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Classifiability of crossed products by nonamenable groups

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Abstract. We show that all amenable, minimal actions of a large class of nonamenable countable groups on compact metric spaces have dynamical comparison. This class includes all nonamenable hyperbolic groups, many HNN-extensions, nonamenable Baumslag–Solitar groups, a large class of amalgamated free products, lattices in many Lie groups, $\widetilde{A}_2$-groups, as well as direct products of the above with arbitrary countable groups. As a consequence, crossed products by amenable, minimal and topologically free actions of such groups on compact metric spaces are Kirchberg algebras in the UCT class, and are therefore classified by K-theory.

1. Introduction

One of the most remarkable achievements in C^*-algebra theory in the last decade was the completion of the classification programme initiated by George Elliott over 30 years ago. The outcome is the combination of the work of a large number of mathematicians over several decades, and can be phrased as follows:

Theorem (Classification). *Simple, separable, unital, nuclear, $\mathcal{Z}$-stable C^*-algebras satisfying the Universal Coefficient Theorem (UCT) are classified by the Elliott invariant (K-theory and traces).*

By Kirchberg's dichotomy, [44, Theorem 4.1.10], a C^*-algebra satisfying the assumptions of the above theorem (also called classifiable) is either stably finite or purely infinite. The

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currently available proof of the classification theorem considers the stably finite and purely infinite cases separately: while the purely infinite case was settled over twenty years ago by Kirchberg and Phillips (see [42]), the stably finite case was only settled in the last five years as a combination of [10, 17, 47]. We refer the reader to Winter's ICM address [51] for a recent survey and further references on the topic.

With such a powerful classification theorem at our disposal, it becomes an imperative task to identify interesting classes of C^*-algebras to which it can be applied. One of the most natural families of C^*-algebras arises from topological dynamics via the crossed product construction. In recent years, a lot of work has been done to establish dynamical criteria for an action $G \curvearrowright X$ of a countable group on a compact metric space that ensure that the associated crossed product $C(X) \rtimes G$ satisfies the assumptions of the classification theorem. Unitality and separability of $C(X) \rtimes G$ are automatic, while nuclearity of $C(X) \rtimes G$ is equivalent to amenability of $G \curvearrowright X$ (see Definition 2.1). Moreover, if $G \curvearrowright X$ is amenable, then $C(X) \rtimes G$ automatically satisfies the UCT by [49, Theorem 10.9], and it is simple if and only if $G \curvearrowright X$ is minimal and topologically free by [5, Theorem 2]. In particular, amenability, minimality, and topological freeness are *necessary* conditions for classifiability of $C(X) \rtimes G$, and it remains to decide when $C(X) \rtimes G$ is $\mathcal{Z}$-stable. Kirchberg's dichotomy takes a particularly nice form in this setting, since a nuclear crossed product $C(X) \rtimes G$ is stably finite if and only if G is amenable. (This is a combination of celebrated results in [7, 12, 25], as well as Lemma 2.2). Not surprisingly, the techniques used to establish $\mathcal{Z}$-stability of $C(X) \rtimes G$ are quite different depending on whether the group G is amenable or not.

On the amenable side, one of the first results in this direction is due to Toms and Winter, who showed in [48] that $C(X) \rtimes \mathbb{Z}$ is $\mathcal{Z}$-stable whenever $\mathbb{Z} \curvearrowright X$ is free and minimal, and $\dim(X) < \infty$. The efforts to extend this result to more general groups led Kerr to introduce the notion of *almost finiteness* for topological actions of amenable groups in [29], and prove that crossed products by free, minimal and almost finite actions are $\mathcal{Z}$-stable. Almost finiteness has been verified in a number of cases of interest [31], and the most recent result in this setting is by Kerr and Naryshkin [30], who proved that free actions of elementary amenable groups on finite-dimensional spaces are automatically almost finite. It is not possible to completely remove the finite-dimensionality assumption in these: Giol and Kerr constructed in [23] a free, minimal homeomorphism of an infinite dimensional space X such that $C(X) \rtimes \mathbb{Z}$ is not $\mathcal{Z}$-stable. The dividing line regarding $\mathcal{Z}$-stability for crossed products by amenable groups is expected to be mean dimension zero (or the conjecturally equivalent notion of the small boundary property), which is weaker than finite-dimensionality of the space. In this direction, Elliott and Niu showed in [18] that $C(X) \rtimes \mathbb{Z}$ is $\mathcal{Z}$-stable whenever $\mathbb{Z} \curvearrowright X$ is free and minimal, and has mean dimension zero; this was generalized by Niu [40] to $\mathbb{Z}^d$-actions. It has been conjectured by Phillips and Toms that the converse should also be true, and there have been some partial results in this direction; see [27].

Much less seems to be known in the nonamenable setting, although certain classes of actions have been successfully studied from this point of view. For example, Laca and Spielberg proved in [35] that crossed products by minimal, topologically free, *strong boundary* actions are purely infinite. As a consequence, for actions as above which are in addition amenable, such crossed products are nuclear and thus $\mathcal{O}_\infty$-stable by Kirchberg's absorption theorem [32, Theorem 3.15], so they are in particular $\mathcal{Z}$-stable. Similar results were obtained independently by Anantharaman-Delaroche in [2]. In [28], Jolissaint and Robertson proved analogous results for the larger class of *n-filling actions*.

A property that is key in the study of dynamical systems is Kerr's notion of dynamical comparison. Given nonempty open sets $U, V \subseteq X$, we write $U \prec V$ if every closed subset of U admits a finite open cover whose elements can be transported via the group action to pairwise disjoint subsets of V (see Definition 2.4). A system $G \curvearrowright X$ has dynamical comparison if $U \prec V$ whenever $\mu(U) < \mu(V)$ for all G-invariant probability measures μ. Establishing dynamical comparison is a powerful tool for proving $\mathcal{Z}$-stability of crossed products, both in the amenable and in the nonamenable settings. For amenable groups, the small boundary property implies almost finiteness (and thus $\mathcal{Z}$-stability) in the presence of dynamical comparison; see [31, Theorem A]. As it turns out, dynamical comparison has been verified in many interesting cases: for free actions of groups with subexponential growth on Cantor spaces in [16], and for arbitrary minimal actions of groups with polynomial growth in [39]. The latter result gives a large class of groups for which the small boundary property implies $\mathcal{Z}$-stability. For amenable, minimal, topologically free actions of nonamenable groups, Ma proved that comparison implies pure infiniteness of the crossed product (see [36], and see Theorem 2.8 for a simple proof). Not surprisingly, establishing dynamical comparison is often very challenging.

In this work, we prove that all amenable and minimal actions of a large class of nonamenable groups automatically satisfy dynamical comparison. As a consequence, for actions which are additionally topologically free, the crossed products are purely infinite (and thus satisfy the assumptions of the classification theorem). The following is the main definition of this work.

Definition A. Given $n \in \mathbb{N}$, we say that a countable group G admits *n-paradoxical towers*, if for every finite subset $D \subseteq G$ there are $A_1, \dots, A_n \subseteq G$ and $g_1, \dots, g_n \in G$ such that:

(1) the sets dA_i, for $d \in D$ and $i = 1, \dots, n$, are pairwise disjoint,

(2) $G = \bigcup_{i=1}^{n} g_i A_i$.

We say that G admits *paradoxical towers* if it admits n-paradoxical towers for some $n \in \mathbb{N}$.

It is easy to see that a group admitting paradoxical towers is necessarily nonamenable. Elementary methods allow one to show that the free group $\mathbb{F}_n$ admits paradoxical towers; see Proposition 3.2, and see Theorem C for more examples. There exist nonamenable groups which do not admit paradoxical towers, such as $\mathbb{F}_2 \times \mathbb{Z}$; see Example 4.16.

We show that *every* amenable and minimal action of a group with paradoxical towers has dynamical comparison. In fact, our methods allow us to deal with products of such groups with arbitrary groups; see Theorem 3.6.

Theorem B. *Let H be a countable group with paradoxical towers, let K be an arbitrary countable group, and set $G = H \times K$. Then every amenable, minimal action $G \curvearrowright X$ on a compact metrizable space has dynamical comparison. If $G \curvearrowright X$ is moreover topologically free, then the crossed product $C(X) \rtimes G$ is a Kirchberg algebra satisfying the UCT.*

By [45, Theorem 6.11], every nonamenable exact group admits a large family of minimal, amenable, topologically free actions on compact metrizable spaces.

The above result shows an unexpected phenomenon in the nonamenable setting: classifiability of $C(X) \rtimes G$ does not require finite dimensionality of X or any version of mean dimension zero for actions of nonamenable groups. There is thus a genuine difference between the amenable and the nonamenable cases. For a nonamenable group G not covered by Theorem B, we do not know if a simple, nuclear crossed product of the form $C(X) \rtimes G$ can be finite, although we strongly suspect that this is not the case[1)]. If G contains $\mathbb{F}_2$, we show in Theorem 3.9 that a simple, nuclear crossed product of the form $C(X) \rtimes G$ is always properly infinite.

We complement Theorem B by proving that large classes of nonamenable groups admit paradoxical towers; see Section 4. We summarize some of the results:

Theorem C. *The following classes of groups admit paradoxical towers:*

(1) *Acylindrically hyperbolic groups; see Proposition* 4.7. *In particular, all nonamenable hyperbolic groups and thus all nonabelian free groups.*

(2) *Highly transitive faithful non-ascending HNN-extensions; see Proposition* 4.9. *In particular, Baumslag–Solitar groups* $\mathrm{BS}(m,n)$ *with* $|m|, |n| > 1$ *and* $|m| \neq |n|$*; see Example* 4.10.

(3) *All free products* $G * H$ *of nontrivial groups with* $|H| > 2$*; see Example* 4.11 *for a larger class.*

(4) *Lattices in a real connected semisimple Lie group without compact factors and with finite center (such as* $\mathrm{SL}_n(\mathbb{Z})$ *for* $n \geq 3$*); see Example* 4.13.

(5) $\widetilde{A}_2$*-groups; see Example* 4.14.

(6) *Discrete subgroups of isometries of a visibility manifold with finite covolume; see Example* 4.15.

After these results appeared on the arXiv, further examples of groups with paradoxical towers and purely infinite crossed products were obtained by Ma and Wang in [37]. Moreover, some of the techniques developed here have also been successfully used in the study of actions on simple C*-algebras; see [21].

Based on the evidence provided in this work, we expect that the conclusion of Theorem B should hold for arbitrary nonamenable groups:

Conjecture D. Let G be a countable nonamenable group and let X be a compact metrizable space. Then every amenable, minimal action $G \curvearrowright X$ has dynamical comparison.

A positive solution to the above conjecture would imply that crossed products by amenable, minimal and topologically free actions of nonamenable groups are *always* classifiable. Our conjecture would also imply a strengthening of Kirchberg's dichotomy for crossed products: if $C(X) \rtimes G$ is simple and nuclear, then it is either stably finite (if and only if G is amenable) or purely infinite (if and only if G is nonamenable), *regardless* of whether it is $\mathcal{Z}$-stable or not.

[1)] Lemma 2.2 implies that such crossed products are never stably finite, and Conjecture D below predicts that such crossed products are always purely infinite.

2. Amenable actions and dynamical subequivalence

In this section, we collect a number of elementary definitions and results that will be needed in the rest of the work. All countable groups will be endowed with the discrete topology. All measures on locally compact spaces are assumed to be regular Borel measures. If G is a discrete group, we denote by $\mathrm{Prob}(G) \subseteq \ell^1(G)$ the set of all probability measures on it. If $\mu \in \mathrm{Prob}(G)$ and $g \in G$, we denote by $g \cdot \mu$ the probability measure given by

$$(g \cdot \mu)(E) = \mu(g^{-1}E)$$

for $g \in G$ and $E \subseteq G$.

The following definition, introduced by Anantharaman-Delaroche and Renault in [3], is standard by now.

Definition 2.1. An action $G \curvearrowright X$ of a countable group G on a compact metrizable space X is said to be *amenable* if there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ of continuous maps

$$\mu_n \colon X \to \mathrm{Prob}(G)$$

such that for all $g \in G$ we have

$$\sup_{x\in X} \|\mu_n(g \cdot x) - (g \cdot \mu_n)(x)\|_1 \xrightarrow{n\to\infty} 0.$$

Note that a countable group is amenable if and only if it acts amenably on the one point space. More generally, an action $G \curvearrowright X$ on a compact Hausdorff space is amenable if and only if $C(X) \rtimes_r G$ is nuclear; see [3, Corollary 6.2.14, Theorem 3.3.7], in which case the full and reduced crossed products of $G \curvearrowright X$ agree. The following lemma is folklore, and we include the proof for the convenience of the reader.

Lemma 2.2. *Let $G \curvearrowright X$ be an amenable action of a countable group on a compact metrizable space. Then G is amenable if and only if there exists a G-invariant probability measure on X.*

Proof. For the "only if" implication, assume that G is amenable and fix a G-invariant mean $\phi \colon \ell^\infty(G) \to \mathbb{C}$. Let η be any probability measure on X. The Poisson map

$$P_\eta \colon C(X) \to \ell^\infty(G)$$

defined by

$$P_\eta(f)(g) = \int_X f(g \cdot x)\, d\eta(x)$$

for $f \in C(X)$ and $g \in G$, is a unital positive G-equivariant map. Then $\phi \circ P_\eta \colon C(X) \to \mathbb{C}$ is a G-invariant state giving rise to a G-invariant probability measure on X. For the "if" implication, let $(\mu_n)_{n\in\mathbb{N}}$ be as in Definition 2.1, and let ν be a G-invariant probability measure on X. For $n \in \mathbb{N}$, define $\rho_n \in \mathrm{Prob}(G)$ by

$$\rho_n(E) = \int_X \mu_n(x)(E)\, d\nu(x)$$

for all $E \subseteq G$. Then $\|\rho_n - g \cdot \rho_n\|_1 \xrightarrow{n\to\infty} 0$ for all $g \in G$, and thus G is amenable. $\square$

Remark 2.3. Recall that any trace on $C(X) \rtimes G$ induces a G-invariant probability measure on X by restriction, and conversely any such measure induces a trace on $C(X) \rtimes G$ via the canonical conditional expectation $C(X) \rtimes G \to C(X)$. It thus follows from Lemma 2.2 that a nuclear crossed product $C(X) \rtimes G$ has a trace if and only if G is amenable.

We need the notion of dynamical subequivalence for tuples of sets, which is the partial order used to define the type semigroup of a dynamical system.

Definition 2.4. Let $G \curvearrowright X$ be an action of a discrete group on a compact Hausdorff space. Let $U_1, \dots, U_n, V_1, \dots, V_m$ be open subsets of X. We say that the family $(U_i)_{i=1}^n$ is *dynamically subequivalent* to $(V_j)_{j=1}^m$, and write $(U_i)_{i=1}^n \prec (V_j)_{j=1}^m$, if for any closed subsets $A_i \subseteq U_i$, for $i = 1, \dots, n$, there exist finite open covers $\mathcal{W}_i$ of A_i, elements $g_W^{(i)} \in G$ for $W \in \mathcal{W}_i$, and a partition

$$\mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_m = \{(i, W) : i = 1, \dots, n,\ W \in \mathcal{W}_i\},$$

such that, for each $j = 1, \dots, m$ the sets $g_W^{(i)} \cdot W$, for $(i, W) \in \mathcal{C}_j$, are pairwise disjoint and contained in V_j. Given a nonnegative integer r, we write

$$(U_i)_{i=1}^n \prec_r (V_j)_{j=1}^m$$

if $(U_i)_{i=1}^n \prec (V_j)_{j=1,\dots,m,k=1,\dots,r+1}$; in other words, if the family $(U_i)_{i=1}^n$ is subequivalent to $r+1$ disjoint copies of the family $(V_j)_{j=1}^m$.

We will identify tuples containing one element with their unique element, and will thus write $U \prec_r V$ instead of $(U) \prec_r (V)$. Note that this definition of dynamical r-subequivalence for open sets agrees with Kerr's [29, Definition 3.1]. We record here the observation that $\prec$ is transitive.

Lemma 2.5. *Let $G \curvearrowright X$ be an action of a discrete group on a compact Hausdorff space, and let $U_1, \dots, U_n, V_1, \dots, V_m, W_1, \dots, W_r \subseteq X$ be open sets satisfying*

$$(U_i)_{i=1}^n \prec (V_j)_{j=1}^m \quad \textit{and} \quad (V_j)_{j=1}^m \prec (W_k)_{k=1}^r.$$

Then $(U_i)_{i=1}^n \prec (W_k)_{k=1}^r$.

Proof. Let $A_i \subseteq U_i$, for $i = 1, \dots, n$, be closed subsets. Using that $(U_i)_{i=1}^n \prec (V_j)_{j=1}^m$, find open covers $\mathcal{W}_i$ of A_i and elements $g_W^{(i)} \in G$, for $i = 1, \dots, n$ and $W \in \mathcal{W}_i$, and a partition

$$\mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_m = \{(i, W) : i = 1, \dots, n,\ W \in \mathcal{W}_i\}$$

such that for each $j = 1, \dots, m$, the sets $g_W^{(i)} W$ for $(i, W) \in \mathcal{C}_j$ are pairwise disjoint and contained in V_j. By shrinking the elements of the open covers $\mathcal{W}_i$ if necessary, we can without loss of generality assume that for each $j = 1, \dots, m$, the set

$$B_j := \bigcup_{(i,W) \in \mathcal{C}_j} g_W^{(i)} \overline{W}$$

is contained in V_j as well. Using that $(V_j)_{j=1}^m \prec (W_k)_{k=1}^r$, find open covers $\mathcal{Y}_j$ of B_j and elements $h_Y^{(j)} \in G$, for $j = 1, \dots, m$ and $Y \in \mathcal{Y}_j$, and a partition

$$\mathcal{D}_1 \sqcup \cdots \sqcup \mathcal{D}_r = \{(j, Y) : j = 1, \dots, m,\ Y \in \mathcal{Y}_j\}$$

such that for each $k = 1, \dots, r$, the sets $h_Y^{(j)} Y$ for $(j, Y) \in \mathcal{D}_k$ are pairwise disjoint and contained in W_k. For $i = 1, \dots, n$, $W \in \mathcal{W}_i$, $j = 1, \dots, m$, with $(i, W) \in \mathcal{C}_j$, and $Y \in \mathcal{Y}_j$, we define an open set

$$Z_{W,Y} := W \cap \big(g_W^{(i)}\big)^{-1}(Y) \subseteq A_i \cap \big(g_W^{(i)}\big)^{-1}(B_j).$$

Then for each $i = 1, \dots, n$, the family

$$\mathcal{Z}_i := \{Z_{W,Y} : W \in \mathcal{W}_i, Y \in \mathcal{Y}_j, (i, W) \in \mathcal{C}_j\}$$

is an open cover of A_i. For $k = 1, \dots, r$, we define

$$\mathcal{E}_k := \{(i, Z_{W,Y}) : W \in \mathcal{W}_i,\ Y \in \mathcal{Y}_j,\ (i, W) \in \mathcal{C}_j,\ (j, Y) \in \mathcal{D}_k\}.$$

Note that

$$\mathcal{E}_1 \sqcup \dots \sqcup \mathcal{E}_r = \{(i, Z) : i = 1, \dots, n,\ Z \in \mathcal{Z}_i\}.$$

For $i = 1, \dots, n$, $j = 1, \dots, m$ and $Z_{W,Y} \in \mathcal{Z}_i$ with $Y \in \mathcal{Y}_j$, set

$$t_{W,Y}^{(i)} := h_Y^{(j)} g_W^{(i)}.$$

For fixed $k = 1, \dots, r$, it easily follows from the construction that the sets $t_{W,Y}^{(i)} Z_{W,Y}$ for $(i, Z_{W,Y}) \in \mathcal{E}_k$ are pairwise disjoint and contained in W_k. This shows that

$$(U_i)_{i=1}^n \prec (W_k)_{k=1}^r,$$

as desired. □

We will ultimately only be interested in comparing individual open sets, but the perspective using tuples will be helpful in the proof of Theorem 3.6, since it will allow us to decrease the number of colors we need to obtain comparison. The reason for this is that $U \prec (V_j)_{j=1}^m$ is a much stronger condition than $U \prec_{m-1} \bigcup_{j=1}^m V_j$. For instance, it follows from the previous lemma that $U \prec (V_j)_{j=1}^m \prec_r W$ implies $U \prec_r W$, while the direct argument using $\bigcup_{j=1}^m V_j$ instead of $(V_j)_{j=1}^m$ would only yield $U \prec_{(r+1)m-1} W$.

Definition 2.6 ([29, Definition 3.2]). Let X be a compact space, and let r be a nonnegative integer. An action $G \curvearrowright X$ of a discrete group G is said to have *dynamical r-comparison*, if for any two nonempty open subsets $U, V \subseteq X$ satisfying $\mu(U) < \mu(V)$ for all G-invariant probability measures μ on X, we have $U \prec_r V$.

If $r = 0$, we say that $G \curvearrowright X$ has *dynamical comparison*.

When $G \curvearrowright X$ is an amenable action of a nonamenable group, we have seen in Lemma 2.2 that there are no G-invariant probability measures on X. In particular, $G \curvearrowright X$ has dynamical r-comparison precisely when $U \prec_r V$ for all nonempty open sets $U, V \subseteq X$. By transitivity, it suffices to check this for $U = X$, since every open set is subequivalent to the whole space.

While we will be interested in establishing dynamical comparison, the tools and arguments we use will only give dynamical r-comparison. For actions of nonamenable groups without invariant probability measures, the following lemma shows that the two properties are in fact equivalent. This should be compared with [39, Lemma 2.3], where it is shown that r-comparison implies comparison for minimal actions of amenable groups.

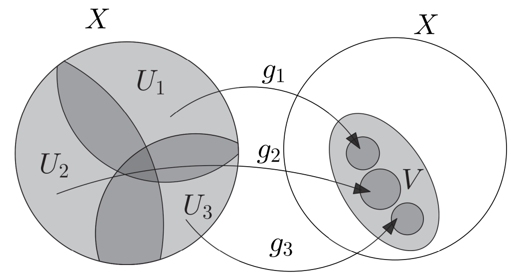

Figure 1. $X \prec V$.

Lemma 2.7. *Let $G \curvearrowright X$ be an action of a discrete group on a compact Hausdorff space with no invariant probability measures, and let r be a nonnegative integer. Then $G \curvearrowright X$ has dynamical r-comparison if and only if $G \curvearrowright X$ has dynamical comparison.*

Proof. We prove the nontrivial implication, so we assume that $G \curvearrowright X$ has dynamical r-comparison. One readily shows, arguing as in the discussion after [36, Definition 2.4], that $G \curvearrowright X$ is minimal and X has no isolated points. As explained above, it suffices to fix a nonempty open set $V \subseteq X$ and show that $X \prec V$. Fix $x \in V$, and note that $V \cap G \cdot x$ is an infinite set. Find $t_1, \dots, t_{r+1} \in G$ such that $t_k \cdot x \in V$ for all $k = 1, \dots, r+1$ and $t_k \cdot x \neq t_\ell \cdot x$ whenever $k \neq \ell$. Using that X is Hausdorff, find an open set $W \subseteq X$ such that $x \in W$ and $\{t_k \cdot W\}_{k=1}^{r+1}$ are pairwise disjoint sets in V. Since $X \prec_r W$ by assumption, there exist a finite open cover $\mathcal{O} = \mathcal{O}_1 \sqcup \cdots \sqcup \mathcal{O}_{r+1}$ of X, and $g_O \in G$ for $O \in \mathcal{O}$, such that the sets $g_O \cdot O$, for $O \in \mathcal{O}_k$ are pairwise disjoint subsets of W, for every $k = 1, \dots, r+1$. Now, $\{t_k g_O \cdot O\}_{O \in \mathcal{O}_k, k=1,\dots,r+1}$ is a collection of pairwise disjoint sets in V, verifying that $X \prec V$ as desired. □

We close this section by giving a simple proof of [36, Theorem 1.1], which avoids the use of scaling elements and hereditary subalgebras in favor of Cuntz semigroup techniques. (We refer the reader to [4, Chapter 2] or [22, Sections 2 and 3] for an introduction to these.) Given positive elements a and b in a C^*-algebra A, we say that a is *Cuntz subequivalent* to b in A, written $a \precsim b$ in A, if there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in A such that $\lim_{n\to\infty} c_n^* b c_n = a$. We write $a \sim b$ if $a \precsim b$ and $b \precsim a$. We will use the fact that if A is abelian, then $a \precsim b$ if and only if the open support of a is contained in that of b. In particular, in a general C^*-algebra, if a and b commute and $0 \leq a \leq b$, then $a \precsim b$.

Theorem 2.8 (Ma). *Let $G \curvearrowright X$ be a minimal and topologically free action of a discrete group on a compact Hausdorff space. Assume that $G \curvearrowright X$ has dynamical comparison and admits no invariant probability measures. Then $C(X) \rtimes_r G$ is simple and purely infinite.*

Proof. Simplicity follows from [5, Theorem 2], so we prove pure infiniteness. To this end, let $a, b \in C(X) \rtimes_r G$ be nonzero positive contractions. We show that $b \precsim a$ in $C(X) \rtimes_r G$, which implies that $C(X) \rtimes_r G$ is purely infinite by [33, Defintion 4.1]. Since $b \precsim 1_{C(X)}$, it is enough to show that $1_{C(X)} \precsim a$ in $C(X) \rtimes_r G$. By [43, Lemma 7.9], there exists a nonzero positive contraction $f \in C(X)$ such that $f \precsim a$ in $C(X) \rtimes_r G$. Set $W = \{x \in X : f(x) > 0\}$, and choose U to be a nonempty open subset of W such that $\overline{U} \subseteq W$. There exists a positive contraction $g \in C(X)$ such that $g = 0$ on $X \setminus W$ and $g = 1$ on $\overline{U}$. Since $X \prec U$, it follows from [29, Lemma 12.3] that $1_{C(X)} \precsim g$ in $C(X) \rtimes_r G$. On the other hand, we have $g \precsim f$ since $\{x \in X : g(x) > 0\} \subseteq W = \{x \in X : f(x) > 0\}$. Transitivity of Cuntz subequivalence gives $1_{C(X)} \precsim f$, and so $1_{C(X)} \precsim a$, as desired. □

3. Paradoxical towers give dynamical comparison

In this section, we introduce the notion of *paradoxical towers*, which is the main technical tool in this work. For (certain extensions of) groups admitting paradoxical towers, we show that amenable, minimal actions always have dynamical comparison; see Theorem 3.6. Using this, we establish classifiability of a large class of crossed products in Corollary 3.8. Examples are discussed in Section 4.

Definition 3.1. Let $n \in \mathbb{N}$. We say that a countable group G admits *n-paradoxical towers* if for every finite subset $D \subseteq G$ there are $A_1, \dots, A_n \subseteq G$ and $g_1, \dots, g_n \in G$ such that:

(1) the sets dA_i, for $d \in D$ and $i = 1, \dots, n$, are pairwise disjoint,

(2) $G = \bigcup_{i=1}^{n} g_i A_i$.

We say that G admits *paradoxical towers* if there is $n \in \mathbb{N}$ such that G admits n-paradoxical towers.

It is easy to see that a group admitting paradoxical towers is necessarily nonamenable. The class of groups admitting paradoxical towers is very large, but it does not exhaust all non-amenable groups; for example, $\mathbb{F}_2 \times \mathbb{Z}$ does not admit paradoxical towers (see Example 4.16). We postpone this discussion until Section 4, and we only present here the following basic example (see Proposition 4.7 for a much larger class).

Proposition 3.2. *The free group $\mathbb{F}_2$ admits 2-paradoxical towers. In fact, given a finite subset $D \subseteq \mathbb{F}_2$ there are nonempty subsets $A_1, A_2, A_3 \subseteq \mathbb{F}_2$ and $g_1, g_2, g_3 \in \mathbb{F}_2$ such that:*

(1) *the sets dA_j, for $d \in D$ and $j = 1, 2, 3$, are pairwise disjoint,*

(2) *the sets $\mathbb{F}_2 \setminus g_j A_j$, for $j = 1, 2, 3$, are pairwise disjoint.*

Proof. We begin by observing that the property in the statement implies that $\mathbb{F}_2$ has 2-paradoxical towers. Indeed, condition (2) implies that

$$\emptyset = (\mathbb{F}_2 \setminus g_1 A_1) \cap (\mathbb{F}_2 \setminus g_2 A_2),$$

and by taking complements we get $\mathbb{F}_2 = g_1 A_1 \cup g_2 A_2$. In particular, A_1, A_2 and g_1, g_2 satisfy the conditions of Definition 3.1 for $n = 2$.

Denote by a, b the generators of $\mathbb{F}_2$, and set $L = \{a, b, a^{-1}, b^{-1}\}$. For $h \in \mathbb{F}_2$, we write $W(h) \subseteq \mathbb{F}_2$ for the set of all reduced words with letters from L which start with h. If $x \in L$ is the last letter of h, then a generic element of $W(h)$ has the form hg with $g \notin W(x^{-1})$. In particular,

$$\mathbb{F}_2 \setminus h^{-1} W(h) = W(x^{-1}). \tag{3.1}$$

For $r \geq 0$, we write B_r for the set of all reduced words of length at most r; note that we have $B_r B_s = B_{r+s}$ for all $r, s > 0$. Let $D \subseteq \mathbb{F}_2$ be a finite set. Find $m \geq 0$ with $D \subseteq B_m$, and define

$$h_1 = a^{2m}ba, \quad h_2 = a^{2m}ba^{-1}, \quad h_3 = a^{2m}b^2.$$

For $i = 1, 2, 3$, set $A_i = W(h_i)$. We claim that these sets satisfy condition (1). Since every element of D has length at most m, it suffices to check that whenever $e \in B_{2m}$ and $i, j = 1, 2, 3$ satisfy $eA_i \cap A_j \neq \emptyset$, then $e = 1$ and $i = j$. Given e, i, j as above, if eA_i intersects A_j, then there is $g \in A_i$ such that eg, after reduction, starts with $2m$ copies of a. Since the $(2m+1)$-st letter of g is not an a, it follows that the product of e and g cannot have any cancelations, and thus $e = a^k$ for some $0 \leq k \leq m$. On the other hand, $a^k h_i$ never has h_j as an initial segment unless $k = 0$ and $i = j$. This proves the claim.

For $i = 1, 2, 3$, set $g_i = h_i^{-1}$. Using (3.1), we get

$$\mathbb{F}_2 \setminus g_1 A_1 = W(a^{-1}), \quad \mathbb{F}_2 \setminus g_2 A_2 = W(a), \quad \mathbb{F}_2 \setminus g_3 A_3 = W(b^{-1}),$$

and these sets are clearly pairwise disjoint. This finishes the proof. □

We will need some auxiliary lemmas. In the following, for $m = 1$ we just get the definition of paradoxical towers. In general, the strengthening is that the towers $A_i^{(j)}$ are jointly (and not just separately) D-free.

Lemma 3.3. *Let $n \in \mathbb{N}$ and let G be a countable group with n-paradoxical towers, and let $D \subseteq G$ be a finite subset. For every $m \in \mathbb{N}$, there exist subsets $A_i^{(j)} \subseteq G$ and group elements $g_i^{(j)} \in G$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, such that:*

(1) *the sets $dA_i^{(j)}$, for $d \in D$, $i = 1, \ldots, n$, and $j = 1, \ldots, m$, are pairwise disjoint,*

(2) *$G = \bigcup_{i=1}^n g_i^{(j)} A_i^{(j)}$ for every $j = 1, \ldots, m$.*

Proof. Let $D \subseteq G$ be a finite subset. Since G is infinite, there exist $s_1, \ldots, s_m \in G$ such that Ds_j, for $j = 1, \ldots, m$, are pairwise disjoint sets. Set $\widetilde{D} = \bigsqcup_{j=1}^m Ds_j$, which is a finite subset of G. Since G admits n-paradoxical towers, there are sets $A_1, \ldots, A_n \subseteq G$ and elements $g_1, \ldots, g_n \in G$ such that the sets $\widetilde{d} A_i$, for $\widetilde{d} \in \widetilde{D}$ and $i = 1, \ldots, n$, are pairwise disjoint, and $G = \bigcup_{i=1}^n g_i A_i$.

For $i = 1, \ldots, n$ and $j = 1, \ldots, m$, set

$$A_i^{(j)} = s_j A_i \quad \text{and} \quad g_i^{(j)} = g_i s_j^{-1}.$$

One readily checks that conditions (1) and (2) in the statement are satisfied. □

Given a metric space (X, d), a set $U \subseteq X$ and $\varepsilon > 0$, we set

$$U^{-\varepsilon} = \{x \in U : d(x, X \setminus U) > \varepsilon\}.$$

Lemma 3.4. *Let $G \curvearrowright X$ be an action of a countable group on a compact metric space X, let n be a nonnegative integer, let $\varepsilon > 0$, and let $D \subseteq G$ be a finite symmetric set.*

(1) *Let $V, U_0, \ldots, U_n \subseteq X$ be open sets and let $R : X \to [0, \infty)$ be a function satisfying*

$$|\{g \in D^2 : g \cdot x \in V\}| + R(x) < \sum_{k=0}^{n} |\{g \in D : g \cdot x \in U_k^{-\varepsilon}\}|$$

for all $x \in X$. Then for every closed subset $A \subseteq V$ there exist $0 < \widetilde{\varepsilon} < \varepsilon$, a finite open cover $\mathcal{O}$ of A, group elements $s_O \in G$, for $O \in \mathcal{O}$, and a partition $\mathcal{O} = \mathcal{O}_0 \sqcup \cdots \sqcup \mathcal{O}_n$ satisfying the following properties:

(a) *for every $k = 0, \ldots, n$, the family $\{s_O \cdot O \colon O \in \mathcal{O}_k\}$ consists of pairwise disjoint subsets of U_k,*

(b) *with B_k denoting the closure of $\bigcup_{O \in \mathcal{O}_k} s_O \cdot O$ and $\widetilde{U}_k = U_k \setminus B_k$, we have*

$$R(x) < \sum_{k=0}^{n} |\{g \in D \colon g \cdot x \in \widetilde{U}_k^{-\widetilde{\varepsilon}}\}|$$

for all $x \in X$.

(2) *Let $V_1, \ldots, V_m, U \subseteq X$ be open sets satisfying*

$$\sum_{j=1}^{m} |\{g \in D^2 \colon g \cdot x \in V_j\}| < (n+1)|\{g \in D \colon g \cdot x \in U^{-\varepsilon}\}|$$

for all $x \in X$. Then $(V_j)_{j=1}^{m} \prec_n U$.

Proof. (1) This is proved exactly as [39, Lemma 3.1]. We omit the proof.

(2) We prove this by repeatedly applying part (1). For each $j = 1, \ldots, m$, let $A_j \subseteq V_j$ be a closed subset. Set

$$\varepsilon^{(1)} = \varepsilon \quad \text{and} \quad U_0^{(1)} = \cdots = U_n^{(1)} = U.$$

For $j = 1, \ldots, m$, let $R^{(j)} \colon X \to [0, \infty)$ be given by

$$R^{(j)}(x) = \sum_{i=j}^{m} |\{g \in D^2 \colon g \cdot x \in V_i\}|$$

for $x \in X$. By construction, we have

$$|\{g \in D^2 \colon g \cdot x \in V_1\}| + R^{(1)}(x) < \sum_{k=0}^{n} |\{g \in D \colon g \cdot x \in (U_k^{(1)})^{-\varepsilon^{(1)}}\}|$$

for all $x \in X$. By part (1), there exist $0 < \varepsilon^{(2)} < \varepsilon^{(1)}$, an open cover $\mathcal{O}^{(1)}$ of A_1, group elements $s_O^{(1)} \in G$, for $O \in \mathcal{O}^{(1)}$, and a partition $\mathcal{O}^{(1)} = \mathcal{O}_0^{(1)} \sqcup \cdots \sqcup \mathcal{O}_n^{(1)}$ satisfying the following two properties:

(a.1) for every $k = 0, \ldots, n$, the family $\{s_O \cdot O \colon O \in \mathcal{O}_k^{(1)}\}$ consists of pairwise disjoint subsets of $U_k^{(1)}$,

(b.1) with $B_k^{(1)}$ denoting the closure of $\bigcup_{O \in \mathcal{O}_k^{(1)}} s_O \cdot O$ and $U_k^{(2)} = U_k^{(1)} \setminus B_k \subseteq U$, we have

$$R^{(1)}(x) < \sum_{k=0}^{n} |\{g \in D \colon g \cdot x \in (U_k^{(2)})^{-\varepsilon^{(2)}}\}|$$

for all $x \in X$.

By construction, we get

$$|\{g \in D^2 \colon g \cdot x \in V_2\}| + R^{(2)}(x) < \sum_{k=0}^{n} |\{g \in D \colon g \cdot x \in (U_k^{(2)})^{-\varepsilon^{(2)}}\}|$$

for all $x \in X$. One continues applying part (1) inductively. After m steps, we will have constructed, for each $j = 1, \dots, m$, an open cover $\mathcal{O}^{(j)}$ of A_j, group elements $s_O^{(j)} \in G$, for $O \in \mathcal{O}^{(j)}$, and a partition $\mathcal{O}^{(j)} = \mathcal{O}_0^{(j)} \sqcup \dots \sqcup \mathcal{O}_n^{(j)}$ such that for every $k = 0, \dots, n$, the family $\{s_O \cdot O : O \in \mathcal{O}_k^{(j)}\}$ consists of pairwise disjoint subsets of $U_k^{(j)} \subseteq U$. For $j = 1, \dots, m$, set

$$\mathcal{C}_j = \{(k, O) : k = 1, \dots, n, O \in \mathcal{O}_k^{(j)}\},$$

and note that

$$\mathcal{C}_1 \sqcup \dots \sqcup \mathcal{C}_n = \{(k, O) : k = 1, \dots, n, O \in \mathcal{O}_k^{(j)}, j = 1, \dots, m\}.$$

Since the sets $U_k^{(j)}$, for $j = 1, \dots, m$ are pairwise disjoint subsets of U, it follows that the above choices witness the fact that $(V_j)_{j=1}^m \prec_n U$, as desired. □

In the following lemma, note that we cannot demand that the sets eB_j for $e \in E$ *and* $j = 1, \dots, m$ be pairwise disjoint, as this cannot happen if K is amenable.

Lemma 3.5. *Let K be a countable group and let $E \subseteq K$ be a finite symmetric subset containing the unit of K. Set $m = |E^2|$. Then there is a finite partition*

$$K = B_1 \sqcup \dots \sqcup B_m$$

such that for each $j = 1, \dots, m$, the sets eB_j, for $e \in E$, are pairwise disjoint.

Proof. Consider the Cayley graph $\mathcal{G} = \mathrm{Cay}(K, E^2)$ whose vertices are the elements of K and whose edges are of the form (k, gk), for $k \in K$ and $g \in E^2 \setminus \{1\}$. Note that every vertex in $\mathcal{G}$ has exactly $m-1$ edges coming out of it, and that there are no loops in $\mathcal{G}$. The greedy coloring algorithm then implies that we can color the vertices of $\mathcal{G}$ using at most m colors, in such a way that every two adjacent vertices have different colors.[2)] For $j = 1, \dots, m$, let $B_j \subseteq K$ denote the vertices with the j-th color. Then $B_1 \sqcup \dots \sqcup B_m = K$. Moreover, for $j = 1, \dots, m$, we have $gB_j \cap B_j = \emptyset$ for all $g \in E^2 \setminus \{1\}$. Thus the sets eB_j, for $e \in E$, are pairwise disjoint for each $j = 1, \dots, m$. □

The following is the main result of this work. The main consequence is the classifiability of the associated crossed products; see Corollary 3.8. In its proof, we will work with doubly-indexed sets $V_{i,j}$ for $i = 1, \dots, n$ and $j = 1, \dots, m$. To lighten the notation, we will write $(V_{i,j})_{i,j=1}^{n,m}$ for $(V_{i,j})_{i=1,\dots,n, j=1,\dots,m}$, and similarly for their union $\bigcup_{i,j=1}^{n,m}$, or for sums $\sum_{i,j=1}^{n,m}$ indexed both by i and j.

Theorem 3.6. *Let H be a countable group admitting paradoxical towers, let K be any countable group, and set $G = H \times K$. Then any amenable, minimal action of G on a compact metrizable space has dynamical comparison.*

Proof. Let X be a compact metrizable space and let $G \curvearrowright X$ be an amenable, minimal action. Let $n \in \mathbb{N}$ be such that H admits n-paradoxical towers. By Lemma 2.2 and Lemma 2.7, and since G is nonamenable, it suffices to show that $G \curvearrowright X$ has dynamical n-comparison.

2) One way to do this is to enumerate the vertices and then color them inductively.

Let $U \subseteq X$ be a nonempty open set. Fix a metric on X inducing its topology, and choose $\varepsilon > 0$ such that $U^{-\varepsilon} \neq \emptyset$. By minimality of $G \curvearrowright X$, there is a finite set $F_0 \subseteq G$ such that $F_0^{-1} \cdot U^{-\varepsilon} = X$. Without loss of generality, we assume that F_0 contains the unit of G, and has the form $F_0 = D_0 \times E$ for finite sets $D_0 \subseteq H$ and $E \subseteq K$ with $E = E^{-1}$. Set $m = |E^4|$.

Since H is infinite, we can find $t_1, \dots, t_m \in H$ such that the sets $D_0 t_j$, for $j = 1, \dots, m$, are pairwise disjoint. Let D be any finite symmetric subset of H containing $\bigsqcup_{j=1}^m D_0 t_j$, set $F = D \times E$, which is finite and symmetric. With $s_j = (t_j, 1) \in G$, note that the sets $F_0 s_j$ are pairwise disjoint and contained in F.

Claim 1. *For all $x \in X$, we have*

$$|\{g \in F : g \cdot x \in U^{-\varepsilon}\}| \geq m. \tag{3.2}$$

To prove the claim, fix $x \in X$ and $j = 1, \dots, m$. Denote by F_x the set in the left-hand side of the displayed equation above. Since

$$s_j^{-1} \underbrace{F_0^{-1} \cdot U^{-\varepsilon}}_{=X} = X,$$

there is $f_j \in F_0$ such that $f_j s_j \cdot x \in U^{-\varepsilon}$, and thus $f_j s_j$ belongs to F_x (in addition to $F_0 s_j$). Hence $|F_x \cap F_0 s_j| \geq 1$ for all $j = 1, \dots, m$, and since the sets $F_0 s_j$ are pairwise disjoint, this shows that $|F_x| \geq m$, as desired.

Since H admits n-paradoxical towers, use Lemma 3.3, to find $A_i^{(j)} \subseteq H$ and $h_i^{(j)} \in H$ for $i = 1, \dots, n$ and $j = 1, \dots, m$ satisfying:

(a.1) the sets $dA_i^{(j)}$, for $d \in D^2$, $i = 1, \dots, n$, and $j = 1, \dots, m$, are pairwise disjoint,

(a.2) $\bigcup_{i=1}^n h_i^{(j)} A_i^{(j)} = H$ for every $j = 1, \dots, m$.

Use Lemma 3.5, with E^2 in place of E, to find subsets $B_1, \dots, B_m \subseteq K$ such that

(b.1) for each $j = 1, \dots, m$, the sets eB_j, for $e \in E^2$, are pairwise disjoint,

(b.2) $K = B_1 \sqcup \dots \sqcup B_m$.

For $i = 1, \dots, n$ and $j = 1, \dots, m$, set

$$C_{i,j} = A_i^{(j)} \times B_j \subseteq G \quad \text{and} \quad g_{i,j} = (h_i^{(j)}, 1) \in G.$$

We proceed to show the following:

(i) the sets $fC_{i,j}$ for $f \in F^2$, $i = 1, \dots, n$, and $j = 1, \dots, m$, are pairwise disjoint,

(ii) $\bigcup_{i,j=1}^{n,m} g_{i,j} C_{i,j} = G$.

To check part (i), let $i, i' = 1, \dots, n$, let $j, j' = 1, \dots, m$, and let $f, f' \in F^2$. Write $\pi_H : G \to H$ for the projection onto the first coordinate, and $\pi_K : G \to K$ for the projection onto the second one. Assume that

$$fC_{i,j} \cap f'C_{i',j'} \neq \emptyset. \tag{3.3}$$

Apply π_H to (3.3) to get $\pi_H(f)A_i^{(j)} \cap \pi_H(f')A_{i'}^{(j')} \neq \emptyset$. Since $\pi_H(F^2) = D^2$, condition (a.1) above implies that $i = i'$, $j = j'$ and $\pi_H(f) = \pi_H(f')$. Applying π_K to (3.3) now gives $\pi_K(f)B_j \cap \pi_K(f')B_j \neq \emptyset$. Since $\pi_K(F^2) = E^2$, it thus follows from (b.1) above that $\pi_K(f) = \pi_K(f')$ and thus $f = f'$. This proves (i).

Part (ii) is immediate from (a.2) and (b.2).

Now fix $0 < \delta < (2nm(nm+1))^{-1}$. Use amenability of $G \curvearrowright X$ to find a continuous map $\mu\colon X \to \mathrm{Prob}(G)$ satisfying

$$\sup_{x \in X} \|\mu(g \cdot x) - g \cdot \mu(x)\|_1 < \delta \tag{3.4}$$

for all $g \in F^2 \cup \{g_{i,j}\}_{i,j=1}^{n,m}$. For $i = 1, \ldots, n$ and $j = 1, \ldots, m$, set

$$V_{i,j} = \left\{ x \in X \colon \mu(x)(C_{i,j}) > \frac{1}{nm+1} + \delta \right\},$$

and note that $V_{i,j}$ is an open subset of X.

Claim 2. *We have* $X \prec (V_{i,j})_{i,j=1}^{n,m}$.

For $i = 1, \ldots, n$ and $j = 1, \ldots, m$, define

$$W_{i,j} = \left\{ x \in X \colon \mu(x)(g_{i,j} C_{i,j}) > \frac{1}{nm+1} + 2\delta \right\}.$$

Then $W_{i,j}$ is open in X. Fix $x \in X$. Since $\mu(x)$ is a probability measure on G, by condition (ii) there are $i_x \in \{1, \ldots, n\}$ and $j_x \in \{1, \ldots, m\}$ such that

$$\mu(x)(g_{i_x,j_x} C_{i_x,j_x}) \geq \frac{1}{nm} > \frac{1}{nm+1} + 2\delta.$$

In other words, $x \in W_{i_x,j_x}$, and thus $X = \bigcup_{i,j=1}^{n,m} W_{i,j}$. Using (3.4) again, we get

$$g_{i,j}^{-1} W_{i,j} \subseteq \left\{ x \in X \colon \mu(x)(C_{i,j}) > \frac{1}{nm+1} + \delta \right\} = V_{i,j}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. This shows that $X \prec (V_{i,j})_{i,j=1}^{n,m}$, as desired.

Claim 3. *We have* $(V_{i,j})_{i,j=1}^{n,m} \prec_n U$.

To prove the claim, note that by (3.4) and the fact that $F = D \times E$ is symmetric, we have

$$f V_{i,j} \subseteq \left\{ x \in X \colon \mu(x)(f C_{i,j}) > \frac{1}{nm+1} \right\}$$

for all $f \in F^2$, $i = 1, \ldots, n$, and $j = 1, \ldots, m$. Since the sets $f C_{i,j}$ are pairwise disjoint by condition (i) above, for any $x \in X$ at most nm of them can have $\mu(x)$-measure more than $\frac{1}{nm+1}$. We deduce that each $x \in X$ belongs to at most nm of the sets $f V_{i,j}$, for $f \in F^2$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. That is, for all $x \in X$, we get by (3.2),

$$\sum_{i,j=1}^{n,m} |\{ f \in F^2 \colon f \cdot x \in V_{i,j} \}| \leq nm < (n+1) |\{ f \in F \colon f \cdot x \in U^{-\varepsilon} \}|.$$

By part (2) of Lemma 3.4, we conclude that $(V_{i,j})_{i,j=1}^{n,m} \prec_n U$.

Combining Claims 2 and 3, we get $X \prec (V_{i,j})_{i,j=1}^{n,m} \prec_n U$, which implies $X \prec_n U$ by Lemma 2.5. This concludes the proof. □

Remark 3.7. The above proof does not show that $H \times K$ admits paradoxical towers. Indeed, although the sets $C_{i,j}$ satisfy the conditions for nm-paradoxical towers, the number

m depends on the finite subset $E \subseteq K$. In fact, one can show that $G = H \times K$ never has paradoxical towers if K is infinite and amenable; see Example 4.16.

By [45, Theorem 6.11], every exact nonamenable group admits a large family of amenable, minimal, free actions on compact metric spaces. In particular, actions satisfying the assumptions of Theorem 3.6 always exist.

We obtain the following corollary:

Corollary 3.8. *Let H be a group admitting paradoxical towers, let K be any countable group, and set $G = H \times K$. Let $G \curvearrowright X$ be an amenable, minimal and topologically free action on a compact metrizable space X. Then the crossed product $C(X) \rtimes G$ is a Kirchberg algebra satisfying the UCT.*

Proof. It is well known that $C(X) \rtimes G$ is simple, separable, unital and nuclear, and it satisfies the UCT by [49, Theorem 10.9]. Finally, it is purely infinite by the combination of Theorem 3.6 and Theorem 2.8. □

Corollary 3.8 reveals an unexpected phenomenon in the nonamenable setting: classifiability of $C(X) \rtimes G$, for the groups G to which the corollary applies, does not require any finite dimensionality assumption on X, or any version of mean dimension zero. There is thus a genuine difference between the amenable and the nonamenable case.

It is an interesting and challenging problem to compute the possible K-groups of Kirchberg algebras arising as in Corollary 3.8. Some progress in this direction has been made in [19].

For a nonamenable group G, a simple, nuclear crossed product of the form $C(X) \rtimes G$ cannot be stably finite by Lemma 2.2, but we do not know if it can ever be finite if G is not covered by Corollary 3.8 (although Conjecture D predicts that this can never happen). Using a weak version of paradoxical towers, we show below that if G contains $\mathbb{F}_2$, then a simple, nuclear crossed product $C(X) \rtimes G$ is automatically properly infinite; see Theorem 3.9. Recall (see [44, Definition 1.1]) that a unital C^*-algebra A is *properly infinite* if there exist two mutually orthogonal projections in A, each of which is Murray–von Neumann equivalent to the unit.

Theorem 3.9. *Let G be a countable group containing a nonabelian free group. If $G \curvearrowright X$ is an amenable, minimal and topologically free action on a compact metrizable space, then $C(X) \rtimes G$ is a properly infinite, simple, separable, nuclear, unital C^*-algebra.*

Proof. We only need to show that $C(X) \rtimes G$ is properly infinite. By [11, Proposition 2.2], it suffices to show that there is an isometry in $C(X) \rtimes G$ which is not a unitary. Let $H \subseteq G$ be a nonabelian free subgroup of rank 2. Let $h \in H \setminus \{1\}$ and set $D = \{1, h\}$. By Proposition 3.2, there exist nonempty sets $B_1, B_2, B_3 \subseteq H$ and $h_1, h_2, h_3 \in H$ such that

(1) the sets dB_j, for $d \in D$ and $j = 1, 2, 3$, are pairwise disjoint, and

(2) the sets $H \setminus h_j B_j$, for $j = 1, 2, 3$, are pairwise disjoint.

Observe that the above conditions imply

(3) h_1, h_2, h_3 are pairwise distinct.

Indeed, if $h_j = h_k$ with $j \neq k$, then $h_j B_j$ and $h_j B_k$ are disjoint sets with disjoint complements. This implies that $G = h_j B_j \sqcup h_j B_j$, and hence $G = B_j \sqcup B_k$, which contradicts the fact that B_1, B_2, B_3 are pairwise disjoint and nonempty.

Let $S \subseteq G$ be a set containing exactly one representative of each right coset in $H\backslash G$, so that $G = \bigsqcup_{s\in S} Hs$. For $j = 1, 2, 3$, set $A_j = \bigsqcup_{s\in S} B_j s$. We claim that

(a) the sets dA_j, for $d \in D$ and $j = 1, 2, 3$, are pairwise disjoint, and

(b) the sets $G \setminus h_j A_j$, for $j = 1, 2, 3$, are pairwise disjoint.

To see (a), let $d, e \in D$ and let $j, k \in \{1, 2, 3\}$. Using that $Hs \cap Ht = \emptyset$ for $s, t \in S$ with $s \neq t$, we get

$$\begin{aligned} dA_j \cap eA_k &= \Big(\bigsqcup_{s\in S} dB_j s\Big) \cap \Big(\bigsqcup_{t\in S} eB_k t\Big) = \bigsqcup_{s,t\in S} \underbrace{dB_j s \cap eB_k t}_{\subseteq Hs\cap Ht} \\ &= \bigsqcup_{s\in S} (dB_j \cap eB_k)s. \end{aligned}$$

If the above intersection is nonempty, then we must have $dB_j \cap eB_k \neq \emptyset$, which implies $d = e$ and $j = k$ by (1). To check condition (b), let $j, k \in \{1, 2, 3\}$. Arguing as above, we have

$$\begin{aligned} (G \setminus h_j A_j) \cap (G \setminus h_k A_k) &= \Big(\bigsqcup_{s\in S}(Hs \setminus h_j B_j s)\Big) \cap \Big(\bigsqcup_{t\in S}(Ht \setminus h_k B_k t)\Big) \\ &= \bigsqcup_{s\in S} \big((H \setminus h_j B_j) \cap (H \setminus h_k B_k)\big)s, \end{aligned}$$

which is empty if $j \neq k$ by (2).

Set $\varepsilon = \frac{1}{24}$, and use amenability of $G \curvearrowright X$ to find a continuous function

$$\mu\colon X \to \mathrm{Prob}(G)$$

satisfying

$$\sup_{x\in X} \|\mu(g\cdot x) - g\cdot \mu(x)\|_1 < \varepsilon \tag{3.5}$$

for all $g \in \{h^{-1}, h_1, h_2, h_3\}$. For $j = 1, 2, 3$, set

$$V_j = \Big\{x \in X : \mu(x)(A_j) > \frac{1}{2} + \varepsilon\Big\},$$

and define $V = V_1 \cup V_2 \cup V_3$. By (3.5), for $d \in D = \{1, h\}$ we have

$$dV_j \subseteq \Big\{x \in X : \mu(x)(dA_j) > \frac{1}{2}\Big\}$$

for $j = 1, 2, 3$. Since the sets dA_j, for $d \in D$ and $j = 1, 2, 3$, are pairwise disjoint, it follows that for each $x \in X$ at most one of these sets can have $\mu(x)$-measure more than $\frac{1}{2}$. We deduce that the sets dV_j, for $d \in D$ and $j = 1, 2, 3$, are pairwise disjoint. In particular, we have:

(i) $V = V_1 \sqcup V_2 \sqcup V_3$.

(ii) $V \neq X$, since $hV \cap V = \emptyset$.

We will now show that $X \prec V$. For $j = 1, 2, 3$, set

$$W_j = \left\{x \in X : \mu(x)(G \setminus h_j A_j) < \frac{1}{2} - 2\varepsilon\right\}.$$

We claim that

(iii) $W_1 \cup W_2 \cup W_3 = X$,

(iv) $h_j^{-1} W_j \subseteq V_j$ for $j = 1, 2, 3$.

Note that $\frac{1}{2} - 2\varepsilon > \frac{1}{3}$. By condition (b) above, for every $x \in X$ at least one of either $G \setminus h_1 A_1$, $G \setminus h_2 A_2$ or $G \setminus h_3 A_3$ must have $\mu(x)$-measure less than $\frac{1}{2} - 2\varepsilon$. This implies (iii). To show (iv), fix $j = 1, 2, 3$. Given $x \in X$, the fact that $\mu(x)$ is a probability measure on G implies that

$$\mu(x)(G \setminus A_j) = 1 - \mu(x)(A_j).$$

Using the above at the second step, we get

$$\begin{aligned} h_j^{-1} W_j &\overset{(3.5)}{\subseteq} \left\{x \in X : \mu(x)(G \setminus A_j) < \frac{1}{2} - \varepsilon\right\} \\ &= \left\{x \in X : \mu(x)(A_j) > \frac{1}{2} + \varepsilon\right\} = V_j, \end{aligned}$$

as desired.

To simplify the notation, we set $g_j = h_j^{-1}$ for $j = 1, 2, 3$. Let $f_1, f_2, f_3 \in C(X)$ be a partition of unity subordinate to the open cover $\{W_1, W_2, W_3\}$ of X; see (iii) above. Denote by α the action of G on $C(X)$ induced by the given action $G \curvearrowright X$. For $g \in G$, we write $u_g \in C(X) \rtimes G$ for the canonical unitary satisfying $u_g f = \alpha_g(f) u_g$ for all $f \in C(X)$. We denote by $E : C(X) \rtimes G \to C(X)$ the canonical conditional expectation, which is determined by $E(u_g) = 0$ whenever $g \in G \setminus \{1\}$.

Set $v = \sum_{j=1}^{3} \alpha_{g_j}(f_j^{1/2}) u_{g_j}$, and note that

$$v = \sum_{j=1}^{3} u_{g_j} f_j^{1/2}.$$

Since $\alpha_{g_j}(f_j^{1/2})$ is supported on $g_j W_j$ and $g_j W_j \cap g_k W_k = \emptyset$ whenever $j \neq k$ by (iv) and (i) above, we have

$$\alpha_{g_j}(f_j^{1/2}) \alpha_{g_k}(f_k^{1/2}) = 0 \tag{3.6}$$

whenever $j \neq k$. Using this, we get

$$\begin{aligned} v^* v &= \sum_{j,k=1}^{3} u_{g_j}^* \alpha_{g_j}(f_j^{1/2}) \alpha_{g_k}(f_k^{1/2}) u_{g_k} \\ &\overset{(3.6)}{=} \sum_{j=1}^{3} u_{g_j}^* \alpha_{g_j}(f_j) u_{g_j} \\ &= \sum_{j=1}^{3} f_j = 1. \end{aligned}$$

Thus v is an isometry. On the other hand, we have

$$vv^* = \sum_{j,k=1}^{3} u_{g_j} f_j^{1/2} f_k^{1/2} u_{g_k}^* = \sum_{j,k=1}^{3} \alpha_{g_j}(f_j^{1/2} f_k^{1/2}) u_{g_j g_k^{-1}}.$$

We will show that $vv^* \neq 1$ by showing that $E(vv^*) \neq 1$. We apply E to the expression above and use (3) to get

$$E(vv^*) = \sum_{j=1}^{3} \alpha_{g_j}(f_j).$$

In particular, $E(vv^*)$ is supported on $\bigsqcup_{j=1}^{3} g_j W_j \subseteq V$. As $V \neq X$ by (ii), the above expression cannot equal 1, and hence v is not a unitary, as desired. □

We point out that the argument used in the theorem above is somewhat different from the one used to prove Theorem 3.6. Indeed, the reasoning used in Theorem 3.6 would only give the existence of a nontrivial open set V satisfying $X \prec_1 V$, which suffices to show infiniteness of $M_2(C(X) \rtimes G)$, but not of $C(X) \rtimes G$. In order to obtain $X \prec_0 V$, we need the strengthening of 2-paradoxical towers proved for $\mathbb{F}_2$ in Proposition 3.2.

4. Examples of groups with paradoxical towers

In this section we exhibit large classes of nonamenable groups which admit paradoxical towers; see Theorem C in the introduction. In addition to proving some preservation properties for the class of groups admitting paradoxical towers, the main tool to construct such groups is given in Proposition 4.6, where we show that one can produce paradoxical towers in groups admitting *some* topologically free n-filling action on a completely metrizable (but not necessarily locally compact) space. Using this, we give several concrete and explicit examples.

We begin by looking at extensions of groups with paradoxical towers, both by finite groups (Proposition 4.1) and by other groups with paradoxical towers (Proposition 4.2).

Proposition 4.1. *Let $n \in \mathbb{N}$, let G be a group, and let $K \leq G$ a finite normal subgroup such that G/K has n-paradoxical towers. Then G has $n|K|$-paradoxical towers.*

Proof. Denote by $\pi\colon G \to G/K$ the quotient map, let $F \subseteq G$ be a finite subset, and set $D_0 = \pi(F)$ and $m = |K|$. Since G/K is infinite, there exist $t_1, \dots, t_m \in G/K$ with $t_1 = 1$ such that $D_0 t_j$, for $j = 1, \dots, m$, are pairwise disjoint sets. Set

$$D = \bigsqcup_{j=1}^{m} D_0 t_j, \tag{4.1}$$

which is a finite subset of G/K. Since G/K admits n-paradoxical towers, it follows that there are sets $A_1, \dots, A_n \subseteq G/K$ and elements $h_1, \dots, h_n \in G/K$ such that

(i) the sets dA_i, for $d \in D$ and $i = 1, \dots, n$, are pairwise disjoint, and

(ii) $G/K = \bigcup_{i=1}^{n} h_i A_i$.

Fix an arbitrary enumeration $K = \{k_1, \dots, k_m\}$. Let $s\colon G/K \to G$ be a section for π. For $i = 1, \dots, n$ and $j = 1, \dots, m$, set

$$C_{i,j} = s(t_j h_i{}^{-1})s(h_i A_i) \subseteq G \quad \text{and} \quad g_{i,j} = k_j s(t_j h_i{}^{-1})^{-1} \in G. \tag{4.2}$$

We claim that the above are paradoxical towers in G for F. To check the first condition in Definition 3.1, let $d, d' \in F$, let $i, i' = 1, \dots, n$ and let $j, j' = 1, \dots, m$ satisfy

$$dC_{i,j} \cap d'C_{i',j'} \neq \emptyset. \tag{4.3}$$

Applying π in the equation above and using (4.2), we obtain

$$\pi(d)t_j A_i \cap \pi(d')t_{j'} A_{i'} \neq \emptyset.$$

Note that $\pi(d)t_j$ and $\pi(d')t_{j'}$ belong to D. By condition (i), we deduce that $\pi(d)t_j = \pi(d')t_{j'}$ and $i = i'$. The identity in (4.1) implies that $j = j'$ and $\pi(d) = \pi(d')$. In other words, $d^{-1}d'$ belongs to K. Combining this with (4.3) and (4.2), we get

$$ds(t_j h_i{}^{-1})s(h_i A_i) \cap d's(t_j h_i{}^{-1})s(h_i A_i) \neq \emptyset,$$

and thus

$$s(h_i A_i) \cap \underbrace{s(t_j h_i{}^{-1})^{-1} \overbrace{d^{-1}d'}^{\in K} s(t_j h_i{}^{-1})}_{\in K,\text{ since } K \text{ is normal}} s(h_i A_i) \neq \emptyset.$$

Since s is a section, we have $ks(G/K) \cap s(G/K) \neq \emptyset$ for some $k \in K$ if and only if $k = 1$. It follows that $s(t_j h_i{}^{-1})^{-1}d^{-1}d's(t_j h_i{}^{-1}) = 1$ and thus $d = d'$, as desired.

We check the second condition in Definition 3.1. By condition (ii), we have

$$s(G/K) = \bigcup_{i=1}^{n} s(h_i A_i).$$

Using that $G = \bigsqcup_{j=1}^{m} k_j s(G/K)$ at the last step, we obtain

$$\begin{aligned}\bigcup_{i,j=1}^{n,m} g_{i,j} C_{i,j} &\overset{(4.2)}{=} \bigcup_{i,j=1}^{n,m} k_j s(t_j h_i{}^{-1})^{-1} s(t_j h_i{}^{-1}) s(h_i A_i) \\ &= \bigcup_{i,j=1}^{n,m} k_j s(h_i A_i) = G.\end{aligned}$$

This proves that G has nm-paradoxical towers, as desired. □

Proposition 4.2. *Let G be a group, let $K \leq G$ be a normal subgroup. Assume that G/K has n-paradoxical towers and that K has m-paradoxical towers. Then G has nm-paradoxical towers.*

Proof. Let $\pi\colon G \to G/K$ be the canonical quotient map, and let $s\colon G/K \to G$ be any section for it. Let $F \subseteq G$ be a finite subset, and set $E_0 = F^2 \cap K$ and $D = \pi(F)$. Without loss of generality, we assume that F is symmetric and contains the identity of G. Since

G/K has n-paradoxical towers, there exist subsets $A_1, \dots, A_n \subseteq G/K$ and group elements $h_1, \dots, h_n \in G/K$ such that

(a.1) the sets dA_i, for $d \in D$ and $i = 1, \dots, n$, are pairwise disjoint,

(a.2) $G/K = \bigcup_{i=1}^n h_i A_i$.

Set $E = \bigcup_{i=1}^n s(h_i) E_0 s(h_i)^{-1}$, which by normality is a (finite) subset of K. As K has m-paradoxical towers, there exist subsets $B_1, \dots, B_m \subseteq K$ and group elements $k_1, \dots, k_m \in K$ such that

(b.1) the sets eB_j, for $e \in E$ and $j = 1, \dots, m$, are pairwise disjoint,

(b.2) $K = \bigcup_{j=1}^m k_j B_j$.

For $i = 1, \dots, n$ and $j = 1, \dots, m$, set

$$C_{i,j} = s(h_i)^{-1} B_j s(h_i A_i) \quad \text{and} \quad g_{i,j} = k_j s(h_i).$$

Note that $\pi(C_{i,j}) = A_i$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$.

We claim that the above are paradoxical towers in G for F. It is immediate to check that

$$\bigcup_{i,j=1}^{n,m} g_{i,j} C_{i,j} = G$$

using (a.2) and (b.2). Given $f, f' \in F$, $i, i' = 1, \dots, n$ and $j, j' = 1, \dots, m$, suppose that

$$f C_{i,j} \cap f' C_{i',j'} \neq \emptyset.$$

Applying π gives $\pi(f) A_i \cap \pi(f') A_{i'} \neq \emptyset$, which by condition (a.1) implies that $i = i'$ and $f^{-1} f' \in K$. Set $e = f^{-1} f' \in E_0$. Substituting in the equation above, we get

$$s(h_i)^{-1} B_j s(h_i A_i) \cap e s(h_i)^{-1} B_{j'} s(h_i A_i) \neq \emptyset.$$

Choose $a, a' \in A_i$, $b \in B_j$ and $b' \in B_{j'}$ such that

$$s(h_i)^{-1} b s(h_i a) = e s(h_i)^{-1} b' s(h_i a').$$

Applying π to the identity above gives $a = a'$, so we get

$$s(h_i)^{-1} b = e s(h_i)^{-1} b',$$

which implies that $B_j \cap (s(h_i) e s(h_i)^{-1}) B_{j'} \neq \emptyset$ since F contains the identity of G. Since $s(h_i) e s(h_i)^{-1}$ belongs to E, condition (b.1) implies that $s(h_i) e s(h_i)^{-1} = 1$ and $j = j'$. Thus $e = 1$ and $f = f'$, as desired. □

Lemma 4.3. *Let $n \in \mathbb{N}$, and let G be a group that can be expressed as an increasing union $G = \bigcup_{k \in \mathbb{N}} G_k$ of groups G_k that admit n-paradoxical towers. Then G admits n-paradoxical towers.*

Proof. Let $D \subseteq G$ be a finite subset, and find $k \in \mathbb{N}$ such that $D \subseteq G_k$. Find subsets $B_1, \dots, B_n \subseteq G_k$ and $g_1, \dots, g_n \in G_k$ such that

(1) the sets dB_i, for $d \in D$ and $i = 1, \dots, n$, are pairwise disjoint, and

(2) $\bigcup_{i=1}^n g_i B_i = G_k$.

Let $S \subseteq G$ be a set containing exactly one representative of each right coset in $G_k \backslash G$, so that $G = \bigsqcup_{s \in S} G_k s$. For $i = 1, \ldots, n$, set $A_i = \bigsqcup_{s \in S} B_i s$. It is then immediate to check that $A_1, \ldots, A_n \subseteq G$ and $g_1, \ldots, g_n \in G$ satisfy the conditions of Definition 3.1. □

The following notion was introduced in [28] for actions on compact spaces.

Definition 4.4. Let $n \in \mathbb{N}$. An action $G \curvearrowright Z$ of a countable group on a Hausdorff space Z is said to be *n-filling* if for any nonempty open sets $U_1, \ldots, U_n \subseteq Z$, there exist $g_1, \ldots, g_n \in G$ such that $\bigcup_{j=1}^n g_i U_i = Z$.

In the definition above, we do not assume the space Z to be compact, or even locally compact. It is not hard to see that if $G \curvearrowright Z$ is n-filling and Z is locally compact, then Z must in fact be compact. There exist, however, interesting n-filling actions on spaces that are not locally compact; see Proposition 4.7.

Remark 4.5. Actions that are 2-filling are also called *strong boundary actions*, and their C^*-algebraic crossed products were studied in [35]. They have also been studied under the name *extremely proximal actions* by Glasner in [24] and are called *extreme boundary actions* in [8].

Recall that a topological space is called *Baire* if the conclusion of the Baire category theorem holds: a countable intersection of open dense subsets is dense. The class of Baire spaces includes all locally compact Hausdorff spaces as well as all completely metrizable ones.

Proposition 4.6. *Let $n \in \mathbb{N}$ and let G be a countable, infinite group. Assume that there exists a topologically free n-filling action of G on a Hausdorff Baire space. Then G admits n-paradoxical towers.*

Proof. Let Z be a Hausdorff Baire space and let $G \curvearrowright Z$ be a topologically free n-filling action. Let $D \subseteq G$ be a finite subset, and assume without loss of generality that D contains the unit 1_G of G. Since $G \curvearrowright Z$ is topologically free, for every $g \in G \setminus \{1_G\}$, the open set

$$U_g = \{z \in Z : g \cdot z \neq z\}$$

is dense in Z. Since G is countable and Z is Baire, the set

$$Y := \bigcap_{g \in G \setminus \{1_G\}} U_g$$

is dense in Z, and in particular nonempty. Fix $z_1 \in Y$, so that $\mathrm{Stab}_G(z_1) = \{1_G\}$. Using that G is infinite, for $i = 2, \ldots, n$, we can choose $z_i \in Z$ recursively satisfying

$$z_i \in G \cdot z_1 \setminus (D^{-1} D \cdot z_1 \cup \cdots \cup D^{-1} D \cdot z_{i-1}).$$

Note that $dz_i = d' z_{i'}$ for $d, d' \in D$ and $i, i' = 1, \ldots, n$ implies $d = d'$ and $i = i'$. Since Z is Hausdorff, there exist open neighborhoods $U_1, \ldots, U_n$ of $z_1, \ldots, z_n$, respectively, such that dU_i, for $d \in D$ and $i = 1, \ldots, n$, are pairwise disjoint sets in Z. Since $G \curvearrowright Z$ is n-filling, there exist $g_1, \ldots, g_n \in G$ such that

$$\bigcup_{i=1}^n g_i U_i = Z.$$

For $i = 1, \dots, n$, set

$$A_i = \{g \in G : g \cdot z_1 \in U_i\}.$$

One checks that $A_1, \dots, A_n$ and $g_1, \dots, g_n$ satisfy the conditions of Definition 3.1. □

We turn to examples of groups which admit paradoxical towers. The first class we will consider is that of acylindrically hyperbolic groups, as introduced by Osin in [41, Definition 1.3]; see Proposition 4.7. This class includes all nonamenable hyperbolic groups (in particular, all nonabelian free groups), all but finitely many mapping class groups, the outer automorphism group $\mathrm{Out}(\mathbb{F}_n)$ of $\mathbb{F}_n$, nonelementary CAT(0)-groups containing a rank-one element, and many fundamental groups of hyperbolic 3-manifolds; see [41, Appendix].

Proposition 4.7. *Let G be an acylindrically hyperbolic group. Then G admits paradoxical towers.*

Proof. Let us first assume that G has no nontrivial finite normal subgroups. By definition, G admits a nonelementary acylindrical action by isometries on a (not necessarily proper) Gromov-hyperbolic space Y. By [41, Theorem 1.2], we can even assume that the action is cobounded (as we can take the space to be the Cayley graph associated to a suitable generating set). Let Z denote the (not necessarily compact) Gromov boundary ∂Y of Y. Since G has no nontrivial finite normal subgroups, [1, Proposition 4.1] ensures that the induced action $G \curvearrowright Z$ is minimal and topologically free.

We claim that $G \curvearrowright Z$ is 2-filling.[3)] Let $U, V \subseteq Z$ be nonempty open subsets. Since the action $G \curvearrowright Y$ is nonelementary, there exists a loxodromic element $h \in G$ (see [41, Theorem 1.2]). Denote by $h^{-\infty}$ the repelling point of h. By the minimality of $G \curvearrowright Z$, there is $t \in G$ with $t \cdot h^{-\infty} \in U$. By definition (see the paragraph before [41, Theorem 1.1]), there is $y \in Y$ such that $(h^{-n} \cdot y)_{n \in \mathbb{N}}$ converges to $h^{-\infty}$, and $h^{-\infty}$ is independent of y. In particular, G acts on the limit points of loxodromic elements by conjugation on the group, namely, for the loxodromic element $g = tht^{-1}$ we have $t \cdot h^{-\infty} = g^{-\infty}$.

Again by minimality, there is an element $s \in G$ and an open neighborhood W of the attracting point $g^{+\infty} \in Z$ such that $s \cdot W \subseteq V$. Since g is loxodromic, there is $n \in \mathbb{N}$ such that $g^n \cdot (Z \setminus U) \subseteq W$ (see [26, Lemma 4.3] or the proof of [50, Theorem 2B]). In particular, we have $sg^n(Z \setminus U) \subseteq V$. Thus $U \cup g^{-n}s^{-1} \cdot V = Z$, which proves that $G \curvearrowright Z$ is 2-filling.

Note that by [14, Proposition 3.4.18] $Z = \partial Y$ is a completely metrizable space. By Proposition 4.6, G admits 2-paradoxical towers.

If G is an arbitrary acylindrically hyperbolic group, then G contains a unique maximal finite normal subgroup $K(G)$ by [13, Theorem 6.14]. We claim that $G/K(G)$ is acylindrically hyperbolic. To see this, use [41, Theorem 1.2] to find a proper, infinite, hyperbolically embedded subgroup $H \hookrightarrow_h G$ (see [41, Definition 2.9]). By [13, Theorem 6.14], we have $K(G) \subseteq H$. By [13, Lemma 8.3], the map $H/K(G) \to G/K(G)$ induced by $H \hookrightarrow_h G$ is a hyperbolic embedding. By [41, Theorem 1.2], this shows that $G/K(G)$ is acylindrically hyperbolic.

Since $G/K(G)$ clearly contains no nontrivial finite normal subgroups, it follows from the claim above and the first part of the proof that $G/K(G)$ admits 2-paradoxical towers. We conclude from Proposition 4.1 that G admits $2|K(G)|$-paradoxical towers. □

3) This argument is inspired by [35, Example 2.1], but note that ∂Y is not necessarily compact.

The following is a concrete application of the example above to the work of Klisse [34].

Example 4.8. Let W be a nonamenable finite rank irreducible right-angled Coxeter group. Denote by ∂W its boundary in the sense of [34, Definition 3.1]. Then the crossed product $C(\partial W) \rtimes W$ is classifiable Kirchberg algebra.

Proof. We claim that W is acylindrically hyperbolic. By [46, Theorem 1.3], it suffices to show that W has a rank-one isometry and acts properly and cocompactly on a proper CAT(0)-space. The fact that a nonamenable (also called nonaffine) Coxeter group acts properly and cocompactly on a proper CAT(0)-space is a classical theorem of Moussong, and the fact that W contains a rank-one isometry follows from [9, Corollary 4.3 and Proposition 4.5]. This proves that W is acylindrically hyperbolic, and thus it admits paradoxical towers by Proposition 4.7.

The action $W \curvearrowright \partial W$ is amenable by [34, Theorem 0.2], minimal by [34, Theorem 3.19], and topologically free by [34, Lemma 3.25]. Thus $C(\partial W) \rtimes W$ is purely infinite by Corollary 3.8. □

Recall that an *HNN-triple* (G, H, θ) consists of a group G, a subgroup H, and an injective group homomorphism $\theta\colon H \to G$. The *HNN-extension* $\mathrm{HNN}(G, H, \theta)$ associated to (G, H, θ) is the quotient of the free product $G * \mathbb{Z} = \langle G, x\rangle$ by the relation $xh = \theta(h)x$ for all $h \in H$. The HNN-extension $\Gamma = \mathrm{HNN}(G, H, \theta)$ is said to be *faithful* if its natural action on the associated Bass–Serre tree is faithful. Moreover, Γ is said to be *ascending* if either $G = H$ or $G = \theta(H)$.

For the definition of a highly transitive group, we refer the reader to the introduction of [20].

Proposition 4.9. *Every faithful highly transitive non-ascending HNN-extension has* 2*-paradoxical towers.*

Proof. Let (G, H, θ) be a non-ascending HNN-triple with $\Gamma = \mathrm{HNN}(G, H, \theta)$ faithful and highly transitive. By [8, Proposition 4.16], the natural action of Γ on the Bass–Serre tree T associated to (G, H, θ) is strongly hyperbolic. Since this action is always minimal (see the comments before [8, Proposition 4.16]), it follows from [8, Lemma 3.5] that the induced action $\Gamma \curvearrowright \overline{\partial T}$ is 2-filling (see Remark 4.5). By [20, Theorem B], the action $\Gamma \curvearrowright \partial T$ is topologically free, and therefore also $\Gamma \curvearrowright \overline{\partial T}$ is topologically free. The result thus follows from Proposition 4.6. □

A concrete and relevant class of groups covered by Proposition 4.9 is that of Baumslag–Solitar groups. These groups are never acylindrically hyperbolic by [20, Remark 8.4], and are thus not covered by Proposition 4.7.

Example 4.10. Let $m, n \in \mathbb{Z}$ with $|m|, |n| > 1$ and $|m| \neq |n|$. Then the associated Baumslag–Solitar group

$$\mathrm{BS}(m, n) = \langle a, b\colon ab^m = b^n a\rangle$$

has 2-paradoxical towers.

Proof. We identify the group $\mathrm{BS}(m,n)$ as an HNN-extension as in [8, Example 4.21]: we take $G = \mathbb{Z} = \langle a \rangle$, with $H = m\mathbb{Z} = \langle a^m \rangle$ and $\theta \colon m\mathbb{Z} \to \mathbb{Z}$ determined by $\theta(a^m) = a^n$; we denote this map by $\cdot\frac{n}{m}$. Since $|m|, |n| > 1$, the HNN-extension is non-ascending. Moreover, $\mathrm{BS}(m,n) = \mathrm{HNN}(\mathbb{Z}, m\mathbb{Z}, \cdot\frac{n}{m})$ is highly transitive by [20, Proposition 8.8], and is faithful by [38] (see also [15, Remark (iii) before Proposition 19]). Thus the claim follows from Proposition 4.9. □

A further class of groups we can treat with our methods is that of amalgamated free products. Given groups A and B containing a common subgroup C, for each $k \geq 0$ the subgroup $C_k \subseteq C$ is defined after [15, Corollary 2].

Example 4.11. Let A and B be groups containing a common subgroup C. Assume that the following conditions hold:

(1) $[A : C] > 1$ and $[B : C] > 2$.

(2) There is $k \geq 1$ such that $C_k = \{1\}$.

Then the amalgamated free product $\Gamma = A *_C B$ has 2-paradoxical towers. In particular, a free product $G * H$ of nontrivial groups with $|H| > 2$ always has 2-paradoxical towers.

Proof. Let T denote the Bass–Serre tree of Γ. By [15, Proposition 19], the action of Γ on T is minimal and strongly hyperbolic. Therefore, by [8, Lemma 3.5], the action of Γ on $\overline{\partial T}$ is 2-filling. By [15, Proposition 19] and [8, Proposition 3.8], the action of Γ on ∂T is topologically free, which implies that the action of Γ on $\overline{\partial T}$ is topologically free as well. The claim follows from Proposition 4.6. □

There is a generalization of Proposition 4.9 and Example 4.11 to groups acting on trees, as follows:

Remark 4.12. Let a group Γ act on a tree T. Assume that the action $\Gamma \curvearrowright T$ is minimal and strongly hyperbolic. Then the action $\Gamma \curvearrowright \overline{\partial T}$ is 2-filling by [8, Lemma 3.5]. Assume moreover that the fixator subgroup of every half-tree of T is trivial, which is equivalent to topological freeness of the action of Γ on $\overline{\partial T}$, by [8, Proposition 3.8] and [8, Remark 2.1]. Then Γ has 2-paradoxical towers by Proposition 4.6.

Next, we establish the existence of paradoxical towers in certain lattices in Lie groups. Note that the following example covers $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$ (while $\mathrm{SL}_2(\mathbb{Z})$ is covered by Proposition 4.7).

Example 4.13. Let Γ be a lattice in a real connected semisimple Lie group G without compact factors and with finite center. Then Γ admits paradoxical towers.

Proof. Let us first assume that G has trivial center. Then the action of Γ on the Furstenberg boundary G/P of G is topologically free and n-filling for some $n \in \mathbb{N}$ by [28, Proposition 2.5 and Remark 2.6] and the proof of [2, Proposition 3.4]. Thus Γ has paradoxical towers by Proposition 4.6.

We now treat the general case, so suppose that the center K of G is finite, and set $K_\Gamma = \Gamma \cap K$, which is a finite normal subgroup of Γ. Then G/K is a real connected semisimple Lie group without compact factors and with trivial center, and it is easy to see that $\Gamma/K_\Gamma \subseteq G/K$ is a lattice. Then Γ/K_Γ admits paradoxical towers by the paragraph above, and hence Γ admits paradoxical towers by Proposition 4.1. □

The following, in combination with Corollary 3.8, generalizes [28, Proposition 4.2]. We refer the reader to [28, Sections 3 and 4] and references therein for the definitions of buildings of type $\widetilde{A}_2$ and actions on them.

Example 4.14. Let G a group which acts simply transitively in a type rotating manner on the vertices of a building Δ of type $\widetilde{A}_2$. (Such groups are called $\widetilde{A}_2$-*groups* in [28].) Then G admits paradoxical towers.

Proof. The action of the group G on the boundary of Δ is topologically free and 6-filling by [28, Theorem 3.8 and Proposition 4.1]. Thus G admits paradoxical towers by Proposition 4.6. □

For the following example, we refer the reader to the discussion just before [2, Proposition 3.5].

Example 4.15. Let G be a countable subgroup of isometries of a visibility manifold X with $\mathrm{vol}(X/G) < \infty$. Then G admits paradoxical towers.

Proof. It follows from [28, Proposition 2.5] and [6, Theorem 2.8, Theorem 2.2] (see also [2, p. 218]) that the action of G on ∂X is topologically free and n-filling for some n. Therefore G admits paradoxical towers by Proposition 4.6. □

To conclude, we give examples of nonamenable groups that do not have paradoxical towers. Observe, however, that many of these groups are covered by Theorem B.

Example 4.16. Let H be a nonamenable group and let K be an infinite amenable group. Then $G = H \times K$ is a nonamenable group that does not have paradoxical towers.

Proof. Assume by contradiction that G has n-paradoxical towers for some $n \in \mathbb{N}$. Let $D \subseteq K$ be a subset with $|D| \geq n+1$. We canonically identify D with $\{1\} \times D \subseteq G$. Since G has n-paradoxical towers, it follows that there are subsets $A_1, \ldots, A_n \subseteq G$ and elements $g_1 = (h_1, k_1), \ldots, g_n = (h_n, k_n) \in G$ such that:

(1) the sets dA_i, for $d \in D$ and $j = 1, \ldots, n$, are pairwise disjoint,

(2) $G = \bigcup_{j=1}^n g_j A_j$.

Fix a finitely additive left-invariant probability measure μ on K, and denote by $\pi_K \colon G \to K$ the canonical projection. Given $h \in H, d \in D$ and $j = 1, \ldots, n$, we have

$$\pi_K((\{h\} \times K) \cap dA_j) = d\pi_K((\{h\} \times K) \cap A_j).$$

By condition (1) above, for different $d \in D$ and $j = 1, \ldots, n$, the above sets are pairwise disjoint. Since μ is left-invariant, we get

$$\mu\big(\pi_K((\{h\} \times K) \cap A_j)\big) \leq \frac{1}{n+1}$$

for all $h \in H$ and $j = 1, \ldots, n$. Denote by 1_G the unit of G. Using condition (2) at the second step, we get

$$\begin{aligned}
1 &= \mu(K) \\
&= \mu\left(\pi_K\left((\{1_G\} \times K) \cap \left(\bigcup_{j=1}^{n} (h_j, k_j)A_j\right)\right)\right) \\
&= \mu\left((\pi_K\left(\bigcup_{j=1}^{n} (h_j, k_j)((\{h_j^{-1}\} \times K) \cap A_j)\right)\right) \\
&= \mu\left(\bigcup_{j=1}^{n} k_j \pi_K((\{h_j^{-1}\} \times K) \cap A_j)\right) \\
&\leq \sum_{j=1}^{n} \mu\big(\pi_K((\{h_j^{-1}\} \times K) \cap A_j)\big) \\
&\leq \frac{n}{n+1},
\end{aligned}$$

a contradiction. □

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