ABSTRACT

Title of dissertation: MINIMAL SURFACES, HYPERBOLIC 3-MANIFOLDS, AND RELATED DEFORMATION SPACES

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Given a closed, oriented, smooth surface Σ of negative Euler characteristic, the relationships between three deformation spaces of geometric structures are compared: the space of minimal hyperbolic germs $\mathcal{H}$, the space of representations $\mathcal{R}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{C}))$, and the space $\mathcal{M}(X)$ of solutions to the self-duality equations on a rank-2 complex vector bundle over a Riemann surface $X \simeq \Sigma$.

Inside both $\mathcal{H}$ and $\mathcal{R}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{C}))$ lies the space $\mathcal{AF}$ of almost-Fuchsian manifolds comprised of quasi-Fuchsian 3-manifolds $M \simeq \Sigma \times \mathbb{R}$ which contain an immersed closed minimal surface whose principal curvatures lie in the interval $(-1, 1)$. The structure of these manifolds is explored through a study of the domain of discontinuity of the associated almost-Fuchsian holonomy group. It is proved that there are no doubly degenerate geometric limits of almost-Fuchsian manifolds.

Next, the space $\mathcal{H}$ is studied through an analysis of a smooth real valued function which records the topological entropy of the geodesic flow arising from a minimal hyperbolic germ. Estimates on this function are obtained which culminate

in a new lower bound on the Hausdorff dimension of the limit set of a quasi-Fuchsian group. As a corollary we obtain a new proof of Bowen's theorem on quasi-circles: a quasi-Fuchsian group is Fuchsian if and only if the Hausdorff dimension of its limit set is equal to 1.

Lastly, we recall a construction of Donaldson which shows how each minimal hyperbolic germ gives rise to a solution of the self-duality equations. In this context, we compare various deformations of a Fuchsian representation $\pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$, finally obtaining an explicit formula for a deformation arising from minimal surfaces in terms of Fuchsian and bending deformations. Interestingly, the hyperkähler structure on the moduli space $\mathcal{M}$ of solutions to the self-duality equations makes an appearance here.

MINIMAL SURFACES, HYPERBOLIC 3-MANIFOLDS, AND RELATED DEFORMATION SPACES

by

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Dedication

To the stars, from which all the beautiful people who made this possible have come.

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This thesis owes its completion to numerous things and people who have occupied and enriched my life since I began my graduate study.

Firstly, I thank my advisor and friend, William Goldman. Without his endless encouragement, guidance, and sweeping knowledge of mathematics, I would have never had the opportunity nor the ability to complete this thesis. Additionally, I thank Richard Wentworth for getting me interested in gauge theory and Higgs bundles in the first place, and for constant encouragement throughout the years. Zheng Huang was the first person to point out that the zeroes of holomorphic quadratic differentials should have special significance for the structure of minimal surfaces, a fact which is exploited in chapter 3 of this thesis. He also pointed me to the Epstein papers on the hyperbolic Gauss map which are also key to the arguments in chapter 3. I am very grateful for his interest in my work and lucky he was around to give me such insightful advice.

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Notation

The following is a list of symbols used frequently throughout the text.

- Σ = closed, oriented, smooth surface of genus greater than 1.
- $\chi(\Sigma)$ = the Euler characteristic of Σ.
- $\pi_1(\Sigma)$ = the fundamental group of Σ.
- $\widetilde{\Sigma}$ = the universal cover of Σ.
- f = an immersion from Σ into some 3-manifold.
- g = a Riemannian metric on Σ.
- K_g = the sectional curvature of g.
- $[g]$ = the conformal class of g.
- B = the second fundamental form of an immersion f of Σ into some 3-manifold.
- λ_i = the eigenvalues of B. Also called the principal curvatures.
- σ = a complex structure on Σ.
- $X = (\Sigma, \sigma)$ = a Riemann surface diffeomorphic to Σ.
- α = a holomorphic quadratic differential on X.
- $\|\alpha\|_g$ = the L^∞ norm of α with respect to a metric g.
- K_X = the canonical line bundle on X.
- $\mathcal{T}$ = the Teichmüller space of isotopy classes of complex structures on Σ compatible with the orientation.
- $\mathcal{F}$ = the Fuchsian space of isotopy classes of Riemannian metrics on Σ of constant sectional curvature -1.
- $\mathcal{H}$ = the space of minimal hyperbolic germs.
- $\mathcal{AF}$ = the space of almost-Fuchsian germs, groups, or manifolds.
- $\mathcal{QF}$ = the space of quasi-Fuchsian representations, groups, or manifolds.
- $\mathcal{R}(\pi_1, G)$ = the space of conjugacy classes of representations of $\pi_1(\Sigma)$ into a Lie group G.
- ∇ = the covariant derivative of a connection on a vector bundle.
- ∇^h = the Chern connection of a hermitian metric h on a holomorphic vector bundle.

- $F(\nabla)$ = the curvature 2-form of a connection ∇.
- Δ_g = the Laplace-Beltrami operator of a metric g.
- $\mathcal{M}(X)$ = the moduli space of solutions to the self-duality equations on a degree zero, rank 2 complex vector bundle over X.
- ϕ = a Higgs field.
- $\mathrm{End}_0(V)$ = the bundle of traceless endomorphisms of a vector bundle V.
- V^* = the dual vector bundle of V. Written V^{-1} if V is a line bundle.
- $\mathbb{H}^3$ = hyperbolic 3-space.
- $\mathrm{Isom}(X)$ = the space of isometries of a metric space (X, d).
- $\partial_\infty Y$ = the geometric/visual boundary of a CAT(-1) metric space Y.

Chapter 1

Introduction

This thesis is concerned with the mutual interrelationships between three deformation spaces prominent in the study of geometric structures on a closed, smooth, oriented surface Σ : the Taubes space [Tau04] of minimal hyperbolic germs $\mathcal{H}$, the space of representations $\mathcal{R}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{C}))$, and the moduli space $\mathcal{M}(X)$ ([Hit87]) of solutions to the self-duality equations on a rank 2, degree 0 complex vector bundle over a Riemann surface $X \simeq \Sigma$. These three spaces provide different viewpoints and have varying levels of intrinsic structure. For example, $\mathcal{H}$ is a smooth manifold, (the smooth part of) $\mathcal{R}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{C}))$ is a complex symplectic manifold (see [Gol04]), and $\mathcal{M}(X)$ has a hyperkähler structure (see [Hit87]).

The space $\mathcal{H}$ of minimal hyperbolic germs is a deformation space whose typical element (g, B) consists of the induced metric and second fundamental form of a minimal immersion of a closed surface Σ into a (potentially incomplete) hyperbolic 3-manifold. There is a special subset $\mathcal{AF} \subset \mathcal{H}$, the *almost-Fuchsian* germs, which directly give rise to a distinguished subset of quasi-Fuchsian 3-manifolds. In Chapter 3, we prove that an almost-Fuchsian 3-manifold, and more directly the almost-Fuchsian holonomy group of such a manifold, has a very special structure which is not enjoyed by an arbitrary quasi-Fuchsian group. Roughly speaking, the domain of discontinuity of an almost-Fuchsian group shows some of the vestiges of the perfect symmetry exhibited by a Fuchsian group. As an application, we prove that a sequence of almost-Fuchsian groups can never converge to a doubly-degenerate Kleinian surface group. This is achieved via a careful study of the domain of discontinuity.

In Chapter 4, a direct study of the space $\mathcal{H}$ is initiated through the study of a dynamically defined function

$$E : \mathcal{H} \to \mathbb{R},$$

which records the topological entropy of the geodesic flow arising from the metric g occurring in the pair $(g, B) \in \mathcal{H}$. Every such g has sectional curvature bounded above by -1, in this setting the topological entropy of the geodesic flow is a much studied, smoothly varying function. Lower bounds are derived on E in terms of the L^1 norm of the quantity $\|B\|_g$. Furthermore, in the space $\mathcal{AF} \subset \mathcal{H}$ of almost-Fuchsian germs, the function E is shown to have no critical points and its growth rate along certain paths is explicitly computed. The final result of chapter 4 shows that the Hessian of E can be used to produce a metric on the Fuchsian space $\mathcal{F}$ of metrics with constant sectional curvature -1 on the surface Σ. This metric is shown to have norm bounded below by 2π times the norm induced by the Weil-Petersson metric. It is possible that this metric is in fact equal to Weil-Petersson, though we have not been able to prove this.

Chapter 5 concerns the interplay between $\mathcal{H}$ and the space of representations $\mathcal{R}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{C}))$. The high point here is a new lower bound on the Hausdorff dimension of the limit set of a quasi-Fuchsian group in terms of the geometry of

π_1-injective immersed minimal surfaces in the quotient quasi-Fuchsian manifold. This relies in a key way on an estimate proved in chapter 4. As a corollary, we obtain a new proof of Bowen's theorem on quasi-circles [Bow79] stating that the Hausdorff dimension of a quasi-Fuchsian group is equal to one if and only if the group is Fuchsian. In fact, the previously mentioned estimate is a quantified version of Bowen's theorem. We close the chapter by comparing a natural involution on the space $\mathcal{AF}$ with the involution on the space of quasi-Fuchsian representations given by switching the conformal boundaries in Bers' simultaneous uniformization parameterization [Ber60]. There is a geometrically defined mapping (first given by Uhlenbeck in [Uhl83]),

$$\Phi : \mathcal{H} \to \mathcal{R}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{C}))$$

whose restriction to $\mathcal{AF}$ intertwines these two involutions. The failure of this map to intertwine the involutions on the whole of $\mathcal{H}$ arises from the failure of Φ to be an immersion everywhere.

In the sixth Chapter, a construction of Donaldson [Don03] is recalled, which begins with an element $(g, B) \in \mathcal{H}$ and produces an element $\overline{\Psi}(g, B) \in \mathcal{M}([g])$. Precisely, $\overline{\Psi}(g, B)$ is a solution to the self-duality equations on a rank-2 complex vector bundle over the Riemann surface with conformal structure induced by the metric g. From the Fuchsian point $(h, 0) \in \mathcal{F}$, there are a number of interesting deformations to consider, those in the Fuchsian direction, those coming from the bending construction of Thurston (see [Bon96]), and also those arising from rays $X(t) = (e^{2u_t}h, tB) \subset \mathcal{H}$. All of these deformations can be viewed jointly once we are in the context of the self-duality equations. Even more interestingly, the complex structure of the character variety, hence that of quasi-Fuchsian space, becomes tractable at this special point since we have explicit formulas representing the objects in question. In particular, we show that the initial direction $\dot{X}(0)$ of the path arising from minimal surfaces is not a pure bending direction, but an explicit formula exhibits $\dot{X}(0)$ as a sum of a pure bending direction and a tangent vector to the Fuchsian space. While such a sum must exist for formal reasons, the explicit sum involves one of the mysterious hyperkähler structures (the one which Hitchin calls K) on $\mathcal{M}([h])$ and thus gives some geometric interpretation of the effect of the complex structure K. Our final result in this chapter is an explicit computation of the length of the U(1)-orbit of a Higgs bundle $\overline{\Psi}(g, B)$, which involves the area of the minimal surface. In appendix A.1, conventions regarding the relation between Riemmanian metrics and hermitian metrics are delineated. These conventions are different, for example, than those taken by Hitchin in [Hit87].

Lastly, Chapter 2 contains preliminary information about hyperbolic geometry and minimal surfaces which will be utilized throughout the text. Previous to this introduction is a notation guide which records symbols which recur throughout the text.

Chapter 2

Preliminaries

Throughout this thesis, Σ always denotes a smooth, closed, oriented surface of genus greater than 1. The universal cover of Σ is denoted $\widetilde{\Sigma}$. Whenever Σ is endowed with a Riemannian metric (or any tensor for that matter), we equip $\widetilde{\Sigma}$ with the pull-back metric so that the covering projection is a local isometry.

2.1 Basics of hyperbolic space

The n-dimensional hyperbolic space $\mathbb{H}^n$ is the unique complete, 1-connected Riemannian manifold with constant negative sectional curvature -1. We will utilize two models of hyperbolic space: the upper half-space model consists of the smooth manifold

$$\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n | x_n > 0\}$$

together with the Riemannian metric

$$dh_n^2 = \frac{\delta_{ij} dx^i dx^j}{x_n^2}.$$

The Poincaré ball model is the smooth manifold

$$\mathbb{H}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n |\ x_1^2 + ... + x_n^2 < 1\}$$

with the Riemannian metric

$$dh_n^2 = \frac{4\delta_{ij} dx^i dx^j}{(1 - |x|^2)^2}.$$

For $\mathbb{H}^3$, we use coordinates $(z,t) \in \mathbb{H}^3$ with $z = x + iy \in \mathbb{C}$ and $t > 0$. A Kleinian group is a discrete (torsion-free) subgroup $\Gamma < \mathrm{Isom}^+(\mathbb{H}^3) \simeq \mathrm{PSL}(2,\mathbb{C})$ of orientation-preserving isometries of hyperbolic 3-space. Given a Kleinian group Γ, the action on $\mathbb{H}^3$ extends to an action on the conformal boundary $\partial_\infty(\mathbb{H}^3) \simeq \mathbb{C} \cup \{\infty\}$ by Möbius transformations. This action divides $\partial_\infty(\mathbb{H}^3)$ into two disjoint subsets: $\Lambda(\Gamma)$ and $\Omega(\Gamma)$. The *limit set* $\Lambda(\Gamma)$ is defined to be the smallest non-empty, Γ-invariant closed subset of $\partial_\infty(\mathbb{H}^3)$. The *domain of discontinuity* $\partial_\infty(\mathbb{H}^3)\backslash\Lambda(\Gamma) = \Omega(\Gamma)$ is the largest open set on which Γ acts properly discontinuously. The quotient $M = \mathbb{H}^3/\Gamma$ is a complete hyperbolic 3-manifold with *holonomy* group Γ.

Given a sequence of Kleinian groups Γ_n, we say that Γ_n converges *geometrically* to the group Γ if,

1. For all $\gamma \in \Gamma$ there exists $\gamma_n \in \Gamma_n$ such that $\gamma_n \to \gamma$ in $\mathrm{Isom}(\mathbb{H}^3)$.

2. If $\gamma_n \in \Gamma_n$ and $\gamma_{n_j} \to \gamma$, then $\gamma \in \Gamma$.

We denote geometric convergence by $\Gamma_n \to \Gamma$. It is well known [CEG87] that the geometric convergence of Kleinian groups is equivalent to the base-framed, Gromov-Hausdorff convergence of the associated quotient manifolds.

Equipping $\partial_\infty(\mathbb{H}^3) = \mathbb{C} \cup \{\infty\}$ with the spherical metric $d_{\mathbb{S}^2} = \frac{|dz|^2}{(1+|z|^2)^2}$, a sequence of closed subsets $A_n \subset \mathbb{C}\cup\{\infty\}$ converges to $A \subset \mathbb{C}\cup\{\infty\}$ in the *Hausdorff topology* if $A_n \to A$ with respect to the distance

$$d(A_n, A) = \inf\left\{ r : A_n \subset \bigcup_{x\in A} B_{\mathbb{S}^2}(x,r) \text{ and } A \subset \bigcup_{x\in A_n} B_{\mathbb{S}^2}(x,r) \right\}.$$

If a sequence of Kleinian groups Γ_n converges geometrically to Γ, it is a simple exercise to verify that the limit set of Γ satisfies $\Lambda(\Gamma) \subset \lim_{n\to\infty} \Lambda(\Gamma_n)$ where the limit is taken with respect to the Hausdorff topology.

2.1.1 Kleinian surfaces groups

Now we restrict to the case that there is an isomorphism $\rho : \pi_1(\Sigma) \to \Gamma$. The representation ρ is *quasi-Fuchsian* if and only if $\Omega(\Gamma)$ consists of precisely two invariant, connected, simply-connected components. The quotient $\Omega(\Gamma)/\Gamma = X^+ \cup \overline{X^-}$ is a disjoint union of two marked Riemann surfaces $(X^+, \overline{X^-})$, each diffeomorphic to Σ, where the bar over X^- denotes the surface with the opposite orientation. The marking, which is a choice of homotopy equivalence $f^\pm : \Sigma \to X^\pm$, is determined by the requirement that $f^\pm_* = \rho$. Conversely, we have the Bers' simultaneous uniformization theorem [Ber60],

Theorem 2.1.1. *Given an ordered pair of marked Riemann surfaces* $(X^+, \overline{X^-})$ *each diffeomorphic to* Σ, *there exists an isomorphism* $\rho : \pi_1(\Sigma) \to \Gamma$ *onto a quasi-Fuchsian group* Γ, *unique up to conjugation in* $\mathrm{PSL}(2,\mathbb{C})$, *such that* $\Omega(\Gamma)/\Gamma = X^+ \cup \overline{X^-}$.

The space of all conjugacy classes of representations of $\pi_1(\Sigma)$ into $\mathrm{Isom}^+(\mathbb{H}^3)$ is denoted

$$\mathcal{R}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3)).$$

For the details concerning the following discussion see [Gol04]. Via the identification

$\mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3)$, the set of homomorphisms

$$\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3))$$

has the structure of an affine algebraic set. The set of irreducible representations is comprised of those $\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3))$ which do not fix a point in $\partial_\infty(\mathbb{H}^3)$. The set of irreducible representations

$$\mathrm{Hom}^{irr}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3))$$

is a complex manifold of complex dimension $-3\chi(\Sigma) + 3$ upon which the action of $\mathrm{Isom}^+(\mathbb{H}^3)$ by conjugation is free and proper. The quotient,

$$\mathrm{Hom}^{irr}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3))/\mathrm{Isom}^+(\mathbb{H}^3) \subset \mathcal{R}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3))$$

is a complex manifold of complex dimension $-3\chi(\Sigma)$. Quasi-Fuchsian space $\mathcal{QF}$, which consists of conjugacy classes of all quasi-Fuchsian representations, lies in the subspace of irreducible representations as an open subset

$$\mathcal{QF} \subset \mathrm{Hom}^{irr}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3))/\mathrm{Isom}^+(\mathbb{H}^3),$$

and thus inherits a complex structure. With respect to this complex structure, the bijection provided by theorem 2.1.1 becomes a biholomorphism

$$\mathcal{QF} \simeq \mathcal{T} \times \overline{\mathcal{T}}.$$

The complex structure on $\mathcal{T}$ is the one arising from Kodaira-Spencer deformation theory (see [Kod05]).

It is a striking feature of the diversity of hyperbolic 3-manifolds that there exist isomorphisms $\rho : \pi_1(\Sigma, p) \to \Gamma$ onto Kleinian groups which are not quasi-Fuchsian.

A Kleinian surface group $\Gamma \simeq \pi_1(\Sigma, p)$ is called *doubly degenerate* if the domain of discontinuity for Γ is empty; $\Omega(\Gamma) = \emptyset$. The first explicit examples of doubly-degenerate groups were free groups of rank 2 discovered by Jorgenson [Jor77].

Due to the resolution of the ending lamination conjecture, the isometry classification of doubly-degenerate manifolds is now well understood. Nonetheless, their fine scale geometry is extremely intricate and interesting. Bonahon showed [Bon86] that they are diffeomorphic to $\Sigma \times \mathbb{R}$, then the ending lamination theorem (for surface groups) [Min10], [BCM] in conjunction with Thurston's double limit theorem [Thu] can be applied to see that every doubly degenerate group is a geometric limit of quasi-Fuchsian groups.

2.2 Minimal surfaces in 3-manifolds

Let (M, h) be a 3-dimensional Riemannian manifold. Given an immersion $f : \Sigma \to M$, the area of f is the area of the Riemannian manifold (Σ, g) where $g = f^*(h)$. Suppose $f(\Sigma)$ is two sided in M and let ν be a globally defined unit normal vector field. Then given X and Y tangent vectors to $f(\Sigma)$, the second fundamental form B is defined as,

$$B(X, Y) = h(\nabla_X \nu, Y)$$

where ∇ is the Levi-Civita connection of h. The second fundamental form B defines a symmetric, contravariant 2-tensor on Σ. By the spectral theorem for self-adjoint operators, B is diagonalizable with eigenvalues λ_1, λ_2 whose product $\lambda_1\lambda_2$ is a smooth function on Σ. They are called the *principal curvatures* of the immersion and $\|B\|_g^2 = \lambda_1^2 + \lambda_2^2$.

Contracting with the metric g yields the mean curvature of the immersion,

$$H = g^{ij} B_{ij}$$

where repeated upper and lower indices are to be summed over. The immersion $f : \Sigma \to M$ is *minimal* if $H = \lambda_1 + \lambda_2 = 0$. Equivalently, f is a critical point for the area functional defined on the space of immersions of Σ into M. A minimal surface is said to be *least area* if $f : \Sigma \to M$ has least area among all maps homotopic to f; least area maps are minimal.

The following existence theorem combines the work of Meeks-Simon-Yau [MSY82] and Gulliver [Gul77] (see also [SU82] and [FHS83]).

Theorem 2.2.1. *Let (M, g) be a compact 3-dimensional Riemannian manifold with $\pi_2(M) = 0$. If M has boundary, then assume that ∂M is mean convex. Then given $f : \Sigma \to M$ such that $f_* : \pi_1(\Sigma) \to \pi_1(M)$ is injective, there exists an area minimizing immersion $g : \Sigma \to M$ in the homotopy class of f.*

Remark: The condition that ∂M is mean convex means that any deformation of ∂M in the outward normal direction is area non-decreasing. The hypothesis also allows that the boundary be non-smooth, provided it satisfies a natural convexity condition (see [MY82] for details).

Now suppose M is a hyperbolic 3-manifold. If $f : \Sigma \to M$ is an immersion as above, the Gauss equation of the immersion is

$$K_g = -1 + \lambda_1\lambda_2 \tag{2.2.1}$$

where K_g is the sectional curvature of the metric g.

If f is a minimal immersion, it follows that $K_g = -1 - \lambda^2$ where we have packaged the principal curvatures into a single smooth function $\lambda^2 = \lambda_1^2 = \lambda_2^2$ since they satisfy $\lambda_1 = -\lambda_2$. Then $\frac{1}{2}\|B\|_g^2 = \lambda^2$ and $K_g = -1 - \frac{1}{2}\|B\|_g^2$.

2.2.1 The space of minimal hyperbolic germs

Suppose $f : \widetilde{\Sigma} \to \mathbb{H}^3$ is an immersion. If g denotes the pullback of the metric on $\mathbb{H}^3$ via the immersion f, then the equations of Gauss and Codazzi relate g and B via:

$$K_g = -1 + \lambda_1\lambda_2 \tag{2.2.2}$$

$$(\nabla_{\partial_i} B)_{jk} - (\nabla_{\partial_j} B)_{ik} = 0. \tag{2.2.3}$$

Here, K_g denotes the sectional curvature of the metric g and ∇ its Levi-Civita covariant derivative.

Now we can introduce the space of minimal hyperbolic germs constructed by Taubes in [Tau04].

Let $(g, B) \in \Gamma(S^2_{>0}T^*\Sigma) \oplus \Gamma(S^2T^*\Sigma)$ be a pair consisting of a Riemannian metric and symmetric 2-tensor on Σ. Such a pair is called a *minimal hyperbolic germ* if B is traceless with respect to g and the Gauss-Codazzi equations (2.2.2) and (2.2.3) are satisfied. Letting $\mathrm{Diff}_0(\Sigma)$ be the space of orientation preserving diffeomorphisms of Σ isotopic to the identity, the space $\mathcal{H}$ of minimal hyperbolic germs is the quotient

$$\mathcal{H} = \{\text{minimal hyperbolic germs}\}/\text{Diff}_0(\Sigma),$$

with $\text{Diff}_0(\Sigma)$ acting by pullback on the pair of tensors (g, B). By abuse of notation, we write $(g, B) \in \mathcal{H}$ to indicate that the orbit of the pair belongs to $\mathcal{H}$. The following fundamental theorem of surface theory [Uhl83] shows that every element $(g, B) \in \mathcal{H}$ can be integrated to an immersed minimal disk in $\mathbb{H}^3$ with first and second fundamental form (g, B).

Theorem 2.2.2. *Let $(g, B) \in \mathcal{H}$. Then there exists an immersion $f : \widetilde{\Sigma} \to \mathbb{H}^3$ whose induced metric and second fundamental form coincide with the lifts of g and B to $\widetilde{\Sigma}$. Furthermore, if $O \in \mathbb{H}^3$ is chosen along with a preferred orthonormal frame $\{F_1, F_2, N\} \subset T_O\mathbb{H}^3$, then the map f is uniquely determined by fixing $p \in \widetilde{\Sigma}$ and an orthonormal frame $\{E_1, E_2\} \subset T_p\widetilde{\Sigma}$ and requiring that*

- $f(p) = O$,
- $df(E_i) = F_i$.

The following fundamental theorem is due to Taubes [Tau04],

Theorem 2.2.3. *The space of minimal hyperbolic germs $\mathcal{H}$ is a smooth, oriented manifold of dimension $12g - 12$ where g is the genus of Σ.*

The Teichmüller space $\mathcal{T}$ is the space of isotopy classes of complex structures agreeing with the orientation of Σ, which by the Koebe-Poincaré uniformization theorem can also be described as the space of isotopy classes of Riemannian metrics of constant sectional curvature -1. For the sake of context, the space of isotopy classes of metrics of constant curvature -1 will be called the Fuchsian space denoted by $\mathcal{F}$. As such, the Fuchsian space embeds into $\mathcal{H}$ via the map

$$\mathcal{F} \longrightarrow \mathcal{H}$$

$$g \mapsto (g, 0).$$

Given a complex structure $\sigma \in \mathcal{T}$, Kodaira-Spencer deformation theory identifies the fiber of the holomorphic cotangent bundle $T^*\mathcal{T}$ over σ as the space of holomorphic quadratic differentials $\alpha = \alpha(z)dz^2$ on the Riemann surface (Σ, σ).

The space $\mathcal{H}$ admits an important map to $T^*\mathcal{T}$: given $(g, B) \in \mathcal{H}$, let $[g] \in \mathcal{T}$ denote the conformal structure induced by the Riemannian metric g. If (x_1, x_2) are local isothermal coordinates for the metric g, Hopf observed in [Hop54] that the Codazzi equations along with the fact that B is trace-free imply that the expression:

$$\alpha(g, B) = (B_{11} - iB_{12})(x_1, x_2)dz^2$$

defines a holomorphic quadratic differential on $(\Sigma, [g])$ where $z = x_1 + ix_2$. This assignment defines a smooth mapping

$$\Psi : \mathcal{H} \longrightarrow T^*\mathcal{T}$$

$$(g, B) \mapsto ([g], \alpha).$$

Furthermore, $\Re(\alpha) = B$. The obvious action of the circle U(1) on the fibers of $T^*\mathcal{T}$ induces an action of U(1) on $\mathcal{H}$ making the above mapping equivariant, under this action the metric g is left completely unchanged. Hence, the U(1)-orbits of minimal surfaces are all mutually isometric. This collection is often called the *associated family* corresponding to an element $(g, B) \in \mathcal{H}$. Furthermore, the action is free if and only if $B \neq 0$.

2.2.2 Almost-Fuchsian germs

The space $\mathcal{H}$ contains a very special subset, the *almost-Fuchsian* hyperbolic germs, which were initially studied by Uhlenbeck [Uhl83] and give rise to bona fide complete hyperbolic 3-manifolds.

Definition 2.2.4. *A minimal hyperbolic germ $(g, B) \in \mathcal{H}$ is almost-Fuchsian if*

$$\|B\|_g^2(x) < 2$$

for all $x \in \Sigma$.

An application of the implicit function theorem shows (see [Uhl83]) that the space of almost-Fuchsian germs is actually an open subset of $\mathcal{H}$ which we record in the following fashion.

Theorem 2.2.5. *Let h be a hyperbolic metric on Σ and α a holomorphic quadratic differential on the Riemann surface $X = (\Sigma, [h])$. Then there exists $\varepsilon(\alpha) > 0$ and $u_t \in C^\infty(\Sigma)$ smooth functions such that for all $|t| < \varepsilon$,*

$$(e^{2u_t} h, tB) \in \mathcal{H}$$

where the real part of α is equal to B.

The following classification theorem is due to Uhlenbeck [Uhl83].

Theorem 2.2.6. *Suppose $(g, B) \in \mathcal{H}$ is almost-Fuchsian. Then,*

- *The metric on $M = \Sigma \times \mathbb{R}$ expressed as*

$$dt^2 + g(\cosh(t)\ \mathbb{I}(\cdot) + \sinh(t)\ \mathbb{S}(\cdot), \cosh(t)\ \mathbb{I}(\cdot) + \sinh(t)\ \mathbb{S}(\cdot))$$

  is a complete metric of constant sectional curvature -1. The corresponding holonomy representation $\rho : \pi_1(\Sigma) \to Isom^+(\mathbb{H}^3)$ is quasi-Fuchsian. Here $\mathbb{I}$ is the identity operator and $\mathbb{S}_i^j = g^{jk} B_{ik}$ is the shape operator associated to the second fundamental form B.

- *The inclusion $f : \Sigma \to \Sigma \times \{0\}$ is a minimal surface in M with induced metric g and second fundamental from B. It is the only closed minimal surface of any kind in M.*

- *$f : \Sigma \to M$ is incompressible, i.e. it is a π_1-injective, smooth embedding.*

We will denote the space (of isotopy classes) of almost-Fuchsian germs by $\mathcal{AF} \subset \mathcal{H}$. By theorem 2.2.6 every $(g, B) \in \mathcal{AF}$ gives rise to a quasi-Fuchsian representation which we will call the holonomy representations of $(g, B) \in \mathcal{AF}$, the image of which will be called an almost-Fuchsian group.

Given $(g, B) \in \mathcal{AF}$ with holonomy group Γ, we call the unique embedded Γ-invariant minimal disk $\widetilde{\Sigma} \subset \mathbb{H}^3$ an *almost-Fuchsian disk* with data (g, B). An almost-Fuchsian disk $\widetilde{\Sigma}$ is *normalized* if

1. The point $p = (0, 0, 1) \in \widetilde{\Sigma}$ and the oriented unit normal to $\widetilde{\Sigma}$ at p is $-\frac{\partial}{\partial t} \in T_p\mathbb{H}^3$

2. The principal curvatures vanish at $p: \; B(p) = 0$.

Note that since B is the real part of a holomorphic quadratic differential on Σ, there always exists $p \in \Sigma$ such that $B(p) = 0$.

If $(g, B) \in \mathcal{AF}$ with holonomy group Γ, we can always select $I \in \mathrm{Isom}(\mathbb{H}^3)$ such that the $I\Gamma I^{-1}$-invariant almost-Fuchsian disk is normalized. Such a choice of I is unique up to the action of the circle group $U(1) < \mathrm{Isom}(\mathbb{H}^3)$ of rotations about the t-axis in the upper half-space model of hyperbolic space.

Chapter 3

Domains of Discontinuity of Almost-Fuchsian Groups

The systematic study of closed minimal surfaces in hyperbolic 3-manifolds began with the work of Uhlenbeck in the early 1980's [Uhl83]. There, she identified a class of quasi-Fuchsian hyperbolic 3-manifolds, the almost-Fuchsian manifolds, which contain a unique closed, incompressible minimal surface which has principal curvatures in $(-1, 1)$. The structure of almost-Fuchsian manifolds has been studied considerably by a number of authors [GHW10], [HW]. In particular, the invariants arising from quasi-conformal Kleinian group theory (e.g. Hausdorff dimension of limit sets, distance between conformal boundary components) are controlled by the principal curvatures of the unique minimal surface.

Given an almost-Fuchsian manifold M, this chapter further explores the relationship between the geometry of the unique minimal surface and the conformal structure at infinity. As a result we will show that there are no doubly-degenerate geometric limits of almost-Fuchsian groups. This will be achieved through a careful study of the hyperbolic Gauss map from the minimal surface, which serves to communicate information from the minimal surface to the conformal structure at infinity.

We briefly summarize the strategy: if Γ is the holonomy group of an almost-Fuchsian manifold $M = \mathbb{H}^3/\Gamma$, consider the Γ-invariant minimal disk $\widetilde{\Sigma} \subset \mathbb{H}^3$ which projects to the unique closed, minimal surface in M. We locate disks $D_i \subset \widetilde{\Sigma}$ which are very close to being totally geodesic; the hyperbolic Gauss map from these disks is nicely behaved and in particular satisfies a Koebe-type theorem [AG85], namely its image contains disks of bounded radius. Epstein [Eps86] has studied the hyperbolic Gauss map extensively. In particular we apply his work to show that the images of the D_i under the hyperbolic Gauss map are contained in the domain of discontinuity for Γ. These images form barriers for the limit set of Γ. Since the limit set of a doubly-degenerate group is equal to $\partial_\infty(\mathbb{H}^3)$, there are no doubly-degenerate limits.

If Σ is immersed in some hyperbolic manifold M, the immersion induces a conformal structure σ on Σ which underlies the induced Riemannian metric g. Provided the immersion is minimal, Hopf [Hop54] showed that the second fundamental form

$$B = B_{11}dx^2 + 2B_{12}dxdy + B_{22}dy^2$$

appears as the real part of

$$\alpha = (B_{11} - iB_{12})\ dz^2,$$

which is a holomorphic quadratic differential on (Σ, σ). The norm $\|\alpha\|_g$ measures how much Σ bends inside of M. In § 3.3.1, we prove a Harnack inequality for $\|\alpha\|_g$ satisfying some bound $\|\alpha\|_g \leq K$. First we show that the norm of a bounded holomorphic quadratic differential on the hyperbolic plane satisfies a Harnack inequality. Then, we prove the induced metric g is uniformly comparable to the hyperbolic metric in that conformal class. Therefore, the growth of the principal curvatures of a minimal immersion is bounded; the surface cannot bend too much, too quickly. The disks D_i mentioned in the previous paragraph are obtained by taking balls around the zeros of α; the Harnack inequality ensures we may pick balls of a uniform radius.

In § 3.3.2, we begin the study of the hyperbolic Gauss map: given an oriented surface $\widetilde{\Sigma} \subset \mathbb{H}^3$ in hyperbolic 3-space with oriented unit normal field N, the pair of hyperbolic Gauss maps $\mathcal{G}^{\pm} : \widetilde{\Sigma} \to \partial_\infty(\mathbb{H}^3)$ are defined by recording the endpoint of the geodesic ray in the direction of $\pm N$. We show that the images of the disks $D_i \subset \widetilde{\Sigma}$ under the hyperbolic Gauss map contain disks of a bounded radius in $\partial_\infty(\mathbb{H}^3)$. To achieve this, we utilize the generalization of the Köebe $\frac{1}{4}$-theorem to quasiconformal maps due to Gehring and Astala [AG85].

Finally, in § 3.4 we prove the main result that the domain of discontinuity of an almost-Fuchsian group Γ contains a disk (in fact many) of fixed radius in $\mathbb{C}$. As we mentioned above, this is an application of work of Epstein who showed [Eps86] that the hyperbolic Gauss map from the minimal Γ-invariant disk $\widetilde{\Sigma}$ is a (quasi-conformal) diffeomorphism onto the domain of discontinuity for Γ. In particular, we obtain definite regions R_i of $\partial_\infty(\mathbb{H}^3)$ into which the limit set of Γ can not penetrate. This leads to the following theorem.

Theorem 3.0.7. *There are no doubly-degenerate geometric limits of almost-Fuchsian groups.*

There is a technical issue in the proof of the above theorem; in order to estimate the size of the regions R_i into which the limit set cannot penetrate, we must conjugate the group Γ by some element of $\mathrm{Isom}(\mathbb{H}^3)$ to put the surface $\widetilde{\Sigma}$ into a normalized position. It is possible that given a sequence of almost-Fuchsian groups

Γ_n, the elements I_n by which we conjugate leave every compact set of $\mathrm{Isom}(\mathbb{H}^3)$. Since the action of $\mathrm{Isom}(\mathbb{H}^3)$ on $\partial_\infty(\mathbb{H}^3)$ is not isometric, this could destroy any control we had gained on the size of the R_i. Proving that we may always conjugate the group into a normalized position by an isometry with some bounded translation distance relies strongly on the geometry of the minimal surface.

3.1 Historical context

In [Uhl83], Uhlenbeck conjectured[1] that any doubly-degenerate hyperbolic 3-manifold M contains infinitely many distinct incompressible (stable) minimal surfaces which are homotopy equivalent to M. Examples of this phenomenon are provided by the doubly-degenerate manifolds M which arise as cyclic covers of closed hyperbolic 3-manifolds fibering over the circle. In this case, there is an infinite cyclic group of isometries of M. The general existence theory ([FHS83], [SU82] and [SY79]) yields an incompressible minimal surface Σ in the closed manifold which M covers. Σ lifts to an incompressible minimal surface in M. The translates of Σ by the infinite cyclic group of deck transformations yields infinitely many distinct incompressible minimal surfaces in M. Theorem 1.1 supports a general philosophy that the number of closed incompressible (stable) minimal surfaces in a quasi-Fuchsian

[1]It is more appropriate to say that this conjecture is implicit in the work of Uhlenbeck. The exact quote in [Uhl83] is, "If the area minimizing surfaces in M are isolated, we expect a large number of minimal surfaces in quasi-Fuchsian manifolds near M." The M she refers to is the doubly-degenerate manifold which arises as a cyclic cover of a closed manifold fibering over the circle.

manifold should serve as a measure of how far that manifold is from being Fuchsian.

In [HW], the authors show that if $M = \mathbb{H}^3/\Gamma$ is an almost-Fuchsian manifold and λ is the maximum positive principal curvature of the unique closed minimal surface in M, then the Hausdorff dimension of the limit set of Γ is at most $1 + \lambda^2$. The question remains whether there exists a sequence of almost-Fuchsian groups Γ_n such that the Hausdorff dimension of the limit set approaches 2. Theorem 3.0.7 rules out the most naive way in which this might occur.

3.2 The hyperbolic Gauss map

Details on the material in this section may be found in the paper of Epstein [Eps86]. Let $S \subset \mathbb{H}^3$ be an oriented, embedded surface. Let N be a global unit normal vector field on S such that if $\{X, Y\} \subset TS$ is an oriented basis of the tangent space of S, then $\{X, Y, N\}$ extends to an oriented basis of $T\mathbb{H}^3$. Given $p \in S$, let $\gamma_p(t)$ be the unit speed geodesic ray with initial point $\gamma_p(0) = p$ and initial velocity $\frac{d\gamma_p(t)}{dt}|_{t=0} = N$.

Definition 3.2.1. *The forward hyperbolic Gauss map associated to S is the map $\mathcal{G}_S^+ : S \to \partial_\infty(\mathbb{H}^3)$ defined by*

$$\mathcal{G}_S^+(p) = \lim_{t \to +\infty} \gamma_p(t).$$

If we use $-N$ in the definition we shall call the associated map $\mathcal{G}_S^-$ the backwards hyperbolic Gauss map.

We quickly recall the results we shall utilize from [Eps84], [Eps86]. Let $f : \mathbb{D} \to \mathbb{H}^3$ be an immersion of the unit disk $\mathbb{D}$ such that the principal curvatures are always contained in $(-1,1)$. Then we have,

Theorem 3.2.2 ([Eps84], [Eps86]). *Let $S = f(\mathbb{D})$. The immersion $f : \mathbb{D} \to \mathbb{H}^3$ satisfies:*

1. *f is a proper embedding.*

2. *f extends continuously to an embedding $\overline{f} : \overline{\mathbb{D}} \to \mathbb{H}^3 \cup \partial_\infty(\mathbb{H}^3)$. In particular $\partial_\infty(S) = \overline{f}(\partial\mathbb{D} \setminus \mathbb{D})$ is a Jordan curve.*

3. *Each hyperbolic Gauss map $\mathcal{G}_S^{\pm} : S \to \partial_\infty(\mathbb{H}^3)$ is a quasi-conformal homeomorphism onto a component of $\partial_\infty(\mathbb{H}^3) \setminus \partial_\infty(S)$. Furthermore, if the principal curvatures lie in $(-\beta, \beta)$ for some $0 < \beta < 1$, then $\mathcal{G}_S^{\pm}$ is a diffeomorphism. Note that the Jordan curve theorem implies $\partial_\infty(\mathbb{H}^3) \setminus \partial_\infty(S)$ consists of two components.*

Recall that given a homeomorphism $f : D_1 \to D_2$ between domains $D_1, D_2 \subset \mathbb{C}$, we say that f is a *quasiconformal* homeomorphism if:

- $f \in H^1_{loc}(D_1)$, that is both f and its distributional derivatives $f_z,\ f_{\overline{z}}$ are locally square-integrable on D_1.

- There exists $\mu \in L^\infty(D_1)$ with $\|\mu\|_{L^\infty} < 1$ such that $f_{\overline{z}} = \mu f_z$ in the sense of distributions.

Defining the *dilatation* $K = \frac{1+\|\mu\|_{L^\infty}}{1-\|\mu\|_{L^\infty}}$, f is said to be K-quasiconformal.

From Theorem 3.2.2 it immediately follows that if $\widetilde{\Sigma} \subset \mathbb{H}^3$ is an almost-Fuchsian disk invariant under an almost-Fuchsian group Γ, each hyperbolic Gauss map

$$\mathcal{G}^{\pm}_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \partial_\infty(\mathbb{H}^3)$$

is a (quasiconformal) diffeomorphism onto one component of the domain of discontinuity Ω of Γ. Furthermore, $\pi_1(\Sigma, p_0) \simeq \Gamma$ implies $\partial_\infty(\widetilde{\Sigma}) = \Lambda(\Gamma)$.

3.3 Technical estimates

3.3.1 Growth of bounded holomorphic quadratic differentials

In this section we prove some technical results which give us control on the growth of the principal curvature function for a minimal surface immersed in a hyperbolic 3-manifold.

We begin with a basic result establishing a bound on the growth of the L^∞-norm of a bounded, holomorphic quadratic differential $\alpha = f(z)\ dz^2$ on $\mathbb{H}^2$. We denote the canonical bundle of holomorphic 1-forms on $\mathbb{H}^2$ by $K_{\mathbb{H}^2}$. In this section we will utilize the Poincaré disk model of the hyperbolic plane consisting of the unit disk $\mathbb{D} \subset \mathbb{C}$ with the metric

$$\frac{4|dz|^2}{(1-|z|^2)^2}.$$

Proposition 3.3.1. *Let $\alpha \in H^0(\mathbb{H}^2, K^2_{\mathbb{H}^2})$ be a holomorphic quadratic differential on the hyperbolic plane and assume there exists a $C > 0$ such that $\sup_{z \in \mathbb{H}^2} \|\alpha\|_{\mathbb{H}^2}(z) \leq C$. Assume there is $z_0 \in \mathbb{H}^2$ such that $\alpha(z_0) = 0$. Then for all $\varepsilon > 0$, there exists $r(\varepsilon, C) > 0$ such that*

$$\|\alpha\|_{\mathbb{H}^2}(x) < \varepsilon$$

for all $x \in B_{\mathbb{H}^2}(z_0, r)$ where $B_{\mathbb{H}^2}(z_0, r)$ is the ball of hyperbolic radius r centered at z_0.

Proof. Without loss of generality, assume $z_0 = 0 \in \mathbb{D}$. Writing $\alpha(z) = f(z)\, dz^2$, the condition

$$\sup_{z \in \mathbb{H}^2} \|\alpha\|_{\mathbb{H}^2}(z) \leq C$$

becomes

$$\frac{(1 - |z|^2)^2}{4} |f(z)| \leq C \tag{3.3.1}$$

for all $z \in \mathbb{D}$. Define an auxiliary holomorphic function defined on $\mathbb{D}$ by

$$g(z) = C' f\left(\frac{1}{2}z\right)$$

with

$$C' = \frac{\left(1 - \frac{1}{4}\right)^2}{4C}.$$

Utilizing (3.3.1),

$$|g(z)| = \frac{\left(1 - \frac{1}{4}\right)^2}{4C} \left| f\left(\frac{1}{2}z\right)\right| \leq \frac{\left(1 - \frac{1}{4}\right)^2}{\left(1 - \frac{|z|^2}{4}\right)^2} < 1$$

for all $|z| < 1$. Thus,

$$|g(z)| < 1$$

for all $|z|z < 1$. By construction $g(0) = 0$, hence the Schwarz lemma implies

$$|g(z)| \leq |z|.$$

Thus,

$$\|\alpha\|_{\mathbb{H}^2}\left(\frac{1}{2}z\right) = \frac{(1-\frac{1}{4}|z|^2)^2}{4}\left|f\left(\frac{1}{2}z\right)\right| \leq \frac{(1-\frac{1}{4}|z|^2)^2}{4C'}|z| < \varepsilon$$

provided $|z| < 4C'\varepsilon$. Thus, if

$$r' = 2C'\varepsilon,$$

select r so that

$$B_{\mathbb{H}^2}(0,r) = B_{\mathbb{C}}(0,r').$$

Then $\|\alpha\|_{\mathbb{H}^2}(z) < \varepsilon$ for all $z \in B_{\mathbb{H}^2}(0,r)$. This completes the proof. □

Our next task is to port the bounds achieved in Proposition 3.3.1 to the case of the norm of a quadratic differential arising from a minimal surface in an almost-Fuchsian manifold. Fortunately, the next lemma shows that the induced metric on the surface is uniformly comparable to the hyperbolic metric. Recall that given a Riemannian metric g on Σ, the uniformization theorem provides a unique hyperbolic metric in the conformal class of g.

Lemma 3.3.2. *Let $(g,\alpha) \in \mathcal{AF}$ and suppose h is the unique hyperbolic metric in the conformal class of g. If we write $g = e^{2u}h$, then*

$$\frac{-\ln(2)}{2} < u \leq 0.$$

Proof. Since $g = e^{2u}h$ and the sectional curvature of h is equal to -1,

$$K_g = e^{-2u}(-\Delta_h u - 1)$$

where K_g is the sectional curvature of g. Further applying the Gauss equation (2.2.1),

$$-1 - \|\alpha\|_g^2 = e^{-2u}(-\Delta_h u - 1). \tag{3.3.2}$$

Since $(g, \alpha) \in \mathcal{AF}$, $-1 - \|\alpha\|_g^2 > -2$. Let $p \in \Sigma$ be a local minimum for u. Then,

$$-2 < e^{-2u}(-\Delta_h u - 1) \leq -e^{-2u}$$

since $-\Delta_h u(p) \leq 0$. Thus, $\frac{-\ln(2)}{2} < u$.

Now apply the maximum principle directly to the equation (3.3.2),

$$\Delta_h u + 1 - e^{2u} - e^{-2u}\|\alpha\|_h^2 = 0.$$

If $p \in \Sigma$ is a local maximum of u, then $\Delta_h u(p) \leq 0$ which implies $-e^{2u(p)} + 1 \geq 0$, so $u \leq 0$ and the proof is complete. $\square$

In lieu of the previous lemma, the following is a direct consequence of Proposition 3.3.1.

Proposition 3.3.3. *Let $(g, \alpha) \in \mathcal{AF}$ and $p \in \Sigma$ such that $\alpha(p) = 0$. Then for all $\varepsilon > 0$, there exists $r = r(\varepsilon) > 0$ such that*

$$\|\alpha\|_g(x) < \varepsilon$$

for all $x \in B_g(p, r)$, where $B_g(p, r)$ is the ball of radius r centered at p as measured in the metric g.

Remark: Note that α is a holomorphic quadratic differential on a closed Riemann surface of genus $g > 1$. Thus α is a holomorphic section of the square of the canonical bundle K^2 which has degree $4g-4$. By standard Riemann surface theory, α has $4g-4$ zeros counting multiplicity.

Remark: The above proposition is equally true if we work in the metric universal cover $\widetilde{\Sigma}$ of Σ. This will be important in our applications.

Proof. Let $g = e^{2u}h$ where h is the unique hyperbolic metric in the conformal class of g. Then $\|\alpha\|_g = e^{-2u}\|\alpha\|_h$. Combined with Lemma 3.3.2 this implies $\|\alpha\|_g \leq 2\|\alpha\|_h$. Let $p \in \Sigma$ be such that $\alpha(p) = 0$. Since $\|\alpha\|_g < 1$, Proposition 3.3.1 implies that for all $\varepsilon > 0$ we can find $r' > 0$ such that $\|\alpha\|_h(x) < \frac{\varepsilon}{2}$ for all $x \in B_h(p, r')$. Now, $u > \frac{-\ln(2)}{2}$ implies $B_g(p, r) \subset B_h(p, r')$ where $r = \frac{r'}{\sqrt{2}}$. Therefore

$$\|\alpha\|_g(x) \leq 2\|\alpha\|_h(x) \leq \varepsilon$$

for all $x \in B_g(p, r) \subset B_h(p, r')$. $\square$

By the Gauss equation, on $B_g(p, r)$ the sectional curvature K_g of g satisfies $-1-\varepsilon^2 \leq K_g \leq -1$. Thus the disks $B_g(p, r)$ are very close to being totally geodesic. These are the nearly geodesic regions referred to in the introduction.

3.3.2 Thick regions in the domain of discontinuity

In this section we show that the nearly geodesic regions obtained in the previous section are mapped, via the hyperbolic Gauss map, to regions in $\partial_\infty \mathbb{H}^3$ which have a uniformly bounded diameter. This will rely on the generalization of the Köebe $\frac{1}{4}$-theorem due to Gehring and Astala [AG85].

Let (g, α) be the data for a normalized almost-Fuchsian disk $\widetilde{\Sigma} \subset \mathbb{H}^3$. By the uniformization theorem, there exists a conformal isomorphism

$$\phi : \mathbb{D} \to (\widetilde{\Sigma}, g)$$

satisfying $f(0) = p$. Since ϕ is conformal,

$$\phi^* g = e^{2u} h$$

where h is the hyperbolic metric on $\mathbb{D}$. If dV_g is the volume element arising from the metric g, then

$$\phi^* dV_g = e^{2u} dV_h. \tag{3.3.3}$$

Convention: The following **notation** is in place for the rest of this section: $\widetilde{\Sigma} \subset \mathbb{H}^3$ is a normalized almost-Fuchsian disk with data (g, α). Recall, this means $(0, 0, 1) = p \in \widetilde{\Sigma}$ is such that $\alpha(p) = 0$. Choose an $\varepsilon > 0$, then Proposition 3.3.3 yields $r > 0$ such that the norm of α is less than ε on $B_g(p, r)$.

Proposition 3.3.4. *Let $\widetilde{\Sigma}$ be a normalized almost-Fuchsian disk with data (g, α). Let*

$$\phi : \mathbb{D} \to (\widetilde{\Sigma}, g)$$

be a uniformization such that $\phi(0) = p$. Then there exists $r_1 = r_1(r) > 0$ such that

$$B_{\mathbb{C}}(0, r_1) \subset \phi^{-1}(B_g(p, r)).$$

Furthermore,

$$J_\phi(z) > 2$$

for all $z \in \mathbb{D}$. Here $J_\phi(z)$ is the Jacobian of ϕ at z calculated with respect to the Euclidean metric on $\mathbb{D}$.

Proof. All of the conclusions are direct consequences of Lemma 3.3.2. On a fixed compact subset of $\mathbb{D}$, the metric g is uniformly comparable to the hyperbolic metric and the hyperbolic metric is uniformly comparable to the Euclidean metric, thus there exists $r_1 > 0$ such that

$$B_{\mathbb{C}}(0, r_1) \subset \phi^{-1}(B_g(p, r)).$$

Writing $\phi^*(g) = e^{2u}h$, then by (3.3.3), the Jacobian of ϕ (with respect to the Euclidean metric on $\mathbb{D}$) is given by

$$J_\phi(z) = \frac{4e^{2u(z)}}{(1 - |z|^2)^2}.$$

Applying the estimate in Lemma 3.3.2 reveals

$$J_\phi(z) > 2$$

for all $z \in \mathbb{D} = \mathbb{H}^2$. This completes the proof. □

The following proposition which controls the distortion of the hyperbolic Gauss map is due to Epstein [Eps86].

Proposition 3.3.5. *Let $\widetilde{\Sigma}$ be a normalized almost-Fuchsian disk with data (g, α). Then:*

1. *The hyperbolic Gauss map*

$$\mathcal{G}^+_{\widetilde{\Sigma}} : B_g(p, r) \to \mathbb{C}$$

is quasiconformal with dilatation $K \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\frac{1}{2}}$.

2. There exists a universal constant $C > 0$ such that

$$J_{\mathcal{G}^+_{\tilde{\Sigma}}}(x) > C(1 - \varepsilon^2) \tag{3.3.4}$$

for all $x \in B_g(p, r)$ where the Jacobian of $\mathcal{G}^+$ is computed with respect to the Euclidean metric on $\mathbb{C}$.

Proof. These facts can be found on pages $120 - 121$ of [Eps86]. Epstein uses the spherical metric on $\mathbb{C}$, but in a bounded neighborhood of zero the spherical metric and Euclidean metric are uniformly comparable. Thus, the C we obtain in (3.3.4) is some multiple of that obtained by Epstein. □

Next, we introduce the generalization of Köebe's $\frac{1}{4}$-theorem due to Gehring and Astala [AG85]. Let $U, V \subset \mathbb{C}$ be open domains and

$$f : U \to V$$

a K-quasiconformal mapping with Jacobian J_f. Then $\log J_f$ is locally integrable and the quantity

$$(\log J_f)_B = \frac{1}{|B|} \int_B \log J_f \; dx,$$

with $B \subset U$ a ball, is well defined. For each $x \in U$, define

$$B(x) = B_{\mathbb{C}}(x, d_{\mathbb{C}}(x, \partial U))$$

to be the largest ball centered at x which remains in U. Lastly, define

$$a_f(x) = \exp\left(\frac{1}{2}(\log J_f)_{B(x)}\right).$$

The promised generalization follows.

Theorem 3.3.6 ([AG85]). *Suppose U and V are open domains in $\mathbb{C}$ and $f : U \to V$ is K-quasiconformal. Then there exists a constant $C = C(K)$ such that*

$$\frac{1}{C}\frac{d_{\mathbb{C}}(f(x), \partial V)}{d_{\mathbb{C}}(x, \partial U)} \leq a_f(x) \leq C\frac{d_{\mathbb{C}}(f(x), \partial V)}{d_{\mathbb{C}}(x, \partial U)}.$$

Given $\widetilde{\Sigma} \subset \mathbb{H}^3$ a normalized almost-Fuchsian disk with data (g, α), fix a uniformization

$$\phi : \mathbb{D} \to (\widetilde{\Sigma}, g)$$

such that $\phi(0) = p$ as in Proposition 3.3.4. Consider the composition

$$\Phi := \mathcal{G}^+_{\widetilde{\Sigma}} \circ \phi : \mathbb{D} \to \mathbb{C}$$

where we have identified $\partial_\infty(\mathbb{H}^3)$ with $\mathbb{C} \cup \{\infty\}$. Our strategy is to apply Theorem 3.3.6 to the function Φ. We collect the necessary properties of Φ below.

Proposition 3.3.7. *The map $\Phi : \mathbb{D} \to \mathbb{C}$ above satisfies:*

1. $\Phi(0) = 0$
2. *The restriction of Φ to the sub-disk $B_{\mathbb{C}}(0, r_1)$ from Proposition 3.3.4 is K-quasiconformal with K independent of (g, α).*
3. *There exists $\beta(r_1) = \beta > 0$ such that $J_\Phi(z) > \beta$ for all $z \in B_{\mathbb{C}}(0, r_1)$ with β independent of (g, α).*

Proof. First,

$$\Phi(0) = 0$$

is a direct consequence of the fact that $\widetilde{\Sigma}$ is normalized. Next, Φ is a composition of a conformal map ϕ with the hyperbolic Gauss map $\mathcal{G}^+_{\widetilde{\Sigma}}$. Thus, the restriction of Φ to $B_{\mathbb{C}}(0, r_1)$ is K-quasiconformal if $\mathcal{G}^+_{\widetilde{\Sigma}}$ is K-quasiconformal on $B_g(p, r)$. This is proved in Proposition 3.3.5 which verifies (2). Lastly, the Jacobian is multiplicative:

$$J_\Phi(z) = J_{\mathcal{G}^+_{\widetilde{\Sigma}}}(\phi(z))\, J_\phi(z).$$

Combining Propositions 3.3.4 and 3.3.5, the product on the right is bounded below by $\beta = 2C(1 - \varepsilon^2)$. This proves (3) and the proof is complete.

□

Proposition 3.3.7 and Theorem 3.3.6 combine to show that the image of the Gauss map contains disks of a definite size.

Proposition 3.3.8. *Let $\widetilde{\Sigma}$ be a normalized almost-Fuchsian disk. Then there exists $R > 0$ such that $B_{\mathbb{C}}(0, R) \subset \mathcal{G}^+_{\widetilde{\Sigma}}(\widetilde{\Sigma})$ where $B_{\mathbb{C}}(0, R)$ is the Euclidean disc of radius R centered at $0 \in \mathbb{C}$.*

Proof. Consider the map

$$\Phi := \mathcal{G}^+_{\widetilde{\Sigma}} \circ \phi : B_{\mathbb{C}}(0, r_1) \to \mathbb{C}.$$

By Proposition 3.3.7(3), there exists $\beta > 0$ such that

$$a_\Phi(0) \geq \exp\left(\frac{1}{2}\log\beta\right).$$

Noting that Φ is K-quasiconformal by Proposition 3.3.7(2), Theorem 3.3.6 yields

$$\sqrt{\beta} \leq \frac{C}{r_1} d_{\mathbb{C}}\Big(0, \partial\Big(\Phi\big(B_{\mathbb{C}}(0, R_1)\big)\Big)\Big).$$

Taking $R = \frac{r_1\sqrt{\beta}}{C}$ completes the proof. □

3.4 Main results

In this section, we present the main results concerning the domain of discontinuity of an almost-Fuchsian group. An almost-Fuchsian group Γ is normalized if the Γ-invariant almost-Fuchsian disk is normalized.

Theorem 3.4.1. *Let Γ be a normalized almost-Fuchsian group. Then the domain of discontinuity Ω of Γ contains $B_{\mathbb{C}}(0, R) \in \mathbb{C}$ for some $R > 0$.*

Proof. Let $\widetilde{\Sigma}$ be the normalized Γ-invariant almost-Fuchsian disk. By Theorem 3.2.2, the forward hyperbolic Gauss map $\mathcal{G}^{+}_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \partial_\infty(\mathbb{H}^3)$ is a (quasi-conformal) diffeomorphism onto one connected component Ω^+ of the domain of discontinuity. By Proposition 3.3.8, $B_{\mathbb{C}}(0, R) \subset \mathcal{G}^{+}_{\widetilde{\Sigma}}(\widetilde{\Sigma}) = \Omega^+$ for some $R > 0$. $\square$

Note that given an almost-Fuchsian group Γ, conjugating Γ by a rotation so that $0 \in \Omega$ followed by a translation with arbitrarily large translation distance in the direction of the positive t-axis, we obtain an almost-Fuchsian group whose domain of discontinuity contains a disk around zero of arbitrarily large radius. Of course, the resulting almost-Fuchsian group is not normalized. This is the essential point of the above theorem.

We can actually do better than the previous theorem. Because the almost-Fuchsian disk $\widetilde{\Sigma}$ is the universal cover of a closed minimal surface whose non-negative principal curvature equals the norm of a holomorphic quadratic differential, the principal curvatures of $\widetilde{\Sigma}$ are zero at a countably infinite set of points. Thus, the arguments given above can be applied one by one to each such point. Furthermore, we may use the opposite pointing normal to obtain the same results for the opposing domain of discontinuity. With this in mind, we can refine the above theorem in the following way.

Theorem 3.4.2. *Let* Γ *be a normalized almost-Fuchsian group and* $\widetilde{\Sigma}$ *the almost-Fuchsian* Γ*-invariant disk with data* $(g, \alpha) \in \mathcal{AF}$*. For a fixed compact set* $E \subset \mathbb{H}^3$ *containing* $p = (0,0,1)$*, consider any set of points* $\{p_i\} \subset \widetilde{\Sigma}$ *contained in* $E \cap \widetilde{\Sigma}$ *such that* $\|\alpha\|_g(p_i) = 0$ *for each* i*. Then there exists an* $R'(E) > 0$ *such that* $\bigcup_i B_{\mathbb{S}^2}\left(\mathcal{G}^{\pm}_{\widetilde{\Sigma}}(p_i), R'\right) \subset \Omega$ *where* Ω *is the domain of discontinuity of* Γ*.*

Remark: By Theorem 3.2.2, $\widetilde{\Sigma}$ is properly embedded so that $E \cap \widetilde{\Sigma}$ is compact. As any set of zeros $\{p_i\}$ is necessarily discrete (they are zeros of a holomorphic quadratic differential), any compact set of zeros is finite.

Proof. Firstly, we claim there exists a positive integer $K(E)$ such that the cardinality of the set of zeros of α contained in $\widetilde{\Sigma} \cap E$ is always less than K. As observed in the remark above, $\widetilde{\Sigma} \cap E$ is compact. Because the metric on the minimal surface is uniformly comparable to the hyperbolic metric, there exists a uniform constant $C > 0$ such that the area of $\widetilde{\Sigma} \cap E$ is bounded above by C. By the Gauss equation the area of the closed minimal surface which $\widetilde{\Sigma}$ covers is at least $\pi(2g-2)$ where g is the genus, thus a fundamental domain for the action of the almost-Fuchsian group has area at least $\pi(2g-2)$. Then there are at most $\frac{C}{\pi(2g-2)}$ fundamental domains which lie in $\widetilde{\Sigma} \cap E$. Since each fundamental domain contains at most $4g-4$ zeros of α, we take $K(E)$ equal to the nearest integer greater than $(4g-4)\frac{C}{\pi(2g-2)} = \frac{2C}{\pi}$.

For any set of zeros $\{p_i\}_{i=1}^n \subset \widetilde{\Sigma} \cap E$, each member of a collection of normalizing elements $\{I_i\}_{i=1}^n \subset \mathrm{Isom}(\mathbb{H}^3)$ (i.e. I_i satisfies $I_i(p_i) = p = (0,0,1)\big)$ lies in a fixed compact set in $\mathrm{Isom}(\mathbb{H}^3)$. This follows from the fact that each I_i will be a composition of a hyperbolic element of uniformly bounded translation distance (specifically bounded by the distance from p to ∂E) and a rotation. Fixing the $R > 0$ of Theorem 3.4.1, we set R' equal to the minimum of the spherical radii of the disks $I\left(B_{\mathbb{C}}(0,R)\right)$ where $I \in \mathrm{Isom}(\mathbb{H}^3)$ ranges over the finite set of all words of length at most n in the I_i and their inverses. Remember that we first established that there exists a universal constant such that $n < K(E)$ and so this is a bounded list over all almost-Fuchsian disks. Each I_i is a conformal transformation so indeed $I(B_{\mathbb{C}}(0,R))$ is a disk. Then, the argument in Theorem 3.4.1 implies that $\bigcup_i B_{\mathbb{S}^2}\left(\mathcal{G}^{+}_{\widetilde{\Sigma}}(p_i), R'\right) \subset \Omega$.

Repeating the same argument after initially conjugating the group by a reflection which reverses the orientation of $\widetilde{\Sigma}$ proves that there exists $R'' > 0$ such that $\bigcup_i B_{\mathbb{S}^2}\left(\mathcal{G}^{-}_{\widetilde{\Sigma}}(p_i), R''\right) \subset \Omega$. Letting $R' = \min\{R', R''\}$ completes the proof. □

Finally, we arrive at the result that no sequence of almost-Fuchsian groups converges geometrically to a doubly-degenerate group.

Theorem 3.4.3. *Suppose Γ_n is a sequence of almost-Fuchsian groups and $\Gamma_n \to \Gamma$. Then Γ is not doubly-degenerate.*

Proof. Recall that the limit set of a doubly-degenerate group Γ equals $\partial_\infty(\mathbb{H}^3)$. Suppose Γ_n is a sequence of almost-Fuchsian groups converging geometrically to Γ. By Theorem 3.4.1 there exists an $R > 0$ and a sequence of transformations $I_n \in \mathrm{Isom}(\mathbb{H}^3)$ such that

$$\Lambda(I_n \Gamma_n I_n^{-1}) \cap B_{\mathbb{C}}(0,R) = \emptyset. \tag{3.4.1}$$

We shall refer to any such sequence I_n as a normalizing sequence.

First suppose that the I_n remain in a compact subset of $\mathrm{Isom}(\mathbb{H}^3)$. Choose a subsequence such that $I_n \to I$ where we have relabeled the indices. Then $I_n \Gamma_n I_n^{-1} \to I\Gamma I^{-1}$ geometrically. Since $\Lambda(I\Gamma I^{-1}) = I\Lambda(\Gamma)$, the limit set of $I\Gamma I^{-1}$ is also equal to $\partial_\infty(\mathbb{H}^3)$. We conclude that

$$\partial_\infty(\mathbb{H}^3) = \Lambda(I\Gamma I^{-1}) \subset \lim_{n\to\infty} \Lambda(I_n \Gamma_n I_n^{-1})$$

and so

$$\lim_{n\to\infty} \Lambda(I_n \Gamma_n I_n^{-1}) = \partial_\infty(\mathbb{H}^3)$$

in the Hausdorff topology on closed subsets of $\partial_\infty(\mathbb{H}^3)$. In particular, since the spherical and Euclidean metrics are comparable on compact subsets in the Euclidean topology (they nearly agree around $0 \in \mathbb{C}$), given any $R > 0$ there exists an $N \in \mathbb{N}$ such that

$$\Lambda(I_n \Gamma_n I_n^{-1}) \cap B_{\mathbb{C}}(0, R) \neq \emptyset$$

for every $n \geq N$. This contradicts equation (3.4.1).

Now suppose that no normalizing sequence I_n remains in a compact subset of $\mathrm{Isom}(\mathbb{H}^3)$. We work towards a contradiction. Let $\widetilde{\Sigma_n}$ be the unique Γ_n-invariant almost-Fuchsian disk. We select a sequence of points $p_n \in \widetilde{\Sigma_n}$ satisfying the following properties,

1. Each $p_n \in \widetilde{\Sigma_n}$ is such that the principal curvatures of $\widetilde{\Sigma_n}$ vanish at p_n.

2. There exists $x \in \mathbb{C} \subset \partial_\infty(\mathbb{H}^3)$ such that $p_n \to x$ where the convergence is in the Euclidean topology on the closed upper half-space $\mathbb{H}^3 \cup \partial_\infty(\mathbb{H}^3)$.

A sequence can be found satisfying both conditions since we have assumed every sequence p_n of vanishing principal curvature leaves every compact subset of $\mathbb{H}^3$. Next, conjugate each group Γ_n by a parabolic isometry, represented by a Möbius transformation

$$J_n = z + a_n,$$

which maps p_n to some point $(0, 0, p_n) \in \mathbb{H}^3$ where we have abused notation in calling the z-coordinate by the same name. Note that assumption (2) on the sequence p_n implies $|a_n| \leq K$ for some $K > 0$. As the next step, conjugate each group further by an elliptic isometry to make the downward pointing unit normal vector to $J_n(p_n)$ directed at $0 \in \partial_\infty(\mathbb{H}^3)$. Further abusing notation, we label the new sequence of groups as Γ_n and denote the unique Γ_n-invariant almost-Fuchsian disks by $\widetilde{\Sigma_n}$.

It remains true that $\Lambda(\Gamma_n) \to \partial_\infty(\mathbb{H}^3)$ since the parabolic elements were chosen from a compact set of isometries and elliptic elements act as isometries of the spherical metric from which the Hausdorff topology was induced.

Given the above prerequisites, a $\widetilde{\Sigma_n}$-normalizing sequence I_n takes the form

$$I_n = \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n^{-1} \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{Isom}^+(\mathbb{H}^3)$$

for some $\lambda_n \to \infty$.

Since $I_n(\widetilde{\Sigma_n})$ is a sequence of normalized almost-Fuchsian disks, by Theorem 3.4.2 there exists an $R' > 0$ such that

$$\Lambda(I_n \Gamma_n I_n^{-1}) \cap B_{\mathbb{S}^2}(\infty, R') = \emptyset. \tag{3.4.2}$$

Since $\lambda_n \to \infty$, there exists an $N \in \mathbb{N}$ such that

$$\begin{aligned} \partial_\infty(\mathbb{H}^3) \setminus I_n^{-1}(B_{\mathbb{S}^2}(\infty, R')) &= (\mathbb{C} \cup \{\infty\}) \setminus I_n^{-1}(B_{\mathbb{S}^2}(\infty, R')) \\ &\subset B_{\mathbb{S}^2}(0, R'). \end{aligned} \tag{3.4.3}$$

for all $n > N$. Applying I_n^{-1} to equation (3.4.2),

$$\Lambda(\Gamma_n) \cap I_n^{-1} B_{\mathbb{S}^2}(\infty, R') = \emptyset.$$

By (3.4.3) this implies that for all $n > N$

$$\Lambda(\Gamma_n) \subset B_{\mathbb{S}^2}(0, R'),$$

but this is impossible since $\Lambda(\Gamma_n) \to \partial_\infty(\mathbb{H}^3)$. This contradiction implies that we may always find a normalizing sequence which remains in a compact subset of $\mathrm{Isom}(\mathbb{H}^3)$. Since this case is handled by the initial arguments in the proof, the proof is complete.

□

Chapter 4

Entropy of Minimal Hyperbolic Germs

In this chapter, we begin a direct study of the space of minimal hyperbolic germs $\mathcal{H}$ through the analysis of a smooth function

$$E : \mathcal{H} \to \mathbb{R}$$

which assigns to a pair $(g, B) \in \mathcal{H}$ the topological entropy of the geodesic flow arising from the metric g. We derive a lower bound on E in terms of the L^1 norm of $\|B\|_g$ which will be crucial for applications in the subsequent chapter. In addition, we show that the restriction of E to the subset of almost-Fuchsian germs $\mathcal{AF}$ has critical points exactly at the Fuchsian germs $\mathcal{F}$ by explicitly calculating its growth rate along certain paths. Since E is critical at the Fuchsian locus, its Hessian is a well defined bilinear form on the tangent space to $\mathcal{F}$; this form is non-degenerate inducing a norm which is bounded below by 2π times the norm induced by the Weil-Petersson metric.

In § 4.1 we introduce the necessary formal background from the theory of limit sets of discrete groups acting on negatively curved metric spaces which will be utilized in this chapter and Chapter 5.

In § 4.2 we define the function E and prove the aforementioned properties.

4.1 Limit sets of discrete groups acting on CAT(-1) spaces

In this section we introduce the necessary ingredients from the theory of discrete groups acting on negatively curved spaces developed by Patterson [Pat76], Sullivan [Sul84], Bourdon [Bou95] and Coornaert [Coo93]. Details concerning generalities of CAT(-1) spaces can be found in the book by Bridson and Haefliger [BH99].

Let (X, d) be a proper CAT(-1) metric space (for our needs we may assume this to be a simply connected Riemannian manifold of sectional curvature ≤ -1). Given $p \in X$, the geometric (or visual) boundary $\partial_{p,\infty}(X)$ of X is the space of equivalence classes of geodesic rays based at X. Two geodesic rays $\gamma, \eta : [0, \infty) \to X$ parameterized by arc length are equivalent if there exists $K > 0$ such that $d(\gamma(t), \eta(t)) < K$ for all t. The *Gromov product* at p is defined by

$$(x, y)_p = \frac{1}{2}(d(x, p) + d(y, p) - d(x, y)).$$

In a proper CAT(-1) space, this product extends to the geometric boundary via

$$(\eta, \gamma)_p = \lim_{t \to \infty} (\eta(t), \gamma(t))_p$$

for $\eta, \gamma \in \partial_{p,\infty}(X)$. Using the Gromov product, we define the visual metric on the geometric boundary by

$$d_p(\eta, \gamma) = \begin{cases} e^{-(\eta,\gamma)_p} & : \eta \neq \gamma \\ 0 & : \text{else} \end{cases}$$

As $p \in X$ varies, the visual metrics are all bi-Lipschitz equivalent.

Suppose $\Gamma < \text{Isom}(X)$ is a discrete, convex-cocompact subgroup; this means there is a geodesically convex, Γ-invariant subset of X upon which Γ acts cocompactly. Define the orbit counting function associated to Γ by

$$N_\Gamma(x, R) = |\{\gamma \in \Gamma \mid d(x, \gamma(x)) < R\}|.$$

Then the volume entropy of Γ is

$$\delta(\Gamma) = \limsup_{R\to\infty} \frac{\log(N_\Gamma(R, x))}{R}.$$

This number is independent of $x \in X$ and measures the complexity of the action of the group Γ. The limit set Λ_Γ of Γ is the set of accumulation points in $\partial_\infty(X)$ of the Γ-orbit of a selected point $x \in X$. The limit set is a closed, Γ-invariant subset of $\partial_\infty(X)$. We use the following theorem (see [Coo93]).

Theorem 4.1.1. *Let $\Gamma < Isom(X)$ be a discrete, convex-cocompact subgroup of isometries of a proper $CAT(-1)$ metric space X. Then*

$$\delta(\Gamma) = H.dim(\Lambda_\Gamma).$$

Here the Hausdorff dimension is computed using any of the Gromov products on $\partial_\infty(X)$.

4.2 The entropy function

In this section, we define the entropy function on the space of minimal hyperbolic germs and discuss some interesting properties.

Given $(g, B) \in \mathcal{H}$, the Gauss equation reads:

$$K_g = -1 - \frac{1}{2}\|B\|_g^2.$$

Thus, every minimal hyperbolic germ refines the structure of a closed Riemannian surface with sectional curvature bounded above by -1. Such Riemannian surfaces are proper CAT(-1) metric spaces and the theory described in section 4.1 applies.

Definition 4.2.1. *Given $(g, B) \in \mathcal{H}$, let $B_g(p, R)$ be the metric ball in the universal cover $\widetilde{\Sigma}$ of radius R centered at a basepoint $p \in \widetilde{\Sigma}$. Define the volume entropy as the quantity*

$$E(g, B) = \limsup_{R\to\infty} \frac{\log|B_g(p, r)|}{R},$$

where $|B_g(p, R)|$ is the Riemannian volume of the ball centered at p of radius R.

Manning introduced this quantity in [Man79] and showed the limit exists and is independent of basepoint. Furthermore, in the case where the manifold in question has negative curvature, Manning showed that this quantity equals the topological entropy of the geodesic flow defined on the unit tangent bundle of Σ. Katok, Kneiper and Weiss [KKW91] show that given a C^∞-perturbation of a metric of negative curvature, the topological entropy of the geodesic flow also varies smoothly. Hence:

Proposition 4.2.2. *The volume entropy*

$$E : \mathcal{H} \to \mathbb{R}$$

is a smooth, non-negative function on the space of minimal hyperbolic germs. This function equals the topological entropy h_{top} of the geodesic flow on the unit tangent bundle.

We will simply refer to this function as the *entropy* of the minimal hyperbolic germ.

We begin with an important lower bound on the entropy.

Theorem 4.2.3. *The entropy function satisfies*

$$E(g,B) \geq \frac{1}{Vol(g)} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|_g^2}\, dV_g$$

and $E(g,B) = 1$ *if and only if* $B = 0$*. Furthermore, equality is achieved if and only if* $E(g,B) = 1$.

Before beginning the proof, we introduce an estimate of Manning which easily yields the theorem. Let (Σ, g) be a Riemannian surface with strictly negative sectional curvature. On the unit tangent bundle of Σ, the (normalized) *Liouville measure* m_L is a probability measure invariant under the geodesic flow. In a local trivialization, m_L is a constant multiple of the product of Riemannian volume on Σ with the standard angle measure on the circle giving it total measure 2π. The measure theoretic entropy of a metric of constant sectional curvature -1 with respect to Liouville measure equals $\sqrt{-2\pi\chi(\Sigma)}$.

Theorem 4.2.4 (Manning, [Man81])**.** *Let* (Σ, g) *be a Riemannian surface of negative curvature. Then*

$$\frac{1}{\sqrt{Vol(g)}} \int_\Sigma \sqrt{-K_g}\, dV_g \leq h(m_L)$$

where $h(m_L)$ *is the measure theoretic entropy of the geodesic flow with respect to Liouville measure.*

Remark: Note that if $K = -1$ in the above formula, the inequality becomes equality: $h(m_L) = \sqrt{\mathrm{Vol}(g)} = \sqrt{-2\pi\chi(\Sigma)}$.

We now give the proof of Theorem 4.2.3.

Proof. Let (g, B) be a minimal hyperbolic germ. By the Gauss equation

$$K_g = -1 - \frac{1}{2}\|B\|_g^2.$$

Applying theorem 4.2.4 and inserting the above expression for K_g, we obtain the inequality,

$$\frac{1}{\sqrt{\mathrm{Vol}(g)}} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|_g^2}\, dV_g \leq h(m_L) \tag{4.2.1}$$

where $h(m_L)$ is the measure theoretic entropy of the geodesic flow with respect to Liouville measure for the metric g. By the variational principle (see [KH95]),

$$h(m_L) \leq h_{top}\left(\frac{1}{\mathrm{Vol}(g)} g\right)$$

where $h_{top}\left(\frac{1}{\mathrm{Vol}(g)} g\right)$ is the topological entropy of the geodesic flow for the normalized Riemannian metric $\frac{1}{\mathrm{Vol}(g)} g$. But, since g has negative curvature

$$E\left(\frac{1}{\mathrm{Vol}(g)} g, B\right) = h_{top}\left(\frac{1}{\mathrm{Vol}(g)} g\right)$$

by [Man79]. Furthermore, the entropy scales via

$$E\left(\frac{1}{\mathrm{Vol}(g)} g, B\right) = \sqrt{\mathrm{Vol}(g)} E(g, B).$$

Returning to line (4.2.1) the previous lines imply,

$$\begin{aligned}
\frac{1}{\sqrt{\mathrm{Vol}(g)}} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|_g^2}\, dV_g &\leq h(m_L) \\
&\leq h_{top}\left(\frac{1}{\mathrm{Vol}(g)} g\right) \\
&= E\left(\frac{1}{\mathrm{Vol}(g)} g, B\right) \\
&= \sqrt{\mathrm{Vol}(g)} E(g, B).
\end{aligned}$$

Dividing by $\sqrt{\text{Vol}(g)}$ proves the inequality asserted in theorem 4.2.3.

If $E(g, B) = 1$, then

$$\frac{1}{\text{Vol}(g)} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|_g^2}\, dV_g \leq 1.$$

This implies $\|B\|_g^2 = 0$.

For the other direction, if $\|B\|_g^2 = 0$, then the surface has constant sectional curvature -1 and one may compute directly that $E(g, B) = 1$. In this case the volume of a ball of radius R in the universal cover is asymptotically e^R.

For the final statement, we invoke a deep theorem of Katok [Kat82]. For a closed surface of genus greater than 1, equality holds in

$$h(m_L) \leq h_{top}\left(\frac{1}{Vol(g)} g\right)$$

if and only if the metric is constant negative curvature. This completes the proof. □

Next we show that critical points of the restriction of entropy to the space of almost-Fuchsian germs occur precisely at the Fuchsian germs.

Theorem 4.2.5. *Consider the restriction of the entropy to the space of almost-Fuchsian hyperbolic germs,*

$$E : \mathcal{AF} \to \mathbb{R}.$$

This function is critical at (g, B) if and only if $B = 0$, hence if and only if the germ is Fuchsian. Furthermore, the entropy increases monotonically to first order along rays $(e^{2u_t}h, tB)$ provided $\|tB\|_{g_t}^2 < 2$. Here h is the hyperbolic metric corresponding to the germ $(h, 0)$, so $u_0 = 0$.

Before we prove the theorem, we need to introduce a very useful formula due to Katok, Knieper and Weiss [KKW91] for the first variation of the topological entropy of the geodesic flow on a manifold with negative curvature. Throughout the rest of this section, dots over a function dependent on a single real parameter $t \in \mathbb{R}$ represent successive derivatives with respect to t.

Theorem 4.2.6. *Let g_t be a smooth path of negatively curved Riemannian metrics on a closed manifold M. If $h_{top}(g_t)$ is the topological entropy of the geodesic flow on $T^1(M)$ for the metric g_t then,*

$$\frac{d}{dt} h_{top}(g_t)\big|_{t=0} = -\frac{h_{top}(g_0)}{2} \int_{T^1(M)} \frac{d}{dt} g_t(v,v)\big|_{t=0} d\mu_0.$$

Here, μ_0 is the Bowen-Margulis measure of maximal entropy for the geodesic flow arising from the metric g_0.

Remark: Since we will use none of its properties, we will not define the Bowen-Margulis measure. Details about its properties and construction can be found in [Mar04], although a considerably easier construction mirroring [Pat76] can be used for negatively curved Riemannian manifolds.

Proof of theorem 4.2.5. We already know from Theorem 4.2.3 that the entropy function is critical at Fuchsian hyperbolic germs. We have two expressions for the sectional curvature of $g_t = e^{2u_t} h$,

$$-1 - t^2 e^{-4u_t} \|B\|_h^2 = K_{g_t} = e^{-2u_t}(-\Delta_h u_t - 1)$$

where Δ_h is the Laplace-Beltrami operator associated to the metric h. Taking the time derivative and evaluating at t_0 reveals,

$$-\Delta_h \dot{u}_t = e^{2u_{t_0}}\dot{u}_t(\|t_0 B\|^2_{g_{t_0}} - 2) - t_0 e^{-2u_{t_0}}\|B\|^2_h. \tag{4.2.2}$$

At a maximum $-\Delta_h \dot{u}_t \geq 0$ which implies that the right hand side of (4.2.2) is non-negative. The hypothesis $\|t_0 B\|^2_{g_{t_0}} < 2$ implies that $\dot{u}_t \leq 0$. Futhermore, if $\dot{u}_t = 0$ everywhere then equation (4.2.2) implies that $B = 0$. We have shown that

$$\frac{d}{dt} g_t = 2\dot{u}_t g_t$$

is negative definite. Applying Theorem 4.2.6,

$$\frac{d}{dt}E(g_t, tB)|_{t=t_0} = \frac{d}{dt}h_{top}(g_t)\big|_{t=t_0} = -\frac{h_{top}(g_{t_0})}{2}\int_{T^1(M)} 2\dot{u}_t d\mu_{t_0} \geq 0$$

with equality if and only if $t_0 = 0$. This completes the proof. □

Now we show that the entropy function yields a metric on Teichmüller space $\mathcal{T}$ whose norm is bounded below by the Weil-Petersson norm. Recall that given a point $\sigma \in \mathcal{T}$, the cotangent space to $\mathcal{T}$ at σ is identified, via Kodaira-Spencer deformation theory (see [Kod05]), with the space of holomorphic quadratic differentials on the Riemann surface (Σ, σ). The uniformization theorem furnishes a unique hyperbolic metric h_σ in the conformal class of metrics defined by σ. Given two holomorphic quadratic differentials α and β, the Weil-Petersson Hermitian pairing is defined by,

$$\langle \alpha, \beta \rangle_{WP} = \int_\Sigma \frac{\alpha\overline{\beta}}{h_\sigma}.$$

This defines a Kähler metric on the Teichmüller space whose geometry has been intensely studied (for a nice survey see [Wol10]). A number of geometrically defined potential functions for the Weil-Petersson metric have been found, it seems probable, although we have not found a proof, that the entropy function defined here is yet another potential. Before we prove this theorem, we need to describe a key formula due to Pollicott [Pol94] from which the theorem will follow easily.

Theorem 4.2.7. *Let g_t be a smooth path of Riemannian metrics of negative curvature on a closed manifold M. Then,*

$$\frac{d^2}{dt^2} h_{top}(g_t)|_{t=0} \geq h(g_0) \left(Var\left(\frac{\dot{g}(v,v)}{2}\right) + 2\left(\int_{T^1(M)} \frac{\dot{g}(v,v)}{2} d\mu_0\right)^2 \right) + \\ + \left(- \int_{T^1(M)} \frac{\ddot{g}(v,v)}{2} d\mu_0 + \frac{1}{4}\int_{T^1(M)} (\dot{g}(v,v))^2 d\mu_0 \right).$$

Here μ_0 is the Bowen-Margulis measure of maximal entropy for the geodesic flow associated to the metric g_0. Further, dots refer to t derivatives evaluated at $t = 0$.

In the above, formula, we have not defined the term $\left(Var\left(\frac{\dot{g}(v,v)}{2}\right)\right)$. The reader should see [Pol94] for details and definitions, for us the only thing we will need is that $Var(0) = 0$.

Theorem 4.2.8. *The Hessian of the entropy function defines a metric on the Fuchsian space $\mathcal{F} \subset \mathcal{H}$. Furthermore, the norm of this metric is bounded below by 2π times the norm defined by the Weil-Petersson metric.*

Proof. By Theorem 4.2.5, the entropy function is critical along the Fuchsian locus $\mathcal{F}$, thus its Hessian is a well-defined non-negative quadratic form on the tangent space. Given a holomorphic quadratic differential α on a Riemann surface (Σ, σ), for small enough $t > 0$ we have the almost-Fuchsian germ $(e^{2u_t}h, t\alpha) \in \mathcal{AF}$ where h is the hyperbolic metric uniformizing (Σ, σ). Recalling (4.2.2),

$$-\Delta_h \dot{u}_t = e^{2u_{t_0}} \dot{u}_t (2\|t_0\alpha\|^2_{g_{t_0}} - 2) - 2t_0 e^{-2u_{t_0}} \|\alpha\|^2_h, \qquad (4.2.3)$$

the maximum principle implies that $\dot{u}_t = 0$ at $t = 0$. Hence, all terms in Theorem 4.2.7 vanish except for the third containing a second derivative. Differentiating (4.2.3) again with respect to t and evaluating at $t = 0$ yields

$$-\Delta_h \ddot{u}_0 = -2\ddot{u}_0 - 2\|\alpha\|_h^2.$$

Integrating with respect to the Riemannian volume form of h shows

$$\int_\Sigma \ddot{u}_0 \, dV_h = -\int_\Sigma \|\alpha\|_h^2 \, dV_h.$$

Now, letting $g_t = e^{2u_t}h$, the fact that $\dot{u}_0 = 0$ implies that

$$\ddot{g}_0 = 2\ddot{u}_0 h.$$

Moreover, the Bowen-Margulis measure for the hyperbolic metric h is simply the Liouville measure on the unit tangent bundle $T^1\Sigma$. Thus

$$\begin{aligned}\int_{T^1\Sigma} \frac{\ddot{g}_0(v,v)}{2} \, d\mu_0 &= \int_{T^1\Sigma} \ddot{u}_0 h(v,v) \, d\mu_0 \\ &= -\int_{T^1\Sigma} \|\alpha\|_h^2 \, d\mu_0 \\ &= -2\pi \int_\Sigma \|\alpha\|_h^2 \, dV_h \\ &= -2\pi \|\alpha\|_{WP}^2.\end{aligned}$$

Thus, Theorem 4.2.7 reveals

$$\frac{d^2}{dt^2} E(g_t, tB)|_{t=0} = \frac{d^2}{dt^2} h_{top}(g_t)|_{t=0} \geq 2\pi \|\alpha\|_{WP}^2$$

which completes the proof. □

Chapter 5

From Minimal Germs to Hyperbolic 3-Manifolds

This chapter studies the interplay between the geometry of minimal surfaces and the geometry of hyperbolic 3-manifolds. In § 5.2, we prove a new lower bound on the Hausdorff dimension of the limit set of a quasi-Fuchsian group in terms of the data of π_1-injective minimal surfaces in the quotient 3-manifold. The key is an application of the estimate proved in Theorem 4.2.3. In § 5.3 we compare the action of multiplication by -1 on $\mathcal{H}$ with the anti-holomorphic involution on quasi-Fuchsian space: a map

$$\Phi : \mathcal{H} \to \mathcal{R}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3))$$

which we will introduce shortly is shown to intertwine these actions when restricted to the space of almost-Fuchsian germs. The failure of Φ to be globally equivariant is explained. Lastly, we show that the set of quasi-Fuchsian representations which contain a π_1-injective minimal surface which minimizes area to second order has measure zero in $\mathcal{QF}$.

5.1 Mapping germs to representations

As recorded in Theorem 2.2.2, every $(g, B) \in \mathcal{H}$ can be integrated to an immersed minimal surface in $\mathbb{H}^3$ with induced metric and second fundamental form (g, B). Furthermore, this immersion is unique up to an isometry of $\mathbb{H}^3$. Since the data arises from tensors on a closed surface Σ, this minimal immersion is equivariant for a representation $\rho : \pi_1(\Sigma) \to \mathrm{Isom}^+(\mathbb{H}^3)$.

Given $(g, B) \in \mathcal{H}$ we first describe how to obtain a representation $\rho : \pi_1(\Sigma) \to \mathrm{Isom}^+(\mathbb{H}^3)$. Let

$$f(g, B) : \widetilde{\Sigma} \to \mathbb{H}^3$$

be an immersion described above and select $\tilde{p} \in \widetilde{\Sigma}$ such that

$$f(\widetilde{p}) = O \in \mathbb{H}^3$$
$$df(\widetilde{p})(E_i) = F_i \in T_O\mathbb{H}^3$$

where $\{E_i\}$ constitute an orthonormal frame at $\widetilde{p} \in \widetilde{\Sigma}$ and $\{F_i\}$ are an orthonormal frame at $O \in \mathbb{H}^3$. Now let $\gamma \in \pi_1(\Sigma)$. Then $f \circ \gamma$ defines a new immersion also with induced metric and second fundamental form (g, B). Thus, by Theorem 2.2.2 there exists a unique $\rho(\gamma) \in \mathrm{Isom}^+(\mathbb{H}^3)$ such that

$$f(\gamma(\widetilde{p})) = \rho(\gamma) f(\widetilde{p})$$
$$df \circ d\gamma(X) = d(\rho(\gamma)) \circ df(X)$$

for all $X \in T_{\widetilde{p}}\widetilde{\Sigma}$. This assignment defines a map

$$\Phi : \mathcal{H} \longrightarrow \mathcal{R}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3)) \tag{5.1.1}$$

where $\mathcal{R}(\pi_1(\Sigma), \mathrm{Isom}^+(\mathbb{H}^3))$ is the space of conjugacy classes of representations of $\pi_1(\Sigma)$ into $\mathrm{Isom}^+(\mathbb{H}^3)$. Note that Φ is well defined since changing a pair (g, B) by a diffeomorphism isotopic to the identity produces a conjugate representation. Taubes proved [Tau04]:

Theorem 5.1.1. *The image of Φ consists solely of irreducible representations.*

As a result of the discussion in subsection 2.1.1, the above theorem shows that the map Φ takes values in a smooth manifold.

5.2 Limit sets of quasi-Fuchsian groups

If $\rho \in \mathcal{QF}$ is a quasi-Fuchsian representation, the fiber $\Phi^{-1}(\rho)$ of the map Φ from (5.1.1) consists of $\pi_1(\Sigma)$-injective minimal immersions $\Sigma \to \rho(\pi_1(\Sigma))\backslash\mathbb{H}^3$. The general existence theorem 2.2.1 guarantees that this set is always non-empty.

We first show the dynamics of a quasi-Fuchsian representation ρ is at least as complicated as the induced dynamics on an invariant minimal surface in $\Phi^{-1}(\rho)$.

Theorem 5.2.1. *Let $\rho \in \mathcal{QF}$ be a quasi-Fuchsian representation and $(g, B) \in \Phi^{-1}(\rho)$. Then*

$$\frac{1}{Vol(g)} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|_g^2}\, dV_g \leq \text{H.dim}(\Lambda_\Gamma)$$

with equality if and only if B is identically zero.

Proof. Given $(g, B) \in \Phi^{-1}(\rho)$, let $\widetilde{\Sigma} \subset \mathbb{H}^3$ be the $\rho(\pi_1(\Sigma)) = \Gamma$-invariant minimal disk with induced metric and second fundamental form (g, B) and fix $x \in \widetilde{\Sigma}$. Given $R > 0$, let $\widetilde{N}_\Gamma(R)$ denote the number of Γ-orbits within distance R from x with the distance computed in the induced metric g. Every point distance R from x in the metric g is distance less than or equal to R in the hyperbolic metric. Thus

$$\widetilde{N}_\Gamma(R) \leq N_\Gamma(R)$$

where $N_\Gamma(R)$ is the number of Γ orbits in a ball of hyperbolic radius R centered at $x \in \mathbb{H}^3$. Taking logarithms of each side, dividing by R, and then letting $R \to \infty$, the left hand side converges to the entropy $E(g, B)$. Since quasi-Fuchsian groups are convex-cocompact, Theorem 4.1.1 implies that the right hand side converges to the Hausdorff dimension of the limit set of Γ. This proves:

$$E(g, B) \leq \text{H.dim}(\Lambda_\Gamma).$$

Applying Theorem 4.2.3,

$$\frac{1}{Vol(g)} \int_\Sigma \sqrt{1 + \frac{1}{2}\|B\|_g^2}\, dV_g \leq E(g, B) \leq \text{H.dim}(\Lambda_\Gamma).$$

Furthermore, another appeal to Theorem 4.2.3 shows that the first inequality is an equality if and only if $B = 0$. This completes the proof. □

As a corollary we obtain a new proof of Bowen's theorem on the Hausdorff dimension of quasi-circles proved in [Bow79].

Corollary 5.2.2. *A quasi-Fuchsian representation $\rho \in \mathcal{QF}$ is Fuchsian if and only if* $\text{H.dim}(\Lambda_\Gamma) = 1$.

Proof. If ρ is Fuchsian the result is immediate. Meanwhile, if $\text{H.dim}(\Lambda_\Gamma) = 1$ then Theorem 5.2.1 forces $B = 0$ for any $(g, B) \in \Phi^{-1}(\rho)$. Since there must exist some $(g, B) \in \Phi^{-1}(\rho)$, this implies that ρ leaves invariant a totally geodesic surface; thus ρ is Fuchsian. □

A final corollary is the following L^1 bound on the norm of the second fundamental form of a π_1-injective minimal surface in a quasi-Fuchsian manifold.

Corollary 5.2.3. *If* $\rho \in \mathcal{QF}$ *is a quasi-Fuchsian representation and* $(g, B) \in \Phi^{-1}(\rho)$, *then*

$$\frac{1}{\mathrm{Vol}(g)} \int_{\Sigma} \sqrt{1 + \frac{1}{2}\|B\|_g^2}\, dV_g < 2.$$

Proof. This follows from Theorem 5.2.1 since $\mathrm{H.dim}(\Lambda_\Gamma) < 2$. □

This is a necessary condition for a minimal hyperbolic germ to arise as a closed minimal surface in a quasi-Fuchsian manifold; it is unknown to what extent this condition is sufficient.

5.3 Comparing actions on $\mathcal{QF}$ and $\mathcal{H}$.

The space $\mathcal{QF}$ of quasi-Fuchsian representations possesses an anti-holomorphic involution ι which acts by,

$$\iota(X, \overline{Y}) = (Y, \overline{X})$$

where $(X, \overline{Y}) \in \mathcal{T} \times \overline{\mathcal{T}} \simeq \mathcal{QF}$. Meanwhile, the space of minimal germs $\mathcal{H}$ also carries an involution given by the restriction of the U(1)-action to multiplication by -1 sending a germ (g, B) to $(g, -B)$. The following theorem shows that these actions are actually intertwined, at least on the almost-Fuchsian germs.

Theorem 5.3.1. *For all* $(g, B) \in \mathcal{AF}$,

$$\Phi((g, -B)) = \iota \circ \Phi((g, B)).$$

where Φ *is the map defined in 5.1.1*

Proof. Theorem 2.2.6 explicitly expresses the almost-Fuchsian metric on $\Sigma \times \mathbb{R}$ corresponding to the germ $(g, B) \in \mathcal{AF}$. Fock [Foc07] discovered a complex analytic way to express this metric. Namely, write $g = e^{2u}|dz|^2$ in conformal coordinates and let $\alpha = \Psi((g, B))$ be the quadratic differential whose real part is B. Then

$$G((g, B)) = dt^2 + e^{2u}|\cosh(t)dz + \sinh(t)e^{-2u}\overline{\alpha}d\overline{z}|^2$$

expresses the almost-Fuchsian metric (expanding out this expression, it is equal to the one appearing in Theorem 2.2.6). The Beltrami differential

$$\mu = e^{-2u}\overline{\alpha}$$

has the property that the metrics

$$|dz \pm \mu d\overline{z}|^2$$

furnish conformal metrics on the two components of the domain of discontinuity for the almost-Fuchsian group corresponding to (g, B). Now, sending (g, B) to $(g, -B)$ sends α to $-\alpha$ which changes μ to $-\mu$. Hence, the mapping ι interchanges the conformal structures on the domain of discontinuity. This completes the proof. □

If the mapping Φ were an immersion everywhere, then an analytic continuation argument would show that these actions intertwine on the whole quasi-Fuchsian space. However, as we now explain this is not the case and is responsible for some very complicated bifurcation behavior.

A minimal hyperbolic germ is non-degenerate if the second variation of area has zero nullity. More precisely, given the minimal immersion $f : \widetilde{\Sigma} \to \mathbb{H}^3$ corresponding to a minimal hyperbolic germ $(g, B) \in \mathcal{H}$, let ν be a unit normal vector field to the image of f and let $u \in C^\infty(\Sigma)$. Then lifting u periodically to the universal cover $\widetilde{\Sigma}$, take a normal variation f_t of f such that

$$\frac{d}{dt} f_t|_{t=0} = u\nu.$$

Then the well known formula for the second variation of area (see [CM11]) gives

$$\frac{d^2}{dt^2} Area(f_t^* G_{\mathbb{H}^3})|_{t=0} = \int_\Sigma -u\Delta_g u - (\|B\|_g^2 - 2)u^2 dV_g.$$

Here $G_{\mathbb{H}^3}$ is the Riemannain metric on $\mathbb{H}^3$. A minimal hyperbolic germ is non-degenerate if and only if the *Jacobi operator*

$$L_{(g,B)} = -\Delta_g - (\|B\|_g^2 - 2)$$

has no non-zero eigenfunctions with eigenvalue 0, that is there are no non-zero solutions to $L_{(g,B)} u = 0$. Solutions to $L_{(g,B)} u = 0$ are called *Jacobi fields.*

The following theorem of Taubes [Tau04] shows the special significance of degenerate minimal germs.

Theorem 5.3.2. *The vector space of Jacobi fields for the operator $L_{(g,B)}$ is in bijection with the kernel of the differential of the map*

$$\Phi : \mathcal{H} \to \mathcal{R}(\pi_1(\Sigma), PSL(2, \mathbb{C}))$$

at the germ (g, B).

An argument using the continuity method in [Uhl83] shows that there do exist minimal germs for which the associated Jacobi operator admits nontrivial zero eigenfunctions. Using the above theorem, an application of Sard's theorem shows that in terms of representations, this phenomena is non-generic.

Theorem 5.3.3. *The set of quasi-Fuchsian representations* $\rho : \pi_1(\Sigma) \to Isom^+(\mathbb{H}^3)$ *which contain a degenerate closed minimal surface of genus* g *is measure zero.*

Proof. By Theorem 5.3.2, a quasi-Fuchsian representation is a critical value of the map Φ if and only if it contains a degenerate closed minimal surface of genus g. Since Φ is a map between the smooth manifolds (this is a consequence of Theorem 5.1.1), Sard's theorem applies and the set of critical values of the smooth map Φ has measure zero. $\square$

Chapter 6

Higgs Bundles and Deformations of Quasi-Fuchsian Groups

This chapter is somewhat different in flavor than the preceding ones. Here we discuss the interaction between the theory of minimal surfaces and the complex analytic theory of Higgs bundles. We begin by reviewing a construction of Donaldson which produces out of a minimal hyperbolic germ $(g, B) \in \mathcal{H}$ a solution to the self-duality equations. Via the non-abelian Hodge correspondence, these correspond to representations of the fundamental group $\pi_1(\Sigma)$ into $\mathrm{SL}(2, \mathbb{C})$. We show that this mapping coincides with the more synthetic mapping Φ constructed in the previous chapter. Then, we use this setting to compare deformations of a Fuchsian group arising from hyperbolic geometry, namely bending and shearing along totally geodesic planes, with those arising from a ray $(g_t, tB) \in \mathcal{H}$ of minimal hyperbolic germs. An explicit formula is obtained for the initial tangent vector in the direction of the ray (g_t, tB) in terms of tangent vectors to the Fuchsian group. Lastly, we compute the length of the U(1)-orbit of a Higgs bundle arising from a minimal hyperbolic germ in the hyperkähler metric on the space of solutions to the self-duality equations.

6.1 From minimal germs to Higgs bundles

We begin this chapter with a description of another map

$$\overline{\Phi} : \mathcal{H} \to \mathcal{R}(\Sigma, \mathrm{PSL}_2(\mathbb{C})).$$

Our description is due to Donaldson [Don03] and uses Hitchin's theory of Higgs bundles [Hit87]. Given a minimal hyperbolic germ $(g, B) \in \mathcal{H}$, consider its image $([g], \alpha)$ under the map $\Psi : \mathcal{H} \to T^*\mathcal{T}$ where $[g]$ denotes the conformal class of the metric g; as usual let $X = (\Sigma, [g])$ denote the Riemann surface. Given the canonical bundle of holomorphic 1-forms K_X, a choice of spin structure on X is equivalent to the choice of a square root $K_X^{\frac{1}{2}}$ of the canonical bundle. We will consider a holomorphic structure $\overline{\partial}_E$ on a rank-2 complex vector bundle E over X. The holomorphic bundle arises as an extension of $K_X^{\frac{1}{2}}$ by $K_X^{-\frac{1}{2}}$,

$$1 \to K_X^{-\frac{1}{2}} \to (E, \overline{\partial}_E) \to K_X^{\frac{1}{2}} \to 1.$$

The metric g induces a Hermitian metric h on K_X^{-1} and hence on the bundles $K_X^{-\frac{1}{2}}$ and $K_X^{\frac{1}{2}}$. Let a and a^- denote the Chern connections for these metrics (see appendix A.1 for the definition of h and important related conventions). This pair of metrics equips E with a Hermitian metric H and induces a smooth orthogonal splitting of the smooth bundle $E \simeq K_X^{-\frac{1}{2}} \oplus K_X^{\frac{1}{2}}$. With respect to this splitting, the unitary connection A given by

$$A = \begin{pmatrix} a & \frac{1}{4}\alpha^* \\ -\frac{1}{4}\alpha & a^- \end{pmatrix}$$

induces a holomorphic structure on E. This holomorphic structure is classified by the extension class $\frac{1}{4}\alpha^* \in H^1(X, \mathrm{Hom}(K_X^{\frac{1}{2}}, K_X^{-\frac{1}{2}})) \simeq H^0(X, K_X^2)^*$ with the isomorphism given by Serre duality. Writing the metric $g = e^{2u}|dz|^2$ in local conformal coordinates, the form α^* is defined by

$$\alpha^* = 4e^{-2u}\overline{\alpha}.$$

Finally, consider the holomorphic section $\phi \in H^0(X, K_X \otimes \mathrm{End}_0(E))$ given by the matrix

$$\phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where 1 represents the canonical section of the bundle $K_X \otimes \mathrm{End}(K_X^{\frac{1}{2}}, K_X^{-\frac{1}{2}}) = \mathcal{O}_X$ which is the trivial line bundle. Donaldson recognized (see [Don03]) that the Gauss equation (2.2.2) is equivalent to the pair (A, ϕ) satisfying Hitchin's self-duality equations,

$$F(A) = -[\phi, \phi^{*_H}] \tag{6.1.1}$$

$$\overline{\partial}_A \phi = 0$$

where the endomorphism ϕ^{*_H} is the adjoint of ϕ relative to the Hermitian metric H. The holomorphicity of ϕ is immediate. The first equation (6.1.1) reads,

$$\begin{pmatrix} F(a) - \frac{1}{16}\alpha^* \wedge \alpha & 0 \\ 0 & F(a^-) - \frac{1}{16}\alpha \wedge \alpha^* \end{pmatrix} = \begin{pmatrix} -hdz \wedge d\overline{z} & 0 \\ 0 & hdz \wedge d\overline{z} \end{pmatrix}. \tag{6.1.2}$$

In the Appendix we recall that $F(a) = \frac{1}{2}F(\nabla^h)$ where ∇^h is the Chern connection of h. Furthermore, $iF(\nabla^h) = K_g dV_g$. Expanding out the formula in (6.1.2) using the previous identities shows that the left and right hand side of (6.1.2) are equal if and only if the *Gauss* equation

$$K_g = -1 - \frac{1}{2}\|B\|_g^2$$

holds.

Let $X \simeq \Sigma$ be a Riemann surface and E a rank 2, degree 0 complex vector bundle over X. Given a unitary connection A on E preserving a Hermitian metric

H, the holomorphic bundle $(V, \overline{\partial}_A)$ is said to have *trivial determinant* if the induced connection on the determinant line bundle is just the exterior differential d acting on complex valued functions on X. The rank 2, degree 0 moduli space of solutions to the self-duality equations with trivial determinant $\mathcal{M}(X)$ is defined to be the space of unitary gauge equivalence classes of pairs (A, ϕ) where A is a unitary connection on E, $(E, \overline{\partial}_A)$ has trivial determinant, and $\phi \in H^0(X, K_X \otimes \mathrm{End}_0(E))$ is a holomorphic endomorphism-valued 1-form such that

$$F(A) = -[\phi, \phi^{*_H}].$$

The bundle $\mathrm{End}_0(E)$ consists of traceless endomorphisms of E. The field ϕ is called the *Higgs field*. A pair $(A, \phi) \in \mathcal{M}(X)$ is called a Higgs bundle. For details about this space and relations with complex geometry the initial sections of [DWW10] give a nice introduction. The original paper of Hitchin [Hit87] is still a premier resource.

That the pair (A, ϕ) satisfies the self-duality equations is equivalent to the flatness of the linear connection

$$B = A + \phi + \phi^{*_H}.$$

The holonomy of this flat connection defines a homomorphism,

$$\rho : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C}).$$

The association $(A, \phi) \mapsto B$ defines a mapping

$$\mathcal{M}(X) \to \mathcal{R}(\Sigma, \mathrm{SL}_2(\mathbb{C}))$$

which is commonly called the *non-abelian Hodge correspondence* (see [DW07] or [DWW10] for a definitive statement).

Gauge equivalent connections define conjugate representations so the above mapping is well defined. Thus the previous construction yields a map

$$\overline{\Phi} : \mathcal{H} \to \mathcal{R}(\Sigma, \mathrm{PSL}_2(\mathbb{C}))$$

where we compose the representation with the projection

$$\mathrm{SL}(2, \mathbb{C}) \to \mathrm{PSL}(2, \mathbb{C}).$$

Different choices of spin structure yield different representations in $\mathrm{SL}(2, \mathbb{C})$, but they all project to the same representation into $\mathrm{PSL}(2, \mathbb{C})$. The space $\mathcal{M}(X)$ is equipped with a circle action which sends $(A, \phi) \to (A, e^{i\theta}\phi)$. Define the map

$$\Xi : \mathcal{H} \to \coprod_{X \in \mathcal{T}} \mathcal{M}(X)$$

by the above recipe.

Proposition 6.1.1. *The map Ξ is $U(1)$-equivariant.*

Proof. The image of $([g], e^{i\theta}\alpha)$ under the map Ξ consists of the pair

$$A_\theta = \begin{pmatrix} a & \frac{e^{-i\theta}}{4}\alpha^* \\ -\frac{e^{i\theta}}{4}\alpha & a^- \end{pmatrix}$$

and

$$\phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Consider the unitary gauge transformation

$$U_\theta = \begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{-\frac{i\theta}{2}} \end{pmatrix}.$$

Conjugating we see that

$$A_0 = U_\theta A_\theta U_\theta^{-1}$$

and

$$e^{i\theta}\phi = U_\theta \phi U_\theta^{-1}.$$

Hence (A_θ, ϕ) is equivalent to $(A_0, e^{i\theta}\phi)$ which proves that the map is $U(1)$-equivariant.

□

In the previous chapter we constructed a map Φ from $\mathcal{H}$ to the space of representations $\mathcal{R}(\Sigma, \mathrm{Isom}^+(\mathbb{H}^3))$. It is well-known that orientation-preserving isometries of $\mathbb{H}^3$ identify with the group $\mathrm{PSL}(2, \mathbb{C})$. After this identification, the maps Φ and $\overline{\Phi}$ agree.

Theorem 6.1.2. *The pair of maps* $\Phi : \mathcal{H} \to \mathcal{R}(\Sigma, Isom^+(\mathbb{H}^3))$ *and* $\overline{\Phi} : \mathcal{H} \to \mathcal{R}(\Sigma, PSL(2, \mathbb{C}))$ *are equal.*

Proof. By definition, $\Phi(g, B) = \rho$ is a representation for which a minimal embedding of a disk $f : \widetilde{\Sigma} \to \mathbb{H}^3$ with induced metric and second fundamental form (g, B) is equivariant.

Given the same $(g, B) \in \mathcal{H}$, we need to show that $\overline{\Phi}(g, B)$ is also equal to the representation ρ. This is explained in [Don03] but we produce a sketch of the argument here. Given the pair (A, ϕ) constructed in this chapter, the connection

$$A + \phi + \phi^*$$

is flat and the holonomy of this flat connection projects to the representation $\rho' =$

$\overline{\Phi}(g, B)$. By [Don87], there exists a unique ρ'-equivariant harmonic map

$$f : (\widetilde{\Sigma}, [g]) \to \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$$

where we identify

$$\mathbb{H}^3 = \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2).$$

The pair (A, ϕ) is constructed out of this map f. The special form of (A, ϕ) implies that f is a conformal harmonic immersion with second fundamental form B. Hence, the mapping f is equivariant for both ρ and ρ'. The proof is finished by the following simple lemma. □

Lemma 6.1.3. *Let $f : \widetilde{\Sigma} \to \mathbb{H}^3$ be an immersion. If f is equivariant for two representations ρ and ρ' from $\pi_1(\Sigma)$ into Isom$^+(\mathbb{H}^3)$, then $\rho = \rho'$.*

Proof. Let $f : \widetilde{\Sigma} \to \mathbb{H}^3$ be such a mapping. Then,

$$\rho(\gamma)\widetilde{f}(p) = \widetilde{f}(\gamma(p)) = \rho'(\gamma)\widetilde{f}(p)$$

for all $p \in \widetilde{\Sigma}$. Thus, for all $p \in \widetilde{\Sigma}$,

$$\rho'(\gamma)^{-1}\rho(\gamma)\widetilde{f}(p) = \widetilde{f}(p).$$

Since f is an immersion, its image is a 2-dimensional submanifold. A hyperbolic isometry which stabilizes a two dimensional submanifold is the identity, thus $\rho'(\gamma)^{-1}\rho(\gamma) = Id$. Hence $\rho = \rho'$ and ρ is in the image of Φ. □

6.1.1 Relation to bending deformations

Because we can faithfully represent an open neighborhood of Fuchsian representations in $\mathcal{QF}$ via minimal surfaces, it is reasonable to try to compare deformations arising from the minimal surface theory with the more hands-on deformations built by bending and shearing Fuchsian representations along geodesic laminations. We begin by reviewing the latter construction.

Given a Fuchsian representation $\rho : \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{R})$ and the corresponding equivariant totally geodesic plane $\mathbb{H}^2 \to \mathbb{H}^3$, we wish to build a path of quasi-Fuchsian representations called a *bending deformation* of ρ. Equipping Σ with the hyperbolic structure induced by ρ, let c be a homotopically non-trivial oriented simple closed geodesic on Σ. Taking the complete set of lifts of c to the metric universal cover of Σ one obtains a ρ-equivariant discrete set of geodesics on $\mathbb{H}^2$. Standing along one of the lifts $\widetilde{c}$ of c, the orientation of c plus the orientation of the surface gives a well defined notion of the region to the left and to the right of $\widetilde{c}$. Given a small real number $t > 0$, one builds a piecewise geodesic plane in $\mathbb{H}^3$ by the following recipe: every time you are standing facing along a lift of c, bend the part of the plane to the right counter-clockwise through an angle t. Doing this iteratively for each lift of c, the resulting object is a convex, piecewise geodesic surface, an example of a *pleated surface*, which is bent along the *discrete geodesic lamination* given by the closed curve c. The bending gives a measure on arcs transverse to the lamination which assigns mass t every time an arc crosses the curve c. For t small enough, there is a unique path of quasi-Fuchsian representations ρ_t for which this new piecewise geodesic surface is equivariant. That is, there is a path-isometric continuous (not smooth) mapping

$$f_t : \mathbb{H}^2 \to \mathbb{H}^3$$

such that f_t is ρ_t equivariant.

More generally, a measured geodesic lamination on a hyperbolic surface is a closed subset which is a disjoint union of complete simple geodesics (the leaves of the lamination) equipped with a positive Borel measure on (isotopy classes of) arcs transverse to the leaves of the lamination. It is well known (see [Sig74]) that every geodesic lamination can be approximated (in the weak* topology) by a sequence of discrete measured geodesic laminations. Bending deformations can be defined for an arbitrary measured lamination via an approximation procedure.

Complementary to bending is the notion of *twisting* and, more generally, that of an *earthquake* deformation. Again, let c be a essential simple closed geodesic on a hyperbolic surface Σ and let $t > 0$ be a real number. Cutting Σ open along c, rotating the component to the right of c to the right by distance t and then regluing yields a new hyperbolic surface said to be obtained from the original surface by a *right twist* along c. As with bending, these twisting deformations can be extended via approximation to define a *right earthquake* along a general measured lamination. An infinitesimal form of the earthquake theorem due to Thurston (unpublished) and Kerckhoff [Ker83] is recorded in the following theorem.

Theorem 6.1.4. *Every tangent vector to the space of Fuchsian representations is tangent to a unique right earthquake path.*

There is a beautiful relationship between bending deformations and earthquake deformations which involves the complex structure of quasi-Fuchsian space, denoted by $J : T\mathcal{QF} \to T\mathcal{QF}$.

Theorem 6.1.5 (see [Bon96]). *Let $X \in T\mathcal{QF}$ be a tangent vector to Fuchsian space and λ the measured geodesic lamination such that X is obtained by the infinitesimal right earthquake along λ. Then $J(X) \in T\mathcal{QF}$ is the tangent vector corresponding to infinitesimal bending along λ.*

The setting of minimal germs provides another interesting method of obtaining deformations of Fuchsian groups. Namely, let $(e^{2u_t}h, tB) \in \mathcal{H}$ be a path of minimal hyperbolic germs where at $t = 0$ we have the Fuchsian germ $(h, 0)$ with h a hyperbolic metric on Σ. This path is guaranteed to exist by Theorem 2.2.5. In certain ways, this deformation has similarities to bending deformations, we will explore some of these connections here. We begin by noting that both deformations are curve-shortening.

Proposition 6.1.6. *Let ρ be a Fuchsian representation and let $X \in T_{[\rho]}\mathcal{QF}$ be either tangent to a bending path or to a path of the form $(e^{2u_t}h, tB)$. Let ρ_t be the representations corresponding to these paths and let $\gamma \in \pi_1(\Sigma)$. If $\ell_t(\gamma)$ is the length of the unique closed geodesic homotopic to γ, then*

$$\frac{d}{dt}\ell_t(\gamma)|_{t=0} \leq 0.$$

Proof. If ρ_t is a bending path, then each of the quasi-Fuchsian manifolds M_{ρ_t} contains a closed surface which is path isometric to a single hyperbolic surface, namely that one which is uniformized by ρ_0. This is because the bent piecewise-geodesic plane which ρ_t leaves invariant does not change intrinsically. Hence,

$$\ell_t(\gamma) \leq \ell_0(\gamma)$$

which implies that

$$\frac{d}{dt}\ell_t(\gamma)|_{t=0} \leq 0.$$

If instead ρ_t is a path given by the almost-Fuchsian germs $(e^{2u_t}h, tB)$, then (the proof of) Theorem 4.2.6 implies $u_t \leq 0$ and $\dot{u}_t \leq 0$. Just as above,

$$\ell_t(\gamma) \leq \ell_0(\gamma)$$

completing the proof. □

Despite some similarity in the qualitative features of these deformations, we now show that the initial tangent vector we get from the almost-Fuchsian deformation is not a pure bending vector; nonetheless there is an interesting relationship.

In order to express the result we need to delve more deeply into the properties of the moduli space of solutions to the self duality equations. Let $(h, 0) \in \mathcal{F}$ be a Fuchsian germ, we consider the rank 2, degree 0 moduli space $\mathcal{M}([h])$ of solutions to the self-duality equations with trivial determinant over the Riemann surface $(\Sigma, [h])$.

The space of unitary connections with trivial determinant on a complex vector bundle E is an affine space modelled on the vector space $\Omega^{0,1}(X, End_0(E))$ since the $(0, 1)$-part determines the full connection form. Thus the tangent space to the moduli space $\mathcal{M}([h])$ at a pair (A, ϕ) is a certain subspace of the infinite dimensional complex vector space

$$\Omega^{0,1}(X, \mathrm{End}_0(E)) \oplus \Omega^{1,0}(X, \mathrm{End}_0(E)) \tag{6.1.3}$$

with the complex structure just given by componentwise multiplication by the imaginary unit i. This space carries a Hermitian pairing

$$G((\Phi,\Psi),(\Phi,\Psi)) = 2i\int_{\Sigma} Tr(\Phi^*\Phi + \Psi\Psi^*).$$

The tangent space is given by those pairs $(\dot{A},\dot{\phi}) \in \Omega^1(X,\mathrm{End}(V))\oplus\Omega^{(1,0)}(X,\mathrm{End}(V))$ which satisfy the trio of equations,

$$d_A\dot{A} + [\dot{\phi},\phi^*] + [\phi,\dot{\phi}^*] = 0,$$

$$\overline{\partial}_A\dot{\phi} + [\dot{A}^{(0,1)},\phi] = 0,$$

$$d_A^*\dot{A} + \mathrm{Re}[\phi^*,\dot{\phi}] = 0,$$

for a representative pair $(A,\phi) \in \mathcal{M}([h])$. The first two equations above linearize the self-duality equations, while the third equation expresses the G-orthogonality of $(\dot{A},\dot{\phi})$ to the gauge orbit.

This vector space actually admits an action of the quaternions with respect to which the moduli space (at least its smooth part) becomes a hyperkähler manifold. For us, the important other complex structure will be denoted J, and its action relative to the splitting (6.1.3) is given by

$$J(\dot{A},\dot{\phi}) = (i\dot{\phi}^*, -i\dot{A}^*). \tag{6.1.4}$$

The import of this lies in the following theorem of Hitchin [Hit87].

Theorem 6.1.7. *The restriction of the non-abelian Hodge correspondence to the quasi-Fuchsian space is a holomorphic embedding*

$$(\mathcal{QF},J) \to (\mathcal{M}([h]),J)$$

where $\mathcal{M}([h])$ *is equipped with the complex structure* J *from 6.1.4.*

Remark: In fact, $\mathcal{M}([h])$ is homeomorphic to the space of conjugacy classes of all reductive representations from $\pi_1(\Sigma)$ into $\mathrm{SL}_2(\mathbb{C})$. We word the above theorem in this way since we are only interested in studying quasi-Fuchsian representations.

We must be careful here because we have been forced to fix a conformal structure on the surface Σ. That provided, we now have a formula for examining the effect of the complex structure of quasi-Fuchsian space. We also must utilize Hitchin's description of the Fuchsian space in terms of solutions to the self-duality equations; let $H^0(X, K_X^2)$ be the space of holomorphic quadratic differentials on X.

Theorem 6.1.8 (Hitchin, [Hit87]). *Every solution of the self-duality equations which corresponds to a Fuchsian representation consists of a $U(1)$-connection on $K_X^{-\frac{1}{2}} \oplus K_X^{\frac{1}{2}}$ with a Higgs field of the form*

$$\phi = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}.$$

for some $\alpha \in H^0(X, K_X^2)$.

Now consider the path of Fuchsian Higgs bundles $(B(t), \phi(t))$ with

$$\phi(t) = \begin{pmatrix} 0 & 1 \\ t\alpha & 0 \end{pmatrix},$$

and $B(t)$ the requisite U(1)-connection provided by Theorem 6.1.8. We need the following lemma.

Lemma 6.1.9. *The connection $B(t)$ above satisfies:*

$$\frac{d}{dt} B(t)|_{t=0} = 0.$$

Proof. As $B(t)$ is a U(1) connection, it has the form

$$B(t) = \begin{pmatrix} b(t) & 0 \\ 0 & b^{-}(t) \end{pmatrix}.$$

Let $h(t)$ be the Hermitian metric on K_X^{-1} preserved by $2b(t)$. If $F(t)$ denotes the curvature of the Chern connection of $h(t)$, then the self-duality equations become

$$iF(t) = -2i\left(1 - t^2\frac{\alpha\overline{\alpha}}{h(t)^2}\right)h(t)dz \wedge d\overline{z}. \tag{6.1.5}$$

Now, $iF(t) = K_t dV(t)$ where K_t is the Gauss curvature of the Riemannian metric $g(t)$ defined by $h(t)$ and $V(t)$ its volume form. Let h_0 be the unique hyperbolic metric in the conformal class defined by the Riemann surface X, then $g(t) = e^{2u_t}h_0$. Then,

$$K_t = -e^{2u_t}(\Delta_{h_0}u_t + 1).$$

The self-duality equations (6.1.5) become

$$-e^{2u_t}(\Delta_{h_0}u_t + 1) = -(1 - t^2e^{-4u_t}\|\alpha\|_{h_0}^2). \tag{6.1.6}$$

Since $g(0) = h_0$ we have $u_0 = 0$. Differentiating (6.1.6) and evaluating at $t = 0$ yields

$$2\dot{u}_0 - \Delta_{h_0}\dot{u}_0 = 0.$$

The maximum principle implies that $\dot{u}_0 = 0$ which implies that $\dot{h}(0) = \frac{\dot{g}(0)}{4} = 0$. Since the Chern connection of h depends on h and its first derivatives, and since h vanishes to first order, so does the connection $B(t)$. This completes the proof. □

As a result of the above lemma, the path $(B(t), \phi(t))$ has initial tangent vector at $t = 0$,

$$X = \left(0, \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}\right). \tag{6.1.7}$$

Applying the complex structure J from (6.1.4) to this tangent vector yields

$$J\left(0, \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}\right) = \left(i\begin{pmatrix} 0 & \alpha^* \\ 0 & 0 \end{pmatrix}, 0\right). \tag{6.1.8}$$

By Theorem 6.1.4 the initial tangent vector X is the infinitesimal right earthquake along a unique measured lamination, so Theorem 6.1.5 implies that JX is given by infinitesimal bending along the same lamination.

Now consider the path of almost-Fuchsian Higgs bundles given by

$$\left(A(t) = \begin{pmatrix} a(t) & \frac{t}{4}\alpha^* \\ -\frac{t}{4}\alpha & a^-(t) \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \tag{6.1.9}$$

which we constructed at the beginning of the chapter. The following lemma allows us to compute the initial tangent vector to this path as well.

Lemma 6.1.10. *Let $A(t)$ be as above. Then,*

$$\frac{d}{dt}A(t)|_{t=0} = \begin{pmatrix} 0 & \frac{1}{4}\alpha^* \\ -\frac{1}{4}\alpha & 0 \end{pmatrix}$$

Proof. Remember that in this context $a(t)$ is one half the Chern connection corresponding to the metric $g(t) = e^{2u_t}h$ where $g(t)$ is the induced metric coming from the minimal immersion with induced metric $g(t)$ and second fundamental form corresponding to the holomorphic quadratic differential $t\alpha$. The proof of Theorem 4.2.5

implies $\dot{u}_0 = 0$ whereby $\dot{g}(0) = 0$. Since $a(t)$ is a constant multiple of the Chern connection associated to $g(t)$, as before the vanishing to first order of $g(t)$ at $t = 0$ implies that $\dot{a}(0) = 0$. Since $a^-(t) = -a(t)$, we conclude that the full variation of connection is

$$\frac{d}{dt}A(t)|_{t=0} = \begin{pmatrix} 0 & \frac{1}{4}\alpha^* \\ -\frac{1}{4}\alpha & 0 \end{pmatrix},$$

completing the proof. □

By the previous lemma, the initial tangent vector to the almost-Fuchsian path from (6.1.9) is given by,

$$Y = \left(\begin{pmatrix} 0 & \frac{1}{4}\alpha^* \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -\frac{1}{4}\alpha & 0 \end{pmatrix} \right) \tag{6.1.10}$$

with respect to the splitting in (6.1.3). Finally, the following proposition compares the directions of these deformations.

Proposition 6.1.11. *The ray of almost-Fuchsian representations corresponding to the data $\gamma(t) = (g_t, t\alpha) \in \mathcal{AF}$ is not initially tangent to a pure bending direction. Moreover, if X is the tangent vector to the Fuchsian space from* (6.1.7)*) and $Y = \dot{\gamma}(0)$ (from* (6.1.10)*), then they are related by the formula,*

$$-4Y = X + IJX = X + KX$$

where I, J and K are the complex structures yielding the hyperkähler structure on the Higgs bundle moduli space.

Remark: The complex structure K on the Higgs bundle moduli space is quite mysterious and geometric incarnations of it rather elusive. The above proposition shows it indeed has geometric significance, namely, twisting minus K(twisting) yields a deformation along a ray of almost-Fuchsian representations. Unfortunately, in this case there seems to be no simple relationship between the lamination giving rise to the vector X and any properties of the deformation Y.

Proof. The value of JX from line (6.1.8) shows that

$$-IJX = -KX = \left(\begin{pmatrix} 0 & \alpha^* \\ 0 & 0 \end{pmatrix}, 0 \right).$$

Comparing the value of X from line (6.1.7) and the value of Y from line (6.1.10) shows that

$$-4Y = X + KX$$

which completes the proof. □

6.1.2 Geometry of moduli of Higgs bundles via minimal germs

The explicit formula for the almost-Fuchsian Higgs bundles lets us compute some geometric quantities coming from the hyperkähler metric. Specifically, given a tangent vector $(\Phi, \Psi) \in T\mathcal{M}([h])$ recall the Hermitian pairing,

$$G((\Phi, \Psi), (\Phi, \Psi)) = 2i \int_\Sigma Tr(\Phi^*\Phi + \Psi\Psi^*).$$

This defines a Riemannian metric on $\mathcal{M}([h])$ which together with the complex structures I, J and K defines the hyperkähler structure. Consider the loops of Higgs bundles defined by,

$$\gamma(\theta) = \left(\begin{pmatrix} a & \frac{e^{-i\theta}}{4}\alpha^* \\ -\frac{e^{i\theta}}{4}\alpha & a^- \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right).$$

Recall that by 6.1.1 this path is tangent to the $U(1)$-orbit of the Higgs bundle corresponding to $\gamma(0)$. We can compute the length of γ and find that,

Proposition 6.1.12. *The length of the $U(1)$-orbit of the Higgs bundle*

$$\gamma(0) = \left(\begin{pmatrix} a & \frac{1}{4}\alpha^* \\ -\frac{1}{4}\alpha & a^- \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$$

is $2\pi\sqrt{2}\left(-2\pi\chi(\Sigma) - Area(g)\right)^{\frac{1}{2}}$. *Here,* $-\chi(\Sigma) = 2 \times (genus) - 2$ *is the Euler characteristic of* Σ.

Proof. The U(1)-action is isometric so we may compute just the initial tangent vector. Recalling that the circle action leaves the induced metric invariant (hence the Chern connections a and a^- are constant) yields

$$\dot{\gamma}(0) = \left(\begin{pmatrix} 0 & \frac{-i}{4}\alpha^* \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \frac{i}{4}\alpha & 0 \end{pmatrix} \right).$$

Direct computation shows that the length of $\dot{\gamma}(0)$ is,

$$\left(2i \int_\Sigma \frac{1}{8}\alpha \wedge \alpha^*\right)^{\frac{1}{2}} = \left(2 \int_\Sigma \|\alpha\|_g^2 dV_g\right)^{\frac{1}{2}}. \tag{6.1.11}$$

Denoting the Riemannian metric on Σ by g, the Gauss equation reads

$$\|\alpha\|_g^2 = -K_g - 1.$$

Thus an application of the Gauss-Bonnet theorem combined with (6.1.11) reveals the length of $\dot{\gamma}(0)$ is,

$$\left(2 \int_\Sigma -K_g - 1 dV_g\right)^{\frac{1}{2}} = \sqrt{2}\left(-2\pi\chi(\Sigma) - \text{Area}(g)\right)^{\frac{1}{2}}.$$

Thus, the full orbit length is the above quantity multiplied by 2π which completes the proof. □

It is interesting to remark that the inequality $\text{Area}(g) \leq -2\pi\chi(\Sigma)$, which is a direct consequence of the Gauss equation, is a corollary of the above theorem since the length must be positive.

A.1 Conventions for Hermitian metrics

Due to the large number of conventions concerning various normalizations of Hermitian metrics derived from Riemannian metrics, in this appendix we will quickly make explicit the choices made in the present paper. Ours differ, for example, from those taken by Hitchin in [Hit87].

If J is an (almost) complex structure on a closed, oriented surface Σ, let $X = (\Sigma, J)$ be the compact Riemann surface which J defines and pick a metric $g = g|dz|^2$ compatible with J. The complexification of the real tangent bundle to Σ splits as a direct sum of eigenspaces of J;

$$T\Sigma \otimes_{\mathbb{R}} \mathbb{C} \simeq T_{(1,0)}\Sigma \oplus T_{(0,1)}\Sigma$$

Now, $T_{(1,0)}\Sigma$ is a complex line bundle holomorphically equivalent to the holomorphic tangent bundle K_X^{-1} (usually called the anti-canonical bundle). Taking the real part defines a bundle isomorphism,

$$\mathrm{R} : T_{(1,0)}\Sigma \to T\Sigma. \tag{A.1.1}$$

For example,

$$\mathrm{R}\left(\frac{\partial}{\partial z}\right) = \frac{1}{2}\frac{\partial}{\partial x}$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

The map R is equivariant for the action of the imaginary unit on the fibers of $T_{(1,0)}\Sigma$ and the action of J on the fibers of $T\Sigma$. That is to say, the mapping R identifies $T_{(1,0)}\Sigma$ and $T\Sigma$ as holomorphic vector bundles over X.

On $T\Sigma$, define the Hermitian pairing

$$h(x, y) = g(X, Y) - ig(J(X), Y)$$

whose imaginary part is the Kähler form of the Kähler manifold (Σ, J, g). The mapping R from (A.1.1) is made fiberwise unitary by defining a Hermitian pairing q on $T_{(1,0)}\Sigma$ via the recipe

$$q(U, V) = h(\mathrm{R}(U), \mathrm{R}(V)).$$

Observe that with these conventions, the metrics q and g are related via,

$$q\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = \frac{1}{4} g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right).$$

As an example of how this arises, consider a holomorphic quadratic differential α considered as a holomorphic 1-form with values in the holomorphic cotangent bundle $K_X = T^*_{(1,0)}\Sigma$ (i.e. the canonical bundle). Then the associated dual form to α is

$$\alpha^* = q^{-1}\overline{\alpha} = 4g^{-1}\overline{\alpha}$$

where

$$q = q\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = \frac{1}{4} g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \frac{g}{4}.$$

Lastly, we note the relation between the Chern connection of the Hermitian metric q and the Levi-Civita connection of the Riemannian metric g. Let K_g denote the sectional curvature of g and ∇^q the Chern connection of q. Recall this is the unique connection such that $\nabla^q q = 0$ and the $(0, 1)$ part of ∇^q defines the holomorphic structure on K_X^{-1}. The curvature of the Chern connection satisfies:

$$iF(\nabla^q) = i\overline{\partial}\partial(\log(q)) = i\overline{\partial}\partial(\log(g)) = K_g \, dV_g$$

where dV_g is the volume form of g.

Bibliography

[AG85] K. Astela and F. W. Gehret. Quasiconformal analogues of theorems of Koebe and Hardy-Littlewood. *Michigan Math. J.*, **32**(1):99–107, 1985.

[BCM] Jeffrey Brock, Richard Canary, and Yair Minsky. The classification of Kleinian surface groups, II: The ending lamination conjecture. *Annals of Math. (to appear).* arXiv:math/0412006v2.

[Ber60] Lipman Bers. Simultaneous uniformization. *Bull. Amer. Math. Soc*, **66**:94–97, 1960.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1999.

[Bon86] Francis Bonahon. Bouts de varieties hyperboliques de dimension 3. *Annals of Math.*, **2**(124):71–158, 1986.

[Bon96] Francis Bonahon. Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form. *Ann. Fac. Sci. Toulouse Math. (6)*, 5(2):233–297, 1996.

[Bou95] Marc Bourdon. Structure conforme au bord et flot géodésique d'un CAT(−1)-espace. *Enseign. Math. (2)*, 41(1-2):63–102, 1995.

[Bow79] Rufus Bowen. Hausdorff dimension of quasi-circles. *Publ. Math. de L'Inst. Hautes Études Sci.*, 50(1):11–25, 1979.

[CEG87] Richard Canary, David Epstein, and P. Green. Notes on notes of Thurston. In *Analytical and geometric aspects of hyperbolic space*, pages 3–92. Cambridge University Press, 1987.

[CM11] Tobias Holck Colding and William P. Minicozzi, II. *A course in minimal surfaces*, volume 121 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2011.

[Coo93] Michel Coornaert. Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov. *Pacific J. Math.*, 159(2):241–270, 1993.

[Don87] S. K. Donaldson. Twisted harmonic maps and the self-duality equations. *Proc. London Math. Soc. (3)*, 55(1):127–131, 1987.

[Don03] S. K. Donaldson. Moment maps in differential geometry. In *Surveys in differential geometry, Vol. VIII (Boston, MA, 2002)*, volume 8 of *Surv. Differ. Geom.*, pages 171–189. Int. Press, Somerville, MA, 2003.

[DW07] Georgios D. Daskalopoulos and Richard A. Wentworth. Harmonic maps and Teichmüller theory. In *Handbook of Teichmüller theory. Vol. I*, volume 11 of *IRMA Lect. Math. Theor. Phys.*, pages 33–109. Eur. Math. Soc., Zürich, 2007.

[DWW10] Georgios D. Daskalopoulos, Richard A. Wentworth, and Graeme Wilkin. Cohomology of $\mathrm{SL}(2,\mathbb{C})$ character varieties of surface groups and the action of the Torelli group. *Asian J. Math.*, 14(3):359–383, 2010.

[Eps84] Charles L. Epstein. Envelopes of horospheres and weingarten surfaces in hyperbolic 3-space. Unpublished, 1984.

[Eps86] Charles L. Epstein. The hyperbolic Gauss map and quasiconformal reflections. *J. Reine Angew. Math.*, **372**:96–135, 1986.

[FHS83] Michael Freedman, Joel Hass, and Peter Scott. Least area incompressible surfaces in 3-manifolds. *Invent. Math.*, **71**(3):609–642, 1983.

[Foc07] V. V. Fock. Cosh-Gordon equation and quasi-Fuchsian groups. In *Moscow Seminar on Mathematical Physics. II*, volume 221 of *Amer. Math. Soc. Transl. Ser. 2*, pages 49–58. Amer. Math. Soc., Providence, RI, 2007.

[GHW10] Ren Guo, Zheng Huang, and Biao Wang. Quasi-Fuchsian three-manifolds and metrics on Teichmuller space. *Asian J. Math.*, **14**(2):243–256, 2010.

[Gol04] William M. Goldman. The complex-symplectic geometry of $\mathrm{SL}(2,\mathbb{C})$-characters over surfaces. In *Algebraic groups and arithmetic*, pages 375–407. Tata Inst. Fund. Res., Mumbai, 2004.

[Gul77] Robert Gulliver. Branched immersions of surfaces and reduction of topological type. II. *Math. Ann.*, 230(1):25–48, 1977.

[Hit87] N. J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc. (3)*, 55(1):59–126, 1987.

[Hop54] Heinz Hopf. *Differential geometry in the large*, volume 1000 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1954.

[HW] Zheng Huang and Biao Wang. On almost Fuchsian manifolds. *Trans. A.M.S. (to appear)*. arXiv:0907.2899v4.

[Jor77] Troels Jorgensen. Compact 3-manifolds of constant negative curvature fibering over the circle. *Annals of Math.*, 106(1):61–72, 1977.

[Kat82] A. Katok. Entropy and closed geodesics. *Ergodic Theory Dynam. Systems*, 2(3-4):339–365 (1983), 1982.

[Ker83] Steven P. Kerckhoff. The Nielsen realization problem. *Ann. of Math. (2)*, 117(2):235–265, 1983.

[KH95] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[KKW91] Anatole Katok, Gerhard Knieper, and Howard Weiss. Formulas for the derivative and critical points of topological entropy for Anosov and geodesic flows. *Comm. Math. Phys.*, 138(1):19–31, 1991.

[Kod05] Kunihiko Kodaira. *Complex manifolds and deformation of complex structures*. Classics in Mathematics. Springer-Verlag, Berlin, english edition, 2005. Translated from the 1981 Japanese original by Kazuo Akao.

[Man79] Anthony Manning. Topological entropy for geodesic flows. *Ann. of Math. (2)*, 110(3):567–573, 1979.

[Man81] Anthony Manning. Curvature bounds for the entropy of the geodesic flow on a surface. *J. London Math. Soc.*, 24, 1981.

[Mar04] Grigoriy A. Margulis. *On some aspects of the theory of Anosov systems*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004. With a survey by Richard Sharp: Periodic orbits of hyperbolic flows, Translated from the Russian by Valentina Vladimirovna Szulikowska.

[Min10] Yair Minsky. The classification of Kleinian surface groups I: Models and bounds. *Annals of Math.*, **171**(1):1–107, 2010.

[MSY82] William Meeks, III, Leon Simon, and Shing Tung Yau. Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature. *Ann. of Math. (2)*, 116(3):621–659, 1982.

[MY82] William W. Meeks, III and Shing Tung Yau. The existence of embedded minimal surfaces and the problem of uniqueness. *Math. Z.*, 179(2):151–168, 1982.

[Pat76] S. J. Patterson. The limit set of a Fuchsian group. *Acta Math.*, 136(3-4):241–273, 1976.

[Pol94] Mark Pollicott. Derivatives of topological entropy for Anosov and geodesic flows. *J. Differential Geom.*, 39(3):457–489, 1994.

[Sig74] Karl Sigmund. On dynamical systems with the specification property. *Trans. Amer. Math. Soc.*, 190:285–299, 1974.

[SU82] J. Sacks and K. Uhlenbeck. Minimal immersions of closed Riemann surfaces. *Amer. Math. Soc.*, **271**(2):639–652, 1982.

[Sul84] Dennis Sullivan. Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups. *Acta Math.*, 153(3-4):259–277, 1984.

[SY79] Richard Schoen and Sing-Tung Yau. Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature. *Ann. of Math.*, **110**(1):127–142, 1979.

[Tau04] Clifford Henry Taubes. Minimal surfaces in germs of hyperbolic 3-manifolds. *Proceedings of the Casson Fest, Geom. Topol, Monogr.*, **7**:69–100, 2004. Electronic.

[Thu] William Thurston. Hyperbolic structures on 3-manifolds, II: surface groups and manifolds which fiber over the circle. *Preprint.* arXiv:math.GT/9801045.

[Uhl83] Karen K. Uhlenbeck. *Closed minimal surfaces in hyperbolic 3-manifolds*, volume **103**. Princeton Univ. Press, Princeton, NJ, 1983.

[Wol10] Scott A. Wolpert. *Families of Riemann surfaces and Weil-Petersson geometry*, volume 113 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2010.