SOME CHARACTERISTICS OF BROADBAND DELTA-SIGMA MODULATION

By

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APPROVAL SHEET

Title of Thesis: Some Characteristics of Broadband Delta-Sigma Modulation

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ABSTRACT

Title of Thesis: Some Broadband Characteristics of Delta-Sigma Modulation

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This paper presents an analysis using correlation techniques of an idealized Delta-Sigma Modulation system. An analytical assumption of errors with a marginally Gaussian distribution is shown to yield accurate results for broadband modulation with a maximum input-output cross correlation. It is also shown that this maximum is greatest for the degenerate case of only "hard limiting" with no feedback and no integration. A case of highly correlated inputs for Delta-Sigma Modulation is also discussed to compare it with broadband performance and "hard limiting."

ACKNOWLEDGMENT

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INTRODUCTION

Delta modulation techniques have been proposed for many communication tasks from space telemetry to walkie-talkie systems. The basic feedback technique for analog to binary conversion was introduced by F. de Jager [1] who was primarily interested in the transmission of speech signals. O'Neil [2] has investigated the use of delta modulation for transmitting Gaussian and television signals. Higher order quantization and prediction have also been studied by O'Neil [3]. Other modifications have been studied by Winkler [4], Halijuk and Tripp [5], and others. These systems have proved useful for the transmission of signals whose power density spectrum decreases with increasing frequency.

A significantly different modification to delta modulation was introduced by Inose, Yasuda, and Murakami [6] - [7] which they called Delta-Sigma (Δ-Σ) modulation. Δ-Σ modulation is significantly different in that it can transmit the dc component of a signal, its dynamic range and signal-to-noise ratio are independent of signal frequency rather than inversely proportional to the signal frequency, and transmission errors are not cumulative.

The objective of this paper is to investigate the characteristics of a Delta-Sigma Modulation system for wide sense stationary signals and to compare these characteristics with a clipper modulator alone. A discrete time Delta-Sigma Modulation system is studied for wideband noise inputs and a theoretical model is developed on the assumption of statistical linearization. The validity of this model is determined by computer simulation.

Idealized Model

An idealized Δ-Σ modulation system is shown in Figure 1 with an input random stationary process $X(t)$ and an output waveform $y(t)$. The input is periodically sampled with a period T. An output from the clipper is multiplied by h and subtracted from the next input sample. These differences are then integrated with an integration factor $\beta \leq 1$. The output samples y_n are passed through a boxcar circuit which may be considered a filter with an impulse response $f(t)$ where $f(t)$ is given by (1) for the purposes of discussion.

$$f(t) = \begin{cases} 1 & 0 < t < T \\ 1/2 & t = 0,\ T \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

The clipper output is defined as

$$y_n = [Z_n] = \begin{cases} +1 & Z_n > 0 \\ 0 & Z_n = 0 \\ -1 & Z_n < 0 \end{cases}$$

The purpose of the theory presented here is to use correlation techniques to study the system performance for various feedback and integration values and to compare these results with the case of no feedback and no integration, [8], (i.e., $\beta = h = 0$).

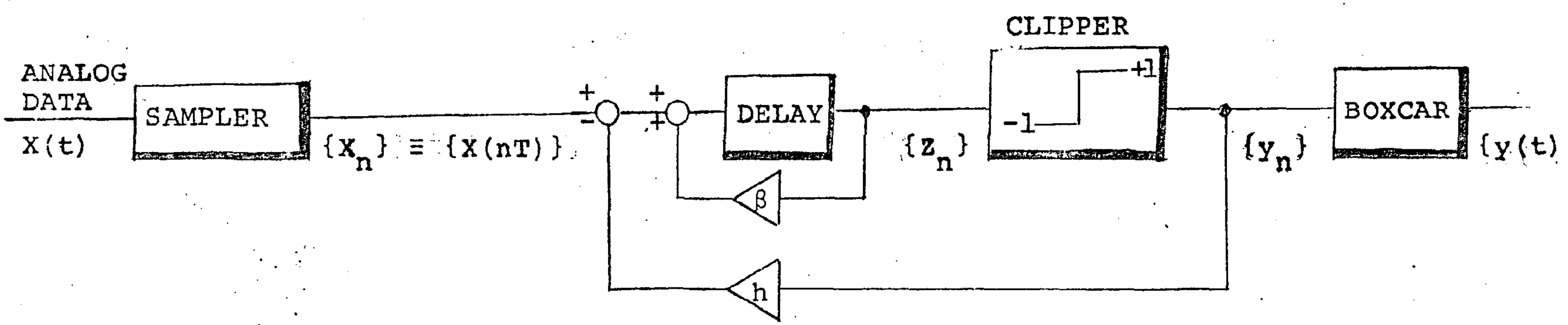


FIGURE 1. IDEALIZED DELTA-SIGMA MODULATION

Consider a long sequence of pulses of width T and magnitude 1 with a random fluctuation of sign. The auto-correlation function of the sequence is given in the statistical sense [9] by

$$\phi_{yy}(\tau) = \overline{y(t_o)\ y(t_o + \tau)} \qquad (2)$$

where y(t) is a function derived from a sample sequence, holding each sample value from the clipper for an interval T. To obtain $\phi_{yy}(\tau)$, the function y(t) and this function shifted by an amount τ is taken at an arbitrary point t_o, and the statistical average is taken over the ensemble of all sequences. For values of τ which are integral multiples of the sample interval T (say $\tau = nT$) the correlation will be the average product of the samples separated by n sample intervals.

The values of the auto-correlation function between the integral values nT will be a linear function connecting the discrete points at nT. The auto-correlation function is known to be an even function of τ, so $\phi_{yy}(nT) = \phi_{yy}(-nT)$. The complete function may then be written as

$$\phi_{yy}(\tau) = \sum_{n=-\infty}^{\infty} \phi_{yy}(n)\ \phi_{ff}(\tau-n\tau) \qquad (3)$$

where

$$\phi_{ff}(\tau) = \begin{cases} (T-|\tau|) & 0 \le |\tau| \le T \\ 0 & |\tau| > T \end{cases}$$

$\phi_{ff}(t)$ is the auto-correlation function of the impulse response of f(t) defined above and $\phi_{yy}(n)$ is the correlation of the samples separated by n sample intervals.

Then the power density spectrum of y(t) is given by

$$\Phi_{yy}(\omega) = |F(\omega)|^2 \sum_{n=-\infty}^{\infty} \phi_{yy}(n)\, e^{-j\omega nT} \tag{4}$$

where F(w) is the Fourier transform of the impulse response of the boxcar circuit. For f(t) as defined above:

$$F(\omega) = T e^{-j\omega T/2}\left(\frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}}\right)$$

$$|F(\omega)|^2 = T^2\left(\frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}}\right)^2 \tag{5}$$

So the problem at hand reduces to finding the correlation between the samples, y(mT)y((m + n)T) where m and n are integers. This correlation will be found for the idealized systems, then the results of testing the validity of the theoretical model will be presented.

The correlation between the samples can be found as follows.

Referring to Figure 1, Z_n is given by the recursive equation:

$$Z_{n+1} = X_n - h y_n + \beta Z_n \tag{6}$$

or

$$X_m = Z_{m+1} - \beta Z_m + h y_m \tag{7}$$

also

$$X_{m+n} = Z_{m+n+1} - \beta Z_{m+n} + h y_{m+n} \tag{8}$$

Multiplying the respective sides of (7) and (8) and indicating the statistical average by $(\bar{\ })$ results in

$$\overline{X_m X_{m+n}} = \overline{Z_{m+1} Z_{m+n+1}} - \beta \overline{Z_{m+1} Z_{m+n}} + h \overline{Z_{m+1} y_{m+h}} - \beta \overline{Z_m Z_{m+n+1}} + \beta^2 \overline{Z_m Z_{m+n}} - \beta h \overline{Z_m y_{m+n}} + h \overline{y_m Z_{m+n+1}} - \beta h \overline{y_m Z_{m+n}} + h^2 \overline{y_m y_{m+n}} \tag{9}$$

Now X(t) is assumed to be at least wide-sense stationary [10] and here we will assume Z_n and Y_n to also be wide-sense stationary. With this assumption then we let:

$$\overline{u_m v_{m+n}} = \phi_{uv}(n) \tag{10}$$

Equation (9) is

$$\phi_{xx}(n) = \phi_{zz}(n) - \beta \phi_{zz}(n-1) + h \phi_{zy}(n-1) - \beta \phi_{zz}(n+1) + \beta^2 \phi_{zz}(n) - \beta h \phi_{zy}(n) + h \phi_{yz}(n+1) - \beta h \phi_{yz}(n) + h^2 \phi_{yy}(n) \tag{11}$$

SECTION II

CROSSCORRELATION STATISTICS

An examination of Figure 1 shows that for values of β near 1.0 the $\{Z_n\}$ are the sums of a large number of random variables. In particular if $\{X_n\}$ are the samples from a broadband source with a reasonably flat spectrum it is expected that the distribution of $\{Z_n\}$ due to the $\{X_n\}$ can be approximated by a Gaussian probability distribution. The $\{y_n\}$ are also expected to approximate the spectra of $\{X_n\}$ so that $\{Z_n\}$ becomes the sum of a large number of weakly correlated samples and might therefore be distributed approximately as a Gaussian distribution. The same type of reasoning does not apply to the joint distributions of $\{Z_n\}$. To the contrary it is expected that the joint distribution of $\{Z_n\}$ separated by one sample interval will be strongly influenced by the sign of the earlier sample in a manner described by Papoulis [11]. In order to solve equation (11) for $\phi_{yy}(n)$ we must have some knowledge of the crosscorrelation function $\phi_{zy}(n)$ which will involve some assumptions on the joint distribution of $\{Z_n\}$. A method of determining some properties of $\phi_{xy}(n)$ will be described below. This approach is based on an approach taken by Brown [12] and his notation will be used here.

Let $p\ (X_1,\ X_2)$ denote the second-order joint probability distribution of X_1 and X_2 where X_1 and X_2 are derived from the same stationary random process a time interval τ apart.

The first order distributions are given by

$$p(X_1) = \int p(X_1, X_2)\, dX_2$$
$$p(X_2) = \int p(X_1, X_2)\, dX_1 \tag{12}$$

where integrals without limits denote integrals from $-\infty$ to $+\infty$. It is assumed $p(X_1, X_2)$ can be expressed with a set of orthogonal polynomials $\{\Theta(X)\}$ so that

$$p(X_1, X_2) = p(X_1)\, p(X_2) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}(\tau)\, \Theta_m(X_1)\, \Theta_n(X_2) \tag{13}$$

where

$$a_{mn}(\tau) = \iint p(X_1, X_2)\, \Theta_m(X_1)\, \Theta_n(X_2)\, dX_1\, dX_2 \tag{14}$$

and the orthonormality conditions are

$$\int p(X)\, \Theta_m(X)\, \Theta_n(X)\, dX = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \tag{15}$$

the first two polynominals are given by

$$\Theta_0 = 1$$
$$\Theta_1 = \frac{X - \mu}{\sigma} \tag{16}$$

where μ is the mean of $X(t)$ and σ the standard deviation. The term a (τ) is then the normalized autocorrelation function of the process $X(t)$

$$\rho(\tau) = a_{11}(\tau) = \iint \Theta_1(X_1)\, \Theta_1(X_2)\, p(X_1, X_2)\, dX_1\, dX_2 \tag{17}$$

and it can be shown that $a_{00} = 1$ and $a_{0n} = a_{n0} = 0$.

An instantaneous non-linear device can also be expressed in terms of $\{\Theta_k(X)\}$ as

$$f(X_1) = \sum_{k=0}^{\infty} C_k\, \Theta_k(X_1) \tag{18}$$

with

$$C_k = \int f(X_1)\, \Theta_k(X_1)\, p(X_1)\, dX_1 \tag{19}$$

For convenience we let $C_o = 0$, $\mu = 0$.

The crosscorrelation of $f(X_1)$ and X_2 is then given by

$$\phi_{12}(\tau) = \iint \left[\sum_{k=1}^{\infty} C_k \theta_k(X_1)\right] \cdot X_2 \left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}(\tau)\,\theta_m(X_1)\theta_n(X_2)\right] \cdot$$

$$p(X_1)\,p(X_2)\; dX_1\, dX_2 \qquad (20)$$

Since $X_2 = \sigma\,\theta_1(X_2)$

$$\phi_{12}(\tau) = \sigma \sum_{k=1}^{\infty} a_{k1}(\tau)\, C_k \qquad (21)$$

Consider the expression

$$\int X_2\, p(X_1, X_2)\; dX_2 = p(X_1)\; E[X_2/X_1] \qquad (22)$$

$$E[X_2/X_1] = \int X_2\, p(X_2/X_1)\; dX_2 \qquad (23)$$

$$p(X_1)\, E[X_2/X_1] = \int X_2 \left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}(\tau)\,\theta_m(X_1)\,\theta_n(X_2)\right] p(X_1)\,p(X_2)\; dX_1\, dX_2 \qquad (24)$$

For $X_2 = \sigma\,\theta_1(X_2)$ and $a_{01} = 0$

$$p(X_1)\, E[X_2/X_1] = \sigma \sum_{k=1}^{\infty} a_{k1}(\tau)\,\theta_k(X_1)\, p(X_1) \qquad (25)$$

For those values of X_1 where $p(X_1) \neq 0$

$$E[X_2/X_1] = \sigma \sum_{k=1}^{\infty} a_{k1}(\tau)\;\theta_h(X_1) \qquad (26)$$

In the case where X_1 and X_2 are jointly normal variables

$$E[X_2/X_1] = X_1\, \rho(\tau) \qquad (27)$$

Then

$$X_1\, \rho(\tau) = \sigma \sum_{k=1}^{\infty} a_{k1}(\tau)\;\theta_k(X_1) \qquad (28)$$

but $\sigma a_{11} \theta_1(x_1) = x_1 \rho(\tau)$

so that $a_{k1}(\tau) = 0 \quad k \neq 0 \quad$ and $\phi_{12}(\tau) = \sigma c_1 \rho(\tau)$ (29)

For the clipper with a Gaussian input

$$c_1 = \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty \frac{x}{\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{\frac{2}{\pi}} \tag{30}$$

Obtaining the conditional expectations to obtain $\phi_{zy}(h)$ is technically much more difficult for Δ-Σ modulation. For the Gaussian assumption of $\{Z_n\}$, $\phi_{zy}(0)$ is given directly by

$$\phi_{zy}(0) = \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty z e^{-\frac{z^2}{2\sigma^2}} dz = \sigma\sqrt{\frac{2}{\pi}} \tag{31}$$

If $\{Z_n\}$ were not Gaussian $\phi_{zy}(0)$ would be given by the general form:

$$\phi_{zy}(0) = \sigma c_1 \tag{32}$$

The conditional expectation $E[Z_{n+1}/Z_n]$ can be obtained through the use of equation (6)

$$z_{n+1} = x_n - h y_n + \beta z_n \tag{6}$$

$$z_{n+1} = x_n - h \operatorname{Sgn}[z_n] + \beta z_n \tag{33}$$

Let the probability density function of X_n be given by $P_x(X_n)$ and let $E[X] = 0$. Then for independent input samples:

$$p(Z_{n+1}/Z_n) = p_x(Z_{n+1} + h \operatorname{Sgn}[Z_n] - \beta Z_n) \tag{34}$$

$$E[Z_{n+1}/Z_n] = \int Z_{n+1}\, p(Z_{n+1} + h \operatorname{Sgn}[Z_n] - \beta Z_n)\, dZ_{n+1} \tag{35}$$

$$E[Z_{n+1}/Z_n] = \beta Z_n - h \operatorname{Sgn}[Z_n] \tag{36}$$

As in equation (18) we let

$$\mathrm{Sgn}[Z_n] = \sum_{k=1}^{\infty} c_k \, \Theta_k(Z_n) \tag{37}$$

Substituting (36) and (37) into (26) we have

$$\beta Z_n - h \sum_{k=1}^{\infty} c_k \, \Theta_k(Z_n) = \sigma \sum_{k=1}^{\infty} a_{k1}(1)\, \Theta_k(Z_n) \tag{38}$$

with the set of solutions

$$\begin{aligned} \sigma\beta - h c_1 &= \sigma\, a_{11}(1) \\ a_{k1}(1) &= -\frac{h c_k}{\sigma} \qquad k = 2, 3, 4, \cdots \end{aligned} \tag{39}$$

This also gives a solution for the autocorrelation function at $\phi_{zz}(1) = \sigma^2 a_{11}(1)$

$$\phi_{zz}(1) = \sigma(\sigma\beta - h c_1) \tag{40}$$

From equation (21) one obtains

$$\phi_{yz}(+1) = c_1 \sigma \beta - h c_1^2 - h \sum_{k=2}^{\infty} c_k^2 \tag{41}$$

$$\therefore \phi_{yz}(+1) = \frac{c_1}{\sigma} \phi_{zz}(1) - h \sum_{k=2}^{\infty} c_k^2 \tag{42}$$

Since $E[f^2(x_1)] = 1 = \sum_{k=1}^{\infty} c_k^2$ then under the Gaussian assumption

$$\sum_{k=2}^{\infty} c_k^2 = 1 - \frac{2}{\pi} = .3634 \tag{43}$$

$$\phi_{yz}(+1) = \frac{c_1}{\sigma} \phi_{zz}(1) - .3634 h \tag{44}$$

$$\phi_{yz}(+1) = c_1 \sigma \beta - h \tag{45}$$

Also since $\phi_{zy}(0) = c_1 \sigma$ the normalized correlation $\rho_{yz}(+1)$ will be given by

$$\rho_{yz}(+1) = \frac{\phi_{yz}}{\sigma c_1} = \frac{\phi_{zz}(1)}{\sigma^2} - \frac{(1-c_1^2)h}{c_1 \sigma} \tag{46}$$

$$\rho_{yz}(+1) = \beta - \frac{h}{\sigma c_1} \tag{47}$$

$$\rho_{yz}(+1) = \beta - 1.253\left(\frac{h}{\sigma}\right) \tag{48}$$

Substituting equations (32), (40), and (41) in equation (11) for n = 0 yields a solution for σ^2. Thus

$$\sigma_x^2 = \sigma^2 - 2\beta(\sigma^2\beta - \sigma h c_1) + 2h\left(\frac{c_1}{\sigma}\right)(\sigma^2\beta - \sigma h c_1) - 2h^2(1-c_1^2) + \beta^2\sigma^2 - 2\beta h \sigma c_1 + h^2 \tag{49}$$

$$\sigma_x^2 = \sigma^2(1-\beta^2) + 2\beta h c_1 \sigma - h^2 \tag{50}$$

Therefore σ is given by

$$\sigma = -\frac{\beta h c_1}{1-\beta^2} + \sqrt{\left(\frac{c_1 h \beta}{1-\beta^2}\right)^2 + \frac{h^2 + \sigma_x^2}{1-\beta^2}} \tag{51}$$

and for $\beta = 1$

$$\sigma = \frac{\sigma_x^2 + h^2}{2 h c_1} \tag{52}$$

For $\beta = h = \sigma_x = 1$

$$\sigma = 1/c_1 \tag{53}$$

The expression for σ can be minimized with respect to h by solving the equation $\frac{\partial\sigma}{\partial h} = 0$ for h with the following result

$$h^2 = \sigma_x^2 \frac{c_1^2\beta}{1-\beta^2+c_1\beta^2} \tag{54}$$

This expression shows that the feedback gain which will minimize the variance of $\{z_n\}$ is linearly related to σ_x and is a function of both C_1 and β. For $\beta = 1$, $h = \sigma_x$ will minimize σ with the result that

$$\sigma_{min} = \sigma_x / C_1 \quad (55)$$

It should be pointed out that this result is theoretically not dependent on the probability distribution of the input but requires only that the input consists of independent samples with zero mean. A plot of minimum σ, and the corresponding feedback gain, h, as a function of the integration factor, β, is given in Figure 2 for an input variance of 1. For other values of the input variance, both minimum σ and feedback gain, h, are linearly related to the input standard deviation, σ_x. Quite appropriately, the minimum of the minimum σ occurs when $\beta = h = 0$ which is the case of the clipper only. To find $\phi_{zz}(2)$ the conditional expectation of $E[Z_{n+2}/Z_n]$ is needed. Using (6)

$$Z_{n+2} = X_{n+1} - h\,Y_{n+1} + \beta Z_{n+1} \quad (56)$$

$$E[Z_{n+2}/Z_n] = \beta E[Z_{n+1}/Z_n] - h\,E[\mathrm{Sgn}[Z_{n+1}]/Z_n] \quad (57)$$

$E[Z_{n+1}/Z_n]$ is given in equation (36).

From equation (34)

$$p(Z_{n+1}/Z_n) = p_x(Z_{n+1} + h\,y_n - \beta Z_n) \quad (58)$$

$$E[\mathrm{Sgn}[Z_{n+1}]/Z_n] = \int_0^{\infty} p_x(Z_{n+1} + h\,y_n - \beta Z_n)\,dZ_{n+1} - \int_{-\infty}^{0} p_x(Z_{n+1} + h\,y_n - \beta Z_n)\,dZ_{n+1} \quad (59)$$

For $P_x(x)$ being Gaussian

$$E[\mathrm{Sgn}[Z_{n+1}]/Z_n] = -\frac{2}{\sigma_x\sqrt{2\pi}} \int_0^{h y_n - \beta Z_n} e^{-\frac{x^2}{2\sigma_x^2}}\,dx \quad (60)$$

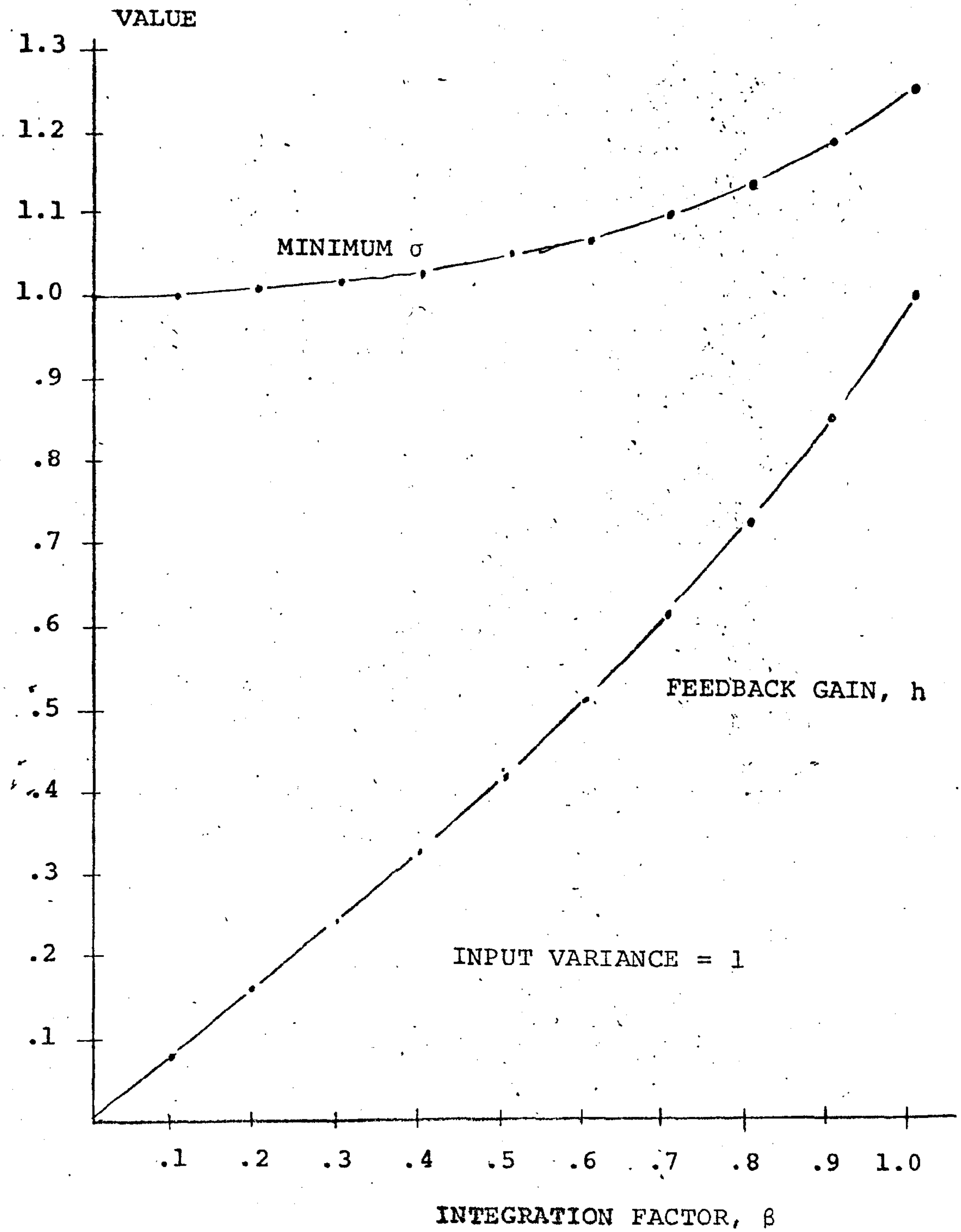


FIGURE 2. MINIMUM σ VS INTEGRATION FACTOR

Let this last expression be represented by the expression

$$E[Sgn[Z_{n+1}]/Z_n] = \sum_{k=0}^{\infty} d_k \theta_k(Z_n) \tag{61}$$

where

$$d_k = \int \theta_k(Z_n) E[Sgn[Z_{n+1}]/Z_n] p(Z_n)\, dZ_n \tag{62}$$

$$d_1 = \int_{-\infty}^{\infty} \frac{Z_n}{\sigma^2\sqrt{2\pi}} \int_0^{h\,Sgn[Z_n]-\beta Z_n} \frac{-2\, e^{-\frac{X^2}{2\sigma_X^2}}}{\sigma_X\sqrt{2\pi}}\, dX\, e^{-\frac{Z^2}{2\sigma^2}} dZ_n \tag{63}$$

$$d_1 = \int_0^{\infty} \frac{-4Z}{\sigma^2\sigma_X 2\pi} \int_0^{h-\beta Z} e^{-\frac{X^2}{2\sigma_X^2}} e^{-\frac{Z^2}{2\sigma^2}} dx\, dZ \tag{64}$$

For $\sigma_x = \beta = h = 1$

$$d_1 = -c_1^4 \int_0^{\infty}\int_0^{1-Z} z\, e^{-x^2/2} e^{-\frac{c_1^2 Z^2}{2}} dx\, dz \tag{65}$$

This was solved by integration by parts and the use of probability tables to give $d_1 = .418$.

Since $\theta_1(Z_n) = \frac{Z_n}{\sigma}$

$$d_1 = \frac{1}{\sigma} E[Z_n E[Sgn[Z_{n+1}]/Z_n]] \tag{66}$$

$$d_1 = \frac{1}{\sigma} \phi_{zy}(1) \tag{67}$$

The expression for $E[Z_{n+2}/Z_n]$ can then be expressed as

$$E[Z_{n+2}/Z_n] = \beta^2 Z_n - h\beta\, Sgn[Z_n] - h\sum_{k=0}^{\infty} d_k \theta_k(Z_n) \tag{68}$$

Since $E[Z_{n+2}/Z_n] = \sigma \sum_{k=1}^{\infty} a_{k1}(2)\,\theta_k(Z_n)$

we have the set of solutions

$$\sigma a_{11}(2) = \beta[\sigma\beta - c_1 h] - \frac{h}{\sigma}\phi_{zy}(1) \tag{69}$$

$$\sigma a_{k1}(2) = -\beta h c_k - h d_k \qquad k = 2, 3, 4, \cdots \tag{70}$$

As before the autocorrelation $\phi_{ZZ}(2)$ is given by $\sigma^2 a_{11}(2)$

$$\phi_{ZZ}(2) = \sigma\beta(\sigma\beta - c_1 h) - h\phi_{zy}(1) \tag{71}$$

$$\phi_{zz}(2) = \beta\phi_{zz}(1) - h\phi_{zy}(1) \tag{72}$$

This last expression can be generalized to give for $n \geq 1$

$$\phi_{zz}(n) = \beta\,\phi_{zz}(n-1) - h\,\phi_{zy}(n-1) \tag{73}$$

The crosscorrelation $\phi_{Zy}(2)$ is given by

$$\phi_{yz}(2) = \sigma\sum_{k=1}^{\infty} a_{k1}(2)\,c_k \tag{74}$$

$$\phi_{yz}(2) = \beta[\sigma\beta c_1 - h c_1^2] - \frac{hc_1}{\sigma}\phi_{zy}(1) - \beta h\sum_{k=2}^{\infty} c_k^2 - h\sum_{k=2}^{\infty} c_k d_k \tag{75}$$

$$\phi_{yz}(2) = \sigma\beta^2 c_1 - \beta h - h c_1 d_1 - h\sum_{k=2}^{\infty} c_k d_k \tag{76}$$

To obtain $\phi'_{yz}(2)$ it was necessary to determine $\phi_{zy}(1)$. In doing so it is observed that $\phi_{yz}(1) \neq \phi_{zy}(1)$; also whereas $\phi_{yz}(1)$ depended only on the mean of $\{x_n\}$, $\phi_{zy}(1)$ is a strong function of $P_x(n)$. Other values of $\phi_{yz}(n)$ can be determined only on a per case basis and then using involved integrals, but they will not grant any further insight into the basic feedback process.

The next theoretical item is to determine the input-output crosscorrelation for the independent input case. Because of independent inputs $\phi_{xy}(n) \equiv 0$ for $n < 1$. Because of the unit delay in the formation of $\{z_n\}$, $\phi_{xy}(1)$ is the first non zero crosscorrelation coefficient and for a Δ-Σ

modulation system with a "flat" response $\phi_{xy}(1)$ will be the maximum value of $\phi_{xy}(n)$. $\phi_{xy}(1)$ can be computed by first considering the joint distribution of X_n, Z_n, Z_{n+1}.

Formally we have:

$$p(X_n, Z_n, Z_{n+1}) = p_X(X_n)\, p(Z_n/X_n)\, p(Z_{n+1}/Z_n, X_n) \qquad (77)$$

Since the $\{X_n\}$ are independent

$$p(Z_n/X_n) = p_Z(Z_n) \qquad (78)$$

and since $X_n = Z_{n+1} + hY_n - \beta Z_n$

$$p(Z_{n+1}/Z_n, X_n) = \delta(Z_{n+1} - X_n + hy_n - \beta Z_n) \qquad (79)$$

where $\delta(x)$ is the Dirac Delta function.

The value of $\phi_{xy}(1)$ is expressed as

$$\phi_{xy}(1) = \iiint X_n \,\mathrm{Sgn}[Z_{n+1}]\, p(X_n, Z_n, Z_{n+1})\, dX_n\, dZ_n\, dZ_{n+1} \qquad (80)$$

Integrating Z_{n+1} first gives:

$$\phi_{xy}(1) = \iint X_n \,\mathrm{Sgn}[X_n - hy_n + \beta Z_n]\, p_X(X_n)\, p_Z(Z_n)\, dX_n\, dZ_n \qquad (81)$$

Next integrating with respect to X_n yields

$$\phi_{xy}(1) = \sigma_X \sqrt{\frac{2}{\pi}} \int p_Z(Z_n)\, e^{-\frac{(\beta Z_n - hX_n)^2}{2\sigma_X^2}}\, dZ_n \qquad (82)$$

$$\phi_{xy}(1) = \frac{2}{\pi}\frac{\sigma_X}{\sigma}\int_0^{\infty} e^{-\left(\left(\frac{1}{2\sigma^2}+\frac{\beta^2}{2\sigma_X^2}\right)Z_n^2 - 2\left(\frac{\beta h}{2\sigma_X^2}\right)Z_n + \frac{h^2}{\sigma_X^2}\right)} dZ_n \qquad (83)$$

This expression was evaluated for $\beta = h = \sigma_x = 1$ to yield

$$\phi_{xy}(1) = .6477 \qquad (84)$$

From the expression $X_n = Z_{n+1} + hY_n - \beta Z_n$ we have that

$$\phi_{xy}(n) = \phi_{zy}(n-1) + h\,\phi_{yy}(n) - \beta\,\phi_{zy}(n) \qquad (85)$$

Therefore:

$$h\,\phi_{yy}(1) = \phi_{xy}(1) + \beta\,\phi_{zy}(1) - \phi_{zy}(0) \qquad (86)$$

For the case of $\beta = h = \sigma_x = 1$ we have from equations (84), (67), and (55)

$$\phi_{yy}(1) = -.0177 \qquad (87)$$

A few general properties of the correlation functions can be found by working directly with the joint distribution functions as follows. Since for independent inputs $P_x(X_n/Z_n) = P_x(Z_{n+1} + hY_n - Z_n/Z_n)$ and $p(Z_n)$ is Gaussian by assumption, the second order joint distribution is given by

$$p(Z_n, Z_{n+1}) = p(Z_n)\, p_X(Z_{n+1} + hY_n - \beta Z_n) \qquad (88)$$

This process can be extended to have

$$p(X_{n+m}/Z_{n+m}, Z_{n+m-1}, \cdots Z_n) = p_X(Z_{n+m+1} - \beta Z_{n+m} + hY_{n+m}/Z_{n+m}) \qquad (89)$$

so that the m+1 joint distribution is given by

$$p(Z_n, Z_{n+1} \cdots Z_{n+m}) = p(Z_n) \prod_{i=0}^{m-1} p_X(Z_{n+i-1} - \beta Z_{n+i} + hY_{n+i}) \qquad (90)$$

The integral for $\phi_{zz}(m)$ then looks like

$$\phi_{ZZ}(m) = \iint \cdots \int Z_n Z_{n+m}\, p(Z_n) \prod_{i=0}^{m-1} p_X(Z_{n+i-1} - \beta Z_{n+i} + hY_{n+i})\, dZ_n\, dZ_{n+1} \cdots dZ_{n+m} \qquad (91)$$

Integrating with respect to Z_{n+m} we have

$$\phi_{ZZ}(m) = \iint \cdots \int Z_n(\beta Z_{n+m-1} - hY_{n+m-1})\, p(Z_n \cdots Z_{n+m-1})\, dZ_n \cdots dZ_{n+m-1} \qquad (92)$$

which yields the relation

$$\phi_{ZZ}(m) = \beta\, \phi_{ZZ}(m-1) - h\, \phi_{Zy}(m-1) \qquad (93)$$

In the same manner it is found that

$$\phi_{yZ}(m) = \beta\, \phi_{yZ}(m-1) - h\, \phi_{yy}(m-1) \qquad (94)$$

These last two equations are valid only for m > 1 and they also are a solution to equation (11)

SECTION III

EXPERIMENTAL RESULTS

The Delta-Sigma Modulation system shown in Figure 1 was simulated on a digital computer. An approximately gaussian random sequence of samples was derived from a uniform number generator. This sequence was used as an input to the modulator. All the correlation functions in equation (11) were computed for n =-10 to 10.

A histogram of the first 5000 uniform number generator samples is shown in Figure 3. Two methods of generating normally distributed random variables from the uniform samples were tested. One method was to use the sum of twelve uniformly distributed samples and the other was the direct method discussed by Dillard [13] where if U and V are independent samples from a uniform distribution then two independent samples T and Y from a normal distribution are given by

$$T = \sqrt{-2 \ln u} \cos (2\pi v)$$

$$Y = \sqrt{-2 \ln u} \sin (2\pi v)$$

Figure 4 shows the resulting histograms of the first 5000 samples generated by the two methods. Neither method is clearly superior but the sum of twelve samples from a uniform distribution was chosen for the simulation process because it seemed to have a smoother histogram for smaller sample sizes

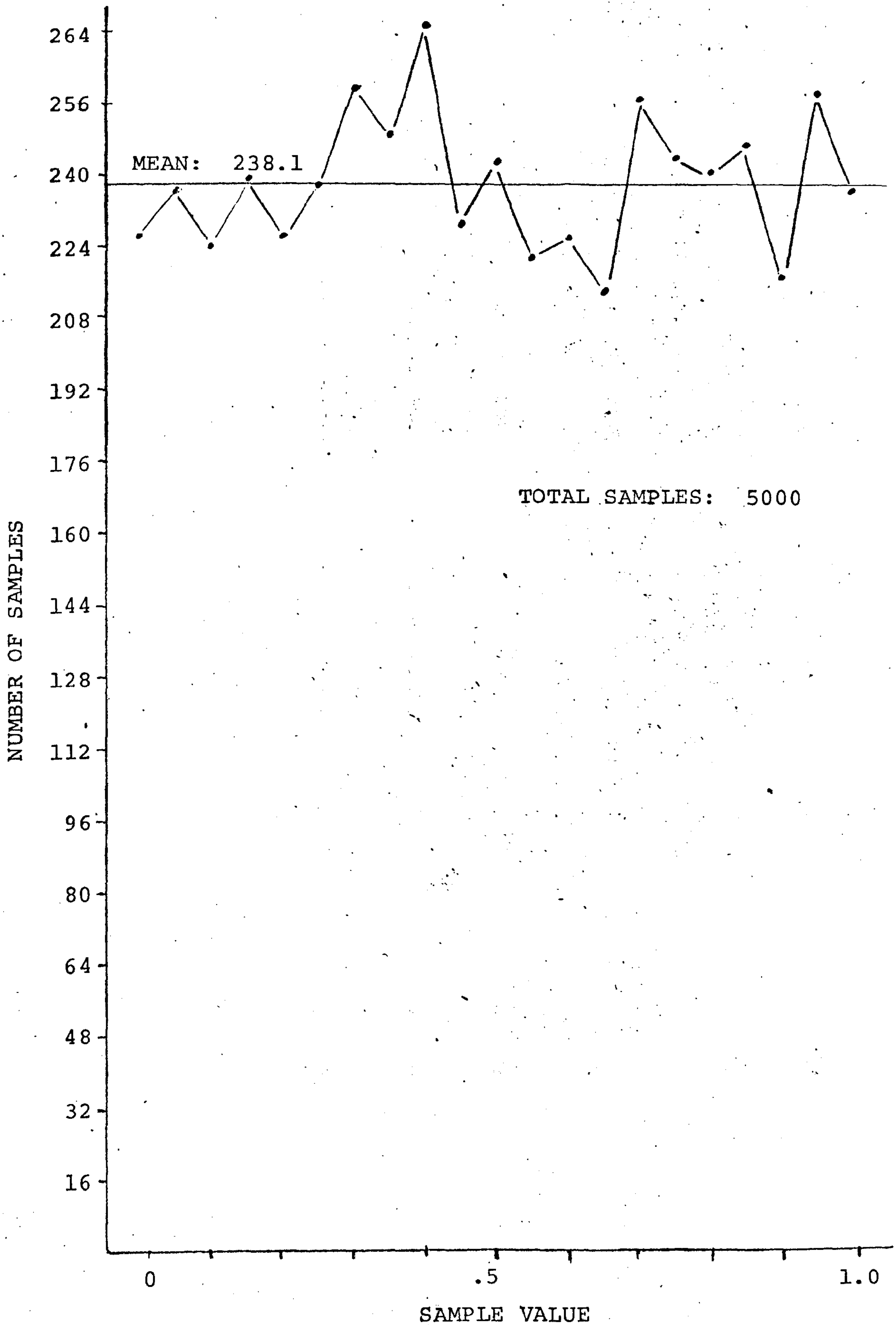


FIGURE 3. HISTOGRAM OF UNIFORM NUMBER GENERATOR

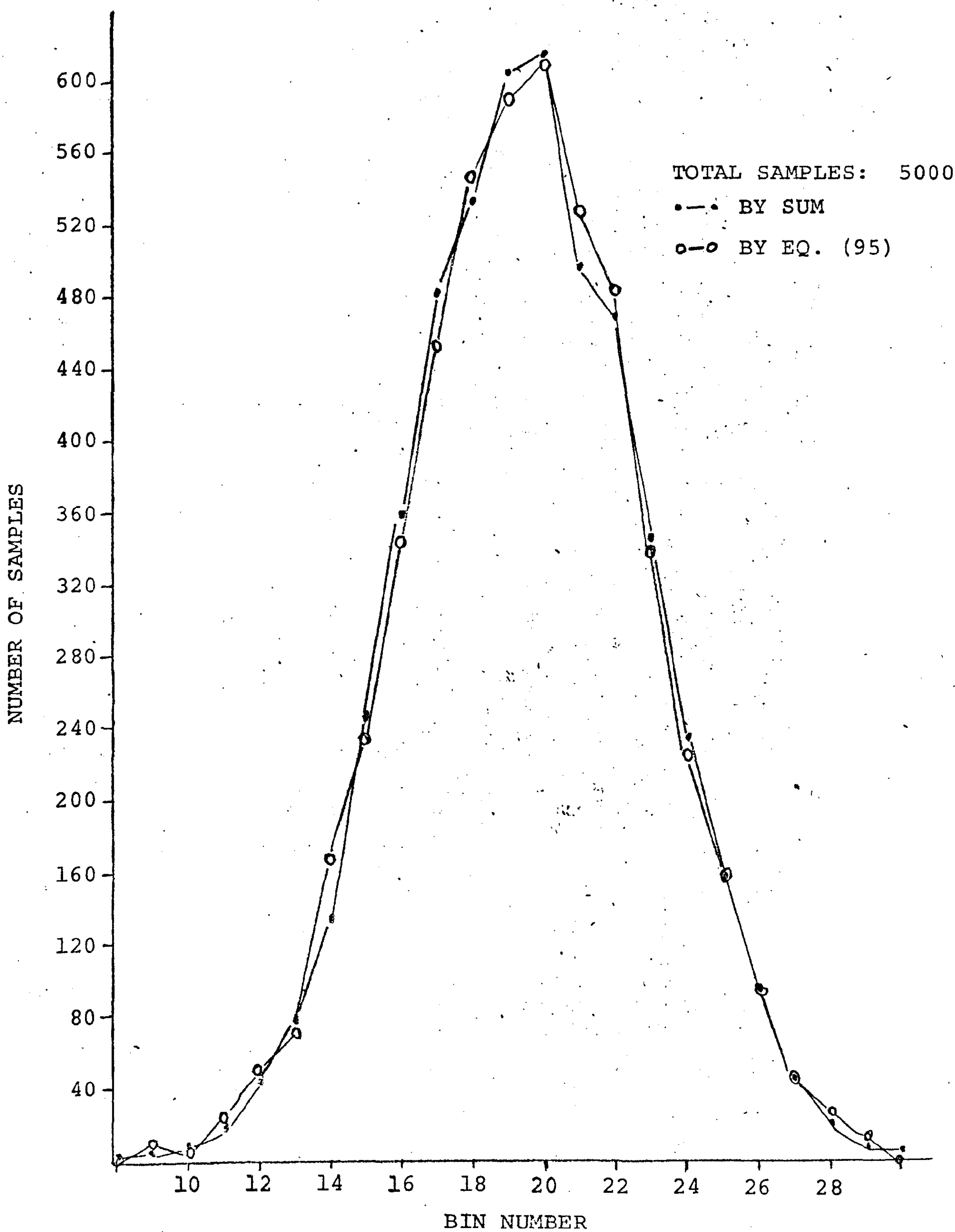


FIGURE 4. HISTOGRAM OF NORMAL SAMPLES

as shown in the histograms for the first 100 samples in Figure 5.

Clipper Correlation Statistics

The crosscorrelation of the clipper input and output for a sum of twelve uniformly distributed variables as an approximation for the normally distributed variable was also investigated. For 13000 independent samples the input variance (assuming zero mean) was .992. The crosscorrelation coefficient for zero delay was .7955 which is within 1% of the theoretical value given by equation (32) of $C_1\sigma = .7914$. Other correlation coefficients for delays different than zero were less than .02.

Correlated input samples were generated by filtering the input samples with the equation:

$$Z_n = X_n + .8\ Z_{n-1} \tag{96}$$

where the $\{X_n\}$ are the independent samples and $\{Z_n\}$ are the clipper inputs. The input variance for 13000 samples was 2.774 and the crosscorrelation coefficient for zero was 1.337. The theoretical value, $C_1\sigma = 1.329$, was again within 1% of the experimental value. Table 1 lists the normalized input autocorrelation, crosscorrelation, theoretical output autocorrelation, and measured output autocorrelation functions. From equation (29) the normalized crosscorrelation and input autocorrelation are theoretically equal. The table shows that for sample sizes on the order of 13000 the correlation functions can be expected to be accurate to within 3% when

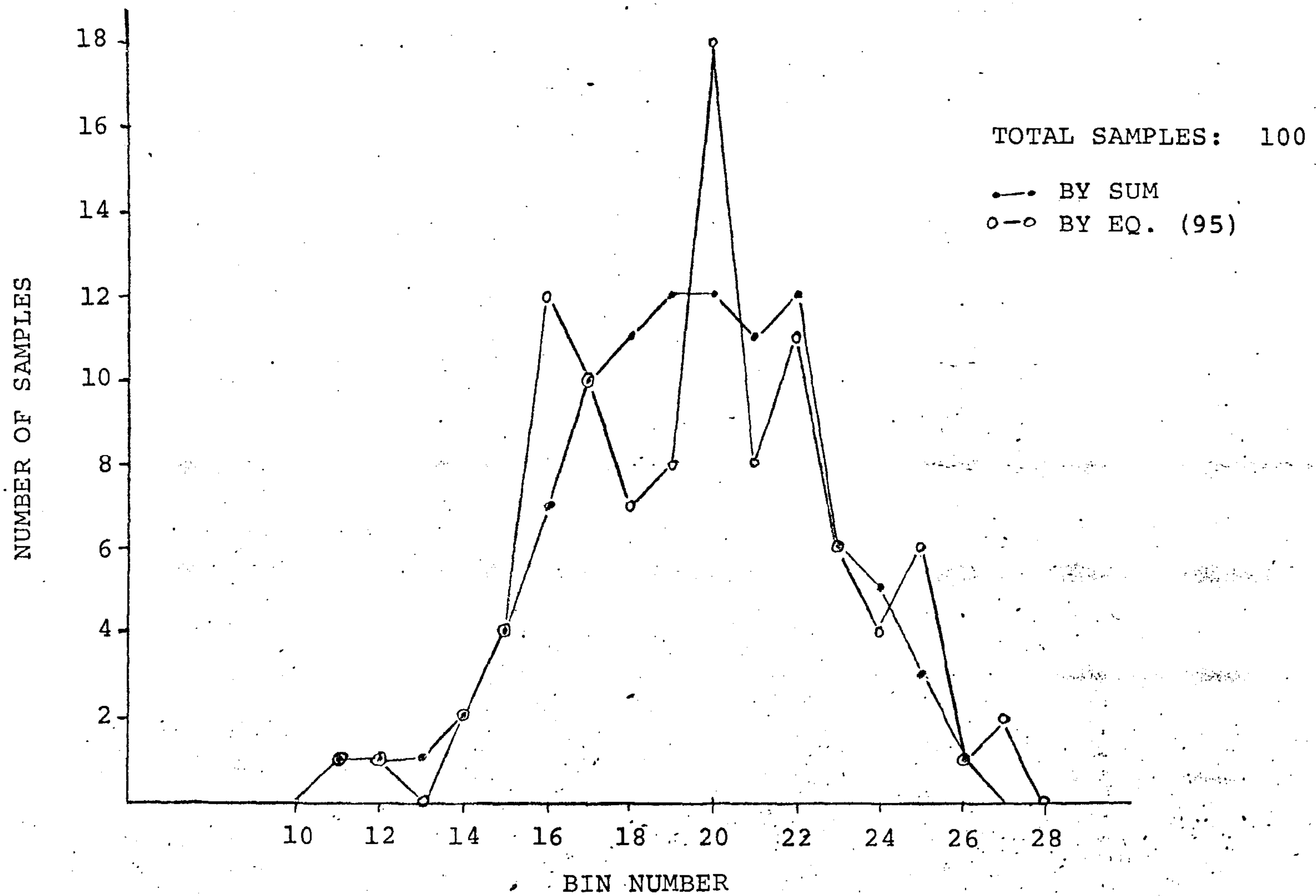


FIGURE 5. HISTOGRAM OF GENERATED SAMPLES

Table I

CORRELATION VALUES FOR CLIPPER

Sample Delay	Input Auto Corre- lation	Cross- Corre- lation	Output Auto- Corre- lation $\frac{2}{\pi} \sin^{-1} \rho$	Output Auto- Corre- lation Measured
-10	.102	0.99		
- 9	.120	.114		
- 8	.156	.150		
- 7	.203	.198		
- 6	.255	.251		
- 5	.320	.318		
- 4	.403	.407		
- 3	.506	.510		
- 2	.638	.641		
- 1	.801	.804		
0	1	1	1	1
1	.801	.798	.590	.595
2	.638	.630	.441	.438
3	.506	.497	.338	.334
4	.403	.399	.263	.269
5	.320	.310	.207	.202
6	.255	.243	.164	.154
7	.203	.193	.130	.120
8	.156	.144	.100	.085
9	.121	.110	.077	.068
10	.103	.096	.066	.057

they have a normalized value of .2 or more. Smaller values of correlation coefficients are less accurate.

Distribution of Z

The next experimental result is the histogram of the samples at the input of the clipper in the Δ-Σ modulator. These are the samples indicated in the theoretical section by $\{Z_n\}$. For the feedback gain, h, equal to the standard deviation of the independent approximately normal input samples and for $\sigma_x = .98$, the histogram of $\{Z_n\}$ shown in Figure 6 was obtained. The data points are the histogram samples for a total sample set of 49000 and the cross marks indicate the standard deviations of the data set crossed with the appropriate number of samples at the same standard deviation if the set were normally distributed. The agreement is quite good at the 1σ and 2σ points but when compared with a normal distribution having the same standard deviation the sample histogram of $\{Z_n\}$ was noticeably more flat at the peak of the distribution. The normalized cumulative distribution of the sample set is plotted in Figure 7 for the same set of data. On the probability graph of Figure 7 a Gaussian distribution would form a straight line. The data set seems to give a very straight line with no discernable flattening out at the peak of the distribution. No statistical tests were made to determine confidence limits on the Gaussian assumption of this distribution. It was reasoned that the appropriate test would be in the accuracy of the crosscorrelation coefficient of $\{Z_n\}$

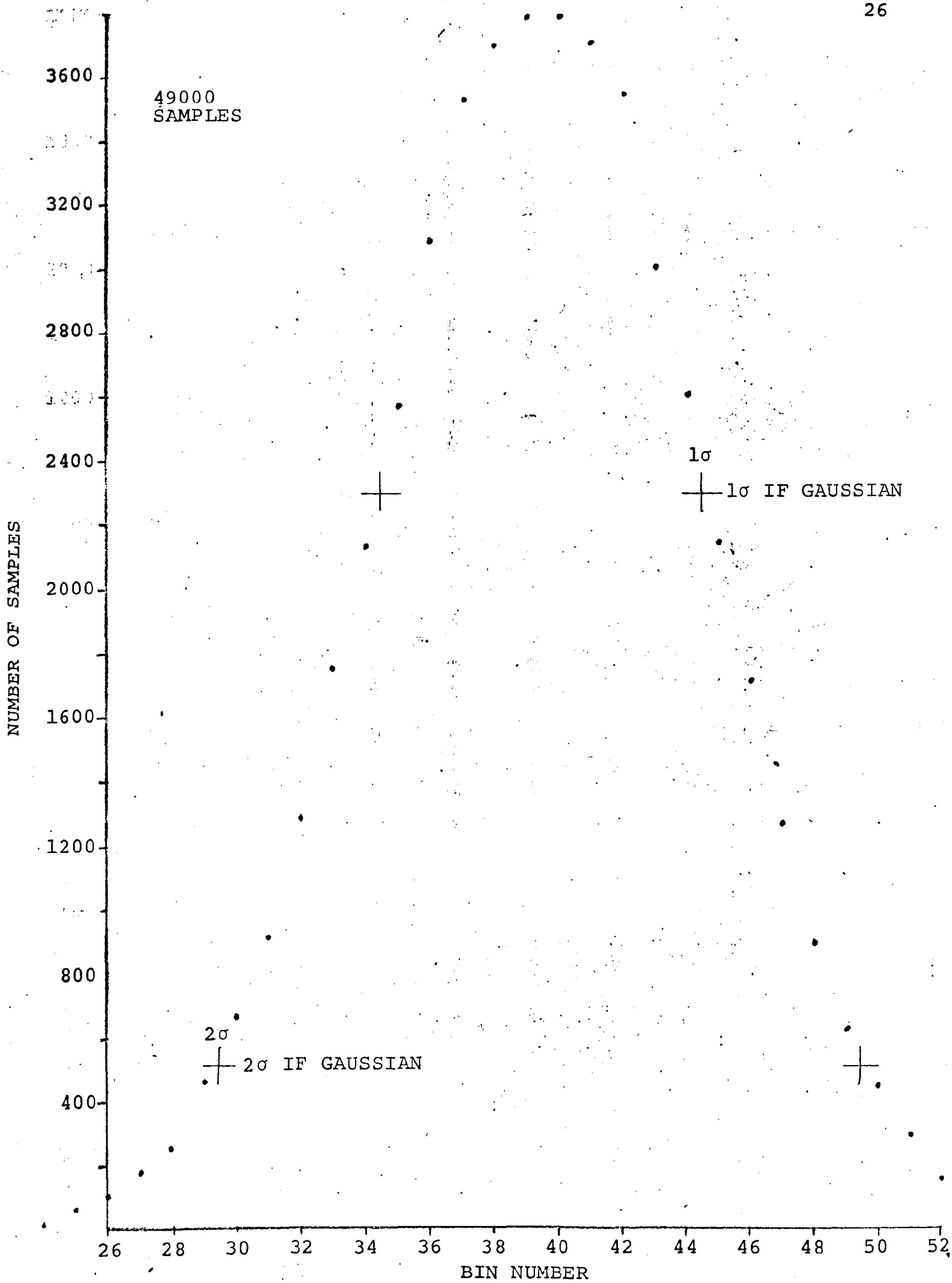
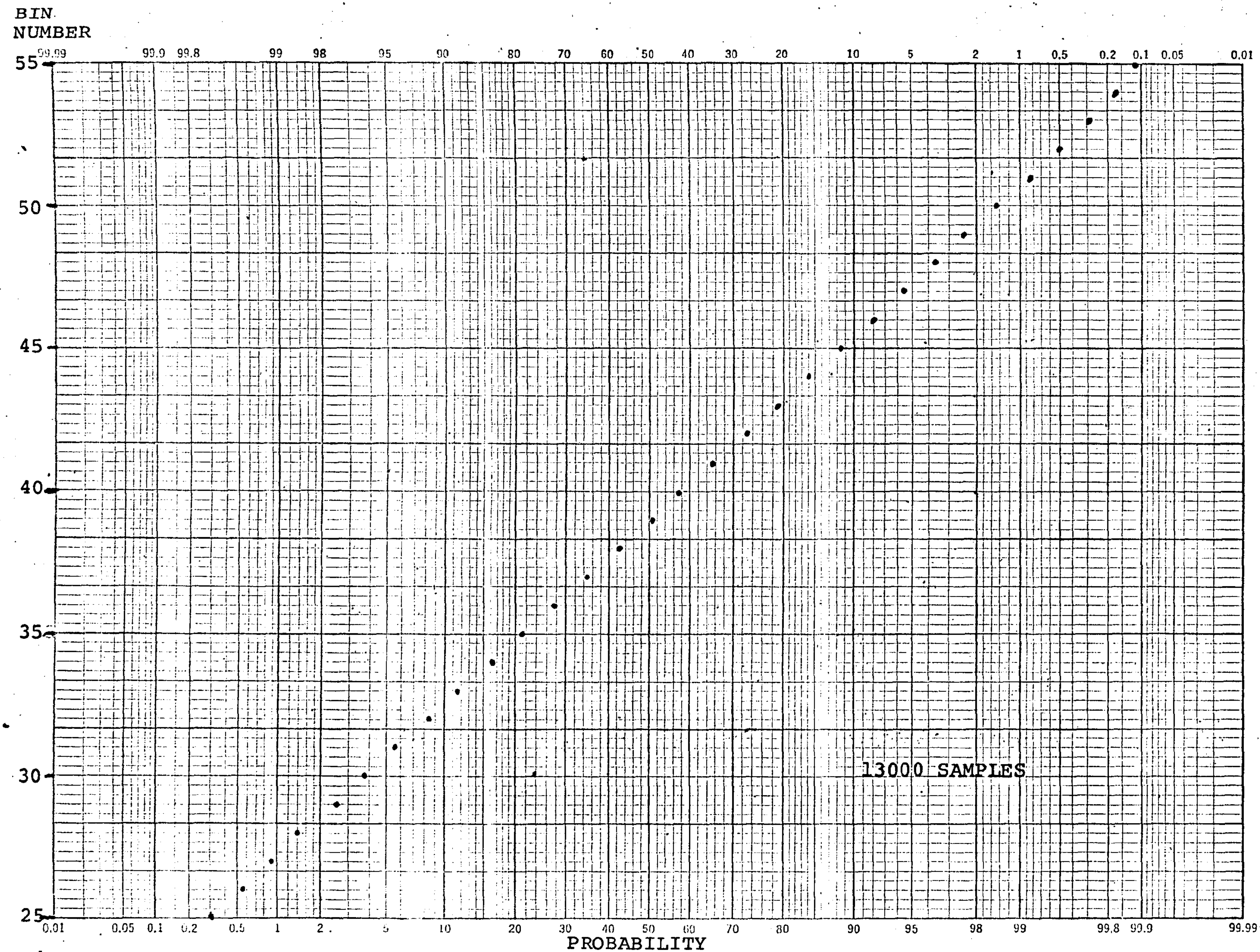


FIGURE 6. SAMPLE HISTOGRAM OF z_n


BIN NUMBER
55
50
45
40
35
30
25
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0.01 0.05 0.1 0.2 0.5 1 2 5 10 20 30 40 50 60 70 80 90 95 98 99 99.8 99.9 99.99
13000 SAMPLES
PROBABILITY

and $\{Y_n\}$ to determine the coefficient C_1 since it is the main consequence of the Gaussian assumption.

Simulation for $\beta = 1$

The Δ-Σ modulation was simulated with independent inputs and with the correlated inputs used to generate Table I. For the independent inputs, the variance of the input (σ_x^2) was .992. With this input and an integration factor (β) of 1, $\phi_{zz}(0)$, $\phi_{yz}(0)$, and $\phi_{xy}(1)$ were determined with different feedback gains, h, for a sample size of 13000. The linear gain coefficient C_1 was computed as $C_1 = \phi_{zy}(0)/\sigma_z$. Figure 8 shows this estimate of C_1 and the input-output cross-correlation coefficient $\phi_{xy}(1)$, as a function of feedback gain. These results indicate that C_1 is monotonically increasing function of the feedback gain with the value of $\sqrt{\frac{2}{\pi}} \doteq .7979$ occurring very near unity feedback. This is also where the input-output crosscorrelation also achieves a maximum. For $\{Z_n\}$ being a Gaussian process, there is only one value of C_1, namely $\sqrt{\frac{2}{\pi}}$; therefore for feedback gains other than the one which gives C_1 as $\sqrt{\frac{2}{\pi}}$, the $\{Z_n\}$ cannot be derived from a gaussian process.

As indicated in the theoretical section, for $\beta = \sigma_x = 1$ the minimum value for the variance of $\{Z_n\}$ occurs with a feedback gain of 1. This can be seen in Figure 9 which shows $\phi_{zz}(0)$ as a function of h. It is just at this minimum value of $\phi_{zz}(0)$ where $\phi_{xy}(2)$ obtains its maximum and $C_1 = \sqrt{\frac{2}{\pi}}$ indicating a normal distribution. This result might be informally

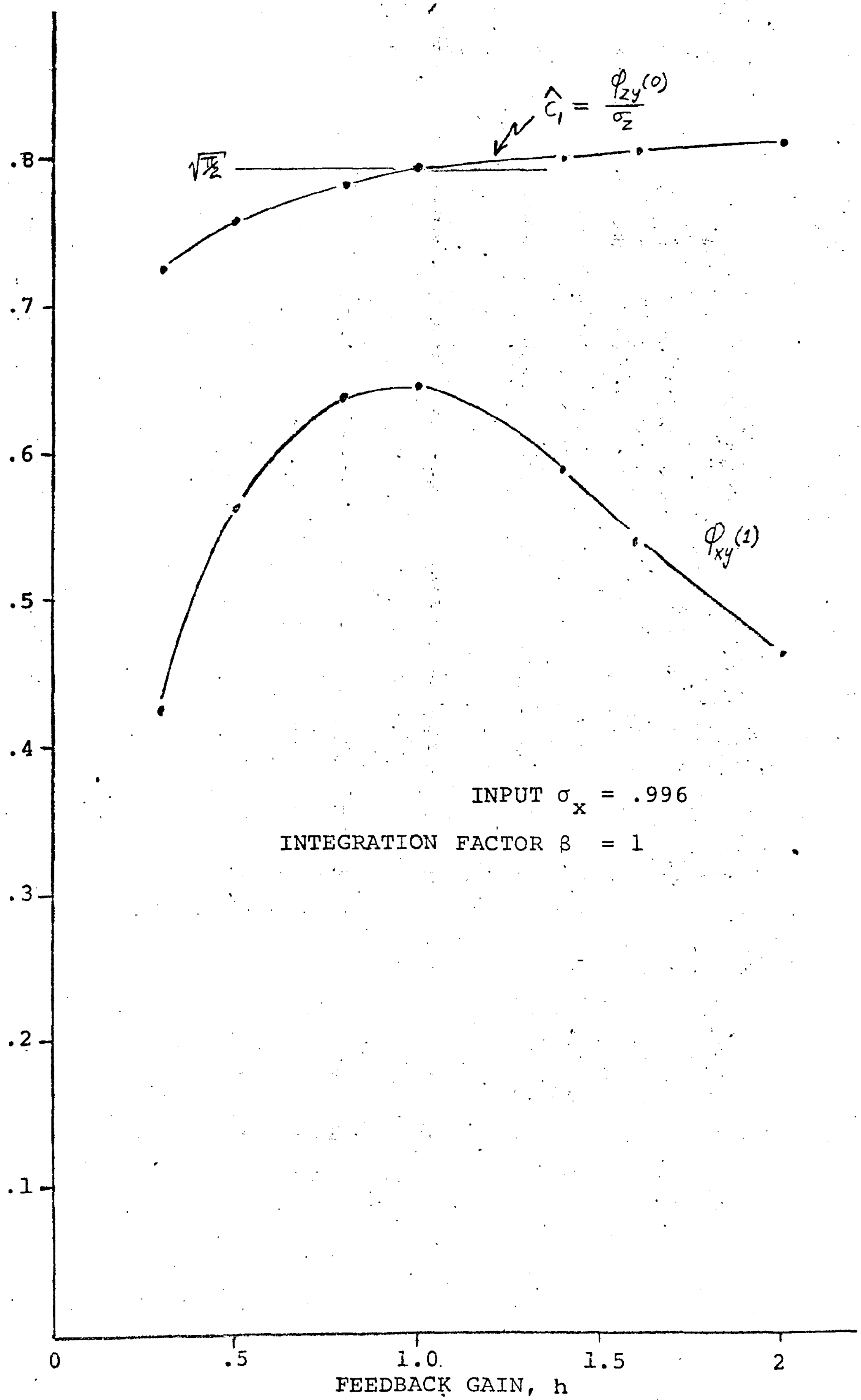


FIGURE 8. LINEAR GAIN COEFFICIENT C_1 AND INPUT-OUTPUT CROSSCORRELATION FOR β = 1

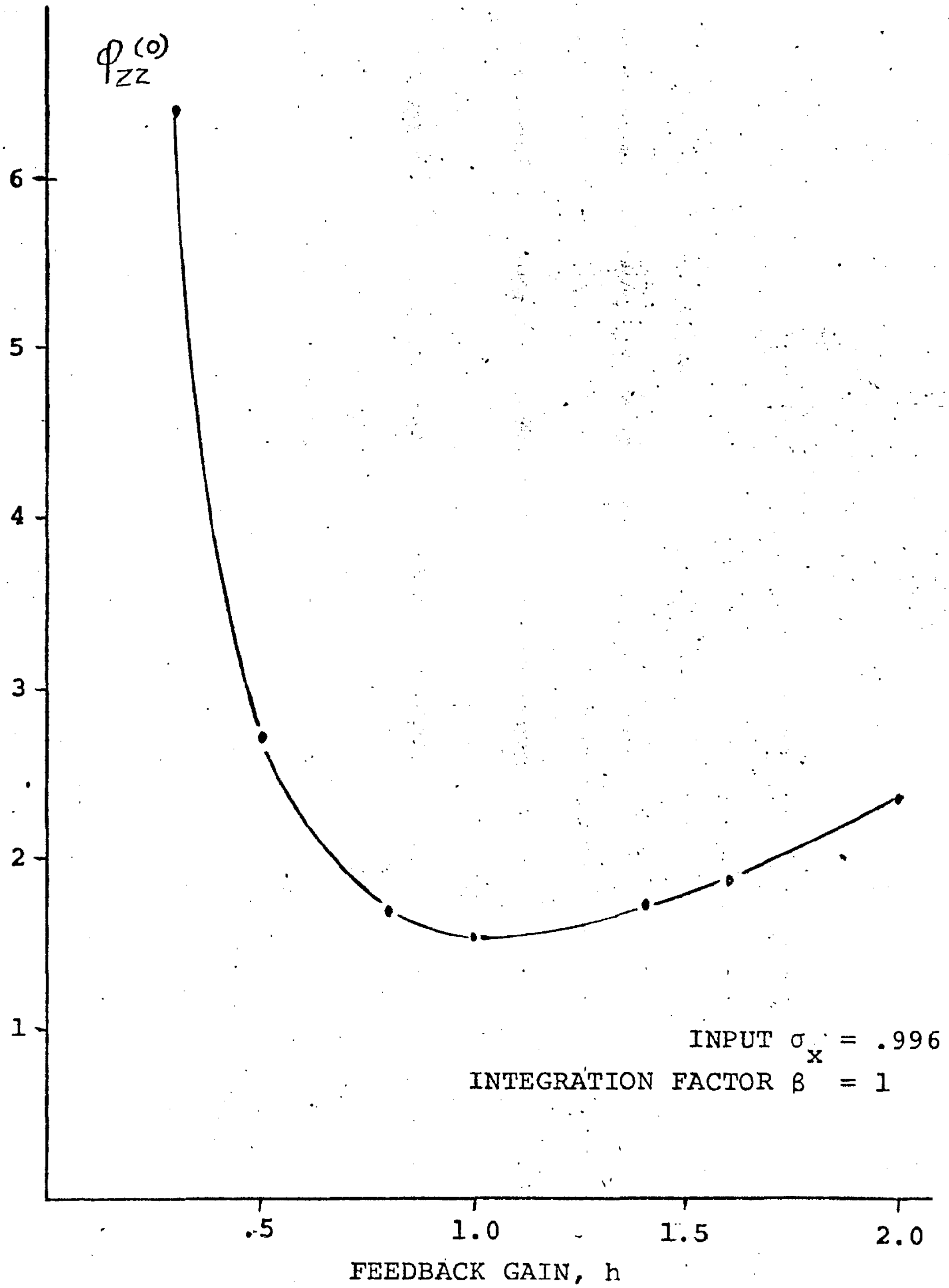


FIGURE 9. VARIANCE OF z_n VS FEEDBACK GAIN

explained by viewing $\{Z_n\}$ as the sum of the weighed errors between the Δ-Σ modulation output and its input. When these errors are made a minimum such that the variance of $\{Z_n\}$ is a minimum, it is expected that the input-output cross-correlation would be a maximum. Along with this minimum error, we have the philosophical notion that when the errors of a system are minimized by a proper adjustment of all the variables (there are only 2, h and β, in Δ-Σ modulation) the residue error is likely to be of a Gaussian nature.

Table II shows the normalized correlation values $(\phi_{uv}(n)/\phi_{uv}(0))$ for a 13000 sample simulation with $h = \beta = 1$. Table III shows a comparison of the simulation results and the theoretical values derived for $\beta = h = \sigma_x = 1$. The experimental numbers agree well with the theoretical results. For those functions which are theoretically zero, the standard deviation of the error due to finite sample size is .01. For example the function $\phi_{xy}(n+1)$ for $n = 0$ is theoretically zero. For the values of this function in Table II the standard deviation of the error is $.01/\phi_{xy}(1) \simeq .015$. A plot of the normalized functions of $\phi_{zz}(n)$ and $\phi_{zy}(n)$ is shown in Figure 10. From the erratic behavior of the functions at small values, the errors due to finite sample size would be estimated as .01 to .02. This is about the size of the errors found in the simulation of the clipper results which was shown in Table I.

Table II

NORMALIZED CORRELATION VALUES

$\beta = 1 \quad h = 1$

13000 Samples

Shift n	$\frac{\phi_{zz}(n)}{\phi_{zz}(0)}$	$\frac{\phi_{zy}(n)}{\phi_{zy}(0)}$	$\phi_{yy}(n)$	$\frac{\phi_{xy}(n+1)}{\phi_{xy}(1)}$
-10	-.007	-.008	.004	.007
- 9	-.002	-.025	-.013	-.026
- 8	-.008	-.010	-.002	-.024
- 7	.009	-.001	-.004	.006
- 6	.012	.009	.011	.005
- 5	.015	-.002	-.008	-.005
- 4	.037	.004	.003	-.004
- 3	.071	.019	.012	-.006
- 2	.152	.026	.003	.001
- 1	.364	.006	-.019	.03
0	1.0	1.0	1.0	1.0
1		.329		.325
2		.121		.121
3		.054		.039
4		.032		.025
5		.008		.009
6		.013		.002
7		.008		.020
8		-.007		.000
9		-.020		-.029
10		.002		.000

Table III

SIMULATION AND THEORETICAL RESULTS

Function	Simulation Result (σ_x=.996 h=β=1)	Theoretical Result (β=h= σ_x=1)	Theoretical Equation Used
C_1	.7950	.7979	(30)
σ	1.239	1.253	(55)
$\phi_{zz}(1)$	.559	.571	(40)
$\phi_{zz}(2)$	.234	.236	(72)
$\phi_{zy}(0)$	.985	1.0	(32)
$\phi_{yz}(1)$	.006	0	(45)
$\phi_{yz}(2)$	.026	.018	(76) and (87)
$\phi_{zy}(1)$	.329	.3346	(67)
$\phi_{yy}(1)$	-.019	-.018	(87)
$\phi_{xy}(0)$	.021	0	(independence)
$\phi_{xy}(1)$	.641	.648	(84)

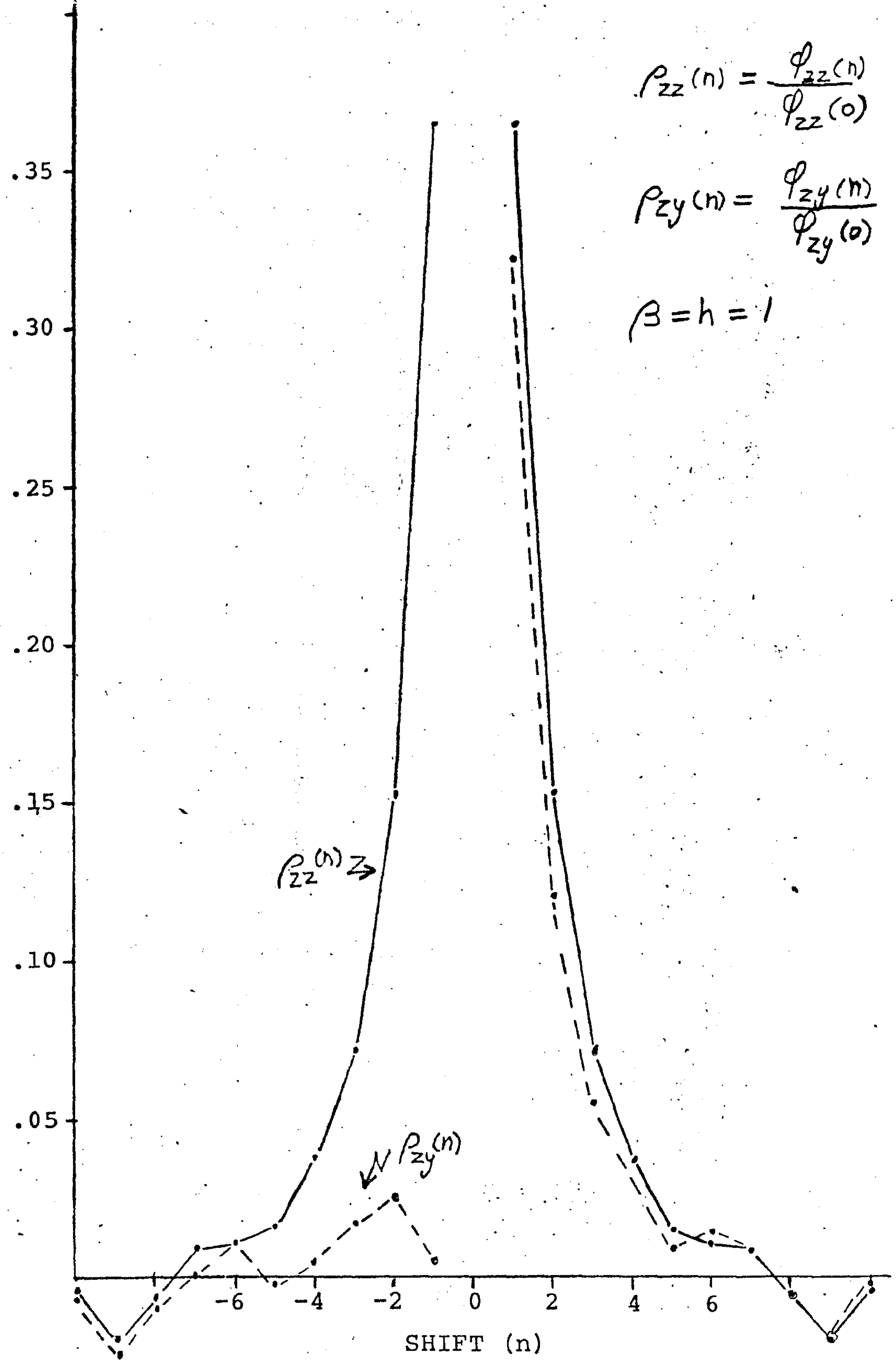


FIGURE 10. NORMALIZED CORRELATION FUNCTIONS OF $\phi_{zz}(n)$ AND $\phi_{zy}(n)$ FOR $\beta = 1$

Simulation for $\beta = .5$

The Δ-Σ modulator was also simulated with an integration factor $\beta = .5$ for the same independent inputs samples as in the simulation above. For this case Figure 11 shows the estimate of C_1 and the input-output crosscorrelation coefficients as a function of feedback gain. It can be seen that C_1 is not a very strong function of feedback gain. In the interval of .1 to 1 the difference between the maximum estimate of C_1 and the minimum is .0044. The function of the input-output crosscorrelation coefficient does have a maximum very near the theoretical gain for minimum variance of $\{Z_n\}$. For $\beta = 0.5$, the feedback gain for a minimum of $\phi_{zz}(0)$ is given by equation (54) as .418. At this value, $\phi_{xy}(1)$ does seem to be a maximum, and the estimate of C_1 is .798 which is the value expected for the gaussian process of $\sqrt{\frac{2}{\pi}}$ rounded to 3 significant figures.

Figure 12 shows the variance of $\{Z_n\}$ as a function of feedback gain for $\beta = .5$. As in Figure 8, $\phi_{zz}(0)$ also has a definite minimum near the point where the theoretical equation (54) indicates. The theoretical minimum occurs at $h = .418$. For this value of h, the simulation yielded a standard deviation of $\{Z_n\}$ of 1.044. This compares well with the theoretical value of 1.049.

Table IV shows the normalized correlation values $(\phi_{uv}(n)/\phi_{uv}(0))$ for a 13000 sample simulation with $\beta = .5$ and $h = .418$. Table V shows a comparison of the simulation

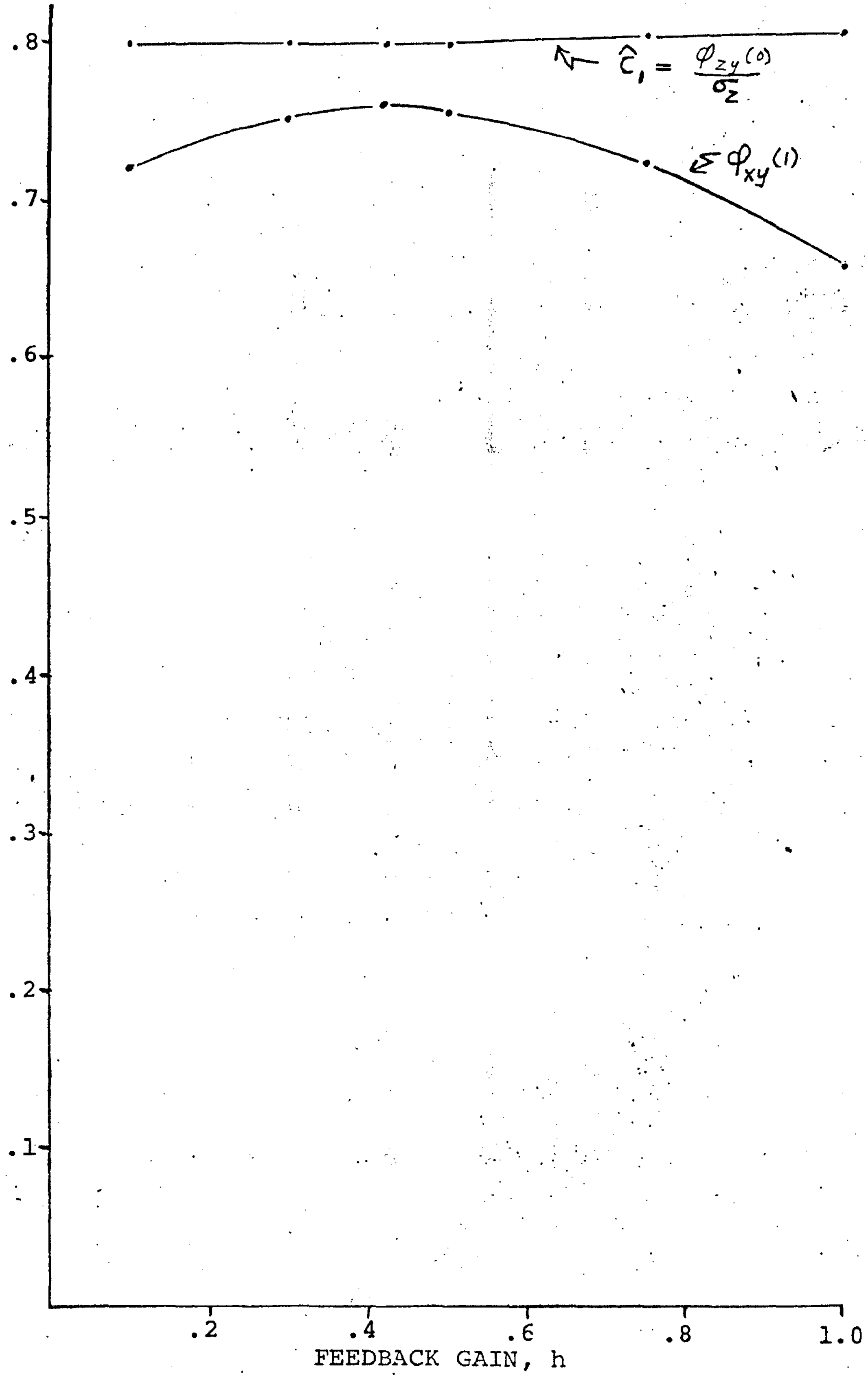


FIGURE 11. LINEAR GAIN COEFFICIENT C_1 AND INPUT-OUTPUT CORRELATION FOR $\beta = .5$

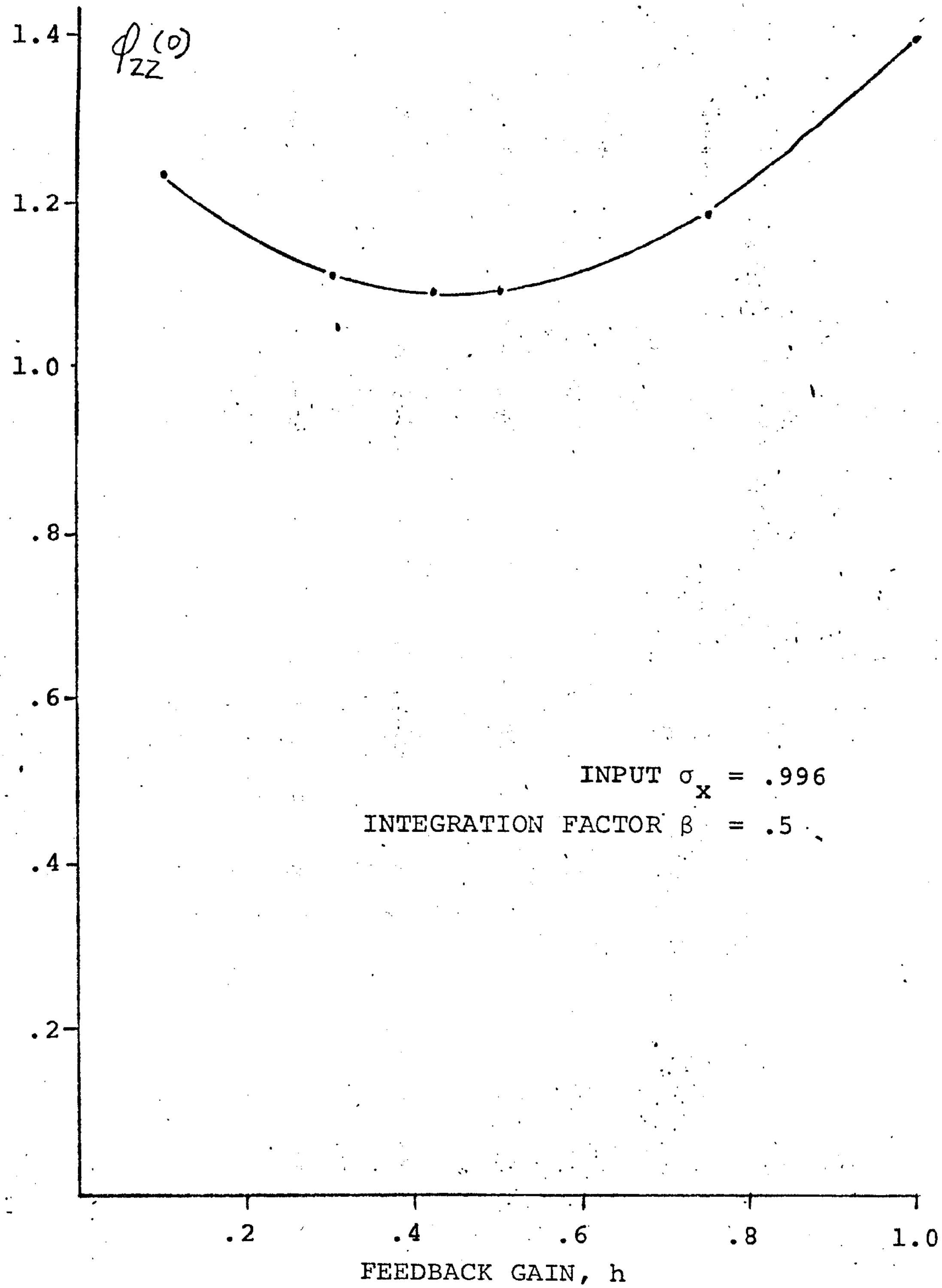


FIGURE 12. $\phi_{zz}(0)$ VS FEEDBACK GAIN

Table IV

NORMALIZED CORRELATION VALUES

$\beta = .5 \quad h = .418$

13000 Samples

Shift n	$\frac{\phi_{zz}(n)}{\phi_{zz}(0)}$	$\frac{\phi_{zy}(n)}{\phi_{zy}(0)}$	$\phi_{yy}(n)$	$\frac{\phi_{xy}(n+1)}{\phi_{xy}(1)}$
-10	-.003	-.009	-.013	-.005
- 9	-.025	-.019	-.008	-.015
- 8	-.019	-.012	-.002	-.007
- 7	.002	.000	.012	.005
- 6	-.006	-.007	.001	.000
- 5	-.004	-.008	.005	-.008
- 4	.003	.005	.007	.010
- 3	-.002	-.014	-.007	-.019
- 2	.037	.007	.001	.001
- 1	.190	.017	.005	.021
0	1.0	1.0	1.0	1.0
1		.184		.185
2		.033		.034
3		-.004		-.005
4		.008		.005
5		.012		.011
6		.006		.001
7		.022		.031
8		-.016		-.011
9		-.020		-.026
10		-.005		-.006

Table V

SIMULATION AND THEORETICAL RESULTS

$\beta = .5 \quad h = .418$

Function	Simulation Results	Theoretical Results	Theoretical Equation Used
C_1	.7981	.7979	(30)
σ	1.0433	1.0488	(51)
$\phi_{zz}(1)$	.207	.200	(40)
$\phi_{zz}(2)$	.0399	.0386	(93)
$\phi_{zy}(0)$	.8331	.8368	(32)
$\phi_{yz}(1)$	.014	.001	(45)
$\phi_{yz}(2)$	.0056	.0025	(94)
$\phi_{zy}(1)$	.153	.147	(67)
$\phi_{yy}(1)$	.005	-.00455	(86)
$\phi_{xy}(0)$	.016	0.0	(independence)
$\phi_{xy}(1)$	.7586	.7614	(83)

results and the theoretical values derived for $\beta = .5$ and $h = .418$. The accuracy of these results are about the same as in the case of $\beta = h = 1$ shown previously. A plot of the normalized functions of $\phi_{zz}(n)$ and $\phi_{zy}(n)$ is shown in Figure 13.

It was demonstrated in both simulations that a minimum variance of $\{Z_n\}$ corresponded to a maximum input-output cross-correlation. Figure 14 shows a plot of the theoretical minimum $\phi_{zz}(0)$ and the corresponding input-output correlation, $\phi_{xy}(1)$, as a function of the integration factor β. The results of the two simulations are indicated in the figure by an "X". Figure 14 shows that a maximum input-output cross-correlation for independent input samples is achieved by the clipper alone with $\beta = h = 0$. It was also observed that the effects of the clipper non-linearity were not unique. When the clipper alone is used with $\beta = h = 0$, an input with a jointly normal distribution yields a unique output such that the output autocorrelation function or power density spectrum indicates a unique input autocorrelation function or power density spectrum in a one-to-one manner. In the Δ-Σ modulator, the distribution of $\{Z_n\}$ is such that the autocorrelation function of the output does not in itself indicate the autocorrelation function of $\{Z_n\}$. This inverse transformation is dependent on the integration factor, β, and feedback gain h. Figure 15 shows the output power density spectrum for the two independent input sample simulation cases discussed along with the power density spectrums of $\{Z_n\}$ for these cases as a

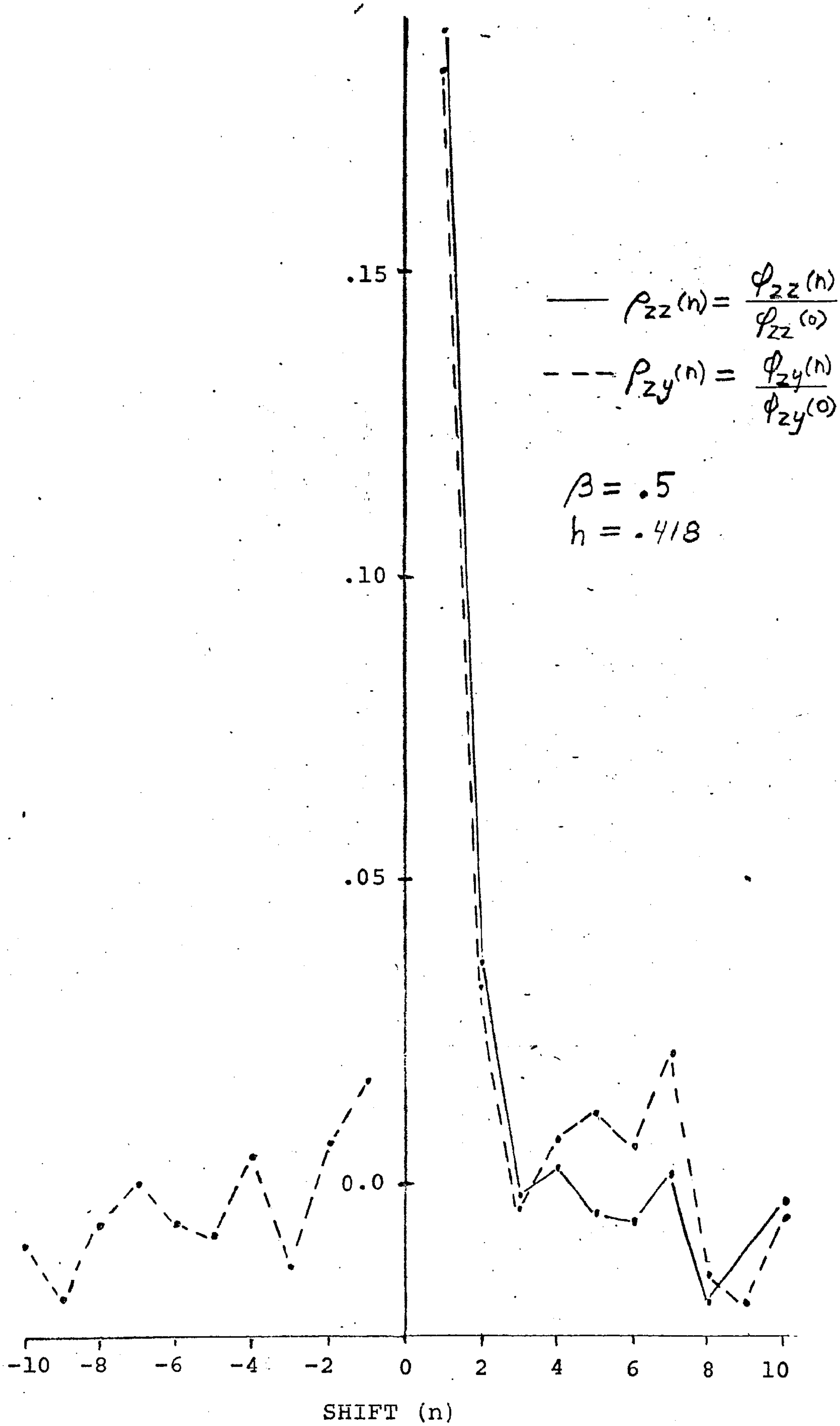


FIGURE 13. NORMALIZED CORRELATION VALUES OF $\phi_{zz}(n)$ AND $\phi_{zy}(n)$ FOR $\beta \doteq .5$

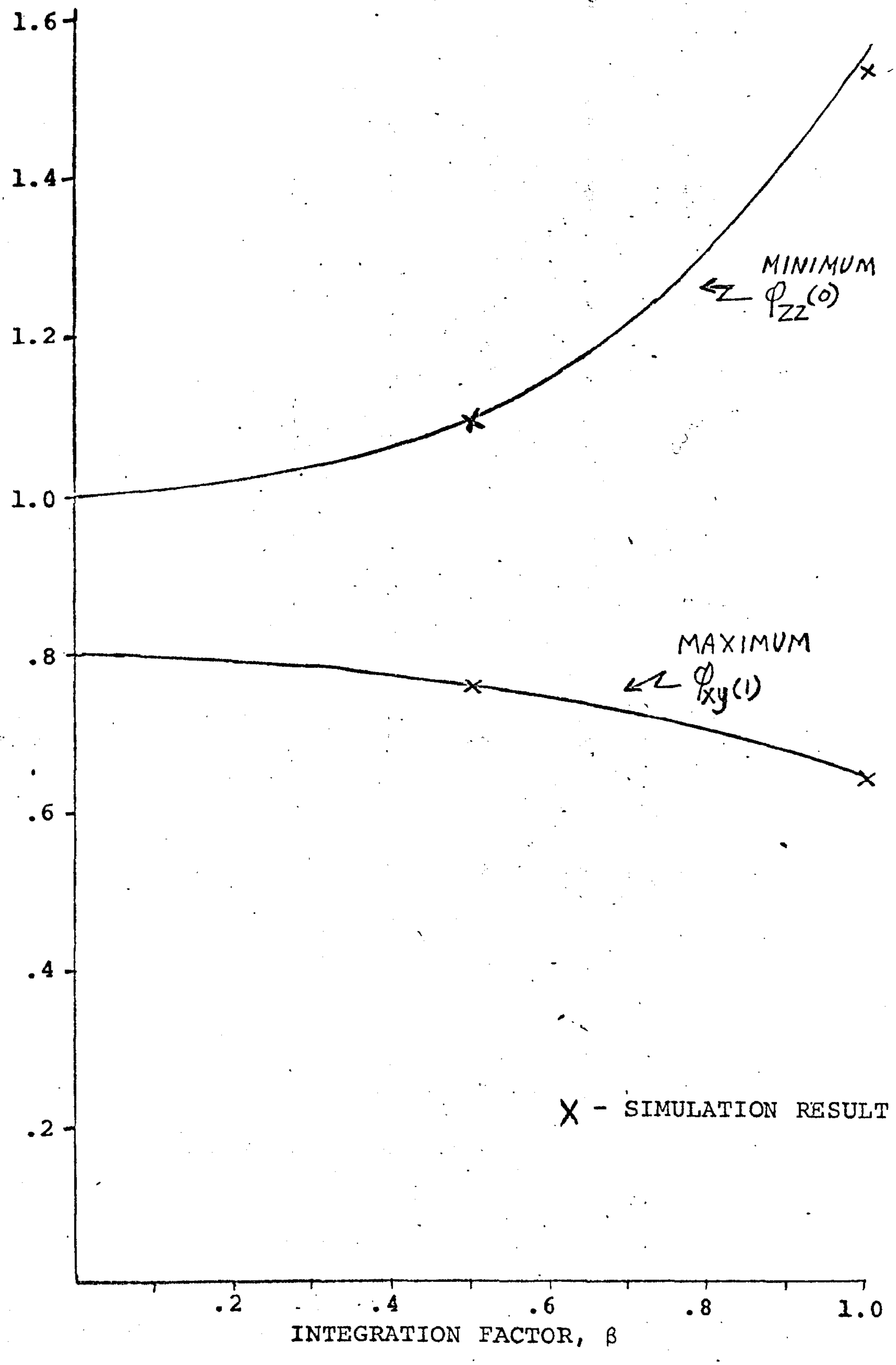


FIGURE 14. $\phi_{zz}(0)$ AND $\phi_{xy}(1)$ AS A FUNCTION OF β

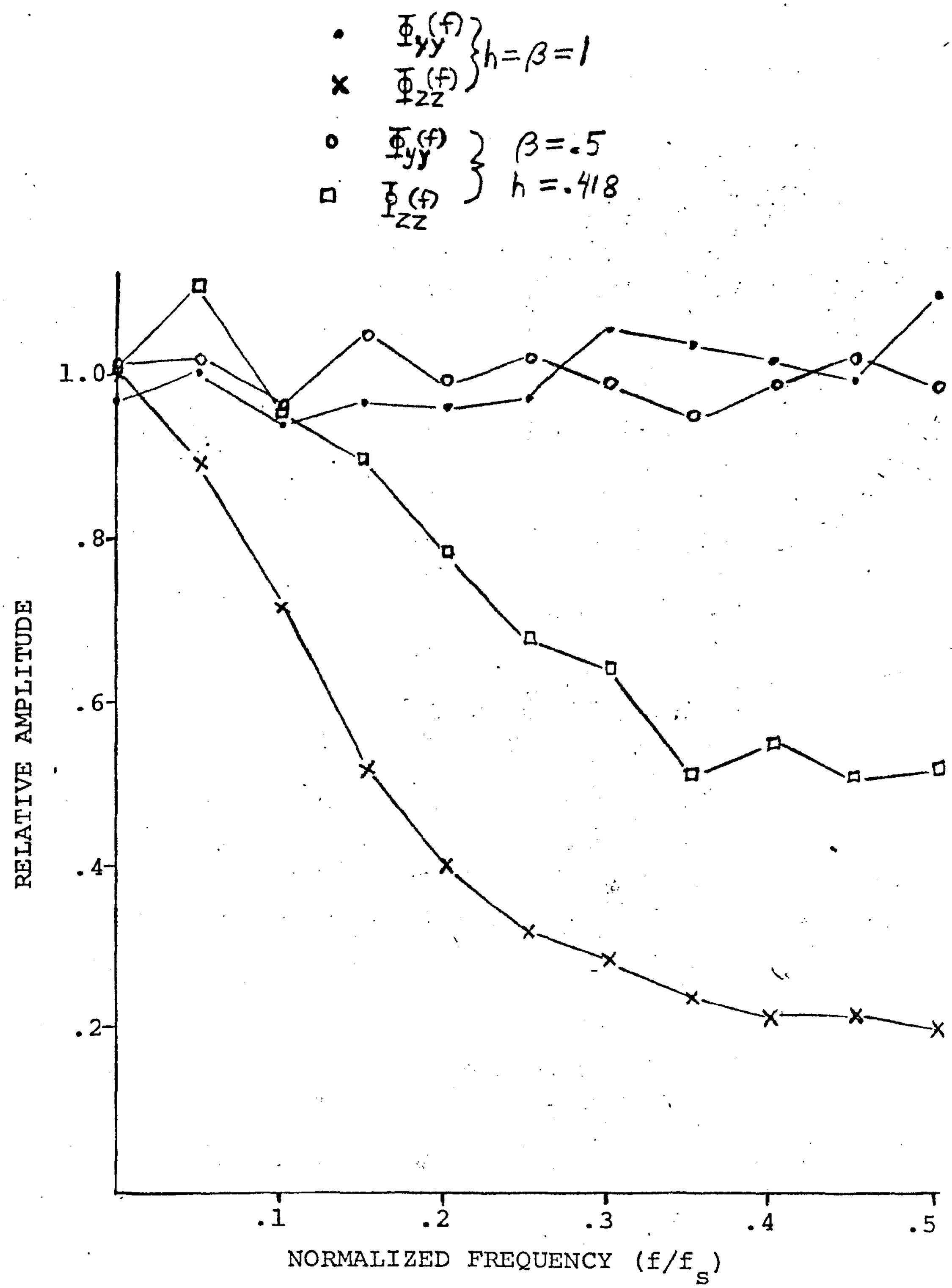


FIGURE 15: POWER DENSITY SPECTRUM OF $\Phi_{yy}(n)$ AND $\phi_{zz}(n)$

function of frequency normalized to the sample rate. From this figure it can be seen that although the output spectrums were similar, the spectrums of $\{Z_n\}$ for the two simulation cases differ.

Simulation for Correlated Inputs

The operation of the Δ-Σ modulation system for correlated inputs was simulated to observe what properties of the independent input case might apply. In this simulation the correlated inputs were those used to generate Table I. For an integration factor $\beta = 1$, the linear gain coefficient, C_1, was estimated as a function of feedback gain, h, using 13000 samples. The results are plotted in Figure 16. These results show that C_1 has a minimum near the value of feedback gain equal to the standard deviation of the input samples, 1.66. After this C_1 increases monotonically with feedback gain and reaches the value for Gaussian samples, .798, near h = 3.4. The experimental data indicates a slight discontinuity in the estimate of C_1 in this region with C_1 = .794 for h = 3.4 and C_1 = .806 for h = 3.45.

The input-output crosscorrelation, $\phi_{xy}(1)$, as a function of feedback gain for the same set of data is shown in Figure 17. It can be seen that this crosscorrelation is a maximum near the value of feedback gain equal to the standard deviation of the input samples. The variance of the samples $\{Z_n\}$ at the input of the clipper is a minimum near the value of h = 3.4 where the estimate of C_1 indicates the possibility of Gaussian samples. The variance of $\{Z_n\}$, $\phi_{zz}(0)$, as a function

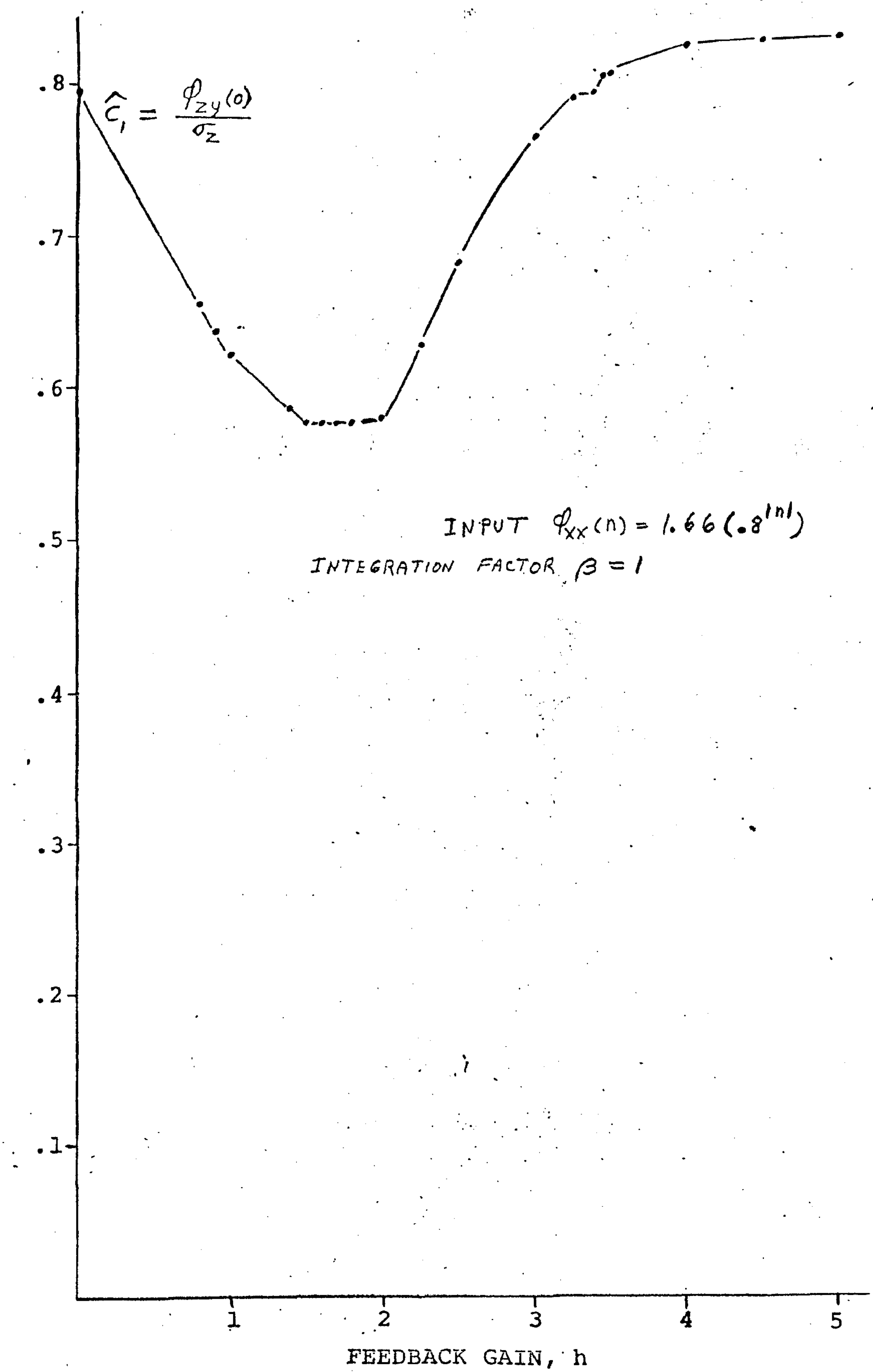


FIGURE 16. ESTIMATE OF LINEAR GAIN COEFFICIENT

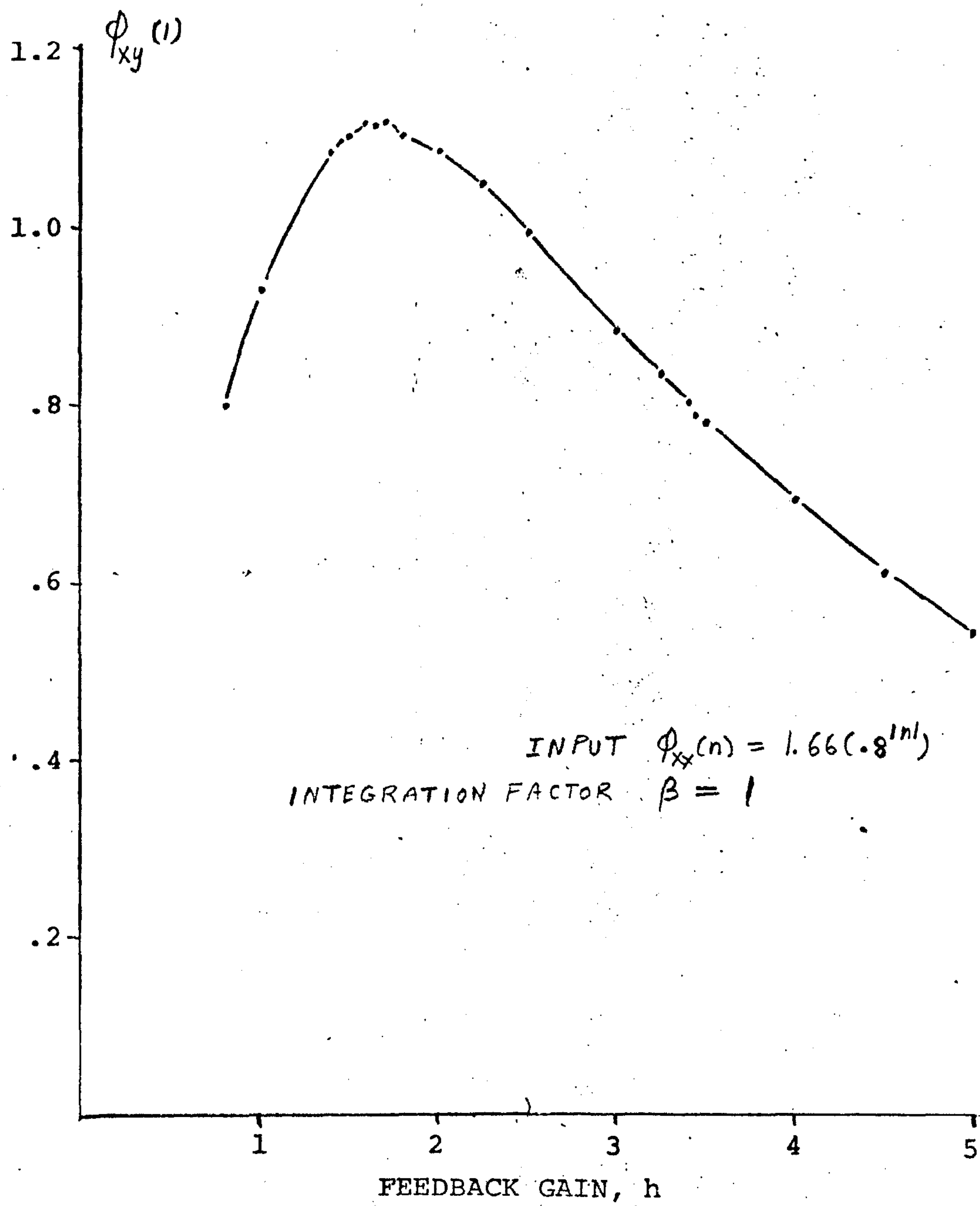


FIGURE 17. INPUT-OUTPUT CROSSCORRELATION COEFFICIENT

of feedback gain is shown in Figure 18. With the independent input samples shown earlier, the conditions for minimum variance of $\{Z_n\}$ coincided with a maximum input-output crosscorrelation, whereas for this low pass filtered sample set the maximum of $\phi_{xy}(1)$ and the minimum of $\phi_{zz}(0)$ occur at different values of feedback gain. However, the minimum of $\phi_{zz}(0)$ is consistant with the hypothesis that $\{Z_n\}$ has a marginal distribution which is a normal distribution.

Table VI lists the normalized correlation values of $\phi_{zz}(n)$, $\phi_{zy}(n)$, $\phi_{yy}(n)$, and $\phi_{xy}(n)$ for the situation where $\phi_{xy}(1)$ is a maximum in Figure 17. This is the condition where the feedback gain was equal to the standard deviation of the input samples. These results do not compare favorably with what could be achieved by using the clipper alone. Using only polarity sampling the input-output crosscorrelation is 1.33 which is approximately 20% greater than the 1.11 maximum $\phi_{xy}(1)$ for the case when $\beta = 1$. A comparison of the output correlation as a function of input correlation for the clipper alone and the simulated output is shown in Figure 19. For the low pass filtered data set, the accuracy of the power density spectrum at low frequencies is determined by the accuracy of the smaller values of the autocorrelation function which occur at the greater sample intervals, n. Figure 19 shows that although the output contains more lower frequency content that the output of the clipper alone, the clipper output autocorrelation is approximately linearly related to the input autocorrelation for small values.

Table VI

NORMALIZED CORRELATION VALUES

$\beta = 1 \quad h = 1.66$

Shift n	$\frac{\phi_{zz}(n)}{\phi_{zz}(0)}$	$\frac{\phi_{zy}(n)}{\phi_{zy}(0)}$	$\phi_{yy}(n)$	$\frac{\phi_{xy}(n+1)}{\phi_{xy}(1)}$
-10	.591	.476	.146	.121
- 9	.629	.509	.158	.139
- 8	.670	.549	.189	.174
- 7	.174	.588	.207	.214
- 6	.759	.626	.234	.257
- 5	.805	.672	.281	.322
- 4	.852	.718	.324	.399
- 3	.898	.767	.381	.510
- 2	.940	.820	.466	.635
- 1	.972	.802	.382	.795
0	1.0	1.0	1.0	1.0
1		.880		.921
2		.817		.833
3		.744		.740
4		.672		.649
5		.609		.554
6		.551		.489
7		.501		.435
8		.458		.375
9		.420		.327
10		.389		.295

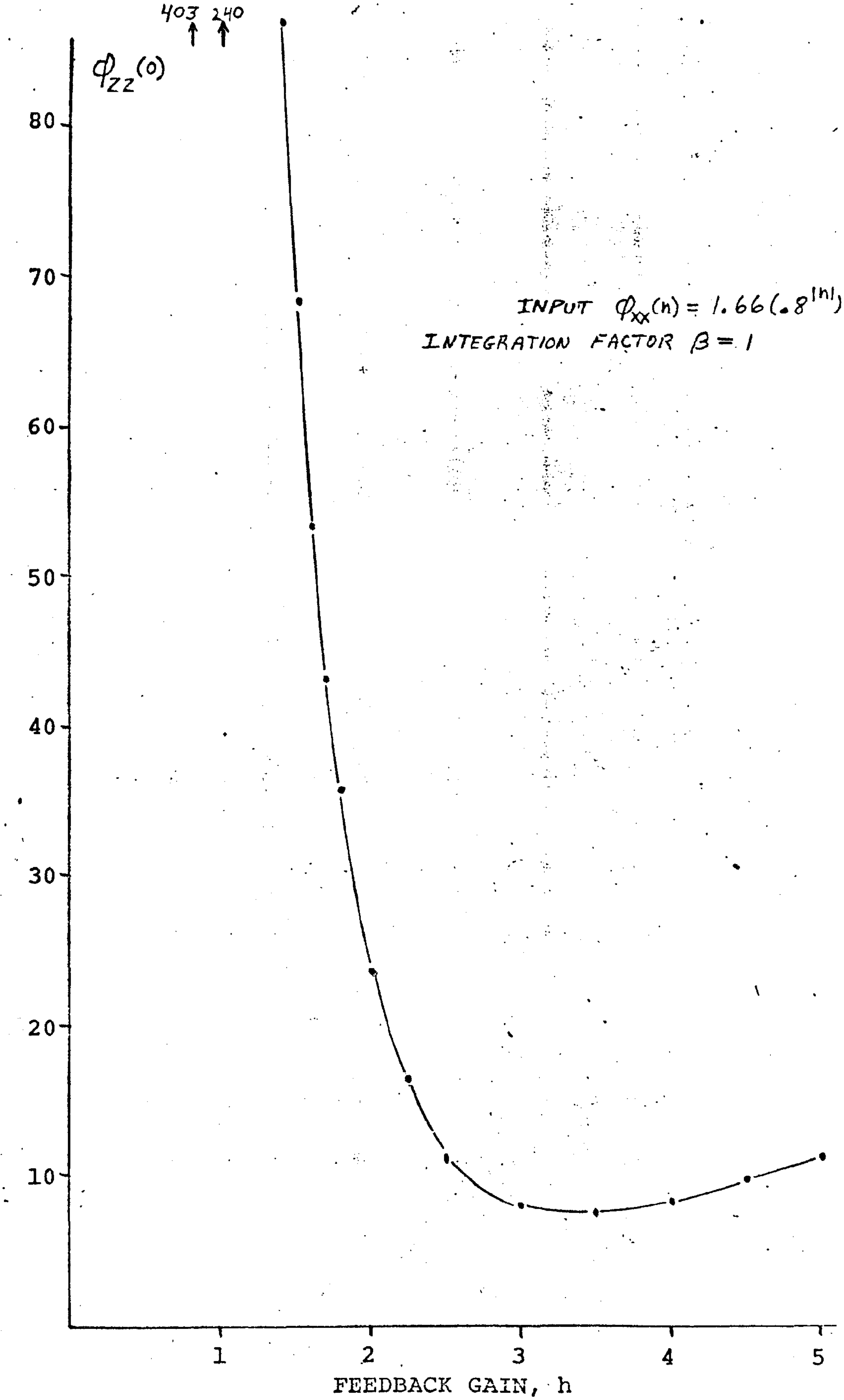


FIGURE 18. VARIANCE OF $\{z_n\}$ VS FEEDBACK GAIN

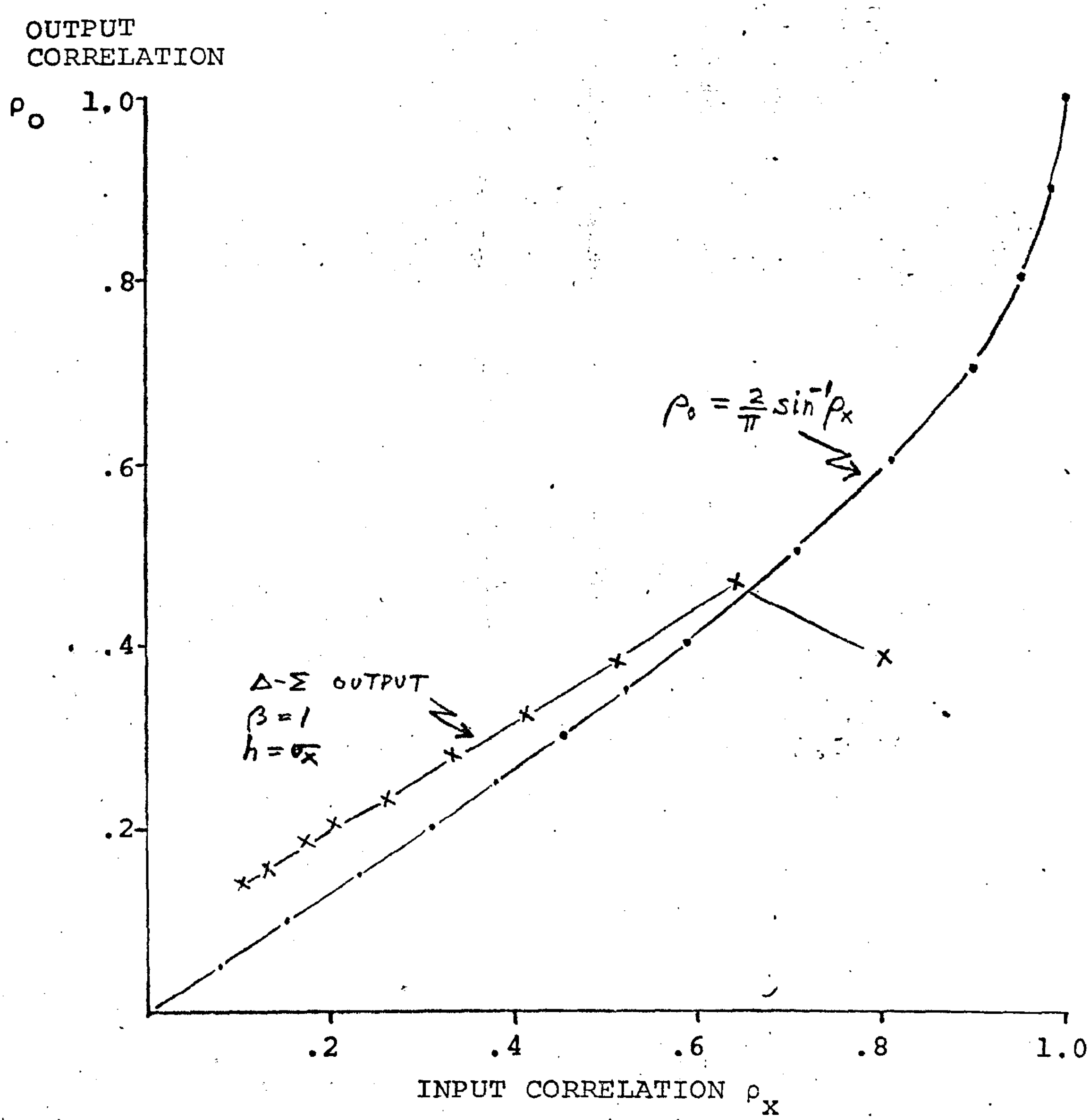


FIGURE 19. COMPARISON OF Δ-Σ OUTPUT AND CLIPPER OUTPUT

The region where the estimate of C_1 was closest to the value obtained for a Gaussian process was also investigated. This is where the variance of $\{Z_n\}$ was found to be a minimum. Tables VII and VIII show the normalized correlation values of $\phi_{zz}(n)$, $\phi_{zy}(n)$, $\phi_{yy}(n)$, and $\phi_{xy}(n)$ for $\beta = 1$ and for $h = 3.4$ and $h = 3.45$ respectfully. These are the values of feedback gain before and after the slight discontinuity in the estimate of C_1 shown in Figure 16. It can be seen from these tables that for $\beta = 1$ and for $h = 3.4$ and 3.45, the input-output crosscorrelation function is nearly symmetric about $\phi_{xy}(1)$ indicating the modulator is doing very little integration or differentiation of the input signal. It can also be observed from the input-output crosscorrelation function, that for these conditions the Δ-Σ modulator is behaving much like a simple non-linear device with no memory such that, like the clipper alone, the input-output crosscorrelation function is linearly related to the input autocorrelation function and as a consequence the output modulation noise, or that which is equivalent to quantization noise, will be orthogonal to the input signal.

Assuming the output "noise" is orthogonal to the input "signal," some approximations of the output signal-to-signal plus noise ratios, $^S/(S+N)$, can easily be made. The total signal in the output bandwidth from 0 to one half the sampling rate is given by $(\phi_{xy}(1)/\sigma_x)^2$. This is also the output signal-to-signal plus noise ratio because the total output power is 1. Figure 20 shows the output power spectrum as a

Table VII

NORMALIZED CORRELATION COEFFICIENTS

$\beta = 1 \quad h = 3.40$

Shift n	$\frac{\phi_{zz}(n)}{\phi_{zz}(0)}$	$\frac{\phi_{zy}(n)}{\phi_{zy}(0)}$	$\phi_{yy}(n)$	$\frac{\phi_{xy}(n+1)}{\phi_{xy}(1)}$
-10	.086	.064	.031	.110
- 9	.094	.063	.025	.122
- 8	.117	.090	.047	.156
- 7	.135	.100	.043	.204
- 6	.162	.129	.067	.270
- 5	.188	.130	.064	.313
- 4	.252	.196	.117	.395
- 3	.288	.208	.100	.502
- 2	.366	.367	.221	.629
- 1	.342	-.262	-.257	.790
0	1.0	1.0	1.0	1.0
1		.235		.801
2		.284		.659
3		.200		.516
4		.191		.434
5		.132		.335
6		.113		.259
7		.086		.222
8		.077		.160
9		.058		.129
10		.059		.122

Table VIII

NORMALIZED CORRELATION COEFFICIENTS

$\beta = 1 \quad h = 3.45$

Shift n	$\frac{\phi_{zz}(n)}{\phi_{zz}(0)}$	$\frac{\phi_{zy}(n)}{\phi_{zy}(0)}$	$\phi_{yy}(n)$	$\frac{\phi_{xy}(n+1)}{\phi_{xy}(1)}$
-10	.047	.040	.619	.106
- 9	.063	.053	.032	.132
- 8	.081	.067	.039	.160
- 7	.106	.086	.048	.199
- 6	.134	.114	.063	.250
- 5	.162	.128	.666	.314
- 4	.201	.165	.095	.418
- 3	.264	.215	.126	.509
- 2	.319	.352	.202	.649
- 1	.290	-.296	-.259	.818
0	1.0	1.0	1.0	1.0
1		.225		.808
2		.253		.656
3		.217		.533
4		.174		.419
5		.127		.340
6		.104		.267
7		.085		.220
8		.067		.165
9		.059		.133
10		.041		.116

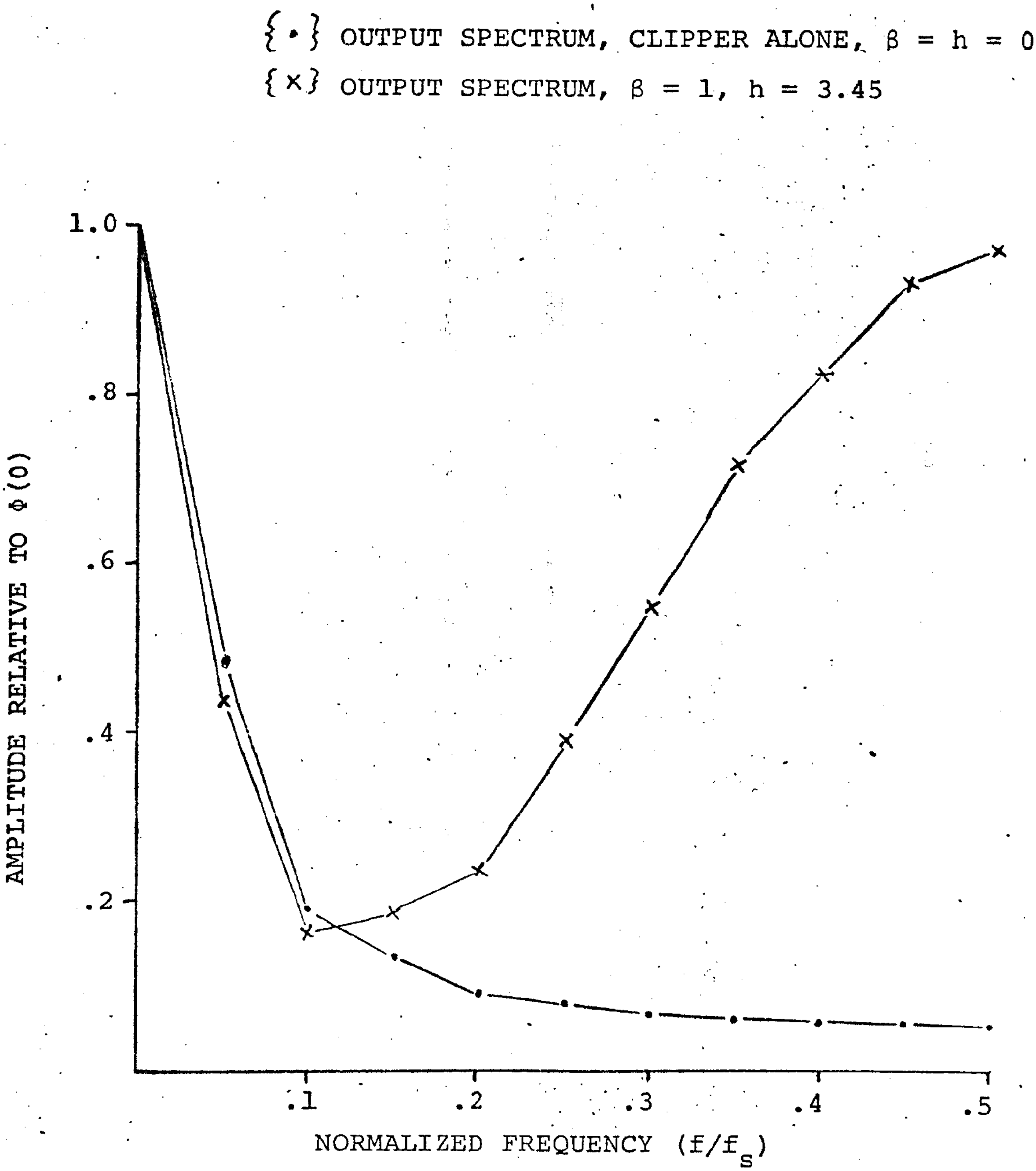


FIGURE 20. OUTPUT POWER DENSITY SPECTRUMS

function of frequency normalized to the sampling rate calculated for the output of the clipper alone with $\beta = h = 0$ and for the modulator with $\beta = 1$ and $h = 3.45$. It is apparent that the Δ-Σ modulator output has a large amount of its power in the higher frequencies. The input signal for this case has approximately 90% of its power in the bandwidth from 0 to 1/8 the sampling rate. The output power in this bandwidth was approximated from the sample points in Figure 20 as the ratio

$$\frac{.5\Phi(0) + \Phi(.05) + \Phi(.1)}{.5(\Phi(0) + \Phi(.5)) + \sum_{i=1}^{9} \Phi(.05i)} \tag{96}$$

Table IX shows the results of the signal and signal plus noise calculations in the total output bandwidth and in the first quarter bandwidth. Results are shown for the clipper alone and for the three cases of Δ-Σ modulation discussed. The case where $\beta = 1$ and $h = 1.66$ was computed for comparison even though it is expected that the modulation noise will be correlated with the input signal. The signal "loss" for the clipper alone was 2 db in the total bandwidth compared to 6.5 db for Δ-Σ with a minimum $\phi_{zz}(0)$ and 3.5 db for Δ-Σ with a maximum input-output crosscorrelation. For the lower quarter bandwidth where approximately 90% of the input power exists the Δ-Σ modulator with a minimum variance of $\{Z_n\}$ has approximately .7 db less "loss" than the clipper alone

The assumption for uncorrelated output noise can be strengthened by showing it leads to an appropriate value for

Table IX

OUTPUT SIGNAL AND SIGNAL PLUS NOISE CALCULATIONS

INPUT

	Clipper Alone $\beta=h=0$	$\Delta-\Sigma$ $\beta=1$ $h=3.4$	$\Delta-\Sigma$ $\beta=1$ $h=3.45$	$\Delta-\Sigma$ $\beta=1$ $h=1.66$
Input-Output Correlation	1.33	.802	.785	1.11
Total Output Signal Power	.6366	.232	.2223	.444
Total $\frac{S}{S+N}$	-1.96 db	-6.35 db	-6.54 db	-3.53 db
S+N in 0 to $\frac{f_s}{8}$ Bandwidth	.67	.21	.20	.57
Signal in 0 to $\frac{f_s}{8}$ Band (.88x Total Signal)	.56	.204	.196	.391
$\frac{S}{S+N}$ in 0 to $\frac{f_s}{8}$ Bandwidth	-.87 db	-.13 db	-.09 db	-1.64 db

the variance of $\{Z_n\}$. The general case for uncorrelated noise can be shown by representing a non-linear device with no memory as a linear gain, K, followed by the addition of noise, d(t). For an input x(t) and an output y(t):

$$y(t) = K\,x(t) + d(t) \tag{97}$$

$$\overline{d^2(t)} = K^2\,\overline{x^2(t)} - 2K\,\overline{x(t)y(t)} + \overline{y^2(t)} \tag{98}$$

Finding the minimum value of $\overline{d^2(t)}$ as a function of K by setting $\frac{\partial \overline{d^2(t)}}{\partial K} = 0$ yields the result

$$K = \frac{\phi_{xy}(0)}{\Phi_{xx}(0)}$$

Using this value for linear gain in the model gives

$$\phi_{dd}(0) = \phi_{yy}(0) - \frac{\phi_{xy}(0)^2}{\phi_{xx}(0)} \tag{100}$$

From equation (97)

$$\phi_{yy}(0) = K^2\,\phi_{xx}(0) - 2K\,\phi_{xd}(0) + \phi_{dd}(0) \tag{101}$$

and substituting the above value of K gives the uncorrelated noise result

$$\phi_{xd}(0) = 0 \qquad (102$$

Using the simular value for K in Δ-Σ modulation gives a correct value for the variance of $\{Z_n\}$. Using the relation,

$$Z_{n+1} - X_n = -\,hY_n - \beta Z_n \tag{103}$$

Squaring and averaging both sides of the equation gives

$$2\,\phi_{xz}(1) - \phi_{xx}(0) = (1-\beta^2)\,\phi_{zz}(0) + 2\beta h\,\phi_{zy}(0) - h^2\,\phi_{yy}(0) \tag{104}$$

Since

$$Z_{n+1} = \sum_{i=0}^{\infty} \beta^i (X_{n-i} - hY_{n-i}) \quad (105)$$

Then

$$\phi_{xz}(1) = \sum_{i=0}^{\infty} \beta^i (\phi_{xx}(i) - \phi_{yx}(i)) \quad (106)$$

For the Δ-Σ modulation case with minimum variance we get as an approximation that

$$\phi_{xy}(n) = \frac{\phi_{xy}(1)}{\phi_{xx}(0)} \phi_{xx}(n-1) \quad (107)$$

$$\phi_{yx}(n) = K \phi_{xx}(n+1) \quad (108)$$

Using equation (108) and (106) in equation (104) gives

$$\phi_{xx}(0) + 2 \sum_{i=1}^{\infty} \beta^i (\phi_{xx}(i) - \frac{hK}{\beta} \phi_{xx}(i)) =$$

$$(1-\beta^2) \phi_{zz}(0) + 2\beta h \phi_{zy}(0) - h^2 \phi_{yy}(0) \quad (109)$$

$$\sigma_x^2 + 2(1-\frac{hK}{\beta}) \sum_{i=1}^{\infty} \beta^i \phi_{xx}(i) = (1-\beta^2) + \sigma^2 \quad (110)$$

$$+ 2\beta hc\sigma - h^2$$

For the case with minimum $\phi_{zz}(0)$, $\beta = 1$, and $hK \simeq 1$

$$\sigma = \frac{\sigma_x^2 + h^2}{2hc} \quad (111)$$

Equation (111) gives standard deviations of 2.7 and 2.6 for the simulations where h = 3.4 and h = 3.45 which gave standard deviations of 2.8 and 2.7 respectfully.

SECTION IV

CONCLUSIONS

The assumption that the integrated differences between the input and output signals, $\{Z_n\}$, have a Gaussian distribution leads to a correct solution of Δ-Σ modulator performance only under a strictly limited set of circumstances. This set of circumstances occur when the variance of $\{Z_n\}$ is a minimum. However, the case where the variance of $\{Z_n\}$ is a minimum may be the most important one from a signal processing point of view because, for independent sample inputs, the input-output crosscorrelation is a maximum, and, for the simple "low pass" sample inputs, the input-output crosscorrelation function is approximately linearly proportional to the input autocorrelation function.

When the Δ-Σ modulator input consists of independent inputs, the conditions for minimum variance of $\{Z_n\}$, σ^2_{min}, can be determined analytically to give appropriate values for the integration factor, β, and the feedback gain, h. The values of the Δ-Σ modulator parameters, β and h, which gave σ^2_{min} did not depend on the probability distribution but only on the independence of the input samples. For independent input samples, the n^{th} order joint probability distribution of $\{Z_n\}$ can be defined and if the values of β and h for σ^2_{min} are used, the assumption of a marginal Gaussian distribution for $\{Z_n\}$ provides correct analytical results for the steady state performance of Δ-Σ modulation. The results from simulation with independent inputs demonstrated that when the variance of $\{Z_n\}$ is a minimum the input-output crosscorrelation is a maximum. This experimental result together with the analytical results leads to the conclusion that both the minimum of the minimum variance of $\{Z_n\}$ and the maximum of the maximum input-output crosscorrelation occur when $\beta = h = 0$. In terms of the performance factors of minimum error, or maximum input-output crosscorrelation, a simple clipper performs better than a Δ-Σ modulator for independent input samples.

Δ-Σ modulator performance for correlated inputs was examined using an input with a simple "low pass" spectrum. The first result noted was that the values of h for $\beta = 1$ which gave a minimum value for the variance of $\{Z_n\}$ was different from the value which gave a maximum input-output crosscorrelation. The maximum input-output correlation was less than could be obtained with a simple clipper alone. At the value of h where the variance of $\{Z_n\}$ was a minimum two interesting results were found: the crosscorrelation of $\{Z_n\}$ and the Δ-Σ modulator out-out was approximately what could be expected if the distribution of $\{Z_n\}$ were Gaussian and the input-output crosscorrelation function was approximately linearly proportional to the input autocorrelation function. These observations were shown to be consistent with the operation of a non-linear devise with no memory so that the output modulator noise, or that which is equivalent to quantization noise, will be orthogonal to the input signal.

This property of a non-linear device with no memory made simple the calculation of signal-to-signal plus noise ratio for the Δ-Σ modulator. These calculations indicated that although the Δ-Σ modulator output had a worse signal-to-noise ratio than a simple clipper in the total system bandwidth from 0 to one-half the sample rate, the Δ-Σ modulator output had a better signal-to-noise ratio than a simple clipper in the lower quarter system bandwidth from 0 to one-eight the sample rate. This is the spectrum region which contained approximately 90% of the signal energy. This performance of the Δ-Σ modulator where the variance of $\{Z_n\}$ is a minimum was found to be consistant with the general notion used in practice that Δ-Σ modulation tends to displace quantization noise into the higher frequency spectrum. For the case simulated, the signal-to-signal plus noise ratio was .7 db better than a clipper alone in the lower quarter bandwidth at the cost of more than 4 db worse signal-to-signal plus noise ratio than a clipper alone in the total system bandwidth.

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