ABSTRACT

Title of Document: THE EFFECTS OF BLENDED INSTRUCTION AND VISUAL REPRESENTATIONS ON AREA PROBLEMS INVOLVING QUADRATIC EXPRESSIONS FOR SECONDARY STUDENTS WITH MATHEMATICS LEARNING DIFFICULTIES

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The current study examined the effect of an instructional package on the algebra performance of secondary students with mathematics learning disabilities or difficulties (MD) when applied to the grade-appropriate algebra content of quadratic expressions. The instructional package included a blend of research-based instructional practices for secondary students with LD (i.e., concrete-representation-abstract instruction, graphic organizers, and components of explicit instruction) and the process standards recommended by the National Council of Teachers of Mathematics process standards (i.e., problem solving, reasoning and proof, communication, connections, and representations). A concurrent embedded mixed methods design was utilized with the quantitative data representing the main strand, while qualitative data provided supplemental data (Creswell & Clark, 2011). Specifically, the quantitative data were collected from a multiple-probe design across two groups replicated over five students. The participants were five high school students identified as a learning disability or difficulty in mathematics. The qualitative analysis of transcriptions from instructional sessions, field notes, and work samples was completed on one participant, who represented a critical case (Creswell, 2007). Results of the study indicated that all five participants improved their algebraic accuracy on tasks involving quadratic expressions embedded within an area context. Further, providing multiple representations allowed participants to make connections to algebraic content and enhanced their metacognition. Additionally all participants maintained their performance up to six weeks following intervention. Three participants also transferred the performance to novel and more complex tasks. The study suggests that students with MD may be successful with higher-level algebra content when provided blended instruction and visual representations.

# THE EFFECTS OF BLENDED INSTRUCTION AND VISUAL REPRESENTATIONS ON AREA PROBLEMS INVOLVING QUADRATIC EXPRESSIONS FOR SECONDARY STUDENTS WITH MATHEMATICS LEARNING DIFFICULTIES

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## Chapter 1: Introduction

Algebra has often been referred to as a gatekeeper to postsecondary education and employment opportunities. Many colleges and universities require mathematics placement tests that include a minimum of Algebra 1 with essential domain content including representing algebraic expressions, solving algebraic word problems and linear equations, and representing and applying polynomials (ACT, 2011; The College Board, 2009). Additionally, admissions to apprenticeships, such as the National Joint Apprenticeship and Training Committee (NJATC) of National Electrical Contractors Association (NECA) and International Brotherhood of Electrical Workers (IBEW), require one year of high school algebra (NJACT, 2009). Success in algebra provides a foundation for higher mathematics, which increases the likelihood of completing college and a obtaining a higher quality of living as adults. Adults who participate in higher mathematics courses in high school are more likely to have a higher income, use new technology, vote, and engage in civic leadership (National Mathematics Advisory Panel [NMAP], 2008). Additionally, some claim that a greater number of students with mathematics backgrounds are necessary for the United States (U.S.) to maintain its position as an international leader in science, technology, engineering, or mathematics (STEM), as only 12% of postsecondary students graduate with a degree in STEM fields (Chen & Weko, 2009; NMAP, 2008).

Given the presumed importance of math competency, secondary schools are implementing more rigorous mathematics requirements for *all* students, including students with disabilities. To date, 45 states and the District of Columbia have adopted the rigorous mathematics standards set forth by the Common Core State Standards Initiative, a state-led effort coordinated by the National Governors Association Center for Best Practices and the Council of Chief State School Officers (Achieve, 2011). These standards represent the knowledge and skills students need to be prepared for college and work. Within the mathematics standards, all students are expected to progress through Algebra I, Geometry, and Algebra II (Common Core State Standards Initiative, 2010). Additionally, 26 states require a mathematics exit exam containing algebra content as a requirement for high school graduation (Colasanti, 2007; Zabala, Minnici, McMurrer, Hill, Bartley, & Jennings, 2007).

Two pieces of federal legislation that affect the mathematics education of students with disabilities are the Individuals with Disabilities Education Improvement Act (IDEIA) and the No Child Left Behind Act (NCLB). IDEIA (2004) provides students with disabilities access to the general education curriculum and NCLB (2001) requires states and districts to assess students with disabilities on the general education content with the use of accommodations or an alternate assessment, if necessary. As algebra is a required component of the high school curriculum, students with disabilities have access to and are held accountable for this rigorous mathematics content.

Approximately 6% of the school population is identified as having a learning disability (Mazzocco, 2007). A learning disability (LD) is defined as a disorder in one or more of the basic psychological functions involved in understanding or in using language and manifests itself in an imperfect ability to listen, speak, read, write, spell, or do mathematical calculations (IDEA, 2004). Although 62% of secondary students with LD participate in mathematics courses in the general education setting (Newman, 2006), on average they are enrolled in less rigorous mathematics courses that focus on basic math rather than age-appropriate mathematics content (Kortering, deBettencourt, & Braziel, 2005; Maccini & Gagnon, 2002; Wagner, et al., 2003). Additionally, students with LD take fewer mathematics courses as they progress through high school (i.e., 98% of freshman with disabilities participate in a high school mathematics course compared to 85% of juniors) (Wagner, et al., 2003). On average, Algebra 1 is the highest level mathematics course for students with disabilities, while Trigonometry is the highest average mathematics course for students without disabilities (Wilson, 2008). Access to high-level mathematics courses is critical for successful postsecondary outcomes for students with LD.

In this chapter, I discuss the challenges students with and without LD experience as they develop algebraic proficiency, followed by a brief description of the impact of characteristics of students with LD on algebra competency. Next, I summarize the results from international, national and state assessments, which provide an overview of the current status of algebra proficiency in the U.S. I then discuss the impact of advocacy groups and mathematics reform on the mathematics curriculum in the U.S. The concluding sections review the existing research, leading to the purpose of this study, the guiding research questions, and definitions of terminology.

**Development of Algebraic Proficiency**

In 2008, the National Mathematics Advisory Panel (NMAP) released their final report summarizing the state of mathematics education in the U.S, focusing on the preparation of students for proficiency in algebra. To obtain algebraic proficiency, students must develop: (a) conceptual understanding; (b) procedural knowledge; and

(c) problem-solving skills associated with algebra. Conceptual understanding is defined as the knowledge of relations and connections (Star, 2005), while procedural knowledge refers to actions and manipulations involving rules, algorithms, and strategies (de Jong & Ferguson-Hessler, 1996). In-depth procedural knowledge coincides with in-depth conceptual knowledge, and vice versa (Baroody, Feil, & Johnson, 2007). Together, conceptual understanding and procedural fluency support effective and efficient problem solving (NMAP, 2008), as students apply previously learned concepts and skills to novel problems in which the solution method is unknown (National Council of Teachers of Mathematics [NCTM], 2000).

In general, U.S. students struggle to reach procedural and conceptual proficiency in algebra due to a poor understanding of whole numbers, fractions (including decimals, percents, and negative fractions), and aspects of geometry and measurement (i.e., perimeter, area, similar triangles, slope of a line) (NMAP, 2008). Additionally, students lack an understanding of foundational algebra concepts such as negativity (Kieran, 1989; Vlassis, 2004), variables (Kieran, 1989; NMAP, 2008; Russel, O'Dwyer & Miranda, 2009; MacGregor & Stacey, 1997; Wagner, 1981), equality and the equal sign (Kieran, 1989; NMAP, 2008; Russel, et al., 2009; Stacey & MacGregor, 1997), commutative and distributive properties (Carraher & Schliemann, 2007; Saul, 2008), and algebraic expressions (Clements, 1982; Kieran, 1989; NMAP, 2008). Additionally, students struggle with complex algebra processes, such as solving algebraic equations (Clements, 1982; Kieran, 1989; NMAP, 2008) and factoring quadratics (Kotsopoulos, 2007; Nataraj & Thomas, 2006).

**Characteristics of Students with LD**

Students with LD experience many of the aforementioned difficulties and exhibit additional characteristics that may impede their algebra performance. Although this is a heterogeneous population, students with LD may exhibit one or more of the following characteristics. Students with LD have difficulty recalling math facts as a result of long term memory deficits (Garnett, 1998; Geary, 2004) and struggle to compute arithmetic problems as a result of poor sequential memory with procedures (Calhoon, Emerson, Flores, & Houchins, 2007; Garnett, 1998; Geary, 2004). Students with LD may also have visual spatial deficits, as evidenced by difficulty spatially representing and interpreting mathematical information (Garnett, 1998; Geary, 2004). Additionally, students with LD perform significantly lower than their non-disabled peers in problem solving as a result of poor recall and generalization of previously learned materials (Bley & Thorton, 2001; Bryant, Bryant, & Hammill, 2000; Gagnon & Maccini, 2001). Language deficits may interfere with the association of words to the symbols of algebra, which causes significant difficulties in classrooms that rely heavily on comprehension of texts and the spoken language of teachers and peers (Bley & Thornton, 2001; Ives, 2007). Poor metacognition impedes their ability to solve complex, multi-step problems and engage in problem solving (Bley & Thornton, 2001; Vaidya, 1999). Additionally, students with LD struggle to conceptualize abstract algebraic concepts (i.e., abstract symbols such as variables) and algebraic tasks (i.e., solving complex equations) (Bley & Thorton, 2001; Garnett, 1998; Geary, 2004; Witzel, 2005). Affective measures may also negatively impact performance of in algebra, such low motivation, self-esteem, and/or passivity in the classroom (Gagnon & Maccini, 2001; Mazzocco, 2007).

**Status of Algebra Proficiency in the United States**

Due to the difficulties that students with and without LD experience in algebra, U.S. students have not demonstrated algebraic proficiency to the level expected from mathematics educators, researchers, and policy makers, as evidenced by their performance on both international and national assessments. The results of one international study in particular heighten this concern. The Program for International Student Assessment (PISA) is sponsored by the Organization for Economic Cooperation and Development (OECD) and measures the mathematics literacy of 15-year olds mathematics literacy. The most recent PISA in 2009 reported that U.S. 15-year olds scored 9 points below the OECD average, were outperformed by 24 of the 34 countries, and scored measurably higher than only five OECD countries (Fleischman, Hopstock, Pelczar,& Shelley, 2010).

A second international assessment, the Trends in International Mathematics and Science Study (TIMSS), measures the performance of students in mathematics and science, based on classroom curricula. In regard to the algebra content domain, U.S. eighth-grade students scored in the average range, outperforming 37 countries, scoring lower than seven countries, and demonstrating no measurable difference in scores with the remaining three countries (Gonzales, Williams, Jocelyn, Roey, Kastberg, & Brenwald, 2008).

Whereas the TIMSS focuses on curricula outcomes, PISA focuses on the application of knowledge in real-life contexts. A general conclusion may be made that U.S. students are improving in their knowledge of the concepts and skills taught in the classroom, based on the TIMSS data, but are not improving in their application of this knowledge, based on the PISA data. Data for students with disabilities was not disaggregated; therefore, no comparisons can be made between students with disabilities and their international counterparts.

In addition to the overall underperformance of U.S. students in mathematics on international assessments, U.S. students have not made the achievement gains expected on national assessments, such as the National Assessment of Educational Progress (NAEP), which covers a wide range of mathematics content areas, including algebra. The NAEP places a greater emphasis on algebra as the assessments advance through grade levels. For instance, only 15% of the questions from the fourth grade NAEP are related to algebra as compared to 30% of the questions on the eighth grade assessment. Additionally, algebra problems represent 35% of the total twelfth-grade mathematics assessment. Overall, approximately two-thirds of eighth graders and three-fourths of twelfth-graders in the general education population scored below the proficient level (i.e., competency in grade level material) on the most recent NAEP assessments. Based on the 2009 NAEP, 65% of eighth-graders and 76% of twelfth graders are performing below a *proficient* level (National Center for Education Statistics [NCSE], 2009; NCES, 2010).

An even larger number of secondary students with disabilities performed below the proficient level. Specifically, 91% of eighth-graders with disabilities and 94% of twelfth-graders with disabilities performed below the proficient level (NCES, 2009; NCES, 2010). Additionally, students without disabilities outperformed their peers in special education on algebra tasks. Eighth-grade students without disabilities outperformed peers with disabilities in the algebra content domain as demonstrated by mean scale scores of 291 and 251, respectively, while twelfth grade students without disabilities outperformed their peers in special education with mean scores of 157 and 118, respectively (NCES, 2011).

State level data also indicates poor performance for students with disabilities. Only three states reported that 50% or more of their students with disabilities demonstrated proficiency on their high school mathematics assessment compared to 35 states reporting 50% or more of their general student population demonstrated proficiency (VanGetson & Thurlow, 2007). For example, in the state of Maryland, 88% of the general education population passed the algebra/data analysis exit exam by twelfth grade compared to only 57% of students in special education (Maryland State Department of Education, 2010).

**Math Reform and Algebra**

The data from the international, national, and state assessments warrant concern regarding the mathematic education of students in the U.S. students, particularly for students with disabilities. To address this concern, advocacy groups, such the National Council of Teachers of Mathematics (NCTM), Achieve, Inc., the National Mathematics Advisory Panel (NMAP), and the Common Core State Standards Initiative (CCSSI), a state-led effort coordinated by the National Governors Association Center for Best Practices and the Council of Chief State School Officers, have promoted equality programs and access to a rigorous algebra curriculum for *all* learners. Additionally, these organizations promote the inclusion of student-centered instructional practices, in which students learn mathematics through activities, often with groups of peers, which explore mathematical concepts. During student-centered instruction, the teacher acts as a facilitator, asking guiding questions that enable learners to explore the nature of the target concept or skill. For instance, the NCTM's *Principles and Standards for School Mathematics* (2000) promotes student acquisition and application of the algebra content through the processes of problem solving, reasoning and proof, communication, connections, and representations. Additionally, the NCTM advocates for *equity* for *all* students to meet high expectations and excel in mathematics through these processes. Student-centered instruction promotes these process standards. Currently, all but one state in the U.S. has aligned their mathematics curriculum standards to the NCTM Standards (Woodward, 2004).

More recently, the CCSSI (2010) released a focused and coherent set of rigorous standards that define the mathematics *all* "students should understand and be able to do" (p.5). The high school algebra standards contain algebra content found in typical Algebra I and Algebra II courses, ranging from algebraic expressions to trigonometric functions. In addition to content, the common cores state standards (CCSS) promote specific standards related to mathematical practice, which are based on NCTM process standards and the National Research Council's strands of mathematical proficiency (i.e., adaptive reasoning, strategic competence, conceptual understanding, procedural fluency, and productive disposition). Specifically, mathematics educators should develop in all of their students the ability to: (a) make sense of problems and persevere in solving them; (b) reason abstractly and quantitatively; (c) construct viable arguments and critique the reasoning of others; (d) model with mathematics; (e) use appropriate tools strategically; (f) attend to precision; (g) look for and make use of structure; and (h) look for and express regularity in repeating reasoning. These standards promote a balanced combination of procedural fluency and conceptual understanding (CCSSI, 2010).

Additionally, these standards may be developed through a combination of teacher-directed and student-centered activities.

Similarly to NCTM and the CCSS, the American Diploma Project (ADP, 2004), and the NMAP (2008) advocate for a more rigorous high school mathematics curriculum for *all* students and have developed benchmarks describing specific mathematics knowledge and skills that high school graduates must master to succeed in postsecondary education or employment. These benchmarks include topics addressed in Algebra I and Algebra II courses, including algebraic expressions, linear equations, quadratic equations, polynomials, and functions. Although the ADP and the CCSSI do not promote a specific instructional practice, the NMAP recommends a blended model of student-centered and teacher-directed instruction, which is consistent with expert recommendations in the field of special education (Hudson, Miller, & Butler, 2006; Jones & Southern, 2003; Woodward & Montague, 2002).

Regardless of the instructional practice, learning requires engagement and social interactions. According to Vygotsky, social interactions within a classroom create a *zone of proximal development*, in which students learn new concepts by interacting with a teacher and/or other students (Gurganus, 2007; Kozulin, 1998). Questioning and discussion are important components of student-centered and teacher-directed instruction, with teachers providing scaffolds (i.e., guiding questions, models) to assist struggling students. As all learning is a constructive process (Gurganus, 2007; Kieran, 1994; Moshman, 1982), student-centered and teacher-directed instructional practices provide opportunities for *all* students to build knowledge through the use of appropriate supports and scaffolds to address individual student needs.

**Existing Research**

To help students with LD develop proficiency in rigorous algebra content, certain instructional strategies, which will be defined and analyzed in chapter 2, have been identified in previous reviews of algebra interventions for secondary students with LD (Foegen, 2008; Maccini, McNaughton, & Ruhl, 1999; Strickland & Maccini, 2010). Strategies include cognitive strategy instruction (Foegen, 2008; Maccini, et al., 1999), components of explicit instruction (Foegen, 2008; Maccini et al., 1999; Strickland & Maccini, 2010), graphic organizers (Strickland & Maccini, 2010) and the concrete-representation-abstract (CRA) instructional sequence (Foegen, 2008; Maccini, et al., 1999; Strickland & Maccini, 2010). Many studies in these reviews investigated the effects of an instructional package, which included two or more of the above strategies, therefore, the impact of specific components can not be determined; however, the package produced positive effects. Additionally, findings from the literature reviews are limited due to the relatively basic algebra content (i.e., integers, one-variable equations) addressed in the reviewed studies. Future research should address interventions that focus on more complex aspects of algebra, including simplifying polynomial and rational expressions, computation and factoring polynomials expressions, and solving and graphing quadratic equations (Foegen, 2008; Maccini, et al., 1999; Strickland & Maccini, 2010).

The critical topic of algebraic expressions (e.g., linear and quadratic expressions) is absent from the current literature on algebra interventions for students with LD. This algebraic topic is critical for three reasons: a) algebraic expressions introduce students to the abstract, symbolic notation of mathematics; b) many students experience difficulty when encountering this set of topics, which may result in an inadequate understanding of higher mathematics (MacDonald, 1986); and c) students are required to represent, add, subtract, multiply and factor algebraic expressions on many state algebra exit exams, as well as on SAT's, ACT's, and other college mathematics placement exams. Additionally, organizations such Achieve (2008) and the NMAP (2008) identify skills associated with polynomials as a benchmark for secondary mathematics and this content is integrated in the CCSSI (2010). As students with LD are being held accountable for the general education curriculum through NCLB (2001), they will need to develop complex algebraic knowledge.

**Statement of Purpose**

Algebra is a gatekeeper to postsecondary education and employment opportunities for *all* students, including students with LD. Overall, U.S. students are not demonstrating proficiency in algebra as evidenced by performance on international, national, and state assessments. Students with disabilities are performing significantly below their non-disabled peers on national and state assessments as specific characteristics, such deficits in long term and sequential memory (Calhoon, et al., 2007; Garnett, 1998; Geary, 2004), poor generalization of previously learned material (Bley & Thornton, 2001), and difficulty understanding abstract algebraic concepts and tasks (Garnett, 1998; Geary, 2004; Witzel, 2005) impede their progress toward proficiency in algebra.

The current study was developed in light of the reform efforts and recent legislative mandates that call for research-based methods to support the algebra development of students with LD. By expanding the existing research literature, this study investigated an intervention that consisted of an instructional package containing the concrete-representational-abstract integration strategy, explicit instruction, graphic organizers, and real-world problem solving on the completion of tasks involving multiple representations of quadratic expressions for secondary students with mathematics disabilities or difficulties (MD) . This study addressed algebra content that aligns with the NCTM standards, the CCSSI, and the NMAP and ADP benchmarks for secondary mathematics for *all* learners, as well as the NCTM process standards of problem-solving, representations, reasoning, communication, and connections.

**Research Questions and Hypotheses**

The overarching research question relevant to this study was to determine if blending instructional practices from both the special education and the mathematics education literature would lead to improved academic performance of students with MD on general education algebra content. The study was guided by the following specific research questions.

When provided blended instruction with visual representations:

1. To what extent do secondary students with mathematics disabilities or difficulties (MD) increase their accuracy on algebraic tasks involving quadratic expressions embedded within area problems?
2. To what extent do secondary students with MD maintain performance on algebraic tasks involving quadratic expressions embedded within area problems four to six weeks after the end of the intervention?
3. To what extent do secondary students with MD transfer their knowledge of quadratic expressions to problem-solving tasks?

4. To what extent do secondary students with MD find blended instruction with visual representations beneficial (i.e., social validity)?
5. How do the qualitative data findings provide an enhanced understanding of the quantitative results? Specifically, what connections and disconnections to the algebra content emerge as a result of the intervention and how can these findings improve future instruction? In what ways does the intervention enhance aspects of metacognition?

Based on current findings in both the special education and mathematics education research communities that will be summarized in the Chapter 2, I have developed hypotheses to address each research questions.

1. Secondary students with MD will increase their accuracy on algebraic tasks involving quadratic expressions embedded within area problems by scoring a minimum average of 80% or greater on the post test domain probes.
2. Secondary students with MD will maintain performance on of transforming quadratic expressions as evidenced by scoring a minimum of 80% or greater on the maintenance measure.
3. Secondary students with MD will transfer their conceptual and procedural knowledge of quadratic expressions to problem-solving tasks as evidenced by scoring a minimum of 80% or greater on the transfer measure.
4. Secondary students with MD will find blended instruction beneficial and enjoyable.
5. Using the multiple representations will provide support (i.e., flexibility across instructional phases, multiple modes of representations, opportunities for

reasoning and sense-making) for students with MD to access the algebra content.

**Definition of Terms**

This section provides definitions of terms used in this study.

*Abstract phase* refers to an instructional phase that utilizes only symbolic numbers and notations.

*Algebra* includes many definitions, such as: a) the study of relationships among quantities using symbolic notation (NCTM, 2001; Usiskin, 1988); b) generalized arithmetic (Kieran, 1989; Saul, 2008; Usiskin, 1988); c) the study of procedures (Kieran, 1989); d) a representation system (Wagner & Kieran, 1989); and e) the study of structures (Saul, 2008; Usiskin, 1988).

*Blended Instruction* refers to instruction that incorporates instructional practices from the special education literature (e.g., CRA instruction, graphic organizers, and explicit instruction) with NCTM process standards (i.e., representations, communication, connections, problem solving, and reasoning and proof).

*Conceptual knowledge* refers to the idea that logical relationships are constructed internally and are constructed in the mind as part of a network of ideas.

*Concrete phase* refers to an instructional phase that utilizes physical manipulatives.

*Concrete- representational-abstract integration strategy (CRA-I)* refers to an instructional strategy that simultaneously introduces physical manipulatives, sketches of manipulatives and symbolic notation while gradually fading the manipulatives and the sketches.

*Constructivist Theory of Learning* refers to learning that is actively created by the student.  Ideas are made meaningful when the student integrates them into their existing structures of knowledge (Clements & Battista, 1990; Greeno, et al., 1996).

*Contextualized problems* refer to problems that are based on real world examples and are presented in a narrative form.

*Explicit instruction* refers to a highly structured, teacher-directed method for presenting new information that incorporates key variables such as curriculum-based assessment, advanced organizer, teacher modeling, guided practice, independent practice, and review for maintenance.

*Graphic organizers* refer to visual representations that depict the relationship between facts or ideas within a learning task (i.e. graphs, charts).

*Learning disability* refers to a disorder in one or more of the basic psychological processes involved in understanding or in using language, which may manifest itself in the imperfect ability to listen, think, speak, read, write, spell, or do mathematical calculations (IDEA, 2004).

*Manipulatives* are physical objects that support mathematical thinking and represent a mathematical concept (i.e., counters, beads, algebra tiles, geoboards).

*Peer-assisted instruction* involves a student assisting a peer to learn a skill or concept, under the supervision of a teacher.

*Procedural fluency* refers to the flexible, accurate, and efficient use of mathematical procedures and algorithms (NRC, 2001).

*Problem solving* refers to the process of applying previously learned concepts and skills to novel situations (NCTM, 2000).

*Representational phase* refers to an instructional phase that utilizes visual representations, such as drawings and virtual manipulatives, to represent abstract mathematical concepts and is synonymous with the term semi-concrete.

*Secondary students* are students in grades 6 through 12 and/or of middle school to high school ages (11-21).

*Strategy instruction* provides students with a plan for solving a problem, i.e. providing students with a memory device or a cue card.

*Student-center instruction* refers to instruction in which the students are primarily responsible for their learning.

*Teacher-directed instruction* refers to instruction in which the teacher is primarily communicating the mathematics to students.

# Chapter 2: Review of the Literature

Despite the increasing demand for proficiency in algebra, international assessments indicate that secondary students in the United States (U.S.) experience difficulty in mathematics and score below many of their international counterparts. Data from the Trends in International Mathematics and Science Study (TIMSS) and the Program for International Student Assessment (PISA) indicate that students in the U.S. are performing below the level of many other industrialized countries in mathematics. Algebra problems comprise 30% of the TIMSS assessment, which is the highest percentage of questions of all content domains (Gonzales et al., 2008; Baldi, Jin, Skemer, Green, & Herget, 2007). Experts express concern that the U.S. will lose its standing as an international leader because U.S. students are not excelling in mathematics (National Mathematics Advisory Panel, 2008).

Additionally, U.S. students have not made the achievement gains expected on national assessments. For example, data from the National Assessment of Educational Progress (NAEP) indicates that 65% of eighth graders without disabilities scored below the *Proficient* level, defined as solid academic performance and competency over challenging subject matter. The performance of eighth-grade with disabilities is even more unsettling as 91% of these students performed below the proficiency level (NCES, 2009). An even greater percentage of twelfth-grade students, both within the general student population and specifically students with disabilities, scored below the proficiency level, 77% and 95% respectively. Similarly to the TIMSS, the majority of questions on the NAEP mathematics assessment address the algebra domain for the eighth-grade and the twelfth-grade assessments, 30% and 35% respectively. Clearly, algebra is a priority at the national level and students are not performing at an acceptable level.

In response to these concerns, the National Council of Teachers of Mathematics (NCTM, 2000) addressed the need for more rigorous standards by publishing the *Principles and Standards for School Mathematics.* The NCTM Standards focus on conceptual understanding and real-world problem solving and reflect a belief in the importance of mathematics for *all* students, including students with disabilities. The *Content Standards* describe the content that *all* students should learn from pre-kindergarten through twelfth grade (i.e., number and operations, algebra, geometry, measurement, data analysis, and probability). The *Process Standards* (i.e. problem solving, reasoning and proof, communication, connections, and representations) describe ways of learning the content knowledge. Additionally, the high school mathematics curriculum should emphasize mathematical reasoning (i.e. drawing conclusions based on evidence) and sense making (i.e. developing an understanding of a concept by connecting it to existing knowledge) in all courses, for students of varying abilities (NCTM, 2009).

In addition to the NCTM standards, reports from the National Mathematics Advisory Panel (NMAP; 2008) and the American Diploma Project (2004) emphasize a rigorous curriculum for *all* learners. Authors of both reports suggest mathematics benchmarks which include foundational skills to prepare elementary age learners for algebra (i.e., fluency and conceptual understanding of whole numbers, fractions, and certain aspects of geometry) (NMAP, 2008) and algebra skills necessary for completion of Algebra during secondary education (i.e., linear equations, quadratic equations, functions, and polynomials) (ADP, 2004; NMAP, 2008).

In addition to the higher math standards set forth by NCTM, NMAP, and ADP, the Individuals with Disabilities Education Act (IDEA) of 1997 requires that all students, including students with disabilities, have access to grade-appropriate curriculum. Additionally, the 2004 Individuals with Disabilities Education Improvement Act (IDEIA) regulations require states to establish academic standards for students with disabilities that are comparable to those for their nondisabled peers and for states to annually report progress toward meeting these goals.

The No Child Left Behind (NCLB) Act of 2001 also mandates access to the general education curriculum and accountability for content proficiency for students with disabilities. Under Title I of the No Child Left Behind Act (2001), all states must adopt challenging academic content standards and student academic achievement standards. Title I identifies mathematics as a content area that must be aligned with academic standards. Additionally, states must develop and implement a statewide accountability system that is based on their academic standards and academic assessments and be held accountable for making Adequate Yearly Progress (AYP). AYP includes applying the same high standards of academic achievement to all public elementary and secondary schools. All students need to demonstrate continuous and substantial improvements, including students with disabilities, who are consistently outperformed by their non-disabled peers (NCES, 2009).

In compliance with NCLB, *all* students are assessed in mathematics annually in grades 3-8, and at least once more between grades 10 – 12. These assessments must align with the state's academic content and student academic achievement standards and must assess higher-order thinking skills and understanding. Results from assessments are reported with disaggregation of scores for students with disabilities, therefore, students with disabilities are being held accountable for the general education secondary mathematics content. As such, 62% of secondary students with LD participate in mathematics courses in the general education classroom (Newman, 2006); however, only 12% of students with disabilities participate in advanced mathematics courses such as Algebra I (Kortering, deBettencourt, & Braziel, 2005). Of the students with disabilities participating in advanced mathematics courses, many do not perform well. For example, students with disabilities demonstrate mathematics proficiency at a rate of 50% or greater in only three states as measured by high school assessments (VanGetson & Thurlow, 2007). Further, only 13.6% of students with LD perform on grade level during secondary school (Wagner, et al., 2003).

Certain characteristics of students with LD may impede their algebra performance, including difficulty recalling of math facts (Garnett, 1998; Geary, 2004), poor computations (Calhoon, et al., 2007; Garnett, 1998; Geary, 2004), language deficits (Bley & Thorton, 2001; Garnett, 1998; Ives, 2007), visual spatial deficits (Garnett, 1998; Geary, 2004), difficulty understanding abstract symbols (Bley & Thorton, 2001; Garnett, 1998; Geary, 2004; Witzel, 2005), and poor conceptual understanding of procedures (Geary, 2004). Students with LD may also have difficulties processing information and self-monitoring during problem solving tasks. Additionally, they may experience low motivation, poor self-esteem, and/or passivity in the classroom (Gagnon & Maccini, 2001). To help students with LD meet the algebra requirements necessary for high school graduation and preparation for post-secondary education and occupational opportunities, teachers look to research for effective strategies to successfully instruct these students (Scheuermann, Deshler, & Schumaker, 2009; The Access Center, 2004).

**Organization of the Review of the Literature**

In this chapter, I present a comprehensive review of the current research involving algebra interventions for secondary students with LD. This review serves two purposes: (a) to determine the current status of and the need for effective algebra interventions for secondary students with LD; and (b) to examine empirically-based instructional variables to inform the current study. Studies meeting the following criteria were included in this review: (a) included students identified as LD; (b) examined the effects of an instructional intervention on the performance of secondary students with LD in algebra; (c) used an experimental, quasi-experimental, or single-subject design; and (d) been published in a peer reviewed journal between 1989 and 2009, to reflect the origin of the NCTM standards. An electronic search of the following five databases was conducted; ERIC, Education Research Complete (EBSCO), JSTOR, PsycINFO, and Social Sciences Citations Index. The descriptors "algebra" and "learning disabilities" were used in this search. Initially, 12 articles were identified as meeting all the criteria for inclusion (Allsopp, 1997; Bottge, Heinrichs, Chan, & Serlin, 2001; Bottge, Rueda, LaRoque, Serlin, & Kwon, 2007a; Bottge, Rueda, Serlin, Hung, & Kwon, 2007b; Hutchinson, 1993; Ives, 2007; Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Mayfield & Glenn, 2008; Scheuermann, Deshler, & Schumaker, 2009; Witzel, 2005; Witzel, Mercer, & Miller, 2003). However, two of the articles (Witzel, 2005; Witzel, et al., 2003) reported the same study and the first study (Witzel et al., 2003) will be reported in the current review.

**Overview of Studies**

A total of 11 studies met the criteria for inclusion in this literature review (see Table 1). Out of the total sample of 1009 participants, 262 (26%) were identified as LD. Five of the 10 studies included only students with LD (Bottge, et al., 2007a; Hutchinson, 1993; Ives, 2007; Maccini & Hughes, 2000; Maccini & Ruhl, 2000), while five studies included students with and without disabilities (i.e., Allsopp, 1997; Bottge, et al., 2001; Bottge, et al., 2007b; Mayfield & Glenn, 2008; Witzel, et al., 2003) and two studies (Allsopp, 1997; Witzel, et al., 2003) included students at risk for mathematics failure. Six studies utilized a group design (Allsopp, 1997; Bottge, et al., 2001; Bottge, et al., 2007a; Bottge, et al., 2007b; Ives, 2007; Witzel, et al., 2003) and four studies utilized a single subject design (Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Mayfield & Glenn, 2008; Scheuermann, et al., 2009). Additionally, one study incorporated both a group design and a single subject design (Hutchinson, 1993). The following review of the literature is divided into three major sections: (a) nature of the sample, (b) instructional content and focus, and (c) instructional activities.

Table 1
*Algebra Interventions for Secondary Students with LD*

| Author (year) | Sample and Setting | Instructional Content / Instructional Focus | Research Design/ Intervention | Dependent Variable | Results | M / G |
|---|---|---|---|---|---|---|
| Allsopp (1997) | N = 262; LD = 8; middle school; ages 12 -15<br><br>Inclusive classrooms | division equations and word problems<br><br>Procedural Focus | Pretest/posttest group design<br><br>(a) classwide peer tutoring<br>(b) independent practice | Unit test | No significant difference between groups. | M |
| Bottge, Heinrichs, Chan, & Serlin (2001) | N = 75;<br>LD = 16;<br>grade 8;<br>ages 13 – 15<br><br>1 remedial inclusive class; 3 pre-algebra classes | Linear function, line of best fit, variables, rate of change (slope)<br><br>Conceptual Focus | Pretest/posttest group design<br><br>(a) video-based instruction<br>(b) text-based instruction | Researcher developed problem solving test; WRAT-III arithmetic subtest | (a) = (b) on problem solving measure, greater gains for students in remedial class;<br>(a) = (b) on computation measure for students in pre-algebra classes;<br>Decline in computation scores for students in remedial class receiving (b). | M |
| Bottge, Rueda, LaRoque, Serlin, & Kwon (2007a) | N = 100;<br>LD = 100;<br>grades 6 - 12;<br>ages NS<br><br>Special ed. Classroom | linear function, line of best fit, variables, rate of change (slope)<br><br>Conceptual Focus | Pretest/posttest group design<br><br>(a) video-based instruction<br>(b) typical instruction | Researcher developed problem solving test; ITBS computation and problem solving | (a) > (b) on problem solving measure; effect size was large (1.08) | M |
| Bottge, Rueda, Serlin, Hung, & Kwon, 2007b) | N = 128;<br>LD = 12;<br>grade 7;<br>ages NS<br><br>1 inclusive classroom, 1 pre-algebra classroom, 4 typical classrooms | linear function, line of best fit, variables, rate of change (slope)<br><br>Conceptual Focus | Quasi-experimental group design<br><br>(a) video-based instruction | Researcher developed problem-solving test | All students demonstrated improvements with students with LD demonstrating larger improvements on algebraic tasks.<br>No difference between students with LD and without LD on maintenance. | M |

| Author (year) | Sample and Setting | Instructional Content / Instructional Focus | Research Design/ Intervention | Dependent Variable | Results | M / G |
|---|---|---|---|---|---|---|
| Hutchinson (1993) | N = 20;<br>LD = 20;<br>grades 8 - 10;<br>ages 12 – 15<br><br>Special ed. classroom (individual instruction) | word problems involving equations in one and two variables<br><br>Conceptual + Procedural Focus | Multiple baseline across participants; Pretest/ posttest group design<br><br>(a) strategy instruction<br>(b) regular instruction | Assessment measures, multiple choice and problem solving tests, metacognitive interview, think alouds | (a) > (b) on representation and solution posttests, metacognitive interview; medium to large effect sizes (0.52 – 10.34) | M, G |
| Ives (2007) | N = 40;<br>LD and/or ADHD = 40;<br>grades 7 – 12;<br>ages 13 – 19<br><br>Special ed. Classroom | linear systems of equations in two variables (Study 1) and three variables (Study 2)<br><br>Procedural Focus | Two group comparison<br><br>(a) graphic organizer<br>(b) regular instruction | Researcher developed test, teacher-generated test | (a) > (b) in Study 1 and Study 2 posttests<br>(a) > (b) in Study 1 maintenance test; large effect sizes for both studies | M |
| Maccini & Hughes (2000) | N = 6; LD = 6; grades 9 – 12; ages 14 – 18<br><br>Individualized instruction | word problems involving addition, subtraction, multiplication, and division of integers<br><br>Conceptual + Procedural Focus | Multiple-probe single subject design across participants<br><br>instructional package: strategy instruction + CSA instructional sequence | Researcher develop problem representation and problem solution measures | Significant improvements in problem representation and problem solution across participants and problem types; Large effect sizes (PND = 90% for representation; PND = 72% for solution) | M, G |
| Maccini & Ruhl (2000) | N = 3; LD = 3; grade 8; ages 14 – 15<br><br>Individualized instruction | word problems involving subtraction of integers<br><br>Conceptual + Procedural Focus | Multiple-probe single subject design across participants<br><br>instructional package: strategy instruction + CSA instructional sequence | Researcher developed problem representation and problem solution measures | Significant improvements in problem representation and problem solution across participants; Large effect size (PND = 67% for representation; PND = 94% for solution) | M,G |

| Author (year) | Sample and Setting | Instructional Content / Instructional Focus | Research Design/ Intervention | Dependent Variable | Results | M / G |
|---|---|---|---|---|---|---|
| Mayfield & Glenn (2008) | N = 3; LD = 2; grades 4 – 8; ages 9 – 14<br><br>Residential | problem solving involving target skills of variables, exponents, and linear equations<br><br>Problem Solving Focus | Single subject design across skills replicated across three participants<br><br>(a) cumulative practice<br>(b) tiered feedback<br>(c) feedback + solution sequence<br>(d) review<br>(e) transfer training | Researcher developed target skills tests, problem solving test | Limited increases in accuracy after (a) and (c). Consistent improvements in all participants after (e) with large effect size (PND = 96%) | -- |
| Scheuermann, Deshler, & Schumaker (2009) | N=14; LD =14; Middle school; ages 11-14<br><br>Special ed. classroom | One-variable equations<br><br>Conceptual + Procedural Focus | Multiple probe across students<br><br>Explicit Inquiry Routine = content sequencing, scaffold inquiry, modes of representation | Researcher-developed word problem test and concrete manipulation test; KeyMath revised | Mean of 95% accuracy on final instructional word problem probe; large effect size (PND = 93%);<br><br>Mean of 88.6% accuracy on manipulation<br><br>Significant improvement on KeyMath (effect size .54) | M,G |
| Witzel, Mercer, & Miller (2003) | N = 358;<br>LD = 41;<br>grades 6 – 7;<br>ages NS<br><br>Inclusive classroom | Linear equations<br><br>Conceptual + Procedural Focus | Pretest/posttest/ follow-up design with random assignment of clusters<br><br>(a) CRA instruction<br>(b) abstract instruction | Researcher-developed assessment | (a) > (b); Effect size was large (0.97) | M |

N = total number of participants; LD = number of participants identified with a learning disability; NS = not specified; CSA= concrete-semiconcrete-abstract; CRA = concrete-representational-abstract; PND = percentage of nonoverlapping data; WRAT-III = Wide Range Achievement Test, 3rd edition; ITBS = Iowa Test of Basic Skills

**Nature of the Sample**

In this section, I review the literature for participant descriptions, including identification criteria for LD status, gender, demographic information, age, grade level, and setting. These variables were chosen based on previous analyses of mathematics interventions for students with disabilities (Mulcahy, 2007; Templeton, Neel, & Blood, 2008).

**Identification criteria of LD status.** Six (55%) studies (Allsopp, 1997; Bottge, et al., 2001; Ives, 2007; Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Mayfield & Glenn, 2008) did not report specific criteria for identifying students with LD, although two of these studies (Maccini & Hughes, 2000; Maccini & Ruhl, 2000) stated that students met state criteria. The remaining five studies (Bottge, et al., 2007a; Bottge, et al., 2007b; Hutchinson, 1993; Scheuermann, et al., 2009; Witzel, et al., 2003) identified students as LD based on a discrepancy between intellectual ability and academic achievement.

**Gender.** Based on reported gender data within the total sample size, 354 (54%) of the participants were males and 297 (46%) were females. One study (Witzel, et al., 2003) did not include gender data. Six studies (Hutchinson, 1997; Ives, 2007; Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Mayfield & Glenn, 2008; Scheuermann, et al., 2009) identified gender specific to students with LD. Those samples included a total of 56 (66%) males with LD and 29 (34%) females with LD. Authors of the remaining studies did not identify the gender of students with LD.

**Demographic data.** Authors of six studies (Allsopp, 1997; Bottge, et al., 2007a; Ives, 2007; Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Scheuermann, et al., 2009) reported the race and/or ethnicity of participants. The combined number of students in these studies included 311 (77%) White students, 71 (17%) Black students, 18 (4%) Hispanic students, 3 (<1%) Native American students, and 3 (<1%) Asian American students. The remaining five studies (Bottge, et al., 2001; Bottge, et al., 2007b; Hutchinson, 1997; Mayfield & Glenn, 2008; Witzel, et al., 2003) did not identify the race and/or ethnicity of the participants. Nine of the studies (Allsopp, 1997; Bottge, et al., 2001; Bottge, et al., 2007b, Hutchinson 1993; Ives, 2007; Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Scheuermann et al., 2009; Witzel, et al., 2003) provided information regarding geographic location with representation from the southeastern, Midwestern, and mideastern locations of the U.S., as well as Canada. Socioeconomic status of participants was identified in only four studies (Allsopp, 1997; Bottge, et al., 2007a; Ives, 2007; Maccini & Hughes, 2000). Two studies (Allsopp, 1997; Bottge, et al., 2007a) identified 36% of the combined participants as receiving Free and Reduced Meals (FARMS), while Maccini and Hughes identified one student's socioeconomic status as "below average." Additionally, Ives measured socioeconomic status by the highest educational degree completed by a parent and reported 85% of participants had a parent with a bachelor's degree or higher, which would indicate a high socioeconomic position.

**Age, grade level, and setting.** Students' age was reported in seven studies and ranged from 12 years to 19 years with one outlier of age 9. Six studies (55%) took place in middle schools, one study (9%) in a high school, and three studies (27%) in both middle and high schools. Settings included inclusive general education classrooms (n=4), special education classrooms (n=4), and a residential setting (n=1). Additionally, three studies (Hutchinson, 1993; Maccini & Hughes, 2000; Maccini & Ruhl, 2000) were conducted in a resource room setting utilizing individualized instruction.

**Summary: Nature of sample.** The majority of the current research on algebra interventions for students with LD has been conducted on white males in middle school special education settings. However, almost four times the number of studies in the present review examined the effects of the intervention within the general education classroom, as compared to an earlier review conducted by Maccini, et al. (1999). Interventions that demonstrated positives effects within this environment are important as more students with LD are participating in general education classrooms. Future research should include a more diverse sample to include more female students with LD, minority students, and students from a variety of socioeconomic backgrounds. Data on eighth grade students from the most recent NAEP (National Center for Educational Statistics, 2009) exemplifies the need to address a diverse group of students. Although there is only a two point gender gap favoring males, substantial achievement gaps exists among racial and ethnic groups, with White students outperforming Black students, Hispanic students, and students from Native American and Native Alaskan descent (percentage of students at or above proficiency equaling 43%, 12%, 17%, and 20%, respectively). Additionally, students from impoverished socioeconomic backgrounds, as measured by eligibility for FARMS, are outperformed by students who are not eligible (percentage of students at or above proficiency equaling 17% and 45%, respectively).

The majority of the current research does not report specific identification criteria for students with LD. For the five studies reporting criteria, students were identified as having LD based on a discrepancy between intellectual ability and academic achievement. This is problematic for four reasons. First, intellectual ability and academic achievement are not independent of each other (Fuchs, Fuchs, Compton, & Bryant, 2005). Second, each study utilizes differing criteria to establish a discrepancy (i.e., 1.5 standard deviations, 1.75 standard errors, three years below on standard achievement test, severe discrepancy). Third, discrepancy-based identification of students with LD is often an inaccurate measure as students who meet this criterion may not have a specific LD, while students who do not meet this criterion may have LD (Mazzocco, 2007). Finally, as suggested by the 2004 reauthorization of IDEA, special education policy is moving away from the discrepancy model and toward a Response to Intervention (RtI) model, in an effort to address the rising number of LD diagnoses, particularly of minority students (Hollenbeck, 2007; Mellard, 2004).

Typically, RtI involves three-tiers of instruction: a) general classroom instruction, b) specialized small group instruction for students not responding to classroom instruction, and c) intensive individualized instruction for students who do not respond to small group instruction (Fuch, et al., 2005). This process holds the general education environment accountable for the success of all students, as only children who do not respond to research-based instruction through these tiers may be diagnosed as LD. Only through the RtI process can ineffective teaching be eliminated as a factor in determining a specific learning disability, thus resulting in accurate diagnoses (Hollenbeck, 2007). Therefore, future research on algebra interventions for students with LD should include students who have been identified as LD through the RtI process.

**Instructional Content and Focus**

In this section, interventions are identified by instructional content and focus of algebra abilities (see Table 1). Instructional content refers to the algebra skills and concepts that interventions intend to improve. The instructional focus refers to three abilities (i.e., conceptual knowledge, procedural fluency, and problem solving) necessary for proficiency in algebra (Hudson & Miller, 2006; NCES, 2009; NMAP, 2008; NRC, 2001).

**Instructional content.** Similar to the review of algebra interventions for secondary students with LD conducted by Maccini et al. (1999), the majority of the interventions (91%) in the current review address basic algebra content, including a) integers (Maccini & Hughes, 2000; Maccini & Ruhl, 2000), b) equations in one variable (Allsopp, 1997; Hutchinson, 1993; Mayfield & Glenn, 2008; Scheuermann, et al., 2009; Witzel, et al., 2003), and c) multiple areas, including variables, slope, and linear functions (Bottge, et al., 2001; Bottge, et al., 2007a; Bottge, et al., 2007b). Only one study (Ives, 2007) focused on complex algebra content (systems of equations in two and three variables), while an additional study (Hutchinson, 1993) expanded the intervention to include equations in two variables.

**Instructional focus**. In addition to examining instructional content, algebra interventions can be identified by instructional focus. In this review, the instructional focus refers to three abilities (i.e., conceptual knowledge, procedural fluency, and problem solving) necessary for proficiency in algebra (Hudson & Miller, 2006; NMAP, 2008; NRC, 2001). Conceptual knowledge involves an integrated and functional understanding of mathematical ideas (NRC, 2001) and allows students to develop a meaningful understanding of abstract mathematical objects (Hudson & Miller, 2006). The NRC refers to procedural fluency as the flexible, accurate, and efficient use of mathematical procedures (i.e., regrouping during computations of whole numbers, data analysis, simplifying algebraic expressions). Lastly, problem solving refers to the process of applying previously learned concepts and skills to novel situations (NCTM, 2000). Although problem solving has been limited in focus to solving word problems (Hudson & Miller, 2006), experts in math education (Hudson & Miller, 2006; NCTM, 2000; NRC, 2001) recommend a broader definition to include solving word problems as well as problems that involve abstract notation only.

Although these three mathematical abilities are defined separately, they need to develop simultaneously to ensure proficiency in algebra (Hudson & Miller, 2006; NMAP, 2008). A certain degree of procedural knowledge is necessary to develop conceptual understanding, as procedures strengthen conceptual development (NRC, 2001). Additionally, conceptual knowledge goes beyond memorization of facts and procedures to provide students with an in-depth understanding of a mathematic idea (Hudson & Miller, 2006; NRC, 2001), while supporting retention of procedures (NRC, 2001). Both procedural fluency and conceptual understanding are necessary to establish a network of skills and concepts to link new material in the development of problem solving abilities (NRC, 2001). In the section below, studies are categorized by the type of ability addressed. As previously stated, these abilities are intertwined; therefore, the studies are grouped based on the prominent features of the interventions.

***Procedural fluency.*** Authors of two studies (Allsopp, 1997; Ives, 2007) focused on developing procedural fluency. Allsopp utilized peer tutoring to develop students' abilities to solve division equations. In the tutoring situation, the tutor used an answer key to determine if the tutee's solution was correct. In Ives' study, students were taught a procedure for solving systems of equations using a graphic organizer.

***Conceptual knowledge.*** Authors of three students (Bottge, et al., 2001; Bottge, et al., 2007a; Bottge, et al., 2007b) focused on developing conceptual understanding. Bottge and his colleagues focused on developing understanding of algebra concepts such as variables, linear functions, and slope, using a videodisc anchor, hands-on activities, and group discussions. For example, in order to predict the speed of a car at the end of a stratightaway when released from various heights on a ramp, students timed cars on a video. Students then graphed the heights and times and used the graphs to make their predictions.

***Conceptual knowledge and procedural fluency.*** Five studies (Hutchinson, 1993; Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Scheuermann, et al., 2009; Witzel, et al., 2003) incorporated instructional features that developed both conceptual knowledge and procedural fluency. Four of these studies (Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Scheuermann, et al., 2009; Witzel, et al., 2003) incorporated the use of the CRA sequence to represent and solve problems involving integers and linear equations in one variable. The CRA sequence represents algebraic ideas in a variety of ways (i.e., concrete objects, pictorial drawings, abstract notation). Connections between representations allow students to develop conceptual understanding (NRC, 2001). Maccini and colleagues also utilized strategy instruction (i.e., a mnemonic) to develop accurate representations and solutions of contextualized problems involving integers. Through the use of a cue card, Hutchinson taught students to solve various types of word problems involving equations, by representing the problem structure. Lastly, Scheurermann and colleagues investigated the effects of the Explicit Inquiry Routine (EIR), which incorporated explicit content sequencing, scaffolded inquiry, and systematic use of various modes of illustration, using concrete manipulatives, pictorial representations, and abstract notation.

***Problem solving.*** Mayfield and Glenn (2008) used explicit instruction (EI) to teach target skills involving variables, exponents, and linear equations, then examined the effects of various forms of practice and review on the students' ability to solve novel problems incorporating at least two of the target skills. Although this was the only study that focused exclusively on problem solving as defined in this review, four additional studies (Hutchinson, 1993; Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Scheuermann, et al., 2009) included a generalization or transfer measure, which assessed participants' ability to solve problems that they were not directly exposed to during the intervention (Gersten, Chard, Jayanthi, Baker, Morphy, & Flojo, 2009b). Although not the primary focus of the studies, these measures represented the ability to problem solve.

**Summary of instructional content and focus.** The studies in this review included a narrow range of algebra content with most of the studies focusing on basic algebra concepts and skills, including integers, variables, exponents, linear equations/functions (see Table 1). Basic algebra content is appropriate for the majority of participants, as they are in middle school classrooms (82% of participants without LD, 69% of participants with LD) and not participating in a formal algebra course. However, future research should include high school students who are participating in formal Algebra I and/or Algebra II courses.

Five studies (Hutchinson, 1993; Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Scheuermann, et al., 2009; Witzel, et. al, 2003) focused on a combination of conceptual knowledge and procedural fluency and authors of four studies (Hutchinson, 1993; Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Scheuermann, et al., 2009) incorporated all three abilities (i.e., conceptual knowledge, procedural fluency, and problem solving) necessary for proficiency in algebra (Hudson & Miller, 2006; NCES, 2009; NMAP, 2008; NRC, 2001). Additionally, studies that targeted all three algebra abilities incorporated a single-subject design, while the majority of studies that incorporated a group design focused on either conceptual understanding or procedural fluency. The use of single subject designs to investigate interventions focusing on the conceptual and procedural development of complex algebra content may provide a springboard for future studies incorporating experimental or quasi-experiment group design.

**Instructional Activities**

This section reviews studies based on instructional practices and instructional materials (see Table 2). The organization of this section is adopted from a current analysis of math interventions for students with LD (Gersten, et al., 2009b). For each instructional activity, a definition is provided followed by a review of the current studies incorporating these activities. The section concludes with a discussion of the potential benefits of the instructional activities to assist students with LD in accessing the algebra content.

Table 2

*Algebra Interventions by Practice, Method of Delivery, and Materials Used*

| Author (year) | Practice | Method of Delivery | Materials |
|---|---|---|---|
| Allsopp (1997) | Peer-assisted | Student-centered | prompt card |
| Bottge, Heinrichs, Chan, & Serlin (2001) | Enhanced Anchored Instructions | Student-centered | Videodisc, model cars, model ramp |
| Bottge, Rueda, LaRoque, Serlin, & Kwon (2007a) | Enhanced Anchored Instructions | Student-centered | Videodisc, model cars, model ramp |
| Bottge, Rueda, Serlin, Hung, & Kwon, 2007b) | Enhanced Anchored Instructions | Student-centered | Videodisc, model cars, model ramp |
| Hutchinson (1993) | Strategy Instruction | Teacher-directed | Self-questioning prompt card, structured worksheet |
| Ives (2007) | Visual Representation | Teacher-directed | Graphic organizer |
| Maccini & Hughes (2000) | Explicit instruction+ Strategy instruction + visual representation + sequencing of examples | Teacher-directed | Manipulatives (Algebra Lab Gear), structured worksheet |
| Maccini & Ruhl (2000) | Explicit instruction+ Strategy instruction + visual representation + sequencing of examples (CRA) | Teacher-directed | Manipulatives (Algebra Lab Gear), structured worksheet |
| Mayfield & Glenn (2008) | Explicit instruction | Teacher-directed | Prompt card, structured worksheet |
| Scheuermann, Deshler, & Schumaker (2009) | Sequencing of examples (CRA + range) + visual representation | Teacher-directed | Manipulatives (beans, buttons, unifix cubes) |
| Witzel, Mercer, & Miller (2003) | Sequencing of example (CRA) + Visual representation | Teacher-directed | Manipulatives (algebra tiles) |

**Instructional practices.** Instructional practices refer to the methods of instruction that promote student access to the algebra content. The type of instructional practice is categorized as method of delivery (i.e. teacher-directed or student-centered), sequence and range of examples, strategy instruction (SI), visual representations, peer-assisted math instruction, and Enhanced Anchored Instruction (EAI).

***Method of delivery.*** Teacher-directed instruction occurs when the teacher is primarily communicating the mathematics to students; however, this varies greatly, ranging from scripted lessons to interactive lessons (NMAP, 2008). Authors of seven studies (Hutchinson, 1993; Ives, 2007; Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Mayfield & Glenn, 2008; Scheuermann, et al., 2009; Witzel, et al., 2003) used teacher-directed instruction. In one study (Hutchinson, 1993), the author referred to using scripted lessons, while the authors of the remaining studies incorporate elements of explicit instruction (EI).

EI is a specific type of teacher-directed instructional method that incorporates the following components: an advanced organizer, teacher demonstration, guided practice, independent practice, progress monitoring with corrective feedback, and distributed reviews for maintenance (Hudson & Miller, 2006). EI is a systematic approach to teaching that involves a step-by-step plan for solving the problem (Gersten, et al., 2009b). Authors of three studies (Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Mayfield & Glenn, 2008) incorporated EI in their instructional intervention package.

Maccini and colleagues investigated the effects of an instructional package including EI on the problem representation and problem solution of contextualized word problems involving integers. Specifically, each lesson consisted of six EI components: (a) advance organizer, (b) teacher demonstration,(c) guided practice, (d) independent practice, (e) posttest, and (f) feedback. Both studies utilized a single subject design across subjects with six high school students with LD (Maccini & Hughes, 2000) and three eighth-grade students with LD (Maccini & Ruhl, 2000). In both of these studies, students reached criterion level of 80% accuracy or greater on two consecutive probes.

However, it is unclear the extent to which EI is responsible for the academic improvements in these studies, as EI was a component of the instructional package.

Mayfield and Glenn (2008) investigated the effect of five types of review and practice on students' performance with multiplying and dividing variables with coefficients and exponents and solving linear equations. The purpose of this study was to determine which of the intervention phases (cumulative practice, tiered feedback, feedback plus solution sequence instruction, review practice, and transfer training) helped students reach criterion on the problem solving tasks that embedded the target skills. A single subject design across skills replicated across three participants was used to examine the effects of the five intervention phases. Two middle school students with LD enrolled in special education classes and one fourth-grade student enrolled in general education classes participated in the intervention within a residential setting.

Initially, the participants received target skill training via EI. All participants reached criterion of 100% on three nonconsecutive tests on all target skills. Researchers then implemented the intervention phases and determined that transfer training produced the most significant effects on students' problem solving performance with participants meeting criterion of 100% accuracy on three nonconsecutive problem solving tests. Transfer training involved presenting each step of the problem solving task as an individual target skill prior to presenting the original problem solving task in its entirety. No modeling or feedback was provided and prompts were added and faded as needed. Future research should include the use of transfer training on various multi-step algebra problems, such as multiplying and factoring polynomials.

In contrast to teacher-directed instruction, student-centered instruction occurs when students are primarily responsible for their learning (NMAP, 2008) with little or no guidance from the teacher (Kirschner, Sweller, & Clark, 2006; Mayer, 2004; NMAP, 2008). Authors of four studies (Allsopp, 1997; Bottge, et al., 2001; Bottge, et al., 2007a; Bottge, et al., 2007b) in the current review used student-centered instruction. In one study (Allsopp, 1997), the author systematically arranged peer tutoring pairs and provided the tutor with an answer key to guide instruction provided by the tutor. Additional participants engaged in independent practice. No significant differences in posttest scores were found between students who participated in peer tutoring and students who engaged in independent practice.

Bottge and colleagues incorporated student-centered learning in their instructional intervention entitled Enhance Anchored Instruction (EAI). Students worked in groups to solve problems based on a videodisc anchor. Two studies (Bottge, et al., 2001; Bottge, et al., 2007b) were conducted in general education classrooms with students with LD and their non-disabled peers. The authors noted that at times the group process collapsed and students with LD merely copied the work of more mathematically able students. Additionally, one study (Bottge, et al., 2007a) took place in a special education classroom where authors observed students with LD struggling to learn key concepts in student groups (i.e., time-speed-distance relationship, plotting variables). Therefore, teachers provided explicit instruction on key concepts needed to complete the hands-on activities. Bottge and colleagues' work appears to support the NMAP (2008) finding that high-quality research in mathematics does not support the exclusive use of either student-centered or teacher-directed instruction.

Teacher directed instruction, such as EI, has proven to be an effective mathematics intervention for students with LD (Hudson & Miller, 2006; Maccini & Hughes, 1997; Maccini, et al., 2008) and their nondisabled peers (Kirschner, et. al., 2006; Mayer, 2004). Specifically, EI is beneficial for students with LD as it compensates for deficits in retention and recall through the review of prerequisite skills, multiple practice opportunities, and cumulative reviews. The effect of student-centered instruction on the performance of students with LD is uncertain. For example, after receiving EAI, students in Bottge's studies improved their performance on problem solving posttests, but performed worse on computation posttests. Additionally, Allsopp (1997) reported similar improvements on posttest measures for both the treatment group (i.e., CWPT) and the comparison group (i.e., independent practice).

Appropriate scaffolds may be necessary to help students with LD access the curriculum during student-centered instruction. Bottge and colleagues incorporated student-centered instruction within EAI which developed the problem-solving skills of students with LD. However, computational skills frequently declined from pretest to posttest, despite the inclusion of formal, teacher-directed instruction on targeted skills. To address this area of concern, Bottge, Rueda, Grant, Stephens, and Laroque (2010) investigated the effects of providing students with math LD formal instruction (i.e., teacher-directed instruction) on targeted computational skills involving fractions prior to student participation in problem solving tasks embedded within EAI that required. Bottge and colleague found that students who received this formal instruction made greater gains in computations involving fractions as well as gains in problem solving compared to students who only received EAI. Future research is needed to determine if formal instruction of computational and/or procedural skills followed by engagement in student-centered problem solving tasks produces greater achievements for students with LD than only student-centered activities in other mathematics content areas, such as algebra. This is a critical area of research, as more students with LD are placed in the general education classroom, which is likely to incorporate student-centered instruction (Woodward & Montague, 2002).

***Sequence and range of examples.*** To be included in this category, interventions needed to include: (a) a specified sequence or pattern of examples, such the CRA sequence; and/or (b) a systematic variation in the range of examples, such as teaching only proper fractions followed by improper fractions versus teaching proper and improper fractions simultaneously (Gersten, et al., 2009b). Four studies investigated the effectiveness of the CRA sequence as part of an instructional package to improve students' ability to solve problems involving integers (Maccini & Hughes, 2000; Maccini & Ruhl, 2000) and to solve linear equations (Scheuermann, et al., 2009; Witzel, et al., 2003). The sequence of examples included physical objects (i.e., algebra tiles, buttons, unifix cubes), pictorial representations (i.e., drawings of algebra tiles, tallies, dots, pictures), and abstract notation. Additionally, one study (Scheuermann, et al., 2009) included the systematic planning of the range of examples (i.e., introducing one-step equations prior to teaching two-step equations).

Maccini and colleagues investigated the effects of an instructional package on the representation and solution of contextualized word problems involving integers. The authors incorporated the CRA instructional sequence and elements of EI and strategy instruction to address the target areas. In the concrete phase of instruction, students used

Algebra Lab Gear (Picciotto, 1990) to represent and solve problems. In the semi-concrete phase, students drew picture representations of the algebra tiles. After using concrete manipulatives and representational drawings, students were able to abstractly solve similar problems. In both studies, students reached criterion level of 80% accuracy or greater on two consecutive probes.

Similar to Maccini and colleagues, Scheuermann, et al. (2009) used a multiple baseline design across 14 middle school students with LD to investigate the effects of an instructional package entitled Explicit Inquiry Routine (EIR) on the performance of students' representation and solution of one-variable equations embedded in word problems. EIR consists of three components: a) explicit sequencing, b) scaffold inquiry, and c) various modes of illustration. Within explicit sequencing, the essential concept or process (e.g. solving equations in one variable) was broken down into instructional bites that were taught in a predetermined sequence, from simple to more complex (e.g., simple equations consisting of adding a constant to complex equations consisting of multiplying by a constant and adding and/or subtracting two constants). Scaffold inquiry provided opportunities for students to verbalize and illustrate how to represent and solve problems to the teacher, to their classmate, and to themselves. Lastly, various modes of illustration incorporated the CRA instructional sequence. After instruction, all participants demonstrated substantial gains with a mean of 95% across all probes. Additionally, students scored significantly better on a standardized assessment (KeyMath-Revised) posttest. EIR incorporated both a sequencing pattern of examples through the CRA sequence and systematic variation in the range of examples through the introduction of simple to more complex equations. These components provided students will LD multiple representations of the concept and a foundation of previous knowledge to build upon as they progress to more complex problems.

In addition, Witzel et al. (2003) investigated the impact of the CRA sequence on student performance in solving equations in one variable. Using a pretest/posttest/follow-up design with random assignment of classrooms including sixth and seventh graders with LD, Witzel and colleagues compared the performance gains of students receiving instruction utilizing the CRA instructional strategy to students receiving instruction via abstract notation only. During CRA instruction, students first solved equations by using physical manipulatives. After successfully solving the equations using the manipulatives, students drew pictures of the manipulatives to aid in the solution process, and then solved equations using abstract notation only. The CRA group significantly outperformed the group receiving instruction in abstract notation only. However, future research is needed given that neither group performed to mastery level, indicated by low mean scores on post-test measures (27% and 11%, respectfully).

The CRA sequence holds promise as an effective instructional practice as physical and pictorial representations scaffold students' learning of abstract concepts, which are often challenging for students with LD to understand (Bley & Thorton, 2001; Garnett, 1998; Geary, 2004; Witzel, 2005). Additionally, a systematic variation in the range of examples builds foundational skills for more complex problems and supports generalization to novel situations, which is critical area of need for students with LD (Bley & Thornton, 2001; Bryant, et al., 2000; Gagnon & Maccini, 2001).

***Strategy instruction.*** SI includes the use of memory aids (i.e., mnemonics, cue cards) and graphic organizers (i.e., graphs and charts) that provide students with a strategic plan to solving problems (Gersten, et al., 2009b; Maccini, Strickland, Gagnon, & Malmgren, 2008). The use of a strategy involves a general approach to solving a wide range of problems (Gersten, et al., 2009b) and has been found to be effective for students with LD in a wide range of educational settings, including general education classrooms and alternative settings (Maccini, et al., 2008). In the current review, authors of three studies (Hutchinson, 1993; Maccini & Hughes, 2000; Maccini & Ruhl, 2000) investigated the effectiveness of SI.

Hutchinson (1993) investigated the effectiveness of SI for students with LD in grades eight through ten with relational and proportional word problems in one and two variables. The author used a mixed methods research design including two research methodologies: (a) a single subject multiple baseline design across 12 students with LD; and (b) a pretest/posttest comparison group design including the 12 students from the single subject design and an additional eight students with LD as a comparison group. Students who received SI were provided a self-questioning prompt card and a structured worksheet to assist in representing and solving a variety of word problems (i.e., relational, proportional) involving equations in one and two variables. The representation of problems focused on structural elements of relational (i.e., one unknown quantity in terms of its relationship to another) and proportional (i.e., ratios) problems. The author modeled how to use the self-questions to represent and solve word problems through thinking alouds (i.e., reading and answering questions aloud). All students in the SI group reached criterion (80% accuracy on three consecutive assessments), although for differing instructional phases, as students progressed at their own pace. Additionally, students in the SI group significantly outperformed students in the treatment group on the posttest.

Similarly to Hutchinson (1993), Maccini and colleagues incorporated SI in their instructional package to assist students with LD in the problem representation and solution of word problems containing integers. The authors incorporated the use of a mnemonic strategy, STAR, which cued students to *Search, Translate, Answer,* and *Review* when representing and solving word problems involving integers for each phase of the CRA sequence. In the *search* phase, students were prompted to read the problem carefully and write down what is known and what needs to be found out. Next, students *translated* the words into an algebraic equation by choosing a variable, identifying the operation(s), and representing the problem via manipulatives, pictures, or algebraic notation. To *answer* the problem, students used the manipulatives or followed rules for the necessary symbolic manipulation. Lastly, students *reviewed* the problem by rereading it, asking if the answer made sense, and checking the answer for accuracy. As a component of the instructional package including EI, the CRA sequence, and SI, participants demonstrated significant improvements in their ability to represent and solve contextualized problems involving integers.

SI is a promising instructional practice that supports metacognitive processes (i.e., self-regulation, strategic planning, monitoring, and evaluating a learning task) that are often deficient in students with LD (Bley & Thornton, 2001; Vaidya, 1999). In the studies above, students were able to plan, monitor, and evaluate their progress in the representation and solution of algebraic word problems while using a mnemonic device (i.e., STAR) or self-questioning worksheet. Future research should include the use of SI to support students' metacognition during complex algebra content.

***Visual representations.*** In this review visual representations include the use of graphic organizers (GO), such as diagrams and charts, that depict the relationship between facts or ideas within a learning task (Hall & Strangman, 2002) to clarify ideas, support reasoning, and build understanding (NRC, 2001). One article (Ives, 2007) in the present review explored the use of a GO as a tool for solving systems of linear equations.

Ives (2007) reported the results of two related studies that utilized a two-group comparison experimental design. Participants included a total of 40 middle and high school students with LD who attended a private school for students with LD and attention disorders. The first study consisted of 14 participants in the treatment group and 16 participants in the control group and addressed the effects of the GO on the solution of systems of two linear equations in two variables. The researcher found no significant difference in solving for the solution of systems of equations between the treatment and comparison groups, perhaps due to students' inconsistent use of the GO. However, both the treatment and the comparison groups performed poorly on the posttest with approximately 40% accuracy. Ten high school students with LD participated in the second study, which extended use of the GO to the solution of three linear equations in three variables. Participants in the treatment group demonstrated greater gains, significantly outperforming the comparison group, with average accuracy of approximately 61% to 42%, respectively.

The use of the GO in Ives' (2007) study is a promising instructional practice for students with LD who have language deficits. The use of the GO emphasized a nonverbal approach to teaching systems of linear equations that relied on visual-spatial skills rather than language skills. This instructional practice reduced the emphasis on language while accessing a higher-level algebra skill. Future research should include various forms of graphic organizers that access additional higher-level algebra content, such as the use of expansion boxes for multiplying polynomials.

***Peer-assisted math instruction.*** Peer assisted instruction, or peer tutoring, involves a student, under the supervision of the teacher, assisting a peer to learn a skill or concept. These student partnerships occur during structured math study sessions, after receiving instruction from the teacher. Peer tutoring can involve both cross-age tutoring, in which an older student tutors a younger student, within-classroom tutoring, which involves a higher performing student tutoring a lower performing student (Gersten, et al., 2009b; The Access Center, 2004). One study (Allsopp, 1997) in the present review examined the effects of Classwide Peer Tutoring (CWPT), using within classroom tutoring.

Using a pretest/posttest experimental design, Allsopp (1997) examined the effects of Classwide Peer Tutoring (CWPT) verses individual student practice on beginning algebra problem-solving skills (i.e. division equations and word problems). Students in both the CWPT treatment group and the comparison group received 12 instructional lessons consisting of direct instruction, mnemonics, and the use of concrete manipulatives. After direct instruction of the skill or concept by the teacher, students in the CWPT group worked with a partner to practice while students in the treatment group completed worksheets independently for practice. Within the CWPT classrooms, students were divided into four quarters based on their class grade. A ranking procedure was used to control for large differences in student ability levels within tutoring pairs. Students from the top quarter were paired with students from the second quarter, while students from the lowest quarter were paired with students from the third quarter. In each pair, students took a turn being the tutor and being the tutee. The researcher determined that both groups demonstrated higher mean scores from pretest to posttest and pretest to the maintenance measure with no significant difference between the CWPT treatment group and the comparison group.

Although CWPT did not produce an increase in performance over independent practice, students demonstrated equal improvements and indicated a favorable opinion of the CWPT. Peer tutoring may provide other benefits in the algebra classroom, such as an opportunity for students to engage in the NCTM (2000) process standards of communication and reasoning. Additionally, tutoring partnerships may promote active engagement of students with LD, who have become passive learners (Gagnon & Maccini, 2001). Overall, future research is needed to determine the effectiveness of peer tutoring on the algebra performance for students with LD.

***Enhanced Anchored Instruction.*** Enhanced Anchored Instruction (EAI) is a video-based instructional program aimed at developing the computation and problem solving skills for students with LD through authentic contexts. After watching a video portraying a real life situation, students solve related problems. For example, after watching a videodisc entitled *Fraction of a Cost*, students determined the cost of materials needed to build a skateboard ramp. Students may view the video as many times as necessary, scanning the video to find the pertinent information. Three studies in the present review (Bottge, et al., 2001; Bottge, et al., 2007a; Bottge, et al., 2007b) investigated the use of EAI.

Using quasi-experimental group designs, Bottge and colleagues (Bottge, et al., 2001; Bottge, et al., 2007a; Bottge, et al., 2007b) examined the effects of EAI on students' algebra performance with linear functions, lines of best fit, variables, and slope. EAI involved the use of video based problems that are solved through hands-on activities within student groups. Participants in the EAI groups watched a videodisc entitled *Kim's Komet* about two girls entering a model car soapbox derby. Participants solved problems from the video, such as determining the fastest model car and constructing a graph to predict the speed of a car at the end of a straightaway when released from any height on a soapbox derby ramp. Additionally, participants built model cars to release on a ramp and solved additional problems similar to those on the video. In contrast, participants in the comparison groups solved a variety of standard textbook word problems involving distance, rate, and time, as well as completing tables and graphing the information. The researchers in all three studies determined that students with disabilities improved their problem solving skills when provided EAI, although outcomes on computational skills were mixed, with students frequently performing worse on computational posttests. Further, the performance of students with LD receiving EAI in an inclusive classroom matched or exceeded the performance of their non-disabled peers on problem solving measures (Bottge, et al., 2001; Bottge, et al., 2007b).

The NCTM (2000) recognizes technology (i.e., videos, calculators, and computers) as an essential and influential principle for school mathematics that can help students *learn* and *do* mathematics. Technology enables students to conceptually *learn* mathematics by providing multiple representations. Additionally, technology, such as calculators, enables students to *do* computations and procedures that may be laborious without the use of technology (NCTM, 2000). EAI, which included video anchors, positively affected students' problem solving abilities, although additional research is needed to determine the effects of EAI on students' computation skills.

**Instructional materials.** Instructional materials are equipment used in the classroom to support construction of mathematical ideas (Reys, Suydam, & Lindquist, 1992). The use of materials in the mathematics classrooms has led to increased student achievement in mathematics (Bley & Thornton, 2001; Hudson & Miller, 2006; Reys, Suydam, & Lindquist, 1992). Instructional materials included in this review are manipulatives, prompt cards and instructional worksheets, and graphic organizers.

***Manipulatives.*** Manipulatives are physical objects that support mathematical thinking (NRC, 2001) and include any physical object that represents a mathematic concept. Examples include counters, beads, blocks, fraction bars, pattern blocks, Cuisenaire rods, algebra tiles, and geoboards (Maccini, et al., 2008).

In the current review, authors of seven studies (Bottge, et al., 2001; Bottge, et al., 2007a; Bottge, et al., 2007b; Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Scheuermann, et al., 2009; Witzel, et al., 2003) incorporated the use of manipulatives in their interventions. Maccini and colleagues used Algebra Lab Gear (Picciotto, 1990), which are plastic colored blocks of varying size to represent numeric and variable constants, to help students represent problems involving integers. Witzel and colleagues developed a set of manipulatives to represent numbers in units of ones and tens (small sticks and large sticks, respectively), and variables and operators (pictured on square tiles). These manipulatives were used to solve a variety of equations in one variable. Further, Scheuermann and colleagues used buttons, beans, and unifix cubes to teach students to solve equations in one variable. In contrast, Bottge and colleagues incorporated the use of model cars and ramps in hands-on activity that encouraged student exploration of concepts involving distance, rate, and time. Students in all seven studies demonstrated gains in target skills and concepts, which supports previous research that promotes the use of manipulatives in mathematics classrooms (Bley & Thornton, 2001; Hudson & Miller, 2006; Maccini & Gagnon, 2000; Reys, et al., 1992).

The use of manipulatives is a promising instructional activity for students with LD as it addresses various areas of deficit. For example, manipulatives provide students with a referent to the abstract symbolism of mathematics (Reys, et al., 1992). Through the CRA sequence, manipulatives develop conceptual knowledge (Hudson & Miller, 2006) and provide a bridge to the development of abstract ideas (Reys, et al., 1992). Additionally, manipulatives provide students with opportunities for active engagement as they explore mathematic relationships (Gurganus, 2007). Further, the use of manipulatives has been found to support retention of mathematical ideas (Reys, et al., 1992). Future research should include the use of manipulatives for teaching higher-level algebra content (i.e., computing and factoring of polynomials) to help students with LD develop a conceptual understanding of symbolic manipulation necessary for advanced mathematics (Banchoff, 2008).

***Prompt cards and structured worksheets.*** Prompt cards and structured worksheets provide students with cues to complete a task. Additionally, these materials help students to develop a strategic plan to solve problems (Maccini, et al., 2008) as they prompt students to think about important components of the problem and to ask themselves questions regarding know and unknown information. Authors in five studies (Allsopp, 1997; Hutchinson, 1993; Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Mayfield & Glenn, 2008) used prompt cards and/or structured worksheets.

Allsopp (1997) provided an answer sheet to assist tutors in determining correct responses from tutees. Therefore, tutors were able to provide accurate feedback. Maccini and colleagues (2000) provided students with a structured worksheet that aligned with the STAR strategy. The worksheet cued students to follow the strategy when representing and solving contextualized problems involving integers. The cue card included the four main steps of the STAR strategy (i.e., search, translate, answer, review). Additionally, the cue card contained prompting questions for each step. For example, the cue card prompted students to "Search the word problem." Additionally, students were prompted to read the question carefully and write down facts.

Hutchinson (1993) used both a prompt card and a structured worksheet as part of the strategy intervention. The prompt card included self-questions for representing and solving algebra word problems. Examples of self-questions for representing the word problem included, "Have I read and understood each sentence? Are there any words whose meaning I have to ask?" and "Have I written down my representation on the worksheet?" (p.39). Examples of self-questions for solving the problems included "Have I written an equation?" and "Have I written out the steps of my solution on the worksheet? (p.39). Additionally, Hutchinson provided students with a structured worksheet in which they needed to fill in information such as the goal, known information, unknown information, an equation, and solution.

Further, Mayfield and Glenn (2008) included a prompt card in the feedback plus solution sequence instructional phase, which provided written directions for solving novel problems, such as beginning the problem with a known part of the problem. Additionally, in the transfer training intervention phase, the authors included the use of a structured worksheet which broke down a novel problem into familiar steps and then presented the original novel problem.

When completing an algebra problem, students with LD often have difficulty identifying important information, understanding the nature of the problem, and organizing an efficient strategy for solving the problem (Gurganus, 2007). Prompt cards and structured worksheets guide students through these important elements to reach a reasonable problem solution. Additionally, they support students' metacognition by assisting with self-regulation. Future research should include the use of these materials in high school mathematics, including Algebra I and Algebra II, as students are required to solve complex, multi-step problems.

**Summary of instructional activities.** Authors of the current studies incorporated a variety of instructional practices and instructional materials to improve the algebra performance of secondary students with LD (see Table 2). Instructional practices include method of delivery, sequence and/or range of examples (i.e., CRA sequence), SI, visual representations, peer-assisted instruction (i.e., CWPT), and EAI. Six studies incorporated an instructional package consisting of more than one approach. This is important as each instructional practice addresses specific areas of need for students with LD. For example, EI compensates for memory deficits that affect recall and retention, by providing students with multiple opportunities for practice and teacher scaffolds to promote content mastery

(Hudson & Miller, 2006). SI promotes metacognition by providing students with a plan to solve various problems (Bley & Thornton, 2001; Vaidya, 1999). The CRA instructional sequence provides a bridge toward developing understanding of abstract concepts. Finally, visual representations, such as graphic organizers, allow students to access algebra content using visual-spatial skills and are beneficial for students with language deficits. Therefore, an instructional package can address multiple areas of deficit to support student achievement.

Overall, seven of the current algebra intervention studies for students with LD incorporated teacher-directed instruction with all but one of these studies occurring in a special education setting. Teacher-directed instruction has a long history in special education (Hudson & Miller, 2006); however, as more students with LD are included in general education classrooms (Newman, 2006), future research should include interventions involving student-center approaches. A blended approach to instruction has been proposed by experts (Hudson, Miller, & Butler, 2006; Jones & Southern, 2003; NMAP, 2008) to address the specific needs of students with LD and to promote the NCTM standards for a rigorous mathematics for all students. For example, by providing EI addressing foundational skills and concepts (i.e., multiplication of binomials), students will be prepared to engage in student-centered instruction to further develop understanding of the topic (factoring a trinomial).

Materials in the current studies included the use of manipulatives (i.e., algebra tiles, unifix cubes, model cars), prompt cards and structured worksheets, and technology (i.e., videodisc). Five studies included more than one material; therefore, a promising intervention includes a variety of materials that provide multiple representations necessary for generalization and supports various areas of deficits (i.e., abstract notation, metacognition, memory) for students with LD.

## Summary

The current review of the literature on algebra interventions for secondary students with LD identified instructional approaches, instructional practices, and instructional materials which lead to improved performance on a variety of algebra skills and concepts. In the following section, limitations of the current literature and suggestions for future research are summarized.

**Limitations.** Overall, the authors of the studies in this review employed sound methodology and data analysis to reach their conclusions. However, there are several limitations to the current research.

1) Few studies exist that examine the effects of an intervention on students' algebra performance and the exiting studies focus primarily on basic algebra content, rather than critical algebra tasks as recommended by NCTM, NMAP, and ADP.
2) Few studies include participants in high school algebra courses, therefore, the effectiveness of the instructional practices in this review are uncertain for this population of students in this content.
3) There is no consistent criterion for identifying students with LD and socioeconomic status which limits generalization of current findings for these populations.
4) Three studies (Allsopp, 1997; Bottge, et al., 2001; Witzel, et al., 2003) do not disaggregate outcome data for students with LD.

**Future research.** Historically, research of reading disabilities far surpasses research for mathematics disabilities, while the current number of studies researching reading disabilities continues to exceed the number of studies examining mathematics disabilities by a ratio of 14 to 1 (Gersten, Clarke, & Mazzocco, 2007). In a recent meta-analysis of mathematical interventions for students with LD, Gersten and colleagues (2009b) identified a variety of instructional practices that produce improvements in the mathematical performance of students with LD, including EI, sequencing and range of examples, visual representations, and SI. However, these findings are limited as the meta-analysis includes only one algebra intervention, as a result of restricting inclusion to studies that employ an experimental or quasi-experimental design. More comprehensive reviews of algebra interventions for students with LD (Foegen, 2008; Maccini, et al., 1999; Strickland & Maccini, 2010) support Gersten's findings of effective instructional practices, however, findings from these reviews are limited due to the relatively basic algebra content (i.e., integers, one-variable equations) addressed in the reviewed studies. Given the limitations in the existing research base, future research should examine:

1) Student performance on complex algebra content, including content from Algebra II.
2) Inclusion of a greater number of high school students with LD.
3) Performance of secondary students with LD in general education classes, including Algebra I, Geometry, Algebra II, and high school courses that integrate this content.
4) Performance of secondary students with LD when participating in student-centered instruction.

5) The use of an instructional package containing instructional practices and materials supported by the current research.

## Conclusion

The current literature review synthesizes research findings involving algebra interventions for secondary students with LD.  Theses interventions address instructional practices and materials that provide students with disabilities access to the general education algebra curriculum.  This is a critical area of research, as more secondary students with LD are included in the general algebra classes (Newman, 2006). Additionally, algebra intervention research for this population needs to be grounded in general education algebra research.  For example, the way in which teachers and researchers view algebra (i.e., generalized arithmetic, polynomial-based, or functions-based) impacts students' learning (Chazan, 2008).  However, the authors of the studies from the special education literature reviewed above do not focus on defining *algebra*. Additionally, beliefs regarding the ways in which students learn affect the algebra classroom by influencing curriculum choices and pedagogy. Therefore, a theoretical grounding is essential for developing an instructional intervention. Traditionally, special education research has been based on behaviorist theories of learning that emphasize teacher directed instruction with repeated practice and reinforcement (Woodward & Montague, 2002).  In contrast, the NCTM (2000) promotes student centered instruction rooted in a form of constructivism that supports guided inquiry.  The conclusion of this literature review: (a) discusses the multiple definitions of algebra; (b) identifies characteristics of typical students relating to algebra; and (c) summarizes the theory of the constructivist continuum and the theory of reification.  This section ends with a rationale for the current study, which is influenced by the multiple definitions of algebra, the constructivist and reification theories of learning, and existing research findings from the special education and mathematics education communities.

**Defining school algebra.** According to Usiskin (1988), algebra can be defined in four ways: (a) generalized arithmetic, (b) the study of procedures, (c) the study of relationships between or among quantities, and (d) the study of structures, such as groups, domains, fields, and vector spaces. Similarly, Saul (2008) discusses three milestones for secondary students to reach as they progress through school algebra. First, students view algebra as generalized arithmetic, in which algebra identities are generalizations of arithmetic sentences. Next, students view algebra as the study of binary relations on sets. Students understand expressions as mathematical objects of their own right. Equality of two expressions focuses on operations, not numbers as previously viewed in the first milestone. Lastly, algebra is viewed for algebra's sake. Algebra becomes the study of structures (i.e., groups, rings, fields). This is the work of mathematicians and many typically developing students never reach this milestone.

Currently, a debate exists in mathematics education regarding the most appropriate approach to school algebra. Algebra as generalized arithmetic, the study of procedures, and the study of structures may be categorized as a traditional algebra program that focuses on symbolic manipulations involving equations and polynomials. In contrast, algebra as the study of relationships is the basis of a reformist algebra program that focuses on functions (Kieran, 2007). I briefly describe these two approaches below.

According to Cuoco (2008), "polynomial algebra sits at the historical core of algebra" (p.57). School algebra that focuses on polynomial expressions and their manipulation aligns with traditional algebra courses from the 1960's and 1970's. Topics that are taught within this type of algebra course include simplification of expressions, solving equations, inequalities, systems of equations, factoring polynomials, and rational expressions. Functions play a minor role in polynomial-based algebra courses (Kieran, 2007); instead symbolic forms and manipulations are emphasized. Many mathematicians, mathematics educators, and researchers continue to value polynomial-based algebra, believing that through symbolic manipulations and computational processes, students develop a deeper understanding of mathematical objects (Kieran, 2007; Kilpatrick & Izsak, 2008; Sfard & Linchevski, 1994). Specifically, Banchoff (2008) asserts that the heart of algebra involves procedures. Additionally, to advance through Saul's (2008) milestones, students must demonstrate both procedural and conceptual knowledge of polynomials.

In contrast, reformist algebra courses emphasize functions and the various ways of representing them (i.e., graphs, tables, narrative, algebraically). Students are encouraged to solve real-world problems by methods other than manual symbolic manipulations (Kieran, 2007). Due to technological advancements, calculators and computer algebra systems can perform complex symbolic manipulation, which has lead many mathematics educators and researchers to question the purpose of teaching topics such as factoring polynomials (Kieran, 2007; Kilpatrick & Izask, 2008). Instead, they propose changing the focus of algebra to functions and minimizing instructional time spent on symbolic manipulation (Kieran, 2007). Kilpatrick and Izsak (2008) suggest three reasons for moving away from symbolic manipulation toward a functions-based approach to algebra. First, moving to functions may be more motivating for students. Next, a shift to functions represents a shift from memorizing procedures toward a conceptual understanding of problem situations. Lastly, students may use technology (i.e., calculators and computer software) to create graphs and to perform symbolic manipulations. Yerushalmy and Gafni (1993) believe that, "function in its multiple representations is *the* [italics added] fundamental object of algebra which ought to be present through the learning and teaching of any algebra topic" (p.319).

The current debate between traditional polynomial-based algebra and reformist functions-based algebra has lead to wide variations in the content of school algebra between states in the U.S. and between countries worldwide (Kieran, 2007). As a result, students are often required to demonstrate competency in both contexts. Advocacy groups, such as NCTM, ADP, NMAP, and CCSSI promote aspects of both polynomial-based and functions-based school algebra. For example, the NCTM (2000) promotes functions-based algebra by defining algebra as the study of relationships among quantities, while acknowledging the use of symbolic notation in this process. The CCSS (2010) for high school mathematics include a category entitled *Algebra*, (i.e., expressions, polynomials, and equations) and a category entitled *Functions*, (i.e., linear, quadratic, exponential, and trigonometric). Both the ADP and NMAP include functions and polynomials in their benchmarks (ADP, 2004; NMAP, 2008). Additionally, college placement tests, such as ACT and SAT, include content from both a functions-based and a polynomial based algebra curriculum (ACT, 2011; The College Board, 2009).

Therefore, to support all students in algebra, effective instructional practices addressing topics from both approaches toward algebra need to be investigated.

**Algebra and typical student learning.** Many typically developing students struggle with the content found in secondary algebra courses. Challenges arise from a poor understanding of foundational algebra ideas such as the negative sign (Kieran, 1989; Vlassis, 2004), variables (Kieran, 1989; NMAP, 2008; Russel, O'Dwyer & Miranda, 2009; MacGregor & Stacey, 1997; Wagner, 1981), equality and the equal sign (Kieran, 1989; NMAP, 2008; Russel, et al., 2009; Stacey & MacGregor, 1997), commutative and distributive properties (Carraher & Schliemann, 2007; Saul, 2008), an implicit coefficient of one (Anderson, 1995), and algebraic expressions (Clements, 1982; Kieran, 1989; NMAP, 2008). Because of poor understanding of these foundational ideas, students struggle with complex algebra processes, such as quadratics (Kotsopoulo, 2007; MacDonald, 1986).

Specifically, according to Kotsopoulo (2007), students have difficulty with quadratics because of poor retrieval of multiplication facts and their inability to recognize and understand varied representations of the same quadratic relationship (i.e., standard form and factored form). Additionally, traditional teaching tools may interfere with students' development of an in-depth conceptual and procedural understanding of quadratics. For example, FOIL is a traditional instructional tool for teaching students to multiply binomials and produce a quadratic expression. However, many mathematics educators and researchers oppose the use of this mnemonic as it does not apply to multiplying polynomials of varying terms and it does not generalize to factorization (Kennedy, Curtin, & Warshauer, 1991; Nataraj & Thomas, 2006; Rauff, 1994; Tanner &

Hale, 2007). Instead, experts in mathematics education promote the use of *general-purpose tools* (Cuoco, 2008). For example, an expansion box is a general-purpose tool that can be used to multiply polynomials with any number of terms (CME Project, 2009; Tanner & Hale, 2007). Future research is needed to determine if general purpose tools, such as an expansion box, improve student performance on tasks involving quadratics.

**Constructivist theory of learning.** According to the constructivist theory of learning, students learn by an active process of constructing their own understanding of the subject matter (Greeno, Collins, & Resnick, 1996; Woodward & Montague, 2002). This active construction of knowledge is influenced by their environment, interactions with others, and their previous experiences and understandings (Gurganus, 2007). According to Gurganus (2007), teaching is constructivist when students are active learners, the curriculum is relevant, the curriculum connects to previous learning, and the teacher actively facilitates and monitors learning. These elements are found in variety of pedagogies. For instance, in the explicit instruction model, students are positioned as active learners, the curriculum connects to previous and future learning with the use of an advanced organizer, and the teacher actively facilitates and monitors learning through guided practice.

According to Mushman (1982), constructivism is a continuum including three variations (See Appendix A). Endogenous constructivism is at one end of the continuum and promotes pure discovery learning with student-centered instruction and the teacher taking a peripheral role. On the other end of the continuum is exogenous constructivism, which includes teacher-directed instruction with active student engagement. In between, there is dialectical constructivism, which involves the teacher guiding the students as they discover the targeted skills and concepts. Choosing the type of constructivist approach depends on the content being taught and the previous experiences and knowledge of the students. Regardless of instructional practice (i.e., student-centered instruction, teacher-directed instruction), all students must construct their own understanding of a skill or concept (Kieran, 1994).

***Reification.*** In this study, reification means to regard an abstract mathematical object, such as a quadratic function, as concrete. Sfard and Linchevski (1994) suggests that reification occurs through the process-object theory, in which computational processes, such as determining the output value of $f(x) = x^2 + 3x + 2$ when $x = 1$, proceeds students' understanding of the quadratic expression as a mathematical object. In fact, students' understanding of the object (i.e., quadratic function) is strengthened by practicing computational techniques, even if these techniques are not yet fully understood (Sfard, 1995). A particular representation, such as an input-output table for a quadratic function, may be perceived as both a process (i.e., filling in missing values) and an object (i.e., the quadratic function). The process and the object are complementary views and mutually depend on one another for reification. As algebra is hierarchical, concepts that are understood as a process at one level must be understood as an object at a higher level (Sfard & Linchevski, 1994). For example, functions may be initially perceived as a series of processes (i.e. filling in tables, graphing, and transforming equations); however, functions must be perceived as an object within the study of calculus (Thorpe, 1989).

Additionally, the reification of a mathematical object may be developed through examining its properties. Slavit (1997) states that a property-oriented view of functions begins with an awareness of specific functional properties and is followed by the ability to recognize and analyze functions by identifying the presence or absence of these properties. Through this process, students reify a function as a mathematical object. Slavit also blends this theory with Sfard's process-object theory by stating that reification occurs when students shift from identifying properties of processes to identifying properties of objects.

When focusing on quadratic functions, students may experience various examples of quadratic functions and then develop an understanding of this type of function as an object either possessing or not possessing the required properties (i.e., square term is the highest degree). As students identify critical properties, they are developing an understanding of a quadratic function as a mathematic object either with or without the necessary properties (Ronda, 2009). As reification is difficult to achieve (Sfard & Linchevski, 1994), appropriate levels of scaffolds may support learners during this process.

**Rationale for Current Study**

Competency in algebra is required for *all* secondary students' success in school. Additionally, *all* students who wish to attend college must demonstrate knowledge of algebra, including quadratics. Therefore, to prepare students with LD for post-secondary education, instructional interventions for accessing quadratics within the algebra curriculum are essential. Currently, no studies within the special education literature examine interventions that address algebra tasks involving quadratics expressions for secondary students with LD and specific mathematics difficulties. To develop competency in quadratics, students need to develop both procedural and conceptual understanding. Procedural skills focus on the syntactic aspects of algebra, such as the manipulation of symbols (Yerushalmy & Gafni, 1993). Syntactic aspects are important as they represent general relations and procedures in concise and unambiguous terms, which is a primary goal of algebra (Booth, 1989). Specifically in this study procedural skills refer to the process of multiplying linear expressions to form a quadratic expression and factoring quadratic expressions into two linear expressions.

Conceptual knowledge is linked to semantic aspects, or the structural properties of an algebraic objective such as a quadratic expression (Yerushalmy & Gafni, 1993). Semantic aspects of algebra provide the rationale and the justification for the manipulations and provide an understanding of what the algebraic expressions represents (Booth, 1989). In this study, conceptual knowledge refers to student recognition of the equality of a quadratic expression in standard form and in factored form and recognizing that a quadratic expression is a generalizing statement reflecting a context such as area.

Conceptual knowledge and procedural fluency are equally important (CCSSI, 2010). Based on process-object theory, practicing procedures, such as factoring quadratic expressions, will strengthen students understanding of quadratics as an object (Sfard & Linchevski, 1994). Additionally, identifying semantic properties of quadratics, such as understanding that the quadratic expression represents the area context, will support students' conceptual understanding of quadratics (Slavit, 1997; Yerushalmy & Gafni, 1992). The semantic properties of quadratics may be represented through various representations such as narrative contexts and tables of data, which corresponds to a functions-based approach to algebra. Both procedural fluency and conceptual understanding of quadratic functions are necessary as students are required to demonstrate competency in both symbolic manipulation and functions-based content on state exams and college placement tests.

The current study examined the effects of blended instruction and multiple visual representations of quadratics on the performance of secondary students with LD or at-risk for LD and mathematics difficulties (MD) to complete tasks associated with quadratic expressions. Blended instruction incorporated components of EI and the NCTM process standards. Multiple visual representations included Algebra Lab Gear, sketches of Lab Gear, drawings of areas using discrete numbers, tables of data, and graphic organizers (e.g., expansion box). Constructivist theory, as outlined by Gurganus (2007) and Sfard's theory of reification provided the theoretical groundwork. The intervention included components of instruction that were found to be effective in this review, including the use of components of explicit instruction, a specified sequence and range of examples, and graphic organizers. A combination of teacher-directed and student-directed instruction were implemented, emphasizing a procedural and conceptual understanding of quadratic expressions.

# Chapter 3: Methodology

The current study was developed in light of reform efforts and recent legislative mandates that espouse research-based methods to support the algebra development of students with mathematics disabilities or difficulties (MD). By expanding the existing special education research literature, this study focused on higher level algebra content (i.e., quadratic expressions) with the use of blended instruction and multiple visual representations. Critical supports for secondary students with MD embedded in the instructional package included: (a) components of explicit instruction; (b) concrete to representational to abstract integration strategy (CRA-I); (c) graphic organizers; and (d) graphing calculators. This study addressed algebra content that aligns with the NCTM, NMAP, ADP, CCSS, and the State of Maryland standards and benchmarks for algebra for *all* learners (see Appendix B), as well as the NCTM process standards of problem-solving, representations, reasoning, communication, and connections (see Appendix C). The specific algebra content addressed included: (a) the procedural fluency of transforming quadratic expressions in standard form to factored-form and vice versa; and (b) the conceptual understanding of quadratic expressions through the exploration of multiple visual representations and tabular data embedded in area word problems. See Appendix D for unit objectives.

This study employed a concurrent embedded mixed methods design, which incorporated a quantitative single-subject design and supplemental qualitative data from a case study design (Creswell & Clark, 2011). Single-subject research has a long, productive history in the field of special education (Tawney & Gast, 1984) and is particularly well suited to establish evidence-based practices for students with disabilities

(Horner, Carr, Halle, McGee, Odom, & Wolery, 2005) who represent a small percentage of the student population. Single-subject research is experimental and documents causal or functional relationships between independent and dependent variables (Horner, et al., 2005). The qualitative strand provided a secondary, supportive role to the predominately single-subject design study (Creswell & Clark, 2011). Qualitative research involves a systematic approach to understanding the nature of a phenomenon within a particular context (Brantlinger, Jimenez, Klinger, Pugach, & Richardson, 2005). In this investigation, the phenomenon under investigation is the participants' understanding of quadratic expressions as an outcome of blended instruction and multiple visual representations within the context of the intervention. Qualitative data validate the quantitative outcomes by representing the voices of the participants. Additionally, the qualitative data help the investigators to understand the impact of the intervention on participants and depicts the processes experienced by the participants (Creswell & Clark, 2011).

The joint use of the single-subject design and the qualitative analysis were complementary as the single-subject design allowed the investigator to determine if the intervention was effective in teaching the participants how to accurately complete tasks involving quadratic expressions, while the qualitative data provided insight into their thinking and understanding that impacted their change in performance. Therefore, using both designs concurrently within an embedded mixed method design provided a thorough representation of the phenomenon. See Appendix E for the organization of the research design.

This chapter provides a description of the: (a) participants and setting; (b) instructional package and materials; (c) concurrent mixed method design including single subject design, (i.e., experimental design, independent variable, dependent variable, and data analysis) and supplemental case study, (i.e., procedures, data collection, and data analysis).

## Participants and Setting

This section provides an overview of the participants and setting of the study. Additionally, obtaining Internal Review Board approvals, informed consent from parents/legal guardians, and informed assent from participants are discussed.

**Participation eligibility.** Five high school female students participated in this study. Three participants had a documented learning disability as determined by a discrepancy between scores on achievement and aptitude tests and their school performance. The other two participants were at-risk of failure in algebra. Administrators suggested that these two participants were in fact LD, however, as these participants had always attended private schools that met their learning needs, parents never pursued a formal evaluation. All participants shared a common learning history of difficulty in the algebra content domain throughout their secondary school experience, as evidenced by input from teachers, the math department chair, the learning specialist, and the principal. Additionally, participants demonstrated a need for this intervention as evidenced by low scores (range = 0% - 25%) on an investigator-developed domain probe (see Appendix F). Demographic data including gender, age, grade, race, disability status, and scores from intelligence and achievement tests, are reported in Table 3.

Table 3 Demographic Information

| Characteristics | Participants | | | | |
|---|---|---|---|---|---|
| | Cheryl | Cindy | Sasha | Anna | Marcia |
| Demographic: | | | | | |
| Gender | Female | Female | Female | Female | Female |
| Race | White | White | White | White | White |
| Age | 16 yrs | 16 yrs | 16 yrs | 17 yrs | 16 yrs |
| Grade | $11^{th}$ | $11^{th}$ | $11^{th}$ | $11^{th}$ | $11^{th}$ |
| Disability | SLD | At-Risk | SLD; ADHD | At–Risk | SLD; ADHD |
| Achievement: | | | | | |
| Broad Reading | SS = 97 (WIAT) | NA | NA | NA | 103 (WJ – III) |
| Broad Math | SS = 97 (WIAT) | NA | NA | NA | 75 (WJ – III) |
| Aptitude: **Wechsler Intelligence Scale for Children** $4^{th}$ edition | | | | | |
| Full Scale | 116 | NA | 94 | NA | 107 |
| Verbal | 110 | NA | 77 | NA | 104 |
| Perceptual Reasoning | 100 | NA | NA | NA | 102 |
| Processing Speed | 112 | NA | 98 | NA | 94 |
| Working Memory | 132 | NA | 93 | NA | 120 |
| IEP Math Goals: | Yes | No | Yes | No | Yes |

SLD = specific learning disability; ADHD = attention deficit with hyperactivity disorder; NA= not available; GE = Grade Equivalent; SS = Standard Score which classifies relative standings; NA = Not available; WIAT = Wechsler Individual Achievement Test; WJ-III = Woodcock Johnson, third edition

**Instructor and setting.** The intervention was implemented by the investigator who assumed the role of teacher-researcher for the duration of the study. Teacher-researcher is defined as "systematic and intentional inquiry carried out by teachers" (Cochran-Smith & Lytle, 1990, p. 3). Specific to this study, data were systematically gathered and all activities were planned rather than spontaneous. Additionally, this research resulted from previous experiences I had as an algebra teacher of 12 years to students with high incidence disabilities and low mathematic achievers. Specifically, I observed that students with disabilities struggle to demonstrate competency in algebra content when taught using symbolic manipulation only. Additionally, I had a pre-existing relationship with two of the participants (Marcia and Anna) as their seventh-grade mathematics teacher.

The study took place in a private school located in the greater Washington, DC area. During the baseline condition of the study, participants were removed from the classroom to complete the pretest domain probes and then returned to their typical mathematics classroom but received no instruction in the content covered in the study. During the intervention, participants were removed from their mathematics class and received instruction in small groups. Primarily, the intervention occurred in a conference room, which contained a table and a white board. On occasion, we were moved to a classroom in which we utilized a table and the white board for instruction.

**Internal Review Board.** Prior to the beginning of the investigation, I submitted plans for approval to the Internal Review Board at the University of Maryland, College Park.

**Informed consent.** Participants and their parents/legal guardian received a letter (see Appendix G) that stated the purpose of the study, the algebra content addressed during the study, and the risks and benefits of the study. Additionally, I asked for access to each participant's educational records (i.e., IEP, scores from IQ and achievement tests, grades in previous math courses). Participants and their parents/legal guardians were informed that participants may withdraw from the study at any time without penalty. Parents/legal guardians signed a permission form (see Appendix H) and participants signed an assent form (see Appendix I).

## Instructional Materials

This section provides a description of the instructional materials that were used to develop students understanding of quadratics expressions. The intervention included an investigator-developed instructional unit that incorporated materials and instructional supports to help students with MD access the curriculum.

**Manipulative materials.** Concrete manipulatives were used in this intervention by way of Algebra Lab Gear (Picciotto, 1990), a comprehensive manipulative program designed for the teaching of algebra concepts. Color-coded blocks included constants, x-bars, and $x^2$-blocks.

**Graphic organizer.** Students were provided a graphic organizer, specifically an expansion box (see Appendix J) to expand and factor polynomials of varying terms. An expansion box emphasizes the distributive property and may be used to multiply whole numbers as well as polynomials of various terms (CME Project, 2009), thus creating arithmetic to algebra connections. Additionally, graphic organizers are beneficial instructional materials for students with language-based learning disabilities (Ives, 2007).

**Calculators.** Throughout the instructional unit, participants used calculators, which served two purposes. First, as lesson objectives were not related to computation, participants were provided calculators to perform computations for discrete numbers. For example, they used a calculator to determine the area of a square bedroom with each side 10 meters long. Second, when transforming quadratic expressions from standard form to factored form and vice versa, participants created tables of data. If they successfully transformed the expressions, then the values in the tables were identical. Additionally, participants graphed each expression to determine if their transformations were correct. If they successfully transformed the expressions, then the lines on the graph were identical. If they incorrectly transformed the expression, then two different lines appeared and participants used this visual feedback to correct their work. Yerushalmy (1991a) found graphic feedback to be a beneficial and motivational tool for checking transformations, particularly since students did not notice if they made a mistake with symbolic transformations unless provided feedback.

## Instructional Unit and Lesson Plans

The investigator-developed instructional unit included lesson plans that addressed age- and grade-level appropriate algebra content consistent with the NCTM Standards (2000), the Maryland Voluntary State Curriculum (2007), the American Diploma Project Benchmarks (2004), and the Common Core State Standards (2010). Appendix B outlines the NCTM standards, the state curriculum, the ADP benchmarks, and the CCSS. The goal of the instructional unit was to promote the conceptual understanding of quadratic expression which supported the procedural fluency of multiplying linear expressions and factoring quadratics. Additionally, the NTCM (2000)

Process Standards (see Appendix C), which describe ways students acquire and apply content knowledge, were included to promote values held by the mathematics education community. The NCTM Process Standards included: (a) problem solving; (b) reasoning and proof; (c) communication; (d) mathematic connections; and (e) representations of mathematics ideas and concepts.

Additionally, the lesson plans incorporated instructional practices that have demonstrated positive effects for the acquisition of various mathematics processes for students with LD within the special education research. These instructional practices include visual representations (i.e., concrete manipulatives, sketches, graphic organizers), and components of explicit instruction (e.g., teacher-directed investigations, multiple practice opportunities).

Many general educators do not receive training on adapting or modifying their curricula or instructional methods for students with LD (Maccini & Gagnon, 2002), despite the recommendation from the National Council of Teachers of Mathematics (2001) to do so. As part of their *Principles and Standards for School Mathematics,* NCTM has identified an *equity principle* as a critical theme of school mathematics. This principle identifies equity as high expectations and strong support for all students (NCTM, 2000, p.11). "Equity does not mean that every student should receive identical instruction; instead, it demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students" (p. 11). Incorporating instructional practices identified by the special education research community into the mathematics classroom promotes the equity principle. Therefore, this intervention incorporated instructional practices from the special education literature and the NCTM's five process standards. See Appendix K for an example of a lesson plan.

**Single-Subject Design**

The primary research design within Embedded Mixed Methods was a single subject research design. Single subject research incorporates an experimental design that documents causal or functional relations between independent and dependent variables (Horton, et. al., 2005; Kennedy, 2005) through documenting patterns of performance during phases of the study (i.e., baseline and intervention). Many variations of single subject design (i.e., reversal, multiple baseline, multiple probe) are included in this broad category of research, however, all designs share three common characteristics: (a) continuous assessment over time; (b) replication of intervention effects over multiple participants, behaviors, or settings; and (c) data evaluated through visual analysis (Kazdin, 1982).

Single subject research is largely based in the field of behavior analysis and includes three underlying assumptions. First, this research approaches the subject matter by understanding how individuals behave, not by describing mathematical averages of groups (Kennedy, 2005). The individual remains the focus, unlike group research designs where the individual's performance may be lost when only the group average is reported (Kazdin, 1982). The second assumption is that the variables being studied must be operationalized, or concretely described (Kennedy, 2005). Operational definitions must be provided for independent and dependent variables, as well as participants and settings (Horton, et al., 2005). In fact, quantifying a behavior is fundamental. In educational research, the academic performance on a particular task is the quantified behavior. This behavior must be (a) defined; (b) measureable; and (c) physically recorded. The third assumption is that single-subject research uses an inductive approach to understanding human behavior as single-subject designs explore the nature of behavior and develop theories from the data collected (Kennedy, 2005).

I choose to use a single-subject design for two reasons. First, this design has a history of establishing evidence-based practices for students with disabilities (Horton, et al., 2005), which is critical as federal mandates require the use of scientifically-validated practices (IDEIA, 2004). Although experimental group designs also establish evidence-based instructional practices, group designs report only group performance and neglect the individual. I am most interested in the individual; therefore, my second reason for choosing a single-subject design is that this design focuses on the individual and will support a detailed analysis of nonresponders as well as responders to the intervention (Horton, et al., 2005). The following section will describe the: (a) experimental design; (b) independent variable; (c) dependent variables and measurement procedures; and (d) data analysis that will be used in the single-subject component of the proposed study.

**Experimental design and study procedures.** This study used a multiple probe design across two groups over five participants. This design consisted of two phases, baseline and intervention, (although maintenance and transfer data were also collected) and involved the systematic and sequential introduction of the independent variable to one group at a time (Tawney & Gast, 1984). In this design, baseline data are collected intermittently prior to the introduction of the intervention. A stable baseline followed by a change in performance after the introduction of the intervention replicated over multiple participants establishes internal validity (Tawney & Gast, 1984). A multiple probe design is appropriate and advantageous when prolonged baseline measures are unnecessary, reactive, or impractical (Horner & Baer, 1978; Tawney & Gast, 1984).

***Baseline phase.*** The baseline condition consisted of the participants in their typical mathematics classroom with no instruction relating to quadratic functions. During this phase, I had no contact with the participants. Randomly chosen parallel versions of domain probes were administered by school staff, specifically the learning specialist and the math department chair. Participants were removed from their typical classroom to complete the baseline domain probes. According to Kennedy (2005), “baseline needs to be as long as necessary, but no longer” (p. 38). Therefore, a minimum number of baseline probes were administered to establish stability. During probe sessions, the participants were given pencil, paper, calculator, and algebra blocks. If asked, the word problems were read to participants, as my goal was to assess algebraic ability not reading ability. However, the word problems were read verbatim and without additional explanation or prompting. If a participant asked a question regarding a problem, the staff member was asked to respond by saying “Do the best you can.” Probes were collected for scoring. When the performance of all group members reflected stability in level and trend (i.e., at or below 60% accuracy for at least two data points, little variability in scores, and trends in data did not reflect a significant increase in scores), the intervention was introduced to the first group.

***Intervention phase.*** During the intervention phase, participates were removed from their mathematics class in groups of two or three to receive instruction via the independent variable. I provided all of the instruction during the intervention phase; however all domain probes were administered by school personnel. Two groups of students participated in the study. Group 1 consisted of Cheryl and Cindy and Group 2 consisted of Sasha, Anna, and Marcia. The introduction of the independent variable was staggered and each group of participants received the intervention after demonstrating stability in level and trend during baseline probes. Specifically, an initial probe was collected for each group during the baseline phase. Then an additional consecutive probe was collected for Group 1. Stable baseline was demonstrated, therefore, I introduced the intervention to Group 1 only. When Group 1 completed the intervention, three additional probes were collected on participants in Group 2 so that there was one more consecutive probe than the number of probes collected for Group 1. I then introduced the intervention to Group 2.

During the intervention phase, participants received instruction on target objectives as outlined in the scope and sequence of objectives (see Appendix D). The group of participants advanced to the next lesson after each participant scored 80% or higher on the objective probe during the lesson (Hudson & Miller, 2006). Additionally, an error analysis was completed on each probe to confirm that participants mastered the skills and concepts necessary to move forward in the instructional unit. The intervention ended after the participants reached criterion on all lesson objectives. Parallel versions of posttest domain probes were administered over the next three consecutive sessions immediately following the end of the intervention.

***Interrater reliability.*** Interrater reliability was obtained on 33% of the domain probes, lesson probes, and transfer measures to monitor the consistency in which the dependent variables are being measured. A trained graduate student and I independently scored each probe to allow for an objective comparison. Each domain probe was scored using an answer sheet which stated specific point assignments for each task. The investigator provided mock domain probes with the answer sheet to the graduate student to practice scoring. The graduate student was considered trained after scoring a minimum of three mock probes with at least 90% agreement with the investigator. The percentage of scorer agreement was determined by: (a) summing the total number of correct responses recorded by each observer, (b) dividing the smaller total by the larger total; and (c) multiplying by 100 (Kennedy, 2005).

***Fidelity of treatment.*** Fidelity of treatment refers to the extent to which critical components of the intervention are implemented as planned (O'Donnell, 2008). An independent observer (i.e., trained graduate student) conducted fidelity of treatment observations by using a checklist that included the components of the intervention (see Appendix L). Training was provided by the investigator via explanation and review of scripted lesson plans and accompanying video recorded instructional sessions. The graduate student was considered trained after successfully identifying components of the intervention from three scripted lesson plans and accompanying video recorded instructional sessions with at least 90% agreement with the investigator. Fidelity observations were conducted on 33% of the instructional sessions, via video recordings. Fidelity of treatment was calculated by dividing the number of components present by the number of total components and multiplying the quotient by 100 (Kennedy, 2005).

Additionally, interobserver agreement was obtained for one of every three fidelity observations. During these sessions, two independent observers will conduct fidelity of treatment observations. Specifically, an additional independent observer (i.e., doctoral student) viewed three of the eight fidelity observations and completed a checklist that included the components of the intervention (see Appendix L). Training was provided by the investigator via explanation and review of the scripted lesson plans and the fidelity checklist. The doctoral student was considered trained after successfully identifying components of the intervention from one scripted lesson plans with at least 90% agreement with the investigator. Percentage of interobserver agreement will be calculated by (a) summing the total number of correct components recorded by each observer; (b) dividing the smaller total by the larger total; and (c) multiplying by 100 (Kennedy, 2005).

**Independent variable.** In single-subject research, the independent variable is the intervention which is actively manipulated during the study (Horton, et al., 2005). In this study, the independent variable combined the process standards promoted by the mathematics education community (NCTM, 2000) and critical instructional practices identified by the special education research community. Specifically, the independent variable combined the use of blended instruction and multiple visual representations (e.g., manipulatives and graphic organizers) to develop conceptual knowledge of quadratic expressions and the procedural fluency in transformations of quadratics expressions. The following sections describe elements of the instructional package.

***Instructional procedures.*** Critical instructional practices identified by the special education research community and the process standards promoted by the mathematics education community (NCTM, 2000) were used in instructional delivery. Components included providing a(n):

1) Advanced organizer, which consisted of a review of pre-requisite skills, the objective of the current lesson, and motivation for learning the skill;

2) Investigation, which consisted of the investigator facilitating the completion of a new task using critical instructional practices and materials from both the special education and mathematic education research;
3) Multiple practice opportunities with appropriate scaffolds, which included opportunities for students to work on similar problems with the teacher, a peer, or individually.
4) Closure, which consisted of a review of the lesson and assessment.

Lesson plans were developed to include each component to ensure a systematic implementation of the lessons. Only one target skill/concept was presented for each lesson as recommended by Hudson and Miller (2006). Additionally, the NCTM process standards were embedded in each lesson plan. Lesson plans included word problems reflecting real-world situations and problems using symbolic notation only, as suggested by the National Mathematics Advisory Panel (2008).

Throughout the instructional sessions, the teacher-researcher primarily acted as a facilitator. Specifically, participants were guided toward concepts and skills through discussions with the teacher-researcher and other participants in the group. For example, when participants completed a task such as factoring a quadratic expression, they were asked to compare their responses with their group members and to confirm using the instructional methods embedded within the intervention (i.e., Lab Gear, Box Method). The process was emphasized rather than the answer, therefore, students were required to communicate with their peers and justify their responses when their answers were correct or incorrect. Specific questions included, “Explain how you got your solution” and “Confirm your solution by using another representation.” These prompts were provided for correct, partially correct, and incorrect responses. Participants were provided explicit instruction in the form of teacher modeling and teacher think alouds if they were unable to attain the correct response through questioning and discussions.

***Blended instruction.*** Procedural fluency and conceptual knowledge were targeted through the components in the blended instructional format. Components of explicit instruction in blended instruction included: (a) an advanced organizer; (b) teacher-directed investigation; (c) multiple practice opportunities; and (d) explicit sequencing of tasks. An advanced organizer provided students with a review of prerequisite skills, the objective of the current lesson, and rationale for learning the skill. The teacher-directed investigation involved maximizing students' engagement via questions and prompts, while modeling the thinking and action procedures needed to solve the problem. Multiple practice opportunities included a variety of activities including hands-on activities, completion of real-world problems, group work, or individual work. I continually checked for student understanding of the content being taught and provided corrective feedback, which took the form of prompting questions or reteaching. Explicit sequencing referred to the break down of a mathematical concept into instructional bites that were taught in a predetermined sequence (Scheuermann, et. al., 2009). See Appendix D for a break down of the content for this intervention. Additionally, the quadratics were first introduced as discrete, fixed values and then as continuous functions, as suggested by experts in mathematics education (Picciotto, 2010; Sfard & Linchevski, 1994). Additionally, the NCTM process standards were embedded throughout blended instruction as students were provided opportunities to: (a) problem solve; (b) demonstrate reasoning by formulating conjectures and justifying solutions; (c) communicate using mathematics language; (d) connect new algebra content to previously learned mathematics and/or to other mathematical content (e.g., geometry); and (e) work with various representations of the task.

***Multiple visual representations.*** In this study, multiple visual representations of quadratics included concrete manipulatives, sketches of manipulatives, sketches of qualitative representations on graph boards, and an expansion box. Multiple representations of a mathematic concept is recommended by NCTM (2000) and special education researchers (Bryant, Bryant, Kethley, Kim, Pool, & Seo, 2008; Jitendra, Salmento, & Haydt, 1999; Jitendra, Griffin, Deatline-Buchman, Sczesniak, Sokol, & Xin, 2005). Research in special education has identified the concrete-representational-abstract graduated instructional sequence as an effective strategy for teaching algebraic procedures and concepts such as integers (Maccini & Hughes, 2000; Maccini & Ruhl, 2000) and linear equations (Witzel, 2003). In the current study, the CRA sequence was modified to and simultaneously introduce multiplying linear expressions through concrete manipulatives, sketches of the manipulatives, and symbolic notation as suggested by Pashler and colleagues (2007).

Procedural fluency was targeted through the use of concrete manipulatives (i.e., algebra tiles), which provided a visual representation of the algebraic expressions, promoting a meaningful understanding of the abstract symbolism involved in the procedures. Additionally, procedural knowledge at the abstract level was targeted through the use of a graphic organizer in the form of an expansion box (see Appendix J). The expansion box (i.e., the Box Method) supported the transition from using algebra blocks to using only abstract notation when factoring a quadratic expression and when multiplying two linear expressions (Picciotto, 1995).

Prior to the intervention, the independent variable (i.e. blended instruction and multiple visual representations) was piloted with three high school students that met the inclusion criteria listed above and were not part of the actual study. The purpose of piloting the independent variable was to gather input from the pilot participants on the enjoyment and benefit of the intervention which lead to adjustments and revisions.

**Dependent variables and measurement procedures.** Dependent measures consisted of investigator-developed probes and included: (a) domain probes; (b) objective probes; and (c) transfer probes. Each probe was reviewed by experts in mathematics education and mathematics special education to determine internal validity. Prior to the proposed study, the probes were piloted with three high school students meeting the inclusion criteria stated above but who were not included in the actual study. The purpose of piloting these dependent measures was to evaluate their reliability and obtain feedback from the pilot participants regarding their ease of use.

***Domain probes.*** The investigator developed 3 parallel versions of the domain probe that were identical in content and addressed all objectives in the instructional unit. Content validity of the parallel measures was established by expert review (Huck, 2008). Specifically, two experts in the field of mathematics special education reviewed each version of the domain probes and determined that they contained the same algebra content and were of the same level of difficulty. Domain probes were used to: (a) establish baseline performance prior to the intervention; (b) determine performance after intervention; and (c) determine maintenance of performance four to six weeks after the intervention. To address both conceptual and procedural knowledge, question types included contextualized and non-contextualized problems, tables of data, and open-ended questions. See Appendix F for an example domain probe containing examples of each question type.

Parallel versions of the domain probe were developed and administered randomly as a pretest, posttest, and maintenance test. Pretest domain probes were administered during baseline for all participants. Posttest probes were administered the three sessions immediately following the end of the intervention. Maintenance probes were administered four to six weeks following intervention. Participants in Group 1 completed the maintenance probe six weeks after intervention and participants in Group 2 completed the maintenance probe four weeks after intervention. All participants completed the same version of the domain probe for their maintenance measure. All domain probes were administered to participants by school personnel. See Appendix F for a sample domain probe.

***Lesson probes.*** Lesson probes contained items related to the objective of a specific lesson. Objective probes were given to participants at the end of each session in which the specific objective was taught. When all participants in a group met criterion (80% accuracy) on the objective probe, the next consecutive target objective was addressed the following session. See Appendix D for the scope and sequence of the unit objectives. See Appendix M for a sample lesson probe.

***Transfer probes.*** Transfer probes addressed the participants' ability to transfer, or generalize, knowledge learned during the intervention to additional algebra tasks. In this study, participants applied knowledge and procedures learned through blended instruction with multiple visual representations of quadratics to three tasks. For task one, participants completed a table of data incorporated perimeter and volume contexts. Participants multiplied a trinomial by a four-term polynomial for task two. The final task required participants to factor a quadratic expression with a coefficient of three. Transfer probes were administered by school personnel the session immediately following the session in which participants complete the posttest probe. Additionally, the investigator observed this session. See Appendix N for a sample transfer probe.

**Data analysis procedures.** The traditional approach to analyzing single-subject research data involves visual analysis of graphic displays that provides a detailed summary and description of the participant's performance (Horton, et al., 2005; Kennedy, 2005; Tawney & Gast, 1984). Data were collected and graphed continually throughout the study for each individual participant. Patterns in data were continuously analyzed to determine the next step in the study. For example, all participants in a group needed to meet criterion on a lesson probe before advancing to the next objective in the sequence. Additionally, visual analysis allowed for independent analysis of the data to determine the reliability of the findings (Tawney & Gast, 1984). Specifically, I focused on: (a) within-phase patterns; and (b) between-phase patterns (Kennedy, 2005). Information from these analyses determined if a functional relationship existed between the independent and dependent variables (Horner, et al., 2005).

***Within-phase patterns.*** Within a given phase (i.e., baseline or intervention), data points were analyzed in regard to: (a) level; (b) trend; and (c) variability. Level referred to the average of the data points within a phase (Kennedy, 2005). Trend referred to the slope of the data points (Tawney & Gast, 1984). Variability was the degree to which individual data points deviate from the overall trend (Kennedy, 2005). For this study, within-phase patterns for baseline consisted of: (a) a low average level of performance; (b) stable or decelerating trend; and (c) low variability in data points (i.e., 80-90% of the data points within a 15% range of the mean level) (Tawney & Gast, 1984). The within-phase patterns for the intervention phase consisted of: (a) a higher average level of performance; (b) stable or accelerating trend; and (c) low variability in data points with 80-90% of the data points within a 15% range of the mean level.

***Between-phase patterns.*** The between-phase pattern determined the functional relation between the independent and dependent variables by an immediate change in level and trend (Kennedy, 2005; Tawney & Gast, 1984). Following the introduction of the intervention, the visual analysis of the graphs was analyzed to determine: (a) an increase in level; (b) an accelerating trend; (c) the magnitude of the trend; and (d) low variability of data points within each phase.

**Social validity.** Social validity represents the importance, effectiveness, appropriateness, and/or satisfaction the participants' experience in relation to the intervention. Kennedy (2005) identified three steps in determining social validity: (a) choosing a consumer; (b) choosing an assessment strategy; and (c) analyzing the data. At the end of the study, participants completed an investigated-developed questionnaire which assessed their perceptions regarding their learning of the content, the helpfulness of instructional tools (i.e., manipulatives, Box Method, word problems, tables of data) and their likes and dislikes of the intervention (see Appendix O). The instrument was developed from other social validity measures within the field of mathematics special education research (Maccini, 1998; Mulcahy, 2007).

The math department chair administered the questionnaire to all of the participants who were asked to share their thoughts on the intervention. The measure consisted of 10 questions on a five-point Likert scale concerning the effectiveness of various aspects of the intervention. Participants indicated a score of “1” if they strongly disagreed with a statement, “2” if they disagreed, “3” if they felt neutral, “4” if they agreed, and “5” if they strongly agreed. Additionally, participants responded to six open-ended questions concerning their opinion of quadratic expressions and suggestions for improving the intervention. Participants were permitted to dictate or write their responses to these questions. Data were analyzed by determining the mean score for each item on the Likert scale and reporting themes from the responses to the open-ended questions.

**Qualitative Design**

The overarching goal of the qualitative design for this study was to produce descriptive knowledge that will supplement the findings from the single-subject design to answer the research questions identified in chapter 1. Although the probes from the single-subject design provided quantitative data indicating to what extent students with LD develop, maintain, and transfer procedural and conceptual knowledge associated with quadratic expressions, single-subject designs purposively avoid cognitive references (Kennedy, 2005). Therefore, a qualitative design was necessary to provide a thorough understanding of participants’ cognitive processes regarding the algebraic tasks.

Similar to single-subject research, qualitative research focuses on the individual with the goal of understanding the phenomenon from the participants’ perspective. However, qualitative research provides an opportunity to explore a variety of factors that may influence a situation (Hancock & Algozzine, 2006), but are often overlooked in single-subject designs. Factors, such as past math experiences, relationships with teachers and peers, and overall happiness in school, are extraneous variables that effect student performance (Noddings, 2003) but are not easily controlled in single-subject designs (Kennedy, 2005). Qualitative research methods provide an opportunity to identify these factors and explore their possible contributions to participant performance. Specifically, the qualitative method used in this embedded design was a case study focusing on one critical case, Marcia, who provided a rich data source that was representative of the group (Creswell, 2007). The use of a single case study to highlight students understanding of mathematics programs has made significant contributions in the field of mathematics education (Erlwanger, 1973). The purpose of this case study was not to generalize the data, but to elucidate the specific features of the participants of this study (Creswell, 2007). The case study focused on Marcia's thinking and understanding of quadratic expressions through the instructional practices and tools embedded within the intervention. Qualitative data provided greater insight than only the quantitative data collected from the probes. Understanding how students think about a task is extraneous to the single-subject design. The following section describes procedures for: (a) the data collection; (b) the data analysis; and (c) data validation of the case study of Marcia.

**Data collection.** Qualitative data were collected through: (a) transcriptions of video recorded sessions; (b) work samples; (c) investigator field notes of direction observations (Creswell, 2007). All instructional sessions were video recorded. After viewing all recordings, segments that describe the participants' cognitive processes were transcribed. Video recordings were transcribed to document: (a) participants' spoken words verbatim; and (b) participants' behaviors (i.e., manipulation of algebra blocks). Additionally, work samples were collected from Marcia for analysis. In addition, the investigator wrote write field notes after each section to address Marcia's progress and participation during the intervention sessions. Session recordings, work samples, and field notes provided descriptive data to support the research findings from the single-subject design.

**Data analysis.** My method for data analysis was based on Creswell's (2007) data analysis procedure. Specifically, I progressed through four stages of data analysis: (a) data managing; (b) reading and memoing; (c) describing, classifying, and interpreting the data; and (d) representing the data. To manage my data, I transcribed verbatim relevant sections of all instructional sessions. Specifically, I transcribed sections in which participants demonstrated their cognitive processes through verbalizations and/or behaviors. Additionally, I focused on the overarching themes of representations and metacognition, which directly linked to my research questions and were themes that were evident from the pilot study. At this point, I decided to focus my qualitative analysis on only one student (Marcia). This decision was made through discussion with three members of my dissertation committee and with the rationale that Marcia provided a rich data source that was representative of the group. To fulfill the requirements of the embedded mixed method design, Marcia's data provided the supplementary qualitative data to support the quantitative data from the single subject research design. Specifically, Marcia's data provided insight into why all participants demonstrated significant gains on the domain probes from pre-intervention to post-intervention.

Next, I read and re-read the transcripts of Group 2 (i.e., Marcia's group), while making notes (i.e., memoing). My notes reflected initial analysis and possible codes and/or themes. For example, I observed the impact of specific components of the intervention on Marcia's mathematic development and her metacognition. I chose to personally code the data rather than use a computer program, which would have caused an uncomfortable distance between me and the data (Creswell, 2007). Additionally, reliability, or dependability, of codes was established through confirmation from a second coder (Creswell & Clark, 2011). Throughout this stage, I continually triangulated (i.e., cross-checked) my memos with my field notes and with Marcia's work samples.

In the describing, classifying, and interpreting phase, I developed possible codes based on my memos. Specifically, codes focused on the multiple representations (i.e., area context, Lab Gear, and Box Method) included in the intervention and the impact of the intervention on metacognition (i.e., self regulation, strategic planning, disposition, socially shared metacognition). Then I read through the transcripts again searching for support of these codes. Additionally, I discussed my initial codes with two members of my dissertation committee. Based on these discussions and the support from transcripts, field notes, and Marcia's work samples, I organized the data into codes (See Appendix P for coding description). Through interpretation of codes, I developed two overarching themes (i.e., representations and metacognition) and subthemes (Creswell, 2007). Figure 1 displays the coding and theme development process.

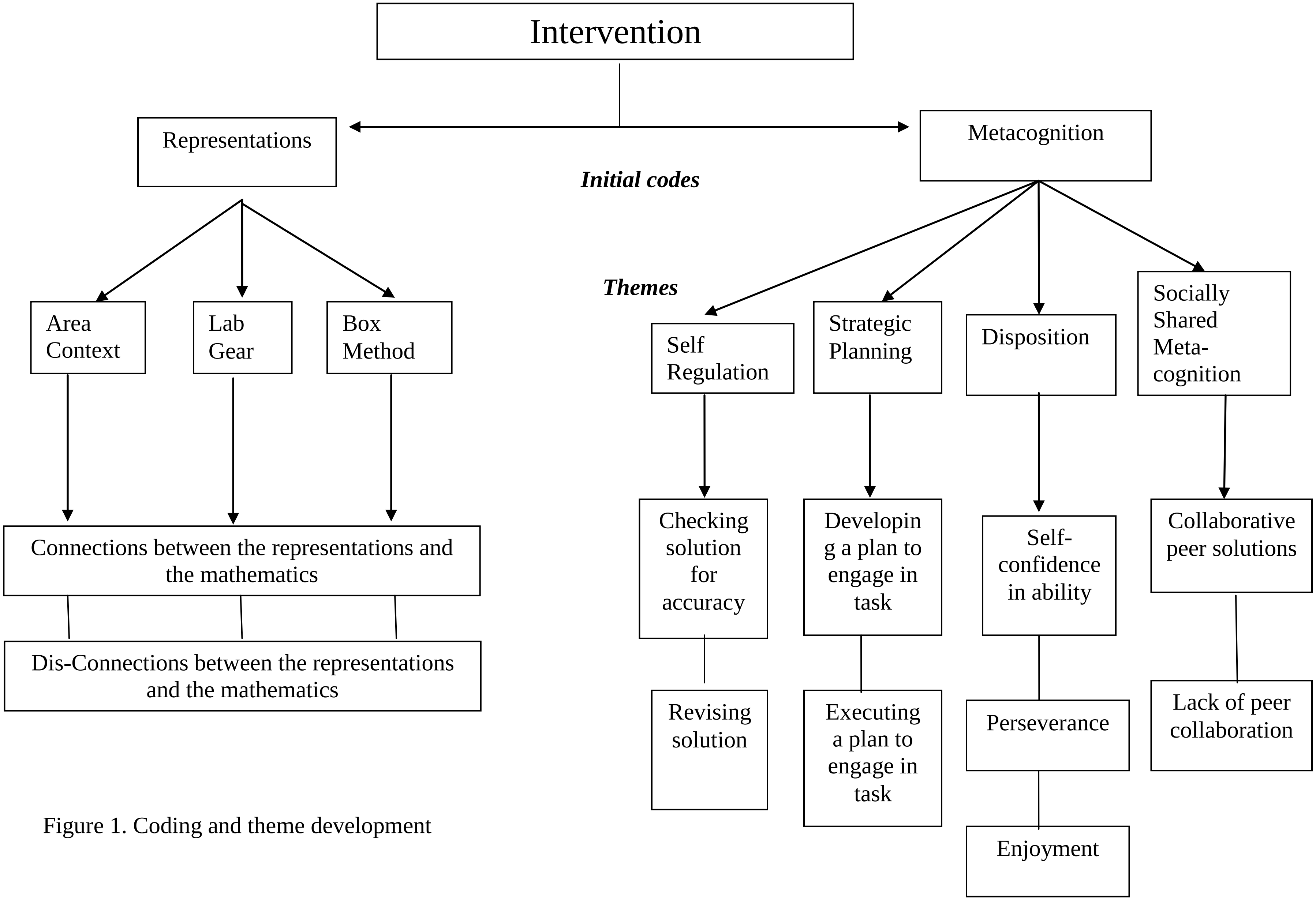



Figure 1. Coding and theme development

**Data validation.** In qualitative research, validation refers to the attempt to assess the accuracy of the findings as described by the researcher and the participant (Creswell, 2007; Creswell & Clark, 2011). Creswell and Clark (2011) recommend video recording intervention sessions as a way to collect unobtrusive data, which minimizes bias and threats to internal validity. A variety of additional validation strategies may be employed while conducting qualitative research and Creswell (2007) suggests that qualitative researchers use a minimum of two strategies. The current study contained four validation strategies based on Creswell's procedures of validation. Specifically, through triangulation, I obtained collaborating evidence of themes found in the transcripts from my field notes and Marcia's work samples. Additionally, throughout the data analysis process, I continually engaged in peer debriefing sessions with my advisor. I also provided rich, descriptive data in Chapter 5 to allow readers to draw their own conclusions. Lastly, I had an external auditor examine both the process and the product of the account to assess for accuracy. The external auditor had no connections to the study, but had experience with mixed methods research designs. After reading the relevant sections (i.e., sections pertaining to qualitative research) of this dissertation, the external auditor stated that I clearly described what was happening during the study. Additionally, she confirmed that the themes, interpretations, and conclusions were supported by the data through the examples from the transcripts, work samples, and field notes.

# Chapter 4: Quantitative Results

In this chapter I report results relative to the research questions addressed by the single subject design. Specifically, the following research questions are addressed. When provided blended instruction with visual representations:

1. To what extent do secondary students with mathematics disabilities or difficulties (MD) increase their accuracy on algebraic tasks involving quadratic expressions embedded within area problems?
2. To what extent do secondary students with MD maintain performance on algebraic tasks involving quadratic expressions embedded within area problems two to four weeks after the end of the intervention?
3. To what extent do secondary students with MD transfer their knowledge of quadratic expressions to problem-solving tasks?
4. To what extent do secondary students with MD find blended instruction with visual representations beneficial (i.e., social validity)?

## Results on Academic Outcomes

**Research Question 1: Increase in Accuracy on Algebraic Tasks.** Increases in accuracy on algebraic tasks involving quadratic expressions were measured by performance on domain probes and lesson probes. As shown in Figure 2 all participants substantially increased their overall accuracy on domain probes from an average of 10% during baseline to an average of 93% after the intervention. Specifically, baseline scores ranged from 0% - 25% and scores ranged from 84% - 100% following intervention, indicating that all participants met criterion (i.e., 80% or greater).

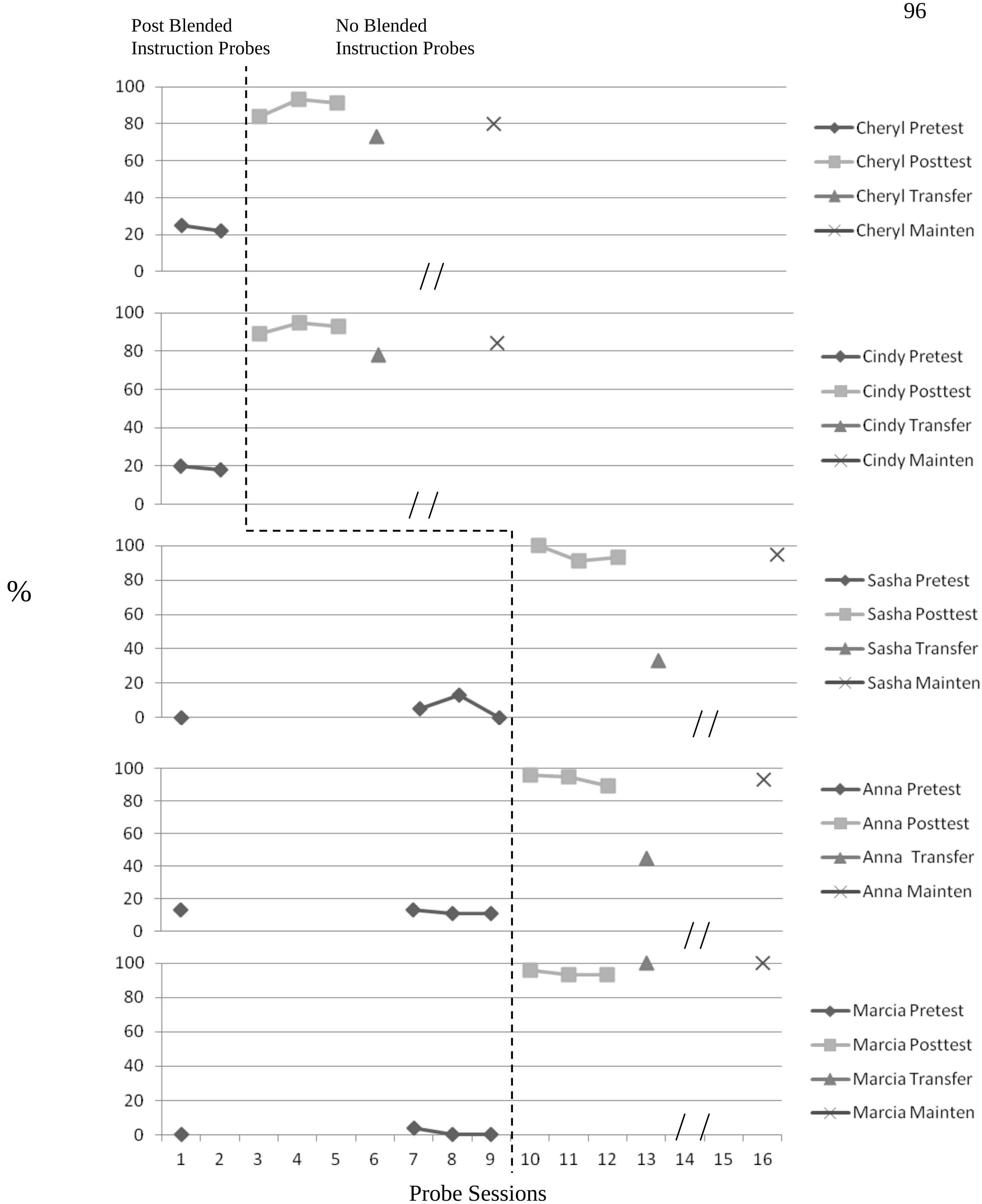



Figure 2. Domain Probes

Visual analysis of the graph indicates within-phase patterns and between-phase patterns. Specifically, an analysis of within-phase patterns indicates stability in level and trend with little variability of data points. An analysis of between-phase patterns indicates a dramatic increase in level with low variability within each phase (i.e., baseline and intervention), reflecting stability of performance. Table 4 provides summary data for each participant.

Table 4
*Average Percentage of Accuracy and Increases in Percentages for Domain Probes*

| | Baseline | Post-Intervention | Increase |
|---|---|---|---|
| Cheryl | 24%<br>(r = 22% - 25%) | 89%<br>(r = 84% - 91%) | 65 % points |
| Cindy | 19%<br>(r = 18% - 20%) | 92%<br>(r = 89% - 95%) | 73% points |
| Sasha | 5%<br>(r = 0% - 13%) | 95%<br>(r = 91% - 100%) | 90% points |
| Anna | 12%<br>(r = 11% - 13%) | 93%<br>(r = 89% - 93%) | 81% points |
| Marcia | 1%<br>(r = 0% - 4%) | 94%<br>(r = 93% - 96%) | 93% points |

To establish a greater understanding of the participant performance as relative to the 80% accuracy criterion, an analysis of post intervention domain probes was completed. This analysis revealed that all five participants successfully transformed quadratic expressions in factored form to standard form and vice versa. Specifically, the participants mastered the task of multiplying linear expressions and factoring quadratic trinomials. Additionally, all participants demonstrated competency in explaining the impact on the area after renovations (i.e., the area increased or decreased). Additionally, all participants demonstrated an understanding of the equality between the standard form and factored form of the quadratic expressions by comparing tables of data and graphs for each form of a specific quadratic expression.

An error analysis revealed that all participants missed points on the open ended questions based on three reasons: (a) they incorrectly responded to the question; (b) they provided faulty justification; or (c) they neglected to provide an explanation. Two participants responded incorrectly to an open ended question. When asked if the office workers would have more, less, or the same amount of classroom space after renovations, Cheryl responded, “The same. The shape just changed to a rectangle.” Additionally, Sasha missed points on an open ended question that asked if the shape of the area will change after the renovation and to justify the response. Sasha incorrectly responded that the shape remained a square, just a bigger square; however, the renovations changed the shape from a square to a rectangle.

Cheryl also provided faulty justifications. Specifically, when responding to the question, *What can be said about the shape and size of each family’s backyard? Explain how you know this*, Cheryl consistently referred to the backyards as getting bigger. She explained that “the y-axis graph shows us” and “There are a lot of positives.”

The other four participants neglected to provide justifications for some of the open ended questions. For example, in response to the question *Do the renovations change the shape or the squareness of the area of the dorm? Justify your answer*, Marcia wrote, “shape & area is diff – bigger area & now rectangle.” but, she did not provide justification for her response. Cindy and Anna wrote "They are equal" in response to the question, *What can you tell me about the standard form and the factored form of quadratic expressions?,* however, no explanation was provided. Sasha did not provide any justification for her incorrect response listed above. In addition to the error from the open ended questions, Anna missed points on the third post intervention domain probe as a result of not multiplying the binomials to complete the table. Specifically, she indicated that the area of any square dorm after renovations was represented as $(x + 5)(x + 4)$.

Increases in accuracy on algebraic tasks involving quadratic expressions were also measured by performance on lesson probes. Lesson probes assessed content specific to each lesson objective and were administered at the end of each lesson. All participants demonstrated high performance on each lesson probe and consistently met or exceeded the criterion (80% accuracy or greater). Cheryl earned a mean score of 99% (r = 95% - 100%), Cindy earned a mean score of 97% (r = 82% - 100%), Sasha earned a mean score of 96% (r = 82% - 100%), Anna earned a mean score of 96% (r = 89% - 100%), and Marcia earned a mean score of 95% (r = 83% - 100%).

**Research Question 2: Maintenance.** Maintenance of performance on algebraic tasks involving quadratic expressions was measured by a domain probe given four to six weeks after intervention. All participants demonstrated a high degree of retention of the content taught during intervention and reached the criterion score of 80% accuracy or greater. The mean score across participants equaled 90% with a range of 80% - 100%. Scores for participants in Group 1 averaged 82% accuracy. Specifically, Cheryl earned a score 80% and Cindy scored 84% accuracy. Scores for participants in Group 2 averaged

96% accuracy. Specific scores for Sasha, Anna, and Marcia were 95%, 93%, and 100% respectively.

**Research Question 3: Transfer.** Transfer of the participants knowledge of quadratic expressions was measured by their performance on a transfer measure that consisted of three tasks: (a) determining perimeter and volume expressions from a contextualized problem with accompanying tabular data; (b) multiplication of a trinomial by a four term polynomial; and (c) factoring a trinomial containing a quadratic term with a coefficient other than one. Participants' performance varied as evidenced by the percentage of accuracy ranging from 33% to 100% with a mean score of 66%. Scores for participants in Group 1 averaged 76% accuracy. Specifically, Cheryl earned a score 73% and Cindy scored 78% accuracy. Scores for participants in Group 2 averaged 59% accuracy, but contained a wide discrepancy among scores. Specific scores for Sasha, Anna, and Marcia were 33%, 45%, and 100% respectively.

**Research Question 4: Social Validity.** The mean score from the social validity measure equaled 4.3 (r = 2 – 5; mode = 4; See Table 5). All participants reported that they found the intervention to be beneficial and would recommend this intervention for other students. Results from the social validity measure indicated that the participants agreed or strongly agreed that the use of manipulatives and the box method helped them to multiply binomials and factor quadratic trinomials. Responses were mixed when asked about the benefit of word problems, tables of data, and talking about the problems with the teacher and/or peers.

Overall, participants responded positively on open-ended questions from the social validity measure. For example, Cindy commented, "It was visual and I actually understand it"   and Anna stated, "It helped me, the repetitiveness and the visual / hands-on thing."  When asked what they liked least about the intervention, Cheryl indicated that the intervention "lasted too long" as she "understood it very quickly."  Additionally, Sasha did not like "answering the problem after the actual work problem, like asking what happened to the shape."

Table 5

*Participants' Responses on Social Validity Measure*

| Questions | Cheryl | Cindy | Sasha | Anna | Marcia | Mean |
|---|---|---|---|---|---|---|
| I learned to multiply binomial expressions to form a quadratic expression. | 5 | 4 | 5 | 4 | 4 | 4.4 |
| I learned to factor quadratic trinomials. | 5 | 4 | 5 | 4 | 4 | 4.4 |
| The use of manipulatives helped me to multiply binomials expressions and factor quadratic expressions. | 5 | 4 | 5 | 5 | 5 | 4.8 |
| The use of the *box* helped me to multiply binomials expressions and factor quadratic expressions. | 5 | 5 | 5 | 5 | 5 | 5 |
| The word problems helped me understand what the expressions represented. | 4 | 3 | 3 | 4 | 5 | 3.8 |
| The data tables helped me understand what the expressions represented. | 5 | 3 | 4 | 4 | 5 | 4.2 |
| Talking about the problems with the teacher and/or classmates helped me understand. | 2 | 3 | 4 | 3 | 5 | 3.4 |
| This intervention was worth my time. | 4 | 4 | 5 | 4 | 5 | 4.4 |
| I would recommend this intervention to other students. | 4 | 4 | 4 | 4 | 5 | 4.2 |
| As a result of the intervention, I feel better about my algebra skills. | 3 | 4 | 4 | 4 | 5 | 4 |

**Interrater reliability.** Initial reliability on lesson probes, domain probes, and transfer probes was 97% (range 96-100%), with a reliability of 100% following discussion of differences. Disagreement occurred on the scoring of the open-ended tasks, specifically, when to assign full credit verses partial credit for a justification.

**Treatment fidelity.** According to an independent observer, the intervention was implemented as intended as 100% of the instructional components were identified. A second independent observer also viewed videos of three fidelity sessions. The interobserver agreement averaged 93% (range 80 – 100%) with fidelity equaling 100% following discussion of differences. Disagreements occurred on one session in which the objective, rationale, and review of big ideas at end of the lesson was initially overlooked by the second observer.

# Chapter 5: Qualitative Results

In this chapter I report results relative to the research questions that addressed the qualitative data. Specifically, the following research question is addressed.

When provided blended instruction with visual representations:

> How do the qualitative findings provide an enhanced understanding of the quantitative results? Specifically, what connections and disconnections to the algebra content emerge as a result of the intervention and how can these findings improve future instruction? In what ways does the intervention enhance aspects of metacognition?

To supplement the quantitative findings from the single subject design, a qualitative analysis of work samples, transcriptions of participation in instructional sessions, and my field notes were completed on one participant (Marcia). I decided to focus my qualitative analysis on Marcia because she provided a rich data source that was representative of the group, which Creswell (2007) refers to as a *critical case*. According to Creswell, a critical case refers to a significant case which permits generalization and application of information to other similar cases. Through a coding process and thematic analysis of these qualitative sources, two categories of themes (representations and metacognition) emerged from the data and are described in the section below (see Appendix P for the coding descriptions and examples; Figure 1 in Chapter 3 for the coding and theme development process).

## Representations

Representations referred to processes and products that were externally observable within Marcia's work samples, as well as to those that occurred internally as evidenced by statements made while she engaged in doing mathematics (NCTM, 2000). Processes included the steps that Marcia progressed through as she engaged with a task, while products referred to her final solution or response to a task, whether externally on a work sample or internally via her verbalizations. Internal and external aspects of processes and products are important to consider in school mathematics relating to connections and disconnections between elements of the intervention (i.e., area context, Lab Gear, and Box Method) and the algebra content. Through the Connections process standard, NCTM (2000) stresses the importance of students making connections or associations within and among mathematical ideas as well as connections to the real world and to other disciplines. However, Marcia did not always form these connections. At times she demonstrated a disconnection between the representations and the algebra content (i.e., a misconception regarding the representation and the algebra content). The following sections describe three themes that emerged under the category of representations: (a) connections and disconnections between the area context and the algebraic knowledge; (b) connections and disconnections between the Lab Gear and the algebra content; and (c) connections and disconnections between the Box Method and the algebra content.

**Connections and disconnection between the area context and the algebra content.** The area context referred to the representation of the *length · width = area* in three ways. First, the instructional unit incorporated contextualized word problems involving area situations. For example, square bedrooms were renovated so that the lengths and widths would increase or decrease. Second, tables of data were included that contained area problem containing discrete number examples and generalizing statements using variables (see Appendices F and K). Third, all non-contextualized problems were written using the area formula *length* · *width* = *area.*

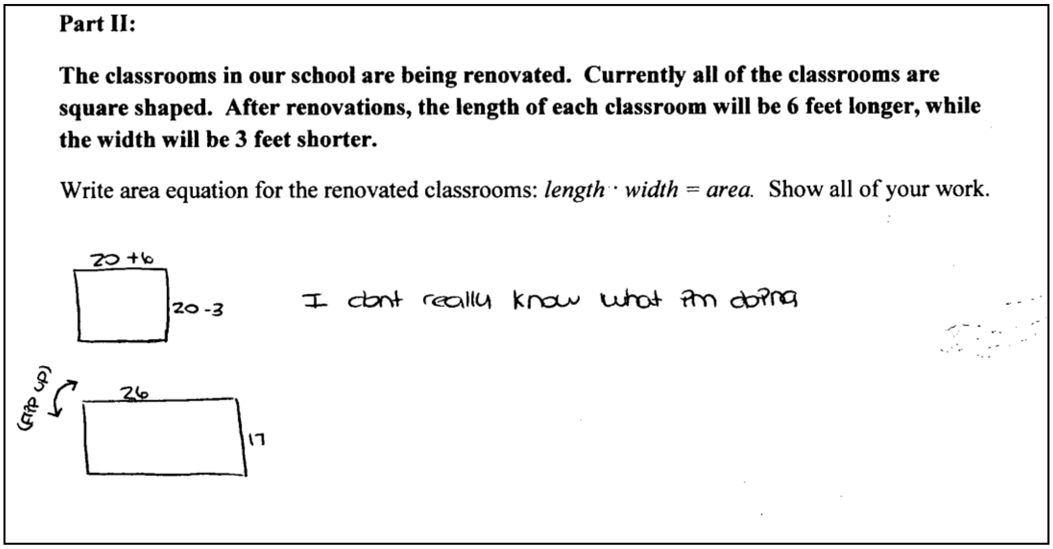
Figure 3 displays the first task on the first pre-test domain probe Marcia was given. As reported in the quantitative section, her percentage correct was 0% (see Figure 2 in Chapter 4). A further analysis of the qualitative data illuminates her difficulties. Marcia responded to this task by writing, “I kind of understand what this is talking about, but there is too many words and I can’t break it down.” She frequently complained about word problems and stated that she never understood what “they want me to do.” However, the area word problems, along with the tables of data, served as an anchor to the algebraic content as she completed tasks involving multiplying linear expressions to form a quadratic expression. Through this anchoring, Marcia demonstrated connections between the area context and three areas of algebraic content: (a) existing knowledge; (b) abstract symbolism; and (c) conceptualization of area quantity. Additionally, Marcia demonstrated disconnections between the area context the qualitative process of factoring and the purpose of completing tables of data in standard form and factored form. The connections and disconnection are described below in the order in which the tasks were presented within the instructional unit.

**Many hotels in the cities have square swimming pools that are surrounded by a deck. The deck increases the length of the pool area by 8 feet and increases the width of the pool area by 6 feet. The dimensions of two area hotel swimming pool are recorded in the table below.** Fill in the table below to determine the area of the swimming pools with the surrounding deck.

| | Side of swimming pool in feet | Length of pool + deck in feet | Width of pool + deck in feet | Area of pool + deck in square feet |
|---|---|---|---|---|
| Hilton Pool | 10 | | | |
| Double Tree Pool | 15 | | | |
| Any square pool | x | | | |

Figure 3. First task presented to Marcia during pretesting.

***Connection between area context and existing knowledge.*** On her second pretest domain probe (i.e., 4% accuracy, see Figure 2), Marcia appeared to be using the area context to anchor her existing knowledge with the new content. When presented with the task in Figure 4, Marcia wrote, “I don’t really know what I ‘m doing” although, her drawings showed a connection with the area context. As shown in Figure 3, she chose a discrete number (i.e., 20) to represent the sides of the original square bedroom. Then she drew an example using discrete numbers; however, she did not know how to write generalizing statements of the algebraic expressions to represent the dimensions and the area. However, she demonstrated critical foundational knowledge (i.e., adding 6 to the length and 3 to the width) necessary for engaging in this instructional unit.


**Part II:**

**The classrooms in our school are being renovated. Currently all of the classrooms are square shaped. After renovations, the length of each classroom will be 6 feet longer, while the width will be 3 feet shorter.**

Write area equation for the renovated classrooms: *length* · *width* = *area*. Show all of your work.



Figure 4. Response on Marcia's pretest domain probe.

***Connection between area context and abstract symbolism.*** The discrete numbers in the tables served as an important component within the area context as it provided Marcia with a connection to abstract symbols. Marcia immediately experienced success completing tables of data such as shown in Figure 3 as evidenced by her performance on posttest and maintenance domain probes in which she scored 100% accuracy on completing the tables of data (see Figure 2). She realized that, "whatever I do to the numbers, I do to the variable" so she easily created the linear algebraic expressions. For example, in Figure 5, Marcia completed the table and indicated that $(x + 3)\ (x + 2) = x^2 + 5x + 6$ was the area for any renovated classroom in this context. She accepted the quadratic expression as the product of the dimensions $(x + 3) \quad (x + 2)$ because she was able to "see" the original classroom and then the renovation within the Lab Gear representation. For example, when asked where in the quadratic expression the original classroom was represented, Marcia pointed to the x-squared block. However, she struggled to accept that the quadratic expression was "an answer" to the multiplication problem when no context was provided. For example, at the end of lesson 1, participants were provided non-contextualized problems to practice using the Lab Gear to solve without the use of accompanying data tables:

> Use the algebra blocks to multiply the following binomial expressions. Sketch the blocks. Then complete the area equation: *length · width = area.*

When presented with her first non-contextualized problem, *(x + 3) (x + 5),* to multiply using the Lab Gear, Marcia immediately responded, “I don’t get this” despite having successfully multiplied two linear expressions a few minutes earlier.

Marcia: I don’t get this.

TS: Show me what x + 3 is going to look like.

*Marcia correctly represents this with blocks.*

TS: Show me what x + 5 is going to look like.

*Marcia correctly represents this with blocks.*

TS: Now I want you to find the area. Because this is length times width (pointing to blocks). So multiply these binomials.

Marcia: I know how to make a picture, I don’t know how to actually do this.

TS: Well, you are doing it.

Marcia: no you said to add to multiply these. I don’t know how to do that.

TS: You are doing it.

Marcia: no I am making a picture. I’m not equaling, I’m not giving you an answer. I’m just drawing a picture.

TS: What would be an answer?

Marcia: Like plus like adding it together. Making it an answer. I don’t know.

TS: Your answer is inside (pointing to the blocks). This represents the area.

Marcia: You asked me to tell you what (x + 3) (x+5) equals. I don't get it. This is frustrating me.

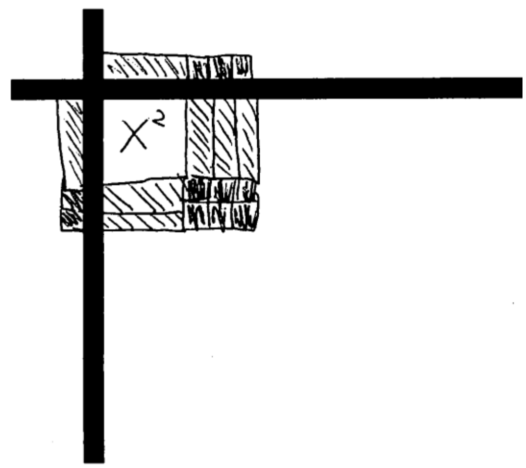
Marcia's connection to the concrete area context representation interfered with her ability to generalize to symbolic notation. She wanted to solve for *X* and have a number as an answer. When previously presented with tables of data, she visually saw that the value of *X* varied and accepted a quadratic expression as an answer. Without the area word problem and table of data, she did not recognize that multiplication of the linear expressions to form a quadratic expression was the completion of the task, or in her words "an answer." Marcia was not ready to work in the purely abstract, despite given concrete manipulatives and the area equation. She needed the area context to justify her quadratic expression as representative of the area.

**All of the classrooms at the Yeshiva are currently square shaped. All of these classrooms will be renovated so that the length will be increased by 3 feet and the width will increase by 2 feet. The dimensions of two classrooms are recorded in the table below.**

Fill in the table below to determine the area of the renovated classrooms at the Yeshiva. To find the area, multiply the *length* · *width.* Show all of your work.

| | Side of the current classroom in feet | Length of new classroom in feet | Width of new classroom in feet | Area of new classroom in square feet |
|---|---|---|---|---|
| Room 205 | 10 | 13 | 12 | 156 |
| Room 206 | 11 | 14 | 13 | 154 |
| Any Yeshiva classroom after renovations | x | $x+3$ | $x+2$ | $(x+3)(x+2)$ |

To multiply the two linear expressions, use algebra blocks and the corner piece. Then sketch the blocks.




Write the area equation: *length* · *width* = *area*

$(x+3)\ (x+2) = x^2+5x+6$

Figure 5. Lesson 1

***Connection between area context and area quantity.*** The area context was also an anchor to help Marcia conceptualize what was happening mathematically to the area. When responding to the open-ended questions embedded in lesson and domain probes, she connected the area context to quantity as demonstrated in her explanations with a satisfactory degree of accuracy. Upon disaggregating data found in Figure 2, she scored and an average of 79% accuracy on open-ended questions relating to area quantity on her posttest and maintenance domain probes and an average of 80% accuracy on open-ended questions on lesson probes. However, she may have scored higher if she had chosen to dictate her responses as written language difficulties may have affected her score. Marcia's explanation in class clearly demonstrated a connection between the area context and the area quantity. For example, when multiplying linear expressions with positive terms, she easily recognized that the area became larger with more space. However, after the introduction of negative terms, answering the question of more, less, or the same amount of space became more difficult. Marcia relied on the area context, specifically the use of discrete numbers, to explore this as evidenced by the transcription below.

> TS: I am going to sketch Latanya's office. Her current office is 5 by 5. Her new office will be 4 feet longer in length and 2 feet shorter in width. (I am drawing this on the graph board). Will she have more or less room in her new office?
>
> Anna: more
>
> TS: How do you know that?
>
> Anna: She has 2 feet more. You added 4 and took away 2.
>
> TS: Let's talk about that.
>
> Marcia: Multiply the diameter and see what area is bigger. *(She means dimensions not diameter)*
>
> Anna: You need to take these 2 feet and move them.
>
> TS: Feet is a linear term. I can represent feet as a straight line. Area is what type of unit?
>
> *Pause*
>
> TS: When we count area are we counting straight lines?

All: squares

TS: So we are talking square units. Marcia, where were you going with this?

Marcia: Multiply the diameters to find the areas see the areas for comparisons. One area was 5 times 5 and the area was 25. Now you added 4 so it's 9 times 3 which is 27.

TS: So in this case it got bigger. (I draw on graph board)

Anna: By 2 feet.

TS: Yes, but it's not always going to be 2. The other one was 6. I increase it by 4 and decrease it by 2. So now I have 4 times 10,and it gives me 40. (I am drawing on graph board)

Sasha: So is it more or less?

TS: Let's keep talking about it. So now I have the teeny tiny office. It's only 3 feet by 3 feet. I only have 9 square feet. I can barely squeeze my desk in it. If I add 4 feet in length and take away 2 feet in width, I have a 7 by 1. What's the area? (I am drawing this on the graph boards)

All: 7

TS: So here, I have less space. What do you think the difference is? Why were these more space and this one less space? (I am pointing to the graph boards with the sketches)

Anna: It depends upon how much space you had to begin with. The smaller office will lose space and the bigger ones will have more space.

Marcia responded to a similar type of question on her post-test domain probe by drawing different size rooms on graph paper and wrote

> I thought the same amount of space but diff shape but when I drew it out it got all messed up- it also depended on how long the square is.

She incorporated the area context to connect mathematically to what was happening to the amount of space per area when multiplying linear expressions containing positive and negative terms. Despite the mathematical connections between the area context and multiplying linear expressions, these connections did not generalize to factoring quadratic expressions.

***Disconnection between area context and qualitative process of factoring.*** Participants were given a specific x-value to substitute in a quadratic expression when learning to factor quadratic expressions, and then drew sketches of rectangles that had the given area. For example, when presented the quadratic expression of $x^2 + 3x + 2$, Marcia was given an x-value of 4, Sasha had an x-value of 3, and Anna had an x-value of 5. The participants drew rectangles for their specific areas. For example, Marcia had an area of 30 so she drew a 1 x 30 rectangle, a 2 x 15 rectangle, and a 5 x 6 rectangle. Sasha had an area of 20 and she drew rectangles that were 1 x 20, 2 x 10, and 4 x 5. Anna had an area of 42 and she drew four rectangles (1 x 42, 2 x 21, 3 x 14, 6 x 7). Using the discrete numbers and the area context, participants examined the qualitative representations to discover the generalizing algebraic expressions that represented the dimensions of a given quadratic expression (i.e., area). For example, Marcia used her x-value of 4 and determined the dimensions of the 5 x 6 rectangle could be represented as x + 1 and x + 2.

> TS: What happens to YOUR x value for each of the dimensions?
>
> Marcia: What does this have to do with the dimensions? I don’t like this at all.
>
> TS: OK ladies, do we have anything that is the same? *(no response from the students)*

TS: So I have x + 17 and x – 2. Does anyone have that?

Sasha: I have x-2

TS: But do you have both of the dimensions?

Sasha shakes her head.

Marcia: How is that possible? We all have different areas.

TS: Marcia, give me one of your dimensions.

Marcia: x + 2 and x + 11

TS: Does anyone else have x + 2 and x + 11?

Sasha and Anna: No

TS: Ok, Sasha what do you have?

Sasha: x + 3, x – 5

TS: Anyone have that?

Marcia and Anna: no

TS: Anna, give me one of yours.

Anna: x + 1, x + 2

TS: Does anyone have x + 1, x + 2?

Sasha: I do!

TS: I do too! Marcia do you?

Marcia: maybe, yes I do

TS: Right there is what we all have in common. That's our dimensions for the area $x^2 + 3x + 2$.

Despite demonstrating mastery of this process on the lesson probe (i.e., 94%), Marcia was not convinced that this was the best way to find the dimensions of our area. The following lesson she stated; “What’s the point of drawing all of those rectangles? It takes so much time.” She was missing the connection of the area model as a means of finding dimensions when given the area in the standard quadratic form. This process was laborious partially because Marcia was not fluent at her math facts, which may have interfered with her ability to make this connection and a common characteristic of students with math LD (Geary, 2004).

***Disconnection between area context and factoring.*** Additionally, Marcia found the word problem and tables of data for the quadratic expressions as meaningless. When presented the task in Figure 6, she stated, “All those words don’t matter.” Prior to this, Marcia had already used the algebra blocks to discover the rules for factoring quadratic expressions. Therefore, she did not want to waste her time reading the word problem and completing the table, despite the contextualize problems helping her as previously noted. She was completely disinterested in completing this task as evidenced by the following transcript.

**All of the dorms at Tower 1 at University of Silver Spring have an area that can be represented as** $x^2 + 9x + 20$.

*Fill in the table to represent the areas of the specific students' dorms in Tower 1. Remember to use your cue card to help you use the graphing calculator.*

| *Name* | *x-value in yards* | *Area of dorm using $x^2 + 9x + 20$* |
|---|---|---|
| *Tzivia* | *4* | |
| *Tova* | *5* | |
| *Tricia* | *6* | |

*Find the dimensions of each student's dorm by factoring the quadratic expression that represents the area.* ***You may draw your own*** **box**. *Complete the area equation: area = length · width.*

Figure 6. Contextualized problem for a quadratic expression.

Marcia: All those words don't matter.

TS: It gives us context. Remember when we multiplied the length times the width, we got the area. Now we have the area. Now I want you to do is use the calculators, and using the y= plug $x^2 + 9x + 20$ into the calculator and fill in the table.

*Girls successfully complete table and look at each others to compare.*

TS: Now I want you to tell me the dimensions. What are dimensions Sasha?

Sasha: The outside

Marcia: the perimeter

Sasha: length times width

TS: In order find the dimensions, what do we have to do? (pause). We are given $x^2 + 9x + 20$

*Marcia Yawns*

Sasha: So, oh my gosh Marcia!

Marcia: What paper are we on?

Sasha: So we are going to do x + 4

Marcia: Why are we doing x plus 4?

Sasha: We want to find the dimensions.

When completing the lesson probe, Marcia skipped the narrative and worked only with the abstract symbols. Additionally, when asked to compare tables of data containing discrete numbers and different forms of the quadratic (i.e., standard or factored), Marcia only filled in the data while ignoring the context. Despite this, she scored 100% on the lesson probe and only observations from the instructional session revealed this disconnection.

Initially, Marcia appeared overwhelmed by the word problems and tables of data used within the area context. For example, when asked to look at an area problem with the accompanying table of data, she responded by saying "Wow, there's a lot going on here." She also stated that she did not "do" word problems. Although the area context provided an anchor to help her develop an understanding of representations of quadratic expressions, this connection was not evident when factoring quadratic expressions. In fact, Marcia felt that the area context "pointless."

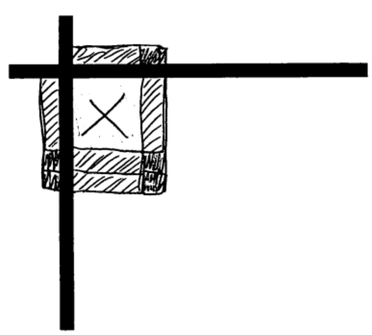
**Connections and disconnections between the Lab Gear and the algebra content.** The Lab Gear was incorporated into the instructional unit to support the process of multiplying linear expressions and factoring quadratic expressions. As shown in Figure 7, participants physically manipulated the Lab Gear to represent the area context, sketched the blocks, and then symbolically wrote the area equation. They also used the Lab Gear to discover rules for factoring quadratic expressions and to factor the expressions using procedures based on algebraic properties and concepts. Specifically, the following sections describe Marcia's connections with the Lab Gear to the Zero Principle, negativity, rules for factoring quadratic expressions, and the comparison of quantity. Disconnections between the Lab Gear in relation to the distributive property and making literal interpretations are also described in the order in which the content was addressed within the unit.

*All three of us will have our bedrooms renovated. After renovation, the length will increase by 2 feet and the width will increase by 1 foot. What are the measurements of our renovated bedroom?*

| | Side of original bedroom in feet | New length in feet | New width in feet | New Area in square feet |
|---|---|---|---|---|
| Ms. Tricia | 9 | 11 | 10 | 110 |
| Student 1 | 10 | 12 | 11 | 132 |
| Student 2 | 11 | 13 | 12 | 156 |
| Any square bedroom | x | x + 2 | x + 1 | $x^2 + 3x + 2$ |

Use algebra blocks to determine the area of any square bedroom after it is expanded. Sketch the blocks.

To multiply the two linear expressions, we will use algebra blocks and our corner piece. Then sketch the blocks.



Write area equation: *length* · *width* = *area*_________________________________________

Figure 7. Sample task using Algebra Lab Gear.

***Disconnection between Lab Gear and distributive property.*** Participants were introduced to the procedure of multiplying binomials through the use of the Lab Gear. Specifically, participants were guided to use the visual cues associated with the Lab Gear program, such as all blocks inside the corner piece form a rectangle. Below is an excerpt from Lesson 1 which demonstrates Marcia's initial procedural fluency with the Lab Gear.

> Marcia: Can I see my paper from yesterday? I forget how to do this. Oh, I remember, you do the two things in parentheses. I know it's x plus 3 times x plus 2 equals. Then don't you do the equals with this? *(Marcia is pointing to the corner piece to where the blocks would go).*
>
> *Sasha is sitting there not knowing what to do.*
>
> TS: Marcia, you are doing such a great job. Can you tell Sasha what you are doing?
>
> Marcia: I'm writing out my problem over here *(pointing to the manipulatives.)* I have an equation and I am breaking it up and multiplying because this is a multiplying bar and this is x and this is x so I have x plus 3 so $x + 3$ times and this is timesing it plus $x + 2$ and that's going to equal x, so now I have x *(pointing to x squared).*
>
> TS: what's this?
>
> Marcia: x squared. I don't know if I am doing it right.
>
> TS: You are doing beautifully.
>
> Marcia: So now I have x squared. So now it's x plus 3 times x plus 2 equals x squared plus 5 x plus 6. *(Marcia is writing symbolically as she points to the manipulatives and talks out loud).*

Although Marcia referred to "timesing" the linear expressions, she was not able to articulate why the blocks were placed in the position in which they were placed, other than to say that they "fit" and that she was "making a picture." Although she consistently scored 100% accuracy on lesson probes that assessed this content (i.e., Lesson Probes 1 and 2), Marcia's initial procedural fluency did not demonstrate a connection to the Distributive Property, which was the critical algebraic concept in multiplying expressions.

***Connection between Lab Gear and the Zero Principle.*** During Lesson 2, Marcia continued to demonstrate procedural fluency when multiplying linear expressions; however, she also connected this procedure to the algebraic concept of the Zero Principle. The Zero Principle refers to two opposites equaling zero. The transcript below demonstrates Marcia's connection between the procedure of multiplying $(x + 4)\ (x - 2)$ with the Lab Gear and the Zero Principle.

> Marcia: So I am doing x plus 4 times x minus two so, now I have an issue. This doesn't really work. So this is negative and this is positive so it cancels. 4 times negative 2 is negative 8. Negative 8. Ok, it's negative 8. So it's x squared minus 2x plus 4x minus 8. I am confused.
>
> TS: You already told me what we are going to do with these.
>
> Marcia: So I cancel now. Is this right? I don't think it is.
>
> TS: Look at the middle, linear term. Think about what you said to me.
>
> *Marcia works a little longer.*
>
> TS: So we have 4 positive x's and 2 negative x's.
>
> Marcia: Cancel each other out.

TS: You are absolutely right. So how many x's are we left with?

Marcia: 2 x's

In this example, Marcia recognized that positive 2 and negative 2 were zero pairs that cancelled out and then she was left with 2x. She physically removed the blocks from the representation. However, she was initially concerned that by removing the blocks she was violating the rules for making the picture, (i.e., the blocks inside the corner piece needed to form a rectangle). Marcia stopped physically removing the blocks, despite being reassured that she was still following the rules, but still correctly applied the Zero Principle when completing tasks on lesson probes and domain probes which assisted her in achieving high scores on her probes (see Figure 2).

***Connection between Lab Gear and negativity.*** Marcia also connected the Lab Gear representation to the concept of negativity. The concept of negativity refers to the relationship between taking away (i.e., subtraction) and adding a negative number (i.e., a number less than zero). When Marcia first encountered tasks involving linear expressions with negative terms, she initially struggled with the Lab Gear representation.

TS: Using your lab gear, how do you think we would multiply x times x minus 2?

Sasha: I don't know what x minus to is?

TS: Remember your negative blocks.

Marcia: This? *(Holding up a constant with a negative sign)*

TS: This is a minus 1. I like that. What about this? *(Pointing to a negative yellow block)*

Marcia: Well if we did minus, minus it would be plus.

TS: Remember we are just counting them.

Marcia: That really makes sense because I don't agree with this. I am still adding. Look, I am adding blocks not subtracting them.

TS: But remember you can also write a subtraction problem as an addition.

Marcia: So this is x plus negative 2.

*Girls set up blocks and multiple x(x-2)*

TS: Now the thing that is kind of misleading with the blocks is that even though these are negatives, it still makes it physically bigger.

Marcia: Yeh, that's what I was saying!

TS: Good. Some students like to do this. *(I place the negatives on top of the x square block)*

Marcia: That's even more confusing because it looks like a square.

TS: You don't have to do this. The thought is that only what is showing represents the area. But if you don't like to stack this, don't stack. It's just another way of using the blocks.

Marcia was provided two ways to represent negative terms. First, blocks had a negative sign and she could "add" the negative blocks to the x-bar to indicate an expression such as $x - 2$. Additionally, she was shown how to "stack" blocks. For example, two constant blocks could be stacked on top of the x-bar to represent $x - 2$. This provided students with a representation of the x-value decreasing in size. Marcia chose to use the blocks with the negative sign and conceptualized subtraction as adding a negative as evidenced by the transcript above. However, on all lesson and domain probes, she consistently wrote the expressions with a subtraction sign (e.g., x – 2) rather than and addition sign (e.g., x + - 2), which was the choice Cindy, Anna, and Sasha.

***Disconnection between Lab Gear and literal interpretation.*** Although Marcia was procedurally fluent at manipulating the Lab Gear to multiply linear expressions; her connections to the Lab Gear were not always linked to mathematical concepts. Often Marcia was simply "making a picture" as described previously. Additionally, she demonstrated a disconnection between the Lab Gear representation and the area context. For example, when comparing the representation of $x^2 + 5x + 6$ with drawings on graph boards of bedrooms using discrete numbers, Marcia demonstrated a desire for the Lab Gear to literally represent the area of the bedroom.

> TS: How is the Lab Gear representation the same as what you drew on your graph boards using discrete numbers?
>
> Anna: Because it's a length plus 3 and a width plus 2.
>
> Marcia: 3 x-bars and 2 x-bars. This is our original room *(pointing to the $x^2$ block).* This doesn't fully work because that's bigger. Like the 3 x-bars are bigger. So if I had a room that was 5 feet by 5 feet, and I added that, that's more than a foot in comparison to that so that's off.
>
> TS: But we can not say anything about the relationship between this to this. We can't say 4 x-bars equals one x square block. We can't say 3 constants equals an x. Remember x could be 2 or 200. It's just a representation.

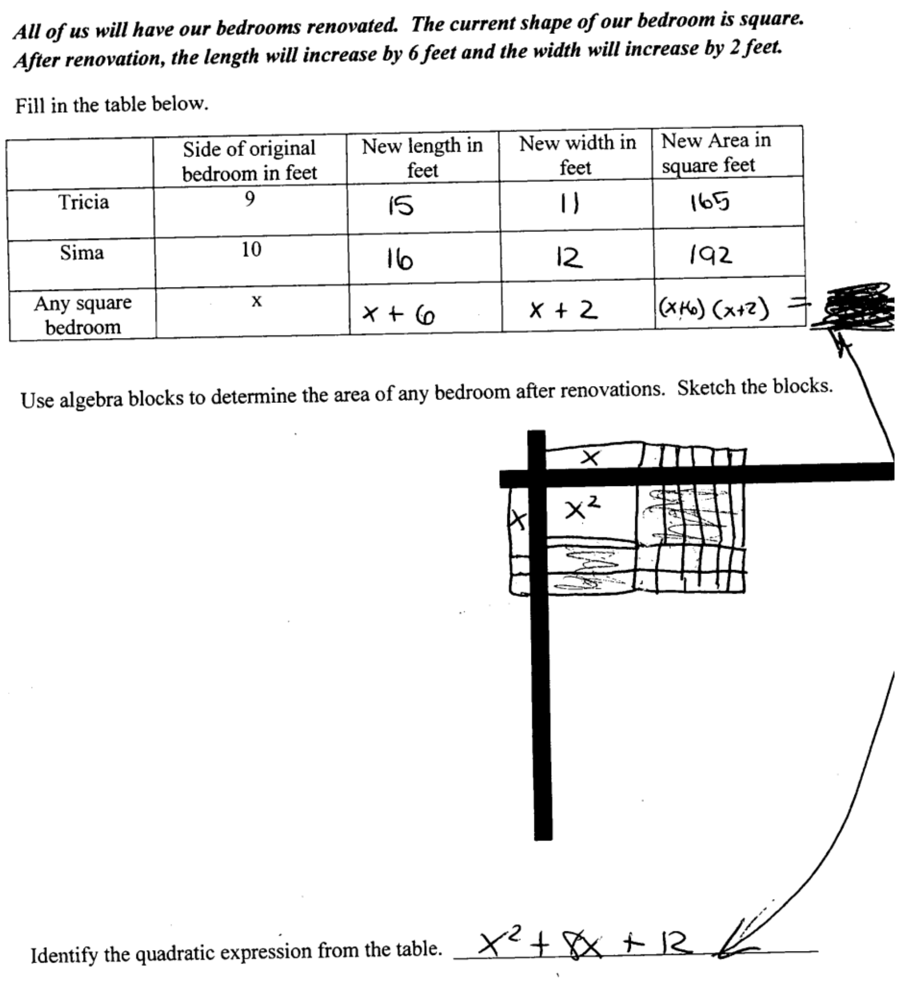
This disconnection resurfaced the following day when she was presented with the word problem below.

> *Everyone in our class just won the opportunity to have our bedrooms renovated for the TV show Extreme Makeover. Currently, all of our bedrooms are shaped like a square and all of us have different size bedrooms. After the renovations, the dimensions of our all of our bedrooms will be increased by 4-feet in length and 3-feet in width.*

Marcia successfully completed a table of data with discrete numbers and wrote the algebraic expressions generalizing the context. She also used the Lab Gear to correctly multiply the linear expressions to produce the quadratic expression. However, a disconnection emerged during our discussion of the Lab Gear representation.

> TS: Where in this representation *(the blocks)* is our original bedrooms?
>
> *All girls point to the $x^2$ block.*
>
> Sasha: And here is the extra 4 feet and the 3 feet. *Pointing to the other blocks.*
>
> TS: Yeh great. This is a representation of how we could generalize what happened to all of our bedrooms.
>
> Marcia: It's not an accurate representation, like when you make a blue print.
>
> TS: You are absolutely right. It's not like a blueprint. It's a generalized statement saying that we all are going to have this expansion in the length and the width in the same way. You are absolutely right, it's not like when we were using discrete numbers.
>
> Marcia: Why didn't they make these *(pointing to x-bar)* like half the size so that it would be?

Again Marcia demonstrated a desire for the Lab Gear to literally represent the area of the bedroom, in other words, she was very literal, which is a common characteristic of students with LD (Bley & Thorton, 2001; Garnett, 1998; Geary, 2004; Witzel, 2005). Although these were disconnections between the Lab Gear and the algebra content, Marcia demonstrated logic and reasoning while processing how the Lab Gear served as a generalized representation of the area context, as evidenced in the previous transcriptions. As the intervention progressed, she developed an understanding of the Lab Gear as a representative tool and did not attempt to make a “blueprint" for the algebraic expressions. Her sketches reflected the sizes of blocks, not proportions reflective of the expressions. For example, in Figure 8, Marcia’s sketched the six constants on top of the corner piece as longer than the x-bar, despite the given values of the x being 9 and 10. Marcia was developed an understanding of the Lab as a generalized representation.



***All of us will have our bedrooms renovated. The current shape of our bedroom is square. After renovation, the length will increase by 6 feet and the width will increase by 2 feet.***

Fill in the table below.

| | Side of original bedroom in feet | New length in feet | New width in feet | New Area in square feet |
|---|---|---|---|---|
| Tricia | 9 | 15 | 11 | 165 |
| Sima | 10 | 16 | 12 | 192 |
| Any square bedroom | x | x + 6 | x + 2 | (x+6) (x+2) = |

Use algebra blocks to determine the area of any bedroom after renovations. Sketch the blocks.



Identify the quadratic expression from the table. $x^2 + 8x + 12$

Figure 8. Lesson 1 Probe.

***Connection between Lab Gear and rules for factoring quadratic expressions.***

Participants were taught to factor quadratic expressions by following the same process used when multiplying linear expressions, (i.e., the blocks inside the corner must form a rectangle). At this point in the instructional unit, Marcia was given the quadratic expression (i.e., the area), and she realized that she needed to "work backwards" to find the linear expressions (i.e., the dimensions). She reached this conclusion by referencing the area context (i.e., the area was provided and she needed to determine the dimensions) and also by "undoing" the Lab Gear representation.

During lesson 7, participants analyzed representations of the Lab Gear and looked for patterns that would lead to discovering rules for factoring. Initially, Marcia only wanted to sketch the Lab Gear, but could not determine how to arrange the five *x'*s to form the rectangle inside the corner piece.

*Marcia working on* $x^2 + 5x + 4$

Marcia: Makes 4 and 5? 3 and 2? No. How do you get 5?

TS: Use the tiles if you need to.

Marcia: Why won't you help me? I'm trying to figure out using 5 bars how do I get 4 inside.

TS: Use the blocks. *(giving Marcia blocks)* Play around a little bit.

Marcia wanted me to tell her the answer, but I wanted to her to discover her own patterns with use of the tiles. Participants appeared uncomfortable with having a teacher whose role was facilitator. Similarly to Marcia, participants were most interested in getting the correct answer rather than the process. At times participants would become frustrated when asked to justify their responses and confirm their answers using another method (i.e., Lab Gear, Box Method). However, with encouragement they discovered the patterns related to transforming quadratic expressions.

Marcia: Oh I am so stupid! Marcia, you're an idiot! 4 and 5 I have a picture. Can I tell you my trick that I just figured out?

TS: Yeh, yeh, what's your trick?

Marcia: That, whenever you have a number, that is one more than the amount, than the constant, then you are multiplying basically by 1. All the bars go here except for one.

TS: Marcia that's a great pattern.

Marcia observed that the factors of the constant in the quadratic added to the coefficient of the linear term in the quadratic. The Lab Gear provided a visual representation that supported this discovery across two instructional sessions. Marcia was able to see that the x-bars needed to be "split up" to fit into the rectangle inside the corner piece. Below is an excerpt when discussing the dimension of the quadratic expression $x^2 + 4x + 3$.

TS: How do we determine the middle term in the quadratic, your linear term?

Marcia: That goes to how many constants are on the outside. They're 3 constants here *(pointing on top of corner piece)* and 3 bars here *(pointing inside of corner piece).* There's 1 constant here *(pointing to side of corner piece)* and 1 bar here *(pointing inside corner piece).*

Anna: three plus one

TS: does that work for every single one?

Anna: Yeh

TS: Let's confirm.

Although Marcia previously noticed that the constants in the linear expressions multiply to equal the constant in the quadratic, she had difficulty discerning the relationship between those factors and the coefficient of the linear term in the quadratic. This was developed with the Lab Gear as evidenced in the above transcription. Marcia's discovery of the x-bars "splitting up" to fit into the Lab Gear representation further developed into the rules she used for factoring with only abstract symbols.

***Connection between Lab Gear and comparison of quantity.*** In addition to the Lab Gear supporting Marcia's development of rules for factoring, the use of manipulatives supported her conceptual understanding of the quantities of various quadratic expressions. For example, when comparing the Lab Gear representation of $x^2 + 4x + 3$ and $x^2 + 4x + 4$, Marcia initially had difficulty seeing that one area was larger.

Marcia: Well it's the same shape. It's the same size and same shape.

Anna: Like it got longer

Marcia: No, this one got a little this way and a little that way. It's the same amount of area. How is that possible? Can I tell you why this doesn't make sense? These are fillers *(constants).* So now look. Now it's 2 and 2. I took this and moved it here. It's the same amount of x bars. The only thing that changes is that we need one more box to fill. But this is the same amount. *(pointing to number of xbars)*

TS: That part is the same. But what is not the same, is the constant.

Marcia: The area is still the same.

TS: Almost the same, but we have gone form having 3 of these to having 4 *(pointing to constants)*. So what happened to the quantity?

Marcia: Ok, more, but I'll tell you why it's confusing me. It's a very low number. It only went up by 1 so you can like assume like when you look at the picture there 3 here there's 1 here. There's 2 here, there's 2 here. Ok, move that one here and they're the same. Ok one makes a difference but if it was bigger you would be able to see that better.

Marcia realized that the quantities were very similar with one area only "up by one." The Lab Gear also supported Marcia's comparison of quadratic expressions with a different coefficient for the linear term. As described below, we discussed the representations of the quadratic expressions $x^2 + 4x + 4$ and $x^2 + 5x + 4$, which occurred the same day as our previous discussion.

Marcia: It's different because when there's more x's there's more constants. When there's more x's the whole thing changes.

TS: Now Marcia brought up a really good point earlier. When we went looked at these two top ones, there was really very little change in area. ($x^2 + 4x + 3$ *and* $x^2 + 4x + 4$). What do you think adding one extra

Marica: It can do a lot depending on where you add it.

T: So when we change this x, depending upon what x equals when we change out linear term that will tell us what our change in our product is. Or the change in our final answer. So, if 8 is x and we change the linear coefficient from 4 to 5. It will increase by 8, because that is what x is. If x equaled 20, what do you think the change would be?

Anna: 20

TS: But if I change the constant, literally, the quantity is only being changed by one. Changing this middle term has a greater impact in the overall area than changing the constant.

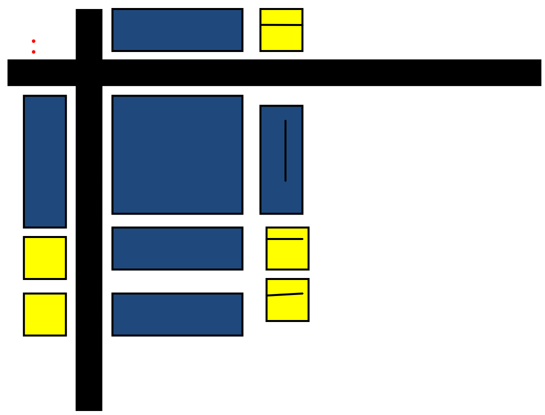
Marcia: Well yeh, you can just look at it and see that it is bigger than this.

Marcia relied on the visual representation from the Lab Gear to discuss the impact of increasing the linear term on the area. She saw that this increase resulted in a larger picture than the increase of only one constant. Her analysis was based on a literal interpretation of the Lab Gear and the x-bar being physically bigger than the constant block. Therefore, I used discrete numbers to help concretize the impact of changing the linear term of the quadratic; however, she still reverted to the visual representation of the Lab Gear. These behaviors mirrored her disconnections observed while multiplying linear expressions. Specifically, she interpreted the Lab Gear literately (i.e., and x-bar was bigger than a constant) and she gained a fuller understanding of the quantities represented by each quadratic expression when discrete numbers were substituted for the variable.

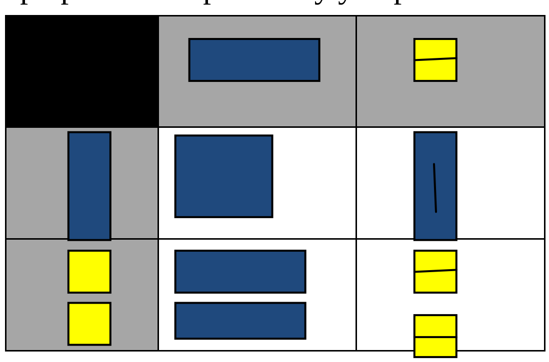
Throughout the intervention, Marcia used the Lab Gear to strengthen her procedural fluency with multiplying linear expressions and factoring quadratic expressions by supporting her discovery of the rules for factoring. Additionally, the Lab Gear assisted Marcia in conceptually understanding quadratic expressions as she engaged in discussions regarding the quantity of the area represented by similar quadratic expressions. The procedural fluency and conceptual understanding were evidenced by high scores on posttest and maintenance domain probes (see Figure 2). Despite these connections between the Lab Gear and the algebraic content, Marcia demonstrated a disconnection as she attempted to interpret the Lab Gear literally. She did, though, develop an understanding of the Lab Gear as a generalized representation as the intervention progressed.

**Connections and Disconnections between the Box Method and the Algebra Content.** The graphic organizer used during the intervention was the Box Method, which was based on similar methods found in *Lab Gear Activities for Algebra I* (Picciotto, 1995) and the *CME Project Algebra I* (2009) textbooks (See Figure 9). The purpose for incorporating the Box Method into the instructional unit was two-fold: (a) to support participants' procedural fluency of multiplying linear expressions and factoring quadratic expressions when using only abstract symbols; and (b) to model the Lab Gear representation to ensure a successful transition to symbolic notation (see Figure 9). The following sections describe Marcia's connections and disconnections between the Lab Gear and the Box Method in the order in which the tasks were presented within the instructional unit. Specific subthemes that emerged include (a) connections and disconnections with Lab Gear; (b) connections to the Distributive Property; and (c) connections to factoring.

*Our neighborhood swimming pool has a length that is 2-meters longer than the width. On Saturday mornings, a section of the swimming pool is roped off for swimming laps. That section makes the pool 1-meter shorter in width.*
Use algebra blocks to determine the area of the pool available for free swim after the lap lanes are roped off. Sketch the blocks.



Place the algebra blocks into the box in the spaces that you think make sense. Be prepared to explain why you placed the blocks in the spaces.



Complete the box below using only symbolic notation. Write the area equation.

| | $x$ | $-1$ |
|---|---|---|
| $x$ | $x^2$ | $-x$ |
| $+2$ | $2x$ | $-2$ |

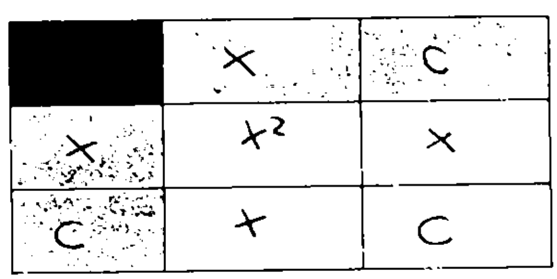
Figure 9. Transition from Lab Gear to Box Method

***Connection between Box Method and Lab Gear.*** Participants were introduced to the Box Method as they transitioned from working with the Lab Gear to working with only abstract symbolism. Initially, Marcia was reluctant to move away from the blocks, as she exclaimed, “Whoa Whoa, Whoa. What do you mean moving away from the blocks?” However, she demonstrated a connection between the Lab Gear and the box template I provided.

> TS: I have created this template. So what would go in the first white box? Marcia: x squared!
>
> TS: Yeh, that’s what you showed me.
>
> Marcia: So, 2x, negative x, and the constant
>
> TS: So what do you think will go in those gray spots?
>
> Marcia: x, no wait
>
> TS: Let’s put our blocks back together.
>
> Marcia: x plus the numbers
>
> TS: You are right!

Additionally, Marcia developed her own graphic representation that generalized the components within the box template (see Figure 10).

> Marcia: Look (showing me her paper)




> TS: Marcia, this is an excellent generalization. Show the girls what you did.

Figure 10. Marcia’s first Box.

Similarly, she initially labeled the sketches of the Lab Gear (see Figure 11). Marcia used these labels to help her remember what the blocks represented. However, by the end of Lesson 1, Marcia did not need to label the blocks in order to score 100% accuracy on the probe.

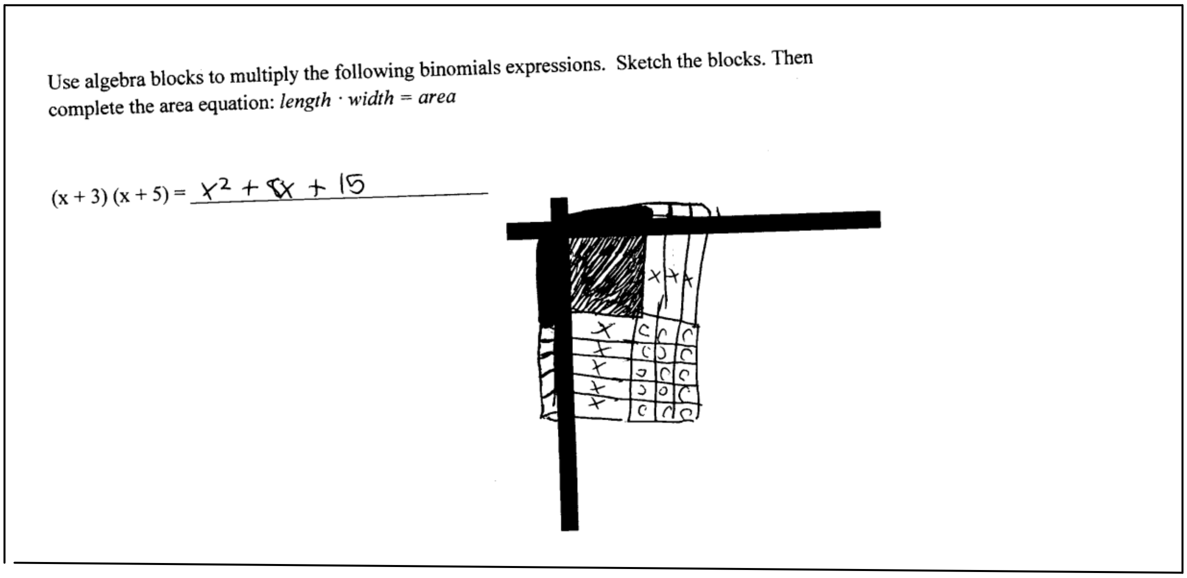



Figure 11. From lesson 1, Marcia labeled the sketches of Lab Gear.

***Disconnections between Box Method and algebra content.*** Marcia developed disconnections with the algebraic content when confronted with linear expressions with the constant before the variable and with linear expressions that had a coefficient other than one. For example, when given ( – 13 + 2x) (10 + x), Marcia wanted the terms with the variables first in the expression when using the Box Method.

TS: Just set this up. Don’t solve it yet.

Marcia: Can I do the numbers after the x’s?

TS: Show me what you mean?

Marcia: Can I do 2x – 13?

TS: yes, why do you want to do that?

M: (2x – 13)(x + 10). X's are always in this box *(pointing to the top left box of her graphic organizer).*

Marcia was extremely faithful to her first Box and consistently rearranged the expressions so that the variables would always be first. This worked well for her as she was also attentive to the appropriate sign for each the term. Marcia was rule driven and her Box provided the visual cues she needed to follow her rules. However, her desire to understand why these rules worked enabled her to develop a deeper conceptual understanding of quadratics.

An additional disconnection with Marcia's original Box involved expressions with coefficients other than one. When presented with *(3x + 15) (x -2)*, Marcia became confused as her Box did not have a coefficient with the quadratic term.

Marcia: I did x – 2 and then on the down I did 3x + 15

TS: So, what's going to go in this top white box?

Anna: x squared

TS: It's not x squared. Why not?

Anna: Because there's a 3x there.

TS: So what is it?

Marcia: 3x, x

TS: What was x times x using our blocks?

Sasha: Oh, $x^2$

Marcia: Oh, right.

Sasha: So, $3x^2$ because there's 3

Marcia: How are you getting this? This is not working for me. We never did it like that. It was always $x^2$. I need the blocks. I can't do this.

TS: Well, let's practice a little bit. We are going to multiply exactly like we multiplied the blocks but we are only using symbols. What is the area for this problem?

Marcia: I have no idea.

*Marcia gets the lab gear and sets up the problem x times 3x.*

Marcia: We never did problems like this so I didn't know.

As illustrated above, Marcia displayed difficulty with generalizing the Lab Gear representation to the Box representation. In fact, Marcia multiplied linear expressions with coefficients greater than one when using the Lab Gear in the previous session; however, she did not recall doing so. However, $3x^2$ as the first term of her quadratic expression did not fit with Marcia's initial Box in Figure 10, as this representation contained only the term $x^2$. After confirming with the Lab Gear that $3x$ times $x$ equaled $3x^2$, she consistently multiplied similar terms correctly on probes.

***Connection between Box Method and the Distributive Property.*** Although Marcia learned to accurately fill in the provided Box Method template (see Figure 9), she did not draw this template when given the choice. Additionally, her first Box (see Figure 10) was not being used as an appropriate tool to help her make the mathematical connections. Marcia needed to accurately conceptualize the process of multiplying linear expressions, which is based on the Distributive Property. Therefore, her graphic organizer needed to evolve. The transcript below describes this evolution of Marcia's Box.

TS: Draw your own box. Use the box method to multiply the length and the width. You won't have the template with you, so you need to be able to draw your own.

*Marcia begins to draw tiles.*

TS: I don't want you to draw tiles. That's just too many to draw.

Marcia: I'm not drawing tiles. I'm drawing numbers.

Marcia: Is this wrong? *(See Figure 12)*

TS: It's not wrong. It's incomplete. Can I add something?

Marcia: Yeh

TS: What about this box and this box? *(I draw squares in missing sections)*

Marcia: I know, but I already used them.

TS: But we have to use them twice. Remember how we did it with the algebra blocks?

Marcia: Oh, I forgot about that.

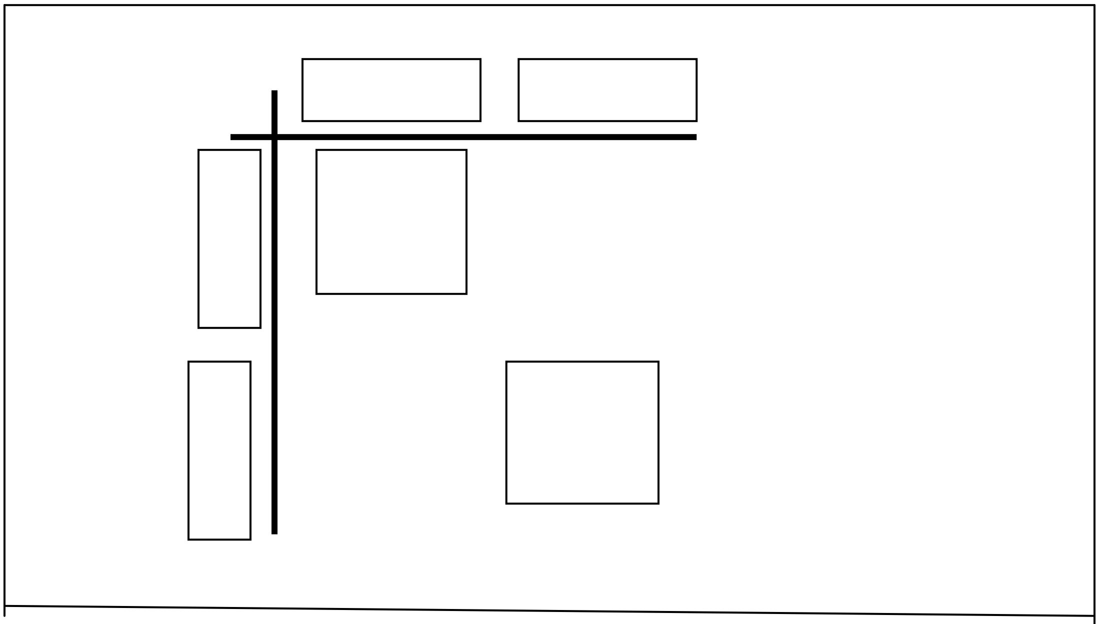


Figure 12. Reproduction of Marcia's attempt at modifying her original Box.

At this point in the intervention, Marcia was not associating the Distributive Property with multiplying linear expressions, as evidenced by her misrepresentation above. Similarly, Marcia demonstrated a disconnection between the Lab Gear and the procedural fluency of multiplying the linear expressions when she was "making a picture." She was not successfully transitioning from the Lab Gear to the abstract symbols. Instead, she was consistently multiplying linear expressions and only getting the quadratic term and the constant, such as, $(2x - 10)\ (x - 12) = 2x^2 + 120$. The $2x$ was being distributed to the x, but not to the $-12$. Additionally, the $-10$ was distributed to the -12, but not to the x. This disconnection was apparent in her Box in Figure 12. Marcia needed a Box that prompt her to distribute each term in one expression to each term in the other, thus connecting to the Distributive Property.

Figure 13 displays an emerging connection between her Box Method and the Distributive Property. She understood that she needed "to use each term twice" through our conversations surrounding her incomplete Box in Figure 12. Initially, Marcia drew a variety of arrows to help her remember to distribute each term of one expression to each term of the other, as shown in Figure 13; however, she became very confused by the arrows.

> Marcia: The arrows are confusing me. How can I remember to do all of this?
>
> TS: Think of it as a grid. Go up and over. Those are the terms you are multiplying to fill in those boxes.
>
> Marcia: That's helpful!

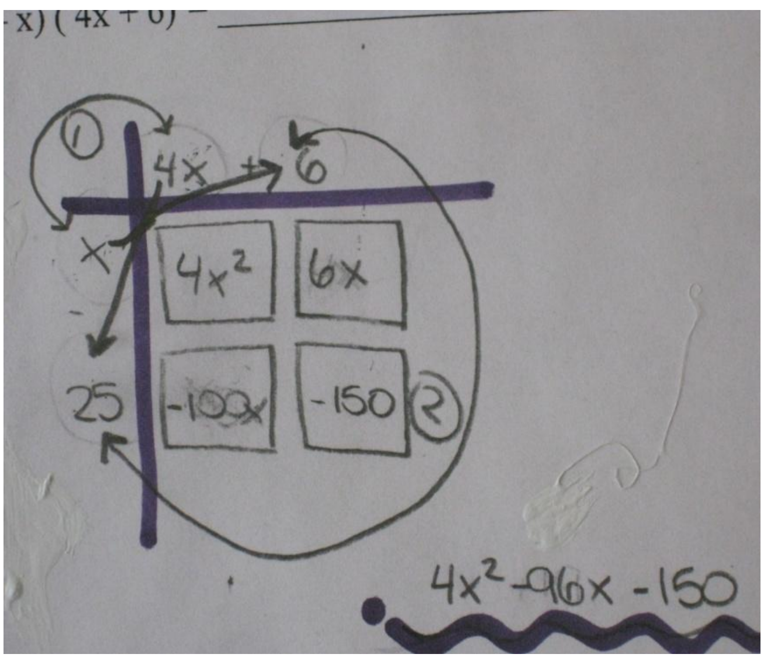



Figure 13. Marcia's connection between the Distributive Property and her Box.

Marcia needed a context that would remind her to distribute the terms. She tried arrows, but that was visually confusing, so the context of a grid provided Marcia with the rules that supported her use of the Box.

At the end of Lesson 4, Marcia developed the graphic organizer pictured in Figure 13, but without the arrows. She used this Box for her posttest domain probes with 100% accuracy. Additionally, she successfully adapted her graphic organizer to multiply a trinomial by a four-term polynomial on the transfer measure (see Figure 14). During this test, Marcia referred back to her original Box that generalized the position of the $x^2$ term, the x terms, and the constant (see Figure 10). She noticed that the diagonal terms (i.e., the x terms) were combined and she generalized this to assist her in indentifying terms to combine in this transfer task, which enabled her to multiply these polynomials with 100% accuracy.

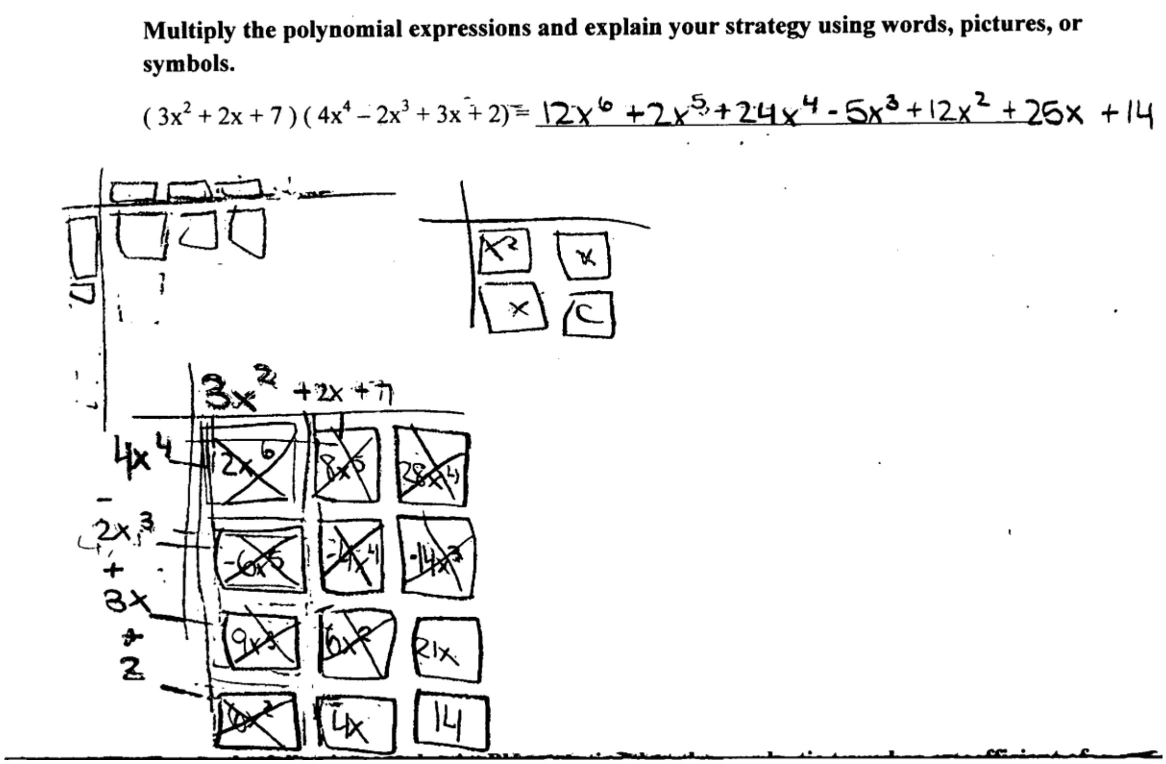



Figure 14. Transfer Task using Box Method.

***Connections between the Box Method and factoring.*** Participants applied the rule for factoring (i.e., find factors of the constant that add to equal the coefficient of the linear term) to the Box Method when working with abstract symbols. Instruction was scaffolded so that participants were initially provided a template of the box with the quadratic term and the constant filled in (see Figure 15). Then students made their own box to complete independently in its entirety.

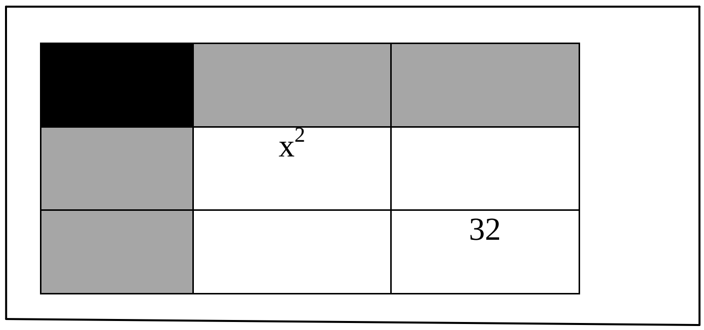



Figure 15. Box Method template with scaffolds.

Marcia's transition to the Box Method for factoring quadratics without the aid of the Lab Gear representation was also impacted by: (a) her desire to continue using the Lab Gear representations; (b) her deficits in understanding the concept of factors; and (c) her lack of automaticity with math facts. Examples of these challenges are apparent in the following excerpts.

Similarly to her use of the Box Method for multiplying expressions, Marcia struggled to factor quadratics when initially given the template of the Box Method without the use of the Lab Gear. She needed an explicit link between the two representations. Specifically, Marcia first factored a quadratic expression by sketching the Lab Gear. Then she sketched each block into the template (see Figure 16). This graduated process helped her establish a link to the Lab Gear representations that she transferred to the Box Method. When asked to factor $x^2 + 9x + 20$, Marcia indicated that the x-bars were split up into the two inside squares of the template.

TS: So if I am doing my box, what's going to go here *(pointing to top left inside square)*

Marcia: Think about what the blocks look like *(said to self).* X squared.

TS: What's going to go here? *(pointing to lower right inside square)*

Marcia: 20.

TS: What will go in the remaining two squares?

Marcia: Some x and some x. This is where my x-bars go. So these x's add up to 9x.

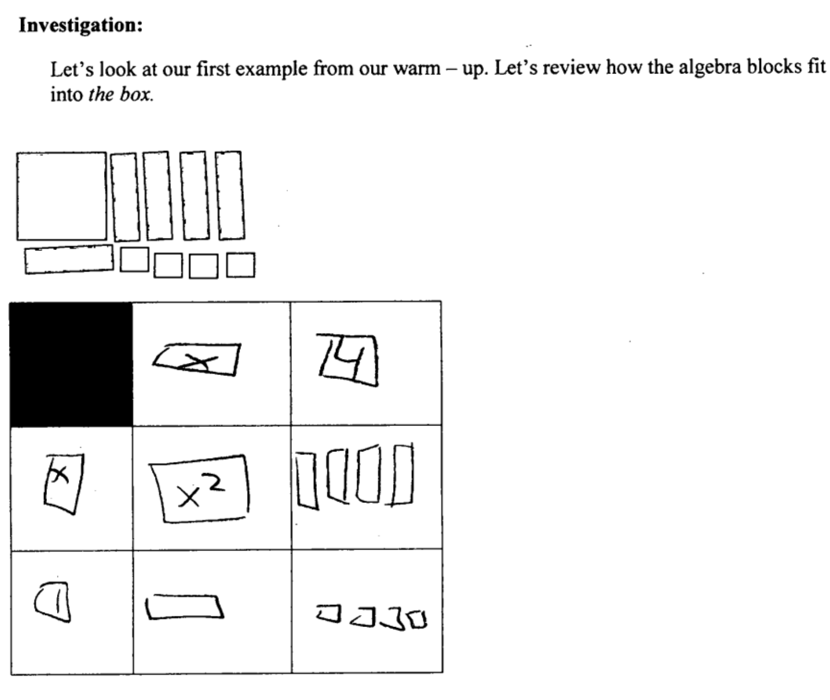



Figure 16. Marcia's Transition to Box Method

As Marcia developed competency in using the Box Method, she frequently referred to splitting up the x-bars to fill in the template. Although she wrote the symbolic notation, Marcia was visualizing the manipulatives being divided into each section of the Box. This was helpful because she competently and comfortably factored quadratics with the Lab Gear (e.g., scoring 100% accuracy on those tasks in lesson probes). Additionally, Marcia demonstrated a connection to the Lab Gear when factoring quadratic expressions containing negative terms, such as $x^2 - 4x - 5$.

> Sasha: I still don't get how -5 plus 1 is negative 4
>
> TS: Let me draw these blocks. What did we have to do with our Lab Gear when we had a positive and a negative, Sasha?
>
> *Sasha doesn't respond*
>
> TS: What does a positive x-bar and a negative x-bar equal?
>
> Sasha: negative
>
> Marcia: Remember when there's positives and negatives, negatives do what to positives?
>
> Sasha: make them negative
>
> TS: Remember we are adding these.
>
> Marcia: Negative means less. So you are taking out. They cancel each other. The negative is going to take one of the x-bars out so the one goes away and one of the 5 goes away *(pointing to sketches on board)*

In the above transcription, Marcia was prompted to visualize the Lab Gear by my sketching because she was reluctant to move beyond the manipulatives. I facilitated the connection between the Lab Gear and the Box Method to aid in the transition to using only symbolic notation. The connection between the Lab Gear and the Box Method also was important for understanding that the process for factoring with Lab Gear also applied when using the Box Method.

Marcia demonstrated a connection between the Box Method and the rule for factoring quadratics, when using Lab Gear representations for support. However, she experienced difficulty with foundational skills, such as determining the factors of 36, when factoring $x^2 + 12x + 36$.

TS: Alright. Let's factor $x^2 + 12x + 36$. What's going to go here? *(pointing to the top left square and the bottom right square in template)*

Marcia: x squared and 32

T: Ok, that's my easy stuff. It doesn't matter if there are negatives in the quadratic, I always know these two boxes. So these two need to add to equal what?

Sasha: 12

TS: So I have to think of factors of 32. And I can use my calculator. I see this is an even number, so I know what will divide into it?

Marcia: So, 2

TS: So I could plug that into my calculator. 32 divided by 2

Marcia: That's 12 + 12 + 12

TS: But I'm dividing it by 2.

Marcia: Why?

TS: Because 32 is even.

Marcia: Oh, so the 32 doesn't have to go into the 12. The numbers that you use to divide 32 you add up to 12, those don't, the numbers that you are using are connecting 32 and 12. They don't have to do with each other.

TS: They are connecting in that the numbers you are using will multiply to get 32 and add to get 12.

Marcia: The 32 and the 12 don't have anything in common except those numbers.

TS: So what are my factors of 32.

Marcia: 32 and 2

TS: 2 times what equals 32?

Marcia: 16. Oh.

As Marcia continued to search for factors of 32, she also demonstrated a lack of automaticity with multiplication facts.

TS: Will 3 go into it?

Marcia: 3, 6,9, 12, 15, 18, 21, 24, 27, 30, 33 (counting using her fingers) No.

Sasha: I was using my calculator to divide.

TS: Ok, does 4 go into 32?

Marcia: *(counting using her fingers)* 4,8, 16, 20, 24, 28, 32 *(she does this really quickly)* Yes. 4 times 8.

TS: Ok what about 5?

Marcia: No because it doesn't end in 0 or 5.

TS: Ok, 6?

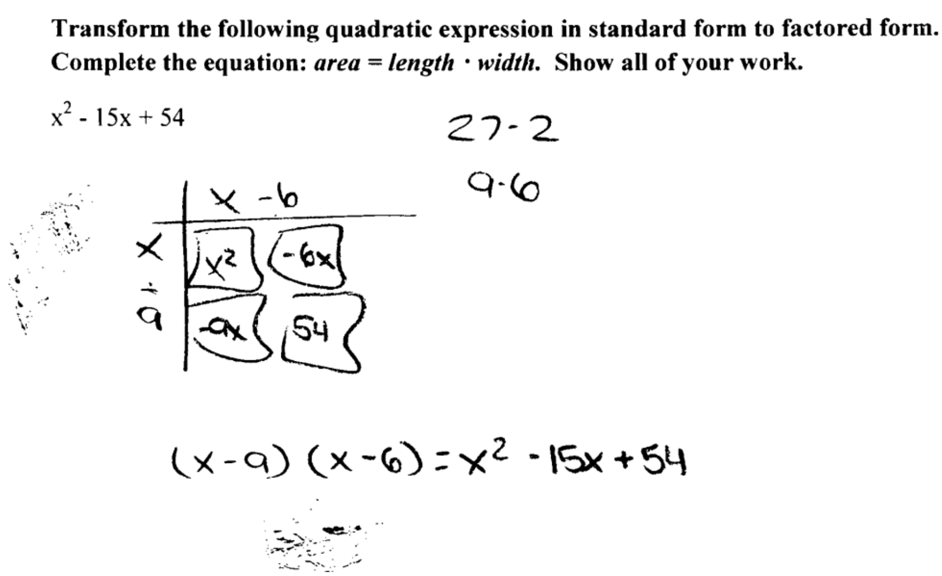
Marcia: Yeh, 6 times, hold on. Well 4 and 8 work anyway.

TS: Why?

Marcia: 4 plus 8 is 12

TS: So 4x and 8x

These deficits in her understanding of factors and her weak automaticity of math facts was not an issue when using Lab Gear representations, however, it initially impacted her success with the Box Method. However, she was able to develop her understanding of factors through engaging in the tasks of factoring quadratics and she compensated for her lack of rote memory of math facts by counting with her fingers and using the calculator. In Figure 17, Marcia first used the calculator to determine factor pairs for 54 and then again used the calculator to determine which pair would equal 15 when added. She also demonstrated an understanding of the rules for integers by identifying negative 6 and negative 9 add to equal -15 and multiply to equal 54.



**Transform the following quadratic expression in standard form to factored form.**
**Complete the equation: *area = length · width*. Show all of your work.**

$x^2 - 15x + 54$

27·2
9·6

| | x | -6 |
|---|---|---|
| x | $x^2$ | -6x |
| -9 | -9x | 54 |

$(x-9)(x-6) = x^2 - 15x + 54$

Figure 17. Sample Posttest Domain Probe factoring task.

Although Marcia successfully factored quadratic expressions using the provided Box template, she independently moved away from the template and worked with her self-developed Box as shown in Figure 18, which again linked with the Lab Gear representation. Marcia confidently switched to her Box method as she did not ask for permission to use her graphic organizer nor did she discuss this with me, which was in contrast to her use of her Box to multiply linear expressions when she consistently wanted my consent. Ironically, Marcia used both her Box and the Box template on her posttest domain probe with 100% accuracy.

Throughout the intervention, Marcia developed her own unique Box that would support her processes for multiplying expressions and factoring quadratic expressions. Marcia initially was reluctant to give up her Lab Gear to work with abstract symbols only. Although she successfully multiplied linear expressions and factored quadratic expressions using the template I provided, she developed her own Box method which linked to the Lab Gear representation, but more importantly, solidly connected to the Distributive Property. See Appendix Q for a summary of the evolution of Marcia's Box, which provided the support needed for Marcia's transition to abstract symbols.

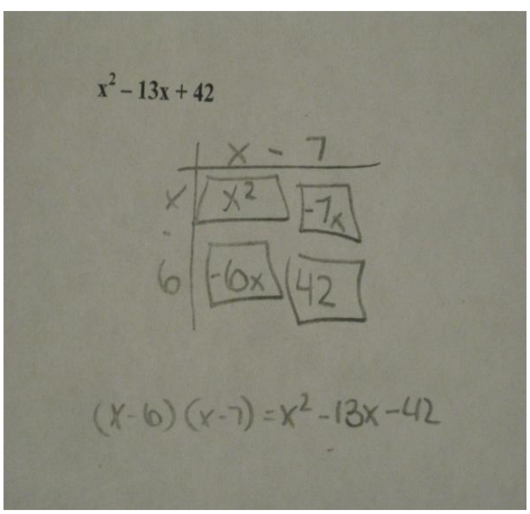



Figure 18. Marcia's Box Method for factoring.

**Summary.** Marcia demonstrated connections and disconnections between the representations and the algebraic concept. Specifically, she used the area concept as an anchor for developing her existing knowledge (i.e., determining area with discrete numbers). Additionally, she connected the area context to abstract symbolism (i.e., expressions as generalized statements of area context) and area quantity (i.e., determining if there was more, less, or the same amount of space after renovations). Although the area context was beneficial when multiplying linear expressions (i.e., multiplying length times width), Marcia did not demonstrate a connection between the area context and factoring (i.e., finding the dimensions when given the area).

Marcia also demonstrated connections and disconnections to the algebra content when working with the Lab Gear. At times, she attempted to literally interpret the blocks for example stating that three constants equaled an x-bar and she did not recognize the Distributive Property in the process of multiplying expressions. However, Marcia connected the visual representation of the Lab Gear to critical mathematical concepts such as the Zero Principle, negativity, and rules for factoring. Additionally, she used the Lab Gear to compare quantities of quadratic expressions.

Lastly, Marcia displayed only connections between the Box Method and the algebraic content. This resulted from the connections she made between the Lab Gear and the Box Method, from which emerged her understanding of the Distributive Property and the process of factoring.

## Metacognition

In addition to themes relating to representations, themes also emerged from the qualitative data sources (i.e., transcripts, work samples, field notes) relating to metacognition, which is a complex phenomenon with many different aspects including strategic behavior, disposition, and socially shared experiences. Metacognition refers to a person's self-awareness of their cognitive abilities, steps and strategies used during a task, self-monitoring of task completion, and appraisal of task completion through checking the accuracy of work (Bley & Thornton, 1995; Mazzocco, 2007). Additionally, metacognition refers to a student's ability to make accurate predictions of future performance (Mazzocco, 2007), which is linked to disposition, such as feelings of difficulty, confidence, and satisfaction (Iiskala, Vauras, Lehtinen, Salonen, 2011). Lastly, socially shared metacognitive experiences refer to those shared within a collaborative endeavor such as students confirming one another's correctness through reciprocal turns when in a problem-solving process (Iiskala, et al., 2011). Marcia demonstrated strategic planning, self-regulation, future planning, and socially shared metacognition as she progressed through the intervention: Several themes emerged from each of these categories as described below.

**Strategic Planning.** Strategic planning refers to developing a plan to engage in a task and executing the plan to successfully complete the task. Development and execution of plans of action occurred simultaneously and therefore are described concurrently below in the order in which the tasks occurred within the instructional unit.

Marcia's scores on her pretest domain probes were extremely low (0% - 4%), partially because she was unable to develop a plan to engage in the tasks. When presented with a word problem and table of data, she wrote on her pretest, "I think if it was broken down I would be able to do it. The problem is that there a lot of words and a lot of steps and once I understand what to do with one part I forget the other – I guess I'm not good blending the steps." Additionally, when asked to transform a quadratic expression from standard form to factored form, Marcia wrote, "as I said this kind of stuff turns me off BUT I think that parts of it I really might know so again if it was explained and broken down I think there may be some hope." On an additional pretest domain probe, Marcia also wrote "I can't break it down." However, during the intervention, she stated that the Lab Gear and the Box Method helped her to break down the tasks and develop of a plan of action that she executed to successfully complete the tasks. Examples of this process are described below.

The Lab Gear served as a tool for "breaking up" the procedure of multiplying linear expressions. Marcia explained that she could "break up" did she say this? If not, use another term the problem into steps by using the manipulatives.

> I'm writing out my problem over here *(pointing to the manipulatives)*. I have an equation and I am breaking it up and multiplying because this is a multiplying bar *(pointing to the corner piece)* and this is x and so I have x plus 3 so x plus 3 times and this is timesing it plus x plus 2 and that's going to equal x squared. So now its x plus 3 times x plus 2 equals *(manipulating the blocks)* x squared plus 5x plus 6.

Marcia used the Lab Gear to both develop and execute a plan for multiplying linear expressions. First she represented her dimensions (i.e., linear expressions) using the manipulatives and placed them on the outside of the corner piece. Then she filled in the corner piece with the appropriate manipulatives to form the required rectangle to correctly determine the area (i.e., quadratic expression). Marcia was pleased with her ability to multiply linear expressions using the Lab Gear and therefore resisted giving up the Lab Gear. When told that we were moving away from the blocks to use only abstract symbols, Marcia replied:

> It's so much more hard because it's not broken up then. Like what I do is I see this *(pointing to xbar)* and this *(pointing constant blocks)* and I read it and I write it then I move it. And then it's all broken up and I see the whole problem happening. But when it's all numbers then I forgot where to break it up and what's what.

Although resistant to giving up the manipulatives, Marcia developed a graphic organizer (i.e., the Box) that was closely linked to the Lab Gear representation which further assisted with strategic planning. This was evident when presented with the task of multiplying *(-13 +2x) (10 + x):*

> Marcia: Can I do the numbers after the x's?
>
> TS: Show me what you mean.
>
> Marcia: Can I do 2x – 13? The x's are always in this box *(pointing to the top left box of her graphic organizer)*

Marcia was able to develop her own plan of action and switch the order of the terms so that the terms with the variables were always in the position of the Lab Gear representation. She then was able to successfully complete the task.

When factoring quadratic expressions, the Lab Gear also supported Marcia's plan of action, which was to "go backward." She was able to arrange the blocks into a rectangle inside the corner piece and visualize, or as Marcia stated "see" the dimensions of this area. When transitioning to the abstract notation, she again used her Box method to develop her plan of action. She always placed the quadratic term in the top left of her organizer and the constant in the bottom right square. She then wrote out all of the factors of the constant to find a pair that equaled the coefficient of the linear term (see Figure 19).

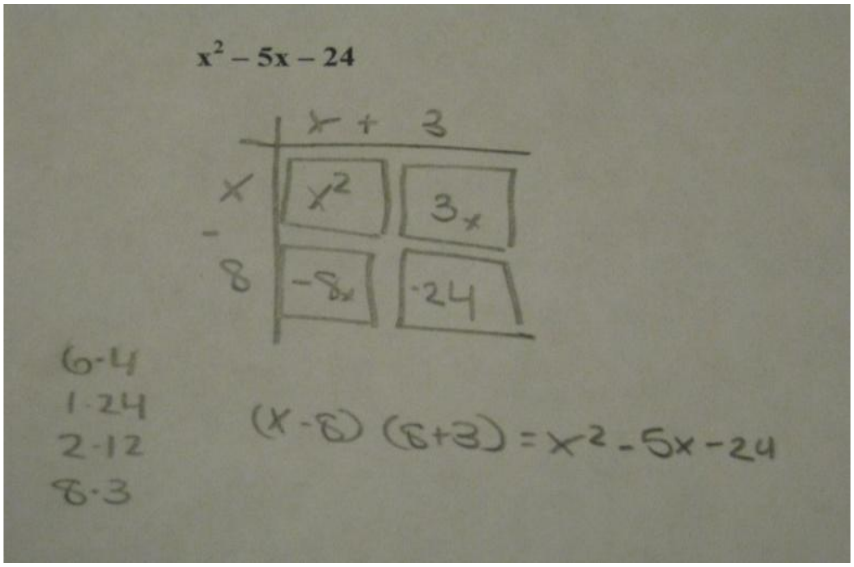
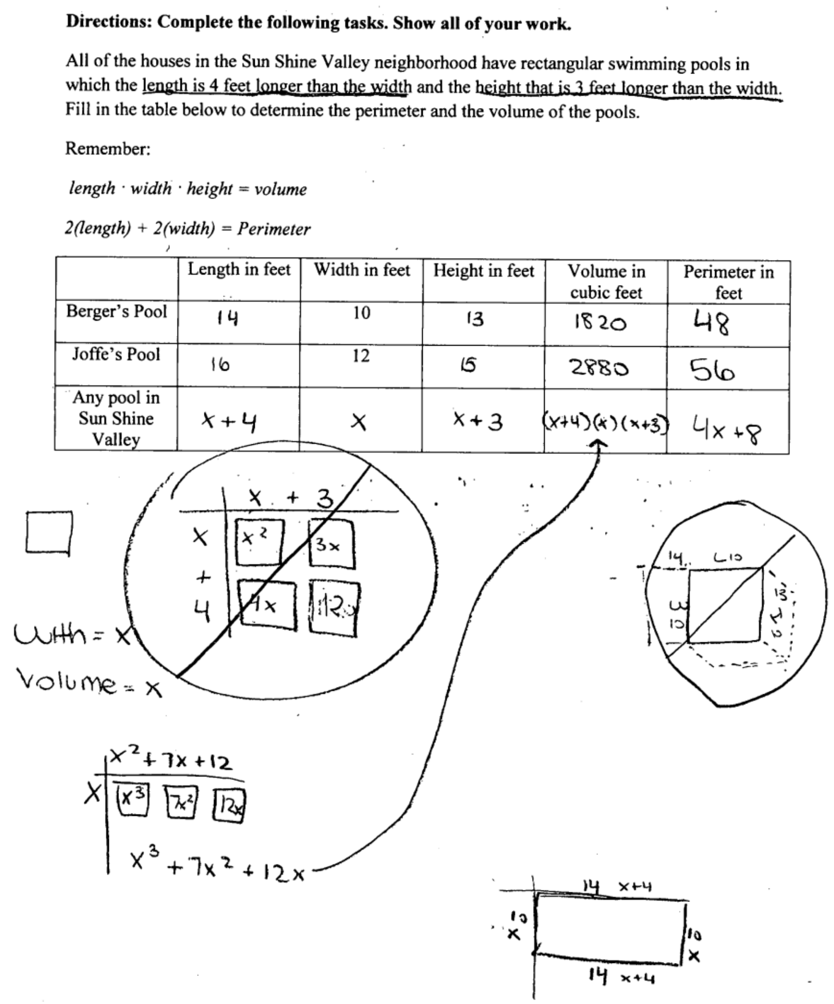



Figure 19. Factoring a quadratic expression

The Lab Gear and the Box Method also served as valuable tools when Marcia completed her Transfer measure (see Appendix N). Despite being presented with tasks that differed from those in the instructional unit, Marcia developed a plan of action and successfully solved each task using the Lab Gear and the Box Method representations. In the first transfer task, Marcia relied on the Box Method for planning and executing her solution strategy by completing a table of data for determining the perimeter and volume for specified numbers and for a generalized statement (see Figure 20).



**Directions: Complete the following tasks. Show all of your work.**

All of the houses in the Sun Shine Valley neighborhood have rectangular swimming pools in which the length is 4 feet longer than the width and the height that is 3 feet longer than the width. Fill in the table below to determine the perimeter and the volume of the pools.

Remember:

*length · width · height = volume*

*2(length) + 2(width) = Perimeter*

| | Length in feet | Width in feet | Height in feet | Volume in cubic feet | Perimeter in feet |
|---|---|---|---|---|---|
| Berger's Pool | 14 | 10 | 13 | 1820 | 48 |
| Joffe's Pool | 16 | 12 | 15 | 2880 | 56 |
| Any pool in Sun Shine Valley | x + 4 | x | x + 3 | (x+4)(x)(x+3) | 4x + 8 |



Figure 20. Marcia's Transfer task 1

For the second transfer task, Marcia also used the Box Method to develop and implement a plan for multiplying a trinomial by a four-term polynomial (see Figure 21). At first, she sketched the $3x^2$ inside the corner piece, as evidenced by the sketch in the upper left. Marcia realized that this was a multiplication problem so the polynomials must be on the outside of the corner piece, which lead to the bottom representation. After distributing all of the terms in that sketch, Marcia was confused how to combine terms. She then drew her representation on the top right and she recognized that she combined terms that were diagonal (i.e., the x-terms). Marcia used that process of looking at diagonals terms to begin the process of simplifying like terms.

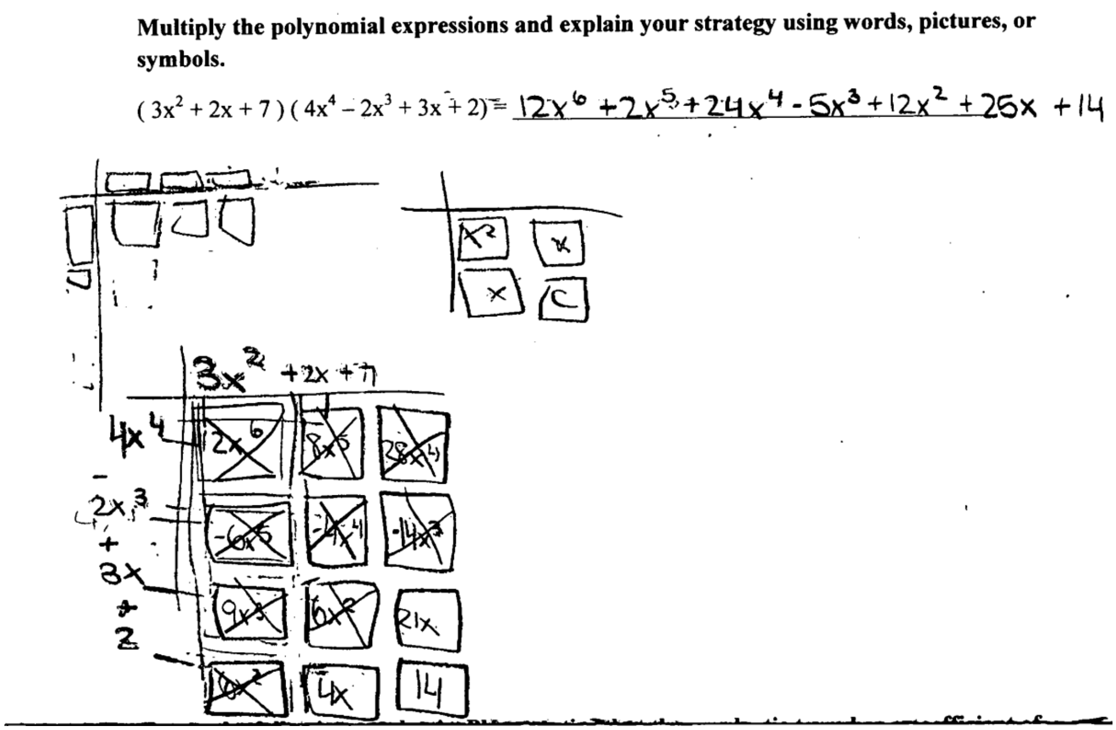



Figure 21. Marcia’s Transfer task 2.

When completing the third task on the transfer measure (see Appendix N), Marcia initially attempted to use the Box Method to factor a quadratic expression with a coefficient of 3. She chose to use the template graphic organizer, rather than her unique form. When Marcia realized that “having the 3 doesn’t let us just add anymore” she abandoned the Box Method and instead sketched the Lab Gear to successfully find the dimensions (see Figure 22). This exemplified Marcia’s ability give up a faulty plan and develop and execute an appropriate revised plan of action.

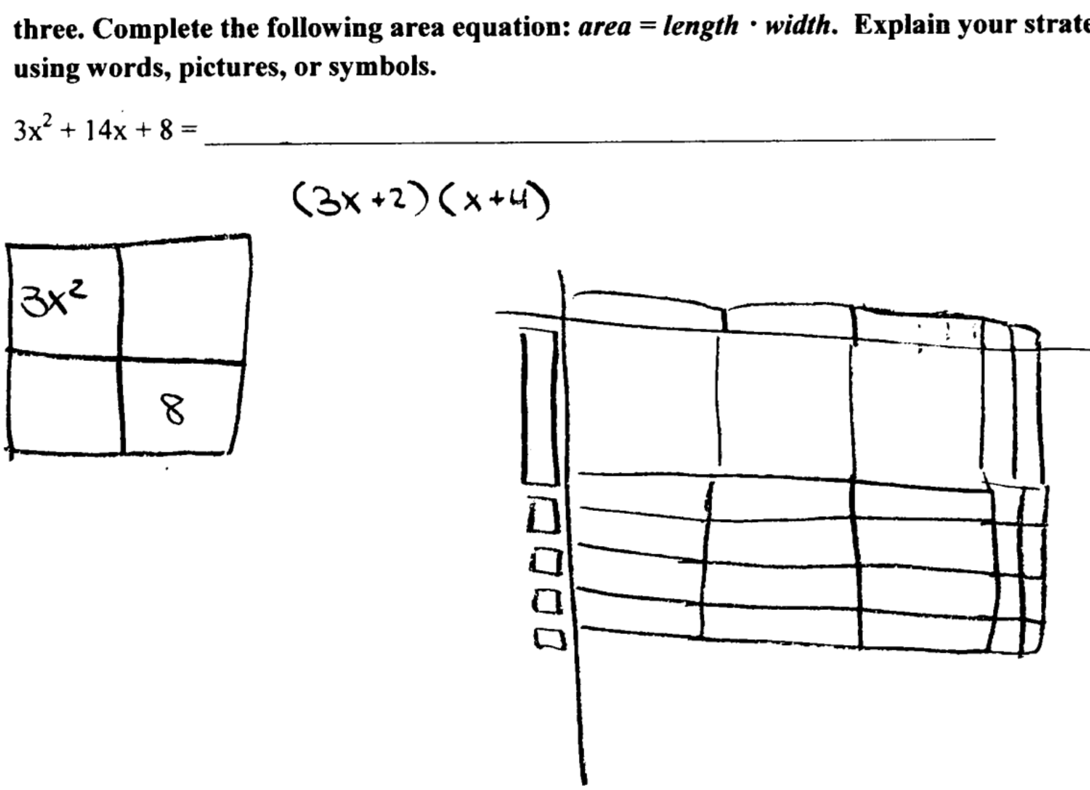



Figure 22. Marcia's Transfer task 3

Initially, Marcia was unable to develop a plan to complete tasks on the pretest domain probes stating that she did not know how to "break it down." Throughout the instructional unit, she used the Lab Gear and the Box Method as tools for strategic planning. These tools provided Marcia with the means for "breaking up" the tasks on the posttest domain probes, which she stated that she needed. Additionally, she used the Lab Gear and her Box to successfully complete tasks on the transfer test.

**Self-regulation.** Self-regulation refers to monitoring and evaluating one's performance during a problem solving task (Fuchs & Fuchs, 2007). Marcia displayed self-regulation behaviors as she routinely checked the accuracy of her and revised as necessary, and evaluated the accuracy of her solution. These two themes are discussed below in the order in which they occurred in the intervention.

***Monitoring performance.*** Throughout the intervention, Marcia consistently monitored the accuracy of her solutions by using the tools provided, such as the Lab Gear, the Box Method, or the tables of data. For example, Marcia used the visual cues embedded in the Lab Gear to determine if she correctly multiplied binomials (e.g., blocks must form a perfect rectangle) and referred to this process as "making a picture." Additionally, Marcia frequently returned to the Lab Gear for verification of solutions to tasks involving abstract notation. For example, when using the Box Method to multiply *(3x + 15) ( x – 2)*, Marcia confirmed that 3x times x equaled $3x^2$ by setting up the Lab Gear. Additionally, she wanted to explore other examples of multiplying algebraic terms with coefficients other than one by using the Lab Gear.

Marcia: I have a question.

TS: Yes

Marcia: So if I add more here *(she places two x-bars on each side of corner piece)* I would multiply and get $4x^2$ ?

TS: Yes, that's exactly right. You got it.

Marcia: ok *(pushing away the blocks)*

Marcia reverted to using the Lab Gear to confirm the process for multiplying linear expressions with coefficients other than one. After determining that her responses were correct, Marcia returned to working in symbolic notation.

Marcia frequently moved back and forth between the Lab Gear and the abstract notation when monitoring the accuracy of her solutions. For example, when multiplying *(x + 3) (x + 5)* using only abstract symbolism, she first responded $x^2 + 15$. When asked to explain her response using the blocks, she realized her solution was incorrect and revised her solution to $x^2 + 8x + 15$. Therefore, the Lab Gear also provided Marcia with a way to check the accuracy of her work and to revise incorrect solutions.

When factoring quadratic expressions, Marcia monitored her solution in three ways: (a) analyzing visual cues; (b) comparing tables of data; and (c) comparing graphs. First, she relied on visual cues from the Lab Gear or sketches of Lab Gear. Marcia stated that she, "made a rectangle and then fit blocks up top and to the side" of the corner piece to factor a quadratic expression. Although "making a rectangle" did not link to algebraic reasoning, she later used the Distributive property to check her factoring when using the Lab Gear, which also transferred to the Box Method. After using the Box Method to factor $x^2 - 4x - 5$, she checked her work by using the Distributive Property.

TS: Explain how you got this? $(x^2 - 4x - 5) = (x - 5)(x + 1)$

Marcia: It checked out. X times x is x squared. X times one is one x. Negative 5 times x is negative 5x. Negative 5 times positive 1 is negative 5.

Throughout this explanation, Marcia pointed to the squares within the Box template. She demonstrated that multiplying the binomials was an appropriate method for checking her factoring.

The second method for monitoring accuracy of factoring tasks involved comparing tables of data. The last lesson of the instructional unit embedded factoring within a contextualized problem or task and two tables of data to complete within each task. One table incorporated the quadratic expression in standard form and the second table incorporated the quadratic expression in factored form (see Figure 23). Marcia first compared their tables of data to determine that the quadratic expression in standard form was equal to the quadratic expression in factored form. She then used a graphing calculator to complete the tables to determine if she factored correctly and contained the same values.

**Part III:**

**In a neighborhood Columbia, the area of all the backyards can be represented as** $x^2 + 8x + 15$.

Fill in the table to represent the areas of the specific families' backyards.

| Name | x-value in yards | Area of backyards using standard form |
|---|---|---|
| Jacoby's backyard | 10 | 195 |
| Brown's backyard | 11 | 224 |
| Strickland's backyard | 12 | 255 |

| | x | + 3 |
|---|---|---|
| x | $x^2$ | 3x |
| + 5 | 5x | 15 |

1 · 15
3 · 5

(x+3)(x+5)

Complete the table below

| Name | x-value in yards | Area of backyards using factored-form |
|---|---|---|
| Jacoby's backyard | 10 | 194 |
| Brown's backyard | 11 | 224 |
| Strickland's backyard | 12 | 255 |

What can you tell me about the standard form and the factored form of quadratic expressions? Explain.

they are the same - my table\numbers are the same in factored and standard form

Figure 23. Comparing standard form and factored form expressions.

Initially, she conscientiously completed each table by inputting each form of the quadratic into the calculator and then graphed each expression. However, by the end of the lesson, Marcia filled in the second table without inputting the factored form into the calculator.

TS: You can't just fill in the numbers. You have to check.

Marcia: It's pointless.

TS: It's how you check and make sure you have factored the quadratic correctly.

Marcia: It's the same numbers.

Marcia seldom compared numbers within the tables for each form of the quadratic unless she completed the tasks with her peers.

The third method for monitoring her solution was to examine the graphs of each form of the quadratic expression. Using a graphing calculator, Marcia initially graphed each form of a quadratic, however, upon realizing that one graph appeared if they were factored correctly, she stopped graphing the expressions for practice tasks. However, she compared the tables of data and the graphs when completing her Lesson 9 probe, in which she completed the tasks with 100% accuracy.

***Evaluating solutions.*** Despite having tools for monitoring her work, Marcia often made faulty evaluations of the accuracy of her solutions. For example, she would often make comments such as "I'm not good at that" and "I don't know if I am doing this right" and yet Marcia would have an accurate solution and be able to justify her answer. For example, during Lesson 4 Marcia was transitioning from the Lab Gear to using abstract symbols only with the Box Method.

TS: Marcia, what do we have to do to find the area of something?

Marcia: Multiply. So x times x is x squared.

TS: Well, do it down here using the box.

Marcia: Oh, the parenthesis. Oh you do the inside outside. This is supposed to be x – 3 times x. I don't get this.

*Marcia accurately completes the Box.*

Marcia: This is all wrong *(handing me her paper)*

TS: This is all right!

*Marcia looks at me disbelieving.*

TS: I'm serious.

Marcia: No way!

She often needed confirmation from me before she would acknowledge that she successfully completed a task. I regularly encouraged Marcia to relay on the tools more than me; however, she was resistant and accused me of not helping her. Marcia was often surprised by her success as exemplified in the above transcript.

**Disposition.** Disposition refers to students beliefs about their ability to do mathematics including self-confidence, perseverance, and enjoyment (Van de Walle, Karp, & Bay-Williams, 2010). Marcia's disposition evolved throughout the intervention, as times demonstrating a positive disposition and at other times a negative as described below.

***Self-confidence.*** Self -confidence refers to students' beliefs about their ability to do mathematics. Throughout the instructional unit, Marcia's self-confidence ebbed and flowed. She began the unit with low self-confidence, as she repeatedly stated that she didn't believe that she her responses to tasks were accurate. However, the area context, Lab Gear, and Box Method built her belief in her abilities and at the end of the intervention she felt confident that she could accurately complete every task on her maintenance domain probe and she was correct (see Figure 2). With every new situation, Marcia's confidence wavered and she returned to making statements that she didn't know anything and that she "sucked at math." However, after repeated exposure to tools such as the Lab Gear and the Box, Marcia was able to confidently engage in the tasks demonstrating mastery on all lesson and posttest domain probes.

On the first day of the intervention, Marcia offered an explanation for her responses on her pretest domain probes, "I suck at math, but I am very good at telling why I can't do it." In many ways this was an accurate statement. She had a history of poor performance in mathematics and yet she was very articulate during the intervention when explaining why she was confused, as evidenced by the many transcriptions provided thus far.

In addition, Marcia clearly lacked confidence whenever a new task appeared. For example, during the introduction to Lab Gear lesson, Marcia repeatedly stated "I'm not good at this" while pulling out blocks that represented a given algebraic expression. However, during Lesson 1, Marcia never questioned her ability to represent the algebraic expressions; however, she did question her ability to multiply the binomials with the Lab Gear, which was the new objective for the lesson.

Marcia: So now I have x *(pointing to x square block)*

TS: What's this called?

Marcia: X squared. I don't know if I am doing it right.

TS: You are doing beautifully

Marcia: So now I have x squared. So now its' x plus 3 times x plus 2 equals x squared plus 5x plus 6 *(writing the area equation symbolically)*

Throughout this lesson, she continued to make comments showing her lack of confidence, such as "Anna's smarter, ask her." However, by lesson 2, Marcia developed confidence in her ability to multiply binomials and engage in the discussions surrounding the area context. After receiving praise for providing an accurate solution, Marcia stated, "Thanks. I feel good about myself."

Marcia transitioned to the Box Method during Lesson 3 and immediately stated, "I'm not good at that" at the mention of the objective. After participating in the activity that transitioned from the blocks to the Box template (see Figure 7), Marcia initially remarked, "That's pretty cool." But when told to practice one, she lost confidence.

Marcia: That's pretty cool

TS: Alrighty, I want you to practice one!

Marcia: Ahh, I 'm not good at that.

TS: I want you to set up the problem in the box

Marcia: I can't. My brain's fried

Marcia's lack of self-confidence produced a visceral reaction. She put her head down and needed a lot of encouragement to practice using the Box Method.

Marcia: I don't like this.

TS: I'm going to give you one more to practice.

Marcia: No, I can't do this. It hurts my brain.

TS: Please, just one more then you are done for today.

Marcia: I need the blocks.

TS: No, you don't. The numbers are too big for blocks.

Marcia: I can't do it without the blocks. I'm going to draw them

TS: OK. Use sketches to help you.

*Marcia sketches*

Marcia: I'm stuck. No I'm not.

TS: Marcia, thank you for sticking with it. For all that complaining, you're problem is perfect.

Although still not confident, Marcia completed the additional practice problem by reverting back to using sketches of the Lab Gear, a tool in which she confidently used to multiply linear expressions previously. She also needed repeated exposure to the Box method and then needed to develop her own model of the Box before confidently using it to multiply linear expressions.

Marcia appeared confident when discovering the rules for factoring in Lesson 7. She was actively engaged with the Lab Gear, offering suggestions for her peers, and exclaiming, "This is the first thing that I understand!" This lesson was student-centered with minimal interaction between the participants and me. She took a leadership role by encouraging her peers to "work together" and offering encouraging statements such as "Sasha, you're a genius!"

Her self-confidence again wavered when presented with the task of factoring a quadratic expression via the Box Method. She particularly struggled when the quadratic expression contained negative terms. "This is too far-fetched for me." During the teacher modeling of factoring a quadratic expression containing negative terms, Marcia withdrew and put her head down on the desk and continually stated that she didn't "know anything." However, after reviewing an additional problem containing all positives, Marcia again became engaged and even helped Sasha with a problem.

***Enjoyment.*** Enjoyment refers to students' feelings of pleasure and satisfaction while participating in the intervention. Throughout the intervention, Marcia's appearance of enjoyment wavered, often as her confidence ebbed and flowed. She did not enjoy activities in which she did not feel confident. The key to Marcia's enjoyment was the development of her self-confidence and the supports embedded in the intervention such as the Lab Gear and the Box Method cultivated her confidence in her math ability. Marcia began the intervention by saying "Math class is horrible. We don't learn anything" and ended by writing on the social validity measure "I hate math" when asked if she was interested in learning more about quadratics. Despite these negative statements, during the intervention Marcia showed glimmers of enjoyment.

On the second day of the intervention Marcia stated, "I hate math. The bad thing when I'm with you I can't cut. I always cut math." Ironically, Marcia could choose not to come to class, as participation in this study was voluntary. I reminded her of this and that she was not receiving a grade. She responded by saying "I'll learn something from you." I was Marcia's math teacher when she was in 7$^{th}$ grade, so we had a pre-existing relationship for her to make this assumption. Additionally, I believe she enjoyed not receiving a grade. Marcia was perceptive regarding the procedures of the study. For example, she inferred which activities I was scoring during the lessons (i.e., lesson probes).

> TS: I want you to sketch the blocks for this. Sometimes I don't care, but this time I want you to.

Marcia: So it's a quiz

TS: It not really a quiz. You aren't getting grades for this.

Marcia: True. We shouldn't ever get grades. It would be less stress.

TS: I agree

Marcia: We should start a new school.

This study was less stressful than her typical math class because she was not being graded and therefore more enjoyable. These factors may also have contributed to her high achievement on the probes.

She was smiling and engaging during activities in which she felt comfortable. For example, in Lesson 2, Marcia was confident in her ability to multiply linear expressions with the Lab Gear and replied, "Thanks, I feel so good about myself" when praised. Marcia continued throughout the lesson actively engaged in discourse regarding representations of negativity, such as $x - 2$ equaling $x + -2$. Her body language was also reflective of enjoyment, as she smiled and laughed.

Marcia's enjoyment was also evident when she created a "Starbucks commercial" at the beginning of class. As I hit the record button on the video recording, Marcia began acting.

> Here we do intense math as you can see with this teacher over there. She's crazy. She makes us do this stuff with these blocks which is why we need our coffee. She makes us do this intense work because she thinks we are some kind of rocket science genius people. But we love her!

Marcia was smiling throughout her commercial and all the participants were laughing by the end. Marcia felt that the algebraic tasks were difficult, but she also felt confident that she could be successful and that pleased her.

However, when presented with new tasks, Marcia visibly became uneasy. Marcia enjoyed working with the Lab Gear and was not eager to transition to only abstract symbols. When I first introduced our Lesson 3 objective as transitioning from the blocks to using only pencil paper to multiply linear expressions, Marcia's responded, "We should be able to do that though" when it was noted she could not take the blocks to an SAT testing site. The participants gradually transitioned from using the Lab Gear to the Box Method by physically placing the blocks into a template I created for the Box Method (see Figure 9); however, Marcia was resistant.

TS: This is exactly what we are going to be doing now. We are going to be moving away from the blocks to using the box method.

Marcia: Whoa, whoa, whoa. What do you mean moving away from the blocks?

TS: We aren't going to use them anymore to multiply binomials.

Marcia: I'm not good at that.

*Marcia yawns really loudly.*

She immediately assumed that she would not be good at multiplying binomials unless she had the Lab Gear. She also began to withdraw as she yawned, put her head down on her desk, and checked her phone for the time. These were typical behaviors when Marcia encountered a new algebraic task.

Despite her wavering self-confidence, Marcia always wanted to understand the why of the solution. For example, she was not content simply learning the procedure for multiplying linear expressions with negative terms, but wanted to understand the representation. Her favorite lessons involved exploration, for example discovering the rules for factoring using Lab Gear. In this lesson, she visually saw why the rules worked, and exclaimed, “I understand!” Marcia worked extremely hard during the instruction unit and at the end stated, “I am proud of myself.” In fact, she did not want the study to end. The following exchange occurred as she turned in her maintenance test, which officially ended the study.

> Marcia: Is this over?
>
> TS: Yes
>
> Marcia: I don’t want this to be over! I like to do this stuff. I can’t believe I’m saying this about math.

Marcia’s behavior during this intervention was completely different from her behavior reported by her teacher and her mother. Before and after this study, Marcia consistently refused to attend her math class and refused to complete assignments. In contrast, during this study, Marcia arrived on time, stayed late, and completed all of her tasks with a high degree of accuracy.

***Perseverance.*** Perseverance refers to a students’ ability to continue working on a task until completion. Marcia demonstrated perseverance on task completion throughout the study, specifically relating to attendance, completion of transfer probe, and completion of maintenance domain probe. She consistently attended sessions, arrived on time and stayed late to ask questions. After our session, Marcia had a 15 minute break before her next class and she consistently stayed late to discuss unresolved questions from the instructional session. For example, at the end of Lesson 4, Marcia was struggling with the transition from Lab Gear to the Box Method. During the lesson she developed her own graphic organizer (see Figure 11) but continued to have questions regarding how to use it. Marcia used her entire break to discuss her graphic organizer with me.

Marcia's perseverance was also displayed on the transfer measure and the maintenance domain probe. The transfer measure consisted of three novel tasks in which she needed to apply knowledge and strategies learned from the instructional unit (see Appendix N). Marcia spent one hour and twenty minutes completing this task, although she was provided opportunities to end the task at the end of our scheduled session and at the end of her break time. She continued working through the next class period and stated, "I'm not stopping till I finish this test. Can I get a note to miss my next class?" Her perseverance paid off as she scored 100% accuracy on the transfer measure.

Marcia was equally committed to her performance on the maintenance domain probe that she completed four weeks following the intervention. The learning specialist at the school administered the test to the participants in a conference room while I waited in the adjacent cafeteria. After approximately 30 minutes, Marcia brings her test to me.

Marcia: So, can you grade this and give me a new graph?

TS: I will email you a new graph.

Marcia: When other people's graphs went down, was it because they made mistakes on this? Maybe I should look over mine. I want mine to stay up.

Marcia then returned to the conference room to continue working on the test. All participants received a graph of their pretest and posttest scores. She appeared pleased with her progress and did not want her graph to show a decline in her performance (she scored 100%).

**Socially shared metacognition.** Socially shared metacognition refers to students sharing the problem-solving process by confirming one another's correctness through reciprocal turns. Marcia, Anna, and Sasha formed group 2 of the study and progressed through the instructional unit together. Throughout the 13 instructional sessions for Marcia's group, there were only four incidents of socially shared metacognition between peers. I intentionally chose to work with pairs or triads of students to facilitate discourse between peers, which proved challenging. I noted this challenge in my field notes.

> Marcia and Sasha are doing their own problem and then looking at each other's work. This is their idea of working together, doing their own work and then checking to see if they get the same answer. If they do not have the same answer they ask me which is correct, each saying their answer is wrong as they begin to erase their problem.

Working together usually consisted of the participants telling their answer to one another to see if they had the same answer. Their interaction ended if they had the same solution or they looked to me for guidance if they had varying responses. At this point, I prompted participants to share their solutions and strategies with one another in hopes that they would challenge or confirm each other's ideas, or discover the solution together. Additionally, I often provided explicit guidance.

> Sasha: So what does it equal?
>
> TS: Marcia and Anna, explain to Sasha how you write the equation length times width equals area.
>
> Marcia: I don't know how to explain it.
>
> TS: So what is your length?
>
> Marcia: x + 3
>
> Sasha: width is x + 2

TS: You have your length and your width. What is inside your representation is your area.

Sasha: So you have x squared plus 5 um

TS: Show us where you are getting the 5.

Sasha: here *(pointing to 5 x-bars).* But then what are these things *(pointing to 6 constants)*

TS: You said $x^2$ plus 5x when you pointed to your x-square and x-bars and now?

Sasha: plus 6? Yeh?

TS: Marcia and Anna, do you agree?

*Marcia and Anna nod.*

In addition to my explicit guidance in the exchange above, Marcia frequently demonstrated the ability to tutor her peers. For example, in the excerpt below, Marcia explained to Sasha how to set up the Lab Gear to represent length that is two meters longer than the width.

Marcia: So this is the length *(an x-bar and 2 constants)* and this is the width *(an x-bar)*

TS: So how do you know that?

Marcia: Because we don't know what the width is.

Sasha: What are you saying? How do you know what is the length?

Marcia: We don't have an answer so we can use X. Remember when we had the square, we used X. Because the square wasn't a size.

Sasha: So this is the length and this is the width (setting up blocks for each on corner piece)

However, the socially shared aspect was limited in these examples as the participants did not engage in any turn taking of ideas, nor did they develop a solution or response based on sharing their ideas. In the first excerpt I asked leading questions and in the second excerpt Marcia provided her explanation but Sasha did not offer any of her ideas to revise that solution.

The first incident of socially shared metacognition occurred during Lesson 3 when the participants were multiplying *(x – 2) (3x + 15)* using the Box Method.

Sasha: -2 times 3x. Negative 6x.

Marcia: -2 times 3 is -6

Sasha: x times 15 is 15x. -2 times 15 is -30.

Marcia: What? 13

TS: How did you get 13?

Marcia: 15 – 2

Sasha: we're multiplying

Marcia: Multiplying. Oh, I really don't like this at all. I need the blocks. I can't do this. I need to make my own chart. No, my own blocks.

TS: Well, let's practice a little bit. We are going to multiply exactly like we multiplied the blocks but we are only using symbols. What is the area for this problem?

Marcia: I have no idea

*Sasha looks at Marcia and sighs.*

Sasha: $3x^2$-6x+15x-30.

Marcia: How are you doing this?

Although I provided some probing questions, the majority of this exchange occurred between Sasha and Marcia. Ultimately, Marcia revised her solution strategy by multiplying the terms rather than adding. Unfortunately, Sasha became impatient with Marcia and then Marcia switched her attention away from Sasha toward me.

Marcia and Sasha displayed another attemptat shared metacognition during Lesson 4. Participants were provided with the following directions for a warm-up task which was similar to a problem from the previous lesson.

> Today we are going to continue our transition from using the blocks to using the box method to multiply binomials. We want to multiply x plus 5 times x plus 3. Put the blocks into the box. Then write the area equation somewhere for me. So work on this together while I go get pencils from Mrs. Brown.

I then left the room, but continued video recording. All participants began working individually on the problem; however, Sasha and Marcia looked at each other's work.

> Marcia: This is hard.
>
> Sasha: How'd you get a negative?
>
> Marcia: Oh, I thought it was negative three.
>
> *Marcia makes revisions to her work.*
>
> Marcia: Do we have the same?
>
> Marcia and Sasha: Yeh! *(give each other high five)*

Since I left the room, the participants did not have me to provide prompts or to confirm responses. They needed to rely on each other for verification. Marcia applied Sasha comment, "How'd you get a negative?" to reevaluate her solution and then made the necessary revision.

Lesson 7 involved participants discovering the rules for factoring while looking for patterns in a series of factoring tasks using the Lab Gear representations. This was a student-centered activity and I was hoping to see evidence of socially shared metacognition. In the beginning, Marcia encouraged her peers to "help each other out."

They compared answers and Marcia praised Sasha for successfully arranging a set of blocks in the corner piece.

Marcia: We don't have ones.

TS: you don't have any ones.

Anna: You can add ones.

TS: You can't add ones.

Marcia: So what do we do?

TS: Why don't you look and see what other people are doing.

*Marcia looks at Sasha's paper*

Marcia: Oh you are a genius.

TS: Your *(Sasha's)* area is perfect. Now you have to figure out the dimensions. You can help each other.

Marcia: Ok let's help each other. Ok it would have to be an x and an x and 3 ones.

Sasha: No there are no ones

Marcia: Yes, because if you put 3 here *(pointing to the x-bars inside the corner piece)* there has to be 3 here *(pointing to constants above the corner piece)*

Sasha revised her Lab Gear representation based on Marcia's explanation. Unfortunately, Marcia proceeded to complete this activity independently and much quicker than the other two participants, exclaiming, "Done! I was on a roll there!" when she completed the activity. Then Marcia took on the role of tutor and explained her strategy for factoring with the blocks. For example, she told Sasha, "Don't make the blocks, it takes too much time. Look at this." Marcia explained her strategies and observations but did not give Sasha and Anna the opportunity to respond.

> Marcia: Yes, so we had 7xbars so technically if there wasn't any of these (constants) I could just go 7 across. Since you have the constants you are basically making a multiplication problem that the answer is the amount of squares that is needed. So now what it really is I am doing 12 divided by, 4 times 3 is 12 so my bars fit in. That's how I figured out what each thing is. And also when, when, yeh, I had another thing, but I don't remember.
>
> TS: Do you hear what she is saying about the constants? When she was looking at the 12 here, she said 4 times 3 is 12.
>
> Marcia: Yeh, that's how I had to figure it out (*said something I can't understand)* because one could go here and 12 across, and 12 times 1 is 12. But since you have this x bar here, this gets all filled up , since everything has to connect, everything has to have a place to go but if I would have gone 5 across and only 2 down that wouldn't have been even space to fill out because there wouldn't be enough numbers. You have to make sure that you are distributing everything evenly so that it fills everything.
>
> Anna: Yeh, I did notice this a little bit but I didn't put it into words. I just followed my instincts.

Marcia monopolized all conversation, speaking very quickly and so I attempted to restate what she said to give the other participants an opportunity to provide feedback. However, they continued to work independently and share their ideas only when I asked specific questions.

One final example of socially shared metacognition occurred during Lesson 8 when participants were factoring a quadratic expression with negative terms (i.e., $x^2 - 4x – 5$) using the Box Method. They had agreed on the factor pair but did not agree which factor needed to be the negative.

Marcia: negative 5

TS: Good! I'm glad you said negative 5. So we need to figure out factors of what?

Anna and Sasha: -5

TS: Give me factors of negative 5

Marcia: one times 5

Anna: Negative 5. 1 times 5 is 5. One times -5 is -5 and 1 plus -5 is -4.

Sasha: Why can't it be the one?

Marcia: Because 5 is the bigger number.

Anna: Because that problem is -4.

Marcia: if you look at a number line, 5 is way down lower on the number line.

Sasha: No if you do -1 times 5 it's the same number

Marcia: No cause then the 5's not negative. So if you look at the number line, the 5 is here and if you add one you move it one this way to the negative 4 (using fingers to draw on table).

TS: So tell me what to put here.

Marcia: plain old 1 and negative 5

Sasha: I still don't get how -5 plus 1 is negative 4

TS: Let me draw these blocks. What did we have to do with our Lab Gear when we had a positive and a negative, Sasha?

*Sasha doesn't respond.*

TS: What does a positive x-bar and a negative x-bar equal?

Sasha: negative

TS: Marcia, help her out.

Marcia: Remember when there's positives and negatives, negatives do what to positives?

Sasha: make them negative

TS: Remember we are adding these.

Marcia: Negative means less. So you are taking out. They cancel each other. The negative is going to take one of the x-bars out so the one goes away and one of the 5 goes away (referring to sketch on board)

Sasha confused the rules of adding integers and multiplying integers. When Marcia asked, "negatives do what to positives?" Sasha replied, "make them negative." However, through discourse with Marcia and Anna, Sasha was able to revise her solution for the factors and agreed with the group that "- 4x and one x" completed the box, therefore, the area equation was $x^2 - 4x - 5 = (x - 4)(x + 1)$.

**Summary.** Marcia demonstrated enhanced metacognition throughout the intervention. Enjoyment appeared linked to her confidence whereas perseverance consistently improved throughout the intervention, peaking on the transfer and maintenance probes. By the end of the intervention, she was very confident in her ability to complete tasks associated with the content in the instructional unit and when met with a new challenge in the transfer measure, persevered for over an hour to complete the tasks. Marcia believed she could solve the transfer tasks and persevered despite never receiving explicit instruction on the tasks. This is in contrast to her pretests in which, instead of mathematically attempting to solve the tasks, she wrote explanations for why she could not solve them. Additionally, by the end of the intervention, Marcia appeared to enjoy learning about quadratic functions and requested that I stay and continue to work with her. Although Marcia enjoyed the intervention overall, stating that, "I don't want this to be over! I like to do this stuff" her enjoyment of math did not transfer to her regular math class. Her teacher reported that Marcia refused to return to her math class.

Socially shared metacognition is the only area that did not develop as the intervention progressed, despite my attempts to facilitate the process. Marcia revised her solution or strategy based on the feedback of a peer on one occasion during the intervention. However, she was willing to discuss with peers and provided insight into tasks which assisted Sasha in revising her solutions as noted above. In those examples, Marcia was very confident in her knowledge and eagerly tutored her peers. However, when not confident, Marcia appeared to prefer to engage in discourse with me rather than her peers as evidenced by her choosing to remain after class to work with me.

# Chapter 6: Discussion

The purpose of this study was to investigate the effectiveness of blended instruction and visual representations on area problems involving quadratic expressions for secondary students with mathematics disabilities or difficulties (MD). Overall, participants learned to multiply linear expressions to form a quadratic expression and to factor a quadratic expression to form two linear expressions. They all demonstrated an understanding of the relationship between the linear expressions (i.e., dimensions) and the quadratic (i.e., area). Additionally, three participants transferred their knowledge to problems involving more complex procedures and to problems involving perimeter and volume. In the first section of this chapter, I summarize the findings from the study and discuss their importance relative to the current literature. Next, I interpret the major findings as they relate to the research questions. Finally, I discuss the limitations of the study and implications for research and practice.

## Summary of the Study Results Relative to Current Research Literature

Since the publication the National Council of Teacher's of Mathematics (NCTM) *Curriculum and Evaluation Standards for School Mathematics* in 1989, only 11 studies have been published examining algebraic interventions for secondary students with LD or at-risk of LD. Additionally, this research focused on more foundational content including integers (Maccini & Hughes, 2000; Maccini & Ruhl, 2000) and linear equations (Allsopp, 1997; Bottge, Heinrichs, Chan, & Serlin, 2001; Bottge, Rueda, LaRoque, Serlin, & Kwon, 2007a; Bottge, Rueda, Serlin, Hung, & Kwon, 2007b; Hutchinson, 1993; Mayfield & Glenn, 2008; Scheuermann, Deshler, & Schumaker, 2009; Witzel, Mercer, & Miller, 2003). These studies did not examine student performance on more advanced algebra content that is also inline with the content proposed by NMAP (2008), Achieve (2004), and CCSSI (2010). The current study was designed to address this gap in the research literature by applying a package of research-based instructional practices to quadratic expressions, which is a common algebra topic from the high school math curriculum. This was the first documented study with secondary students with LD or at-risk for LD that: (a) addressed quadratic expressions from both a polynomial-based approach and a functions-based approach, which aligns with Common Core Standards (CCSSI, 2010); and (b) incorporated a mixed methods research design that included qualitative data to enhance the understanding of the quantitative results. Specifically, the quantitative data revealed the extent of the participants' academic performance, while, the qualitative data explains why they performed the way they did.

An exhausted review of the current literature led to the development of an instructional package which incorporated the following research-based practices: (a) contextualized instruction (Allsopp, 1997; Bottge, et. al., 2001; Bottge, et.al., 2007a; Bottge, et. al., 2007b; Hutchinson, 1993; Maccini & Hughes, 2000; Maccini & Ruhl, 2000); (b) CRA instruction (Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Scheuermann, et.al., 2009; Witzel, et.al., 2003); and (c) a graphic organizer (Ives, 2007). Additionally, the instructional package was delivered through blended instruction, which included components of explicit instruction (Hudson & Miller, 2006) and the NCTM process standards (2000). As recommended by the NMAP (2008), students received some explicit instruction while having opportunities to engage in constructivist-based activities that required them to problem-solve, justify their solutions, and make connections between mathematics topics (i.e., algebra and geometry). The current study incorporated research-supported practices to address the advanced algebra content of quadratic expressions.

**Emphasis on quadratics.** Many typical students have difficulty with quadratics because of poor understanding of foundational ideas (e.g., negativity, equality, variables) and their inability to recognize and understand varied representations of the same quadratic relationship (i.e., standard form and factored form) (Kotsopoulo, 2007; MacDonald, 1986). The topic of quadratics was introduced in the current study via a functions-based approach as students were provided tables of data relating to the contextualized problem situation. This aligns with recommendations from experts in mathematics education that algebra should be functions-based (Kilpatrick & Izsak, 2008) and be present in the learning and teaching of any algebra topic (Yerushalmy & Gafni, 1993). Additionally, the current study included polynomial-based algebraic manipulations of the quadratic expression from factored-form to standard form and vice versa. Many experts in mathematics education continue to value polynomial-based algebra, believing that through symbolic manipulations and computational processes, students develop a deeper understanding of mathematical objects (Kieran, 2007; Kilpatrick & Izsak, 2008; Sfard & Linchevski, 1994). Addressing quadratics through a functions-based approach and a polynomial-based approach is critical as these topics are aligned with state and national standards (CCSSI, 2010; NMAP, 2008). In addition both approaches are needed for students to obtain a deeper understanding of the content.

**Contextualized instruction.** The instructional package implemented in this study included contextualized problems situated in the context of area. Many problem situations included renovations of a square-shaped room in which the lengths and widths were increased or decreased by a specified amount. Participants completed tables that included examples using discrete numbers and were required to write an algebraic expression to generalize each dimension (i.e., linear expression) and the area (i.e., quadratic expression). The use of contextualized instruction has proven to be an effective practice for teaching algebraic concepts such as linear equations (Allsopp, 1997; Bottge, et. al., 2001; Bottge, et.al., 2007a; Bottge, et. al., 2007b; Hutchinson, 1993) and integers (Maccini & Hughes, 2000; Maccini & Ruhl, 2000) to secondary students with LD and students at-risk. Furthermore, Nathan and Koedinger (2000) found that typically developing secondary students performed better on algebraic word problems than on symbolic-equation problems. When presented with word problems, students used informal strategies such as guess-and-test and unwinding rather than symbolic manipulation to solve the problem. Nathan and Koedinger suggest that use of contextualized word problems may serve as models for developing algebraic reasoning,

**Use of multiple visual representations.** The instructional package implemented in this study included the use of multiple visual representations of quadratics specifically in the form of concrete manipulatives (i.e., concrete representation using the Lab Gear), sketches of the manipulatives (representational using pictures), and a graphic organizer (abstract notation), which also formed the three components of the CRA strategy. The use of manipulatives as an effective tool for teaching mathematics to students is documented in the special education literature (Gersten, et al., 2009a; Hudson & Miller, 2006) and the mathematics education literature (NRC, 2001; Van de Walle, 2010). Additionally, four previous studies in the special education literature determined that the CRA instructional sequence was a beneficial strategy for teaching algebraic content such as integers with use of the Lab Gear (Maccini & Hughes, 2000; Maccini & Ruhl, 2000) and linear expressions (Scheuermann, et al., 2009; Witzel, et al., 2003) to secondary students with LD.

Maccini and colleagues (2000) and Witzel et al. (2003) implemented the CRA sequence as three separate stages with each stage tied to mastery performance (i.e., concrete, representational, abstract). However, Pashler and colleagues (2007) recommended integrating and connecting the concrete and the abstract representations during instruction to support student's conceptual understanding and to promote transfer of the concept to a range of novel situations. Integrating the phases expedites the progress from the concrete to the abstract which is recommended by Gersten and colleagues (2009a). Therefore, this study modified the CRA sequence and simultaneously introduced the concrete manipulatives, sketches of the manipulatives, and the abstract symbols, which is also referred to as the CRA Integration strategy (Strickland & Maccini, 2011).

A graphic organizer, the Box Method, was used to support participants' transition from the concrete to the abstract, as students with LD often find this challenging (Hudson & Miller, 2006). Specifically, many students with LD have experienced difficulty generalizing learned material (Gagnon & Maccini, 2001) and conceptualizing abstract algebraic concepts and algebraic tasks (Witzel, 2005). The Box Method was incorporated in numerous mathematics curricula such as *Lab Gear Activities for Algebra 1* (Picciotto, 1995) and the CME project *Algebra 1* (2009) for multiplying linear expressions and factoring quadratics.

**Use of blended instruction.** This study blended instructional practices found within the special education literature (i.e., CRA instruction, graphic organizers, contextualized tasks, and explicit instruction) with the NCTM process standards emphasized in mathematics education. Specifically, the CRA sequence and graphic organizers have been identified as effective strategies for teaching mathematics and algebra to students with LD (Maccini et al., 2008; Strickland & Maccini, 2010). These instructional strategies link to the NCTM process standard of multiple representations. Additionally, the use of area problems offered another representation of the quadratic expression and allowed participants to make connections between mathematical ideas. Further, the NCTM emphasizes reasoning and sense-making to support the mathematics progress for all students, including students with disabilities (Dieker, Maccini, Strickland, & Hunt, 2011). The NCTM process standards of reasoning and communication offer opportunities for students with LD to engage in sense-making.

In addition, this study blended components of explicit teacher-directed instruction with student-centered activities. For example, the investigator modeled the process of multiplying linear expressions using the Lab Gear and provided participants with multiple practice opportunities to develop mastery. Participants then applied their knowledge of multiplying linear expressions and the Lab Gear representations to verify the process of factoring quadratic expressions. Specifically, participants worked together to discover the process of factoring quadratic expressions in standard form (i.e., determine factors of the constant that add to equal the coefficient of the linear term) by comparing various quadratic expressions containing the same linear term but different constant or containing the different linear terms but the same constant. Participants were provided guiding questions as needed, however, no direct modeling was given. Using both explicit instruction and student-center activities aligns with the recommendations from the NMAP for students with learning disabilities and mathematics difficulties (2008). Blending the instructional practices supported by special education research and the NCTM process standards supported by mathematics education research is critical as more students with LD participate in general education classrooms.

The current study investigated the effects of blended instruction and visual representations on participants' accuracy on algebraic tasks involving quadratic expressions embedded within the area context and the extent to which their knowledge was maintained over time and transferred to more complex tasks. Additionally, a qualitative analysis provided insight into why the participants made substantial gains on their accuracy on algebraic tasks involving quadratic expressions.

## Interpretations of Findings Relative to Research Questions

The research questions were addressed using an embedded mixed methods design. The quantitative data served as the primary source while the qualitative data provided supplementary information. Specifically, a multiple probe design across two groups replicated over 5 students provided the quantitative data which addressed the first four research questions. Transcripts of instructional sessions, field notes, and the work samples of one representative participant (Marcia) provided the qualitative data which addressed the fifth research question.

**Research Question 1.** Research Question 1 was: When provided blended instruction with visual representations, to what extent do secondary students with mathematics difficulties (MD) increase their accuracy on algebraic tasks involving quadratic expressions embedded within area problems? The effectiveness of this intervention on the algebraic accuracy is evident in the drastic changes each participant demonstrated from pre-to-posttest on the domain probes (see Figure 2). All participants scored below the inclusionary criterion on pretests with scores ranging from 0% - 25% accuracy; however they demonstrated mastery of the content on all post-intervention domain probes with scores ranging from 84% - 100% accuracy. Increases in percentage points on domain probes from baseline to post-intervention ranged from 65 percentage points to 93 percentage points.

The ability of each of participant to demonstrate mastery on post-intervention domain probes suggests that the instructional package, including blended instruction and multiple visual representations through CRA-I instruction, positively affected performance on algebraic tasks involving quadratic expressions embedded within area problems. Furthermore, between-phase patterns indicate an increase in scores only after the intervention, which establishes a functional relationship (Kennedy, 2005). Replication of the findings across both groups and all participants demonstrated experimental control and generality to other participants (Horton, et. al., 2005; Kennedy, 2005).

The results of this study related to algebraic accuracy are similar to previous research in which students with LD demonstrated increased performance as a result of contextualized problems, the CRA sequence, and explicit instruction (Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Scheuermann, et al., 2009) as participants receiving this package of instructional practices demonstrated mastery of the content. The results of this study differed from previous research in which students with LD participated in student-centered activities with explicit instruction provided as needed. For example, Bottge and colleagues found that students with LD demonstrated improvements on problem solving tasks, however they did not improve performance (Bottge, et. al., 2007a; Bottge, et. al., 2007b) or declined in performance (Bottge, et. al., 2001) on computational measures. Unlike previous research, participants in this study also demonstrated improved performance on both problem-solving and computational (i.e., symbolic manipulation) tasks. A possible explanation may relate to the design of the instructional unit in the current study. Specifically, the use of explicit instruction was an intended practice rather than remediation for students who were unsuccessful in the student-centered activities (Bottge, et al., 2007a; Bottge, et. al., 2007b; Bottge, et. al., 2001). Additionally, the current study included multiple visual representations of the targeted algebraic concept (i.e., quadratic expressions) that were linked to a familiar geometric concept of area.

All participants demonstrated mastery of the content as evidenced by earning 80% or greater on post-intervention domain probes. This criterion is recommended by experts in special education (Hudson & Miller, 2006). An error analysis provided meaning to this quantitative percent. An analysis of post intervention domain probes revealed that all five participants successfully transformed quadratic expressions in factored form to standard form and vice versa. Specifically, the participants mastered the task of multiplying linear expressions and factoring quadratic trinomials. This is an important finding as many typically developing students find these manipulations challenging (Banchoff, 2008; Cuoco, 2008; MacDonald, 1986). Participants' mastery performance resulted from their ability to check their work through the use of the Lab Gear and the

Box Method. Both methods provided students with visual cues to check the accuracy of their work (i.e., making a rectangle with the blocks and completed all sections of the Box). Students' ability to implement procedures, such as those used to multiply linear expressions and factor quadratic expressions, and to check their work throughout the procedure is a desired mathematical behavior for all students (Banchoff, 2008). Only one participant (Anna) missed points on tasks relating to transformation of quadratic expression as a result of not checking her work. Anna did not multiply the two binomials as requested when completing the table of data. This oversight significantly impacted her score on the final post-intervention domain probe as the scoring rubric contained points for showing the process in which the binomials were multiplied. She demonstrated her ability to multiply binomials on another task on this specific domain probe as well as on the other post intervention probes.

An error analysis of post-intervention domain probes revealed that all participants missed points on the open-ended questions based on three reasons: (a) they responded incorrectly to the question; (b) they provided faulty justification; or (c) they neglected to provide an explanation. Participants' incorrect responses reflected a misconception between rectangles and squares. For example, Sasha missed points on an open ended question that asked if the shape of the area changed after the renovation. She incorrectly responded that the shape remained a square, just a bigger square; however, the renovations changed the shape from a square to a rectangle. When these errors occurred, participants were focusing on the appearance of the shape not the properties of the shape, which indicates immature geometric thinking based on van Hiele's level of geometric thought (Van de Walle, et al., 2010). Additionally, students with LD often demonstrate visual-spatial deficits (Garnett, 1998; Geary, 2004) which may have also impacted Sasha's visual perception of the pictures of the rooms.

Participants did not always make connections with the tasks embedded within the instructional sessions and the tasks on the domain probes, which lead to faulty justifications. Faulty justifications were frequently linked to an inappropriate reasoning based on linear measurements. For example, if the length of the room increased by four feet and decreased by two feet, then participants assumed that the renovated room increased by two feet. However, during the instructional sessions, participants drew numerous area models for discrete numbers on graph boards to determine if a particular set of similarly renovated rooms had more, less, or the same amount of space. When encountering this question on the domain probes, they did not consistently generalize that method when responding to the question. This is consistent with previous research which reports that students with LD have difficulty in problem solving tasks as a result of poor recall and generalization of previously learned materials (Bley & Thorton, 2001; Bryant, et. al., 2000; Gagnon & Maccini, 2001).

Finally, participants lost points for not provided explanations for their responses on the open-ended tasks. Although the school staff members who administered the post-intervention domain probes were informed that students could dictate responses, no participant did so. The reason for this is unknown; perhaps participants did not feel comfortable dictating to the school staff members. Possibly they would have scored additional points on these tasks if they had dictated their explanations, as one-half to two-thirds of students with mathematics difficulties also have co-morbid language deficits (Jordan, 2007).

**Research Question 2.** Research Question 2 was: When provided blended instruction with visual representations, to what extent do secondary students with MD maintain performance on algebraic tasks involving quadratic expressions embedded within area problems four to six weeks after the end of the intervention? All participants demonstrated a high degree of maintenance of the content taught during intervention and reached the criterion score of 80% accuracy or greater four to six weeks following the intervention. The mean score across participants equaled 90% accuracy with a range of 80% - 100%. This finding is consistent with previous research in which students with LD demonstrated maintenance of mastery performance as a result of contextualized problems, the CRA sequence, and explicit instruction (Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Scheuermann, et al., 2009).

Group 1 participants outperformed Group 2 participants as evidenced by mean group scores on the maintenance domain probe (82% and 96%, respectively). Two possible explanations for this include: (a) the number of instructional sessions per group; and (b) level of engagement. Although both groups completed the exact same tasks, Group 1 completed the unit in 10 instructional sessions while Group 2 completed the unit in 13 sessions. Group 1 participants were consistently on time and worked steadily. In contrast, one participant (Sasha) in Group 2 was consistently late by 10 – 15 minutes the first six sessions. Sasha was informed that she would not be permitted to continue in the study if she continued to be late. Fortunately, she was consistently on time beginning session seven. However, Sasha's tardiness resulted in the Lab Gear introductory lesson and Lessons 1 through 3 extending into two days rather than completing during one session. Perhaps spreading the instructional unit over the additional three sessions may have improved students' retention of the content.

The level of engagement as determined by the amount of comments and questions asked by participants may have also contributed to the mean level difference across participants in Group 1 and Group 2. Although participants in both groups struggled to work cooperatively (i.e., finished tasks independently and then checked their response with their peers), participants in Group 2 may have benefited from the verbal participation of Marcia. She consistently asked questions and spoke aloud while engaging in the task, even though she did not consistently engage in tasks with her peers. Other research has shown similar results with peers benefiting from hearing the verbalizations of other group members (Van de Walle, et al., 2010). Additionally, according to Vygotsky, social interactions within a classroom create a *zone of proximal development,* in which students learn new concepts by interacting with a teacher and/or other students (Gurganus, 2007; Kozulin, 1998; Van de Walle, et al., 2010).

An analysis of maintenance domain probes revealed all participants recalled how to multiply linear expressions and how to factor a trinomial quadratic expression with a high degree of accuracy. However, Cheryl, Cindy, and Sasha missed points for not providing explanations for their responses on the open ended tasks. Anna again missed points as a result of not multiplying the linear expressions to complete the table as requested. Instead she indicated that the area of any square classroom after renovations was represented as $(x + 6)(x + 3)$. The oversight was a result of not following written directions, rather than her algebraic ability, which significantly impacted her score as the scoring rubric contained points for showing the process in which the linear expressions were multiplied. She demonstrated an ability to multiply linear expressions on another task on the maintenance domain probe.

Although participants in Group 2 outperformed participants in Group 1, all participants demonstrated a high level of retention on the maintenance domain probe. This is an important finding as poor retention is characteristic of students with LD (Bley & Thornton, 2001; Geary, 2004; Hudson & Miller, 2006). To compensate for memory issues, this intervention provided participants with ways of experiencing the algebra content through the manipulatives, which is a recommended practice when teaching students with LD (Bley & Thornton, 1995). Through a combination of explicit instruction and exploratory activities, participants were able to construct their own understanding of quadratics which supported maintenance.

**Research Question 3.** Research Question 3 was: When provided blended instruction with visual representations, to what extent do secondary students with MD transfer their knowledge of quadratic expressions to problem-solving tasks? Results on the transfer measures were mixed as two participants (Anna and Sasha) demonstrated minimal ability to transfer the algebra content to novel situations, while two additional participants (Cheryl and Cindy) demonstrated a satisfactory performance. Only one participant (Marcia) demonstrated competency on all three transfer tasks.

For the first task, all participants successfully found the perimeter and the volume using discrete numbers and wrote algebraic expressions to represent the length, width, and height for the set of data. This task included a contextualized word problem and tabular data, which was a format that participants recognized from the instructional unit. Although all participants accurately represented the dimensions using an algebraic expression, only Marcia performed the accurate computations with the algebraic expressions to determine the volume and perimeter. The other participants did not generalize the Lab Gear representations and the Box Method to determining perimeter and volume when represented by algebraic expressions. Difficulty with generalization of mathematical concepts and skills is characteristic for students with LD (Bley & Thornton, 1995; Fuchs & Fuchs, 2007).

For task two, Sasha replied, “Can’t put it in the Box Method too many numbers.” She did not attempt to modify the Box Method and gave up easily. Therefore, Sasha demonstrated two characteristics of students with LD, poor generalization (Bley & Thornton, 1995; Fuchs & Fuchs, 2007) and passivity (Gagnon & Maccini, 2001). Anna did not attempt to use the Box Method and incorrectly used the distributive property. Instead she multiplied the first term of the trinomial with the first two terms of the polynomial, then multiplied the second term of the trinomial with the second term of the polynomial, and finally multiplied the last terms of each expression. Anna demonstrated procedural deficits which are characteristic of students with MD who have deficits in working memory (Geary, 2004). Marcia, Cindy, and Cheryl modified the Box Method to multiply a trinomial by a polynomial with four terms. Cindy and Cheryl made computational mistakes when multiplying expressions with exponents and when simplifying like terms, which is characteristic of students with LD who may have semantic memory deficits which interfere with the ability to retrieve facts and skills from long term memory (Geary, 2004). Marcia was the only participant to complete the task correctly.

For the third task, all participants attempted to use the Box Method to factor a trinomial quadratic expression with a coefficient of three. Sasha gave up quickly when she could not generalize the learned procedure (i.e., find factors of the constant that add to the coefficient of the linear term). Anna incorrectly applied this procedure and recognized that she did not have the correct response as she wrote "wrong" on her paper. Cindy, Cheryl, and Marcia reverted back to sketching the Lab Gear when they had difficulty with the Box Method and then were able to accurately factor the quadratic expression. The movement between representations demonstrates a depth of understanding (Van de Walle, et al., 2010).

Difficulty with transferring mathematical concepts and skills to novel situations is characteristic for students with LD (Bley & Thornton, 1995; Fuchs & Fuchs, 2007). In fact, many typical students fail to transfer learned material to new problems (Greeno, Collins, & Resnick, 1996). Fuchs and Fuchs (2007) suggest explicitly teaching transfer to increase students' awareness of the connections between novel and familiar problems. In the current study, no explicit instruction was provided on the three transfer tasks. Despite this, all participants were able to transfer their knowledge on the first transfer task. Hudson and Miller (2006) suggest that teaching conceptually aids in transfer. The first transfer task was conceptually linked to familiar concepts of volume and perimeter. In contrast, the other transfer tasks were procedural tasks involving only abstract symbols. Previous research shows that the students with LD have difficulty understanding abstract symbols (Bley & Thorton, 2001; Garnett, 1998; Geary, 2004; Witzel, 2005), and poor understanding of procedures (Garnett, 1998; Geary, 2004). Regardless, three of the five participants were able to transfer their knowledge from the instructional unit to tasks 2 and 3 on the transfer measure with some degree of accuracy (see Figure 2). To do so, they used the Box Method which is a general purpose tool that can be used to multiply polynomials of varying degrees (Cuoco, 2007). Participants' recognition of the Box as a general method for solving tasks with similar features promoted their success on the transfer measure, which is consistent with research in mathematics education (Greeno, et al., 1996).

**Research Question 4.** Research Question 4 was: When provided blended instruction with visual representations, to what extent do secondary students with MD find blended instruction with visual representations beneficial (i.e., social validity)? The math department chair at the participating school administered the social validity measure to each group of participants. The measure consisted of two parts: (a) a 5-point Likert scale; and (b) open-ended questions. All participants reported that they found the intervention to be beneficial and would recommend this intervention for other students. They particularly enjoyed the Algebra Lab Gear, which is consistent with previous research (Maccini & Hughes, 2000; Maccini & Ruhl, 2000). Responses were mixed when asked about the benefit of word problems, tables of data, and talking about the problems with the teacher and/or peers, which may have been influenced by their previous experiences with these instructional tasks. Specifically, students with LD characteristically struggle to complete word problems (Bryant, et. al., 2000). Additionally, the participants of this study had no experience with tabular data nor were they required to discuss and justify their responses in their current math class.

**Research Question 5.** Research Question 5 was: How do the qualitative data findings provide an enhanced understanding of the quantitative results? Specifically, what connections and disconnections to the algebra content emerge as a result of the intervention and how can these findings improve future instruction? In what ways does the intervention enhance aspects of metacognition? An analysis of Marcia's transcriptions, field notes, and work samples provided insight into why participants' performance on domain probes increased. Specifically, two main themes emerged: representations and metacognition.

***Representations.*** Within the theme of representations, numerous subthemes surfaced involving Marcia's use of the area context, the Lab Gear, and the Box Method. Specifically, she developed connections and disconnections between these types of representations and the algebraic content.

*Area context.* Marcia established connections between the area context and her previous knowledge, abstract symbolism, and the concept of area as a quantity. Additionally, a disconnection was observed between the area context and factoring. First, the area context provided a connection to Marcia's previous knowledge of *area* as *length times width* using discrete numbers. The ability to build on existing knowledge is a critical component in both explicit instruction (Hudson & Miller, 2006) and constructivist teachings (Clements & Battista, 1990; Knight, 2002). The area context also served as an anchor to the abstract symbols. After discussing the area context through the word problem and the tabular data, Marcia developed a generalized algebraic expression to represent the area context and was able to perform the symbolic manipulation to transform the factored-form (i.e., dimensions) into the standard-form (i.e., the area). However, when presented with a noncontextualized quadratic in factored-form, she struggled to associate the symbolic manipulation as "the answer" or the final result to the task. Although we continued to write the symbolic-equation using the area formula, there was no story situation to anchor her response. This finding is consistent with previous research (Nathan & Koedinger, 2000) as the researchers found that typically developing students solved contextualized algebraic word problems with a higher degree of accuracy than symbolic-equation problems because of their use of informal strategies. For example, students frequently used the common informal strategy of guess and check when provided a contextualized problem. Nathan and Koedinger stated that informal methods may facilitate performance when technical skills are lacking, however, informal strategies were insufficient for solving complex algebraic tasks.

In addition Marcia also made connections between the area context and the area quantity. When completing open-ended questions asking if the renovated rooms now had more, less, or the same amount of space, she consistently substituted discrete numbers into the factored-form of the quadratic expression to confirm her answer. Although she developed an understanding of quantity when comparing quadratic expressions later in the unit, she always reverted back to the area context to confirm her responses to these types of opened-ended questions on the post-intervention and maintenance domain probes. Marcia was more comfortable working with discrete numbers rather than the abstract algebraic expressions, which is a typical response as many students with LD struggle to conceptualize abstract algebraic concepts (i.e., abstract symbols such as variables) and algebraic tasks (i.e., solving complex equations) (Bley & Thorton, 2001; Garnett, 1998; Witzel, 2005).

Although the area context enhanced Marcia's connections to algebraic content, she also demonstrated a disconnection between the area context and factoring. Marcia stated that “All those words don’t matter” when presented with a contextualized task involving factoring. As factoring is often considered a difficult topic for students (Banchoff, 2008; Cuoco, 2008; MacDonald, 1986), Marcia’s inability to see a connection between the area context and the process of factoring is consistent with previous research. Additionally, the use of word problems for more complex algebraic equations is not as helpful as informal strategies such as guess-and-check break down (Nathan & Koedinger, 2000).

*Lab Gear*. Marcia established connections between the Lab Gear representations and the Zero Property, negativity, rules for factoring, and comparison of quantity between different quadratic expressions. She also demonstrated disconnections to the Distributive Property and the abstract nature of the Lab Gear representation. Marcia recognized many connections between the Lab Gear and algebraic concepts. Specifically, she was able to act on the manipulatives to demonstrate the Zero Principle by removing equal numbers of positive blocks and negatives blocks. Additionally, she recognized this principle in her drawings when she no longer needed to physically manipulate the blocks, indicating a transfer of this concept to a sketch. She also successfully modeled the concept of negativity as adding a negative by using the negative blocks. This is an important observation as many typically developing students struggle with concept of negativity (Kieran, 1989; Vlassis, 2004).

Marcia also established a connection between the Lab Gear and the rules for factoring. She identified factoring as “working backwards” and “undoing” the multiplication of the linear expressions (i.e., the dimensions). During this process, Marcia identified patterns that developed into the process of factoring quadratics with a coefficient of one, such as the factors of the constant must add to equal the coefficient of the linear term. Additionally, she was able to use a sketch of the Lab Gear to modify her process on the third task of the Transfer measure, which required her to factor a quadratic with a coefficient of three. The use of multiple representations provided Marcia with a deeper understanding of factoring (NCTM, 2000; Van de Walle, 2010) which helped her connect the procedural rules with conceptual understanding of "working backwards" and "undoing." These are important findings as typical students struggle to factor (Banchoff, 2008; Cuoco, 2008; MacDonald, 1986).

The last connection Marcia made between the Lab Gear and the algebra content involved comparing quantities of various quadratic expressions. For example, when comparing the Lab Gear representation of $x^2 + 4x + 3$ and $x^2 + 4x + 4$, Marcia initially had difficulty seeing that one area was larger. She realized that the quantities were very similar with one area only "up by one." When presented with the representations of the quadratic expressions $x^2 + 4x + 4$ and $x^2 + 5x + 4$, Marcia observed that $x^2 + 5x + 4$ resulted in a larger picture than the increase of only one constant in the example above. Her analysis was based on a literal interpretation of the Lab Gear and the x-bar being physically bigger than the constant block. Therefore, I used discrete numbers to help concretize the impact of changing the linear term of the quadratic; however, she still reverted to the visual representation of the Lab Gear. When combining the visual representation and the area context, Marcia developed an understanding of the quantity of given quadratic expressions. The use of multiple representations to develop an algebraic concept is consistent with best practices for teaching math to students with math difficulties (Maccini & Gagnon, 2000).

In addition to the connections between the Lab Gear and the algebra content, Marcia also developed two disconnections involving the Distributive Property and literal interpretations of the Lab Gear representation. Although she was procedurally fluent manipulating the Lab Gear to multiply linear expressions; she stated that she was "making a picture." At this point in the intervention, she was not focusing on the process of the Distributive Property that was represented with use of the Lab Gear. Van de Walle, Karp, and Bay-Williams (2010) warn teachers about this "mindless procedure" (p.29) when using manipulatives. Additionally, Marcia wanted the Lab Gear to literally represent measurements "like a blueprint." Despite being a concrete manipulative, the Lab Gear representation required students to make abstract interpretations. For example, students can not assume that three constant blocks equals an x-bar, even though they are almost the same size. This was initially challenging for all participants, as students with LD often demonstrate difficulty understanding abstract algebraic concepts (Garnett, 1998; Geary, 2004; Witzel, 2005).

*Box Method.* The Box Method was used to support the transition from concrete representations of the Lab Gear to abstract symbolic representations. Initially, Marcia developed disconnections between the Lab Gear and the Box Method. However, as the intervention progressed, Marcia established connections between the Box Method and the Distributive Property and factoring. Explicit instruction regarding how to apply principles from the Lab Gear to the Box Method was provided, which aligns with suggestions from Fuchs and Fuchs (2007) to explicitly teach transfer to increase student's awareness of the connections between novel and familiar tasks.

Despite this explicit instruction, disconnections emerged. For example, when presented with tasks such as *(– 13 + 2x) (10 + x),* Marcia placed the terms with the variables first in the expression because the linear term always appeared first in the expressions when using the Lab Gear. Expressions with the constant as the first term were not presented when using the Lab Gear, which was a limitation in the instruction unit. The instructional unit should have included multiply opportunities for participants to multiply linear expressions such as *(– 13 + 2x) (10 + x)* with the use of the Lab Gear so as not to reinforce a misconception that the term with the variable must come before the constant. An additional disconnection between the Lab Gear and the Box Method revolved around multiplying linear expressions with a coefficient greater than one. For example, when presented with *(3x + 15) (x -2)*, Marcia became confused as she was accustomed to having $x^2$ as the first term in the quadratic expression when working with the Lab Gear. The instructional unit should have included additional problems in this form when using the Lab Gear to promote an easier transition to the Box Method.

Despite these initial disconnections, Marcia used the Box Method to make many connections to the algebra content including the Distributive Property and factoring. Although explicit instruction, including a template, was provided, Marcia chose to develop her own form of the Box which was closely linked to the Lab Gear representation. The connection between the two representations helped her generalize the procedures from using the Lab Gear to using the Box Method. This was an important criterion for her Box Method as generalizing was challenging for Marcia, which is typical of students with LD (Gagnon & Maccini, 2001). As she developed her unique Box Method, the connection to the Distributive Property emerged. When using only abstract symbols, Marcia consistently multiplied linear expressions to generate only the quadratic term and the constant, such as $(2x - 10)(x - 12) = 2x^2 + 120$. The $2x$ was being distributed to the x, but not to the *-12*. Additionally, the *-10* was distributed to the -12, but not to the x. Marcia then began thinking of the Box as a grid, which supported the Distributive Property. Her process of developing a Box that accurately represented the Distributive Process exemplifies Vygotsky's zone of proximal development in that she developed a personal tool that she use to make sense of the process of multiplying linear expressions. She needed scaffolds, such as the context of the grid, to develop her full understanding of this process of distributing. Marcia also developed connections between the Box Method and factoring. Again explicit instruction was provided to demonstrate how the Lab Gear representation generalized to the Box Method template; however, she continued to use her own model of the Box Method.

Although Marcia demonstrated an understanding of factoring quadratic expressions as "working backwards" to find the dimensions, deficits in prerequisite skills impacted her progress as she did not have an understanding of the concept of factors as they relate to whole numbers. Therefore, a review of this concept was needed during the instructional sessions. A review of fundamental skills is important for preparing students with LD for success in new mathematical topics (Hudson & Miller, 2006).

Furthermore, Marcia demonstrated a lack of automaticity with multiplication facts, which is a characteristic of students with mathematics LD (Garnett, 1998; Geary, 2004). Although Marcia experienced these deficits, she compensated by efficiently using her fingers and the calculator to determine factors. At times this appeared laborious; however, these deficits did not impact her performance as evidenced by her high percentage of accuracy on probes. This is consistent with research reporting that students with MD may learn to compensate for their disability and may achieve proficiency with laborious struggles (Mazzocco, 2007). Additionally, the use of calculators helped Marcia access grade-appropriate content, which is mandated by federal legislation (IDEIA, 2004; NCLB, 2001) for students with disabilities. Furthermore, NCTM views calculators as essential tools for doing and learning mathematics and allows students to focus on the mathematical ideas (NCTM, 2000; Van de Walle,et al., 2010).

The transition from the concrete manipulatives to abstract symbols is challenging for typically developing students (Pashler, et. al., 2007) and for students with LD (Hudson & Miller, 2006). Therefore, use of the Box Method supported the transition from the concrete to the abstract phase as it provided organizational support and prompted Marcia to use the Distributive Property. This is consistent with previous research which identifies the use of graphic organizers as tools that help students to understand the procedures necessary to solve algebraic tasks (Maccini, et. al., 2008) This is particularly beneficial for students with LD who may have semantic memory deficits, which is characterized by difficulty remembering procedures (Geary, 2004). Additionally, graphic organizers support working memory deficits, which are also characteristic of students with LD (Strickland & Maccini, 2010).

***Metacognition.*** In addition to representations, the theme of metacognition emerged from the qualitative data. Within the theme of metacognition, the following subthemes emerged: (a) strategic planning; (b) self-regulation; (c) disposition; and (d) socially shared metacognition.

*Strategic planning.* Marcia indicated on her pretest domain probes that she did not know how to break up the task into steps that would enable her to reach a solution. On three of the four pre-test domain probes, she did not attempt to solve any of the tasks. This is typical behavior of students with LD as they are characteristically passive in their learning and do not actively attack a problem (Gagnon & Maccini, 2001; Hudson & Miller, 2006). An explanation for this may be that students with LD have procedural and working memory deficits (Geary, 2004) which interfere with strategic planning.

During the intervention, Marcia stated that the Lab Gear and the Box Method helped her break down the tasks and develop a plan of action that she executed to successfully complete the tasks, as evidenced by her high scores on post-intervention domain probes. Additionally, she was able to incorporate her knowledge from the instructional unit to develop and execute strategic plans for solving tasks on the transfer measure. This is an important finding as students with LD (Bley & Thornton, 1995; Fuchs & Fuchs, 2007) and without LD (Greeno, Collins, & Resnick, 1996) typically struggle to transfer learned material to novel situations. However, Marcia used multiple ways of expressing the algebraic content (i.e., sketches of Lab Gear, the Box Method, and symbolic notation) which supported her strategic planning (CAST, 2008).

*Self-regulation.* Marcia demonstrated self-regulation when monitoring her performance on tasks and when evaluating her solutions. Specifically, she relied on visual cues from the Lab Gear to help monitor her performance on tasks involving multiplication of linear expressions. After transitioning to using only abstract symbols in the Box Method, she frequently returned to the blocks to verify the answer from the Box Method. The integration of the concrete and abstract representation is recommended in the mathematics literature (Pashler, et. al., 2007), although previous research has shown that a graduated approach from the concrete, semi-concrete, to abstract representations is also beneficial (Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Scheuermann, et. al., 2009). Additionally, Gersten and colleagues (2009) recommend that use of manipulatives with older students should be expeditious as the goal should be facility in abstract symbolism. Therefore, there are benefits to both the graduated and the integrated approach to CRA instruction and the determination of which approach to use should depend on the characteristics of the students and the mathematics topic.

Marcia did not always monitor her performance, especially when factoring quadratic expressions. For example, when factoring quadratic expressions, she was instructed to monitor her solution in three ways: (a) analyze visual cues; (b) compare tables of data; and (c) compare graphs. First, she relied on visual cues from the Lab Gear or sketches of Lab Gear (i.e., a perfect rectangle represented the quadratic expression). Next, when transitioning to the Box Method, she was instructed to compare tables of data for each form of the quadratic (standard-form and factored-form). Third, she compared graphs created on the graphing calculator (i.e., correct transformation produced one graph). After determining that the tables and graphs would be the same, Marcia only completed a table for one form of the quadratic and then copied the numbers into the second table. Additionally, she stopped graphing the quadratics expressions. Marcia's lack of self-monitoring her performance is characteristic of students with LD (Gagnon & Maccini, 2001).

Another component of self-monitoring involved the evaluation of accuracy of one's solutions. Marcia often made faulty evaluations of the accuracy of her performance and would often say, “This is all wrong” and yet she would have an accurate solution and be able to justify her answer. This is consistent with previous research which reported that students with mathematics LD were less accurate than their non-disabled peers when evaluating the accuracy of their solutions (Mazzacco, 2007).

*Disposition.* Aspects of Marcia’s disposition observed during the intervention included self-confidence, perseverance, and enjoyment. Throughout the instructional unit, Marcia’s self-confidence ebbed and flowed. She began the unit with low self-confidence, as she repeatedly stated that she didn’t believe that her responses to tasks were accurate. This lack of confidence returned with every new objective. Lack of confidence in mathematics is characteristic of students with LD (Gagnon & Maccini, 2001). However, her use of the area context, Lab Gear, and Box Method built Marcia’s belief in her abilities and at the end of the intervention she felt confident that she could accurately complete every task on her maintenance domain probe. In this case, Marcia accurately evaluated her ability as she scored 100% accuracy on her maintenance domain probe.

Marcia’s enjoyment of the intervention coincided with her self-confidence. The key to Marcia’s enjoyment was the development of her self-confidence and the supports embedded in the intervention, such as the Lab Gear and the Box Method, which cultivated her confidence in her math ability. Marcia’s behavior during this intervention was completely different from her behavior reported by her teacher and her mother. Before and after this study, Marcia consistently refused to attend her math class and refused to complete assignments. In contrast, during this study, she arrived on time, stayed late, and completed all of her tasks with a high degree of accuracy. Sasha and Cheryl also displayed similar behaviors in that they did not consistently attend math class prior to the intervention and upon completion of the intervention continued to miss class and not complete assignments. In the beginning of the intervention, Sasha was consistently late and was told that she would be removed from the study if this continued. She was adamant in her desire to continue with the intervention and arrived on-time or early for all other sessions. Additionally, Cheryl's teacher reported that she was frequently off-task and disruptive in class, which were behaviors that were not observed during the intervention.

In a survey study by Kortering, deBettencourt, and Braziel (2005), over half of high school students with LD identified math as their least liked subject and stated the type, complexity, and amount of work as the most difficult part of algebra class. Additionally, these students stated that more help, a change in teacher style, and making the class more enjoyable would enable them succeed in algebra. Perhaps the instructional practices embedded in the intervention provided Marcia with the assistance she needed to succeed. Additionally, we had a very good rapport and she enjoyed my teaching style. Marcia felt a sense of pride in her achievements in the content covered in the instructional unit as she was engaging in grade appropriate algebra content. In contrast, her math teacher informed me that she was teaching word problems from a fourth grade textbook. Marcia felt that her regular math class was not preparing her for her future, as she asked me to stay and teach her the material that will be on the mathematics placement test for community college. Marcia did not see a value in the material that was being taught in her typical math class and therefore she refused to attend.

Students who value mathematics and feel confident in their abilities are more likely to persevere and accurately complete algebra problems than their counterparts

(Van de Walle, et al., 2010). During the intervention, Marcia valued the algebra that she was learning and developed confidence in her ability to improve her performance. As a result, Marcia demonstrated perseverance as she consistently attended sessions, arrived on time and stayed late to ask questions. In contrast to her pretest domain probes in which she only attempted one problem on four probes, Marcia completed each question on the post-intervention and maintenance probes with a high degree of accuracy. Additionally, she persevered for over an hour to complete the transfer measure with 100% accuracy. Marcia's perseverance is not typical of students with LD who characteristically demonstrate low motivation when engaging in mathematics (Gagnon & Maccini, 2001), however, this perseverance was representative of the participants in this study. Additionally, Marcia's motivation is not characteristic of female high school students in general education, who display less motivation in mathematics and report more negative attitudes regarding mathematics than males (Royer & Walles, 2007). However, her motivation was characteristic of the participants in this study.

*Socially shared metacognition.* Previous research shows that socially shared metacognition is likely to occur in situations where peers are at approximately the same ability level and given problems that are difficult but within students' zone of proximal development (Goos, Galbraith, & Renshaw, 2002; Iiskala, Vauras, Lehtinen, Salonen, 2011). Although the grouping of students and the problem difficulty were aligned with this research, there were only four incidents of socially shared metacognition among peers in Marcia's group. Rather than engaging in reciprocal problem solving, Marcia and her peers engaged in peer tutoring or ignored each other completely and relied on my responses. Marcia revised her solution or strategy based on the feedback of a peer on only one occasion during the intervention. When confident in her knowledge, she eagerly tutored her peers. However, when not confident, Marcia appeared to prefer to engage in discourse with me rather than her peers.

As students with disabilities typically have poor metacognitive skills, it is not surprising that Marcia did not demonstrate socially shared metacognition. Research suggests that high-achieving students are more likely to engage in socially shared metacognition (Iiskala, et.al., 2011). Students' lack of confidence in their mathematics abilities and their lack of confidence in their peer's abilities may impede their engagement in socially shared metacognition.

**Summary**

This study suggests that blended instruction with visual representations can improve the algebra performance of secondary students with mathematics learning disabilities and difficulties when working with quadratic expressions embedded in area problems. Each participant substantially improved their percent accuracy from baseline to post-intervention and consistently performed at 80% or above on post-intervention domain probes. Additionally, all participants performed at 82% or above on all lesson probes throughout the intervention. All participants demonstrated a high degree of retention of the content taught during intervention and reached the criterion score of 80% accuracy or greater on the maintenance domain probe four to six weeks following intervention.

The intervention yielded mixed results on the transfer measure. Participants' performance varied as evidenced by the percentage of accuracy ranging from 33% to 100%. A possible explanation for this mixed result on the transfer measure may stem from individual differences regarding strategic planning and perseverance. Anna and Sasha scored the lowest on the transfer measure and they spent the least amount of time completing the test. When confronted with an unfamiliar task, they attempted to use the tools (i.e., the Box Method) from the intervention. When their method was unsuccessful, they did not attempt to revise their strategy, but instead immediately proceeded to the next task. In contrast, the other participants persevered with the task, revising their strategies many times and providing a solution that varied in accuracy.

A qualitative analysis of transcribed instructional sessions, work samples, and field notes focusing on Marcia supplemented the quantitative findings. Marcia was chosen because she provided a rich source data that was representative of the group. Specifically, she demonstrated connections between the instructional practices (i.e., Lab Gear, Box Method) and the algebraic concept. Additionally, the intervention enhanced metacognition as demonstrated by her improved strategic planning, self-regulation, and disposition. Similar to all participants, she did not demonstrate any growth in socially shared metacognition and relied on discourse with her teacher rather than her peers to revise her strategy or solution.

**Limitations and Future Research**

Although the results of this study are promising, limitations and suggestions for future researcher should to be addressed. First, participant attrition led to a design change. The original study proposal involved a multiple probe design across three pairs of participants. Although six participants were selected based on the eligibility requirements, one participant left the study due to an extended medical leave after completing one baseline probe. Therefore, the study design was revised and a multiple probe across two groups replicated over five students was implemented. While three groups (i.e., an initial demonstration replicated by two additional demonstrations) may be desirable (Horner, et. al., 2005), two groups (i.e., an initial demonstration with only one replication) is acceptable to establish a functional relationship (Kennedy, 2005). Future research should include an increased number of participants with additional replications.

Additionally, the low number of baseline probes warrants discussion. Tawney and Gast (1984) recommend a minimum of three probes per participant during baseline. However, Kennedy (2005) states that “baseline needs to be as long as necessary but no longer” (p. 38). As the pretest scores for Cheryl and Cindy (Group 1) demonstrated stability in level, low variability, and a slight decelerating trend, only two baseline probe were collected. To demonstrate experimental control, one more additional consecutive probe was collected on the participants in Group 2 as recommended by Horner & Baer (1978).

The current study was conducted in a small group setting with a student-to-teacher ratio of 2-1 for Group 1 and 3-1 for Group 2. This small student-teacher ratio provided participants with the focused attention of the researcher. Additionally, the location of the study was an isolated conference room which limited distractions. Therefore, replication with a large group of students in a typical classroom is needed to generalize the results to a typical classroom environment.

This study contained five participants who displayed common characteristics including a history of difficulty in mathematics and a need for the intervention as evidenced by the low scores on baseline domain probes. However, the participants also differed on characteristics such as being identified with a specific disability (e.g., learning disability, attention deficit disorder, at-risk for a learning disability). Therefore, caution must be taken when generalizing these findings to other students with disabilities. Future research is needed to determine the effectiveness of the intervention for students with specific mathematics learning disabilities.

The current study utilized dependent measures that were developed by the investigator and aligned with the instructional unit. Therefore, familiarity with the tasks may be an additional explanation for the improvements in domain probe scores from baseline to post-intervention. Additionally, dependent measures in future replications should include released test questions from the National Assessment of Educational Progress, States' high school exit exams, and the Silicon Valley Initiative, as well as standardized tests (i.e., KeyMath).

An additional limitation to the study was the investigator served as the teacher. Therefore, my biases may have influenced the study. I attempted to control for biases by following the designed lesson plans and collecting treatment fidelity data. Additionally, school personnel administered all domain probes while I waited outside of the room. Transfer measures were also administered by school personnel, although I observed the session to collect qualitative data. To address the potential bias of the investigator in this study, future replications should include trained teachers, not investigators, as the implementers of the intervention. Replications with trained teachers should occur prior to scaling up the study.

A possible limitation involving the qualitative method involved the analysis of only Marcia's data. Case studies typically include more than one participant (Creswell, 2007). As the qualitative data was supplemental to the quantitative data, I chose to focus on Marcia as she provided a rich source of data which was representative of the group of participants. However, each participant experienced the intervention in their own way and themes that emerged from analyzing Marcia may not be generalizable to all participants. Future qualitative research should include a larger sample so that common themes among participants may emerge.

The intervention was successful for participants in pairs and triads when used with algebra content focusing on quadratic expressions embedded in area problems. Replicating the package with different algebra topics (i.e., linear or exponential functions) and with other students with mathematics difficulties would be necessary for generalization (Horton, et. al., 2005; Kennedy, 2005). Additionally, a component analysis would be worthwhile to determine if any particular aspect of the intervention was more effective than other components. Finally, although single-subject research is a viable option for establishing research-based practices (Horton, et. al., 2005), scaling up to a group design investigation is recommended by experts in educational research (Gersten, et. al., 2009b).

**Implications for Practice**

The present study contributes to the literature in five ways: (a) addresses age appropriate algebra content for high school students with LD and at-risk for LD; (b) includes procedural fluency and conceptual understanding of an algebraic concept; (c) incorporates research-supported instructional practices for greater accessibility; (d) shows promise as a Tier II intervention for Response to Intervention; and (f) addresses ease of use in terms of affordability and feasibility.

First, this study addressed algebra content consistent with state high school exit exams and college placement exams and aligned with Common Core Standards and benchmarks suggested by the NMAP and ADP. Most of the previous research in algebra for students with LD focused on basic concepts and skills, such as integers (Maccini & Hughes, 2000; Maccini & Ruhl, 2000) and linear equations (Allsopp, 1997; Hutchinson, 1993; Mayfield & Glenn, 2008; Scheuermann, et al., 2009; Witzel et al., 2003). The current study extends to more complex concepts and skills and fills the gap in the literature base. This is critical as students with LD are being held accountable for the general education curriculum; therefore, researching strategies to support these students in demonstrating competency in high-level algebra content is critical for their success in high school and beyond.

Second, this study addressed both procedural fluency related to multiplying linear expressions and factoring quadratic expressions, as well as conceptual understanding of the quadratic expression. Procedural knowledge is necessary to develop conceptual understanding, as procedures strengthen conceptual development. In fact, students' understanding of the object (i.e., quadratic function) is strengthened by practicing computational techniques, even if these techniques are not yet fully understood (Sfard, 1995). Both procedural fluency and conceptual understanding are necessary to establish a network of skills and concepts to link new material in the development of problem solving abilities (NRC, 2001). Participants developed procedural fluency through the use of the Lab Gear and the Box Method. Additionally, the Box Method served as a method for organizing symbolic notation and sequencing of multiple steps in abstract algebraic manipulations. Graphic organizers may be easily developed by either the teacher or the student for a variety of algebraic tasks (Strickland & Maccini, 2010).

Additionally, participants developed a conceptual understanding of the quadratic expressions by embedding the task within area problems and providing participants with opportunities to explore quadratic expressions via tabular data, algebraic expressions, and qualitative data (i.e., drawings on graph paper). Conceptual knowledge goes beyond memorization of procedures to provide students with an in-depth understanding of a mathematic idea (Hudson & Miller, 2006; NRC, 2001), while supporting retention of procedures (NRC, 2001).

Third, this study blended instructional practices (i.e., CRA instruction, explicit instruction, graphic organizers) and the NCTM process standards. The Lab Gear and the Box Method link to the NCTM process standard of multiple representations. The use of area problems offered an additional representation of the quadratic expression and allowed participants to make connections between mathematical ideas. Further, the NCTM emphasizes reasoning and sense-making to support the mathematics progress for *all* students while engaging in mathematics communication (Dieker, Maccini, Strickland, & Hunt, 2011). Additionally, tasks within the instructional unit included both research-based teacher-directed investigations (Maccini, et. al., 2008; Strickland & Maccini, 2010) and student-centered activities (NCTM, 2000; Van de Walle, et. al., 2010). This blending of instructional delivery is recommended by the NMAP (2008) for students with disabilities and mathematics difficulties.

The results of the current study are promising and should be further examined as a Tier 2 intervention for Response to Intervention (RtI). RtI is a systematic and data-based method for identifying and resolving students' academic difficulties (Brown-Chidsey & Steege, 2005). Typically, RtI consists of three tiers. Tier 1 is the general education classroom. If students demonstrate difficulty in Tier 1, then they may be provided small group instruction in Tier 2. Students who do not respond to the small group instruction, then proceed to Tier 3, which may include individualized instruction and special education services (Fuchs, Fuchs, & Stecker, 2010). Unfortunately, little research has been conducted addressing Tier II mathematics interventions at the high school level. Participants in the current study demonstrated a need for small group instruction as all were at risk of failing their algebra course. Additionally, they all made substantial gains in their performance on post-intervention domain probes. Furthermore, the single-subject design is a recommend method for collected data within the RtI process (Brown-Chidsey & Steege, 2005).

Finally, this intervention is both affordable and feasible for teachers to implement. This study exemplified action research as it incorporated a teacher-researcher. Action research is a critical form of inquiry conducted by the teacher to gather information regarding student learning and teacher effectiveness (Gay, Mills, & Airasian, 2006). Additionally, this study incorporated materials that were affordable and feasible. Specifically, commercial algebra blocks such as Algebra Lab Gear (Picciotto, 1990) may be purchased; however, teachers may also make their own algebra blocks using laminated construction paper. Further, participants in this study learned to multiply linear expressions in four lessons and learned to factor quadratic expressions in five lessons using the CRA – I strategy. This is an important finding as experts in special education research previously suggested spending a minimum of three days at each stage of the

CRA sequence (Hudson & Miller, 2006), while, Gersten and colleagues (2009a) recommended the expeditious use of manipulatives as abstract symbolism is the goal for older students.

**Conclusion**

Federal legislation mandates that students with disabilities have access to the general education curriculum and are held accountability for proficiency of the content (IDEIA, 2004; NCLB, 2001). These mandates are necessary as students with disabilities and those at-risk are not demonstrating proficiency of the algebra content. The use of research-based instructional practices may assist secondary students with MD progress through the algebra curriculum. The current study investigated the effects of blended instruction and visual representations on area problems involving quadratic expressions for secondary students with MD. Prior to the study, no research existed that targeting quadratic expressions with secondary students with MD. The results of this study provide promising evidence that students with MD can improve their performance on grade-level algebra content when instruction is delivered through an instructional package. Additionally, participants maintained a high level of performance and transferred at least some of the strategies to novel problems.

Additional research is critical to identify instructional practices to make the general education algebra curriculum accessible and achievable for all students. Students with disabilities must demonstrate competency in rigorous algebra content in order to graduate from high school with a standard diploma and to pursue postsecondary education and employment opportunities. Therefore, a set of research-supported instructional practices for students with mathematics difficulties may contribute to more favorable outcomes, such as improved performance in high algebra courses, increased enrollment in higher level mathematics courses, increased high school completion rates, increased college completion rates, and improvements on state, national, and international mathematics assessments.

# Appendix A

## Constructivist Continuum

| | **Exogenous** | **Dialectical** | **Endogenous** |
|---|---|---|---|
| Current Examples | Bandura | Vygotsky | Piaget |
| Major Emphasis | Learning | Dynamic Interactionism | Development |
| Source of Knowledge | External Structures (environment) | Interaction (subjective experience) | Internal Coordinations (previous knowledge) |
| Role of Teacher | Directive | Supportive | Peripheral |
| Role of Learner | Engaged | Interactive | Self-Regulated |
| Content | Explicit skills | Skills, concepts, relationships | Concepts, relationships |

Adopted from Gurganus (2007) and Moshman (1982)

# Appendix B

## Common Core State Standards, NCTM Standards, ADP Benchmarks, Maryland Voluntary State Curriculum

### Common Core State Standards

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Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.
    a. Interpret parts of an expression, such as terms, factors, and coefficients.
    b. Interpret complicated expressions by viewing one or more of their parts as a single entity.
2. Use the structure of an expression to identify ways to rewrite it.
3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
    a. Factor a quadratic expression to reveal the zeros of the function it defines.

Perform arithmetic operations on polynomials

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### ADP Benchmarks

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J1. Perform basic operations on algebraic expressions fluently and accurately.

J1.1. Understand the properties of integer exponents and roots and apply these properties to simplify algebraic expressions.

J1.3. Add, subtract and multiply polynomials; divide a polynomial by a low-degree polynomial.

J1.4. Factor polynomials by removing the greatest common factor; factor quadratic polynomials.

J5. Solve problems by converting the verbal information given into an appropriate mathematical model involving equations or systems of equations; apply appropriate mathematical techniques to analyze these mathematical models; and interpret the solution obtained in written form using appropriate units of measurement.

J5.3. Recognize and solve problems that can be modeled using a quadratic equation.

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## Maryland Voluntary State Curriculum

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Pre-requisites
Standard 1.0 Knowledge of Algebra, Patterns, or Functions

Write, simplify, and evaluate expressions
Describe a real-world situation represented by an algebraic expression

Algebra/Data Analysis VSC
CLG1. The student will demonstrate the ability to investigate, interpret, and communicate solutions to mathematical and real-world problems using patterns, functions, and algebra.

1.1.3 The student will apply addition, subtraction, multiplication, and/or division of algebraic expressions to mathematical and real-world problems.

Additional Topics
Polynomial expressions in one or two variables.

1.1.3.3 The student will add, subtract, and multiply polynomials.
1.1.3.4 The student will divide a polynomial by a monomial.
1.1.3.5 The student will factor polynomials:

- Using the greatest common factor
- Using the form $ax^2 + bx + c$
- Using special product patterns
    1. Difference of squares
       $a^2 + b^2 = (a\text{-}b)(a\text{+}b)$
    2. Perfect square trinomials
       $a^2 + 2ab + b^2 = (a + b)^2$
       $a^2 - 2ab + b^2 = (a - b)^2$

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## NCTM Standards

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Students will represent and analyze mathematical situations and structures using algebraic symbols.

Students will use mathematical models to represent and understand quantitative relationships.

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# Appendix C

## National Council of Teacher's of Mathematics Process Standards

## Embedded within Blended Instruction

| NCTM Standard | Definition |
|---|---|
| Problem Solving | Engaging in a task for which the solution method is not known in advance |
| Reasoning | Making mathematical conjectures, justifying answers, and/or using alternative solution methods |
| Communication | Using the language of mathematics (i.e., binomial, polynomial, product, etc.) to express mathematical ideas and/or articulating reasoning to peers and/or teacher |
| Connections | Making connections among mathematical ideas (i.e. geometry) and/or applying mathematics in contexts outside of mathematics (i.e. language arts, science, social studies) |
| Representations | Problems requiring students to generate, select, or apply text about a representation to solve a problem. Representations include diagrams, graphs, models, symbolic notation construed from these representations |

## Appendix D

## Unit Objectives

Lesson 1: Given algebra blocks, students will represent and sketch area problems involving linear expressions containing positive terms only to produce a quadratic expression with 80% accuracy.

Lesson 2: Given algebra blocks, students will represent and sketch area problems involving linear expressions containing positive and negative terms to produce a quadratic expression with 80% accuracy.

Lesson 3: Given a graphic organizer (*the box)*, students will solve area problems involving linear expressions to produce a quadratic expression with 80% accuracy.

Lesson 4: Given a graphic organizer (*the box)*, students will solve area problems involving linear expressions to produce a quadratic expression with 80% accuracy.

Lesson 5: Given algebra blocks, students will identify quadratic expressions as a product of two linear expressions with 80% accuracy.

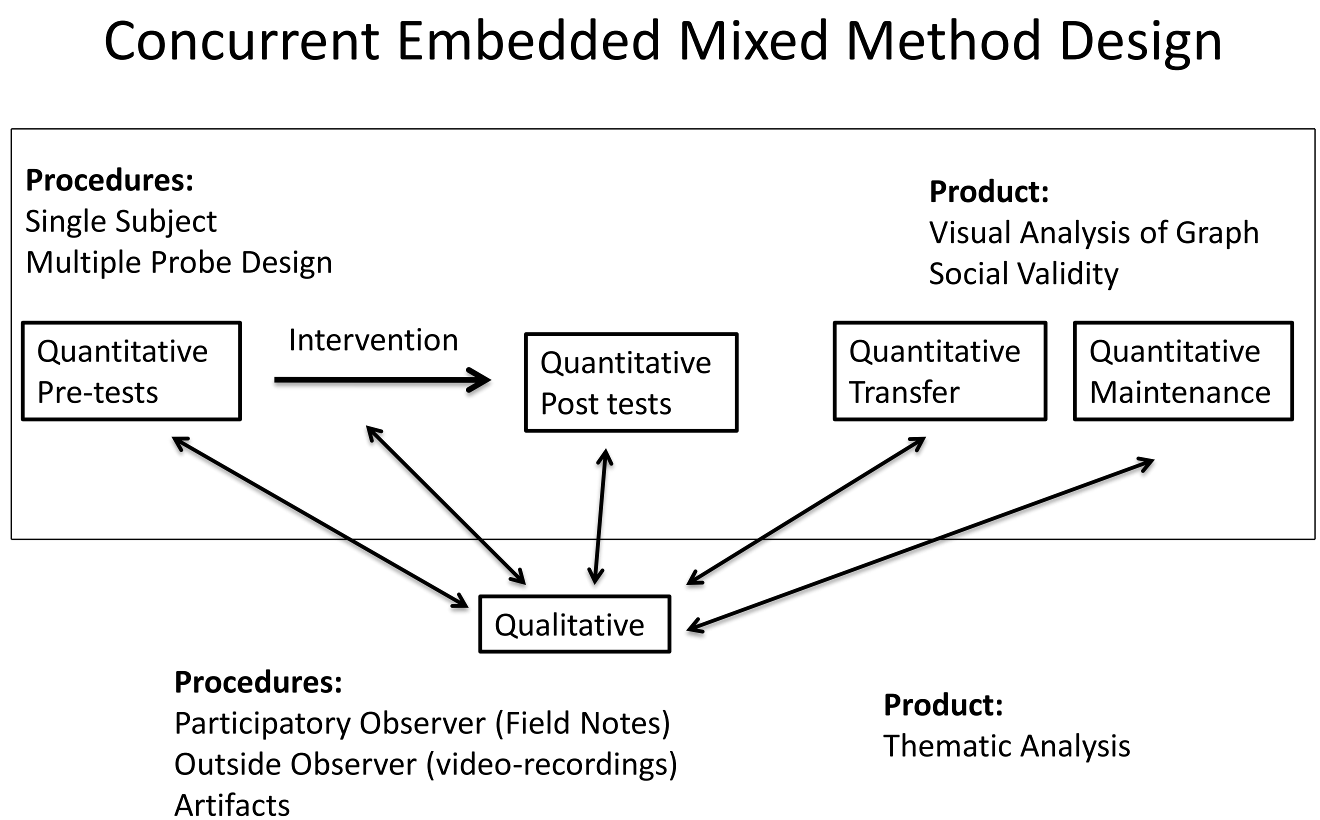
Lesson 6: Given an area problem involving a quadratic expression, students will use algebra blocks to determine the dimensions (length and width) with 80% accuracy.

Lesson 7: Given algebra blocks, students will explore the relationship between quadratic expressions that share the same quadratic and linear terms and discover the following rules for factoring.

Lesson 8: Given a partially completed graphic organizer (*the box)*, students will factor quadratic trinomials with 80% accuracy.

Lesson 9: Given an area problem with narrative and tabular data, students will transform a quadratic expression in standard-form to factored-form using the box method with 80% accuracy.

# Appendix E


Concurrent Embedded Mixed Method Design
Procedures:
Single Subject
Multiple Probe Design
Product:
Visual Analysis of Graph
Social Validity
Quantitative
Pre-tests
Intervention
Quantitative
Post tests
Quantitative
Transfer
Quantitative
Maintenance
Qualitative
Procedures:
Participatory Observer (Field Notes)
Outside Observer (video-recordings)
Artifacts
Product:
Thematic Analysis

## Appendix F

### Sample Domain Probe

Directions: Answer each question. If you choose, you may have the word problems read to you and you may dictate your responses for the essay questions. You may use a calculator to assist with computations. Show all of your work.

**Part I:**
**All of the classrooms at the Yeshiva are currently square shaped. All of these classrooms will be renovated so that the length will be increased by 6 feet and the width will increase by 3 feet. The dimensions of two classrooms are recorded in the table below.**

Fill in the table below to determine the area of the renovated classrooms at the Yeshiva. To find the area, multiply the *length* · *width*. Show all of your work.

| | Side of the current classroom in feet | Length of new classroom in feet | Width of new classroom in feet | Area of new classroom in square feet |
|---|---|---|---|---|
| Room 205 | 8 | | | |
| Room 206 | 9 | | | |
| Any Yeshiva classroom after renovations | x | | | |

Do the renovations change the *shape* or the *squareness* of the area of the classrooms? Justify your answer.

**Part II:**
**All dorms in Tower 1 of the University of Silver Spring are being renovated to include a bathroom. Currently, all the dorms are square shaped. After renovations, the length of each dorm will be 4 feet longer, while the width will be 3 feet shorter.**

Write area equation for the renovated dorms: *length* · *width* = *area*. Show all of your work.

Will the residents of Tower 1 have more, less, or the same amount of classroom space after renovations? Explain your answer.

**Part III:**

**In a neighborhood Columbia, the area of all the backyards can be represented as $x^2 + 8x + 15$.**

Fill in the table to represent the areas of the specific families' backyards.

| Name | x-value in yards | Area of backyards using standard form |
|---|---|---|
| Jacoby's backyard | 10 | |
| Brown's backyard | 11 | |
| Strickland's backyard | 12 | |

What can be said about the shape and size of each family's backyard? Explain how you know this.

Find the dimensions of each backyard by factoring the quadratic expression that represents the area. Complete the area equation: *area = length · width.* Show all of your work.

*area* = ______________________________________________________________

Complete the table below

| Name | x-value in yards | Area of backyards using factored-form |
|---|---|---|
| Jacoby's backyard | 10 | |
| Brown's backyard | 11 | |
| Strickland's backyard | 12 | |

What can you tell me about the standard form and the factored form of quadratic expressions? Explain.

**Part IV:**

**Transform the following quadratic expression in standard form to factored form. Complete the equation: *area = length · width*. Show all of your work.**

$x^2 - 16x + 54$

**Appendix G**

**Letter to Parents and Students**

Date: October 18, 2010

Dear Parents and Students:

We are conducting a study on the effectiveness of an algebra instructional package for high school students with learning disabilities and/or difficulties in mathematics. The instructional package will target algebra skills and content aligned with the state mathematics curriculum and college placements exams. The instructional package will be taught by Ms. Strickland who is a certified special educator and graduate student from the University of Maryland, College Park.
We are looking for students to participate in this study. The study will last about four weeks. Students will be taught every day for approximately 45 minutes during normal school hours. Ms. Strickland will also access confidential student education records to obtain pertinent data from the IEP, as well as IQ and achievement scores. All data regarding your child will be kept confidential and only accessed by the researcher. Data will be destroyed two years after the study ends.

Risks associated with this study include possible frustration with difficult tasks and the possibility of your child's likeness being viewed in research presentations, publications, and/or teacher trainings, if permission for video recording is granted. Participation will not affect your child's grades. Benefits may include improvements in understanding and performance of grade level algebra objectives.

By signing the attached permission form, you are agreeing to allow your child to participate in this study, if your child meets all of the eligibility requirements.

If you have questions about this study, please contact Tricia Strickland at: 1308 Benjamin Building, College Park, MD 20742; (office telephone) 301-405-6498; (mobile telephone) 443-604-1963; (email) tstrickl@umd.edu.
Sincerely,

Tricia K. Strickland
Student Investigator

Dr. Paula Maccini
Faculty Advisor
Associate Professor

# Appendix H

## PARENT / LEGAL GUARDIAN PERMISSION FORM

*Page 1 of 4* *Initials _______ Date ______*

| | |
|---|---|
| **Project Title** | **The Effects of Blended Instruction and Visual Representations on Area Problems Involving Quadratic Expressions for Secondary Students with Mathematics Difficulties** |
| **Why is this research being done?** | This is a research project being conducted by Tricia Strickland, a doctoral student at the University of Maryland, College Park, under the supervision of Dr. Paula Maccini. We are inviting your child to participate in this research because he or she has a history of difficulty in mathematics, particularly algebra. The purpose of this research project is to advance current knowledge on effective algebra interventions for secondary students with learning disabilities and/or difficulties in algebra. |
| **What will my child be asked to do?** | The procedures involve the collection of information from your child's confidential school file, including IQ scores, achievement scores, and grades from past and current mathematics courses to determine if your child is eligible for the intervention.<br>Your child will be asked to complete a minimum of three pretests before instruction is provided.<br>Your child will be asked to participate in daily instructional sessions in algebra 45 minutes per day for a period of 4 weeks. Sessions will be scheduled during your child's regular school day and content will be directly related to the algebra curriculum.<br>After completing all instructional sessions, your child will complete a minimum of three posttests and one test to determine if he or she is able to apply what was learned to new algebraic questions.<br>Two to four weeks after the end of the intervention, your child will be asked to complete a short test to see if he or she remembers the content he or she has been taught.<br>Your child will be asked his or her opinion regarding the instruction. For example, your child will be asked if the intervention helped him/her learn the targeted algebra topics. Additionally, your child will be asked what he/she liked most and least about the intervention. |

| | |
|---|---|
| **Project Title** | **The Effects of Blended Instruction and Visual Representations on Area Problems Involving Quadratic Expressions for Secondary Students with Mathematics Difficulties** |
| **What will my child be asked to do?** | During this study, we will be video recording the instructional sessions only. We would like your permission to use portions of these videos in three ways:<br>1. To determine your child's thinking about the algebra topics<br>2. To determine if the intervention is being implemented as planned<br>3. In research presentations, publications, and/or teacher trainings<br>If you choose not to have your child video recorded, he or she may still participate in the study. |
| **What about confidentiality?** | We will do our best to keep your child's personal information confidential. To help protect your confidentiality, all information collected in this study is confidential to the extent permitted by law. The data obtained about your child will be grouped with data from other students for reporting and presentation and your child's name will not be used. All data collected will be kept in a file cabinet in a locked office at the University of Maryland. Two years after the conclusion of the study, data from student records, test results, and other data will be destroyed by the student investigator. If we write a report or article about this research project, your child's identity will be protected to the maximum extent possible. Your information may be shared with representatives of the University of Maryland, College Park or governmental authorities if you or someone else is in danger or if we are required to do so by law. |
| **What are the risks of this research?** | There may be some risks from participating in this research study. Risks associated with this study include possible frustration with difficult tasks and the possibility of your child's likeness being viewed in research presentations, publications, and/or teacher trainings. |

| | |
|---|---|
| **Project Title** | **The Effects of Blended Instruction and Visual Representations on Area Problems Involving Quadratic Expressions for Secondary Students with Mathematics Difficulties** |
| **What are the benefits of this research?** | This research is not designed to help your child personally, but the results may help the investigator learn more about algebra instruction for students with learning disabilities and/or difficulties in algebra. We hope that, in the future, other people might benefit from this study through improved understanding of instructional practices in algebra. Your child may benefit by participating because the study is designed to improve algebra competencies. |
| **Does my child have to be in this research? Can my child stop participating at any time?** | Your child's participation in this research is completely voluntary. You may choose not to have your child take part at all. If you decide to have your child participate in this research, he/she may stop participating at any time. If you decide not to have your child participate in this study or if he/she stops participating at any time, your child will not be penalized or lose any benefits to which he/she otherwise qualifies. Your child's participation or nonparticipation in this study will not affect his or her grades. |
| **What if I have questions?** | This research is being conducted by Tricia K. Strickland, a doctoral student from the Department of Special Education at the University of Maryland, College Park, under the supervision of Dr. Paula Maccini. If you have any questions about the research study itself, please contact Dr. Maccini at: 1308 Benjamin Building, University of Maryland, College Park, MD 20742, (telephone) 301-405-7443, (email) maccini@umd.edu. If you have questions about your rights as a research subject or wish to report a research-related injury, please contact: **Institutional Review Board Office, University of Maryland, College Park, Maryland, 20742; (e-mail) irb@umd.edu; (telephone) 301-405-0678** This research has been reviewed according to the University of Maryland, College Park IRB procedures for research involving human subjects. |

| | | |
|---|---|---|
| **Project Title** | **The Effects of Blended Instruction and Visual Representations on Area Problems Involving Quadratic Expressions for Secondary Students with Mathematics Difficulties** | |
| **Statement of Age of Subject and Consent** | Your signature indicates that:<br>you are at least 18 years of age and you hereby give permission for your child or legal ward to participate in an educational study;<br>the research has been explained to you;<br>your questions have been fully answered; and<br>you freely and voluntarily choose to participate in this research project. | |
| **Signature and Date** | **Printed Name of Child** | |
| | **I agree to:** | _______ have my child video recorded to determine his or her thinking processes about the algebra topics.<br>_______ have my child video recorded to determine if the intervention is being implemented as planned.<br>_______ have my child's video likeness used in research presentations, publications, and/or teacher trainings. |
| | **Printed Name of Parent** | |
| | **Parent Signature** | |
| | **Date** | |

# Appendix I

## STUDENT ASSENT FORM

### The Effects of Blended Instruction and Visual Representations on Area Problems Involving Quadratic Expressions for Secondary Students with Mathematics Difficulties

We are requesting your participation in an educational project done by Ms. Tricia Strickland from the University of Maryland, College Park. You are under 18 years of age, and your parent or legal guardian has agreed that you can participate in this study.

The purpose of this study is to learn more about good algebra instruction for high school students with learning disabilities and/or difficulties in math. You will participate in daily 45 minute sessions for about four weeks. Instruction will take place at school, during your regular school hours and the instructional sessions will be videotaped. Video recordings may be used for three reasons: (1) to determine how you think about the algebra questions; (2) to determine how I am teaching the algebra topics; and (3) to use your likeness in research presentations, publications, and/or teacher trainings. If you do not want to be video recorded, you may still participate in the study. You will complete tests before, during, and after the study. You will also be asked your opinion about the study, such as what you liked best and what you would change. Ms. Strickland will also collect information from your confidential school records, such as IQ scores, achievement scores, and current math grades. Any information collected by Ms. Strickland will be confidential, which means it will not be shared with anyone.

Participation in this study will not affect your math grade. You may feel frustrated with some of the algebra work. You may benefit from this study because the project is designed to improve your algebra skills. You are free to ask questions anytime and you may stop participating at any time. If you stop participating, your grades in class will not be affected.

________________________________ ____________________
*Print Name* *Date*

________________________________
*Signature*

# Appendix J

## The Box Method

**Directions:**

1. Use the box method to multiply the following polynomials.
2. Complete each equation.

$(x – 8)\ (3x + 7) = \underline{\quad\quad\ 3x^2 - 17x - 56}$

| | x | -8 |
|---|---|---|
| 3x | $3x^2$ | -24x |
| 7 | 7x | -56 |

# Appendix K

## Sample Lesson Plan

**Lesson 1 Objective:**
Given algebra blocks, students will represent and sketch area problems involving linear expressions containing positive terms only to produce a quadratic expression with 80% accuracy.

I. **Launch:** Using the following information, draw a diagram of each bedroom on graph paper. Then fill out the table below.

*Student 1's bedroom is a square with each side 10 feet. Student 2's bedroom is a square with each side measuring 11 feet. Ms. Tricia's bedroom is a square with each side measuring 9 feet.*

| | X = Side of bedroom in feet | Side · Side | Y = Area of bedroom in square feet |
|---|---|---|---|
| Ms. Tricia | 9 | 9 · 9 | 81 |
| Student 1 | 10 | 10 · 10 | 100 |
| Student 2 | 11 | 11 · 11 | 121 |

Discussion questions include:
How did you determine the area of each of our bedrooms?

**Lesson Objective:** Today we are beginning our unit on quadratics. We will explore quadratic expressions by using a functions approach, meaning the variable, *x*, will truly be a variable. The value *x* will truly vary. We will not be solving for *x*. We need to review some vocabulary before we begin our unit. A quadratic function is a product of two linear expressions.

II. **Investigation:** Refer back to the warm up question

Refer back at the warm – up situation: A quadratic expression involves a square as the highest power. It is an expression in the second degree. What number did we square in the warm-up? We will learn how to square algebraic expressions. *All three of us will have our bedrooms renovated. After renovation, the length will increase by 2 feet and the width will increase by 1 foot. What are the measurements of our renovated bedroom?*

Make the renovations to your drawings on the graph paper.

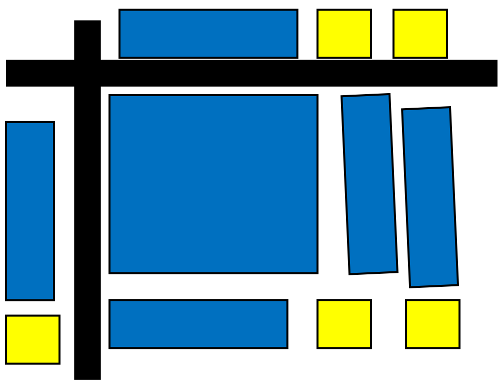
What happened to the size of each of our bedrooms? What about the shape?

Fill in the table below.

| | Side of original bedroom in feet | New length in feet | New width in feet | New Area in square feet |
|---|---|---|---|---|
| Ms. Tricia | 9 | 11 | 10 | 110 |
| Student 1 | 10 | 12 | 11 | 132 |
| Student 2 | 11 | 13 | 12 | 156 |
| Any square bedroom | x | x + 2 | x + 1 | (x + 2)(x +1) |

Write an algebraic expression to represent the new length and width of any size bedroom going through this renovation.

To multiply the two linear expressions, we will use algebra blocks and our corner piece. Teacher models problem representation and solution using Algebra Lab Gear, emphasizing the use of the Distributive Property. Discussion questions include:



1. Write out the area equation: *length · width = area*
2. Where are the linear expressions in the equation? Explain.
3. Which part of the equation represents the quadratic expression? Explain
4. How is the distributive property involved in the procedure?
5. How do the tiles represent the area of the new bedroom?
6. What happened to the area of the bedroom? Is the bedroom bigger or smaller? What about the shape?
7. How does the shape of the tiles compare to your drawings of our renovated bedrooms?

***Throughout the unit, during discussions students will be asked to: (a) share their approaches and strategies; (b) agree or disagree with their peers' approaches and strategies; and (c) share their preference for a particular strategy. All responds will require an explanation or justification.***

**III. Additional Practice:**

*We all have a square shaped rec room where we like to hang out with our friends. We would like to buy a ping pong table but will need to expand all of our basements* *so that the length is increased by 4 feet and the width is increased by 2 feet.* Fill in the table below.

| | Side of original rec room in feet | New length in feet | New width in feet | New Area in square feet |
|---|---|---|---|---|
| Ms. Tricia | 9 | 13 | 11 | 143 |
| Student 1 | 10 | 14 | 12 | 168 |
| Student 2 | 11 | 15 | 13 | 195 |
| Any square bedroom | x | x + 4 | x + 2 | (x + 4)(x +2) |

Use algebra blocks to determine the area of any rec room after it is expanded. Sketch the blocks.

Write area equation: *length · width = area*________________________________________

Use algebra blocks to multiply the following binomials expressions. Sketch the blocks. Then complete the area equation: *length · width = area*

(x + 3) (x + 5) = ________________________________

**IV. Closure:**

**Big Ideas:**

Review vocabulary. Have students explain each vocabulary in their own words or giving their own example.

Ask students to identify which property we use to multiply two linear expressions. Ask students “How did we form a quadratic expression?”

**Link to Future Instruction:** We will not always have positive numbers in the linear expressions so tomorrow we will learn how to multiply linear expressions with negative integers to form quadratic expression.

## Appendix L

## Treatment Fidelity Checklist

| **Observer:** | | **Date:** | **Time:** |
|---|---|---|---|
| **Directions:**<br>**Indicate the observed behaviors by placing a check mark in the spaces below.** | | | |
| **Item** | **Description** | **Observed ?** | **Notes** |
| **Advanced Organizer:** | | | |
| **1** | Review of prerequisite mathematics skills | | |
| **2** | Objective stated at the beginning of the lesson | | |
| **3** | Rationale for learning the skill/concept as it connects to other mathematical concepts | | |
| **Teacher-lead Investigation** | | | |
| **4** | Maximizing students’ engagement via questions and prompts | | |
| **5** | Modeling the thinking and action procedures needed to solve the problem. | | |
| **6** | Prompting questions to facilitate student exploration of topic. | | |
| **Multiple Practice Opportunities** | | | |
| **7** | Opportunities for students to practice tasks completed during teacher-lead investigation. Teacher serves as facilitator. May include group work or individual work. | | |

| | | | |
|---|---|---|---|
| **Visual Representations:** | | | |
| **8** | Visual representations presented simultaneously with abstract notation. Visual representations include algebra blocks, sketches of algebra blocks, or expansion box | | |
| **NCTM Process Standards:** | | | |
| **9** | Representations, generating, selecting, or applying text about a representation to solve a problem. Representations include diagrams, graphs, models, symbolic notation construed from these representations | | |
| **10** | Communication, in which the language of mathematics (i.e., binomial, polynomial, product, etc.) is used to express mathematical ideas and/or articulating reasoning to peers and/or teacher | | |
| **11** | Making connections among mathematical ideas (e.g., geometry) and/or applying mathematics in contexts outside of mathematics (e.g., language arts, science, social studies) | | |
| **12** | Reasoning, which includes making mathematical conjectures, justifying answers, and/or using alternative solution methods | | |
| **13** | Problem solving, which includes engaging in a task for which the solution method is not known in advance | | |
| **Closure:** | | | |
| **14** | Review big ideas at end of lesson. | | |
| **15** | Assessment, which includes students completing an independent task or orally responding to teacher's questions. | | |

**Percentage of observed behaviors _________/15 = _______________%**

## Appendix M

### Lesson 1 Probe

**Student Name: ___________________________________**

**All of us will have our bedrooms renovated. The current shape of our bedroom is square. After renovation, the length will increase by 6 feet and the width will increase by 2 feet.**

Fill in the table below.

| | Side of original bedroom in feet | New length in feet | New width in feet | New Area in square feet |
|---|---|---|---|---|
| Ms. Tricia | 9 | | | |
| Sima | 10 | | | |
| Any square bedroom | x | | | |

Use algebra blocks to determine the area of any bedroom after renovations. Sketch the blocks.

Identify the quadratic expression from the table ______________________________

After renovations, will we have more, less, or the same amount of area in our bedrooms? Explain your answer.

## Appendix N

## Transfer Probe

**Directions: Complete the following tasks. Show all of your work.**

All of the houses in the Sun Shine Valley neighborhood have rectangular swimming pools in which the length is 4 feet longer than the width and the height that is 3 feet longer than the width. Fill in the table below to determine the perimeter and the volume of the pools.
Remember:
*length · width · height = volume*
*2(length) + 2(width) = Perimeter*

| | Length in feet | Width in feet | Height in feet | Volume in cubic feet | Perimeter in feet |
|---|---|---|---|---|---|
| Berger's Pool | | 10 | | | |
| Joffe's Pool | | 12 | | | |
| Any pool in Sun Shine Valley | | | | | |

**Multiply the polynomial expressions and explain your strategy using words, pictures, or symbols.**
$( 3x^2 + 2x + 7 ) ( 4x^4 – 2x^3 + 3x + 2) =$ ____________________________________________

**Factor the following quadratic. Please notice that the quadratic term has a coefficient of three. Complete the following area equation: *area = length · width*. Explain your strategy using words, pictures, or symbols.**
$3x^2 + 14x + 8 =$ ___________________________________________________________

**Appendix 0**
**Social Validity Measure**

**Part 1:**
Please indicate the degree to which you agree with the following statements.

| **Strongly Disagree** | **Disagree** | **Neutral** | **Agree** | **Strongly Agree** | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | I learned to multiply binomial expressions to form a quadratic expression. |
| 1 | 2 | 3 | 4 | 5 | I learned to factor quadratic trinomials. |
| 1 | 2 | 3 | 4 | 5 | The use of manipulatives helped me to multiply binomials expressions and factor quadratic expressions. |
| 1 | 2 | 3 | 4 | 5 | The use of the *box* helped me to multiply binomials expressions and factor quadratic expressions. |
| 1 | 2 | 3 | 4 | 5 | The word problems helped me understand what the expressions represented. |
| 1 | 2 | 3 | 4 | 5 | The data tables helped me understand what the expressions represented. |
| 1 | 2 | 3 | 4 | 5 | Talking about the problems with the teacher and/or classmates helped me understand. |
| 1 | 2 | 3 | 4 | 5 | This intervention was worth my time. |
| 1 | 2 | 3 | 4 | 5 | I would recommend this intervention to other students. |

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | As a result of the intervention, I feel better about my algebra skills. |

**Part 2:**
Are you interested in learning more about quadratics? Why or Why not?

How did the intervention help you understand quadratic expressions?

What did you like best about the intervention?

What did you like least about the intervention?

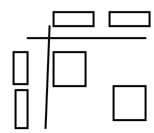
What suggestions do you have for improvement?

# Appendix P
# Codes

***Representations***

| Code | Description | Example |
|---|---|---|
| | ***Lab Gear*** | |
| Connections | Student constructed mathematically sound connections through the use of the Algebra Lab Gear. Student generalized the Lab Gear to rules and/or concepts. | *Marcia working on $x^2 + 5x + 4$*<br>Marcia: Makes 4 and 5? 3 and 2? No How do you get 5?<br>TS: Use the tiles if you need to.<br>Marcia: Why won't you help me? I'm trying to figure out using 5 bars how do I get 4 inside.<br>TS: Use the blocks. (giving Marcia blocks) Play around a little bit.<br>Marcia: Oh I am so stupid! Marcia you're an idiot. 4 and 5. I have a picture. Can I tell you my trick that I just figured out?<br>TS: Yeh, yeh, what's your trick?<br>Marcia: That, whenever you have a number, that is one more than the amount, than the constant, then you are multiplying basically by 1. All the bars go here except for one. |
| Dis -Connections | Student constructed mathematically inappropriate connections (i.e., misconceptions) when using the Lab Gear. | Marcia: This doesn't fully work because that's bigger. Like the 3 x-bars are bigger. So if I had a room that was 5 feet by 5 feet, and I added that, that's more than a foot in comparison to that so that's off. |

***Representations***

<table>
<tr><th>Code</th><th>Description</th><th>Example</th></tr>
<tr><td></td><td>Box Method</td><td></td></tr>
<tr><td>Connections</td><td>Student constructed mathematically sound connections through the use of the Box Method. Student generalized the Box Method to rules and/or concepts.</td><td>Marcia: Look (showing me her paper).<br>TS: Marcia this is an excellent generalization. Show the girls what you did.<br><table><tr><td>$X^2$</td><td>X</td></tr><tr><td>X</td><td>C</td></tr></table></td></tr>
<tr><td>Dis -Connections</td><td>Student constructed mathematically inappropriate connections (i.e., misconceptions) when using<br>the Box Method.</td><td>Marcia’s first attempt at her own design of the box method:<br>Marcia: Is this wrong?<br><br>TS: It’s not wrong. It’s incomplete. Can I add something?<br>Marcia: Yeh<br>TS: What about this box and this box?<br>Marcia: I know, but I already used them.<br>TS: But we have to use them twice. Remember how we did it with the algebra blocks? Now we have to multiply x times -10.<br>Marcia: Oh, I forgot about. that.</td></tr>
</table>

***Representations***

| Code | Description | Example |
|---|---|---|
| ***Area Context*** | | |
| Connections | Students construct mathematically sound connections to the Area context, (i.e., word problems, tables of data, and discrete numbers). | TS: What is a situation within a word problem that we would need to subtract?<br>Marcia: Jodi needed room to make a bathroom. She added 4 feet to the length and she subtracted, brought the room in 2 feet. |
| Dis -Connections | Students construct mathematically inappropriate connections to the Area context, (i.e., word problems, tables or data, and discrete numbers). | TS: What happens to YOUR x value for each of the dimensions?<br>Marcia: What does this have to do with the dimensions? I don't like this at all.<br>TS: OK ladies, do we have anything that is the same? So I have x + 17 and x – 2. Does anyone have that?<br>Sasha: I have x-2<br>TS: But do you have both of the dimensions?<br>Sasha shakes her head.<br>Marcia: How is that possible? We all have different areas. |

# *Metacognition*

| Code | Description | Example |
|---|---|---|
| | ***Self Regulation*** | |
| Checking solution for accuracy | Students use the intervention to check the accuracy of their problem solution. | *When presented with (3x + 5) (x – 2) using the box method:*<br>TS: It's not $x^2$ Why not?<br>Anna: because there's a 3x there.<br>TS: so what is it?<br>Marcia: 3x, x<br>TS: what was x times x using our blocks?<br>Sasha: Oh, $x^2$<br>Marcia: Oh, right.<br>Sasha: So, $3x^2$ because there's 3<br>Marcia: I need the blocks. I can't do this. This is not working for me. We never did it like that. It was always $x^2$<br>*Marcia gets the lab gear and sets up the problem x times 3x to confirm.* |
| Solution Revisions | Student uses the intervention to revise her solution. | TS: Will teachers and students have more, less, or the same amount of classroom space after renovations?<br>Marcia: I thought the same amount of space but different shape but when I drew it out it got all messed up. It also depended on how big the square is. They could have more space or less space. |

***Metacognition***

| Code | Description | Example |
|---|---|---|
| | ***Strategic Planning*** | |
| Planning | Students use the intervention to develop a plan to engage in an algebraic task. | *When presented with (-13 +2x) (10 + x):*<br>Marcia: Can I do the numbers after the x's?<br>TS: Show me what you mean.<br>Marcia: Can I do 2x – 13? The x's are always in this box *(pointing to the top left box of her graphic organizer)* |
| Executing | Students use the intervention to execute a plan to engage in an algebraic task. | Marcia: I'm writing out my problem over here (pointing to the Lab Gear). I have an equation and I am breaking it up and multiplying because this is a multiplying bar and this is $x$ and this is $x$ so I have $x + 3$ so $x + 3$ times and this is timesing *(pointing to the corner piece of Lab Gear)* $x + 2$ and it's going to equal $x^2 + 3x + 6$ (*pointing to blocks as she states her quadratic expression)* |

***Metacognition***

| Code | Description | Example |
|---|---|---|
| ***Disposition*** | | |
| Self-confidence | Student's beliefs about their ability to do mathematics. | TS: I want you to practice using the box to multiply binomials.<br>Marcia: Ah, I'm not good at that.<br>TS: Just set up the problem in the box.<br>Marcia: I can't. My brain is fried.<br>TS: Is this (pointing to a binomial) going to go in the grey or the white boxes?<br>Sasha: White, no grey because the grey represents the length and the width.<br>Marcia: I don't really get this.<br>*Marcia is correctly filling in the box.* |
| Perseverance | Student's ability to continue working on a task until completion. | Marcia worked steadily for one hour and twenty minutes to complete the transfer test with 100% accuracy.<br>Marcia: I'm not stopping till I finish this test. Can I get a note to miss my next class?<br>*After the completion of the intervention, Marcia has refused to attend her regular math class.* |
| Enjoyment | Student's feeling of pleasure and satisfaction while participating in the intervention. | *After completing the maintenance test.*<br>Marcia: Is this over?<br>TS: Yes<br>Marcia: I don't want this to be over! I like to do this stuff. I can't believe I am saying this about math. |

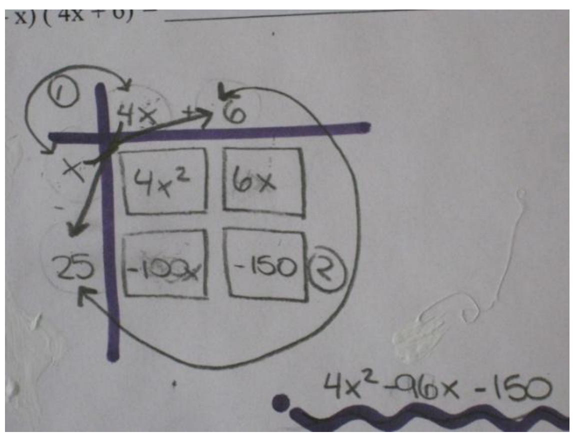
## *Metacognition*

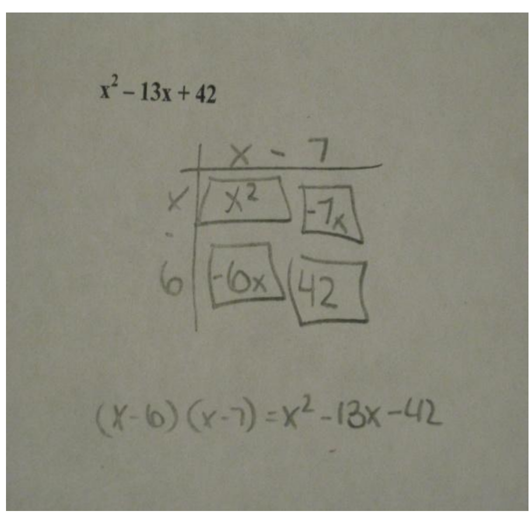
| Code | Description | Example |
| --- | --- | --- |
| ***Socially shared metacognition*** | | |
| Collaborative peer solutions. | Students share the problem-solving process, by confirming one another's correctness through reciprocal turns. | (x – 2) (3x + 15)<br>Sasha: -2 times 3x. Negative 6x.<br>Marcia: -2 times 3 is -6<br>Sasha: x times 15 is 15x. -2 times 15 is -30.<br>Marcia: What? 13<br>TS: How did you get 13?<br>Marcia: 15 – 2<br>Sasha: we're multiplying<br>Marcia: Multiplying. Oh, I really don't like this at all. I need the blocks. I can't do this. I need to make my own chart. No, my own blocks.<br>TS: Well, let's practice a little bit. We are going to multiply exactly like we multiplied the blocks but we are only using symbols. What is the area for this problem?<br>Marcia: I have no idea<br>*Sasha looks at Marcia and sighs.*<br>Sasha: $3x^2$-6x+15x-30.<br>Marcia: How are you doing this? |
| Lack of peer collaboration | Students do not confirm one another's correctness through reciprocal turns. | Field journal entry: Marcia and Sasha are doing their own problem and then looking at each other's work. This is their idea of working together, doing their own work and then checking to see if they get the same answer. If they do not have the same answer they ask me which is correct, each saying their answer is wrong as they begin to erase their problem. |

## Appendix Q

### The Evolution of Marcia's Box

| | | |
|---|---|---|
| | x | c |
| x | $x^2$ | x |
| c | x | c |



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