ABSTRACT

Title of dissertation: ESSAYS ON PACKAGE AUCTIONS

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The recent auctions literature has devoted much attention to mechanisms that allow package bidding: all-or-nothing bids for sets of items. Introducing package bids can improve efficiency by reducing the bidders "exposure" risk of winning undesirable combinations of items. However, package bids can also create a free-rider problem for relatively small bidders since they need to compete jointly against their larger opponents, potentially reducing efficiency. The inherent asymmetry among different package bids significantly complicates an equilibrium analysis of the costs and benefits of allowing package bids in auctions.

The first chapter makes progress in solving for Bayesian-Nash equilibria of the first-price package auction. We develop a new computational method which is based on a complementarity formulation of the system of equilibrium inequalities. Additionally, we establish existence of equilibrium for special cases. Our analysis shows that introducing package bidding can significantly improve efficiency when the exposure risk faced by bidders is large, but it can reduce efficiency otherwise. We

also compare the first-price package auction with other leading package alternatives. Surprisingly, in the environment considered, the first-price package auction performs reasonably well, with respect to both revenue and efficiency, despite the presence of a strong free-rider problem.

The second chapter studies the core-selecting auctions that were proposed recently as alternatives to the famous Vickrey-Clarke Groves (VCG) mechanism for environments with complementarities. The existing literature on core-selecting auctions is limited to only a complete-information analysis. We consider a simple incomplete-information model which allows us to do a full equilibrium analysis, including closed-form solutions for some distributions, for four different core-selecting auction formats suggested in the literature. Our model also admits correlations among bidders values. We find that the revenues and efficiency from core-selecting auctions improve as correlations among bidders values increase, while the revenues from the Vickrey auction worsen. Thus, there may be good reasons for policymakers to utilize a core-selecting auction rather than a VCG mechanism in realistic environments.

ESSAYS ON PACKAGE AUCTIONS

by

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It is impossible to remember all, and I apologize to those I have inadvertently left out.

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List of Abbreviations

α	alpha
β	beta
γ	gamma
VCG	Vickrey-Clarke-Groves mechanism *or Vickrey Auction*
CSA	Core-Selecting Auctions
FCC	Federal Communication Commission
FAA	Federal Aviation Administration

Introduction to Package Auctions

What are package auctions?

Package auctions are auctions in which bidders are allowed to place bids on combinations of items, or "*packages*". A "package" bid is interpreted as an all-or-nothing offer to buy (or sell) all items that compromise the package at the specified price. Various forms of package auctions have been in active use for decades. In estate and bankruptcy auctions, for example, a typical procedure is to sell all lots one by one with the understanding that in the end, these lots are offered again as part of larger packages. In case the package bid on a combination of items exceeds the sum of individual bids on items in the package obtained earlier, the items are sold as a package. Auctions where bidders can bid on packages are also known as *combinatorial auctions*.

Despite the straightforward package idea, up until recently, package auctions have been used only in trivial applications. This is quite understandable. While such auctions allow bidders to fully express and accommodate their preferences over all possible combinations of items, they also bring extremely high level of complexity. It is quite instructive to think about a simple auction with just 10 heterogeneous items. In such auction, any bidder can easily submit up to $2^{10} = 1024$ different mutually-exclusive package bids! Then, even a simple task, faced by the auctioneer, of finding the winners and their corresponding winning packages is highly unattractive and elaborate endeavor. One can only imagine the size of computations required to perform such calculation when hundreds or thousands of objects are involved. The obvious question is then why we need to overcome such complexity and use package auctions instead of well-known non-package alternatives where all calculations that have to be made by the auctioneer are trivial. For example, auctioneer can easily sell all items separately one after another or simultaneously using a sealed-bid auction format.

The fundamental reason behind the growing interest to package auctions is the superior ability of package bids to fully express bidders' preferences over any set of items. Any bidder who demands many items and has no value for smaller sets will find package bids highly attractive as she can not be forced into buying undesired allocations. Equally plausible, package bids will allow a bidder who is indifferent among several items but needs only one of them to submit an appropriate set of package bids that makes it impossible to win more than one object. In a non-package alternative, such bidder will be unable to express her indifferences without running a risk of winning too many items. The former situation is the leading example of the environment for which auction experts strongly recommend the use of package bids in practical applications. Such environments are characterized by the presence of strong *complementarities* (i.e., synergetic valuations for combinations of items that exceed the sum of valuations of the standalone components) that differ across bidders.

The numerous examples of such environments include radio spectrum, truck-load transportation, industrial procurement, public transportation and airport departure/arrival slots. In spectrum domain, complementarities that bidders face can be exceptionally severe. For example, a bidder can be a new entrant for the telecommunication industry that needs to win a large amount of bandwidth. Winning any smaller amount of bandwidth can be completely worthless for this bidder since it will not be enough to achieve the minimum scale of operations. At the same time, incumbents would not mind winning smaller amounts of spectrum that will add capacity to their current holdings. Accommodating bidders' preferences in such applications is extremely important as the combined value of the items auctioned sometimes can be as high as tens of billions of dollars.

Fortunately, recent advancements in technology and science made it possible to overcome some complexity of package auctions. As a result, the package designs that can allocate hundreds of items at once are a materialized reality that rapidly spreads around the globe. Several countries including Austria, Denmark, Netherlands, UK, Canada and Australia[1] have already adopted a state of the art package clock design[2] to allocate spectrum that will be used for the next generation LTE (4G) wireless networks.

Literature review

Literature on package auctions is extensive and quite diverse as researchers from several fields including economics, computer science and operations research tackle it from different angles. Economists are primarily interested in auctions (and package auctions in particular) as market mechanisms that can allocate objects effi-

[1] Canada and Australia are in the review process of adopting a package design at the time of this writing.

[2] Package clock auction is also referred to as combinatorial clock auction.

ciently. Computer science studies different approaches to express bidder preferences over the large number of objects in the most economical way. Finally, optimization algorithms from operations research are used to solve the winner-determination problem which is the cornerstone of any package design.

A volume edited by Peter Cramton, Yoav Shoham and Richard Steinberg (2006) [19] is an excellent source for a unified treatment on combinatorial auctions. It contains a series of articles from economics, computer science and operations research literature with the state of the art contributions. The book also provides deep discussion on the recent practical applications of combinatorial auctions.

The current thesis looks at the package auctions from an economist perspective. Starting with the seminal work of William Vickrey (1961) [52], economists traditionally are interested in incentives, revenue and efficiency characteristics of different auction procedures. There are several general treatments that talk about auction theory and package auctions in particular. Krishna (2002) [37], Klemperer (2004) [36] and Milgrom (2004) [44] give a thorough comprehensive review of the current state of the auction theory.

More detailed literature review related to the specific research question of each chapter can be found in the introduction sections 1.1 and 2.1. The overall structure of the thesis and a short preview of contributions are outlined next.

Summary of contributions and the organization of the thesis

The current thesis contributes to the growing body of literature on package auctions in several important ways. First of all, all equilibrium analysis is done within *incomplete information* environment. Except for a few notable examples, the majority of game-theoretic treatments of package auctions has been routinely done using complete information framework, i.e., assuming that all auction participants know the intrinsic structure of preferences and exact values of ALL auction participants. This is quite understandable given the level of complexity the large number of possible package bids create even in the smallest non-trivial environments. However, economists traditionally prefer incomplete information models for auctions as it allows for a much better fit with the reality and provides compelling motivation for the use of auctions as a market allocation mechanism.

Given the multiplied complexity of package auctions with incomplete information, a simple model is considered that, on the one hand, allows tractable equilibrium analysis of multiple package designs and, on the other hand, contains several realistic features like complementarity and correlation of values. The main contribution is the comprehensive comparison of different package auctions among themselves and some of their non-package alternatives in terms of bidder incentives, seller revenue and overall economic efficiency. Our analysis allows to identify the important elements and features of package environments that can be extremely important in applications. There are also some methodological contributions that include new numerical and analytical techniques for finding solutions of package auctions.

The content of the thesis is organized in two separate chapters that are based on Baranov(2010) [12] and Ausubel and Baranov (2011) [5]. Chapters do not rely on each other and can be read on its own in any order.

The first chapter considers the first-price package auction and compares it with the first-price auction where package bids are not allowed. We solve for Bayesian-Nash equilibria of both auctions by developing a new computational method which is based on a complementarity formulation of the system of equilibrium inequalities. Additionally, we establish existence of equilibrium for special cases.

The second chapter looks at the "second price like" core-selecting auctions that were proposed recently as alternatives to the famous Vickrey-Clarke Groves (VCG) mechanism for environments with complementarities. We perform a full equilibrium analysis, including closed-form solutions for some distributions, for four different core-selecting auction formats suggested in the literature. Then we compare all designs with each other and discuss how our findings can be used in applications.

Chapter 1

Exposure vs. Free-Riding in Auctions with Incomplete Information

by Oleg V. Baranov

1.1 Introduction

The recent auction literature has devoted a lot of attention to auction mechanisms that can be used to allocate multiple items simultaneously. Such strong interest is not surprising since the majority of recent high-stake auctions involved selling many items at once to participants with non-trivial interests in different combinations of these items. For example, in spectrum auctions, governments sell licenses to use the airwave spectrum in many geographical regions to telecommunication companies whose business plans only match some particular subsets of licenses. Other examples include treasury auctions, bankruptcy auctions and numerous private and public procurement auctions.

Bidders in these auctions can often have very specific preferences for some bundles of items offered in the auction. For example, a buyer's value for a pair of objects can be higher than the standalone values of the individual components because the combination of items generates additional synergetic value. In other words, a bidder may value some objects as complements. In such environments, standard independent single-item auctions might fail to produce an efficient allocation because they do not allow bidders to express their possible synergies across multiple items.

For the ease of exposition, consider a simple illustrative example with just two items, "East" and "West" (which might represent spectrum licenses for the Eastern and Western parts of a country), offered for sale and three bidders. The "global" bidder regards both East and West as perfect complements, receiving positive value from the package {East, West} but obtains no value from either item when acquiring them individually. The other bidders, usually referred to as "local" bidders, obtain value only from one item. Local Bidder 1 is interested only in East and obtains no value for West while the local bidder 2 obtains value only for West and has no interest in East. In case the auctioneer decides to sell the items by means of two independent auctions, the global bidder is exposed to a positive chance of being forced to buy just East or just West when she wins only in one auction but loses in the other one. Since the global bidder views both items as perfect complements, acquiring just one of them is totally worthless for her.

This phenomenon, known as the *exposure problem*, can negatively affect efficiency of the auctions because exposed bidders will strategically underbid or overbid in an attempt to avoid paying positive prices for good-for-nothing packages.

The auctioneer can easily accommodate bidders with synergies and avoid the exposure problem altogether by simply auctioning all items simultaneously and allowing bidders to submit package bids, i.e., all-or-nothing bids for subsets of items specified by bidders themselves. In the example from the previous paragraph, the global bidder will be able to submit a package bid for the actually desired combination of items, i.e., for both East and West bundled together. Such package bid completely eliminates any exposure risk because the global bidder can win either both items or no items at all with zero probability of getting just one of them.

However, while package bids solve all exposure concerns, they also can introduce a *free-rider problem*, sometimes referred to as the threshold problem in the auction literature. The issue can be easily demonstrated using the illustrative example introduced above. Consider a first-price auction where the global bidder submits a package bid B for the bundle of East and West while local bidders bid b_1 and b_2 on the corresponding items. The local bidders win their items whenever their total bid is higher than the package bid of the global bidder ($b_1 + b_2 > B$). Without an ability to explicitly coordinate their actions, both local bidders have incentives to free ride on each other by reducing their individual bids. Observe that it is possible for a local bidder to get the desired item for free when the other local bidder outbids the package bid of the global bidder alone[1]. Such free-riding motives can easily mitigate any possible efficiency and revenue gains achieved by the package bidding and further degrade the performance of the package design.

Therefore, the overall effect of the package bidding on the auction performance characteristics is ambiguous. On the one hand, package bids might increase efficiency and revenue by eliminating the exposure problem for bidders who view objects as complements. On the other hand, the presence of package bids might negatively

[1]Consider the following bid data: $B = 5,\ b_1 = 0,\ b_2 = 6$. The local bidder 1 wins her item and pays zero since $b_1 + b_2 = 6 > 5 = B$.

affect efficiency and revenue when some bidders without a fundamental conflict of interest have incentives to free ride on other bidders' bids.

The most widely used package design in applications is the sealed-bid first-price package auction. Virtually all procurement auctions which allow bidders to submit at least some package bids use the pay-as-bid pricing rule. Examples include auctions of bus routes in London,[2] auctions for milk providers in Chile[3] and IBM-Mars procurement auctions.[4] Such features as resistance to collusion, participation encouragement and transparency (winners pay what they bid) explain the popularity of the sealed-bid pay-as-bid auctions.[5] Therefore, the first-price auction is the most relevant auction format to study the consequences of package bidding from a policy-making perspective. However, an inherent structural asymmetry between bidders' values and bids is crucial for having non-trivial versions of both the exposure and free-rider problems. This requirement is unfortunate since a general theoretical analysis of first-price auctions in asymmetric environments has proved to be very tedious.

In this paper, we consider a simple stylized model with two items in which one bidder values them as perfect complements. First, we perform a Bayesian-Nash equilibrium analysis for both the package and non-package versions of the first-price auction, including existence proofs for some instances of the model. Second, we develop a novel numerical technique that is used to approximate equilibrium

[2]See Cantillon and Pesendorfer (2006) [14].

[3]See Epstein, Henríques, Catalán, Weintraub and Martínez (2002) [26].

[4]See Hohner, Rich, Ng, Reed, Davenport, Kalagnanam, Lee and An (2003) [35].

[5]See Cramton (1998) [17] for discussion.

bidding functions in both auctions. Finally, we investigate the impact of package bidding on the performance of the first-price auctions and compare the first-price package auction with other leading package alternatives.

Results on existence of the monotone Bayesian-Nash equilibrium in the first-price package auctions are related to the large body of literature on existence of the equilibrium in first-price auctions for a single item (Maskin and Riley (2000) [43], Lebrun(1999) [40])[6] and in games of incomplete information in general (Radner and Rosenthal (1982) [49], Milgrom and Weber (1985) [45], Athey (2001) [3]). The numerical technique developed here in order to approximate equilibrium strategies is related to the literature on numerical methods designed to solve asymmetric first-price auctions (Marshall et al (1994) [42], Riley and Li (1997) [50], Bajari (2001) [11]) and general games of incomplete information (Armantier and Richard (1997) [1]).

This paper is a contribution to the literature on package auctions. The best-known package auction is the Vickrey-Clarke-Groves (VCG) mechanism, which was introduced in the classic theory of auctions and public choice. Vickrey (1961) [52] looked into multiple-unit auctions for homogeneous goods while Clarke (1971) [16] and Groves (1973) [33] studied public choice models in potentially heterogeneous environments.

Other auction formats with package bids have been extensively studied in complete information settings. Bernheim and Whinston (1986) [13] developed a general theory of the first-price package auctions. Day and Raghavan (2007) [24] and Day and Milgrom (2008) [23] introduced a class of alternative package designs

[6]See also Lebrun(1996) [39], Bajari(1997) [10], Lizzeri and Persico (2000) [41].

which came to be known as *core-selecting auctions*. A particular interesting subclass of core-selecting auctions are auctions that minimize seller's revenue subject to the core constraints. They showed that such *minimum-revenue core auctions* maximize bidders' incentives for truthful reporting[7]. One recently-proposed example of such mechanisms is the nearest-Vickrey package auction suggested by Day and Cramton (2009) [22]. Despite its relatively short existence, this rule has already been used in several important applications.[8] The "ascending proxy auction" introduced by Ausubel and Milgrom (2002, 2006)[9] [8] [9] does not explicitly minimize revenue but often can result in a price vector that corresponds to the minimum-revenue outcome.

Some recent papers have begun to explore the comparison among core-selecting auctions in an incomplete information environment using the simple model similar to the one used in this paper. Erdil and Klemperer (2010) [27] focused on a subclass of minimum-revenue core payment rules, which are referred to as "reference rules," and argued that such payment rules perform better than any other minimum-revenue core rules because they minimize bidders' marginal incentives to deviate. Sano (2010) [51] analyzed the "ascending proxy auction" in a simple setup with in-

[7]Note that the first-price package auction belongs to the class of core-selecting auctions, but in this case the seller's revenue is maximized subject to the core constraints, unlike in "second-price-like" minimum-revenue core auctions.

[8]The package clock (or "combinatorial clock") auction with the nearest-Vickrey pricing rule has been used for spectrum auctions in the UK (two auctions: February and May 2008), the Netherlands (April 2010), Denmark (May 2010) and Austria (September 2010).

[9]A closely related auction procedure was developed independently by Parkes and Ungar (2000) [48] and Parkes (2001) [47].

dependent values. Goeree and Lien (2009) [31] showed that in general environments with complementarities a core-selecting auction that shares the dominant strategy property of the VCG does not exist. Ausubel and Baranov (2011) [5] compared a variety of minimum-revenue core payment rules and showed that a positive correlation among bidders' values can have a dramatic impact on the magnitude of the similar free-rider problem which also plagues all core-selecting auctions in environments with complementarities.

Chernomaz and Levin (2010) [15] studied the effects of package bidding on the sealed-bid first-price auctions in the experimental setting. They considered a similar model with the global bidder, who has positive synergies, and two local bidders. Regretfully, they made a simplifying assumption about the perfect correlation between local bidders' values. While this assumption significantly simplifies their theoretical analysis, it also effectively eliminates the exposure problem[10] from their consideration. As has been noted above, the exposure worries arising from synergies, not synergies without exposure risks, are the leading motivator for using package designs. For that reason, our model explicitly allows exposure outcomes to be part of the equilibrium.

The model adapted here is a variation of the original model introduced by Krishna and Rosenthal (1996) [38] to study the exposure problem in second-price auctions with synergies. The main model of the paper is an incomplete-information

[10] The exposure problem still exists in the experiment part of their paper. However, the lack of the exposure problem in the underlying theoretical model provides an alternative explanation for their main results.

version of the illustrative example model used throughout this introduction. The global bidder obtains value u from winning both items but gets zero value from getting only one of them. Each local bidder i values her corresponding item at v_i and obtains no value from the other item. As in Ausubel and Baranov (2011) [5], the local bidders' values are perfectly correlated with probability γ and independently distributed with probability $1 - \gamma$. Moreover, local bidders only get to observe their values, but they are unaware of whether their values are perfectly correlated ($v_1 = v_2$) or independent. Thus, γ parameterizes a family of distributions that permits the correlation between local bidders' values to be varied continuously from zero to one.

There are several reasons for considering a class of models with positive correlation among bidders' values. First, in important applications such as spectrum auctions, the correlations among bidders' values can be significant since similar bidders are likely to use spectrum licenses to deploy the same telecommunication technology. Second, Ausubel and Baranov (2011) [5] found that positive correlation can considerably affect the performance of different package-bidding designs. Finally, when package bids are not allowed, the positive correlation can also mitigate the exposure problem. For instance, in case the values of local bidders are perfectly correlated ($\gamma = 1$) and both local bidders follow the same bidding strategy[11], the global bidder can completely avoid any exposure risk by submitting the same bid in both

[11] The solution concept is Bayesian-Nash equilibrium. Since the joint distribution of values will be symmetric with respect to the two local bidders, and we will limit attention to Bayesian-Nash equilibria that are symmetric with respect to the two local bidders.

auctions even though her preferences exhibit the extreme form of complementarities.

The model clearly demonstrates the mechanics of the trade-off between the exposure and free-rider problems. When the exposure concerns are weak, package bids can significantly hurt efficiency, but they can sharply improve it when the exposure risk is relatively high. Further, we show that package bids can substantially improve efficiency, even when there is no exposure problem at all, simply because the bid asymmetries introduced by package bids might compensate for the prior distributional asymmetries. Surprisingly, the first-price package auction is more efficient when the global bidder has a distributional advantage over the local bidders. Finally, several examples demonstrate that the package auction can also generate higher revenues in more competitive environments.

The paper proceeds as follows. The model is described in Section 1.2. Section 1.3 contains results of the Bayesian-Nash equilibrium analysis of the first-price auction with and without package bids. The numerical approach which is used to approximate equilibrium bidding functions is described in Section 1.4. Several examples that compare the relative performance of the package and non-package versions of the first-price auction are presented in Section 1.5. We compare the first-price package auction with the other leading package alternatives in Section 1.6. Section 1.7 concludes. Most proofs are relegated to the Appendix 1.A.

1.2 Model

The model used here closely follows the model developed in Ausubel and Baranov (2011) [5]. It consists of two items offered for sale, two local bidders and one global bidder. Local bidders, denoted 1 and 2, are interested only in one item and receive no extra utility from acquiring the second item. Their private values are denoted v_1 and v_2, respectively. The global bidder wants to acquire two items and gets zero utility from owning just one item. The value she receives in case she gets both items is denoted u. All bidders are risk-neutral with quasilinear preferences. Thus, the payoff of the local bidder i if she wins an item at price p_i, is $v_i - p_i$. The payoff of the global bidder, if she wins two items for a total price p, is $u - p$; while, if the global bidder wins only one item at price p, her payoff is simply $-p$.

With probability $\gamma \in [0, 1]$, both local bidders have exactly the same value v, which is drawn from the distribution on $[0, \bar{v}]$ defined by a cumulative distribution function $F(v)$ with atomless probability density function $f(v)$. With probability $1 - \gamma$, the values, v_1 and v_2, of the local bidders are drawn from the same distribution $F(v)$ independently from each other. The value of the global bidder, u, is independently drawn from the distribution on a $[0, \bar{u}]$ described by a cumulative distribution function $G(u)$ with atomless density $g(u)$. For the ease of exposition, we assume that $\bar{u} = 2\bar{v}$.

The assumption of independence between values of the global bidder and local bidders is reasonable since many bidder-specific characteristics such as cost structure and the scale of operations may be substantially different for the global bidder and local bidders. Meanwhile, both local bidders are alike in a sense of demanding only one item, so it is likely that their values are similar. For example, in a spectrum auction, the local bidders might be two firms that plan to put the same telecommunication technology in operation in two different geographic areas with similar demographic characteristics.

Parameter γ controls the amount of correlation between local bidders' values. For example, $\gamma = 0$ and $\gamma = 1$ correspond to the cases of independent values and perfectly correlated values respectively. The local's bidder value model can be summarized by the conditional cumulative distribution function of the local bidder i that defines her probability assessment of bidder's j valuations given her value v_i:

$$F_L(v_j|v_i = s) = \begin{cases} (1-\gamma)F(v_j) & 0 \leq v_j < s \\ (1-\gamma)F(v_j) + \gamma & s \leq v_j \leq \bar{v} \end{cases} \qquad i \neq j$$

Without loss of generality, our attention is limited to the first-price package auction where any bidder is allowed to submit only one bid. While impractical in general environments, this limitation has no implications for the analysis of the model because of the perfect complementarity nature of the bidders' preferences[12] in our model. For example, the global bidder values only a bundle of two items and her bid B is interpreted as a package bid for two items[13]. Each local bidder i is interested only in one item and her bid b_i expresses her willingness to pay b_i for that item.

[12] All bidders weakly prefer to bid for their desired bundles.

[13] Since the global bidder can only submit one bid, there are no inefficiencies arising from the strategic price discrimination in the sense of Cantillon and Pesendorfer (2006) [14].

The first-price package auction proceeds in the following manner. First, all bidders submit their bids to the auctioneer who then chooses an allocation which maximizes total welfare with respect to the bids. In this simple model, only two outcomes are possible. If the package bid of the global bidder is greater than the sum of the local bidders' bids ($B > b_1 + b_2$), the global bidder receives both items and pays B. The local bidders win the auction and receive one item each whenever the sum of their bids is higher than the package bid of the global bidder ($B < b_1 + b_2$). In this event, both local bidders are required to pay their respective bids. Ties are resolved using a fair randomizing device.

When package bids are not allowed, both items are auctioned simultaneously using two independent first-price auctions (or, equivalently, a pay-as-bid auction where bidders submit demand curves in case items are homogeneous). Naturally, the global bidder participates in both auctions by submitting two separate bids, b_1^g and b_2^g, each for the corresponding item. There are several possible outcomes. First, the global bidder can win both items when her bids are higher than the corresponding bids of the local bidders ($b_1^g > b_1$ and $b_2^g > b_2$) and pay $b_1^g + b_2^g$. Second, the global bidder can end up winning only one item, in which case she receives no value from the acquired item but pays the amount of her winning bid. Finally, the global bidder can lose in both auctions and pay nothing if both of her bids are smaller than that of local bidders ($b_1^g < b_1$ and $b_2^g < b_2$). A local bidder i wins the desired item when her bid b_i is higher than the corresponding bid of the global bidder b_i^g and pays b_i. Ties are resolved independently across auctions using a fair randomizing device.

The majority of proofs in Section 1.3 and the numerical technique discussed in Section 1.4 are based on the discrete bidding regime. In the discrete bidding regime all bids by the global bidder are constrained to a countable set of points $\Delta^G = B^0 < B^1 < \ldots < B^j < \ldots$ and bids by local bidders are restricted to a similar countable set of points $\Delta^L = b^0 < b^1 < \ldots < b^i < \ldots$ where $B^0 = b^0 = 0$.

In most applications, the bidding sets Δ^G and Δ^L are equally spaced bid grids characterized by a certain increment, like a dollar or a penny. However, it is possible that the global bidder, when bidding for a package of items, is restricted to bid in larger increments, say twice the minimum increment on any individual item. It is also assumed that Δ^G and Δ^L are unbounded.

Some proofs are based on a continuous bidding regime where $\Delta^G = [0, +\infty)$ and $\Delta^L = [0, +\infty)$.

We proceed with the equilibrium analysis of the first-price package auction.

1.3 Equilibrium Analysis

This section develops the equilibrium existence results for the first-price auction with and without package bids.

1.3.1 Equilibrium Analysis of the First-Price Package Auction

Since the bidding sets Δ^G and Δ^L are unbounded, sufficiently high bids from these sets will never be a part of any equilibrium of the first-price auction. Without loss of generality, the global bidder selects her bid from a finite set $S^G = \{B^0, B^1, ..., B^{k_G}\}$ and local bidders choose their actions from a finite set $S^l = \{b^0, b^1, ..., b^{k_l}\}$ where k_G and k_l are defined as follows:

$$k_G = \{j \in \mathbb{N} : B^j \leq \bar{u},\ B^{j+1} > \bar{u}\} \qquad k_l = \{i \in \mathbb{N} : b^i \leq \bar{v},\ b^{i+1} > \bar{v}\}$$

A pair of functions, $B(u) : [0, \bar{u}] \to S^G$ and $\beta(v) : [0, \bar{v}] \to S^l$, forms a pure symmetric Bayesian-Nash equilibrium if the following two conditions hold:

$$\begin{gathered} \forall v \in [0, \bar{v}] \quad \exists i \in \mathbb{N} : \pi_L(v, b^i) \geq \pi_L(v, b^k) \quad \forall k \in \mathbb{N} \\ (\text{i.e.} \quad \beta(v) = b^i) \end{gathered} \tag{1.3.1}$$

$$\begin{gathered} \forall u \in [0, \bar{u}] \quad \exists j \in \mathbb{N} : \pi_G(u, B^j) \geq \pi_G(u, B^k) \quad \forall k \in \mathbb{N} \\ (\text{i.e.} \quad B(u) = B^j) \end{gathered} \tag{1.3.2}$$

where $\pi_L(v, b^i)$ denotes the expected payoff of a local bidder with value v and bid b^i and $\pi_G(u, B^j)$ denotes the expected payoff of the global bidder with value u and a package bid B^j.

Conditions (1.3.1) and (1.3.2) are incentive compatibility (IC) constraints for local bidders and the global bidder respectively. Note that since $\pi_L(v, b^0) \geq 0$ for any $v \in [0, \bar{v}]$ and $\pi_G(u, B^0) \geq 0$ for any $u \in [0, \bar{u}]$ individual rationality (IR) constraints follow naturally from the IC constraints.

1.3.1.1 Independent Values ($\gamma = 0$)

When values of local bidders are independent, the expected probability of winning depends only on the bidder's bid since her value does not provide any inference about her opponents' values. The expected profits in this case are given by:

$$\pi_L(v, b^i) = (v - b^i)Pr(\text{Locals win}|b^i) = (v - b^i)Pr^i_L$$

$$(1.3.3)$$

$$\pi_G(u, B^j) = (u - B^j)Pr(\text{Global wins}|B^j) = (u - B^j)Pr^j_G$$

For a fixed profile of the opponents' strategies, an increase in a bidder's bid never reduces her probability of winning. Therefore, the following inequalities hold:

$$Pr^i_L \leq Pr^{i+1}_L \quad \forall i \in \mathbb{N}$$

$$(1.3.4)$$

$$Pr^j_G \leq Pr^{j+1}_G \quad \forall j \in \mathbb{N}$$

Lemma 1.1 establishes the monotonicity property for equilibrium bidding functions.

Lemma 1.1. *If local bidders' values are independent ($\gamma = 0$), equilibrium bidding functions $\beta(v)$ and $B(u)$ are nondecreasing.*

Proof. See Appendix 1.A. □

Note that Lemma 1.1 guarantees that sets $\mathbf{b}^i = \{v \in [0, \bar{v}] : \beta(v) = b^i\} \quad \forall i$ and $\mathbf{B}^j = \{u \in [0, \bar{u}] : B(u) = B^j\} \quad \forall j$ are convex.

Lemma 1.2 characterize a Bayesian-Nash equilibrium in pure monotone strategies.

Lemma 1.2. *If local bidders' values are independent ($\gamma = 0$), a pure-strategy Bayesian-Nash symmetric equilibrium is characterized by a pair of step-functions with the following functional forms:*

$$\beta(v) = \begin{cases} b^i & \textit{if } v \in [s_i, s_{i+1}) \quad 0 \le i \le r \\ b^r & \textit{if } v = s_{r+1} \end{cases} \tag{1.3.5}$$

and

$$B(u) = \begin{cases} B^j & \textit{if } u \in [t_j, t_{j+1}) \quad 0 \le j \le q \\ B^q & \textit{if } u = t_{q+1} \end{cases} \tag{1.3.6}$$

where

1. $0 = s_0 < s_1 \le ... \le s_{r+1} = \bar{v}$

$0 = t_0 < t_1 \le ... \le t_{q+1} = \bar{u}$

2. $0 \le r \le \min[k_L, r^*(q)]$ *where* $r^*(q) = \{i \in \mathbb{N} : b^{i-1} \le B^q,\ b^i > B^q\}$

$0 \le q \le \min[k_G, q^*(r)]$ *where* $q^*(r) = \{j \in \mathbb{N} : B^{j-1} \le 2b^r,\ B^j > 2b^r\}$

Proof. By Lemma 1.1, any equilibrium involves a pair of nondecreasing functions. Giving finite discrete strategy sets, the step function is the only possible functional form. Note that we have fixed the actions of all bidders at "jump" points. Technically, there are a lot of equilibria that assign other actions to "jump" points but otherwise they are equivalent to the one described in (1.3.5) and (1.3.6) since densities $f(v)$ and $g(u)$ are atomless. Indexes r and q are the highest bid levels played in the equilibrium with positive probability. Indexes $r^*(q)$ and $q^*(r)$ are defined such that the bidder with the highest type never bids above the bid that outbids the maximum possible bid from the opposing side. This is a standard conclusion for the first-price auctions. The fact that all probabilities of winning, Pr^i_L and Pr^j_G are strictly positive is reflected in strict inequalities: $s_0 < s_1$ and $t_0 < t_1$. □

Proposition 1.1 (Discrete Bidding). *There exists a symmetric (across local bidders) Bayesian-Nash equilibrium of the first-price package auction in pure nondecreasing strategies when values of the local bidders are independent, i.e.,* $\gamma = 0$.

Proof. Existence of the equilibrium when local bidders draw their values independently is readily established by application of Theorem 1 from Athey (2001) [3] since the case of independent locals ($\gamma = 0$) is the only instance of the model when the *Single Crossing Condition for games of incomplete information (SCC)* is satisfied. However, this theorem cannot be applied directly since it does not guarantee existence of the symmetric equilibrium for symmetric players. In order to tackle this challenge, a modified game where one of the locals is replaced with a player with the same strategy set but with a different objective has to be considered. The modified game satisfies all assumptions from Athey (2001) [3] required to apply existence theorem for games of incomplete information. The equilibrium strategies of the modified game form a symmetric equilibrium of the actual game. See appendix for the complete proof. □

Rough intuition behind the single crossing condition is as follows: whenever all opponents of a player use nondecreasing strategies (in the sense that higher types select higher actions), the player's best response strategy is also nondecreasing.

Clearly, when the correlation between local bidders' values is positive, the SCC is not satisfied. A nondecreasing strategy of the local bidder 1 implies a higher probability of winning for the local bidder 2. As a result, the local bidder 2, when she gets a higher value, might prefer to bid lower since the local bidder 1 is already likely to bid higher. Therefore, the single crossing condition is an exceptionally strong property for the environments such as the one considered here.

Fortunately, as shown in the next section, the existence of a Bayesian-Nash equilibrium in monotone strategies when local bidders' values are perfectly correlated can be established using other methods developed in the literature on existence of the equilibrium in the asymmetric first-price auctions for a single-item.

1.3.1.2 Perfectly Correlated Values ($\gamma = 1$)

Chernomaz and Levin (2010) [15] discussed existence of the equilibrium for the model of the first-price package auction where local bidders always have the same value that corresponds to the case of perfect correlation in our model.

In order to prove the existence of the equilibrium in the case of perfect correlation, we consider the continuous bidding regime and also assume that F and G are differentiable over $(0, \bar{v}]$ and $(0, \bar{u}]$ respectively, and that their derivatives , f and g are locally bounded away from zero on these intervals.

Some additional notation is convenient. A per-unit value of the global bidder when she acquires both items is denoted s, i.e., $s = u/2$. Note that v and s are distributed on the same interval $[0, \bar{v}]$. Then, the distribution of s is described by a CDF $\widehat{G}(s) = G(2s)$ with density $\widehat{g}(s) = 2g(2s)$. The strategy of the global bidder is $\widehat{B}(s)$ - a per-unit bid giving her per-unit value s. The actual package bid is then $B(u) = 2\widehat{B}(u/2) = 2\widehat{B}(s)$.

Lemma 1.3 states that a pair of bidding functions that satisfy first-order conditions do form a Bayesian-Nash equilibrium for the first-price package auction.

Lemma 1.3. *When local bidders' values are perfectly correlated ($\gamma = 1$), a pair of strictly increasing bidding functions $\beta(v)$ and $\widehat{B}(s)$ forms a Bayesian-Nash equilibrium of the first-price package auction if there exists $\bar{b} \in (0, \bar{v})$ such that the inverses $\alpha = \beta^{-1}, A = \widehat{B}^{-1}$ form a solution of the following system of differential equations over $(0, \bar{b}]$:*

$$\frac{d}{db}\alpha(b) = \frac{F(\alpha(b))}{(A(b)-b)f(\alpha(b))} \qquad \frac{d}{db}A(b) = \frac{2\widehat{G}(A(b))}{(\alpha(b)-b)\widehat{g}(A(b))} \tag{1.3.7}$$

$$\alpha(\bar{b}) = \bar{v} \quad \alpha(0) = 0 \qquad A(\bar{b}) = \bar{v} \quad A(0) = 0$$

Proof. See Appendix 1.A. □

Proposition 1.2 (Continuous Bidding)**.** *There exists a symmetric (across local bidders) Bayesian-Nash equilibrium of the first-price package auction in pure strictly increasing bidding strategies when values of the local bidders are perfectly correlated, i.e., $\gamma = 1$.*

Proof. See Appendix 1.A. □

The central idea of the proof is straightforward. First, observe that the system of differential equations (1.3.7) also defines an equilibrium of the certain single-item asymmetric first-price auction with two bidders. Then, using the existence results from the extensive literature on existence in the first-price auctions, the existence of the solution to the system (1.3.7) can be established and, by Lemma 1.3, this solution is an equilibrium of the first-price package auction.

1.3.1.3 Positively Correlated Values ($\gamma \in [0, 1]$)

Unfortunately, the methods developed in the equilibrium existence literature can not be applied in general case. However, the existence on both extremes and visual continuity of the approximated bidding functions strongly suggest that the equilibrium exists for any level of correlation between local bidders' values.

In general cases, we provide some equilibrium characterizations for continuous bidding regime assuming that the symmetric equilibrium exists and that the equilibrium bidding functions $B(u)$ and $\beta(v)$ satisfy the following conditions:

1. $B(u) = \underline{B} = 0 \quad \forall u \in [0, \hat{u}) \quad 0 \leq \hat{u} < \bar{u}$
2. $\beta(v) = \underline{b} = 0 \quad \forall v \in [0, \hat{v}) \quad 0 \leq \hat{v} < \bar{v}$
3. $B(u)$ is strictly increasing on $[\hat{u}, \bar{u}]$
4. $\beta(v)$ is strictly increasing on $[\hat{v}, \bar{v}]$

Conditions 1 - 4 state that equilibrium bidding functions have to be strictly increasing except for maybe having flat segments starting at the lowest value. Lemma 1.4 shows that in any equilibrium that satisfies these properties, the global bidder always has a strictly increasing strategy while the local bidders equilibrium strategy always includes a non-trivial flat segment unless local bidders' values are perfectly correlated ($\gamma = 1$).

Lemma 1.4 (Continuous Bidding)**.** *In any equilibrium satisfying conditions 1-4, the following properties hold:*

1. *$B(u)$ is strictly increasing on $[0, \bar{u}]$ (i.e. $\hat{u} = 0$).*

2. *If* $\gamma = 1$ *then* $\beta(v)$ *is strictly increasing on* $[0, \bar{v}]$ *(i.e.* $\hat{v} = 0$*).*

3. *If* $\gamma < 1$ *then* $\beta(v) = 0$ *on* $[0, \hat{v}]$ *where* $\hat{v} > 0$*.*

Proof. See Appendix 1.A. □

The next section provides the existence result for the first-price auction without package bids.

1.3.2 Equilibrium Analysis of the First-Price Auction without Package Bids

When package bids are not available, the global bidder has to compete for items separately in two simultaneous first-price auctions using two separate bids, b_1^g and b_2^g. However, Lemma 1.5 shows that as long as the local bidders follow the same bidding strategy the global bidder is always better off by submitting exactly the same bid in both auctions. Intuitively, such bidding strategy reduces her exposure risk by reducing probability of winning just one item but not both of them.

Lemma 1.5. *If local bidders follow the same nondecreasing strategy* $\beta_l(v)$*, the global bidder prefers to submit the same bid in both auctions, i.e.,* $b_1^g = b_2^g$*.*

Proof. See Appendix 1.A. □

Similar to the first-price package auction considered in the previous section, we are unaware of any results on the existence of the equilibrium in such an environment. The model is complicated by the possibility of the ex post negative payoff of the global bidder. This is a very distinctive feature since in the model where complementarities are not extreme, like in Chernomaz and Levin (2010) [15], the global bidder can guarantee nonnegative ex-post payoff similar to the standard setup of the first-price auctions.

However, it is straightforward to establish existence of the Bayesian-Nash equilibrium in this model for the discrete bidding regime. Unlike the first-price package auction, a positive correlation between local bidders' values does not lead to a failure of the single-crossing condition in this game since local bidders have to win their items independently from each other. Therefore, monotonicity, characterization and existence results can be easily established for any correlation between local bidders' values ($\forall\gamma \in [0,1]$). The proofs are omitted since they are virtually the same as in section 1.3.1.1 where the single-crossing condition for games of incomplete information holds because local bidders' values are independent ($\gamma = 0$).

Without loss of generality, local bidders choose their actions from a finite set $S^l = \{b^0, b^1, ..., b^{k_l}\}$ and the global bidder selects her bid from a finite set $S^g = \{B^0, B^1, ..., B^{k_g}\}$ where $k_g = \{j \in \mathbb{N} : B^j \leq \bar{v},\ B^{j+1} > \bar{v}\}$. By Lemma 1.5, in a symmetric equilibrium the global bidder submits the same bid in both auctions. Her equilibrium bid function is denoted $\beta_g(u)$ where u is the value she obtains if she wins both items.

Lemma 1.6. *In a symmetric equilibrium, both bidding functions $\beta_l(v)$ and $\beta_g(u)$ are nondecreasing.*

Proof. Similar to the proof of Lemma 1.1. □

Lemma 1.7. *A pure-strategy Bayesian-Nash symmetric equilibrium is characterized* *by a pair of step-functions with the following functional forms:*

$$\beta_l(v) = \begin{cases} b^i & \text{if } v \in [s_i, s_{i+1}) \quad 0 \leq i \leq r \\ b^r & \text{if } v = s_{r+1} \end{cases}$$

and

$$\beta_g(u) = \begin{cases} B^j & \text{if } u \in [t_j, t_{j+1}) \quad 0 \leq j \leq q \\ B^q & \text{if } u = t_{q+1} \end{cases}$$

where

1. $0 = s_0 < s_1 \leq ... \leq s_{r+1} = \bar{v}$

   $0 = t_0 < t_1 \leq ... \leq t_{q+1} = \bar{u}$

2. $0 \leq r \leq \min[k_l, r^*(q)]$ *where* $r^*(q) = \{i \in \mathbb{N} : b^{i-1} \leq B^q,\ b^i > B^q\}$

   $0 \leq q \leq \min[k_g, q^*(r)]$ *where* $q^*(r) = \{j \in \mathbb{N} : B^{j-1} \leq b^r,\ B^j > b^r\}$

Proof. Similar to the proof of Lemma 1.2. □

Proposition 1.3 (Discrete Bidding)**.** *There exists a symmetric (across local bidders) Bayesian-Nash equilibrium of the first-price auction without package bids in pure nondecreasing strategies.*

Proof. Similar to the proof of Proposition 1.1. □

In the next section we describe the numerical approximation technique which can be effectively used to solve both versions of the first-price auction.

1.4 Numerical Approach

There are a lot of numerical methods suggested in the literature for solving first-price auctions for a single-item. A pioneering contribution was made in Marshall et al. (1994) [42], and further, this topic was expanded by Riley and Li (1997) [50] and Bajari (2001) [11][14].

One way to compute equilibrium bidding functions is a simple best-response iteration technique. The method provides a certain degree of robustness but tends to be very slow and the convergence is not guaranteed. Other methods approximate equilibrium bidding functions by assigning them some flexible parametric functional forms such as low-order polynomials or piece-wise linear functions and solving the first-order conditions. They are often found to produce highly accurate approximations for the unknown bidding functions in view of their typical smoothness and can be reasonably fast, especially with a good starting guess.

One of the most effective ways to solve the asymmetric first-price auction for a single-item is the backward shooting algorithm, which does not rely on any functional form assumptions. The only disadvantage of the backward shooting routines in a single-item environment is the need for the explicit search for the starting value (the maximum bid), which often results in a slow convergence.

However, in the package environment, any effective use of the shooting methods is highly unlikely. Consider our model of the first-price package auction. The system of equations, which defines a pair of unknown bidding functions, is no longer

[14] See also Armantier and Richard (1997) [1] and Gayle and Richard (2008) [30].

formed by ordinary differential equations[15], which is the crucial part of the backward shooting algorithms. The key idea behind any shooting routine is the possibility to recover unknown bidding functions from the system of equations in a step-by-step manner relying exclusively on the information received at the previous steps of the routine. In contrast, the system of equations for the first-price package auction modeled in the paper necessarily includes integral terms that represent the two-way nature of the optimal bidding decision on the local side of the market. Thus, any shooting-type routine requires an initial guess for the unknown bidding functions as well as an explicit search for several variables ($\bar{b}, \bar{B}$ and $\hat{v}$). While the explicit search only affects the computational speed, the need for the initial guess of the unknown bidding functions makes the shooting algorithms completely impractical even in the simple package environments such as ones studied in this paper.

We suggest a new numerical technique that makes use of the discrete formulation of the model[16]. According to lemmas 1.2 and 1.7, any symmetric equilibrium bid functions for local bidders and the global bidder are just step functions that are fully characterized by two sets of "jump" points (values at which a bidder prefers to switch from one bidding level to a higher bidding level), $\mathbf{s} = (s_0, s_1, ..., s_r, s_{r+1})$ and $\mathbf{t} = (t_0, t_1, ..., t_q, t_{q+1})$.

The main challenge associated with the system of equilibrium equations and

[15] According to Lemma 1.4, the system can be represented as a system of ODEs in case of perfect correlation ($\gamma = 1$)

[16] The best-response iteration method based on a discrete bidding regime for solving asymmetric first-price auctions for a single-item was suggested in Athey (1997) [2] and used in Athey, Coey and Levin (2011) [4].

inequalities is the possibility that some bid levels are not played in equilibrium. For example, some bidding levels ($b^i > b^r$ for local bidders and $B^j > B^q$ for the global bidder) are not part of the equilibrium just because they are too high. The maximum equilibrium bid levels (b^r and B^q) have to be determined simultaneously with calculation of vectors $\mathbf{s}$ and $\mathbf{t}$. Another complication might arise when a bidder skips a bidding level (or several bidding levels) that is smaller than the maximum bid level ($b^i < b^r$ for local bidders and $B^j < B^q$ for the global bidder). Such jumps are difficult to handle since there is no way to know which levels are skipped in the equilibrium. We are going to ignore such possibility and look for an equilibrium where all bidding levels up to b^r and B^q are played with positive probability, i.e., $s_i < s_{i+1}$ for $0 \leq i \leq r$ and $t_j < t_{j+1}$ for $0 \leq j \leq q$. While this assumption is restrictive, the typical bidding functions in first-price auctions do satisfy it.

The usual equilibrium system for the local bidders consists of r equations, which determine a vector of "jump" points $\mathbf{s}$, and one inequality, which ensures that the maximum-type local bidder does not profit from bidding b^{r+1}.

$$\begin{cases} \pi_L(s_i, b^{i-1}) & = & \pi_L(s_i, b^i) \qquad 1 \leq i \leq r \\ \pi_L(\bar{v}, b^r) & \geq & \pi_L(\bar{v}, b^{r+1}) \end{cases} \tag{1.4.1}$$

One way to solve for the equilibrium of the system (1.4.1) is the "guess and verify" method used in the backward-shooting algorithms where the initial guess for maximum bid is updated until convergence. In the discrete version, it is equivalent to an explicit search for b^r. The novelty of our numerical technique is the way to endogenize the search for the maximum bids b^r and B^q by using a complementarity formulation instead of the standard one. There are numerical methods specifically designed to solve complementarity problems[17]. We have used NEOS Server [21] [32] [25] facility to perform all numerical approximations.

The complementarity formulation of the equilibrium system for the local bidders consists of $2k_l$ inequalities that complement each other pairwise. The bid level b^{k_l} is the maximum bid level the local bidders can use in any equilibrium.

$$\begin{cases} \pi(s_i, b^{i-1}) - \pi(s_i, b^i) \geq 0 \\ \qquad \textit{complements} \qquad\qquad 1 \leq i \leq k_l \\ \qquad \bar{v} \geq s_i \end{cases}$$

or (1.4.2)

$$\begin{cases} \pi(s_i, b^{i-1}) - \pi(s_i, b^i) & \geq & 0 \\ \bar{v} - s_i & \geq & 0 \qquad 1 \leq i \leq k_l \\ [\pi(s_i, b^{i-1}) - \pi(s_i, b^i)]\,[\bar{v} - s_i] & = & 0 \end{cases}$$

Note that the system (1.4.2) does not depend on r, but instead depends on k_l which is given. Thus, the whole system for both types of bidders can be solved directly without guessing maximum bids b^r and B^q.

Examples of the equilibrium bidding functions in the package and non-package first-price auctions when bidders' values are uniformly distributed ($F(v) = v$ on

[17]For example, the PATH solver developed by Steven Dirkse, Michael Ferris and Todd Munson and the FilterMPEC solver by Roger Fletcher and Sven Leyffer

[0, 1] and $G(u) = u/2$ on [0, 2]) are provided in Figure 1.1 and Figure 1.2. Both bidding grids Δ_L and Δ_G are uniform with 200 bidding levels on [0, 1] interval. While approximated bidding functions are step-functions, the examples, as shown, are smoothed for a better exposition.

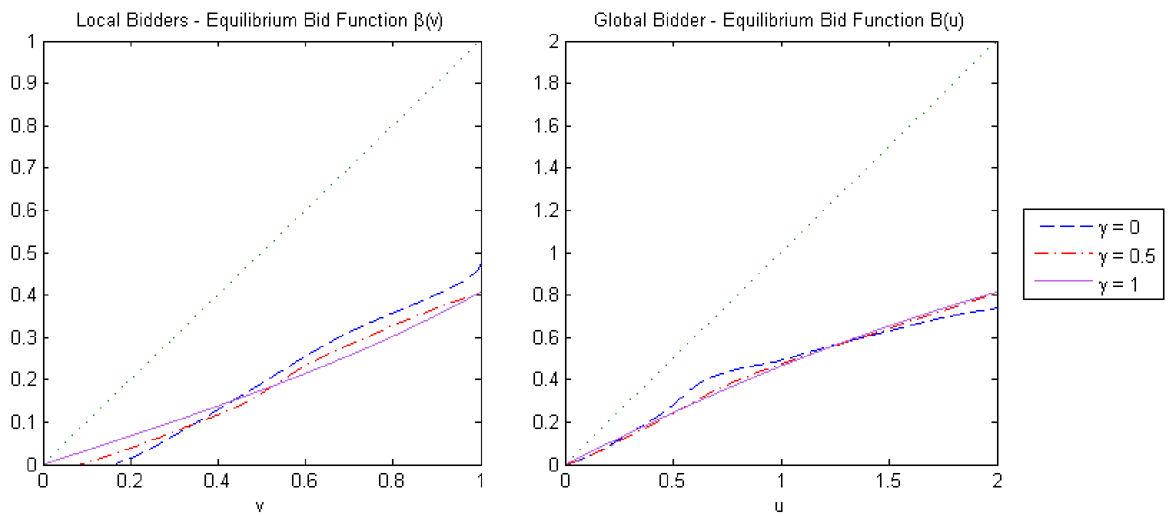

Figure 1.1: Equilibrium Bids: First-Price Auction with Package Bids

As can be seen from Figure 1.1, all bidding functions are in perfect accord with the Lemma 1.4 which describes the shapes and patterns of the equilibrium bid functions in the first-price package auction. When local bidders' values are independent, a local bidder bids zero when her type is sufficiently low expecting a high enough bid from the other local bidder. With an increase in correlation, the size of the zero-bid interval decreases. Intuitively, when correlation is high, a local bidder with low value no longer expects a sufficiently high bid from the other local bidder since with high probability the other local bidder has the same low value. Also, observe that that the maximum total bid from local bidders is around 0.9 while the global bidder maximum bid is around 0.7 when local bidders' values are independent ($\gamma = 0$). However, as correlation between local bidders' values goes up ($\gamma \uparrow$), the distance between maximum bids diminishes quickly.

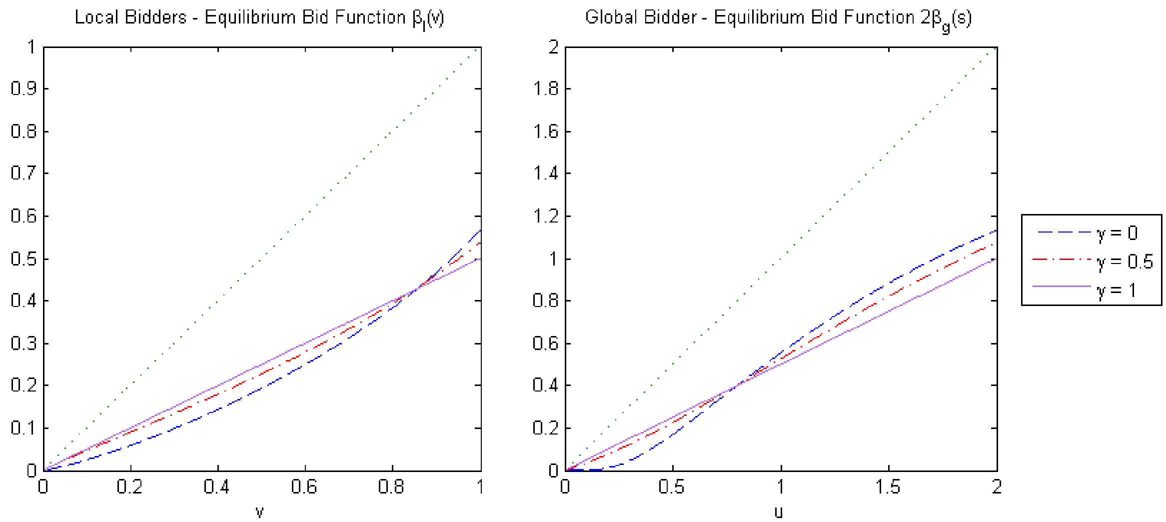

Figure 1.2: Equilibrium Bids: First-Price Auction without Package Bids

Figure 1.2 demonstrates equilibrium bid functions for the first-price auction without package bids. When package bids are not allowed, the global bidder has to compete for items separately and face the exposure risk. When local bidders' values are independent, the global bidder underbids when her value is low and bids more aggressively when her value is high in an attempt to avoid winning just one out of two items. However, with substantial positive correlation between local bidders' values, the impact of the exposure problem is limited. For example, when values of the local bidders are perfectly correlated, the global bidder is never exposed and so she bids accordingly.

The quantitative analysis of the free-rider and exposure problems is presented in the next section.

1.5 Free-Rider Problem versus Exposure Problem

This section presents several illustrative examples that show the mechanics of the trade-off between the free-rider and exposure problems. The numerical technique developed in the previous section is used to approximate equilibrium bidding functions in all considered examples. Various relevant auction characteristics, such as revenue and efficiency, are calculated using simulations.

In addition to the two first-price auctions discussed in the paper, we also consider a first-price auction where both items are sold together as one lot and both local bidders are replaced with one bidder whose value for items is exactly the sum of the local bidders' values, $v_1 + v_2$. Formally, with probability γ, this bidder values both items at $2v$ where v is drawn from $F(v)$ and with a probability $1-\gamma$, her value is $v_1 + v_2$ where v_1, v_2 are drawn from $F(v)$ independently from each other. This auction is a standard first-price auction for a single object. It provides a convenient benchmark for evaluating the impacts of the exposure and free-rider problems since it is completely immune to both of them.

The following distributions are assumed for all examples of this section. The global bidder's value is uniformly distributed on [0,2]. The underlying distribution function for local bidders' values is $F(v) = v^{\alpha}$, $\alpha > 0$ on $[0, 1]$. When $\alpha = 1$ (uniform distribution), the value distributions are symmetric in a sense that for any γ, the value distribution of the global bidder is the mean-preserving spread of the total value distribution of local bidders. For example, when $\gamma = 0$, the total value $v_1 + v_2$ is distributed according to the triangular distribution on $[0, 2]$, and if local bidders' values are perfectly correlated ($\gamma = 1$), the total value $v_1 + v_2$ is distributed uniformly on $[0, 2]$.

The parameter α can be interpreted in the following way. When α is less than 1, the sum of the local bidders' values is expected to be small in comparison with the expected value of the global bidder, implying that the local bidders lose more frequently under full efficiency. When α is greater than 1, the situation is reversed, with the global bidder winning less frequently under truthful bidding. In other words, a high α makes the local bidders' distribution more advantageous in comparison with that of the global bidder.

The first example, based on the model with uniform distributions ($\alpha = 1$), demonstrates the mechanics of the efficiency trade-off between the free-rider and exposure problems (the right panel of Figure 1.3). When γ is high, the global bidder easily avoids any exposure risk by submitting the same bid in both markets since local bidders' bids are likely to be very close to each other. In such environments, the first-price auction without package bids achieves high efficiency similar to the efficiency of the benchmark first-price auction. Meanwhile, the first-price package auction is relatively inefficient because of the local bidders' free-riding incentives. Therefore, package bids can hurt efficiency when the exposure risk faced by bidders with complementarities is relatively small.

However, the bids submitted by local bidders can be extremely unequal when their values are slightly correlated or independent from each other ($\gamma = 0$). If this is the case, the exposure risk of the global bidder is high. In the equilibrium, she adjusts her bidding strategy accordingly, by underbidding when her value is low and overbidding when her value is high (see Figure 1.2). Despite these adjustments, she often wins only one item. Such an outcome is the major inefficiency driver in the first-price auction without package bids.[18] Meanwhile, the package auction is completely immune to such outcomes. Changes in γ do affect the equilibrium bid functions, but the overall efficiency of the first-price package auctions stays relatively constant at high levels. Therefore, package bids can significantly improve efficiency performance of the first-price auction, especially in environments with a high exposure risk.

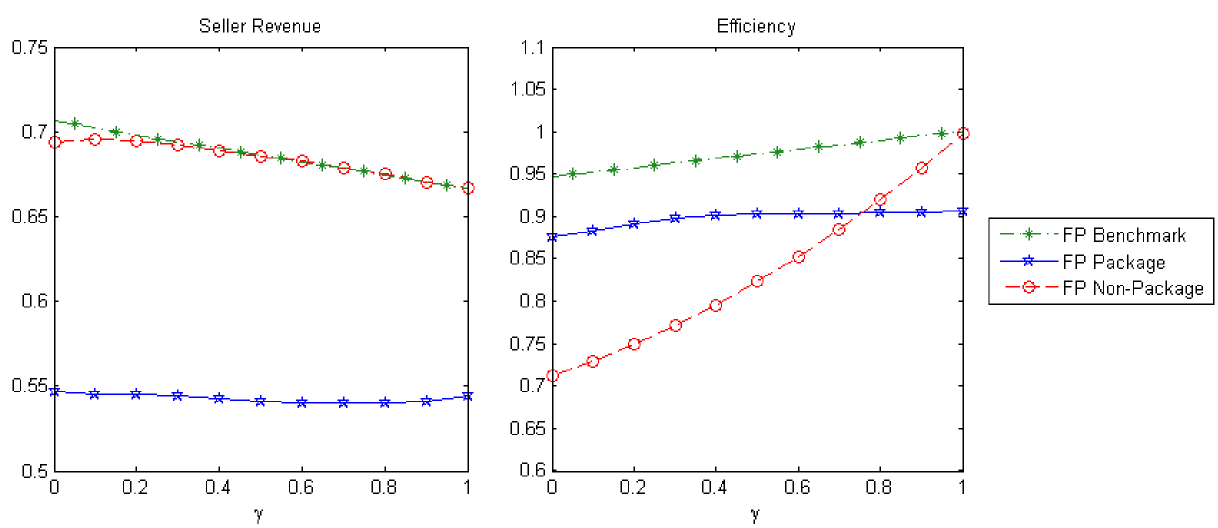

Figure 1.3: Seller Revenue and Efficiency: $F(v) = v$ $(\alpha = 1)$

A differential bid shading, an inevitable property of all first-price auctions, is responsible for inefficiencies arising from bidder asymmetries. In the standard first-price auctions for a single-item the numerous dimensions of asymmetry are limited to a simple distributional asymmetry. Package bids introduce yet another degree of

[18]When local bidders' values are independent ($\gamma = 0$), nearly 26% out of 28% of inefficiencies arise from outcomes where the global bidder wins in one auction but loses in the other one.

asymmetry to the first-price auctions by allowing bidders with different value structures to choose package bids that better fit their preferences. Such bid asymmetries can reduce or increase inefficiencies generated by other forms of asymmetries.

The next example demonstrates that the free-rider problem in the first-price package auction can actually improve efficiency of the benchmark first-price auction. Consider an environment where the underlying value distribution of the local bidders, say $F(v) = \sqrt{v}$ ($\alpha = 0.5$), is comparatively worse than the uniform value distribution of the global bidder. The efficiency performance of all auctions in this environment can be found in the right panel of Figure 1.4. Both the benchmark first-price auction and the non-package first-price auction are inefficient because of the distributional asymmetry. Meanwhile, the bidding asymmetry of the package auction induces the free-rider problem that helps mitigate the distributional differences. As a result, the first-price package auction is highly efficient in spite of the distributional asymmetry. In fact, in this particular example it is fully efficient when local bidders' values are perfectly correlated ($\gamma = 1$). Intuitively, the free-rider problem between local bidders results in a significant bid shading on their part. Using terminology from Maskin and Riley (2000) [43], the global bidder is a *weak bidder* who bids more aggressively while both local bidders together represent a *strong bidder* who bids less aggressively.[19] Therefore, a more advantageous value distribution of the global bidder evens out the differential shading incentives by making the bidding of the global bidder less aggressive while also promoting more

[19]In Maskin and Riley (2000) [43], a strong bidder bids less aggressively because her value distribution first-order stochastically dominates the value distribution of the weak bidder.

competitive bidding from the local bidders by reducing their free-rider incentives.

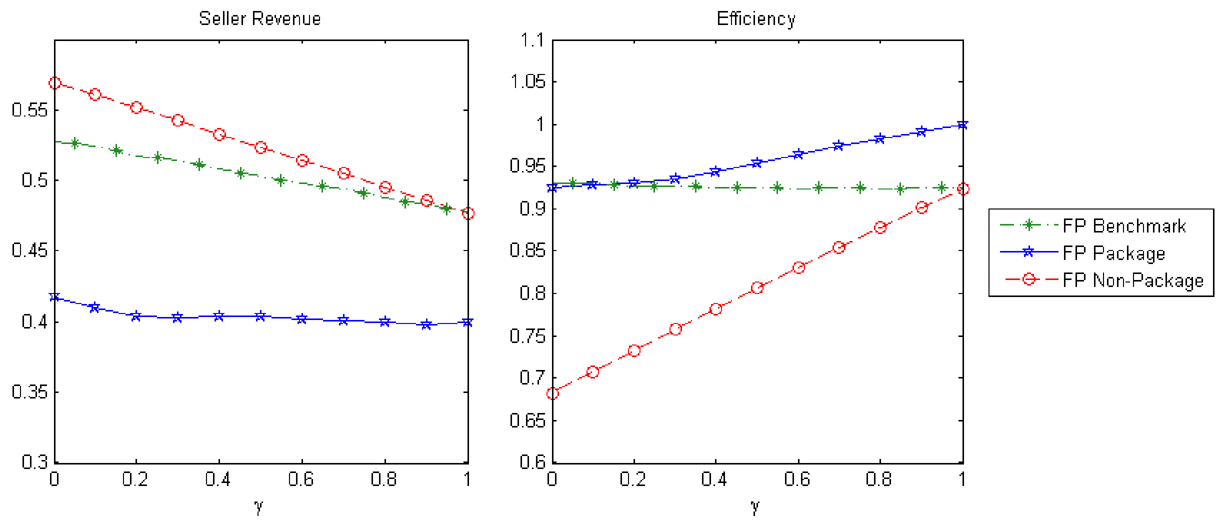

Figure 1.4: Seller Revenue and Efficiency: $F(v) = \sqrt{v}$ ($\alpha = 0.5$)

Note that such asymmetric environments might be empirically relevant. For example, in procurement auctions, big (global) suppliers might have a better cost distribution than small (local) suppliers.

In both examples considered above, the first-price package auction generates significantly lower revenue than the other two formats. Intuitively, the free-rider problem negatively affects revenue since all bidders have incentives to bid lower in comparison with the benchmark first-price auction. Local bidders reduce their bids in an attempt to free-ride on each other, while the global bidder submits a lower bid in response. Meanwhile, the exposure problem can potentially lead to either an increase or a decrease in revenue. On the one hand, the global bidder shades her bid when her value is low since the probability that she wins both items is small. On the other hand, she can substantially overbid when her value is high in order to reduce her exposure risk. Figures 1.3 and 1.4 provide examples when the non-package first-price auction generates lower and higher revenues than the benchmark first-price auction. However, as the next example shows, the revenue performance of the first-price package auction highly depends on the environment.

Consider the following modification of the original model. It consists of two homogeneous items, one global bidder and three local bidders. With homogeneous items, local bidders do not care which particular items they win. However, all local bidders still have value only for one unit and the global bidder still needs both units. In this setup, the first-price auction without package bids is a simple pay-as-bid auction where bidders submit demand curves.

In such environments, the negative revenue impact of the free-rider problem is relatively small since local bidders have to compete not only with the global bidder but also with each other. This competition among local bidders partially mitigates free-rider incentives and substantially improves revenues.

Under the continuous bidding regime, this model does not have a pure Bayesian-Nash equilibrium if local bidders' values are positively correlated[20] in the same way as in the original model of Section 1.2. Therefore, it is assumed that all values are independent ($\gamma = 0$). For the distribution functions used in the previous example, the first-price package auction achieves almost 4% higher revenue and 23% higher efficiency than the first-price auction without package bids.

To sum up, the first-price package auction in environments with sufficient

[20] With probability $\gamma > 0$ a local bidder can perfectly predict the exact bids of the other local bidders and can easily increase her bid by a small amount such that she always gets an item whenever the local bidders win.

competition among free-riding bidders seems to result in revenue and efficiency improvements. Thus, the frequent use of the first-price package design in practical applications can be explained not only by its simple description but also by its performance characteristics.

In the next section we compare the first-price package auction with other package alternatives suggested in the literature.

1.6 Core-Selecting Auctions

Core-selecting auctions have recently been suggested as alternatives to the Vickrey-Clarke-Groves (VCG) mechanism. While the mechanism has the attractive property that truth-telling is a dominant strategy — and truth-telling by all participants in the VCG mechanism implies full efficiency — there are several problems with the VCG in environments with complementarities. This is ironic since exactly these environments are used to motivate the use of package bids. The list of problems includes a possibility of extremely low revenue for the seller (sometimes even zero revenue!), a non-monotonicity of the seller revenue with respect to the bidders' values and high vulnerability to exotic bidders' strategies such as shill-bidding. The main reason for all mentioned disadvantages of the VCG mechanism in environments with complementarities is that sometimes its payment vector falls outside the core in a sense that there exists a coalition of bidders who cumulatively offered to pay more for the same subset of items.

Such practical flaws of the VCG mechanism have trigged both theoretical and applied interests in alternative mechanisms that came to be known as *core-selecting auctions*. The main ingredient of the novel auction design is the *core property*, which guarantees that the payments collected by the auctioneer are always "sufficiently high." In general, payments greater than those of the VCG mechanism are unable to support truthful bidding as the dominant strategy equilibrium and potentially may lead to substantial underbidding. In order to *minimize* bidders' incentives to deviate from truthful bidding, Day and Raghavan (2007) [24] and Day and Milgrom (2008) suggested to minimize seller's revenue subject to the core constraints. Core-selecting auctions that minimize the seller's revenue are known as *minimum-revenue core* auctions.[21] The total payment to the seller in such auctions necessarily coincides with that of the VCG when the VCG payment vector belongs to the core[22] and is strictly greater when it lies outside the core.

As has been noted above, even the minimum-revenue core auctions, in general, cannot induce truth-telling incentives to all bidders in environments with complementarities. Consider our model where the global bidder submits a package bid B for both items and local bidders bid b_1 and b_2 on individual items. In minimum-revenue core auctions, local bidders have to pay the global bidder's bid B whenever

[21]Sometimes minimum-revenue core auctions are referred to as core-selecting auctions.

[22]In environments without complementarities, the VCG payment vector is always in the core. For example, consider a simple single-item private-value environment with several bidders. In this setup, a core-selecting auction is an auction that allocates the item to the highest bidder and requires the winner to pay an amount between her bid and the second-highest bid. Also note that the minimum-revenue core auctions, VCG mechanism and the second-price auction are all equivalent in this environment.

they win. However, the exact split of the total payment B between local bidders necessarily depends on their individual bids b_1 and b_2. As a result, local bidders face the free-rider problem, which is similar to the one they have in the first-price package auction.

Intuitively, the free-rider problem in first-price package auctions is worse. First, bidders have stronger incentives to free-ride on each other since any decrease in the bidder's bid is matched one-to-one by a decrease in the bidder's payment if she wins. Meanwhile, payments in minimum-revenue core auctions are either unaffected or decrease partially (50 cents per a $1 drop) in response to bid reductions. Second, the free-rider problem in the first-price package auction triggers optimal response by other bidders who try to take advantage of their opponents' low bids. In the context of our model, the global bidder bids less competitively because of the free-rider problem between local bidders. In contrast, in minimum-revenue core auctions the global bidder incentives are not distorted.[23] However, the stronger free-rider problem of the first-price package auction does not necessarily imply lower revenue because of the usual difference between first-price and second-price auctions. The global bidder is more likely to win in "second-price-like" minimum-revenue core auctions and pay very little because her payment is a sum of local bidders' bids affected by their free-rider problem. Therefore, revenue generated by minimum-revenue auctions can be extremely low (sometimes even zero revenue) while revenues in the first-price package auctions are always positive.

[23]Truth-telling is a weakly dominant strategy for the global bidder in minimum-revenue core auctions.

Several core-selecting auction rules have been analyzed by Ausubel and Baranov (2011) [5] in an incomplete information environment using the same model with two items and three bidders. Under some distributional assumptions, authors derived closed-form solution for all considered minimum-revenue core auctions including the proxy rule introduced by Ausubel and Milgrom (2002, 2006) [8] [9] and the nearest-Vickrey rule suggested by Day and Cramton (2009) [22]. Using their solutions, we can compare the revenue and efficiency performance of the first-price package auction with those of the leading minimum-revenue core designs. The results are summarized in Figure 1.5.

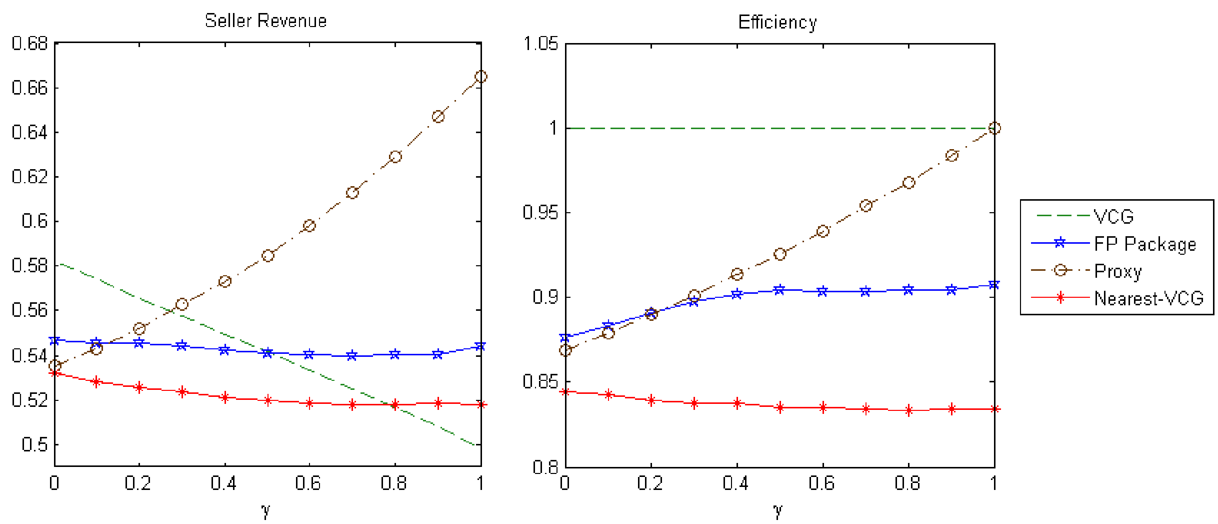

Figure 1.5: Seller Revenue and Efficiency: CSA and First Price Package Auction

The first-price package auction performs reasonably well in comparison with the proxy auction and the nearest-Vickrey auction, in terms of both revenue and efficiency, despite the more serious free-rider problem. For example, the first-price package auction generates higher revenue and efficiency than the nearest-Vickrey auction for any γ. At the same time, the proxy rule generates higher revenue and achieves higher efficiency than the first-price package auction when correlation between local bidders' values is sufficiently high. This is due to the fact that a substantial positive correlation effectively mitigates the free-rider problem in the proxy auction. Corresponding numbers for revenue, efficiency and profits of bidders for all auction rules can be found in Table 1.1.

γ	Statistics	VCG	Proxy*	N-VCG*	FP Pack	FP Non-Pack
$\gamma = 0$	*Revenue*	0.5833	0.5360	0.5327	0.5471	0.6940
	Efficiency	1	0.8679	0.8431	0.8762	0.7126
	Profit Global	0.2916	0.4642	0.4673	0.4269	0.2027
	Profit Local	0.2087	0.1342	0.1335	0.1498	0.1501
$\gamma = 0.5$	*Revenue*	0.5417	0.5852	0.52	0.5412	0.6857
	Efficiency	1	0.9261	0.8356	0.9039	0.8236
	Profit Global	0.3126	0.4148	0.4798	0.4304	0.2604
	Profit Local	0.2295	0.1523	0.1415	0.1647	0.1555
$\gamma = 1$	*Revenue*	0.5	0.6667	0.5185	0.5445	0.6666
	Efficiency	1	1	0.8334	0.9073	1
	Profit Global	0.3335	0.3335	0.4816	0.4270	0.3324
	Profit Local	0.2499	0.1666	0.1481	0.1754	0.1665

* - Based on Ausubel and Baranov (2011) [5]

Table 1.1: Revenue, Efficiency and Profits

1.7 Conclusion

This paper contributes to the quickly expanding literature on the use of combinatorial bids in multi-object auctions. In environments with complementarities, non-package auction designs can easily fail to achieve efficient allocations and generate low revenues for the seller because they do not allow bidders to express their synergies across items. At the same time, package auctions easily handle complementarities of any complexity, but instead introduce free-riding incentives that can decrease or completely mitigate any gains attained from exploiting synergies. Moreover, combinatorial auctions are also more complex technically and computationally, mainly because of the large number of possible packages even for a small number of items. Therefore, a careful cost-benefit analysis of package designs is an important market design question.

The impact of package bids on first-price auctions is of substantial interest since they are the most frequently used package auctions in applications, especially in the public procurement. Unfortunately, an inherent asymmetry among bidders' bids and values, required for a non-trivial comparison between package and non-package designs, complicates the analysis of first-price auctions which are proved to be exceptionally tedious in asymmetric environments.

In the simple model with two types of bidders, we demonstrate the impact of the package bids on the first-price auction. The model, while simple and intuitive, includes a number of realistic features that motivate the use of package auctions, such as the presence of substantial complementarities in bidders' preferences and a positive correlation of bidders' values.

We perform a Bayesian-Nash equilibrium analysis of the first-price auction with and without package bids. For the package auction, we prove the existence of the equilibrium in the case when all bidders' values are independent and also when some bidders' values are perfectly correlated. For the non-package auction, we provide a general equilibrium existence result for our model. A discrete bidding regime, i.e., all bids are restricted to some discrete bidding grids, is used in the majority of the proofs.

We also develop a novel numerical technique, based on the discrete bidding regime, which can be effectively used to approximate equilibrium bidding strategies in first-price auctions. The key element of this procedure is the complementarity formulation for the system of equilibrium inequalities. Our experience suggests that this technique can also be successfully applied in other areas. For example, it is a well-behaved and exceptionally fast alternative for a lot of different numerical methods developed for approximating unknown bidding strategies in asymmetric single-item first-price auctions.

Armed with the numerical method, we take a close look at the forces behind the exposure and free-rider problems in the first-price auctions. Using several examples, we demonstrate the exact mechanics of the efficiency trade-off between package and non-package designs. Even in the environments with extreme forms of complementarities, package bids can be very harmful in terms of revenue and efficiency when the exposure risk faced by the bidders with complementarities is rather small. However, when bidders face a high probability of exposure, package bids can dramatically improve efficiency.

Moreover, a flexibility of package bids introduces bid asymmetries in the first-price auction. Such bid asymmetries can also increase efficiency of the package auction by reducing inefficiency of the non-package design related to the distributional asymmetries. For example, the first-price package auction is highly efficient when bidders with large demands have a more advantageous per-unit distribution than the bidders with small demands.

Finally, we show that in environments that are more competitive than the one considered in the paper, the first-price package auction can be superior to the first-price non-package auction in both revenue and efficiency. We also compare the first-price package auction with the leading package alternatives such as the Vickrey-Clarke-Groves mechanism and core-selecting auctions. In the environment considered, the first-price package auction demonstrates very strong performance characteristics.

These findings suggest that in multi-object environments with complementarities, package designs, and the first-price package auction in particular, indeed can deliver a better performance in comparison with their non-package alternatives.

1.A Appendix A - Proofs

1.A.1 Proof of Lemma 1.1

We prove that $\beta(v)$ is nondecreasing. The proof for $B(u)$ is similar. First, observe that the probabilities of winning with the lowest possible bids (b^0 or B^0) are strictly positive for all bidders in any equilibrium, i.e., $Pr_L^0 > 0$ and $Pr_G^0 > 0$. Therefore, in any equilibrium all probabilities of winning are strictly positive according to (1.3.4).

Now suppose that $\beta(v) = b^i$. Then, using IC constraints (1.3.1), for all $k \in \mathbb{N}$ we get:

$$\pi_L(v, b^i) \geq \pi_L(v, b^k)$$

$$(v - b^i)Pr_L^i \geq (v - b^k)Pr_L^k$$

$$v(Pr_L^i - Pr_L^k) \geq b^i Pr_L^i - b^k Pr_L^k$$

Note that above inequalities still hold if v is replaced with $v' > v$ as long as

$Pr_L^i \geq Pr_L^k$. Thus, $\forall k \in \mathbb{N} : k \leq i$

$$v'(Pr_L^i - Pr_L^k) \geq b^i Pr_L^i - b^k Pr_L^k$$

$$(v' - b^i)Pr_L^i \geq (v' - b^k)Pr_L^k$$

$$\pi_L(v', b^i) \geq \pi_L(v', b^k)$$

It is easy to argue that a stronger version of the last inequality holds, i.e.:

$$\pi_L(v', b^i) > \pi_L(v', b^k) \quad \forall k \in \mathbb{N} : k < i$$

Suppose that expected profits from playing b^i and $b^k : b^k < b^i$ at v' are the same, but then $\pi_L(v, b^i) < \pi_L(v, b^k)$ since $Pr_L^i > Pr_L^k > 0$ which contradicts our initial assumption that $\beta(v) = b^i$. Consequently, $\beta(v') \geq b^i = \beta(v) \quad \forall v' \geq v$.

□

1.A.2 Proof of Proposition 1.1

Consider a modified model with the following payoffs where the global bidder and one of the local bidders have exactly the same payoff functions while the other local bidder has a different payoff function.

$$\pi_G(B,u) = (u-B)Pr(\beta^1(v_1)+\beta^2(v_2) < B) \qquad B \in S_G$$

$$\pi_L^1(b_1,v_1) = (v_1-b_1)Pr(b_1+\beta^2(v_2) > B(u)) \qquad b_1 \in S_L \tag{1.A.1}$$

$$\pi_L^2(b_2,v_2) = -(\beta^1(v_2)-b_2)^2 \qquad b_2 \in S_L$$

The idea behind the last payoff in (1.A.1) is that the local bidder two just tries to match the strategy of the local bidder 1 at her value. Thus, in any equilibrium of the modified game: $\beta^1(v) = \beta^2(v) \quad \forall v \in [0, \bar{v}]$ by construction. By Theorem 1 from Athey (2001), the modified game has an equilibrium since all conditions required by this theorem, including SCC, are satisfied. Any equilibrium of the modified game is an equilibrium of the original game since in the original model the local bidders are symmetric. Therefore, there exists a symmetric equilibrium of the original game.

□

1.A.3 Proof of Lemma 1.3

A local bidder solves the following optimization problem:

$$\begin{aligned}\pi_L(v,b) &= (v-b)Pr(b+\beta(v) \geq 2\widehat{B}(s)) = \\ &= (v-b)Pr\left(s \leq A\left(\tfrac{b+\beta(v)}{2}\right)\right) = \\ &= (v-b)\widehat{G}\left(A\left(\tfrac{b+\beta(v)}{2}\right)\right)\end{aligned}$$

F.O.C.:

$$dA(b) = \tfrac{2\widehat{G}(A(b))}{(\alpha(b)-b)\widehat{g}(A(b))} \qquad A(\bar{b}) = \bar{v} \qquad A(\underline{b}) = 0 \tag{1.A.2}$$

The global bidder faces the following optimization problem:

$$\begin{aligned}\pi_G(s,b) &= (2s-2b)Pr(2b \geq 2\beta(v)) = \\ &= 2(s-b)Pr(v \leq \alpha(b)) = \\ &= 2(s-b)F(\alpha(b))\end{aligned}$$

F.O.C:

$$d\alpha(b) = \tfrac{F(\alpha(b))}{(A(b)-b)f(\alpha(b))} \qquad \alpha(\bar{b}) = \bar{v} \qquad \alpha(0) = 0 \tag{1.A.3}$$

Rewrite (1.A.2) and (1.A.3) as:

$$\tfrac{d}{db}\ln F(\alpha(b)) = \tfrac{1}{A(b)-b} \qquad \tfrac{d}{db}\ln \widehat{G}\left(A\left(\tfrac{b+\beta(v)}{2}\right)\right) = \tfrac{1}{\alpha(b)-b}$$

$$\text{for all} \quad b \in (0, \bar{b}] \tag{1.A.4}$$

First, a bid larger than $\bar{b}$ is never a best response for both types of bidders. Second, a bid of 0 is a best response for any bidder with value 0. Third, if bidders value is above 0, bidding above bidder's value is strictly dominated by bidding , say, $b = (v+0)/2$.

Suppose a local bidder has a value of $v > 0$ and bids $b < v$. Then the logarithm of her positive expected profit ant its derivative are:

$$\ln \pi_L(v,b) = \ln(v-b) + \ln \widehat{G}\left(A\left(\tfrac{b+\beta(v)}{2}\right)\right)$$

$$\ln \pi_L(v,b)' = \tfrac{-1}{(v-b)} + \tfrac{1}{\alpha(b)-b} \tag{1.A.5}$$

Note that the derivative is strictly negative when $v < \alpha(b)$ (or $\beta(v) < b$) and it is strictly positive when $\alpha(b) < v$ (or $b < \beta(v)$). Therefore, $b = \beta(v)$ is the global maximum.

Same technique can be used to show that bid $b = \widehat{B}(s)$ is the global maximum for the global bidder with per unit value s.

□

1.A.4 Proof of Proposition 1.2

Now define $H(x) = \sqrt{\widehat{G}(x)}$ and $h(x) = \frac{\widehat{g}(x)}{2\sqrt{\widehat{G}(x)}}$ and note that:

$$\frac{H(x)}{h(x)} = \frac{2\widehat{G}(x)}{\widehat{g}(x)} \tag{1.A.6}$$

Plugging (1.A.6) into (1.A.2) and (1.A.3) we get the following system of differential equations:

$$\begin{aligned} dA(b) &= \tfrac{H(A(b))}{(\alpha(b)-b)h(A(b))} \qquad A(\bar{b}) = \bar{v} \qquad A(\underline{b}) = 0 \\ d\alpha(b) &= \tfrac{F(\alpha(b))}{(A(b)-b)f(\alpha(b))} \qquad \alpha(\bar{b}) = \bar{v} \qquad \alpha(0) = 0 \end{aligned} \tag{1.A.7}$$

Observe that the system of ODEs formed by (1.A.7) is a standard system for a single item first-price auction with two bidders with their valuations distributed according to cumulative distribution functions $F(.)$ and $H(.)$. Consequently, the existence results from asymmetric first-price literature can be applied.

By Theorem 3 from Lebrun (1997), there exists equilibrium of the asymmetric first-price auction. This is equivalent to the existence of the solution of the system (1.A.7) by Theorem 2 from the same paper.

Since G is atomless, so does H. All assumptions of Theorem 3 in Lebrun (1997) are satisfied, and consequently, the system (1.A.7) has a solution. By Lemma 1.3, this solution is an equilibrium of the first-price package auction when values of local bidders are perfectly correlated.

□

1.A.5 Proof of Lemma 1.4

1. *$B(u)$ is strictly increasing on $[0, \bar{u}]$ (i.e. $\hat{u} = 0$)*

   By characterization assumptions, $B(u)$ is strictly increasing on $[\hat{u}, \bar{u}]$ and constant on $[0, \hat{u})$. The only way $B(u)$ is strictly increasing on the whole interval is when $\hat{u} = 0$. Assume that $\hat{u} > 0$. When $\hat{v} > 0$, a tie at the minimum bid ($\underline{B} = 2\underline{b} = 0$) occur with positive probability which can not be part of equilibrium by standard arguments. When $\hat{v} = 0$ the global bidder with value $0 < u < \hat{u}$ has a profitable deviation since bidding anything above minimum bid and below his value generates a strictly positive expected payoff while bidding minimum bid delivers zero.

2. *If $\gamma = 1$, $\beta(v)$ is strictly increasing on $[0, \bar{v}]$ (i.e. $\hat{v} = 0$)*

   Similar to the previous case, I need to show that $\hat{v} = 0$. Assume that $\hat{v} > 0$. A local bidder with value $0 < v < \hat{v}$ has zero probability of winning since the total bid from the local side is $2\underline{b} = 0$. Bidding anything above minimum bid and below her value gives a strictly positive expected payoff.

3. *If $\gamma < 1$, $\beta(v) = 0$ on $[0, \hat{v}]$ where $\hat{v} > 0$*

   Denote $\Phi_i(b_i, v_i)$ and $\phi_i(b_i, v_i)$ the probability of winning and marginal probability of winning for a local bidder who submits a bid assuming all other bidders follow their equilibrium strategies, i.e.:

$$\Phi_i(b_i, v_i) = \gamma \int\limits_{b_i+\beta(v_i)>B(u)} g(u)du + (1-\gamma) \iint\limits_{b_i+\beta(v_j)>B(u)} f(v_j)g(u)du$$

$$\phi_i(b_i, v_i) = \frac{\partial \Phi_i(b_i, v_i)}{\partial b_i}$$

If $\gamma < 1$, the probability of winning for the local bidder with the lowest bid $\underline{b}$ is greater than zero for all values, i.e. $\Phi(\underline{b}, v) > 0 \quad \forall v \in [0, \bar{v}]$.

Consider the Taylor expansion of $\Phi(\underline{b}, v)$ around $\underline{b}$ for all v:

$$\Phi(b, v) = \Phi(\underline{b}, v) + \phi(\underline{b}, v)(b - \underline{b}) + o((b - \underline{b})^2)$$

Then the problem of the local bidder is as follows:

$$\pi_L(v, b) = (v - b)Pr(b + \beta(v_j) > B(u)) = (v - b)\Phi(b, v)$$

$$\pi_L(v, b) = (v - b) * (\Phi(\underline{b}, v) + \phi(\underline{b}, v)(b - \underline{b}) + o((b - \underline{b})^2))$$

The unrestricted optimal bid b^* is given by:

$$b^* \approx \underline{b} + \frac{(v - \underline{b})\phi(\underline{b}, v) - \Phi(\underline{b}, v)}{2\phi(\underline{b}, v)} \tag{1.A.8}$$

Since $\phi(\underline{b}, v)$ is positive and bounded, there exist $v > 0$ such that the second term in (1.A.8) is negative. Thus, the unconstrained optimal bid b^* is less then $\underline{b} = 0$ since the term $(v - \underline{b})\phi(\underline{b}, v)$ goes to zero when $v \to 0$ while the term $\Phi(\underline{b}, v)$ converges to a positive number.

□

1.A.6 Proof of Lemma 1.5

Denote $F_1 = F(\beta_l^{-1}(b_1^g))$ and $F_2 = F(\beta_l^{-1}(b_2^g))$. Without loss of generality, assume that $b_1^g \geq b_2^g$ such that $F_1 \geq F_2$. Further denote p_1 and p_2 expected payments of the global bidder in case she wins either item. The proof is general and does not have to for the first-price auction only. For the first-price auction $p_1 = b_1^g$ and $p_2 = b_2^g$.

Expected profit of the global bidder when she submits bids b_1^g and b_2^g is given by:

$$\begin{aligned}\Pi(u, b_1^g, b_2^g) &= \gamma\left[(u - p_1 - p_2)F_2 - p_1(F_1 - F_2)\right] + \\ &\quad +(1-\gamma)\left[(u - p_1 - p_2)F_1F_2 - p_1F_1(1 - F_2) - p_2F_2(1 - F_1)\right] \\ &= \gamma\left[uF_2 - p_1F_1 - p_2F_2\right] + (1-\gamma)\left[uF_1F_2 - p_1F_1 - p_2F_2\right]\end{aligned}$$

Then, the following inequality holds:

$$\Pi(u, b_1^g, b_1^g) + \Pi(u, b_2^g, b_2^g) - 2\Pi(u, b_1^g, b_2^g) =$$

$$= \gamma\left[u(F_1 - F_2)\right] + (1-\gamma)\left[u(F_1 - F_2)^2\right] \geq 0$$

Or, equivalently:

$$\Pi(u, b_1^g, b_1^g) \geq \Pi(u, b_1^g, b_2^g) \quad \text{or} \quad \Pi(u, b_2^g, b_2^g) \geq \Pi(u, b_1^g, b_2^g)$$

Therefore, the expected profit from submitting equal bids is higher then submitting different bids.

□

Chapter 2

Core-Selecting Auctions with Incomplete Information

Lawrence M. Ausubel and Oleg V. Baranov

2.1 Introduction

Core-selecting auctions have recently been proposed as alternatives to the Vickrey-Clarke-Groves (VCG) mechanism. In the VCG mechanism, the items are allocated so as to maximize revenues subject to the feasibility of the selected bids and each bidder is charged the opportunity cost of receiving the allocated items. While the mechanism has the attractive property that truth-telling is a dominant strategy – and truth-telling by all participants in the VCG mechanism implies efficient outcomes – there are several reasons to be wary of VCG in environments with complementarities. First, the VCG mechanism may generate *low revenues* (and, in environments with extreme complementarities, the revenues may equal zero). Second, VCG outcomes may be *non-monotonic* in the sense that increasing the number of bidders or increasing their valuations may reduce the seller's revenues. Third, the VCG mechanism may be especially vulnerable to unusual forms of *collusive behavior*, including collusion by losing bidders and shill bidding.

The simplest environment in which these issues can arise has just two items, "East" and "West" (which may be thought of as spectrum licenses for the Eastern half and Western half of a country), and three bidders. The "global" bidder views East and West as perfect complements, valuing the package {East, West} at 1, but obtaining no value from either item individually. Meanwhile, local bidder 1 values East at 1, but obtains no value from West; and local bidder 2 values West at 1, but obtains no value from East. Observe that the VCG mechanism[1] allocates East to local bidder 1 and West to local bidder 2, maximizing social surplus at 2. However, the mechanism charges a price of zero to each bidder.[2] The VCG outcome is non-monotonic in that, if each of the local bidders' values declined from 1 to 1/2, the seller's revenues would increase from 0 to 1. The explanation for this non-monotonicity, as well as for the opportunities present for loser collusion and shill bidding, is that the VCG outcome may lie outside the core;[3] with the data of this paragraph, a coalition of the seller and the global bidder can block the allocation at zero prices to the local bidders.[4]

Observe that the potential deficiencies of the VCG mechanism are likely to be empirically relevant. In the first place, much of the motivation for allowing pack-

[1]The VCG mechanism was developed in the work of Vickrey (1961) [52], Clarke (1971) [16] and Groves (1973) [33]. Throughout this paper, we will use the terms "VCG mechanism" and "Vickrey auction" interchangeably.

[2]Observe that the total surplus when local bidder 1 is absent equals 1, and so the incremental surplus created by local bidder 1 equals 1. Similarly, local bidder 2's incremental surplus also equals 1. In the VCG mechanism, each bidder is permitted to retain the entire incremental surplus that she creates, implying that the price paid by each local bidder is zero.

[3]The core is the subset of allocations in payoff space that are feasible and unblocked by any coalition.

[4]See Ausubel and Milgrom (2002) [8].

age bidding in auctions arises from environments where there appear to be strong complementarities among items. Furthermore, in the area of telecommunications spectrum auctions, empirical work suggests that there exist substantial synergies among licenses covering different geographic areas.[5] Similarly, there is a growing interest in auctions with package bidding for financial assets, and this again occurs in environments where there are apparent complementarities among assets.

As a result of this critique, researchers and auction practitioners recently began to explore a class of alternative mechanisms that have become known as *core-selecting auctions*. As in the VCG mechanism, buyers submit bids associated with various subsets of the set of all items, and the auctioneer determines the combination of bids which maximizes total revenues subject to feasibility. However, as seen two paragraphs above, applying the VCG payment rule in a complements environment may yield a profit allocation that lies outside the core. Instead, a core-selecting auction uses a different pricing rule – a rule always requiring the same or higher payments – which assures that the outcome is always in the core relative to the reported values.

Despite the very recent development of core-selecting auctions, they have already been selected for some important applications. At this writing, five major spectrum auctions have been conducted using a "package clock" auction format: a two-stage auction procedure in which a simultaneous ascending clock phase is followed by a sealed-bid package auction. For the second stage, these auctions have

[5] See, for example, Ausubel, Cramton, McAfee and McMillan (1997) [6] and Fox and Bajari (2009) [29].

utilized a core-selecting auction with the nearest-Vickrey pricing rule.[6],[7] Package clock auctions have also been suggested for the US Federal Aviation Administration (FAA) recently planned slot auctions for landing rights at the three New York City airports. While these auctions were stopped by an airline-industry lawsuit, the published regulations included the use of a core-selecting auction with the nearest-Vickrey pricing rule.[8]

However, to date, most studies of package bidding have been limited to complete-information analyses. This is not a particularly satisfying state of affairs, as much of the motivation for using VCG or other package-bidding mechanisms is that bidders possess incomplete and asymmetric information. At the same time, it is easy to understand why the shortcut of assuming complete information has typically been taken: analyses of auctions under incomplete information can be extremely intricate, except when truth-telling is an equilibrium. Moreover, the typical sort of environment motivating package bidding inherently includes asymmetries, as some bidders desire smaller sets of items and other bidders desire larger sets of items. Researchers have found that asymmetric auctions are particularly difficult to analyze.

A few contemporaneous papers have introduced explicit incomplete-information analyses of package bidding, but they are limited to considering in-

[6]The package clock (or "combinatorial clock") auction has been used for spectrum auctions in the UK (two auctions: February and May 2008), the Netherlands (April 2010), Denmark (May 2010) and Austria (September 2010).

[7]See Cramton (2009) [18].

[8]See Federal Aviation Administration (2008) [28].

dependent valuations. Independence is an extremely confining assumption in an auction environment. In many of the most important applications of package bidding, such as spectrum auctions, we would expect there to be significant correlations among bidders' signals - and correlation among bidders' signals has been one of the important ingredients in the theory of auctions of single items. Moreover, a central message of auction theory and mechanism design is that, when correlations are present, particular choices of auction format may enhance the ability of the seller to extract revenues from bidders.[9]

The current paper seeks to advance the analysis of package bidding. We consider a very simple and stylized class of models in which one bidder values the items as perfect complements. We compare and contrast a variety of package bidding formats, including the core-selecting auctions in the literature, as well as the VCG mechanism.

Our model is an incomplete-information version of the auction environment with two items and three bidders that is described in the second paragraph of this Introduction. The global bidder obtains value u from winning both the Eastern and Western licenses, but gets zero value from having only East or West. Local bidder 1 values East at v_1, but obtains no value from West; while local bidder 2 values West at v_2 , but obtains no value from East. The game is a standard Bayesian game in which each player knows the realization of her own value, but only the distribution from which her opponents' values were drawn. The players simultaneously and independently submit bids, where b_1 denotes the bid submitted by local bidder 1 for

[9] See, for example, Milgrom and Weber (1982) [46] and Crémer and McLean (1985) [20].

East, b_2 denotes the bid submitted by local bidder 2 for West, and B denotes the package bid submitted by the global bidder for $\{\text{East, West}\}$. The solution concept is Bayesian-Nash equilibrium.[10]

One of the novel aspects of our analysis is the family of distributions that we treat. The local bidders' values are perfectly correlated with probability γ and independently distributed with probability $1-\gamma$. (Moreover, at the time that the local bidder selects her bid, she is unaware of whether the values are perfectly correlated or independent.) Thus, we consider a parameterized family of distributions that permits the correlation between local bidders' signals to be varied continuously from zero to one. Surprisingly, despite the private information and correlated signals, we are able to obtain explicit closed-form solutions for the core-selecting auction formats considered - for all $\gamma \in [0,1]$. And the possibility of positive correlation has a quite substantial impact on our comparison of the various package-bidding formats.

To see the various package-bidding mechanisms that we compare and contrast, suppose that the bids submitted by the respective bidders are $b_1 = 6, b_2 = 8$ and $B = 10$. In any of the mechanisms, the auctioneer first solves the winner determination problem of finding the allocation which maximizes revenues subject to the feasibility constraint. This bid data clearly results in local bidder 1 winning East and local

[10]Further, the joint distribution of values will be symmetric with respect to the two local bidders, and we will limit attention to Bayesian-Nash equilibria that are symmetric with respect to the two local bidders. In addition, in all of the core-selecting auctions, the global bidder will have a weakly-dominant strategy, and we will then limit attention to Bayesian-Nash equilibria in which the global bidder plays her weakly-dominant strategy.

bidder 2 winning West, as $6 + 8 = 14 > 10$. The payments, p_1 and p_2, of local bidders 1 and 2, respectively, remain to be defined. The various mechanisms to be discussed in this paper will differ in their payment rules. Our analysis will consider the following package-bidding mechanisms:

VICKREY-CLARKE-GROVES (VCG): Payments are determined such that each winner receives a payoff equal to the incremental surplus that she brings to the system. The incremental surplus of local bidder 1 equals 4, as surplus (evaluated using the bidders' bids) equals 14 if local bidder 1 is present, and 10 (the global bidder's value) if local bidder 1 is absent. Thus, $b_1 - p_1 = 4 \Rightarrow p_1 = 2$. Similarly, the incremental surplus of local bidder 2 equals 4, so $b_2 - p_2 = 4 \Rightarrow p_2 = 4$. Thus, the VCG payments are $(p_1, p_2) = (2, 4)$. However, this outcome is not in the core, as the seller and the global bidder form a *blocking coalition*: together, they can realize surplus of 10 (the global bidder's value), while in the VCG outcome, the seller receives payoff of $6 = p_1 + p_2$ and the global bidder receives payoff of 0.

NEAREST-VICKREY: Payments are determined such that the profit allocation is the bidder-optimal core allocation that minimizes the Euclidean distance from the VCG outcome. In order to avoid the presence of any blocking coalitions, the payments, p_1 and p_2, must sum to at least 10; and in a bidder-optimal core allocation, the payments must sum to exactly 10. The payments that minimize the distance from the VCG payments of $(2, 4)$ are $(p_1, p_2) = (4, 6)$.

PROXY AUCTION: Payments are determined that reflect the outcome of "proxy agents" competing in a simultaneous ascending auction with package bidding and arbitrarily small bid increments, ϵ. The bids b_1, b_2 and B are reinterpreted as limit prices that the bidders have given their respective proxy agents. Each proxy agent must bid in the "virtual auction" whenever it is not a provisionally-winning bidder. In the initial round, all three proxy agents submit bids of ϵ, making the two local bidders provisional winners. In round two, the proxy agent for the global bidder raises its bid to 3ϵ, making the global bidder the provisional winner; in round three, the proxy agents for each of the local bidders raise their bids to 2ϵ, making the two local bidders provisional winners; and the process repeats until the proxy agent for the global bidder drops out of the auction at a price of essentially $B = 10$. Thus, the local bidders win the virtual auction at prices of essentially $(p_1, p_2) = (5, 5)$, an alternative bidder optimal core outcome.

PROPORTIONAL PRICING: Payments are determined such that the bids are scaled down, proportionally, until the bidder-optimal frontier of the core is reached. In the above example, the bids of the local bidders sum to 14, and so they can each be scaled down by a factor of 5/7 in order to sum to 10. Thus, the payments are $(p_1, p_2) = (30/7, 40/7)$.

NEAREST-BID: Payments are determined such that the profit allocation is the bidder-optimal core allocation that minimizes the Euclidean distance from the vector of winning bids. In the above example, the bidder-optimal core payments that minimize the distance from the winning bids of $(6, 8)$ are $(p_1, p_2) = (4, 6)$, coinciding with the nearest-Vickrey outcome in this example.

FIRST-PRICE PACKAGE: Payments simply correspond to the amounts of the winning bids. That is, the auctioneer first solves the winner determination problem of finding the allocation which maximizes revenues subject to the feasibility constraint; and the winning bidders' required payments are simply the amounts of their winning bids. If the same bids were submitted as in the above example, then the payments would be $(p_1, p_2) = (b_1, b_2) = (6, 8)$. Obviously, since this is a "first price" rather than a "second price" auction format, it should be expected that bids would be substantially different from those in the other core-selecting auctions.

Each of these package-bidding pricing rules, as applied to the bid data $b_1 = 6, b_2 = 8$ and $B = 10$, is illustrated in Figure 2.1. Observe that the set of prices associated with core allocations is the shaded triangle of this figure, while the set of bidder-optimal core prices is the hypotenuse of this triangle.

For all but the last of the core-selecting auctions listed above, if the marginal distribution of each bidder's value is the uniform distribution, then we are able to derive explicit closed-form solutions for equilibria, for all $\gamma \in [0, 1]$. However, for the first-price package auction, the methodology of this paper does not yield a solution. To compare the performance of the first-price package auction with the other core-selecting auctions, we report the revenues and efficiency as computed using a numerical technique for approximating equilibria that is introduced in Baranov

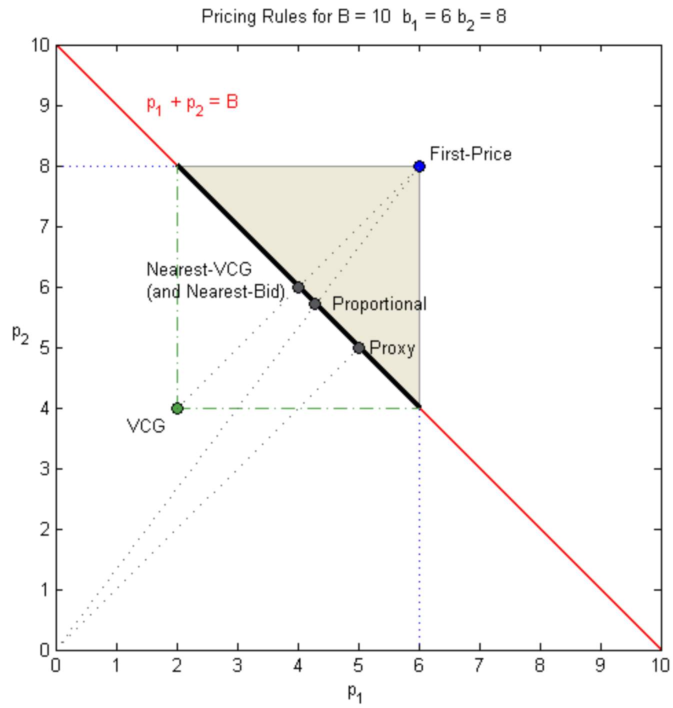

Figure 2.1: Package-Bidding Pricing Rules (as applied to example bid data)

(2010) [12].

The VCG mechanism was introduced in the classic theory of auctions and public choice. William Vickrey (1961) [52] treated auctions with multiple units of a homogeneous product, while Edward Clarke (1971) [16] and Theodore Groves (1973) [33] treated public choice problems. The Clarke-Groves treatment subsumed the environment that Vickrey studied as well as auctions of multiple heterogeneous objects. We use the terminology "VCG mechanism" and "Vickrey auction" interchangeably.

The study of second-price-like auction mechanisms generating core allocations originated with Ausubel and Milgrom (2002, 2006) [8] [9]. They described an "ascending proxy auction" mechanism which was proven to yield core allocations with respect to bidders' reports and whose complete-information equilibrium allocations (subject to a refinement) coincided with the set of bidder-optimal core allocations. A closely-related auction procedure was developed independently by Parkes and Ungar (2000) [48] and Parkes (2001) [47]. Stated somewhat imprecisely, the outcome of the ascending proxy auction coincides with that of the VCG mechanism if and only if all bidders have substitutes preferences.[11] Ausubel, Cramton and Milgrom (2006) [7] then proposed a two-stage auction procedure comprising a (multi-round) ascending-clock auction followed by a single proxy auction round. This became the basis for the "package clock" (or "combinatorial clock") auction design recently adopted by the UK and other governments for spectrum auctions (Cramton, 2009) [18]. Hoffman, Menon, van den Heever and Wilson (2006) [34] introduced acceleration techniques for computing the proxy auction.

Day and Raghavan (2007) [24] and Day and Milgrom (2008) [23] independently introduced the notion of core selecting auctions. Each pair of authors proposed a generalization of the proxy auction where the "virtual" auctions of the proxy are superseded by a direct consideration of core allocations relative to bidders' reports, and each demonstrated the incentive advantages of selecting the bidder-optimal core allocation. Day and Raghavan introduced a core constraint generation algorithm

[11] More precisely, if all bidders have substitutes preferences, then the VCG allocation is in the core. Conversely, if there are at least four bidders, if the set of each bidder's possible valuations includes the additive valuation functions and if at least one bidder's possible valuations includes non-substitutes preferences, then there exists a profile of bidder valuations such that the VCG allocation is not in the core. (See Ausubel and Milgrom (2002) [8].)

which is an especially effective method for calculating bidder-optimal core allocations and, in particular, advocated the bidder-optimal core allocation that minimizes the maximum deviation from the VCG payments. Day and Milgrom proved important economic properties, including that an efficient direct mechanism is immune from shill bidding if and only if it is a core-selecting auction, and that bidder-optimal core selecting auctions (in contrast to the Vickrey auction) exhibit monotonicity of revenues in the number of bidders and their bids. Meanwhile, Day and Cramton (2009) [22] proposed the nearest-Vickrey pricing rule and demonstrated how to compute it efficiently.

Three other recent papers have begun to explore the comparison among core-selecting auctions. Erdil and Klemperer (2010) [27] define a class of payment rules referred to as "reference rules" - the proxy auction's payment rule is one example, while the nearest-Vickrey rule is not – and they argue that reference rules reduce the marginal incentive to deviate as compared to other payment rules. While their paper does not explicitly contain incomplete-information analysis, their conclusions foreshadow the results of the current paper. Goeree and Lien (2009) [31] consider the incomplete-information game with a global bidder and two local bidders whose valuations are independent and uniformly-distributed. Simultaneously and independently from the current paper, they solve for the Bayesian-Nash equilibrium of the nearest-Vickrey pricing rule for independent uniform distributions and they find that the VCG mechanism dominates it in expected revenues as well as efficiency. Sano (2010) [51] considers the incomplete-information game with a global bidder and two local bidders whose valuations are independent and uniformly-distributed. Simultaneously and independently from the current paper, he solves for the Bayesian-Nash equilibrium of the proxy auction under independence, finding that high-value local bidders submit almost their true values, while low-value local bidders shade considerably.

In the current paper, we too analyze incomplete-information games with a global bidder and two local bidders. We formulate the game and solve for equilibria, allowing independence ($\gamma = 0$) or correlation ($\gamma > 0$) between the local bidders' values. We consider four different core-selecting auctions – the nearest-Vickrey, the proxy, the proportional and the nearest-bid pricing rule - and for each $\gamma \in [0, 1]$, we are able to obtain explicit closed-form solutions under certain assumptions on the distributions.[12] For the case where the marginal distributions are uniform, we obtain Figure 2.2, which summarizes the expected seller revenues and efficiency in the equilibrium.

Counter to Goeree and Lien [31], we find that the choice between a core-selecting auction or the VCG mechanism is sensitive to the information structure. As shown in Figure 2.2, the relative performance of the alternative mechanisms changes substantially as the correlation γ increases from zero to 1. When $\gamma = 0$, the VCG mechanism raises 8.9% higher expected revenues than the proxy auction, and it achieves 9.6% higher revenue than the nearest-Vickrey rule, despite achieving greater efficiency. However, at the opposite extreme, when $\gamma = 1$, the proxy auction attains 33.3% higher expected revenues than the VCG mechanism while also realizing full

[12]Only three solutions are required, as the equilibrium for the proportional pricing rule coincides with the equilibrium for the nearest-Vickrey rule in the model we consider.

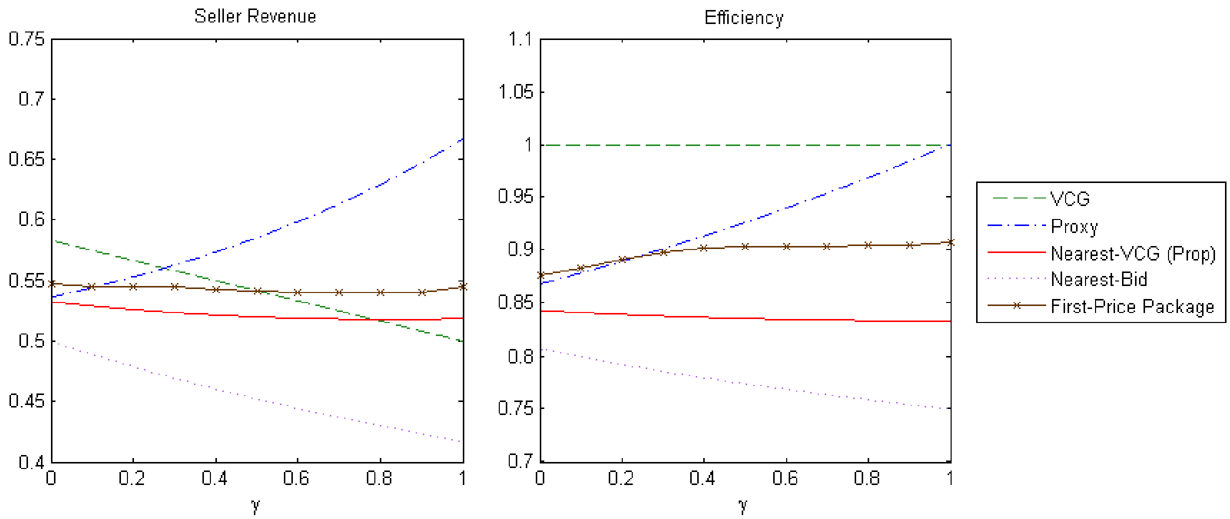

Figure 2.2: Seller Revenue and Efficiency for $\alpha = 1$ and all $\gamma \in [0, 1]$

efficiency. Clearly, the choice of whether to use the VCG mechanism or a core-selecting auction depends on the likely informational environment.

We also consider the effect of varying the uniform distribution on the local bidders' values. While the distribution $F(v) = v^{\alpha}$ does not generally admit a closed-form solution, it does when $\alpha = 2$. Stable numerical simulations can be found for other α - for symmetry with $\alpha = 2$, we also consider $\alpha = 1/2$. In this formulation, α controls the relative frequency of local bidders' winnings under full efficiency. For example, when $\alpha = 1$ (uniform distribution) local bidders are expected to win with probability 1/2 while they only expected to win with probability 1/3 when $\alpha = 1/2$. We find that the comparison among the VCG mechanism and the various core-selecting auctions changes in α. In particular, the case of $\alpha = 2$ reverses the revenue ranking of the proxy auction and the VCG mechanism, while the case of $\alpha = 1/2$ enhances the revenue advantage of VCG emphasized by Goeree and Lien [31].

This paper proceeds as follows. In Section 2.2, we present the model, including the family of distributions that allow partial correlation, and we detail the package-bidding mechanisms to be considered. In Section 2.3, we introduce the pivotal pricing property and we establish Lemma 2.3, which provides local optimality conditions for local bidders in any mechanism satisfying the pivotal pricing property. We solve for explicit closed-form solutions for the various mechanisms under consideration, for all correlation parameters, in Section 2.4. In Section 2.5, we discuss extensions to the basic model, and we conclude in Section 2.6. Most proofs are relegated to Appendix 2.A, and the solutions for different values of parameters γ and α are summarized in Appendix 2.B.

2.2 Model

Two items are offered for sale. There are two local bidders, 1 and 2, who are interested in only one item and receive no extra utility from acquiring the second item. Their values are denoted v_1 and v_2, respectively. There is one global bidder who wants to acquire both items and obtains no utility from owning just one item. Her value for the pair of items is denoted u. The bidders are risk neutral and have quasilinear utilities: the payoff of local bidder i, if she wins one unit at price p_i, is $v_i - p_i$; and the payoff of the global bidder, if she wins both units at a total price of p, is $u - p$.

The value, u, of the global bidder is independently drawn from the distribution on $[0, 2]$ described by a cumulative distribution function $G(u)$ with atomless probability density function $g(u)$. With probability γ, the values, $v_i (i = 1, 2)$, of local bidders are perfectly correlated and drawn from a distribution on the interval $[0, 1]$, defined by a cumulative distribution $F(v)$ function with atomless density $f(v)$. With probability $1 - \gamma$, the values of the local bidders are independently drawn from the same distribution $F(v)$.

The assumption of independence between value realizations of the global bidder versus the local bidders seems reasonable enough - the scale of operations, cost structure and other bidder-specific characteristics of the global and local bidders may be substantially unrelated. However, it seems likely that there might be positive correlation between one local bidder's value and another. For example, in a spectrum auction, the local bidders might be two firms that intend to deploy identical telecommunications technologies in different geographic regions.

Parameter γ controls the amount of correlation between the local bidders' values. The local bidders' value model is summarized by the conditional cumulative distribution function of the local bidder i given her value v_i:

$$F_L(v_j|v_i = s) = \begin{cases} (1-\gamma)F(v_j) & 0 \leq v_j < s \\ (1-\gamma)F(v_j) + \gamma & s \leq v_j \leq \bar{v} \end{cases} \qquad i \neq j$$

It is interesting to observe that v_i and v_j are not affiliated random variables for any $\gamma > 0$.[13] Nevertheless, for $y \geq x$, $F_L(\bullet|y)$ (first-order) stochastically dominates $F_L(\bullet|x)$. The failure of affiliation would prevent some of the results in the theory of single-item auctions from going through, but note that the structure of winning

[13] Consider $x > y > z$ and let $\Xi(\cdot, \cdot)$ denote the joint probability of v_i and v_j. Then $(y, y) \vee (x, z) = (x, y)$ and $(y, y) \wedge (x, z) = (y, z)$, but $\Xi(x, y)\Xi(y, z) < \Xi(y, y)\Xi(x, z)$, contradicting the affiliation inequality.

is different in the current package-bidding context: local bidder 1's bid need not exceed the bid of local bidder 2; rather the sum of the bids of local bidders 1 and 2 needs to exceed the bid of the global bidder.

Our model handles both homogeneous and heterogeneous environments. In the former interpretation, local bidder i derives positive utility v_i from winning either item.[14] In the latter interpretation, there are two heterogeneous items, East and West; local bidder 1 obtains positive utility only from East and local bidder 2 obtains positive utility only from West.[15] Our equilibrium solutions are fully consistent with either interpretation.

All of the auction mechanisms that we analyze in this paper, other than the VCG mechanism, satisfy the following definition:

Definition 2.1. *A core-selecting auction is a mapping from bids to allocations and payments such that the payoffs resulting from every bid profile are elements of the core.*[16]

Without loss of generality we limit our attention to the restricted auctions in which each bidder is allowed to submit only one bid. While impractical in a general environment, this limitation does not affect efficiency in any way because of the perfect complementarity nature of the bidders' preferences in the model we consider here. For example, the global bidder has value for a package of two items and her

[14]Then the global bidder exhibits classic increasing returns to scale.

[15]Then the global bidder is intending to implement a technology which (for technical or marketing reasons) is only economical if deployed on a nationwide basis.

[16]This definition is taken from Day and Milgrom (2008) [23].

bid B is interpreted as a package bid for two items. Each local bidder i is interested only in one item and her bid b_i expresses her willingness to pay up to b_i for the one item.

All auctions considered in the paper proceed in the following manner. First, all bidders submit their bids to the auctioneer who then chooses an allocation which maximizes total welfare with respect to the bids. In our simple model, only two outcomes are possible. If the package bid of the global bidder is greater than the sum of the local bids, i.e., $B > b_1 + b_2$, the global bidder wins the auction and receives both items. The local bidders win the auction and receive one item each whenever the sum of their bids is higher than the package bid of the global bidder, i.e., $B < b_1 + b_2$. Ties are resolved using a fair randomizing device. The payment each winner is required to make depends on a specific pricing rule.

We consider the VCG mechanism and several core-selecting pricing rules. Denote V_1 and V_2, the VCG payments of local bidders in case of winning, i.e., $V_1 = \max\{0, B - b_2\}$, $V_2 = \max\{0, B - b_1\}$. Additionally, we use $p(b_1, b_2, B)$ to denote a payment vector associated with the corresponding bids b_1, b_2 by local bidders and a bid B by the global bidder.

Without loss of generality, we will assume that $b_1 \geq b_2$.

(1) VCG Mechanism (Benchmark Rule)

This is a well-known pricing rule which is motivated by its dominant strategy property. Under this rule, the payment of the particular bidder does not depend upon her bid and only affects the allocation.

$$p(b_1, b_2, B) = \begin{cases} (V_1, V_2, 0) & \text{if} \quad B \leq b_1 + b_2 \\ (0, 0, b_1 + b_2) & \text{if} \quad B > b_1 + b_2 \end{cases}$$

(2) Proxy Auction

The ascending proxy auction was suggested by Ausubel and Migrom (2002) [8]. Given our simple model, it can be summarized using the following formula:

$$p(b_1, b_2, B) = \begin{cases} (\frac{1}{2}B, \frac{1}{2}B, 0) & \text{if} \quad B < 2b_2 \\ (B - b_2, b_2, 0) & \text{if} \quad 2b_2 \leq B < b_1 + b_2 \\ (0, 0, b_1 + b_2) & \text{if} \quad B > b_1 + b_2 \end{cases}$$

(3) Nearest-VCG Rule

The nearest-VCG pricing rule was introduced by Day and Cramton (2009) [22], superseding Day and Raghavan's (2007) [24] suggestion of minimizing the maximum deviation from the VCG payments. The central idea of this rule is to select the bidder-optimal core allocation that minimizes the Euclidean distance to the VCG point:

$$p(b_1, b_2, B) = \begin{cases} (V_1 + \Delta, V_2 + \Delta, 0) & \text{if} \quad B \leq b_1 + b_2 \\ (0, 0, b_1 + b_2) & \text{if} \quad B > b_1 + b_2 \end{cases}$$

where $\Delta = \dfrac{B - V_1 - V_2}{2}$

(4) Proportional Rule

This is a natural rule to consider in this environment. Whenever the local side wins the auction, they split the amount they are required to pay proportionally to their bids.

$$p(b_1, b_2, B) = \begin{cases} \left(\dfrac{b_1}{b_1+b_2}B, \dfrac{b_2}{b_1+b_2}B, 0\right) & \text{if} \quad B \leq b_1 + b_2 \\ (0, 0, b_1 + b_2) & \text{if} \quad B > b_1 + b_2 \end{cases}$$

(5) Nearest-Bid Rule

The "nearest-bid" description corresponds to the point in a minimum-revenue core which is the closest to the winners' bids. This rule can be motivated by a simple description of the payment procedure. In case of winning each local bidder pays her bid and then gets a refund. The amount of the refund is just half of the "money left on the table", i.e., $b_1 + b_2 - B$. As with Proxy Rule, if bids are too different, the amount of refund might be higher than the smallest of the locals' bids. Since payments can not be negative, the local bidder i with the small bid ($b_i << b_j$) is reimbursed completely while the local bidder j pays the global bidder's bid alone. This rule is intuitive and easy to explain to the bidders.

$$p(b_1, b_2, B) = \begin{cases} (B, 0, 0) & \text{if} \quad B < b_1 - b_2 \\ (b_1 - \Delta, b_2 - \Delta, 0) & \text{if} \quad b_1 - b_2 \leq B < b_1 + b_2 \\ (0, 0, b_1 + b_2) & \text{if} \quad B > b_1 + b_2 \end{cases}$$

where $\Delta = \dfrac{b_1 + b_2 - B}{2}$

2.3 Initial Analysis

Definition 2.2. *A bid b by bidder i is pivotal if, for any $\epsilon > 0$, a bid $b + \epsilon$ yields bidder i a non-empty set of items, while a bid of $b - \epsilon$ yields bidder i the empty set.*

Note that, in auctions with pricing rules (1) - (5), any bid (b_1, b_2, B) is pivotal if and only if $b_1 + b_2 = B$.

Definition 2.3. *An auction satisfies the pivotal pricing property with respect to a given bidder if, whenever the bidder's bid is pivotal, the price that she pays (if she wins) equals her bid.*

The pivotal pricing property is very natural and is satisfied for the most of the reasonable auction formats. Consider standard single-item auction with at least three bidders. First-price and all-pay auctions necessarily satisfy the pivotal pricing property since the winner always pays her bid. In a second-price auction a winning bid is pivotal only if top two bids are equal to each other in which case the winner pays her bid precisely. However, some auctions do not satisfy this property. For example, in a third-price auction a winner with pivotal bid in general pays less then her bid.

Lemma 2.0. *VCG mechanism satisfies the pivotal pricing property with respect to all bidders.*

Proof. If bidder i's bid, b_i, is pivotal, then the incremental surplus contributed by bidder i is zero. By the specification of the VCG mechanism, bidder i's payoff in the mechanism equals zero. Consequently, bidder i pays a price of b_i. □

Lemma 2.1. *Every core-selecting auction satisfies the pivotal pricing property with respect to all bidders.*

Proof. Let p_i denote the price paid by bidder i when her bid, b_i, is pivotal, and let S_i denote the set of winning bidders if bidder i had instead submitted a bid of $b_i - \epsilon$.

By the definition of a pivotal bid, $i \notin S_i$. Suppose that $p_i < b_i$. Then the allocation can be blocked by the coalition comprising the seller and set S_i. Suppose instead that $p_i > b_i$. Then the allocation can be blocked by the coalition comprising bidder i alone. We conclude that $p_i = b_i$. □

Lemma 2.2. *The global bidder has a weakly dominant strategy to bid her value in auctions with pricing rules (1) - (5).*

Proof. For each of these pricing rules, the global bidder wins if and only if her package bid, B, satisfies $B \geq b_1 + b_2$, and her payment is then $b_1 + b_2$. Consequently, the exact same argument holds as in the standard second-price auction for a single item. □

In what follows, we assume that the global bidder bids according to her weakly dominant strategy, i.e., $B(u) = u$.

With a slight abuse of notation let $\beta(.)$ denote the symmetric equilibrium bid function of the local bidders for all pricing rules. Additionally, denote $\Phi_i(b_i, v_i)$ and $\phi_i(b_i, v_i)$ the probability of winning and marginal probability of winning for a local bidder i who submits a bid b_i assuming all other bidders follow their equilibrium strategies, i.e.:

$$\begin{aligned}
\Phi_i(b_i, v_i) &= Pr(b_i + b_j > B) \\
&= \gamma \int\limits_{b_i+\beta(v_i)>B(u)} g(u)du + (1-\gamma) \iint\limits_{b_i+\beta(v_j)>B(u)} f(v_j)g(u)dv_j du \\
&= \gamma G(b_i + \beta(v_i)) + (1-\gamma) \int_{v_j} f(v_j)G(b_i + \beta(v_j))dv_j \\
\phi_i(b_i, v_i) &= \frac{\partial \Phi_i(b_i, v_i)}{\partial b_i} \\
&= \gamma g(b_i + \beta(v_i)) + (1-\gamma) \int_{v_j} f(v_j)g(b_i + \beta(v_j))dv_j
\end{aligned}$$

Let $P_i(b_i, v_i)$ and $MP_i(b_i, v_i)$ denote the expected payment and the expected marginal payment, respectively, for a local bidder who submits a bid b_i assuming all other bidders follow their equilibrium strategies, i.e.:

$$\begin{aligned}
P_i(b_i, v_i) &= Ep_i(b_i, b_j, B) \\
&= \gamma \int_u p_i(b_i, \beta(v_i), B) g(u) du + (1-\gamma) \iint_{u,v_j} p_i(b_i, \beta(v_j), B) f(v_j) g(u) dv_j du \\
MP_i(b_i, v_i) &= Ep_i'(b_i, b_j, B) \\
&= \gamma \int_u p_i'(b_i, \beta(v_i), B) g(u) du + (1-\gamma) \iint_{u,v_j} p_i'(b_i, \beta(v_j), B) f(v_j) g(u) dv_j du
\end{aligned}$$

Lemma 2.3. *For an auction satisfying the pivotal pricing property, the optimality conditions for a local bidder i are given by:*

$$(v_i - b_i)\phi_i(b_i, v_i) \leq MP_i(b_i, v_i) \quad b_i \geq 0$$

$$b_i \left[(v_i - b_i)\phi_i(b_i, v_i) - MP_i(b_i, v_i)\right] = 0$$

Proof. See Appendix 2.A □

Lemma 2.3 just simplifies the Karush-Kuhn-Tucker conditions for the local bidders' profit maximization problem taking into account the pivotal pricing property. Intuitively, an infinitely small increase in a bid affects costs by increasing expected payment in non-pivotal states and adding a new payment in the pivotal state (when the increase results in a pivotal bid, or the state in which bidder wins only because she increased her bid by a small amount). The latter payment equals to the player's bid according to the pivotal pricing property.

In case $\phi_i(b_i, v_i) > 0$, the optimality conditions in Lemma 2.3 can be rewritten as:

$$b_i = \max\left(0, v_i - \frac{MP_i(b_i, v_i)}{\phi_i(b_i, v_i)}\right)$$

Note that a local bidder shades her bid when the expected marginal payment is positive. We formalize this general functional form of the equilibrium local's bidder bid function in a Corollary 2.1.

Corollary 2.1. *The general functional form of the locals' equilibrium bid function is:*

$$\beta(v) = \begin{cases} 0 & v \leq d(\gamma) \\ c(v) & v > d(\gamma) \end{cases}$$

where

- $$d(\gamma) = d \quad \gamma < 1$$
  $$d(1) = 0 \quad \gamma = 1$$
- $d \geq 0$ *such that* $\phi(0,d)d = MP(0,d)$
- $c(v)$ *is strictly increasing on* $[d(\gamma), 1]$

The equilibrium bid function potentially has a flat segment in the beginning. Intuitively, the local bidder might find it optimal to free-ride on the other local bidder because the probability of winning is strictly greater than zero for a local bidder with a zero bid.

Proposition 2.0. *The equilibrium bid function of local bidders under the VCG pricing rule is given by* $\beta(v) = v$.

Proof. Well known. □

2.4 Main Results

This section contains our main results. In order to derive equilibrium bids explicitly, we assume uniform distributions for all values. Namely, $f(.)$ is a uniform density on $[0,1]$ and $g(.)$ is a uniform density on $[0,2]$. Under this assumption, there is symmetry between global and local sides of the market because under full efficiency the global and local sides are expected to win equally often.

We start by considering the Proxy Rule.

Proposition 2.1. *The equilibrium bid function of local bidders (in symmetric Bayesian-Nash equilibria) under the Proxy Rule is given by:*

$$\beta(v) = \begin{cases} 0 & v \leq d(\gamma) \\ 1 + \dfrac{\ln(\gamma + (1-\gamma)v)}{1-\gamma} & v > d(\gamma) \end{cases} \quad \textit{if} \quad \gamma < 1^{17}$$

and

$$\beta(v) = v \quad \textit{if} \quad \gamma = 1$$

where $d(\gamma) = \dfrac{e^{-(1-\gamma)} - \gamma}{1-\gamma} > 0 \quad \forall \gamma < 1.$

Proof. See Appendix 2.A. □

Figure 2.3 (left panel) provides examples of equilibrium bid functions for the proxy rule. In equilibrium, local bidders with low values prefer to bid zero in an attempt to free-ride. Moreover, the size of the zero-bid interval is magnified by the proxy rule itself because a local bidder with a sufficiently small bid would be

[17] A symmetric Bayesian-Nash equilibrium for the Proxy Rule with local bidders having independent values, i.e., $\gamma = 0$, was derived independently in Sano (2010) [51]

required to pay her bid whenever the locals win with the other local bidder paying the rest. To put it differently, a local bidder with a small bid in the proxy auction has shading incentives which are similar to that of the first-price package auction. In sharp contrast, a high-type local bidder bids almost truthfully because she expects to be the highest bidder from the local side in which case her payment is independent from her bid.

With the increase in correlation, the zero-bid interval vanishes since a low-type local bidder no longer expects a sufficiently high bid from the other local bidder. Instead she expects a comparably low bid which makes her reluctant to shade. At the extreme case of perfect correlation, both local bidders bids truthfully in a symmetric equilibrium. The case of perfect correlation is very interesting since the proxy rule is able to achieve the first-best by combining equilibrium truthful-bidding property with the core property.

However, the proxy rule model with perfect correlation also has a multiplicity of other, asymmetric equilibria where revenue and efficiency performance is undermined. Specifically, one of the asymmetric equilibria results in truthful bidding by one of the local bidders and bidding zero by the other local bidder.

Proposition 2.2. *The equilibrium bid function of local bidders under the Nearest-Vickrey Rule is given by:*

$$\beta(v) = \begin{cases} 0 & v \leq d(\gamma) \\ k(\gamma)v - d(\gamma) & v > d(\gamma) \end{cases} \quad \textit{if} \quad \gamma < 1^{18}$$

and

[18] A symmetric Bayesian-Nash equilibrium for Nearest-Vickrey Rule with local bidders having

$$\beta(v) = 2/3v \quad if \quad \gamma = 1$$

$$where\ k(\gamma) = \frac{2}{2+\gamma} \qquad d(\gamma) = \frac{2(\sqrt{-\gamma^2+2\gamma+8}-3)}{\gamma^2+\gamma-2} > 0 \quad \forall\gamma < 1.$$

Proof. See Appendix 2.A. □

The equilibrium bid functions for the nearest-Vickrey rule are shown in Figure 2.3 (central panel). The size of the zero-bid interval is smaller when comparing to the proxy rule which, as was mentioned above, induces first-price incentives to bidders with low valuations. In contrast with the proxy rule, a high-type local bidder has no incentive to bid truthfully anymore since her bid affects the price considerably. It is worth highlighting a nice linear functional form of the equilibrium bids in case of nearest-Vickrey rule where correlation parameter γ defines the slope and intercept coefficients. For example, a local bidder shades uniformly across all values when there is no correlation between local bidders, i.e., $\gamma = 0$. Positive correlation has an ambiguous effect on revenue and efficiency since it reduces bid-shading for low-type bidders and increases bid-shading for high-type bidders.

Proposition 2.3. *The equilibrium bid function of local bidders under the Proportional Rule is given by:*

$$\beta(v) = \begin{cases} 0 & v \leq d(\gamma) \\ k(\gamma)v - d(\gamma) & v > d(\gamma) \end{cases} \quad if \quad \gamma < 1$$

and

$$\beta(v) = 2/3v \quad if \quad \gamma = 1$$

$$where\ k(\gamma) = \frac{2}{2+\gamma} \qquad d(\gamma) = \frac{2(\sqrt{-\gamma^2+2\gamma+8}-3)}{\gamma^2+\gamma-2} > 0 \quad \forall\gamma < 1.$$

independent values, i.e., $\gamma = 0$, was derived independently in Goeree and Lien (2009) [31]

Proof. See Appendix 2.A. □

Surprisingly, the equilibrium bid strategies for our model are the same under the nearest-Vickrey pricing rule and the proportional pricing rule. This result is mainly driven by two of our modeling assumptions: uniform distribution of the global bidder's value with the zero lower bound and the number of local bidders. In a model with more than two items for sale (discussed in greater detail in Section ?), this rule results in different equilibrium bid functions.

Proposition 2.4. *The equilibrium bid function of local bidders under the Nearest-Bid Rule is given by:*

$$\beta(v) = \frac{1}{1-\gamma}[\ln(2) - \ln(2 - (1-\gamma)v)] \quad \textit{if} \quad \gamma < 1$$

and

$$\beta(v) = 1/2v \quad \textit{if} \quad \gamma = 1$$

Proof. See Appendix 2.A. □

Figure 2.3 (right panel) demonstrates examples of equilibrium bid functions for the nearest-bid rule. The bidding behavior under this rule is very different from the rules already considered. First, the equilibrium bid functions are strictly increasing for all correlation levels. The absence of the zero-bid interval for low-type local bidders is easily explained by the nature of the nearest-bid rule. Conditional on winning, the expected payment of a low-type local bidder is close to zero since the half of the refund to which the bidder is entitled almost surely exceeds the amount of her bid. At the same time, a high-type bidder shades substantially, since her payment depends heavily on the amount of her bid. Second, correlation has a strong negative impact on the equilibrium bidding functions.

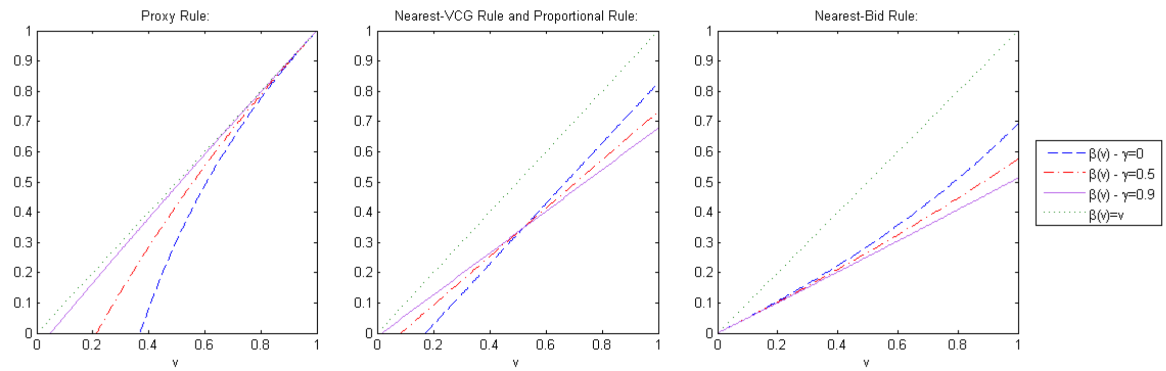

Figure 2.3: Equilibrium Bids: Proxy Auction Rule(left), Nearest-Vickrey(center) and Nearest-Bid (right)

Figure 2.2, already seen in the Introduction, summarizes the expected revenue and efficiency results for all pricing rules. The revenue of the VCG rule is negatively affected by positive correlation. By contrast, the performance of the proxy rule improves rapidly as the correlation increases, allowing the proxy rule to outperform the VCG rule in terms of revenue for a substantial range of values. Moreover, the proxy rule achieves full efficiency when the locals' values are perfectly correlated. The performance of the nearest-Vickrey rule seems to be robust to correlation. This suggests that the seller interested in stable revenue and efficiency outcome across different correlation levels might have a good reason to use the nearest-Vickrey rule. However, for this particular model and distributions the proxy rule dominates other core-selecting rules including nearest-Vickrey rule. The performance of the nearest-bid rule falls with correlation, which makes the rule inferior and impractical for this environment. Corresponding numbers for revenue, efficiency and profits of bidders can be found in Table 2.1. Expressions used to compute revenue and efficiency for all pricing rules are provided in Appendix A. We use simulation results from Baranov (2010) [12] in order to compare all mechanisms considered here with the first-price package auction. Interestingly, the first-price package auction consistently beats the nearest-Vickrey and nearest-bid formats in terms of both revenue and efficiency for any positive correlation value. However, its expected revenue is lower than that of VCG and proxy auction for low and high correlation values respectively.

γ	Statistics	VCG	Proxy	Nearest-VCG	Nearest-Bid	First Price*
$\gamma = 0$	*Revenue*	0.5833	0.5360	0.5327	0.5	0.5471
	Efficiency	1	0.8679	0.8431	0.8069	0.8762
	Profit Global	0.2916	0.4642	0.4673	0.5	0.4269
	Profit Local	0.2087	0.1342	0.1335	0.1253	0.1498
$\gamma = 0.5$	*Revenue*	0.5417	0.5852	0.52	0.4521	0.5412
	Efficiency	1	0.9261	0.8356	0.7739	0.9039
	Profit Global	0.3126	0.4148	0.4798	0.5479	0.4304
	Profit Local	0.2295	0.1523	0.1415	0.1252	0.1647
$\gamma = 1$	*Revenue*	0.5	0.6667	0.5185	0.4167	0.5445
	Efficiency	1	1	0.8334	0.75	0.9073
	Profit Global	0.3335	0.3335	0.4816	0.5834	0.4270
	Profit Local	0.2499	0.1666	0.1481	0.125	0.1754

* - Based on Baranov (2010) [12]

Table 2.1: Revenue, Efficiency and Profits

2.5 Some Extensions

2.5.1 Non-Uniform Model

The main results of this paper were derived under the assumption of uniform distributions for bidders' values. In this subsection we consider a more general model where the underlying distribution for the local bidders' values allows varying the full-efficiency frequency of winning between the global bidder and local bidders. Specifically, we assume that the cumulative distribution function for local bidders is $F(v) = v^{\alpha}, \alpha > 0$ on the interval $[0, 1]$. We continue to assume that the values of the local bidders are perfectly correlated with probability γ and that the global bidder draws her value independently from the uniform distribution on $[0, 2]$.

The parameter α of the local bidders' distribution function can be interpreted in the following way. When α is less than one, the sum of the local bidders' values is expected to be small in comparison with the expected value of the global bidder, implying that the local bidders lose more frequently under full efficiency. When α is greater than one, the situation is reversed, with the global bidder winning less frequently under truthful bidding. In other words, a high α makes the local bidders the stronger side in terms of their expected value.

In general, there are no closed-form solutions for this model, but it can be easily solved by appropriate numerical methods. For example, the equilibrium bidding function for the nearest-Vickrey rule, as in the uniform model, is linear with the slope coefficient being derived explicitly while the intercept term is determined from a non-linear equation which can be solved by a standard numerical procedure like the Newton method. The equilibrium bidding functions for the proxy rule and nearest-bid rule can be easily approximated by numerical methods for solving ODEs. Appendix B contains some equilibrium bidding functions and corresponding equations for numerical approximations for all pricing rules and all correlation levels.

For the various second-price-like core-selecting auction formats, an increase in α leads to an increase in bid shading by the local bidders. Intuitively, a local bidder expects a higher bid from the other local bidder and tries to free-ride, reducing her bid accordingly. Symmetrically, smaller α results in more truthful bidding since opportunities for free-riding are reduced.

Figures 2.4 and 2.5 contain revenue and efficiency calculations for scenarios where $\alpha = 2$ and $\alpha = 0.5$, respectively. As can be seen in Appendix B, the calculations for $\alpha = 2$ for the proxy auction and nearest-bid rule are based on explicit closed-form solutions and for the nearest-VCG rule are based on "almost-closed-form" solutions. Meanwhile, the value $\alpha = 0.5$ was chosen for symmetric comparison with $\alpha = 2$; most of the associated calculations are based on numerical simulations. Even though a low value for α generates more sincere bids by local bidders, the expected total bid from them is smaller than in case of $\alpha = 1$ (uniform distribution) and so expected seller revenue is lower. The seller revenue is affected positively by an increase in α for all core-selecting rules. On the other hand, an increase in α negatively affects revenue of the VCG auction since it leads to an increase in the probability of low revenue and zero revenue outcomes. As can be seen from Figures 2.4 and 2.5, the revenue performance of Vickrey rule relative to any core-selecting rule falls with α. For example, the proxy rule and the nearest-Vickrey rule generate higher revenues for any correlation level ($\forall\gamma$) when $\alpha = 2$.

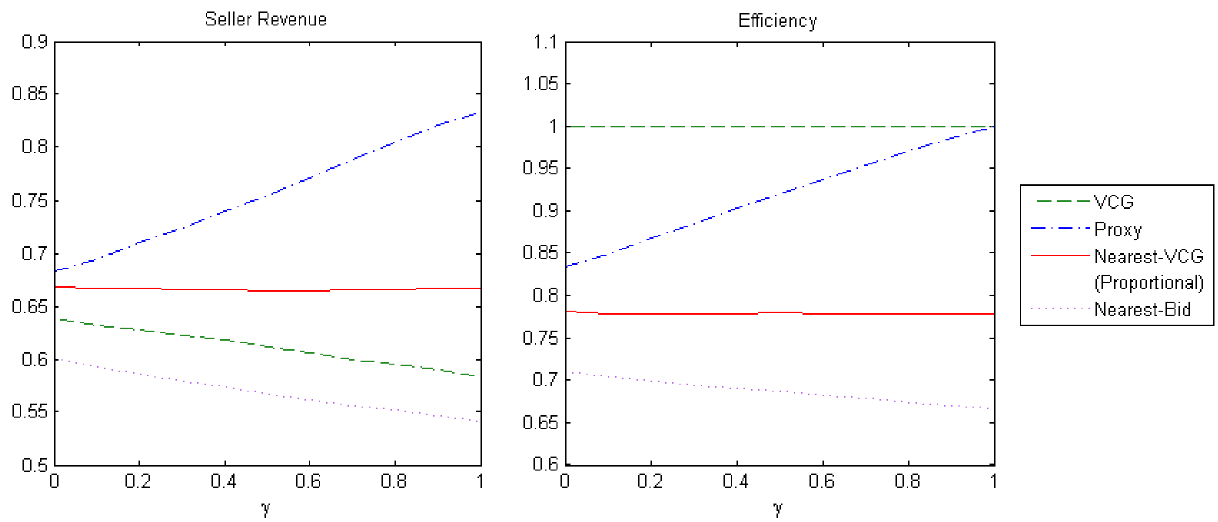

Figure 2.4: Seller Revenue and Efficiency for $\alpha = 2$ and all $\gamma \in [0, 1]$

2.5.2 Number of Bidders

Here we look into a question of robustness of our results with respect to an increase in the number of bidders. There are several interesting modifications of our model one can consider.

First of all, an increase in the number of global bidders can be modeled as a replacement of the distribution function of the global bidder with the extreme value distribution function of values. For example, if there are two global bidders who draw their values from distribution described by a function $H(u)$ independently from each other and local bidders, a version of the model with one global bidder can be used instead with distribution function of the global bidder being equal to the

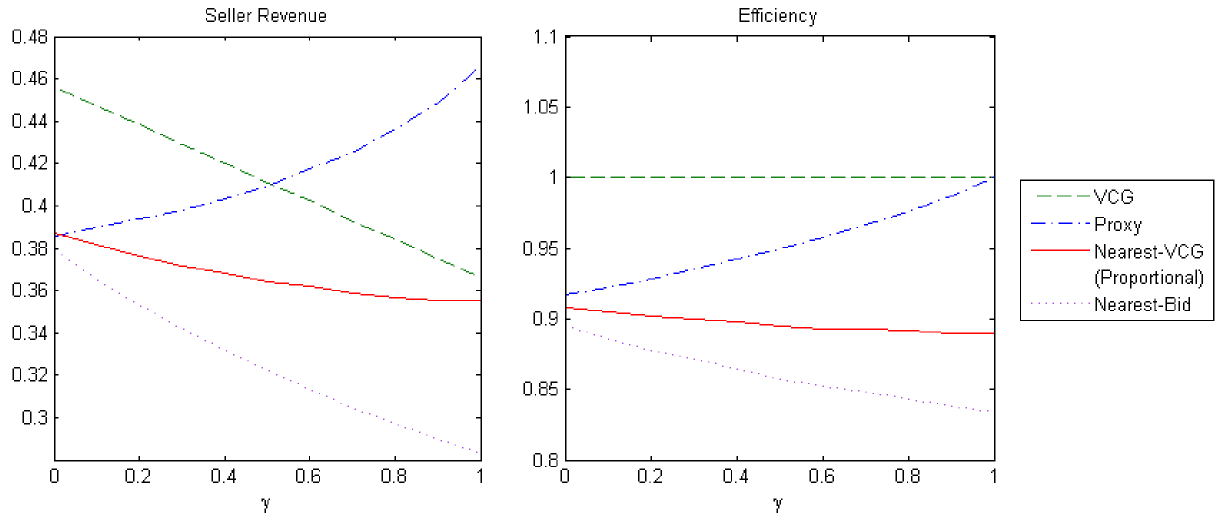

Figure 2.5: Seller Revenue and Efficiency for $\alpha = 0.5$ and all $\gamma \in [0, 1]$

product of individual distributions, i.e., $G(u) = H(u)^2$. This replacement works because global bidders still have a weakly dominant strategy to bid truthfully. Since the distribution of this pseudo global bidder is no longer uniform, the equilibrium bidding functions of local bidders in general have to be approximated numerically.

Another interesting comparative statics exercise is to increase the number of local bidders together with the number of items offered for sale. For example, consider an auction where three items are offered, with three local bidders who only wish to acquire one item each and a global bidder who is interested only in winning all three items. Keeping a similar value structure, it is possible to solve this model for some correlation levels and some pricing rules. For example, for the proportional rule this model can be solved in closed form for all levels of correlation among local bidders' values. Unfortunately, some pricing rules such as the nearest-Vickrey rule become inherently complex in this environment. Luckily, the solution for the proportional rule sheds some light on the revenue and efficiency performance of the core-selecting rules relative to that of the Vickrey rule. An increase in the number of local bidders leads to a more severe coordination problem among them, more bid shading, and lower seller revenues and efficiency in comparison with the Vickrey rule. This finding suggests that any core-selecting rule may be a poor choice for environments where the presence of a coordination problem is significant.

Finally, one can think of increasing the number of local bidders without increasing the number of products offered in the auction. In such environments, local bidders face competitors for their own item or market and they bid more aggressively. For example, a zero-bid interval (interval of values for which local bidder submits zero bid) no longer exists.

2.5.3 Robustness Check

In this subsection we demonstrate numerically that the partial correlation model for local bidders' values used in this paper results in equilibrium bidding functions which are qualitatively very general. Consider the following modification to the original model of Section 2.2. Let M be a common unknown distributional factor for local bidders which is distributed on the interval $[0, 1]$ with some positive density $f_M(m)$. Conditional on a particular realization, m, of the distributional factor, values for local bidders are drawn independently from a truncated logit distribution on $[0, 1]$ with parameters (m, σ), where $\sigma > 0$ is a known scale factor, i.e.:

$$f_L(v_j|M=m) = \frac{A(m,\sigma)e^{-(v_j-m)/\sigma}}{\sigma(1+e^{-(v_j-m)/\sigma})^2} \qquad \text{for} \quad v_j \in [0,1]$$

where $A(m,\sigma)$ is a normalizing constant.

Since m is not observable, both local bidders make inferences about the distribution of the other local bidder's value using their own values as signals about m. It is not hard to show that the conditional density takes the following form:

$$f_L(v_j|v_i = s) = \frac{\int_0^1 f_L(v_j|m)f_L(s|m)f_M(m)dm}{\int_0^1 f_L(s|m)f_M(m)dm}$$

Figure 2.6 contains approximations of conditional densities for different values of the signal and different levels of parameter σ which controls the correlation in this model (taking the role of γ in the main model of this paper). Levels of σ are chosen such that the correlation between local bidders' values in the main model with gamma values 0, 0.5 and 0.9 and the model considered here are approximately equal to each other.

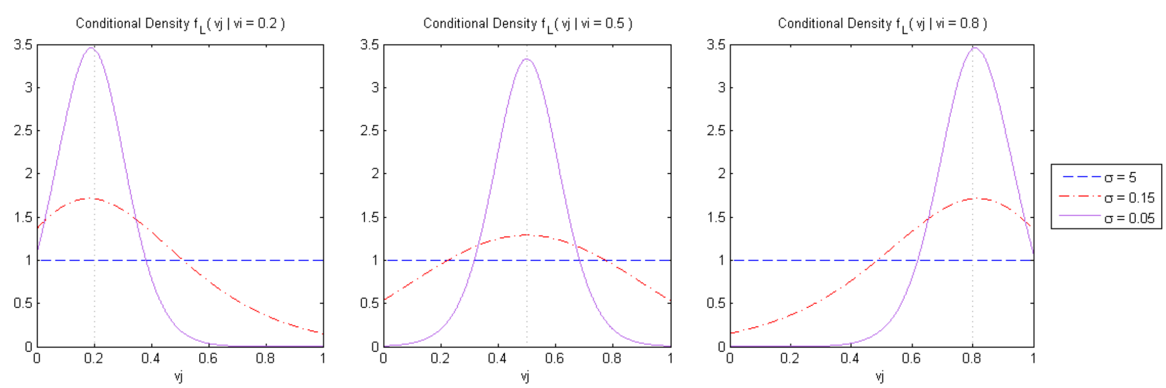

Figure 2.6: Conditional Densities

Given conditional densities, we approximate first-order conditions for different pricing rules derived in Lemma 2.3. The corresponding equilibrium bidding functions can be found at Figure 2.7. The numerical solutions exhibit qualitatively similar shapes and patterns as the closed-form equilibrium bidding functions derived in Section 2.4 (i.e., Figure 2.3). These results are very encouraging for the future use of this paper's partial correlation model in other contexts, since in some environments it allows us to generate closed-form solutions or extremely stable and easy numerical solutions without introducing any qualitative distinctions from smoother and more plausible partial-correlation models.

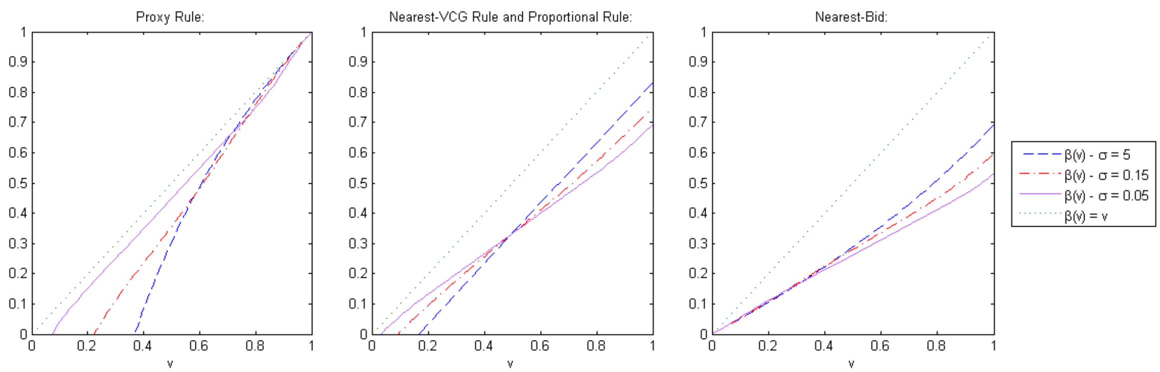

Figure 2.7: Approximations of Equilibrium Bids

2.6 Conclusion

The past literature has shown the VCG mechanism to have a variety of shortcomings in environments with complementarities, including the possibility of low or even zero revenues, non-monotonicity of revenues with respect to bids and number of bidders, and vulnerability to unusual forms of collusion such as shill-bidding and collusion by losing bidders. This list of drawbacks may help to explain why this auction format - despite its attractive dominant-strategy property - is seldom used in practice: to date, we are not aware of any examples of auctions employing the Vickrey payment rule in an environment with multiple heterogeneous items. At the same time, interest in core-selecting auctions appears to be rising, with two high-stakes auctions already conducted using a two-stage version of the nearest-Vickrey pricing rule. Nevertheless, the existing literature on core-selecting auctions primarily studies complete-information environments and, to the extent that incomplete information is introduced, bidders' values are assumed to be independent.

This paper develops a model of package auctions in an environment with private information. The model considered, while simple and intuitive, includes a number of realistic features that motivate the use of package auctions, such as the presence of substantial complementarities in bidders' preferences and a positive correlation of bidders' values. We were able to derive explicit closed-form solutions for all considered payment rules and all correlation levels, under certain assumptions on distributions. Our analysis shows that core-selecting payment rules create strong incentives for bidders without a conflict of interests to shade their bids in equilibrium. At first glance, this equilibrium property might discourage the use of core-selecting auctions, since they do not achieve full efficiency and their expected revenue might be even smaller than the revenue of the corresponding VCG auction. However, the presence of positive correlation may dramatically improve the performance of core-selecting auctions relative to the VCG mechanism. In fact, positive correlation significantly improves the performance of the proxy rule while affecting negatively the performance of the VCG. The nature of the proxy rule makes shading profitable only if the local bidder expects a sufficiently high complementary bid from the other local bidder, which becomes increasingly unlikely as the correlation increases. On the other hand, positive correlation increases the probability of low-revenue or zero-revenue outcomes in the VCG mechanism - these occur when both local bidders' are a substantial fraction of the global bidder's value.

Furthermore, the VCG mechanism has a lot of other potential drawbacks for practical applications. The full efficiency property of the VCG is actually a result of the best-case scenario analysis in a more general model of package auction where bidders can effectively use several identities (or shills) to represent their interests. If the seller has no control over identities of the bidders, the efficiency and revenue of the VCG may be significantly lower. For example, in a simple model with two items, two global bidders and a VCG pricing rule, the truth-telling strategies no longer form equilibria when one or both global bidders can enter the auction using two local shills. In contrast, any bidder-optimal core-selecting auction has an equilibrium in sincere strategies with full efficiency, while VCG does not. Moreover, in important applications such as spectrum auctions, it is very likely that substantial correlations in bidders' valuations may be present. Thus, unlike Goeree and Lien (2009) [31], we conclude that there may be good reasons for policymakers to select a core-selecting auction rather than a VCG mechanism.

A curious reader might notice that our paper only considers the case of positive correlation between local bidders' values without considering the case of negative correlation. This treatment seems to be satisfactory since we are not aware of any reasonable practical application for an auction model with negative correlations.

The received wisdom in auction theory is that the higher the value of the object to one bidder, the higher the value of the object to any other bidder. Nevertheless, from a methodological viewpoint, a similar model of negative correlation between local bidders' values can be easily constructed. We envision that the presence of negative correlations will improve the performance of the VCG and nearest-bid pricing rules while hurting the proxy rule. This conclusion is based on the intuition developed in Section 4 on the effect of increasing the correlation, only applied in the opposite direction.

2.A Appendix A - Proofs

2.A.1 Proof of Lemma 2.3

The profit function of a local bidder i is given by:

$$\pi_i(b_i, v_i) = v_i \Phi_i(b_i, v_i) - P_i(b_i, v_i)$$

The first-order optimality conditions are:

$$\frac{\partial \pi_i(b_i^*, v_i)}{\partial b_i} \leq 0 \quad b_i^* \geq 0 \quad b_i^* \left(\frac{\partial \pi_i(b_i^*, v_i)}{\partial b_i} \right) = 0 \quad \forall v_i \in [0, 1]$$

In order to get the desired form of the first-order conditions, we first compute the marginal cost of winning with a bid b_i:

$$\begin{aligned} \frac{\partial P_i(b_i, v_i)}{\partial b_i} &= \gamma \frac{\partial}{\partial b_i} \left(\int\limits_{b_i + \beta(v_i) > u} p_i(b_i, \beta(v_i), u) g(u) du \right) \\ &+ (1-\gamma) \frac{\partial}{\partial b_i} \left(\iint\limits_{b_i + \beta(v_j) > u} p_i(b_i, \beta(v_j), u) f(v_j) g(u) du \right) \end{aligned}$$

or

$$\begin{aligned} \frac{\partial P_i(b_i, v_i)}{\partial b_i} &= \gamma \, p_i(b_i, \beta(v_i), b_i + \beta(v_i)) g(b_i + \beta(v_i)) \\ &+ \gamma \int\limits_{b_i + \beta(v_i) > u} p_i'(b_i, \beta(v_i), u) g(u) du \\ &+ (1-\gamma) \int\limits_{v_j} p_i(b_i, \beta(v_j), b_i + \beta(v_j)) f(v_j) g(b_i + \beta(v_j)) dv_j \\ &+ (1-\gamma) \iint\limits_{b_i + \beta(v_j) > u} p_i'(b_i, \beta(v_j), u) f(v_j) g(u) du \end{aligned}$$

Note that by Lemma 2.2 and the pivotal pricing property we have the following:

$$p_i(b_i, \beta(v_i), b_i + \beta(v_i)) = p_i(b_i, \beta(v_j), b_i + \beta(v_j)) = b_i \qquad \forall v_i, v_j$$

Plugging the last equality back to the marginal cost of winning we get a short and intuitive form for this term:

$$\begin{aligned}\frac{\partial P_i(b_i, v_i)}{\partial b_i} &= b_i \left[\gamma g(b_i + \beta(v_i)) + (1-\gamma) \int_{v_j} f(v_j) g(b_i + \beta(v_j)) dv_j \right] + MP_i(b_i, v_i) \\ &= b_i \phi_i(b_i, v_i) + MP_i(b_i, v_i)\end{aligned}$$

Finally, the desired form of the first-order optimality conditions:

$$\frac{\partial \pi_i(b_i^*, v_i)}{\partial b_i} = v_i \phi_i(b_i^*, v_i) - \frac{\partial P_i(b_i^*, v_i)}{\partial b_i} = (v_i - b_i^*)\phi_i(b_i^*, v_i) - MP_i(b_i^*, v_i)$$

$$(v_i - b_i^*)\phi_i(b_i^*, v_i) - MP_i(b_i^*, v_i) \leq 0 \quad b_i^* \geq 0$$

$$b_i^* \left[(v_i - b_i^*)\phi_i(b_i^*, v_i) - MP_i(b_i^*, v_i)\right] = 0 \quad \forall v_i \in [0, 1]$$

□

2.A.2 Proof of Proposition 2.1

The following table summarizes marginal payments for a local bidder in all possible situations:

$\beta(v_j), \beta(v_i)$	$u \leq 2\underline{\beta}$	$2\underline{\beta} < u \leq \beta(v_i) + \beta(v_j)$	$u > \beta(v_i) + \beta(v_j)$
Perfect Correlation ($v_i = v_j$)	0	N/A	0
Independence ($v_i < v_j$)	0	1	0
Independence ($v_i \geq v_j$)	0	0	0

($*\underline{\beta} = \beta(\min(v_i, v_j))$)

The expected marginal payment for a local bidder in equilibrium is:

$$\begin{aligned} MP(\beta(v), v) &= \tfrac{1-\gamma}{2}\int_v^1 (\beta(v_j) - \beta(v))dv_j \\ (v - \beta(v)) &= (1-\gamma)\int_v^1 (\beta(v_j) - \beta(v))dv_j \end{aligned}$$

Note that $\beta(v) = v$ in case of $\gamma = 1$. The associated expected revenue and efficiency are

$$R^{Proxy} = 2/3 \qquad Ef^{Proxy} = 1$$

For the case of $\gamma < 1$ the equivalent differential equation and the terminal condition are given by:

$$\beta' = \frac{1}{\gamma + (1-\gamma)v} \qquad \beta(1) = 1$$

The solution for this differential equation yields the equilibrium bid function. The associated expected revenue and efficiency for $\gamma < 1$ are given by:

$$\begin{aligned} R^{Proxy} &= \frac{6e^(- (1-\gamma) - e^(- 2(1-\gamma) - (1 + 5\gamma - 2\gamma^2 + \gamma^3)}{2(1-\gamma)^3} \\ Ef^{Proxy} &= \frac{2e^(- (1-\gamma) + (1 - 4\gamma + \gamma^2}{2(1-\gamma)^2} \end{aligned}$$

□

2.A.3 Proof of Proposition 2.2

The following table summarizes marginal payments for a local bidder in all possible situations:

$\beta(v_j), \beta(v_i)$	$u \le \beta(v_i)$	$\beta(v_i) < u \le \beta(v_i) + \beta(v_j)$	$u > \beta(v_i) + \beta(v_j)$
Perfect Correlation ($v_i = v_j$)	0	1/2	0
Independence	0	1/2	0

The expected marginal payment for a local bidder in equilibrium is:

$$\begin{aligned} MP(\beta(v), v) &= \tfrac{1}{4}\left[\gamma\beta(v) + (1-\gamma)\int_0^1 \beta(v_j)dv_j\right] \\ (v - \beta(v)) &= \tfrac{1}{2}\left[\gamma\beta(v) + (1-\gamma)\int_0^1 \beta(v_j)dv_j\right] \end{aligned}$$

Note that $\beta(v) = \frac{2}{3}v$ in case of $\gamma = 1$. The associated expected revenue and efficiency are

$$R^{Nearest-VCG} = 14/27 \qquad Ef^{Nearest-VCG} = 5/6$$

For the case of $\gamma < 1$:

$$\beta(v) = \frac{2}{2+\gamma}\left[v - \frac{1-\gamma}{2}\int_0^1 \beta(v_j)dv_j\right] = k(\gamma)v - d(\gamma)$$

The associated expected revenue and efficiency are given by:

$$R^{Nearest-VCG} = \frac{(d-1)(-3d^2(1-\gamma)(1+\gamma)^2 + 13d^2(1+\gamma) + d(7+25\gamma+12\gamma^2) - 11\gamma - 17)}{6(2+\gamma)^2}$$

$$Ef^{Nearest-VCG} = \frac{2d^2(1+\gamma) - 2d(2+\gamma) + \gamma + 4}{2(2+\gamma)}$$

where $d(\gamma) = \dfrac{2(\sqrt{-\gamma^2+2\gamma+8} - 3)}{\gamma^2+\gamma-2}$ and $\gamma < 1$

□

2.A.4 Proof of Proposition 2.3

The following table summarizes marginal payments for a local bidder in all possible situations:

$\beta(v_j), \beta(v_i)$	$0 \leq u \leq \beta(v_i) + \beta(v_j)$	$u > \beta(v_i) + \beta(v_j)$
Perfect Correlation ($v_i = v_j$)	$\dfrac{\beta(v_i)u}{4\beta(v_i)^2}$	0
Independence	$\dfrac{\beta(v_j)u}{(\beta(v_i) + \beta(v_j))^2}$	0

The expected marginal payment for a local bidder in equilibrium is:

$$\begin{aligned} MP(\beta(v), v) &= \frac{1}{4}\left[\gamma\beta(v)\frac{(2\beta(v))^2}{4\beta(v)^2} + (1-\gamma)\int_0^1 \beta(v_j)\frac{(\beta(v)+\beta(v_j))^2}{(\beta(v)+\beta(v_j))^2}dv_j\right] \\ &= \tfrac{1}{4}\left[\gamma\beta(v) + (1-\gamma)\int_0^1 \beta(v_j)dv_j\right] \end{aligned}$$

Note that the expected marginal payment is exactly the same as the one for Nearest-VCG Payment Rule. Therefore, equilibrium bid function, expected revenue and efficiency are exactly the same.

□

2.A.5 Proof of Proposition 2.4

The following table summarizes marginal payments for a local bidder in all possible situations:

$\beta(v_j), \beta(v_i)$	$u \leq \overline{\beta} - \underline{\beta}$	$\overline{\beta} - \underline{\beta} < u \leq \beta(v_i) + \beta(v_j)$	$u > \beta(v_i) + \beta(v_j)$
Perfect Correlation ($v_i = v_j$)	N/A	1/2	0
Independence	0	1/2	0

($*\underline{\beta} = \beta(\min(v_i, v_j))$ $\quad \overline{\beta} = \beta(\max(v_i, v_j))$)

The expected marginal payment for a local bidder in equilibrium is:

$$\begin{aligned} MP(\beta(v), v) &= \tfrac{\gamma}{2}\beta(v) + \tfrac{1-\gamma}{2}\left[\int_0^v \beta(v_j)dv_j + \int_v^1 \beta(v)dv_j\right] \\ &= \tfrac{\gamma}{2}\beta(v) + \tfrac{1-\gamma}{2}\left[\int_0^v \beta(v_j)dv_j + \beta(v)(1-v)\right] \\ (v - \beta(v)) &= \gamma\beta(v) + (1-\gamma)\left[\int_0^v \beta(v_j)dv_j + \beta(v)(1-v)\right] \end{aligned}$$

Note that $\beta(v) = 1/2v$ in case of $\gamma = 1$. The associated expected revenue and efficiency are

$$R^{Proxy} = 5/12 \qquad Ef^{Proxy} = 3/4$$

For the case of $\gamma < 1$ the equivalent differential equation and the terminal condition are given by:

$$\beta' = \frac{1}{2 - (1-\gamma)v} \qquad \beta(0) = 0$$

The solution for this differential equation yields the equilibrium bid function. The associated expected revenue and efficiency for $\gamma < 1$ are given by:

$$R^{Proxy} = \frac{1 + \gamma(\gamma(5 + 4\ln 2) - 6 + 4\ln 2) - 4\gamma(1+\gamma)\ln(1+\gamma)}{2(1-\gamma)^3}$$
$$Ef^{Proxy} = \frac{3 - 4\gamma + \gamma^2 + 2(1+\gamma)(\ln(1+\gamma) - \ln 2)}{2(1-\gamma)^2}$$

□

2.B Appendix B - Solution Summary

Notation reminder:

- α - parameter of the distribution function for local bidders' values

  $(F(v) = v^{\alpha}, \alpha > 0)$

- γ - correlation parameter of the joint distribution for local bidders' values

  $(0 \leq \gamma \leq 1)$

2.B.1 Proxy Pricing Rule

$\alpha > 0$	$0 \leq \gamma \leq 1$	Equilibrium Bid Function $\beta(v)$
$\alpha > 0$	$\gamma = 1$	$\beta(v) = v$
$\alpha = 1$	$0 \leq \gamma < 1$	$\beta(v) = \max\left(0, 1 + \frac{\ln(\gamma+(1-\gamma)v)}{1-\gamma}\right)$
$\alpha \neq 1$	$\gamma = 0$	$\beta(v) = \max\left(0, \frac{v^{1-\alpha}-\alpha}{1-\alpha}\right)$
$\alpha = 2$	$0 < \gamma < 1$	$\beta(v) = \max\left(0, \frac{1}{\sqrt{\gamma(1-\gamma)}} \tan^{-1}\left(\sqrt{\frac{1-\gamma}{\gamma}} v + C\right)\right)$ $C = 1 - \frac{1}{\sqrt{\gamma(1-\gamma)}} \tan^{-1}\left(\sqrt{\frac{1-\gamma}{\gamma}}\right)$
$\alpha \neq 1$, $\alpha \neq 2$	$0 < \gamma < 1$	No Closed-Form Solution Differential Equation for approximations: $\beta' = \frac{1}{\gamma+(1-\gamma)v^{\alpha}} \quad \beta(1) = 1$

2.B.2 Nearest-Vickrey Pricing Rule (and Proportional Pricing Rule)

$\alpha > 0$	$0 \leq \gamma \leq 1$	Equilibrium Bid Function $\beta(v)$
$\alpha > 0$	$\gamma = 1$	$\beta(v) = \frac{2}{3}v$
$\alpha = 1$	$0 \leq \gamma < 1$	$\beta(v) = \begin{cases} kv - d & v > \frac{d}{k} \\ 0 & v \leq \frac{d}{k} \end{cases}$ $k = \frac{2}{2+\gamma} \quad d = \frac{3k^2 - 2k\sqrt{3k-1}}{3k-2}$
$\alpha \neq 1$	$0 \leq \gamma < 1$	Almost Closed-Form Solution: $\beta(v) = \begin{cases} kv - d & v > \frac{d}{k} \\ 0 & v \leq \frac{d}{k} \end{cases}$ $k = \frac{2}{2+\gamma}$ d is defined by the equation: $\frac{d^{\alpha+1}}{(1+\alpha)k^{\alpha}} - \frac{3k}{3k-2}d + \frac{\alpha k}{\alpha+1} = 0$

2.B.3 Nearest-Bid Pricing Rule

$\alpha > 0$	$0 \leq \gamma \leq 1$	Equilibrium Bid Function $\beta(v)$
$\alpha > 0$	$\gamma = 1$	$\beta(v) = \frac{1}{2}v$
$\alpha = 1$	$0 \leq \gamma < 1$	$\beta(v) = \frac{1}{1-\gamma}\left[\ln(2) - \ln(2-(1-\gamma)v)\right]$
$\alpha = 2$	$0 \leq \gamma < 1$	$\beta(v) = \frac{1}{\sqrt{2(1-\gamma)}}\left\lvert\frac{(1-\gamma)v+\sqrt{2(1-\gamma)}}{(1-\gamma)v-\sqrt{2(1-\gamma)}}\right\rvert$
$\alpha \neq 1$, $\alpha \neq 2$	$0 < \gamma < 1$	No Closed-Form Solution Differential Equation for approximations: $\beta' = \frac{1}{2-(1-\gamma)v^{\alpha}} \quad \beta(0) = 0$

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