

We seek $E(f) = \int_{-1}^{1}[f(x) - F_{n-1}(x)]\,dx = \int_{-1}^{1}\varphi(x)\,G_{n-1}(x)\,dx = 0.$

This is precisely (2.11), and $\varphi(x) = P_n(x)$, within a constant factor. Thus only in the case of Gauss quadrature will $\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} A_i F(\alpha_i)$ hold exactly if f(x) is a polynomial of degree $\leq 2n-1$. And if

$$f(x) = K_0 + K_1x + K_2x^2 + \cdots + K_{2n}x^{2n} + \cdots, \qquad E(f) = K_{2n}E(x^{2n}) + \cdots.$$

As the problem is solved originally by Gauss, no mention is made of Legendre Polynomials and the entire question is related to continued fractions and hypergeometric functions.[42] An earlier paper of Gauss gives the continued fraction representation for a hypergeometric function.[43] The problem of quadrature is solved by writing $\log\frac{1+t}{1-t}$ as a hypergeometric function and then as a continued fraction; and by taking for the abscissas

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42.) We reserve discussion of this aspect of Gaussian quadrature until we come to the work of Christoffel and others.

43.) C.F. Gauss - Disquisitiones generales circa seriem infinitam $1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots$ pars prior, Comment. soc. reg. scient. Gotting 2(1813), Werke 3(1866)125-162.

the zeros of the denominators of the successive convergents of the continued fraction.[44] These convergents are listed, and it is obvious that they differ from the usual Legendre Polynomials only by constant factors.

In 1826, Jacobi[45] like Gauss is concerned with evaluating $\int_0^1 y\,dt$. After showing that the α's should be the zeros of $\varphi(x)$, a polynomial of degree n, he reduces the problem to finding a polynomial such that $\int_0^1 \varphi(x)x^k dx=0$ for $k=0,1,2,\dots,n-1$. Integrating by parts, he arrives at $\underbrace{\int_0^x\cdots\int_0^x}_{(n)} \varphi(x)\,dx\cdots dx = \Pi(x)$ where $\Pi(x)$ has roots of multiplicity n at $x=0$ and $x=1$. Thus $\varphi(x) = M\dfrac{d^n[x^n(1-x)^n]}{dx^n}$. where M must be constant if $\varphi(x)$ is to be of degree n. In 1827,[46] he considers the interval $(-1,1)$ and arrives at the nth differential coefficient form of the Legendre Polynomial as a solution to the problem of finding a polynomial satisfying $\int_{-1}^1 x^K\varphi(x)\,dx = 0$, $K=0,1,2,\dots,n-1$.

This same problem is the concern of Rev. Murphy in a series of articles on the "inverse method" of definite integrals.[47] He defines

$\psi(K)=\int_0^1 \varphi(t)t^K dt = 0$ for $k=0,1,2,\dots,n-1$ and seeks to determine a polynomial satisfying this condition. Let $\varphi(t) = A_n t^n + A_{n-1}t^{n-1}+\dots+A_1 t+1$, and $\psi(K) = \frac{1}{K+1}+\frac{A_1}{K+2}+\dots+\frac{A_n}{K+n+1} = \frac{P}{Q}$. Since $\psi(K)$ vanishes for $k=0,1,2,\dots,n-1$,

$$P= CK(K-1)(K-2)\cdots(K-n+1), \qquad Q=(K+1)(K+2)\cdots(K+n+1)$$

Now $c=(-1)^n$ and the A's can be found, giving $\varphi(t)$. Observation of the polynomial term by term leads to $\varphi(t) = \frac{1}{n!}\frac{d^n[t^n(1-t)^n]}{dt^n}$, which is the coefficient of z^n in $\left[1-2z(1-2t)+z^2\right]^{-\frac{1}{2}}$. Thus, within a constant factor the Legendre Polynomial, translated so that $x=1-2t$, is determined from the conditions on $\psi(K)$.

~~Murphy makes several applications of the polynomials to problems of~~

44.) C.F. Gauss - Methodus Nova Integralium Valores per approximationem inveniendi, Comment. soc. reg. scient. Gott. 3(1816), Werke 3(1866)165-206.

45.) C.G.J. Jacobi - Ueber Gauss' neue Methode, die Werthe der Integrale näherungsweise zu finden, Jour. für Math. 1'1826)301-8, Werke 6(1891)3-11.

46.) See p. 22.

47.) Rev. R. Murphy - On the Inverse Method of Definite Integrals with Physical Applications, Trans. Camb. Phil. Soc. 4(1833)353-408, 5(1835)113-148, 315-393.

Murphy makes several applications of the polynomials to problems of electrical potential on a sphere, and he discusses several properties of the zeros of the polynomials. These are based on a relation between successive members of the sequence $\{P_n(t)\}$, which Murphy is not the first to produce. Gauss gave the continued fraction of which the $\{P_n(t)\}$ were denominators of the successive convergents, and he also wrote the usual recurrence relation for the denominators of the convergents.[48] Thus

$$(3.9)\qquad nP_n(x) - (2n-1)xP_{n-1}(x) + (n-1)P_{n-2}(x) = 0, \qquad P_0 = 1, \quad P_1 = x,$$

which can be independently derived from equating the coefficients of like powers of x in the easily verified differential equation satisfied by the generating function:

$$(1-2xz+z^2)\frac{\partial T}{\partial z} + (z-x)T = 0.$$

The Legendre Polynomials therefore form a sequence of Sturm functions, and the zeros of $P_{n-1}(x)$ separate those of $P_n(x)$ and are in turn separated by those of $P_{n-2}(x)$.

48.) Gauss-Nova Methoda...paragraph 17, 18, 19.

SECTION IV

STURM - LIOUVILLE

Immediately above is one example of the application of the advances of Sturm to developments in the theory of Legendre Polynomials. It is only fitting to turn to the early volumes of Liouville's Journal and examine their pertinent contents. Prior to the founding of the Journal de Mathematiques, Liouville had already published a memoir on the linear differential equation of second order

(4.1) $(ax^2+bx+c)\frac{d^2y}{dx^2}+(dx+e)\frac{dy}{dx}+fy=0,$ $(a,b,c,d,e,f \text{ constant})$

of which the Legendre differential equation is an example.[49] Legendre gave the procedure for integrating (4.1) in the case where $ax^2+bx+c=K(px+q)^2$, but Liouville is concerned with equations in which this is not necessarily so. By means of generalized differentiation and integration he obtains a méthod for solving the equation.

More important, however, are the papers which appeared in the first two volumes of Liouville's own journal, containing Sturm and Liouville's analysis of the linear differential equation of second order. Sturm's first paper[50] transformed the equation $L(x)\frac{d^2V}{dx^2}+M(x)\frac{dV}{dx}+N(x)V(x)=0$, in which L,M,N are known functions of x, continuous in $[\alpha,\beta]$, such that L does not vanish in $[\alpha,\beta]$, into the equation

(4.2) $\frac{d}{dx}\left\{K(x)\frac{dV}{dx}\right\}+G(x)V(x)=0,$

where $K(x)=e^{\int\frac{M}{L}dx}$ and $G(x)=NK/L$ are continuous functions of x in $[\alpha,\beta]$. If $V=V'=0$ at some point in $[\alpha,\beta]$, V is identically zero. Sturm excludes this possibility, and thus V changes sign each time that it vanishes in $[\alpha,\beta]$.

49.) J. Liouville - Mémoire sur l'intégration de l'équation $(mx^2+nx+p)y''+(qx+r)y'+sy=0$, à l'aide des différentielles à indices quelconques, Jour. de l'École Polytech. 21(1832)163-186.

50.) C. Sturm - Mémoire sur les équations différentielles linéaires du second ordre, Jour. de Math. (1)1(1836)106-186.

The solution V depends on G, K and two arbitrary constants A and B. The chief result of this paper is the Sturm Theorem: V will change sign more often, if K is decreased or G increased.

Liouville applies Sturm's results to the system

(4.3a) $\frac{d}{dx}\left\{K\frac{dV}{dx}\right\} + (gr - l)V = 0,$

(4.3b) $\begin{cases} \frac{dV}{dx} - hV = 0 & \text{for } x = \alpha, \\ \frac{dV}{dx} + HV = 0 & \text{for } x = \beta, \end{cases}$

where g, k, l are continuous functions of x in $[\alpha,\beta]$, g>0 and k>0 in $[\alpha,\beta]$, h and H non-negative constants.[51] This system occurred in his researches on the flow of heat in a heterogeneous bar and attracted the attention of Poisson. For every r, there exists a solution of (4.3a). And in order that (4.3b) hold, it is necessary that r be a root of a certain transcendental equation $\omega(r)=0$. Let the solutions be $\{V_r(x)\}$. Liouville seeks to show that the value of

(4.4) $$\sum_r \left\{\frac{V_r\int_\alpha^\beta g V_r f(x)\,dx}{\int_\alpha^\beta g V_r^2\,dx}\right\}$$

summed over all the values of r satisfying the transcendental equation, is f(x) itself if x is in $[\alpha,\beta]$ and f(x) is arbitrary.[52]

First of all, the roots of $\omega(r)=0$ are infinite in number, all real, distinct, and positive, say $0 < r_1 < r_2 < \cdots < r_m < \cdots < r_n < \cdots$. Let V_1, V_2, ... correspond to the r's. Then

(4.5) $$\int_\alpha^\beta g V_m V_n\,dx = 0 \qquad (m \neq n).$$

There are n-1 zeros of V_n in (α,β) and they are distinct and separate the n zeros of V_{n+1}. Also, $\sum_{i=m}^{n} A_i V_i$, where the A's are constants not all zero,

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51.) Liouville, J. - Mémoire sur le développement des fonctions ou parties de fonctions en séries dont le divers termes sont assujettis à satisfaire une même équation différentielle du second ordre contenant un paramètre variable, Jour. de Math. (1)1(1836)253-265.

52.) The importance of the work of Liouville and Sturm from the standpoint of Legendre Polynomials is evident when the Legendre differential equation is viewed as a special case of (4.2) and (4.3).

has at least m-1 roots and at most n-1 roots in (α,β) but they are not necessarily distinct. Some of these properties are consequences of Sturm's prior memoir; others, as Liouville takes pains to make clear, are contained in a later Sturm memoir with which Liouville as editor and friend of Sturm was already familiar.[53] As for the value of (4.4), he does not really prove the convergence of the series to f(x) but confuses the problem with that of finding the coefficients when f(x) is assumed to have an expansion in series of V's. If an expansion for f(x) in series of V's exists and is uniformly convergent for x in $[\alpha,\beta]$,

(4.6) $f(x)=A_0V_0+A_1V_1+\dots+A_iV_i+\dots = \sum_r A_rV_r$,

and $A_i = \dfrac{\int_\alpha^\beta g\,V_i f(x)\,dx}{\int_\alpha^\beta g\,V_i^2\,dx}$.

Hence (4.4) yields f(x). But this does not prove that the expansion exists, converges, and has the value f(x).

The Sturm memoir mentioned above gives the results quoted by Liouville.[54] The theory of the second order linear differential equation is extended. Bounds for the number of zeros of a solution in a given interval and for the distance between zeros are given. It is to be remarked that Sturm believes Liouville showed that the solution could be obtained in the form of a convergent series. Quite correctly, Sturm maintained that Fourier and others generally confused the problem of the possibility of expressing an arbitrary f(x) by a convergent series of the form (4.6) with the problem of determining the coefficients when the expansion does exist. He declares that Liouville proved that the sum (4.4), if it converges for all x in $[\alpha,\beta]$, can only be f(x).

53.) Actually, most of these are proved in different ways than does Sturm by Liouville in a succeeding memoir, Démonstration d'une théorème du à M. Sturm et relatif à une classe de fonctions transcendantes, Jour de Math. (1)1(1836)269-277.

54.) Sturm, C. - Mémoire sur une classe d'équations à différens partielles, Jour. de Math. (1)1(1836)373-444.

Liouville's argument is insufficient for this conclusion.

Apparently, both Liouville and Sturm shortly recognized this circumstance, for Liouville the next year wrote with dissatisfaction of his attempts to prove the convergence of (4.4) to f(x) unrestricted.[55] He claims to have had a proof for some time that (4.4) can only converge to f(x) for x in $[\alpha,\beta]$ if g,k,f(x) and their first and second derivatives are bounded in $[\alpha,\beta]$. Sturm had also communicated a proof to him which is similar to the one he gives. As a matter of fact, after showing the absolute value of the nth term to be $O(1/n^2)$, Liouville concludes (4.4) is convergent. Nothing is done to show that it converges to f(x).

Writing together in a point paper, Sturm and Liouville hasten to meet this difficulty.[56] They seek the sum of (4.4), which they call F(x). They wish to prove that F(x)=f(x) under the assumption that f(x) is arbitrary but bounded. Before the demonstration goes far, virtually all the assumptions made by Liouville in his previous considerations are implicitly involved; and even so, the proof is no more satisfactory. Liouville himself calls the proof not general enough.[57] He points out that it demands that F(x),g,k, have bounded second derivatives and that f(x) satisfy the conditions for V in (4.3b). Although he distinguishes between the convergence and the sum

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55.) Liouville, J. - Second Mémoire sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujettis à satisfaire à une même équation différentielles du second ordre contenant un paramètre variable, Jour. de Math. (1)2(1837)16-35.

56.) Sturm, C. and Liouville, J. - Extrait d'une Mémoire dur le développement des fonctions en séries dont les différentes termes sont assujettis à satisfaire à une même équation différentielle linéaire, contenant un paramètre variable, Jour. de Math. (1)2(1837)220-235 (abstract, Comptes Rendus 4(1837)675-7)

57.) Liouville, J. - Troisième Mémoire sur le développement des fonctions ou parties de fonctions en séries dont le divers termes sont assujettis à satisfaire à une même équation différentielle du second ordre contenant un paramètre variable, Jour. de Math. (1)2(1837)418-436 (abstract, Comptes Rendus 5(1837)205-7).

of (4.4), he still maintains that when (4.4) is convergent its sum is $f(x)$. The convergence of (4.4), he insists, holds for any bounded $f(x)$; and $f(x)$ may have "jumps" for one or more values of x. However, to show $F(x)=f(x)$ it is necessary to exclude this possibility. Otherwise $F(x)=f(x)$ only for the points of continuity of $f(x)$. The proof is not really valid.

Thus, Sturm discussed the properties of a solution to the differential equation $L(x)V''+M(x)V'+N(x)V=0$,
and showed that these properties are analogous to those of sines and cosines. And Liouville, using these results, generalized Fourier Series, showing the relation between the expansion of a function in series of other functions and the properties of solutions of linear differential equations of the second order. Their work is the basis for the theory of orthogonal functions, of which the classical orthogonal polynomials are a particular instance. The Legendre Polynomials were the first of these latter to be presented in analysis.

SECTION V

INTEGRALS AND SERIES

In the theory of Fourier Series, in the theory of series of which the coefficients are Legendre Polynomials or Laplace Coefficients, in the theory of the expansion of an arbitrary function in terms of any other functions, there is always the fundamental question: What are the restrictions which must be imposed on the given function in order that the series one forms in a preassigned way should converge and should represent the given function? It is this question, limited to expansion in which the coefficients satisfy a linear differential equation of second order, which concerned Liouville. After showing that solutions of the linear differential equation of second order resembled sines and cosines or exponentials in their behavior, he was led to consider the analog of the Fourier Series.

Previously, the same question was considered by Poisson in the case of Laplace Series.[58] Poisson was aware of the relation of Laplace Series to Fourier Series, and his work on Fourier Series encouraged him to attempt to improve the theory of Laplace Series as well. To him perhaps falls the honor for recognizing that the series may not represent the given function, although its coefficients are those one would have if the series did converge and represent the function. In his monumental work on the theory of heat, he apparently was aware of this possibility.[59] Poisson's analysis was directed to showing that if the given function is of class C', the Laplace Series converges and represents the function. While his proof does not meet the demands of modern rigor, the conclusions reached are in

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58.) See p. 26.

59.) S.D. Poisson - Théorie Mathématique de la Chaleur, (Paris, Bachelier 1837), Chap. VIII, 212-32, (also Connaissance des Temps 1829, 1831).

general valid. However, his restrictions are more than is necessary to insure the result.

One phase of the problem is treated in a short note by Liouville on Legendre Series.[60] Liouville has no difficulty showing on the basis of (2.12) that $\Psi(x)$, a rational integral function of degree n in x, can be represented by a finite sum $\sum_{i=0}^{n} A_i P_i(x)$, where $A_i = \frac{2i+1}{2}\int_{-1}^{1}\Psi(t)P_i(t)dt$. But the significant task is to show that if f(x) is bounded and if

$$(5.1) \qquad \sum_{n=0}^{\infty}\left\{\frac{2n+1}{2} P_n(x)\int_{-1}^{1} f(t)P_n(t)dt\right\}$$

exists and converges to F(x) for x in $[-1,1]$, then F(x)=f(x). Liouville multiplies (5.1) by P_n, and integrates term by term with respect to x from -1 to 1, which is tantamount to assuming (5.1) uniformly convergent. By virtue of (2.8), $\int_{-1}^{1}F(x)P_n(x)dx = \int_{-1}^{1}f(x)P_n(x)dx$. Thus $\int_{-1}^{1}[F(x)-f(x)]P_n(x)dx=0$ for all positive integral n, or equally well $\int_{-1}^{1}[F(x)-f(x)]x^n dx=0$. Now an earlier paper of Liouville asserts that F(x)-f(x)=0 in this event.[61] This is his desired result. He also wishes to show that if S_n is the sum of the first n terms of (5.1), then S_n-f(x) changes sign at least n times in (-1, 1) , of utility perhaps in proving that $\lim_{n\to\infty}[S_n-f(x)]=0$.

No careful treatment of Laplace or Legendre Series was really given before Dirichlet.[62] After a magnificent paper in which he gave the famous Dirichlet conditions and integrals for Fourier Series, he turned his attention

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60.) J. Liouville, Sur la sommation d'une série, Jour. de Math. (1)2(1837)107-8.

61) J. Liouville, Solution d'une Problème d'Analyse, Jour. de Math. (1)2(1837)1-2. If $\int_a^b\varphi(x)x^k dx=0$ for all integral k, $\varphi(x)$ bounded and defined in $[a,b]$, then $\varphi(x)=0$. Otherwise $\varphi(x)$ must change sign in (a,b) at least once. Suppose it changes sign at $\alpha_1, \alpha_2, \dots, \alpha_m$. Let $\Psi(x)=(x-\alpha_1)(x-\alpha_2)\dots\dots(x-\alpha_m)$. Then $\int_a^b\Psi(x)\varphi(x)dx\geq 0$, equality holding only when $\varphi(x)=0$.

62.) Dirichlet, P.L., Sur les séries dont le terme général dépend de deux angles, et qui servent à exprimer des fonctions arbitraires entre des limites données, Jour. für Math. 17(1837)35-56.

to Laplace Series. Let $f(\theta,\omega)$ be an arbitrary function of the two angles θ and ω, where $0 \leqq \theta \leqq \pi$ and $0 \leqq \omega \leqq 2\pi$. Laplace asserts that the function can be represented by a series in which the general term is Y_n, a rational integral function of degree n in $\cos\theta$, $\sin\theta \cdot \cos\omega$ and $\sin\theta \cdot \sin\omega$, determined by

$$Y_n = \frac{2n+1}{4\pi}\int_0^{\pi} \sin\theta' d\theta' \int_0^{2\pi} P_n f(\theta',\omega') d\omega',$$

P_n being the coefficient of z^n in $\{1 - 2z[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\omega-\omega')] + z^2\}^{-\frac{1}{2}}$. Ivory and Lagrange pointed out the insufficiency of Laplace's reasoning, except for rational integral functions of $\cos\theta$, $\sin\theta\cos\omega$ and $\sin\theta\sin\omega$. Poisson showed the assertion would hold if $f(\theta,\omega)$ was of class C'. But it was Dirichlet who relaxed the restrictions so that continuous or discontinuous functions could be represented by a Laplace Series.

Dirichlet's attention is centered about showing the convergence and finding the sum of

(5.2a) $$\sum_n \left\{\frac{2n+1}{4\pi}\int_0^{\pi} \sin\theta' d\theta' \int_0^{2\pi} P_n f(\theta',\omega') d\omega'\right\}$$ and

(5.2b) $$\sum_n \left\{\frac{(2n+1)^2}{4\pi}\int_0^{\pi} \sin\theta' d\theta' \int_0^{2\pi} P_n f(\theta'\omega') d\omega'\right\},$$

where $x=\cos\theta \cdot \cos\theta' + \sin\theta \cdot \sin\theta' \cos(\omega-\omega')$ and $f(\theta',\omega')$ is bounded between the limits of integration. The approach is through a consideration of the order of magnitude of the terms in the series. In this connection, Dirichlet develops an entirely new definite integral representation of $P_n(x)$.

Thus he turns to the properties of $P_n(x)$ itself, defining $P_n(x)$ by means of the generating function. Several interesting expressions of $P_n(\cos\theta)$ are then given, all of which appear earlier in Murphy's "Electricity".[63] A trivial form, of course, is that in powers of $\cos\theta$ obtained from (2.2). The expression involving cosines of multiples of θ

63.) See footnote p. 12.

we have already given (2.5). Suppose, however, that $T=(1-2xz+z^2)^{-1/2}$ $=(1-2z\cos\theta+z^2)^{-1/2}=\{(1-z)^2+2z(1-\cos\theta)\}^{-1/2}=(1-z)^{-1}\left\{1+\frac{4z}{(1-z)^2}\sin^2\frac{\theta}{2}\right\}^{-1/2}$ $(|z|<1)$ is expanded in powers of z. We obtain[64]

(5.3) $P_n(\cos\theta)=F(n+1,-n,1,\sin^2\theta/2)$.

Similarly, $T=\{(1+z)^2-2z(1+\cos\theta)\}^{-1/2}=(1+z)^{-1}\left\{1-\frac{4z}{(1+z)^2}\cos^2\frac{\theta}{2}\right\}^{-1/2}$, whence

(5.4) $P_n(\cos\theta)=(-1)^nF(n+1,-n,1,\cos^2\frac{\theta}{2})$.

Again $P_n(x)=\frac{1}{2^n}\frac{1}{n!}\frac{d^n}{dx^n}\{(x+1)^n(x-1)^n\}=\frac{1}{2^n}\{(x+1)^n+\binom{n}{1}^2(x+1)^{n-1}(x-1)+\binom{n}{2}^2(x+1)^{n-2}(x-1)^2+\cdots\}$ by Leibnitz's formula.

Set $x=\cos\theta$, $\frac{1+x}{2}=\cos^2\frac{\theta}{2}$, and $\frac{1-x}{2}=\sin^2\frac{\theta}{2}$,

(5.5) $P_n(\cos\theta)=\cos^{2n}\frac{\theta}{2}\,F(-n,-n,1,-\tan^2\frac{\theta}{2})$.

Murphy also writes

(5.5a) $\frac{4}{\pi}P_n(\cos\theta)=\frac{2^{2n}(n!)^2}{(2n+1)!}\left\{\sin(n+1)\theta+\frac{1\cdot(n+1)}{1\cdot(2n+3)}\sin(n+3)\theta+\frac{1\cdot3\cdot(n+1)(n+3)}{1\cdot2\cdot(2n+3)(2n+5)}\sin(n+5)\theta+\cdots\right\}$.

These representations are not so important as the definite integral expression for $P_n(\cos\theta)$ which accompanies them in Dirichlet's work.

In $T=\{1-2z\cos\theta+z^2\}^{-1/2}=\sum_{n=0}^{\infty}z^nP_n(\cos\theta)$, let $z=e^{i\phi}$. Then

$T=\sum_{n=0}^{\infty}\cos n\phi\,P_n(\cos\theta)+i\sum_{n=0}^{\infty}\sin n\phi\,P_n(\cos\theta)=R+iS$.

At the same time $1-2e^{i\phi}\cos\theta+e^{2i\phi}=e^{i\phi}(e^{i\phi}+e^{-i\phi})-2e^{i\phi}\cos\theta=2e^{i\phi}(\cos\phi-\cos\theta)$, $0<\theta<\pi$, $0<\phi<\pi$.

If $\theta>\phi$, then $\sqrt{\cos\phi-\cos\theta}$ is real and $T=\frac{e^{-\frac{i\phi}{2}}}{\sqrt{2(\cos\phi-\cos\theta)}}=\frac{\cos\frac{\phi}{2}-i\sin\frac{\phi}{2}}{\sqrt{2(\cos\phi-\cos\theta)}}=R_1+iS_1$

If $\theta<\phi$, then $\sqrt{\cos\theta-\cos\phi}$ is real and $T=\frac{ie^{-\frac{i\phi}{2}}}{\sqrt{2(\cos\theta-\cos\phi)}}=\frac{\sin\frac{\phi}{2}+i\cos\frac{\phi}{2}}{\sqrt{2(\cos\theta-\cos\phi)}}=R_2+iS_2$

Now $P_n(\cos\theta)=\frac{2}{\pi}\int_0^\pi R\cos n\phi\,d\phi$ $(n\neq0)$,

$P_0(\cos\theta)=\frac{2}{\pi}\int_0^\pi R\,d\phi$,

$P_n(\cos\theta)=\frac{2}{\pi}\int_0^\pi S\sin n\phi\,d\phi$ $(n\neq0)$.

Hence,

(5.6a) $P_n(\cos\theta)=\frac{2}{\pi}\int_0^\theta\frac{\cos n\phi\cos\frac{\phi}{2}\,d\phi}{\sqrt{2(\cos\phi-\cos\theta)}}+\frac{2}{\pi}\int_\theta^\pi\frac{\cos n\phi\sin\frac{\phi}{2}\,d\phi}{\sqrt{2(\cos\theta-\cos\phi)}}$,

(5.6b) $P_n(\cos\theta)=-\frac{2}{\pi}\int_0^\theta\frac{\sin n\phi\sin\frac{\phi}{2}\,d\phi}{\sqrt{2(\cos\phi-\cos\theta)}}+\frac{2}{\pi}\int_\theta^\pi\frac{\sin n\phi\cos\frac{\phi}{2}\,d\phi}{\sqrt{2(\cos\theta-\cos\phi)}}$.

64.) These formulas are derived differently by P. A. Hansen, Entwickelung der negativen und ungeraden Potenzen der Quadratwurzel der Function $\{r^2+r'^2-2rr'(\cos U\cos U'+\sin U\sin U'\cos U)\}^{-1/2}$. Abhandl. k. sächs. Ges. der Wiss. Leipzig (math.) 2(1855)285-376. We use the hypergeometric function form for brevity.

Add (5.6a) and (5.6b),

(5.6c) $$P_n(\cos\theta) = \frac{1}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})\phi\,d\phi}{\sqrt{2(\cos\phi-\cos\theta)}} + \frac{1}{\pi}\int_\theta^\pi \frac{\sin(n+\frac{1}{2})\phi\,d\phi}{\sqrt{2(\cos\theta-\cos\phi)}}\,.$$

Multiply R by sin $\frac{\phi}{2}$ and S by cos $\frac{\phi}{2}$ and add, obtaining

(5.7a) $$\sum_0^\infty \sin(n+\tfrac{1}{2})\phi\; P_n(\cos\theta) = \begin{cases} R_1\sin\frac{\phi}{2} + S_1\cos\frac{\phi}{2} = 0 & (\theta > \phi) \\ R_2\sin\frac{\phi}{2} + S_2\cos\frac{\phi}{2} = \dfrac{1}{\sqrt{2(\cos\theta-\cos\phi)}} & (\theta < \phi)\,. \end{cases}$$

Multiply R by cos $\frac{\phi}{2}$ and S by sin $\frac{\phi}{2}$ and subtract. Similarly,

(5.7b) $$\sum_0^\infty \cos(n+\tfrac{1}{2})\phi\; P_n(\cos\theta) = \begin{cases} R_1\cos\frac{\phi}{2} - S_1\sin\frac{\phi}{2} = \dfrac{1}{\sqrt{2(\cos\phi-\cos\theta)}} & (\theta > \phi) \\ R_2\cos\frac{\phi}{2} - S_2\sin\frac{\phi}{2} = 0 & (\theta < \phi)\,. \end{cases}$$

The latter play a part in the study of the Laplace Series.

Instead of considering Dirichlet's problem in regard to (5.2a) and (5.2b), we restrict ourselves to the similar problem for the Legendre Series.[65] We seek to show the convergence of (5.1) by proving the convergence of the dominating series

(5.8) $$\sum_{n=0}^\infty \left\{\frac{2n+1}{2}\int_{-1}^1 f(x)\,P_n(x)\,dx\right\} = \frac{1}{2}\int_{-1}^1 f(x)\sum_{n=0}^\infty (2n+1)\,P_n(x)\,dx\,.$$

First, we use the result already known to Legendre, that every rational integral function of x can be represented in a Legendre Series, to find the series for $P'_{n+1}(x)$, a polynomial of degree n containing either only odd or only even powers of x,

$$\frac{dP_{n+1}}{dx} = a_n P_n + a_{n-2}P_{n-2} + a_{n-4}P_{n-4} + \cdots;$$

$$a_{n-i} = \frac{2(n-i)+1}{2}\int_{-1}^1 P_{n-i}\,\frac{dP_{n+1}}{dx}\,dx = \frac{2(n-i)+1}{2}\Big[P_{n-i}P_{n+1}\Big]_{-1}^1 - \frac{2(n-i)+1}{2}\int_{-1}^1 P_{n+1}\,\frac{dP_{n-i}}{dx}\,dx \quad (i \text{ even})$$

Now $P_{n+1}P_{n-i}$ is of odd degree and $\Big[P_{n+1}P_{n-i}\Big]_{-1}^1 = 2$, hence $a_{n-i} = 2(n-i)+1$. Thus[66]

(5.9) $$\frac{dP_{n+1}}{dx} = (2n+1)P_n + (2n-3)P_{n-2} + (2n-7)P_{n-4} + \cdots,$$

$$\frac{dP_{n-1}}{dx} = \qquad (2n-3)P_{n-2} + (2n-7)P_{n-4} + \cdots.$$

Hence,

65.) We apply Dirichlet's results to Legendre Series. The proofs here are variations of Darboux's interpretation of Dirichlet's work.

66.) Dirichlet was not aware of (5.9) which appears for the first time in Christoffel's disseration, "De Motu Permanenti Electricitatis in Corporibus Homogeneis", Berlin 1856. He made use of identities built on his integral formulas (5.6) and (5.7).

(5.10) $(2n+1)P_n = \frac{dP_{n+1}}{dx} - \frac{dP_{n-1}}{dx}$ and

(5.11) $\sum_{i=0}^{n}(2i+1)P_i = \frac{dP_{n+1}}{dx} + \frac{dP_n}{dx}$.

Thus (5.8) yields

(5.12) $S_n = \frac{1}{2}\int_{-1}^{1} f(x)\left(\frac{dP_{n+1}}{dx} + \frac{dP_n}{dx}\right)dx.$

If f'(x) exists and is bounded, we can integrate (5.12) by parts,

$$S_n = \tfrac{1}{2}\left[f(x)(P_{n+1}+P_n)\right]_{-1}^{1} - \tfrac{1}{2}\int_{-1}^{1} f'(x)(P_{n+1}+P_n)dx = f(1) - \tfrac{1}{2}\int_{-1}^{1} f'(x)(P_{n+1}+P_n)dx.$$

To evaluate $\int_{-1}^{1} f'(x)(P_{n+1}+P_n)dx$, Dirichlet writes for arbitrarily small $\epsilon>0$,

$$\int_{-1}^{-1+\epsilon} f'(x)(P_{n+1}+P_n)dx + \int_{-1+\epsilon}^{1-\epsilon} f'(x)(P_{n+1}+P_n)dx + \int_{1-\epsilon}^{1} f'(x)(P_{n+1}+P_n)dx.$$

In the first and third integrals, $|f'(x)| < k$ (a fixed positive constant), for f'(x) is bounded. We apply (2.6) also, obtaining

$$\left|\int_{-1}^{-1+\epsilon} f'(x)(P_{n+1}+P_n)dx + \int_{1-\epsilon}^{1} f'(x)(P_{n+1}+P_n)dx\right| \leq \int_{-1}^{-1+\epsilon}|f'(x)||P_{n+1}+P_n|dx + \int_{1-\epsilon}^{1}|f'(x)||P_{n+1}+P_n|dx < 2K\epsilon + 2K\epsilon = 4K\epsilon.$$

As for the middle integral, from Laplace's asymptotic form (2.21), for preassigned ϵ and $\eta = \epsilon > 0$ $\exists n > N_{\epsilon,\eta} \ni |P_n(x)| < \eta$ as long as $|x| < 1-\epsilon$.

Thus $|S_n - f(1)| < 4K\epsilon + \int_{-1+\epsilon}^{1-\epsilon} 2K\eta\, dx < 4K(\epsilon+\eta) = 8K\epsilon$ for $n > N_{\epsilon,\eta}$.

Hence (5.12) converges to f(1).

Actually, Dirichlet proves the convergence of (5.1) in the case when f(x) has at most a finite number of finite discontinuities in [-1, 1] and a finite number of maxima and minima. In the above discussion, the points of discontinuity and the extrema can be isolated. Let them be $l_1, l_2, l_3, \ldots, l_p$.

Then (5.12) becomes

$$S_n = \tfrac{1}{2}\left[\int_{-1}^{l_1} + \int_{l_1}^{l_2} + \cdots + \int_{l_{p-1}}^{l_p} + \int_{l_p}^{1}\right] f(x)(P'_{n+1}+P'_n)dx,$$

and in each interval of integration f(x) is monotone and continuous. Here Dirichlet seems to imply f'(x) exists in each subinterval. If f'(x) is also bounded in each subinterval, he immediately integrates by parts, obtaining

$$S_n = \tfrac{1}{2}\sum_{i=0}^{p} f(x)(P_{n+1}+P_n)\Big]_{l_i}^{l_{i+1}} - \tfrac{1}{2}\sum_{i=0}^{p}\int_{l_i}^{l_{i+1}} f'(x)(P_{n+1}+P_n)dx \qquad (l_0=-1,\ l_{p+1}=1).$$

In the limit as $n \to \infty$, the first sum reduces to $\frac{1}{2}\{f(1)[P_{n+1}(1)+P_n(1)] - f(-1)[P_{n+1}(-1)+P_n(-1)]\} = f(1)$ as long as f(x) has only finite discontinuities and in view of (2.3) and (2.21). As for the second sum, exclusive of the integral over $[-1, -1+\epsilon]$ and $[1-\epsilon, 1]$, all the other integrals vanish as $n \to \infty$, if f'(x) is bounded.

And now, the excluded integrals can be treated as we treated the similar integrals in the simpler case. Thus, again $S_n \rightarrow f(1)$.

If $f'(x)$ is unbounded at finite many points, we can isolate these as well. In carrying the above through, we must consider specially the intervals $(K_i-\eta, K_i+\eta)$ $\underset{(\eta>0)}{(i=1,2,\cdots,q)}$ in which $f'(x)$ is unbounded at k_i. If there are only finite many maxima and minima of $f(x)$, we can arrive at a subdivision of $(K_i-\eta, K_i+\eta)$ such that $f'(x)$ does not change sign in any subinterval of $(K_i-\eta, K_i+\eta)$. Thus

$$\frac{1}{2}\left[\int_{K_i-\eta}^{K_i}+\int_{K_i}^{K_i+\eta}\right]f'(x)(P_{n+1}+P_n)\,dx=\frac{1}{2}\left[\int_{K_i-\eta}^{K_i-\eta_1}+\int_{K_i-\eta_1}^{K_i-\eta_2}+\cdots+\int_{K_i-\eta_\lambda}^{K_i}+\int_{K_i}^{K_i+\eta_{\lambda+1}}+\int_{K_i+\eta_{\lambda+1}}^{K_i+\eta_{\lambda+2}}+\cdots+\int_{K_i+\eta_s}^{K_i+\eta}\right]f'(x)(P_{n+1}+P_n)\,dx.$$

Again, all the integrals vanish, excluding those over $[K_i-\eta_\lambda, K_i]$ and $[K_i, K_i+\eta_{\lambda+1}]$. We have $\frac{1}{2}\left|\int_{K_i-\eta_\lambda}^{K_i}f'(x)(P_{n+1}+P_n)dx\right|+\frac{1}{2}\left|\int_{K_i}^{K_i+\eta_{\lambda+1}}f'(x)(P_{n+1}+P_n)dx\right| < \left|f(K_i-0)-f(K_i-\eta_\lambda)\right|+\left|f(K_i+\eta_{\lambda+1})-f(K_i+0)\right|$.

Thus, if for an arbitrarily small $\epsilon>0$ we can select an $\eta_\lambda, \eta_{\lambda+1}$ so small that this last quantity be less than ϵ, Dirichlet's theorem will hold. This is possible, for $f(x)$ is continuous both to the right and to the left of k_i, bounded, and both $\lim_{h\to 0} f(K_i+h)$ and $\lim_{h\to 0} f(K_i-h)$ exist.

Todhunter raised the objection that the integrals $\int_{-1}^{-1+\epsilon} f'(x)(P_{n+1}+P_n)dx$ and $\int_{1-\epsilon}^{1} f'(x)(P_{n+1}+P_n)\,dx$ could not be dismissed as lightly as Dirichlet did.[67]

The basis of his objection is the use of (2.21) to show $P_n(x)\rightarrow 0$ as $n\rightarrow\infty$ for $x\neq\pm 1$. He distrusts Laplace's formula, as we have remarked (p.24), because it does not hold for $x=\pm 1$ and is not a good approximation for x in the neighborhood of ± 1. However, it is clear that with a preassigned $\epsilon>0$, however small, the argument (p.43) is perfectly valid. This is all we need to assert.

67.) Todhunter, I., An Elementary Treatise on Laplace's Functions, Lamé's Functions, and Bessel's Functions, London, MacMillan and Co., 1875, Cap. XI, paragraph 152.

Before we continue with equestions of convergence of Legendre Series, it would be well to return to some further formal developments. The most interesting of these are due to Jacobi.[68] Not only does he compare

$$\frac{\sin(n \arccos x)}{n} = \frac{(-1)^{n-1}2^n \cdot n!}{(2n)!}\frac{d^{n-1}}{dx^{n-1}}(1-x^2)^{n-\frac{1}{2}}$$

with the Legendre Polynomial given as an nth differential coefficient, but he also introduces some transformations of definite integrals which lead him from $\int_0^\pi \frac{\cos m\theta\, d\theta}{\sqrt{1-2z\cos\theta+z^2}}$ to a new integral formula for P_n. Poisson had already discussed a similar integral and its relation to Legendre's elliptic integrals.[69] These preliminary works are only of relevance in that in them is contained the theory of the transformations used to obtain the Jacobi Integral Formula,[70]

$$P_n(x) = \pm\frac{1}{\pi}\int_0^\pi \frac{d\phi}{(x \pm \sqrt{x^2-1}\cos\phi)^{n+1}}.$$

Perhaps the simplest way to arrive at this formula is to begin with

$$\int_0^\pi \frac{d\phi}{u \pm v\cos\phi} = \frac{\pm\pi}{\sqrt{u^2-v^2}}, \qquad |u-\sqrt{u^2-v^2}| < |v|,$$

where v/u is real. Then

$$\int_0^\pi \frac{d\phi}{1-xz \pm z\sqrt{x^2-1}\cos\phi} = \frac{\pm\pi}{\sqrt{1-2xz+z^2}} = \frac{\pm\pi}{z\sqrt{1-2x\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2}}.$$

Choose z so large that $|z||x \pm \sqrt{x^2-1}\cos\phi| > 1 \quad (0 \leq \phi \leq \pi)$. Expand both sides

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68.) C.G.H. Jacobi, Formula transformationis integralium definitorum, Jour. f. Math. 15(1836)1-38.

69.) Poisson, S.D., Suite du Mémoire sur les intégrales définies, Jour. de l'Ec. Polytech., 17(1815) 612-631.

70.) C.G.J. Jacobi, Ueber die Entwickelung des Ausdrucks $\{a^2-2aa'[\cos\omega\cos\varphi+\sin\omega\sin\varphi\cos(\theta-\theta')]+a'^2\}^{-\frac{1}{2}}$ Journ. f. Math. 26(1843)81-87. ;- Sopra le funzioni di Laplace che risultano della sviluppo dell'espressione $\{a^2-2aa'[\cos\omega\cos\varphi+\sin\omega\sin\varphi\cos(\theta-\theta')+a'^2\}^{-\frac{1}{2}}$ Giorn. Arcadico di Scienze (Roma) 98(1844)59-66.

in descending powers of z and equate coefficients of like powers,

$$P_n(x) = \pm \frac{1}{\pi}\int_0^\pi \frac{d\phi}{(x \pm \sqrt{x^2-1}\cos\phi)^{n+1}}.$$

The question of sign arises. Since $P_n(1)=1$, the upper sign must be taken. Similarly, $P_n(-1)=(-1)$ and the lower sign is taken. In fact

(5.13) $$P_n(x) = +\frac{1}{\pi}\int_0^\pi \frac{d\phi}{(x \pm \sqrt{x^2-1}\cos\phi)^{n+1}} \qquad (x>0)$$

$$P_n(x) = -\frac{1}{\pi}\int_0^\pi \frac{d\phi}{(x \pm \sqrt{x^2-1}\cos\phi)^{n+1}} \qquad (x<0).$$

Both Jacobi and Heine[71] write

$$(x+\sqrt{x^2-1}\cos\phi)^n = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n + \frac{2}{2^n}\sum_{m=0}^{n}\frac{(x^2-1)^{\frac{m}{2}}}{(n+m)!}\cos m\phi\,\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n.$$

The coefficient of $\cos m\phi$ multiplied by a constant Heine calls the associated Legendre function $P_m^n(x)$, i.e.

(5.14) $$P_m^n(x) = \frac{(n-m)!}{(2n)!}(x^2-1)^{\frac{m}{2}}\frac{d^{n+m}}{dx^{n+m}}(x^2-1)^n,$$

satisfying the orthogonality property

(5.15) $$\int_{-1}^{1} P_m^n(x)P_r^n(x)\frac{dx}{1-x^2} = \begin{cases} 0 & (m \neq r) \\ \frac{(-1)^m}{m}\frac{(n-m)!\,(n+m)!\,2^{2n}(n!)^2}{[(2n)!]^2} & (m=r). \end{cases}$$

Heine and Neumann give recurrence relations for these functions, relating $P_m^{n+1}, P_m^n, P_m^{n-1}$ or relating $P_{m+1}^n, P_m^n, P_{m-1}^n$. They also write concerning the relation of Lamé Functions to Legendre Polynomials, but we shall not be concerned with this subject.[72]

70.) Heine, E. - Berlin Dissertation (1842)15-18, Ueber einige Aufgaben, welche auf partielle Differentialgleichungen führen, Jour. f. Math. 26(1843)185-216.

71.) Todhunter (see p.44ff) treats this in an elementary and adequate fashion. See also E. Heine, Handbuch der Kugelfunctionen, vol. I, p. 259, and F. Neumann, Beiträge zur Kegelfunktionen, p. 73.

SECTION VI

SOME GENERALIZATIONS

While generalizations of the Legendre Polynomials are beyond the domain of the present work, a brief consideration of the most important developments is not out of place. The method of generalization has been standard. Some one of the avenues of approach to the development of Legendre Polynomials is considered in a more general form. For example, one could consider, asdid Liouville and Sturm, the Legendre differential equation and from it pass to the general linear differential equation of second order with variable coefficients, obtaining in this way the theory of orthogonal functions. One could observe, further, that the Legendre differential equation is a particular case of the type considered by Liouville in (4.1). As a matter of fact, this equation has often been taken as the point of departure in the development of the theory of classical orthogonal polynomials.[72]

Just such an extension appears in an 1843 work of Jacobi, posthumously published in 1859 by Heine.[73] In this instance, Jacobi considers the Legendre differential equation to be a special case of the hypergeometric differential equation,

$$x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0.$$

72.) See Holmgren, H., Sur l'intégration de l'équation différentielle $(a_2+b_2x+c_2x^2)y''+(a_1+b_1x)y'+a_0y=0$, Kongl. Svenska Vetenskaps-Akad. Handl. (2)7(1868)#9pp58; E. Routh, On some Properties of certain Solutions of a Differential Equation of the Second Order, Proc. Lond. Math. Soc. 16(1884)245-261; G. Humbert, Sur l'équation différentielle lineaires du second ordre, Jour. de l'Ec. Polytech. (1)48(1880)207-228; J.A. Shohat, Theorie générale des polynomes orthgonaux de Tchebichef, Mem. des Sc. Math. 66(1934)31-3.

73.) C.G.J. Jacobi, Untersuchungen über die Differentialgleichungen der hypergeometrischen Reihe, Jour. f. Math., 56(1859)149-165.

We have already written (5.3). In another form, it becomes

(6.1) $P_n(x) = F(-n, n+1, 1, \frac{1-x}{2})$.

Jacobi considers similarly $\frac{1}{n!}\frac{d^n}{dx^n}\{x^n(1-x)^n\} = F(-n, n+1, 1, x)$ and

(6.2) $$\frac{x^{1-\gamma}(1-x)^{\gamma-\alpha}(\gamma-1)!}{(\gamma+n-1)!}\frac{d^n}{dx^n}\left\{x^{\gamma+n-1}(1-x)^{\alpha+n-\gamma}\right\} = (-n, n+\alpha, \gamma, x).$$

These must be considered as more than mere extension of his observations concerning sin (n arc cos x) (p. 45). In fact, there is defined a sequence of polynomials satisfying the orthogonality property

(6.3) $$\int_0^1 J_m J_n x^{\alpha-1}(1-x)^{\alpha-\gamma}\,dx = \begin{cases} 0 & (m \neq n) \\ f(n) \neq 0 & (m = n) \end{cases} \quad (\alpha > \gamma-1, \alpha > 0),$$

and a hypergeometric differential equation as well. These are the famous Jacobi Polynomials.

One could equally well generalize the generating function (2.1). As a particular instance, Jacobi considers the coefficient of z^n in the expansion of $(1-2xz+z^2)^{-\nu} = \sum_{n=0}^{\infty} z^n C_n^\nu(x)$. In hypergeometric form

(6.4) $C_n^\nu(x) = \frac{(2\nu+n-1)!}{(2\nu)!\,n!} F(-n, 2\nu+n, \frac{2\nu+1}{2}, \frac{1-x}{2})$.

The polynomial $C_n^\nu(x)$ has been considered carefully by Most.[74] He gives the orthogonality property,

(6.5) $$\int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x) C_m^\nu(x)\,dx = \begin{cases} 0 & (m \neq n) \\ f(n) \neq 0 & (m = n). \end{cases}$$

These polynomials satisfy the differential equation[75]

(6.6) $$(1-x^2)\frac{d^2 C_n^\nu}{dx^2} - (1+2\nu)\frac{d C_n^\nu}{dx} + n(n+2\nu)C_n^\nu = 0;$$

and hence, one can use series of them in the same manner as Fourier Series.

74.) R. Most, Ueber die Differentialgleichungen der Kugelfunctionen, Jour. f. Math. 70(1869)163-8.

75.) R.R. Webb, on Legendre's Coefficients, Mess. of Math. (2)9(1879)125-6.

Most gives also relations analogous to (3.9) between $C_n^{\nu}, C_n^{\nu+1}, C_n^{\nu+2}$ and also between $C_n^{\nu}, C_{n+1}^{\nu}, C_{n+2}^{\nu}$. A peculiar representation for $C_n^{\nu}(x)$ appears in a note by Glaisher,[76]

(6.7) $$C_n^{\nu}(x) = \frac{1}{n!(\nu-1)!}\int_0^{\infty} t^{\nu-1} e^{-(1-x^2)t}\left(-\frac{d}{dx}\right)^n e^{-x^2 t}\,dt.$$

The polynomials $C_n^{\nu}(x)$ are frequently and unjustly called Gegenbauer Polynomials. Gegenbauer, however, is responsible for considerable investigation into their properties.[77]

Escary[78] considers closely the special case $\nu = -\frac{2l+1}{2}$, and writes

(6.8) $$(1-2zx+a^2z^2)^{-\frac{2l+1}{2}} = \sum_{n=0}^{\infty}\frac{z^n}{2^n n!}\frac{d^n}{dx^n}(x^2-a^2)^n.$$

76.) J.W.L. Glaisher, Notes on Laplace's Coefficients, Proc. Lond. Math. Soc. 6(1875)126-136.

77.) L. Gegenbauer, Generalizzazione di Alcune Relazioni contenute nella nota del Prof. Modera "Sui polinomii di Legendre", Rend. Circ. Mat. Palermo 12(1898)21-22. - Generalizzazione di alcuni teoremi intorno alle funzioni sferiche contenuti in una Nota del Prof. Paci, Rend. Circ. Mat. Palermo 13(1899)92-4.
For further consideration of $C_n^{\nu}(x)$, $P_n^{\nu}(x)$ of any degree n, any order ν, and any argument μ (n and ν rational and μ real or complex) see E.W. Hobson - On a type of spherical harmonics of unrestricted degree, order, and argument, Phil. Trans. (A)187(1896)443-531.

78.) Escary, M. - Sur les fonctions qui naissent du développement de l' expression $(1-2\alpha x+a^2\alpha^2)^{-\frac{2l+1}{2}}$, Comptes Rendus 86(1878)114-6, 1451-3. I here correct some errors in the formulas given by Escary.

Escary, M. - Généralisations des fonctions X_n de Legendre, Jour. de Math. (3)5(1879)47-68.

The results of Escary are more than mere generalizations of Gegenbauer Polynomials. Escary gives recurrence relations involving his functions for varying ν as well as for varying n. He discusses quite completely the zeros of his polynomials, giving various separation theorems and bounds for the maximum and minimum zeros. He also treats the problem of the number of zeros of certain linear combinations of his functions. Concerning this latter problem, his results are much like those of Laguerre in the case of Legendre Polynomials, which we shall soon discuss in the next section.

This leads then to a generalized Legendre Polynomial with real and distinct roots in (-a,a), satisfying $(a^2-x^2)y'' - 2x(\ell+1)y' + n(n+2\ell+1)y = 0,$

and $nC_n^\nu - (2n+2\ell-1)x\, C_{n-1}^\nu + (n+2\ell-1)a^2 C_{n-2}^\nu = 0,$

and such that $\int_{-a}^{a} C_n^\nu C_m^\nu (a^2-x^2)^\ell dx = \begin{cases} 0 & (m \neq n) \\ f(n) > 0 & (m = n). \end{cases}$

Heine[79] uses the generating function $\log(1-2xz+z^2)$ to obtain the sequence $\{\cos(n \arccos x)\}$. Pincherle[80] obtains new functions as coefficients of the ascending powers of t in $(t^3-3tx+1)^{-1/2}$.

More general differential equations of type (4.1) and more general generating functions $\varphi(x,z) = \sum_0^\infty z^n P_n(x)$ are not alone in leading to extensions. With Jacobi, we can consider more general nth differential coefficients $\frac{d^n}{dx^n}\{(x-a)^\alpha (x-b)^{n-\beta}\}$. Further, we can seek a sequence of polynomials satisfying an orthogonality property:

(6.9) $\int_a^b p(x) P_n(x) x^m dx = 0 \quad (m = 0, 1, 2, \ldots, n-1),$

p(x) being a function which does not change sign in (a,b) and $P_n(x)$ being a polynomial of degree n. The latter is often taken as the foundation of the theory of orthogonal polynomials in general.[81]

Another type of extension is due primarily to Hermite, who considers functions of more than one variable.[82] For nth differential coefficient he uses $\frac{d^n}{dx^\alpha dy^\beta}(x^2+y^2-1)^n$ and $\frac{d^n}{dx^\alpha dy^\beta}(ax^2+2bxy+cy^2-1)^n, \; (\alpha+\beta=n)$.

79.) Heine, E. - Die speciellen Laméschen Functionen erster Art von beliebiger Ordnung, Jour. f. Math. 62(1863) 110-141. See also C. Hermite - Lettre à M. P. Gordan, Math. Annalen 10(1876)287-8.

80.) S. Pincherle - Memorie Istituto Bologna (5)1(1890)337-369.

81.) J. Shohat - Mem. des Sciences Math. 66(1934)7-8.

82.) C. Hermite - Sur quelques développements en séries de fonctions de plusieurs variables, Comptes Rendus 60(1865)370-8, 432-440, 461-6, 512-8. Extrait d'une lettre à M. Borchardt, Jour. f. Math. 64(1865)294-6.

Likewise, the generating function is extended to

$$\frac{1}{1-2ax-2by+a^2+b^2} = \sum_{\alpha,\beta=0}^{\infty} a^{\alpha} b^{\beta} U_{\alpha,\beta},$$

where $\iint U_{\alpha,\beta} U_{\gamma,\delta}\, dx\, dy = 0 \qquad (x^2+y^2 \leqq 1,\ \alpha+\beta \neq \gamma+\delta).$

Subsequently, he proves that $\{V_{m,n}\}$ is generated by

$$\{1-2ax-2by+a^2(1-y^2)+2abxy+b^2(1-x^2)\}^{-1/2},$$

where $V_{m,n} = \frac{1}{m!\,n!\,2^{m+n}} \frac{d^{m+n}}{dx^{m+n}}(x^2+y^2-1)^{m+n}.$

The orthogonality properties, upper and lower bounds, and the expansion of functions in series of these polynomials of two variables are considered. Extensions have been made to a higher number of variables and to the complex domain.

SECTION VII

FURTHER FORMAL PROPERTIES

We come now to a further development of the formal properties of the Legendre Polynomials. We shall see that almost without exception every new development was related to the then current problems in Gauss quadrature or Legendre Series. Although we will here treat the properties apart from their applications, we do not intend to neglect referring to their significance.

The remarkable properties of Legendre Polynomials thru which Jacobi revived interest in Gauss quadrature, we have already considered (p. 31). We saw how the orthogonality property led Jacobi to the nth differential coefficient form of the Legendre Polynomials. This same property led Rouché also to a new representation of the polynomials.[83] Every function of x, $V_n(x)$, rational and integral of degree n, which satisfies the relation

(7.1) $$\int_{-1}^{1} x^K V_n(x)\,dx = 0 \qquad (K = 0, 1, 2, \cdots n-1),$$

differs from the Legendre Polynomial $P_n(x)$ only by an arbitrary constant. If we seek to determine $V_n = A_n + A_{n-1}x + A_{n-2}x^2 + \cdots + A_1 x^{n-1} + x^n$, it is sufficient to determine the A's by the n linear equations resulting from (7.1). If we let $\alpha_K = \frac{1}{2}\int_{-1}^{1} x^K dx = \begin{cases} 0 & (K \text{ odd}) \\ \frac{1}{K+1} & (K \text{ even}) \end{cases}$, the system of equations is

(7.2) $$\begin{cases} A_n\alpha_0 + A_{n-1}\alpha_1 + \cdots + A_1\alpha_{n-1} + \alpha_n = 0 \\ A_n\alpha_1 + A_{n-1}\alpha_2 + \cdots + A_1\alpha_n + \alpha_{n+1} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ A_n\alpha_{n-1} + A_{n-1}\alpha_n + \cdots + A_1\alpha_{2n-2} + \alpha_{2n-1} = 0. \end{cases}$$

The determinant of the coefficients is

(7.3) $$\Delta_n = \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ \\ \alpha_{n-1} & \alpha_n & \alpha_{n+1} & \cdots & \alpha_{2n-2} \end{vmatrix}.$$

83.) E. Rouché - Sur les fonctions X_n de Legendre, Comptes Rendus 47(1858) 917-921.

Solving for the A's, and substituting in the expression for V_n, we obtain

(7.4)
$$V_n = \frac{1}{\Delta_n}\begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 \cdots & \alpha_n \\ \alpha_1 & \alpha_2 & \alpha_3 \cdots & \alpha_{n+1} \\ & & & \\ \alpha_{n-1} & \alpha_n & \alpha_{n+1} \cdots & \alpha_{2n-1} \\ 1 & x & x^2 \cdots & x^n \end{vmatrix}.$$

Thus the Legendre Polynomial P_n, which is within a constant factor equal to V_n, can be written as the quotient of two determinants.[84]

These determinants can be given still another form when we consider P_{2n} and P_{2n+1} separately. Because P_{2n} is an even function of x, (7.2) becomes

(7.2a)
$$\begin{cases} A_{2n}\alpha_0 + A_{2n-2}\alpha_2 + \cdots + A_2\alpha_{2n-2} + \alpha_{2n} = 0 \\ A_{2n}\alpha_2 + A_{2n-2}\alpha_4 + \cdots + A_2\alpha_{2n} + \alpha_{2n+2} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ A_{2n}\alpha_{2n-2} + A_{2n-2}\alpha_{2n} + \cdots + A_2\alpha_{4n-4} + \alpha_{4n-2} = 0, \end{cases}$$

from which

(7.5)
$$P_{2n} = \frac{1}{K_{2n}}\begin{vmatrix} \alpha_0 & \alpha_2 & \cdots & \alpha_{2n-2} & \alpha_{2n} \\ \alpha_2 & \alpha_4 & \cdots & \alpha_{2n} & \alpha_{2n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{2n-2} & \alpha_{2n} & \cdots & \alpha_{4n-4} & \alpha_{4n-2} \\ 1 & x^2 & \cdots & x^{2n-2} & x^{2n} \end{vmatrix} = \frac{1}{K_{2n}}\begin{vmatrix} 1 & \frac{1}{3} & \frac{1}{5} & \cdots & \frac{1}{2n+1} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2n+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2n-1} & \frac{1}{2n+1} & \frac{1}{2n+3} & \cdots & \frac{1}{4n-1} \\ 1 & x^2 & x^4 & \cdots & x^{2n} \end{vmatrix}.$$

Similarly,

(7.5b)
$$P_{2n+1} = \frac{1}{K_{2n+1}}\begin{vmatrix} \alpha_2 & \alpha_4 & \cdots & \alpha_{2n+2} \\ \alpha_4 & \alpha_6 & \cdots & \alpha_{2n+4} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{2n} & \alpha_{2n+2} & \cdots & \alpha_{4n} \\ x & x^3 & \cdots & x^{2n+1} \end{vmatrix} = \frac{1}{K_{2n+1}}\begin{vmatrix} \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2n+3} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \cdots & \frac{1}{2n+5} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2n+1} & \frac{1}{2n+3} & \frac{1}{2n+5} & \cdots & \frac{1}{4n+1} \\ x & x^3 & x^5 & \cdots & x^{2n+1} \end{vmatrix}.$$

These formulas can be verified a posteriori. Multiply each determinant by x^k and integrate from -1 to 1. For $k=0,1,2,\ldots m-1$, where m is the degree of the polynomials, the integration will produce two like rows in the determinant. Thus, the orthogonality property will be satisfied with the vanishing of the determinant.

From the theory of determinants, Rouché is able to write still other representations for $P_n(x)$. For example,

(7.6)
$$P_n(x) = \frac{1}{\Delta_n}\begin{vmatrix} \alpha_1 - \alpha_0 x & \alpha_2 - \alpha_1 x & \cdots & \alpha_n - \alpha_{n-1}x \\ \alpha_2 - \alpha_1 x & \alpha_3 - \alpha_2 x & \cdots & \alpha_{n+1} - \alpha_n x \\ & & & \\ \alpha_n - \alpha_{n-1}x & \alpha_{n+1} - \alpha_n x & \cdots & \alpha_{2n-1} - \alpha_{2n-2}x \end{vmatrix}.$$

84.) Rouché does not remark that it is essential to prove that $\Delta_n \neq 0$, which he fails to do. Actually $\Delta_n > 0$, for it is the discriminant of a positive definite form. (See Shohat, J. - Mem. des Sc. Math. 66(1934) p.19.

Another determinant for $P_n(x)$ appears in a memoir by Glaisher.[85] In a previous note,[86] he establishes the interesting equality

$$(7.7) \qquad \sum_{i=0}^{n} P_{n-i}\, x^i (-1)^i \binom{n}{i} = \begin{cases} 0 & (n \text{ odd}) \\ \dfrac{n!}{2^n \left(\frac{n}{2}!\right)^2} (x^2-1)^{n/2} & (n \text{ even}), \end{cases}$$

and from this, he obtains

$$(7.8) \qquad P_n = \begin{vmatrix} 1 & 1 & 0 & 0 \cdots\cdots \\ 0 & x & 1 & 0 \cdots\cdots \\ \frac{1}{2}(x^2-1) & x^2 & \binom{2}{1}x & 1 \cdots\cdots \\ 0 & x^3 & \binom{3}{1}x^2 & \binom{3}{2}x \cdots\cdots \\ \frac{1\cdot 3}{2\cdot 4}(x^2-1)^2 & x^4 & \binom{4}{1}x^3 & \binom{4}{2}x^2 \cdots\cdots \\ 0 & x^5 & \binom{5}{1}x^4 & \binom{5}{2}x^3 \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots \;\; \vdots \;\cdots \end{vmatrix},$$

(in the first column the 2rth term is 0 and the (2r+1)th term is $\frac{(2r)!}{2^{2r}(r!)^2}(x^2-1)^r$; in the other columns, the coefficients are the binomial coefficients and the law is evident).

Glaisher is also responsible for another peculiar relation.[87] Christoffel and Bauer gave (5.9) simultaneously in 1858. Later the expression of $P_n^{(r)}(x)$ in a finite Legendre Series was given. Glaisher considers the rth integral of $P_n(x)$. The result is a sum $\sum_{i=n-r}^{n+r} A_i P_i$ in which the coefficients are formed according to a complex law.

The similar problem of expressing the product of any two Legendre Polynomials by means of a Legendre Series led also to some curious integrals, in which the integrand consisted of the product of three Legendre Polynomials. By making use of the recurrence relation (3.9) and the explicit expression for the polynomials, Adams attempts to set up an induction which will lead him to

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85.) J.W.L. Glaisher, Expressions for Laplace's Coefficients, Bernoullian and Eulerian Numbers, etc.; as Determinants, Mess. of Math. (2)6(1877)49-63.

86.) See Glaisher p.49ff.

87.) J.W.L. Glaisher - Formulae for the rth Integral of a Legendrian Coefficient and of the Logarithm Integral, Mess. of Math. (2)12(1883)120-5.

the desired expression for $P_m P_n$.[88] His efforts are successful, but his method cannot compare in simplicity with that of Todhunter.[89]

There is a great number of formulas relating the elements of the sequence $\{P_n(x)\}$. Some of these we have already considered (2.19), (3.9), (5.10). There are also many integral relations involving Legendre Polynomials. From several of these relations it is possible to obtain results applicable to number theory and to combinatorial analysis; others are useful in the study of Legendre Series. These formulas are, however, so numerous and, on the whole, their derivations are so devoid of mathematical subtlety that there is nothing to be gained from reviewing several hundred such relations. It will suffice to remark that in a series of memoirs Catalan has produced approximately three hundred such relations. These memoirs are[90]

a. Mémoire sur les fonctions X_n de Legendre (1881),

b. Note sur les fonctions X_n de Legendre (1882),

c. Sur les fonctions X_n de Legendre (Second Mémoire) (1882),

d. Sur les fonctions X_n de Legendre (Troisième Mémoire) (1886),

e. Seconde Note sur les fonctions X_n (1889),

f. Nouvelles proprietes des fonctions X_n (avec Supplément) (1889),

g. Sur quelques formules d'Analyse (1893)

h. Sur les polynomes de Legendre, d'Hermite, et de Polignac (1893).

88.) J. C. Adams - On the expression of the product of any two Legendre's Coefficients by means of a Series of Legendre's Coefficients, Proc. Roy. Soc. Lond. 27(1878)63-71.

89.) I. Todhunter - Note on Legendre's Coefficients, Proc. Roy. Soc. London 27(1878)381-3.

90.) More detailed references are given in the bibliography.

As an example of a relation useful in combinatorial analysis, we consider $T = (1-2xz+z^2)^{-1/2} = \sum_{i=0}^{\infty} P_i z^i$ and $T^3 = (1-2xz+z^2)^{-3/2} = \sum_0^\infty \frac{dP_i}{dx} z^{i-1}$ or $\left(\sum_0^\infty P_i z^i\right)^3 = \sum_0^\infty \frac{dP_i}{dx} z^{i-1}$. Thus $\frac{dP_{n+1}}{dx} = \sum_{\alpha,\beta,\gamma} P_\alpha P_\beta P_\gamma$ where α, β, γ range over all integral values such that $\alpha+\beta+\gamma = n$. Since $P_n(1)=1$, $\left.\frac{dP_{n+1}}{dx}\right]_{x=1}$ gives the number of solutions of $\alpha+\beta+\gamma = n$. Now from (5.9) $\left.\frac{dP_{n+1}}{dx}\right]_{x=1} = \frac{(n+1)(n+2)}{2}$, so we have not only the solution of the partition problem but also the multinomial expansion

$$(1+z+z^2+z^3+\cdots)^3 = 1+3z+6z^2+10z^3+\cdots.$$

Among the diverse expressions of P_n appear

$$(7.9)\quad P_n = \frac{1}{2^n}\sum_{q=0}^{k}(-1)^q\binom{n}{q}\binom{2n-2q}{n}x^{n-2q} = \frac{1}{2^n}\sum_{q=0}^{n}(-1)^q\binom{n}{q}^2(1+x)^{n-q}(1-x)^q,$$

(k is either n/2 or (n-1)/2 according as n is even or odd) and

$$(7.10)\quad P_n = x^n - \frac{n(n-1)}{2^2}x^{n-2}(1-x^2) + \frac{n(n-1)(n-2)(n-3)}{2^2 4^2}x^{n-4}(1-x^2)^2 - \cdots.$$

If (7.10) and the latter part of (7.9) are equated and the coefficients compared, we obtain $1+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n-1}^2+1 = \binom{2n}{n}$.

Another property of the Legendre Polynomial is due to Bauer.[91]

From (6.1)

$$(7.11)\quad P_n(x) = 1 + \frac{n(n+1)}{1^2}\left(\frac{x-1}{2}\right) + \frac{(n-1)n(n+1)(n+2)}{1^2, 2^2}\left(\frac{x-1}{2}\right)^2 + \cdots + A_n^{(r)}\left(\frac{x-1}{2}\right)^r + \cdots,$$

where $A_n^{(r)} = \frac{(n-r+1)(n-r+2)\cdots(n+r)}{(r!)^2} = \frac{(n+r)!}{(n-r)!\,n!\,r!}$.

Since $A_n^{(r)}$ can be considered to be the number of permutations of n+r things of which n-r are of one kind, r of another kind, and r of still another kind, all the A's are integers. Thus the coefficients of the explicit form for the Legendre Polynomial have only powers of two in their denominators. Bauer also proves that if x is an odd integer, $P_n(x)$ and all its derivatives as well as $\int_{-1}^{x} P_n(x)dx$ are integers.

91.) G. Bauer - Bemerkungen über zahlentheoretische Eigenschaften der Legendre'schen Polynome, Sitzungsber. k. bay Ak. Wiss. (math. - phys.) zu Munchen 24(1894) 343-359.

There are other algebraic theorems which are more significant. The first of these is in regard to the discriminant of the equation $P_n(x)=0$. Stieltjes demonstrates that the expression

(7.12) $$(1-\xi_1^2)(1-\xi_2^2)\cdots(1-\xi_n^2)\prod_{\substack{k,l=1\\k<l}}^{n}(\xi_k-\xi_l)^2$$

is maximized if the ξ's are the zeros of $P_n(x)$.[92] The value then obtained for the expression is $\frac{2^4\cdot 3^6\cdot 4^8\cdots n^{2n}}{3^3\cdot 5^5\cdot 7^7\cdots(2n-1)^{2n-1}}$,

and the discriminant of $P_n(x)=0$ is then

(7.13) $$\prod_{\substack{k,l=1\\k<l}}^{n}(x_k-x_l)^2=\frac{2^2\cdot 3^4\cdot 4^6\cdots n^{2n-2}}{3\cdot 5^3\cdot 7^5\cdots(2n-1)^{2n-3}}.$$

He further shows that of all the equations of degree n with real zeros all in (-1,1), that which has the maximum discriminant is $V_n(x)=0$ where $\sqrt{1-2xz+z^2}=\sum_0^\infty V_n(x)z^n$. This discriminant has the value $\frac{1^1\cdot 2^2\cdot 3^3\cdots(n-2)^{n-2}\cdot 2^2\cdot 3^3\cdot 4^4\cdots n^n}{1^1\cdot 3^3\cdot 5^5\cdot 7^7\cdots(2n-3)^{2n-3}}$.
When n is very great, the relation of this discriminant to that of $P_n(x)=0$ is about $\frac{n\pi}{2}$. These results are verified by Hilbert.[93]

There are many theorems concerning the zeros of $P_n(x)$. We have seen already that these zeros are all real, distinct, and in (-1,1); that they separate the zeros of P_{n+1} and are in turn separated by those of P_{n-1}. Tchebycheff goes further and states a very general theorem for certain classes of polynomials,[94] which when put in terms of Legendre Polynomials results in P_n having a zero in (a,b) if $\int_a^b P_n(x)\,dx \leq \frac{2n}{n+1}\pi\left(\frac{b-a}{4}\right)^{n+1}$.
Another theorem, similarly interpreted, asserts the existence of a zero of P_n in $\left[t\,,\,t\pm\left\{\frac{|P_n(t)|}{2(n-1)\pi}\right\}^{1/n}\right]$ where the sign of the readical is opposite to that of $P_n(t)/P_n'(t)$.

92.) T.J. Stieltjes - Sur quelques théorèmes d'Algebre, Comptes Rendus 100(1885)439-440.

93.) D. Hilbert - Ueber die Discriminante der im Endlichen abbrechenden hypergeometrischen Reihe, Jour. f. Math. 103(1888)337-345.

94.) P. Tschebichef - Sur les fonctions qui différent le moins possible de zéro, Jour. de Math. (2)19(1874)3 19-346.

Theorems on the upper bound of the zeros of $P_n(x)$ are due to Laguerre.[95] If $F(x)$ is a polynomial of degree n in x with all its zeros real and distinct, and if $F(x)$ satisfies the linear differential equations

$$(7.14)\qquad \begin{cases} L(x)\,y'' + M(x)\,y' + N(x)\,y = 0 \\ L\,y''' + (L'+M)\,y'' + (M'+N)\,y' + N'y = 0, \end{cases}$$

then the polynomial

$$(7.15)\qquad \Omega(x) = LN + LM' - ML' - \frac{n+2}{4(n-1)}M^2 \geqq 0.$$

when x is a zero of $F(x)$. If we can find the maximum value of x which will make $\Omega(x) \geqq 0$, we can find an upper bound for the zeros of $F(x)$. This is the problem Laguerre solves for Legendre Polynomials. In this case $L=x^2-1$, $M=2x$, $N=-n(n+1)$ and $\Omega(x) = (n+2)(n-1) - \frac{n(n^2+2)}{n-1}x^2$.

Therefore, the absolute value of the greatest zero of P_n is less than or equal to $(n-1)\left\{\frac{n+2}{n(n^2+2)}\right\}^{1/2}$. We can obtain an even better approximation by considering separately P_{2m} and P_{2m+1}. In the former case, Laguerre finds that the absolute values of the zeros of P_{2m} lie between two positive roots of $\frac{m+2}{4(m-1)}(3x^2-1)^2 + 2m(m+1)(x^4-x^2) + 6x^4 - 4x^2 + 2 = 0$.

In the latter case, the absolute value of the zeros of P_{2m+1} lie between two positive roots of $\frac{m+2}{4(m-1)}(5x^2-3)^2 + 2m(2m+3)(x^4-x^2) + 10x^4 - 12x^2 + 6 = 0$.

One can see more briefly that $x_{1,n}$, the greatest zero of P_n, approaches one very rapidly as n increases. Newton's approximation applied at x=1 results in $x_{1,n} \sim 1 - \frac{P_n(1)}{P_n'(1)} = 1 - \frac{2}{n(n+1)}$.

Let the zeros of P_n be $(1>)\ x_{1,n} > x_{2,n} > x_{3,n} > \cdots > x_{k,n} > \cdots > x_{n,n}\ (>-1)$.

More complex considerations lead Stieltjes[96] to an approximation for $x_{k,n}$.

95.) E. Laguerre - Sur les équations algébriques dont le premier membre satisfait à une équation différentielle linéaire du second ordre, Comptes Rendus 90(1880)809-812.

96.) T.J. Stieltjes - Sur les polynomes de Legendre, Ann. Fac. Sc. Toulouse 4(1890)G1-17.

He finds $\quad x_{k,n} \sim \left[1 - \frac{1}{2(2n+1)^2}\right] \cos\frac{(4k-1)\pi}{4n+2}$.

Laguerre[97] also points out that if

(7.16) $\quad F(x) = A_1 P_{n_1} + A_2 P_{n_2} + A_3 P_{n_3} + \cdots + A_k P_{n_k} \quad (n_1 < n_2 < \cdots < n_k),$

then the number of zeros of F(x) which are greater than or equal to one is at most equal to the number of variations in the ordered set $A_1, A_2, A_3, \quad \cdot \quad \cdot \quad \cdot, A_k$. Obviously, if there are no variations in the set, the theorem holds because $|P_n(x)| \geqq 1$ for $x \geqq 1$. We assume the theorem true if there are v-1 variations in the sequence $\{A_i\}$. We now seek an inductive process. Suppose there are v variations in the sequence. Consider $\frac{F(x)}{P_{n_i}(x)}$ which vanishes at the same time as F(x) and remains finite and continuous for all $x \geqq 1$. Set $\frac{d}{dx}\left\{\frac{F(x)}{P_{n_i}(x)}\right\} = \frac{f(x)}{P^2_{n_i}(x)}$.

Then Rolle's Theorem states that $(F) \leqq (f)+1$, where (F) represents the number of zeros of F(x) which are greater than or equal to 1. Also,

$$f(x) = P_{n_i} F'(x) - P'_{n_i} F(x) = \sum_{j=1}^{k} A_j (P_{n_i} P'_{n_j} - P'_{n_i} P_{n_j}).$$

From (2.7) $\quad (x^2-1) P''_{n_j} + 2x P'_{n_j} = n_j(n_j+1) P_{n_j},$

$$(x^2-1) P''_{n_i} + 2x P'_{n_i} = n_i(n_i+1) P_{n_i},$$

whence $\quad \frac{d}{dx}\{(x^2-1)(P_{n_i}P'_{n_j} - P'_{n_i}P_{n_j})\} = \{n_j(n_j+1) - n_i(n_i+1)\} P_{n_j} P_{n_i},$

and $\quad \frac{d}{dx}\{(x^2-1)f(x)\} = P_{n_i} \sum_{j=1}^{k} \{n_j(n_j+1) - n_i(n_i+1)\} A_j P_{n_j} = P_{n_i} \Phi(x).$

Now the sign of the coefficients of $\Phi(x)$ differ from those of the sequence $\{A_i\}$ in that the coefficient of P_{n_i} is zero and all those preceding it preserve the same sign as the corresponding A's, while all those following it have their signs opposite to the corresponding A's. Consequently, $(\Phi) \leqq$ v-1. Applying Rolle's Theorem for $x \geqq 1$ to $(x^2-1)f(x)$, we obtain $(f) \leqq (\Phi)$ and hence $(F) \leqq v$.

97.) E. Laguerre - Sur une propriété des polynomes P_n de Legendre, Comptes Rendus 91(1880)849-851.

There are some obvious extensions. If we wish to consider the number of zeros of $F(x)$ which are $\leqq -1$, we seek the number of variations of the sequence $\{B_i\}$ where $B_i = (-1)^{n_i} A_i$. If all the zeros are real, we can determine the minimum number of zeros in $(-1,1)$. If the sequence of non-negative integers $\{n_i\}$ has lacunae, their extent gives bounds on the number of zeros which are imaginary or in $(-1,1)$. For example, if a term is missing in the sequence of the P's used to form $F(x)$ and if the neighboring terms are of the same sign, the equation has at least two zeros imaginary or in $(-1,1)$.

Bounds for $x_{k,n}$, the kth greatest zero of P_n, have also been given first by Bruns[98] and then improved by Markoff[99]. The derivation of Stieltjes is preferable for its simplicity.[100] Stieltjes proceeds on the basis of

some theorem on quadratic form. His paper contains a most peculiar dynamical problem, closely associated with the results to be obtained. If at -1 and 1 on the x-axis are situated point masses α and β respectively (both positive), if there are n material points of unit mass which move freely on the x-acis between -1 and 1, and if any two points act upon each other with Newtonian forces, then there will be a unique position of equilibrium for the n points between -1 and 1. If their abscissas are denoted by $x_1 > x_2 > \dots > x_n$, then the x's are zeros of $\phi(x)$, a polynomial of degree n in x satisfying the linear differential equation

(7.17) $$(1-x^2)\phi''(x) + 2[\alpha-\beta-(\alpha+\beta)x]\phi'(x) + n(n+2\alpha+2\beta-1)\phi(x) = 0.$$

98.) Bruns, H. - Zur Theorie der Kugelfunctionen, Jour. f. Math. 90(1881)322-8.

99.) A. Markoff - Sur les racines de certaines équations, Math Ann. 27(1886)177-182.

100.) T. J. Stieltjes - Sur les racines de l'équation $X_n=0$, Acta Math. 9(1886)385-400.

Furthermore, the x's are shown to be continuous functions of α and β. If $\alpha = \beta$, $x_i = -x_{n-i+1}$ (with $x_{\frac{n+1}{2}} = 0$ if n is odd). For $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$ we have the Legendre differential equation

(2.7) $(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$.

for $\alpha = 3/4$ and $\beta = \frac{1}{4}$ and for $\alpha = \frac{1}{4}$ and $\beta = 3/4$, we obtain respectively

(7.18) $(1-x^2)\bar{\Phi}'' + (1-2x)\bar{\Phi}' + n(n+1)\bar{\Phi} = 0$ and

(7.19) $(1-x^2)\bar{\bar{\Phi}}'' - (1+2x)\bar{\bar{\Phi}}' + n(n+1)\bar{\bar{\Phi}} = 0$.

In view of the dynamical relations, one might presume that the zeros of the solution of (7.18) will be closer than the zeros of P_n to 1. Similarly, the zeros of the solution of (7.19) should be closer than the zeros of P_n to -1. Solutions of (7.18) and (7.19) are respectively

$$\bar{\Phi} = \frac{\cos[(n+\frac{1}{2})\arccos x]}{\cos[\frac{1}{2}\arccos x]} \quad \text{and} \quad \bar{\bar{\Phi}} \; \frac{\sin[(n+\frac{1}{2})\arccos x]}{\sin[\frac{1}{2}\arccos x]},$$

with zeros at $\bar{x}_i = \cos\frac{(2i-1)\pi}{2n+1}$ and $\bar{\bar{x}}_i = \cos\frac{2i\pi}{2n+1}$ respectively. Stieltjes proves, by considering the x's as functions of α and β, that actually $\cos\frac{(2i-1)\pi}{2n+1} = \bar{x}_i > x_{i,n} > \bar{\bar{x}}_i = \cos\frac{2i\pi}{2n+1}$.
This result was already known to Bruns.

Even more remarkable are restrictions imposed by Markoff. Stieltjes arrives independently at the same result by considering (7.17) for $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{4}$ and for $\alpha = 3/4$ and $\beta = 3/4$, for which we have respectively

(7.20) $(1-x^2)\bar{\Psi}'' - x\bar{\Psi}' + n^2\bar{\Psi} = 0$ and

(7.21) $(1-x^2)\bar{\bar{\Psi}}'' - 3x\bar{\bar{\Psi}}' + n(n+2)\bar{\bar{\Psi}} = 0$.

Solutions of (7.20) and (7.21) are

$$\bar{\Psi} = \sin[n \arccos x] \quad \text{and} \quad \bar{\bar{\Psi}} = \frac{\sin[(n+1)\arccos x]}{\sin[\arccos x]},$$

and the corresponding zeros are

$$\bar{\xi}_i = \cos\frac{2i-1}{2n}\pi \quad \text{and} \quad \bar{\bar{\xi}}_i = \cos\frac{i\pi}{n+1}.$$

Again from dynamical considerations, one might presume that the zeros of

$\overline{\Psi}$ and $\overline{\overline{\Psi}}$ will be symmetric with respect to the origin, that the positive zeros of $\overline{\Psi}$ will be closer to 1 and the negative zeros of $\overline{\Psi}$ closer to -1 than the corresponding zeros of P_n, that the positive zeros of $\overline{\overline{\Psi}}$ will be closer to -1 and the negative zeros of $\overline{\overline{\Psi}}$ closer to 1 than the corresponding zeros of P_n. Stieltjes actually establishes this striking property. Thus

(7.22) $$\begin{cases} \cos \frac{i\pi}{n+1} = \overline{\overline{\xi}}_i < x_{i,n} < \overline{\xi}_i = \cos \frac{2i-1}{2n}\pi & \text{for } x_{i,n} > 0. \\ \cos \frac{i\pi}{n+1} = \overline{\overline{\xi}}_i > x_{i,n} > \overline{\xi}_i = \cos \frac{2i-1}{2n}\pi & \text{for } x_{i,n} < 0. \end{cases}$$

It will be wise to recall that

if $n=2m$, $1 > x_{1,n} > x_{2,n} > \cdots x_{m,n} > 0 > x_{m+1,n} > \cdots > x_{n,n} > -1$ and

if $n=2m+1$, $1 > x_{1,n} > x_{2,n} > \cdots > x_{m,n} > x_{m+1,n} = 0 > x_{m+2,n} > \cdots > x_{n,n} > -1$. Note also that if $i = \frac{n+1}{2}$, $\cos \frac{i\pi}{n+1} = \cos \frac{2i-1}{2n}\pi$; if $i < \frac{n-1}{2}$, $\frac{2i-1}{2n}\pi > \frac{i}{n+1}\pi$ and $\overline{\overline{\xi}}_i > \overline{\xi}_i$; if $i < \frac{n-1}{2}$, $\frac{2i-1}{2n}\pi < \frac{i}{n+1}\pi$ and $\overline{\overline{\xi}}_i < \overline{\xi}_i$.

Very often new representations for $P_n(x)$ led to discovery of new properties of the Legendre Polynomials. We have already discussed the Laplace Integral (2.20), the Jacobi Integral (5.13) and the Dirichlet Integral (5.6) . From the latter, Mehler has deduced a further integral.[101]

If (5.6a) and (5.6b) are added,

(5.6c) $$P_n(\cos\theta) = \frac{1}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})\phi\, d\phi}{\sqrt{2(\cos\phi - \cos\theta)}} + \frac{1}{\pi}\int_\theta^\pi \frac{\sin(n+\frac{1}{2})\phi\, d\phi}{\sqrt{2(\cos\theta - \cos\phi)}} ;$$

while the result of subtraction is

$$0 = \frac{1}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})\phi\, d\phi}{\sqrt{2(\cos\phi - \cos\theta)}} - \frac{1}{\pi}\int_\theta^\pi \frac{\sin(n+\frac{1}{2})\phi\, d\phi}{\sqrt{2(\cos\theta - \cos\phi)}}$$

As a consequence, $P_n(\cos\theta)$ permits two new representations,

(7.23a) $$P_n(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})\phi\, d\phi}{\sqrt{2(\cos\phi - \cos\theta)}}$$

(7.23b) $$P_n(\cos\theta) = \frac{2}{\pi}\int_\theta^\pi \frac{\sin(n+\frac{1}{2})\phi\, d\phi}{\sqrt{2(\cos\theta - \cos\phi)}}$$

Very similar to these is an integral which Catalan[102] derives from the

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101.) F.G. Mehler - Notiz über Dirichlet'schen Integralausdrücke fur Kugelfunction $P_n(\cos\theta)$ und über eine analoge Integralform für die Cylinderfunction J(x), Math. Anna. 5(1872)141-4.

102.) See p. 55 a.

Laplace Integral

(2.20) $P_n(\cos\alpha) = \frac{1}{\pi}\int_0^{\pi} (\cos\alpha + i \sin\alpha \cos\omega)^n d\omega$.

Set $\cos\omega = \frac{\tan\beta}{\tan\alpha}$, $\sin\omega = \frac{\sqrt{\sin^2\alpha - \sin^2\beta}}{\sin\alpha\cos\beta}$ and $d\omega = -\frac{\cos\alpha\, d\beta}{\cos\beta\sqrt{\sin^2\alpha - \sin^2\beta}}$.

The integral becomes

$$P_n(\cos\alpha) = -\frac{1}{\pi}\int_{\alpha}^{-\alpha} \left(\frac{\cos\alpha}{\cos\beta}\right)^{n+1} (\cos n\beta + i \sin n\beta)\frac{d\beta}{\sqrt{\sin^2\alpha - \sin^2\beta}}.$$

We neglect the imaginary part of the integrand, obtaining

$$P_n(\cos\alpha) = \frac{2}{\pi}\cos^{n+1}\alpha \int_0^{\alpha} \frac{\cos n\beta}{\cos^{n+1}\beta}\, \frac{d\beta}{\sqrt{\sin^2\alpha - \sin^2\beta}}$$

Let $\alpha = \frac{\theta}{2}$, $\beta = \frac{\phi}{2}$; this last integral becomes

(7.24) $P_n\left(\cos\frac{\theta}{2}\right) = \frac{2}{\pi}\cos^{n+1}\frac{\theta}{2}\int_0^{\theta/2} \frac{\cos n\frac{\phi}{2}}{\cos^{n+1}\frac{\phi}{2}}\,\frac{d\phi}{\sqrt{2(\cos\phi - \cos\theta)}}$.

Still other definite integral representations have been given by Laurent.[103] His results are based on Cauchy's Residue Theorem. Let $f(z) = \frac{1}{z^{n+1}\sqrt{1-2xz+z^2}}$, where n is a positive integer. We seek $\int_c f(z)dz$ taken around a small circle with the center at the origin. Now $f(z) = \frac{T}{z^{n+1}} = \sum_{i=0}^{\infty} \frac{P_i}{z^{n-i+1}}$, and the residue of f(z) at the origin is thus P_n. Consequently,

(7.25) $P_n(x) = \frac{1}{2\pi i}\int_c \frac{dz}{z^{n+1}\sqrt{1-2xz+z^2}}$

Similarly, Laurent shows that the residue of $f(z) = \frac{(z^2-1)^n}{(z-x)^{n+1}}$ at z=x is $\frac{1}{n!}\frac{d^n(x^2-1)^n}{dx^n}$. Thus

(7.26) $P_n(x) = \frac{1}{2\pi i}\,\frac{1}{2^n}\int_{\gamma} \frac{(z^2-1)^n}{(z-x)^{n+1}}\,dz$,

taken around a small circle with center at z=x.

Laurent's paper is distinguished in several other respects. He derives many known results without giving credit to the original authors. For example, he considers anew the convergence of the generating function expansion (2.1). Writing $x = y + \frac{1}{y}$, the generating function becomes

$$T = (1-2xz+z^2)^{-\frac{1}{2}} = (1-zy)^{-\frac{1}{2}}\left(1-\frac{z}{y}\right)^{-\frac{1}{2}}$$

103.) M. H. Laurent - Mémoire sur les fonctions de Legendre, Jour. de Math. (3)1(1875)373-398.

If $|z||y|<1$ and $|z/y|<1$, the expansion of T is convergent. Thus if ω is the minimum of $\{|y|, |1/y|\}$ and $|z|<\omega$, the condition is met. This is equivalent to z varying inside a circle of radius ω with the origin as center, while x varies inside an ellipse having $\frac{\omega+\frac{1}{\omega}}{2}$ and $\frac{\omega-\frac{1}{\omega}}{2}$ for semi-axes and foci at ± 1. The totality of ellipses for varying ω form a homofocal family. This result is actually due to Neumann.[104]

One of Laurent's most interesting results is a relation between Legendre Polynomials and Bessel's Functions. We can write Murphy's Series

(5.5) $$P_n(\cos\theta) = \cos^{2n}\frac{\theta}{2}\left[1-\binom{n}{1}^2\tan^2\frac{\theta}{2}+\binom{n}{2}^2\tan^4\frac{\theta}{2}-\cdots\right].$$

Now, $\lim \cos^{2n}\theta/2 = 1$ ($\epsilon \le \theta \le \pi-\epsilon$, $\epsilon>0$ arbitrarily small). We consider then the series in tangents of the half angle. If $x=\cos\theta$, $y=\frac{1-x}{1+x}$ and $x=\frac{1-y}{1+y}$, the series becomes

$$1-\binom{n}{1}^2 y+\binom{n}{2}^2 y^2-\binom{n}{3}^2 y^3+\cdots$$

Set $y=z^2/(2n)^2$ and the series is

$$1-\frac{z^2}{2^2(1!)^2}+\frac{z^4}{2^4(2!)^2}\left(1-\frac{1}{n}\right)^2-\frac{z^6}{2^6(3!)^2}\left(1-\frac{1}{n}\right)^2\left(1-\frac{2}{n}\right)^2+\cdots$$

Thus, for n very large

(7.27) $$P_n\left(\frac{4n^2-z^2}{4n^2+z^2}\right) \sim J_0(z)$$

where $J_0(z)$ is the Bessel function of zero order. This result differs somewhat from Laurent's and it is Laurent who is in error.

We can arrive at a smilar result by still another method.[105] From Laplace's integral

(2.20) $$\pi P_n(x) = \int_0^\pi \{x+i\sqrt{1-x^2}\cos\phi\}^n\, d\phi$$

Let $\frac{z}{n}=\sqrt{1-x^2}$, $x=\sqrt{1-\frac{z^2}{n^2}}$; then $\lim_{n\to\infty}\{x+i\sqrt{1-x^2}\cos\phi\}^n = e^{iz\cos\phi}$.

104.) C. Neumann- Ueber die Entwickelung einer Function mit imaginärem Argument nach den Kugelfunctionen, Halle, 1862 (Thesis). (See Heine, Haundbuch der Kugelfunctionen, vol. I, p.40).

105.) J. Strutt (Lord Rayleigh) - On the relation between the Functions of Laplace and Bessel, Proc. Lond. Math. Soc. 9(1878)61-4.

But the real part of $\int_0^{\pi} e^{iz\cos\phi}\,d\phi$ is $\int_0^{\pi}\cos(z\cos\phi)\,d\phi = \pi J_0(z)$. Thus $P_n\left(\sqrt{1-\frac{z^2}{n^2}}\right)\sim J_0(z)$.

Previously, we have had asymptotic expressions for the Legendre Polynomial given by Laplace (2.21), by Heine (2.21a), and by Poisson (2.24). A skillful paper by Bonnet adds still another expression.[106] The method is long, although not complex, and it will suffice to write

(7.28) $$P_n(\cos\theta)\sim 2\frac{\cos\left[(n+\frac{1}{2})\theta-\frac{\pi}{4}\right]}{\sqrt{2n\pi\sin\theta}}+(-1)^n\frac{\cos\left[(n+\frac{1}{2})\theta-\frac{\pi}{4}\right]}{2n\sqrt{2n\pi\sin\theta}}+\frac{\cot\theta\sin\left[(n+\frac{1}{2})\theta-\frac{\pi}{4}\right]}{4n\sqrt{2n\pi\sin\theta}}+\frac{p}{n^2\sqrt{n}}$$

where $\epsilon\le\theta\le\pi-\epsilon$, $\epsilon>0$ arbitrarily small, and p is a bounded function of θ and n.

The results of Bonnet are verified by Darboux and considerably improved.[107] First of all, Darboux shows that the error in Laplace's formula (2.21) is of the order of $\frac{1}{n\sqrt{n}}$ and writes

(7.29) $$P_n(\cos\theta)\sim\frac{2\cos\left[(n+\frac{1}{2})\theta-\frac{\pi}{4}\right]}{\sqrt{2n\pi\sin\theta}}+\frac{p}{n\sqrt{n}},$$

(7.30) $$P_n'(\cos\theta)\sim\frac{-2\sqrt{n}\sin\left[(n+\frac{1}{2})\theta-\frac{\pi}{4}\right]}{\sqrt{2n\pi\sin\theta}}+\frac{p'}{\sqrt{n}},$$

(7.31) $$\frac{d}{d\theta}(P_n+P_{n+1})\sim-2\sqrt{\frac{n}{\pi}}\sin\left[(n+\tfrac{1}{2})\theta-\tfrac{\pi}{4}\right]\sqrt{\cot\tfrac{\theta}{2}}+\frac{p''}{n\sqrt{n}}$$

($0<\theta<\pi$; p, p', p" finite). These formulas are of considerable importance in Darboux's treatment of Legendre Series, and we shall return to them later.

Let $T=(t-e^{i\theta})^{-\frac{1}{2}}(t-e^{-i\theta})^{-\frac{1}{2}}=\frac{(t-e^{i\theta})^{-\frac{1}{2}}}{\sqrt{2i\sin\theta}}\left(\frac{t-e^{-i\theta}}{2i\sin\theta}\right)^{-\frac{1}{2}}$.

Now $\frac{t-e^{-i\theta}}{2i\sin\theta}=1+\frac{t-e^{i\theta}}{2i\sin\theta}$, and therefore

$$T=\frac{(t-e^{i\theta})^{-\frac{1}{2}}}{\sqrt{2i\sin\theta}}\left[1-\frac{1}{2}\left(\frac{t-e^{i\theta}}{2i\sin\theta}\right)+\frac{1\cdot3}{2\cdot4}\left(\frac{t-e^{i\theta}}{2i\sin\theta}\right)^2-\cdots+(-1)^p\frac{1\cdot3\cdots(2p-1)}{2\cdot4\cdots(2p)}\left(\frac{t-e^{i\theta}}{2i\sin\theta}\right)^p+\cdots\right];$$

Whence, for very large n,

(7.32) $$P_n(\cos\theta)\sim2\cdot\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\frac{1}{\sqrt{2\sin\theta}}\left\{\cos\left[(n+\tfrac{1}{2})\theta-\tfrac{\pi}{4}\right]-\frac{1}{2}\frac{1}{(2n-1)}\cos\left[(n-\tfrac{1}{2})\theta+\tfrac{\pi}{4}\right]+\frac{1\cdot3}{2\cdot4}\frac{1\cdot3}{(2n-1)(2n-3)}\cos\left[(n-\tfrac{3}{2})\theta+\tfrac{3\pi}{4}\right]-\cdots\right\}$$

106.) - Sur le développement des fonctions en séries ordonnées suivant les fonctions X_n et Y_n, Jour. de Math. (1)17(1852)265-300. The paper includes a misstatement. Under Theorem VIII, Bonnet asserts that the Legendre differential equation determines $P_n(x)$ to within a constant factor. This is not true, as we shall see, when we present the Legendre function of the second kind.

107.) G. Darboux - Mémoire sur l'approximation des fonctions de très-grands nombres, et sur une class étendue de développements en série, Jour. de Math. (3)4(1878)377-416. See also p 65 ff.; G. Ascoli - Sulle serie $\sum_{n=0}^{\infty} A_nX_n$, Annali di Mat. (2)7(1875)258-344 verifies Bonnet's result also.

This formula is a particular achievement, for if the approximation is taken to a finite number of terms, the error is always of the order of the first term neglected; and if p terms are taken, the error then will be of the order of $\frac{1}{n^{p+\frac{1}{2}}}$. If only the first term is considered, we obtain

$$P_n(\cos\theta) \sim \sqrt{\frac{2}{n\pi\sin\theta}}\cos\left[(n+\tfrac{1}{2})\theta - \frac{\pi}{4}\right],$$

since $\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \sim \frac{1}{\sqrt{n\pi}}$ by Stirling's Formula. This result is the same as that of Laplace (2.21).

Bruns[108] and Heine[109] concern themselves with the behavior of $P_n(\cos\theta)$ as $n \to \infty$. If $0<\theta<\pi$, $P_n(\cos\theta)\to 0$; but suppose that θ is not fixed but approaches zero with $n \to \infty$. In this case, Bruns shows that $P_n(\cos\theta) \to 0$ as $n \to \infty$ if $\theta = \frac{K}{n^\alpha}$ $(0 \le \alpha \le \frac{1}{4})$ and further this will be true even if $\frac{1}{4} \le \alpha < 1$. In short, as long as $n\theta \to \infty$ as $n \to \infty$, $P_n(\cos\theta) \to 0$. Both Bruns and Heine use the Mehler-Dirichlet Integral to obtain the result, but we can use equally well

(5.3) $$P_n\left(\cos\frac{\theta K}{n}\right) = F\left(n+1, -n, 1, \sin^2\frac{\theta K}{2n}\right)$$
$$= 1 - \binom{n+1}{1}\binom{n}{1}\sin^2\frac{\theta K}{2n} + \binom{n+2}{2}\binom{n}{2}\sin^4\frac{\theta K}{2n} - \cdots.$$

As $n \to \infty$, $\sin\frac{\theta K}{2n} \to \frac{\theta K}{2n}$ and

(7.33) $$P_n\left(\cos\frac{\theta K}{n}\right) \sim 1 - \frac{\theta^2K^2}{2^2(1!)^2}\left(1+\frac{1}{n}\right) + \frac{\theta^4K^4}{2^4(2!)^2}\left(1-\frac{1}{n^2}\right)\left(1+\frac{2}{n}\right) - \frac{\theta^6K^6}{2^6(3!)^2}\left(1-\frac{4}{n^2}\right)\left(1-\frac{1}{n^2}\right)\left(1+\frac{3}{n}\right) + \cdots,$$

whence

(7.34) $$\lim_{n\to\infty} P_n\left(\cos\frac{\theta K}{n}\right) = J_0(\theta K).$$

From this last result, Giuliani[110] obtains the theorems of Heine and Bruns.

108.) See p. 60 ff

109.) E. Heine - Ueber die Kugelfunction $P_n(\cos\gamma)$ für ein unendlichen n, Jour. f. Math. 90(1881)329-331.

110.) G. Giuliani - Sopra la funzione $P_n(\cos\gamma)$ per n infinito, Gior. di Mat. 22(1884)236-9.

As Frischauf points out,[111] the convergence of Legendre Series depends on the fact that $P_n(\cos\theta) \to 0$ as $n \to \infty$ if $n\theta \to \infty$.

Stieltjes,[112] too, was interested in the nature of $P_n(\cos\theta)$ for large n. He obtains a series comparable to that of Darboux,

$$(7.35)\quad P_n(\cos\theta) = \frac{4}{\pi}\,\frac{2^{2n}(n!)^2}{(2n+1)!}\left\{\frac{\cos\left[(n+\frac{1}{2})\theta-\frac{\pi}{4}\right]}{\sqrt{2\sin\theta}} + \frac{1^2}{2(2n+3)}\,\frac{\cos\left[(n+\frac{3}{2})\theta-\frac{3\pi}{4}\right]}{\sqrt{(2\sin\theta)^3}} + \frac{1^2\cdot 3^2}{2\cdot 4}\,\frac{\cos\left[(n+\frac{5}{2})\theta-\frac{5\pi}{4}\right]}{(2n+3)(2n+5)\sqrt{(2\sin\theta)^5}} + \cdots\right\}.$$

This series converges and represents $P_n(\cos\theta)$ when $2\sin\theta > 1$, i.e. $\pi/6 < \theta < 5\pi/6$. It further has the peculiar property that if the first p terms of the series are taken, the error is in absolute value less than twice the (p+1)th term in which the cosine is replaced by unity in the numerator.

Callandreau has proved the results of Darboux and Stieltjes in a different way and examined the remainder of the Darboux expression.[113] The integral related to the Bessel Function is

$$J_0(x) = J_0(-x) = \frac{1}{\pi}\int_0^\pi e^{-ix\cos\phi}\,d\phi,$$

and we can write for $p>0$

$$\int_0^\infty e^{-px} J_0(bx)\,x^n\,dx = \frac{1}{\pi}\int_0^\pi d\phi\int_0^\infty e^{-px}\,e^{-ibx\cos\phi}\,x^n\,d\phi$$

$$= \frac{n!}{\pi}\int_0^\pi \frac{d\phi}{(p+ib\cos\phi)^{n+1}}.$$

111.) J. Frischauf - Zur Theorie der Kugelfunctionen, Jour. f. Math. 107(1890)87-8.

112.) T.J. Stieltjes - Sur la valeur asymptotique des polynomes de Legendre, Comptes Rendus 110(1890)1026-7; Sur les polynomes de Legendre, Ann. Fac. Sc. Toulouse (1)4(1890)G1-17.

113.) O. Callandreau - Sur le calcul des polynomes $X_n(\cos\theta)$ de Legendre pour les grandes valeurs de n, Bull des Sci Math. (2)15(1891)121-4.

If $p = \cos\theta$ and $b = \sin\theta$, we have the Jacobi Integral on the right and

$$P_n(\cos\theta) = \frac{1}{n!}\int_0^\infty e^{-x\cos\theta} J_0(x\sin\theta)\, x^n\, dx.$$

On the basis of this and Poisson's expansion (2.24) which is essentially $J_0(\theta)$, Callandreau announces a theorem. The remainder of the Darboux expansion can be considered as the real part of $\rho\, e^{i[\phi + \frac{2p+1}{4}\pi - (n-p-\frac{1}{2})\theta]}$ of which the affix falls inside the circle with center at the origin and radius equal to the absolute value of the (p+1)th term (and in the fourth quadrant if $\cos\theta > 0$). By this we are enabled to say something about the sign of the remainder as well.

SECTION VIII

INTERPOLATION AND MECHANICAL QUADRATURE

It is rather surprising that, just as there was a lapse of fifteen years between Gauss' paper on evaluating definite integrals and Jacobi's revival of interest in Gauss quadrature, there was once again, after the three memoirs of Murphy, a period of more than fifteen years in which no apparent progress was made in the problem of evaluating definite integrals by means of Gauss quadrature. There were at best a few attempts to apply the results of Jacobi and Murphy to specific problems.[114]

It is Tchebycheff who heralded the development of the modern theory of interpolation and mechanical quadrature. With Tchebycheff's early memoirs we shall not be concerned in any detail. It will be sufficient to remark the nature of the problems solved. His first two papers deal with the development of the sum $\sum_{i=0}^{n} w^2(x_i)\frac{1}{x-x_i}$ in continued fractions, where $x_0, x_1, \ldots, x_n$ are n+1 real and distinct numbers for which $w(x)$ is a defined, non-vanishing weight function.[115] The continued fraction development has as the denominators of the successive convergents a sequence of functions $\{\psi_i(x)\}$. The sequence is distinguished by the fact that, among all the functions of the same degree having the same coefficient for the highest power of x, the sequence of ψ's renders minimum the sum

$$\sum_{i=0}^{n} \psi_j^2(x_i)\, w^2(x_i) \qquad \text{for } j=1,2,\ldots,n.$$

114.) For example, F. Neumann, "Über eine neue Eigenschaft der Laplaceschen $Y^{(n)}$ und ihre Anwendung zur analytischen Darstellung derjenigen Phänomene, welche Functionen der geographischen Länge und Breite sind, Astron. Nachr. 15(1838)313-324 (reprint Math. Annalen 14(1879)567-576).

115.) P Tchebichev - Sur une formule d'Analyse, Bull. Ac. Imp. Sc. St. Petersbourg (phys. -math.) 13(1853)210-211; Extrait d'une Mémoire sur les fractions continues, ibid, 287-288.

That the problem is related to interpolation and quadrature problems of the kind with which we are more immediately concerned is a little more evident from a third paper.[116] In the latter, $f(x)$, a rational integral function of degree n in x whose value is known for $x_0, x_1, x_2, \ldots, x_n$ ($n+1$ real and distinct values of x), is written by the method of Lagrange as

$$f(x) = \sum_{i=0}^{n} \frac{L(x)\,f(x_i)}{(x-x_i)L'(x_i)}, \quad L(x) = (x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n).$$

Also $$f(x) = q_1\sum_{i=0}^{n} f(x_i) - q_2\,\psi_1(x)\sum_{i=0}^{n}\psi_1(x_i)f(x_i) + q_3\psi_2(x)\sum_{i=0}^{n}\psi_2(x_i)f(x_i) + \cdots,$$

where the q's are the coefficients of x in the Q's of the continued fraction $\frac{1|}{|Q_1} + \frac{1|}{|Q_2} + \frac{1|}{|Q_3} + \cdots$ development of $\sum_{i=0}^{n} \frac{1}{x-x_i}$ and $\psi_r(x)$ is the denominator of the rth convergent. If only the first m terms of this approximation are taken, $f(x)$ is approximated by a polynomial of degree $m-1$, and this approximation will be best in the sense of least squares, if the known values $f(x_0), f(x_1), \ldots, f(x_n)$ are considered to be of equal weight. The problems are solved in great detail by first Tchebycheff[117] and then Hermite.[118]

A systematic exposition of the use of series of the denominators of the convergents of a continued fraction development to approximate a given function is given by Rouché.[119] Before describing his work in detail, we will mention the general nature of his results. Let $f(x)$ be a polynomial of degree n in x, and $F(x)$ a polynomial of degree $m+1$ in x with zeros at $x_0, x_1, x_2, \ldots, x_m$ and $m \geq n$. Denote $\frac{f(x_i)}{F'(x_i)}$ by p_i, then $\frac{f(x)}{F(x)} = \sum_{i=0}^{n} \frac{p_i}{x-x_i}$.

116.) P. Tchebichev, Sur une formulé d'Analyse, Jour. f. Math. 53(1857)123-5.

117.) P. Tchebichef, Sur les fractions continues, Jour. de Math. (2)3(1858)289-323.

118.) C. Hermite - Sur l'interpolation, Comptes Rendus, 48(1859)62-67. (F. Brioschi, Intorno ad una formola di interpolazione, Annali di Sc. Mat. e Fis. (1)2(1859)132-4, reviews the work of Tchebycheff and Hermite).

119.) E. Rouché, Mémoire sur le développement des fonctions en séries ordonnées suivant les dénominateurs des réduites d'une fraction continue, Comptes Rendus 46(1858)1221-4, Jour. de l'Ec. Plytech. (1)37(1858)1-34.

Form the Sturm sequence

$$(8.1)\quad \begin{cases} \dfrac{F}{f} = Q_1 - \dfrac{R_1}{f} \\ \dfrac{f}{R_1} = Q_2 - \dfrac{R_2}{R_1} \\ \dfrac{R_1}{R_2} = Q_3 - \dfrac{R_3}{R_2} \\ - - - - - \\ \dfrac{R_{n-2}}{R_{n-1}} = Q_n - \dfrac{R_n}{R_{n-1}} \\ \dfrac{R_{n-1}}{R_n} = Q_{n+1}. \end{cases}$$

and the continued fraction

$$(8.2)\quad \frac{f(x)}{F(x)} = \frac{1|}{|Q_1} - \frac{1|}{|Q_2} - \frac{1|}{|Q_3} - \cdots - \frac{1|}{|Q_{n+1}}.$$

Rouché's demonstrations demand that R_i be a polynomial of degree n-i, Q_1 of degree m-n+1 and all the other Q's linear. Write

$$R_i = r_i x^{n-i} + \cdots \qquad (i = 1,2,3,\ldots,n),$$
$$Q_1 = q_1 x^{m-n+1} + \cdots \qquad ,$$
$$Q_i = q_i x + k_i \qquad (i = 2,3,4,\ldots,n+1).$$

Let N_k/D_k be the kth convergent of (8.2), then N_k is a polynomial of degree k-1 and D_k is of degree m-n+1+k-1=m-n+k. Given m+1 values $\varphi(x_0), \varphi(x_1), \varphi(x_2), \ldots, \varphi(x_m)$ of $\varphi(x)$, a polynomial of degree m, we seek to develop this function in series of D's and to study the properties of this expansion. We shall see that there are essentially two cases. For m=n, there will exist a solution

$$(8.3)\quad \varphi(x) = \sum_{k=0}^{m} \Big[q_{k+1}\, D_k(x) \sum_{i=0}^{m} p_i\, D_k(x_i)\, \varphi(x_i) \Big] \qquad \big(D_0(x) = 1 \big).$$

For m>n, the D's will depend on an excess number of linear equations and the problem will be "overdetermined".

The most fruitful discussion comes from seeking an approximation which will be best in the sense of least squares, considering the $\varphi(x_i)$'s to be equally weighted. In this case, of all the rational functions which lead to series giving exact representations of $\varphi(x)$ (m = n), the fraction

$\lambda \frac{F'(x)}{F(x)}$ (λ constant) has the following property: If in

(8.4) $$\varphi(x) = \sum_{k=0}^{m} \left[\lambda q_{k+1} D_k(x) \sum_{i=0}^{m} D_k(x_i)\, \varphi(x_i)\right]$$

we take only the first r+1 terms (r arbitrary), we obtain an approximation

(8.5) $$Z(x) = \sum_{k=0}^{r} \left[\lambda q_{k+1} D_k(x) \sum_{i=0}^{m} D_k(x_i)\, \varphi(x_i)\right]$$

which is such that the sum $\sum_{i=0}^{m} \left[\varphi(x_i) - Z(x_i)\right]^2$ is minimum. This is the result obtained by Tchebycheff in the case of $\lambda = 1$, for $\frac{F'(x)}{F(x)} = \sum_{i=0}^{m} \frac{1}{x - x_i}$ and (8.4) becomes

(8.6) $$\varphi(x) = \sum_{k=0}^{m} \left[q_{k+1} D_k(x) \sum_{i=0}^{m} D_k(x_i)\, \varphi(x_i)\right].$$

If the given x's are equally spaced, i.e. $x_i - x_{i-1} = 2/m$ in $[-1,1]$, consider $\lim_{m\to\infty} \frac{1}{m} \frac{F'(x)}{F(x)} = \lim_{m\to\infty} \frac{1}{2} \sum_{i=0}^{m} \frac{2}{m} \frac{1}{x - x_i} = \frac{1}{2}\int_{-1}^{1} \frac{dy}{x-y} = \frac{1}{2} \log \frac{x+1}{x-1}$. Then, we consider the continued fraction development of $\frac{1}{2}\log\frac{x+1}{x-1}$ of which the denominators of the convergents are Legendre Polynomials. In this case, (8.6) becomes

(8.7) $$\varphi(x) = \sum_{k=0}^{m} \left[\frac{2k+1}{2} P_k(x) \int_{-1}^{1} \varphi(t)\, P_n(t)\, dt\right] = \sum_{k=0}^{m} A_k P_k.$$

One could easily show that such an approximation is best in the sense of least squares.

Let us seek now the relations regarding the rational fraction and its continued fraction. Consider, as before, $f(x)/F(x)$, with $p_i = f(x_i)/F'(x_i)$.

Let $S_\mu = p_0 x_0^\mu + p_1 x_1^\mu + p_2 x_2^\mu + \cdots + p_m x_m^\mu = \sum_{i=0}^{m} p_i x_i^\mu$, $\omega = m - n + k$,

$$\Delta_k = \begin{vmatrix} S_0 & S_1 & \cdots & S_\omega \\ S_1 & S_2 & \cdots & S_{\omega+1} \\ - & - & & - \\ S_\omega & S_{\omega+1} & \cdots & S_{2\omega} \end{vmatrix}, \qquad T_k = \begin{vmatrix} S_0 & S_1 & \cdots & S_\omega \\ S_1 & S_2 & \cdots & S_{\omega+1} \\ - & - & & - \\ S_{\omega-1} & S_\omega & \cdots & S_{2\omega-1} \\ 1 & x & \cdots & x^\omega \end{vmatrix}.$$

$T_k(x)$ is a polynomial of degree ω. In its expansion on the last row,

$$T_k(x) = \frac{d\Delta_k}{dS_\omega} + x\frac{d\Delta_k}{dS_{\omega+1}} + x^2 \frac{d\Delta_k}{dS_{\omega+2}} + \cdots + x^\omega \frac{d\Delta_k}{dS_{2\omega}}.$$

Replace x by x_i, multiply by $p_i x_i^\mu$ and sum. Then

(8.8) $$\sum_{i=0}^{m} p_i x_i^\mu T_k(x_i) = S_\mu \frac{d\Delta_k}{dS_\omega} + \cdots + S_{\mu+\omega}\frac{d\Delta_k}{dS_{2\omega}} = \begin{vmatrix} S_0 & S_1 & \cdots & S_\omega \\ S_1 & S_2 & \cdots & S_{\omega+1} \\ & & & \\ S_{\omega-1} & S_\omega & \cdots & S_{2\omega-1} \\ S_\mu & S_{\mu+1} & \cdots & S_{\mu+\omega} \end{vmatrix}.$$

For $\mu=\omega$, this gives

(8.9) $\sum_{i=0}^{m} p_i x_i^{\omega} T_k(x_i) = \Delta_k$.

For $\mu = \omega - j < \omega \quad (j=1,2,3,\dots,\omega)$, the last determinant will contain twice the row $S_\mu, S_{\mu+1}, S_{\mu+2}, \dots, S_{\mu+\omega}$, therefore

(8.10) $\sum_{i=0}^{m} p_i x_i^{\mu} T_k(x_i) = 0 \quad (\mu = 0,1,2,\dots,\omega-1)$.

Furthermore,

(8.11) $R_k = f\cdot D_k - F\cdot N_k$.

From (8.1) and (8.2) this is obviously true for k=1,2. In general

$R_i = R_{i-1} Q_i - R_{i-2}$ from (8.1); $D_i = D_{i-1} Q_i - D_{i-2}$

and $N_i = N_{i-1} Q_i - N_{i-2}$ from well known continued fraction relationships.

We assume $R_{i-1} = f\cdot D_{i-1} - F\cdot N_{i-1}$ and $R_{i-2} = f\cdot D_{i-2} - F\cdot N_{i-2}$. Now

$$R_i = (f\cdot D_{i-1} - F\cdot N_{i-1}) Q_i - (f\cdot D_{i-2} - F\cdot N_{i-2}) = f(D_{i-1}Q_i - D_{i-2}) - F(N_{i-1}Q_i - N_{i-2}).$$

Hence, $R_i = f\cdot D_i - F\cdot N_i$ and (8.11) follows from an obvious induction. Since $F(x_i)=0$,

(8.12) $R_k(x_i) = f(x_i) D_k(x_i)$.

A known result in the theory of rational fractions is

(8.13) $\sum_{i=0}^{m} \frac{R_k(x_i)}{F'(x_i)} x_i^{j} = 0 \quad (j=0,1,2,\dots,\omega-1), \sum_{i=0}^{m} x_i^{\omega} \frac{R_k(x_i)}{F'(x_i)} = \frac{r_k}{A}$,

(r_k, A constants; A is coefficient of highest power of x in F); by virtue of (8.12) this becomes

(8.14) $\sum_{i=0}^{m} p_i D_k(x_i) x_i^{j} = 0 \quad (j=0,1,2,\dots,\omega-1), \sum_{i=0}^{m} p_i x_i^{\omega} D_k(x_i) = \frac{r_k}{A}$.

These $\omega+1$ relations determine the $\omega+1$ coefficients of $D_k(x)$ which is of degree ω. Now (8.9) and (8.10) determine $T_k(x)$ which is also of degree ω, and it is clear that $T_k(x)$ differs from $D_k(x)$ only by a constant factor. In fact

(8.15) $D_k(x) = \frac{1}{A}\frac{r_k}{\Delta_k} T_k(x) = \frac{1}{A} r_k \frac{\Delta_{k-1}}{\Delta_k} x^{\omega} + \dots$.

Further, write (8.11) for k-1 and k, multiply the first by D_k and the second by D_{k-1}, subtract to eliminate f. Thus

$$R_{k-1} D_k - R_k D_{k-1} = F(N_k D_{k-1} - N_{k-1} D_k) = F,$$

the parenthetic quantity in the middle expression reducing to unity by a well known continued fraction property. Using (8.15), equate the coefficients of the highest power of x(x^{m+1} here). Then $\frac{1}{A} R_k R_{k-1} \frac{\Delta_{k-1}}{\Delta_k} = A$.
But the coefficient of x in Q_{k+1} is the coefficient of x in R_{k-1}/R_k, and so

(8.16) $$q_{k+1} = \frac{R_{k-1}}{R_k} = \frac{R_k R_{k-1}}{R_k^2} = \frac{A^2}{R_k^2} \frac{\Delta_k}{\Delta_{k-1}}$$

In particular, for Q_1 the coefficient of the highest power of x is that of the highest power of x in F/f. If we let a be the coefficient of the highest power of x in f(x), we have $q_1 = A/a$.

We now exhibit a most important property of the D's. From (8.13) and (8.15), for $h < k$ two integers between 0 and m+1,

$$\sum_{i=0}^{m} D_k(x_i) \frac{R_k(x_i)}{F'(x_i)} = \frac{1}{A^2} R_k^2 \frac{\Delta_{k-1}}{\Delta_k},$$

$$\sum_{i=0}^{m} D_h(x_i) \frac{R_k(x_i)}{F'(x_i)} = 0$$

By virtue of (8.12) and (8.16) these become

(8.17a) $$\sum_{i=0}^{m} p_i D_k^2(x_i) = \frac{1}{q_{k+1}},$$

(8.17b) $$\sum_{i=0}^{m} p_i D_h(x_i) D_k(x_i) = 0 .$$

Even for h=0, (8.17b) holds. For k=0, (8.17a) is $\sum_{i=0}^{m} \frac{f(x_i)}{F'(x_i)}$ and 0 or a/A according as $n < m$ or $n = m$. Thus (8.17a) is true for $k = 0$ and $n = m$ also.

We now seek to represent a rational integral function by a series of polynomials, which are denominators of the convergents of a continued fraction. Given $\varphi(x_0), \varphi(x_1), \varphi(x_2), \dots, \varphi(x_m)$, m+1 values of $\varphi(x)$, a rational integral function of degree m, and the continued fraction development from which we obtain the sequence $\{D_k(x)\}$. We write

$y = u_0 D_0(x) + u_1 D_1(x) + \cdots + u_n D_n(x)$ where $y_i = \varphi(x_i)$ $(i = 0, 1, 2, \dots, m)$

Suppose $n < m$. We can then write

$$\begin{cases} u_0 D_0(x_0) + u_1 D_1(x_0) + \cdots + u_n D_n(x_0) = \varphi(x_0) \\ u_0 D_0(x_1) + u_1 D_1(x_1) + \cdots + u_n D_n(x_1) = \varphi(x_1) \\ \text{---} \qquad \text{---} \qquad \text{---} \\ u_0 D_0(x_m) + u_1 D_1(x_m) + \cdots + u_n D_n(x_m) = \varphi(x_m) \end{cases}$$

an overdetermined system of m+1 equations in n+1 unknowns ($m > n$). We can impose, however, some added conditions, say that the approximation be best in the sense of least squares. Thus, let us minimize

$$(8.18) \qquad \Omega = \sum_{i=0}^{m} \left[\varphi(x_i) - \sum_{j=0}^{n} u_j D_j(x_i)\right]^2$$

in the usual way, setting

$$(8.19) \qquad \frac{\partial \Omega}{\partial u_0} = 0, \quad \frac{\partial \Omega}{\partial u_1} = 0, \quad \ldots, \quad \frac{\partial \Omega}{\partial u_n} = 0 .$$

Let

$$(8.20) \qquad \begin{cases} \sum_{i=0}^{m} D_k(x_i) D_j(x_i) = \delta_k^{(j)} \\ \sum_{i=0}^{m} D_k(x_i)\ \varphi(x_i) = \pi_k \end{cases}$$

and (8.19) then gives a system of n+1 linear equations

$$(8.21) \qquad \begin{cases} u_0 \delta_0^{(0)} + u_1 \delta_0^{(1)} + u_2 \delta_0^{(2)} + \cdots + u_n \delta_0^{(n)} = \pi_0 \\ u_0 \delta_1^{(0)} + u_1 \delta_1^{(1)} + u_2 \delta_1^{(2)} + \cdots + u_n \delta_1^{(n)} = \pi_1 \\ \text{—} \qquad \text{—} \qquad \text{—} \\ u_0 \delta_n^{(0)} + u_1 \delta_n^{(1)} + u_2 \delta_n^{(2)} + \cdots + u_n \delta_n^{(n)} = \pi_n \end{cases}$$

for determining the u's, the n+1 coefficients of the series.

The case of more immediate interest occurs when $n=m$. In this case, multiply each y_i by $p_i D_k(x_i)$ and sum on i, then

$$\sum_{i=0}^{m} p_i D_k(x_i)\varphi(x_i) = u_k \sum_{i=0}^{m} D_k^2(x_i) p_i = \frac{1}{g_{k+1}} u_k$$

by virtue of (8.17). Hence the result previously announced in (8.3) will be obtained. How is this related to the best approximation in the sense of least squares? If we take $\sum_{k=0}^{h} u_k D_k(x)$ ($h < m$) to approximate $\varphi(x)$, the coefficients

$$(8.22) \qquad u_k = g_{k+1} \sum_{k=0}^{m} p_i D_k(x_i)\varphi(x_i)$$

depend on all $m+1$ values of $\varphi(x)$, but the approximation will in general not be best in the sense of least squares. In order to achieve the latter, we must take the u's as in (8.21). The same is true for $f(x) = \lambda F'(x)$, i.e. the rational fraction $\lambda F'(x)/F(x)$, in which $p_i = \lambda$ and $\delta_k^{(j)} = 0$ ($k \neq j$). Then (8.21) reduces to $u_k = \pi_k / \delta_k^{(k)}$.

We are now ready to seek the more intimate relations of the preceding discussion to Legendre Polynomials. Gauss[120] had already given the continued fraction associated with the hypergeometric function, and had written $\log \frac{1+t}{1-t} \qquad 2tF(\frac{1}{2}, 1, 3/2, t^2)$. We are going to prove that the denominators of the convergents of this continued fraction development differ from Legendre Polynomials only by constant factors. We recall first that

(3.9) $(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0 \quad , P_0 = 1, \; P_1 = x$

completely determines the Legendre Polynomials. Now the continued fraction

$\psi = \frac{v_0|}{|w_0} - \frac{v_1|}{|w_1} - \frac{v_2|}{|w_2} - \cdots$ has convergents $\frac{V_i}{W_i} = \frac{v_0|}{|w_0} - \frac{v_1|}{|w_1} - \cdots - \frac{v_i|}{|w_i}$.

We list some well known continued fraction relations which will be helpful

$$V_n = w_{n-1} V_{n-1} - v_{n-1} V_{n-2}, \qquad V_{n+1} W_n - V_n W_{n+1} = v_0 v_1 v_2 \cdots v_n,$$

$$W_n = w_{n-1} W_{n-1} - v_{n-1} W_{n-2}, \qquad \frac{V_{n+1}}{W_{n+1}} - \frac{V_n}{W_n} = \frac{v_0 v_1 v_2 \cdots v_n}{W_n W_{n+1}},$$

$$\psi = \frac{v_0}{W_0 W_1} + \frac{v_0 v_1}{W_1 W_2} + \frac{v_0 v_1 v_2}{W_2 W_3} + \cdots + \frac{v_0 v_1 v_2 \cdots v_n}{W_n W_{n+1}} + \cdots .$$

Let $\psi_1 = \frac{1|}{|Q_1} - \frac{1|}{|Q_2} - \frac{1|}{|Q_3} - \cdots$ have the convergents N_i/D_i. Then

$$N_{n+1} = N_n Q_{n+1} - N_{n-1}, \qquad N_{n+1} D_n - N_n D_{n+1} = 1,$$

$$D_{n+1} = D_n Q_{n+1} - D_{n-1}, \qquad Q_{n+1} = \frac{D_{n+1} + D_{n-1}}{D_n}$$

$$\psi_1 = \frac{1}{D_1} + \frac{1}{D_1 D_2} + \cdots + \frac{1}{D_n D_{n+1}} + \cdots .$$

Further, Gauss gives

$$F(\alpha, 1, \gamma, t) = \frac{1|}{|1} - \frac{a_0 t|}{|1} - \frac{b_0 t|}{|1} - \frac{a_1 t|}{|1} - \frac{b_1 t|}{|1} - \cdots ,$$

where $a_K = \frac{(\alpha+K)(\gamma+K-1)}{(\gamma+2K-1)(\gamma+2K)}$, $b_K = \frac{(K+1)(\gamma+K-\alpha)}{(\gamma+2K)(\gamma+2K+1)}$

Thus $\frac{1}{2}\log \frac{1+t}{1-t} = tF(\frac{1}{2}, 1, \frac{3}{2}, t^2) = \frac{t|}{|1} - \frac{\frac{1}{3}t^2|}{|1} - \frac{\frac{2\cdot 2}{3\cdot 5}t^2|}{|1} - \frac{\frac{3\cdot 3}{5\cdot 7}t^2|}{|1} - \cdots$.

Let $t = 1/x$, this becomes

$$\frac{1}{2}\log \frac{x+1}{x-1} = \frac{1|}{|x} - \frac{\frac{1}{3}|}{|x} - \frac{\frac{2\cdot 2}{3\cdot 5}|}{|x} - \frac{\frac{3\cdot 3}{5\cdot 7}|}{|x} - \cdots ,$$

in which $v_n = \frac{n \cdot n}{(2n-1)(2n+1)}$, $w_n = x$, $W_0 = 1$, $W_1 = x$. Now

$$W_{n+1} = xW_n - \frac{n^2}{(2n-1)(2n+1)} W_{n-1},$$

120.) The discussion here resembles in its basic form that of Gass (see p 29). There are, of course, important embellishments, for Gauss was apparently unaware of the relation of his results to Legendre Polynomials.

in which set $U_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{n!} W_n$, $U_0 = 1$, $U_1 = x$. Thus $(n+1)U_{n+1} - (2n+1)xU_n + nU_{n-1} = 0$.
Comparing with (3.9),[121] we see that $U_n = P_n$ and $W_n = \frac{2^n (n!)^2}{(2n)!} P_n$. Thus

(8.23) $\frac{1}{2}\log\frac{x+1}{x-1} = \frac{1}{P_1} + \frac{1}{2P_1P_2} + \frac{1}{3P_2P_3} + \cdots + \frac{1}{(n+1)P_nP_{n+1}} + \cdots$

$$= \frac{1|}{|Q_1} - \frac{1|}{|Q_2} - \cdots = \frac{N}{D}$$

with $D_1 = P_1$, $D_2 = 2P_2$, $D_3 = \frac{1\cdot3}{2}P_3$, $D_4 = \frac{2\cdot4}{1\cdot3}P_4$, etc. In general,

(8.24) $D_n = C_nP_n$, $C_n = \frac{2^n\left(\frac{n}{2}!\right)^2}{n!}$ for n even and $C_n = \frac{n!}{2^{n-1}\left[\left(\frac{n-1}{2}\right)!\right]^2}$ for n odd.

Now $Q_{n+1} = q_{n+1}x + k_{n+1} = \frac{D_{n+1} + D_{n-1}}{D_n}$. The coefficient of x in $\frac{D_{n+1}}{D_n}$ is $\frac{2n+1}{C_n^2}$, and so

(8.25) $q_{n+1}C_n^2 = 2n+1.$

Let $x_i = \alpha + \frac{i}{m}(\beta - \alpha)$ in (α, β) be equidistant abscissas. Then
$\sum_{i=0}^{m} \frac{1}{m}\theta(x_i) = \frac{1}{\beta-\alpha}\sum_i \theta(x_i)(x_{i+1}-x_i)$ and $\lim_{m\to\infty}\frac{1}{\beta-\alpha}\sum_{i=0}^{m}\theta(x_i)(x_{i+1}-x_i) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \theta(x)\,dx.$
In the same way, for $\lambda = \frac{1}{m}$ and q, D, λ arising from $\frac{f(x)}{F(x)} = \frac{1}{m}\frac{F'(x)}{F(x)} = \sum_i \frac{1}{m}\frac{1}{x-x_i}$,

(8.4) becomes

(8.26) $\varphi(x) = \sum_{k=0}^{m}\left[q_{k+1}D_k(x)\sum_{i=0}^{m}\frac{1}{m}D_k(x_i)\,\varphi(x_i)\right]$.

Let x_i increase by 2/m in [-1,1], i.e. $x_i = -1 + i\frac{2}{m}$. Let $m \to \infty$, then

$$\varphi(x) = \sum_{k=0}^{m}\left[\frac{1}{2}q_{k+1}D_k(x)\int_{-1}^{1} D_n(t)\,\varphi(t)\,dt\right]$$

where q and D are determined from $\lim_{m\to\infty}\sum_{i=0}^{m}\frac{1}{m}\frac{1}{x-x_i} = \frac{1}{2}\int_{-1}^{1}\frac{dt}{x-t} = \frac{1}{2}\log\frac{x+1}{x-1}$.

We assume, of course, that the expression converges and represents the function. From (8.24) and (8.25), we obtain

(8.27) $\varphi(x) = \sum_{n=0}^{\infty}\left[\frac{1}{2}(2n+1)P_n(x)\int_{-1}^{1}P_n(t)\varphi(t)\,dt\right] = \sum_{n=0}^{\infty} A_n P_n(x).$

The mean square error, if only the first r+1 terms of (8.26) are taken, is $\frac{1}{m}\sum_i\left[\varphi(x_i) - \zeta_n(x_i)\right]^2$. This expression is minimum and has $\frac{1}{2}\int_{-1}^{1}\left[\varphi(x) - \zeta_n\right]^2 dx$ as limit as $m \to \infty$. Similarly, if $S_r(x)$ is the sum of the first r +1 terms of (8.27), then, among all the rational and integral functions {Z} of x of the same degree, $S_r(x)$ is the one which minimizes $\Omega = \int_{-1}^{1}\left[\varphi(x) - Z\right]^2 dx$.

121.) Other proof given by:
Hermite, C. - Extrait d'une lettre addressée à F. Gomes Teixeira, Jour. de Ciencias Math. e Astron. 6(1885)81-4, Oeuvres 4(1917)169-171.
Catalan apparently believed no proof had ever been given. Most authors considered it obvious from the work of Gauss. Catalan gives a formal proof (see p. 55 g.).

The proof is simple.[122] Since Z is a rational integral function of degree r, set $Z = \sum_{n=0}^{r} A_n P_n$. The condition for minimum is $\frac{\partial \Omega}{\partial A_n} = 0 \quad (n = 0, 1, 2, \dots, r)$,

i.e. $\int_{-1}^{1} [\varphi(x) - Z] P_n \, dx = 0$ or $\int_{-1}^{1} [\varphi(x) - \sum_{n=0}^{r} A_n P_n] P_n \, dx = 0$.

In view of the orthogonality property of Legendre Polynomials, this becomes

$\int_{-1}^{1} \varphi(x) P_n \, dx = \frac{2}{2n+1} A_n$ or $A_n = \frac{2n+1}{2} \int_{-1}^{1} P_n(x) \varphi(x) dx$. Hence the Z pertaining to the minimum is of the form $Z = \sum_{n=0}^{r} \left[\frac{2n+1}{2} P_n(x) \int_{-1}^{1} \varphi(t) P_n(t) dt \right] = S_r(x)$.

The orthogonality property upon which the proof depends could be obtained, it may be remarked, from the limiting case of (8.17) for $p_i = 1/m$ and $x_i = -1 + i\frac{2}{m} \quad (i = 0, 1, 2, \dots, m)$ as $m \to \infty$, if we make use of the relations which we have already seen to result, $D_n = C_n P_n$ and $q_{n+1} c_n^2 = 2n+1$.

Returning now to the earlier discussion of Gauss quadrature (Section III), we can extend the results there obtained so that the role of the continued fraction development of $\log \frac{x+1}{x-1}$ is more obvious. We evaluated $\int_{-1}^{1} f(x) dx$, when $f(x)$ was known for $\alpha_1, \alpha_2, \dots, \alpha_n$ all in $(-1,1)$, by writing

(3.6) $\int_{-1}^{1} f(x) dx = \sum_{i=1}^{n} A_i f(\alpha_i)$ where

(3.5) $A_i = \frac{1}{P_n'(\alpha_i)} \int_{-1}^{1} \frac{P_n(x)}{x - \alpha_i} dx$

and the error function is

(3.8) $E(x^m) = \int_{-1}^{1} x^m dx - \sum_{i=1}^{n} A_i \alpha_i^m = \begin{cases} -\sum_{i=1}^{n} A_i \alpha_i^m & (m \text{ odd}) \\ \frac{2}{m+1} - \sum_{i=1}^{n} A_i \alpha_i^m & (m \text{ even}) \end{cases}$

If $\log \frac{t+1}{t-1} - t \sum_{i=1}^{n} \frac{A_i}{t^2 - \alpha_i^2}$ is expanded in descending powers of t, the coefficient of t^{-2k-1} will be precisely $E(x^{2k})$. Let

(8.28) $\psi(t) = \int_{-1}^{1} \frac{P_n(x) - P_n(t)}{x - t} dx, \quad \psi(\alpha_i) = \int_{-1}^{1} \frac{P_n(x)}{x - \alpha_i} dx$.

Then $A_i = \frac{\psi(\alpha_i)}{P_n'(\alpha_i)}$. Now $\psi(t)$ is a rational integral function of t of degree n-1; and therefore, by the method of Lagrange,

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122.) One is given by Plarr, G. - Note sur une propriété commune aux séries dont le terme général dépend des fonctions X_n de Legendre, ou des cosinus et sinus des multiples de la variable, Comptes Rendus 44(1857)984-6.

$$\psi(t) = P_n(t)\sum_{i=1}^{n} \frac{\psi(\alpha_i)}{P_n'(\alpha_i)(t-\alpha_i)} = P_n(t)\sum_{i=1}^{n} \frac{A_i}{t-\alpha_i}.$$

By virtue of the symmetry of the α's and A's (see p. 28), $A_{n-m+1} = A_m$ and $\alpha_{n-m+1} = -\alpha_m$. Thus $\psi(t) = P_n(t)\sum_{i=1}^{n}\frac{A_i}{t+\alpha_i}$ also. Adding this to the preceding expression for $\psi(t)$, we have $\psi(t) = tP_n(t)\sum_{i=1}^{n}\frac{A_i}{t^2-\alpha_i^2}$ and $\frac{\psi(t)}{P_n(t)} = t\sum_{i=1}^{n}\frac{A_i}{t^2-\alpha_i^2}$. Thus $E(x^{2k})$ is the coefficient of t^{-2k-1} in the expansion of $\log\frac{t+1}{t-1} - \frac{\psi(t)}{P_n(t)}$ in descending powers of t.

Before we rewrite this result, we introduce another function, $Q_n(x)$, which is the solution of the Legendre differential equation in descending powers of x. It has the representation

(8.29) $$Q_n(x) = \frac{(n!)^2 2^n}{(2n+1)!}\frac{1}{x^{2n+1}}F\left(\frac{n+1}{2}, \frac{n+2}{2}, \frac{3}{2}+n, \frac{1}{x^2}\right).$$

If the equation for Q_n is multiplied by P_n and subtracted from the product of Q_n and the equation for P_n, we obtain

$$(1-x^2)\frac{d}{dx}\{P_nQ_n' - Q_nP_n'\} = 2x\{P_nQ_n' - Q_nP_n'\}$$

and therefore, $P_nQ_n' - Q_nP_n' = \frac{k}{x^2-1}$. Divide this last result by P_n^2,

$\frac{1}{P_n}Q_n' - \frac{Q_n}{P_n^2}P_n' = \frac{d}{dx}\left\{\frac{Q_n}{P_n}\right\} = \frac{k}{(x^2-1)P_n^2}$. Hence

$$Q_n = kP_n\int\frac{dx}{(x^2-1)P_n^2} = kP_n\int\left[\frac{c_1}{x-1} + \frac{c_2}{x+1} + \sum_i\frac{B_i}{(x-\alpha_i)^2} + \sum_i\frac{B_i'}{x-\alpha_i}\right]dx,$$

where $c_1 = \left.\frac{1}{(x+1)P_n^2}\right]_{x=1} = \frac{1}{2}$, $c_2 = \left.\frac{1}{(x-1)P_n^2}\right]_{x=-1} = -\frac{1}{2}$, $B_i = \left.\frac{(x-\alpha_i)^2}{(x^2-1)P_n^2}\right]_{x=\alpha_i}$,

$B_i' = \left.\frac{d}{dx}\left\{\frac{(x-\alpha_i)^2}{(x^2-1)P_n^2}\right\}\right]_{x=\alpha_i}$. Now $P_n(x) = (x-\alpha_i)S(x)$, and

$B_i' = \left.\frac{d}{dx}\left\{\frac{1}{(x^2-1)S^2}\right\}\right]_{x=\alpha_i} = -2\left.\frac{(x^2-1)S'+Sx}{S^3(x^2-1)^2}\right]_{x=\alpha_i}$. From the differential equation for P_n, $(1-x^2)[(x-\alpha_i)S''+2S'] - 2x[(x-\alpha_i)S'+S] + n(n+1)(x-\alpha_i)S = 0$.

Hence if $x=\alpha_i$, $(1-x^2)S' - Sx = 0$ and $B_i' = 0$. Thus

$Q_n = -kP_n\left[\frac{1}{2}\log\frac{x+1}{x-1} + \sum_{i=1}^{n}\frac{B_i}{x-\alpha_i} + k'\right]$. Comparison with (8.29) will justify taking $k=-1$ and $k'=0$, and writing

(8.30) $$Q_n = P_n\int_x^\infty\frac{dx}{(x^2-1)P_n^2} = \frac{1}{2}P_n(x)\log\frac{x+1}{x-1} - R_n(x).$$

This Q_n is the Legendre function of the second kind.

Heine[123] already knew that if $\frac{1}{x-t}$ is developed in increasing powers

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123.) Heine, E. - Theorie der Anziehung eines Ellipsoids, Jour f. Math. 42(1851)70-82.

of t/x, the powers of t replaced by their expressions in Legendre Series (2.12), and the whole series collected according to the indices of the Legendre Polynomials, then

(8.31) $$\frac{1}{x-t} = \sum_{n=0}^{\infty} (2n+1) P_n(t)\, Q_n(x) \quad , \quad |t-\sqrt{t^2-1}| > |x-\sqrt{x^2-1}| .$$

Multiply both sides of (8.31) by $P_n(t)$ and integrate with respect to t,

(8.32) $$Q_n(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)}{x-t}\, dt$$

by virtue of the orthogonality property of P_n.[124] Now

$$Q_n(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(x)}{x-t}\, dt - \frac{1}{2}\int_{-1}^{1} \frac{P_n(x) - P_n(t)}{x-t}\, dt$$

(8.33) $$Q_n(x) = \frac{1}{2} P_n(x) \log \frac{x+1}{x-1} - \frac{1}{2}\int_{-1}^{1} \frac{P_n(x)-P_n(t)}{x-t}\, dt$$

and thus, by comparison with (8.30)

(8.34) $$R_n(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(x)-P_n(t)}{x-t}\, dt.$$

Finally we write

(8.35) $$Q_n(x) = P_n(x)\, L(x) - R_n(x) \; ; \quad L(x) = \frac{1}{2}\int_{-1}^{1} \frac{dt}{x-t}$$

We showed that $E(x^{2k})$ is the coefficient of t^{-2k-1} in the expansion of $\log \frac{t+1}{t-1} - \frac{\psi(t)}{P_n(t)}$ in descending powers of t. From (8.28), we see that $\psi(t) = 2R_n(t)$. Thus, $E(x^{2k})$ is the coefficient of t^{-2k-1} in $2Q_n(t)/P_n(t)$. In Gauss quadrature all the preceding coefficients must vanish, and the leading term in $Q_n(t)$ is therefore that in t^{-n-1}. We may also remark that the denominators of the convergents of the continued fraction development of L(x) are the polynomials whose zeros we use in Gaussian quadrature.

Let us return to (8.30) and use the fact that P_n and Q_n satisfy (2.7), then R_n satisfies $\frac{d}{dx}\{(x^2-1)R_n'\} = n(n+1)R_n - 2P_n'$

Note also that R_n like $\psi(x)$ is a polynomial of degree n-1, and (8.34) shows

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124.) J. Neumann- Entwickelung der in elliptischen Coordinaten ausgedrückten reciproken Entfernung zweier Puncte in Reihen, welche nach den Laplaceschen Y_n fortschreiten; und Anwendung dieser Reihen zur Bestimmung des magnetischen Suztandes eines Rotations-Ellipsoids, welcher durch vertheilende Kräfte erregt ist, Jour. f. Math. 37(1848)21-50.

that $R_n(x) = (-1)^{n-1} R_n(-x)$. The Legendre Series for R_n is thus

$$R_n(x) = a_1 P_{n-1} + a_3 P_{n-3} + a_5 P_{n-5} + \dots$$

If this is put in the differential equation for R_n and (5.9) used to eliminate P_n', then $a_1 = 2 \cdot \frac{2n-1}{1 \cdot n}$, $a_3 = 2 \cdot \frac{2n-5}{3 \cdot (n-1)}$, $\dots$ and

(8.36) $$R_n(x) = 2\left[\frac{2n-1}{1 \cdot n} P_{n-1} + \frac{2n-5}{3(n-1)} P_{n-3} + \frac{2n-9}{5(n-2)} P_{n-5} + \dots\right]$$

This is the result given by Christoffel [125] and Bauer [126]. Bauer also gives

(8.37) $$R_n(x) = \sum_{i=0}^{n-1} \frac{P_i P_{n-i-1}}{n-i}, \quad \int_{-1}^{1} R_n P_{n-m}\, dx = \begin{cases} 0 & (m \text{ even} \\ \dfrac{4}{(2m-1)(n-m+1)} & (m<n, \text{ odd}) \end{cases}$$

and this can be used to evaluate $\int_{-1}^{1} P_m P_n P_q dx$ and to obtain P_n^2 in a Legendre Series.

In his paper on Gauss quadrature, Christoffel gives a startling summation formula, of fundamental importance in the theory of Legendre Series. From (3.9), we obtain

$$(2n+1)\, x P_n(x) P_n(t) = (n+1) P_{n+1}(x) P_n(t) + n P_{n-1}(x) P_n(t)$$
$$(2n+1)\, t P_n(x) P_n(t) = (n+1) P_{n+1}(t) P_n(x) + n P_{n-1}(t) P_n(x)$$

and
$$(2n+1)(x-t) P_n(x) P_n(t) = (n+1)\left[P_{n+1}(x) P_n(t) - P_{n+1}(t) P_n(x)\right] - n\left[P_n(x) P_{n-1}(t) - P_n(t) P_{n-1}(x)\right]$$

sum this and the series on the right will telescope,

(8.38) $$\sum_{i=0}^{n} (2i-1) P_i(x) P_i(t) = \frac{(n+1)}{x-t}\left[P_{n+1}(x) P_n(t) - P_{n+1}(t) P_n(x)\right].$$

This is the famous Christoffel Summation Formula.

It would be well to consider some steps which were taken to generalize Gaussian quadrature. In this respect, Mehler[127] considers an analytic function $f(x) = a_0 + a_1 x + a_2 x^2 + \dots$ and seeks to evaluate a definite

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125.) E.B. Christoffel, Über die Gaussische Quadratur und eine Verallgemeinerung derselben, Jour. f. Math. 55(1858)61-82.

126.) G. Bauer, Bemerkungen über Reihen nach Kugelfunktionen und insbesondere auch über Reihen, welche nach Produkten oder Quadraten von Kugelfunktionen, Sitzber. k. bay. Ak. Wiss. Nunchen (math. - phys.) 5(1875)247-272.

127.) Mehler, F.G., Bemerkungen zur Theorie der mechanischen Quadratur, Jour. für Math. 63(1864)152-7.

integral by a mechanical quadrature

(8.39) $\int_{-1}^{1} f(x)(1-x)^{\lambda}(1+x)^{\mu}\,dx \sim A_1 f(\alpha_1) + A_2 f(\alpha_2) + \cdots + A_n f(\alpha_n)$.

(λ and μ > -1), where $w(x) = (1-x)^{\lambda}(1+x)^{\mu}$ we will call the weight function and the A's and α's are to be determined in such a way as to obtain absolute precision if f(x) is a polynomial of degree $\leq 2n-1$, the error in general being $E(f) = a_{2n}E(x^{2n}) + a_{2n+1}E(x^{2n+1}) + \cdots$. The procedure in the solution is identical with that used in Section III. We take a polynomial $J_n(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ and write the Lagrange Interpolation Polynomial for f(x), $f(x) = \sum_{i=1}^{n} \frac{J_n(x) f(\alpha_i)}{(x-\alpha_i) J_n'(\alpha_i)}$.

Then the A's in (8.39) have the form

$$A_i = \frac{1}{J_n'(\alpha_i)} \int_{-1}^{1} \frac{J_n(x)(1-x)^{\lambda}(1+x)^{\mu}}{x-\alpha_i}\,dx .$$

If we specify absolute precision in the case of f(x) of degree 2n-1, we may write $f(x) = J_n(x)Q(x) + R(x)$, Q and R of degree n-1 and $R(\alpha_i) = f(\alpha_i)$, $J_n(\alpha_i) = 0$.

If $\int_{-1}^{1} (1-x)^{\lambda}(1+x)^{\mu} J_n(x)\, x^k\,dx = 0 \quad (k = 0,1,2,\dots,n-1)$ then

$$\int_{-1}^{1} w(x) f(x)\,dx = \int_{-1}^{1} R(x)\,w(x)\,dx = \sum_{i=1}^{n} A_i R(\alpha_i) = \sum_{i=1}^{n} A_i f(\alpha_i),$$

the desired result. But this condition will be met only if the J's are Jacobi Polynomials and the α's are zeros of these polynomials (see p. 47)

In the case of Mehler, $J_n(x) = \frac{1}{(1-x)^{\lambda}(1+x)^{\mu}} \frac{d^n\left[(1-x)^{n+\lambda}(1+x)^{n+\mu}\right]}{dx^n}$

to within a constant factor. We can obtain results analogous to (8.33), by writing $K_n(x) = \int_{-1}^{1} \frac{J_n(t)\,w(t)}{x-t}\,dt$, $R_n(x) = \int_{-1}^{1} \frac{J_n(t) - J_n(x)}{t-x} w(t)\,dt$ and $K_n(x) = J_n(x) \int_{-1}^{1} \frac{w(t)\,dt}{x-t} - R_n(x)$.

Of course, Jacobi had already considered the case where $w(x) = (1-x^2)^{-\frac{1}{2}}$ in which event the α's should be zeros of cos $\left[(n+1) \text{ arc cos } x\right]$.[128]

Remarkably enough, it turns out that the A's are all equal in this case,

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128.) See p. 46 ff and F. Tisserand - Sur l'interpolation, Comptes Rendus 68(1869)1101-4. Other special w(x) leading to Hermite and Laguerre Polynomials appear in the works of Hermite, Tchebycheff, Stieltjes, Laurent and others. We shall treat the question more abstractly in the manner of Stieltjes.

simplifying computation considerably and making the quadrature

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx = \frac{\pi}{n+1}\sum_{i=0}^{n} f\left(\cos\frac{2i\pi}{2[n+1]}\pi\right).$$

The notions developed here are perfectly general. We can consider a weight function $w(x) \geq 0$ and integrable in $[a,b]$ and such that $\int_a^b w(x)dx$ has a meaning. Further, we require that $w(x) > 0$ in some finite subinterval of $[a,b]$, so that $\int_a^b w(x)dx > 0$. Here, we deal with $[a,b]$ finite. First we seek a polynomial of degree n, $X_n(x) = x^n + a_1 x^{n-1} + \ldots + a_n$, such that

(8.40a) $\int_a^b X_n(x)\, w(x)\, x^k\, dx = 0 \quad (k = 0, 1, 2, \ldots, n-1)$

If we let $\alpha_k = \int_a^b w(x) x^k\, dx$, we get n linear equations to determine the a's, $\alpha_{n+k} + a_1\alpha_{n+k-1} + \ldots + a_n\alpha_n = 0 \quad (k = 0, 1, 2, \ldots, n-1)$. We can solve for the a's and there will consequently exist a X_n satisfying the orthogonality property (8.40a), if the determinant of the α's, i.e. of the coefficients of the a's in the linear system, does not vanish. Stieltjes actually proves that this determinant is positive.[129] We can restate (8.40a), writing

(8.40b) $\int_a^b w(x)\, X_n\, G_{n-1}(x)\, dx = 0$ (G_{n-1} an arbitrary polynomial of degree n-1)

(8.40c) $\int_a^b w(x)\, X_n\, X_m\, dx = 0 \quad (m \neq n)$

These conditions determine $X_n(x)$ uniquely to within a constant factor.

The zeros of X_n are all real, distinct and in (a,b), the proof following the pattern of the Legendre case (p. 20). Also, there exists a recurrence relation analogous to (3.9). For, let $X_n - (x - c_{n-1})X_{n-1} = R(x)$. Then $\int_a^b R(x)\, w(x)\, G_{n-3}\, dx = 0$ and $R(x) = \lambda_{n-1} X_{n-2}$, λ's and c's being constants. Thus,

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129.) T.J. Stieltjes - Quelques recherches sur la théorie des quadratures dites mécaniques, Ann. Sc. l'Ec. Norm. Sup. (3)1(1884)409-426.

(8.41a) $\mathfrak{X}_n = (x - c_{n-1})\mathfrak{X}_{n-1} - \lambda_{n-1}\mathfrak{X}_{n-2}$, $\mathfrak{X}_1 = x - c_0$, $\mathfrak{X}_2 = (x-c_1)\mathfrak{X}_1 - \lambda_1$

Multiplying (8.41) by $w(x)X_{n-1}$, integrating and applying (8.40), we obtain

(8.41b)
$$c_{n-1} = \frac{\int_a^b x\,\mathfrak{X}_{n-1}^2\, w(x)\,dx}{\int_a^b \mathfrak{X}_{n-1}^2\, w(x)\,dx} \quad \text{and} \quad a < c_{n-1} < b$$

Similarly, multiplying by $w(x)X_{n-2}$, since $xX_{n-1} = X_n + G_{n-1}$,

(8.41c)
$$\lambda_{n-1} = \frac{\int_a^b \mathfrak{X}_{n-1}^2\, w(x)\,dx}{\int_a^b \mathfrak{X}_{n-2}^2\, w(x)\,dx} > 0$$

The conditions (8.41) completely determine the sequence of the X's. From (8.41) it is also clear that we have a set of Sturm functions, and the zeros of X_{n-1} separate those of X_n and are in term separated by those of X_{n-2}.

As for the application to mechanical quadrature, let f(x) be a polynomial of degree 2n-1, and $\frac{f(x)}{\mathfrak{X}_n} = U_{n-1} + \frac{V_{n-1}}{\mathfrak{X}_n}$ where U and V are of degree n-1. Then $f(x) = X_n U_{n-1} + V_{n-1}$ and $\int_a^b f(x)\,w(x)\,dx = \int_a^b V_{n-1}\,w(x)\,dx$ because of (8.40b).

Let $x_1 > x_2 > \cdots > x_n$ be the zeros of X_n, then

$$V_{n-1}(x) = \mathfrak{X}_n(x)\sum_{i=1}^{n} \frac{V_{n-1}(x_i)}{(x-x_i)\,\mathfrak{X}_n'(x_i)} = \mathfrak{X}_n(x)\sum_{i=1}^{n}\frac{f(x_i)}{(x-x_i)\,\mathfrak{X}_n'(x_i)}$$

Consequently, $\int_a^b f(x)\,w(x)\,dx = \sum_{i=1}^{n}\left[f(x_i)\int_a^b \frac{\mathfrak{X}_n(x)\,w(x)\,dx}{(x-x_i)\,\mathfrak{X}_n'(x_i)}\right] = \sum_{i=1}^{n} A_i\, f(x_i)$

Note that the A's are completely independent of f(x). Thus, we can evaluate the A's for $f(x) = \left[\frac{\mathfrak{X}_n(x)}{x - x_j}\right]^2$ by writing $\int_a^b w(x)\left[\frac{\mathfrak{X}_n(x)}{x-x_j}\right]^2 dx = \sum_{i=1}^{n}\left[\frac{\mathfrak{X}_n(x_i)}{x_i - x_j}\right]^2 A_i = A_i\left[\mathfrak{X}_n'(x_i)\right]^2 > 0$

As a result, $A_i > 0$. By taking $f(x) = 1$, it is clear that $\sum_{i=1}^{n} A_i = \int_a^b w(x)\,dx$.

And in general,

(8.42)
$$A_1 + A_2 + \cdots + A_k > \int_a^{x_k} w(x)\,dx \qquad (k = 1, 2, \ldots, n)$$
$$A_1 + A_2 + \cdots + A_k < \int_a^{x_{k+1}} w(x)\,dx \qquad (k = 1, 2, \ldots, n-1)$$

In the case of Gauss quadrature, where $-1 < x_1 < x_2 < \cdots < x_n < 1$ are the zeros of the Legendre Polynomial, these become

$$-1 + A_1 + A_2 + \cdots + A_k > x_k \qquad \text{and} \quad \sum_{i=1}^{n} A_i = 2.$$
$$-1 + A_1 + A_2 + \cdots + A_k < x_{k+1}$$

Again in the case of Gauss quadrature, the zeros of P_n are in $(-1,-1+A_1), (-1+A_1,-1+A_1+A_2), \ldots, (-1+A_1+A_2+\cdots+A_{n-1}, 1)$.

Then applying quadrature to an arbitrary integrable function $f(x)$, we have

$$\int_{-1}^{1} f(x)\,dx \sim A_1 f(x_1) + A_2 f(x_2) + \cdots + A_n f(x_n)$$

where A_k represents also the length of the kth subinterval of $[-1,1]$ and x_k is a point in this interval. Thus, the quadrature formula becomes in this sense a Riemann sum, and $\int_{-1}^{1} f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} A_i f(x_i)$, $A_i \to 0$ as $n_i \to \infty$. In order to substantiate that $A_i \to 0$ as $n\to\infty$, Stieltjes proves that if (α,β) is any subinterval of (a,b) for which $\int_\alpha^\beta w(x)dx > 0$, then $\exists$ $n > N \ni$ a zero of X_n lies in (α,β); and hence, $|x_i - x_{i-1}| \to 0$ as $n \to \infty$. From (8.42), $A_k < \int_{x_{k-1}}^{x_{k+1}} w(x)dx$. Since A_k is a continuous function of the limits of integration and $|x_{k+1} - x_{k-1}| \to 0$, then $A_k \to 0$ as $n \to \infty$. This result holds for general mechanical quadrature.

We might ask whether the error in the quadrature formula is decreased with increasing n. The answer is definitely in the affirmative. Let $f(x)$ be a function satisfying conditions so that it can be represented by a Legendre Series, $f(x) = \sum_{i=0}^{2n-1} B_i P_i(x) + \sum_{2n}^{\infty} B_i P_i(x) = S_{2n} + R_{2n}$, where $\epsilon > 0$ $\exists$ $n > N_\epsilon$ $\ni$ $|R_{2n}| < \epsilon$. Apply quadrature to $f(x)$,

$$\int_{-1}^{1} f(x)dx - \sum_{i=1}^{n} A_i f(x_i) = \int_{-1}^{1} S_{2n}\,dx - \sum_{i=1}^{n} A_i S_{2n}(x_i) + \int_{-1}^{1} R_{2n}\,dx - \sum_{i=1}^{n} A_i R_{2n}(x_i).$$

But S_{2n} is a polynomial of degree 2n-1, and its quadrature is exact; thus, the first two terms on the right vanish. Now for a preassigned, arbitrarily small $\epsilon > 0$ $\exists$ $n > N_\epsilon \ni |R_{2n}| < \epsilon$, i.e. $\left|\int_{-1}^{1} R_{2n}dx\right| < 2\epsilon$ and $\left|\sum_{i=1}^{n} A_i R_{2n}(x_i)\right| < \epsilon \sum_{i=1}^{n} A_i = \epsilon \int_{-1}^{1} dx = 2\epsilon$. Thus $\left|\int_{-1}^{1} f(x)dx - \sum_{i=1}^{n} A_i f(x_i)\right| < 4\epsilon$, and the error approaches zero as $n \to \infty$.[130]

130.) This result of Stieltjes (Sur l'évaluation approchée des intégrales, Comptes Rendus 97(1883)740-2) is readily extensible to the general case where $w(x)$ is not necessarily 1 and (a,b) not necessarily (-1,1).

We have already seen how Gauss quadrature was related to the continued fraction development of $L(x)$ of (8.35). We might ask whether there are not similar relations of general mechanical quadrature. At any rate, one might suspect that in the work of Stieltjes and Tchebycheff some such relations would invariably appear. In a more general case,[131] let

$$L(x)=\int_a^b \frac{w(t)\,dt}{x-t} = \frac{b_0}{x}+\frac{b_1}{x^2}+\frac{b_2}{x^3}+\cdots,\ Z_n(x)=\int_a^b \frac{X_n(x)-X_n(t)}{x-t}w(t)\,dt,\ W_n(x)=\int_a^b \frac{X_n(t)w(t)\,dt}{x-t}.$$

Then, as in (8.35), $W_n(x) = X_n(x)L(x)-Z_n(x)$. Again, as before, Z_n is a polynomial of degree n-1. If $L(x)$ is expanded in descending powers of x, the integral part of LX_n is evidently Z_n and the terms in the negative powers of x form W_n. In the Legendre case Q_n begins not with $1/x$ but with x^{-n-1}. The same condition imposed here leads to the orthogonality property $\int_a^b X_n G_{n-1} w(x)\,dx = 0$ and the fact that all the roots of X_n are real, distinct and in (a,b), separate those of X_{n+1} are are separated by those of X_{n-1}. Finally, the quadrature formula associated with X_n is

$$\int_a^b f(x)w(x)\,dx \sim \sum_{i=1}^{n} f(x_i)\int_a^b \frac{X_n(x)w(x)}{(x-x_i)X_n'(x_i)}\,dx = \sum_{i=1}^{n} f(x_i)\frac{Z_n(x_i)}{X_n'(x_i)}$$

an exact formula if $f(x)$ is a polynomial of degree $\leq 2n-1$. Christoffel shows that $X_{n-1}Z_n - X_n Z_{n-1} = a_n B_{n-1}$ and $\left[\frac{X_n(x)X_{n-1}(t)-X_n(t)X_{n-1}(x)}{x-t}\right] = a_n B_{n-1}\sum_{i=0}^{n-1}\frac{X_i(x)X_i(t)}{a_i B_i}$ where a_n is the coefficient of x^n in X_n and B_{n-1} is the coefficient of x^{-n} in W_{n-1}. These can be compared with the earlier Christoffel Summation Formula. But even more to the point is the continued fraction development written by Stieltjes, $L(x)=\frac{a_0|}{|x-c_0} - \frac{\lambda_1|}{|x-c_1} - \frac{\lambda_2|}{|x-c_2} - \frac{\lambda_3|}{|x-c_3} - \cdots$

131.) Christoffel, E.B. - Sur une classe particulière de fonctions entières et de fractions continues, Annali di Mat. (2)8(1877)1-10; T.J. Stieltjes - Sur l'évaluation approchée des intégrales, Comptes Rendus 97(1883)798-9.

with convergents N_k/D_k given by

$$\begin{cases} N_0 = 0 \\ N_1 = \lambda_0 \\ N_2 = (x-c_1)N_1 - \lambda_1 N_0 \\ \quad - \quad - \quad - \\ N_{n+1} = (x-c_n)N_n - \lambda_n N_{n-1} \end{cases} \qquad \begin{aligned} &D_0 = 1 \\ &D_1 = x \\ &D_2 = (x-c_1)D_1 - \lambda_1 D_0 \\ &\quad - \quad - \quad - \quad - \\ &D_{n+1} = (x-c_n)D_n - \lambda_n D_{n-1} \end{aligned}$$

Thus, the D's form really the sequence of the X's and the N's are actually the Z's.

These results were in a measure known earlier and may be obtained much differently. For example, Humbert[132] showed that the equation of Liouville

$$(4.1) \quad (a x^2 + bx + c)y'' + (dx + e)y' + fy = 0$$

has polynomial solutions which are denominators of the convergents of $\int_\alpha^\beta \frac{w(t)\,dt}{x-t}$ where α and β are the zeros of ax^2+bx+c and

$w(x) = \frac{1}{ax^2+bx+c} e^{\int \frac{dx+e}{ax^2+bx+c}dx}$. Orthogonality of the denominators of the convergents and their applicability to quadrature in (α,β) would follow immediately. Some notable examples may be given:[133]

$$(1) \quad (1-x^2)y'' - xy' + n^2 y = 0, \; (\alpha,\beta) = (-1,1), \; w(x) = (1-x^2)^{-\frac{1}{2}};$$
$$X_n = \cos(n \operatorname{arc\,cos} x), \quad x_i = \cos\frac{2i-1}{2n}\pi;$$

$$(2) \quad (1-x^2)y'' - 3xy' + n(n+2)y = 0, \; (\alpha,\beta) = (-1,1), \; w(x) = (1-x^2)^{\frac{1}{2}}$$
$$X_n = \frac{\sin[(n+1)\operatorname{arc\,cos} x]}{\sqrt{1-x^2}}, \quad x_i = \cos\frac{i\pi}{n+1}$$

A peculiar generalization of mechanical quadrature to evaluating definite double integrals has also been made. Appell uses the polynomials due to Hermite obtained from $\frac{\partial^p(x^2+y^2-1)}{\partial x^p}$ and $\frac{\partial^p(x^2+y^2-1)}{\partial y^p}$ (see p. 50).[134]

132.) G. Humbert, Sur l'équation différentielle linéaire du second ordre, Jour. de l'Ec. Polytech. (1)48(1880)207-228.

133.) See A. Berger - Sur l'évaluation approchée des intégrales définies simples, Nova Acta reg. Soc. Sc. Upsaliensis (3)16(1893)#4, pp.52.

134.) H. Bourget, Sur une extension de la methode de quadrature de Gauss, Comptes Rendus 126(1898)634-6; P. Appell, Sur une classe de polynomes, Ann. Fac. Sc. Toulouse (1)4(1890)H1-20.

SECTION IX

LEGENDRE SERIES

In Section V we discussed Legendre Series to some extent, chiefly in the course of reviewing the work of Dirichlet. We proved then that if $f(x)$ has at most finite many points of finite discontinuity and finite many maxima and minima in $[-1,1]$, it could be represented by a Legendre Series which would converge and represent the function. Dirichlet had established this result in the case of Fourier Series, but his work on Legendre Series was not quite so satisfactory; and the proof we gave was an extension of Darboux's interpretation of Dirichlet's results.

Here, we are going to consider some minor considerations first. Legendre already knew that any integral power of x can be expanded in a Legendre Series. Consequently, any rational integral function can be expanded in a Legendre Series. Even more than that, if $f(x)$ has a power series representation, it can be expanded in a Legendre Series. Cayley considered the expansion in this case and gave relations between the coefficients in the power series and the coefficients in the Legendre Series by means of a symbolic operator.[135] Echols put the relations in determinant form.[136] Echols even writes the coefficients of the Legendre Series and the remainder after n terms in determinant form.[137] Curiously enough, if $f(x)$ is a hypergeometric function, its

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135.) A. Cayley, On the expansion of Integral Functions in a Series of Laplace's Coefficients, Camb. and Dublin Math. Jour. 3(1848)120-1.

136.) W.H. Echols, On certain determinant forms and their applications, Annals of Math. 7(1892)11-59.

137.) W. H. Echols, On the expansion of an arbitrary function in terms of Laplace's Functions, Annals of Math. 12(1898-9)164-9.

expansion in Legendre Series has coefficients which are hypergeometric functions also. But such considerations are not those which will occupy us. Our chief concern will be in examining attempts to minimize the restrictions on f(x) and to set up necessary and sufficient conditions for the existence and convergence of a Legendre Series.

We have already pointed out (see p. 48) that if f(x) does permit of representation by a Legendre Series in $[-1, 1]$, and if the series converges uniformly and represents the function, then the coefficients can be obtained from

$$A_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)\, P_n(x)\, dx \tag{9.1}$$

Also, we have shown (p.77,78) that if the first r terms of the series are taken, S_r will approximate f(x) best in the sense of least squares if the A's are chosen as in (9.1). We are going to show that as r increases, the A's will still be chosen in the same way and the approximation will be improved. Let

$$J = \int_{-1}^{1}\Big[f(x) - \sum_{\lambda=0}^{n} A_\lambda P_\lambda\Big]^2 dx = \int_{-1}^{1} f^2(x)\,dx - 2\sum_{\lambda=0}^{n} A_\lambda \int_{-1}^{1} f(x) P_\lambda(x)\,dx + \sum_{\lambda=0}^{n} A_\lambda^2 \int_{-1}^{1} P_\lambda^2\,dx$$

$$= \int_{-1}^{1} f^2(x)\,dx - 2\sum_{\lambda=0}^{n}\frac{2A_\lambda^2}{2\lambda+1} + \sum_{\lambda=0}^{n}\frac{2A_\lambda^2}{2\lambda+1}$$

$$= \int_{-1}^{1} f^2(x)\,dx - \sum_{\lambda=0}^{n}\frac{2A_\lambda^2}{2\lambda+1} \geq 0 \;;$$

and J will be clearly less if the negative terms are numerically greater. Thus, if we take more terms, J will not increase; and if the Legendre Series does not terminate, J will decrease. Note also that, since $\int_{-1}^{1} f^2(x)\,dx \geq \sum_{\lambda=0}^{n}\frac{2A_\lambda^2}{2\lambda+1}$ holds regardless of the value of r, $\sum_{\lambda=0}^{\infty}\frac{2A_\lambda^2}{2\lambda+1}$ converges.

Another question which we can settle quickly enough is that of the uniqueness of the expansion $f(x) = \sum_{\lambda=0}^{\infty} A_i P_i$. The expansion of f(x) in a

Legendre Series is possible in only one way. For, if $f(x) = \sum_{i=0}^{\infty} B_i P_i$ also, then the coefficients in the expansion of zero would be $A_i - B_i = \frac{2i+1}{2} \cdot 0 = 0$ and $A_i = B_i$. Further, if $F(x) = \sum_{0}^{\infty} \left[\frac{2n+1}{2} P_n(x) \int_{-1}^{1} P_n(t) f(t)\,dt\right]$ and the Legendre Series is uniformly convergent in $[-1,1]$, then $F(x) = f(x)$.

The first to attempt to improve Dirichlet's results was Bonnet.[138] His work is based on an asymptotic expression (p. 65) first given by him and on the relation $\frac{1}{n+1}(1-x^2)P_n' = xP_n - P_{n+1}$. The latter is used in deriving the asymptotic expansion. Although f(x) is permitted to have at most finite many discontinuities in $[-1,1]$, it may not be discontinuous nor have more than finite many extrema in $[-1, -1+\epsilon]$ or $[1-\epsilon, 1]$. Since Bonnet's result is no real improvement of the work of Section V, we will enter into no detail concerning his methods.

Dini was another who failed to do as well with Legendre Series as with Fourier Series.[139] In fact, his work was completed and modified by Heine.[140] However, Dini does give, for perhaps the first time, a rigorous treatment of the question of differentiation and integration of Legendre Series.[141] Darboux adopts essentially the same point of view as Dini.[142]

138.) See p. 65 ff.

139.) Dini, U., Sopra le serie di funzioni sferiche, Ann. di Mat. (2)6(1874)112-140, 208-225.

140.) E. Heine, Handbuch der Kugelfunktionen, 2nd ed., vol. 1(1878)435; vol. 2(1881)361.

141.) U. Dini, Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale, Pisa, 1880.

142.) G. Darboux, Sur les séries dont le terme général dépend de deux angles et qui servent à exprimer des fonctions arbitraires entre des limites données, Jour. de Math. (2)19(1874)1-18.

In the case of Darboux, (5.12), we have already shown that if $f(x)$ is bounded, continuous except for finite many points, and has a bounded derivative except for finite many points, then $\lim_{n\to\infty} S_n = f(1)$. If $f(x)$ is unbounded within the limits of integration, we have a new difficulty. Darboux's contribution consists of relaxing the conditions on $f(x)$ so that it may have at most finite many points of infinite discontinuity. It will be no less general to consider a single point x_0 in $(-1,1)$. In this case, Darboux writes (5.12) as

$$S_n = \frac{1}{2}\int_0^{\pi} F(\theta)\frac{d}{d\theta}\left[P_n(\cos\theta) + P_{n+1}(\cos\theta)\right] d\theta$$

where $F(\theta) = f(\cos\theta)$, and uses his approximation formulas (p. 65). Let θ_0 be the value of θ for which $F(\theta)$ becomes infinite. Darboux writes $F(\theta) = F_1(\theta) + F_2(\theta)$, where F_1 is zero outside of $(\theta_0 - \eta_1,\ \theta_0 + \eta_2)$ $(\eta_1, \eta_2$ two positive numbers) and F_2 is finite for $\theta = \theta_0$. Then

$$S_n = \frac{1}{2}\int_{\theta_0-\eta_1}^{\theta_0+\eta_2} F_1(\theta)\frac{d}{d\theta}(P_n + P_{n+1})\,d\theta + \frac{1}{2}\int_0^{\pi} F_2(\theta)\frac{d}{d\theta}(P_n + P_{n+1})\,d\theta$$

For $n \to \infty$, the second integral approaches $F_2(0)$ or $F(0)$. The approximation formulas transform the first integral into

$$\sqrt{\frac{n}{\pi}}\int_{\theta_0-\eta_1}^{\theta_0+\eta_2} F_1(\theta)\sin\left[(n+\tfrac{1}{2})\theta - \tfrac{\pi}{4}\right]\sqrt{\cot\frac{\theta}{2}}\;d\theta$$

If $F_1(\theta)$ is of the order $\leq \sqrt{n}$, this last integral will approach zero as n increases. Later the results are extended.[143] Then $f(x)$ will be developable in a Legendre Series if the integrals have a meaning; if $f(x)$ is unbounded at x_0, it remains less than $O(n^{3/4})$; this condition must also be met at ± 1. It is interesting that Darboux believed his conditions to be necessary as well as sufficient, for satisfactory necessary conditions for neither Fourier nor Legendre Series have ever been given.

143.) G. Darboux, Mémoire sur l'approximation des fonctions, Jour. de Math. (3)4(1878)5-56, 377-416; Comptes Rendus 82(1876)365-8, 404-6.

Integration by parts of (5.12) is not essential to the attainment of the results. If $[-1,1]$ can be broken up into subintervals in each of which f(x) is continuous and monotone, we can separate the integral of (5.12) into a sum of integrals over these subintervals and make use of the second mean value theorem of Integral Calculus. The proof will go through as before, and this is the method usually used today in treating Dirichlet's conditions.

Dirichlet limited himself to functions with finite many extrema. Lipschitz showed, in the case of Fourier Series, that f(x) may have infinitly many extrema. Concerning the latter problem, du Bois Reymond gave examples of functions with infinitely many extrema not having Fourier expansions. The theory of Legendre Series in these respects lagged behind the theory of Fourier Series. Dirichlet requires at most finite many points of finite discontinuity. Darboux permits infinite discontinuities as long as they remain finite in number and provided that $\int_{-1}^{1} f(x)dx$ is bounded. Today we usually say f(x) satisfies "Dirichlet's Conditions" if f(x) has a finite number of infinite discontinuities in $[-1, 1]$; if, when arbitrarily small neighborhoods of the points of discontinuity are excluded, f(x) is bounded in the remainder of the interval; if the remainder can be broken up into a finite number of open subintervals in each of which f(x) is monotone; and finally, if the integral $\int_{-1}^{1} f(x)dx$ is absolutely convergent.

In Jordan's "Cours d'Analyse" is a statement that f(x) will have a Legendre Series for $x = x_0$ in $(-1,1)$ if it is continuous at and in the neighborhood of x_0 and if it is of bounded variation.[144] Jordan's conditions for Fourier Series applied to Legendre Series would require the

144.) C. Jordan, Cours d'Analyse de l'École Polytechnique, Gauthier-Villars, Paris, (1894) vol. 2, pp. 245-260.

existence and absolute convergence of $\int_{-1}^{1} f(x)dx$ as well as bounded variation in the neighborhood of an interior point. Continuity is non-essential. Such an extension to Legendre Series by Jordan has not, to the best of our knowledge, been made during the period with which we are dealing.

It may seem rather arbitrary to stop our investigations suddenly with the end of the nineteenth century. And it is no doubt true that, particularly in the theory of Legendre Series, major developments have appeared during the last thirty-five years. (One might place in the forefront the work of Fejer.) At the same time, it is exceedingly doubtful whether a study of the properties of the Legendre Polynomials should be so much dissociated from a study of the properties of Orthogonal Polynomials in general, as we have done. Such an approach to the literature of the last thirty-five years would involve considerable difficulty. Particularly in the theory of Legendre Series has it become of dubious value to dissociate the polynomials from orthogonal functions. And perhaps, we have already pressed the development of the theory of Legendre Polynomials beyond the point where it is justifiably separable from the theory of Orthogonal Polynomials. For these reasons, we do not take on further problems.

SECTION X

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$$(a^2 - 2aa'[\cos\omega\cos\varphi + \sin\omega\sin\varphi\cos(\theta-\theta')] + a'^2)^{-1/2}$$

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