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Christoforos Somarakis and John S. Baras

The Institute for Systems Research

A. JAMES CLARK SCHOOL OF ENGINEERING

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A FIXED POINT THEORY APPROACH TO MULTI-AGENT CONSENSUS DYNAMICS WITH DELAYS

Christoforos Somarakis[1] and John S. Baras[2]

The Institute For Systems Research
Univerisity Of Maryland
College Park, MD 20740, USA

Abstract. The classic linear time-invariant multi-agent consensus scheme is revisited in the presence of constant and distributed bounded delays. We create a fixed point argument and prove exponential convergence with specific rate that depends both on the topology of the communication graph and the upper bound of the allowed delay.

1. **Introduction.** Distributed consensus dynamics have, over the past decade, carried the beacon of research in the control community. Starting from the seminal work of Tsitsiklis [18] the subject was reheated with the work of Jadbabaie et al. [7] who gave a rigorous proof of the leaderless co-ordination in a flocking model proposed by Viscek et al. [19].

Since then, an enormous amount of works has been produced from different fields of Applied Science (Engineering, Phsysics, Mathematics) concerning types of coordination among autonomous agents who exchange information in a distributed way, different frameworks (e.g. deterministic or stochastic) and various communication conditions. See for example [1, 11, 10, 4, 16, 9, 15, 13, 14, 12] and references therein. All of the proposed models are mainly based on a specific type of dynamic evolution of the agents' states known as *consensus schemes*. Each agent evolves it's state by some type of convex averaging of the states of it's 'neighbours'. Each new state lies, typically, in the convex hull of the previous averaged ones so that the limit value is common to all the agents under certain communication criteria (see for example [1]).

In this work, we revisit the fundamental linear time consensus dynamics model in the presence of communication delays. We argue that a Lyapunov-based approach for the stability of the network to the convergence subspace is not only restricting on the assumptions for the communication graph; but also does not shed light upon the critical quantities associated with the asymptotic behavior of the system as it is for instance, the consensus point or the rate of convergence. On the other hand a Fixed Point Theory approach fits better to these types of problems where robust results can be obtained in the price of extensive analysis and perhaps somehow conservative assumptions.

Key words and phrases. Consensus Dynamics, Flocking Networks, Fixed Point Theory, Contraction Mappings.

[1]csomarak@umd.edu
[2]baras@umd.edu

1.1. **Introduction to the model and related literature.** The model we will discuss in this work is of the form

$$\dot{x}_i(t) = \sum_j a_{ij}(x_j(t-\tau) - x_i(t)) \tag{MDL}$$

Each agent evolves according to the dynamics of it's own state as well as a retarded measurement of the states of it's neighbouring agents. Surprisingly enough compared to other and more complex models, this model has not received that much attention. To the best of our knowledge we mention four relative works.

A simple delayed consensus algorithm was proposed and discussed in the work of Olfati-Saber et. al [14] where the model

$$\dot{x}_i(t) = \sum_j a_{ij}(x_j(t-\tau) - x_i(t-\tau)) \tag{1.1}$$

With $\tau > 0$ constant and uniform for all agents, a frequency method analysis was carried through. The problem with this method is that it is over simplistic and cannot be generalized in case the weights are time varying or the delays are incommensurate.

In [18, 1] the authors consider a discrete time version of (MDL) with, in fact, time varying delays $\tau = \tau(t)$. On condition that the delay is uniformly bounded from above, the strategy of attacking the problem is to extend the state space by adding artificial agents which played no actual role in the dynamics other than transmitting a pre-described delayed version of an agent's state. This method although applicable in the discrete time, it is unclear how it would work in a continuous time system, unless the latter one is discretized and solved numerically.

In [15] the authors discuss the convergence properties of a non-linear model which has the form

$$\dot{x}_i = \sum_j a_{ij} f_{ij}(x_j(t-\tau) - x_i) \tag{1.2}$$

Using passivity assumptions on f_{ij} they apply invariance principles to derive delay-independent convergence results both in static and switching topologies. The main setback of this approach, however, is that nothing can be said for either the rate of convergence to the consensus or synchronization space or the consensus point itself. This is also noted by the authors themselves.

The last type of models have to do with rendezvouz type of algorithms. For example in [13] the authors propose a second order consensus based algorithms, where agents asymptotically meet in a common place as their speed vanishes to zero. This algorithm is of the form

$$\dot{v}_i(t) = -cv_i(t) + \sum_{i=1}^{N} a_{ij}(r_j(t-\tau_i^j) - r_i(t)) \tag{1.3}$$

The authors make a Lyapunov-Krasovskii argument on the base that the delayed quantities act only as perturbations to the main dynamical equation. Again, little can be said about the rate of convergence of this system.

1.2. **Organization of the Report.** This work is organized as follows. In section 2, we introduce the main notations and definitions that we will use throughout the paper. In section 3 we introduce our models, pose the sufficient assumptions and the main results. In section 4, we take a digression to a Lyapunov-based approach outlining the difficulties of such an attempt. In section 5, we introduce the family of metric spaces we are interested in and prove an important result which will help us make use of Contraction Mappings. The analysis of linear time invariant weights will be carried out in section (6) using undirected connected network communication (i.e. symmetric weights $a_{ij} = a_{ji}$). In section (6.2) we will briefly explain how these results can be adapted for the case of asymmetric weights (i.e. graphs with a spanning tree).

The analysis of linear time varying weights will be discussed in section (7).

2. **Notations and Definitions.** In this section we will explain the preliminary notations and definitions which will be used in this work. By $N < \infty$ we denote the number of agents. The set of agents is denoted by $[N] := \{1, \dots, N\}$. Each agent $i \in [N]$ is associated with a real quantity $x_i \in \mathbb{R}$.

In general, unless otherwise stated, by $|a|$ we understand the absolute value of a scalar number, $||\mathbf{a}||$ the norm of a vector, $||A||$ the (induced) norm of a matrix and by $|\mathbf{a}|$ the norm of a function.

The notation $\exists \alpha \in [0, 1)$ will be abused for different conditions throughout this work. By this condition, one understands that the proposed quantity should be strictly less than one.

The Euclidean vector space $\mathbb{R}^N$ is the state space of the system with state vectors $\mathbf{x} = (x_i, \dots, x_N)^T$ and it is equipped with the $p = 1$ norm $|| \cdot ||$, where for $\mathbf{x} \in \mathbb{R}^N$ we get $||\mathbf{x}|| = \sum_{i=1}^N |x_i|$.

It is noted that it is this norm that establishes the inequality

$$\left|\left| \int_a^b \mathbf{f}(t)dt \right|\right| \leq \int_a^b ||\mathbf{f}(t)||dt$$

for $a, b \in \mathbb{R}$ and $\mathbf{f}$ a vector valued integrable function. By the equivalence of norms in finite dimensional spaces:

$$||\mathbf{x}||_2 \leq ||\mathbf{x}|| \leq \sqrt{N}||\mathbf{x}||_2$$

where $|| \cdot ||_2$ is the Euclidean norm.

For a square $N \times N$ matrix A the induced norm is defined as $||A|| = \sup_{||\mathbf{x}||=1} ||A\mathbf{x}||$. By $\mathbb{1}$ we understand the column vector of all ones. The subspace of $\mathbb{R}^N$ of interest is defined by

$$\Delta = \{\mathbf{y} \in \mathbb{R}^N : \mathbf{y} = \mathbb{1}c \text{ for some } c \in \mathbb{R}\} = \{\mathbf{y} \in \mathbb{R}^N : y_1 = \cdots = y_N\}$$

and it is called the *consensus (sub)-space*.

2.1. **Algebraic Graph Theory.** In this subsection we review some tools from algebraic and spectral graph theory. For more information on the subject the interested reader is refered to [5, 2, 10].

The mathematical object which will be used to model the communication structure among the N agents is the weighted directed graph. This is defined as the triple $G = (V, E, W)$ where V is the set of nodes (here $[N]$), E is a subset of $V \times V$ which characterizes the established communication connections and W is a set associating a positive number (the weight) with any member of E. So by a_{ij} we will denote the weight in the connection from node j to node i and this is amount of the effect that j has on i. If $a_{ij} = 0$ then $(j, i) \notin E$. This is a directed graph and in this work we will be interested in the family routed out branching graphs or strongly connected directed graphs. A rooted out branching graph is a graph that contains a spanning tree (i.e. at least one node is a root) and a strongly connected graph is a graph that each node is a root. Moreover, in case of symmetric communicating weights $a_{ij} = a_{ji}$ the graph is called undirected; hence simple connectivity suffices for our results. Given E, each agent i has a neighborhood of nodes, to which it is adjacent. We denote by N_i the subset of V such that $(j, i) \in E$ and by $|N_i|$ it's cardinality. The notation $\sum_{i,j}$ stands for $\sum_{i=1}^N \sum_{j \in N_i}$. The degree of any node i, denoted by d_i, is the sum of the weights with which each of it's adjacent nodes affects him, i.e. $d_i = \sum_{j \in N_i} a_{ij}$. If the weights are time varying we write the $d_i(t)$ and the vector $\mathbf{d}_t = (d_1(t), \dots, d_N(t))$ of overall degree influence.

A matrix representation of G is through the adjacency matrix $A = [a_{ij}]$, the degree matrix $D = \text{Diag}[d_i]$ and the Laplacian $L := D - A$. If G is directed we name it as in-degree Laplacian. If G is undirected it is simply known as graph Laplacian. The spectral properties of L are of interest. In case of undirected network the L is a symmetric positive semi-definite matrix. So there is an eigensystem of real eigenvalues and mutually orthogonal eigenvectors such that

$$0 = \lambda_1(L) \leq \lambda_2(L) \leq ... \leq \lambda_N(L) = ||L||_2$$

and $\mathbf{u}_i|_{i=1}^N$ is the family of eigenvectors such that $\mathbf{u}_j^T \mathbf{u}_i = 0$ for $i \neq j$ and $||\mathbf{u}_i||_2 = 1$ for all $i \in [N]$. An important result is that a (directed) graph is assumed to be (strongly) connected if and only if $\lambda_2(L) > 0$. The term *underlying graph* (or associated graph) will be used to characterize the connectivity conditions of a weighted graph. The underlying graph of a weighted graph is connected if all the positive weights of the latter graph are set to 1 the the resulting (topological) graph is connected.

3. **The model, the assumptions and the statement of the results.** In this section we will state the two main results, the one with linear time invariant weights and the other with time varying weights.

Given $N < \infty$, $0 < \tau < \infty$ and the initial functions $\phi_i(t) : [-\tau, 0] \to \mathbb{R}|_{i=1}^{N}$, we consider two systems of delayed differential equations

3.1. **Linear Time Invariant Weights.**

$$\begin{aligned} \dot{x}_i &= \sum_{j \in N_i} a_{ij}(x_j^i - x_i),\ t \geq 0 \\ x_i(t) &= \phi_i(t),\ t \in [-\tau, 0] \end{aligned} \tag{IVP1}$$

where $x_j^i := x_j(t - \tau_i^j)$ for some constants τ_i^j. The notation signifies the delay with which agent i receives the signal from agent j , $j \in N_i$.

(H.1.1) $\forall i, j$ we have $\tau_i^j \geq 0$ such that $\tau = \max_{i,j} \tau_i^j$.
(H.1.2) The initial functions ϕ_i are given, continuous functions of time.
(H.1.3) The weights a_{ij} are non-negative constants such that $a_{ij} = a_{ji}$. The associated graph is simply connected.

3.1.1. *Main Result.*

Theorem 3.1. *Consider an undirected connected graph $G = (N, E, W)$ with the associated combinatorial Laplacian matrix L and it's spectrum. Let $A := \sum_{i=1}^{N} \sum_{j \in N_i} a_{ij}$ denote the sum of all the communication weights. Then there exist constant $k \in \mathbb{R}$ and $d > 0$ such that under assumptions $(H.1.1-3)$,(IVP1) converges to a common value, k, exponentially fast with rate d if the additional two conditions hold:*

$$d < \lambda_2(L) \tag{H.1.4}$$

$$A \frac{e^{d\tau} - 1}{d} \left(1 + \frac{\sqrt{N} \lambda_N(L)}{\lambda_2(L) - d} \right) \leq \alpha \tag{H.1.5}$$

for some $0 \leq \alpha < 1$.

Remark 1. A first comment on the assumptions $(H.1.4, 5)$ is that at one hand the rate of convergence of the delayed system cannot be faster than the rate of convergence of the un-delayed system while at the other hand, $(H.1.5)$ establishes a stability condition associated with the topological connectivity of the graph, the weights, the rate of convergence and the maximum allowed delay.

Remark 2. The assumption $(H.1.5)$ is rather restrictive since A, is the sum of all the weights. This is the price one pays for not using the Lyapunov approach for this model. This assumption can be significantly improved if assumptions on the symmetry of the delays are taken. For example, if $\tau_i^j \equiv \tau > 0$ then A can be replaced with $||A||$, that is the induced norm of the adjacency matrix of the communication graph G.

Remark 3. The consensus point, k, has an analytical expression and is defined in (CNS). It is, as expected, a function of $N, a_{ij}, \tau_i^j, \phi_i$.

3.2. **Linear Time Varying Weights.** For fixed $t_0 > 0$ we consider the time varying model:

$$\begin{aligned} \dot{x}_i &= \sum_{j \in N_i} a_{ij}(t)(x_j^i - x_i),\ t \geq t_0 \\ x_i(t) &= \phi_i(t),\ t \in [t_0 - \tau, t_0] \end{aligned} \tag{IVP2}$$

where $x_j^i := x_j(t-\tau_i^j)$ and τ_i^j is the delay constant, non-negative and bounded and as above $\tau := \max_{i,j} \tau_i^j < \infty$. We impose the following assumptions:

(H.2.1) The transition matrix of the linear system

$$\dot{\mathbf{x}} = -L(t)\mathbf{x}\ ,\ t > t_0 \tag{LTV}$$

is denoted by $\mathbf{\Phi}(t_1, t_2)$ where $t_1, t_2 \geq t_0$ and satisfies the following relation: For fixed $t_0 \in \mathbb{R}$, there exist $\gamma > 0$ (independent of t_0), $\Gamma > 0$ and $\mathbf{c} \in \mathbb{R}^N$ (possibly dependent on t_0) with $\sum_{i=1}^N c_i = 1$ such that

$$|\mathbf{\Phi}(t, t_0) - \mathbb{1}\mathbf{c}^T|_1 \leq \Gamma e^{-\gamma(t-t_0)}$$

(H.2.2) The weights $a_{ij}(t) : \mathbb{R}_+ \to \mathbb{R}_+$ are C^1, bounded functions of time, with uniform bound $|a_{ij}| \leq a$.

(H.2.3) There is $\beta \in (0, \gamma)$ such that

$$|\dot{a}_{ij}(s)|e^{-\beta s} \in L^1_{[t_0,\infty)}$$

(H.2.4) There exist $M > 0$ and $\delta > 0$ such that $\forall i \in [N]$:

$$\left| \sum_{j \in N_i} c_i a_{ij}(t) - c_j a_{ji}(t) \right| \leq M e^{-\delta t}\ ,\ t > t_0$$

(H.2.5) There exists $\alpha \in [0, 1)$ such that

$$\sup_{t \geq t_0} \tau |\mathbf{c}^T \mathbf{d}_t| \leq \alpha$$

(H.2.6) Set

$$A := \sup_t \sum_{i,j} a_{ij}(t),\ \dot{A} := \sup_t \sum_{i,j} |\dot{a}_{ij}(t)|,\ L := \sup_t ||L(t)||,\ D := \sup_t \left| \sum_{i=1}^N c_i \sum_{j \in N_i} a_{ij}(t) \right|$$

For fixed $d \in (\beta, \gamma)$, there exists $\alpha \in (0, 1)$ such that

$$\frac{e^{d\tau} - 1}{d} \left(A \left(1 + \frac{\Gamma L}{\gamma - d} + \frac{M}{d + \delta} \right) + \frac{\Gamma \dot{A}}{\gamma - d} + \frac{D}{d} \right) \leq \alpha \tag{3.2}$$

The imposed hypotheses, although numerous and seemingly restricting are in fact the result of the drop of symmetry assumptions usually considered in consensus dynamics. For example we do not consider symmetry in a_{ij}. Moreover very little is actually known about the weights $a_{ij}(t)$. Below, we make a few comments on the assumptions, reviewing them one by one.

3.2.1. *First Remarks.* Assumption $(H.2.1)$ describes the dynamics of (LTV) and imposes the asymptotic consensus of the agents at the rate of $\gamma > 0$. To simplify the analysis we assume not failure of connectivity (i.e. $a_{ij} > 0$ if and only if $a_{ij}(t) > 0$ for some t). In the discussion of the result we will discuss the possibility of connectivity failures. The condition $\sum_i c_i = 1$ is necessary so that Δ is an (LTV)-invariant subspace.

Assumption $(H.2.2)$ characterizes the dynamics of the communication weights. The boundedness of a_{ij} is imposed as a reasonable assumption based on applications of the Control & Communication area and as an expected consideration which makes independent the communication framework from the dynamics. It also follows that $|L| < \infty$ and we denote this bound by $||L||$.

Assumption $(H.2.3)$ characterizes the dynamics of $\dot{a}_{ij}$ and asks for certain smoothness properties. Although not very important for the stability results of (LTV), it seems that under the Fixed Point Theory Approach these dynamics are important. The assumption is readily be fulfilled if for instance $|\dot{a}_{ij}| \in L^1[t_0, \infty)$. In such case, it is implied that the transmission weights asymptotically "freeze".

Assumption $(H.2.4)$ is one way to bridge the gap between $\mathbf{c}$ and $a_{ij}(t)$. Indeed $[H.1]$ says nothing about the connection between the weights and the consensus value. Note that in the case of weight symmetry If $a_{ij}(t) \equiv a_{ji}(t)$ it is readily fulfilled and it is thus obsolete.

Assumption $(H.2.5)$ is necessary to prove existence and uniqueness of the consensus point. It's need is due to the fact that the dynamical system is non-autonomous and thus constant information of the weights and thus the solution is needed. This assumption can be significantly relaxed if the system was non-autonomous (time-invariant linear or non-linear) or if it was periodic.

Assumption $(H.2.6)$ includes two assumptions. The reasonable one, that the rate of convergence of the (IVP2) cannot be faster than (LTV) and also the crucial condition so that the solution operator P (to be introduced below) is a contraction in $(\mathcal{M}, \rho_d)$.

We are now ready to state our result.

3.2.2. *Main Result.*

Theorem 3.2. *Consider the* (IVP2) *and the assumptions* $(H.2.1-6)$. *Then there exists a unique* $k \in \mathbb{R}$ *which is a function of the initial conditions and the connectivity weights of the graph such that the solution of* (IVP2) *converges to, exponentially fast, with rate* d.

3.3. **Preliminary results.** We end this section with two preliminary results which will be used as tools for the analysis to follow for both (IVP1) and (IVP2).

Boundedness of solutions.

Proposition 1. *If* $\max_{i\in[N]} \sup_{t\geq t_0-\tau} |\phi(t)| = c$, *then for* $\mathbf{x}$ *being the solution of* (IVP1) *or* (IVP2) *it holds* $|x_i| \leq c \ \forall i \in [N]$.

Proof. Assume that the condition does not hold. Then there exists a first time of escape for some $i \in [N]$, say $\bar{t} > 0$ such that

$$\left\{x_i(\bar{t}) = c, \dot{x}_i(\bar{t}) > 0\right\} \text{ or } \left\{x_i(\bar{t}) = -c, \dot{x}_i(\bar{t}) < 0\right\}$$

The first case for example, however yields

$$\dot{x}_i(\bar{t}) = \sum_j a_{ij}(\bar{t})(x_j(\bar{t} - \tau_i^j) - c) \leq 0$$

a contradiction. A similar contradiction arises for the second case. □

Non-existence of non-trivial periodic solutions.

Proposition 2. *The undelayed system bears no periodic solutions.*

Proof. It is a simple exercise to show that the un-delayed system

$$\dot{\mathbf{x}} = -L(t)\mathbf{x} \tag{3.3}$$

does not sustain non-trivial periodic solutions when $L(t)$ is T-periodic and when the assumption $(H.2.1)$ does not hold (but the smoothness assumptions on the weights and certain connectivity criteria hold.

To see this assume for the sake of contradiction the existence of a periodic solution $\mathbf{x}$ of period $T > 0$, that is $\mathbf{x}(t+T) = \mathbf{x}(t)$ for all $t > 0$. Obviously this cannot be a consensus solution since $\dot{\mathbf{x}}(t) \neq 0$. It follows that there exists $i \in [N]$ and $t_1 > 0$ such that

$$x_i(t_1) = \max_{j \in [N]} x_j(t_1) \tag{3.4}$$

(If there are more than one agents choose another t_1). By continuity of the solutions and the finite number of agents, i will attain the maximum value in the open $[t_1, t_1 + \epsilon)$ for some $\epsilon > 0$. Since $x_i(t_1) = x_i(t_1 + T) = c_1$ and x_i is non constant we assume without loss of generality that

$$x_i(t) < c_1 \tag{3.5}$$

for $t \in (t_1, t_1 + T)$ (if one does not want to assume that, one is free to consider the very next time $t_2 > t_1$ where $x(t_2) = c_1$). In view of $x_i(t) < c_1$ in $(t_1, t_1 + T)$ there must exist $j \in N_i$ such that $x_j(\hat{t}) > c_1$ for some $t \in (t_1, t_1 + T)$. The same line of arguments should hold for j, so that some $l \in N_j$ must satisfy $x_l(t) > c_1$ for $t \in (t_1, \hat{t})$. In view of the dynamics (3.3), this argumentation will yield contradiction to either (3.4) or (3.5).

□

4. **Digression: A Lyapunov approach.** We take a digression in order to discuss a possible Lyapunov approach to the problem. We argue that such a strategy is both demanding on the connectivity assumptions one needs to take and yields to results which are not very useful. In this section we will prove convergence of (IVP1) using Lyapunov-Krasovksii functionals with the completion of squares method.

We begin with a simple, yet illustrative, example

Example 1. *Consider the system*

$$\begin{aligned} \dot{x} &= -ax + ay_t \\ \dot{y} &= -by + bx_t \end{aligned} \tag{Ex.1}$$

where $a, b > 0$ are constants and $x_t = x(t-\tau), y_t = y(t-\tau)$ for some constant $\tau > 0$. Given $\phi_1, \phi_2 : [-\tau, 0] \to \mathbb{R}$ such that $\phi_1(t) = x(t), \phi_2(t) = y(t),\ -\tau \le t \le 0$. We consider the Lyapunov-Krasovski functional

$$V(x,y) = 2\int_0^x bx dx + 2\int_0^y ay dy + ab\int_{t-\tau}^t x^2 + y^2 dt$$

This functional is a Lyapunov function on $\mathbb{R}^2$ relative to (Ex.1) in this example since it is obviously continuous and

$$\begin{aligned} \dot{V} &= 2bx\dot{x} + 2ay\dot{y} + ba(x^2+y^2) - ba(x_t^2 + y_t^2) \\ &= -2abx^2 + 2baxy_t - 2aby^2 + 2abyx_t + ba(x^2+y^2) - ba(x_t^2+y_t^2) \\ &= -ab[(x-y_t)^2 + (y-x_t)^2] \end{aligned}$$

Then the set, S, such that $\dot{V}(t)0$ is the one where $x(t) = y(t-\tau)$ and $y(t) = y(t-\tau)$. The largest subset of S that is invariant with respect to the dynamics is Δ. Then standard Invariance Theory Arguments yield asymptotic convergence to the consensus subspace (see section 5.3 of [6]).

To generalize the above result for N agents and multiple delays we need some elementary algebra:

Lemma 4.1. *Given the connectivity weights $a_{ij} \ge 0$ such that the underlying graph is connected, if the following assumptions hold $\forall i, j$[3]:*

$$\frac{a_{ji}^2}{d_j^2} + \frac{4}{|N_i||N_j|} - \frac{a_{ij}^2}{d_i^2} > 0 \text{ and } \left(\frac{a_{ji}^2}{d_j^2} + \frac{4}{|N_i||N_j|} - \frac{a_{ij}^2}{d_i^2}\right)^2 \ge \frac{16a_{ij}^2}{|N_j||N_i|d_i^2} \tag{4.1}$$

Then there exist $0 < B_{ij} < \frac{2}{|N_j|}$, $0 < B_{ji} < \frac{2}{|N_i|}$ such that

$$\left(\frac{2}{|N_i|} - B_{ji}\right)B_{ij} = \frac{a_{ij}^2}{d_i^2} \tag{4.2}$$

Proof. Notice that by connectivity assumption $d_i > 0$. Set, $\Gamma := \frac{a_{ji}^2}{d_j^2} + \frac{4}{|N_i||N_j|} - \frac{a_{ij}^2}{d_i^2}$. Take (4.2) interchange i, j and solve the pair of algebraic equations for B_{ij}, B_{ji}. Then B_{ij} satisfies:

$$\frac{2}{|N_i|}(B_{ij})^2 - \Gamma B_{ij} + \frac{2a_{ij}^2}{|N_j|d_i^2} = 0$$

The function $f(z) = \frac{2}{|N_i|}z^2 - \Gamma z + \frac{2a_{ij}^2}{|N_j|d_i^2} = 0$ is decreasing at 0 if $\Gamma > 0$ and has a global minimum at $\frac{|N_i|}{4}\Gamma$ where $f\left(\frac{|N_i|}{4}\Gamma\right) < 0$ if the second condition in (4.2) holds.

[3] See section 2 for notations.

This implies the existence of two positive roots of B_{ij} both of which are bounded above by $z^* := \frac{2}{|N_j|}$ since it can be easily verified that $f(z^*) > 0$ and also $f'(z^*) > 0$ exactly because the first condition in (4.2) holds. Then z^* lies after the second acceptable root for B_{ij}. Similar analysis holds for B_{ji} and the proof of the lemma is complete. □

Examples of cases where (4.2) is fulfilled :

1. All to all connectivity with symmetric weights.
2. Certain families of balanced or regular graphs.

4.1. **Convergence.**

Theorem 4.2. *Under assumptions (H1) and (H2), (IVP1) admits consensus solutions each of which is uniformly asymptotically stable.*

Proof. In view of the discussion in section 5, it suffices to check the stability of the origin.

Set $d_i := \sum_{j \in N_i} a_{ij}$ and for and i, j such that $i \neq j$ consider positive constants B_{ij}, B_{ji} such that

Consider the Lyapunov-Krasovksii functional

$$V(\phi) = \sum_{i=1}^{N} \frac{1}{d_i} \phi_i(0)^2 + \sum_{i=1}^{N} \sum_{j \in N_i} \int_{-\tau_i^j}^{0} B_{ij} \phi_j^2(s) ds \tag{4.3}$$

for which for some $D_1 = D_1(a_{ij}, \tau)$ and $D_2 = \frac{N}{\min_i d_i}$ it holds that

$$D_2 ||\phi(0)||_2^2 \leq V(\phi) \leq D_1 \sup_{s \in [-\tau, 0]} ||\phi(s)|| \tag{4.4}$$

Then for $x_t = x(t + \theta), \ -\tau \leq \theta \leq 0$

$$V(x_t) = \sum_{i=1}^{N} \frac{1}{d_i} x_i^2 + \sum_{j \in N_i} \int_{t-\tau_i^j}^{t} B_{ij} x_j^2(s) ds \tag{4.5}$$

which differentiating throughout a solution we get:

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{N} \frac{2}{d_i} x_i \dot{x}_i + \sum_{i=1}^{N} \sum_{j \in N_i} B_{ij} x_j^2 - B_{ij} (x_j^i)^2 \\
&= \sum_{i=1}^{N} -2x_i^2 + \sum_{i=1}^{N} \sum_{j \in N_i} 2 \frac{a_{ij}}{d_i} x_j^i x_i + \sum_{i=1}^{N} \sum_{j \in N_i} B_{ij} x_j^2 - B_{ij} (x_j^i)^2 \\
&= -\sum_{i=1}^{N} \sum_{j \in N_i} \left(\left[\frac{2}{|N_i|} - B_{ji} \right] x_i^2 - 2 \frac{a_{ij}}{d_i} x_j^i x_i + B_{ij} (x_j^i)^2 \right) \\
&= -\sum_{i=1}^{N} \sum_{j \in N_i} \left(\left[\sqrt{\frac{2}{|N_i|} - B_{ji}} \right] x_i - \sqrt{B_{ij}} x_j^i \right)^2 \leq 0
\end{aligned} \tag{4.6}$$

Note that in principal it holds that $\sqrt{\frac{2}{|N_i|} - B_{ji}} \neq \sqrt{B_{ij}}$.

Then the invariance principle identifies $\dot{V} \equiv 0$ as the invariant set where $x_i \equiv x_j^i \equiv 0$ for all i, j, i.e. the consensus solution, which is asymptotically stable. □

This approach bears no significant difference from [15]. Actually, the results are very similar, as no assumptions on the delays were taken and the stability is uniform and asymptotic. (Delay independent results). However the price is heavy balancing assumptions (4.1) and it tells us nothing about either the consensus point or the rate of convergence. Finally, it is not applicable in case of time varying or state-dependent weights or delays. Notice also that the case of 2 agents with identical constant delay (Example 1) satisfy (4.1) with equality in the second condition.

One may attempt to consider another Lyapunov functional, but this is the main difficulty with the method. The delayed systems are too asymmetrical to support an easy choice of V.

5. **Fixed Point Theory.**

The stability problems we discuss are through contraction mappings and they are thus formulated in complete metric spaces.

5.1. **Topological and Metric Spaces.**

A topological space is an abstract space of mathematical objects together with a set of axioms that define the structure of open sets. The collection of open sets, in turn, determines the notions of a neighbourhood and convergence in this space. In applications one needs a strong notion of a topological space and this is provided by the use of *metrics*. Metrics are particular types of functions which generate useful topologies by explicitly evaluating the distance between the points in this space. The prototypical metric space is the real line $\mathbb{R}$ together with the function

$$d(x,y) := |x-y| \qquad \forall x, y \in \mathbb{R}$$

An arbitrary metric space is defined axiomatically.

Definition 5.1. A pair $(\mathcal{S}, \rho)$ is a metric space if $\mathcal{S}$ is a set and $\rho : \mathcal{S} \times \mathcal{S} \to [0, \infty)$ such that when x, y, z are in $\mathcal{S}$ then

- $\rho(x,z) \geq 0$, $\rho(y,y) = 0$ and $\rho(y,z) = 0$ implies $y = z$.
- $\rho(y,z) = \rho(z,y)$ and
- $\rho(y,z) \leq \rho(y,x) + \rho(x,z)$

The following definitions determine the types of convergence and the properties of metric spaces that are particular useful in this work.

Definition 5.2.
1. A sequence $\{x_n\}$ in a metric space $(\mathcal{S}, \rho)$ is said to *converge* if $\exists x \in \mathcal{S}$ such that $\lim_n \rho(x_n, x) = 0$.
2. A sequence $\{x_n\} \in \mathcal{S}$ is *Cauchy* if for each $\epsilon > 0$ there exists N such that $n, m > N$ imply $\rho(x_n, x_m) < \epsilon$.
3. The metric space is *complete* if every Cauchy sequence in $(\mathcal{S}, \rho)$ has a limit in that space.
4. A set L in a metric space $(\mathcal{S}, \rho)$ is *open* if for every $x \in L$ there exists $r > 0$ such that $\{y : \rho(x,y) < r\} \subset L$ and it is closed if $\mathcal{S} \backslash L$ is open.
5. A set L in a metric space $(\mathcal{S}, \rho)$ is *compact* if each sequence $\{x_n\} \in L$ has a subsequence with limit in L.

The next proposition is essential in proving completeness in subsets of complete metric spaces.

Proposition 3. *Any compact subset L of a metric space $(\mathcal{S}, \rho)$ is a closed set. Any closed subset of a complete metric space defines a complete metric space with the same metric, ρ.*

Proof. For the first part, see [8] page 276. For the second part, it is reminded that a closed set contains all of it's limit points, hence each Cauchy sequence converges in it. □

In this work, we consider only complete metric spaces.

However, in the next subsection, together with the metric spaces, we will briefly mention related *Normed Vector Spaces* and discuss their completeness properties as well. It is reminded that a vector space is an abstract space closed under the addition of it's elements and multiplication of an element by scalars. The most common method for defining a topology on a vector space is to specify a length for each vector, i.e. a norm.

Definition 5.3. A normed vector space $\mathcal{X}$ is a vector space with a real valued function $||\cdot||$ with the properties that for all $x, y \in \mathcal{X}$ and $\dashv \in \mathbb{R}$

i. $||x|| \geq 0$
ii. $||x|| = 0 \Leftrightarrow x = 0$
iii. $||\alpha x|| = |\alpha| \cdot ||x||$
iv. $||x + y|| \leq ||x|| + ||y||$

Every normed vector space is a metric space and consequently a topological space as there is a natural metric ρ defined by the norm, i.e. $\rho(x, y) = ||x-y||$. A complete normed vector space is called *Banach* space.

5.2. **Examples of Metric and Normed Spaces.** We are interested in function spaces each member of which is a vector valued function, that is bounded (in the sup norm) such that it asymptotically converges to a point in Δ. The underlying normed vector space is this of continuous functions defined in $[-\tau, \infty)$ for $0 \leq \tau < \infty$ and taking values in $\mathbb{R}^N$. More formally, let $\mathbf{f} : [-\tau, \infty) \to \mathbb{R}^N$ continuous and define the function $|\mathbf{f}| := \sup_{t \geq -\tau} ||\mathbf{f}(t)||$. Then

$$\mathcal{C} = \{\mathbf{y} : [-\tau, \infty) \to \mathbb{R}^N | \quad \mathbf{y} \text{ continuous and } \quad |\mathbf{y}| < \infty\}$$

One of the standard results is the following:

Proposition 4. *$(\mathcal{C}, |\cdot|)$ is a Banach space.*

Proof. See for example [8] page 278. □

The technique in the proof of Proposition (4) will guide in establishing our results too.

5.2.1. *The space $\mathcal{B}_\Delta$.* A subset of $\mathcal{C}$ of main interest is the following

$$\mathcal{B}_\Delta = \{\mathbf{y} \in \mathcal{C} : \quad \lim_t \mathbf{y}(t) \in \Delta\}$$

Proposition 5. *The pair $(\mathcal{B}_\Delta, |\cdot|)$ consists a Banach space.*

Proof. It is a trivial exercise to show that $\mathcal{B}_\Delta$ is a vector space, under the addition and scalar multiplication of functions and that $|\cdot|$ is a norm that generates a topology. Consider the metric

$$\rho(\phi_1, \phi_2) := |\phi_1 - \phi_2| = \sup_{t \geq -\tau} ||\phi_1(t) - \phi_2(t)||$$

and let $\{\phi_k\}$ be a Cauchy sequence in $\mathcal{B}_\Delta$. Then

$$|\phi_j(t) - \phi_k(t)| \leq \rho(\phi_1, \phi_2)$$

implies that $\phi_k(t)$ is a Cauchy sequence in $(\mathbb{R}^N, |\cdot|)$ for all t, so $\phi_k(t) \to \phi(t)$. We need to show that $\phi \in \mathcal{B}_\Delta$.

Claim: ϕ is continuous. Given $\varepsilon > 0$ there exists N such that $|\phi_j - \phi_k| < \varepsilon$ for $j, k > N$. Fix $k > N$ and let $j \to \infty$ from above we get

$$||\phi(t) - \phi_k(t)|| < \varepsilon$$

for all t. So $\phi_k \rightrightarrows \phi$ and thus ϕ is continuous.

Claim: ϕ is Bounded.

$$||\phi(t)|| \leq ||\phi(t) - \phi_k(t)|| + ||\phi_k(t)|| < \varepsilon + |\phi_k| < \infty \tag{5.1}$$

Claim: $\phi \to \Delta$ For any $\varepsilon > 0$ take $N > 0$ such that $k > N$ implies $||\phi(t) - \phi_k(t)|| < \frac{\varepsilon}{2}$ Fix $k > N$ and $t > T$ such that $|\phi - \phi_k| < \frac{\varepsilon}{2}$ for $k > N$ and $||\phi_k(t) - \lim_t \phi_k(t)|| < \frac{\varepsilon}{2}$ for $t > T$. Then

$$||\phi(t) - \lim_t \phi_k(t)|| \leq ||\phi(t) - \phi_k(t)|| + ||\phi_k(t) - \lim_t \phi_k(t)|| < \varepsilon$$

and the result follows.

□

5.2.2. *The space $\mathcal{B}_\Delta^{Lip}$.* Another subset of $\mathcal{C}$ (and of B_Δ as well) is the following:

$$\mathcal{B}_\Delta^{Lip} = \{\mathbf{y} \in \mathcal{B}_\Delta : \quad \mathbf{y} \quad \text{is Lipschitz continuous}\}$$

Proposition 6. *The pair $(\mathcal{B}_\Delta^{Lip}, |\cdot|)$ consists a Banach space.*

Proof. Let ϕ_k a Cauchy sequence in $\mathcal{B}_\Delta^{Lip}$ which converges uniformly in ϕ. Along with all the claims in Proposition (5) we have
Claim: ϕ is Lipschitz. For any t_1, t_2 and k large enough so that $||\phi(t) - \phi_k(t)|| < \frac{\varepsilon}{2}$ we write

$$\begin{aligned} ||\phi(t_1) - \phi(t_2)|| &\leq ||\phi(t_1) - \phi_k(t_1) + \phi_k(t_1) + \phi_k(t_2) - \phi_k(t_2) - \phi(t_2)|| \\ &\leq ||\phi(t_1) - \phi_k(t_1)|| + ||\phi_k(t_2) - \phi(t_2)|| + ||\phi_k(t_1) - \phi_k(t_2)|| \\ &< \varepsilon + L_k|t_1 - t_2| \end{aligned}$$

for ε arbitrary small the result follows. □

Remark 4. It should be noted that the following subset of $\mathcal{B}_\Delta^{Lip}$

$$\mathcal{B}_\Delta^{Lip(L)} := \{\mathbf{y} \in \mathcal{B}_\Delta : \quad ||\mathbf{y}(t_1) - \mathbf{y}(t_2)|| \leq L|t_1 - t_2|, \quad t_1, t_2 \geq -\tau\}$$

is not a vector space, but it is a complete metric space with the natural metric ρ. This is an easy proof and goes exactly as Proposition (6) after switching L_k to the uniform L.

Rate functions. In order to yield convergence results with prescribed convergence rate we will make use of special functions, called *rate functions* with which we will establish *weighted metrics*.

Definition 5.4. A continuous function $h(t) : [0, \infty) \to [1, \infty)$ with the properties that

1. $h(0) = 1$
2. $\lim_t h(t) = \infty$

will be called *rate function*.

Typical example of a rate function is the exponential $h(t) = e^{\gamma t}$, $\gamma > 0$

5.2.3. *The space of solutions, $\mathcal{M}$.* Initially consider the space

$$\mathcal{M} = \{\mathbf{y} \in \mathcal{B}_\Delta : \sup_{t \geq 0} h(t)||\mathbf{y}(t) - \mathbb{1}k_\mathbf{y}|| < \infty\}$$

this is a subset of $\mathcal{B}_\Delta$, each member $\mathbf{y}$ of which, converges to a point $\mathbb{1}k_\mathbf{y} \in \Delta$ with rate $h(t)$. It is reminded that any $\mathbf{y}$ is bounded and so is $k_\mathbf{y}$. The norm, however, is replaced with the following:

$$|\mathbf{y}|_\Delta := |k_\mathbf{y}| + \sup_t h(t)||\mathbf{y}(t) - \mathbb{1}k_\mathbf{y}||$$

and this is due to the fact that the usual supremum function in $\mathcal{M}$ becomes a pseudonorm.

Proposition 7. *The pair $(\mathcal{M}, |\cdot|_\Delta)$ consists a Banach space.*

Proof. At first we need to show that $|\cdot|_\Delta$ is indeed a norm. For this it suffices to check only that $|\mathbf{y}|_\Delta = 0$ if and only if $\mathbf{y} \equiv 0$ and this is readily true. Then, the new natural metric is

$$\rho_\Delta(\mathbf{y}_1, \mathbf{y}_2) := |\mathbf{y}_1 - \mathbf{y}_2|_\Delta = |k_{\mathbf{y}_1} - k_{\mathbf{y}_2}| + \sup_t h(t)\big|\big|[\mathbf{y}_1(t) - \mathbb{1}k_{\mathbf{y}}] - [\mathbf{y}_2(t) - \mathbb{1}k_{\mathbf{y}}]\big|\big|$$

and it is indeed a metric by the properties of the Definition of the Metric Space. Using the same line of arguments as in the proof of Proposition (5) we can show that if $\{\mathbf{y}_i\}$ is a Cauchy sequence in $(\mathcal{M}, \rho_\Delta)$ then

- For any fixed $t > 0$, $\mathbf{y}_i(t)$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$, consequently
- $k_{\mathbf{y}_i}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$, and of course
- $\sup_t h(t)||\mathbf{y}_i(t) - \mathbb{1}k_{\mathbf{y}_i}||$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$.

All these sequences converges uniformly in a function that is eligible to be a member of $\mathcal{M}$. □

Let us consider now a subset of $\mathcal{M}$ of particular interest is this where all member of this space agree on a (initial) point. That is:

$$\mathcal{M}_\phi := \big\{\mathbf{x} \in \mathcal{M} : \mathbf{x} \equiv \phi|_{[-\tau,0]}\big\} \tag{5.2}$$

This is obviously not a vector space and in order to prove completeness of this metric space we need an appropriate metric function and a completeness argument.

Proposition 8. *The space $\mathcal{M}_\phi$ together with the function*

$$\rho_h(\mathbf{y}_1, \mathbf{y}_2) := \sup_{t \geq 0} h(t)\big|\big|\big[\mathbf{y}_1(t) - \mathbb{1}k_{\mathbf{y}_1}\big] - \big[\mathbf{y}_2(t) - \mathbb{1}k_{\mathbf{y}_2}\big]\big|\big|$$

constitutes a complete metric space.

We will need two auxiliary lemmas:

Lemma 5.5. *The function ρ_h constitutes a metric in $\mathcal{M}_\phi$.*

Proof. From Definition (5.1) the only property that needs a little work is the first one. Indeed for $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{M}_\phi$ such that $\rho_h(\mathbf{y}_1, \mathbf{y}_2) = 0$ we have:

$$h(t)\big|\big|\big[\mathbf{y}_1(t) - \mathbb{1}k_{\mathbf{y}_1}\big] - \big[\mathbf{y}_2(t) - \mathbb{1}k_{\mathbf{y}_2}\big]\big|\big| \equiv 0 \Rightarrow \mathbf{y}_1(t) - \mathbb{1}k_{\mathbf{y}_1} = \mathbf{y}_2(t) - \mathbb{1}k_{\mathbf{y}_2}\ \forall t$$

since for $t = 0$: $\mathbf{y}_1(0) = \mathbf{y}_2(0)$, it follows that $k_{\mathbf{y}_1} = k_{\mathbf{y}_2}$ and thus $\mathbf{y}_1 \equiv \mathbf{y}_2$. □

Lemma 5.6. *For every $\epsilon > 0$ and $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{M}_\phi$, the following relation holds:*

$$\rho_h(\mathbf{x}_1, \mathbf{x}_2) < \epsilon \Rightarrow ||\mathbb{1}k_1 - \mathbb{1}k_2|| < \epsilon \Rightarrow |\mathbf{x}_1 - \mathbf{x}_2| < 2\epsilon$$

Proof. Note that $\rho_h(\mathbf{x}_1, \mathbf{x}_2) < \epsilon$ implies $h(t)\big|\big|\big[\mathbf{y}_1(t) - \mathbb{1}k_{\mathbf{y}_1}\big] - \big[\mathbf{y}_2(t) - \mathbb{1}k_{\mathbf{y}_2}\big]\big|\big| < \epsilon$ $\forall t$ and for $t = 0$ it follows that $||\mathbb{1}k_1 - \mathbb{1}k_2|| < \epsilon$. The final result follows from the triangular inequality. □

Proof of Proposition (8). In view of the above lemmas the process of proving this proposition is no different than the proof in Propositions (5) and (7). □

5.3. **The Complete Metric Spaces $(\mathcal{M}, \rho)$.** In this work we will use two particular types of metric spaces, for the case of linear time-invariant (LTI) and the case of linear time varying (LTV) systems. It is noted that the supremum norm is taken over $[-\tau, \infty)$.

LTI case.

$$\mathcal{M}_{LTI} = \big\{\mathbf{y} \in C([-\tau,\infty],\mathbb{R}^N) : \mathbf{y} = \phi|_{[-\tau,0]}, |\mathbf{y}| < \infty, \sup_{t\geq-\tau} e^{dt}||\mathbf{y}(t) - \mathbb{1}k|| < \infty\big\} \tag{LTIFS}$$

together with the metric

$$\rho_d(\mathbf{y}_1,\mathbf{y}_2) := \sup_{t\geq 0} e^{dt}\big|\big|\mathbf{y}_1(t) - \mathbf{y}_2(t)\big|\big| \tag{LTIMF}$$

LTV case.

$$\mathcal{M}_{LTV} = \big\{\mathbf{y} \in C([t_0-\tau,\infty],\mathbb{R}^N) : \mathbf{y} = \phi|_{[t_0-\tau,t_0]}, |\mathbf{y}| < \infty, \sup_{t\geq-\tau} e^{dt}||\mathbf{y}(t) - \mathbb{1}k_{\mathbf{y}}|| < \infty\big\} \tag{LTVFS}$$

together with the metric

$$\rho_d(\mathbf{y}_1,\mathbf{y}_2) := \sup_{t\geq 0} e^{dt}\big|\big|\big[\mathbf{y}_1(t) - \mathbb{1}k_{\mathbf{y}_1}\big] - \big[\mathbf{y}_2(t) - \mathbb{1}k_{\mathbf{y}_2}\big]\big|\big| \tag{LTVMF}$$

These metric spaces are complete with metrics as was proved in Proposition (8) and each member of them is a function which converge to a unique consensus point exponentially fast. While in the LTI case the consensus point is based on the constant connectivity weights, the fixed delays and the fixed initial conditions; in LTV systems along with the rest, one needs the information of the whole orbit and not just the initial conditions, since the weights vary with time. So the space of solutions needs to be expanded to allow functions that converge to their own consensus point, characterized implicitly by the flow of information that comes out of the time varying connectivity weights.

5.3.1. *Contraction Mapping Principle.* Given two metric spaces $(\mathcal{M}_i, \rho_{\mathcal{M}_i})$ for $i = 1,2$ an operator $P : \mathcal{M}_1 \to \mathcal{M}_2$ is a *contraction* if there exists a constant $\alpha \in [0,1)$ such that $x_1, x_2 \in \mathcal{M}$ imply

$$\rho_{\mathcal{M}_2}(Px_1, Px_2) \leq \alpha\rho_{\mathcal{M}_1}(x_1,x_2) \tag{5.3}$$

The next celebrated theorem will be used in proving our main results.

Theorem 5.7. *[Contraction Mapping Principle] Let $(\mathcal{M},\rho)$ be a complete metric space and $P : \mathcal{M} \to \mathcal{M}$ a contraction operator. Then there is a unique $x \in \mathcal{M}$ with $Px = x$. Furthermore, if $y \in \mathcal{M}$ and if $\{y_n\}$ is defined inductively by $y_1 = Py$ and $y_{n+1} = Py_n$ then $y_n \to x$, the unique fixed point. In particular, the equation $Px = x$ has one and only one solution.*

The proof of the theorem can be found in any advanced analysis or ordinary differential equations book. We refer the reader to [3] which is closest to our work. We are now ready to introduce our model, pose the assumptions and state the main result.

6. **The LTI Case.** In this section we will prove convergence to the consensus point and the rate at which this occurs for the case of linear time invariant symmetric communication weights. It is reminded that $||\cdot||$ stands for the $p=1$ norm in $\mathbb{R}^N$.

6.1. **Preliminary Results.** In this section, we review preliminary results which will be used as tools for the analysis to follow for both (IVP1) and (IVP2).

6.1.1. *The undelayed dynamics.* Equations (IVP1) without delays is a simple and well-studied system (see for example [10]). What is of importance to recall for this work is that the solution kernel e^{-Lt} takes any vector $\mathbf{z} \in \mathbb{R}^N$ which can be uniquely decomposed as $\mathbf{z} = \mathbf{z}_{//} + \mathbf{z}_c := \mathbb{1}\frac{1}{N}z_i + \mathbf{z}_c$ for some $\mathbf{z}_c \in \Delta^c$ and "suppress" the "magnitude" of $\mathbf{z}_c$ by $e^{-\lambda_2(L)t}$ so that $\lim_t e^{-Lt}\mathbf{z} = \mathbb{1}\frac{1}{N}z_i$. Another interesting view is that the quantity $\mathbb{I}(t) := \frac{1}{N}\sum_{i=1}^N x_i(t)$ is an integral of motion.

Next we state two technical lemmas to be used in the proof of the main result. We only prove the first one due to space limitations.

Lemma 6.1. *Let $\mathbf{z}(t) \in \mathbb{R}^N$ such that $\lim_{t\to\infty}\mathbf{z}(t)$ exists and is finite. Then for L the combinatorial Laplacian of an undirected connected graph we have:*

$$\lim_{t\to\infty} \mathbf{z}(t) - \int_0^t Le^{-L(t-s)}\mathbf{z}(s)ds = \mathbb{1}\frac{1}{N}\sum_{i=1}^N z_i(\infty) \tag{6.1}$$

Proof. Write $\mathbf{z}(t)$ as the sum of the vector projected onto the consensus subspace and it's complement, i.e.

$$\mathbf{z}(t) := \mathbf{z}_{//}(t) + \mathbf{z}_c(t) = \mathbb{1}\frac{1}{N}\sum_i z_i(t) + \mathbf{z}_c(t)$$

Then, since L and e^{sL} commute, the integral is equal to

$$\begin{aligned}\int_0^t Le^{Ls}\mathbf{z}(s)ds &= \int_0^t e^{Ls}L\mathbf{z}(s)ds \\ &= \int_0^t e^{Ls}L\mathbf{z}_c(s)ds = \int_0^t Le^{Ls}\mathbf{z}_c(s)ds\end{aligned}$$

and integration by parts yields

$$\int_0^t d(e^{Ls})\mathbf{z}_c(s) = e^{Lt}\mathbf{z}_c(t) - \mathbf{z}_c(0) - \int_0^t e^{Ls}\dot{\mathbf{z}}_c(s)ds$$

In view of the whole expression the integral

$$\begin{aligned}Q :&= \int_0^t e^{-L(t-s)}\dot{\mathbf{z}}_c(s)ds \\ &= \int_0^t \left(e^{-L(t-s)} - \frac{\mathbb{1}\mathbb{1}^T}{N}\right)\dot{\mathbf{z}}_c(s)ds \to 0\end{aligned} \tag{6.2}$$

by the standard argument that the convolution of an L^1 function (that is, $[e^{-L(t-s)} - \frac{\mathbb{1}\mathbb{1}^T}{N}]$) with a function that goes to zero (that is, $\dot{\mathbf{z}}(t)$), vanishes as well. So the whole expression converges to

$$\mathbf{z}_{//}(t) + \mathbf{z}_c(t) - \mathbf{z}_c(t) + e^{-Lt}\mathbf{z}_c(0) + Q \to \mathbf{z}_{//}(t)$$

∎

Lemma 6.2 (Bounds)**.** *For any $r \le s \le t$ the following relations hold:*

$$||e^{-L(t-s)}L|| \le \sqrt{N}\lambda_N e^{-\lambda_2(t-s)} \tag{6.3}$$

$$\int_r^t \left|\left|e^{-L(t-s)} - \frac{1}{N}\mathbb{1}\mathbb{1}^T\right|\right| ds \le \frac{\sqrt{N}}{\lambda_2}\left(1 - e^{-\lambda_2(t-r)}\right) \tag{6.4}$$

where $\lambda_2 > 0$ is the second largest eigenvalue of the weighted Laplacian L of a simply connected graph.

Proof. The integral I is upper bounded as follows: Let $\{\mathbf{u}_i\}_{i=1}^n$ be the set of normalized orthogonal eigenvectors of L corresponding to it's ordered eigenvalues with respect to the Euclidean Norm $||\cdot||_2$. Then any vector $\mathbf{x} \in \mathbb{R}^N$ can be written as $\mathbf{x} = \sum_{i=1}^n c_i\mathbf{u}_i$, where $||\mathbf{x}||_2 = \sqrt{\sum_{i=1}^N c_i^2}$. On the other hand,

$$e^{-L(t-s)} = \sum_{i=1}^N e^{-\lambda_i(t-s)}\mathbf{u}_i\mathbf{u}_i^T \tag{6.5}$$

where we take $\mathbf{u}_1 = \mathbb{1}/\sqrt{N}$ and $\lambda_1 = 0$. Then

$$e^{-L(t-s)}L\mathbf{x} = e^{-L(t-s)}L\sum_{i=1}^N c_i\mathbf{u}_i = \sum_{i=1}^N e^{-L(t-s)}Lc_i\mathbf{u}_i = \sum_{i=1}^N \lambda_i e^{-L(t-s)}c_i\mathbf{u}_i =$$

$$= \sum_{i=1}^N \lambda_i \sum_{j=1}^N e^{-\lambda_j(t-s)}(\mathbf{u}_j\mathbf{u}_j^T)c_i\mathbf{u}_i = \sum_{i=2}^N \lambda_i e^{-\lambda_i(t-s)}c_i\mathbf{u}_i$$

since $\max_i \lambda_i e^{-\lambda_i(t-s)} \le \lambda_N e^{-\lambda_2(t-s)}$ so

$$||e^{-L(t-s)}L\mathbf{x}||_2 = \left|\left|\sum_{i=2}^N \lambda_i e^{-\lambda_i(t-s)}c_i\mathbf{u}_i\right|\right|_2 = \sqrt{\sum_{i=2}^N c_i^2\lambda_i^2 e^{-2\lambda_i(t-s)}}$$

$$\le \lambda_N e^{-\lambda_2(t-s)}\sqrt{\sum_{i=2}^N c_i^2} \le \lambda_N e^{-\lambda_2(t-s)}||\mathbf{x}||_2$$

so for $0 \le s < t$ and the equivalence of norms on $\mathbb{R}^N$ we get

$$||e^{-L(t-s)}L|| = \max_{||\mathbf{x}||=1}||e^{-L(t-s)}L\mathbf{x}|| \le \sqrt{N}\max_{||\mathbf{x}||_2=1}||e^{-L(t-s)}L\mathbf{x}||_2 \le \sqrt{N}\lambda_N e^{-\lambda_2(t-s)}$$

This concludes the proof of (6.3). For (6.4) we use (6.5) and similar analysis to conclude. □

6.1.2. *Consensus Point.* At this moment, we will make an **Ansatz**:

All solutions of (IVP1) *and* (IVP2) *tend to some constants in Δ, $\mathbb{1}k_{(IVP1)}$ and $\mathbb{1}k_{(IVP2)}$, respectively.*

In view of this educated guess,we take the limit $t \to \infty$ so that $x_i(t) \to k$ for all i and solve for k to obtain (CNS).

Proposition 9. *An integral of motion for* (IVP1) *is*

$$\mathbb{I}_{(\text{IVP1})}(t) = \sum_{i=1}^N x_i(t) + \sum_{i,j} a_{ij}\int_{t-\tau_i^j}^t x_j(s)ds \tag{6.6}$$

Proof. For (IVP1) take

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^{N} x_i =& \sum_{i=1}^{N}\sum_{j\in N_i} a_{ij}(x_j - x_i) - \frac{d}{dt}\sum_{i=1}^{N}\sum_{j\in N_i} a_{ij}\int_{t-\tau_i^j}^{t} x_j(s)ds \\ &= -\frac{d}{dt}\sum_{i=1}^{N}\sum_{j\in N_i} a_{ij}\int_{t-\tau_i^j}^{t} x_j(s)ds \Rightarrow\end{aligned}$$

$$\sum_{i=1}^{N} x_i(t) - \sum_{i=1}^{N} x_i(0) = -\sum_{i=1}^{N}\sum_{j\in N_i} a_{ij}\int_{t-\tau_i^j}^{t} x_j(s)ds + \sum_{i=1}^{N}\sum_{j\in N_i} a_{ij}\int_{-\tau_i^j}^{0} \phi_j(s)ds$$

to derive $\mathbb{I}_{(\text{IVP1})}(t)$. $\square$

From $\mathbb{I}_{(\text{IVP1})}(t)$ one can calculate the consensus point k for (IVP1), assuming for a moment that all solutions converge to $\mathbb{1}k$. Then k must satisfy:

$$k_{(\text{IVP1})} := \frac{\sum_i^N \phi_i(0) + \sum_{i,j} a_{ij}\int_{-\tau_i^j}^{0}\phi_j(s)ds}{N + \sum_{i,j} a_{ij}\tau_i^j} \tag{CNS}$$

6.1.3. *Convergence.* We rewrite (IVP1) as follows

$$\dot{x}_i(t) = \sum_{j\in N_i} a_{ij}(x_j - x_i) - \frac{d}{dt}\sum_{j\in N_i} a_{ij}\int_{t-\tau_i^j}^{t} x_j(s)ds$$

In the vector form, we get for $\mathbf{x} = (x_1, \ldots, x_N)^T \in \mathbb{R}^N$

$$\dot{\mathbf{x}} = -L\mathbf{x} - \frac{d}{dt}\sum_{i,j}\int_{t-\tau_i^j}^{t} A_i^j \mathbf{x}(s)ds \tag{6.7}$$

where L is the Laplacian matrix and $A_i^j = [A_i^j]_{m,n} = [a_{m,n}\delta_{i,j}]$ are matrices with zero elements that are not in the i,j position.

The general solution using the variation of constants formula and integration by parts becomes:

$$\begin{aligned}\mathbf{x}(t) &= e^{-Lt}\mathbf{x}(0) - \int_0^t e^{-L(t-s)}\frac{d}{ds}\sum_{i,j}\int_{s-\tau_i^j}^{s} A_i^j\mathbf{x}(u)duds \\ &= e^{-Lt}\mathbf{x}(0) - \sum_{i,j}\int_{t-\tau_i^j}^{t} A_i^j\mathbf{x}(u)du \\ &+ e^{-Lt}\sum_{i,j}\int_{-\tau_i^j}^{0} A_i^j\phi(u)du \\ &+ e^{-Lt}\int_0^t Le^{Ls}\sum_{i,j}\int_{s-\tau_i^j}^{s} A_i^j\mathbf{x}(u)duds \\ &=: I_1(t) + I_2(t) + I_3(t) + I_4(t)\end{aligned}$$

We will consider the weighted metric space $(\mathcal{M}_k, \rho_d)$ as it was defined in (LTIFS) (and use the notation $(\mathcal{M}, \rho_d)$). The fixed point argument is the implementation of the Contraction Mapping Principle (Theorem (5.7)) and consists of the following steps: We define an appropriate operator with prescribed smoothness properties, we show that this operator maps $\mathcal{M}$ onto itself and we show that this operator is a contraction in $(\mathcal{M}, \rho_d)$. So for $\mathbf{y} \in \mathcal{M}$ and $t \geq -\tau$, we define the operator P by

$$(P\mathbf{y})(t) := \begin{cases} \phi(t), \ -\tau \leq t \leq 0 \\ \\ e^{-Lt}\big(\phi(0) + \sum_{i,j} \int_{-\tau_i^j}^{0} A_i^j \phi(u)du\big) - \sum_{i,j} \int_{t-\tau_i^j}^{t} A_i^j \mathbf{y}(u)du + \\ +e^{-Lt} \int_0^t Le^{Ls} \sum_{i,j} \int_{s-\tau_i^j}^{s} A_i^j \mathbf{y}(u)duds, t > 0 \end{cases}$$

For convenience, set $\mathbf{z}(t) := -\sum_{i,j} \int_{t-\tau_j^i}^{t} A_i^j \mathbf{x}(u)du$.

Proposition 10. *The operator P possesses the following properties:*

1. *P is a continuous function of time for any $t > 0$.*
2. *$P : \mathcal{M} \to \mathcal{M}$ under assumption (H.1.4).*
3. *P is a contraction under assumption (H.1.5).*

Proof of Proposition (10). The first statement follows trivially by the definition. The second requires to prove that that P is bounded and that it converges to a value $\mathbb{1}k \in \Delta$ exponentially fast with rate at most d. Since by hypothesis the communication graph is simply connected, the terms of P for $t > 0$ converge as follows:

$$\lim_t (I_1 + I_3)(t) = \mathbb{1}\frac{1}{N}\bigg(\sum_{i=1}^{N} \phi_i(0) + \sum_{i,j} a_{ij} \int_{-\tau_i^j}^{0} \phi_j(s)ds\bigg)$$

by the discussion in section (IV.A), and by Lemma (6.1)

$$\lim_t (I_2 + I_4)(t) = -\mathbb{1}\frac{1}{N}\sum_{i,j} a_{ij}\tau_j^i k$$

if $\mathbf{x}(t) \to \mathbb{1}k$. Combine the results above to conclude that the operator $\mathcal{P}$ indeed converges to $\mathbb{1}k$ just like all the members of $\mathcal{M}$ only if k is defined as in (CNS). Then another useful expression for k for any $\mathbf{x} \in \mathcal{M}$ comes from the variation of constants formula:

$$\mathbb{1}k = \mathbb{1}\frac{1}{N}\sum_i \phi_i(0) + \frac{\mathbb{1}\mathbb{1}^T}{N}\int_0^\infty \frac{d}{ds}\sum_{i,j}\int_{s-\tau_i^j}^{s} A_i^j \mathbf{x}(u)duds \tag{6.8}$$

It remains to show that $\sup_t h(t)||(P\mathbf{y})(t) - \mathbb{1}k|| < \infty$. We break the components of k in the form of (6.8) and use the triangular inequality to obtain the estimates.

$$\begin{aligned} ||(P\mathbf{y})(t) - \mathbb{1}k|| &\leq \left|\left|e^{-Lt} - \frac{\mathbb{1}\mathbb{1}^T}{N}\right|\right| \cdot ||\phi(0)|| \\ &+ \int_0^t \left|\left|e^{-L(t-s)} - \frac{\mathbb{1}\mathbb{1}^T}{N}\right|\right| \cdot \left|\left|\frac{d}{ds}\sum_{i,j}\int_{s-\tau_i^j}^{s} A_i^j \mathbf{x}(u)du\right|\right| ds \\ &+ \int_t^\infty \left|\left|\frac{\mathbb{1}\mathbb{1}^T}{N}\frac{d}{ds}\sum_{i,j}\int_{s-\tau_i^j}^{s} A_i^j \mathbf{x}(u)du\right|\right| ds \end{aligned}$$

It is an easy exercise to see that the first two parts converge to 0 with rate $\lambda_2(L)$ whereas the last part converges with rate d.

So the claim follows under the assumption that

$$d < \lambda_2(L) \tag{H.1.4}$$

The last part is to show that $(P\mathbf{y})(t)$ is a contraction in $(\mathcal{M}, \rho_d)$. Recall the notations from sections (III) and (IV) and consider $t > 0$, $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{M}$. Then

$$\begin{aligned}
&||(P\mathbf{y}_1)(t) - (P\mathbf{y}_2)(t)|| \leq \\
&\leq \left|\left| \sum_{i,j} A_i^j \int_{t-\tau_i^j}^t \big(\mathbf{y}_1(s) - \mathbf{y}_2(s)\big) ds \right|\right| + \\
&+ \left|\left| e^{-Lt} \int_0^t L e^{Ls} \sum_{i,j} A_i^j \int_{s-\tau_i^j}^s (\mathbf{y}_1(w) - \mathbf{y}_2(w)) dw ds \right|\right|
\end{aligned}$$

The first term can be bounded as follows:

$$\begin{aligned}
&\left|\left| \sum_{i,j} A_i^j \int_{t-\tau_i}^t \big(\mathbf{y}_1(s) - \mathbf{y}_2(s)\big) ds \right|\right| \leq \sum_{i,j} |a_{ij}| \int_{t-\tau}^t |\big(\mathbf{y}_1(s) - \mathbf{y}_2(s)\big)| ds \\
&\leq e^{-dt} \frac{e^{d\tau} - 1}{d} \left(\sum_{i,j} a_{ij} \right) \rho_d(\mathbf{y}_1, \mathbf{y}_2)
\end{aligned}$$

The second term can be bounded as follows:

$$\begin{aligned}
&\left|\left| \int_0^t e^{-L(t-s)} L \sum_{i,j} A_i^j \int_{s-\tau_i^j}^s (\mathbf{y}_1(w) - \mathbf{y}_2(w)) dw ds \right|\right| \leq \\
&\leq \sum_{i,j} a_{ij} \int_0^t ||e^{-L(t-s)} L|| \int_{s-\tau}^s ||\mathbf{y}_1 - \mathbf{y}_2|| ds \\
&\leq e^{-dt} \frac{e^{d\tau} - 1}{d} \sum_{i,j} a_{ij} \left[\frac{\sqrt{N} \lambda_N(L)}{\lambda_2(L) - d} \right] \rho_d(\mathbf{y}_1, \mathbf{y}_2)
\end{aligned}$$

The result follows on condition that there exists $0 \leq \alpha < 1$ such that

$$A \frac{e^{d\tau} - 1}{d} \left(1 + \frac{\sqrt{N} \lambda_N(L)}{\lambda_2(L) - d} \right) \leq \alpha \tag{H.1.5}$$

for some $d < \lambda_2(L)$ according to (H.1.4). □

Solving (H.1.5) for τ we get:

$$\tau < \frac{1}{d} \ln \left(1 + \frac{d}{A} \frac{\lambda_2(L) - d}{\lambda_2(L) - d + \sqrt{N} \lambda_N(L)} \right) \tag{6.9}$$

6.2. **The non-symmetric static case:** Following the discussion in [10], our result can be generalized in the case of non-symmetric constant weights at the expense of more analysis. This is the case of directed networks and the sufficient assumption is for the corresponding graph to contain a routed out sub-graph. Such is the case when the G is strongly connected. Then the (in-degree) Laplacian for the digraph G can be Jordan decomposed as

$$L(G) = PJ(\Lambda)P^{-1} = P \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & J(\lambda_2) & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & J(\lambda_n) \end{pmatrix} P^{-1} \tag{6.10}$$

where n is not necessarily equal to N and λ_i have positive real parts for $i \geq 2$ and the absolute value of which can be ordered just like the case of the combinatorial

Laplacian. Suppose that $\mathrm{Re}(\lambda_2) = \max_i \mathrm{Re}(\lambda_i)$Then the operator $e^{-L(G)t} \to \mathbf{p}_1\mathbf{q}_1^T$ where $\mathbf{p}_1$ is the first column of P and $\mathbf{q}_1$ is the first row of P^{-1} such that $\mathbf{q}_1^T\mathbf{p}_1 = 1$.

Choosing P such that $\mathbf{p}_1 = \mathbb{1}$ it follows that $e^{-L(G)t}$ converges to a rank 1 matrix.

The quantity $\mathbf{q}^T\mathbf{x}$ is conserved, where $\mathbf{q}$ is the left eigenvector of the in-degree Laplacian associated with the zero eigenvalue. The analysis is not much different than the symmetric case. There are two parts one needs to pay attention.

1. The consensus point in this case is
$$k := \frac{\sum_i^N q_i\phi_i(0) + \sum_{i=1}^N \sum_{j\in N_i} q_i a_{ij} \int_{-\tau_i^j}^0 \phi_j(s)ds}{1 + \sum_{i=1}^N \sum_{j\in N_i} q_i a_{ij}\tau_i^j} \tag{6.11}$$
2. The bounds in Lemma (6.2) needs to be taken special care of. From (6.10) we get
$$e^{-L(t-s)}L = P \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & J(\lambda_2)e^{J(-\lambda_2)(t-s)} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & J(\lambda_n)e^{J(-\lambda_n)(t-s)} \end{pmatrix} P^{-1} \tag{6.12}$$
A tedious manipulation of this matrix will reveal that there exists a constant $R > 0$ which effectively depends on the maximum dimension of the generalized eigenspace such that
$$||e^{-L(t-s)}L|| < R|\lambda_n(L)|e^{-\mathrm{Re}(\lambda_2(L))(t-s)} \tag{6.13}$$
The calculation of R is rather unclear and a sharper upper bound must be established especially when we talk about actual calculations on graphs.

7. **Time Varying Weights and Constant delays.** We restate the initial value problem and recall the assumptions $(H.2.1-3)$

$$\begin{aligned}\dot{x}_i &= \sum_{j\in N_i} a_{ij}(t)(x_j^i - x_i),\ t\geq t_0 \\ x_i(t) &= \phi_i(t),\ t\in[t_0-\tau_i^j, t_0]\end{aligned}\tag{IVP2}$$

We rewrite (IVP2)

$$\dot{\mathbf{x}}(t) = -L(t)\mathbf{x}(t) - \sum_{i=1}^N\sum_{j\in N_i} A_i^j(t)\frac{d}{dt}\int_{t-\tau_i^j}^t \mathbf{x}(s)ds =: -L(t)\mathbf{x}(t) - \mathbf{w}(t)\tag{7.1}$$

$$\mathbf{x}(t) = \mathbf{\Phi}(t,t_0)\mathbf{x}(t_0) - \int_{t_0}^t \mathbf{\Phi}(t,s)\mathbf{w}(s)ds\tag{7.2}$$

and integrating by parts [4]

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{\Phi}(t,t_0)\mathbf{x}(t_0) - \int_{t_0}^t \mathbf{\Phi}(t,s)\sum_{i,j} A_i^j(s)\frac{d}{ds}\int_{s-\tau_i^j}^s \mathbf{x}(w)dwds \\
&= \mathbf{\Phi}(t,t_0)\mathbf{x}(t_0) - \int_{t_0}^t \sum_{i,j}\mathbf{\Phi}(t,s) A_i^j(s)\frac{d}{ds}\int_{s-\tau_i^j}^s \mathbf{x}(w)dwds \\
&= \mathbf{\Phi}(t,t_0)\mathbf{x}(t_0) - \sum_{i,j} A_i^j(t)\int_{t-\tau_i^j}^t \mathbf{x}(s)ds + \sum_{i,j}\mathbf{\Phi}(t,t_0)A_i^j(t_0)\int_{t_0-\tau_i^j}^{t_0}\phi(s)ds+ \\
&+ \sum_{i,j}\int_{t_0}^t \frac{d}{ds}\Big[\mathbf{\Phi}(t,s)A_i^j(s)\Big]\int_{s-\tau_i^j}^s \mathbf{x}(w)dwds \\
&= \mathbf{\Phi}(t,t_0)\Big(\mathbf{x}(t_0) + \sum_{i,j} A_i^j(t_0)\int_{t_0-\tau_i^j}^{t_0}\phi(s)ds\Big) - \sum_{i,j} A_i^j(t)\int_{t-\tau_i^j}^t \mathbf{x}(s)ds+ \\
&+ \sum_{i,j}\int_{t_0}^t \mathbf{\Phi}(t,s)\Big[L(s)A_i^j(s) + \dot{A}_i^j(s)\Big]\int_{s-\tau_i^j}^s \mathbf{x}(w)dwds \\
&= \mathbf{\Phi}(t,t_0)\Big(\mathbf{x}(t_0) + \sum_{i,j} A_i^j(t_0)\int_{t_0-\tau_i^j}^{t_0}\phi(s)ds\Big) - \sum_{i,j} A_i^j(t)\int_{t-\tau_i^j}^t \mathbf{x}(s)ds+ \\
&+ \int_{t_0}^t \mathbf{\Phi}(t,s)L(s)\sum_{i,j} A_i^j(s)\int_{s-\tau_i^j}^s \mathbf{x}(w)dwds + \int_{t_0}^t \mathbf{\Phi}(t,s)\sum_{i,j}\dot{A}_i^j(s)\int_{s-\tau_i^j}^s \mathbf{x}(w)dwds\end{aligned}$$

set $\mathbf{y}(t) := \sum_{i,j} A_i^j(t)\int_{t-\tau_i^j}^t \mathbf{x}(s)ds$ and $\mathbf{z}(t) := \sum_{i,j}\dot{A}_i^j(t)\int_{t-\tau_i^j}^t \mathbf{x}(s)ds$ so that the expression becomes

$$\mathbf{x}(t) = \mathbf{\Phi}(t,t_0)\big(\mathbf{x}(t_0)+\mathbf{y}(t_0)\big) - \mathbf{y}(t) + \int_{t_0}^t \mathbf{\Phi}(t,s)L(s)\mathbf{y}(s)ds + \int_{t_0}^t \mathbf{\Phi}(t,s)\mathbf{z}(s)ds\tag{7.3}$$

Consider the complete metric space $(\mathcal{M},\rho_d)$ as defined in (LTVFS) and (LTVMF). Recall that $\mathcal{M} = \mathcal{M}_{LTV}$ is the function space of all continuous functions $\mathbf{x}:[t_0-\tau,\infty)\to\mathbb{R}^N$ such that $\mathbf{x}(t)=\phi(t)$ for $t_0-\tau\leq t\leq t_0$, $\sup_{t\geq t_0-\tau} e^{dt}||\mathbf{x}(t)-\mathbb{1}k_{\mathbf{x}}||<\infty$

[4] It is reminded that $\sum_{i,j}$ stands for $\sum_{i=1}^N\sum_{j\in N_i}$

for some unique $k_{\mathbf{x}} \in \mathbb{R}$ and some $d > 0$ to be determined. Define the operator P by

$$(P\mathbf{x})(t) := \begin{cases} \phi(t), \ t_0 - \tau \leq t \leq t_0 \\ \mathbf{\Phi}(t, t_0)\big(\mathbf{x}(t_0) + \mathbf{y}(t_0)\big) - \mathbf{y}(t) + \int_{t_0}^{t} \mathbf{\Phi}(t, s) L(s)\mathbf{y}(s)ds + \int_{t_0}^{t} \mathbf{\Phi}(t, s)\mathbf{z}(s)ds \ , \ t > t_0 \end{cases} \tag{F1}$$

Remark 5. The operator P is the solution expression (7.3) and for any $\mathbf{x} \in \mathcal{M}$ and $t > t_0$ it can be stated in the following equivalent forms:

- The form:

$$(P\mathbf{x})(t) = \phi(0) - \int_{t_0}^{t} L(s)\mathbf{x}(s)ds - \int_{t_0}^{t} \sum_{i,j} A_i^j(s) \frac{d}{ds} \int_{s-\tau_i^j}^{s} \mathbf{x}(w)dwds \tag{F2}$$

  This form comes from direct integration of the vector for (7.2) and will be used as a part of the claim that P takes values in $\mathcal{M}$.
- Another useful from is the following. From (7.2) and for $\mathbf{x} \in \mathcal{M}$ and $t > t_0$ and we write:

$$\begin{aligned}(P\mathbf{x})(t) &= \mathbf{\Phi}(t, t_0)\phi(0) - \int_{t_0}^{t} \mathbf{\Phi}(t, s) \sum_{i,j} A_i^j(s) \frac{d}{ds} \int_{s-\tau_i^j}^{s} (\mathbf{x}(w) - \mathbb{1}k_{\mathbf{x}})dwds \\ &= \mathbf{\Phi}(t, t_0)\phi(0) - \int_{t_0}^{t} \Big(\mathbf{\Phi}(t, s) - \mathbb{1}\mathbf{c}^T\Big) \sum_{i,j} A_i^j(s) \frac{d}{ds} \int_{s-\tau_i^j}^{s} (\mathbf{x}(w) - \mathbb{1}k_{\mathbf{x}})dwds - \\ &\quad - \int_{t_0}^{t} \mathbb{1}\mathbf{c}^T \sum_{i,j} A_i^j(s) \frac{d}{ds} \int_{s-\tau_i^j}^{s} (\mathbf{x}(w) - \mathbb{1}k_{\mathbf{x}})dwds \\ &= \mathbf{\Phi}(t, t_0)\phi(0) - \int_{t_0}^{t} \Big(\mathbf{\Phi}(t, s) - \mathbb{1}\mathbf{c}^T\Big) \sum_{i,j} A_i^j(s) \frac{d}{ds} \int_{s-\tau_i^j}^{s} (\mathbf{x}(w) - \mathbb{1}k_{\mathbf{x}})dwds - \\ &\quad - \mathbb{1}\mathbf{c}^T \sum_{i,j} A_i^j(t) \int_{t-\tau_i^j}^{t} (\mathbf{x}(w) - \mathbb{1}k_{\mathbf{x}})dwds + \mathbb{1}\mathbf{c}^T \sum_{i,j} A_i^j(t_0) \int_{t_0-\tau_i^j}^{t_0} (\mathbf{x}(w) - \mathbb{1}k_{\mathbf{x}})dwds + \\ &\quad + \mathbb{1}\mathbf{c}^T \int_{t_0}^{t} \sum_{i,j} \dot{A}_i^j(s) \int_{s-\tau_i^j}^{s} (\mathbf{x}(w) - \mathbb{1}k_{\mathbf{x}})dwds \end{aligned} \tag{F3}$$

  This expression will be used as a part of the claim that P takes values in $\mathcal{M}$ at the point where $(Px)(t) \to \mathbb{1}k_{(P\mathbf{x})}$.

The first crucial step is to describe the sufficient conditions under which P is an operator that takes values in $\mathcal{M}$.

Proposition 11. *Under Hypothesis (H.2.5) and $d < \gamma$, for any $\mathbf{x} \in \mathcal{M}$, and $t \geq t_0 - \tau$, the function $(P\mathbf{x})(t)$ as defined in (F1) with the equivalent forms (F2) and (F3), takes values in $\mathcal{M}$.*

To prove this crucial step we need to show that $(P\mathbf{x})(t)$ is a function that meets the requirements of $\mathcal{M}$. Before doing so we need the following analysis on the asymptotic behaviour of $(P\mathbf{x})(t)$ in t.

7.1. **Asymptotic behaviour of $(P\mathbf{x})$.** From (F3) we see that for $t \to \infty$ the first term converges to

$$\mathbb{1}\mathbf{c}^T\phi(0)$$

the second term converges to zero as a convolution of an L^1 function (that is $\mathbf{\Phi}(t,s)-\mathbb{1}\mathbf{c}^T$ and a function that goes to zero. The third term is handled by the following integration by parts:

$$\mathbb{1}\mathbf{c}^T\Big(\sum_{i,j}A_i^j(t)\int_{t-\tau_i^j}^{t}(\mathbf{x}(w)-\mathbb{1}k_{\mathbf{x}})dw-\sum_{i,j}A_i^j(t_0)\int_{t_0-\tau_i^j}^{t_0}(\phi(w)-\mathbb{1}k_{\mathbf{x}})dw\Big)-$$
$$-\mathbb{1}\mathbf{c}^T\sum_{i,j}\int_{t_0}^{t}\dot{A}_i^j(s)\int_{s-\tau_i^j}^{s}(\mathbf{x}(w)-\mathbb{1}k_{\mathbf{x}})dwds$$

as $t\to\infty$ the first part converges to zero, the second is constant and the third converges in Δ in view of Hypothesis (H.2.3). So, all in all,

$$\lim_{t\to}(P\mathbf{x})(t)=\mathbb{1}\mathbf{c}^T\Big(\phi(0)+\sum_{i,j}A_i^j(t_0)\int_{t_0-\tau_i^j}^{t_0}(\phi(w)-\mathbb{1}k_{\mathbf{x}})dw$$
$$+\sum_{i,j}\int_{t_0}^{\infty}\dot{A}_i^j(s)\int_{s-\tau_i^j}^{s}(\mathbf{x}(w)-\mathbb{1}k_{\mathbf{x}})dwds\Big)$$

Since by the definition for any $\mathbf{x}\in\mathcal{M}$ there exists a unique point in Δ to which $\mathbf{x}(t)$ converges with rate e^{dt}, this should be the case for $(P\mathbf{x})(t)$. We will derive, now the sufficient conditions of the existence and uniqueness of this point which will be identical to $\mathbb{1}k_{\mathbf{x}}$, i.e.

$$k_{(P\mathbf{x})}=k_{\mathbf{x}} \tag{7.4}$$

For $a_{ij}(t)$ as defined in the Hypotheses $(H.2.-)$, define the operator $Q:\mathcal{C}([t_0-\tau,\infty),\mathbb{R}^N])\times[t_0,\infty)\times\mathbb{R}\to\mathbb{R}$ by

$$\begin{aligned}Q(\mathbf{x},t,k)=\sum_{i=1}^{N}c_i\Big[\phi_i(0)+\sum_{j\in N_i}a_{ij}(t_0)\int_{t_0-\tau_i^j}^{t_0}(x_j(w)-k)dw\\+\sum_{j\in N_i}\int_{t_0}^{t}\dot{a}_{ij}(s)\int_{s-\tau_i^j}^{s}(x_j(w)-k)dwds\Big]\end{aligned} \tag{7.5}$$

Lemma 7.1. *If there exists $\alpha\in[0,1)$ such that*

$$\sup_{t\geq t_0}\Big|\sum_{i=1}^{N}c_i\sum_{i\in N_i}a_{ij}(t)\tau_i^j\Big|\leq\alpha \tag{7.6}$$

then for any $\mathbf{x}\in\mathcal{C}([t_0-\tau,\infty),\mathbb{R}^N)$ and $t\geq t_0$ there exists a unique solution to the equation

$$Q(\mathbf{x},t,k)=k$$

Proof. To prove this lemma we will again use Fixed Point Theory in the complete metric space $(\mathbb{R},|\cdot|)$ where the metric is the classic absolute value. Obviously Q is a continuous function of k. It only needs to be shown that it is a contraction in $(\mathbb{R},|\cdot|)$. So for any $k_1,k_2\in\mathbb{R}$ we get

$$|Q(\mathbf{x},t,k_1)-Q(\mathbf{x},t,k_2)|\leq\Big|\sum_{i=1}^{N}c_i\sum_{j\in N_i}a_{ij}(t)\tau_i^j\Big||k_1-k_2|$$

which is a contraction by condition (7.6). □

This lemma can be used in our case, since $\mathcal{M}$ is a subset of $\mathcal{C}([t_0-\tau,\infty),\mathbb{R}^N)$ and as $t\to\infty$ the integral in the last part of Q is well defined and convergent so that $\lim_t Q(\mathbf{x},t,k_{\mathbf{x}})=k_{\mathbf{x}}$ makes sense.

7.2. **Convergence analysis.** We now turn to the proof of Proposition 8:

Proof of Proposition (11). To show that $(P\mathbf{x})$ is in $\mathcal{M}$ for any $\mathbf{x}\in\mathcal{M}$ we prove the following assertions

1. $(P\mathbf{x})$ is a continuous function of time. It follows by it's definition.
2. $(P\mathbf{x})(t)\to \mathbb{1}k_{(P\mathbf{x})}$. The existence and uniqueness of the consensus point for the operator is established in Lemma (7.1), under hypothesis $(H.2.5)$ which implies (7.6).
3. $\sup_{t\geq t_0} e^{dt}||(P\mathbf{x})(t)-\mathbb{1}k_{\mathbf{x}}||<\infty$ if $d<\gamma$.

   To prove the last claim we will use (F3). In view of the fact that $k_{\mathbf{x}}$ is the unique solution of

$$k_{\mathbf{x}}=\mathbf{c}^T\phi(0)+\mathbf{c}^T\sum_{i,j}A_{ij}(t_0)\int_{t_0-\tau_i^j}^{t_0}(\phi(w)-\mathbb{1}k_{\mathbf{x}})dw+ \\ +\mathbf{c}^T\sum_{i,j}\int_{t_0}^{\infty}\dot{A}_i^j(s)\int_{s-\tau_i^j}^{s}(\mathbf{x}(w)-\mathbb{1}k_{\mathbf{x}})dwds$$

   so we use (F3) to write

$$\begin{aligned}||(P\mathbf{x})(t)-\mathbb{1}k_{(P\mathbf{x})}||&=||(P\mathbf{x})(t)-\mathbb{1}k_{\mathbf{x}}||\leq\\ &\leq\left|\left|\left(\mathbf{\Phi}(t,t_0)-\mathbb{1}\mathbf{c}^T\right)\phi(0)\right|\right|+\left|\left|\mathbb{1}\mathbf{c}^T\sum_{i,j}A_i^j(t)\int_{t-\tau_i^j}^{t}(\mathbf{x}(w)-\mathbb{1}k_{\mathbf{x}})dw\right|\right|+\\ &+\left|\left|\int_{t_0}^{t}\left(\mathbf{\Phi}(t,s)-\mathbb{1}\mathbf{c}^T\right)\sum_{i,j}A_i^j(s)\frac{d}{ds}\int_{s-\tau_i^j}^{s}(\mathbf{x}(w)-\mathbb{1}k_{\mathbf{x}})dwds\right|\right|+\\ &+\left|\left|\mathbb{1}\mathbf{c}^T\sum_{i,j}\int_{t}^{\infty}\dot{A}_i^j(s)\int_{s-\tau_i^j}^{s}(\mathbf{x}(w)-\mathbb{1}k_{\mathbf{x}})dwds\right|\right|\end{aligned}$$

   The first term is bounded by $e^{-\gamma t}$, the second is bounded by e^{-dt}, the third by $e^{(\gamma-d)t}$ and the fourth by $e^{-d\tau}$. The claim then follows by the imposed condition.

$$d<\gamma \tag{H.2.6a}$$

This concludes the proof of Proposition (11). ☐

The last part is to prove that P is a contraction in $\mathcal{M}$ for every $t\geq t_0-\tau$. For $t\in[t_0-\tau,t_0]$ the result follows by definition. For $t>t_0$ and any $\mathbf{x}_1,\mathbf{x}_2\in\mathcal{M}$, we need to show that

$$\rho_d((P\mathbf{x}_1),(P\mathbf{x}_2))\leq\alpha\rho_d(\mathbf{x}_1,\mathbf{x}_2)$$

for some $\alpha \in [0,1)$. To simplify analysis set $\mathbb{x}_{12}(s) := \big((\mathbf{x}_1(s) - \mathbb{1}k_{\mathbf{x}_1}) - (\mathbf{x}_2(s) - \mathbb{1}k_{\mathbf{x}_2})\big)$. For $t > t_0$

$$
\begin{aligned}
&\big|\big|\big[(P\mathbf{x}_1)(t) - \mathbb{1}k_{(P\mathbf{x}_1)}\big] - \big[(P\mathbf{x}_2)(t) - \mathbb{1}k_{(P\mathbf{x}_2)}\big]\big|\big| \\
&= \Bigg|\Bigg| - \sum_{i,j} A_i^j(t) \int_{t-\tau_i^j}^{t} \mathbb{x}_{12}(s)ds + \int_{t_0}^{t} \big(\mathbf{\Phi}(t,s) - \mathbb{1}\mathbf{c}^T\big) L(s) \sum_{i,j} A_i^j(s) \int_{s-\tau_i^j}^{s} \mathbb{x}_{12}(w)dwds + \\
&+ \int_{t}^{\infty} \mathbb{1}\mathbf{c}^T L(s) \sum_{i,j} A_i^j(s) \int_{s-\tau_i^j}^{s} \mathbb{x}_{12}(w)dwds + \\
&+ \int_{t_0}^{t} \big(\mathbf{\Phi}(t,s) - \mathbb{1}\mathbf{c}^T\big) \sum_{i,j} \dot{A}_i^j(s) \cdot \int_{s-\tau_i^j}^{s} \mathbb{x}_{12}(w)dwds + \mathbb{1}\mathbf{c}^T \int_{t}^{\infty} \sum_{i,j} \dot{A}_i^j(s) \cdot \int_{s-\tau_i^j}^{s} \mathbb{x}_{12}(w)dwds \Bigg|\Bigg|
\end{aligned}
\tag{7.7}
$$

the first term is bounded as follows

$$
\Bigg|\Bigg| \sum_{i,j} A_i^j(t) \int_{t-\tau_i^j}^{t} \mathbb{x}_{12}(s)ds \Bigg|\Bigg| \leq \frac{e^{d\tau} - 1}{d} \bigg(\sum_{i,j} a_{i,j}(t) \bigg) e^{-dt} \rho_d(\mathbf{x}_1, \mathbf{x}_2)
$$

The second and third terms are bounded as follows

$$
\begin{aligned}
&\Bigg|\Bigg| \int_{t_0}^{t} \big(\mathbf{\Phi}(t,s) - \mathbb{1}\mathbf{c}^T\big) L(s) \sum_{i,j} A_i^j(s) \int_{s-\tau_i^j}^{s} \mathbb{x}_{12}(w)dwds \Bigg|\Bigg| + \\
&\Bigg|\Bigg| \int_{t}^{\infty} \mathbb{1}\mathbf{c}^T L(s) \sum_{i,j} A_i^j(s) \int_{s-\tau_i^j}^{s} \mathbb{x}_{12}(w)dwds \Bigg|\Bigg| \leq \\
&\Gamma \frac{e^{d\tau} - 1}{d(\gamma - d)} \sup_t \bigg\{ ||L(t)|| \sum_{i,j} a_{ij}(t) \bigg\} (1 - e^{-(\gamma-d)(t-t_0)}) e^{-dt} \rho_d(\mathbf{x}_1, \mathbf{x}_2) + \\
&M \bigg(\sup_t \big\{ \sum_{i,j} a_{ij}(t) \big\} \bigg) \frac{e^{d\tau} - 1}{d(d+\delta)} e^{-dt} \rho_d(\mathbf{x}_1, \mathbf{x}_2)
\end{aligned}
$$

Finally the last two terms in (7.7) are bounded as follows:

$$
\begin{aligned}
&\Bigg|\Bigg| \int_{t_0}^{t} \big(\mathbf{\Phi}(t,s) - \mathbb{1}\mathbf{c}^T\big) \sum_{i,j} \dot{A}_i^j(s) \cdot \int_{s-\tau_i^j}^{s} \mathbb{x}_{12}(w)dwds \Bigg|\Bigg| \\
&\Bigg|\Bigg| \mathbb{1} \int_{t}^{\infty} \sum_{i,j} \mathbf{c}^T \dot{A}_i^j(s) \cdot \int_{s-\tau_i^j}^{s} \mathbb{x}_{12}(w)dwds \Bigg|\Bigg| \leq \\
&\frac{e^{d\tau} - 1}{d(\gamma - d)} \sup_t \bigg\{ \sum_{i,j} |\dot{a}_{ij}(t)| \bigg\} \Gamma (1 - e^{-(\gamma-d)(t-t_0)}) e^{-dt} \rho_d(\mathbf{x}_1, \mathbf{x}_2) \\
&+ \frac{e^{d\tau} - 1}{d^2} \sup_t \bigg| \sum_{i=1}^{N} c_i \sum_{j \in N_i} \dot{a}_{ij}(t) \bigg| e^{-dt} \rho_d(\mathbf{x}_1, \mathbf{x}_2)
\end{aligned}
$$

Set

$$\begin{aligned}
A &:= \sup_t \sum_{i,j} a_{ij}(t) \\
\dot{A} &:= \sup_t \sum_{i,j} |\dot{a}_{ij}(t)| \\
L &:= \sup_t ||L(t)|| \\
D &:= \sup_t \left| \sum_{i=1}^{N} c_i \sum_{j \in N_i} a_{ij}(t) \right|
\end{aligned}$$

and gather up to conclude that P is a contraction if there exists $\alpha \in [0, 1)$ such that

$$\frac{e^{d\tau} - 1}{d} \left(A \left(1 + \frac{\Gamma L}{\gamma - d} + \frac{M}{d + \delta} \right) + \frac{\Gamma \dot{A}}{\gamma - d} + \frac{D}{d} \right) \leq \alpha \tag{7.8}$$

which is the second part of hypothesis $(H.2.6)$

8. Discussion.

There is none. This is a technical report.

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