ABSTRACT

Title of dissertation: ORDERS OF ACCUMULATION OF ENTROPY AND RANDOM SUBSHIFTS OF FINITE TYPE

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The first portion of this dissertation concerns orders of accumulation of entropy. For a continuous map T of a compact metrizable space X with finite topological entropy, the order of accumulation of entropy of T is a countable ordinal that arises in the context of entropy structure and symbolic extensions. We show that every countable ordinal is realized as the order of accumulation of some dynamical system. Our proof relies on the functional analysis of metrizable Choquet simplices and a realization theorem of Downarowicz and Serafin. Further, if M is a metrizable Choquet simplex, we bound the ordinals that appear as the order of accumulation of entropy of a dynamical system whose simplex of invariant measures is affinely homeomorphic to M. These bounds are given in terms of the Cantor-Bendixson rank of $\overline{\mathrm{ex}(M)}$, the closure of the extreme points of M, and the relative Cantor-Bendixson rank of $\overline{\mathrm{ex}(M)}$ with respect to $\mathrm{ex}(M)$. We also address the optimality of these bounds.

Given any compact manifold M and any countable ordinal α, we also construct a continuous, surjective self-map of M having order of accumulation of entropy α.

If the dimension of M is at least 2, then the map can be chosen to be a homeomorphism. The realization theorem of Downarowicz and Serafin produces dynamical systems on the Cantor set; by contrast, our constructions work on any manifold and provide a more direct dynamical method of obtaining systems with prescribed entropy properties.

Next we consider random subshifts of finite type. Let X be an irreducible shift of finite type (SFT) of positive entropy, and let $B_n(X)$ be its set of words of length n. Define a random subset ω of $B_n(X)$ by independently choosing each word from $B_n(X)$ with some probability α. Let X_ω be the (random) SFT built from the set ω. For each $0 \leq \alpha \leq 1$ and n tending to infinity, we compute the limit of the likelihood that X_ω is empty, as well as the limiting distribution of entropy for X_ω. For α near 1 and n tending to infinity, we show that the likelihood that X_ω contains a unique irreducible component of positive entropy converges exponentially to 1. These results are obtained by studying certain sequences of random directed graphs. This version of "random SFT" differs significantly from a previous notion by the same name, which has appeared in the context of random dynamical systems and bundled dynamical systems.

ORDERS OF ACCUMULATION OF ENTROPY AND RANDOM SUBSHIFTS OF FINITE TYPE

by

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Foreword

The results contained in Chapter 2 and Appendix A of this dissertation are joint work with David Burguet (l'ENS Cachan). The results contained in Chapters 3 and 4 are solely due to Kevin McGoff.

The results in Chapter 2 are to appear with revision in Fundamenta Mathematicae [18]. The results in Chapter 3 are to appear in Journal d'Analyse Mathématique [70]. The results in Chapter 4 are to appear in Annals of Probability [71].

Dedication

To my loving wife, Molly

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I am forever indebted to my advisor, Mike Boyle, without whom this dissertation would not exist. His brilliance has enlightened me; his wisdom has guided me; his patience and encouragement have strengthened me; and his enthusiasm for mathematics has inspired me.

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List of Abbreviations

SFT	(sub)shift of finite type
$\mathbb{N}$	$\{1, 2, 3, \dots\}$

Chapter 1

Introduction

1.1 Overview

The appearance of entropy in the context of sequences of symbols goes back to Shannon [84], for whom it served as a measure of the average information capacity of a communication channel. Soon the concept of entropy appeared in the context of stationary random processes, and measure-theoretic entropy become a centerpiece of ergodic theory following the work of Kolmogorov [61, 62, 63]. The notion of topological entropy was defined a bit later [1], and it is now considered a primary measure of the complexity of a topological dynamical system. See [51] for a survey of the history of entropy in dynamical systems.

In this work, a topological dynamical system is a pair (X, T), where X is a compact metrizable space and T is a continuous mapping of X to itself. For such a system (X, T), the topological entropy $\mathbf{h}_{top}(T)$ provides a well-studied measure of the topological dynamical complexity of the system. We only consider systems with $\mathbf{h}_{top}(T) < \infty$. Let $M(X, T)$ be the space of Borel probability measures on X which are invariant under T. The entropy function $h : M(X, T) \to [0, \infty)$, where $h(\mu)$ is the metric entropy of the measure μ, quantifies the amount of complexity in the system that lies on generic points for μ. In this sense, the entropy function h describes both *where* and *how much* complexity lies in the system. The theory of entropy structures developed by Downarowicz [35] produces a master entropy invariant in the form of a distinguished class of sequences of functions on $M(X,T)$ whose limit is h. The entropy structure of a dynamical system completely determines almost all previously known entropy invariants such as the topological entropy, the entropy function on invariant measures, the tail entropy (or topological conditional entropy [73]), the symbolic extension entropy, and the symbolic extension entropy function. Entropy structure also produces new entropy invariants, such as the order of accumulation of entropy. Furthermore, the theory of entropy structures and symbolic extensions provides a rigorous description of *how entropy emerges on refining scales.* Entropy structures and the closely related theory of symbolic extensions [13] have attracted interest in the dynamical systems literature [4, 17, 15, 33, 35, 36, 37], especially with the intention of using entropy structure to obtain information about various classes of smooth systems. The purpose of the Chapters 2 and 3 is to investigate a new entropy invariant arising from the theory of entropy structures: the order of accumulation of entropy, which is denoted $\alpha_0(X,T)$.

A shift of finite type (SFT) is a dynamical system defined by finitely many local transition rules. These systems have been studied for their own sake [59, 66], and they have also served as important tools for understanding other dynamical systems [53, 12, 32]. Each SFT can be described as the set of bi-infinite sequences on a finite alphabet that avoid a finite list of words over the alphabet. Thus there are only countably many SFTs up to the naming of letters in an alphabet. The purpose of Chapter 4 is to study some typical properties within the class of SFTs. Since there are essentially only countably many SFTs, our notion of typical involves randomly choosing an SFT from certain classes. Loosely speaking, a property is then "typical" if it holds for a random SFT with high probability. The main properties of interest for the SFTs considered in this work are emptiness, entropy, and the number and structure of irreducible components.

1.2 Organization of the dissertation

Chapters 2, 3, and 4 are intended to be mostly self-contained treatments of the relevant results. For that reason, some material in Chapter 3 overlaps with material already presented in Chapter 2.

In Chapter 2, we show that every countable ordinal is realized as the order of accumulation of some dynamical system. Our proof relies on the functional analysis of metrizable Choquet simplices and a realization theorem of Downarowicz and Serafin. Further, if M is a metrizable Choquet simplex, we bound the ordinals that appear as the order of accumulation of entropy of a dynamical system whose simplex of invariant measures is affinely homeomorphic to M. These bounds are given in terms of the Cantor-Bendixson rank of $\overline{\mathrm{ex}(M)}$, the closure of the extreme points of M, and the relative Cantor-Bendixson rank of $\overline{\mathrm{ex}(M)}$ with respect to $\mathrm{ex}(M)$. We also address the optimality of these bounds.

In Chapter 3, given any compact manifold M and any countable ordinal α, we construct a continuous, surjective self-map of M having order of accumulation of entropy α. If the dimension of M is at least 2, then the map can be chosen to be a homeomorphism. The realization theorem of Downarowicz and Serafin cited in Chapter 2 produces dynamical systems on the Cantor set; by contrast, the constructions in Chapter 3 work on any manifold and provide a more direct dynamical method of obtaining systems with prescribed entropy structure properties.

Chapter 4 contains the results on random subshifts of finite type, which we summarize as follows. Let X be an irreducible shift of finite type (SFT) of positive entropy, and let $B_n(X)$ be its set of words of length n. Define a random subset ω of $B_n(X)$ by independently choosing each word from $B_n(X)$ with some probability α. Let X_ω be the (random) SFT built from the set ω. For each $0 \leq \alpha \leq 1$ and n tending to infinity, we compute the limit of the likelihood that X_ω is empty, as well as the limiting distribution of entropy for X_ω. For α near 1 and n tending to infinity, we show that the likelihood that X_ω contains a unique irreducible component of positive entropy converges exponentially to 1. These results are obtained by studying certain sequences of random directed graphs.

Chapter 2

Orders of accumulation of entropy

Given a dynamical system (X, T), one may associate a particular sequence $\mathcal{H}(T) = (h_k)$ to (X, T) with the following properties [35]:

1. (h_k) is a non-decreasing sequence of harmonic, upper semi-continuous functions from $M(X, T)$ to $[0, \infty)$;

2. $\lim_k h_k = h$;

3. $h_{k+1} - h_k$ is upper semi-continuous for every k.

This sequence, or any sequence uniformly equivalent to it (Definition 2.1.18), is called an entropy structure for the system (X, T) [35]. This distinguished uniform equivalence class of sequences is an invariant of topological conjugacy of the system [35]. Consequently, we sometimes refer to the entire uniform equivalence class of $\mathcal{H}$ as *the* entropy structure of the system (X, T).

Associated to a non-decreasing sequence $\mathcal{H} = (h_k)$ of functions $h_k : M \to [0, \infty]$, where M is a compact metrizable space, there is a transfinite sequence of functions $u_\alpha : M \to [0, \infty]$, indexed by the ordinals and defined by transfinite induction as follows. Let $\widetilde{f}$ denote the upper semi-continuous envelope of the function f (Definition 2.1.14; by convention $\widetilde{f} \equiv \infty$ if f is unbounded). Let $\tau_k = h - h_k$. Then

- let $u_0 \equiv 0$;
- if u_α has been defined, let $u_{\alpha+1} = \lim_k \widetilde{u_\alpha + \tau_k}$;
- if u_β has been defined for all $\beta < \alpha$ for a limit ordinal α, let $u_\alpha = \widetilde{\sup_{\beta<\alpha} u_\beta}$.

The sequence (u_α) is non-decreasing in α and does not depend on the particular representative of the uniform equivalence class of $\mathcal{H}$. Since M is compact and metrizable, an easy argument (given in [13]) implies that there exists a countable ordinal α such that $u_\beta \equiv u_\alpha$ for all $\beta \geq \alpha$. The least ordinal α with this property is denoted $\alpha_0(\mathcal{H})$ and is called the order of accumulation of $\mathcal{H}$. In the case when $M = M(X,T)$ and $\mathcal{H}$ is an entropy structure for (X,T), the order of accumulation of entropy of (X,T) is defined as $\alpha_0(\mathcal{H})$. Because the entropy structure of (X,T) is invariant under topological conjugacy, the sequence (u_α) associated to (X,T) and the order of accumulation $\alpha_0(X,T)$ are invariants of topological conjugacy.

To explain the meaning of $\alpha_0(X,T)$ and $u_{\alpha_0(X,T)}$, we discuss symbolic extensions and their relationship to entropy structures. A symbolic extension of (X,T) is a (two-sided) subshift (Y,S) on a finite number of symbols, along with a continuous surjection $\pi : Y \to X$ (the *factor map* of the extension) such that $\pi \circ S = T \circ \pi$. Symbolic extensions have been important tools in the study of some dynamical systems, in particular uniformly hyperbolic systems. A symbolic extension serves as a "lossless finite encoding" of the system (X,T) [35]. If π is the factor map of a symbolic extension (Y,S), we define the extension entropy function $h^\pi_{\text{ext}} : M(X,T) \to [0,\infty)$ for μ in $M(X,T)$ by

$$h^\pi_{\text{ext}}(\mu) = \max\{h(\nu) : \pi^*\mu = \nu\}.$$

The number $h^{\pi}_{\text{ext}}(\mu)$ represents the amount of complexity above the measure μ in the symbolic extension. The symbolic extension entropy function of a dynamical system (X, T), $h_{\text{sex}} : M(X, T) \to [0, \infty]$, is defined for μ in $M(X, T)$ as

$$h_{\text{sex}}(\mu) = \inf\{h^{\pi}_{\text{ext}}(\mu) : \pi \text{ is the factor map of a symbolic extension of } (X, T)\},$$

where the infimum is understood to be ∞ if (X, T) admits no symbolic extensions. The symbolic extension entropy function measures the amount of entropy that must be present above each measure in any symbolic extension of the system. Finally, we define the residual entropy function $h_{\text{res}} : M(X, T) \to [0, \infty]$ as $h_{\text{res}} = h_{\text{sex}} - h$. The residual entropy function then measures the amount of entropy that must be added above each measure in any symbolic extension of the system. The functions h_{res} and h_{sex} give much finer information about the complexity of the system than the entropy function h. These quantities are related to the entropy structure of the system by the following remarkable result of Boyle and Downarowicz.

Theorem 2.0.1 ([13])**.** *Let X be a compact metrizable space and $T : X \to X$ a continuous map. Let $\mathcal{H}$ be an entropy structure for (X, T). Then*

$$h_{\text{sex}} = h + u^{\mathcal{H}}_{\alpha_0(X,T)}.$$

The conclusion of the theorem may also be stated as $u_{\alpha_0(X,T)} = h_{\text{res}}$. In this sense, the order of accumulation $\alpha_0(X, T)$ and the function $u_{\alpha_0(X,T)}$ each measures a residual complexity in the system that is not detected by the entropy function h. The order of accumulation of entropy measures, roughly speaking, *over how many distinct layers residual entropy emerges* in the system [13]. It is then natural to ask the following question.

Question 2.0.2. Which countable ordinals can be realized as the order of accumulation of entropy of a dynamical system?

It is shown in [13] that all finite ordinals can be realized as the order of accumulation of dynamical system. There are constructions in [17, 37] (built for other purposes) that show that some infinite ordinals are realized in this way, but these constructions do not allow one to determine exactly which ordinals appear. Moreover, it is stated without proof in [35] that all countable ordinals are realized.

We prove that all countable ordinals can be realized as the order of accumulation of entropy for a dynamical system (Corollary 2.3.5), answering Question 2.0.2. On account of the realization theorem of Downarowicz and Serafin (restated as Theorem A.1.1 in this work), this result reduces to establishing the following result, which is purely functional analytic.

Theorem 2.0.3. *For every countable ordinal α, there exists a metrizable Choquet simplex M and a sequence of functions $\mathcal{H} = (h_k)$ on M such that*

- (h_k) *is a non-decreasing sequence of harmonic, upper semi-continuous functions from M to $[0, \infty)$;*
- $\lim_k h_k$ *exists and is bounded;*
- $h_{k+1} - h_k$ *is upper semi-continuous for every k;*
- $\alpha_0(\mathcal{H}) = \alpha$.

Building on the approach of Downarowicz and Serafin to reduce questions in the theory of entropy structure to the study of functional analysis, we also consider what constraints, if any, the simplex of invariant measures may place on orders of accumulation of entropy.

Question 2.0.4. Given a metrizable Choquet simplex M, which ordinals can be realized as the order of accumulation of a dynamical system (X, T) such that $M(X, T)$ is affinely homeomorphic to M?

For a metrizable Choquet simplex M, we let $S(M)$ denote the set of all ordinals that can be realized as the order of accumulation of a sequence $\mathcal{H}$ on M satisfying properties (1)-(3). The realization theorem of Downarowicz and Serafin (Theorem A.1.1) reduces Question 2.0.4 to the following question in functional analysis.

Question 2.0.5. Given a metrizable Choquet simplex M, which ordinals are in $S(M)$?

Theorem 2.4.3 answers Question 2.0.5 (and therefore Question 2.0.4) completely in the event that M is a Bauer simplex by giving a precise description of $S(M)$ in terms of the Cantor-Bendixson rank of the extreme points of M. Theorem 2.5.5 addresses the general case, giving constraints on $S(M)$ in terms of Cantor-Bendixson rank of the closure $\overline{E}$ of the space $E = \mathrm{ex}(M)$ of extreme points of M and the relative Cantor-Bendixson rank of $\overline{E}$ with respect to E. Theorems 2.5.6 and 2.5.10 address the optimality of these constraints, and Section 2.5.3 summarizes our progress on this question and poses some remaining questions.

In the language of dynamical systems, if M is a metrizable Choquet simplex, we have found constraints on the orders of accumulation of entropy that appear within the class of all dynamical systems (X, T) such that $M(X, T)$ is affinely homeomorphic to M. These constraints are in terms of the Cantor-Bendixson ranks of the closure $\overline{E}$ of the space E of ergodic measures and the relative Cantor-Bendixson rank of $\overline{E}$ with respect to E.

2.1 Preliminaries

2.1.1 Ordinals

We assume a basic familiarity with the ordinal numbers, ordinal arithmetic, and transfinite induction. The relevant sections in [81] provide a good introduction. Here we briefly recall some notions that are used in this work.

We view the ordinal α as the set $\{\beta : \beta < \alpha\}$. The symbols ω and ω_1 will always be used to denote the first infinite ordinal and the first uncountable ordinal, respectively.

Definition 2.1.1. An ordinal α is **irreducible** if whenever $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \geq \alpha_2$, it follows that $\alpha_2 = 0$.

Recall the well-known Cantor Normal Form of an ordinal.

Theorem 2.1.2. *For every ordinal $\alpha > 0$, there exists natural numbers $n_1, \ldots, n_k$ and ordinals $\beta_1 > \cdots > \beta_k$ such that $\alpha = \omega^{\beta_1} n_1 + \cdots + \omega^{\beta_k} n_k$. Furthermore, the numbers $n_1, \ldots, n_k$ and the ordinals $\beta_1, \ldots, \beta_k$ are unique.*

The following corollary is an easy consequence of the Cantor Normal Form.

Corollary 2.1.3. *An ordinal $\alpha > 0$ is irreducible if and only if there exists an ordinal β such that $\alpha = \omega^\beta$.*

In light of this corollary, one can view the Cantor Normal Form of α as a decomposition of α into a finite sum of irreducible ordinals.

The following corollary is then a simple consequence of Corollary 2.1.3 and the fact that any non-zero ordinal β is either a successor ordinal or a limit ordinal.

Corollary 2.1.4. *If $\alpha > 0$ is countable and irreducible, then either (i) there exists an irreducible ordinal $\widetilde{\alpha} < \alpha$ such that $\sup_{n\in\mathbb{N}} \widetilde{\alpha}n = \alpha$, or (ii) there exists a strictly increasing sequence of irreducible ordinals $(\alpha_k)_{k\in\mathbb{N}}$ such that $\sup_{k\in\mathbb{N}} \alpha_k = \alpha$.*

Any ordinal α can be viewed as a topological space with the order topology (sets of the form $\{\gamma \in \alpha : \gamma < \beta\}$ or $\{\gamma \in \alpha : \beta < \gamma\}$ form a subbase for the topology). With this topology, α is a completely normal, Hausdorff space, and if α is countable, then it is a Polish space (see below for definition). The space α is compact if and only if α is a successor ordinal. The accumulation points in α are exactly the limit ordinals in α.

For ease of notation, if α is a successor ordinal, let $\alpha - 1$ denote the unique ordinal β such that $\alpha = \beta + 1$. Also, for countable ordinals $\alpha \leq \beta$, we will write $[\alpha, \beta]$ to denote the ordinal interval $\{\gamma : \alpha \leq \gamma \leq \beta\}$. If $\beta = \omega_1$, we make the convention that $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma < \beta\}$. We also make use of the notation $]\alpha, \beta[= \{\gamma : \alpha < \gamma < \beta\}$, as well as the other possible "half-open" and "half-closed" notations.

2.1.2 Polish Spaces

A general reference that covers Polish spaces is [86]. We recall that a topological space E is a Polish space if it is separable and completely metrizable. In particular, any compact metrizable space is Polish. Moreover, any closed subset of a Polish space is itself a Polish space. Some of the definitions and statements below hold for more general topological spaces, but we require them only in the case of Polish spaces.

For any Polish space E, let E' denote the set of accumulation points of E,

$$E' = \{x \in E : \exists (x_n) \subset E \setminus \{x\},\ x_n \to x\}.$$

Note that E' is closed in E.

A subset A of a Polish space E is a perfect set if A is a compact subset of E and A contains no isolated points (in the subspace topology). The following result is a special case of the Cantor-Bendixson Theorem.

Theorem 2.1.5. *Let E be a Polish space. Then $E = C \cup A$, where C is countable, A is closed and has no isolated points, and $C \cap A = \emptyset$.*

We will also use the following fact (see [86]). Let $\mathcal{C}$ denote the Cantor space.

Theorem 2.1.6. *Let A be a non-empty Polish space with no isolated points. Then there is an embedding of $\mathcal{C}$ into A.*

The following statement is an immediate corollary of the previous two theorems.

Corollary 2.1.7. *Let E be any uncountable Polish space. Then there is an embedding of $\mathcal{C}$ into E.*

The following corollary is an easy consequence of Corollary 2.1.7.

Corollary 2.1.8. *Let E be an uncountable Polish space. Then for every countable ordinal α and every natural number n, there exists an embedding $g : \omega^\alpha n + 1 \to E$.*

2.1.3 Cantor-Bendixson Rank

Given a Polish space E, we now use transfinite induction to define a transfinite sequence of topological spaces, $\{\Gamma^\alpha(E)\}$. Let $\Gamma^0(E) = E$. If $\Gamma^\alpha(E)$ has been defined, then let $\Gamma^{\alpha+1}(E) = (\Gamma^\alpha(E))' \subset \Gamma^\alpha(E)$. If α is a limit ordinal and $\Gamma^\beta(E)$ is defined for all $\beta < \alpha$, then let $\Gamma^\alpha(E) = \cap_{\beta<\alpha}\Gamma^\beta(E)$. Each set $\Gamma^\alpha(E)$ is closed in E and therefore Polish.

Note that $\Gamma^\alpha(E) = \Gamma^{\alpha+1}(E)$ implies that $\Gamma^\alpha(E)$ has no isolated points (in the subspace topology) and then that $\Gamma^\beta(E) = \Gamma^\alpha(E)$ for all $\beta > \alpha$. For any Polish space E, Theorem 2.1.5 implies that there exists a countable ordinal α such that $\Gamma^\alpha(E) = \Gamma^{\alpha+1}(E)$.

Definition 2.1.9. With the notation above, the **Cantor-Bendixson rank** of the space E, denoted $|E|_{CB}$, is defined to be the least ordinal α such that $\Gamma^\alpha(E) = \Gamma^{\alpha+1}(E)$.

When E is compact, $\Gamma^{|E|_{CB}}(E)$ is a perfect set (which may be the empty set). Now we mention a pointwise version of Cantor-Bendixson rank.

Definition 2.1.10. Let E be a Polish space, and let x be in E. We define the **topological rank** of x, $r(x)$, to be

$$r(x) = \begin{cases} \sup\{\alpha : x \in \Gamma^\alpha(E)\} & \text{if } x \notin \Gamma^{|E|_{CB}}(E) \\ \omega_1 & \text{if } x \in \Gamma^{|E|_{CB}}(E). \end{cases}$$

The following proposition follows directly from the definitions and compactness.

Proposition 2.1.11. *Let E be a countable, compact Polish space. Then*

1. $|E|_{CB}$ *is a successor ordinal.*

2. *If* $|E|_{CB} = \alpha + 1$, *then* $\Gamma^\alpha(E)$ *is a non-empty, finite set, and* $\Gamma^{\alpha+1}(E) = \emptyset$.

3. $|E|_{CB} = \big(\sup_{x \in E} r(x)\big) + 1 = \big(\max_{x \in E} r(x)\big) + 1$.

Now we state a well-known classification of countable, compact Polish spaces, due to Mazurkiewicz and Sierpiński [69, p. 21]. We denote the cardinality of a set E by $|E|$.

Theorem 2.1.12. *Let E and F be countable, compact Polish spaces, and assume that $|E|_{CB} = \alpha + 1$. Then E and F are homeomorphic if and only if $|E|_{CB} = |F|_{CB}$ and $|\Gamma^\alpha(E)| = |\Gamma^\alpha(F)|$.*

Remark 2.1.13. Let α be a countable ordinal. Then $\Gamma^\alpha(\omega^\alpha + 1) = \{\omega^\alpha\}$ and $|\omega^\alpha + 1|_{CB} = \alpha + 1$. It follows from Theorem 2.1.12 that if γ_k is any increasing sequence of ordinals such that $\sup_k \gamma_k = \omega^\alpha$, then $\omega^\alpha + 1$ is homeomorphic to the one-point compactification of the disjoint union of the spaces γ_k, with the point at infinity corresponding to ω^α.

Note that for any countable ordinal α, the space $\omega^\alpha n+1$ has Cantor-Bendixson rank $\alpha+1$ and exactly n points of topological rank α given by $\omega^\alpha k$ for $k = 1, \ldots, n$. Then by the above classification, the space $\omega^\alpha n + 1$ provides a representative of the homeomorphism class of countable, compact Polish spaces with Cantor-Bendixson rank $\alpha + 1$ and n points of topological rank α.

2.1.4 Upper-semicontinuity

Now we consider functions $f : E \to \mathbb{R}$, where E is a metrizable space. For such a function f, we let $||f|| = \sup_{x\in E} |f(x)|$, where the supremum is taken to be $+\infty$ if f is unbounded.

Definition 2.1.14. Let E be a compact metrizable space, and let $f : E \to \mathbb{R}$. Then f is **upper semi-continuous** (u.s.c.) if one of the following equivalent conditions holds:

1. $f = \inf_\alpha g_\alpha$ for some family $\{g_\alpha\}$ of continuous functions;
2. $f = \lim_n g_n$ for some nonincreasing sequence $(g_n)_{n\in\mathbb{N}}$ of continuous functions;
3. For each $r \in \mathbb{R}$, the set $\{x : f(x) \geq r\}$ is closed;
4. $\limsup_{y\to x} f(y) \leq f(x)$, for all $x \in E$.

For any $f : E \to \mathbb{R}$, the **upper semi-continuous envelope** of f, written $\widetilde{f}$, is defined, for all x in E, by

$$\widetilde{f}(x) = \begin{cases} \inf\{g(x) : g \text{ is continuous, and } g \geq f\}, & \text{if } f \text{ is bounded} \\ +\infty, & \text{if } f \text{ is unbounded.} \end{cases}$$

Note that when f is bounded, $\widetilde{f}$ is the smallest u.s.c. function greater than or equal to f and satisfies

$$\widetilde{f}(x) = \max\left(f(x), \limsup_{y\to x} f(y)\right).$$

It is immediately seen that for any $f, g : E \to \mathbb{R}$, $\widetilde{f+g} \le \widetilde{f} + \widetilde{g}$, with equality holding if f or g is continuous.

Definition 2.1.15. Let $\pi : E \to F$ be a continuous map. If $f : F \to \mathbb{R}$ is any function, we define the **lift** of f, denoted ${}^{\pi}f$, to be the function given by $f \circ \pi$.

If $\pi : E \to F$ is a surjection and $f : E \to \mathbb{R}$ is bounded, then the **projection** of f, denoted $f^{[F]}$, is the function defined on F by

$$f^{[F]}(x) = \sup_{y\in\pi^{-1}(x)} f(y).$$

Remark 2.1.16. Let $\pi : E \to F$ be a continuous surjection.

1. If $f : F \to \mathbb{R}$, then $({}^{\pi}f)^{[F]} = f$.
2. If $f : E \to \mathbb{R}$, then ${}^{\pi}(f^{[F]}) \ge f$, and the inequality is strict in general.
3. If $f : E \to \mathbb{R}$ is u.s.c., then $f^{[F]}$ is also u.s.c. and the supremum is attained.
4. If $f : F \to \mathbb{R}$ is u.s.c., then ${}^{\pi}f$ is also u.s.c.

2.1.5 Candidate Sequences

Definition 2.1.17. A **candidate sequence** on a compact, metrizable space E is a non-decreasing sequence $\mathcal{H} = (h_k)$ of non-negative, real-valued functions on E that converges pointwise to a function h. We often write $\lim \mathcal{H} = h$. We always assume by convention that $h_0 \equiv 0$.

A candidate sequence $\mathcal{H}$ **has u.s.c. differences** if $h_{k+1} - h_k$ is u.s.c. for all k. Note that in this case each h_k is u.s.c., since $h_0 \equiv 0$. If $\mathcal{H}$ has u.s.c. differences, we may also refer to $\mathcal{H}$ as a u.s.c.d. candidate sequence, or we may write that $\mathcal{H}$ is u.s.c.d.

Given a candidate sequence $\mathcal{H}$, it is natural to seek a precise description of the manner in which h_k converges to h. For example, is this convergence uniform or not? The notion of *uniform equivalence*, as defined by Downarowicz in [35], captures exactly the manner in which h_k converges to h.

Definition 2.1.18. Let $\mathcal{H}$ and $\mathcal{F}$ be two candidate sequences on a compact, metrizable space E. We say that $\mathcal{H}$ **uniformly dominates** $\mathcal{F}$, written $\mathcal{H} \geq \mathcal{F}$, if for all $\epsilon > 0$, and for each k, there exists ℓ, such that $f_k \leq h_\ell + \epsilon$.

The candidate sequences $\mathcal{H}$ and $\mathcal{F}$ are **uniformly equivalent**, written $\mathcal{H} \cong \mathcal{F}$, if $\mathcal{H} \geq \mathcal{F}$ and $\mathcal{F} \geq \mathcal{H}$.

Note that uniform equivalence is in fact an equivalence relation.

2.2 Basic Constructions

2.2.1 Order Of Accumulation

Definition 2.2.1. Let $\mathcal{H}$ be a candidate sequence on E. The **transfinite sequence** associated to $\mathcal{H}$, which we write as $(u_\alpha^{\mathcal{H}})$ or (u_α), is defined by transfinite induction as follows. Let $\tau_k = h - h_k$. Then

- let $u_0 \equiv 0$;

- if u_α has been defined, let $u_{\alpha+1} = \lim_k \widetilde{u_\alpha + \tau_k}$;

- if u_β has been defined for all $\beta < \alpha$ for a limit ordinal α, let $u_\alpha = \widetilde{\sup_{\beta<\alpha}} u_\beta$.

Note that for each α, either $u_\alpha \equiv +\infty$ or u_α is u.s.c. (since a non-increasing limit of u.s.c. functions is u.s.c.). Furthermore, the sequence (u_α) is non-decreasing in α. It is also sub-additive in the following sense.

Proposition 2.2.2. *Let $\mathcal{H}$ be a candidate sequence on E. Then for any two ordinals α and β,*

$$u_{\alpha+\beta} \leq u_\alpha + u_\beta.$$

Proof. Let α be any ordinal. We prove the statement by transfinite induction on β. For $\beta = 0$, the statement is trivial. Now assume by induction that the statement is true for $\gamma < \beta$. If β is a successor ordinal, then by the inductive hypothesis,

$$u_{\alpha+\beta} = \lim_k (\widetilde{u_{\alpha+(\beta-1)} + \tau_k}) \leq u_\alpha + \lim_k (\widetilde{u_{\beta-1} + \tau_k}) = u_\alpha + u_\beta.$$

If β is a limit ordinal, then by the inductive hypothesis,

$$u_{\alpha+\beta} = \widetilde{\sup_{\gamma<\beta} u_{\alpha+\gamma}} \leq u_\alpha + \widetilde{\sup_{\gamma<\beta} u_\gamma} \leq u_\alpha + u_\beta.$$

□

If $\mathcal{H}$ is a candidate sequence on E, then by Theorem 3.3 in [13], there exists a countable ordinal α such that the associated transfinite sequence satisfies $u_\alpha = u_{\alpha+1}$, which then implies that $u_\beta = u_\alpha$ for all $\beta > \alpha$.

Definition 2.2.3. In this setting, the least ordinal α such that $u_\alpha = u_{\alpha+1}$ is called the **order of accumulation** of the candidate sequence $\mathcal{H}$, which we write as either $\alpha_0(\mathcal{H})$ or $\alpha_0^{\mathcal{H}}$.

Both the transfinite sequence and the order of accumulation are independent of the choice of representative of uniform equivalence class [35].

While it is true that $u_\alpha = u_{\alpha+1}$ implies $u_\alpha = u_\beta$ for all $\beta > \alpha$, it is not true that for a fixed x, $u_\alpha(x) = u_{\alpha+1}(x)$ implies $u_\beta(x) = u_\alpha(x)$ for all $\beta > \alpha$. In fact, in many of the constructions in Section 2.3 there is a point $\mathbf{0}$ and an ordinal α such that $u_\gamma(\mathbf{0}) = 0$ for all $\gamma < \alpha$ and $u_\alpha(\mathbf{0}) = a > 0$. Nonetheless, we make the following definition.

Definition 2.2.4. Let $\mathcal{H}$ be a candidate sequence on E. Then for each x in E, we define the **pointwise order of accumulation** of $\mathcal{H}$ at x, $\alpha_0^{\mathcal{H}}(x)$ or $\alpha_0(x)$, as

$$\alpha_0^{\mathcal{H}}(x) = \inf\{\alpha : u_\beta(x) = u_\alpha(x) \text{ for all } \beta > \alpha\}.$$

Remark 2.2.5. Note that $\alpha_0^{\mathcal{H}}(x)$ is always a countable ordinal, and

$$\alpha_0(\mathcal{H}) = \sup_{x \in E} \alpha_0^{\mathcal{H}}(x).$$

The following proposition relates the pointwise topological rank (Definition 2.1.10) to the pointwise order of accumulation.

Proposition 2.2.6. *Let $\mathcal{H}$ be a candidate sequence on E. Then for any x in E,*

$$\alpha_0(x) \leq \begin{cases} r(x) & \textit{if } r(x) \textit{ is finite} \\ r(x) + 1 & \textit{if } r(x) \textit{ is infinite.} \end{cases}$$

Proof. The proof proceeds by transfinite induction on $r(x)$. If $r(x) = 0$ it is easily seen that $u_\gamma(x) = 0$ for all γ and $\alpha_0(x) = 0$.

Suppose the statement is true for all y with $r(y) < \alpha$, and fix x with $r(x) = \alpha$. If α is finite, let $\epsilon = \alpha$, and if α is infinite, let $\epsilon = \alpha + 1$. We show that for all $\beta > \alpha$, $u_\beta(x) = u_\epsilon(x)$, and here we use transfinite induction on $\beta > \alpha$. Note that there is an open neighborhood U of x such that for all y in U, $r(y) < r(x)$. Thus any real-valued function f on E satisfies $\limsup_{y\to x} f(y) = \limsup_{y\to x,\, r(y)<r(x)} f(y)$.

Suppose $\beta > \alpha$ is a successor. Then

$$(\widetilde{u_{\beta-1} + \tau_k})(x) = \max\Big(\limsup_{\substack{y\to x\\ r(y)<\alpha}} (u_{\beta-1} + \tau_k)(y),\ (u_{\beta-1} + \tau_k)(x)\Big).$$

Applying the induction hypotheses to all y with $r(y) < \alpha$ and $u_{\beta-1}(x)$ gives that

$$(\widetilde{u_{\beta-1} + \tau_k})(x) = \max\Big(\limsup_{\substack{y\to x\\ r(y)<\alpha}} (u_{\epsilon-1} + \tau_k)(y),\ (u_\epsilon + \tau_k)(x)\Big).$$

Letting k tend to infinity, we obtain $u_\beta(x) = u_\epsilon(x)$.

Suppose β is a limit ordinal. Then the inductive hypotheses imply

$$\begin{aligned} u_\beta(x) &= \max\Big(\limsup_{\substack{y\to x\\ r(y)<\alpha}} \sup_{\gamma<\beta} u_\gamma(y),\ \sup_{\gamma<\beta} u_\gamma(x)\Big) \\ &= \max\Big(\limsup_{\substack{y\to x\\ r(y)<\alpha}} u_{\epsilon-1}(y),\ \sup_{\gamma<\beta} u_\epsilon(x)\Big) \\ &= u_\epsilon(x). \end{aligned}$$

□

It follows from the proof of Theorem 2.4.3 that these pointwise bounds on $\alpha_0(x)$ are optimal. Also, combining Remark 2.2.5, Proposition 2.2.6, and Proposition 2.1.11 (3), we obtain the following result.

Corollary 2.2.7. *Let $\mathcal{H}$ be a candidate sequence on a countable, compact Polish space E. Then*

$$\alpha_0(\mathcal{H}) \leq \begin{cases} |E|_{CB} - 1, & \text{if } |E|_{CB} \text{ is finite} \\ |E|_{CB}, & \text{if } |E|_{CB} \text{ is infinite.} \end{cases}$$

2.2.2 Construction of Candidate Sequences

Now we discuss various ways of creating candidate sequences. We first begin with elementary constructions that will be studied later in the context of Choquet simplices.

Definition 2.2.8. Let $\mathcal{H}$ be a candidate sequence on E. If F is a compact subset of E, then we define the **restriction candidate sequence**, $\mathcal{H}|_F$, on F.

Definition 2.2.9. Let $\mathcal{H}$ be candidate sequence on E, and let F be a compact metrizable space with $\pi : F \to E$ a continuous surjection. Then the **lifted candidate sequence** of $\mathcal{H}$ to F, denoted ${}^{\pi}\mathcal{H}$, is the candidate sequence on F given by $({}^{\pi}h_k) = (h_k \circ \pi)$.

Definition 2.2.10. Let $\mathcal{F} = (f_k)$ be a candidate sequence on F, and let $g : F \to E$ be an embedding (continuous injection). The **embedded candidate sequence**, $g\mathcal{F} = (h_k)$, on E is defined to be

$$h_k(x) = \begin{cases} f_k \circ g^{-1}(x) & \text{if } x \in g(F) \\ 0 & \text{if } x \in E \setminus g(F). \end{cases}$$

While all of the constructions in this section will be used, the following two constructions (disjoint union and product candidate sequences) form the basis of the proofs of Theorem 2.3.1 and Corollary 2.3.2.

Definition 2.2.11. Let $(\mathcal{H}_n)$ be a countable collection of candidate sequences, where $\mathcal{H}_n = (h_k^n)$ is defined on E_n. Then we define the **disjoint union candidate sequence**, $\coprod \mathcal{H}_n$, as follows. Let E be the one-point compactification of the disjoint union of the spaces E_n, with the point at infinity denoted $\mathbf{0}$. For each k, let f_k be the function on E such that $f_k|_{E_n} = h_k^n$ and $f_k(\mathbf{0}) = 0$. Then the disjoint union candidate sequence, $\coprod \mathcal{H}_n$, is defined to be (f_k).

Recall that $||f||$ denotes the supremum norm of the real-valued function f.

Lemma 2.2.12. *Let $(\mathcal{H}_n)$ be a sequence of candidate sequences on E_n, where $h^n = \lim \mathcal{H}_n$. Let $\mathcal{H} = \coprod \mathcal{H}_n$. If $||h^n|| \to 0$, then for all β,*

1. $u_\beta^{\mathcal{H}}(\mathbf{0}) = \limsup_n ||u_\beta^{\mathcal{H}_n}||$, *and*

2. $||u_\beta^{\mathcal{H}}|| = \sup_n ||u_\beta^{\mathcal{H}_n}||$.

Proof. For each n, E_n is a clopen subset of E. It follows that $u_\gamma^{\mathcal{H}}(x) = u_\gamma^{\mathcal{H}_n}(x)$ for all ordinals γ, and for all x in E_n. Then (2) follows from the definitions and (1). Also, upper semi-continuity of $u_\beta^{\mathcal{H}}$ implies that $u_\beta^{\mathcal{H}}(\mathbf{0}) \geq \limsup_n ||u_\beta^{\mathcal{H}_n}||$. It remains only to show the reverse inequality.

The hypotheses imply that

$$u_1^{\mathcal{H}}(\mathbf{0}) \leq \widetilde{h}(\mathbf{0}) \leq \lim_n ||h^n|| = 0.$$

Now we use transfinite induction on β. The case $\beta = 0$ is trivial. Suppose β is a successor. By sub-additivity of the transfinite sequence (Lemma 2.2.2) $u_\beta^{\mathcal{H}}(\mathbf{0}) \leq u_{\beta-1}^{\mathcal{H}}(\mathbf{0}) + u_1^{\mathcal{H}}(\mathbf{0}) = u_{\beta-1}^{\mathcal{H}}(\mathbf{0})$, which, along with induction, implies the desired inequality. Now suppose β is a limit ordinal. Monotonicity of the transfinite sequence and induction again imply that

$$u_\beta^{\mathcal{H}}(\mathbf{0}) = \max\Big(\limsup_{y\to\mathbf{0}} u_\beta^{\mathcal{H}}(y),\ \sup_{\gamma<\beta} u_\gamma^{\mathcal{H}}(\mathbf{0})\Big) \leq \limsup_n ||u_\beta^{\mathcal{H}_n}||.$$

□

By a marked space $(E, \mathbf{0})$, we mean a compact, metrizable space E together with a marked point $\mathbf{0}$ in E.

Definition 2.2.13. Let $\mathcal{F} = (f_k)$ and $\mathcal{G} = (g_k)$ be two candidate sequences defined on the marked spaces $(E_1, \mathbf{0}_1)$ and $(E_2, \mathbf{0}_2)$, respectively. Then we define the **product candidate sequence**, $\mathcal{H} = \mathcal{F} \times \mathcal{G}$, on the marked product space $(E_1 \times E_2, (\mathbf{0}_1, \mathbf{0}_2))$ as the sequence

$$h_k(x, y) = \begin{cases} f_k(x) & \text{if } y = \mathbf{0}_2 \\ g_k(y) & \text{if } y \neq \mathbf{0}_2 \end{cases}$$

Note that this definition is not symmetric under transposition of $\mathcal{F}$ and $\mathcal{G}$. In other words, this product is not commutative, but one may check easily that it is associative.

Let $\mathcal{H}$ be a candidate sequence on the marked space $(E, \mathbf{0})$. Define $(\mathcal{H})^{\times p}$ to be the candidate sequence on the product space $(E^p, \mathbf{0}^p)$ given by iterated multiplication: $(\mathcal{H})^{\times p} = \mathcal{H}^{\times(p-1)} \times \mathcal{H}$.

Lemma 2.2.14 (Powers Lemma)**.** *Let $\mathcal{H}$ be a candidate sequence on the marked space $(E, \mathbf{0})$. Suppose that for some limit ordinal α and real number $a > 0$,*

(i) $||u_\gamma|| \leq a$ for all γ, and $||u_\gamma|| < a$ for $\gamma < \alpha$;

(ii) $u_\gamma(\mathbf{0}) = 0$*, for all* $\gamma < \alpha$*, and* $u_\alpha(\mathbf{0}) = a$*;*

(iii) $\alpha_0(x) \le \alpha$*, for all* x *in* E*.*

Then the transfinite sequence associated to $(\mathcal{H})^{\times p}$ *satisfies*

(1) $||u_\gamma^{\mathcal{H}^{\times p}}|| \le pa$ *for all* γ*;*

(2) $||u_{\alpha k}^{\mathcal{H}^{\times p}}|| \le ka$ *and* $||u_\gamma^{\mathcal{H}^{\times p}}|| < ka$*, for all* $\gamma < \alpha k$ *and* $k \le p$*;*

(3) $\alpha_0^{\mathcal{H}^{\times p}}(x) \le \alpha p$*, for all* x *in* E^p*;*

(4) $u_\gamma^{\mathcal{H}^{\times p}}(\mathbf{0}^p) = \ell a$*, for all* $\alpha\ell \le \gamma < \alpha(\ell+1)$*, and* $\ell = 0, \ldots, p$*;*

(5) $\alpha_0^{\mathcal{H}^{\times p}}(\mathbf{0}^p) = \alpha p$*.*

Proof. We argue by induction on p. For $p = 1$, the claims (1)-(5) follow from (i)-(iii).

Assume that (1)-(5) hold for p. We prove that (1)-(5) also hold with $p+1$ in place of p. Let (u_α^p) be the transfinite sequence for $\mathcal{H}^{\times p} = (h_k^p)$, and let $h^p = \lim \mathcal{H}^{\times p}$. Recall that $E^{p+1} = E^p \times E$. The definition of $\mathcal{H}^{\times(p+1)}$ is that

$$h_k^{p+1}(x,y) = \begin{cases} h_k^p(x), & \text{if } y = \mathbf{0} \\ h_k(y), & \text{if } y \ne \mathbf{0}. \end{cases}$$

For all (x,y) in E^{p+1}, $(h^{p+1} - h_k^{p+1})(x,y) \le (h^p - h_k^p)(x) + (h - h_k)(y)$. It follows from transfinite induction that for all γ, $u_\gamma^{p+1}(x,y) \le u_\gamma^p(x) + u_\gamma(y)$. Using the inductive hypotheses, we obtain that $||u_\gamma^{p+1}|| \le ap + a = a(p+1)$ for all γ, proving (1).

It follows from subadditivity that $||u_{\alpha k+\gamma}^{p+1}|| \le k||u_\alpha^{p+1}|| + ||u_\gamma^{p+1}||$, which means that in order to establish (2) we need only show that for all $\gamma < \alpha$, $||u_\gamma^{p+1}|| < a$. Furthermore, since u_γ^{p+1} is u.s.c. and therefore attains its supremum, it suffices to show that for all $\gamma < \alpha$ and all (x, y) in E^{p+1}, $u_\gamma^{p+1}(x, y) < a$. Let $\gamma < \alpha$ and let (x, y) be in E^{p+1}. If $y \neq \mathbf{0}$, then there exists an open neighborhood U of (x, y) in E^{p+1} such that for all (s, t) in U, $t \neq \mathbf{0}$. Then $h_k^{p+1}(s, t) = h_k(t)$ for all (s, t) in U. It follows that $u_\gamma^{p+1}(x, y) = u_\gamma(y) < a$. Now suppose $y = \mathbf{0}$. Let $\epsilon > 0$. Since $u_\gamma(\mathbf{0}) = 0$ and u_γ is u.s.c., there exists an open neighborhood U of $\mathbf{0}$ in E such that for all s in U, $u_\gamma(s) \leq \epsilon$. Then for all (t, s) in the open set $E^p \times U$, $u_\gamma^{p+1}(t, s) \leq u_\gamma^p(t) + u_\gamma(s) \leq u_\gamma^p(t) + \epsilon$. Since ϵ was arbitrary, we obtain that $u_\gamma^{p+1}(x, \mathbf{0}) \leq u_\gamma^p(x)$. Using the induction hypothesis for $\mathcal{H}^{\times p}$, we conclude that $u_\gamma^{p+1}(x, \mathbf{0}) < a$.

For any point (x, y) in E^{p+1} with $y \neq \mathbf{0}$, we have already shown that $u_\gamma^{p+1}(x, y) = u_\gamma(y)$ for all γ. For any point of the form $(x, \mathbf{0})$, we have shown that $u_\alpha^{p+1}(x, \mathbf{0}) \leq a$. Furthermore, by upper-semicontinuity of u_α^{p+1}, we have that

$$u_\alpha^{p+1}(x, \mathbf{0}) \geq \limsup_{y \to \mathbf{0}} u_\alpha^{p+1}(x, y) = \limsup_{y \to \mathbf{0}} u_\alpha(y) = u_\alpha(\mathbf{0}) = a.$$

Thus $u_\alpha^{p+1}(x, \mathbf{0}) = a$ for all points of the form $(x, \mathbf{0})$. This fact, in combination with the fact that $u_\gamma^{p+1}(x, y) = u_\gamma(y) \leq a$ for $y \neq \mathbf{0}$ and all γ, immediately implies that $u_{\alpha+\gamma}^{p+1}(x, \mathbf{0}) = u_\gamma^p(x) + a$ for all x in E^p. Then induction gives statements (3)-(5). $\square$

Definition 2.2.15. For the rest of this chapter, we let $\mathcal{H}^p$ denote the **renormalized product** of $\mathcal{H}$ taken p times: if $\mathcal{H}^{\times p} = (h_k^{\times p})$, then let $\mathcal{H}^p = (h_k^p) = (\frac{1}{p} h_k^{\times p})$.

Now we discuss more general products than just powers of the same candidate sequence. We will only consider products of marked spaces. Let x be a point in the product space $(E_N \times \cdots \times E_1, \mathbf{0})$, where $\mathbf{0} = (\mathbf{0}_N, \ldots, \mathbf{0}_1)$. Let π_i be projection onto

E_i. Then define the function

$$\operatorname{ind}(x) = \begin{cases} \min\{i : \pi_i(x) \neq \mathbf{0}_i\} & \text{if } x \neq \mathbf{0} \\ N & \text{if } x = \mathbf{0}. \end{cases}$$

Also, let $\eta_i(x_N, \ldots, x_1) = (x_N, \ldots, x_i)$. Note that with these notations, if $(h_k) = \mathcal{H}_N \times \cdots \times \mathcal{H}_1$, then $h_k(x) = h_k^{\mathcal{H}_{\operatorname{ind}(x)}}(\pi_{\operatorname{ind}(x)}(x))$ for all x.

Lemma 2.2.16 (Product Lemma)**.** *Let α be any non-zero countable ordinal, and let $\alpha = \omega^{\beta_1} m_1 + \cdots + \omega^{\beta_N} m_N$ be the Cantor Normal Form of α. Let $a > 0$ be a real number, and suppose $a_1 > \cdots > a_N > 0$ such that*

$$\sum_{i=1}^{N} a_i = a,$$

and for each $j = 1, \ldots, N-1$,

$$\frac{a_j}{m_j} \geq \sum_{i=j+1}^{N} a_i. \tag{2.2.1}$$

(Note that for any $a > 0$, such $a_1, \ldots, a_N$ exist.) Now suppose that for each j in $\{1, \ldots, N\}$, $\mathcal{F}_j$ is a candidate sequence on $(E_j, \mathbf{0}_j)$ such that

(i) $||u_\gamma^{\mathcal{F}_j}|| \leq a_j$ for all γ, and $||u_\gamma^{\mathcal{F}_j}|| < a_j$ for $\gamma < \omega^{\beta_j}$;

(ii) $u_\gamma^{\mathcal{F}_j}(\mathbf{0}_j) = 0$, for all $\gamma < \omega^{\beta_j}$;

(iii) $u_{\omega^{\beta_j}}^{\mathcal{F}_j}(\mathbf{0}_j) = a_j$;

(iv) $\alpha_0(x) \leq \omega^{\beta_j}$, for all $x \neq \mathbf{0}_j$;

(v) $\alpha_0(\mathbf{0}_j) = \omega^{\beta_j}$.

Denote $\mathcal{H}_j = \mathcal{F}_j^{m_j}$ and $\alpha_j = \omega^{\beta_j} m_j$. Then the product $\mathcal{H}_N \times \cdots \times \mathcal{H}_1$ satisfies

(1) $||u_\gamma|| \leq a$ *for all* γ*, and* $||u_\gamma|| < a$ *for* $\gamma < \alpha$*;*

(2) $\alpha_0(x) \leq \alpha$*, for all* $x \neq \mathbf{0}$*;*

(3) $\alpha_0(\mathbf{0}) = \alpha$*, and* $u_{\alpha_0}(\mathbf{0}) = a$*. In particular,* $\alpha_0(\mathcal{H}_N \times \cdots \times \mathcal{H}_1) = \alpha$*.*

Proof. The proof proceeds by induction on N. The case $N = 1$ follows from (i)-(v). Now we assume that $N > 1$ and the statement holds for $N - 1$, and we show that it holds for N.

Let $\mathcal{H}_N \times \cdots \times \mathcal{H}_1 = (h_k)$ be as above, with $h = \lim_k h_k$, and let $\mathcal{H}_N \times \cdots \times \mathcal{H}_2 = (h'_k)$ with $h' = \lim_k h'_k$. By the definition of the product candidate sequence, we observe that $(h - h_k)(x) \leq (h' - h'_k)(\eta_2(x)) + (h^1 - h^1_k)(\pi_1(x))$. It follows that $u_\alpha(x) \leq u_\alpha^{\mathcal{H}_N \times \cdots \times \mathcal{H}_2}(\eta_2(x)) + u_\alpha^{\mathcal{H}_1}(\pi_1(x))$ for all x in E and α.

Let x be in E. Then there exists an open neighborhood U in E such that for all y in U, $\operatorname{ind}(y) \leq \operatorname{ind}(x)$.

If $\operatorname{ind}(x) = 1$, the existence of the neighborhood U implies that $u_\gamma^{\mathcal{H}}(x) = u_\gamma^{\mathcal{H}_1}(\pi_1(x))$ for all γ.

Now we prove that for $\gamma < \omega^{\beta_1}$ and x such that $\operatorname{ind}(x) > 1$, we have $u_\gamma^{\mathcal{H}}(x) \leq u_\gamma^{\mathcal{H}_N \times \cdots \times \mathcal{H}_2}(\eta_2(x))$. Since $\mathcal{F}_1$ satisfies the hypotheses $(i) - (v)$, we may apply Lemma 2.2.14 and conclude that $\mathcal{H}_1$ satisfies conclusions (1)-(5) in Lemma 2.2.14. Now let $\gamma < \omega^{\beta_1}$ and let x be in E with $\operatorname{ind}(x) > 1$. By conclusion (4) in Lemma 2.2.14 applied to $\mathcal{H}_1$, $u_\gamma^{\mathcal{H}_1}(\mathbf{0}_1) = 0$. Then for any $\epsilon > 0$, using that $u_\gamma^{\mathcal{H}_1}$ is u.s.c., there exists an open neighborhood V of x such that for all y in V, $u_\gamma^{\mathcal{H}}(y) \leq u_\gamma^{\mathcal{H}_N \times \cdots \times \mathcal{H}_2}(\eta_2(y)) + \epsilon$. Since $\epsilon > 0$ was arbitrary, we have the desired inequality.

By the induction hypothesis on $N - 1$ applied to $\mathcal{H}_N \times \cdots \times \mathcal{H}_2$, we have

$\sup_{\gamma<\omega^{\beta_1}} u_\gamma^{\mathcal{H}_N\times\ldots\mathcal{H}_2}(\eta_2(x)) \leq \sum_{j=2}^N a_j$. By conclusion (2) in Lemma 2.2.14 applied to $\mathcal{H}_1$, $||u_{\omega^{\beta_1}}^{\mathcal{H}_1}|| \leq \frac{a_1}{m_1}$. Hence, for x in E,

$$u_{\omega^{\beta_1}}^{\mathcal{H}}(x) = \big(\widetilde{\sup_{\gamma<\omega^{\beta_1}} u_\gamma^{\mathcal{H}}}\big)(x) \leq \max\Big(\frac{a_1}{m_1}, \sum_{j=2}^N a_j\Big) \leq \frac{a_1}{m_1}.$$

Then by upper semi-continuity of $u_{\omega^{\beta_1}}^{\mathcal{H}}$, we have that for any x with $\operatorname{ind}(x) > 1$,

$$u_{\omega^{\beta_1}}^{\mathcal{H}}(x) \geq \limsup_{\substack{y\to x\\ \operatorname{ind}(y)=1}} u_{\omega^{\beta_1}}^{\mathcal{H}}(y) = \limsup_{\substack{y\to x\\ \operatorname{ind}(y)=1}} u_{\omega^{\beta_1}}^{\mathcal{H}_1}(\pi_1(y)) = u_{\omega^{\beta_1}}^{\mathcal{H}_1}(\mathbf{0}_1) = \frac{a_1}{m_1}.$$

We conclude that for any x with $\operatorname{ind}(x) > 1$, $u_{\omega^{\beta_1}}^{\mathcal{H}}(x) = \frac{a_1}{m_1}$. By sub-additivity (Proposition 2.2.2), we have that $u_{\omega^{\beta_1}m_1}^{\mathcal{H}}(x) \leq a_1$. By upper semi-continuity, for all x with $\operatorname{ind}(x) > 1$,

$$u_{\omega^{\beta_1}m_1}^{\mathcal{H}}(x) \geq \limsup_{\substack{y\to x\\ \operatorname{ind}(y)=1}} u_{\omega^{\beta}m_1}^{\mathcal{H}_1}(\pi_1(y)) = u_{\omega^{\beta}m_1}^{\mathcal{H}_1}(\mathbf{0}_1) = a_1.$$

It follows that $u_{\omega^{\beta_1}m_1}^{\mathcal{H}}(x) = a_1$ for all x with $\operatorname{ind}(x) > 1$, and then $u_{\omega^{\beta_1}m_1+\gamma}^{\mathcal{H}}(x) = a_1 + u_\gamma^{\mathcal{H}_N\times\cdots\times\mathcal{H}_2}(\eta_2(x))$ for all x with $\operatorname{ind}(x) > 1$ and all γ. Now with the induction hypothesis on $N-1$ applied to $\mathcal{H}_N \times \cdots \times \mathcal{H}_2$, the properties (1)-(3) follow immediately. □

We end this section by stating the semi-continuity properties of these new candidate sequences.

Proposition 2.2.17. *(1) If $\mathcal{H}_k$ is a sequence of u.s.c.d. candidate sequences and $||h^k|| \to 0$, then $\mathcal{H} = \coprod \mathcal{H}_k$ is a u.s.c.d. candidate sequence.*

(2) If $\mathcal{H}_1$ and $\mathcal{H}_2$ are u.s.c.d. candidate sequences and $(\lim \mathcal{H}_2)(\mathbf{0}_2) = 0$, then $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ is a u.s.c.d. candidate sequence.

(3) If $\mathcal{H}$ is a u.s.c.d. candidate sequence on E and F is closed subset of E, then $\mathcal{H}|_F$ is a u.s.c.d. candidate sequence.

(4) If $\mathcal{H}$ is a u.s.c.d. candidate sequence on E and ${}^{\pi}\mathcal{H}$ is the lift of $\mathcal{H}$ to F, where $\pi : F \to E$ is a continuous surjection, then ${}^{\pi}\mathcal{H}$ is a u.s.c.d. candidate sequence.

Proof. (1) The condition $||h^k|| \to 0$ implies that $\mathcal{H}$ has u.s.c. differences at $\mathbf{0}$ for all k.

(2) Because $\mathcal{H}_1$ is u.s.c.d., the condition $(\lim \mathcal{H}_2)(\mathbf{0}_2) = 0$ implies that $\mathcal{H}$ has u.s.c. differences at $(x, \mathbf{0}_2)$ for all x and k.

(3) The restriction of any u.s.c. function to a subset is also u.s.c.

(4) The lift of any u.s.c. function under a continuous map is also u.s.c.

□

2.2.3 Choquet Simplices and Candidate Sequences

The relevant chapters of [80] provide a good reference for most of the basic facts about simplices required in this work.

Let K be a metrizable, compact, convex subset of a locally convex topological vector space. Then the extreme points of K, $\mathrm{ex}(K)$, form a non-empty G_δ subset of K. We call a function $f : K \to \mathbb{R}$ **affine** (resp. **convex, concave**) if $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$ (resp. $\leq, \geq$) for all x and y in K and all t in $[0, 1]$.

Definition 2.2.18. Let K be a compact, convex subset of a locally convex topological vector space. Then K is a **Choquet simplex** if the dual of the continuous affine functions on K is a lattice.

For any Polish space E, let $\mathcal{M}(E)$ be the space of all Borel probabilities on E with the weak* topology. If E is compact, then $\mathcal{M}(E)$ is a Choquet simplex, with the extreme points given by the point measures.

Definition 2.2.19. Let K be a Choquet simplex. Then we define the **barycenter map**, bar : $\mathcal{M}(K) \to K$, to be the function given for each μ in $\mathcal{M}(K)$ by

$$\mathrm{bar}(\mu) = \int y \, d\mu(y),$$

where the integral means that for all continuous, affine functions $f : K \to \mathbb{R}$,

$$f(\mathrm{bar}(\mu)) = \int_K f \, d\mu.$$

The barycenter map is well-defined, continuous, affine, and surjective (see [80]).

If K is a metrizable Choquet simplex, then a function $f : K \to \mathbb{R}$ is called **harmonic** (resp. sub-harmonic, sup-harmonic) if for all μ in $\mathcal{M}(K)$,

$$f(\mathrm{bar}(\mu)) = \int_{\mathrm{ex}(K)} f \, d\mu,$$

(resp. $\leq, \geq$). A harmonic (resp. sub-harmonic, sup-harmonic) function is always affine (resp. convex, concave), but an affine (resp. convex, concave) function need not be harmonic (resp. sub-harmonic, sup-harmonic). On the other hand, a continuous affine (resp. convex, concave) function is always harmonic (resp. sub-harmonic, sup-harmonic). Furthermore, by standard arguments, any u.s.c. affine (resp. concave) function is harmonic (resp. sup-harmonic). It is shown in the proof of Fact 2.2.24 (see Appendix B, Section A.2) that any u.s.c. convex function is sub-harmonic.

In the metrizable case, Choquet proved the following characterization of Choquet simplices.

Theorem 2.2.20 (Choquet)**.** *Let K be a metrizable, compact, convex subset of a locally convex topological vector space. Then K is a Choquet simplex if and only if for each point x in K, there exists a unique Borel probability measure $\mathcal{P}_x$ on* $\mathrm{ex}(K)$ *such that for every continuous affine function $f : K \to \mathbb{R}$,*

$$f(x) = \int_{\mathrm{ex}(K)} f \, d\mathcal{P}_x.$$

Definition 2.2.21. If K is a metrizable Choquet simplex and $f : \mathrm{ex}(K) \to \mathbb{R}$ is measurable, the **harmonic extension** $f^{har} : K \to \mathbb{R}$ of f is defined as follows: for x in K, let

$$f^{har}(x) = \int_{\mathrm{ex}(K)} f \, d\mathcal{P}_x.$$

Remark 2.2.22. Using Choquet's characterization of metrizable Choquet simplices, it is not difficult to show that if $f : K \to \mathbb{R}$ is a measurable function and for each x in K,

$$f(x) = \int f \, d\mathcal{P}_x,$$

then f is harmonic. It follows that the harmonic extension of a function on $\mathrm{ex}(K)$ is, in fact, harmonic.

In the metrizable case, the following theorem of Choquet characterizes exactly which topological spaces appear as the set of extreme points of a Choquet simplex.

Theorem 2.2.23 (Choquet [26])**.** *The topological space E is homeomorphic to the set of extreme points of a metrizable Choquet simplex if and only if E is a Polish space.*

The following fact is stated as Fact 2.5 in [36], where there is a sketch of the proof. We include a proof as Appendix B (Section A.2) for the sake of completeness.

Fact 2.2.24. *Let K be a metrizable Choquet simplex, and let $f : K \to [0, \infty)$ be convex and u.s.c. Then $(f|_{\mathrm{ex}(K)})^{har}$ is u.s.c.*

If K is a metrizable Choquet simplex, we denote by $\mathcal{M}(\mathrm{ex}(K))$ the set of measures μ in $\mathcal{M}(K)$ such that $\mu(K \backslash \mathrm{ex}(K)) = 0$. Consider the map $\pi : \mathcal{M}(\mathrm{ex}(K)) \to K$ given by the restriction of the barycenter map to $\mathcal{M}(\mathrm{ex}(K))$. This restriction inherits the continuity and affinity of the barycenter map. Furthermore, this restriction is always bijective (by Choquet's characterization of metrizable Choquet simplices, Theorem 2.2.20), but it may not have a continuous inverse. In fact, π has a continuous inverse if and only if $\mathrm{ex}(K)$ is closed in K. These considerations lead to the study of Bauer simplices.

Definition 2.2.25. A metrizable, compact, convex subset K of a locally convex topological vector space is a **Bauer simplex** if K is a Choquet simplex such that $\mathrm{ex}(K)$ is a closed subset of K.

If E is any compact, metrizable space, then $\mathcal{M}(E)$ is a Bauer simplex with $\mathrm{ex}(\mathcal{M}(E))$ homeomorphic to E. If K is a Bauer simplex, then the restriction of the barycenter map $\pi : \mathcal{M}(\mathrm{ex}(K)) \to K$ has a continuous inverse and is therefore an affine homeomorphism from $\mathcal{M}(\mathrm{ex}(K))$ to K.

Proposition 2.2.26. *If K is a Bauer simplex and $f : K \to [0,\infty)$ is bounded and harmonic, then $\widetilde{f}$ is harmonic and $\widetilde{f}|_{\mathrm{ex}(K)} = \widetilde{f|_{\mathrm{ex}(K)}}$.*

Proof. Since f is harmonic, in particular f is affine. Let x and y be in K, and let $ax+by$ be a convex combination in K. We have $\widetilde{f}(ax+by) \geq f(ax+by) = af(x)+bf(y)$. For fixed a, b, and y, the above formula implies that $\widetilde{f}(ax+by) \geq a\widetilde{f}(x)+bf(y)$. Now fixing a, b, and x, we obtain $\widetilde{f}(ax+by) \geq a\widetilde{f}(x)+b\widetilde{f}(y)$. Now since $\widetilde{f}$ is u.s.c. and concave, it follows that $\widetilde{f}$ is sup-harmonic.

Let $E = \mathrm{ex}(K)$. It follows from the definitions that

$$f(t) = \int_E f|_E \, d\mathcal{P}_t \leq \int_E \widetilde{(f|_E)} \, d\mathcal{P}_t \leq \int_E \widetilde{f} \, d\mathcal{P}_t. \tag{2.2.2}$$

Now consider the two functions $g_1, g_2 : K \to \mathbb{R}$, given for each t in E by

$$g_1(t) = \begin{cases} \widetilde{f|_E}(t), & \text{if } t \in E, \\ 0, & \text{if } t \notin E, \end{cases}$$

$$g_2(t) = \begin{cases} \widetilde{f}|_E(t), & \text{if } t \in E, \\ 0, & \text{if } t \notin E. \end{cases}$$

Since E is closed, g_1 and g_2 are u.s.c. They are also obviously convex. Then by Fact 2.2.24, $G_1 = \big((g_1)|_E\big)^{har}$ and $G_2 = \big((g_2)|_E\big)^{har}$ are u.s.c. Note that for $t \in K$,

$$G_1(t) = \int_E \widetilde{(f|_E)} \, d\mathcal{P}_t, \text{ and } G_2(t) = \int_E \widetilde{f} \, d\mathcal{P}_t.$$

Thus, taking the u.s.c. envelope of the expressions in Equation (2.2.2) and using that G_1 and G_2 are u.s.c., we have that

$$\widetilde{f}(t) \leq \int_E \widetilde{(f|_E)} \, d\mathcal{P}_t \leq \int_E \widetilde{f} \, d\mathcal{P}_t, \tag{2.2.3}$$

which shows that $\widetilde{f}$ is sub-harmonic. Now we have shown that $\widetilde{f}$ is harmonic and the inequalities in Equation (2.2.3) are all equalities. □

A candidate sequence $\mathcal{H} = (h_k)$ on a Choquet simplex is said to be harmonic if each h_k is harmonic. The following proposition relates the transfinite sequence of a candidate sequence $\mathcal{H}$ on a Bauer simplex K to the transfinite sequence of $\mathcal{H}|_{\mathrm{ex}(K)}$.

Proposition 2.2.27. *If $\mathcal{H}$ is a harmonic candidate sequence on the Bauer simplex K, then for each α, $u_\alpha^{\mathcal{H}}$ is harmonic and*

$$u_\alpha^{\mathcal{H}} = (u_\alpha^{\mathcal{H}|_{\mathrm{ex}(K)}})^{har}. \tag{2.2.4}$$

Proof. The proof proceeds by transfinite induction on α. For all k, since h_k and h are harmonic, $\tau_k = h - h_k$ is harmonic.

Suppose $u_\alpha^{\mathcal{H}}$ is harmonic and Equation (2.2.4) holds. Then $u_\alpha^{\mathcal{H}} + \tau_k$ is harmonic. By Proposition 2.2.26, we deduce that $\widetilde{u_\alpha^{\mathcal{H}} + \tau_k}$ is harmonic, and for t in K,

$$(u_\alpha^{\mathcal{H}} + \tau_k)(t) = \int_E (\widetilde{u_\alpha^{\mathcal{H}} + \tau_k})|_E d\mathcal{P}_t = \int_E (\widetilde{u_\alpha^{\mathcal{H}|_E} + \tau_k})|_E d\mathcal{P}_t.$$

Recall that $\{u_\alpha + \tau_k\}_k$ is a non-increasing sequence in k. Thus we can take the limit in k and apply the Monotone Convergence Theorem to obtain that $u_{\alpha+1}^{\mathcal{H}}$ is also harmonic, and for t in K,

$$u_{\alpha+1}^{\mathcal{H}}(t) = \int_E u_{\alpha+1}^{\mathcal{H}|_E} d\mathcal{P}_t,$$

which implies that Equation (2.2.4) holds with $\alpha + 1$ in place of α.

The previous arguments apply in a similar way to the case when α is a limit ordinal.

□

Remark 2.2.28. Let K be a Choquet simplex which is not necessarily Bauer. Even when the candidate sequence $\mathcal{H}$ on K is harmonic, the functions $u_\alpha^\mathcal{H}$ are not in general harmonic. However, we check now that if $\mathcal{H}$ is harmonic, then $u_\alpha^\mathcal{H}$ is concave for all α. Assuming by induction that $u_\alpha^\mathcal{H}$ is concave, we have that $\widetilde{u_\alpha^\mathcal{H} + \tau_k}$ is concave, as it is the u.s.c. envelope of a concave function. Then $u_{\alpha+1}^\mathcal{H}$ is the limit of a sequence of concave functions, and so $u_{\alpha+1}^\mathcal{H}$ is concave. Now for any countable limit ordinal α, there is a strictly increasing sequence (α_n) of ordinals tending to α. Then $\sup_{\beta<\alpha} u_\beta^\mathcal{H} = \lim_n u_{\alpha_n}^\mathcal{H}$ since the sequence $(u_\beta^\mathcal{H})$ is increasing in β. Then $\sup_{\beta<\alpha} u_\beta^\mathcal{H}$ is concave, as it is the limit of a sequence of concave functions (by induction), and thus $u_\alpha^\mathcal{H}$ is concave for any countable limit ordinal as well.

When $\text{ex}(K)$ is not compact, $\mathcal{M}(\text{ex}(K))$ is not a Bauer simplex, and the restriction of the barycenter map to this set is not a homeomorphism. Instead of using this restriction in such cases, we consider the Bauer simplex $\mathcal{M}(\overline{\text{ex}(K)})$ and the continuous surjection $\pi : \mathcal{M}(\overline{\text{ex}(K)}) \to K$, where π is the restriction of the barycenter map to $\mathcal{M}(\overline{\text{ex}(K)})$. In the following two lemmas we consider candidate sequences which may arise as embedded candidate sequences.

Lemma 2.2.29. *Let E be a compact, metrizable space, and let K be a metrizable Choquet simplex. Suppose there exists a continuous injection $g : E \to K$. Let $\mathcal{F}$ be a u.s.c.d. candidate sequence on E, let $\mathcal{H}' = (h'_k)$ be the embedded candidate sequence $g\mathcal{F}$, and let $\mathcal{H}$ be the harmonic extension of $\mathcal{H}'|_{\text{ex}(K)}$ to K. If $h'_{k+1} - h'_k$ is convex for each k, then $\mathcal{H}$ is u.s.c.d. In particular, if $g(E) \subset \text{ex}(K)$ then $\mathcal{H}$ is u.s.c.d.*

Proof. Since $\mathcal{F}$ is u.s.c.d. and $g(E)$ is closed, we have that $h'_{k+1} - h'_k$ is u.s.c. for each k. Then $h'_{k+1} - h'_k$ is convex and u.s.c. for each k. By applying Fact 2.2.24, we obtain that $h_{k+1} - h_k$ is u.s.c. for each k. Thus $\mathcal{H}$ is u.s.c.d.

In particular, if $g(E) \subset \text{ex}(K)$, then $h'_{k+1} - h'_k$ takes non-zero values only on $\text{ex}(K)$. Therefore $h'_{k+1} - h'_k$ is convex for each k, and by the previous argument, $\mathcal{H}$ is u.s.c.d. □

The following lemma is used repeatedly throughout the rest of this work. The utility of this statement lies in the fact that it allows one to compute the transfinite sequence on a (frequently much simpler) subset of the simplex and then write the transfinite sequence on the entire simplex in terms the transfinite sequence on this subset. When K is a Choquet simplex that is not Bauer and $\mathcal{H}$ is a harmonic candidate sequence on K, then this statement takes the place of an integral representation of $u_\alpha^{\mathcal{H}}$.

Lemma 2.2.30 (Embedding Lemma)**.** *Let K be a metrizable Choquet simplex with $E = \text{ex}(K)$. Suppose $\mathcal{H}$ is a harmonic candidate sequence on K and there is a set $F \subset E$ such that the sequence $\{(h - h_k)|_{E \setminus F}\}$ converges uniformly to zero. Let $L = \overline{F}$, and let $\pi : \mathcal{M}(\overline{E}) \to K$ be the restriction of the barycenter map. Then for all ordinals α and for all x in K,*

$$u_\alpha^{\mathcal{H}}(x) = \max_{\mu \in \pi^{-1}(x)} \int_L u_\alpha^{\mathcal{H}|_L} d\mu, \tag{2.2.5}$$

and $\alpha_0(\mathcal{H}) \leq \alpha_0(\mathcal{H}|_L)$. In particular, if F is compact, then $u_\alpha^{\mathcal{H}}|_F = u_\alpha^{\mathcal{H}|_F}$ for all α and $\alpha_0(\mathcal{H}) = \alpha_0(\mathcal{H}|_F)$.

Proof. Note that Equation (2.2.5) implies immediately that $\alpha_0(\mathcal{H}) \leq \alpha_0(\mathcal{H}|_L)$. Further, suppose F is compact. Then $L = F \subset \text{ex}(K)$, and if x is in F, then

$\pi^{-1}(x) = \{\epsilon_x\}$, where ϵ_x is the point mass at x. In this case Equation (2.2.5) implies that $u_\alpha^{\mathcal{H}}|_F = u_\alpha^{\mathcal{H}|_F}$ for all α and $\alpha_0(\mathcal{H}) = \alpha_0(\mathcal{H}|_F)$. We now prove Equation (2.2.5).

Observe that since L is closed and $u_\alpha^{\mathcal{H}|_L}$ is u.s.c., the function $\mathbf{1}_L \cdot u_\alpha^{\mathcal{H}|_L}$ is u.s.c., where $\mathbf{1}_L$ is the characteristic function of the set L. Then the function $\mu \mapsto \int_L u_\alpha^{\mathcal{H}|_L}\, d\mu$ is u.s.c., and therefore by Remark 2.1.16 (3), for each x in K,

$$\sup_{\mu \in \pi^{-1}(x)} \int_L u_\alpha^{\mathcal{H}_L}\, d\mu = \max_{\mu \in \pi^{-1}(x)} \int_L u_\alpha^{\mathcal{H}_L}\, d\mu.$$

Let x be in K. Since $u_\alpha^{\mathcal{H}}$ is concave (see Remark 2.2.28) and u.s.c., it follows that $u_\alpha^{\mathcal{H}}$ is sup-harmonic. Therefore

$$u_\alpha^{\mathcal{H}}(x) \geq \int_K u_\alpha^{\mathcal{H}} d\mu, \qquad \text{for all } \mu \in \pi^{-1}(x).$$

Using the fact that $u_\alpha^{\mathcal{H}}|_L \geq u_\alpha^{\mathcal{H}|_L}$, we obtain, for all $\mu \in \pi^{-1}(x)$,

$$u_\alpha^{\mathcal{H}}(x) \geq \int_K u_\alpha^{\mathcal{H}} d\mu \geq \int_L u_\alpha^{\mathcal{H}} d\mu \geq \int_L u_\alpha^{\mathcal{H}|_L} d\mu.$$

It follows that for each ordinal α,

$$u_\alpha^{\mathcal{H}}(x) \geq \max_{\mu \in \pi^{-1}(x)} \int_L u_\alpha^{\mathcal{H}|_L} d\mu.$$

We now prove using transfinite induction on α that for all α and x in K,

$$u_\alpha^{\mathcal{H}}(x) \leq \max_{\mu \in \pi^{-1}(x)} \int_L u_\alpha^{\mathcal{H}|_L} d\mu, \tag{2.2.6}$$

which will complete the proof of the Lemma.

The inequality in Equation (2.2.6) is trivial for $\alpha = 0$. Suppose Equation (2.2.6) holds for some ordinal α. For the sake of notation, we allow $y = x$ in all expressions involving $\limsup_{y \to x}$ below. First we claim that for any y in K, there exists a measure μ_y supported on $L \cup E$ such that μ_y is in $\pi^{-1}(y)$ and

$$\max_{\mu \in \pi^{-1}(y)} \int_L u_\alpha^{\mathcal{H}|_L} d\mu = \int_L u_\alpha^{\mathcal{H}|_L} d\mu_y. \tag{2.2.7}$$

Indeed, suppose the maximum is obtained by the measure ν. If $\nu(L) = 1$, then we are done. Now suppose $\nu(L) < 1$. Then $\nu = \nu(L)\nu_L + (1-\nu(L))\nu_{\overline{E}\setminus L}$, where ν_S is the zero measure on S if $\nu(S) = 0$ and otherwise $\nu_S(A) = \frac{1}{\nu(S)}\nu(S \cap A)$. Let $z = \text{bar}(\nu_{\overline{E}\setminus L})$, which exists since $\nu_{\overline{E}\setminus L}$ is in $\mathcal{M}(\overline{E})$ (using that $\nu(\overline{E} \setminus L) = 1 - \nu(L) > 0$). Now let $\mu_y = \nu(L)\nu_L + (1-\nu(L))\mathcal{P}_z$. Then μ_y is supported on $L \cup E$, $\text{bar}(\mu_y) = y$, and

$$\int_L u_\alpha^{\mathcal{H}|_L} d\nu \leq \int_L u_\alpha^{\mathcal{H}|_L} d\mu_y.$$

Thus the maximum in Equation (2.2.7) is obtained by the measure μ_y, which is supported on $L \cup E$ and satisfies $\text{bar}(\mu) = y$.

Now let $\epsilon > 0$. Since $\mathcal{H}$ is harmonic, we also have that τ_k is harmonic. Then for any y in K and k large enough (depending only on ϵ),

$$u_\alpha^{\mathcal{H}}(y) + \tau_k(y) = \max_{\mu \in \pi^{-1}(y)} \int_L u_\alpha^{\mathcal{H}|_L} d\mu + \tau_k(y) \tag{2.2.8}$$

$$= \int_L u_\alpha^{\mathcal{H}|_L} d\mu_y + \int \tau_k d\mu_y \tag{2.2.9}$$

$$= \int_L u_\alpha^{\mathcal{H}|_L} d\mu_y + \int_L \tau_k d\mu_y + \int_{E\setminus L} \tau_k d\mu_y \tag{2.2.10}$$

$$\leq \int_L u_\alpha^{\mathcal{H}|_L} d\mu_y + \int_L \tau_k d\mu_y + \epsilon \tag{2.2.11}$$

$$= \int_L (u_\alpha^{\mathcal{H}|_L} + \tau_k) d\mu_y + \epsilon \tag{2.2.12}$$

$$\leq \int_L (\widetilde{u_\alpha^{\mathcal{H}|_L} + \tau_k})|_L d\mu_y + \epsilon \tag{2.2.13}$$

$$\leq \max_{\mu \in \pi^{-1}(y)} \int_L (\widetilde{u_\alpha^{\mathcal{H}|_L} + \tau_k})|_L d\mu + \epsilon. \tag{2.2.14}$$

Then we have (allowing $y = x$ in the limit suprema) that

$$u_{\alpha+1}^{\mathcal{H}}(x) = \lim_k \limsup_{y\to x} u_\alpha^{\mathcal{H}}(y) + \tau_k(y) \tag{2.2.15}$$

$$\le \lim_k \limsup_{y\to x} \max_{\mu\in\pi^{-1}(y)} \int_L (\widetilde{u_\alpha^{\mathcal{H}|_L} + \tau_k})|_L d\mu + \epsilon \tag{2.2.16}$$

$$\le \lim_k \max_{\mu\in\pi^{-1}(x)} \int_L (\widetilde{u_\alpha^{\mathcal{H}|_L} + \tau_k})|_L d\mu + \epsilon \tag{2.2.17}$$

$$\le \max_{\mu\in\pi^{-1}(x)} \int_L u_{\alpha+1}^{\mathcal{H}|_L} d\mu + \epsilon, \tag{2.2.18}$$

where the inequalities in (2.2.17) and (2.2.18) are justified by Lemmas 2.2.33 and 2.2.34, respectively. Since ϵ was arbitrary, we have shown the inequality in Equation (2.2.6) with the ordinal α replaced by $\alpha + 1$.

Now suppose the inequality in Equation (2.2.6) holds for all $\beta < \alpha$, where α is a limit ordinal. Using monotonicity of the sequence $u_\alpha^{\mathcal{H}|_L}$, we see that (allowing $y = x$ in the limit suprema)

$$\begin{aligned} u_\alpha^{\mathcal{H}}(x) &= \widetilde{\sup_{\beta<\alpha} u_\beta^{\mathcal{H}}}(x) \\ &= \limsup_{y\to x} \sup_{\beta<\alpha} \max_{\mu\in\pi^{-1}(y)} \int_L u_\beta^{\mathcal{H}|_L} d\mu \\ &\le \limsup_{y\to x} \max_{\mu\in\pi^{-1}(y)} \int_L u_\alpha^{\mathcal{H}|_L} d\mu \\ &\le \max_{\mu\in\pi^{-1}(x)} \int_L u_\alpha^{\mathcal{H}|_L} d\mu, \end{aligned}$$

where Lemma 2.2.33 justifies the last inequality. Thus we have shown that the inequality in Equation (2.2.6) holds for α, which completes the induction and the proof. □

Remark 2.2.31. Given the assumptions of the Embedding Lemma, if x is in $\text{ex}(K)$, then $\pi^{-1}(x) = \{\epsilon_x\}$, where ϵ_x is the point mass at x. It follows that, if x is in

$L \cap \mathrm{ex}(K)$, then $u_\alpha^{\mathcal{H}}(x) = u_\alpha^{\mathcal{H}|_L}(x)$ for all α. Further, if x is in $\mathrm{ex}(K) \setminus L$, then $u_\alpha^{\mathcal{H}}(x) = 0$ for all α.

Remark 2.2.32. With the notation of the Embedding Lemma, Equation (2.2.5) implies that $||u_\alpha^{\mathcal{H}}|| = ||u_\alpha^{\mathcal{H}|_L}||$ for all α.

Lemma 2.2.33. *Let K be a metrizable Choquet simplex and L a closed subset of K. Let $f : K \to [0, \infty)$ be u.s.c. Then for all x in K,*

$$\limsup_{y \to x} \max_{\mu \in \pi^{-1}(y)} \int_L f d\mu \leq \max_{\mu \in \pi^{-1}(x)} \int_L f d\mu,$$

where π is the restriction of the barycenter map on $\mathcal{M}(K)$ to $\mathcal{M}(\overline{\mathrm{ex}(K)})$.

Proof. Let $T : \mathcal{M}(K) \to \mathbb{R}$ be defined by $T(\mu) = \int_L f d\mu$. We have that $f\chi_L$ is u.s.c. since f is non-negative and u.s.c. and L is closed. It follows that T is u.s.c. Then the result follows from Remark 2.1.16 (3). □

Lemma 2.2.34. *Let K be a metrizable Choquet simplex and L a closed subset of K. Let $\{f_k : K \to [0, \infty)\}$ be a non-increasing sequence of u.s.c. functions, with $\lim_k f_k = f$. Then for all x in K,*

$$\lim_{k \to \infty} \max_{\mu \in \pi^{-1}(x)} \int_L f_k d\mu \leq \max_{\mu \in \pi^{-1}(x)} \int_L f d\mu,$$

where π is the restriction of the barycenter map on $\mathcal{M}(K)$ to $\mathcal{M}(\overline{\mathrm{ex}(K)})$.

Proof. Let x be in K. Define $T_k : \mathcal{M}(K) \to \mathbb{R}$ and $T : \mathcal{M}(K) \to \mathbb{R}$ by the equations

$$T_k(\mu) = \int_L f_k d\mu, \quad \text{and} \quad T(\mu) = \int_L f d\mu.$$

Since $f_k\chi_L$ and $f\chi_L$ are u.s.c., T and T_k are u.s.c. Proposition 2.4 of [13] states (in slightly greater generality) that

$$\lim_k \max_{\mu\in\pi^{-1}(x)} T_k(\mu) = \max_{\mu\in\pi^{-1}(x)} \lim_k T_k(\mu). \tag{2.2.19}$$

By the Monotone Convergence Theorem,

$$T(\mu) = \lim_k T_k(\mu). \tag{2.2.20}$$

Combining Equations (2.2.19) and (2.2.20) concludes the proof. □

Even when the hypotheses of the Embedding Lemma are satisfied, it is possible to have $\alpha_0(\mathcal{H}) < \alpha_0(\mathcal{H}|_L)$, as the next example shows.

Example 2.2.35. This example provides a candidate sequence $\mathcal{H}$ satisfying the hypotheses of the Embedding Lemma and $\alpha_0(\mathcal{H}) < \alpha_0(\mathcal{H}|_L)$, which proves that the inequality $\alpha_0(\mathcal{H}) \leq \alpha_0(\mathcal{H}|_L)$ is not an equality in general. Suppose the set of extreme points of K consists of two points, b_1 and b_2, sequences $\{c_n\}$ and $\{d_n\}$ with $c_n \to b_1$ and $d_n \to b_2$, and a countable collection $\{a_n\}$. Let $b = \frac{1}{2}(b_1 + b_2)$ in K. Suppose further that with the subspace topology inherited from K, the set $\{a_n\} \cup \{b\}$ is homeomorphic to $\omega^2 + 1$, with the homeomorphism given by $g_1 : \omega^2 + 1 \to \{a_n\} \cup \{b\}$ and $g_1(\omega^2) = b$. One may construct such a simplex K as the image of $\mathcal{M}(\{a_n\} \cup \{b, b_1, b_2\} \cup \{c_n\} \cup \{d_n\})$ under a continuous affine map (Lemma 2.5.14). Let $\mathcal{F}_1 = (f_k^1)$ be u.s.c.d. candidate sequence on $\omega^2 + 1$ such that $\alpha_0(\mathcal{F}_1) = 2$, $u_1^{\mathcal{F}_1}(t) = u_2^{\mathcal{F}_1}(t)$ for $t \neq \omega^2$, and $||u_2^{\mathcal{F}_1}|| = 1$. Such a sequence is given by Corollary 2.3.2. Let $\mathcal{F}_2 = (f_k^2)$ be the u.s.c.d. candidate sequence on

$\{c_n\} \cup \{d_n\} \cup \{b_1, b_2\}$ given, for x in $\{c_n\} \cup \{d_n\} \cup \{b_1, b_2\}$ and $k \geq 1$, by

$$f_k^2(x) = \begin{cases} 0 & \text{if } x = c_n \text{ or } x = d_n, \text{ with } k < n \\ 1 & \text{otherwise.} \end{cases}$$

Now consider the candidate sequence $\mathcal{H}' = (h_k')$ on K such that for x in K,

$$h_k'(x) = \begin{cases} f_k^1(g_1^{-1}(x)) & \text{if } x = a_n \\ f_k^2(x) & \text{if } x = c_n, \ d_n \\ 0 & \text{otherwise} \end{cases}$$

Note that $\mathcal{H}'$ is u.s.c.d., convex, and $h_{k+1}' - h_k'$ is convex. Let $\mathcal{H}$ be the harmonic extension of $\mathcal{H}'|_{\text{ex}(K)}$ on K. Then by Lemma 2.2.29, $\mathcal{H}$ is harmonic and u.s.c.d.

Let $F = \text{ex}(K)$ and $L = \overline{F} = \{a_n\} \cup \{b, b_1, b_2\} \cup \{c_n\} \cup \{d_n\}$. Since L is the disjoint union the two (clopen in L) sets $\{a_n\} \cup \{b\}$ and $\{b_1, b_2\} \cup \{c_n\} \cup \{d_n\}$, we see that for t in L,

$$u_\alpha^{\mathcal{H}|_L} = \begin{cases} u_\alpha^{\mathcal{F}_1}(t), & \text{if } t \in \{a_n\} \cup \{b\} \\ u_\alpha^{\mathcal{F}_2}(t), & \text{if } t \in \{b_1, b_2\} \cup \{c_n\} \cup \{d_n\}. \end{cases}$$

Thus $\alpha_0(\mathcal{H}|_L) = \max(\alpha_0(\mathcal{F}_1), \alpha_0(\mathcal{F}_2)) = \alpha_0(\mathcal{F}_1) = 2$ and $||u_2^{\mathcal{H}|_L}|| \leq 1$. Also, for all $t \neq b$, $u_1^{\mathcal{H}|_L}(t) = u_2^{\mathcal{H}|_L}(t)$, and for $t \in \{b_1, b_2\}$, $u_1^{\mathcal{H}|_L}(t) = 1$.

Applying the Embedding Lemma, we have that for all t in K,

$$u_\alpha^{\mathcal{H}}(t) = \max_{\mu \in \pi^{-1}(t)} \int_L u_\alpha^{\mathcal{H}|_L} \, d\mu. \tag{2.2.21}$$

If $\mu \in \pi^{-1}(t)$ and $\mu(\{b\}) > 0$, then let $\nu = \frac{1}{2}\mu(\{b\})(\epsilon_{b_1} + \epsilon_{b_2}) + (1 - \mu(\{b\}))\mu_{L \setminus \{b\}}$, where $\mu_{L \setminus \{b\}}$ is the measure μ conditioned on the set $L \setminus \{b\}$. Then $\nu \in \pi^{-1}(t)$, $\nu(\{b\}) = 0$, and $\int_L u_i^{\mathcal{H}|_L} \, d\mu \leq \int_L u_i^{\mathcal{H}|_L} \, d\nu$ for $i \in \{1, 2\}$. Thus the maximum in

Equation (2.2.21) is obtained by a measure μ with $\mu(\{b\}) = 0$. Now if $\mu \in \pi^{-1}(t)$ and $\mu(\{b\}) = 0$, then $\int_L u_1^{\mathcal{H}|_L}\, d\mu = \int_L u_2^{\mathcal{H}|_L}\, d\mu$ since $u_1^{\mathcal{H}|_L}(s) = u_2^{\mathcal{H}|_L}(s)$ for $s \in L \setminus \{b\}$. From these facts we deduce $u_1^{\mathcal{H}}(t) = u_2^{\mathcal{H}}(t)$ for all t in K, and therefore $\alpha_0(\mathcal{H}) = 1 < \alpha_0(\mathcal{H}|_L)$.

2.3 Realization of Transfinite Orders of Accumulation

Recall that for every countable ordinal α, $\omega^\alpha + 1$ is a countable, compact, Polish space. Then let K_α be the (unique up to affine homeomorphism) Bauer simplex with $\text{ex}(K_\alpha) = \omega^\alpha + 1$. For notation, let $\mathbf{0}_\alpha$ be the point ω^α in K_α, and let $E_\alpha = \text{ex}(K_\alpha)$. In this section we construct, for each countable α, a harmonic, u.s.c.d. candidate sequence $\mathcal{H}_\alpha$ on K_α such that $\alpha_0(\mathcal{H}_\alpha) = \alpha$.

The idea of the following theorem is to construct, for each countable, irreducible ordinal α, a candidate sequence $\mathcal{H}$ such that the transfinite sequence does not converge uniformly at α, in some sense. The main tools of the proof are the disjoint union candidate sequence and the powers candidate sequences.

Theorem 2.3.1. *For all real numbers $0 < \epsilon < a$, and for all countable, irreducible ordinals δ and α, with $\delta < \alpha$, there exists a harmonic, u.s.c.d candidate sequence $\mathcal{H}_\alpha$ on K_α such that*

(1) $||h|| \leq a$ if α is finite, and $||h|| \leq \epsilon$ if α is infinite;

(2) $||u_\delta|| \leq \epsilon$;

(3) $||u_\gamma|| \leq a$ for all γ, and $||u_\gamma|| < a$ for $\gamma < \alpha$;

(4) $h(\mathbf{0}_\alpha) = 0$, $u_\gamma(\mathbf{0}_\alpha) = 0$, *for all* $\gamma < \alpha$, *and* $u_\alpha(\mathbf{0}_\alpha) = a$*;*

(5) $\alpha_0(\mathcal{H}_\alpha) = \alpha$.

Proof. Suppose that we have constructed an u.s.c.d. candidate sequence $\mathcal{H}'$ on $\omega^\alpha+1$ and shown that it possesses properties (1)-(5). Since K_α is Bauer, Proposition 2.2.27 implies that we can let $\mathcal{H}_\alpha$ be the harmonic extension of $\mathcal{H}'$ to K_α and properties (1)-(5) carry over exactly. So without loss of generality, we will define $\mathcal{H}_\alpha$ directly on E_α and work exclusively on E_α.

The rest of the proof proceeds by transfinite induction on the non-zero irreducible ordinals α (α is non-zero because $\delta < \alpha$). This is equivalent, by Proposition 2.1.3, to writing $\alpha = \omega^\beta$ and using transfinite induction on β. The base case is when $\beta = 0$.

Case ($\beta = 0$). In this case $E_{\omega^0} = E_1 = \omega + 1$, the one-point compactification of the natural numbers. Now δ must be 0 and by definition $u_0 \equiv 0$. Let $\mathcal{H} = (h_k)$, where $h_k(n) = 0$ if $k \leq n$, $h_k(n) = a$ if $k > n$, and $h_k(\mathbf{0}_1) = 0$. Then $h \leq a$. Since each n is isolated in E_1, $r(n) = 0$, which implies that $\alpha_0(n) = 0$ and $u_\gamma(n) = 0$ for all γ (by Proposition 2.2.6). The point at infinity, $\mathbf{0}_1$, has topological order of accumulation 1, which implies that $\alpha_0(\mathbf{0}_1) \leq 1$ (by Proposition 2.2.6). It only remains to check that $u_1(\mathbf{0}_1) = a$. Fix k. For any $n > k$, $\tau_k(n) = h(n) - h_k(n) = a$. Thus $\widetilde{\tau}_k(\mathbf{0}_1) \geq a$. Letting k go to infinity gives that $u_1(\mathbf{0}_1) \geq a$. Since $u_1 \leq \widetilde{h} \leq a$, we obtain that $u_1(\mathbf{0}_1) = a$, as desired.

Case (β implies $\beta + 1$). We assume the statement is true for ω^β, and we need to show that it is true for $\omega^{\beta+1} = \sup_n \omega^\beta n$. In this case $E_{\omega^{\beta+1}}$ is homeomorphic to the one-point compactification of the disjoint union of the spaces $(E_{\omega^\beta n})$ (by Theorem 2.1.12). With this homeomorphism, we may assume without loss of generality that $E_{\omega^{\beta+1}}$ is the one-point compactification of the disjoint union of the spaces $E_{\omega^\beta n}$. Fix $0 < \epsilon < a$, and let $\{a_p\}$ be a sequence of positive real numbers such that $a_p < a$ for all p and $\lim_p a_p = a$. Using the induction hypothesis, for each p, we choose a u.s.c.d. candidate sequence $\mathcal{H}_{\omega^\beta}$ on E_{ω^β} which satisfies conditions (1)-(5) with parameters a_p, ϵ, and $\delta < \omega^\beta$. For each p, let $\mathcal{H}^p_{\omega^\beta}$ be the p-power sequence of this $\mathcal{H}_{\omega^\beta}$ restricted to $E_{\omega^\beta p}$ (note that $\omega^{\omega^\beta p} + 1 \subset (\omega^{\omega^\beta} + 1)^p$). Then $||\lim(\mathcal{H}^p_{\omega^\beta})|| \leq \frac{a}{p}$, and $||u^{\mathcal{H}^p_{\omega^\beta}}_{\omega^\beta}|| \leq \frac{a_p}{p}$. Let N be such that $\frac{a}{N} \leq \epsilon$, and define $\mathcal{H}_{\omega^{\beta+1}} = \coprod_{n \geq N} \mathcal{H}^n_{\omega^\beta}$. It remains to check (1)-(5) for $\mathcal{H}_{\omega^{\beta+1}}$.

(1) Using that $h(\mathbf{0}_{\omega^{\beta+1}}) = 0$,

$$||h|| = \sup_{n \geq N} ||\lim \mathcal{H}^n_{\omega^\beta}|| = ||\lim \mathcal{H}^N_{\omega^\beta}|| \leq \frac{a}{N} \leq \epsilon < a.$$

(2) For irreducible $\delta < \omega^{\beta+1}$, we have $\delta \leq \omega^\beta$. Monotonicity of the transfinite sequence implies

$$||u^{\mathcal{H}^n_{\omega^\beta}}_{\delta}|| \leq ||u^{\mathcal{H}^n_{\omega^\beta}}_{\omega^\beta}||,$$

for every n. Also, Lemma 2.2.12 implies

$$||u_\delta|| = \sup_{n \geq N} ||u^{\mathcal{H}^n_{\omega^\beta}}_{\delta}||.$$

Putting these inequalities together gives

$$||u_\delta|| = \sup_{n \geq N} ||u^{\mathcal{H}^n_{\omega^\beta}}_{\delta}|| \leq \sup_{n \geq N} ||u^{\mathcal{H}^n_{\omega^\beta}}_{\omega^\beta}|| \leq \frac{a}{N} \leq \epsilon.$$

(3) For every γ, Lemma 2.2.12 and Lemma 2.2.14 (1) imply

$$||u_\gamma|| = \sup_{n\geq N} ||u_\gamma^{\mathcal{H}^n_{\omega^\beta}}|| \leq a.$$

Further, for any $\gamma < \alpha$, there exists m such that $\gamma < \omega^\beta m$. Using subadditivity (Lemma 2.2.2), $||u_\gamma^{\mathcal{H}^n_{\omega^\beta}}|| \leq ||u_{\omega^\beta m}^{\mathcal{H}^n_{\omega^\beta}}|| \leq \frac{m}{n} a_n$. Then

$$||u_\gamma|| = \sup_{n\geq N} ||u_\gamma^{\mathcal{H}^n_{\omega^\beta}}|| \leq \max\Big(a_1, \ldots, a_m, \sup_{n>m} \frac{m}{n} a_n\Big) < a.$$

(4) By definition, $h(\mathbf{0}_{\omega^{\beta+1}}) = 0$. Let $\gamma < \alpha$. There exists a k such that $\gamma < \omega^\beta k$. Then Lemma 2.2.12, monotonicity, and Lemma 2.2.14 imply

$$u_\gamma(\mathbf{0}_{\omega^{\beta+1}}) \leq \limsup_{n\to\infty} ||u_\gamma^{\mathcal{H}^n_{\omega^\beta}}|| \leq \limsup_{n\to\infty} ||u_{\omega^\beta k}^{\mathcal{H}^n_{\omega^\beta}}|| \leq \limsup_{n\to\infty} \frac{ka}{n} = 0.$$

Also, Lemma 2.2.12 and Lemma 2.2.14 imply

$$u_\alpha(\mathbf{0}_{\omega^{\beta+1}}) \geq \limsup_{n\to\infty} u_\alpha^{\mathcal{H}^n_{\omega^\beta}}(\mathbf{0}_{\omega^\beta n}) = a,$$

which (combining with (3)) implies that $u_\alpha(\mathbf{0}_{\omega^{\beta+1}}) = a$.

(5) For $x \neq \mathbf{0}_{\omega^{\beta+1}}$, there exists n such that $x \in E_{\omega^\beta n}$, which implies that $r(x) \leq \omega^\beta n$. Then Proposition 2.2.6 gives that $\alpha_0(x) \leq \omega^\beta n + 1 < \omega^{\beta+1}$. The fact that $\alpha_0(\mathbf{0}_{\omega^{\beta+1}}) = \omega^{\beta+1}$ then follows immediately from (3) and (4). Thus $\alpha_0(\mathcal{H}) = \omega^{\beta+1}$.

Case (β limit ordinal). We assume the statement is true for all ω^ξ with $\xi < \beta$, and we need to show that it is true for ω^β. In this case there is a strictly increasing sequence of irreducible ordinals (ω^{β_n}) with $\sup_n \omega^{\beta_n} = \omega^\beta$, and E_{ω^β} is homeomorphic to the one-point compactification of the disjoint union of the $E_{\omega^{\beta_n}}$ (by Remark

2.1.13). With this homeomorphism, we may assume without loss of generality that E_{ω^β} is the one-point compactification of the disjoint union of the spaces $E_{\omega^{\beta_n}}$. Fix $0 < \epsilon < a$, and let $\{a_n\}$ be a sequence of positive real numbers with $a_n < a$ for all n and $\lim_n a_n = a$. By the induction hypothesis, for each $n > 1$, there exists a u.s.c.d. candidate sequence $\mathcal{H}_{\omega^{\beta_n}}$ on $E_{\omega^{\beta_n}}$ satisfying (1)-(5) with parameters a_n, $\frac{\epsilon}{n}$, ω^{β_n} and $\delta_n = \omega^{\beta_{n-1}}$. Now fix δ irreducible with $\delta < \omega^\beta$. Since $\sup_n \omega^{\beta_n} = \omega^\beta$, there exists N such that $\omega^{\beta_{N-1}} > \delta$. Let $\mathcal{H}_{\omega^\beta} = \coprod_{n\geq N} \mathcal{H}_{\omega^{\beta_n}}$. All that remains is to verify (1)-(5).

(1) Using that $h(\mathbf{0}_{\omega^\beta}) = 0$, we get

$$||h|| = \sup_{n\geq N} ||\lim \mathcal{H}_{\omega^{\beta_n}}|| \leq \frac{\epsilon}{N} \leq \epsilon.$$

(2) Since $\delta < \omega^{\beta_{N-1}}$, Lemma 2.2.12 and monotonicity imply (as in the previous case)

$$||u_\delta|| \leq \sup_{n\geq N} ||u_\delta^{\mathcal{H}_{\omega^{\beta_n}}}|| \leq \sup_{n\geq N} ||u_{\omega^{\beta_{n-1}}}^{\mathcal{H}_{\omega^{\beta_n}}}|| \leq \sup_{n\geq N} \frac{\epsilon}{n} \leq \epsilon.$$

(3) For any γ, by construction,

$$||u_\gamma|| \leq \sup_{n\geq N} ||u_\gamma^{\mathcal{H}_{\omega^{\beta_n}}}|| \leq a.$$

Further, for $\gamma < \alpha$, there exists m such that $\gamma < \omega^{\beta_m}$. For $n > m$, $||u_\gamma^{\mathcal{H}_{\omega^{\beta_n}}}|| \leq \frac{\epsilon}{n}$. Then

$$||u_\gamma|| \leq \sup_{n\geq N} ||u_\gamma^{\mathcal{H}_{\omega^{\beta_n}}}|| \leq \max\Big(a_1, \ldots, a_m, \sup_{n>m} \frac{\epsilon}{n}\Big) < a.$$

(4) By definition, $h(\mathbf{0}_{\omega^\beta}) = 0$. For any $\gamma < \omega^\beta$, there exists some k such that for all

$n \geq k$, $\omega^{\beta_n} > \gamma$. Then

$$u_\gamma(\mathbf{0}_{\omega^\beta}) \leq \limsup_{n\to\infty} ||u_\gamma^{\mathcal{H}_{\omega^{\beta_n}}}|| \leq \limsup_{n\to\infty} ||u_{\omega^{\beta_{n-1}}}^{\mathcal{H}_{\omega^{\beta_n}}}|| \leq \limsup_{n\to\infty} \frac{\epsilon}{n} = 0.$$

(5) For any $x \neq \mathbf{0}_\alpha$, there exists n such that $x \in E_{\omega^{\beta_n}}$. Then $\alpha_0(x) \leq r(x) \leq \omega^{\beta_n} < \omega^\beta$. By (3) and (4), $\alpha_0(\mathbf{0}_{\omega^\beta}) = \omega^\beta$. Therefore $\alpha_0(\mathcal{H}) = \omega^\beta$.

□

Corollary 2.3.2. *For all positive real numbers a and non-zero countable ordinals α, there exists a harmonic, u.s.c.d. candidate sequence $\mathcal{H}$ on K_α such that the transfinite sequence corresponding to either $\mathcal{H}$ or $\mathcal{H}|_{\text{ex}(K_\alpha)}$ satisfies*

(1) $||u_\gamma|| \leq a$ for all γ, and $||u_\gamma|| < a$ for all $\gamma < \alpha$;

(2) $h(\mathbf{0}_\alpha) = 0$, and $u_\alpha(\mathbf{0}_\alpha) = a$;

(3) $\alpha_0(\mathcal{H}) = \alpha_0(\mathcal{H}|_{\text{ex}(K_\alpha)}) = \alpha$.

Proof. Let α be a non-zero countable ordinal, and suppose the Cantor Normal Form of α (as in Theorem 2.1.2) is given by

$$\alpha = \alpha_1 m_1 + \cdots + \alpha_N m_N.$$

Let $a_1 > \cdots > a_N > 0$ be real numbers such that $\sum a_j = a$ and for each $j = 1, \ldots, N-1$,

$$\frac{a_j}{m_j} \geq \sum_{i=j+1}^{N} a_i.$$

For each $j = 1, \ldots, N$, let $\mathcal{F}_j$ be a harmonic, u.s.c.d. candidate sequence given by Theorem 2.3.1 with parameters a_j and α_j. Define $\mathcal{H}_j$ to be the product sequence

$\mathcal{F}_j^{m_j}$ restricted to $K_{\alpha_j m_j}$, and let $\mathcal{H} = \mathcal{H}_N \times \cdots \times \mathcal{H}_1$ restricted to K_α. By definition of $\mathcal{H}$, $h(\mathbf{0}_\alpha) = 0$. The rest of properties (1)-(3) follow from Lemma 2.2.16.

□

Corollary 2.3.3. *Let $a > 0$, and let α be a countable, infinite ordinal. Then there is a harmonic, u.s.c.d. candidate sequence $\mathcal{H}$ on K_α such that the transfinite sequence corresponding to either $\mathcal{H}$ or $\mathcal{H}|_{\mathrm{ex}(K_\alpha)}$ satisfies*

(1) $||u_\gamma|| \leq a$ for all γ, and $||u_\gamma|| < a$ for $\gamma < \alpha + 1$;

(2) $h(\mathbf{0}_\alpha) = 0$ and $u_{\alpha+1}(\mathbf{0}_\alpha) = a$;

(3) $\alpha_0(\mathcal{H}) = \alpha_0(\mathcal{H}|_{\mathrm{ex}(K_\alpha)}) = \alpha + 1$.

Proof. Using Proposition 2.2.27, we may deal exclusively with u.s.c.d. candidate sequences on E_α (as opposed to K_α), and all properties will carry over to K_α.

The proof is executed in two stages. First we prove the statement for the countably infinite, irreducible ordinals. In the second stage, we prove the statement for all countable, infinite ordinals.

Stage 1. Let α be a countably infinite, irreducible ordinal. Let $\alpha = \omega^\beta$ (since α is infinite, $\beta > 0$). and let $b = \frac{2}{3}a$. Let $\mathcal{F}$ be given by Theorem 2.3.1 with parameters b, α, ϵ, and δ. Recall from the proof of Theorem 2.3.1 that we may take $\mathcal{F} = \sqcup \mathcal{F}_n$, where the exact form of the $\mathcal{F}_n$ is as follows. Let $\{a_n\}$ be a sequence of positive real numbers with $a_n < b$ for all n and $\lim_n a_n = b$. If β is a successor, then we may take $\mathcal{F}_n = \mathcal{G}^n$, where $\mathcal{G}$ satisfies the conclusions of Theorem 2.3.1 with parameters a_n, ϵ, $\omega^{\beta-1}$, and δ. Otherwise, if β is a limit with β_n increasing to β, then $\mathcal{F}_n$ satisfies the conclusions of Theorem 2.3.1 with parameters a_n, ϵ, ω^{β_n}, and δ. Let $\mathcal{F} = (f_k)$, and let $\mathbf{0}_n$ denote the marked point in E_{β_n} (so E_{β_n} is the domain of $\mathcal{F}_n$). Let $\mathcal{H} = (h_k)$ be defined by the rule

$$h_k(x) = \begin{cases} f_k(x) & \text{if } x \neq \mathbf{0}_n \\ 0 & \text{if } x = \mathbf{0}_n,\ k \leq n \\ \frac{b}{2} & \text{if } x = \mathbf{0}_n,\ k > n. \end{cases}$$

By definition, let $h_k(\mathbf{0}_\alpha) = 0$. Note that $\mathcal{H}$ is again an u.s.c.d. sequence on E_α, and $u_\gamma^{\mathcal{H}}(x) = u_\gamma^{\mathcal{F}}(x)$ for all γ and all $x \neq \mathbf{0}_\alpha$. It follows that $u_\gamma^{\mathcal{H}}(x) \leq b$ for all γ and all $x \neq \mathbf{0}_\alpha$. Computing the transfinite sequence at $\mathbf{0}_\alpha$, we see that

$$u_\ell^{\mathcal{H}}(\mathbf{0}_\alpha) = \frac{b}{2},\ \text{for } 1 \leq \ell < \alpha$$

$$u_\alpha^{\mathcal{H}}(\mathbf{0}_\alpha) = b$$

$$u_{\alpha+1}^{\mathcal{H}}(\mathbf{0}_\alpha) = b + \frac{b}{2} = a.$$

Since $\alpha_0(\mathbf{0}_\alpha) \leq r(\mathbf{0}_\alpha) + 1 = \alpha + 1$, we conclude that $\alpha_0(\mathbf{0}_\alpha) = \alpha + 1$. Thus we obtain properties (1)-(3).

Stage 2. Let $\alpha = \omega^{\beta_1} m_1 + \ldots \omega^{\beta_N} m_N$ be the Cantor Normal Form of α.

The construction proceeds by cases. In the first case, suppose ω^{β_N} is infinite. Let $a > 0$, and select $a_1 > \cdots > a_N$ as in Lemma 2.2.16. Let $\mathcal{F}_j$ be given by Lemma 2.3.1 with parameters a_j and ω^{β_j}, for $j = 1, \ldots, N$. Let $\mathcal{F}'_N$ be given by Stage 1 corresponding to $\frac{a_N}{m_N}$ and ω^{β_N}. For $j = 1, \ldots, N-1$, let $\mathcal{H}_j = \mathcal{F}_j^{m_j}$, and for $j = N$, if $m_N > 1$, let $\mathcal{H}_j = \mathcal{F}_N^{m_N - 1}$. Now let $\mathcal{H}'$ be given by the product (where $\mathcal{H}_N$ is omitted if $m_N = 1$)

$$\mathcal{H}' = \mathcal{F}'_N \times (\mathcal{H}_N) \times \cdots \times (\mathcal{H}_1),$$

Let $\mathcal{H}$ be the restriction of $\mathcal{H}'$ to $E_{\omega^\alpha+1}$. Note that $h(\mathbf{0}_\alpha) = 0$. Then using Lemmas 2.2.14 and 2.2.16, we conclude that

$$\alpha_0(\mathcal{H}) = \big(\sum_{i=1}^{N-1} \omega^{\beta_i} m_i\big) + \omega^{\beta_N}(m_N - 1) + (\omega^{\beta_N} + 1) = \alpha + 1.$$

For the second case, we suppose that ω^{β_N} is finite, which implies that $\omega^{\beta_N} = 1$. Let $a > 0$, and select $a_1 > \cdots > a_N$ as in Lemma 2.2.16, with the additional condition that $\frac{a_{N-1}}{3m_{N-1}} \geq a_N$. Let $\mathcal{F}_j$ be given by Lemma 2.3.1 with parameters a_j and ω^{β_j}, for $j = 1, \ldots, N$. Since α is infinite, it follows that $\omega^{\beta_{N-1}}$ is infinite. Let $\mathcal{F}'_{N-1}$ be given by Stage 1 corresponding to $\frac{a_{N-1}}{m_{N-1}}$ and $\omega^{\beta_{N-1}}$ (so that the condition $\frac{a_{N-1}}{3m_{N-1}} \geq a_N$ implies $b/2 \geq a_N$ in the notation of Stage 1). For $j \in \{1, \ldots, N-2, N\}$, let $\mathcal{H}_j = \mathcal{F}_j^{m_j}$. If $m_{N-1} > 1$, let $\mathcal{H}_{N-1} = \mathcal{F}_{N-1}^{m_{N-1}-1}$. Now let $\mathcal{H}'$ be given by the product (where $\mathcal{H}_{N-1}$ is omitted if $m_{N-1} = 1$):

$$\mathcal{H}' = (\mathcal{H}_N) \times \mathcal{F}'_{N-1} \times (\mathcal{H}_{N-1}) \times \cdots \times (\mathcal{H}_1),$$

Let $\mathcal{H}$ be the restriction of $\mathcal{H}'$ to $E_{\omega^\alpha+1}$. Note that $h(\mathbf{0}_\alpha) = 0$. Then the reader may easily adapt the proofs of Lemmas 2.2.14 and 2.2.16 with the additional assumption that $\frac{a_{N-1}}{3m_{N-1}} \geq a_N$ to check that

$$||u^{\mathcal{H}}_{\omega^{\beta_1}m_1+\cdots+\omega^{\beta_{N-1}}m_{N-1}}|| = \sum_{i=1}^{N-2} a_i + \big(\frac{a_{N-1}}{m_{N-1}}(m_{N-1}-1)\big) + \frac{a_{N-1}}{m_{N-1}}(\frac{2}{3})$$
$$||u^{\mathcal{H}}_{\omega^{\beta_1}m_1+\cdots+\omega^{\beta_{N-1}}m_{N-1}+1}|| = \sum_{i=1}^{N-1} a_i$$
$$||u^{\mathcal{H}}_{\omega^{\beta_1}m_1+\cdots+\omega^{\beta_{N-1}}m_{N-1}+1+k}|| = \sum_{i=1}^{N-1} a_i + \frac{a_N}{m_N}k, \text{ for } k = 1, \ldots, m_N,$$

and,

$$\alpha_0(\mathcal{H}) = \big(\sum_{i=1}^{N-2} \omega^{\beta_i} m_i\big) + \omega^{\beta_{N-1}}(m_{N-1} - 1) + (\omega^{\beta_{N-1}} + 1) + m_N = \alpha + 1.$$

□

Remark 2.3.4. In Corollaries 2.3.2 and 2.3.3, one may further require that $\mathcal{H}|_{\mathrm{ex}(K_\alpha)}$ has the following property (P): for any t in $\mathrm{ex}(K_\alpha)$, for any sequence $\{s_n\}$ of isolated points in $\mathrm{ex}(K_\alpha)$ that converges to t, $\limsup_n \tau_k(s_n) = \lim_n \tau_k(s_n)$. Let us prove this fact. In the case $\alpha = 1$, there is only one sequence of isolated points in $\mathrm{ex}(K_1) \cong \omega + 1$, and the candidate sequence $\mathcal{F}$ constructed in the proof of Theorem 2.3.1 satisfies (P). Then we note that if each of the candidate sequences $\mathcal{F}_1, \ldots, \mathcal{F}_N$ satisfies this property, then so does the product $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_N$. To see this fact, note that the projection π_N onto the last coordinate of any isolated point x in the product space is not the marked point $\mathbf{0}_N$, and thus $\mathcal{F}(x) = \mathcal{F}_N(\pi_N(x))$. Hence the product candidate sequence satisfies property (P) because $\mathcal{F}_N$ does. Now suppose there is a sequence $(\mathcal{F}_n)_n$ of candidate sequences such that each $\mathcal{F}_n$ satisfies (P). Let $h^n = \lim \mathcal{F}_n$ and let I_n be the set of isolated points in the domain of $\mathcal{F}_n$. Further suppose that $h^n|_{I_n}$ converges uniformly to 0. Then $\coprod_n \mathcal{F}_n$ satisfies (P) as well (to see this, note that property (P) is satisfied on the domain of each candidate sequence $\mathcal{F}_n$ separately because $\mathcal{F}_n$ has property (P), and then it is satisfied at the point at infinity because $h^n|_{I_n}$ converges uniformly to 0). The constructions used in the proofs of Theorem 2.3.1, Corollary 2.3.2 and Corollary 2.3.3 only rely on these three types of constructions ($\alpha = 1$, product sequences, and disjoint union sequences with $h^n|_{I_n}$ tending uniformly to 0), and thus at each step we may choose candidate sequences satisfying (P). Making these choices yields $\mathcal{H}|_{\mathrm{ex}(K_\alpha)}$ with the desired property.

We conclude this section by stating these results in the language of dynamical systems. The following corollary follows from Corollary 2.3.2 by appealing to the Downarowicz-Serafin realization theorem (Theorem A.1.1).

Corollary 2.3.5. *For every countable ordinal α, there is a minimal homeomorphism T of the Cantor set such that α is the order of accumulation of entropy of T.*

2.4 Characterization of Orders of Accumulation on Bauer Simplices

Definition 2.4.1. For any non-empty countable Polish space E, we define

$$\rho(E) = \begin{cases} |E|_{CB} - 1, & \text{if } |E|_{CB} \text{ is finite} \\ |E|_{CB}, & \text{if } |E|_{CB} \text{ is infinite} \end{cases}$$

For any uncountable Polish space E, we let $\rho(E) = \omega_1$, the first uncountable ordinal.

Definition 2.4.2. For any metrizable Choquet simplex K, we define

$$S(K) = \{\gamma : \text{ there exists a harmonic, u.s.c.d sequence } \mathcal{H} \text{ on } K \text{ with } \alpha_0(\mathcal{H}) = \gamma\}.$$

Recall our conventions that if $\beta < \omega_1$, then $[\alpha, \beta]$ denotes the ordinal interval $\{\gamma : \alpha \leq \gamma \leq \beta\}$, and if $\beta = \omega_1$, then $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma < \beta\}$. We also require the use of "open" or "half-open" intervals, which have the usual definitions.

Theorem 2.4.3. *Let K be a Bauer simplex. Then*

$$S(K) = [0, \rho(\text{ex}(K))].$$

Proof. Let $\mathcal{H}$ be a harmonic, u.s.c.d. candidate sequence on K. Proposition 2.2.6 implies that

$$\alpha_0(\mathcal{H}|_{\text{ex}(K)}) \leq \rho(\text{ex}(K)).$$

and it is always true that $\alpha_0(\mathcal{H}|_{\text{ex}(K)}) < \omega_1$. Then since K is Bauer, Proposition 2.2.27 implies the same bounds for $\alpha_0(\mathcal{H})$. It remains to show that if $\text{ex}(K)$ is countable, then $S(K) \supset [0, \rho(\text{ex}(K))]$, and if $\text{ex}(K)$ is uncountable, then $S(K) \supset [0, \omega_1[$.

Suppose $E = \text{ex}(K)$ is countable. Let $\alpha < |E|_{CB}$. Then by Proposition 2.1.11, there exists x in E such that $r(x) = \alpha$, which implies that x is isolated in $\Gamma^\alpha(E)$. Let U be a clopen neighborhood of x in E such that $U \cap (\Gamma^\alpha(E) \setminus \{x\}) = \emptyset$. Then $|U|_{CB} = \alpha + 1$ and $|\Gamma^\alpha(U)| = 1$. Then by the classification of countable, compact Polish spaces (Theorem 2.1.12), there is a homeomorphism $g : \omega^\alpha + 1 \to U$. Let $\mathcal{H}'$ be the u.s.c.d candidate sequence on $\omega^\alpha + 1$ given by Corollary 2.3.2 with $\alpha_0(\mathcal{H}') = \alpha$. Define $\mathcal{H}$ on K to be harmonic extension of the embedded candidate sequence $g\mathcal{H}'$, which is harmonic and u.s.c.d by Lemma 2.2.29. Since $\mathcal{H}|_{E \setminus g(\omega^\alpha+1)} \equiv 0$, the Embedding Lemma (Lemma 2.2.30) applies. Since $g(\omega^\alpha + 1)$ is a compact subset of $\text{ex}(K)$, we obtain that $\alpha_0(\mathcal{H}) = \alpha_0(\mathcal{H}') = \alpha$. Since $\alpha < |\,\text{ex}(K)|_{CB}$ was arbitrary, this argument shows that $S(K) \supset [0, |\,\text{ex}(K)|_{CB} - 1]$ (note that since K is Bauer, $\text{ex}(K)$ is compact and $|\,\text{ex}(K)|_{CB}$ is a successor). If $|\,\text{ex}(K)|_{CB}$ is infinite, then let $\alpha = |\,\text{ex}(K)|_{CB} - 1$ and repeat the above argument with $\mathcal{H}'$ given by Corollary 2.3.3 so that $\alpha_0(\mathcal{H}) = \alpha + 1$. In this case we obtain that $S(K) \supset [0, |\,\text{ex}(K)|_{CB}]$. In any case, we conclude that $S(K) \supset [0, \rho(\text{ex}(K))]$, as desired.

Now suppose $E = \text{ex}(K)$ is uncountable. Fix $\alpha < \omega_1$. Let $g : \omega^\alpha + 1 \to E$ be the embedding given by Proposition 2.1.8, and let $\mathcal{H}_\alpha$ be the u.s.c.d. candidate sequence on $\omega^\alpha + 1$ given by Corollary 2.3.2. Then the harmonic extension $\mathcal{H}$ of the embedded candidate sequence $g\mathcal{H}_\alpha$ on K is harmonic and u.s.c.d. by Lemma 2.2.29.

Furthermore, $\mathcal{H}$ satisfies $\alpha_0(\mathcal{H}) = \alpha_0(\mathcal{H}_\alpha) = \alpha$, by the Embedding Lemma (as $g(\omega^\alpha+1)$ is a compact subset of $\mathrm{ex}(K)$). Since $\alpha < \omega_1$ was arbitrary, $S(K) \supset [0, \omega_1[$.

□

2.5 Orders of Accumulation on Choquet Simplices

In this section we address the extent to which the orders of accumulation that appear on a metrizable Choquet simplex K are constrained by the topological properties of the pair $(\mathrm{ex}(K), \overline{\mathrm{ex}(K)})$.

We will require a relative version of Cantor-Bendixson rank, whose definition we give here.

Definition 2.5.1. Given a Polish space X contained in the Polish space T, we define the sequence $\{\Gamma_X^\alpha(T)\}$ of subsets of T using transfinite induction. Let $\Gamma_X^0(T) = T$. If $\Gamma_X^\alpha(T)$ has been defined, then let $\Gamma_X^{\alpha+1}(T) = \{t \in T : \exists (t_n) \in \Gamma_X^\alpha(T) \cap X \setminus \{t\} \text{ with } t_n \to t\}$. If $\Gamma_X^\beta(T)$ has been defined for all $\beta < \alpha$, where α is a limit ordinal, then we let $\Gamma_X^\alpha(T) = \cap_{\beta<\alpha} \Gamma_X^\beta(T)$.

Note that $\Gamma_X^\alpha(T)$ is closed in T for all α, and $\Gamma_X^\alpha(T) \subset \Gamma_X^\beta(T)$ for $\alpha > \beta$. For X and T Polish, there exists a countable ordinal β such that $\Gamma_X^\alpha(T) = \Gamma_X^\beta(T)$ for all $\alpha > \beta$.

Definition 2.5.2. The **Cantor-Bendixson rank of T relative to X**, denoted $|T|_{CB}^X$, is the least ordinal β such that $\Gamma_X^\alpha(T) = \Gamma_X^\beta(T)$ for all $\alpha > \beta$.

If X is countable, then $\Gamma_X^\alpha(T) = \emptyset$ if and only if $\alpha \geq |T|_{CB}^X$. If X is countable and T is compact, then by the finite intersection property, $|T|^X_{CB}$ is a successor ordinal.

Definition 2.5.3. For t in T, we also define the pointwise relative topological rank $r_X(t)$ of t with respect to X:

$$r_X(t) = \begin{cases} \sup\{\alpha : t \in \Gamma^\alpha_X(T)\} & \text{if } t \notin \Gamma_X^{|T|^X_{CB}}(T) \\ \omega_1 & \text{if } t \in \Gamma_X^{|T|^X_{CB}}(E). \end{cases}$$

It follows that for X countable, for all t in T, $r_X(t) \leq |X|_{CB}$, and thus $|T|^X_{CB} \leq |X|_{CB} + 1$. Also, $|X|_{CB} \leq |T|^X_{CB} \leq |T|_{CB}$.

For a Polish space T, the usual Cantor-Bendixson rank is obtained from the relative version by taking $X = T$ in the above construction. Thus, we have $|T|^T_{CB} = |T|_{CB}$.

2.5.1 Results for Choquet Simplices

Definition 2.5.4. Let X and T be non-empty Polish spaces, with $X \subset T$. If X is countable, let

$$\rho_X(T) = \begin{cases} |T|^X_{CB} - 1, & \text{if } |T|^X_{CB} \text{ is finite} \\ |T|^X_{CB}, & \text{if } |T|^X_{CB} \text{ is infinite} \end{cases}$$

If X is uncountable, let $\rho_X(T) = \omega_1$.

Now we present bounds on the set $S(K)$ (see Definition 2.4.2) for any metrizable Choquet simplex K. Recall our convention that for a countable ordinal β, $[0, \beta] = \{\alpha : 0 \leq \alpha \leq \beta\}$, but for $\beta = \omega_1$, $[0, \beta] = \{\alpha : 0 \leq \alpha < \omega_1\} = [0, \omega_1[$.

Theorem 2.5.5. *Let K be a metrizable Choquet simplex. Then*

$$[0, \rho_{\text{ex}(K)}(\overline{\text{ex}(K)})\,] \subset S(K) \subset [0, \rho(\overline{\text{ex}(K)})\,].$$

Proof. First we prove the lower bound on $S(K)$.

Suppose $\text{ex}(K)$ is uncountable, then by Corollary 2.1.8, for any countable α, there exists a map $g : \omega^\alpha + 1 \to \text{ex}(K)$, where g is a homeomorphism onto its image. Let $\mathcal{F}$ be a u.s.c.d. sequence on $\omega^\alpha + 1$, and let $\mathcal{H}$ be the harmonic extension of the embedded sequence $g\mathcal{F}$ on K. $\mathcal{H}$ is a harmonic, u.s.c.d. candidate sequence on K by Lemma 2.2.29. Also, $\mathcal{H}|_{\text{ex}(K)\setminus g(\omega^\alpha+1)} \equiv 0$. Thus the Embedding Lemma (Lemma 2.2.30) applies, and then since $g(\omega^\alpha + 1)$ is a compact subset of $\text{ex}(K)$, we obtain that $\alpha_0(\mathcal{H}) = \alpha_0(\mathcal{F})$. Letting $\mathcal{F}$ vary over all u.s.c.d. candidate sequences on $\omega^\alpha + 1$, it follows that $S(\mathcal{M}(\omega^\alpha + 1)) \subseteq S(K)$. By Theorem 2.4.3, $S(\mathcal{M}(\omega^\alpha+1)) = [0, \rho(\omega^\alpha+1)]$. Now $\rho(\omega^\alpha+1) = \alpha$ if α is finite and $\rho(\omega^\alpha+1) = \alpha+1$ if α is infinite. In either case, $\rho(\omega^\alpha+1) \geq \alpha$. Hence $S(K) \supset [0, \alpha]$. Since this inclusion holds for any countable ordinal α, we have that $S(K) \supset [0, \omega_1[$, as desired.

If $\text{ex}(K)$ is countable, then $|\overline{\text{ex}(K)}|_{CB}^{\text{ex}(K)}$ is a successor ordinal. For each ordinal $\alpha < |\overline{\text{ex}(K)}|_{CB}^{\text{ex}(K)}$, we have $\Gamma^\alpha_{\text{ex}(K)}(\overline{\text{ex}(K)}) \neq \emptyset$. Fix $\alpha < |\overline{\text{ex}(K)}|_{CB}^{\text{ex}(K)}$, and let t be in $\Gamma^\alpha_{\text{ex}(K)}(\overline{\text{ex}(K)})$. Since t lies in $\Gamma^\alpha_{\text{ex}(K)}(\overline{\text{ex}(K)})$, there exists a map $g : \omega^\alpha + 1 \to K$, where g is a homeomorphism onto its image, $g(\omega^\alpha+1) \subset \text{ex}(K) \cup \{t\}$ and $g(\mathbf{0}_\alpha) = t$, where $\mathbf{0}_\alpha$ is the point ω^α in $\omega^\alpha + 1$. Given some real number $a > 0$, let $\mathcal{F} = (f_k)$ be a u.s.c.d. candidate sequence on $\omega^\alpha + 1$ with $\alpha_0(\mathcal{F}) = \alpha$ and satisfying (1)-(3) of Corollary 2.3.2. Recall that $f_k(\mathbf{0}_\alpha) = 0$ for all k. Then let $\mathcal{H}' = (h'_k)$ be the embedded candidate sequence $g\mathcal{F}$ on K.

Note that for s in $K \setminus \text{ex}(K)$, $(h'_{k+1} - h'_k)(s) = 0$. Also, for s in $\text{ex}(K)$, $(h'_{k+1} - h'_k)(s) \geq 0$. It follows that $h'_{k+1} - h'_k$ is convex on K.

Now let $\mathcal{H} = (h_k)$, where h_k is the harmonic extension of h'_k on K. By Lemma 2.2.29, $\mathcal{H}$ is a u.s.c.d. candidate sequence on K.

Let $F = g(\omega^\alpha + 1) \cap \text{ex}(K)$, and note that $\mathcal{H}|_{\text{ex}(K)\setminus F} \equiv 0$. Also $\overline{F} = g(\omega^\alpha + 1)$ and $\mathcal{H}|_{\overline{F}} = \mathcal{F} \circ g^{-1}$. Applying the Embedding Lemma (Lemma 2.2.30), we obtain that $\alpha_0(\mathcal{H}) \leq \alpha_0(\mathcal{H}|_{\overline{F}}) = \alpha_0(\mathcal{F}) = \alpha$. We now show the reverse inequality. Recall that $t = g(\mathbf{0}_\alpha)$. For $\gamma < \alpha$, the Embedding Lemma (Lemma 2.2.30) implies that $u^{\mathcal{H}}_\gamma(t) \leq ||u^{\mathcal{F}}_\gamma|| < a$ (where the strict inequality comes from Corollary 2.3.2 (1)). Also, $u^{\mathcal{H}}_\alpha(t) \geq u^{\mathcal{F}}_\alpha(\mathbf{0}_\alpha) = a$. From these facts, we have that $\alpha \leq \alpha^{\mathcal{H}}_0(t) \leq \alpha_0(\mathcal{H})$. Thus $\alpha_0(\mathcal{H}) = \alpha$.

Since $\alpha < |\overline{\text{ex}(K)}|^{\text{ex}(K)}_{CB}$ was arbitrary, we obtain that $S(K) \supset [0, |\overline{\text{ex}(K)}|^{\text{ex}(K)}_{CB}[$. If $|\overline{\text{ex}(K)}|^{\text{ex}(K)}_{CB}$ is infinite, then we may let $\alpha = |\overline{\text{ex}(K)}|^{\text{ex}(K)}_{CB} - 1$ and repeat the above argument with $\mathcal{F}$ given by Corollary 2.3.3 so that $\alpha_0(\mathcal{H}) = \alpha + 1$. Thus we have that $S(K) \supset [0,\, \rho_{\text{ex}(K)}(\overline{\text{ex}(K)})\,]$.

Here we prove the upper bound on $S(K)$. Suppose $\overline{\text{ex}(K)}$ is uncountable. Then $\rho(\overline{\text{ex}(K)}) = \omega_1$. Since the order of accumulation of any candidate sequence on K is countable, we have (trivially) that $S(K) \subset [0, \omega_1)$. Now suppose $\overline{\text{ex}(K)}$ is countable. If $\mathcal{H}$ is a u.s.c.d., harmonic candidate sequence on K, then by Corollary 2.2.7, the restricted sequence $\mathcal{H}|_{\overline{\text{ex}(K)}}$ satisfies

$$\alpha_0(\mathcal{H}|_{\overline{\text{ex}(K)}}) \leq \begin{cases} |\overline{\text{ex}(K)}|_{CB} - 1, & \text{if } |\overline{\text{ex}(K)}|_{CB} \text{ is finite} \\ |\overline{\text{ex}(K)}|_{CB}, & \text{if } |\overline{\text{ex}(K)}|_{CB} \text{ is infinite,} \end{cases}$$

which is exactly the statement that $\alpha_0(\mathcal{H}|_{\overline{\text{ex}(K)}}) \leq \rho(\overline{\text{ex}(K)})$. Also, the Embedding

Lemma (Lemma 2.2.30) implies that $\alpha_0(\mathcal{H}) \leq \alpha_0(\mathcal{H}|_{\overline{\text{ex}(K)}})$. This establishes the upper bound on $S(K)$. □

2.5.2 Optimality of Results for Choquet Simplices

In this section we study the optimality of the results in Theorem 2.5.5.

The following theorem answers a question of Jerome Buzzi, and answers the question of whether the bounds in Theorem 2.5.5 can be improved using only knowledge of the ordinals $\rho_{\text{ex}(K)}(\overline{\text{ex}(K)})$ and $\rho(\overline{\text{ex}(K)})$.

Theorem 2.5.6. *Let $\alpha_1 \leq \alpha_2 \leq \alpha_3$ be ordinals such that α_1 and α_2 are countable successors and α_3 is either a countable successor ordinal or ω_1. Then there exists a metrizable Choquet simplex K such that $\rho_{\text{ex}(K)}(\overline{\text{ex}(K)}) = \alpha_1$, $S(K) = [0, \alpha_2]$, and $\rho(\overline{\text{ex}(K)}) = \alpha_3$.*

We postpone the proof of Theorem 2.5.6 until after the proof of Theorem 2.5.10. The proofs of these theorems are very similar and we prefer not to repeat the arguments unnecessarily.

Now we address the following question: can the bounds in Theorem 2.5.5 be improved with knowledge of the homeomorphism class of the compactification $(\text{ex}(K), \overline{\text{ex}(K)})$? We will need some definitions.

Definition 2.5.7 ([39])**.** If E is a topological space, then a compactification of E is a pair $(\overline{E}, g)$, where $\overline{E}$ is a compact, Hausdorff space and g is a homeomorphism of E onto a dense subset of $\overline{E}$.

If E is a topological space and $(\overline{E}, g)$ is a compactification of E, then we may identify E with $g(E)$ and assume that E is a subset of $\overline{E}$. In such instances, we may refer to $\overline{E}$ as a compactification of E, or we may refer to the pair $(E, \overline{E})$ as a compactification.

Consider compactifications $(E, \overline{E})$, where E is a topological space and $\overline{E}$ is a compactification of E. Suppose there are two such compactifications, $(E_1, \overline{E_1})$ and $(E_2, \overline{E_2})$. We say that the compactifications are homeomorphic, written $(E_1, \overline{E_1}) \simeq (E_2, \overline{E_2})$, if there is a homeomorphism $g : \overline{E_1} \to \overline{E_2}$ such that $g(E_1) = E_2$. Recall that Theorem 2.2.23 may be strengthened as follows.

Theorem 2.5.8 (Choquet [26])**.** *Let E be a topological space and $\overline{E}$ a metrizable compactification of E. Then there exists a metrizable Choquet simplex K such that $(\mathrm{ex}(K), \overline{\mathrm{ex}(K)}) \simeq (E, \overline{E})$ if and only if E is Polish.*

Given a Polish space E and a compactification $\overline{E}$, the proof of Theorem 2.5.10 that is given below involves constructing a metrizable Choquet simplex K such that $(\mathrm{ex}(K), \overline{\mathrm{ex}(K)}) \simeq (E, \overline{E})$ while simultaneously controlling the possible harmonic, u.s.c.d. candidate sequences on K. In this sense Theorem 2.5.10 may be viewed as a partial generalization of Theorem 2.5.8.

Remark 2.5.9. In Theorem 2.5.10, we restrict our attention to metrizable compactifications of Polish spaces. Since we are only interested in studying pairs $(\mathrm{ex}(K), \overline{\mathrm{ex}(K)})$ where K is a metrizable Choquet simplex, Theorem 2.5.8 implies that there is no loss of generality in making this restriction.

Theorem 2.5.10. *Let E be a non-compact, countably infinite Polish space, and let $\overline{E}$ be a metrizable compactification of E.*

(1) If $\overline{E}$ is countable, then for each successor $\beta \in [\rho_E(\overline{E}),\ \rho(\overline{E})]$, there exists a Choquet simplex K such that $(\mathrm{ex}(K), \overline{\mathrm{ex}(K)}) \simeq (E, \overline{E})$ and $S(K) = [0, \beta]$.

(2) If E is countable and $\overline{E}$ is uncountable, then for each countable ordinal $\beta \geq \rho_E(\overline{E})$, there exists a Choquet simplex K such that $(\mathrm{ex}(K), \overline{\mathrm{ex}(K)}) \simeq (E, \overline{E})$ and $S(K) \supset [0, \beta]$.

Observe that when E is uncountable, Theorem 2.5.5 gives that for any metrizable Choquet simplex K with $\mathrm{ex}(K)$ homeomorphic to E, $S(K) = [0, \omega_1[$. The proofs of Theorem 2.5.10 (1) and (2) rely very heavily Lemma 2.5.14, which in turn relies very heavily on Haydon's proof (see [49] or [5, pp. 126-129]) of Theorem 2.2.23.

Proof of Theorem 2.5.10 (1).

2.5.2.1 Setup for proof of Theorem 2.5.10 (1)

Let β be a successor ordinal with $\rho_E(\overline{E}) \leq \beta \leq \rho(\overline{E})$. Let $\beta_0 = \beta$ if β is finite, and let $\beta_0 = \beta - 1$ if β is infinite. For notation, we let $T = \overline{E}$ and $X = E$. Since T is countable and compact, $T \cong \omega^{|T|_{CB}-1}n + 1$ for some natural number n (by Theorem 2.1.12). We may assume without loss of generality that $n = 1$ (if $n > 1$, then T is just the finite disjoint union of the case when $n = 1$, and we may repeat the following constructions independently n times). Using this homeomorphism of T and $\omega^{|T|_{CB}-1} + 1$, we obtain a well-ordering on T such that the induced order topology coincides with the original topology on T. Thus we may assume without loss of generality that $T = \omega^{|T|_{CB}-1} + 1$. Also, we fix a complete metric $d(\cdot, \cdot)$ on T.

Let $Y \subset T$ be the set $\omega^{\beta_0} + 1$ in $T = \omega^{|T|_{CB}-1} + 1$. Let $Z = Y \setminus X$, which may be empty. There are two cases: either $Y = T$ or $Y \subsetneq T$. The case $Y = T$ occurs if and only if $\beta = \rho(T)$, while the case $Y \subsetneq T$ occurs if and only if $\beta < \rho(T)$. If $Y = T$, then one may ignore the constructions in Sections 2.5.2.3, 2.5.2.4, and 2.5.2.5. If $Y \subsetneq T$, then Z may be empty. If Z is empty, then one may ignore the construction in Section 2.5.2.2. We make the convention that an empty sum is zero.

2.5.2.2 Definition of the points z_m, u_m, v_m

Assuming Z is not empty, we will define distinct points $z_m \in Z$ and $u_m, v_m \in X$. In the simplex K, they will satisfy $z_m = \frac{1}{2}(u_m + v_m)$, and it is exactly this formula which allows us to prove that $[0, \beta] \subseteq S(K)$.

Since T is countable, Z is countable, and we may enumerate $Z = \{z_m\}$ (in the case when Z is finite, this sequence is finite). If $z_m < \omega^{|T|_{CB}-1}$ in T, then let $u_m = z_m + 1$ and $v_m = z_m + 2$ (successor ordinals). If $z_m = \omega^{|T|_{CB}-1}$ in T, we let $u_m = 1$ and $v_m = 2$. Since X is dense in T, any isolated point in T must lie in X. Therefore any successor ordinals in T must be in X. It follows that u_m, v_m are points in X.

2.5.2.3 Construction of the sets V_k

Here we will use notations defined previously, such as the relative topological rank, $r_X(x)$, of the point x (Definition 2.5.3) and the relative Cantor-Bendixson derivatives $\Gamma_X^\alpha(Y)$ (Definition 2.5.1). Also, since it is an important hypothesis in this section, we remind the reader that Y is clopen in T.

In this section we assume that $T \setminus Y$ is not empty, which occurs exactly when $\beta < \rho(T)$, and we define certain sets V_k. The construction of the sets V_k and the points x_k and y_k (see section 2.5.2.4) allows one to prove that $S(K) \subseteq [0, \beta]$. In the simplex K, all points in the set V_k will lie in the convex hull of x_k and y_k, which will imply that the order of accumulation cannot be increased by the points in $V_k \setminus \{y_k\}$ (see Lemmas 2.5.12 and 2.5.15).

Below, by an interval in a subset A of T, we mean the intersection of an interval of T (which may be a singleton) with A.

Lemma 2.5.11. *If $T \setminus Y$ is not empty, then there exists a collection $\{V_k\}$ of non-empty subsets of T with the following properties:*

(1) if $V_k \cap V_j \neq \emptyset$, then $k = j$;

(2) for each V_k there exists an ordinal $\alpha_k \geq 1$ such that $r_X(t) = \alpha_k$ for all t in V_k;

(3) each V_k is a clopen interval in $\Gamma_X^{\alpha_k}(T)$;

(4) if $V_k \cap X \neq \emptyset$, then $V_k \cap X = \{\sup(V_k)\}$;

(5) $\Gamma_X^1(T) \setminus Y = \cup_k V_k$.

(6) $\lim_k \operatorname{diam}(V_k) = 0$.

Proof. Suppose $\alpha \in [1, \rho(T)]$ and the set $A_\alpha = \{t \in \Gamma_X^\alpha(T) \setminus Y : r_X(t) = \alpha\}$ is non-empty (which it must be for $\alpha = 1$ since $Y \neq T$). For $x \in X \cap A_\alpha$, let $a(x) = \min\{a \in \Gamma_X^1(T) \setminus Y : [a, x] \cap (X \cap A_\alpha) = \{x\} \text{ and } [a, x] \cap \Gamma_X^{\alpha+1}(T) = \emptyset\}$. Let $U_x = [a(x), x] \cap \Gamma_X^\alpha(T)$ and note that $U_x \subset A_\alpha$. The set $\Gamma_X^{\alpha+1}(T)$ is closed and does not intersect A_α, and the set $X \cap A_\alpha$ has no accumulation points in A_α. Thus each U_x is a clopen interval in $\Gamma_X^\alpha(T)$. Now let $U_\omega^\alpha = A_\alpha \setminus \cup_{x \in X \cap A_\alpha} U_x$, which may be empty.

If U_ω^α is non-empty, then we claim that it is also a clopen interval in $\Gamma_X^\alpha(T)$. Let $y_0 = \sup(X \cap A_\alpha)$. Note that Y is an initial subinterval of T and $X \cap A_\alpha \subset T \setminus Y$, which implies that $[y_0, \max(T)] \subset T \setminus Y$. We also have that y_0 is in $X \cup \Gamma_X^{\alpha+1}(T)$, which implies that y_0 is not in U_ω^α. We will show that $U_\omega^\alpha = [y_0 + 1, \max(T)] \cap \Gamma_X^\alpha(T)$. To see this fact, first note that if $y \leq x$ with $y \in A_\alpha$ and $x \in X \cap A_\alpha$, then $y \in \cup_{x \in X \cap A_\alpha} U_x$. Thus we have that $U_\omega^\alpha \subset [y_0 + 1, \max(T)] \cap \Gamma_X^\alpha(T)$. To show the reverse inclusion, we show that $[y_0 + 1, \max(T)] \cap A_\alpha = [y_0 + 1, \max(T)] \cap \Gamma_X^\alpha(T)$. We assume for the sake of contradiction that there is a point t in $[y_0 + 1, \max(T)] \cap \Gamma_X^{\alpha+1}(T)$. From this assumption and the fact that $[y_0 + 1, \max(T)]$ is open it follows that $[y_0 + 1, \max(T)] \cap \Gamma_X^\alpha(T) \cap X$ has t as an accumulation point (and so, in particular, this set is non-empty). If $[y_0 + 1, \max(T)] \cap \Gamma_X^\alpha(T) \cap X$ contains a single point s with $r_X(s) = \alpha$, then we see that $s \in X \cap A_\alpha$ and $s > y_0$, which contradicts the definition of y_0. Now suppose that for all s in $[y_0 + 1, \max(T)] \cap \Gamma_X^\alpha(T) \cap X$, $r_X(s) > \alpha$. Then $[y_0 + 1, \max(T)] \cap \Gamma_X^\alpha(T) \cap X$ is a non-empty, countable, metrizable space with no isolated points, which implies that it is not Polish. But $[y_0 + 1, \max(T)] \cap \Gamma_X^\alpha(T)$ is closed in T, which implies that it is a G_δ in T, and X is Polish in T, which implies it is a G_δ in T, and the intersection of two G_δ sets is a G_δ. Also, any G_δ set in a Polish space is Polish. Thus, $[y_0 + 1, \max(T)] \cap \Gamma_X^\alpha(T) \cap X$ is Polish, and we arrive at a contradiction.

Let $\{V_k'\}$ be an enumeration of all the non-empty sets U_x and U_ω^α constructed above, for any $\alpha \in [1, \rho(T)]$. The collection $\{V'_k\}$ satisfies properties (1)-(5) but not necessarily (6). However, given V'_k a clopen interval in $\Gamma^\alpha_X(T)$ contained in A_α, we may find a finite collection of pairwise disjoint clopen intervals (in $\Gamma^\alpha_X(T)$) $V'_{k,i}$, contained in A_α, whose union is V'_k, such that each $V'_{k,i}$ has diameter at most $\frac{1}{k}$. Re-enumerating the collection $\{V'_{k,i}\}$, we obtain the required collection $\{V_k\}$. $\square$

Note that since $T \setminus X \subset \Gamma^1_X(T)$, we have that $T \setminus (X \cup Y) = \sqcup_k V_k \setminus X$.

2.5.2.4 Definition of the points x_k and y_k

The points x_k and y_k are part of the construction that allows one to bound the possible orders of accumulation on K from above.

Assuming $\beta < \rho(T)$, we let $\{V_k\}$ be a collection of non-empty subsets of T given by Lemma 2.5.11, and fix a natural number k. There are two cases: either $V_k \cap X = \emptyset$ or $V_k \cap X \neq \emptyset$. Suppose $V_k \cap X = \emptyset$. Then choose a point t_k in V_k. If $t_k = \sup(T)$, let $x_k = \omega^{\beta_0} + 3$ and $y_k = \omega^{\beta_0} + 4$, and otherwise let $x_k = t_k + 1$ and $y_k = t_k + 2$. If $V_k \cap X \neq \emptyset$, then let $y_k = \sup(V_k)$ (which is in X by conclusion (4) of Lemma 2.5.11). If $y_k = \sup(T)$, let $x_k = \omega^{\beta_0} + 5$ and otherwise let $x_k = y_k + 1$. The fact that the V_k are pairwise disjoint implies that the points x_k and y_k are all distinct. Note that for all k, x_k and y_k are in X.

Notice that the points x_k, y_k, z_m, u_m, and v_m and the sets V_k have been chosen so that (i) the quantities $\mathrm{diam}(V_k)$, $\max_{t \in V_k} \mathrm{dist}(x_k, t)$, and $\max_{t \in V_k} \mathrm{dist}(y_k, t)$ each converge to zero as k tends to infinity, (ii) $d(z_m, u_m)$ and $d(z_m, v_m)$ each converge to zero as m tends to infinity, (iii) the points x_k, y_k, z_m, u_m, and v_m are all distinct,

(iv) the points x_k, y_k, u_m, v_m are all in X, and (v) if $V_k \cap X \neq \emptyset$, then $V_k \cap X = \{y_k\}$ (v) for all m and k, $z_m \notin V_k$, and (vi) the sets V_k are pairwise disjoint.

2.5.2.5 Definition of F_k and G_k

Choose Borel measurable functions $F_k : T \to [0, 1]$ and $G_k : T \to [0, 1]$ with the following properties:

(1) $F_k, G_k < 1$ on $T \setminus X$;

(2) F_k and G_k are continuous and injective on V_k and 0 on $T \setminus V_k$;

(3) $F_k + G_k = \chi_{V_k}$;

(4) $F_k(y_k) = 0$ and $G_k(y_k) = 1$.

The existence of such maps follows easily from the fact that T can be order-embedded in $(0, 1)$ and V_k is closed.

2.5.2.6 Conclusion of the proof of Theorem 2.5.10 (1)

Let $C_n = (\cup_{k=1}^{n} V_k) \cup \{z_1, \ldots, z_n\}$ for each n. Consider the collection of points $\{x_k\} \cup \{y_k\} \cup \{u_m\} \cup \{v_m\}$. To each point x_k we associate the function F_k. To each point y_k we associate the function G_k. To each point u_m or v_m, we associate the function $\frac{1}{2}\chi_{z_m}$. Then the hypotheses in Lemma 2.5.14 are satisfied by the countable collection of closed sets $\{C_n\} \cup \{D_n\}$, the countable collection of points $\{x_k\} \cup \{y_k\} \cup \{u_m\} \cup \{v_m\}$ in X, and the associated functions $\{F_k\} \cup \{G_k\} \cup \{\frac{1}{2}\chi_{z_m}\}$. Lemma 2.5.14 gives a metrizable Choquet simplex K and a homeomorphism $\phi : T \to \overline{\mathrm{ex}(K)}$

such that $\phi(X) = \text{ex}(K)$ and such that for all t in $T \setminus X$,

$$\phi(t) = \sum_k F_k(t)\phi(x_k) + G_k(t)\phi(y_k) + \frac{1}{2}\sum_m \chi_{z_m}(t)(\phi(u_m) + \phi(v_m)). \tag{2.5.1}$$

Lemma 2.5.12. *Let X, Y, T, and K be as above. Then for every $t \in T \setminus Y$, there exists an open (in T) neighborhood U_t and points x_t and y_t in $X \setminus Y$ such that for all s in U_t, either $r_X(s) < r_X(t)$ or else $r_X(s) = r_X(t)$ and $\phi(s) = a_s\phi(x_t) + b_s\phi(y_t)$ in K, with $0 \leq a_s, b_s \leq 1$ and $a_s + b_s = 1$.*

Proof. Let $t \in T \setminus Y$. If $r_X(t) = 0$, then t is isolated in T and t is in X, since X is dense in T. In this case we may choose $U_t = \{t\}$ and the requirement is trivially satisfied.

If $r_X(t) \geq 1$, then t is in V_k for some k. Let U_t be any open (in T) neighborhood of t with $\Gamma_X^{r_X(t)} \cap U_t \subseteq V_k$ (such a neighborhood exists since V_k is an open interval in in $\Gamma_X^{r_X(t)}(T)$), and let $x_t = x_k$ and $y_t = y_k$. We have that for each s in U_t, either $r_X(s) < r_X(t)$ or s is in V_k. If s is in V_k, then $r_X(s) = r_X(t)$, and it follows from Equation (2.5.1) that $\phi(s) = F_k(s)\phi(x_k) + G_k(s)\phi(y_k)$ in K. Also, we have that $F_k(s) + G_k(s) = 1$. □

By Lemmas 2.5.15 and 2.5.16, we have that $S(K) \subset [0, \rho(Y)]$. By Lemma 2.5.17, $S(K) \supset [0, \rho(Y)]$. Thus $S(K) = [0, \rho(Y)] = [0, \beta]$.

This concludes the proof of Theorem 2.5.10 (1).

Proof of Theorem 2.5.10 (2).

2.5.2.7 Setup for proof of Theorem 2.5.10 (2)

Let β be a successor ordinal with $\beta \geq \rho_E(\overline{E})$. Let $\beta_0 = \beta$ if β is finite and let $\beta_0 = \beta - 1$ if β is infinite. For notation, we let $T = \overline{E}$ and $X = E$. Fix a metric d on T that is compatible with the topology of T. Since T is uncountable and compact, T contains an uncountable perfect set P. Since β_0 is countable, P contains a set Y that is homeomorphic to $\omega^{\beta_0} + 1$. Let $\{a_\alpha\}_{\alpha=0}^{\omega^{\beta_0}}$ be a transfinite sequence of real numbers a_α such that $0 < a_\alpha \leq 1$ and $\sum_{\alpha \leq \omega^{\beta_0}} a_\alpha = 1$ (such a sequence exists since ω^{β_0} is countable). Let $Z = Y \setminus X$, and choose an enumeration of $Z = \{z_m\}$. Note that Z may be empty or finite. In the construction to follow, if Z is empty then we do not choose points u_m and v_m, and any summation over the index m will be zero by convention.

Let $X_0 = X \sqcup Z = X \cup Y$. Recall that since X is a completely metrizable subset of the the compact metrizable space T, X is a G_δ in T (see, for example, [86]). Y is a G_δ in T because it is compact. Therefore X_0 is a G_δ in T, since it is the union of two G_δ sets in T. Thus we may let $X_0 = \cap_{n \in \mathbb{N}} G_n$, where G_n is open, $G_1 = T$, and $G_{n+1} \subset G_n$. Let $F_n = T \setminus G_n$, which is compact. Fix n. Choose a sequence ϵ_ℓ strictly decreasing to 0. Let $D_\epsilon(F_n) = \{t \in T : \text{dist}(t, F_n) \geq \epsilon\}$, which is compact for any ϵ. Then for each ℓ there exists a countable collection of open sets $\{U_\ell^j\}_{j=1}^\infty$ such that

- $D_{\epsilon_\ell}(F_n) \subset \cup_j U_\ell^j \subset D_{\epsilon_{\ell+1}}(F_n)$;
- $\text{diam}(U_\ell^j) \leq 2^{-(\ell+n)}$ for all j;
- $\text{diam}(U_\ell^j)$ tends to 0 as j tends to infinity;

- the collection $\{U_\ell^j\}$ separates the points in $D_{\epsilon_\ell}(F_n)$.

Then we may enumerate the collection of all sets U_ℓ^j to form the sequence $\{V_k^n\}_{k=1}^\infty$. Repeating this procedure for all n, we obtain a collection of open sets V_k^n such that $\mathrm{diam}(V_k^n) \leq 2^{-n}$ and $\mathrm{diam}(V_k^n)$ converges to 0 as k tends to infinity with n fixed. The sets V_k^n also satisfy $\cup_k V_k^n = G_n$ and separate points in G_n, for each n. For each n and k, let $g_k^n(t) = \min(\mathrm{dist}(t, T \setminus V_k^n), 1)$ and $f_k^n = 2^{-k} g_k^n$. Then for each n, $\sum_k f_k^n$ converges uniformly on T. Now let

$$h_k^n(t) = \begin{cases} 0, & \text{if } \sum_k f_k^n(t) = 0 \\ \frac{f_k^n(t)}{\sum_k f_k^n(t)}, & \text{if } \sum_k f_k^n(t) > 0 \end{cases}$$

The functions h_k^n are all continuous and satisfy $h_k^n(t) > 0$ if and only if $t \in V_k^n$. Furthermore, $\sum_k h_k^n = \chi_{G_n}$. Now we let $p_k^n = h_k^n \cdot \chi_{T \setminus G_{n+1}}$ and notice that $\sum_n \sum_k p_k^n = \chi_{T \setminus (X \cup Y)}$. Also, the collection p_k^n separates points in the sense that if $t \neq s$ with t and s in $T \setminus (X \cup Y)$, then there exists n and k such that $p_k^n(t) > 0$ and $p_k^n(s) = 0$.

Using induction (on m, n, and k simultaneously) and the fact that X is dense in T, we choose points u_m, v_m, x_k^n, and y_k^n in X such that (i) $d(z_m, u_m) \leq a_{\alpha_{z_m}}$ and $d(z_m, v_m) \leq a_{\alpha_{z_m}}$, (ii) for each m, u_m and v_m are not accumulation points of Y (which is possible since the isolated points of Y, corresponding to successors of $\omega^{\beta_0} + 1$, are dense in Y and the set $X \setminus Y$ accumulates at each of the isolated points of Y that is not in X) (iii) x_k^n and y_k^n are in V_k^n, and (iv) the union of all of these points is a disjoint union.

2.5.2.8 Conclusion of the proof of Theorem 2.5.10 (2)

Let $C_n = (T \setminus G_n) \cup \{z_1, \ldots, z_n\}$. To each point x_k^n or y_k^n, we associate the function $\frac{1}{2}p_k^n$. To each point u_m or v_m, we associate the function $\frac{1}{2}\chi_{z_m}$. Then the hypotheses of Lemma 2.5.14 are satisfied by the countable collection of closed sets $\{C_n\}$, the countable collection of points $\{x_k^n\} \cup \{y_k^n\} \cup \{u_m\} \cup \{v_m\}$ in X, and the associated functions$\{\frac{1}{2}p_k^n\} \cup \{\frac{1}{2}\chi_{z_m}\}$. Lemma 2.5.14 gives a metrizable Choquet simplex K and a homeomorphism $\phi : T \to \overline{\text{ex}(K)}$ such that $\phi(X) = \text{ex}(K)$ and such that for all t in $T \setminus X$,

$$\phi(t) = \frac{1}{2}\sum_n \sum_k p_k^n(t)(\phi(x_k^n) + \phi(y_k^n)) + \frac{1}{2}\sum_m \chi_{z_m}(t)(\phi(u_m) + \phi(v_m)).$$

By Lemma 2.5.17, $S(K) \supset [0, \beta]$.

This concludes the proof of Theorem 2.5.10 (2).

Proof of Theorem 2.5.6.

Fix $\alpha_1 \leq \alpha_2 \leq \alpha_3$ as above. Let $X_1 = \omega^{\alpha_1} + 1$, and let $T_1 = X_1$. If α_2 is finite, let $T_2 = \omega^{\alpha_2} + 1$, and if α_2 is infinite, let $T_2 = \omega^{\alpha_2 - 1} + 1$. In either case, let X_2 be all the isolated points (successors) in T_2. Let S be a non-empty compact subset of $(0,1) \times \{0\}$ in $\mathbb{R}^2$, chosen so that if α_3 is finite, then $\rho(S) = \alpha_3 - 1$, and otherwise $\rho(S) = \alpha_3$. Let X_3 be a bounded, countable subset of $\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\})$ whose set of accumulation points is exactly S. Let $T_3 = X_3 \cup S$. Now we let $T = T_1 \sqcup T_2 \sqcup T_3$ and $X = X_1 \sqcup X_2 \sqcup X_3$. Below we will construct a Choquet simplex K such that $(X, T) \simeq (\text{ex}(K), \overline{\text{ex}(K)})$. Let $Y = T_1 \sqcup T_2$, and $Z = Y \setminus X$. Note that Z is actually just the set of accumulation points in T_2. We have

$$\rho_X(T) = \rho(X_1) = \alpha_1, \ \rho(T) = \rho(T_3) = \alpha_3, \ \text{and } \rho(Y) = \rho(T_2) = \alpha_2.$$

Let $Z = \{z_m\}$. If $z_m < \sup(T_2)$, choose $u_m = z_m + 1$ and $v_m = z_m + 2$. If $z_m = \sup(T_2)$, choose $u_m = 1$ and $v_m = 2$ in T_2. Let x_0 and y_0 be a choice of two isolated points in X_3. Let $F : T \to [0,1]$ be the function such that, for a point t in T,

$$F(t) = \begin{cases} s, & \text{if } t = (s,0) \in S \\ 0, & \text{otherwise .} \end{cases}$$

Let $G : T \to [0,1]$ be such that for t in T,

$$G(t) = \begin{cases} 1-s, & \text{if } t = (s,0) \in S \\ 0, & \text{otherwise .} \end{cases}$$

Let $C_n = S \cup \{z_1, \ldots, z_n\}$ for each n. To each point u_m or v_m, we associate the function $\frac{1}{2}\chi_{z_m}$. To the point x_0, we associate the function F, and to the point y_0 we associate the function G. Then the hypotheses in Lemma 2.5.14 are satisfied by the collection of closed sets $\{C_n\}$, the collection of points $\{x_0, y_0\} \cup \{u_m, v_m\}$, and the associated functions $\{F, G\} \cup \{\frac{1}{2}\chi_{z_m}\}$. Lemma 2.5.14 gives a Choquet simplex K and a homeomorphism $\phi : T \to \overline{\text{ex}(K)}$ such that $\phi(X) = \text{ex}(K)$ and such that for all t in $T \setminus X$,

$$\phi(t) = F(t)\phi(x_0) + G(t)\phi(y_0) + \frac{1}{2}\sum_m \chi_{z_m}(t)\big(\phi(u_m) + \phi(v_m)\big). \tag{2.5.2}$$

It follows immediately that $\rho_{\text{ex}(K)}(\overline{\text{ex}(K)}) = \rho_X(T) = \alpha_1$ and $\rho(\overline{\text{ex}(K)}) = \rho(T) = \alpha_3$. We show that $S(K) = [0, \alpha_2]$.

Lemma 2.5.13. *Let X, Y, T, and K be as above. Then for every $t \in T \setminus Y$, there exists an open (in T) neighborhood U_t and points x_t and y_t in $X \setminus Y$ such that for all s in U_t, either $r_X(s) < r_X(t)$ or else $r_X(s) = r_X(t)$ and $\phi(s) = a_s\phi(x_t) + b_s\phi(y_t)$ in K, with $0 \le a_s, b_s \le 1$ and $a_s + b_s = 1$.*

Proof. Let t be in $T \setminus Y = T_3$. If t is in X_3, then t is isolated in T and we let $U_t = \{t\}$. In this case the requirement on U_t is trivially satisfied.

If t is in $T_3 \setminus X_3$, then t is in S and $r_X(t) = 1$. Let U_t be any open neighborhood of t in T_3, and let $x_t = x_0$ and $y_t = y_0$. Let s be in U_t. If s is in X_3, then $r_X(s) = 0 < r_X(t)$. If s is in $T_3 \setminus X_3$, then s is in S, and then we have $r_X(s) = 1$ and by Equation (2.5.2), $\phi(s) = F(s)\phi(x_0) + G(s)\phi(y_0)$, where $F(s) + G(s) = 1$. □

Now by Lemmas 2.5.15 and 2.5.16, we have that $S(K) \subset [0, \rho(Y)]$. By Lemma 2.5.17 we have that $S(K) \supset [0, \rho(Y)]$. Then $S(K) = [0, \rho(Y)] = [0, \alpha_2]$.

This concludes the proof of Theorem 2.5.6.

2.5.2.9 Helpful Lemmas

Recall the following notations. Suppose T is a compact, metrizable space. Let $\mathcal{SM}(T)$ denote the set of all signed, totally finite, Borel measures on T. Recall that $\mathcal{SM}(T) = C_{\mathbb{R}}(T)^*$, and therefore $\mathcal{SM}(T)$ inherits the structure of a normed topological vector space over $\mathbb{R}$. For μ in $\mathcal{SM}(T)$, let $\mu = \mu_1 - \mu_2$ be the Jordan decomposition of μ. Let $|\mu| = \mu_1 + \mu_2$. The norm on $\mathcal{SM}(T)$ is then given by $||\mu|| = |\mu|(T)$. We will use $\mathcal{SM}(T \setminus X)$ to denote the set of measures μ in $\mathcal{SM}(T)$ such that $|\mu|(X) = 0$. We write $\mathcal{SM}_{prob}(T) = \{\mu \in \mathcal{SM} : \mu \geq 0,\ ||\mu|| = 1\}$, and for any subset $\mathcal{M}$ of $\mathcal{SM}(T)$, $\mathcal{M}_1 = \{\mu \in \mathcal{SM} : ||\mu|| \leq 1\}$. Let ϵ_{x_k} be the point mass at x_k.

Lemma 2.5.14. *Let T be a compact, metric space, and let X be a dense, Polish subset of T. Suppose $\{C_n\}$ is a countable collection of closed subsets of T. Suppose*

$\{w_k\}$ is a countable collection of distinct points in X, and to each point w_k there is an associated Borel measurable function $H_k : T \to [0,1]$. Let $W_k = \operatorname{supp}(H_k)$. Furthermore, suppose the following conditions are satisfied:

(i) $C_n \subset C_{n+1}$ *for all* n, $C_0 = \emptyset$, *and* $\cup_n C_n \setminus X = T \setminus X$;

(ii) $\sum_k H_k \leq 1$ *and* $\big(\sum_k H_k\big)|_{T\setminus X} \equiv 1$;

(iii) for all t *in* $T \setminus X$, $H_k(t) < 1$;

(iv) if $H_k(s) = H_k(t)$ *for all* k *with* $t, s \in T \setminus X$, *then* $s = t$;

(v) for each k, *there exists* n_k *such that* $W_k \subset C_{n_k+1} \setminus C_{n_k}$, *and with this notation,* H_k *is continuous on* C_{n_k+1};

(vi) $\max_{t \in W_k} d(t, w_k)$ *converges to* 0 *as* k *tends to infinity;*

(vii) if $H_k(x) > 0$ *for* x *in* X, *then* $x = w_k$ *and* $H_k(w_k) = 1$.

Let $\xi : \mathcal{SM}(T) \to \mathcal{SM}(T)$, *where for* μ *in* $\mathcal{SM}(T)$,

$$\xi(\mu) = \mu - \sum_k \Big(\int H_k \, d\mu\Big)\, \epsilon_{w_k}.$$

Let $\mathcal{M} = \{\xi(\mu) : \mu \in \mathcal{SM}(T\setminus X)\}$, *and let* $q : \mathcal{SM}(T) \to \mathcal{SM}(T)/\mathcal{M}$ *be the natural quotient map. Let* $\psi : T \to \mathcal{SM}_{prob}(T)$ *be* $\psi(t) = \epsilon_t$, *and let* $\phi = q \circ \psi$. *Finally, let* $K = q(\mathcal{SM}_{prob}(T))$. *Then*

(1) $\mathcal{M}$ *is a closed linear subspace of* $\mathcal{SM}(T)$, *and thus* ϕ *is continuous;*

(2) K *is a metrizable Choquet simplex;*

(3) ϕ *is injective on* T;

(4) $\mathrm{ex}(K) = \phi(X)$;

(5) for t *in* $T \setminus X$, $\phi(t) = \sum_k H_k(t)\phi(w_k)$ *in* K.

Proof. This lemma is almost entirely a restatement of Haydon's proof (see [49] or [5, pp. 126-129]) of Theorem 2.2.23. There are two differences. Firstly, we allow H_k to be positive on X, while Haydon does not. Secondly, we claim that ϕ is injective on all of T, whereas Haydon claims injectivity of ϕ only on X. For the proofs of properties (2), (4) and (5), theses differences do not play any role, and one may repeat Haydon's proof. For this reason, we will prove only (1) and (3).

(1) Note that $\mathcal{M}$ is a linear subspace. Recall that $\mathcal{M}$ being closed in the weak* topology is equivalent to $\mathcal{M}_1$ being closed in the weak* topology (a proof of this general fact, which follows from the Banach-Dieudonné Theorem, can be found in [83]). Let σ_i be a sequence of measures in $\mathcal{M}_1$. Since $||\xi(\mu)|| \geq ||\mu||$ for all μ in $\mathcal{SM}(T \setminus X)$, there exist measures μ_i in $\mathcal{SM}(T \setminus X)_1$ such that $\sigma_i = \xi(\mu_i)$. Since each C_n is compact, each $\mathcal{SM}(C_n)_1$ is compact in the weak* topology. Therefore a diagonal argument gives a subsequence $\{\nu_i\}$ of $\{\mu_i\}$ such that there exist measures $\hat{\nu}^n \in \mathcal{SM}(C_{n+1})$ such that $\nu_i|_{C_{n+1}}$ converges to $\hat{\nu}^n$ for each n. (We note that there may not be a measure $\hat{\nu}$ such that $\hat{\nu}|_{C_{n+1}} = \hat{\nu}^n$, since $\hat{\nu}^n|_{C_n}$ may not equal $\hat{\nu}^{n-1}$.)

Let $\nu^n = \hat{\nu}^n|_{C_{n+1}\setminus X}$, and let $\mathbf{1}_A$ be the characteristic function of the set A. Now define

$$\nu = \sum_n \nu^n|_{C_{n+1}\setminus C_n} = \sum_{k,n} H_k \mathbf{1}_{C_{n+1}\setminus C_n} \nu^n,$$

where the second equality follows from hypotheses (i) and (ii). Let $g_k^n = H_k \cdot \mathbf{1}_{C_{n+1}\setminus C_n}$, and note that by hypothesis (v), g_k^n is continuous on C_{n+1} for all k and n. Then

$g_k^n \nu_i$ weak* converges to $g_k^n \hat{\nu}^n$ as i tends to infinity. Since $||\nu_i|| \leq 1$ and $g_k^n \nu_i$ weak* converges to $g_k^n \hat{\nu}^n$, it follows that $||\nu|| \leq 1$ and ν is in $\mathcal{SM}(T \setminus X)$. Let us show that $\xi(\nu_i)$ converges to $\xi(\nu)$ in the weak* topology. Let $f \in C_{\mathbb{R}}(T)$. Then for any μ in $\mathcal{SM}(T \setminus X)$ we have

$$\int f d\xi(\mu) = \int f d\mu - \sum_k \int f(w_k) H_k \, d\mu = \sum_k \int (f - f(w_k)) H_k \, d\mu = \sum_{n,k} \lambda_k^n(\mu),$$

where

$$\lambda_k^n(\mu) = \int (f - f(w_k)) g_k^n \, d\mu.$$

For each k and n, we have that $(f - f(w_k)) g_k^n$ is continuous on C_{n+1} by hypothesis (v). Therefore, by the choice of subsequence ν_i, $\lambda_k^n(\nu_i)$ converges to $\lambda_k^n(\hat{\nu}^n)$. Also, using hypothesis (vii), we have that if $H_k(x) \mathbf{1}_{C_{n+1} \setminus C_n}(x) > 0$ for some x in X, then $x = w_k$. It follows that

$$\begin{aligned}
\lambda_k^n(\hat{\nu}^n) &= \int (f - f(w_k)) H_k \mathbf{1}_{C_{n+1} \setminus C_n} \, d\hat{\nu}^n \\
&= \int (f - f(w_k)) H_k \mathbf{1}_{C_{n+1} \setminus C_n} \, d\nu^n \\
&\quad + (f(w_k) - f(w_k)) H_k(w_k) \mathbf{1}_{C_{n+1} \setminus C_n}(w_k) \hat{\nu}^n(\{w_k\}) \\
&= \int (f - f(w_k)) H_k \mathbf{1}_{C_{n+1} \setminus C_n} \, d\nu^n \\
&= \lambda_k^n(\nu^n) = \lambda_k^n(\nu).
\end{aligned}$$

This calculation shows that $\lambda_k^n(\nu_i)$ converges to $\lambda_k^n(\nu)$. For fixed f in $C_{\mathbb{R}}(T)$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(t) - f(s)| < \epsilon$ whenever $d(t, s) < \delta$, by uniform continuity. Then since $\max_{t \in W_k} d(w_k, t)$ tends to zero as k tends to infinity, there exists k_0 such that for $k \geq k_0$ and $z \in W_k$, $|f(z) - f(w_k)| < \epsilon$. Then for any

μ in $\mathcal{SM}(T \setminus X)$, and $K \geq k_0$ and N,

$$\begin{aligned}\Big|\int f d\xi(\mu) - \sum_{n=1}^{N}\sum_{k=1}^{K}\lambda_k^n(\mu)\Big| &= \Big|\sum_{n>N}\sum_{k>K}\lambda_k^n(\mu)\Big| \\ &\leq \sum_{n>N}\sum_{k>K}\int |f - f(w_k)| g_k^n \, d|\mu| \leq \epsilon||\mu||,\end{aligned}$$

which implies that $\sum_{n=1}^{N}\sum_{k=1}^{K}\lambda_k^n(\mu)$ converges uniformly on $\mathcal{SM}(T\setminus X)_1$ to $\int f \, d\xi(\mu)$. Using this uniform convergence and the fact that $\lambda_k^n(\nu_i)$ converges to $\lambda_k^n(\nu)$, we conclude that $\xi(\nu_i)$ converges to $\xi(\nu)$.

(3) Suppose that $\phi(t) = \phi(s)$, or equivalently, $\epsilon_t - \epsilon_s$ is in $\mathcal{M}$. Thus there exists a measure μ in $\mathcal{M}(T \setminus X)$ such that $\epsilon_t - \epsilon_s = \xi(\mu)$. We consider three cases.

If t and s are both in X, then we notice that $\xi(\mu) = \epsilon_t - \epsilon_s$ has no mass in $T \setminus X$. As w_k are all in X, it follows from the definition of $\xi(\mu)$ that we must have $|\mu|(T \setminus X) = 0$, which implies that μ is the zero measure. Then $\xi(\mu)$ is the zero measure, and we have that $\epsilon_t = \epsilon_s$, which means that $t = s$.

If exactly one of t and s is in X, then we may assume without loss of generality that $t \in X$ and $s \in T \setminus X$. In this case, we notice that $-\epsilon_s = (\epsilon_t - \epsilon_s)|_{T\setminus X} = \xi(\mu)|_{T\setminus X} = \mu|_{T\setminus X} = \mu$. Therefore we conclude that

$$\epsilon_t = \xi(\mu) + \epsilon_s = \xi(\mu) - \mu = \sum_k H_k(s) \cdot \epsilon_{w_k}. \tag{2.5.3}$$

From this equation, we deduce that $t = w_k$ for some k. Then $H_k(s) = 1$, which gives a contradiction since $H_k < 1$ on $T \setminus X$ by hypothesis (iii).

If t and s are both in $T \setminus X$, then we see that $\xi(\mu) = \epsilon_t - \epsilon_s = \xi(\mu)|_{T\setminus X} = \mu$, which implies that $\int H_k d\mu = 0$ for all k. Hence $H_k(t) = H_k(s)$ for all k. By hypothesis (iv), we obtain that $t = s$. $\square$

Lemma 2.5.15. *Let K be a metrizable Choquet simplex. Let X be a Polish subspace of a compact metrizable space T, and let Y be clopen in T. Let $\phi : T \to \overline{\mathrm{ex}(K)}$ be a homeomorphism with $\phi(X) = \mathrm{ex}(K)$. Suppose that for every point t in $T \setminus Y$, there exists an open (in T) neighborhood U_t and points x_t and y_t in $X \setminus Y$ such that for all s in U_t, either $r_X(s) < r_X(t)$ or else $r_X(s) = r_X(t)$ and $\phi(s) = a_s\phi(x_t) + b_s\phi(y_t)$ in K, with $0 \leq a_s, b_s \leq 1$ and $a_s + b_s = 1$. Then for each point t in $T \setminus Y$, and any harmonic, u.s.c.d. candidate sequence $\mathcal{H}$ on K,*

$$\alpha_0^{\mathcal{H}|_{\phi(T)}}(t) \leq \begin{cases} r_X(t) & \text{if } r_X(t) \text{ is finite} \\ r_X(t) + 1 & \text{if } r_X(t) \text{ is infinite.} \end{cases} \tag{2.5.4}$$

Proof. For the sake of notation, we identify X, Y, and T with their images under ϕ. Observe that $T \setminus Y$ is clopen in T. Thus, for every t in $T \setminus Y$, $u_\beta^{\mathcal{H}|_T}(t) = u_\beta^{\mathcal{H}|_{T\setminus Y}}(t)$ for all ordinals β, which implies $\alpha_0^{\mathcal{H}|_T}(t) = \alpha_0^{\mathcal{H}|_{T\setminus Y}}(t)$. For the sake of notation, we assume that $\mathcal{H}$ is defined only on $T \setminus Y$ and $u_\beta^{\mathcal{H}} = u_\beta$.

Now we prove the lemma by transfinite induction on $\alpha = r_X(t)$. For $\alpha = 0$, we have that $r_X(t) = 0$, and thus t is isolated in T. Then $\alpha_0^{\mathcal{H}}(t) = 0$.

Suppose the lemma holds for all t in $T \setminus Y$ such that $r_X(t) < \alpha$. If α is finite, let $\delta = \alpha$. If α is infinite, let $\delta = \alpha + 1$. We now prove that for all t in $T \setminus Y$ with $r_X(t) = \alpha$ and all $\gamma \geq \delta$, $u_\gamma(t) = u_\delta(t)$. The proof of this statement is by transfinite induction on γ.

Let $\gamma > \delta$ be a successor ordinal, and let t be in $T \setminus Y$ with $r_X(t) = \alpha$. Let U_t be an open neighborhood of t, and let x_t and y_t be corresponding to U_t according the hypotheses. Fix $\epsilon > 0$. Choose k_0 such that $\max(\tau_k(x_t), \tau_k(y_t)) \leq \epsilon$ for all $k \geq k_0$. Then if (s_n) is a sequence in U_t with $r_X(s_n) < r_X(t)$ for all n, then using the inductive hypotheses, we get

$$(u_{\gamma-1}+\tau_k)(s_n)=(u_{\delta-1}+\tau_k)(s_n).$$

If (s_n) is a sequence in U_t with $r_X(s_n)\geq r_X(t)$, then by the hypotheses, we have that $r_X(s_n)=r_X(t)=\alpha$ and $s_n=a_{s_n}x_t+b_{s_n}y_t$. Then by the induction hypothesis on γ and the harmonicity of τ_k, we have that

$$(u_{\gamma-1}+\tau_k)(s_n)=u_\delta(s_n)+a_{s_n}\tau_k(x_t)+b_{s_n}\tau_k(y_t)\leq\epsilon.$$

Thus we may conclude that

$$(\widetilde{u_{\gamma-1}+\tau_k})(t)\leq\max\bigl((\widetilde{u_{\delta-1}+\tau_k})(t),u_\delta(t)+\epsilon\bigr).$$

Letting k tend to infinity, we obtain that $u_\gamma(t)\leq u_\delta(t)+\epsilon$. Since ϵ was arbitrary, we have that $u_\gamma(t)=u_\delta(t)$.

Now let $\gamma>\delta$ be a limit ordinal, and let t be in $T\setminus Y$ with $r_X(t)=\alpha$. Fix U_t, x_t, and y_t as in the hypotheses. Note that by the induction hypotheses, if s is in U_t, then $u_\beta(s)=u_\delta(s)$ for all $\beta<\gamma$. Then $\sup_{\beta<\gamma}u_\beta(s)=u_\delta(s)$ for all s in U_t. Taking upper semi-continuous envelope at t, we have that $u_\gamma(t)=u_\delta(t)$.

We conclude that for all t in $T\setminus Y$ with $r_X(t)=\alpha$, $\alpha_0^{\mathcal{H}}(t)\leq\delta$, as desired. □

Lemma 2.5.16. *Let K be a metrizable Choquet simplex. Let X be a Polish subspace of a compact metrizable space T, and let Y be clopen in T. Let $\phi:T\to\overline{\mathrm{ex}(K)}$ be a homeomorphism with $\phi(X)=\mathrm{ex}(K)$. Suppose that for each point t in $T\setminus Y$ and any harmonic, u.s.c.d. candidate sequence $\mathcal{H}$ on K, Equation (2.5.4) holds. Further, suppose that $\rho_X(T)\leq\rho(Y)$. Then $S(K)\subset[0,\rho(Y)]$*

Proof. Let $\mathcal{H}$ be a harmonic, u.s.c.d. candidate sequence on K. For t in Y, we have that $\alpha_0^{\mathcal{H}|_T}(t) = \alpha_0^{\mathcal{H}|_Y}(t)$ since Y is open in T. By Remark 2.2.5, $\alpha_0^{\mathcal{H}|_Y}(t) \leq \alpha_0(\mathcal{H}|_Y)$. By Proposition 2.2.6, $\alpha_0(\mathcal{H}|_Y) \leq \rho(Y)$. Putting these facts together, we obtain $\alpha_0^{\mathcal{H}|_T}(t) \leq \rho(Y)$ for all t in Y.

For t in $T \backslash Y$, Equation (2.5.4) gives that if $r_X(t)$ is finite, then $\alpha_0^{\mathcal{H}|_T}(t) \leq r_X(t)$, and if $r_X(t)$ is infinite, then $\alpha_0^{\mathcal{H}|_T}(t) \leq r_X(t) + 1$. Since X is countable and T is compact, $|T|_{CB}^X$ is a successor, and we have $r_X(t) \leq |T|_{CB}^X - 1$. If $|T|_{CB}^X$ is finite, then for all t in $T \setminus Y$ we have $\alpha_0^{\mathcal{H}|_T}(t) \leq r_X(t) \leq |T|_{CB}^X - 1 = \rho_X(T)$. If $|T|_{CB}^X$ is infinite, then for all t in $T \setminus Y$ we have $\alpha_0^{\mathcal{H}|_T}(t) \leq r_X(t) + 1 \leq |T|_{CB}^X = \rho_X(T) \leq \rho(Y)$.

We have shown that for all t in T, $\alpha_0^{\mathcal{H}|_T}(t) \leq \rho(Y)$. Taking supremum over all t in T, we have that $\alpha_0(\mathcal{H}|_T) \leq \rho(Y)$. Now using the Embedding Lemma (Lemma 2.2.30), we get that $\alpha_0(\mathcal{H}) \leq \alpha_0(\mathcal{H}|_T) \leq \rho(Y)$. Hence $S(K) \subset [0, \rho(Y)]$. □

Lemma 2.5.17. *Let K be a metrizable Choquet simplex. Let X be a Polish subspace of a compact metric space T, and let Y be a subset of T with $Y \cong \omega^{\beta_0} + 1$, where β_0 is a countable ordinal. Let $\phi : T \to \overline{\mathrm{ex}(K)}$ be a homeomorphism with $\phi(X) = \mathrm{ex}(K)$. Let $Y \setminus X = \{z_m\}$. Suppose there is countable collection of distinct points $W = \{u_m\} \cup \{v_m\}$ in X such that each point w in W is isolated in $Y \cup W$ and for each z_m in $Y \setminus X$, $\phi(z_m) = \frac{1}{2}(\phi(u_m) + \phi(v_m))$. Further suppose that $d(u_m, z_m)$ and $d(v_m, z_m)$ both tend to 0 as m tends to infinity. Then $S(K) \supset [0, \rho(Y)]$.*

Proof. For the sake of notation, we identify X, Y, W and T with their images under ϕ and refer to these sets as subsets of K. Then let $g : Y \to \omega^{\beta_0} + 1$ be a homeomorphism. For any γ in $[0, \beta_0]$, there is an u.s.c.d. candidate sequence $\mathcal{F}$ on $\omega^{\gamma} + 1$

given by Corollary 2.3.2 with $\alpha_0(\mathcal{F}) = \gamma$. Since $\omega^\gamma + 1 \subset \omega^{\beta_0} + 1$, we may extend $\mathcal{F}$ to a u.s.c.d. candidate sequence on $\omega^{\beta_0} + 1$ (still denoted $\mathcal{F}$) by letting $\mathcal{F}$ be uniformly 0 off of $\omega^\gamma + 1$. Note that $\mathcal{F}$ on $\omega^{\beta_0} + 1$ still has the properties stated in Corollary 2.3.2. We now construct a harmonic, u.s.c.d. sequence $\mathcal{H}$ on K such that $\alpha_0(\mathcal{F}) = \alpha_0(\mathcal{H})$. Let $\mathcal{F} = (f_k)$ be given as above. Then let $\mathcal{H}' = (h'_k)$ be the candidate sequence on K defined as follows. For t in K, let

$$h'_k(t) = \begin{cases} 0, & \text{if } t \notin Y \cup W \\ f_k(g(t)), & \text{if } t \in Y \setminus W \\ f_k(g(z_m)), & \text{if } t = u_m \text{ or } t = v_m. \end{cases}$$

We claim that for each k, $h'_{k+1} - h'_k$ is convex and u.s.c. Let t be in K. If t is in X, then $(h'_{k+1} - h'_k)(t) = \int_X (h'_{k+1} - h'_k) d\mathcal{P}_t$ since $\mathcal{P}_t = \epsilon_t$. If t is in $K \setminus (Y \cup W)$, then $0 = (h'_{k+1} - h'_k)(t) \leq \int_X (h'_{k+1} - h'_k) d\mathcal{P}_t$. If t is in $(Y \cup W) \setminus X = Y \setminus X = Z$, then $t = z_m$ for some m, and we have that $\mathcal{P}_{z_m} = \frac{1}{2}(\epsilon_{u_m} + \epsilon_{v_m})$. Then

$$\begin{aligned} (h'_{k+1} - h'_k)(z_m) = (f_{k+1} - f_k)(g(z_m)) &= \frac{1}{2}\Big((h'_{k+1} - h'_k)(u_m) + (h'_{k+1} - h'_k)(v_m)\Big) \\ &= \int_X h'_{k+1} - h'_k d\mathcal{P}_t. \end{aligned}$$

We have shown that $h'_{k+1} - h'_k$ is convex.

Let us prove that $h'_{k+1} - h'_k$ is u.s.c. Since $\{u_m\}$, $\{v_m\}$ and $\{z_m\}$ each have the same limit points, which are in Y (since $\{z_m\}$ is in Y and Y is closed), we obtain that $Y \cup W$ is compact in K. Thus if t is in $K \setminus (Y \cup W)$, then $(\widetilde{h'_{k+1} - h'_k})(t) = 0 = (h'_{k+1} - h'_k)(t)$. For t in $Y \setminus W$, assume $\{t_n\}$ is a sequence in $K \setminus \{t\}$ converging to t in K. Since $(h'_{k+1} - h'_k)|_{K \setminus (Y \cup W)} \equiv 0$, we may assume that t_n lies in $Y \cup W$ for all n. For each n, if t_n is not in Y, then there exists a natural number m_n such that $t_n \in \{u_{m_n}, v_{m_n}\}$. If t_n is in Y, then $(h'_{k+1} - h'_k)(t_n) = (f_{k+1} - f_k)(g(t_n))$, and if t_n is not Y, then there exists a natural number m_n such that $t_n \in \{u_{m_n}, v_{m_n}\}$ and $(h'_{k+1} - h'_k)(t_n) = (f_{k+1} - f_k)(g(z_{m_n}))$. By the choice of $\{u_m\}$ and $\{v_m\}$, we have that$\{z_{m_n}\}$ also converges to t. Then since $\mathcal{F}$ is u.s.c.d. and g is continuous, we have that $\limsup_n (h'_{k+1} - h'_k)(t_n) \le (f_{k+1} - f_k)(g(t)) = (h'_{k+1} - h'_k)(t)$. Thus $(\widetilde{h'_{k+1} - h'_k})(t) = (h'_{k+1} - h'_k)(t)$. For t in W, t is isolated in $Y \cup W$, and we conclude that $(\widetilde{h'_{k+1} - h'_k})(t) = (h'_{k+1} - h'_k)(t)$. Thus $(h'_{k+1} - h'_k)$ is u.s.c.

Now for t in K, let $\mathcal{H} = (h_k)$, where h_k is the harmonic extension of h'_k on K. $\mathcal{H}$ is harmonic by definition. Fact 2.2.24 states that the harmonic extension of a non-negative, convex, u.s.c. function on a Choquet simplex K is a harmonic, u.s.c. function on K. Applying this fact to each element in the sequence $(h'_{k+1} - h'_k)$, we obtain that $\mathcal{H}$ is a harmonic, u.s.c.d. candidate sequence.

Let $F = (Y \cap X) \cup W$ and $L = \overline{F} = Y \cup W$. Note that $\mathcal{H}|_{X \setminus F} \equiv \mathcal{H}'|_{X \setminus F} \equiv 0$, which implies that we may apply the Embedding Lemma (Lemma 2.2.30). The Embedding Lemma gives that for all ordinals α and all t in K,

$$u_\alpha^{\mathcal{H}}(t) = \max_{\mu \in \pi^{-1}(t)} \int_L u_\alpha^{\mathcal{H}|_L} \, d\mu.$$

Let us now show that for all t in K,

$$u_\alpha^{\mathcal{H}}(t) = \max_{\mu \in \pi^{-1}(t)} \int_Y u_\alpha^{\mathcal{H}|_L} \, d\mu = \max_{\mu \in \pi^{-1}(t)} \int_Y u_\alpha^{\mathcal{H}|_Y} \, d\mu = \max_{\mu \in \pi^{-1}(t)} \int_Y u_\alpha^{\mathcal{F}} \circ g \, d\mu. \quad (2.5.5)$$

The first equality in Equation (2.5.5) has already been justified as an application of the Embedding Lemma. The second equality in (2.5.5) will be justified by showing that for all ordinals α, $u_\alpha^{\mathcal{H}|_L}|_{L \setminus Y} \equiv 0$ and $u_\alpha^{\mathcal{H}|_L}|_Y = u_\alpha^{\mathcal{H}|_Y}$. Recall that $\mathcal{H}|_Y = \mathcal{F}' \circ g$, where $\mathcal{F}' = (f'_k)$ is the candidate sequence on $\omega^{\beta_0} + 1$ defined in terms of $\mathcal{F} = (f_k)$

as follows. If t is in $(\omega^{\beta_0}+1)\setminus g(W\cap Y)$, then $f'_k(t) = f_k(t)$, and if t is in $g(W\cap Y)$, then $f'_k(t) = f_k(z_m)$ for $t = g(u_m)$ or $t = g(v_m)$. Since g is a homeomorphism, we have that $u_\alpha^{\mathcal{H}|_Y} = u_\alpha^{\mathcal{F}'} \circ g$ for all ordinals α. Then we will justify the third equality in Equation (2.5.5) by proving that $u_\alpha^{\mathcal{F}} = u_\alpha^{\mathcal{F}'}$ for all ordinals α. We proceed with these steps and then conclude the proof of the lemma using Equation (2.5.5).

Notice that for all t in W, $r_L(t) = 0$ (t is isolated in L by hypothesis). Thus, if $t \in W$, then $u_\alpha^{\mathcal{H}|_L}(t) = 0$ for all α.

For t in Y, suppose there is a sequence $s_n \in W$ such that s_n converges to t and $\limsup_{s\to t} \tau_k^{\mathcal{H}|_L}(s) = \lim_n \tau_k^{\mathcal{H}|_L}(s_n)$. Since s_n is in W, for each n there exists m_n such that $s_n \in \{u_{m_n}, v_{m_n}\}$. Then $\tau_k^{\mathcal{H}|_L}(s_n) = \tau_k^{\mathcal{H}|_L}(z_{m_n})$, z_{m_n} also converges to t, and since z_{m_n} is in Y, $\tau_k^{\mathcal{H}|_Y}(z_{m_n}) = \tau_k^{\mathcal{H}|_L}(z_{m_n})$. Thus $\limsup_{s\to t} \tau_k^{\mathcal{H}|_L}(s) = \lim_n \tau_k^{\mathcal{H}|_Y}(z_{m_n})$. By these considerations, we have that for all t in Y, $\widetilde{\tau_k^{\mathcal{H}|_L}}(t) = \widetilde{\tau_k^{\mathcal{H}|_Y}}(t)$. Letting k tend to infinity gives that $u_1^{\mathcal{H}|_L}(t) = u_1^{\mathcal{H}|_Y}(t)$, for all t in Y.

Now we show by transfinite induction that $u_\alpha^{\mathcal{H}|_Y} = u_\alpha^{\mathcal{H}|_L}|_Y$ for all ordinals α. The equality holds for $\alpha = 1$ by the previous paragraph. Suppose by induction that it holds for some ordinal α. For the sake of notation, we allow $s = t$ in the following limit suprema. Also, the limit supremum over an empty set is assumed to be 0 by convention. For t in Y, the induction hypothesis implies that

$$\begin{aligned}
(\widetilde{u_\alpha^{\mathcal{H}|_L} + \tau_k})(t) &= \max\Big(\limsup_{\substack{s\to t\\ s\in W}} u_\alpha^{\mathcal{H}|_L}(s) + \tau_k(s),\ \limsup_{\substack{s\to t\\ s\in Y}} u_\alpha^{\mathcal{H}|_L}(s) + \tau_k(s)\Big) \\
&= \max\Big(\limsup_{\substack{s\to t\\ s\in W}} \tau_k(s),\ \limsup_{\substack{s\to t\\ s\in Y}} u_\alpha^{\mathcal{H}|_Y}(s) + \tau_k(s)\Big)
\end{aligned}$$

Taking the limit as k tends to infinity gives that

$$u_{\alpha+1}^{\mathcal{H}|_L}(t) = \max\Big(u_1^{\mathcal{H}|_L}(t),\, u_{\alpha+1}^{\mathcal{H}|_Y}(t)\Big) = \max\Big(u_1^{\mathcal{H}|_Y}(t),\, u_{\alpha+1}^{\mathcal{H}|_Y}(t)\Big) = u_{\alpha+1}^{\mathcal{H}|_Y}(t).$$

Thus we conclude that $u_{\alpha+1}^{\mathcal{H}|_Y} = u_{\alpha+1}^{\mathcal{H}|_L}|_Y$, proving the successor case of our induction. For the limit case, suppose the equality holds for all ordinals β less than a limit ordinal α. Then for t in Y, we have

$$\begin{aligned}
u_{\alpha}^{\mathcal{H}|_L}(t) =& \max\Big(\limsup_{\substack{s\to t\\ s\in W}} \sup_{\beta<\alpha} u_{\beta}^{\mathcal{H}|_L}(s),\, \limsup_{\substack{s\to t\\ s\in Y}} \sup_{\beta<\alpha} u_{\beta}^{\mathcal{H}|_L}(s)\Big)\\
=& \max\Big(0, \limsup_{\substack{s\to t\\ s\in Y}} \sup_{\beta<\alpha} u_{\beta}^{\mathcal{H}|_Y}(s)\Big)\\
=& u_{\alpha}^{\mathcal{H}|_Y}(t),
\end{aligned}$$

which concludes the limit step of the transfinite induction.

Now we turn our attention towards showing that $u_{\alpha}^{\mathcal{F}'} = u_{\alpha}^{\mathcal{F}}$ for all ordinals α. By Remark 2.3.4, we assume (without loss of generality) that $\mathcal{F}$ has the property (P) that for t in $\omega^{\beta_0}+1$,

$$\limsup_{\substack{s\to t\\ r(s)=0}} \tau_k^{\mathcal{F}}(s) = \lim_{\substack{s\to t\\ r(s)=0}} \tau_k^{\mathcal{F}}(s). \tag{2.5.6}$$

We also require the following topological fact. For every point t in $Y \setminus I$, there is a sequence in $I \setminus W$ that tends to t, where I is the set of isolated points in Y. To prove this fact, let t be a point with $r(t) \geq 1$ and let U be an open (in Y) neighborhood of t. Suppose for the sake of contradiction that $(I \setminus W) \cap U = \emptyset$. Since $Y \cong \omega^{\beta_0}+1$ (a countable, compact Polish space), we have that I is dense in Y and $\Gamma^1(Y) \setminus \Gamma^2(Y)$ is dense in $\Gamma^1(Y)$. Since $\Gamma^1(Y) \setminus \Gamma^2(Y)$ is dense in $\Gamma^1(Y)$, there is a point t' in U with $r(t')=1$. Since I is dense in Y, there is a sequence w_n in $I \cap U$ tending to t'.

Since $(I \setminus W) \cap U = \emptyset$, we must have that w_n is in W and then there is a sequence m_n such that $w_n \in \{u_{m_n}, v_{m_n}\}$ for all n. Then z_{m_n} tends to t'. Note that z_{m_n} is not in W by hypothesis, and since $r(t') = 1$, we must have that z_{m_n} is isolated in Y for all large n. Thus $(I \setminus W) \cap U \neq \emptyset$, a contradiction.

Using that $\mathcal{F}$ satisfies property (P) and the topological fact from the previous paragraph, let us show that for any non-isolated point t in $\omega^{\beta_0}+1$, we have $\widetilde{\tau_k^{\mathcal{F}'}}(t) = \widetilde{\tau_k^{\mathcal{F}}}(t)$. First note that for every sequence s_n converging to t, there is a sequence t_n converging to t such that $\tau_k^{\mathcal{F}'}(s_n) = \tau_k^{\mathcal{F}}(t_n)$: if s_n is not in $g(W \cap Y)$, then let $t_n = s_n$, and if s_n is in $g(W \cap Y)$, then there exists m_n such that $s_n \in \{g(u_{m_n}), g(v_{m_n})\}$, and one may take $t_n = g(z_{m_n})$. It follows that $\limsup_{s \to t} \tau_k^{\mathcal{F}'}(s) \leq \limsup_{s \to t} \tau_k^{\mathcal{F}}(s)$. Also, since t is not isolated, t is not in $g(W \cap Y)$ and $\tau_k^{\mathcal{F}'}(t) = \tau_k^{\mathcal{F}}(t)$. We deduce that $\widetilde{\tau_k^{\mathcal{F}'}}(t) \leq \widetilde{\tau_k^{\mathcal{F}}}(t)$. Now we show the reverse inequality. If s_n is a sequence converging to t with $r(s_n) > 0$, then s_n is not in $g(W \cap Y)$ and thus $\tau_k^{\mathcal{F}'}(s_n) = \tau_k^{\mathcal{F}}(s_n)$. In such a case, we have $\limsup_n \tau_k^{\mathcal{F}'}(s_n) = \limsup_n \tau_k^{\mathcal{F}}(s_n)$. Now let s_n be a sequence converging to t with $r(s_n) = 0$. By the topological fact from the previous paragraph, there is a sequence t_n of isolated points in $\omega^{\beta_0}+1$ that are not in $g(W \cap Y)$ such that t_n converges to t. Using the fact that $\mathcal{F}$ satisfies property (P) (see Equation (2.5.6)), we have $\limsup_n \tau_k^{\mathcal{F}}(s_n) = \limsup_n \tau_k^{\mathcal{F}}(t_n)$. Since the points t_n are not in $g(W \cap Y)$ we also have that $\limsup_n \tau_k^{\mathcal{F}}(t_n) = \limsup_n \tau_k^{\mathcal{F}'}(t_n) \leq \limsup_{s \to t} \tau_k^{\mathcal{F}'}(s)$. We have shown that for every sequence s_n converging to t, $\limsup_n \tau_k^{\mathcal{F}}(s_n) \leq \limsup_{s \to t} \tau_k^{\mathcal{F}'}(s)$. It follows that $\widetilde{\tau_k^{\mathcal{F}'}}(t) \geq \widetilde{\tau_k^{\mathcal{F}}}(t)$, and therefore we have shown that $\widetilde{\tau_k^{\mathcal{F}'}}(t) = \widetilde{\tau_k^{\mathcal{F}}}(t)$.

Finally, we show that for all ordinals α, $u_\alpha^{\mathcal{F}'} = u_\alpha^{\mathcal{F}}$ by transfinite induction on

α. We make the conventions that we allow $s = t$ in the following limit suprema, and the limit supremum over an empty set is 0. Note that if t is isolated in $\omega^{\beta_0}+1$, then $u_\alpha^{\mathcal{F}}(t) = 0 = u_\alpha^{\mathcal{F}'}(t)$ for all α, and thus we need only show the equality at non-isolated points t in $\omega^{\beta_0} + 1$. For the sake of induction, suppose the equality holds for an ordinal α. Let t be in $(\omega^{\beta_0}+1) \setminus g(I)$. For every sequence s_n converging to t, there is a sequence t_n converging to t such that $(u_\alpha^{\mathcal{F}'} + \tau_k^{\mathcal{F}'})(s_n) = (u_\alpha^{\mathcal{F}} + \tau_k^{\mathcal{F}})(t_n)$: if s_n is not in $g(W \cap I)$, then let $t_n = s_n$, and if s_n is in $g(W \cap I)$, then there exists m_n such that $s_n \in \{g(u_{m_n}), g(v_{m_n})\}$, and one may take $t_n = z_{m_n}$. It follows that $\limsup_{s\to t}(u_\alpha^{\mathcal{F}'} + \tau_k^{\mathcal{F}'})(s) \leq \limsup_{s\to t}(u_\alpha^{\mathcal{F}}+\tau_k^{\mathcal{F}})(s)$. Thus we have that $(\widetilde{u_\alpha^{\mathcal{F}'} + \tau_k^{\mathcal{F}'}})(t) \leq (\widetilde{u_\alpha^{\mathcal{F}} + \tau_k^{\mathcal{F}}})(t)$. Now we show the reverse inequality. Let s_n be a sequence in $g(I)$ converging to t. Then $(u_\alpha^{\mathcal{F}} + \tau_k^{\mathcal{F}})(s_n) = \tau_k^{\mathcal{F}}(s_n)$ and so $\limsup_n(u_\alpha^{\mathcal{F}} + \tau_k^{\mathcal{F}})(s_n) = \limsup_n \tau_k^{\mathcal{F}}(s_n) \leq \widetilde{\tau_k^{\mathcal{F}}}(t) = \widetilde{\tau_k^{\mathcal{F}'}}(t)$ (recall that we showed the last equality in the previous paragraph). Now let s_n be a sequence in $(\omega^{\beta_0} + 1) \setminus g(I)$ converging to t. Since s_n is not isolated, s_n is not in $g(W \cap Y)$, and we have $(u_\alpha^{\mathcal{F}} + \tau_k^{\mathcal{F}})(s_n) = (u_\alpha^{\mathcal{F}'} + \tau_k^{\mathcal{F}'})(s_n)$. Also, $\limsup_n(u_\alpha^{\mathcal{F}'} + \tau_k^{\mathcal{F}'})(s_n) \leq (\widetilde{u_\alpha^{\mathcal{F}'} + \tau_k^{\mathcal{F}'}})(t)$. Combining these considerations, we have shown that

$$(\widetilde{u_\alpha^{\mathcal{F}} + \tau_k^{\mathcal{F}}})(t) \leq \max\Big(\widetilde{\tau_k^{\mathcal{F}'}}(t), (\widetilde{u_\alpha^{\mathcal{F}'} + \tau_k^{\mathcal{F}'}})(t)\Big) = (\widetilde{u_\alpha^{\mathcal{F}'} + \tau_k^{\mathcal{F}'}})(t).$$

Then we deduce that $(\widetilde{u_\alpha^{\mathcal{F}} + \tau_k^{\mathcal{F}}}) = (\widetilde{u_\alpha^{\mathcal{F}'} + \tau_k^{\mathcal{F}'}})$. Taking the limit in k gives that $u_{\alpha+1}^{\mathcal{F}} = u_{\alpha+1}^{\mathcal{F}'}$, which concludes the successor step of the transfinite induction. For the limit step, assume that $u_\beta^{\mathcal{F}} = u_\beta^{\mathcal{F}'}$ for all ordinals β less than a limit ordinal α. We show that $u_\alpha^{\mathcal{F}} = u_\alpha^{\mathcal{F}'}$. For t in $\omega^{\beta_0} + 1$, the induction hypothesis gives that

(allowing $s = t$ in the the limit suprema)

$$u_\alpha^{\mathcal{F}}(t) = \limsup_{s\to t} \sup_{\beta<\alpha} u_\beta^{\mathcal{F}}(s) = \limsup_{s\to t} \sup_{\beta<\alpha} u_\beta^{\mathcal{F}'}(s) = u_\alpha^{\mathcal{F}'}(t).$$

We conclude that $u_\alpha^{\mathcal{F}} = u_\alpha^{\mathcal{F}'}$ for all ordinals α. This fact completes the verification of Equation (2.5.5).

It follows immediately from Equation (2.5.5) that $\alpha_0(\mathcal{H}) \leq \alpha_0(\mathcal{F}) = \gamma$. We now show the reverse inequality. Let $\mathbf{0}_\gamma$ be the marked point in Corollary 2.3.2, and let $t = g^{-1}(\mathbf{0}_\gamma)$. Then $u_\gamma^{\mathcal{H}}(t) \geq u_\gamma^{\mathcal{F}}(\mathbf{0}_\gamma) = a$. For an arbitrary $\alpha < \gamma$, we also have that $u_\alpha^{\mathcal{H}}(t) \leq ||u_\alpha^{\mathcal{F}}|| < a$ by Equation (2.5.5) and Corollary 2.3.2 (1). Thus $\gamma = \alpha_0(t) \leq \alpha_0(\mathcal{H})$, and we conclude that $\alpha_0(\mathcal{H}) = \gamma$.

Since $\gamma \leq \beta_0$ was arbitrary, we deduce that $S(K) \supset [0, \beta_0]$. For β finite, $\beta_0 = \beta$ and the proof is finished in this case. On the other hand, if β is infinite, then $\beta_0 = \beta - 1$ and we may repeat the above argument starting with $\mathcal{F}$ on $\omega^{\beta_0} + 1$ given by Corollary 2.3.3 such that $\alpha_0(\mathcal{F}) = \beta_0 + 1$. In this case, we conclude that $S(K) \supset [0, \beta_0 + 1] = [0, \beta]$, which concludes the proof. □

2.5.3 Open Questions

In general, our analysis leaves open the following problem.

Question 2.5.18. For a metrizable Choquet simplex K, what is $S(K)$?

Theorem 2.5.5 completely answers this question when $\rho_{\text{ex}(K)}(\overline{\text{ex}(K)}) = \rho(\overline{\text{ex}(K)})$. In particular, when K is Bauer or when $\text{ex}(K)$ is uncountable, Theorem 2.5.5 gives a complete answer. In general, Theorem 2.5.5 gives upper and lower bounds on $S(K)$.

Theorem 2.5.6 shows that the bounds in Theorem 2.5.5 cannot be improved using only knowledge of the ordinals $\rho_{\mathrm{ex}(K)}(\overline{\mathrm{ex}(K)})$ and $\rho(\overline{\mathrm{ex}(K)}))$. Theorem 2.5.10 (1) shows that if $\overline{\mathrm{ex}(K)}$ is countable, then the bounds in Theorem 2.5.5 cannot be improved using only knowledge of the homeomorphism class of the compactification $(\mathrm{ex}(K), \overline{\mathrm{ex}(K)})$. Theorem 2.5.10 (2) shows that the upper bound in Theorem 2.5.5 cannot be improved using only knowledge of the homeomorphism class the compactification $(\mathrm{ex}(K), \overline{\mathrm{ex}(K)})$. Thus we have the following question remaining.

Question 2.5.19. Let E be a countable, non-compact Polish space, and let $\overline{E}$ be an uncountable metrizable compactification of E. Let β be a successor in $[\rho_E(\overline{E}), \omega_1[$. Must there exist a metrizable Choquet simplex K such that $(E, \overline{E}) \simeq (\mathrm{ex}(K), \overline{\mathrm{ex}(K)})$ and $S(K) = [0, \beta]$?

Also, when E is countable and $\overline{E}$ is uncountable, we do not know whether the upper bound on $S(K)$ may be attained. We state this problem as a question as follows.

Question 2.5.20. Let E be a countable, non-compact Polish space, and let $\overline{E}$ be an uncountable metrizable compactification of E. Must there exist a metrizable Choquet simplex K such that $(E, \overline{E}) \simeq (\mathrm{ex}(K), \overline{\mathrm{ex}(K)})$ and $S(K) = [0, \omega_1[$?

If the answers to Questions 2.5.19 and 2.5.20 are affirmative, then one could conclude that the bounds in 2.5.5 cannot be improved using knowledge of the homeomorphism class of the compactification $(\mathrm{ex}(K), \overline{\mathrm{ex}(K)})$, and furthermore, one could conclude that these bounds are obtained.

Notice that for every simplex K for which we can compute $S(K)$, $S(K)$ is either $[0, \omega_1[$ or $[0, \beta]$ for a countable successor β. This observation leads to the following two questions.

Question 2.5.21. If K is a metrizable Choquet simplex, must $S(K)$ be an ordinal interval?

Question 2.5.22. If K is a metrizable Choquet simplex, must $S(K)$ be either $[0, \omega_1[$ or $[0, \beta]$ for a countable successor β?

If the answers to Questions 2.5.19, 2.5.20, 2.5.21, and 2.5.22 are all affirmative, then these results would give a complete description of the constraints imposed on orders of accumulation by the compactification of the ergodic measures for a dynamical system.

Chapter 3

Orders of accumulation of entropy on manifolds

3.1 Introduction

The purpose of the current chapter is to investigate a new entropy invariant arising from the theory of entropy structure and symbolic extensions: the order of accumulation of entropy, which is a countable ordinal associated to the system (X, T), denoted $\alpha_0(X, T)$ or just $\alpha_0(T)$. The order of accumulation of entropy of the system is an invariant of topological conjugacy that measures, roughly speaking, *over how many distinct "layers" residual entropy emerges* [35]. It is shown in Chapter 2, using a realization theorem of Downarowicz and Serafin [35, 38], that all countable ordinals appear as the order of accumulation for a minimal homeomorphism of the Cantor set. If follows from work of Buzzi [19] that if f is a C^∞ self-map of a compact manifold, then $\alpha_0(f) = 0$ (see Theorem 7.8 in [14]). Our main result, which is contained in Theorem 3.3.3, states that if M is a compact manifold and α is a countable ordinal, then there exists a continuous surjection $f : M \to M$ such that $\alpha_0(f) = \alpha$. Furthermore, if $\dim(M) \geq 2$, then f can be chosen to be a homeomorphism. The proof of this theorem gives a much more concrete construction of dynamical systems with prescribed order of accumulation than the proofs in Chapter 2, which rely on a realization theorem of Downarowicz and Serafin.

This chapter is organized as follows. In Section 3.2, we recall the basic notions and necessary facts in the theory of entropy structures and symbolic extensions. The main result of this chapter is stated in Section 3.3 as Theorem 3.3.3, and a proof of this result, depending on Theorem 3.3.1, is also given in 3.3. At the end of Section 3.3, we outline a proof of Theorem 3.3.1, and the rest of the chapter is essentially devoted to proving that result. Section 3.4 contains some lemmas regarding the behavior of several entropy invariants under certain suspensions and extensions. The proof of Theorem 3.3.1 involves inductively "blowing up" periodic points and "sewing in" more complicated dynamical behavior. The operation of "blowing up" periodic points and "sewing in" more complicated dynamics is carried out in Section 3.5, where we need only work in dimensions 1 and 2. Section 3.6 contains some technical lemmas in which the transfinite sequence is computed for some specific instances of maps resulting from the blow-and-sew construction. The transfinite induction scheme is executed in Section 3.7, which concludes with a proof of Theorem 3.3.1.

3.2 Background

We assume some basic familiarity with ordinals (see, for instance, [81]) and metrizable Choquet simplices (see [80]), but in this section we present the definitions and facts required for the following sections. We will denote by $\mathbb{N}$ the set of positive integers.

Definition 3.2.1. In this work, a dynamical system consists of a pair (X, T), where X is a compact metrizable space and $T : X \to X$ is a continuous surjection.

Furthermore, we assume that the topological entropy of T is finite, $\mathbf{h}_{\text{top}}(T) < \infty$. For references on the ergodic theory of such topological dynamical systems, see [78, 87].

3.2.1 Choquet simplices and $M(X, T)$

Let K be a compact, convex subset of a locally convex topological vector space. Let $\mathcal{M}(K)$ be the space of all Borel probability measures on K with the weak* topology. The barycenter map, bar : $\mathcal{M}(K) \to K$, is defined as follows: for μ in $\mathcal{M}(K)$, let bar(μ) be the unique point in K such that for each continuous affine function $f : K \to \mathbb{R}$,

$$f(\text{bar}(\mu)) = \int_K f \, d\mu.$$

The barycenter map itself is continuous and affine.

Definition 3.2.2 ([5] p. 69)**.** Let K be a metrizable, compact, convex subset of a locally convex topological vector space. Then K is a metrizable **Choquet simplex** if the dual of the continuous affine functions on K is a lattice.

We only need Choquet's characterization of metrizable Choquet simplices (see [80]): a metrizable, compact, convex subset of a locally convex topological vector space is a metrizable Choquet simplex if and only if for each point x in K, there exists a unique measure $\mathcal{P}_x$ in $\mathcal{M}(K)$ such that $\mathcal{P}_x(K \setminus \text{ex}(K)) = 0$ and $\text{bar}(\mathcal{P}_x) = x$, where ex$(K)$ is the set of extreme points of K.

Suppose K is a metrizable Choquet simplex. A Borel measurable function $f : K \to \mathbb{R}$ is called **harmonic** if, for each x in K and each $\mathcal{Q}$ in $\mathcal{M}(K)$ with

$\text{bar}(\mathcal{Q}) = x$, we have

$$f(x) = \int f \, d\mathcal{Q}.$$

Using that $\mathcal{P}_x$ is the unique measure supported on the extreme points of K with barycenter x, one may check that f is harmonic if and only if $f(x) = \int f \, d\mathcal{P}_x$ for each x in K. If f is a real-valued function defined on the extreme points of K, then we define the **harmonic extension** of f to be the function $f^{\text{har}} : K \to \mathbb{R}$ given for x in K by $f^{\text{har}}(x) = \int f d\mathcal{P}_x$. We also define $f : K \to \mathbb{R}$ to be supharmonic if, for each x in K and each $\mathcal{Q}$ in $\mathcal{M}(K)$ such that $\text{bar}(\mathcal{Q}) = x$, it holds that $f(x) \geq \int f d\mathcal{Q}$.

For a dynamical system (X, T), we write $M(X, T)$ to denote the space of Borel probability measures on X that are invariant under T. We give $M(X, T)$ the weak* topology. It is well known that in this setting $M(X, T)$ is a metrizable, compact, convex subset of a locally convex topological vector space (see, for example, [46, 78]). The extreme points of $M(X, T)$ are exactly the ergodic measures, $M_{\text{erg}}(X, T)$. Also, the statement that each invariant measure μ in $M(X, T)$ has a unique ergodic decomposition [46, 78] implies that $M(X, T)$ is a metrizable Choquet simplex (using Choquet's characterization). In other words, we have that for each μ in $M(X, T)$, there exists a unique measure $\mathcal{P}_\mu$ in $\mathcal{M}(M(X, T))$ such that $\mathcal{P}_\mu(M(X, T) \setminus M_{\text{erg}}(X, T)) = 0$ and $\text{bar}(\mathcal{P}_\mu) = \mu$.

3.2.2 Dynamical systems notations

We need some notation.

Notation 3.2.3. Let (X, T) be a dynamical system.

- Let A be a Borel measurable subset of X. We make the convention that $M(A,T) = \{\mu \in M(X,T) : \mu(X \setminus A) = 0\}$.

- Let NW(T) denote the non-wandering set for (X,T).

- A measure μ in $M(X,T)$ as totally ergodic if μ is ergodic for the system (X,T^n), for all $n \in \mathbb{N}$.

- If $\theta = \{x_0, \ldots, x_{n-1}\}$ is a T-periodic orbit, then we let μ_θ denote the periodic measure $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{x_k}$, where δ_x is the point mass concentrated at the point x.

- Let $h : M(X,T) \to [0,\ \infty)$ be the function that assigns to each measure in $M(X,T)$ its metric entropy with respect to the system (X,T). When we wish to emphasize the dependence of h on the system (X,T), we write h^T. Also, if $\mathcal{A}$ is a Borel partition of X, then we denote by $h^T(\mu,\mathcal{A})$ the entropy of the partition $\mathcal{A}$ with respect to the measure-preserving system (X,T,μ).

- If μ is a Borel probability measure on the space X, then supp(μ) is the intersection of all the closed sets C in X such that $\mu(C) = 1$.

  Recall that if μ is in $M(X,T)$, then supp(μ) $\subset$ NW(T).

Definition 3.2.4 ([12])**.** Let T be a continuous self-map of the compact metric space X. Let $\epsilon > 0$, $x \in X$, and $\Phi_\epsilon(x) = \{y \in X : d(T^n x, T^n y) \leq \epsilon \text{ for all } n\}$. If there exists $\epsilon > 0$ such that the topological entropy of T on the set $\Phi_\epsilon(x)$ is 0 for all $x \in X$, then (X,T) is h-expansive.

3.2.3 Upper semi-continuity

If E is a compact metrizable space and $f : E \to \mathbb{R}$, then we denote by $||f||$ the supremum norm of f. For x in E, we define

$$\limsup_{y\to x} f(y) = \max\Big(f(x), \sup\{\limsup_{n\to\infty} f(x_n) : \{x_n\}_{n\in\mathbb{N}} \subset E \setminus \{x\}, \lim_n x_n = x\}\Big).$$

Definition 3.2.5. Let E be a compact metrizable space, and let $f : E \to \mathbb{R}$. Then f is **upper semi-continuous** (u.s.c.) if one of the following equivalent conditions holds for all x in E,

(1) $f = \inf_\alpha g_\alpha$ for some family $\{g_\alpha\}_\alpha$ of continuous functions;

(2) $f = \lim_n g_n$ for some nonincreasing sequence $(g_n)_{n\in\mathbb{N}}$ of continuous functions;

(3) For each $r \in \mathbb{R}$, the set $\{x : f(x) \geq r\}$ is closed;

(4) $\limsup_{y\to x} f(y) \leq f(x)$, for all $x \in E$.

For any $f : E \to \mathbb{R}$, the **upper semi-continuous envelope** of f, written $\widetilde{f}$, is defined by letting $\widetilde{f} \equiv \infty$ if f is unbounded, and otherwise

$$\widetilde{f}(x) = \inf\{g(x) : g \text{ is continuous, and } g \geq f\}, \text{ for all } x \text{ in } E.$$

Note that $\widetilde{f}$ is the smallest u.s.c. function greater than f and satisfies

$$\widetilde{f}(x) = \limsup_{y\to x} f(y).$$

It is immediately seen that for any $f, g : E \to \mathbb{R}$, $\widetilde{f+g} \leq \widetilde{f} + \widetilde{g}$, with equality holding if f or g is continuous. We remark that if $f : E \to [0, \infty)$ is bounded and u.s.c., then f achieves its supremum. Also, if K is a Choquet simplex and $f : K \to \mathbb{R}$ is concave and u.s.c., then f is supharmonic.

3.2.4 Entropy structure and symbolic extensions

Definition 3.2.6. Let M be a compact metrizable space. A **candidate sequence** on M is a non-decreasing sequence (h_k) of functions from M to $[0,\infty)$ such that $\lim_k h_k$ exists and is bounded. We assume by convention that $h_0 \equiv 0$. Given two candidate sequences $\mathcal{H} = (h_k)$ and $\mathcal{F} = (f_k)$ defined on the same space, we say that $\mathcal{H}$ **uniformly dominates** $\mathcal{F}$, written $\mathcal{H} \geq \mathcal{F}$, if for each $\epsilon > 0$, and for each k, there exists ℓ, such that $f_k \leq h_\ell + \epsilon$. The candidate sequences $\mathcal{H}$ and $\mathcal{F}$ are **uniformly equivalent**, written $\mathcal{H} \cong \mathcal{F}$, if $\mathcal{H} \geq \mathcal{F}$ and $\mathcal{F} \geq \mathcal{H}$. Note that uniform equivalence is, in fact, an equivalence relation.

The uniform equivalence relation captures the manner in which sequences converge to their limit. For example, if two sequences converge uniformly to the same limit function, then they are uniformly equivalent. Also, if (h_k) and (f_k) are two candidate sequences on a compact metrizable space, then $\lim_k ||h_k - f_k|| = 0$ implies $(h_k) \cong (f_k)$, but $(h_k) \cong (f_k)$ does not necessarily imply $\lim_k ||h_k - f_k|| = 0$.

Definition 3.2.7 ([35])**.** Let X be a compact metrizable space and $T : X \to X$ a continuous surjection. For any continuous function $f : X \to [0,1]$, let $\mathcal{A}_f$ be the partition of $X \times [0,1]$ consisting of the set $\{(x,t) : f(x) \geq t\}$ and its complement. If $\mathcal{F} = \{f_1, \ldots, f_n\}$ is a finite collection of continuous functions $f_i : X \to [0,1]$, then let $\mathcal{A}_\mathcal{F} = \vee_{i=1}^n \mathcal{A}_{f_i}$. Let $\{\mathcal{F}_k\}_{k\in\mathbb{N}}$ be an increasing sequence of finite sets of continuous functions from X to $[0,1]$ chosen so that the partitions $\mathcal{A}_{\mathcal{F}_k}$ separate points (such sequences exist [35]). Let λ be Lebesgue measure on $[0,1]$. We define $\mathcal{H}^{\text{fun}}(T) = (h_k)$ to be the candidate sequence on $M(X,T)$ given by $h_k(\mu) = h^{T\times \text{Id}}(\mu \times \lambda, \mathcal{A}_{\mathcal{F}_k})$.

Definition 3.2.8 ([35])**.** Let (X, T) be a dynamical system. A candidate sequence $\mathcal{H}$ on $M(X, T)$ is an **entropy structure** for (X, T) if $\mathcal{H} \cong \mathcal{H}^{\text{fun}}(T)$. We may also refer to the entire uniform equivalence class of candidate sequences containing $\mathcal{H}^{\text{fun}}(T)$ as *the* entropy structure of (X, T).

Downarowicz showed that many of the known methods of computing or defining entropy can be adapted to become an entropy structure. For example, suppose (X, T) is a dynamical system with a refining sequence $\{P_k\}_{k \in \mathbb{N}}$ of finite Borel partitions of X such that the boundaries of all partition elements have zero measure for all T-invariant measures. Then the sequence of functions (h_k) defined for μ in $M(X, T)$ by $h_k(\mu) = h^T(\mu, P_k)$ is an entropy structure for (X, T). It may happen, though, that a particular system does not admit such a sequence of partitions (for example, if the system has an interval of fixed points). In such a case, we give another example of an entropy structure, known as the Katok entropy structure [35].

Definition 3.2.9 ([35])**.** For an ergodic measure μ in $M(X, T)$, $\epsilon > 0$ and $0 < \sigma < 1$, let

$$h(\mu, \epsilon, \sigma) = \limsup_n \frac{1}{n} \log \min\{|E| : \mu\big(\cup_{x \in E} B(x, n, \epsilon)\big) > \sigma\},$$

where $B(x, n, \epsilon) = \{y \in X : d(T^k y, T^k x) < \epsilon, \text{ for each } 0 \leq k \leq n-1\}$ and $d(\cdot, \cdot)$ is a metric compatible with the topology of X. For an invariant but non-ergodic measure μ, define $h(\mu, \epsilon, \sigma)$ by harmonic extension. Then for any sequence $\{\epsilon_k\}_{k \in \mathbb{N}}$ tending to 0, the sequence of functions $h_k(\mu) = h(\mu, \epsilon_k, \sigma)$ is an entropy structure (for proof, see [35] if T is a homeomorphism and [16] if T is merely continuous).

Notation 3.2.10. Let $\mathcal{H} = (h_k)$ be a candidate sequence on K, and let $\pi : L \to K$. We write $\mathcal{H} \circ \pi$ to denote the candidate sequence on L given by $h_k \circ \pi$. Also, if S is a subset of K, let $\mathcal{H}|_S$ be the candidate sequence on S given by $(h_k|_S)$.

Definition 3.2.11. Let $\mathcal{H}$ be a candidate sequence. The **transfinite sequence associated to $\mathcal{H}$**, which we write as $(u_\alpha^{\mathcal{H}})$, is defined by transfinite induction as follows. Let $\tau_k = h - h_k$. Then

- let $u_0^{\mathcal{H}} \equiv 0$;
- if $u_\alpha^{\mathcal{H}}$ has been defined, let $u_{\alpha+1}^{\mathcal{H}} = \lim_k \widetilde{u_\alpha^{\mathcal{H}} + \tau_k}$;
- if $u_\beta^{\mathcal{H}}$ has been defined for all $\beta < \alpha$, where α is limit ordinal, then let

$$u_\alpha^{\mathcal{H}} = \widetilde{\sup_{\beta<\alpha} u_\beta^{\mathcal{H}}}.$$

The sequence $(u_\alpha^{\mathcal{H}})$ is non-decreasing in α and does not depend on the choice of representative of uniform equivalence class [35], which allows us to make the following definition.

Definition 3.2.12. Let (X, T) be a topological dynamical system. Then **the transfinite sequence associated to** (X, T) is the sequence $(u_\gamma^{\mathcal{H}(T)})$, where $\mathcal{H}(T)$ is an entropy structure for T.

Note that for each α, the function $u_\alpha^{\mathcal{H}}$ is either identically equal to $+\infty$ or it is u.s.c. into $\mathbb{R}$ (since a non-increasing limit of u.s.c. functions is u.s.c.). The sequence $(u_\alpha^{\mathcal{H}})$ is also sub-additive in the following sense.

Proposition 3.2.13 ([18])**.** *Let $\mathcal{H}$ be a candidate sequence on E. Then for any two ordinals α and β,*

$$u^{\mathcal{H}}_{\alpha+\beta} \leq u^{\mathcal{H}}_{\alpha} + u^{\mathcal{H}}_{\beta}.$$

If $\mathcal{H}$ is a candidate sequence, then by Theorem 3.3 in [13], there exists a countable ordinal α such that $u^{\mathcal{H}}_{\alpha} = u^{\mathcal{H}}_{\alpha+1}$, which then implies that $u^{\mathcal{H}}_{\beta} = u^{\mathcal{H}}_{\alpha}$ for all $\beta > \alpha$.

Definition 3.2.14. If $\mathcal{H}$ is a candidate sequence, then the least ordinal α such that $u^{\mathcal{H}}_{\alpha} = u^{\mathcal{H}}_{\alpha+1}$ is called the **order of accumulation** of $\mathcal{H}$, which we write as $\alpha_0(\mathcal{H})$. If (X, T) is a topological dynamical system, then the **order of accumulation of entropy** of (X, T), written $\alpha_0(X, T)$ or just $\alpha_0(T)$, is defined as $\alpha_0(\mathcal{H}(T))$, where $\mathcal{H}(T)$ is an entropy structure for T.

To understand the meaning of the transfinite sequence and the order of accumulation of entropy of (X, T), we turn to the connection between symbolic extensions and entropy structure.

Definition 3.2.15. Let (X, T) be a topological dynamical system. A **symbolic extension** of (X, T) is a subshift (Y, S) of a (two-sided) full shift on a finite alphabet, along with a continuous surjection $\pi : Y \to X$ such that $T \circ \pi = \pi \circ S$.

Definition 3.2.16. If (Y, S) is a symbolic extension of (X, T) with factor map π, then the **extension entropy function**, $h^{\pi}_{\text{ext}} : M(X, T) \to [0, \infty)$, is defined for μ in $M(X, T)$ by

$$h^{\pi}_{\text{ext}}(\mu) = \sup\{h(\nu) : \pi\nu = \mu\}.$$

The **symbolic extension entropy function** of a dynamical system (X,T), $h_{\text{sex}} : M(X,T) \to [0,\ \infty]$, is defined for μ in $M(X,T)$ by

$$h_{\text{sex}}(\mu) = \inf\{h^{\pi}_{\text{ext}}(\mu) : \pi \text{ is the factor map of a symbolic extension of } (X,T)\},$$

and the **residual entropy function**, $h_{\text{res}} : M(X,T) \to [0,\infty]$, is defined as

$$h_{\text{res}} = h_{\text{sex}} - h.$$

If (X,T) does not admit symbolic extensions, we let $h_{\text{sex}} \equiv \infty$ and $h_{\text{res}} \equiv \infty$, by convention.

We think of a symbolic extension as a "lossless finite encoding" of the dynamical system (X,T) [35]. The symbolic extension entropy function quantifies at each measure the minimal amount of entropy that must be present in such an encoding.

The study of symbolic extensions is related to entropy structures by the following remarkable result of Boyle and Downarowicz.

Theorem 3.2.17 ([13])**.** *Let (X,T) be a dynamical system with entropy structure $\mathcal{H}$. Then*

$$h_{\text{sex}} = h + u^{\mathcal{H}}_{\alpha_0(T)}.$$

Note that the conclusion of Theorem 3.2.17 could be restated as $h_{\text{res}} = u^{\mathcal{H}}_{\alpha_0(\mathcal{H})}$. This theorem relates the notion of how entropy emerges on refining scales to the symbolic extensions of a system, showing that there is a deep connection between these topics. Using this connection, some progress has been made in understanding the symbolic extensions of certain classes of dynamical systems. For examples of these types of results, see [4, 13, 17, 15, 33, 35, 36, 37]. In light of Theorem 3.2.17, we observe that the order of accumulation of entropy measures over how many "layers" residual entropy accumulates in the system.

3.2.5 Background lemmas

The following lemma (Lemma 3.2.18) will be used to compute the transfinite sequence associated to the systems that appear in Sections 3.4 - 3.7. Although the entropy function h is a harmonic function on the simplex of invariant measures, the functions $u_\alpha^{\mathcal{H}}$ are not harmonic in general. Lemma 3.2.18 is useful because it nonetheless provides an integral representation of the functions $u_\alpha^{\mathcal{H}}$. A candidate sequence (h_k) on a Choquet simplex such that each function h_k is harmonic will be referred to as a harmonic candidate sequence. Let K be a metrizable Choquet simplex with $E = \text{ex}(K)$. In Lemma 3.2.18 we identify $\mathcal{M}(C \cup \overline{E})$ with the set $\{\mu \in \mathcal{M}(K) : \text{supp}(\mu) \subset C \cup \overline{E}\}$ in the natural way, where $\overline{E}$ denotes the closure of E in K. Also, if f is a measurable function defined on the measurable subset C of K and μ is a measure on K, then $\int_C f \, d\mu$ is defined to be the integral with respect to μ of the function

$$x \mapsto \begin{cases} f(x), & \text{if } x \in C \\ 0, & \text{if } x \notin C. \end{cases}$$

Lemma 3.2.18 (Embedding Lemma [18])**.** *Let K be a metrizable Choquet simplex with $E = \text{ex}(K)$. Suppose $\mathcal{H}$ is a harmonic candidate sequence on K and there is a set $F \subset E$ such that the sequence $\{(h - h_k)|_{E \setminus F}\}_{k \in \mathbb{N}}$ converges uniformly to zero. Let C be a closed subset of K such that $F \subset C$, and let $\Phi : \mathcal{M}(C \cup \overline{E}) \to K$ be the* *restriction of the barycenter map. Then for all ordinals α and for all x in K,*

$$u_\alpha^{\mathcal{H}}(x) = \max_{\mu \in \Phi^{-1}(x)} \int_C u_\alpha^{\mathcal{H}|_C} \, d\mu, \tag{3.2.1}$$

and $\alpha_0(\mathcal{H}) \leq \alpha_0(\mathcal{H}|_C)$. In particular, if x is an extreme point of K contained in C, then $u_\alpha^{\mathcal{H}}(x) = u_\alpha^{\mathcal{H}|_C}(x)$ for all ordinals α.

The Embedding Lemma (Lemma 3.2.18) was proved in Chapter 2 as Lemma 2.2.30.

We end this section by stating some facts that will be used repeatedly in the following sections. Facts 3.2.19 (1)-(4) are easily checked from the definitions, and Fact 3.2.19 (5), which is proved in [18], follows from the fact that the u.s.c. envelope of a concave function is concave and the limit of concave functions is concave.

Fact 3.2.19. *Let M, M_1, M_2, and K be compact metrizable spaces. Then for all ordinals γ, the following hold.*

(1) If $\mathcal{H}$ is a candidate sequence on M and U is an open neighborhood of x in M, then $u_\gamma^{\mathcal{H}}(x) = u_\gamma^{\mathcal{H}|_U}(x)$.

(2) Suppose that $\mathcal{H}$ is a candidate sequence on M_1 and $M_2 \subset M_1$. Then $u_\gamma^{\mathcal{H}|_{M_2}} \leq u_\gamma^{\mathcal{H}|_{M_1}}$.

(3) Suppose that $\pi : M \to K$ is a continuous surjection, $\mathcal{F}$ is a candidate sequence on K, and $\mathcal{H} = \mathcal{F} \circ \pi$. Then $u_\gamma^{\mathcal{H}} \leq u_\gamma^{\mathcal{F}} \circ \pi$.

(4) Suppose $\pi : M \to K$ is continuous, surjective, and open (which is satisfied, in particular, if π is a homeomorphism), $\mathcal{F}$ is a candidate sequence on K, and $\mathcal{H} = \mathcal{F} \circ \pi$. Then $u_\gamma^{\mathcal{H}} = u_\gamma^{\mathcal{F}} \circ \pi$.

(5) Suppose $\mathcal{H}$ is a harmonic candidate sequence on a metrizable Choquet simplex M. Then $u_\gamma^{\mathcal{H}}$ is concave for all γ, and since $u_\gamma^{\mathcal{H}}$ is u.s.c., $u_\gamma^{\mathcal{H}}$ is also supharmonic. In particular, if (X,T) is a topological dynamical system, then there exists a harmonic entropy structure $\mathcal{H}(T)$ for T [35], and therefore $u_\gamma^{\mathcal{H}(T)}$ is concave and supharmonic for all γ.

3.3 Main Results

The notation $\mathcal{S}(\alpha, d, a)$ is defined in Definition 3.6.1. For our purposes now, it suffices to use the following facts. Let $\mathbb{D}$ be the closed unit ball in dimension $\mathbb{R}^d$, which has boundary $\partial\mathbb{D}$. Suppose a map $F : \mathbb{D} \to \mathbb{D}$ is in $\mathcal{S}(\alpha, d, a)$. Then F is continuous and surjective, and F is a homeomorphism when $d \geq 2$. Also, $F|_{\partial\mathbb{D}} = \mathrm{Id}$, $\mathbf{h}_{\mathrm{top}}(F) < \infty$, $\alpha_0(F) = \alpha$ and $||u_\alpha^{\mathcal{H}(F)}|| = a$.

Theorem 3.3.1. *Let d be in $\{1, 2\}$. For every countable ordinal $\alpha > 0$ and any $a > 0$, there is a map F in $\mathcal{S}(\alpha, d, a)$.*

The formal proof of Theorem 3.3.1 appears at the end of Section 3.7, since it relies on the accumulated results of Sections 3.4 - 3.7. Using Theorem 3.3.1, we obtain the following result.

Corollary 3.3.2. *Let α be a countable ordinal, let $a > 0$ and let d be in $\mathbb{N}$. Let $\mathbb{D}$ be the closed unit ball in $\mathbb{R}^d$. Then there exists a continuous surjection $f : \mathbb{D} \to \mathbb{D}$ such that $f|_{\partial\mathbb{D}} = \mathrm{Id}$, $\alpha_0(f) = \alpha$, and $||u_\alpha^{\mathcal{H}(f)}|| = a$. If $d \geq 2$, then f can be chosen to be a homeomorphism.*

Proof. If d is 1 or 2, then Theorem 3.3.1 implies that there exists g in $\mathcal{S}(\alpha, d, a)$, which satisfies the conclusion of the corollary. We remark that since $\mathbb{D}$ and $[-1,1]^d$ are homeomorphic, the statement of the theorem is equivalent to the same statement with $[-1,1]^d$ in place of $\mathbb{D}$. Thus we may consider all maps defined on $[-1,1]^d$ without loss of generality. The proof proceeds by induction on d. Suppose the corollary holds for some $d \geq 2$. Using this inductive hypothesis, choose a homeomorphism $g : [-1,1]^d \to [-1,1]^d$ such that $g|_{\partial[-1,1]^d} = \mathrm{Id}$, $\alpha_0(g) = \alpha$, and $||u_\alpha^{\mathcal{H}(g)}|| = a$. Then there exists a homeomorphism $f : [-1,1]^{d+1} \to [-1,1]^{d+1}$ such that $f|_{\partial[-1,1]^{d+1}} = \mathrm{Id}$, $f(x, 0) = (g(x), 0)$ for x in $[-1,1]^d$, and $\mathrm{NW}(f) \subset \{(x,t) \in [-1,1]^d \times [-1,1] : t = 0\} \cup \partial[-1,1]^{d+1}$. Such a map f may be constructed as follows. Let

$$V = \{(x_1, \ldots, x_d, t) \in [-1,1]^{d+1} : |x_i| \leq (1 - |t|) \text{ for } 1 \leq i \leq d\}.$$

Also, define $T : [-1,1] \to [-1,1]$ by $T(t) = t + \frac{1}{10}\sin(\pi t)$. For x in $\partial[-1,1]^{d+1}$, let $f(x) = x$. For (x,t) in V (where $x \in [-1,1]^d$) such that $|t| < 1$, let

$$f(x,t) = \left((1 - |T(t)|) g\left(\frac{1}{(1-|t|)} x \right), T(t) \right).$$

We have defined f on $V \cup \partial[-1,1]^{d+1}$. For any point p in $[-1,1]^{d+1} \setminus (V \cup \partial[-1,1]^{d+1})$, let ℓ_p denote the line in $\mathbb{R}^{d+1}$ passing through p and the origin. Let p_1 and p_2 be the points such that $\{p_1\} = \partial V \cap \ell_p$ and $\{p_2\} = \partial[-1,1]^{d+1} \cap \ell_p$. Then let s in $[0,1]$ be such that $p = sp_1 + (1-s)p_2$. Now define $f(p) = sf(p_1) + (1-s)f(p_2)$. With this definition, f is a homeomorphism of $[-1,1]^{d+1}$ (using that $g|_{\partial[-1,1]^d} = \mathrm{Id}$). Furthermore, we have that $f|_{\partial[-1,1]^{d+1}} = \mathrm{Id}$, $f(x,0) = (g(x), 0)$ for x in $[-1,1]^d$, and $\mathrm{NW}(f) \subset \{(x,t) \in [-1,1]^d \times [-1,1] : t = 0\} \cup \partial[-1,1]^{d+1}$. Then $\alpha_0(f) = \alpha_0(g) = \alpha$

and $||u_\alpha^{\mathcal{H}(f)}|| = ||u_\alpha^{\mathcal{H}(g)}|| = a$. In this way we have verified the inductive hypotheses for $d+1$, which finishes the proof of the corollary. □

Todd Fisher asked the following question [45]. Given a countable ordinal α and a compact manifold M, does there exist a continuous surjection (or homeomorphism if $\dim(M) \geq 2$) $f : M \to M$ such that $\alpha_0(f) = \alpha$? Theorem 3.3.3, which we view as our main result, answers this question affirmatively.

Theorem 3.3.3. *Let α be a countable ordinal and let $a > 0$. Let M be a compact manifold. Then there exists a continuous surjection $f : M \to M$ such that $\alpha_0(f) = \alpha$ and $||u_\alpha^{\mathcal{H}(f)}|| = a$. If $\dim(M) \geq 2$, then f can be chosen to be a homeomorphism.*

Proof. Let $d = \dim(M)$, and let $\mathbb{D}$ be the closed unit ball in $\mathbb{R}^d$. By Corollary 3.3.2, there exists a continuous onto map $g : \mathbb{D} \to \mathbb{D}$ such that $g|_{\partial\mathbb{D}} = \mathrm{Id}$, $\alpha_0(g) = \alpha$, $||u_\alpha^{\mathcal{H}(g)}|| = a$, and g is a homeomorphism if $d \geq 2$. We define a map $G : \mathbb{D} \to \mathbb{D}$ as follows. Let $G|_{\overline{B(\mathbf{0},\frac{1}{2})}} = A_{\frac{1}{2},\mathbf{0}} \circ g \circ A_{2,\mathbf{0}}$, where $A_{s,\mathbf{p}}$ is the affine map on $\mathbb{R}^d$ given by $A_{s,\mathbf{p}}(x) = sx + \mathbf{p}$ and $\mathbf{0}$ is the origin. Now parametrize the annulus $\{x \in \mathbb{R}^d : \frac{1}{2} \leq |x| \leq 1\}$ with polar coordinates $(r,\theta) \in [\frac{1}{2},1] \times S^1$. For (r,θ) in $[\frac{1}{2},1] \times S^1$, let $G(r,\theta) = (r + \frac{1}{10}\sin(2\pi r), \theta)$. Now G is continuous and surjective and satisfies $G|_{\partial\mathbb{D}} = \mathrm{Id}$ and $\mathbf{h}_{\mathrm{top}}(G) < \infty$. Also, G is a homeomorphism if $d \geq 2$, and $\overline{B(\mathbf{0},\frac{1}{2})}$ is an isolated set for G. Let $\phi : \mathbb{D} \to M$ be a homeomorphism onto its image (such a map exists since M is a manifold). Define $f : M \to M$ as follows. For x in $\phi(\mathbb{D})$, let $f(x) = \phi(G(\phi^{-1}(x)))$. For x in $M \setminus \phi(\mathbb{D})$, let $f(x) = x$. Then $NW(f) = \phi(\overline{B(\mathbf{0},\frac{1}{2})}) \cup (M \setminus \phi(\mathrm{int}(\mathbb{D})))$. Further, f is topologically conjugate to $G|_{\overline{B(\mathbf{0},\frac{1}{2})}}$ on $\phi(\overline{B(\mathbf{0},\frac{1}{2})})$, and f is the identity on $M \setminus \phi(\mathrm{int}(\mathbb{D}))$. It follows that

$\alpha_0(f) = \alpha_0(G) = \alpha_0(g) = \alpha$ and $||u_\alpha^{\mathcal{H}(f)}|| = ||u_\alpha^{\mathcal{H}(G)}|| = ||u_\alpha^{\mathcal{H}(g)}|| = a$. $\square$

The remainder of this chapter is devoted to proving Theorem 3.3.1. Let us provide an outline of the constructions and statements that follow. The main construction uses a complicated transfinite induction argument, which is similar in format to the induction carried out in [18]. At each stage of the induction, we assume that we have maps in $\mathcal{S}(\gamma, d, b)$ for all $b > 0$ and some collection of ordinals $\gamma < \alpha$ (specified in the formal statements below). The goal is to show that there exists a map in $\mathcal{S}(\alpha, d, a)$ for the ordinal α and an arbitrary $a > 0$. We start by choosing f in $\mathcal{S}(\gamma_0, d, a_0)$ and a sequence of maps $\{\chi_m\}_{m\in\mathbb{N}}$ such that χ_m is in $\mathcal{S}(\gamma_m, d, a_m)$ (for some well-chosen ordinals $\{\gamma_i\}_{i\geq 0}$ and real numbers $\{a_i\}_{i\geq 0}$). Then we perform the "blow-and-sew" operation on f and $\{\chi_m\}_{m\in\mathbb{N}}$, in which we "blow up" a sequence $\{\theta_m\}_{m\in\mathbb{N}}$ of f-periodic orbits into tiny discs and "sew in" a tower over the map χ_m on the discs corresponding to θ_m. This construction is executed in such a way that from the point of view of invariant measures, the resulting map F looks like a countable, disjoint union consisting of a principal extension over f and towers over the maps $\{\chi_m\}_{m\in\mathbb{N}}$. Using this decomposition and the inductive hypotheses, we can prove that F is in $\mathcal{S}(\alpha, d, a)$, as desired. The rest of the body the chapter is organized as follows:

- In Section 3.4, we analyze the entropy structures and transfinite sequences that arise from principal extensions and towers.

- In Section 3.5, the "blow-and-sew" construction is described in general, and many properties of the resulting map F are deduced that will be used in the following sections.

- In Section 3.6, it is shown that if f and the maps $\{\chi_m\}_{m\in\mathbb{N}}$ satisfy certain properties, mostly involving their invariant measures and transfinite sequences, then F also satisfies some desirable properties involving its invariant measures and transfinite sequences.

- Section 3.7 combines Sections 3.4 - 3.6 and actually carries out the transfinite induction scheme.

3.4 Principal extensions and towers

Definition 3.4.1. Let (X,T) be a factor of (Y,S) with factor map π. The system (Y,S) is a **principal extension** of (X,T) if $h^T(\pi\mu) = h^S(\mu)$ for all μ in $M(Y,S)$.

If (Y,S) is a principal extension of (X,T), then we may refer to S as a principal extension of T. The following fact is a basic result in the theory of entropy structures.

Fact 3.4.2 ([35]). *If S is a principal extension of T with factor map π and $\mathcal{H}(T)$ is an entropy structure for T, then $\mathcal{H}(T)\circ\pi$ is an entropy structure for S.*

Lemma 3.4.3. *Let (X,f) be a topological dynamical system. Suppose there exists a compact subset C of $M(X,f)$ such that for each ordinal γ and each measure μ in $M(X,f)$,*

$$u_\gamma^{\mathcal{H}(f)}(\mu) = \int_C u_\gamma^{\mathcal{H}(f)|_C}\, d\mathcal{P}_\mu. \tag{3.4.1}$$

Let (Y,F) be a principal extension of (X,f) with factor map π and induced map $M(Y,F)\to M(X,f)$ also denoted by π. Suppose that $\pi|_{\pi^{-1}(C)}$ is a homeomorphism *onto C. Then for each ordinal γ and each measure ν in $M(Y,F)$,*

$$u_\gamma^{\mathcal{H}(F)}(\nu) = \int_{\pi^{-1}(C)} u_\gamma^{\mathcal{H}(F)|_{\pi^{-1}(C)}} \, d\mathcal{P}_\nu = u_\gamma^{\mathcal{H}(f)}(\pi(\nu)).$$

Proof. Let $\mathcal{H}(f)$ be an entropy structure for f, and let $\mathcal{H}(F) = \mathcal{H}(f) \circ \pi$, which is an entropy structure for F by Fact 3.4.2. By monotonicity (Fact 3.2.19 (2)), $u_\gamma^{\mathcal{H}(F)}(x) \geq u_\gamma^{\mathcal{H}(F)|_{\pi^{-1}(C)}}(x)$ for all x in $\pi^{-1}(C)$. Since $\pi|_{\pi^{-1}(C)}$ is a homeomorphism onto C, $u_\gamma^{\mathcal{H}(F)|_{\pi^{-1}(C)}} = u_\gamma^{\mathcal{H}(f)|_C} \circ \pi$ (Fact 3.2.19 (4)). Combining these facts with Equation (3.4.1) and the fact that $u_\gamma^{\mathcal{H}(F)}$ is concave (Fact 3.2.19 (5)), we obtain that for all ordinals γ and all ν in $M(Y,F)$,

$$\begin{aligned} u_\gamma^{\mathcal{H}(F)}(\nu) &\geq \int_{\pi^{-1}(C)} u_\gamma^{\mathcal{H}(F)|_{\pi^{-1}(C)}} \, d\mathcal{P}_\nu = \int_{\pi^{-1}(C)} u_\gamma^{\mathcal{H}(f)|_C} \circ \pi \, d\mathcal{P}_\nu \\ &= \int_C u_\gamma^{\mathcal{H}(f)|_C} \, d\mathcal{P}_{\pi\nu} = u_\gamma^{\mathcal{H}(f)}(\pi\nu). \end{aligned}$$

Since π is continuous and surjective, $u_\gamma^{\mathcal{H}(F)} \leq u_\gamma^{\mathcal{H}(f)} \circ \pi$ (Fact 3.2.19 (3)). Combining the above inequalities, we obtain that $u_\gamma^{\mathcal{H}(F)} = u_\gamma^{\mathcal{H}(f)} \circ \pi$, and all of the above inequalities are equalities. This concludes the proof of the lemma. □

Now we turn our attention to simple towers. We begin with a definition.

Definition 3.4.4. Let (X,T) be a topological dynamical system. Let n and p be natural numbers with $p \leq n$. Let $Y = X \times \{0,\ldots,n-1\}$. We define a map $S : Y \to Y$ as follows. Let $S(x,i) = (x,i+1)$ for all x in X and $i \in \{0,\ldots,n-2\}$. For each x in X, let $S(x,n-1) = (T^p(x),0)$. We will refer to (Y,S) (or possibly just S) as an (n,p) **tower over** (X,T) (or possibly just T).

Notation 3.4.5. Suppose (Y,S) is an (n,p) tower over (X,T). Let $Y_0 = X \times \{0\}$, and note that Y_0 is invariant under S^n. Let $\pi_1 : M(Y,S) \to M(Y_0, S^n|_{Y_0})$ be the map given by $\mu \mapsto \mu|_{Y_0}$. Let $\pi_2 : Y_0 \to X$ be projection onto X. With π_2 as the factor map, $(Y, S^n|_{Y_0})$ is a principal extension over (X, T^p). Note that the maps π_1 and π_2 on measures are affine homeomorphisms. Further, recall that if μ is in $M(Y, S)$, then the measure-preserving systems (S, μ) and $(T^p, \pi_2 \circ \pi_1(\mu))$ are measure-theoretically isomorphic. Let $\pi_3 : M(Y, T^p) \to M(X, T)$ be the map $\pi_3(\mu) = \frac{1}{p} \sum_{i=0}^{p-1} T^i \mu$.

Definition 3.4.6. If S is a tower over T with notation as above, then the map $\psi = \pi_3 \circ \pi_2 \circ \pi_1$ will be referred to as **the map associated to the tower S over T**.

Lemma 3.4.7. *Let (X, T) be a topological dynamical system with entropy structure $\mathcal{H}(T)$. Then $p\mathcal{H}(T) \circ \pi_3$ is an entropy structure for T^p, where $\pi_3 : M(X, T^p) \to M(X, T)$ is defined by $\pi_3(\mu) = \frac{1}{p} \sum_{i=0}^{p-1} T^i \mu$.*

Proof. It is shown in [13] that every finite entropy dynamical system has a zero-dimensional principal extension. (In fact, [13] deals only with homeomorphisms, but the natural extension $\overline{T}$ of a continuous surjection T is a homeomorphism and a principal extension of T, and then applying [13] to $\overline{T}$ yields a zero-dimensional principal extension of T.) Applying this fact to (X, T), we fix a zero-dimensional principal extension (X', T') of (X, T) with factor map π. Then $(X', (T')^p)$ is a zero-dimensional principal extension of (X, T^p) with factor map π. We let π_3 denote the averaging map from $M(X', (T')^p)$ to $M(X', T')$ as well as the averaging map from $M(X, T^p)$ to $M(X, T)$. Note that $\pi \circ \pi_3 = \pi_3 \circ \pi$. Now let $\mathcal{H}(T)$ be an entropy structure for T and let $\mathcal{H}(T^p)$ be an entropy structure for T^p. We prove the lemma by showing that $\mathcal{H}(T^p)$ is uniformly equivalent to $p\mathcal{H}(T) \circ \pi_3$. Since (X', T') is zero dimensional, there exists a refining sequence $\{P_k\}_{k\in\mathbb{N}}$ of clopen partitions of X' with diameters tending to 0. Let $\mathcal{H}(T') = (h_k^{T'})$ and $\mathcal{H}((T')^p) = (h_k^{(T')^p})$ be the entropy structures (for T' and $(T')^p$ respectively) defined by this sequence of partitions, *i.e.* $h_k^{T'}(\mu) = h^{T'}(\mu, P_k)$ and $h_k^{(T')^p}(\mu) = h^{(T')^p}(\mu, P_k)$. Then for any μ in $M(X', (T')^p)$, we have that $h_k^{(T')^p}(\mu) = h^{(T')^p}(\mu, P_k) = ph^{T'}(\pi_3(\mu), P_k) = ph_k^{T'}(\pi_3(\mu))$. Thus $\mathcal{H}((T')^p)$ is uniformly equivalent to $p\mathcal{H}(T')\circ\pi_3$. Since T' is a principal extension of T and $(T')^p$ is a principal extension of T^p, both with factor map π, we have that $\mathcal{H}(T') \cong \mathcal{H}(T)\circ\pi$ and $\mathcal{H}((T')^p) \cong \mathcal{H}(T^p)\circ\pi$. Combining these facts, we obtain that $p\mathcal{H}(T)\circ\pi\circ\pi_3 \cong \mathcal{H}(T^p)\circ\pi$. Since $\pi\circ\pi_3 = \pi_3\circ\pi$, we see that $p\mathcal{H}(T)\circ\pi_3\circ\pi \cong \mathcal{H}(T^p)\circ\pi$. Using this fact and the definition of uniform equivalence, we see that $p\mathcal{H}(T)\circ\pi_3 \cong \mathcal{H}(T^p)$, which finishes the proof of the lemma. □

Lemma 3.4.8. *Let (Y, S) be an (n, p) tower over (X, T) with associated map $\psi : M(Y, S) \to M(X, T)$, and let $\mathcal{H}(T)$ be an entropy structure for T. Then $\frac{p}{n}\mathcal{H}(T)\circ\psi$ is an entropy structure for S.*

Proof. We use Notation 3.4.5. Note that the maps π_1, π_2 and π_3 are each continuous and affine. For any entropy structure $\mathcal{H}(S^n)$ of S^n, we have that $\frac{1}{n}\mathcal{H}(S^n)\circ\pi_1$ is an entropy structure for S (Theorem 5.0.3 (3) in [35]). If $\mathcal{H}(T^p)$ is an entropy structure for T^p, then $\mathcal{H}(T^p)\circ\pi_2$ is an entropy structure for S^n by Fact 3.4.2, since S^n is a principal extension of T^p with factor map π_2. By Lemma 3.4.7, we have that if $\mathcal{H}(T)$ is an entropy structure for T, then $p\mathcal{H}(T)\circ\pi_3$ is an entropy structure for T^p. Combining these facts, we obtain that if $\mathcal{H}(T)$ is an entropy structure for T, then $\frac{p}{n}\mathcal{H}(T)\circ\psi$ is an entropy structure for S. □

Lemma 3.4.9. *Let (Y,S) be an (n,p) tower over (X,T) with associated map $\psi : M(Y,S) \to M(X,T)$. Let $\{\theta_m\}_{m\in\mathbb{N}}$ be a sequence of periodic orbits of T. Let $C(T) = \cap_{n=1}^{\infty}\overline{\cup_{m\geq n}\{\mu_{\theta_m}\}}$. Suppose that each measure in $C(T)$ is totally ergodic. Further suppose that for all μ in $M(X,T)$ and all ordinals α,*

$$u_\alpha^{\mathcal{H}(T)}(\mu) = \int_{C(T)} u_\alpha^{\mathcal{H}(T)|_{C(T)}}\, d\mathcal{P}_\mu. \tag{3.4.2}$$

Let $\{\Theta_\ell\}$ be an enumeration of the S-periodic orbits in $\cup_m \theta_m \times \{0,\ldots,n-1\}$, and let $C(S) = \cap_{n=1}\overline{\cup_{\ell\geq n}\{\mu_{\Theta_\ell}\}}$. Then

(1) each ν in $C(S)$ is totally ergodic;

(2) ψ maps $C(S)$ homeomorphically onto $C(T)$;

(3) for all ν in $M(Y,S)$ and all ordinals α

$$u_\alpha^{\mathcal{H}(S)}(\nu) = \int_{C(S)} u_\alpha^{\mathcal{H}(S)|_{C(S)}}\, d\mathcal{P}_\nu = \frac{p}{n} u_\alpha^{\mathcal{H}(T)}(\psi(\nu)).$$

Proof. We use Notation 3.4.5. Let μ be in $C(T)$. Since μ is invariant for T, it is also invariant for T^p and we have $\pi(\mu) = \mu$. Further, μ is totally ergodic for T by hypothesis, and therefore μ is totally ergodic for T^p. Hence μ is an extreme point in $M(X,T^p)$. If there were any other measure ν in $\pi_3^{-1}(\mu)$, then we would have $\mu = \frac{1}{p}\sum_{k=0}^{p-1} T^k\nu$, and thus μ would be a non-trivial convex combination of measures in $M(X,T^p)$, which would be a contradiction. Hence $\pi_3^{-1}(\mu) = \{\mu\}$. Since $\pi_2 \circ \pi_1 : M(Y,S) \to M(X,T^p)$ is a homeomorphism, $(\pi_2\circ\pi_1)^{-1}(\mu)$ consists of exactly one measure ν. Since (S,ν) is measure-theoretically isomorphic to (T^p,μ) and μ is totally ergodic with respect to T^p, we have that ν is totally ergodic with respect to

S. Combining these facts, we obtain that $\psi^{-1}(\mu)$ consists of exactly one measure, which is totally ergodic for S.

The fact that $\psi^{-1}(\mu)$ consists of exactly one measure for each μ in $C(T)$ implies that $\psi^{-1}(C(T)) = C(S)$ and that ψ maps $C(S)$ bijectively onto $C(T)$. Since $C(S)$ is compact and ψ is continuous, we conclude that ψ maps $C(S)$ homeomorphically onto $C(T)$, which proves (2). The fact that $\psi^{-1}(\mu)$ is totally ergodic for S implies that each ν in $C(S)$ is totally ergodic for S, proving (1).

Now Lemma 3.4.8 implies that if $\mathcal{H}(T)$ is an entropy structure for T, then $\frac{p}{n}\mathcal{H}(T) \circ \psi$ is an entropy structure for S. Since $\psi|_{C(S)}$ is a homeomorphism onto $C(T)$, Fact 3.2.19 (4) implies that $u_\gamma^{\mathcal{H}(S)|_{C(S)}} = \frac{p}{n} u_\gamma^{\mathcal{H}(T)|_{C(T)}} \circ \psi|_{C(S)}$. Using this fact, as well as Equation (3.4.2) and Facts 3.2.19 (2), (3), and (5), we obtain that for any ν in $M(Y, S)$,

$$\begin{aligned} \frac{p}{n} \int_{C(T)} u_\gamma^{\mathcal{H}(T)|_{C(T)}} \, d\mathcal{P}_{\psi(\nu)} &= \int_{C(S)} u_\gamma^{\mathcal{H}(S)|_{C(S)}} \, d\mathcal{P}_\nu \leq u_\gamma^{\mathcal{H}(S)}(\nu) \\ &\leq \frac{p}{n} u_\gamma^{\mathcal{H}(T)}(\psi(\nu)) = \frac{p}{n} \int_{C(T)} u_\gamma^{\mathcal{H}(T)|_{C(T)}} \, d\mathcal{P}_{\psi(\nu)}. \end{aligned}$$

Thus the above inequalities are all equalities and the proof is complete. □

3.5 "Blow-and-sew"

We now begin building towards the proof of Theorem 3.3.1. The main idea of the proof is that we may "blow-up" periodic points (to intervals in dimension 1 and to discs in dimension 2) and "sew in" more complicated dynamical behavior, and in the process we increase the order of accumulation in a controlled way. In this section we describe and analyze the operation of "blowing up" a sequence of periodic orbits and "sewing in" other maps. The basic idea of this construction appears in Appendix C of [14]. In this section, we assume $d \in \{1, 2\}$.

Notation 3.5.1. Recall that we have adopted the convention that $\mathbb{N}$ denotes the set of positive integers and $\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$. Let $\mathbb{D}$ be the closed unit disc in $\mathbb{R}^d$. For a subset S of $\mathbb{R}^d$, let $\text{int}(S)$ denote the interior of S, and let ∂S be the boundary of S. For $r > 0$ and p in $\mathbb{R}^d$, we let $B(p, r)$ be the open ball of radius r centered at p. Given $s > 0$ and a point p in $\mathbb{R}^d$, let $A_{s,p}$ be the affine map of $\mathbb{R}^d$ given by $A_{s,p}(x) = sx + p$.

We consider maps in the following class.

Definition 3.5.2. Define $\mathcal{C}_d$ to be the class of functions $f : \mathbb{D} \to \mathbb{D}$ with the following properties:

(1) f is a continuous surjection, and if $d = 2$, then f is a homeomorphism;

(2) $f|_{\partial \mathbb{D}} = \text{Id}$;

(3) $\mathbf{h}_{\text{top}}(f) < \infty$.

Definition 3.5.3. Let $f : \mathbb{D} \to \mathbb{D}$ be continuous. Let $\{\theta_m\}_{m \in \mathbb{N}}$ be a sequence of periodic orbits for f, and let $S = \cup_m \theta_m$. We say that f is **ready for operation on** S if the following conditions are satisfied, where $Q = \cup_{k \geq 0} f^{-k}(S)$:

(1) for any ν in $\cap_{n=1}^{\infty} \overline{\cup_{m \geq n} \{\mu_{\theta_m}\}}$, it holds that $\nu(\cup_m \theta_m) = 0$;

(2) the set Q is countable and $Q \subset \text{int}(\mathbb{D})$;

(3) for each point x in Q, the derivative Df_x is invertible, and if $d = 2$, then $\det Df_x > 0$ for each point x in Q.

We remark that if $d = 2$, then in the above notation we have $Q = S$. To get non-zero orders of accumulation of entropy in dimension 1, we must look outside the class of homeomorphisms because a homeomorphism of the circle or the unit interval has zero entropy, and therefore its order of accumulation of entropy is zero.

3.5.1 The "blow-and-sew" construction

Proposition 3.5.4 carries out the "blow and sew" procedure. See Remark 3.5.6 for an informal interpretation of Proposition 3.5.4.

Proposition 3.5.4. *Suppose:*

- *f is a function in $\mathcal{C}_d$;*
- *$\{\theta_m\}_{m\in\mathbb{N}}$ is a sequence of periodic orbits for f, and f is ready for operation on $\cup_m \theta_m$;*
- *$\{\chi_m\}_{m\in\mathbb{N}}$ is a sequence of functions in $\mathcal{C}_d$;*
- *for each natural number m, the sequence $\{\theta_\ell^m\}_{\ell\in\mathbb{N}}$ is a sequence of periodic orbits for χ_m, and χ_m is ready for operation on $\cup_\ell\, \theta_\ell^m$;*
- *$\{\xi_m\}_{m\in\mathbb{N}}$ is a sequence of natural numbers satisfying $1 \le \xi_m \le |\theta_m|$ for each m in $\mathbb{N}$, where $|\theta_m|$ is the length of the periodic orbit θ_m.*
- $\sup_m \frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m) < \infty$

Let $Q = \cup_{m\in\mathbb{N}, j\geq 0} f^{-j}(\theta_m)$*, and let* $\{q_k\}_{k\in\mathbb{N}}$ *be an enumeration of* Q*. Then there exist functions* $F : \mathbb{D} \to \mathbb{D}$ *and* $\pi : \mathbb{D} \to \mathbb{D}$*, a sequence* $\{K_i\}_{i\geq 0}$ *of pairwise disjoint, compact subsets of* $\mathbb{D}$*, and a sequence* $\{\phi_m\}_{m\in\mathbb{N}}$ *of* C^∞ *diffeomorphisms,* $\phi_m : \mathbb{D} \times \{0, \ldots, |\theta_m| - 1\} \to K_m$*, such that the following hold, with* $L_k := \pi^{-1}(\{q_k\})$ *for each* k*:*

(1) F *is in* $\mathcal{C}_d$*;*

(2) π *is a factor map from* $(\mathbb{D}, F)$ *to* $(\mathbb{D}, f)$*;*

(3) $\pi(K_m) = \theta_m$*,* L_k *is* C^∞ *diffeomorphic to* $\mathbb{D}$ *for each* k*,* $\pi|_{\mathbb{D}\setminus(\cup_k L_k)}$ *is injective, and* $K_m \subset \cup_{q_k\in\theta_m} \mathrm{int}(L_k)$ *for each* m *in* $\mathbb{N}$*.*

(4) K_i *is* F*-invariant for each* $i \geq 0$*,* $K_0 = \mathbb{D} \setminus (\cup_{k=1}^\infty \mathrm{int}(L_k))$*, and* $\cup_k \partial L_k$ *is* F*-invariant.*

(5) $\mathrm{NW}(F) \subseteq \bigcup_{i\geq 0} K_i$*;*

(6) $F|_{K_0}$ *is a principal extension of* f *with factor map* $\pi|_{K_0}$*, and for* ν *in* $M(K_0 \setminus \cup_k \partial L_k, F)$*, the map* π *is a measure theoretic isomorphism between the measure preserving systems* (F, ν) *and* $(f, \pi(\nu))$*.*

(7) ϕ_m *is a topological conjugacy between* $F|_{K_m}$ *and a* $(|\theta_m|, \xi_m)$ *tower over* χ_m*, for each* m *in* $\mathbb{N}$*.*

(8) $\cap_{n=1}^\infty \overline{\cup_{m\geq n} M(K_m, F)} = (\pi)^{-1}(\cap_{n=1}^\infty \overline{\cup_{m\geq n}\{\mu_{\theta_m}\}}) \subset M(K_0 \setminus \cup_k \partial L_k, F)$*, and* π *maps* $\cap_{n=1}^\infty \overline{\cup_{m\geq n} M(K_m, F)}$ *homeomorphically onto* $\cap_{n=1}^\infty \overline{\cup_{m\geq n}\{\mu_{\theta_m}\}}$*;*

(9) F *is ready for operation on* $\cup_{m,\ell}\, \phi_m(\theta_\ell^m \times \{0, \ldots, |\theta_m| - 1\})$*.*

Remark 3.5.5. The notation in Proposition 3.5.4 is used repeatedly throughout the rest of this work, including the symbols used to index the various sequences. Notice that the only sequence of objects having index set different from $\mathbb{N}$ is $\{K_i\}_{i\geq 0}$. The symbol m is used to index the sequence of f-periodic orbits $\{\theta_m\}_{m\in\mathbb{N}}$, and any object with index m is associated to the periodic orbit θ_m. For example, for any m in $\mathbb{N}$, the set K_m is mapped onto θ_m under π. The fact that the index set for the sequence $\{K_i\}_{i\geq 0}$ includes 0 highlights the special role that K_0 plays in the construction (as it does not correspond to any periodic orbit). The symbol k is used to index the countable collection of periodic or pre-periodic points in Q, and any object with index k is associated to the point q_k.

Remark 3.5.6. Informally, we interpret Proposition 3.5.4 as follows. We begin with a map f, which has a distinguished sequence of periodic orbits $\{\theta_m\}_{m\in\mathbb{N}}$, and a sequence of maps $\{\chi_m\}_{m\in\mathbb{N}}$, each having a distinguished sequence of periodic orbits $\{\theta_\ell^m\}_{\ell\in\mathbb{N}}$ (because we want to use this statement in an induction). Then the proposition asserts the existence of another map F with the following properties: f is a factor of F (with map π), the non-wandering set of F is contained in $\cup_{i\geq 0}K_i$, F has essentially the same dynamics as f on K_0, and $F|_{K_m}$ is essentially a $(|\theta_m|, \xi_m)$ tower over χ_m. In this sense, we think of "blowing up" each periodic point in $\cup_m\theta_m$ into a disc. Then for each m in $\mathbb{N}$, we "sew in" a tower over χ_m (on K_m) inside the discs associated to θ_m to create the map F.

Proof of Proposition 3.5.4. Let f, $\{\theta_m\}_{m\in\mathbb{N}}$, $\{\chi_m\}_{m\in\mathbb{N}}$, $\{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}$, and $\{\xi_m\}_{m\in\mathbb{N}}$ be given as in the hypotheses. Let $Q = \cup_{m\in\mathbb{N},k\geq 0}f^{-k}(\theta_m)$, and let $Q = \{q_k\}_{k\in\mathbb{N}}$ be a enumeration of Q. The following lemma blows up each of the points in Q into a disc.

Lemma 3.5.7. *Let $Q = \{q_k\}_{k\in\mathbb{N}}$ be a sequence of points in the interior of $\mathbb{D}$. Then there exists a summable sequence $\{\epsilon_k\}_{k\in\mathbb{N}}$ of positive real numbers, a sequence $\{p_k\}_{k\in\mathbb{N}}$ of points in $\mathbb{D}$ such that $\overline{B(p_k,\epsilon_k)}$ is contained in $\operatorname{int}(\mathbb{D})$ for each k in $\mathbb{N}$, and a function $\pi : \mathbb{D} \to \mathbb{D}$ such that*

(1) π is continuous and surjective;

(2) $\pi^{-1}(\{q_k\}) = \overline{B(p_k,\epsilon_k)}$ for each k;

(3) $\pi|_{\mathbb{D}\setminus\cup_k\overline{B(p_k,\epsilon_k)}}$ is a homeomorphism onto its image, $\mathbb{D}\setminus Q$.

Proof. Let $Q = \{q_k\}_{k\in\mathbb{N}}$ be as in the hypotheses. Consider $\mathbb{R}^d \setminus \{0\}$ in polar coordinates: $(r,\theta) \in (0,\infty)\times S^{d-1}$. For n in $\mathbb{N}$ and $\epsilon > 0$, consider the function $R_{\epsilon,n} : \mathbb{R}^d \to \mathbb{R}^d$ given by $R_{\epsilon,n}(0) = 0$ and for (r,θ) in $\mathbb{R}^d\setminus\{0\}$,

$$R_{\epsilon,n}(r,\theta) = \begin{cases} 0, & \text{if } r \le \frac{\epsilon}{n} \\ (\frac{n}{n-1}(r-\frac{\epsilon}{n}),\theta), & \text{if } \frac{\epsilon}{n} \le r \le \epsilon \\ (r,\theta), & \text{otherwise.} \end{cases}$$

Let $S_{\epsilon,n} : \mathbb{R}^d \to \mathbb{R}^d$ be given by $S_{\epsilon,n}(0) = 0$ and $S_{\epsilon,n}(x) = R^{-1}_{\epsilon,n}(x)$ for $x \neq 0$. Now for p in $\mathbb{R}^d$, let $R_{\epsilon,n,p} : \mathbb{R}^d \to \mathbb{R}^d$ be defined by $R_{\epsilon,n,p}(x) = R_{\epsilon,n}(x-p)+p$. Also define $S_{\epsilon,n,p} : \mathbb{R}^d \to \mathbb{R}^d$ to be $S_{\epsilon,n,p}(x) = S_{\epsilon,n}(x-p)+p$. Note that $R_{\epsilon,n,p}$ is continuous on $\mathbb{R}^d$ and $S_{\epsilon,n,p}$ is continuous on $\mathbb{R}^d\setminus\{p\}$. Also, $S_{\epsilon,n,p}|_{\mathbb{R}^d\setminus\{p\}}$ is a homeomorphism onto its image, with inverse given by $R_{\epsilon,n,p}|_{\mathbb{R}^d\setminus\overline{B(0,\frac{1}{n}\epsilon)}}$. Moreover, we have

(i) $d(R_{\epsilon,n,p}(x), R_{\epsilon,n,p}(y)) \le \frac{n}{n-1}d(x,y)$;

(ii) $d(x, R_{\epsilon,n,p}(x)) \le \epsilon$;

(iii) $d(x, S_{\epsilon,n,p}(x)) \le \epsilon$.

Choose a sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that $\prod_{k=1}^{\infty} \frac{n_k-1}{n_k} > 0$. Let $C = \prod_{k=1}^{\infty} \frac{n_k}{n_k-1} < \infty$. We make the following inductive definitions. Let $\delta_1 > 0$ be such that $\operatorname{dist}(q_1, \partial\mathbb{D}) > \delta_1$. Let $f_1 = S_{\delta_1,n_1,q_1}$ and $g_1 = R_{\delta_1,n_1,q_1}$. If δ_k, f_k and g_k are defined, choose $\delta_{k+1} > 0$ such that $\delta_{k+1} < \operatorname{dist}(f_k(q_{k+1}), \partial\mathbb{D} \cup g_k^{-1}(\{q_1, \ldots, q_k\}))$ and let $f_{k+1} = S_{\delta_{k+1},n_{k+1},f_k(q_{k+1})} \circ f_k$ and $g_{k+1} = g_k \circ R_{\delta_{k+1},n_{k+1},f_k(q_{k+1})}$. We also require that $\{\delta_k\}_{k\in\mathbb{N}}$ is summable.

The properties (i)-(iii) above imply that for any $k_1 \le k_2$

(a) $d(g_{k_1}(x), g_{k_1}(y)) \le (\prod_{k=1}^{k_1} \frac{n_k}{n_k-1}) d(x, y)$;

(b) $d(g_{k_1}(x), g_{k_2}(x)) \le \sum_{k=k_1}^{k_2} \delta_k$;

(c) $d(f_{k_1}(x), f_{k_2}(x)) \le \sum_{k=k_1}^{k_2} \delta_k$.

For each k, f_k is continuous on $\mathbb{R}^d \setminus \{q_1, \ldots, q_k\}$ and g_k is continuous on $\mathbb{R}^d$. In fact, f_k is a homeomorphism from $\mathbb{R}^d \setminus \{q_1, \ldots, q_k\}$ to its image, and g_k is its inverse. Note that the sequences $\{f_k\}_{k\in\mathbb{N}}$ and $\{g_k\}_{k\in\mathbb{N}}$ are uniformly Cauchy by properties (b) and (c) above. Therefore the pointwise limits $f(x) = \lim_k f_k(x)$ and $g(x) = \lim_k g_k(x)$ exist for all x in $\mathbb{R}^d$. Since f_k is continuous on $\mathbb{R}^d \setminus Q$ for all k, and since $\{f_k\}_{k\in\mathbb{N}}$ is uniformly Cauchy, f is continuous on $\mathbb{R}^d \setminus Q$. The fact that g_k is continuous on $\mathbb{R}^d$ for each k and the sequence $\{g_k\}_{k\in\mathbb{N}}$ is uniformly Cauchy implies that g is continuous. Using the fact that $\delta_{k+1} < \operatorname{dist}\big(f_k(q_{k+1}), \partial\mathbb{D} \cup g_k^{-1}(\{q_1, \ldots, q_k\})\big)$ for each k, we observe that $f|_{\partial\mathbb{D}} = g|_{\partial\mathbb{D}} = \mathrm{Id}$ and if x is in $g_m^{-1}(\{q_k\})$ where $k \le m$, then $g_n(x) = g_m(x)$ for all $n \geq m$ and therefore $g(x) = g_m(x)$. This last observation means that if $g_m(x)$ is in Q for any m, then $g(x)$ is in Q. We now consider f and g restricted to $\mathbb{D}$, and note that f and g act by the identity map on $\partial\mathbb{D}$. Also, each g_k defines a continuous surjection and therefore g does as well.

Let us check that for x in $\mathbb{D}\backslash Q$, $g(f(x)) = x$. Note that $d(g_k(f_k(x)), g(f_k(x))) \leq \sum_{j=k}^{\infty} \delta_j$. Letting k tend to infinity and using the continuity of g, we obtain that $g(f(x)) = x$.

Finally we check that for x in $g^{-1}(\mathbb{D} \setminus Q)$, $f(g(x)) = x$. Let x be in $g^{-1}(\mathbb{D} \setminus Q)$. Let $\epsilon > 0$. Choose K so large that $\sum_{k \geq K} \delta_k < \epsilon/3$. Since $g(x)$ is not in Q, f_K is continuous at $g(x)$. Since f_K is continuous at $g(x)$, there exists $\delta > 0$ such that $d(y, g(x)) < \delta$ implies $d(f_K(y), f_K(g(x))) < \epsilon/3$. Then choose $M \geq K$ such that $d(g_M(x), g(x)) < \delta$. Then

$$
\begin{aligned}
d(x, f(g(x)) \leq d(f_M(g_M(x)), f_K(g_M(x)))+ \\
+ d(f_K(g_M(x)), f_K(g(x))) + d(f_K(g(x)), f(g(x))) \\
\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned}
$$

Since $\epsilon > 0$ was arbitrary, we have that $x = f(g(x))$.

Now let $\epsilon_k = \frac{\delta_k}{n_k}$, $p_k = f(q_k)$, and $\pi = g$. Note that the conclusions of the lemma are satisfied by these choices. □

3.5.1.1 Setup

Now we proceed with the proof of Proposition 3.5.4. Choose $\{\epsilon_k\}_{k\in\mathbb{N}}$, $\{p_k\}_{k\in\mathbb{N}}$, and π satisfying the assumptions and conclusions of Lemma 3.5.7. These objects will remain fixed throughout the rest of the proof.

For the sake of notation, let $L_k = \overline{B(p_k, \epsilon_k)}$ and $L = \cup_k L_k$. Note that $\mathrm{int}(L) = \cup_k B(p_k, \epsilon_k)$. Also, for each k in $\mathbb{N}$, we define the natural number $j(k)$ as the unique solution to the equation $f(q_k) = q_{j(k)}$.

3.5.1.2 Construction of F

We now construct $F : \mathbb{D} \to \mathbb{D}$. For x in $\mathbb{D} \setminus L$, let

$$F(x) = \pi^{-1} \circ f \circ \pi(x). \tag{3.5.1}$$

Since $L = \pi^{-1}(Q)$ and Q is completely invariant for f, we have that if x is in $\mathbb{D} \setminus L$ then $F(x)$ is in $\mathbb{D} \setminus L$. Note that F is continuous on $\mathbb{D} \setminus L$ as it is a composition of continuous functions (recall that $\pi^{-1}|_{\mathbb{D}\setminus Q}$ is continuous by Lemma 3.5.7 (3)). We now show that the function F can be extended to a continuous map on $\mathbb{D} \setminus \mathrm{int}(L)$ such that $F(\partial B(p_k, \epsilon_k)) = \partial B(p_{j(k)}, \epsilon_{j(k)})$.

Suppose $d = 1$ (the case $d = 2$ is treated below). Then $\partial B(p_k, \epsilon_k)$ is just the two endpoints of an interval. Because Df_{q_k} is invertible, f is either orientation preserving or orientation reversing at q_k. In either case, we extend F continuously at $\partial B(p_k, \epsilon_k)$ so that F maps $\partial B(p_k, \epsilon_k)$ bijectively to $\partial B(p_{j(k)}, \epsilon_{j(k)})$. Now we extend F to the one-dimensional annulus $\{x : \frac{1}{2}\epsilon_k \leq |x - p_k| \leq \epsilon_k\}$ as follows. Let $T^+ : [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1] \to [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$ be given by $T^+(x) = x + \frac{1}{10}\sin(2\pi|x|)$. Also, let $T^- = -x + \frac{1}{10}\sin(2\pi|x|)$. If $Df_{q_k} > 0$, let $\sigma = +$, and if $Df_{q_k} < 0$, let $\sigma = -$. Then for x such that $\frac{1}{2}\epsilon_k \leq |x - p_k| \leq \epsilon_k$, let

$$F(x) = \left(A_{\epsilon_{j(k)}, p_{j(k)}} \circ T^\sigma \circ A^{-1}_{\epsilon_k, p_k}\right)(x), \tag{3.5.2}$$

where $A_{s,x}$ is defined in Notation 3.5.1. We remark that the additional terms involving sine in the functions T^+ and T^- are introduced for technical convenience in proving Claim 3.5.12.

Now suppose $d = 2$. We have that $\det Df|_Q > 0$, which implies that for each k, we may extend F continuously on $\partial B(p_k, \epsilon_k)$ in the following way. There is an orientation preserving homeomorphism T_k of the unit circle such that for x in $\partial B(p_k, \epsilon_k)$, we let $F(x) = (A_{\epsilon_{j(k)},p_{j(k)}} \circ T_k \circ A^{-1}_{\epsilon_k,p_k})(x)$. Recall that any orientation preserving homeomorphism of the unit circle to itself is homotopic to the identity. Let $H_k : [\frac{1}{2}, 1] \times S^1$ be a homotopy such that $H_k(\frac{1}{2}, \cdot) = \text{Id}$, $H_k(1, \cdot) = T_k$, and $H_k(t, \cdot)$ is a homeomorphism for each t in $[\frac{1}{2}, 1]$. Now we extend F to the annulus $\{x : \frac{1}{2}\epsilon_k \leq |x - p_k| \leq \epsilon_k\}$ as follows. We consider the annulus centered at 0 with inner radius $\frac{1}{2}$ and outer radius 1 in polar coordinates: $\{(r, \theta) : r \in [\frac{1}{2}, 1], \theta \in S^1\} \subset \mathbb{R}^2$. For (r, θ) in this annulus, define $U_k(r, \theta) = \big(r + \frac{1}{10}\sin(2\pi r),\ H_k(r, \theta)\big)$. Now for x in $\mathbb{D}$ with $\frac{1}{2}\epsilon_k \leq |x - p_k| \leq \epsilon_k$, let

$$F(x) = \big(A_{\epsilon_{j(k)},p_{j(k)}} \circ U_k \circ A^{-1}_{\epsilon_k,p_k}\big)(x). \tag{3.5.3}$$

Up to this point in the construction, we have defined F on $\mathbb{D} \setminus \cup_k B(p_k, \frac{1}{2}\epsilon_k)$. Now let m be in $\mathbb{N}$ and suppose $\theta_m = \{q_{k_0}, \ldots, q_{k_{|\theta_m|-1}}\}$. Let $g_{k_{|\theta_m|-1}} : \mathbb{D} \to \mathbb{D}$ be $\chi_m^{\xi_m}$, and let g_{k_i} be the identity map on $\mathbb{D}$ for all $i \in \{0, \ldots, |\theta_m| - 2\}$. Making these choices for all m, we define g_k for all k such that q_k is in $\cup_m \theta_m$. For all k such that q_k is not in $\cup_m \theta_m$, let g_k be the identity map on $\mathbb{D}$. Now for each k in $\mathbb{N}$ and x in $B(p_k, \frac{1}{2}\epsilon_k)$, let

$$F(x) = A_{\frac{1}{2}\epsilon_{j(k)},p_{j(k)}} \circ g_k \circ A^{-1}_{\frac{1}{2}\epsilon_k,p_k}(x). \tag{3.5.4}$$

This concludes the construction of F.

3.5.1.3 Properties of F

In this section we prove that F has properties (1)-(9) in Proposition 3.5.4. For the sake of notation, we make some definitions. Let $K_0 = \mathbb{D} \setminus \mathrm{int}(L)$, as in the statement of the proposition. For each m in $\mathbb{N}$, let $K_m = \cup_{q_k \in \theta_m} \overline{B(p_k, \frac{1}{2}\epsilon_k)}$. The following claim follows directly from the construction of F.

Claim 3.5.8 (Part of property (1)). F is a continuous surjection, and if $d = 2$, then F is a homeomorphism. Also, $F|_{\partial\mathbb{D}} = \mathrm{Id}$.

Claim 3.5.9 (Property (2)). π is a factor map from $(\mathbb{D}, F)$ to $(\mathbb{D}, f)$.

Proof. By Lemma 3.5.7, the map π is continuous and surjective. For x in $\mathbb{D} \setminus L$, we have that $\pi(F(x)) = f(\pi(x))$ by definition (Equation (3.5.1)). For x in $\overline{B(p_k, \epsilon_k)}$, we have that $F(x)$ is in $\overline{B(p_{j(k)}, \epsilon_{j(k)})}$ by definition, and then $\pi(F(x)) = q_{j(k)} = f(q_k) = f(\pi(x))$, using property (3) in Lemma 3.5.7. □

Claim 3.5.10 (Property (3)). We have that $\pi(K_m) = \theta_m$ for each m in $\mathbb{N}$, L_k is C^∞ diffeomorphic to $\mathbb{D}$, $\pi|_{\mathbb{D}\setminus\mathrm{int}(L)}$ is injective, and $K_m \subset \mathrm{int}(\cup_{q_k \in \theta_m} L_k)$.

Proof. By property (3) in Lemma 3.5.7, we have $\pi(K_m) = \pi(\cup_{q_k \in \theta_m} \overline{B(p_k, \frac{1}{2}\epsilon_k)}) = \theta_m$. The second assertion follows immediately from the fact that $L_k = \overline{B(p_k, \epsilon_k)}$. The third assertion holds by Lemma 3.5.7 (3). The fourth assertion holds since $K_m = \cup_{q_k \in \theta_m} \overline{B(p_k, \frac{1}{2}\epsilon_k)} \subset \cup_{q_k \in \theta_m} L_k$. □

The following claim follows directly from the construction.

Claim 3.5.11 (Property (4)). K_i is F-invariant for each $i \geq 0$, $K_0 = \mathbb{D} \setminus \text{int}(L)$, and $\cup_k L_k$ is F-invariant.

Claim 3.5.12 (Property (5)). $\text{NW}(F) \subseteq \bigcup_{i \geq 0} K_i$.

Proof. If x is in $B(p_k, \epsilon_k)$ for some k such that q_k is not periodic, then x is wandering because q_k is pre-periodic. Now consider the periodic orbit θ_m. Recall that any point in $(\frac{1}{2}, 1)$ is wandering for the map $T(t) = t + \frac{1}{10}\sin(2\pi t - \pi)$. According to Equations (3.5.2) and (3.5.3), the radial component of F restricted to $\cup_{q_k \in \theta_m} B(p_k, \epsilon_k) \setminus \overline{B(p_k, \frac{1}{2}\epsilon_k)}$ is conjugate to a tower over T. It follows that any x in $\cup_{q_k \in \theta_m} B(p_k, \epsilon_k) \setminus \overline{B(p_k, \frac{1}{2}\epsilon_k)}$ is wandering, which means that $\text{NW}(F) \subset (K_0) \cup \left(\bigcup_m K_m\right)$. □

Claim 3.5.13 (Property (6)). $F|_{K_0}$ is a principal extension of f with factor map $\pi|_{K_0}$, and for ν in $M(K_0 \setminus \cup_k \partial L_k, F)$, it holds that π is a measure theoretic isomorphism between (F, ν) and $(f, \pi(\nu))$.

Proof. Let ν be in $M(K_0 \setminus \cup_k \partial L_k, F)$. By conclusion (3) of Lemma 3.5.7, the factor map π is injective on $K_0 \setminus \cup_k L_k$ and therefore defines a measure theoretic isomorphism between (F, ν) and $(f, \pi(\nu))$. An ergodic measure ν for $F|_{K_0}$ that is not in $M(K_0 \setminus \cup_k \partial L_k, F)$ has $\nu(\cup_k \partial L_k) = 1$, and therefore $h^F(\nu) = 0$. It follows that for every ν in $M(K_0, F)$, we have $h^F(\nu) = h^f(\pi(\nu))$.

□

Let $\theta_m = \{q_{k_0}, \ldots, q_{k_{|\theta_m|-1}}\}$ be a periodic orbit for f labeled in such a way that $f(q_{k_i}) = q_{k_{i+1}}$, where $i+1$ is taken modulo $|\theta_m|$. Let $\phi_m : \mathbb{D} \times \{0, \ldots, |\theta_m| - 1\} \to$

$\cup_{i=0}^{|\theta_m|-1}\overline{B(p_{k_i}, \frac{1}{2}\epsilon_{k_i})}$ be the map given by $\phi_m(x,i) = A_{\frac{1}{2}\epsilon_{k_i}, p_{k_i}}(x)$.

Claim 3.5.14 (Property (7)). $F|_{K_m}$ is topologically conjugate by the map ϕ_m to a $(|\theta_m|, \xi_m)$ tower over χ_m, for each m in $\mathbb{N}$.

Proof. By Equation (3.5.4), for any m and any k such that q_k is in θ_m, $F|_{\overline{B(p_k,\frac{1}{2}\epsilon_k)}} = A_{\frac{1}{2}\epsilon_{j(k)}, p_{j(k)}} \circ g_k \circ A^{-1}_{\frac{1}{2}\epsilon_k, p_k}$. Then by the choice of g_k, we have that F is topologically conjugate to a $(|\theta_m|, \xi_m)$ tower over χ_m, with the conjugacy given by the map ϕ_m. □

Claim 3.5.15 (Property (8)). Let $C_0 = \cap_{n=1}^{\infty}\overline{\cup_{m\geq n} M(K_m, F)}$ and also let $C(f) = \cap_{n=1}^{\infty}\overline{\cup_{m\geq n}\{\mu_{\theta_m}\}}$. Then $C_0 = \pi^{-1}(C(f)) \subset M(K_0 \setminus \cup_k \partial L_k, F)$, and π maps C_0 homeomorphically onto $C(f)$.

Proof. Let $\{\mu_{m_\ell}\}_{\ell\in\mathbb{N}}$ be a sequence of measures in $M(\mathbb{D}, F)$ tending to μ such that $\mu_{m_\ell} \in M(K_{m_\ell}, F)$ for each ℓ. Then the sequence $\{\pi(\mu_{m_\ell}) = \mu_{\theta_{m_\ell}}\}_{\ell\in\mathbb{N}}$ converges to $\pi(\mu)$ by the continuity of π, which shows that $C_0 \subset \pi^{-1}(C(f))$.

Now let μ be in $C(f)$, and let ν be in $\pi^{-1}(\mu)$. By property (1) in the definition of the statement that f is ready for operation on $\cup_m\theta_m$ (Definition 3.5.3), $\mu(\cup_m\theta_m) = 0$, and thus $\nu(L) = 0$. Therefore $\nu \in M(K_0 \setminus \cup_k \partial L_k, F)$, and we have shown that $\pi^{-1}(C(f)) \subset M(K_0 \setminus \cup_k \partial L_k, F)$. Since $\pi|_{\mathbb{D}\setminus\cup_k L_k}$ is a homeomorphism onto its image $\mathbb{D} \setminus Q$, we also have that for any μ in $C(f)$, the set $\pi^{-1}(\mu)$ consists of exactly one measure.

Now let $\mu_{\theta_{m_k}}$ converge to μ in $M(\mathbb{D}, f)$. By the previous statement, there exists a measure ν such that $\{\nu\} = \pi^{-1}(\mu)$. Now choose any sequence of measures $\{\nu_{m_k}\}_{k\in\mathbb{N}}$ such that ν_{m_k} is in $\pi^{-1}(\mu_{\theta_{m_k}})$ for each k. By the sequential compactness of $M(\mathbb{D}, F)$, any subsequence $\{\tau_n\}_{n\in\mathbb{N}}$ of $\{\nu_{m_k}\}_{k\in\mathbb{N}}$ has a subsequence $\{\tau_{n_\ell}\}_{\ell\in\mathbb{N}}$ that converges to some measure τ. By continuity of π, we have $\pi(\tau) = \mu$. Since $\pi^{-1}(\mu) = \{\nu\}$, we see that $\tau = \nu$. Since this holds for any subsequence of $\{\nu_{m_k}\}_{k\in\mathbb{N}}$, it follows that $\{\nu_{m_k}\}_{k\in\mathbb{N}}$ converges to ν. This argument shows that $C_0 \supset \pi^{-1}(C(f))$, and therefore $C_0 = \pi^{-1}(C(f))$ (since we showed the reverse inclusion at the beginning of this proof). Since π is surjective, we also have that $\pi(C_0) = C(f)$.

Now we have that $\pi|_{C_0}$ is a continuous bijective map from a compact space into a Hausdorff space. It follows that π maps C_0 homeomorphically onto its image $C(f)$, which completes the proof. □

Claim 3.5.16 (Property (1)). *F is in $\mathcal{C}_d$.*

Proof. Claims 3.5.12-3.5.14 and the variational principle imply that

$$\mathbf{h}_{\text{top}}(F) = \max\Big(\mathbf{h}_{\text{top}}(f),\ \sup_m \frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m)\Big).$$

The right-hand side of this equation is finite by hypothesis. Combining this fact with Claim 3.5.8, we obtain that F is in $\mathcal{C}_d$. □

Claim 3.5.17 (Property (9)). *F is ready for operation on $\cup_{m,\ell} \cup_{q_k\in\theta_m} A_{\frac{1}{2}\epsilon_k,p_k}(\theta^m_\ell)$.*

Proof. First note that F is in $\mathcal{C}_d$ by Claim 3.5.16. Also, we have that $S = \cup_{m,\ell} \cup_{q_k\in\theta_m} A_{\frac{1}{2}\epsilon_k,p_k}(\theta^m_\ell)$ is a countable collection of periodic points for F by Claim 3.5.14. Let $\{\Theta_i\}_{i\in\mathbb{N}}$ be an enumeration of the periodic points orbits in S, and let $C(F) = \cap_{n=1}^\infty \overline{\cup_{i\geq n}\{\mu_{\Theta_i}\}}$. Now we check that F satisfies the properties (1)-(3) in Definition 3.5.3.

Let ν be in $C(F)$. Let $C_0 = \cap_{n=1}^\infty \overline{\cup_{m\geq n} M(K_m, F)}$. Note that $C(F) \subset C_0 \cup \Big(\cup_{m\geq 1} M(K_m, F)\Big)$. By Claim 3.5.15, we have that $C_0 \subset M(K_0 \setminus \cup_k \partial L_k, F)$.

Thus if ν is in C_0, then $\nu(L) = 0$. Since $\cup_i \Theta_i \subset \cup_{m\geq 1} K_m \subset L$, it follows that $\nu(\cup_i \Theta_i) = 0$, which proves property (1) in Definition 3.5.3 in the case that ν is in C_0. Now suppose ν is in $M(K_m, F)$ for some $m \geq 1$. By Claim 3.5.14, we have that $F|_{K_m}$ is topologically conjugate to a tower over χ_m via the map ϕ_m. Any sequence $\{\Theta_{i_k}\}_{k\in\mathbb{N}}$ such that $\{\mu_{\Theta_{i_k}}\}_{k\in\mathbb{N}}$ converges to ν must eventually lie in K_m, and therefore $\nu(\cup_i \Theta_i) = 0$ because χ_m is ready for operation on $\cup_\ell \theta_\ell^m$.

To check that F satisfies property (2) in Definition 3.5.3, we note that $Q = \cup_{i,k} F^{-k}(\Theta_i)$ is countable and contained in $\text{int}(\mathbb{D})$ because f and χ_m satisfy these properties with their respective sequences of periodic points, $\{\theta_m\}_{m\in\mathbb{N}}$ and $\{\theta_\ell^m\}_{\ell\in\mathbb{N}}$.

To check Property (3) in Definition 3.5.3, we need to check that $DF|_x$ is continuous and invertible at each point x of Q and that $\det DF|_x > 0$ if $d = 2$. For each point x in Q there is an open set $B(p_k, \epsilon_k)$ containing x on which F is either affine or conjugate by affine maps to a tower over χ_m. Property (3) in Definition 3.5.3 is satisfied at x if F is affine on $B(p_k, \epsilon_k)$. If F is conjugate to a tower over χ_m on $B(p_k, \epsilon_k)$, then F satisfies property (3) of Definition 3.5.3 because χ_m satisfies this property, which extends to simple towers. □

3.5.1.4 Conclusion of the proof of Proposition 3.5.4

By Claims 3.5.8-3.5.16, properties (1)-(9) are satisfied for F, π, $\{K_i\}_{i\geq 0}$, and $\{\phi_m\}_{m\in\mathbb{N}}$. This completes the proof of Proposition 3.5.4. □

3.5.2 Additional properties of the blown-up map

Definition 3.5.18. Let f, $\{\theta_m\}_{m\in\mathbb{N}}$, $\{\chi_m\}_{m\in\mathbb{N}}$, $\{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}$, and $\{\xi_m\}_{m\in\mathbb{N}}$ satisfy the hypotheses of Proposition 3.5.4. Define $\mathcal{BL}(f,\{\theta_m\}_{m\in\mathbb{N}},\{\chi_m\}_{m\in\mathbb{N}},\{\theta_\ell^m\}_{m,\ell\in\mathbb{N}},\{\xi_m\}_{m\in\mathbb{N}})$ to be the set of functions F in $\mathcal{C}_d$ such that there exists π, $\{K_i\}_{i\geq 0}$, and $\{\phi_m\}_{m\in\mathbb{N}}$ as in the statement of Proposition 3.5.4. In these terms, Proposition 3.5.4 asserts that $\mathcal{BL}(f,\{\theta_m\}_{m\in\mathbb{N}},\{\chi_m\}_{m\in\mathbb{N}},\{\theta_\ell^m\}_{m,\ell\in\mathbb{N}},\{\xi_m\}_{m\in\mathbb{N}})$ is non-empty.

Lemma 3.5.19. *Let $F : D \to D$ be a continuous surjection of a compact metric space. Suppose that $\mathrm{NW}(F) \subseteq \sqcup_{i\geq 0} K_i$, where each K_i is compact, $F(K_i) = K_i$, and $K_i = \cup_{j=1}^{J_i} K_i^j$, where the sets $\{K_i^j\}_{j=1}^{J_i}$ are compact and pairwise disjoint. Also, suppose that $\lim_i \max_{1\leq j\leq J_i} \mathrm{diam}(K_i^j) = 0$. Then there exists an entropy structure (f_k) for F with the following property: for each k, there exists I such that if $i > I$ then $f_k|_{M(K_i,F)} \equiv 0$.*

Proof. Let (f_k) be the Katok entropy structure (see Definition 3.2.9) corresponding to a sequence $\{\epsilon_k\}_{k\in\mathbb{N}}$ of positive numbers that tends to 0. Let k be given. Since $\lim_i \max_{1\leq j\leq J_i} \mathrm{diam}(K_i^j) = 0$, there exists I such that $i > I$ implies that $\mathrm{diam}(K_i^j) < \epsilon_k$ for all $1 \leq j \leq J_i$. Then for $i > I$ and ergodic μ such that $\mathrm{supp}(\mu) \subset K_i$, we have that $h^F(\mu, \epsilon_k, \sigma) = 0$ because K_i is invariant and $\mathrm{diam}(K_i^j) < \epsilon_k$ for $1 \leq j \leq J_i$. Since this holds for ergodic measures μ with $\mathrm{supp}(\mu) \subset K_i$, it also holds for any invariant measure μ with $\mathrm{supp}(\mu) \subset K_i$ because f_k is harmonic, which completes the proof. □

Lemma 3.5.20. *Let $F : D \to D$ be a continuous surjection of a compact metric space. Suppose that $\mathrm{NW}(F) \subseteq \sqcup_{i\geq 0} K_i$, where each K_i is compact, $F(K_i) = K_i$,* *and* $K_i = \cup_{j=1}^{J_i} K_i^j$, *where the sets* $\{K_i^j\}_{j=1}^{J_i}$ *are compact and pairwise disjoint. Also, suppose that* $\lim_i \max_{1\leq j\leq J_i} \operatorname{diam}(K_i^j) = 0$. *For each* i *in* $\mathbb{Z}_{\geq 0}$ *fix a harmonic entropy structure* $\mathcal{H}^i = (h^i_\ell)$ *for* $F|_{K_i}$.

Then there exists a harmonic entropy structure $\mathcal{H}(F) = (h^F_k)$ *such that* $h^F_k(\mu) = h^0_k(\mu)$ *for* μ *with* $\operatorname{supp}(\mu) \subset K_0$, *and for every* i *in* $\mathbb{N}$, *there is a non-decreasing function* $\ell_i : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ *with the following properties:*

(1) if μ *is in* $M(D, F)$ *and* $\operatorname{supp}(\mu) \subset K_i$, *then* $h^F_k(\mu) = h^i_{\ell_i(k)}(\mu)$ *for every* k *in* $\mathbb{Z}_{\geq 0}$.

(2) for any k *in* $\mathbb{N}$, *there exists* I *in* $\mathbb{N}$ *such that* $\ell_i(k) = 0$ *for all* $i \geq I$.

Proof. Let $\mathcal{F} = (f_k)$ be a harmonic entropy structure for F with the property that for every k there exists I such that if $i > I$ then $f_k|_{M(K_i,F)} \equiv 0$ (such an entropy structure exists by Lemma 3.5.19). Let $\delta_k > 0$ be a sequence tending to 0. Let i be in $\mathbb{N}$. Since $(f_k|_{M(K_i,F)})$ and (h^i_ℓ) are both an entropy structures for $F|_{K_i}$, we have that $(f_k|_{M(K_i,F)})$ and (h^i_ℓ) are uniformly equivalent. Using the definition of uniform equivalence (in particular the fact that $(f_k|_{M(K_i,F)})$ is uniformly dominated by (h^i_ℓ)), we define $\ell_i(k) = \min\{\ell \geq 0 : h^i_\ell \geq f_k|_{K_i} - \delta_k\}$ for each k in $\mathbb{Z}_{\geq 0}$. By construction, ℓ_i is non-decreasing. For ergodic measures μ in $M(K_i, F)$, let $h^F_k(\mu) = h^i_{\ell_i(k)}(\mu)$. For ergodic μ in $M(K_0, F)$, let $h^F_k(\mu) = h^0_k(\mu)$.

Since every ergodic measure for F is in $\cup_i M(K_i, F)$, we have defined h^F_k for all ergodic measures. Define h^F_k on all non-ergodic measures by harmonic extension, and let $\mathcal{H}(F) = (h^F_k)$. Note that since $h^i_{\ell_i(k)}$ is harmonic, for μ in $M(K_i, F)$, we have that $h^F_k(\mu) = h^i_{\ell_i(k)}(\mu)$ (which shows that if (h^F_k) is an entropy structure, then it satisfies property (1) by definition). By construction, $\mathcal{H}(F)$ is harmonic. It remains to check that $\mathcal{H}(F)$ is an entropy structure for F.

We show that $\mathcal{H}(F)$ is uniformly equivalent to $\mathcal{F}$, which implies that $\mathcal{H}(F)$ is an entropy structure for F. Since $\mathcal{F}$ and $\mathcal{H}(F)$ are harmonic, we may restrict attention to ergodic measures. Fix k and $\epsilon > 0$, and choose $k' \geq k$ large enough that $\delta_{k'} < \epsilon$. Then for every ergodic μ, we have that μ is in some $M(K_i, F)$, and $h^F_{k'}(\mu) = h^i_{\ell_i(k')}(\mu) \geq f_{k'}(\mu) - \delta_{k'} \geq f_k(\mu) - \epsilon$. Hence $\mathcal{H}(F) \geq \mathcal{F}$. Again, fix k and $\epsilon > 0$. Choose I such that $f_k|_{M(K_i,F)} \equiv 0$ for all $i > I$ (such an I exists by the choice of the sequence (f_k)). Then it follows from the definition of $\ell_i(k)$ that $\ell_i(k) = 0$ for all $i > I$ (showing property (2)). Using that $(f_k|_{M(K_i,F)})$ and (h^i_ℓ) are uniformly equivalent for each $i \leq I$ (in particular, $(f_k|_{M(K_i,F)})$ uniformly dominates (h^i_ℓ)), there exists k_i such that $f_{k_i}|_{M(K_i,F)} \geq h^i_{\ell_i(k)} - \epsilon$. Let $k' = \max(k_0, \dots, k_I)$. Any ergodic measure μ is in $M(K_i, F)$ for some i. Let μ be ergodic in and contained in $M(K_i, F)$. If $i \leq I$, then $f_{k'}(\mu) \geq f_{k_i}(\mu) \geq h^i_{\ell_i(k)}(\mu) - \epsilon = h^F_k(\mu) - \epsilon$. If $i > I$, then $f_{k'}(\mu) = 0 \geq -\epsilon = h^i_{\ell_i(k)}(\mu) - \epsilon = h^F_k(\mu) - \epsilon$. Since these same bounds hold for all ergodic μ, we have that $f_{k'} \geq h^F_k - \epsilon$, and we have shown that $\mathcal{F}$ uniformly dominates $\mathcal{H}(F)$. Then $\mathcal{F}$ and $\mathcal{H}(F)$ are uniformly equivalent, and we conclude that $\mathcal{H}(F)$ is an entropy structure for F. This concludes the proof of the lemma. □

Proposition 3.5.21. *Let f, $\{\theta_m\}_{m\in\mathbb{N}}$, $\{\chi_m\}_{m\in\mathbb{N}}$, $\{\theta^m_\ell\}_{m,\ell\in\mathbb{N}}$, $\{\xi_m\}_{m\in\mathbb{N}}$, F, π, $\{K_i\}_{i\geq 0}$, and $\{\phi_m\}_{m\in\mathbb{N}}$ all be as in Proposition 3.5.4. For each m in $\mathbb{N}$, let $S_m = \phi_m^{-1} \circ F|_{K_m} \circ \phi_m$ and let ψ_m be the map associated to the tower S_m over χ_m (Definition 3.4.6). For each m in $\mathbb{N}$, let $\mathcal{H}(\chi_m) = (h^{\chi_m}_k)$ be a harmonic entropy structure for χ_m, and let*

$\mathcal{H}(f) = (h_k^f)$ *be a harmonic entropy structure for* f. *Then there exists a harmonic entropy structure* $\mathcal{H}(F) = (h_k^F)$ *for* F *such that*

(1) for μ *with* $\operatorname{supp}(\mu) \subset K_0$, $h_k^F(\mu) = h_k^f(\pi(\mu))$;

(2) for every m *and* k, *there exists* k' *such that for each* μ *with* $\operatorname{supp}(\mu) \subset K_m$, $h_k^F(\mu) = \frac{\xi_m}{|\theta_m|} h_{k'}^{\chi_m}(\psi_m((\phi_m^{-1})(\mu)))$;

(3) for every k *there exists* m_0 *such that if* $m \geq m_0$ *and* $\operatorname{supp}(\mu) \subset K_m$, *then* $h_k^F(\mu) = 0$.

Proof. By Fact 3.4.2, $(h_k^f \circ \pi)$ is an entropy structure for $F|_{K_0}$. By Lemma 3.4.8, $(\frac{\xi_m}{|\theta_m|} h_k^{\chi_m} \circ \psi_m)$ is an entropy structure for S_m. Since ϕ_m is a topological conjugacy between S_m and $F|_{K_m}$, we have that $(\frac{\xi_m}{|\theta_m|} h_k^{\chi_m} \circ \psi_m \circ (\phi_m^{-1}))$ is an entropy structure for $F|_{K_m}$. Then Lemma 3.5.20 gives that these entropy structures can be combined to form an entropy structure for F satisfying properties (1)-(3). □

The following corollary is a consequence of Proposition 3.5.21, but one may also check it directly as in the proof of Claim 3.5.16.

Corollary 3.5.22. *Let* f, $\{\theta_m\}_{m\in\mathbb{N}}$, $\{\chi_m\}_{m\in\mathbb{N}}$, $\{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}$, *and* $\{\xi_m\}_{m\in\mathbb{N}}$ *satisfy the hypotheses of Proposition 3.5.4. Further, let* F *be an element of the set* $\mathcal{BL}(f, \{\theta_m\}_{m\in\mathbb{N}}, \{\chi_m\}_{m\in\mathbb{N}}, \{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}, \{\xi_m\}_{m\in\mathbb{N}})$. *Then*

$$\mathbf{h}_{\mathrm{top}}(F) = \max\Big(\mathbf{h}_{\mathrm{top}}(f),\ \sup_m \frac{\xi_m}{|\theta_m|}\mathbf{h}_{\mathrm{top}}(\chi_m)\Big)$$

The following lemma is used to compute the transfinite sequence associated to some of the systems in Section 3.6. In this lemma we combine our lemma for principal extensions (Lemma 3.4.3) and our lemma for towers (Lemma 3.4.9) with our analysis of the "blow-and-sew" construction (Proposition 3.5.4) to give a precise description of the measures and transfinite sequences of some maps constructed by the "blow-and-sew" operation.

Lemma 3.5.23. *Let f, $\{\theta_m\}_{m\in\mathbb{N}}$, $\{\chi_m\}_{m\in\mathbb{N}}$, $\{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}$, $\{\xi_m\}_{m\in\mathbb{N}}$, F, π, $\{K_i\}_{i\geq 0}$, and $\{\phi_m\}_{m\in\mathbb{N}}$ all be as in Proposition 3.5.4. Let $\{\Theta_k\}_{k\in\mathbb{N}}$ be an enumeration of the F-periodic orbits in $\cup_{m,\ell}\,\phi_m(\theta_\ell^m \times \{0,\dots,|\theta_m|-1\})$. Let $M = \cup_i M(K_i, F)$, $C(f) = \cap_{n=1}^\infty \overline{\cup_{m\geq n}\{\mu_{\theta_m}\}}$, $C(\chi_m) = \cap_{n=1}^\infty \overline{\cup_{\ell\geq n}\{\mu_{\theta_\ell^m}\}}$, and $C(F) = \cap_{n=1}^\infty \overline{\cup_{k\geq n}\{\mu_{\Theta_k}\}}$. Suppose that*

(i) each μ in $C(f)$ is totally ergodic for f;

(ii) for each μ in $M(\mathbb{D}, f)$ and each ordinal γ,

$$u_\gamma^{\mathcal{H}(f)}(\mu) = \int_{C(f)} u_\gamma^{\mathcal{H}(f)|_{C(f)}}\, d\mathcal{P}_\mu;$$

(iii) each μ in $C(\chi_m)$ is totally ergodic for χ_m;

(iv) for each μ in $M(\mathbb{D}, \chi_m)$ and each ordinal γ,

$$u_\gamma^{\mathcal{H}(\chi_m)}(\mu) = \int_{C(\chi_m)} u_\gamma^{\mathcal{H}(\chi_m)|_{C(\chi_m)}}\, d\mathcal{P}_\mu;$$

(v) either $\mathbf{h}_{\text{top}}(F|_{K_m})$ tends to 0 as m tends to infinity, or for each $m \geq 1$, $\alpha_0(F|_{K_m}) = 0$ and $\mathbf{h}_{\text{top}}(F|_{K_m}) = h^{F|_{K_m}}(\mu)$ for μ in $C(F) \cap M(K_m, F)$.

Then

(1) each measure ν in $C(F)$ is totally ergodic for F;

(2) $C(F) = \pi^{-1}(C(f)) \cup \left(\bigcup_m \phi_m(\psi_m^{-1}(C(\chi_m)))\right)$*;*

(3) π *maps* $C(F) \cap M(K_0, F)$ *homeomorphically onto* $C(f)$*;*

(4) $\psi_m \circ (\phi_m)^{-1}$ *maps* $C(F) \cap M(K_m, F)$ *homeomorphically onto* $C(\chi_m)$*.*

(5) for all x *in* M *and all ordinals* γ*,*

$$u_\gamma^{\mathcal{H}(F)|_M}(x) \leq \int_{C(F)} u_\gamma^{\mathcal{H}(F)|_{C(F)}} \, d\mathcal{P}_x.$$

Proof. Let ν be in $C(F)$. Since $\cap_{n=1}^{\infty}\overline{\cup_{m\geq n} M(K_m, F)} \subset M(K_0, F)$ (conclusion (8) in Proposition 3.5.4), we have that ν is in $M(K_i, F)$ for some i.

Suppose ν is in $M(K_0, F)$. Then ν is in $M(K_0 \setminus \cup_k \partial L_k, F)$ and $\pi(\nu)$ is in $C(f)$ by conclusion (8) in Proposition 3.5.4. By conclusion (6) in Proposition 3.5.4, we have that π gives a measure preserving isomorphism between ν and $\pi(\nu)$. The fact that ν is totally ergodic now follows from the hypothesis that $\pi(\nu)$ is totally ergodic (since it is in $C(f)$).

Now suppose that ν is in $M(K_m, F)$ for some m in $\mathbb{N}$. By conclusion (7) in Proposition 3.5.4, the map ϕ_m is a topological conjugacy between $F|_{K_m}$ and a $(|\theta_m|, \xi_m)$ tower over χ_m. By Lemma 3.4.9, $(\phi_m^{-1})(\nu)$ is totally ergodic, and therefore ν is totally ergodic, proving (1).

Property (3) is contained in conclusion (8) of Proposition 3.5.4. Using that ϕ_m is a topological conjugacy between $F|_{K_m}$ and a tower over χ_m, we obtain property (4) from Lemma 3.4.9 (2). Then property (2) follows from properties (3) and (4) the fact that $C(F) = \cup_{i\geq 0} M(K_i, F) \cap C(F)$.

Now we prove (5). First note that for $m \geq 1$, $M(K_m, F)$ is open in M, since $\cap_{n=1}^{\infty} \overline{\cup_{m \geq n} M(K_m, F)} \subset M(K_0, F)$ (conclusion (8) in Proposition 3.5.4). Then Fact 3.2.19 (1) implies that for all x in $M(K_m, F)$, $u_\gamma^{\mathcal{H}(F)|_M}(x) = u_\gamma^{\mathcal{H}(F)|_{M(K_m,F)}}(x)$. Furthermore, Lemma 3.4.9 (2) and monotonicity (Fact 3.2.19 (2)) give that for x in $M(K_m, F)$,

$$u_\gamma^{\mathcal{H}(F)|_{M(K_m,F)}}(x) = \int_{C(F) \cap M(K_m,F)} u_\gamma^{\mathcal{H}(F)|_{C(F) \cap M(K_m,F)}} \, d\mathcal{P}_x \tag{3.5.5}$$

$$\leq \int_{C(F)} u_\gamma^{\mathcal{H}(F)|_{C(F)}} \, d\mathcal{P}_x, \tag{3.5.6}$$

which gives the desired inequality for all x in $\cup_{m \geq 1} M(K_m, F)$.

Next, note that $M(K_0, F) \setminus C(F)$ is open in M (by Proposition 3.5.4 (8)). Then Fact 3.2.19 (1) gives that for all x in $M(K_0, F)$, $u_\gamma^{\mathcal{H}(F)|_M}(x) = u_\gamma^{\mathcal{H}(F)|_{M(K_0,F)}}(x)$. By Lemma 3.4.3 and Fact 3.2.19 (2), we obtain that for all x in $M(K_0, F) \setminus C(F)$,

$$u_\gamma^{\mathcal{H}(F)|_{M(K_0,F)}}(x) = \int_{C(F) \cap M(K_0,F)} u_\gamma^{\mathcal{H}(F)|_{C(F) \cap M(K_0,F)}} \, d\mathcal{P}_x$$

$$\leq \int_{C(F)} u_\gamma^{\mathcal{H}(F)|_{C(F)}} \, d\mathcal{P}_x,$$

which gives the desired inequality for all x in $M(K_0, F) \setminus C(F)$.

Lastly, we show (5) for all x in $C(F) \cap M(K_0, F)$ using transfinite induction. Note that $C(F) \cap M(K_0, F) \subset M_{\text{erg}}(\mathbb{D}, F)$, and therefore $\mathcal{P}_x$ is just the point mass at x. Thus for x in $C(F) \cap M(K_0, F)$ property (5) is equivalent to $u_\gamma^{\mathcal{H}(F)|_M}(x) \leq u_\gamma^{\mathcal{H}(F)|_{C(F)}}(x)$. Property (5) holds trivially for $\gamma = 0$. Now suppose for the sake of induction it holds for an ordinal γ, and we show it holds for $\gamma + 1$. For the sake of notation, let $M_i = M(K_i, F) \setminus C(F)$. Let x be in $C(F) \cap M(K_0, F)$. Then using the induction hypothesis and our computation of the transfinite sequence for y in

$M \setminus (C(F) \cap M(K_0, F))$,

$$\limsup_{\substack{y\to x\\ y\in M}} (u_\gamma^{\mathcal{H}(F)|_M} + \tau_k)(y)$$

$$= \max\Big(\limsup_{\substack{y\to x\\ y\in C(F)}} (u_\gamma^{\mathcal{H}(F)|_M} + \tau_k)(y),\ \limsup_{\substack{y\to x\\ y\in M_0}} (u_\gamma^{\mathcal{H}(F)|_M} + \tau_k)(y),$$

$$\limsup_{\substack{y\to x\\ y\in \cup_{m\geq 1} M_m}} (u_\gamma^{\mathcal{H}(F)|_M} + \tau_k)(y)\Big)$$

$$\leq \max\Big(\limsup_{\substack{y\to x\\ y\in C(F)}} (u_\gamma^{\mathcal{H}(F)|_{C(F)}} + \tau_k)(y),\ \limsup_{\substack{y\to x\\ y\in M_0}} (u_\gamma^{\mathcal{H}(F)|_{M(K_0,F)}} + \tau_k)(y),$$

$$\limsup_{\substack{y_{m_\ell}\to x\\ y_{m_\ell}\in M_{m_\ell}}} (u_\gamma^{\mathcal{H}(F)|_{M(K_{m_\ell},F)}} + \tau_k)(y_{m_\ell})\Big).$$

Letting k tend to infinity in the above expressions gives that

$$u_{\gamma+1}^{\mathcal{H}(F)|_M}(x) \leq \max\Big(u_{\gamma+1}^{\mathcal{H}(F)|_{C(F)}}(x),\ u_{\gamma+1}^{\mathcal{H}(F)|_{M(K_0,F)}}(x), \tag{3.5.7}$$

$$\limsup_{\substack{y_{m_\ell}\to x\\ y_{m_\ell}\in M_{m_\ell}}} u_\gamma^{\mathcal{H}(F)|_{M(K_{m_\ell},F)}}(y_{m_\ell}) + \mathbf{h}_{\text{top}}(F|_{K_{m_\ell}})\Big). \tag{3.5.8}$$

We would like to show that the expression in the right-hand side of Equation (3.5.7)-(3.5.8) is less than or equal to $u_{\gamma+1}^{\mathcal{H}(F)|_{C(F)}}(x)$, and we prove this bound by analyzing each expression in the maximum individually. The bound is trivial for the first expression. By Lemma 3.4.3 (applied to $F|_{K_0}$, which is a principal extension of f, with $C(f)$ in place of C), we have that for x in $M(K_0, F)$,

$$u_{\gamma+1}^{\mathcal{H}(F)|_{M(K_0,F)}}(x) = \int_{C(F)\cap M(K_0,F)} u_{\gamma+1}^{\mathcal{H}(F)|_{C(F)\cap M(K_0,F)}}\, d\mathcal{P}_x. \tag{3.5.9}$$

Since $C(F) \subset M_{\text{erg}}(\mathbb{D}, F)$, the measure $\mathcal{P}_x$ is the point mass at x for any x in $C(F)$. Combining this fact with Equation 3.5.9 and then using Fact 3.2.19 (2) gives that

$$u_{\gamma+1}^{\mathcal{H}(F)|_{M(K_0,F)}}(x) = u_{\gamma+1}^{\mathcal{H}(F)|_{C(F)\cap M(K_0,F)}}(x) \leq u_{\gamma+1}^{\mathcal{H}(F)|_{C(F)}}(x),$$

which gives the desired bound on the second expression in the maximum in Equation (3.5.7)-(3.5.8).

We bound the third expression in the maximum in Equation (3.5.7)-(3.5.8) as follows. By hypothesis (v), either $\mathbf{h}_{\text{top}}(F|_{K_m})$ tends to 0 as m tends to infinity or for each m, $\alpha_0(F|_{K_m}) = 0$ and $\mathbf{h}_{\text{top}}(F|_{K_m}) = h^F(\mu)$ for μ in $C(K_m, F)$. First suppose that $\mathbf{h}_{\text{top}}(F|_{K_m})$ tends to 0. Let $\{y_{m_\ell}\}_{\ell\in\mathbb{N}}$ be any sequence tending to x such that $y_{m_\ell} \in M(K_{m_\ell}, F)$ for each ℓ. Equation (3.5.5) implies that $||u_\gamma^{\mathcal{H}(F)|_{M(K_{m_\ell},F)}}|| = ||u_\gamma^{\mathcal{H}(F)|_{C(F)\cap M(K_{m_\ell},F)}}||$ for each ℓ. Since $u_\gamma^{\mathcal{H}(F)|_{C(F)\cap M(K_{m_\ell},F)}}$ is u.s.c., there exists μ_{m_ℓ} in $C(F)\cap M(K_{m_\ell}, F)$ such that $u_\gamma^{\mathcal{H}(F)|_{C(F)\cap M(K_{m_\ell},F)}}(\mu_{m_\ell}) = ||u_\gamma^{\mathcal{H}(F)|_{C(F)\cap M(K_{m_\ell},F)}}||$, for each ℓ in $\mathbb{N}$. Furthermore, $\{\mu_{m_\ell}\}_{\ell\in\mathbb{N}}$ tends to x because $\{y_{m_\ell}\}_{\ell\in\mathbb{N}}$ tends to x. Then

$$\begin{aligned}\limsup_{\ell} u_\gamma^{\mathcal{H}(F)|_{M(K_{m_\ell},F)}}(y_{m_\ell}) + \mathbf{h}_{\text{top}}(F|_{K_{m_\ell}}) &\leq \limsup_{\ell} u_\gamma^{\mathcal{H}(F)|_{C(F)\cap M(K_{m_\ell},F)}}(\mu_{m_\ell}) \\ &\leq \limsup_{\ell} u_\gamma^{\mathcal{H}(F)|_{C(F)}}(\mu_{m_\ell}) \\ &\leq u_{\gamma+1}^{\mathcal{H}(F)|_{C(F)}}(x).\end{aligned}$$

It follows that

$$\limsup_{\substack{y_{m_\ell}\to x \\ y_{m_\ell}\in M_{m_\ell}}} u_\gamma^{\mathcal{H}(F)|_{M(K_{m_\ell},F)}}(y_{m_\ell}) + \mathbf{h}_{\text{top}}(F|_{K_{m_\ell}}) \leq u_{\gamma+1}^{\mathcal{H}(F)|_{C(F)}}(x). \tag{3.5.10}$$

Now suppose that for each $m \geq 1$, $\alpha_0(F|_{K_m}) = 0$ and $\mathbf{h}_{\text{top}}(F|_{K_m}) = h^F(\mu)$ for μ in $C(F)\cap M(K_m, F)$. Since $\alpha_0(F|_{K_m}) = 0$, we have that $u_\gamma^{\mathcal{H}(F)|_{M(K_m,F)}} \equiv 0$ for each $m \geq 1$. Let $\{y_{m_\ell}\}_{\ell\in\mathbb{N}}$ be a sequence tending to x such that y_{m_ℓ} is in $M(K_{m_\ell}, F)$ for each ℓ. Let μ_{m_ℓ} be in $C(F)\cap M(K_{m_\ell}, F)$, for each ℓ. Note that $\{\mu_{m_\ell}\}_{\ell\in\mathbb{N}}$ tends to x because $\{y_{m_\ell}\}_{\ell\in\mathbb{N}}$ tends to x. By Proposition 3.5.21 (3), for each k, we may assume there exists a natural number m_0 such that for $m \geq m_0$, it holds that $h_k(\mu_m) = 0$, which implies that $\tau_k(\mu_m) = h^F(\mu_m)$. Then

$$\limsup_{\ell} u_\gamma^{\mathcal{H}(F)|_{M(K_{m_\ell},F)}}(y_{m_\ell}) + \mathbf{h}_{\text{top}}(F|_{K_{m_\ell}}) = \limsup_{\ell} \mathbf{h}_{\text{top}}(F|_{K_{m_\ell}})$$
$$= \limsup_{\ell} h^F(\mu_{m_\ell}) = \lim_k \limsup_{\ell} \tau_k(\mu_{m_\ell}) \leq u_1^{\mathcal{H}(F)|_{C(F)}}(x) \leq u_{\gamma+1}^{\mathcal{H}(F)|_{C(F)}}(x).$$

We have shown that in either case given by hypothesis (v), the third expression in the maximum in Equation (3.5.7)-(3.5.8) is bounded above by $u_{\gamma+1}^{\mathcal{H}(F)|_{C(F)}}(x)$, as desired. Thus we have shown that $u_{\gamma+1}^{\mathcal{H}(F)|_M}(x) \leq u_{\gamma+1}^{\mathcal{H}(F)|_{C(F)}}(x)$, which finishes the successor case of our induction.

For the limit case, let γ be a limit ordinal and suppose property (5) holds for all $\beta < \gamma$. Taking the limit supremum over the three sets $C(F)$, $M(K_0, F) \setminus C(F)$, and $\cup_{m\geq 1} M(K_m, F)$ in the definition of $u_\gamma^{\mathcal{H}(F)|_M}(x)$, we obtain

$$u_\gamma^{\mathcal{H}(F)|_M}(x) \leq \max\Big(u_\gamma^{\mathcal{H}(F)|_{C(F)}}(x),\ u_\gamma^{\mathcal{H}(F)|_{M(K_0,F)}}(x),\ \limsup_{\substack{y_{m_\ell}\to x \\ y_{m_\ell}\in M_{m_\ell}}} u_\gamma^{\mathcal{H}(F)|_{M(K_{m_\ell},F)}}(y_{m_\ell})\Big). \tag{3.5.11}$$

By the same arguments as in the successor case, we bound the three expressions in the maximum in Equation (3.5.11) from above by $u_\gamma^{\mathcal{H}(F)|_{C(F)}}(x)$, which shows that $u_\gamma^{\mathcal{H}(F)|_M}(x) \leq u_\gamma^{\mathcal{H}(F)|_{C(F)}}(x)$. This finishes our induction, and thus we have verified property (5). □

Lemma 3.5.24. *Suppose (X, F) is a topological dynamical system with entropy structure $\mathcal{H}(F)$. Suppose there exist closed sets C and M in $M(X, F)$ such that $C \subset M_{erg}(X, F) \subset M$ and for all x in M and all ordinals γ,*

$$u_\gamma^{\mathcal{H}(F)|_M}(x) \leq \int_C u_\gamma^{\mathcal{H}(F)|_C}\, d\mathcal{P}_x.$$

Then for all x in $M(X,F)$ and all ordinals γ,

$$u_\gamma^{\mathcal{H}(F)}(x) = \int_C u_\gamma^{\mathcal{H}(F)|_C}\, d\mathcal{P}_x.$$

Proof. Since M is closed and contains $M_{\text{erg}}(X,F)$, the Embedding Lemma (Lemma 3.2.18) implies that for all x in $M(X,F)$ and all ordinals γ,

$$u_\gamma^{\mathcal{H}(F)}(x) = \max_{\mu\in\Phi^{-1}(x)} \int_M u_\gamma^{\mathcal{H}(F)|_M}\, d\mu,$$

where $\Phi : \mathcal{M}(M) \to M(X,F)$ is the restriction of the barycenter map (which is onto since $M_{\text{erg}}(X,F) \subset M$). By Fact 3.2.19 (2) and the fact that $\mathcal{P}_x \in \Phi^{-1}(x)$,

$$\begin{aligned} \max_{\mu\in\Phi^{-1}(x)} \int_M u_\gamma^{\mathcal{H}(F)|_M}\, d\mu &\geq \max_{\mu\in\Phi^{-1}(x)} \int_C u_\gamma^{\mathcal{H}(F)|_C}\, d\mu \\ &\geq \int_C u_\gamma^{\mathcal{H}(F)|_C}\, d\mathcal{P}_x. \end{aligned}$$

For each ordinal γ, let $g_\gamma : M(X,F) \to [0,\infty)$ be defined by

$$g_\gamma(x) = \begin{cases} u_\gamma^{\mathcal{H}(F)|_C}(x), & \text{if } x \in C \\ 0, & \text{otherwise.} \end{cases}$$

Note that since C is closed and $u_\gamma^{\mathcal{H}(F)|_C}$ is u.s.c. and non-negative on C, we have that g_γ is u.s.c. on $M(X,F)$. Also, g_γ is convex for each γ since it takes positive values only on extreme points (using that $C \subset M_{\text{erg}}(X,F)$). Fact 2.5 in [36] (proved in [18]) states that the harmonic extension of a non-negative, convex, u.s.c. function is u.s.c. and of course harmonic. Applying this fact to g_γ, we obtain that the function $g_\gamma^{\text{har}} : M(X,F) \to [0,\infty)$ defined by

$$g_\gamma^{\text{har}}(x) = \int g_\gamma\, d\mathcal{P}_x = \int_C u_\gamma^{\mathcal{H}(F)|_C}\, d\mathcal{P}_x$$

is harmonic and u.s.c. Then for any μ in $\Phi^{-1}(x)$, since g_γ^{har} is harmonic and μ is supported on M, we have that

$$g_\gamma^{\mathrm{har}}(x) = g_\gamma^{\mathrm{har}}(\mathrm{bar}(\mu)) = \int_M g_\gamma^{\mathrm{har}}\, d\mu.$$

By hypothesis, we have

$$u_\gamma^{\mathcal{H}(F)|_M}(x) \leq \int_C u_\gamma^{\mathcal{H}(F)|_C}\, d\mathcal{P}_x.$$

Combining all of these facts, we see that for x in $M(X, F)$,

$$\begin{aligned}
\int_C u_\gamma^{\mathcal{H}(F)|_C}\, d\mathcal{P}_x \leq u_\gamma^{\mathcal{H}(F)}(x)& \\
&= \max_{\mu\in\Phi^{-1}(x)} \int_M u_\gamma^{\mathcal{H}(F)|_M}\, d\mu \\
&\leq \max_{\mu\in\Phi^{-1}(x)} \int_M \int_C u_\gamma^{\mathcal{H}(F)|_C}\, d\mathcal{P}_\tau\, d\mu(\tau) \\
&= \max_{\mu\in\Phi^{-1}(x)} \int_M g_\gamma^{\mathrm{har}}\, d\mu \\
&= g_\gamma^{\mathrm{har}}(x) \\
&= \int_C u_\gamma^{\mathcal{H}(F)|_C}\, d\mathcal{P}_x.
\end{aligned}$$

Thus the above inequalities are actually equalities, and we have proved the lemma.

□

3.6 Computation of some transfinite sequences

Recall that the notation $\mathcal{C}_d$ was defined in Definition 3.5.2. We will be interested in the following subsets of $\mathcal{C}_d$.

Definition 3.6.1. Let α be a countable ordinal and $a \geq 0$. Let $\mathcal{S}(\alpha, d, a)$ be the class of functions f in $\mathcal{C}_d$ such that there exists a sequence of periodic orbits $\{\theta_m\}_m$ of f such that the following conditions are satisfied, where $C(f) = \cap_{N=1}^{\infty} \overline{\cup_{m \geq N} \{\mu_{\theta_m}\}}$:

(1) f is ready for operation on $\cup_m \theta_m$;

(2) for every μ in $C(f)$, μ is totally ergodic;

(3) if $\alpha = 0$, then $C(f) = \{\nu\}$, where ν is the unique measure of maximal entropy for f;

(4) for all ordinals γ and all points x in $M(\mathbb{D}, f)$,

$$u_\gamma^{\mathcal{H}(f)}(x) = \int_{C(f)} u_\gamma^{\mathcal{H}(f)|_{C(f)}} \, d\mathcal{P}_x;$$

(5) $\alpha_0(f) = \alpha$.

(6) $||u_\alpha^{\mathcal{H}(f)}|| = a$.

Also, let $\mathcal{S}(\alpha, d) = \cup_{a \geq 0} \mathcal{S}(\alpha, d, a)$.

Notation 3.6.2. If $\{\theta_m\}_{m \in \mathbb{N}}$ is a sequence of periodic orbits for f satisfying the conditions in Definition 3.6.1 for f, then we write that f is in $\mathcal{S}(\alpha, d, a)$ with $\{\theta_m\}_{m \in \mathbb{N}}$.

Remark 3.6.3. For some pairs α and $a \geq 0$, the set $\mathcal{S}(\alpha, d, a)$ is trivially empty. Indeed, if $\alpha = 0$ and $a > 0$, then $\mathcal{S}(\alpha, d, a)$ is empty. Also, if $\alpha > 0$ and $a = 0$, then $\mathcal{S}(\alpha, d, a)$ is empty. On the other hand, in the course of proving Theorem 3.3.3, we will show that for every countable ordinal $\alpha > 0$, and every $a > 0$, the set $\mathcal{S}(\alpha, d, a)$ is non-empty.

Lemma 3.6.4. *Let p be a non-negative integer and $a > 0$. Suppose f, $\{\chi_m\}_{m\in\mathbb{N}}$, $\{\xi_m\}_{m\in\mathbb{N}}$, and $\{N_m\}_{m\in\mathbb{N}}$ satisfy the following conditions:*

- *f is in $\mathcal{S}(p, d, \frac{ap}{p+1})$ with $\{\theta_m\}_{m\in\mathbb{N}}$;*
- *$||u_\ell^{\mathcal{H}(f)}|| = \frac{a\ell}{p+1}$ for $\ell = 1, \ldots, p$;*
- *for each m, N_m and ξ_m are natural numbers and $1 \le \xi_m \le |\theta_m|$;*
- *for each m, χ_m is in $\mathcal{S}(0, d)$ with $\{\theta_\ell^m\}_{\ell\in\mathbb{N}}$ and $\mathbf{h}_{\mathrm{top}}(\chi_m) = \log(N_m)$;*
- *the sequence $\{\frac{\xi_m}{|\theta_m|}\log(N_m)\}_{m\in\mathbb{N}}$ is increasing to $\frac{a}{p+1}$.*

Then for any F in $\mathcal{BL}(f, \{\theta_m\}_{m\in\mathbb{N}}, \{\chi_m\}_{m\in\mathbb{N}}, \{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}, \{\xi_m\}_{m\in\mathbb{N}})$, F is in $\mathcal{S}(p+1, d, a)$, $\mathbf{h}_{\mathrm{top}}(F) = \max\big(\mathbf{h}_{\mathrm{top}}(f), \sup_m \frac{\xi_m}{|\theta_m|}\mathbf{h}_{\mathrm{top}}(\chi_m)\big)$, and $||u_k^{\mathcal{H}(F)}|| = \frac{ak}{p+1}$ for each k in the set $\{1, \ldots, p+1\}$.

Proof. Let f, $\{\chi_m\}_{m\in\mathbb{N}}$, $\{\xi_m\}_{m\in\mathbb{N}}$, and $\{N_m\}_{m\in\mathbb{N}}$ be as above. By Proposition 3.5.4, there exists F in $\mathcal{BL}(f, \{\theta_m\}_{m\in\mathbb{N}}, \{\chi_m\}_{m\in\mathbb{N}}, \{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}, \{\xi_m\}_{m\in\mathbb{N}})$ with π, $\{K_i\}_{i\ge 0}$, and $\{\phi_m\}_{m\in\mathbb{N}}$ as in Proposition 3.5.4. Then F is in $\mathcal{C}_d$ and F is ready for operation on the set $S = \cup_{m,\ell}\, \phi_m(\theta_\ell^m \times \{0, \ldots, |\theta_m| - 1\})$. Let Θ_k be an enumeration of the periodic orbits in S. Let

- $C(F) = \cap_{n=1}^\infty \overline{\cup_{k\ge n}\{\mu_{\Theta_k}\}}$;
- $C(f) = \cap_{n=1}^\infty \overline{\cup_{m\ge n}\{\mu_{\theta_m}\}}$;
- $C(\chi_m) = \cap_{n=1}^\infty \overline{\cup_{\ell\ge n}\{\mu_{\theta_\ell^m}\}}$;
- for each $i \ge 0$, $C(K_i, F) = C(F) \cap M(K_i, F)$.

To prove the lemma, we show the following:

(A) $\mathbf{h}_{\text{top}}(F) = \max\big(\mathbf{h}_{\text{top}}(f), \sup_m \frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m)\big)$;

(B) for each μ in $C(F)$, μ is totally ergodic;

(C) for each x in $M(\mathbb{D}, F)$ and each ordinal γ,

$$u_\gamma^{\mathcal{H}(F)}(x) = \int_{C(F)} u_\gamma^{\mathcal{H}(F)|_{C(F)}} \, d\mathcal{P}_x;$$

(D) $\alpha_0(\mathcal{H}(F)) = p + 1$;

(E) $||u_k^{\mathcal{H}(F)}|| = \frac{a\ell}{p+1}$ for $\ell = 1, \ldots, p+1$.

Corollary 3.5.22 gives (A). Lemma 3.5.23 (1) implies (B). Lemma 3.5.23 (5) and Lemma 3.5.24 together imply (C).

Property (C) implies that $\alpha_0(\mathcal{H}(F)) \leq \alpha_0(\mathcal{H}(F)|_{C(F)})$ and also that $||u_k^{\mathcal{H}(F)}|| = ||u_k^{\mathcal{H}(F)|_{C(F)}}||$. Since $C(F) \subset M_{\text{erg}}(\mathbb{D}, F)$, the measure $\mathcal{P}_x$ is just the point mass at x, for all x in $C(F)$. With this fact, (C) implies that $u_\gamma^{\mathcal{H}(F)}(x) = u_\gamma^{\mathcal{H}(F)|_{C(F)}}(x)$ for all x in $C(F)$. It follows that $\alpha_0(\mathcal{H}(F)) \geq \alpha_0(\mathcal{H}(F)|_{C(F)})$, and we conclude that in fact $\alpha_0(\mathcal{H}(F)) = \alpha_0(\mathcal{H}(F)|_{C(F)})$. We now observe that properties (D) and (E) will be satisfied once we show that $\alpha(\mathcal{H}(F)|_{C(F)}) = p + 1$ and $||u_\ell^{\mathcal{H}(F)|_{C(F)}}|| = \frac{a\ell}{p+1}$ for $\ell = 1, \ldots, p+1$. Let us prove these two facts by computing the transfinite sequence for $\mathcal{H}(F)|_{C(F)}$.

Note that for $m \geq 1$, $C(K_m, F)$ is open in $C(F)$ (by Lemma 3.5.23). Then

Fact 3.2.19 (1) and Lemma 3.4.9 give that for all x in $C(K_m, F)$,

$$u_\gamma^{\mathcal{H}(F)|_{C(F)}}(x) = u_\gamma^{\mathcal{H}(F)|_{C(K_m,F)}}(x)$$
$$= \frac{\xi_m}{|\theta_m|} u_\gamma^{\mathcal{H}(\chi_m)|_{C(\chi_m)}}(\psi_m((\phi_m^{-1})(x))).$$

By the hypothesis that χ_m is in $\mathcal{S}(0, d)$, $u_\gamma^{\mathcal{H}(\chi_m)} \equiv 0$ for all ordinals γ, and thus $u_\gamma^{\mathcal{H}(F)|_{C(K_m,F)}}(x) = 0$ for all x in $C(F) \cap M(K_m, F)$.

For x in $C(K_0, F)$, we have that for each k,

$$\limsup_{y\to x} \tau_k(y) = \max\Big(\limsup_{\substack{y\to x \\ y\in C(K_0,F)}} \tau_k(y), \limsup_{\substack{y\to x \\ y\in C(F)\setminus C(K_0,F)}} \tau_k(y)\Big)$$
$$\leq \max\Big(\limsup_{\substack{y\to x \\ y\in C(K_0,F)}} \tau_k(y), \limsup_m \frac{\xi_m}{|\theta_m|} \mathbf{h}_{\mathrm{top}}(\chi_m)\Big).$$

Letting k tend to infinity and using Lemma 3.4.3 (applied to $F|_{K_0}$, which is a principal extension of f with factor map π) gives that

$$u_1^{\mathcal{H}(F)|_{C(F)}}(x) \leq \max\Big(u_1^{\mathcal{H}(F)|_{C(K_0,F)}}(x), \limsup_m \frac{\xi_m}{|\theta_m|} \mathbf{h}_{\mathrm{top}}(\chi_m)\Big) \tag{3.6.1}$$
$$= \max\Big(u_1^{\mathcal{H}(f)}(\pi(x)), \limsup_m \frac{\xi_m}{|\theta_m|} \mathbf{h}_{\mathrm{top}}(\chi_m)\Big). \tag{3.6.2}$$

By hypothesis, $||u_1^{\mathcal{H}(f)}|| = \frac{a}{p+1}$ and $\lim_m \frac{\xi_m}{|\theta_m|} \mathbf{h}_{\mathrm{top}}(\chi_m) = \frac{a}{p+1}$. Then by Equations (3.6.1) and (3.6.2), we obtain $u_1^{\mathcal{H}(F)|_{C(F)}}(x) \leq \frac{a}{p+1}$. Since x is in $C(K_0, F)$, there exist periodic orbits θ_{m_k} such that the sequence $\mu_{\theta_{m_k}}$ converges to $\pi(x)$. Let μ_{m_k} be the measure of maximal entropy for $F|_{K_{m_k}}$, which exists by the fact that χ_{m_k} is in $\mathcal{S}(0, d)$ (property (3) in Definition 3.6.1). Then $\{\mu_{m_k}\}_{k\in\mathbb{N}}$ converges to x, and by the upper semi-continuity of $u_1^{\mathcal{H}(F))|_{C(F)}}$ and Proposition 3.5.21 (3), we have that

$$u_1^{\mathcal{H}(F)|_{C(F)}}(x) \geq \lim_\ell \limsup_k (h^F - h_\ell^F)(\mu_{m_k}) = \lim_\ell \limsup_k h^F(\mu_{m_k})$$
$$= \limsup_k \frac{\xi_{m_k}}{|\theta_{m_k}|} \mathbf{h}_{\mathrm{top}}(\chi_{m_k}) = \frac{a}{p+1}.$$

This argument shows that for all x in $C(K_0, F)$, it holds that $u_1^{\mathcal{H}(F)|_{C(F)}}(x) = \frac{a}{p+1}$. Now we claim that by induction on ℓ, $u_\ell^{\mathcal{H}(F)|_{C(F)}}(x) = \frac{a}{p+1} + u_{\ell-1}^{\mathcal{H}(F)|_{C(K_0,F)}}(x)$ for x in $C(K_0, F)$. The claim holds for $\ell = 1$. Assuming it holds for a natural number ℓ, we have for x in $C(K_0, F)$,

$$\begin{aligned}
&\limsup_{y\to x}(u_\ell^{\mathcal{H}(F)|_{C(F)}} + \tau_k)(y) = \\
&= \max\Big(\limsup_{\substack{y\to x\\ y\in C(K_0,F)}} (u_\ell^{\mathcal{H}(F)|_{C(F)}} + \tau_k)(y), \limsup_{\substack{y\to x\\ y\in C(F)\setminus C(K_0,F)}} (u_\ell^{\mathcal{H}(F)|_{C(F)}} + \tau_k)(y)\Big)\\
&= \max\Big(\limsup_{\substack{y\to x\\ y\in C(K_0,F)}} \big(\frac{a}{p+1} + u_{\ell-1}^{\mathcal{H}(F)|_{C(K_0,F)}} + \tau_k\big)(y), \limsup_{\substack{y\to x\\ y\in C(F)\setminus C(K_0,F)}} \tau_k(y)\Big),
\end{aligned}$$

where the second equality follows from the induction hypothesis on ℓ and the fact that $u_\ell^{\mathcal{H}(F)|_{C(K_m,F)}} \equiv 0$ for $m \geq 1$. Letting k tend to infinity gives that

$$\begin{aligned}
u_{\ell+1}^{\mathcal{H}(F)|_{C(F)}}(x) &= \max\Big(\frac{a}{p+1} + u_\ell^{\mathcal{H}(F)|_{C(K_0,F)}}(x), \frac{a}{p+1}\Big)\\
&= \frac{a}{p+1} + u_\ell^{\mathcal{H}(F)|_{C(K_0,F)}}(x).
\end{aligned}$$

By Lemma 3.4.3, we have $u_\ell^{\mathcal{H}(F)|_{C(K_0,F)}}(x) = u_\ell^{\mathcal{H}(f)}(\pi(x))$ for all x in $C(K_0, F)$. Now the facts $\alpha_0(\mathcal{H}(F)|_{C(F)}) = p + 1$ and $||u_\ell^{\mathcal{H}(F)|_{C(F)}}|| = \frac{a\ell}{p+1}$ for $\ell = 1, \ldots, p + 1$ follow from the hypotheses on f (in particular, $\alpha_0(\mathcal{H}(f)) = p$ and $||u_\ell^{\mathcal{H}(f)}|| = \frac{a\ell}{p+1}$ for $\ell = 1, \ldots, p$). This concludes the proof of the lemma. □

Lemma 3.6.5. *Let $\beta = 0$ or $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_k}$, where $\beta_1 \geq \cdots \geq \beta_k$. Let $\alpha > 1$ be an irreducible ordinal such that $\alpha \geq \omega^{\beta_1}$ if $\beta \neq 0$. Let $a > 0$ and $b \geq 0$. Suppose*

- *$\{\alpha_m\}_{m\in\mathbb{N}}$ is a non-decreasing sequence of ordinals whose limit is α;*
- *$\{\delta_m\}_{m\in\mathbb{N}}$ is a strictly increasing sequence of ordinals whose limit is α;*

- *$\{a_m\}_{m\in\mathbb{N}}$ is a sequence of positive real numbers tending to infinity;*
- *f is in $\mathcal{S}(\beta, d, b)$ with $\{\theta_m\}_{m\in\mathbb{N}}$;*
- *$||u_\alpha^{\mathcal{H}(f)}|| \leq a$;*
- *for each m, ξ_m satisfies $1 \leq \xi_m \leq |\theta_m|$, and the sequence $\{\frac{\xi_m}{|\theta_m|} a_m\}_{m\in\mathbb{N}}$ is increasing to a;*
- *χ_m is in $\mathcal{S}(\alpha_m, d, a_m)$ with $\{\theta_\ell^m\}_{\ell\in\mathbb{N}}$;*
- *$\frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m)$ tends to 0;*
- *$\frac{\xi_m}{|\theta_m|}||u_{\delta_m}^{\mathcal{H}(\chi_m)}||$ tends to 0.*

Then for any F in $\mathcal{BL}(f, \{\theta_m\}_{m\in\mathbb{N}}, \{\chi_m\}_{m\in\mathbb{N}}, \{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}, \{\xi_m\}_{m\in\mathbb{N}})$, F is in $\mathcal{S}(\alpha + \beta, d, a + b)$ and for any ordinal γ,

$$||u_\gamma^{\mathcal{H}(F)}|| = \begin{cases} \max\Big(||u_\gamma^{\mathcal{H}(f)}||,\ \sup_m \frac{\xi_m}{|\theta_m|}||u_\gamma^{\mathcal{H}(\chi_m)}||\Big), & \text{if } \gamma < \alpha \\ a + ||u_{\gamma_0}^{\mathcal{H}(f)}||, & \text{if } \gamma = \alpha + \gamma_0. \end{cases} \tag{3.6.3}$$

Furthermore, if $\beta = 0$, then for any $\delta < \alpha$ and $0 < \epsilon < a$, there exists m_0 such that for any F in $\mathcal{BL}(f, \{\theta_{m+m_0}\}_{m\in\mathbb{N}}, \{\chi_{m+m_0}\}_{m\in\mathbb{N}}, \{\theta_\ell^{m+m_0}\}_{m,\ell\in\mathbb{N}}, \{\xi_{m+m_0}\}_{m\in\mathbb{N}})$, $||u_\delta^{\mathcal{H}(F)}|| \leq \epsilon$.

Proof. Let f, $\{\theta_m\}_{m\in\mathbb{N}}$, $\{\chi_m\}_{m\in\mathbb{N}}$, $\{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}$, and $\{\xi_m\}_{m\in\mathbb{N}}$. By Proposition 3.5.4, there exists F in $\mathcal{BL}(f, \{\theta_m\}_{m\in\mathbb{N}}, \{\chi_m\}_{m\in\mathbb{N}}, \{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}, \{\xi_m\}_{m\in\mathbb{N}})$ with π, $\{K_i\}_{i\geq 0}$, and $\{\phi_m\}_{m\in\mathbb{N}}$ as in Proposition 3.5.4. Then F is in $\mathcal{C}_d$ and F is ready for operation on the set $S = \cup_{m,\ell}\, \phi_m(\theta_\ell^m \times \{0, \ldots, |\theta_m| - 1\})$. Let Θ_k be an enumeration of the periodic orbits in S. Let

- $C(F) = \cap_{n=1}^{\infty} \overline{\cup_{k \geq n} \{\mu_{\Theta_k}\}}$;
- $C(f) = \cap_{n=1}^{\infty} \overline{\cup_{m \geq n} \{\mu_{\theta_m}\}}$;
- $C(\chi_m) = \cap_{n=1}^{\infty} \overline{\cup_{\ell \geq n} \{\mu_{\theta_\ell^m}\}}$;
- for each $i \geq 0$, $C(K_i, F) = C(F) \cap M(K_i, F)$.

To prove the lemma, we will show the following:

(A) $\mathbf{h}_{\text{top}}(F) = \max\big(\mathbf{h}_{\text{top}}(f), \sup_m \frac{\xi_m}{|\theta_m|} \mathbf{h}_{\text{top}}(\chi_m)\big)$;

(B) for each μ in $C(F)$, μ is totally ergodic;

(C) for each x in $M(\mathbb{D}, F)$ and each ordinal γ,

$$u_\gamma^{\mathcal{H}(F)}(x) = \int_{C(F)} u_\gamma^{\mathcal{H}(F)|_{C(F)}} \, d\mathcal{P}_x;$$

(D) $\alpha_0(\mathcal{H}(F)) = \alpha + \beta$;

(E) for any ordinal γ, Equation (3.6.3) holds.

(F) if $\beta = 0$, then for any $\delta < \alpha$ and $0 < \epsilon < a$, there exists m_0 such that for any F in $\mathcal{BL}(f, \{\theta_{m+m_0}\}_{m \in \mathbb{N}}, \{\chi_{m+m_0}\}_{m \in \mathbb{N}}, \{\theta_\ell^{m+m_0}\}_{m,\ell \in \mathbb{N}}, \{\xi_{m+m_0}\}_{m \in \mathbb{N}})$, $||u_\delta^{\mathcal{H}(F)}|| \leq \epsilon$.

Corollary 3.5.22 gives (A). Lemma 3.5.23 (1) implies (B). Lemma 3.5.23 (5) and Lemma 3.5.24 together imply (C).

Suppose that $\beta = 0$ and that $\delta < \alpha$ and $0 < \epsilon < a$ are given. Choose m_0 such that for all $m \geq m_0$, $\delta_m > \delta$ and $\frac{\xi_m}{|\theta_m|} ||u_{\delta_m}^{\mathcal{H}(\chi_m)}|| < \epsilon$ (such m_0 exists by the hypotheses that δ_m tends to α and $\frac{\xi_m}{|\theta_m|}||u_{\delta_m}^{\mathcal{H}(\chi_m)}||$ tends to 0). Then property (F) follows from property (E). It remains to show properties (D) and (E).

Property (C) implies that $\alpha_0(\mathcal{H}(F)) \leq \alpha_0(\mathcal{H}(F)|_{C(F)})$ and that $||u_k^{\mathcal{H}(F)}|| = ||u_k^{\mathcal{H}(F)|_{C(F)}}||$. Since $C(F) \subset M_{\text{erg}}(\mathbb{D}, F)$, the measure $\mathcal{P}_x$ is just the point mass at x, for all x in $C(F)$. Combining this fact with property (C) implies that $u_\gamma^{\mathcal{H}(F)}(x) = u_\gamma^{\mathcal{H}(F)|_{C(F)}}(x)$ for all x in $C(F)$. It follows that $\alpha_0(\mathcal{H}(F)) \geq \alpha_0(\mathcal{H}(F)|_{C(F)})$ and therefore that $\alpha_0(\mathcal{H}(F)) = \alpha_0(\mathcal{H}(F)|_{C(F)})$. We now observe that properties (D) and (E) will be satisfied if we show that $\alpha(\mathcal{H}(F)|_{C(F)}) = \alpha + \beta$ and for all ordinals γ, Equation (3.6.3) holds with $\mathcal{H}(F)$ replaced by $\mathcal{H}(F)|_{C(F)}$. Below we prove these two facts by computing the transfinite sequence for $\mathcal{H}(F)|_{C(F)}$, which will complete the proof.

Note that for $m \geq 1$, the set $C(K_m, F)$ is open in $C(F)$ by Lemma 3.5.23. Then Fact 3.2.19 (1) and Lemma 3.4.9 give that for all x in $C(K_m, F)$,

$$u_\gamma^{\mathcal{H}(F)|_{C(F)}}(x) = u_\gamma^{\mathcal{H}(F)|_{C(K_m,F)}}(x) \tag{3.6.4}$$

$$= \frac{\xi_m}{|\theta_m|} u_\gamma^{\mathcal{H}(\chi_m)|_{C(\chi_m)}}(\psi_m((\phi_m^{-1})(x))). \tag{3.6.5}$$

We show by transfinite induction that for $\gamma < \alpha$ and x in $C(K_0, F)$, we have $u_\gamma^{\mathcal{H}(F)|_{C(F)}}(x) = u_\gamma^{\mathcal{H}(F)|_{C(K_0,F)}}(x)$. The statement is trivially true for $\gamma = 0$. Suppose it holds for $\gamma < \alpha$. Then by the inductive hypothesis and Equation (3.6.4), for x in

$C(K_0, F)$,

$$\begin{aligned}\limsup_{y\to x}(u_\gamma^{\mathcal{H}(F)|_{C(F)}} + \tau_k)(y) &= \max\Big(\limsup_{\substack{y\to x\\ y\in C(K_0,F)}} (u_\gamma^{\mathcal{H}(F)|_{C(F)}} + \tau_k)(y),\\ &\qquad \limsup_{\substack{y\to x\\ y\in C(F)\setminus C(K_0,F)}} (u_\gamma^{\mathcal{H}(F)|_{C(F)}} + \tau_k)(y)\Big)\\ &\le \max\Big(\limsup_{\substack{y\to x\\ y\in C(K_0,F)}} (u_\gamma^{\mathcal{H}(F)|_{C(K_0,F)}} + \tau_k)(y),\\ &\qquad \limsup_m ||u_\gamma^{\mathcal{H}(F)|_{C(K_m,F)}}|| + \frac{\xi_m}{|\theta_m|}\mathbf{h}_{\mathrm{top}}(\chi_m)\Big).\end{aligned}$$

Note that there exists m_0 such that $\delta_m > \gamma$ for all $m \geq m_0$, which implies that $u_\gamma^{\mathcal{H}(F)|_{C(K_m,F)}} \leq u_{\delta_m}^{\mathcal{H}(F)|_{C(K_m,F)}}$ for all large m. Then letting k tend to infinity, we obtain

$$\begin{aligned}u_{\gamma+1}^{\mathcal{H}(F)|_{C(F)}}(x) &\le \max\Big(u_{\gamma+1}^{\mathcal{H}(F)|_{C(K_0,F)}}(x), \limsup_m ||u_{\delta_m}^{\mathcal{H}(F)|_{C(K_m,F)}}|| + \frac{\xi_m}{|\theta_m|}\mathbf{h}_{\mathrm{top}}(\chi_m)\Big)\\ &= \max\Big(u_{\gamma+1}^{\mathcal{H}(F)|_{C(K_0,F)}}(x), \limsup_m \frac{\xi_m}{|\theta_m|}||u_{\delta_m}^{\mathcal{H}(\chi_m)}|| + \frac{\xi_m}{|\theta_m|}\mathbf{h}_{\mathrm{top}}(\chi_m)\Big)\\ &= \max\Big(u_{\gamma+1}^{\mathcal{H}(F)|_{C(K_0,F)}}(x), 0\Big)\\ &= u_{\gamma+1}^{\mathcal{H}(F)|_{C(K_0,F)}}(x),\end{aligned}$$

using the hypotheses that $\frac{\xi_m}{|\theta_m|}||u_{\delta_m}^{\mathcal{H}(\chi_m)}||$ tends to 0 and $\frac{\xi_m}{|\theta_m|}\mathbf{h}_{\mathrm{top}}(\chi_m)$ tends to 0.

By Fact 3.2.19 (2), $u_{\gamma+1}^{\mathcal{H}(F)|_{C(K_0,F)}}(x) \leq u_{\gamma+1}^{\mathcal{H}(F)|_{C(F)}}(x)$, and thus we conclude that $u_{\gamma+1}^{\mathcal{H}(F)|_{C(F)}}(x) = u_{\gamma+1}^{\mathcal{H}(F)|_{C(K_0,F)}}(x)$, which finishes the inductive step for successors.

Now suppose that $u_\beta^{\mathcal{H}(F)|_{C(F)}}(x) = u_\beta^{\mathcal{H}(F)|_{C(K_0,F)}}(x)$ holds for all x in $C(K_0, F)$ and all $\beta < \gamma$, where γ is a limit ordinal such that $\gamma < \alpha$. Recall that for m

sufficiently large, $\delta_m > \gamma$. Then by the induction hypothesis, for each x in $C(K_0, F)$,

$$\begin{aligned}
u_\gamma^{\mathcal{H}(F)|_{C(F)}}(x) &= \limsup_{y\to x} \sup_{\beta<\gamma} u_\beta^{\mathcal{H}(F)|_{C(F)}}(y) \\
&= \max\Big(\limsup_{\substack{y\to x \\ y\in C(K_0,F)}} \sup_{\beta<\gamma} u_\beta^{\mathcal{H}(F)|_{C(F)}}(y), \limsup_{\substack{y\to x \\ y\in C(F)\setminus C(K_0,F)}} \sup_{\beta<\gamma} u_\beta^{\mathcal{H}(F)|_{C(F)}}(y)\Big) \\
&= \max\Big(\limsup_{\substack{y\to x \\ y\in C(K_0,F)}} \sup_{\beta<\gamma} u_\beta^{\mathcal{H}(F)|_{C(K_0,F)}}(y), \limsup_{\substack{y\to x \\ y\in C(F)\setminus C(K_0,F)}} u_\gamma^{\mathcal{H}(F)|_{C(F)}}(y)\Big) \\
&\le \max\Big(u_\gamma^{\mathcal{H}(F)|_{C(K_0,F)}}(x), \limsup_m \frac{\xi_m}{|\theta_m|}||u_{\delta_m}^{\mathcal{H}(\chi_m)}||\Big) \\
&= \max\Big(u_\gamma^{\mathcal{H}(F)|_{C(K_0,F)}}(x), 0\Big) \\
&= u_\gamma^{\mathcal{H}(F)|_{C(K_0,F)}}(x)
\end{aligned}$$

By Fact 3.2.19 (2), $u_\gamma^{\mathcal{H}(F)|_{C(K_0,F)}}(x) \le u_\gamma^{\mathcal{H}(F)|_{C(F)}}(x)$, and we conclude that in fact $u_\gamma^{\mathcal{H}(F)|_{C(F)}}(x) = u_\gamma^{\mathcal{H}(F)|_{C(K_0,F)}}(x)$, which finishes the inductive step for limit ordinals. We have shown that for all ordinals $\gamma < \alpha$, $u_\gamma^{\mathcal{H}(F)|_{C(F)}}(x) = u_\gamma^{\mathcal{H}(F)|_{C(K_0,F)}}(x)$ for all x in $C(F)$. Now by Lemma 3.4.3, for x in $C(K_0, F)$ and $\gamma < \alpha$,

$$u_\gamma^{\mathcal{H}(F)|_{C(F)}}(x) = u_\gamma^{\mathcal{H}(F)|_{C(K_0,F)}}(x) = u_\gamma^{\mathcal{H}(f)}(\pi(x)).$$

At this point we conclude based on the above facts that for $\gamma < \alpha$,

$$||u_\gamma^{\mathcal{H}(F)|_{C(F)}}|| = \max\Big(||u_\gamma^{\mathcal{H}(f)}||, \sup_m \frac{\xi_m}{|\theta_m|}||u_{\delta_m}^{\mathcal{H}(\chi_m)}||\Big).$$

Since α is irreducible and greater than 1, α is a limit ordinal. Thus for any x

in $C(K_0, F)$,

$$\begin{aligned} u_\alpha^{\mathcal{H}(F)|_{C(F)}}(x) &= \limsup_{y\to x} \sup_{\beta<\alpha} u_\beta^{\mathcal{H}(F)|_{C(F)}}(y) \\ &\le \max\Big(\limsup_{\substack{y\to x \\ y\in C(K_0,F)}} \sup_{\beta<\alpha} u_\beta^{\mathcal{H}(F)|_{C(K_0,F)}}(y), \limsup_m ||u_\alpha^{\mathcal{H}(F)|_{C(K_m,F)}}|| \Big) \\ &= \max\Big(\limsup_{\substack{y\to x \\ y\in C(K_0,F)}} \sup_{\beta<\alpha} u_\beta^{\mathcal{H}(F)|_{C(K_0,F)}}(y), \limsup_m \frac{\xi_m}{|\theta_m|} a_m \Big) \\ &\le \max\Big(u_\alpha^{\mathcal{H}(F)|_{C(K_0,F)}}(x), a \Big). \end{aligned}$$

By hypothesis, $||u_\alpha^{\mathcal{H}(F)|_{C(K_0,F)}}|| \le a$, and thus we have that $u_\alpha^{\mathcal{H}(F)|_{C(F)}}(x) \le a$. On the other hand, since x is in $C(K_0, F)$, there exists a sequence of periodic orbits $\{\theta_{m_k}\}_{k\in\mathbb{N}}$ such that $\{\mu_{\theta_{m_k}}\}_{k\in\mathbb{N}}$ converges to $\pi(x)$. If $\{\mu_{m_k}\}_{k\in\mathbb{N}}$ is a sequence of measures such that μ_{m_k} is in $M(K_{m_k}, F)$ for each k, then $\{\mu_{m_k}\}_{k\in\mathbb{N}}$ converges to x, and we have

$$u_\alpha^{\mathcal{H}(F)|_{C(F)}}(x) \ge \limsup_k ||u_\alpha^{\mathcal{H}(F)|_{C(K_{m_k})}}|| = \limsup_k \frac{\xi_{m_k}}{|\theta_{m_k}|} ||u_\alpha^{\mathcal{H}(\chi_{m_k})}|| = a.$$

It follows that for each x in $C(K_0, F)$, $u_\alpha^{\mathcal{H}(F)|_{C(F)}}(x) = a$.

We show by induction that for $\gamma \ge 0$ and x in $C(K_0, F)$,

$$u_{\alpha+\gamma}^{\mathcal{H}(F)|_{C(F)}}(x) = a + u_\gamma^{\mathcal{H}(F)|_{C(K_0,F)}}(x). \tag{3.6.6}$$

Note that Equation (3.6.6) holds for $\gamma = 0$. Now suppose Equation (3.6.6) holds for some ordinal γ. Then for all x in $C(K_0, F)$,

$$\limsup_{y\to x}(u_{\alpha+\gamma}^{\mathcal{H}(F)|_{C(F)}} + \tau_k)(y) = \max\Big(\limsup_{\substack{y\to x\\ y\in C(K_0,F)}} (u_{\alpha+\gamma}^{\mathcal{H}(F)|_{C(F)}} + \tau_k)(y), \limsup_{\substack{y\to x\\ y\in C(F)\setminus C(K_0,F)}} (u_{\alpha+\gamma}^{\mathcal{H}(F)|_{C(F)}} + \tau_k)(y)\Big)$$
$$= \max\Big(\limsup_{\substack{y\to x\\ y\in C(K_0,F)}} (a + u_{\gamma}^{\mathcal{H}(F)|_{C(K_0,F)}} + \tau_k)(y), \limsup_{\substack{m\to\infty\\ y\in C(K_m,F)}} (u_{\alpha}^{\mathcal{H}(F)|_{C(K_m,F)}} + \tau_k)(y)\Big).$$

Taking the limit as k tends to infinity gives

$$u_{\alpha+\gamma+1}^{\mathcal{H}(F)|_{C(F)}}(x) = \max\Big(a + u_{\gamma+1}^{\mathcal{H}(F)|_{C(K_0,F)}}(x), a + \limsup_{m} \frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m)\Big)$$
$$= \max\Big(a + u_{\gamma+1}^{\mathcal{H}(F)|_{C(K_0,F)}}(x), a + 0\Big)$$
$$= a + u_{\gamma+1}^{\mathcal{H}(F)|_{C(K_0,F)}}(x).$$

This completes the inductive step for successor ordinals. Now suppose Equation (3.6.6) holds for all $\gamma < \beta$, where β is a limit ordinal. Then for all x in $C(K_0, F)$,

$$u_{\alpha+\beta}^{\mathcal{H}(F)|_{C(F)}}(x) = \limsup_{y\to x}\sup_{\gamma<\beta} u_{\alpha+\gamma}^{\mathcal{H}(F)|_{C(F)}}(y)$$
$$= \max\Big(\limsup_{\substack{y\to x\\ y\in C(K_0,F)}} \sup_{\gamma<\beta} u_{\alpha+\gamma}^{\mathcal{H}(F)|_{C(K_0,F)}}(y), \limsup_{m} ||u_{\alpha+\beta}^{\mathcal{H}(F)|_{C(K_m,F)}}||\Big)$$
$$= \max\Big(a + u_{\beta}^{\mathcal{H}(F)|_{C(K_0,F)}}(x), a\Big)$$
$$= a + u_{\beta}^{\mathcal{H}(F)|_{C(K_0,F)}}(x),$$

which completes the inductive step for limit ordinals. Combining Equation (3.6.6) with Lemma 3.4.3, we obtain that

$$u_{\alpha+\gamma}^{\mathcal{H}(F)|_{C(F)}}(x) = a + u_{\gamma}^{\mathcal{H}(f)}(\pi(x)). \tag{3.6.7}$$

Then Equation (3.6.3) follows immediately and the equality $\alpha_0(\mathcal{H}(F)|_{C(F)}) = \alpha + \beta$ follows from the fact that $\alpha_0(\mathcal{H}(f)) = \beta$. This concludes the proof of the lemma. $\square$

3.7 Constructions by transfinite induction

The following lemma serves as a base case for the transfinite induction construction in this section. Recall that for any countable ordinal α and any real number $a \geq 0$, the set $\mathcal{S}(\alpha, d, a)$ was defined in Definition 3.6.1.

Lemma 3.7.1. *For any odd natural number $N \geq 3$, there exists f in $\mathcal{S}(0, d, 0)$ such that $\mathbf{h}_{\text{top}}(f) = \log(N)$.*

Proof. In the case $d = 1$, let f be the linear N-tent map on $[0, 1]$. In the case $d = 2$, let f be an adaptation of Smale's N-horseshoe map (for a discussion of horseshoes, see [52]) such that $f : \mathbb{D} \to \mathbb{D}$ is a homeomorphism and $f|_{\partial\mathbb{D}} = \text{Id}$. In either case, we have that f is a continuous surjection, $f|_{\partial\mathbb{D}} = \text{Id}$, and $\mathbf{h}_{\text{top}}(f) = \log(N) < \infty$, which implies that f is in C_d. Recall that f has a unique measure of maximal entropy, which we denote as μ. Also, there exists a sequence $\{\mu_{\theta_m}\}_{m\in\mathbb{N}}$ of periodic measures tending to μ with $\cup_m \theta_m$ contained in $\text{int}(\mathbb{D})$. Fix such a sequence. Let $Q = \cup_k f^{-k}(\theta_m)$. Since f is N-to-one when $d = 1$ and f is injective when $d = 2$, we have that Q is countable. Since f has at most finitely many critical points, we assume without loss of generality that Q contains no critical points, and thus Df_x is invertible and continuous at x for all x in Q. Furthermore, we have that if $d = 2$, then $\det Df_x > 0$ for x in Q. We have shown that f is ready for operation on $\cup_m \theta_m$. Now let $C(f) = \cap_{n=1}^{\infty} \overline{\cup_{m\geq n}\{\theta_m\}}$. Since $\{\theta_m\}_{m\in\mathbb{N}}$ tends to μ, we have that $C(f) = \{\mu\}$. Also note that μ is totally ergodic. Recall that h-expansiveness (Definition 3.2.4) implies that any entropy structure (h_k) converges uniformly to h, which is equivalent to $u_\alpha \equiv 0$ and $\alpha_0(f) = 0$ (see [13, 35]). Since f is h-expansive, we have that $u_\alpha \equiv 0$ for all α and $\alpha_0(f) = 0$. Hence we have shown that f is in $\mathcal{S}(0, d, 0)$. □

Lemma 3.7.2. *Let $c \geq \log(3)$. Then for any p in $\mathbb{N}$ and $a > 0$, there exists F in $\mathcal{S}(p, d, a)$ such that $\mathbf{h}_{\text{top}}(F) \leq \max(c, \frac{a}{p})$ and $||u_k^{\mathcal{H}(F)}|| = \frac{ak}{p}$ for $k = 1, \ldots, p$.*

Proof. The proof is by induction on p. Consider the case $p = 1$. By Lemma 3.7.1, there exists f in $\mathcal{S}(0, d, 0)$ with $\{\theta_m\}_{m\in\mathbb{N}}$ and $\mathbf{h}_{\text{top}}(f) = \log(3)$. Choose N_m and ξ_m such that $1 \leq \xi_m \leq |\theta_m|$, $N_m \geq 3$, N_m is odd, and $\{\frac{\xi_m}{|\theta_m|}\log(N_m)\}_{m\in\mathbb{N}}$ increases to a. By Lemma 3.7.1, there exists χ_m in $\mathcal{S}(0, d, 0)$ with $\{\theta_\ell^m\}_{\ell\in\mathbb{N}}$ and $\mathbf{h}_{\text{top}}(\chi_m) = \log(N_m)$. Then Proposition 3.5.4 implies that there exists a function F in $\mathcal{BL}(f, \{\theta_m\}_{m\in\mathbb{N}}, \{\chi_m\}_{m\in\mathbb{N}}, \{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}, \{\xi_m\}_{m\in\mathbb{N}})$. Lemma 3.6.4 implies that F is in $\mathcal{S}(1, d, a)$. Also, $\mathbf{h}_{\text{top}}(F) = \max(\mathbf{h}_{\text{top}}(f), \sup_m \frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m)) \leq \max(c, a)$.

Now assume the lemma holds for some p. By the induction hypothesis, let f be in $S(p, d, \frac{ap}{p+1})$ with $\{\theta_m\}_{m\in\mathbb{N}}$ such that $\mathbf{h}_{\text{top}}(f) \leq \max(c, \frac{a}{p+1})$ and $||u_k^{\mathcal{H}(f)}|| = \frac{ak}{p+1}$ for $k = 1, \ldots, p$. Choose N_m and ξ_m such that $1 \leq \xi_m \leq |\theta_m|$, $N_m \geq 3$, N_m is odd, and $\{\frac{\xi_m}{|\theta_m|}\log(N_m)\}_{m\in\mathbb{N}}$ increases to $\frac{a}{p+1}$. By Lemma 3.7.1, there exists χ_m in $\mathcal{S}(0, d, 0)$ with $\{\theta_\ell^m\}_{\ell\in\mathbb{N}}$ and $\mathbf{h}_{\text{top}}(\chi_m) = \log(N_m)$. Then Proposition 3.5.4 implies that there exists a function F in $\mathcal{BL}(f, \{\theta_m\}_{m\in\mathbb{N}}, \{\chi_m\}_{m\in\mathbb{N}}, \{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}, \{\xi_m\}_{m\in\mathbb{N}})$. Lemma 3.6.4 implies that F is in $\mathcal{S}(p+1, d, a)$. Also, $\mathbf{h}_{\text{top}}(F) = \max(\mathbf{h}_{\text{top}}(f), \sup_m \frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m)) \leq \max(c, \frac{a}{p+1})$, and $||u_k^{\mathcal{H}(F)}|| = \frac{ak}{p+1}$ for $k = 1, \ldots, p+1$. □

Lemma 3.7.3. *Let $\alpha > 1$ be a countable, irreducible ordinal. Let $C > 0$. Suppose that for any ordinal $\delta < \alpha$, and any real numbers ϵ and a such that $0 < \epsilon < a$, there exists f in $\mathcal{S}(\alpha, d, a)$ such that $\mathbf{h}_{\text{top}}(f) \leq c$ and*

$$||u_\delta^{\mathcal{H}(f)}|| \leq \epsilon.$$

Then for any $a > 0$, and any natural number $p > 1$, there exists F in $\mathcal{S}(\alpha p, d, a)$ such that $\mathbf{h}_{\text{top}}(F) \leq c$ and

$$||u_{\alpha\ell}^{\mathcal{H}(F)}|| = \frac{\ell}{p}a, \text{ for } \ell = 1, \ldots, p.$$

Proof. The proof proceeds by induction on p. We suppose it holds for p and show it holds for $p+1$.

Let f be in $\mathcal{S}(\alpha p, d, \frac{ap}{p+1})$ with $\{\theta_m\}_{m\in\mathbb{N}}$ and satisfying the inductive hypotheses for p. Choose sequences $\{\delta_m\}_{m\in\mathbb{N}}$, $\{\xi_m\}_{m\in\mathbb{N}}$, and $\{a_m\}_{m\in\mathbb{N}}$ such that

- $\{\delta_m\}_{m\in\mathbb{N}}$ is an increasing sequence of ordinals whose limit is α;
- $\{a_m\}_{m\in\mathbb{N}}$ is a sequence of positive real numbers tending to infinity;
- for each m, ξ_m satisfies $1 \leq \xi_m \leq |\theta_m|$, and the sequence $\{\frac{\xi_m}{|\theta_m|}a_m\}_{m\in\mathbb{N}}$ is increasing to $\frac{a}{p+1}$.

Applying the hypothesis of the lemma, for each m in $\mathbb{N}$, there exists χ_m in $\mathcal{S}(\alpha, d, a_m)$ with $\{\theta_\ell^m\}_{\ell\in\mathbb{N}}$ such that $\mathbf{h}_{\text{top}}(\chi_m) \leq c$ and $||u_{\delta_m}^{\mathcal{H}(\chi_m)}|| \leq \min(\frac{a}{p+1}, \frac{1}{m})$. Note that since a_m tends to infinity and $\lim_m \frac{\xi_m}{|\theta_m|}a_m = \frac{a}{p+1}$, the sequence $\{\frac{\xi_m}{|\theta_m|}\}$ tends to 0. It follows that the sequence $\{\frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m)\}_{m\in\mathbb{N}}$ tends to 0. We assume without loss of generality that $\sup_m\{\frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m)\}_{m\in\mathbb{N}} \leq c$ (if this inequality is not satisfied, replace χ_m by

χ_{m+m_0} for sufficiently large m_0). Also, the sequence $\{\frac{\xi_m}{|\theta_m|}||u_{\delta_m}^{\mathcal{H}(\chi_m)}||\}_{m\in\mathbb{N}}$ tends to 0. By Proposition 3.5.4, there exists F in $\mathcal{BL}(f, \{\theta_m\}_{m\in\mathbb{N}}, \{\chi_m\}_{m\in\mathbb{N}}, \{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}, \{\xi_m\}_{m\in\mathbb{N}})$. We have that $\mathbf{h}_{\text{top}}(F) = \max(\mathbf{h}_{\text{top}}(f), \sup_m \frac{\xi_m}{|\theta_m}\mathbf{h}_{\text{top}}(\chi_m)) \leq c$. By Lemma 3.6.5, F is in $\mathcal{S}(\alpha + \alpha p, d, \frac{a}{p+1} + \frac{ap}{p+1}) = \mathcal{S}(\alpha(p+1), d, a)$ and

- for any $\gamma < \alpha$, $||u_\gamma^{\mathcal{H}(F)}|| = \max\big(||u_\gamma^{\mathcal{H}(f)}||, \sup_m ||u_\gamma^{\mathcal{H}(\chi_m)}||\big)$;
- for $\gamma \geq 0$, $||u_{\alpha+\gamma}^{\mathcal{H}(F)}|| = \frac{a}{p+1} + ||u_\gamma^{\mathcal{H}(f)}||$.

Then $||u_\alpha^{\mathcal{H}(F)}|| = \frac{a}{p+1}$, and the inductive hypotheses on f imply that $||u_{\alpha\ell}^{\mathcal{H}(F)}|| = \frac{a\ell}{p+1}$ for $\ell = 1, \ldots, p+1$. Thus F satisfies the induction hypotheses for $p+1$, and by induction the lemma holds for all p.

□

Lemma 3.7.4. *Let $\alpha > 1$ be a countable, irreducible ordinal. Let $c \geq \log(3)$. Then for all ordinals $\delta < \alpha$ and all real numbers ϵ and a such that $0 < \epsilon < a$, there exists F in $\mathcal{S}(\alpha, d, a)$ such that $\mathbf{h}_{\text{top}}(F) \leq c$ and*

$$||u_\delta^{\mathcal{H}(F)}|| \leq \epsilon.$$

Proof. The proof is by transfinite induction on the irreducible ordinals $\alpha > 1$. For notation, we let $\alpha = \omega^\beta$, and use transfinite induction on $\beta \geq 1$.

Case ($\beta = 1$). Let f be in $\mathcal{S}(0, d)$ with $\{\theta_m\}_{m\in\mathbb{N}}$ and $\mathbf{h}_{\text{top}}(f) = \log(3)$ (such a map f exists by Lemma 3.7.1). Let a, ϵ, and δ be as in the statement of the lemma. Choose sequences $\{a_m\}_{m\in\mathbb{N}}$ and $\{\xi_m\}_{m\in\mathbb{N}}$ such that

- $\{a_m\}_{m\in\mathbb{N}}$ tends to infinity;

- ξ_m is a natural number such that $1 \leq \xi_m \leq |\theta_m|$;

- the sequence $\{\frac{\xi_m}{|\theta_m|}a_m\}_{m\in\mathbb{N}}$ increases to a.

By Lemma 3.7.2, there exists χ_m in $\mathcal{S}(m, d, a_m)$, with corresponding sequence $\{\theta_\ell^m\}_{\ell\in\mathbb{N}}$, and such that $\mathbf{h}_{\text{top}}(\chi_m) \leq \max(c, \frac{a_m}{m})$ and $||u_k^{\mathcal{H}(\chi_m)}|| = \frac{a_m k}{m}$ for $k = 1, \ldots, m$. Note that since a_m tends to infinity and $\{\frac{\xi_m}{|\theta_m|}a_m\}_{m\in\mathbb{N}}$ increases to a, we have that $\{\frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m)\}_{m\in\mathbb{N}}$ tends to 0. Thus we assume without loss of generality that $\sup_m \frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m) \leq c$ (by replacing χ_m with χ_{m+m_0} for sufficiently large m_0 if necessary). Let $\delta_m = [\log(m)]$, the integer part of $\log(m)$. Then we obtain that $\{\frac{\xi_m}{|\theta_m|}||u_{\delta_m}^{\mathcal{H}(\chi_m)}||\}_{m\in\mathbb{N}}$ tends to 0 (since $\frac{\xi_m}{|\theta_m|}||u_{\delta_m}^{\mathcal{H}(\chi_m)}|| = \frac{\xi_m a_m[\log(m)]}{|\theta_m|m} \leq \frac{a[\log(m)]}{m}$). By Proposition 3.5.4, there exists F in $\mathcal{BL}(f, \{\theta_m\}_{m\in\mathbb{N}}, \{\chi_m\}_{m\in\mathbb{N}}, \{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}, \{\xi_m\}_{m\in\mathbb{N}})$. Then by Lemma 3.6.4, F is in $\mathcal{S}(\omega, d, a)$. Furthermore, by construction we have that $\mathbf{h}_{\text{top}}(F) = \max(\mathbf{h}_{\text{top}}(f), \sup_m \frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m)) \leq c$, and the final statement in Lemma 3.6.4 gives that for $0 < \epsilon < a$ and $\delta < \alpha$, there exists m_0 such that replacing χ_m with χ_{m+m_0} produces F such that $||u_\delta^{\mathcal{H}(F)}|| \leq \epsilon$.

Case (successor ordinal). Now suppose the lemma holds for the irreducible ordinal ω^β. We show that it also holds for $\omega^{\beta+1}$. Let f be in $\mathcal{S}(0, d)$ with $\{\theta_m\}_{m\in\mathbb{N}}$ and $\mathbf{h}_{\text{top}}(f) = \log(3)$. Choose sequences $\{\alpha_m\}_{m\in\mathbb{N}}$, $\{\delta_m\}_{m\in\mathbb{N}}$, $\{a_m\}_{m\in\mathbb{N}}$ and $\{\xi_m\}_{m\in\mathbb{N}}$ such that

- $\alpha_m = \omega^\beta m$;

- $\delta_m = \omega^\beta[\log(m)]$;

- $\{a_m\}_{m\in\mathbb{N}}$ is a sequence of positive real numbers tending to infinity;

- for each m, ξ_m satisfies $1 \leq \xi_m \leq |\theta_m|$, and the sequence $\{\frac{\xi_m}{|\theta_m|}a_m\}_{m\in\mathbb{N}}$ is increasing to a.

The inductive hypotheses imply that the hypotheses in Lemma 3.7.3 are satisfied for ω^β. Applying Lemma 3.7.3 for each m in $\mathbb{N}$, we obtain that there exists χ_m such that

- χ_m is in $\mathcal{S}(\omega^\beta m, d, a_m)$ with $\{\theta_\ell^m\}_{\ell\in\mathbb{N}}$;
- $\mathbf{h}_{\text{top}}(\chi_m) \leq c$;
- $||u_{\delta_m}^{\mathcal{H}(\chi_m)}|| = \frac{a_m[\log(m)]}{m}$.

Since $\{a_m\}_{m\in\mathbb{N}}$ tends to infinity and $\{\frac{\xi_m}{|\theta_m|}a_m\}_{m\in\mathbb{N}}$ tends to a, $\{\frac{\xi_m}{|\theta_m|}\}_{m\in\mathbb{N}}$ tends to 0. Therefore $\{\frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m)\}_{m\in\mathbb{N}}$ tends to 0. Also, we have that $\{\frac{\xi_m}{|\theta_m|}||u_{\delta_m}^{\mathcal{H}(\chi_m)}||\}_{m\in\mathbb{N}}$ tends to 0. We assume without loss of generality that $\sup_m \frac{\xi_m}{|\theta_m|}||u_{\delta_m}^{\mathcal{H}(\chi_m)}|| \leq \epsilon$. Now let $\delta < \alpha$ and $0 < \epsilon < a$ be arbitrary. There exists m_0 such that $\delta_m > \delta$ for all $m \geq m_0$. Also, there exists m_1 such that $\frac{a[\log(m)]}{m} < \epsilon$ for all $m \geq m_1$. Let $m_2 = \max(m_0, m_1)$. Replace χ_m by χ_{m+m_2}. By Proposition 3.5.4, there exists F in $\mathcal{BL}(f, \{\theta_m\}_{m\in\mathbb{N}}, \{\chi_m\}_{m\in\mathbb{N}}, \{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}, \{\xi_m\}_{m\in\mathbb{N}})$. We have that $\mathbf{h}_{\text{top}}(F) = \max(\mathbf{h}_{\text{top}}(f), \sup_m \frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m)) \leq c$. Then Lemma 3.6.5 implies that F is in $\mathcal{S}(\omega^{\beta+1}, d, a)$ and

$$||u_\delta^{\mathcal{H}(F)}|| = \max\Big(||u_\delta^{\mathcal{H}(f)}||, \sup_m \frac{\xi_m}{|\theta_m|}||u_\delta^{\mathcal{H}(\chi_m)}||\Big) = \sup_m \frac{\xi_m}{|\theta_m|}||u_\delta^{\mathcal{H}(\chi_m)}|| \leq \epsilon,$$

as desired.

Case (β limit ordinal). Now suppose the lemma holds for all irreducible ordinals $\omega^\gamma < \omega^\beta$, where β is a limit ordinal. We show that it also holds for ω^β. Let f be in $\mathcal{S}(0,d)$ with $\{\theta_m\}_{m\in\mathbb{N}}$ and $\mathbf{h}_{\text{top}}(f) \leq c$. Choose a sequence $\{a_m\}_{m\in\mathbb{N}}$ of positive real numbers tending to infinity and an increasing sequence of ordinals $\{\beta_m\}_{m\in\mathbb{N}}$ tending to β. The inductive hypothesis implies that for each m, there exists χ_m in $\mathcal{S}(\omega^{\beta_m}, d, a_m)$ with $\{\theta_\ell^m\}_{\ell\in\mathbb{N}}$ such that $\mathbf{h}_{\text{top}}(\chi_m) \leq c$ and $||u_{\omega^{\beta_{m-1}}}^{\mathcal{H}(\chi_m)}|| \leq \frac{1}{m}$. Now let $\delta < \omega^\beta$ and $\epsilon > 0$ be arbitrary. There exists m_0 such that $\delta_m > \delta$ for all $m \geq m_0$. Also, there exists m_1 such that $\frac{1}{m} < \epsilon$ for all $m \geq m_1$. Let $m_2 = \max(m_0, m_1)$. Then replace χ_m by χ_{m+m_2}. By Proposition 3.5.4, there exists F in $\mathcal{BL}(f, \{\theta_m\}_{m\in\mathbb{N}}, \{\chi_m\}_{m\in\mathbb{N}}, \{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}, \{\xi_m\}_{m\in\mathbb{N}})$. By Corollary 3.5.22, we have that $\mathbf{h}_{\text{top}}(F) = \max(\mathbf{h}_{\text{top}}(f), \sup_m \frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m)) \leq c$. Then Lemma 3.6.5 implies that F is in $\mathcal{S}(\omega^\beta, d, a)$ and

$$||u_\delta^{\mathcal{H}(F)}|| = \max\Big(||u_\delta^{\mathcal{H}(f)}||,\ \sup_m \frac{\xi_m}{|\theta_m|}||u_\delta^{\mathcal{H}(\chi_m)}||\Big) = \sup_m \frac{\xi_m}{|\theta_m|}||u_\delta^{\mathcal{H}(\chi_m)}|| \leq \epsilon,$$

as desired.

□

Proof of Theorem 3.3.1. Let $\alpha = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$, with $\beta_1 \geq \cdots \geq \beta_n$. We argue by induction on n. If $n = 1$, then either Lemma 3.7.2 (if $\beta_1 = 0$) or Lemma 3.7.4 (if $\beta_1 > 0$) implies that there exists F in $\mathcal{S}(\alpha, d, a)$. Suppose the statement holds for n. We show that it holds for $n+1$. If $\beta_1 = 0$, then Lemma 3.7.2 implies that F exists with the desired properties. Now suppose $\beta_1 > 0$. Let $a_1 \geq a_0 > 0$ with $a_1 + a_0 = a$. By the induction hypothesis, there exists f in $\mathcal{S}(\omega^{\beta_2} + \cdots + \omega^{\beta_n}, d, a_0)$ with $\{\theta_m\}_{m\in\mathbb{N}}$. Choose sequences $\{a_m\}_{m\in\mathbb{N}}$, $\{\delta_m\}_{m\in\mathbb{N}}$, and $\{\xi_m\}_{m\in\mathbb{N}}$ such that

- $\{a_m\}_{m\in\mathbb{N}}$ is a sequence of positive numbers tending to infinity;

- $\{\delta_m\}_{m\in\mathbb{N}}$ is an increasing sequence of ordinals tending to ω^{β_1};

- $1 \leq \xi_m \leq |\theta_m|$ and the sequence $\{\frac{\xi_m}{|\theta_m|}a_m\}_{m\in\mathbb{N}}$ increases to a.

Let $c \geq \log(3)$. Then for each m, Lemma 3.7.4 implies that there exists χ_m in $\mathcal{S}(\omega^{\beta_1}, d, a_m)$ with $\{\theta_\ell^m\}_{\ell\in\mathbb{N}}$ such that $\mathbf{h}_{\text{top}}(\chi_m) \leq c$ and $||u_{\delta_m}^{\mathcal{H}(\chi_m)}|| \leq \frac{1}{m}$. Note that $\{\frac{\xi_m}{|\theta_m|}\mathbf{h}_{\text{top}}(\chi_m)\}$ tends to 0 with these choices of parameters. By Proposition 3.5.4, there exists F in $\mathcal{BL}(f, \{\theta_m\}_{m\in\mathbb{N}}, \{\chi_m\}_{m\in\mathbb{N}}, \{\theta_\ell^m\}_{m,\ell\in\mathbb{N}}, \{\xi_m\}_{m\in\mathbb{N}})$. By Lemma 3.6.5, F is in $\mathcal{S}(\omega^{\beta_1}+\omega^{\beta_2}+\cdots+\omega^{\beta_n}, d, a_1+a_0) = \mathcal{S}(\alpha, d, a)$, which completes the induction and the proof. $\square$

Chapter 4

Random subshifts of finite type

4.1 Introduction

A shift of finite type (SFT) is a dynamical system defined by finitely many local transition rules. These systems have been studied for their own sake [59, 66], and they have also served as important tools for understanding other dynamical systems [53, 12, 32].

Each SFT can be described as the set of bi-infinite sequences on a finite alphabet that avoid a finite list of words over the alphabet. Thus there are only countably many SFTs up to the naming of letters in an alphabet.

For the sake of simplicity, we state our results in terms of SFTs in the introduction, even though we prove more general results in terms of sequences of directed graphs in the subsequent sections. Let X be a non-empty SFT (for definitions, see Section 4.2.1). Let $B_n(X)$ be the set of words of length n that appear in X. For α in $[0, 1]$, let $\mathbb{P}_\alpha$ be the probability measure on the power set of $B_n(X)$ given by choosing each word in $B_n(X)$ independently with probability α. The case $\alpha = 1/2$ puts uniform measure on the subsets of $B_n(X)$. For notation, let Ω_n be the power set of $B_n(X)$. To each subset ω of $B_n(X)$, we associate the SFT X_ω consisting of all points x in X such that each word of length n in x is contained in ω. With this association, we view $\mathbb{P}_\alpha$ as a probability measure on the SFTs X_ω that can be built out of the subsets of $B_n(X)$. Briefly, if X has entropy $\mathbf{h}(X) = \log\lambda > 0$ and n is large, then a typical random SFT X_ω is built from about $\alpha\lambda^n$ words, an α fraction of all the words in $B_n(X)$, but not all of these words will occur in any point in X_ω.

Our main results can be stated as follows. Let $\zeta_X(t)$ denote the Artin-Mazur zeta function of X (see Definition 4.2.11). The first theorem deals with the likelihood that a randomly chosen SFT is empty.

Theorem 4.1.1. *Let X be a non-empty SFT with entropy $\mathbf{h}(X) = \log\lambda$. Let $\mathcal{E}_n \subset \Omega_n$ be the event that X_ω is empty. Then for α in $[0,1]$,*

$$\lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{E}_n) = \begin{cases} (\zeta_X(\alpha))^{-1}, & \text{if } \alpha \in [0, 1/\lambda) \\ 0, & \text{if } \alpha \in [1/\lambda, 1], \end{cases}$$

Thus when α is in $[0, 1/\lambda)$, there is an asymptotically positive probability of emptiness. The next theorem gives more information about what happens when α lies in $[0, 1/\lambda)$.

Theorem 4.1.2. *Let X be a non-empty SFT with entropy $\mathbf{h}(X) = \log\lambda$. Let $\mathcal{Z}_n \subset \Omega_n$ be the event that X_ω has zero entropy, and let I_n be the random variable on Ω_n which is the number of irreducible components of X_ω. Then for $0 \le \alpha < 1/\lambda$,*

(1) $\lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{Z}_n) = 1$;

(2) the sequence (I_n) converges in distribution to the random variable I_∞ such that $\mathbb{P}(I_\infty = 0) = (\zeta_X(\alpha))^{-1}$ and for $k \ge 1$,

$$\mathbb{P}(I_\infty = k) = (\zeta_X(\alpha))^{-1} \sum_{\substack{S\subset\mathbb{N} \\ |S|=k}} \prod_{s\in S} \frac{\alpha^{|\gamma_s|}}{1-\alpha^{|\gamma_s|}},$$

where $\{\gamma_i\}_{i=1}^\infty$ is an enumeration of the periodic orbits in X;

(3) the random variable I_∞ has exponentially decreasing tail and therefore finite moments of all orders.

Our next result describes the entropy of the typical random SFT when α lies in $(1/\lambda, 1]$.

Theorem 4.1.3. *Let X be an SFT with positive entropy $\mathbf{h}(X) = \log \lambda$. Then for $1/\lambda < \alpha \leq 1$ and $\epsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}_\alpha(|\mathbf{h}(X_\omega) - \log(\alpha\lambda)| \geq \epsilon) = 0,$$

and the convergence to this limit is exponential in n.

Finally, we have a result concerning the likelihood that a random SFT will have a unique irreducible component of positive entropy when α is near 1.

Theorem 4.1.4. *Let X be an irreducible SFT with positive entropy $\mathbf{h}(X) = \log \lambda$. Let $W_n \subset \Omega_n$ be the event that X_ω has a unique irreducible component C of positive entropy and C has the same period as X. Then there exists $c > 0$ such that for $1 - c < \alpha \leq 1$,*

$$\lim_{n\to\infty} \mathbb{P}_\alpha(W_n) = 1;$$

furthermore, the convergence to this limit is exponential in n.

There have been studies of other objects called random subshifts of finite type in the literature [11, 10, 43, 54, 55, 56, 57, 58], but the objects studied here are rather different in nature. The present work is more closely related to perturbations of SFTs, which have already appeared in works by Lind [64] in dimension 1 and by

Pavlov [77] in higher dimensions. In those works, the main results establish good uniform bounds for the entropy of an SFT obtained by removing any single word of length n from a sufficiently mixing SFT as n tends to infinity. Random SFTs may also be interpreted as dynamical systems with holes [22, 21, 23, 24, 25, 29, 28, 30, 31, 67, 68], in which case the words of length n in X that are forbidden in the random SFT X_ω are viewed as (random) holes in the original system X. The question of whether an SFT defined by a set of forbidden words is empty has been studied in formal language theory and automata theory, and in that context it amounts to asking whether the set of forbidden words is *unavoidable* [7, 20, 50]. Also, the random SFTs considered here can be viewed as specific instances of random matrices (see [6, 72]) or random graphs (see [3, 8, 40, 41, 42, 48, 47, 74]), and the concept of directed percolation on finite graphs has appeared in the physics literature in the context of directed networks [76, 82]. To the best of our knowledge, the specific considerations that arise for our random SFTs seem not to have appeared in any of this wider literature.

This chapter is organized as follows. Section 4.2 contains the necessary background and notation, as well as some preliminary lemmas. The reader familiar with SFTs and directed graphs may prefer to skip Sections 4.2.1 and 4.2.2, referring back as necessary. In Section 4.3 we discuss the likelihood that a random SFT is empty, and in particular we prove Theorem 4.1.1. The remainder of the main results are split into two sections according to two cases: $\alpha \in [0, 1/\lambda)$ and $\alpha \in (1/\lambda, 1]$. The case $\alpha \in [0, 1/\lambda)$ is treated in Section 4.4, and the case $\alpha \in (1/\lambda, 1]$ is addressed in Section 4.5. Section 4.6 discusses some corollaries of the main results.

4.2 Preliminaries

4.2.1 Shifts of finite type and their presentations

For a detailed treatment of SFTs and their presentations, see [66]. In this section we describe three ways to present an SFT: with a finite list of forbidden words over a finite alphabet, with a finite, directed graph, or with a square, non-negative integer matrix.

Let $\mathcal{A}$ be a finite set, which we will call the **alphabet**. An element $b \in \mathcal{A}^n$ is called a **word** of length n. Let $\Sigma = \mathcal{A}^{\mathbb{Z}}$, endowed with the product topology induced by the discrete topology on $\mathcal{A}$. Then Σ is a compact metrizable space, which is called the **full shift** on $\mathcal{A}$. Let $\sigma : \Sigma \to \Sigma$ be the left shift, *i.e.* for $x = (x_i)$ in Σ, let $(\sigma(x))_i = x_{i+1}$. With this definition σ is a homeomorphism of Σ.

A subset X of Σ is called shift-invariant if $\sigma(X) = X$. A closed, shift-invariant subset of Σ is called a **subshift**. For any subshift X, the **language** $\mathcal{B}(X)$ of X is the collection of all finite words (blocks) that appear in some sequence x in X. Note that $\mathcal{B}(X) = \cup B_n(X)$, where $B_n(X)$ is the set of all words of length n that appear in some sequence x in X. (By convention we set $B_0(X) = \{\epsilon\}$, where ϵ denotes the empty word). Given a set $\mathcal{F}$ of words on $\mathcal{A}$, we may define a subshift $X(\mathcal{F})$ as the set of sequences x in Σ such that no word in $\mathcal{F}$ appears in x. One may check that this procedure indeed defines a subshift. If X is a subshift and there exists a *finite* set of words $\mathcal{F} = \{F_1, \ldots, F_k\}$ such that $X = X(\mathcal{F})$, then X is called a **subshift of finite type** (SFT).

The natural notion of isomorphism for SFTs is called conjugacy. Two SFTs X

and Y are **conjugate**, written $X \cong Y$, if there exists a homeomorphism $\phi : X \to Y$ such that $\phi \circ \sigma = \sigma \circ \phi$. An SFT X is **irreducible** if for every two non-empty open sets U and V and every N in $\mathbb{N}$, there exists $n \geq N$ such that $\sigma^n(U) \cap V \neq \emptyset$. An SFT X is **mixing** if for every two non-empty open sets U and V in X, there exists n_0 in $\mathbb{N}$ such that for all $n \geq n_0$, we have $\sigma^n(U) \cap V \neq \emptyset$. Mixing and irreducibility are conjugacy-invariant. We now define the higher block presentations of an SFT.

Definition 4.2.1. Let X be an SFT. The n-**block presentation** of X, denoted $X^{[n]}$ is defined as follows. The alphabet for $X^{[n]}$ is $B_n(X)$. We define the code $\phi_n : X \to B_n(X)^{\mathbb{Z}}$ by the equation

$$\phi_n(x)_i = x[i, i+n-1], \tag{4.2.1}$$

for all x in X. Then $X^{[n]} = \phi_n(X)$. For all $n \geq 1$, we have that $X^{[n]} \cong X$, where the conjugacy is given by ϕ_n.

Definition 4.2.2. The **entropy** of an SFT X is defined as $\mathbf{h}(X) = \lim_n \frac{1}{n} \log |B_n(X)|$.

Alternatively, one may define SFTs in terms of finite directed graphs. A directed graph $G = (V, E)$ consists of a set of vertices V and a set of edges E such that for each edge $e \in E$, there is a unique initial vertex, $i(e) \in V$, and a unique terminal vertex, $t(e) \in V$. We view the edge e as going from $i(e)$ to $t(e)$. We allow self-loops, but for the sake of convenience we assume (without loss of generality for our considerations) that there are no multiple edges. In this chapter, we make the standing convention that "graph" means directed graph. We will collect our standing assumptions in Standing Assumptions 4.2.21.

Definition 4.2.3. Given a directed graph G, we define the **edge shift** X_G to be the set of all bi-infinite (oriented) walks on G, *i.e.* $X_G = \{x \in E^{\mathbb{Z}} : t(x_j) = i(x_{j+1}) \text{ for all } j \in \mathbb{Z}\}$.

Any edge shift is an SFT (trivially). Let us show that any SFT is conjugate to an edge shift. If $X = X(\mathcal{F})$ is an SFT and $\mathcal{F}$ is a finite set of forbidden words, then $X \cong X_G$, where $G = (V, E)$ is defined as follows. Let $n_0 = \max\{|F| : F \in \mathcal{F}\}$. Then let $V = B_{n_0-1}(X)$ and $E = B_{n_0}(X)$. Further, for any edge $e \in B_{n_0}(X)$, we let $i(e) = e[1, n_0 - 1]$ and $t(e) = e[2, n_0]$. The same construction works with n in place of n_0 for any $n \geq n_0$.

If G is a graph such that $X \cong X_G$, we say that X_G is an edge presentation of X, or sometimes just a presentation of X. The **adjacency matrix** A of a directed graph G may be defined as follows. Fix an enumeration of the vertices in G. Then let $A_{k\ell}$ be the number of distinct edges e in G such that $i(e) = v_k$ and $t(e) = v_\ell$. A square, non-negative integral matrix A is **irreducible** if for each pair i, j and each N, there exists $n > N$ such that $(A^n)_{ij} > 0$. A matrix A is **non-degenerate** if it has no zero row and no zero column. If A is non-degenerate, then the edge shift X_G is irreducible if and only if A is irreducible. Also, if A is non-degenerate, then the edge shift X_G is mixing if and only if there exists n_0 such that for all $n \geq n_0$ and all pairs i, j, it holds that $(A^n)_{ij} > 0$. A matrix is **primitive** if it satisfies the latter property. A path in G is a finite sequence $\{e_j\}_{j=1}^n$ of edges such that $t(e_j) = i(e_{j+1})$ for $j = 1, \ldots, n-1$. If $b = b_1 \ldots b_n$ is a path in G, we say that b goes from vertex $i(b_1)$ to vertex $t(b_n)$. We denote by $B_k(G)$ the set of paths of length k in G. By convention, we set $B_0(G) = V$.

Definition 4.2.4. For a path b in G, let $V(b)$ and $E(b)$ be the set of vertices and the set of edges traversed by b, respectively.

Definition 4.2.5. Let X be an SFT. An **irreducible component** Y of X is a non-empty, maximal SFT contained in X such that Y is irreducible. Let G be a graph. An **irreducible component** C of G is a non-empty, maximal subgraph of G such that the adjacency matrix of C is irreducible. The reader should be advised that in some papers the definition of irreducible component includes trivial components (a single vertex with no edges adjacent to it), but the definition given here does not include trivial components.

Definition 4.2.6. Let G be a finite, directed graph. For $n \geq 1$, define $G^{[n]} = (V^{[n]}, E^{[n]})$, the n-**block graph** of G, as follows. Let $V^{[n]} = B_{n-1}(G)$ and $E^{[n]} = B_n(G)$, such that if $e \in E^{[n]}$, then $i(e) = e[1, n-1]$ and $t(e) = e[2, n]$. Note that $G^{[1]} = G$.

If $X = X_G$ for some graph G, then it follows immediately from the definitions that $X^{[n]} = X_{G^{[n]}}$.

Definition 4.2.7. Let $G = (V, E)$ be a graph. For p in $\mathbb{N}$, we define the p-th power graph, $G^p = (V^p, E^p)$ as follows. Let $V^p = V$ and $E^p = B_p(G)$. If $b = b_1 \dots b_p$ is an edge in G^p, then we let $i(b) = i(b_1)$ and $t(b) = t(b_1)$.

Definition 4.2.8. Let $G = (V, E)$ be a graph. Define the transpose graph, $G^T = (V^T, E^T)$, as follows. Let $V^T = V$ and $E^T = E$, where an edge e in G^T goes from $t(e)$

to $i(e)$. In other words, the transpose graph is just the graph formed by reversing the direction of all the edges in G.

Given a square, non-negative, integral matrix A, one may also define an SFT X_A as follows. Let G be a directed graph whose adjacency matrix is exactly A (such a graph always exists). Then let X_A be the edge shift defined by G.

Recall the following basic facts (which may be found in [66]). For an SFT X, we have $\mathbf{h}(X) = \inf_n \frac{1}{n} \log |B_n(X)|$. If X is a non-empty SFT and $X = X_A$ for a square, non-negative integral matrix A, then $\mathbf{h}(X) = \log \lambda$, where λ is the spectral radius of A. By the Perron-Frobenius Theorem, if A is non-negative and irreducible, then there exists a strictly positive (column) vector v such that $Av = \lambda v$, and there exists a strictly positive (row) vector w such that $wA = \lambda w$. Furthermore, v and w are each unique up to a positive scalar.

Definition 4.2.9. For any non-negative integer matrix A, let λ_A be the spectral radius of A, and let χ_A be the characteristic polynomial of A. Then let $\mathrm{Sp}_\times(A)$ be the **non-zero spectrum** of the matrix A, which is defined as the multiset of non-zero roots of χ_A listed according to their multiplicity. If A is the adjacency matrix of the graph G, we define $\lambda_G = \lambda_A$ and $\mathrm{Sp}_\times(G) = \mathrm{Sp}_\times(A)$.

If $X_A \cong X_B$ for two non-negative integral matrices A and B, then $\mathrm{Sp}_\times(A) = \mathrm{Sp}_\times(B)$. Also, if A is primitive, then $\max\{|\beta| : \beta \in \mathrm{Sp}_\times(A) \setminus \{\lambda_A\}\} < \lambda_A$. Finally, if A is irreducible, then there exists a unique σ-invariant Borel probability measure μ on X_A of maximal entropy. Let us describe some basic properties of μ. We associate a word $b = b_1 \dots b_k$ in X to the cylinder set $C_b = \{x \in X : x[1, k] =$

$b\}$. In this way we interpret the measure of words in $\mathcal{B}(X)$ as the measure of the corresponding cylinder set. Let v be a positive right eigenvector of A and w a positive left eigenvector of A, and suppose they are normalized so that $w \cdot v = 1$. Our standing assumption that there are no multiple edges means that that $A_{ij} \leq 1$ for all i, j. Then for a vertex u in V, we have $\mu(u) = w_u v_u$, and for $b \in B_n(X_A)$, we have that

$$\mu(b) = w_{i(b_1)} \lambda_A^{-n} v_{t(b_n)}. \tag{4.2.2}$$

Now we define two objects, the period and the zeta function, which contain combinatorial information about the cycles in a graph G (alternatively, one may refer to the periodic points in an SFT X).

Definition 4.2.10. For an SFT X, let $\mathrm{per}(X)$ be the greatest common divisor of the sizes of all periodic orbits in X. For a graph G, let $\mathrm{per}(G)$ be the greatest common divisor of the lengths of all cycles in G.

Definition 4.2.11. Let X be an SFT and $N_p = |\{x \in X : \sigma^p(x) = x\}|$. Then the Artin-Mazur zeta function of X (see [66]) is, by definition,

$$\zeta_X(t) = \exp\left(\sum_{p=1}^{\infty} \frac{N_p}{p} t^p\right).$$

For a graph G, let $\zeta_G = \zeta_{X_G}$.

For a graph G, note that $|\{x \in X_G : \sigma^p(x) = x\}|$ is the number of cycles of (not necessarily least) period p in G, and

$$\zeta_G(t) = \frac{1}{\det(I - tA)} = \prod_{\lambda \in \mathrm{Sp}_\times(G)} \frac{1}{1 - \lambda t}.$$

Also, ζ_G has radius of convergence $1/\lambda_G$ and $\lim_{t \to 1/\lambda_G^-} \zeta_G(t) = +\infty$.

4.2.2 Sequences of graphs under consideration

In this chapter we consider sequences of graphs (G_n) that grow in some way. A particular example of such a sequence is the sequence of n-block graphs of an SFT X. Indeed, by taking (G_n) to be such a sequence in Theorems 4.3.1, 4.4.2, 4.5.13, and 4.5.15, we obtain the theorems stated in the introduction. Generalizing to the graph setting also allows one to consider sequences of graphs presenting SFTs which are conjugate to a fixed SFT X, where the sequences need not be the n-block sequence for X. To indicate the generality of the arguments further, though, we formulate and prove the results for sequences of graphs that do not necessarily present conjugate SFTs. Before we move on to these results, we need to define several notions regarding the manner of growth of the sequence (G_n).

Let G be a finite, directed graph with adjacency matrix A. We will have use for the following notations.

Definition 4.2.12. Let

$$\mathrm{Per}_p(G) = \{b \in B_p(G) : i(b_1) = t(b_p)\}, \quad \text{and} \quad \mathrm{Per}(G) = \cup_{p=1}^{\infty} \mathrm{Per}_p(G).$$

For b in $\mathrm{Per}_p(G)$, let $\theta(b)$ be the set of all paths c in $\mathrm{Per}_p(G)$ such that there exists a natural number ℓ such that $c = b_{\tau^\ell(1)} \dots b_{\tau^\ell(p)}$, where τ is the permutation of $\{1, \dots, k\}$ defined in cycle notation by $(1 \dots k)$.

Definition 4.2.13. For each vertex u in G, let $d_{\mathrm{out}}(u) = |\{e \in E : i(e) = u\}|$ and $d_{\mathrm{in}}(u) = |\{e \in E : t(e) = u\}|$. Then let

$$d_{\max}(G) = \max\{\max(d_{\mathrm{out}}(u), d_{\mathrm{in}}(u)) : u \in V\}.$$

In order to measure the separation of periodic orbits in G, we make the following definition.

Definition 4.2.14. Let

$$z(G) = \max\{n \geq 0 : \forall b, c \in \cup_{p=1}^{n} \operatorname{Per}_p(G) \text{ with } c \notin \theta(b), V(b) \cap V(c) = \emptyset\},$$

where $V(b)$ is the set of vertices traversed by the path b.

As a measure of the size of G, we consider the following quantity.

Definition 4.2.15. If A has spectral radius $\lambda > 1$, then let

$$m(G) = \lceil \log_\lambda |V| \rceil.$$

To measure a range for uniqueness of paths in G, we make the following definitions.

Definition 4.2.16. Let

$$U_1(G) = \sup\{n : \forall i, j \text{ it holds that } (A^n)_{ij} \leq 1\}$$

$$U_2(G) = \sup\{n : \forall u \in V \text{ and } 1 \leq s < t \leq n,\ |\{b \in B_t(X) : i(b_1) = u, b_s = b_t\}| \leq 1\}$$

$$U(G) = \min(U_1(G), U_2(G)).$$

We use the transition length as a type of diameter of G.

Definition 4.2.17. Let

$$R(G) = \inf\{n : \forall i, j, \exists k \leq n, (A^k)_{ij} > 0\}.$$

Here we briefly recall the notion of the weighted Cheeger constant of an irreducible, directed graph G. The weighted Cheeger constant was defined and studied in [27]. Let μ be the measure of maximal entropy of X_G, and let $F : E \to [0,1]$ be given by $F(e) = \mu(e)$. For any vertex v in V, let $F(v) = \sum_{i(e)=v} F(e) = \sum_{t(e)=v} F(e)$. Then for any subset of vertices $S \subseteq V$, let $F(S) = \sum_{v \in S} F(v)$, and for any two subsets $S, T \subseteq V$, let

$$F(S,T) = \sum_{\substack{i(e) \in S \\ t(e) \in T}} F(e).$$

In general $F(S,T)$ is not symmetric in S and T since G is directed. Let $E(S,T)$ be the set of edges e in G such that $i(e) \in S$ and $t(e) \in T$. Let $\overline{S} = V \setminus S$.

Definition 4.2.18. The weighted Cheeger constant of G is defined as

$$c_w(G) = \inf_{\emptyset \subsetneq S \subsetneq V} \frac{F(S, \overline{S})}{\min(F(S), F(\overline{S}))},$$

and the unweighted Cheeger constant of G is defined as

$$c(G) = \inf_{0 < |S| \leq |V|/2} \frac{|E(S, \overline{S})|}{|S|}.$$

Definition 4.2.19. We say that G is a directed b-expander graph if $c(G) \geq b$. Also, a sequence of directed graphs (G_n) is a **uniform expander sequence**, if there exists a $b > 0$ such that G_n is a directed b-expander for each n.

We will also have use for the following quantity related to the spectral gap of G.

Definition 4.2.20. Let $g(G) = \min\Big\{1 - \frac{|\lambda_i|}{\lambda} : \lambda_i \in \mathrm{Sp}_\times(G) \setminus \{\lambda\}\Big\}$.

We make the following standing assumptions, even though some of the statements we make may hold when these restrictions are relaxed. In particular, Theorems 4.3.1 and 4.4.2 do not require that A_n is irreducible, nor do they require that $\lambda > 1$ (see Remark 4.6.1).

Standing Assumptions 4.2.21. Recall that "graph" means directed graph. Let (G_n) be a sequence of graphs with associated sequence of adjacency matrices (A_n). Unless otherwise stated, we will make the following assumptions:

- for each n, each entry of A_n is contained in $\{0, 1\}$;
- each A_n is irreducible;
- for each n, $\mathrm{Sp}_\times(A_n) = \mathrm{Sp}_\times(A_1)$;
- $\lambda := \lambda_{A_1} > 1$;
- $\lim_n m(G_n) = \infty$.

Remark 4.2.22. Note that $|\operatorname{Per}_p(G_n)| = \operatorname{tr}(A_n^p)$, which depends only on $\mathrm{Sp}_\times(A_n)$ and p. Therefore the standing assumptions imply that $|\operatorname{Per}_p(G_n)|$ does not depend on n, and therefore $\operatorname{per}(G_n)$ and ζ_{G_n} do not depend on n.

Additional conditions that we place on sequences of graphs will come from the following list. (Different theorems will require different assumptions, but the sequence of n-block graphs of an irreducible graph with spectral radius greater than 1 will satisfy conditions (C1)-(C8) below by Proposition 4.2.29.)

Definition 4.2.23. We define the following conditions on a sequence of graphs (G_n) with sequence of adjacency matrices (A_n):

(C1) there exists $\Delta > 0$ such that $d_{\max}(G_n) \leq \Delta$ for all n (bounded degree).

(C2) $z(G_n)$ tends to infinity as n tends to infinity (separation of periodic points);

(C3) there exists $C > 0$ such that $z(G_n) \geq Cm(G_n)$ for all n (fast separation of periodic points);

(C4) there exists $C > 0$ such that $U(G_n) \geq m(G_n) - C$ for all n (local uniqueness of paths);

(C5) there exists $C > 0$ such that $R(G_n) \leq m(G_n) + C$ for all n (small diameter);

(C6) there exists $K > 0$ such that $\max_{u \in V_n} \mu(u) \leq K \min_{u \in V_n} \mu(u)$ for all n (bounded distortion of vertices) and $\max_{e \in E_n} \mu(e) \leq K \min_{e \in E_n} \mu(e)$ for all n (bounded distortion of edges);

(C7) there exists $K > 0$ such that $\max_i w_i^n \leq K \min_i w_i^n$ and $\max_i v_i^n \leq K \min_i v_i^n$ for all n, where w^n is a positive left eigenvector of A_n and v^n is a positive right eigenvector of A_n (bounded distortion of weights);

(C8) (G_n) is a uniform expander sequence, and (G_n^T) is a uniform expander sequence (forward/backward expansion).

Now we establish some lemmas, which will be used in the subsequent sections.

Lemma 4.2.24. *Let (G_n) be a sequence of graphs satisfying the Standing Assumptions 4.2.21. Then (C7) implies (C1) and (C6) for both (G_n) and (G_n^T).*

Proof. First note that if (C7) holds for (G_n), then it also holds for (G_n^T) since a positive left eigenvector for A_n^T is given by $(v^n)^T$ and a positive right eigenvector for

A_n^T is given by $(w^n)^T$. Therefore we only need to show that (C7) for (G_n) implies (C1) and (C6) for (G_n) (since the same argument will apply to (G_n^T)).

Let w^n and v^n be positive left and right eigenvectors for A_n, respectively, and assume that $w^n \cdot v^n = 1$. Recall with this normalization, if u is a vertex in V_n, then $\mu(u) = w_u^n v_u^n$. Then condition (C7) implies that there exists $K > 0$ such that for all n,

$$\max_u \mu(u) \leq \max_u w_u^n \max_u v_u^n \leq K^2 \min_u w_u^n \min_u v_u^n$$
$$\leq K^2 \min_u w_u^n v_u^n = K^2 \min_u \mu(u).$$

Similarly, (C7) implies that there exists $K' > 0$ such that for all n, we have that $\max_{e \in E_n} \mu(e) \leq K' \min_{e \in E_n} \mu(e)$ (recall that $\mu(e) = w_{i(e)}^n \lambda^{-1} v_{t(e)}^n$). Thus (C7) implies (C6).

Note that for e in E_n, we have that

$$\mu(e|i(e)) = \frac{w_{i(e)}^n \lambda^{-1} v_{t(e)}^n}{w_{i(e)}^n v_{i(e)}^n} = \frac{v_{t(e)}^n}{\lambda v_{i(e)}^n}.$$

Then condition (C7) implies that there exists a uniform constant $K > 0$ such that $\mu(e|i(e)) \geq K^{-1}$ for all n and all e in E_n. We also have that

$$\mu(u) = \sum_{e:i(e)=u} \mu(e) \geq \sum_{e:i(e)=u} K^{-1}\mu(u) = |\{e : i(e) = u\}| K^{-1}\mu(u).$$

Since G_n is irreducible (by Standing Assumptions 4.2.21), we know that $\mu(u) > 0$, and therefore we have that for any n, and any u in V_n,

$$|\{e \in E_n : i(e) = u\}| \leq K,$$

which implies that $\max_u d_{\text{out}}(u)$ is uniformly bounded in n. A similar argument shows that $\max_u d_{\text{in}}(u)$ is uniformly bounded in n, which shows that $d_{\max}(G_n)$ is uniformly bounded in n and gives (C1). $\square$

Recall that for a graph G, the quantities $g(G)$ and $c_w(G)$ were defined in Definitions 4.2.20 and 4.2.18, respectively.

Lemma 4.2.25. *Let G be a graph with primitive adjacency matrix A. Then it holds that $c_w(G) \geq \frac{1}{2}g$.*

Proof. This lemma is a consequence of [27, Theorems 4.3 and 5.1], as we now explain. Since A is primitive, there exists a strictly positive vector v and $\lambda \geq 1$ such that $Av = \lambda v$. Let P be the stochastic matrix defined by $P_{ij} = \frac{A_{ij}v_j}{\lambda v_i}$. Then P is the transition probability matrix corresponding to the random walk defined by the measure of maximal entropy μ on X_G. We have that $\mathrm{Sp}_\times(P) = \frac{1}{\lambda}\,\mathrm{Sp}\times(A)$. Given such a transition probability matrix, Chung defines a Laplacian L and proves [27, Theorem 4.3] that the smallest non-zero eigenvalue of L, denoted λ_1, satisfies the following inequality:

$$\min\Big\{1 - |\rho| : \rho \in \mathrm{Sp}_\times(P) \setminus \{1\}\Big\} \leq \lambda_1. \tag{4.2.3}$$

We remark that the left-hand side of the inequality in [27, Theorem 4.3] is equal to the left-hand side of Equation (4.2.3) since A is primitive (not just irreducible). Note that the left-hand side of Equation (4.2.3) equals $g(G)$, as defined in Definition 4.2.20. After defining the weighted Cheeger constant (as in Definition 4.2.18), Chung proves [27, Theorem 5.1] that

$$c_w(G) \geq \frac{1}{2}\lambda_1. \tag{4.2.4}$$

Combining the inequalities in Equations (4.2.3) and (4.2.4), we obtain the desired inequality. □

Recall that the p-th power graph was defined in Definition 4.2.7.

Lemma 4.2.26. *Let G be a graph with irreducible adjacency matrix. Let $p = \text{per}(G)$. Let $G^{p,0}$ be an irreducible component of G^p, the p-th power graph of G. Let $g = g(G^{p,0})$ (which does not depend on the choice of irreducible component in G^p). Then there exists $b > 0$, depending only on g and p, such that $c_w(G) \geq b$.*

Proof. Let G, p, and g be as in the statement of the lemma. If $p = 1$, then Lemma 4.2.25 immediately gives the result. Now we assume $p \geq 2$. The fact that G is irreducible and $\text{per}(G) = p$ implies that there is a partition of the vertices into p non-empty subsets, $V = \cup_{j=0}^{p-1} V^j$, such that for each edge e with $i(e) \in V^j$, it holds that $t(e) \in V^{j+1}$, where the superscripts are taken modulo p. Let $X = X_G$ (Definition 4.2.3), and for each $j = 0, \ldots, p-1$, let $X_j = \{x \in X : i(x_0) \in V^j\}$. For any set $S \subset V$ with $0 < |S| < |V|$ and $j = 0, \ldots, p-1$, define

$$C_S = \{x \in X : i(x_0) \in S\}, \quad \overline{C_S} = X_G \setminus C_S,$$
$$C_S^j = X_j \cap C_S, \qquad \text{and} \qquad \overline{C_S^j} = X_j \cap \overline{C_S}.$$

Recall that we denote by μ the measure of maximal entropy on X, and we may write $c_w(G)$ as follows:

$$\begin{aligned} c_w(G) &= \inf_{\emptyset \subsetneq S \subsetneq V} \frac{\mu(C_S \cap \sigma^{-1}\overline{C_S})}{\min(\mu(C_S), \mu(\overline{C_S}))} \\ &= \inf_{\emptyset \subsetneq S \subsetneq V} \max\left(\frac{\mu(C_S \cap \sigma^{-1}\overline{C_S})}{\mu(C_S)}, \frac{\mu(C_S \cap \sigma^{-1}\overline{C_S})}{\mu(\overline{C_S})} \right). \end{aligned}$$

We also use the following notation:

$$r_i = \frac{\mu(C_S^i)}{\mu(C_S)}, \quad \text{and} \quad \overline{r_i} = \frac{\mu(\overline{C_S^i})}{\mu(\overline{C_S})}. \tag{4.2.5}$$

Let us establish a useful inequality. For $i = 0, \ldots, p-1$ and $1 \leq \ell \leq p$, note that each point x in $C_S^i \cap \sigma^{-\ell}\overline{C_S^{i+\ell}}$ also lies in $C_S^j \cap \sigma^{-1}\overline{C_S^{j+1}}$ for $j = \min\{k > 0 : \sigma^k x \notin C_S\}$. Thus

$$\mu\left(C_S^i \cap \sigma^{-\ell}\overline{C_S^{i+\ell}}\right) \leq \sum_{j=0}^{p-1} \mu\left(C_S^j \cap \sigma^{-1}\overline{C_S^{j+1}}\right) = \mu\left(C_S \cap \sigma^{-1}\overline{C_S}\right). \tag{4.2.6}$$

To complete the proof, we will find $b > 0$ in terms of g and p so that for $S \subset V$ with $0 < |S| < |V|$, we have that

$$b \leq \max\left(\frac{\mu(C_S \cap \sigma^{-1}\overline{C_S})}{\mu(C_S)}, \frac{\mu(C_S \cap \sigma^{-1}\overline{C_S})}{\mu(\overline{C_S})}\right). \tag{4.2.7}$$

The bound b will be the minimum of four bounds, each coming from a particular type of set $S \subset V$.

Consider the following conditions on the set S, which we will use to break our proof into cases:

(I) there exists $i \in \{0, \ldots, p-1\}$ such that $\mu(C_S^i) \in \{0, 1\}$;

(II) $\mu(C_S^i) \leq 1/2p$ for each i, or $\mu(C_S^i) \geq 1/2p$ for each i;

(III) $1/4p \leq \mu(C_S^i) \leq 3/4p$ for each i.

Now we consider cases.

Case: (I) holds, *i.e.* there exists $i \in \{0, \ldots, p-1\}$ such that $\mu(C_S^i) \in \{0, 1\}$. Assume first that $\mu(C_S^i) = 0$, which implies that $\mu(\overline{C_S^i}) = \mu(X_i)$. Choose j such that $\mu(C_S^j) = \max_k \mu(C_S^k)$, and finally choose $1 \le \ell \le p$ such that $j + \ell = i$ (mod p). Then by inequality (4.2.6) and the shift-invariance of μ, we have that

$$\frac{\mu(C_S \cap \sigma^{-1}\overline{C_S})}{\mu(C_S)} \ge \frac{\mu(C_S^j \cap \sigma^{-\ell}\overline{C_S^{j+\ell}})}{\mu(C_S)} \ge \frac{\mu(C_S^j \cap \sigma^{-\ell}X_{j+\ell})}{p \max_k \mu(C_S^k)} = \frac{\mu(C_S^j)}{p\mu(C_S^j)} = \frac{1}{p}.$$

Now assume $\mu(C_S^i) = 1$. Choose j such that $\mu(\overline{C_S^j}) = \max_k \mu(\overline{C_S^k})$, and finally choose $1 \le \ell \le p$ such that $i + \ell = j$ (mod p). Then by (4.2.6) and the shift-invariance of μ,

$$\frac{\mu(C_S \cap \sigma^{-1}\overline{C_S})}{\mu(\overline{C_S})} \ge \frac{\mu(C_S^i \cap \sigma^{-\ell}\overline{C_S^{i+\ell}})}{\mu(\overline{C_S})} \ge \frac{\mu(X_i \cap \sigma^{-\ell}\overline{C_S^j})}{p \max_k \mu(\overline{C_S^k})} = \frac{\mu(\overline{C_S^j})}{p\mu(\overline{C_S^j})} = \frac{1}{p}.$$

Let $b_1 = 1/p$, and note that if condition (I) holds, then the inequality in (4.2.7) holds with b_1 in place of b.

Case: (I) does not hold, but (II) holds, *i.e.* $0 < \mu(C_S^i) \le 1/2p$ for all i, or $1 > \mu(C_S^i) \ge 1/2p$ for all i. Assume first that $0 < \mu(C_S^i) \le 1/2p$ for all i. Since $\sum_i r_i = 1$ and $r_i \ge 0$ for all i, there exists j such that $r_j \ge 1/p$. Then by (4.2.6) and the definition of r_i in (4.2.5),

$$\frac{\mu(C_S \cap \sigma^{-1}\overline{C_S})}{\mu(C_S)} \ge \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C_S^j})}{\mu(C_S)} = r_j \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C_S^j})}{\mu(C_S^j)} \ge \frac{1}{p} \cdot \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C_S^j})}{\mu(C_S^j)}.$$

Let $G^{p,j}$ be the irreducible component of G^p with vertex set V^j. Then $G^{p,j}$ has primitive adjacency matrix, and $g = g(G^{p,j}) > 0$. Lemma 4.2.25 gives that $c_w(G^{p,j}) \ge \frac{1}{2}g$. Since $\mu(C_S^j) \le 1/2p$ and $\mu(C_S^j) + \mu(\overline{C_S^j}) = \mu(X_j) = 1/p$, we have that $\min(\mu(C_S^j), \mu(\overline{C_S^j})) = \mu(C_S^j)$, and thus

$$\frac{\mu(C_S^j \cap \sigma^{-p}\overline{C_S^j})}{\mu(C_S^j)} \ge c_w(G^{p,j}) \ge \frac{1}{2}g.$$

Let $b_2 = g/2p$. We have shown that for S such that $\mu(C_S^i) \le 1/2p$ for each i, the inequality in (4.2.7) holds with b_2 in place of b. For S such that $1 > \mu(C_S^i) \ge 1/2p$

for each i, choose j such that $\overline{r_j} \geq 1/p$. Then an analogous argument gives that the inequality in (4.2.7) holds with b_2 in place of b.

Case: (III) holds, *i.e.* $1/4p \leq \mu(C_S^i) \leq 3/4p$ for all i. A simple calculation yields that $r_i \geq 1/3p$ and $\overline{r_i} \geq 1/3p$ for each i. Using (4.2.6), we see that for each j,

$$\frac{\mu(C_S \cap \sigma^{-1}\overline{C_S})}{\mu(C_S)} \geq \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C_S^j})}{\mu(C_S)} = r_j \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C_S^j})}{\mu(C_S^j)} \geq \frac{1}{3p} \cdot \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C_S^j})}{\mu(C_S^j)}, \tag{4.2.8}$$

and

$$\frac{\mu(C_S \cap \sigma^{-1}\overline{C_S})}{\mu(\overline{C_S})} \geq \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C_S^j})}{\mu(\overline{C_S})} = \overline{r_j} \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C_S^j})}{\mu(\overline{C_S^j})} \geq \frac{1}{3p} \cdot \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C_S^j})}{\mu(\overline{C_S^j})}. \tag{4.2.9}$$

Then since $G^{p,j}$ has primitive adjacency matrix, Lemma 4.2.25 and inequalities (4.2.8) and (4.2.9) give that the inequality in (4.2.7) holds with $b_3 := g/6p$ in place of b.

Case: each of (I), (II), and (III) does not hold, *i.e.* we assume that S is such that $0 < \mu(C_S^i) < 1$ for each i, there exists i_1 and i_2 such that $\mu(C_S^{i_1}) > 1/2p$ and $\mu(C_S^{i_2}) < 1/2p$, and there exists i_3 such that either $\mu(C_S^{i_3}) < 1/4p$ or $\mu(C_S^{i_3}) > 3/4p$. Suppose first that $\mu(C_S^{i_3}) < 1/4p$. Choose j such that $\mu(C_S^j) = \max_k \mu(C_S^k)$, and choose $1 \leq \ell \leq p$ such that $j + \ell = i_3 \pmod{p}$. Calculation gives that $\mu(C_S^{i_3}) < \frac{1}{2}\mu(C_S^j)$. Then by (4.2.6) and the shift-invariance of μ,

$$\begin{aligned}\frac{\mu(C_S \cap \sigma^{-1}\overline{C_S})}{\mu(C_S)} \geq \frac{\mu(C_S^j \cap \sigma^{-\ell}\overline{C_S^{j+\ell}})}{p\mu(C_S^j)} &\geq \frac{\mu(C_S^j) - \mu(C_S^{i_3})}{p\mu(C_S^j)} \\ &\geq \frac{\mu(C_S^j) - \frac{1}{2}\mu(C_S^j)}{p\mu(C_S^j)} = \frac{1}{2p}.\end{aligned}$$

Now assume $\mu(C_S^{i_3}) > 3/4p$. Choose j such that $\mu(C_S^j) = \max_k \mu(C_S^k)$, and choose $1 \leq \ell \leq p$ such that $j + \ell = i_2 \pmod{p}$. Calculation reveals that $\mu(C_S^{i_2}) < \frac{2}{3}\mu(C_S^j)$.

Then by (4.2.6) and the shift-invariance of μ,

$$\frac{\mu(C_S \cap \sigma^{-1}\overline{C_S})}{\mu(C_S)} \geq \frac{\mu(C_S^j \cap \sigma^{-\ell}\overline{C_S^{j+\ell}})}{p\mu(C_S^j)} \geq \frac{\mu(C_S^j) - \mu(C_S^{i_2})}{p\mu(C_S^j)}$$
$$\geq \frac{\mu(C_S^j) - \frac{2}{3}\mu(C_S^j)}{p\mu(C_S^j)} = \frac{1}{3p}.$$

Let $b_4 = 1/3p$. We have shown that for S in this case, the inequality in (4.2.7) holds with b_4 in place of b.

Now let $b = \min(b_1, b_2, b_3, b_4) = \min(1/p, g/2p, g/6p, 1/3p) = g/6p$, which depends only on g and p. We have shown that $c_w(G) \geq b$. □

Recall that the transpose graph G^T of a graph G was defined in Definition 4.2.8.

Lemma 4.2.27. *Let (G_n) be a sequence of graphs satisfying the Standing Assumptions 4.2.21 and such that both (G_n) and (G_n^T) have bounded degrees and bounded distortion of edges and vertices (conditions (C1) and (C6) in 4.2.23). Then (G_n) and (G_n^T) are both uniform expander sequences (condition (C8) in 4.2.23).*

Proof. We check that conditions (C1) and (C6) for (G_n) together imply that (G_n) is a uniform expander sequence, and then the same argument will apply to (G_n^T) since (C1) and (C6) also hold for (G_n^T).

Recall the following notation. Let $F : E_n \to [0,1]$ be given by $F(e) = \mu(e)$, where μ is the measure of maximal entropy on X_{G_n}. Also, $c_w(G_n)$ denotes the weighted Cheeger constant of G_n (Definition 4.2.18). By the Standing Assumption 4.2.21, $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$ for each n. Therefore $\mathrm{per}(G_n)$ does not depend on n, and we let $p = \mathrm{per}(G_1)$. Let $G_n^{p,0}$ be an irreducible component of the p-th power graph of G_n, and let $g_n = g(G_n^{p,0})$. Since g_n only depends on the non-zero spectrum of G_n, which is constant in n by the Standing Assumption 4.2.21, we have the g_n is constant in n. Let $g = g_1$. By Lemma 4.2.26, there exists $b_n > 0$, depending only g_n and $\mathrm{per}(G_n)$, such that $c_w(G_n) \geq b_n$. Since we have that $g_n = g$ and $\mathrm{per}(G_n) = p$ for all n, we may choose $b := b_1$, and we obtain that $c_w(G_n) \geq b > 0$ for all n.

Now we relate $c_w(G_n)$ to $c(G_n)$ (Definition 4.2.18) using properties (C1) and (C6). For notation, let $m = m(G_n)$. Since (G_n) satisfies conditions (C1) and (C6), there exists $K_1, K_2 > 0$ such that for every n and every subset $S \subset V_n$,

$$K_1|S|\lambda^{-m} \leq F(S) \leq K_2|S|\lambda^{-m},$$

and

$$K_1|E_n(S,\overline{S})|\lambda^{-m} \leq F(S,\overline{S}) \leq K_2|E_n(S,\overline{S})|\lambda^{-m}.$$

We already have that $c_w(G_n) \geq b$, which implies that for every S such that $\emptyset \subsetneq S \subsetneq V_n$,

$$b \leq \frac{F(S,\overline{S})}{\min(F(S),F(\overline{S}))} \leq \frac{K_2|E_n(S,\overline{S})|\lambda^{-m}}{\min(F(S),F(\overline{S}))}.$$

Now assume $0 < |S| \leq |V_n|/2$. If $\min(F(S),F(\overline{S})) = F(S)$, then $\min(F(S),F(\overline{S})) = F(S) \geq K_1|S|\lambda^{-m}$. If $\min(F(S),F(\overline{S})) = F(\overline{S})$, then we have $\min(F(S),F(\overline{S})) = F(\overline{S}) \geq K_1|\overline{S}|\lambda^{-m} \geq K_1|S|\lambda^{-m}$. Combining these estimates gives that for all S such that $0 < |S| \leq |V_n|/2$, we obtain that

$$|E_n(S,\overline{S})| \geq b\frac{K_1}{K_2}|S|,$$

which shows that (G_n) is a uniform $(b\frac{K_1}{K_2})$-expander sequence. □

Lemma 4.2.28. *Let (G_n) be a sequence of graphs satisfying the Standing Assumptions 4.2.21 and bounded distortion of weights (condition (C7) in 4.2.23). Then*

(1) there exists $K > 0$ such that for all n, k, and $S \subset B_k(G_n)$,

$$K^{-1}|S| \leq \lambda^{m(G_n)+k}\mu(S) \leq |S|K;$$

(2) there exists a $K > 0$ such that for all n, k, $e \in E_n$, and $S \subset B_k(G_n)$,

$$K^{-1}|S \cap C_e^{n,k}| \leq \lambda^k \mu(S|C_e^{n,k}) \leq K|S \cap C_e^{n,k}|,$$

where $C_e^{n,k} = \{b \in B_k(G_n) : b_1 = e\}$;

(3) there exists $K > 0$ such that for all n, k, and $1 \leq s < t \leq k$, it holds that $\mu(A_{s,t}) \leq K\lambda^{-m(G_n)}$, where $A_{s,t} = \{b \in B_k(G_n) : b_s = b_t\}$;

(4) there exists $K > 0$ such that for all n, $k > U(G_n)$, and $u \in V_n$, it holds that $\mu(\mathrm{Per}_k(G_n)|C_u^{n,k}) \leq K\lambda^{-U(G_n)}$, where $C_u^{n,k} = \{b \in B_k(G_n) : i(b_1) = u\}$ and $U(G_n)$ was defined in Definition 4.2.16.

Proof. For notation, let $m = m(G_n)$ and $U = U(G_n)$.

Proof of (1). We have that

$$1 = \sum_{u \in V_n} \mu(u) = \sum_{u \in V_n} w_u^n v_u^n.$$

Then condition (C7) implies that there exists $K_1 > 0$ such that for each n and u in V_n,

$$K_1^{-1}|V_n|^{-1} \leq w_u^n v_u^n \leq K_1|V_n|^{-1}.$$

By the definition of m, there exists $K_2 > 0$ such that $K_2^{-1}|V_n|^{-1} \leq \lambda^{-m} \leq K_2|V_n|^{-1}$. It follows that there exists $K_3 > 0$ such that for each n and u in V_n,

$$K_3^{-1}\lambda^{-m} \leq w_u^n v_u^n \leq K_3\lambda^{-m}.$$

Then (C7) implies that there exists $K_4 > 0$ such that for any n and any three vertices u, u_1 and u_2 in V_n,

$$K_4^{-1} w_{u_1}^n v_{u_2}^n \leq w_u^n v_u^n \leq K_4 w_{u_1}^n v_{u_2}^n.$$

Finally, we conclude that there exists $K_5 > 0$ such that for each n, k, and b in $B_k(G_n)$, we have that

$$K_5^{-1}\lambda^{-(m+k)} \leq \mu(b) = w_{i(b)}^n \lambda^{-k} v_{t(b)}^n \leq K_5\lambda^{-(m+k)}.$$

The statement in (1) follows.

Proof of (2). The statement in (2) follows from the statement in (1) and the fact that $\mu(C_e^{n,k}) = \mu(e)$.

Proof of (3). Note that from (1) we have that there exists $K > 0$ such that

$$\mu(A_{s,t}) = \sum_{\gamma \in \operatorname{Per}_{t-s}(G_n)} \mu(\gamma) \leq K\lambda^{-(m+t-s)} |\operatorname{Per}_{t-s}(G_n)|.$$

Since $\operatorname{Sp}_\times(A_n)$ does not depend on n by our Standing Assumptions 4.2.21, we have that $|\operatorname{Per}_{t-s}(G_n)|$ does not depend on n. Clearly, $|\operatorname{Per}_{t-s}(G_n)|\lambda^{-(t-s)}$ is bounded as $t-s$ tends to infinity. Therefore there exists K' such that

$$\mu(A_{s,t}) \leq K'\lambda^{-m},$$

as desired.

Proof of (4). By (2), we have that there exists $K_1 > 0$ such that for all n, $k > U$, and u in V_n,

$$\mu(\mathrm{Per}_k(G_n)|C_u^{n,k}) \leq K_1\lambda^{-k}|\,\mathrm{Per}_k(G_n) \cap C_u^{n,k}|.$$

By (2), there exists $K_2 > 0$ such that for all n, $k > U$, and u in V_n,

$$|B_{k-U}(G_n) \cap C_u^{n,k-U}| \leq K_2\lambda^{k-U}.$$

By definition of the uniqueness parameter U, each path in $B_{k-U}(G_n) \cap C_u^{n,k-U}$ can be continued in at most one way to form a path in $\mathrm{Per}_k(G_n) \cap C_u^{n,k}$. Therefore, with $K_3 = K_1K_2 > 0$, we have that for all n, $k > U$, and u in V_n,

$$\mu(\mathrm{Per}_k(G_n)|C_u^{n,k}) \leq K_1K_2\lambda^{-k}\lambda^{k-U} = K_3\lambda^{-U}.$$

□

Proposition 4.2.29. *Let G_1 be a graph with irreducible adjacency matrix A_1 having entries in $\{0,1\}$ and spectral radius $\lambda > 1$. Let $G_n = G_1^{[n]}$ for $n \geq 2$. Then the sequence (G_n) satisfies the Standing Assumptions 4.2.21 and conditions (C1)-(C8). Moreover,*

(i) $d_{max}(G_n) = d_{max}(G_1)$ for all n;

(ii) there exists $C > 0$ such that $|m(G_n) - n| \leq C$ for all n;

(iii) $z(G_n) \geq \frac{1}{2}(n-1)$ for all n;

(iv) $U(G_n) \geq n - 1$ for all n;

(v) $R(G_n) \leq n + R(G_1)$ for all n.

Proof. One may easily check from the definitions that each A_n has entries in $\{0, 1\}$, each A_n is irreducible, and $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$. We show below that $m(G_n)$ tends to infinity as n tends to infinity, which gives that (G_n) satisfies the Standing Assumptions 4.2.21.

The set of in-degrees that appear in G_n is constant in n, and so is the set of out-degrees that appear in G_n. Therefore $d_{\max}(G_n) = d_{\max}(G_1)$, which implies condition (C1).

By definition, $m(G_n) = \lceil \log_\lambda |V_n| \rceil$. Since $G_n = G_1^{[n]}$, we have that $|V_n| = |B_{n-1}(G_1)|$. By standard Perron-Frobenius theory, there exist constants K_1 and K_2 such that $K_1\lambda^n \leq |B_n(G_1)| \leq K_2\lambda^n$. It follows that there exists a constant $C > 0$ such that $|m(G_n) - n| \leq C$, and in particular, $m(G_n)$ tends to infinity.

Recall the higher-block coding map $\phi_n : X_{G_1} \to X_{G_n}$ (see Definition 4.2.1). If x is a point in X_{G_1}, then let $V_n(x)$ be the set of vertices in G_n traversed by $\phi_n(x)$. Let us show that $z(G_n) \geq (n-1)/2$. Recall Fine and Wilf's Theorem [44], which can be stated as follows. Let x be a periodic sequence with period p, and y be a periodic sequence with period q. If $x[i+1, i+n] = y[i+1, i+n]$ for $n \geq p+q-\gcd(p, q)$ and i in $\mathbb{Z}$, then $x = y$. It follows from this theorem that if x and y lie in distinct periodic orbits of X_{G_1} and have periods less than or equal to $(n-1)/2$, then $V_n(x) \cap V_n(y) = \emptyset$. Thus $z(G_n) \geq (n-1)/2$, and in particular (G_n) satisfies conditions (C2) and (C3).

Note that the map ϕ_n gives a bijection between $B_k(G_n)$ and $B_{k+n-1}(G_1)$ for all $k \geq 0$. Using this map, we check that $U(G_n) \geq n-1$ as follows. For any two paths $b, c \in B_{n-1}(G_1)$, there is at most one path of length $2n-2$ in G_1 of the form bc (since every edge in such a path is specified by either b or c). This fact implies that $U_1(G_n) \geq n-1$. Now if b is in $B_{n-1}(G_1)$ and $1 \leq s < t \leq n-1$ are given, then there is at most one path c in $B_{t+n-2}(G_1)$ such that $c[1, n-1] = b$ and $c[s, s+n-2] = c[t, t+n-2]$; indeed, if c is such a path, then $c[1, n-1]$ is determined by b, and $c[n, t+n-1]$ is determined by the periodicity condition $c[s, s+n-2] = c[t, t+n-2]$. This fact implies that $U_2(G_n) \geq n-1$, and thus we have that $U(G_n) \geq n-1$, which, in particular, gives condition (C4).

Let us check that $R(G_n) \leq n + R(G_1)$, which will imply that (G_n) satisfies condition (C5). The statement that $R(G_n) \leq n + R(G_1)$ is equivalent to the statement that for any two paths $b, c \in B_{n-1}(G_1)$, there exists a path d in G_1 of length less than or equal to $R(G_1)$ such that bdc is a path in G_1. In this formulation, the statement is clearly true, since, by the definition of $R(G_1)$, there is a path d from $t(b)$ to $i(c)$ of length less than or equal to $R(G_1)$, and then the concatenation bdc gives a path in G_1.

Let w^1 be a positive left (row) eigenvector for A_1 (corresponding to the eigenvalue λ), and let v^1 be a positive right (column) eigenvector for A_1 (corresponding to the the eigenvalue λ). Let $b \in B_{n-1}(G_1) = V_n$. Then let $w^n_b = w^1_{i(b)}$ and $v^n_b = v^1_{t(b)}\lambda^{-(n-1)}$. Then w^n is a positive left eigenvector for A_n and v^n is a positive right eigenvector for A_n. It follows that (G_n) satisfies conditions (C6) and (C7). In fact, to satisfy (C7), we may choose $K = \max(K_1, K_2)$, where $K_1 = (\max_i w^1_i)(\min_i w^1_i)^{-1}$ and $K_2 = (\max_i v^1_i)(\min_i v^1_i)^{-1}$.

Condition (C8) follows from the fact that (G_n) satisfies condition (C7) (by applying Lemmas 4.2.24 and 4.2.27 in succession). □

4.2.3 Probabilistic framework

Let Ω be the probability space consisting of the set $\{0,1\}^n$ and the probability measure $\mathbb{P}_\alpha$, where $\mathbb{P}_\alpha$ is the product of the Bernoulli measures on each coordinate with parameter $\alpha \in [0,1]$. There is a natural partial order on Ω, given by the relation $\omega \leq \tau$ if and only if $\omega_i \leq \tau_i$ for $i = 1, \ldots, n$. We say that a random variable χ on Ω is **monotone increasing** if $\chi(\omega) \leq \chi(\tau)$ whenever $\omega \leq \tau$. An event A is monotone increasing if its characteristic function is monotone increasing. Monotone decreasing is defined analogously. Monotone random variables and events have been studied extensively [47]; however, we require only a small portion of that theory. In particular, we will make use of the following proposition, a proof of which may be found in [47].

Proposition 4.2.30 (FKG Inequality)**.** *If X and Y are monotone increasing random variables on $\{0,1\}^n$, then $\mathbb{E}_\alpha(XY) \geq \mathbb{E}_\alpha(X)\mathbb{E}_\alpha(Y)$.*

It follows easily from the FKG Inequality that if $\cap F_j$ is a finite intersection of monotone decreasing events, then $\mathbb{P}_\alpha(\cap F_j) \geq \prod \mathbb{P}_\alpha(F_j)$ (use induction and note that if χ_F is the characteristic function of the monotone decreasing event F, then $-\chi_F$ is monotone increasing). In fact, we only use this corollary, but we nonetheless refer to it as the FKG Inequality.

For a finite, directed graph G, we consider the discrete probability space on the set $\Omega_G = \{0,1\}^E$, where $\mathbb{P}_\alpha$ is the product of the Bernoulli(α) measures on each coordinate. The set Ω_G corresponds to the power set of E in the usual way: ω in Ω_G corresponds to the set F in 2^E such that e is in F if and only if $\omega(e) = 1$.

Furthermore, Ω_G corresponds to the space of subgraphs of G: for ω in Ω_G, define the subgraph $G(\omega)$ to have vertex set V and edge set F_ω, where an edge e in E is included in $F_\omega \subset E$ if and only if $\omega(e) = 1$. In the percolation literature, the edges e such that $\omega(e) = 1$ are often called "open," and the remaining edges are called "closed." Since we are interested in studying edge shifts defined by graphs, we will refer to an edge e as "allowed" when $\omega(e) = 1$ and "forbidden" when $\omega(e) = 0$. Finally, each ω in Ω_G can be associated to the SFT X_ω defined as the set of all bi-infinite, directed walks on G that traverse only *allowed* edges (with respect to ω). The probability measure $\mathbb{P}_\alpha$ corresponds to allowing each edge of G with probability α, independently of all other edges. For the sake of notation, we suppress the dependence of $\mathbb{P}_\alpha$ on the graph G.

Definition 4.2.31. In this chapter, we consider the following conjugacy invariants of SFTs. Let $\mathcal{E}$ be the property containing only the empty shift. Let $\mathcal{Z}$ be the property containing all SFTs with zero entropy. By convention, we let $\mathcal{E} \subset \mathcal{Z}$. For any SFT X, let $\mathbf{h}(X)$ be the topological entropy, and let $I(X)$ be the number of irreducible components of X. If X is non-empty, let $\beta(X)$ be defined by the equation $\mathbf{h}(X) = \log(\beta(X))$. If X is empty, let $\beta(X) = 0$. If $\mathcal{S}$ is a property of SFTs and G is a finite directed graph, then let $\mathcal{S}_G \subset \Omega_G$ be the set of ω in Ω_G such that X_ω has property $\mathcal{S}$. If f is a function from SFTs to the real numbers and G is a finite directed graph, then let $f_G : \Omega_G \to \mathbb{R}$ be the function $f_G(\omega) = f(X_\omega)$.

4.3 Emptiness

Recall that $\mathrm{Sp}_\times(G)$, ζ_G, and $z(G)$ were defined in Definitions 4.2.9, 4.2.11, and 4.2.14, respectively.

Theorem 4.3.1. *Let (G_n) be a sequence of graphs such that $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$ for all n and either (i) $\lambda = \lambda_{G_1} = 1$ or (ii) $\lambda = \lambda_{G_1} > 1$ and $z(G_n)$ tends to infinity as n tends to infinity. Let $\zeta = \zeta_{G_1}$. Then*

$$\lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{E}_{G_n}) = \begin{cases} (\zeta(\alpha))^{-1}, & \text{if } \alpha \in [0, 1/\lambda) \\ 0, & \text{if } \alpha \in [1/\lambda, 1]. \end{cases}$$

Remark 4.3.2. Theorem 4.1.1 can be obtained as a corollary of Theorem 4.3.1 by taking (G_n) to be the sequence of n-block graphs of X. Indeed, if the SFT X in Theorem 4.1.1 has zero entropy, then $\lambda = 1$, and the conclusion of Theorem 4.1.1 follows from case (i) in Theorem 4.3.1. If the SFT X in Theorem 4.1.1 has positive entropy, then $\lambda > 1$ and $z(G_n)$ tends to infinity by the exact same argument in the proof of Proposition 4.2.29 (iii), and therefore the conclusion of Theorem 4.1.1 follows from case (ii) in Theorem 4.3.1.

In this section we provide a proof of Theorem 4.3.1. Before proceeding with the proof, we state a fact that will be useful in the investigations that follow. Recall that for a path b, we denote by $V(b)$ the set of vertices traversed by b.

Lemma 4.3.3. *Suppose G is a directed graph. Suppose b is in $\mathrm{Per}(G)$ such that $|V(b)| < per(b)$. Then there exists a path c in $\mathrm{Per}(G)$ such that $per(c) < per(b)$ and $V(c) \subset V(b)$.*

Proof. Let v be in $V(b)$. Then there exists a return path to v following b, and we may choose a shortest return path c to v using only vertices in $V(b)$. Then c is in $\operatorname{Per}(G)$ and $\operatorname{per}(c) < \operatorname{per}(b)$, as desired. □

Proof of Theorem 4.3.1. Recall that an SFT is non-empty if and only if it contains a periodic point (see [66]).

First, assume that case (i) holds, which means that $\lambda = 1$. In this case, each X_{G_n} contains finitely many orbits. Further, the number of periodic orbits of each period in X_{G_n} is constant, and the probability of each periodic orbit being allowed in X_ω is constant. Therefore the conclusion follows immediately, since the sequence $\mathbb{P}_\alpha(\mathcal{E}_{G_n})$ is constant.

Now assume that case (ii) holds. For the moment, consider a fixed natural number n. Let $\{\gamma_j\}_{j \in \mathbb{N}}$ be an enumeration of the periodic orbits of X_{G_n} such that if $i \leq j$ then $\operatorname{per}(\gamma_i) \leq \operatorname{per}(\gamma_j)$. Let $p_i = \operatorname{per}(\gamma_i) = |\gamma_i|$. Let $V_n(\gamma_j)$ be the vertices in G_n traversed in the orbit γ_j and let $E_n(\gamma_j)$ be the edges in G_n traversed in the orbit γ_j.

Now for each j, let A_j be the event that γ_j is allowed, which is the event that all of the edges in $E_n(\gamma_j)$ are allowed. Let F_j be the event that γ_j is forbidden, which is A_j^c, the complement of A_j. Notice that A_j is a monotone increasing event (if ω is in A_j and $\omega \leq \omega'$, then ω' is in A_j), and F_j is a monotone decreasing event. The fact that an SFT is non-empty if and only if it contains a periodic point implies that $\mathcal{E}_{G_n} = \cap F_j$.

Combining the definition of $z(G_n)$ and Lemma 4.3.3, we obtain that if $\operatorname{per}(\gamma_i) \leq$

$z(G_n)$, then $|E_n(\gamma_i)| = p_i$. It follows that $\mathbb{P}_\alpha(F_i) = 1 - \alpha^{p_i}$ for each i such that $p_i \le z(G_n)$. Furthermore, the definition of $z(G_n)$ implies that the events F_i such that $p_i \le z(G_n)$ are all jointly independent. These observations give that

$$\mathbb{P}_\alpha(\mathcal{E}_{G_n}) = \mathbb{P}_\alpha\left(\bigcap_{j\in\mathbb{N}} F_j\right) \le \mathbb{P}_\alpha\left(\bigcap_{p_i \le z(G_n)} F_i\right) \tag{4.3.1}$$

$$= \prod_{p_i \le z(G_n)} \mathbb{P}_\alpha(F_i) = \prod_{p_i \le z(G_n)} (1 - \alpha^{p_i}). \tag{4.3.2}$$

Using Lemma 4.3.3, we see that there is great redundancy in the intersection $\cap F_j$. Eliminating some of this redundancy, we obtain the following:

$$\bigcap_{j\in\mathbb{N}} F_j = \bigcap_{j:|E_n(\gamma_j)|=p_j} F_j. \tag{4.3.3}$$

Then using Lemma 4.3.3 again and the fact that $|E_n(\gamma_j)| \le |E_n|$, we see that the intersection on the right in Equation (4.3.3) is actually a finite intersection. Applying the FKG Inequality, we obtain that

$$\mathbb{P}_\alpha(\mathcal{E}_{G_n}) = \mathbb{P}_\alpha\left(\bigcap_{j\in\mathbb{N}} F_j\right) = \mathbb{P}_\alpha\left(\bigcap_{j:|E_n(\gamma_j)|=p_j} F_i\right) \ge \prod_{j:|E_n(\gamma_j)|=p_j} \mathbb{P}_\alpha(F_i) \tag{4.3.4}$$

$$= \prod_{j:|E_n(\gamma_j)|=p_j} (1 - \alpha^{p_j}) \ge \prod_{j:p_j \le |E_n|} (1 - \alpha^{p_j}). \tag{4.3.5}$$

Combining the inequalities in (4.3.1), (4.3.2), (4.3.4) and (4.3.5) gives that for each n,

$$\prod_{p_j \le |E_n|} (1 - \alpha^{p_j}) \le \mathbb{P}_\alpha(\mathcal{E}_{G_n}) \le \prod_{p_i \le z(G_n)} (1 - \alpha^{p_i}). \tag{4.3.6}$$

By the Standing Assumption that $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$, we have that $|\operatorname{Per}_p(G_n)|$ is independent of n. Since $z(G_n)$ and $|E_n|$ tend to infinity as n tends to infinity, Equation (4.3.6) gives that

$$\lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{E}_{G_n}) = \prod_{j=1}^{\infty} (1 - \alpha^{p_j}).$$

Then Theorem 4.3.1 follows from the well-known product formula for ζ (see [66]), which may be stated as

$$(\zeta(t))^{-1} = \prod_{j=1}^{\infty}(1 - t^{p_j}),$$

along with the fact that $\zeta(t)$ converges for $t < 1/\lambda$ and diverges to $+\infty$ for $t \geq 1/\lambda$.

□

4.4 Subcritical phase

In this section we study random SFTs in the subcritical phase: $0 \leq \alpha < 1/\lambda$. The main result of this section is Theorem 4.4.2. Let us fix some notation for this section. We consider a sequence of graphs (G_n) such that $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$ and $z(G_n)$ tends to infinity as n tends to infinity, with $\lambda = \lambda_{G_1} \geq 1$ and $\zeta = \zeta_{G_1}$. Since $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$, there exist shift-commuting bijections $\phi_n : \mathrm{Per}(X_{G_1}) \to \mathrm{Per}(X_{G_n})$. In other words, there exist bijections ϕ_n from the set of cyclic paths in G_1 to the set of cyclic paths in G_n such that if b is in $\mathrm{Per}_p(G_1)$, then $\phi_n(b)$ is $\mathrm{Per}_p(G_n)$. If b is in $\mathrm{Per}(G)$, then we refer to $\theta(b)$ (recall Definition 4.2.12) as a cycle. Using the fixed bijections ϕ_n, we may refer to a cycle γ as being in G_n for any n. We fix an enumeration of the cycles in G_1, $\{\gamma_i\}_{i \in \mathbb{N}}$, and then since the bijections ϕ_n are fixed, this choice simultaneously gives enumerations of all the cycles in each G_n. For any s in $\mathbb{N}$, let $p_s = \mathrm{per}(\gamma_s)$. Let us begin with a lemma.

Lemma 4.4.1. *Let (G_n) be a sequence of graphs such that $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$ and $z(G_n)$ tends to infinity as n tends to infinity, with $\lambda = \lambda_{G_1} \geq 1$ and $\zeta = \zeta_{G_1}$. Given a non-empty, finite set S in $\mathbb{N}$, let $D_{G_n}(S)$ be the event that the set of allowed cycles* *is* $\{\gamma_s : s \in S\}$. *Then*

$$\lim_{n\to\infty} \mathbb{P}_\alpha(D_{G_n}(S)) = \begin{cases} (\zeta(\alpha))^{-1} \prod_{j\in S} \frac{\alpha^{p_j}}{1-\alpha^{p_j}}, & \text{if } \alpha \in [0, 1/\lambda) \\ 0, & \text{if } \alpha \in [1/\lambda, 1], \end{cases}$$

The proof of Lemma 4.4.1 is an easy adaptation of the proof of Theorem 4.3.1, and we omit it for the sake of brevity.

Recall that $I(X)$ denotes the number of irreducible components in the SFT X, and for any graph G, the random variable $I_G : \Omega_G \to \mathbb{Z}_{\geq 0}$ is defined by the equation $I_G(\omega) = I(X_\omega)$.

Theorem 4.4.2. *Let* (G_n) *be a sequence of graphs such that* $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$ *and either (i)* $\lambda = \lambda_{G_1} = 1$ *or (ii)* $\lambda = \lambda_{G_1} > 1$ *and* $z(G_n)$ *tends to infinity as* n *tends to infinity. Let* $\zeta = \zeta_{G_1}$. *Then for* $0 \leq \alpha < 1/\lambda$,

(1) $\lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) = 1$;

(2) the sequence (I_{G_n}) *converges in distribution to the random variable* I_∞ *such that* $\mathbb{P}(I_\infty = 0) = (\zeta(\alpha))^{-1}$ *and for* $k \geq 1$,

$$\mathbb{P}(I_\infty = k) = (\zeta(\alpha))^{-1} \sum_{\substack{S\subset\mathbb{N} \\ |S|=k}} \prod_{s\in S} \frac{\alpha^{p_s}}{1-\alpha^{p_j}},$$

where $\{\gamma_i\}_{i=1}^\infty$ *is an enumeration of the cycles in* G_1;

(3) the random variable I_∞ *has exponentially decreasing tail and therefore finite moments of all orders.*

Remark 4.4.3. One obtains Theorem 4.1.2 as a consequence of Theorem 4.4.2 by taking (G_n) to be the sequence of n-block graphs of a non-empty SFT X. Indeed, if the SFT X in Theorem 4.1.2 has zero entropy, then $\lambda = 1$, and the conclusions of Theorem 4.1.2 follow from the case (i) in Theorem 4.4.2. If the SFT X in Theorem 4.1.2 has positive entropy, then $\lambda > 1$ and $z(G_n)$ tends to infinity by the exact same argument in the proof of Proposition 4.2.29 (iii), and therefore the conclusions of Theorem 4.1.2 follow from case (ii) in Theorem 4.4.2.

Proof of Theorem 4.4.2. Let (G_n) be as above. Let $0 \leq \alpha < 1/\lambda$.

First, assume that case (i) holds, which means that $\lambda = 1$. Conclusion (1) follows immediately, since for each n, we have that $\mathbb{P}_\alpha(\mathcal{Z}_{G_n}) = 1$ (the random SFT X_ω satisfies $0 = \mathbf{h}(X_\omega) \leq \mathbf{h}(X_{G_n}) = \log \lambda = 0$). Also, the fact that $\lambda = 1$ is equivalent to the fact that G_1 (and therefore G_n) contains only finitely many cycles. Then conclusions (2) and (3) also follow immediately, since the sequence I_{G_n} is constant.

Now assume that case (ii) holds. Recall that we have an enumeration $\{\gamma_i\}_{i \in \mathbb{N}}$ of the cycles in G_1, which we refer to as an enumeration of the cycles in G_n, for any n, using the bijections ϕ_n. Also recall that for any non-empty, finite set $S \subset \mathbb{N}$, we denote by $D_{G_n}(S)$ the event in Ω_{G_n} consisting of all ω such that the set of cycles in $G_n(\omega)$ is exactly $\{\gamma_s : s \in S\}$.

Proof of Theorem 4.4.2 (1). Recall that an SFT has zero entropy if and only if it has at most finitely many periodic points [66]. Then we have that

$$\mathcal{Z}_{G_n} = \mathcal{E}_{G_n} \cup \left(\bigcup_{\substack{S \subset \mathbb{N} \\ 0 < |S| < \infty}} D_{G_n}(S) \right). \tag{4.4.1}$$

Also note that by the definition of $D_{G_n}(S)$, the union in (4.4.1) is a disjoint union.

Thus we have that

$$\mathbb{P}_\alpha(\mathcal{Z}_{G_n}) = \mathbb{P}_\alpha(\mathcal{E}_{G_n}) + \sum_{\substack{S\subset\mathbb{N}\\ 0<|S|<\infty}} \mathbb{P}_\alpha(D_{G_n}(S)).$$

Now let $S_1, \dots S_J$ be distinct, non-empty, finite subsets of $\mathbb{N}$. Then by Theorem 4.3.1 and Lemma 4.4.1 we have that

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) &\geq \lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{E}_{G_n}) + \sum_{j=1}^{J} \lim_{n\to\infty} \mathbb{P}_\alpha(D_{G_n}(S_j)) \\ &= (\zeta(\alpha))^{-1}\Big(1 + \sum_{j=1}^{J}\prod_{s\in S_j} \frac{\alpha^{p_s}}{1-\alpha^{p_s}}\Big).\end{aligned}$$

Since J and $S_1, \dots, S_J$ were arbitrary, we conclude that

$$\liminf_{n\to\infty} \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) \geq (\zeta(\alpha))^{-1}\left(1 + \sum_{\substack{S\subset\mathbb{N}\\ 0<|S|<\infty}} \prod_{s\in S} \frac{\alpha^{p_s}}{1-\alpha^{p_s}}\right).$$

Using the facts that $\alpha^{p_s}/(1-\alpha^{p_s}) = \sum_{k=1}^{\infty}(\alpha^{p_s})^k$ and $\alpha < 1/\lambda$ (which implies that the relevant infinite products and series converge uniformly), one may easily check that

$$\left(1 + \sum_{\substack{S\subset\mathbb{N}\\ 0<|S|<\infty}} \prod_{s\in S} \frac{\alpha^{p_s}}{1-\alpha^{p_s}}\right) = \zeta(\alpha).$$

Thus we have shown that $\liminf_n \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) \geq 1$. Since $\limsup_n \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) \leq 1$, we conclude that $\lim_n \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) = 1$.

Proof of Theorem 4.4.2 (2). Since I_{G_n} takes values in $\mathbb{Z}_{\geq 0}$, the sequence (I_{G_n}) converges in distribution to I_∞ if and only if $\mathbb{P}_\alpha(I_{G_n} = k)$ converges to $\mathbb{P}_\alpha(I_\infty = k)$ for each k in $\mathbb{Z}_{\geq 0}$.

Note that $I_{G_n}(\omega) = 0$ if and only if ω is in $\mathcal{E}_{G_n}$, which implies that $\mathbb{P}_\alpha(I_{G_n} = 0) = \mathbb{P}_\alpha(\mathcal{E}_{G_n})$. Thus for $\alpha < 1/\lambda$, Theorem 4.3.1 implies that $\mathbb{P}_\alpha(I_{G_n} = 0)$ converges to $(\zeta(\alpha))^{-1}$ as n tends to infinity.

Now let k be in $\mathbb{N}$. Recall that $\{\gamma_i\}_{i=1}^{\infty}$ is an enumeration of the cycles in G_1, and we have fixed bijections between these cycles and the cycles in each G_n. By Theorem 4.4.2 (1), we have that $\lim_n \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) = 1$, and therefore $\mathbb{P}_\alpha(I_{G_n} = k) = \mathbb{P}_\alpha(\{I_{G_n} = k\} \cap \mathcal{Z}_{G_n}) + \epsilon_n$, where ϵ_n tends to 0 as n tends to infinity. Thus we need only focus on events of the form $\{I_{G_n} = k\} \cap \mathcal{Z}_{G_n}$ for some k. Now if ω is in $\mathcal{Z}_{G_n}$, then $I_{G_n}(\omega)$ is the number of periodic orbits in X_ω. Thus

$$\mathbb{P}_\alpha(\{I_{G_n} = k\} \cap \mathcal{Z}_{G_n}) = \sum_{\substack{S \subset \mathbb{N} \\ |S|=k}} \mathbb{P}_\alpha(D_{G_n}(S)).$$

For any n in $\mathbb{N}$, let $T_n^0 = \mathbb{P}_\alpha(\mathcal{E}_{G_n})$. For k in $\mathbb{N}$ and n in $\mathbb{N}$, let

$$T_n^k = \sum_{\substack{S \subset \mathbb{N} \\ |S|=k}} \mathbb{P}_\alpha(D_{G_n}(S)).$$

We have that $\sum_{k=0}^{\infty} T_n^k = \mathbb{P}_\alpha(\mathcal{Z}_{G_n})$, and therefore $\lim_n \sum_{k=0}^{\infty} T_n^k = 1$ by Theorem 4.4.2 (1). Also, using Lemma 4.4.1, we have that $\liminf_n T_n^k \geq T^k$, where $T^0 = (\zeta(\alpha))^{-1}$ and for k in $\mathbb{N}$,

$$T^k = (\zeta(\alpha))^{-1} \sum_{\substack{S \subset \mathbb{N} \\ |S|=k}} \prod_{s \in S} \frac{\alpha^{p_s}}{1 - \alpha^{p_s}}.$$

Further, we have that $\sum_{k=0}^{\infty} T^k = 1$. It follows from these facts that $\lim_n T_n^k = T^k$. Thus we have shown that for k in $\mathbb{N}$,

$$\lim_n \mathbb{P}_\alpha\big(I_{G_n} = k\big) = \lim_n \mathbb{P}_\alpha\Big(\{I_{G_n} = k\} \cap \mathcal{Z}_{G_n}\Big) = (\zeta(\alpha))^{-1} \sum_{\substack{S \subset \mathbb{N} \\ |S|=k}} \prod_{s \in S} \frac{\alpha^{p_s}}{1 - \alpha^{p_s}},$$

as desired.

Proof of Theorem 4.4.2 (3). For k in $\mathbb{N}$, let

$$T^k = \mathbb{P}_\alpha(I_\infty = k) = (\zeta(\alpha))^{-1} \sum_{\substack{S \subset \mathbb{N} \\ |S|=k}} \prod_{s \in S} \frac{\alpha^{p_s}}{1 - \alpha^{p_s}}.$$

We show that there for any real number $\delta > 0$, there exists k_0 such that $T^{k+1} \leq \delta T^k$ for all $k \geq k_0$. Let $\delta > 0$. Since $\alpha < 1/\lambda$, we have that

$$\sum_{i \in \mathbb{N}} \frac{\alpha^{p_i}}{1 - \alpha^{p_i}} < \infty.$$

Now choose k_0 such that

$$\sum_{i \geq k_0} \frac{\alpha^{p_i}}{1 - \alpha^{p_i}} < \delta.$$

In the following sums, we will use that any set $S \subset \mathbb{N}$ with $|S| = j$ can be written as $S = \{s_1, \ldots, s_j\}$, where $s_1 < \cdots < s_j$. Note that in this case $s_j \geq j$. Then for $k \geq k_0$ we have

$$\begin{aligned}
(\zeta(\alpha))T^{k+1} &= \sum_{\substack{S \subset \mathbb{N} \\ |S| = k+1}} \prod_{i=1}^{k+1} \frac{\alpha^{p_{s_i}}}{1 - \alpha^{p_{s_i}}} = \sum_{\substack{S \subset \mathbb{N} \\ |S| = k}} \prod_{i=1}^{k} \frac{\alpha^{p_{s_i}}}{1 - \alpha^{p_{s_i}}} \sum_{j > s_k} \frac{\alpha^{p_j}}{1 - \alpha^{p_j}} \\
&\leq \sum_{\substack{S \subset \mathbb{N} \\ |S| = k}} \prod_{i=1}^{k} \frac{\alpha^{p_{s_i}}}{1 - \alpha^{p_{s_i}}} \sum_{j > k_0} \frac{\alpha^{p_j}}{1 - \alpha^{p_j}} \leq \left(\sum_{\substack{S \subset \mathbb{N} \\ |S| = k}} \prod_{i=1}^{k} \frac{\alpha^{p_{s_i}}}{1 - \alpha^{p_{s_i}}} \right) \delta \\
&= (\zeta(\alpha))T^k \delta.
\end{aligned}$$

Since $\alpha < 1/\lambda$, we have that $0 < \zeta(\alpha) < \infty$, and we conclude that $T^{k+1} \leq \delta T^k$ for all $k \geq k_0$. □

We recognize the distribution of I_∞ as the sum of countably many independent Bernoulli trials, where the probability of success of trial $i \in \mathbb{N}$ is given by α^{p_i} for some enumeration $\{\gamma_i\}_{i \in \mathbb{N}}$ of the cycles in G_1 (or any G_n). We record some facts about this distribution in the following corollary.

Corollary 4.4.4. *With the same hypotheses as in Theorem 4.4.2, the characteristic* *function of I_∞ is given by*

$$\varphi_{I_\infty}(t) = (\zeta(\alpha))^{-1} \prod_s \Big(1 + e^{it} \frac{\alpha^{p_s}}{1-\alpha^{p_s}}\Big),$$

where the product is over all periodic orbits in X. It follows that the moment generating function of I_∞ is given by

$$M_{I_\infty}(t) = (\zeta(\alpha))^{-1} \prod_s \Big(1 + e^{t} \frac{\alpha^{p_s}}{1-\alpha^{p_s}}\Big).$$

Remark 4.4.5. In Theorems 4.3.1 and 4.4.2, we assert the existence of various limits to certain values. Beyond the bounds given in our proofs, we do not know at which rates these sequences converge to their limits.

4.5 Supercritical Phase

In this section we study random SFTs in the supercritical phase. The main results are Theorem 4.5.13 and Theorem 4.5.15. On a first reading, the reader may prefer to skip Section 4.5.1 and refer back to it as necessary. Our proof of Theorem 4.5.13 relies, in part, on showing that with large probability the number of allowed words of length k in a random SFT is close to $(\alpha\lambda)^k$, for a particular choice of k. In our proof, we choose k to be polynomial in $m = m(G_n)$ for two reasons. Firstly, we need k to dominate m, so that the k-th root of the number of words of length k gives a good upper bound on the Perron eigenvalue of the random SFT. Secondly, k should be subexponential in m, essentially because most paths in G_n with length subexponential in m are self-avoiding, and we need good bounds on the probability of paths of length k that exhibit "too-soon-recurrence." For context, we recall a result of Ornstein and Weiss [75]. In fact, their result is quite general, but we only recall it in a very specific case. Let X be an irreducible SFT with measure of maximal entropy μ. For x in X, let $R_n(x)$ be the first return time (greater than 0) of x to the cylinder set $x[1,n]$ under σ. Then the result of Ornstein and Weiss implies that for μ-a.e. x in X, $\lim_n n^{-1} \log R_n(x) = h(X)$. It follows from this result that for k polynomial in n, the μ-measure of the set of words of length k with a repeated n-word tends to 0. In the following lemmas, we give some quantitative bounds on the μ-measure of the set of paths of length k in G_n with $k-j$ repeated edges, where the important point for our purposes is that the bounds improve exponentially as j decreases. To get these bounds we employ some of the language and tools of information theory. After getting a handle on the μ-measure of paths in G_n with certain self-intersection properties, our assumption that (G_n) satisfies condition (C7) in 4.2.23 implies that μ-measure on paths is the same as the counting measure up to uniform constants.

4.5.1 Information theory and lemmas

In keeping with the convention of information theory, $\log(x)$ denotes the base 2 logarithm of x.

Definition 4.5.1. A binary n-**code** on an alphabet $\mathcal{A}$ is a mapping $C : \mathcal{A}^n \to \{0,1\}^*$, where $\{0,1\}^*$ is the set of all finite words on the alphabet $\{0,1\}$. We may refer to such mappings simply as codes. A code is **faithful** if it is injective. The function that assigns to each w in $\mathcal{A}^n$ the length of the word $C(w)$ is called the **length function** of the code, and it will be denoted by $\mathcal{L}$ when the code is understood. A code is a **prefix code** if $w = w'$ whenever $C(w)$ is a prefix of $C(w')$. A **Shannon code** with respect to a measure ν on $\mathcal{A}^n$ is a code such that $\mathcal{L}(w) = \lceil - \log \nu(w) \rceil$.

We note that for a measure ν on $\mathcal{A}^n$, there is a prefix Shannon code on $\mathcal{A}^n$ with respect to ν [85]. We will also require the following two lemmas from information theory.

Lemma 4.5.2 ([85]). *Let $\mathcal{A}$ be an alphabet. Let C_n be a prefix-code on $\mathcal{A}^n$, and let μ be a shift-invariant Borel probability measure on $\mathcal{A}^{\mathbb{Z}}$. Then*

$$\mu\Big(\{w \in \mathcal{A}^n : \mathcal{L}(w) + \log \mu(w) \leq -a\}\Big) \leq 2^{-a}.$$

Proof. Let $B = \{w \in \mathcal{A}^n : \mathcal{L}(w) + \log \mu(w) \leq -a\}$. Then for any w in B, we have that $\mu(w) \leq 2^{-\mathcal{L}(w)} 2^{-a}$. The Kraft inequality for prefix codes [85, p. 73] states that since $\mathcal{L}$ is a prefix code, $\sum_{w \in \mathcal{A}^n} 2^{-\mathcal{L}(w)} \leq 1$. Hence

$$\mu(B) = \sum_{w \in B} \mu(w) \leq 2^{-a} \sum_{w \in B} 2^{-\mathcal{L}(w)} \leq 2^{-a}.$$

□

Lemma 4.5.3 ([85]). *There is a prefix code $C : \mathbb{N} \to \{0,1\}^*$ such that $\ell(C(n)) = \log(n) + o(\log(n))$, where $\ell(C(n))$ is the length of $C(n)$.*

Definition 4.5.4. A prefix code satisfying the conclusion of Lemma 4.5.3 is called an **Elias code**.

Recall that if b is a path in the graph $G = (V, E)$, then we denote by $E(b)$ the set of edges traversed by b. Let (G_n) be a sequence of graphs satisfying our Standing Assumptions 4.2.21.

Definition 4.5.5. For each n, k, and $1 \leq j \leq k-1$, let

$$N_{n,k}^{j} = \{b \in B_k(G_n) : |E_n(b)| \leq j\}.$$

Definition 4.5.6. For each n, k, and $1 \leq j \leq 2k-1$, let

$$D_{n,k}^{j} = \{(b,c) \in B_k(G_n) \times B_k(G_n) : E_n(b) \cap E_n(c) \neq \emptyset, |E_n(b) \cup E_n(c)| \leq j\}.$$

Definition 4.5.7. For each n, k, and $1 \leq j \leq k-1$, let

$$Q_{n,k}^{j} = \{b \in \mathrm{Per}_k(G_n) : |E_n(b)| \leq j\}.$$

Definition 4.5.8. For each n, k, and $1 \leq j \leq 2k-1$, let

$$S_{n,k}^{j} = \{(b,c) \in \mathrm{Per}_k(G_n) \times \mathrm{Per}_k(G_n) : E_n(b) \cap E_n(c) \neq \emptyset, |E_n(b) \cup E_n(c)| \leq j\}.$$

For any of the sets defined in Definitions 4.5.5 - 4.5.8, we use a "hat" to denote the set with "$\leq$" replaced by "$=$" in the definition. For example,

$$\hat{N}_{n,k}^{j} = \{b \in B_k(G_n) : |E_n(b)| = j\}.$$

The "hat" notation will only appear in the proof of Theorem 4.5.13. The following four lemmas find bounds on $|N_{n,k}^{j}|$, $|D_{n,k}^{j}|$, $|S_{n,k}^{2k-1}|$, and $|S_{n,k}^{j}|$.

The following lemma bounds the μ-measure (and therefore the cardinality) of the set of paths of length k in G_n that traverse at most $j < k$ edges. The proof relies on a general principle in information theory (made precise by Lemma 4.5.2):

a set of words that can be encoded "too efficiently" must have small measure. In order to use this principle, we find an efficient encoding of the paths of length k in G_n that traverse at most j edges. The basic observation behind the coding is trivial: a path of length k that only traverses $j < k$ edges must have $k-j$ repeated edges. Therefore, instead of encoding each of the $k-j$ repeated edges explicitly, we simply encode some combinatorial data that specifies when "repeats" happen and when the corresponding edges are first traversed.

Lemma 4.5.9. *Let (G_n) be a sequence of graphs satisfying the Standing Assumptions 4.2.21 and such that (G_n) has local uniqueness of paths and bounded distortion of weights (conditions (C4) and (C7) in 4.2.23). Then there exists a polynomial $p_0(x)$ and n_0 such that for each $n \geq n_0$, $k > U(G_n)$ and $1 \leq j \leq k-1$,*

$$\mu(N^j_{n,k}) \leq p_0(k)^{\min(k-j,\, k/U(G_n))} \lambda^{-(m(G_n)+k-j)},$$

and

$$|N^j_{n,k}| \leq p_0(k)^{\min(k-j,\, k/U(G_n))} \lambda^j.$$

Proof. Consider (G_n), n, k, and j as in the hypotheses. Let $m = m(G_n)$ and $U = U(G_n)$. A path b in $N^j_{n,k}$ from vertex s to vertex t contributes $w^n_s v^n_t \lambda^{-k}$ to $\mu(N^j_{n,k})$. The condition (C7) gives a uniform constant K such that $w^n_s v^n_t$ is bounded below by $(K^2|V_n|)^{-1} = (K^2\lambda^m)^{-1}$. Therefore the bound on $|N^j_{n,k}|$ follows from the bound on $\mu(N^j_{n,k})$, since $|N^j_{n,k}| \leq K^2\lambda^{m+k}\mu(N^j_{n,k})$ (as in Lemma 4.2.28 (1)). We now proceed to show the bound on $\mu(N^j_{n,k})$.

Let $r = k - j$. Consider b in $N^j_{n,k}$. Then there exists $1 < t_1 < \cdots < t_r \leq k$ such that $b_{t_i} = b_{s_i}$ for some $1 \leq s_i < t_i$, for each $i = 1, \ldots, r$, where $s_i = \min\{s \geq$

$1 : b_s = b_{t_i}\}$. Now we define a set $\mathcal{I} \subset \{1, \ldots, r\}$ by induction. Let $i_1 = 1$ and $\mathcal{I}_1 = \{i_1\}$. Assuming by induction that i_j and $\mathcal{I}_j$ have been defined and that $i_j < r$, we define i_{j+1} and $\mathcal{I}_{j+1}$ as follows:

- if $t_{i_j+1} - t_{i_j} > U$, let $i_{j+1} = i_j + 1$;
- otherwise, if $t_{i_j+1} - t_{i_j} \leq U$, then let

$$i_{j+1} = \max\{i_j < i \leq r : t_i - t_{i_j} \leq U\}.$$

Let $\mathcal{I}_{j+1} = \mathcal{I}_j \cup \{i_{j+1}\}$. This induction procedure terminates when $i_j = r$ for some $j \leq r$, and we denote this terminal j by j_*. Let $\mathcal{I} = \mathcal{I}_{j_*}$. Note that for each $0 \leq s \leq k - U$, we have that

$$|\{i \in \mathcal{I} : s + 1 \leq t_i \leq s + U\}| \leq 2.$$

It follows that $|\mathcal{I}| \leq \min(r, 2k/U + 2)$.

Having defined the set $\mathcal{I}$, we now decompose the integer interval $\{1, \ldots, k\}$ into subintervals. First, let

$$J = \cup_{j=1}^{j_*}\{t_{i_j}\} \cup \{1 \leq s \leq k : \exists i_j, i_{j+1} \in \mathcal{I}, t_{i_{j+1}} - t_{i_j} \leq U \text{ and } t_{i_j} \leq s \leq t_{i_{j+1}}\}.$$

Let $J_1, \ldots, J_N$ be the maximal disjoint subintervals (with singletons allowed) of $\{1, \ldots, k\}$ such that $J = J_1 \cup \cdots \cup J_N$ and $J_\ell < J_{\ell+1}$. Note that $\sum_{\ell=1}^{N} |J_\ell| = |J| \geq r$ and $N \leq |\mathcal{I}|$. Then let $I_1, \ldots, I_{N+1}$ be the maximal disjoint subintervals of $\{1, \ldots, k\}$ such that

- $I_\ell \subset \{1, \ldots, k\} \setminus J$ for each $\ell = 1, \ldots, N + 1$;

- $\cup_{\ell=1}^{N+1} I_\ell = \{1, \ldots, k\} \setminus J$;

- and for each $\ell = 1, \ldots, N$, we have that I_ℓ is non-empty and $I_\ell < I_{\ell+1}$.

In summary, we have that $\{1, \ldots, k\} = I_1 \cup J_1 \cup \cdots \cup I_N \cup J_N \cup I_{N+1}$, and only I_{N+1} may be empty.

For any $1 \leq s < t \leq k$, let $A_{s,t} = \{b \in B_k(G_n) : b_s = b_t\}$. By Lemma 4.2.28 (3), there exists a uniform constant K_1 such that

$$\mu(A_{s,t}) \leq K_1 \lambda^{-m}. \tag{4.5.1}$$

For notation, if I is a subset of $\{1, \ldots, k\}$, then b_I is b restricted to I. Since μ is 1-step Markov on X_{G_n}, we have that

$$\mu(b|A_{s_1,t_1}) = \mu(b_{I_1}|A_{s_1,t_1}) \prod_{\ell=1}^{N} \mu(b_{J_\ell}|A_{s_1,t_1} \cap b_{I_1 \ldots I_\ell}) \prod_{\ell=2}^{N+1} \mu(b_{I_\ell}|A_{s_1,t_1} \cap b_{I_1 \ldots J_{\ell-1}}) \tag{4.5.2}$$

$$= \mu(b_{I_1}|A_{s_1,t_1}) \prod_{\ell=1}^{N} \mu(b_{J_\ell}|b_{I_1 \ldots I_\ell}) \prod_{\ell=2}^{N+1} \mu(b_{I_\ell}|b_{I_1 \ldots J_{\ell-1}}) \tag{4.5.3}$$

Given b, we may form s_i, t_i, I_ℓ and J_ℓ as above, and then we encode b as follows

(1) encode s_1 and t_1 using an Elias code;

(2) encode b_{I_1} using a prefix Shannon code with respect to $\mu(\cdot|A_{s_1,t_1})$;

(3) assuming $b_{I_1 \ldots I_\ell}$ has been encoded, we encode b_{J_ℓ} by encoding s_i and t_i for each i in $\mathcal{I}$ such that $t_i \in J_\ell$, using an Elias code (and note that this information completely determines b_{J_ℓ} by definition of U and construction of J);

(4) assuming $b_{I_1 \dots J_{\ell-1}}$ has been encoded, we encode b_{I_ℓ} using a prefix Shannon code with respect to $\mu(\cdot|b_{I_1 \dots J_{\ell-1}})$.

Now we analyze the performance of the code. Since the code is a concatenation of prefix codes, it is a prefix code. Since U tends to infinity as n tends to infinity (by (C4)) and $k > U$, there exists n_0 such that for $n \geq n_0$ and $1 \leq s \leq k$, the length of the codeword in the Elias encoding of s is less than or equal to $2 \log k$. Then we have, neglecting bits needed to round up,

$$\mathcal{L}(b) \leq -\log \mu(b_{I_1}|A_{s_1,t_1}) + |\mathcal{I}|(4 \log k) + \sum_{\ell=2}^{N+1} -\log \mu(b_{I_\ell}|b_{I_1 \dots J_{\ell-1}}). \tag{4.5.4}$$

Combining Equations (4.5.2), (4.5.3), and Equation (4.5.4), we have that

$$\mathcal{L}(b) + \log \mu(b) \leq |\mathcal{I}|(4 \log k) + \log \mu(A_{s_1,t_1}) + \sum_{\ell=1}^{N} \log \mu(b_{J_\ell}|b_{I_1 \dots I_\ell}). \tag{4.5.5}$$

Now by Lemma 4.2.28 (2) and (3), there exist uniform constants K_2 and K_3 such that

$$\mathcal{L}(b) + \log \mu(b) \leq |\mathcal{I}|(4 \log k) + K_2 - m \log \lambda + N K_3 - |J| \log \lambda \tag{4.5.6}$$

$$= |\mathcal{I}|(4 \log k) + K_2 + N K_3 - (m + |J|) \log \lambda. \tag{4.5.7}$$

By construction, $|\mathcal{I}| \leq \min(k - j,\, 2k/U + 2)$, $N \leq |\mathcal{I}|$, and $|J| \geq r = k - j$. Then by Lemma 4.5.2, there exists a uniform constant $K_4 > 0$ such that

$$\mu(N_{n,k}^j) \leq (K_4 k^4)^{\min(k-j,\, 2k/U+2)} \lambda^{-(m+k-j)}. \tag{4.5.8}$$

Letting $p_0(x) = K_5 x^{12}$, for some uniform constant $K_5 > 0$, we obtain that

$$\mu(N_{n,k}^j) \leq p_0(k)^{\min(k-j,\, k/U)} \lambda^{-(m+k-j)},$$

which completes the proof. □

The following lemma bounds the $\mu \times \mu$-measure (and therefore the cardinality) of the set of pairs paths of length k in G_n that share at least one edge and together traverse at most $j < 2k$ edges. The general strategy of encoding pairs of paths using combinatorial data and appealing to information theory is similar to that of Lemma 4.5.9. Lemma 4.5.10 involves the additional hypothesis that there exists a uniform bound R such that for any pair of paths (u, w) in G_n, there exists a path uvw in G_n with $|v| \leq R$. Using this hypothesis, one observes that pairs of paths can essentially be concatenated in G_n and then treated as single paths as in Lemma 4.5.9.

Lemma 4.5.10. *Let (G_n) be a sequence of graphs satisfying the Standing Assumptions 4.2.21 and such that (G_n) has local uniqueness of paths, small diameter, and bounded distortion of weights (conditions (C4), (C5) and (C7) in 4.2.23). Then there exists a polynomial $p_1(x)$ and n_1 such that for $n \geq n_1$, $k > R(G_n)$ and $1 \leq j \leq 2k - 1$,*

$$\mu \times \mu(D^j_{n,k}) \leq p_1(k)^{\min(2k-j,\, k/U(G_n))} \lambda^{-(m(G_n)+2k-j)},$$

and

$$|D^j_{n,k}| \leq p_1(k)^{\min(2k-j,\, k/U(G_n))} \lambda^{j+m(G_n)}.$$

Proof. Consider (G_n), n, k, and j as in the hypotheses. Let $m = m(G_n)$, $U = U(G_n)$, and $R = R(G_n)$. Note that the bound on $|D^j_{n,k}|$ follows from the bound on $\mu \times \mu(D^j_{n,k})$, since condition (C7) implies that there exists a uniform constant K such that $|D^j_{n,k}| \leq K\lambda^{2m+2k} \mu \times \mu(D^j_{n,k})$ (as in Lemma 4.2.28 (1)). We now proceed to show the bound on $\mu \times \mu(D^j_{n,k})$.

By the definition of R, for every pair $(b, c) \in B_k(G_n) \times B_k(G_n)$, there exists a path d_1 in G_n such that $|b| \leq R$ and bdc is in $B_{2k+|d_1|}(G_n)$. We choose a single such d_1 for each pair (b, c), and we choose a (possibly empty) path d_2 such that bd_1cd_1 is in $B_{2k+R(G_n)}$ (whose existence is guaranteed by the fact that G_n is irreducible). If $(b, c) \in D^j_{n,k}$, then bd_1cd_2 is in $N^{j+R}_{n,2k+R}$. Using condition (C5), we have that $R \leq m + C$ for a uniform constant C. Then we have that there exist uniform constants K_1, K_2, and K_3 such that for each n, each k, and each pair (b, c) in $B_k(G_n) \times B_k(G_n)$,

$$\mu \times \mu((b, c)) \leq K_1 \lambda^{-(2m+2k)} \leq K_2 \lambda^{-(m+R+2k)} \leq K_3 \mu(bd_1cd_2).$$

Thus Lemma 4.5.9 implies that there exists a polynomial $p_0(x)$ and n_0 such that for $n \geq n_0$,

$$\mu \times \mu(D^j_{n,k}) \leq K_3 \mu(N^{j+R}_{n,2k+R}) \leq K_3 p_0(2k + R)^{\min(2k-j,\,(2k+R)/U)} \lambda^{-(m+2k-j)}.$$

With $n_1 = n_0$ and $p_1(x) = K_4 p_0(3x)^3$ for a uniform constant K_4, we have

$$\mu \times \mu(D^j_{n,k}) \leq p_1(k)^{\min(2k-j,\,k/U)} \lambda^{-(m+2k-j)},$$

which completes the proof. $\square$

The following two lemmas (Lemmas 4.5.11 and 4.5.12) give bounds on the $\mu \times \mu$ measure (and therefore the cardinality) of the set of pairs of periodic paths in G_n with certain overlap properties. The general ideas are similar to those in Lemmas 4.5.9 and 4.5.10, but in order to get precise bounds on the relevant sets, we exploit the fact that these sets consist of pairs of periodic paths. In other words, when we encode paths using their the pattern of "repeats," we also take into account their assumed periodicity.

Lemma 4.5.11. *Let (G_n) be a sequence of graphs satisfying the Standing Assumptions 4.2.21 and bounded distortion of weights (condition (C7) in 4.2.23). Then there exists a polynomial $p_2(x)$ and n_2 such that for each $n \geq n_2$ and $k > U(G_n)$,*

$$\mu \times \mu(S_{n,k}^{2k-1}) \leq p_2(k)\lambda^{-(2m(G_n)+U(G_n))},$$

and

$$|S_{n,k}^{2k-1}| \leq p_2(k)\lambda^{2k-U(G_n)}.$$

Proof. Consider (G_n), n, and k as in the hypotheses. Let $m = m(G_n)$ and $U = U(G_n)$. Note that the bound on $|S_{n,k}^{2k-1}|$ follows from the bound on $\mu\times\mu(S_{n,k}^{2k-1})$, since condition (C7) implies that there exists a uniform constant K such that $|S_{n,k}^{2k-1}| \leq K\lambda^{2m+2k}\mu\times\mu(S_{n,k}^{2k-1})$ (as in Lemma 4.2.28 (1)). We now proceed to show the bound on $\mu \times \mu(S_{n,k}^{2k-1})$.

Let b be in $\mathrm{Per}_k(G_n)$. Let e be in $E_n(b)$. For $i = 1, \dots, k$, let $C_i \subset B_k(G_n)$ be the set of paths c of length k in G_n such that $c_i = e$. Then Lemma 4.2.28 (parts (1) and (4)) implies that there exist uniform constants K_1 and K_2 such that

$$\mu(\mathrm{Per}_k(G_n) \cap C_1) = \mu(C_1)\mu(\mathrm{Per}_k(G_n)|C_1) \leq K_1\lambda^{-m}\mu(\mathrm{Per}_k(G_n)|C_1) \tag{4.5.9}$$

$$\leq K_2\lambda^{-(m+U)}. \tag{4.5.10}$$

Let C be the set of paths c of length k in G_n such that $e \in E_n(c)$. Then $C = \cup_{i=1}^k C_i$, and by shift-invariance of μ,

$$\mu(\mathrm{Per}_k(G_n) \cap C) \leq \sum_{i=1}^{k} \mu(\mathrm{Per}_k(G_n) \cap C_i) \leq K_2 k\lambda^{-(m+U)}. \tag{4.5.11}$$

Since $e \in E_n(b)$ was arbitrary, it follows from inequality (4.5.11) that

$$\mu(\{c \in \mathrm{Per}_k(G_n) : E_n(c) \cap E_n(b) \neq \emptyset\}) \leq \sum_{e \in E_n(b)} \mu(\{c \in \mathrm{Per}_k(G_n) : e \in E_n(c)\})$$

$$\leq K_2 \sum_{e \in E_n(b)} k\lambda^{-(m+U)} \leq K_2 k^2 \lambda^{-(m+U)}.$$

Since $b \in \mathrm{Per}_k(G_n)$ was arbitrary, we conclude that there exists a uniform constant K_3 such that

$$\mu \times \mu(S_{n,k}^{2k-1}) \leq K_2 \mu(\mathrm{Per}_k(G_n)) k^2 \lambda^{-(m+U)} \leq K_3 k^2 \lambda^{-(2m+U)},$$

where the last inequality follows from Lemma 4.2.28 (4). This inequality completes the proof. □

Lemma 4.5.12. *Let (G_n) be a sequence of graphs satisfying the Standing Assumptions 4.2.21 and such that (G_n) has local uniqueness of paths, small diameter, and bounded distortion of weights (conditions (C4), (C5) and (C7) in 4.2.23). Then there exists a polynomial $p_3(x)$ and n_3 such that for $n \geq n_3$, $k > U(G_n)$, and $1 \leq j \leq 2k-1$,*

$$\mu \times \mu(S_{n,k}^j) \leq p_3(k)^{k/U(G_n)} \lambda^{-(m(G_n)+U(G_n)+2k-j)},$$

and

$$|S_{n,k}^j| \leq p_3(k)^{k/U(G_n)} \lambda^{j+m(G_n)-U(G_n)}.$$

Proof. Consider (G_n), n, k, and j as in the hypotheses. Let $m = m(G_n)$, $U = U(G_n)$, and $R = R(G_n)$. Note that the bound on $|S_{n,k}^j|$ follows from the bound on $\mu \times \mu(S_{n,k}^j)$, since condition (C7) implies that there exists a uniform constant K

such that $|S^j_{n,k}| \le K\lambda^{2m+2k}\mu \times \mu(S^j_{n,k})$ (as in Lemma 4.2.28 (1)). We now proceed to show the bound on $\mu \times \mu(S^j_{n,k})$.

Let e be in E_n and let C_1 be the set of paths b of length k in G_n such that $b_1 = e$. Then it follows from Lemma 4.2.28 (4) that there exists a uniform constant K_1 such that

$$\mu(\mathrm{Per}_k(G_n)|C_1) \le K_1\lambda^{-U}. \tag{4.5.12}$$

To each pair (b, c) in $S^j_{n,k}$, let us associate a particular path of length $2k+R$ in G_n, which we construct as follows. Let (b, c) be in $S^j_{n,k}$. By definition of $S^j_{n,k}$, there is at least one edge e in $E_n(b) \cap E_n(c)$. Let τ be the cyclic permutation of $\{1, \dots, k\}$ of order k given by $(12\dots k)$. Let τ act on periodic paths of length k in G_n by permuting the indices: $\tau(b_1 \dots b_k) = b_{\tau(1)} \dots b_{\tau(k)}$. Then let b' be in $\{\tau^\ell(b) : \ell \in \{1, \dots, k\}, \tau^\ell(b)_k = e\}$. Similarly, let c' be in $\{\tau^\ell(c) : \ell \in \{1, \dots, k\}, \tau^\ell(c)_1 = e\}$. Now choose a path d_1 in G_n such that $|b'd_1c'| \le R$ and $b'd_1c'$ is a path in G_n (the existence of such a path d_1 is guaranteed by the definition of R). By irreducibility of G_n we also choose a (possibly empty) path d_2 in G_n such that $b'd_1c'd_2$ is in B_{2k+R}. We associate the path $b'd_1c'd_2$ to the pair (b, c), and note that there exist uniform constants K_2, K_3, and K_4 (by Lemma 4.2.28 (1) and condition (C5)) such that

$$\mu \times \mu((b, c)) \le K_2\lambda^{-(2m+2k)} \le K_3\lambda^{-(m+R+2k)} \le K_4\mu(b'd_1c'd_2). \tag{4.5.13}$$

Now we use the same construction as in the proof of Lemma 4.5.9 with only slight modification. We encode the words $b'd_1c'd_2$ as follows.

(1) Construct $\mathcal{I}$, J, and the partition of $\{1, \dots, 2k+R\}$ as in the proof of Lemma 4.5.9, with the additional condition that $J \cap \{k+1, \dots, k+R\} = \emptyset$. (In other

words, we ignore any "repeats" introduced by d.)

(2) Encode b' as in the proof of Lemma 4.5.9.

(3) To encode the path d_1, we first encode the fact that $b'_k = c'_1$ (by encoding k and $k + |d_1|$ using an Elias code), and then encode d_1 using a prefix Shannon code with respect to $\mu(\cdot|A_{k,k+|d_1|} \cap b')$.

(4) Encode c' as in the proof of Lemma 4.5.9.

(5) Encode d_2 using a prefix Shannon code with respect to $\mu(\cdot|b'd_1c')$.

For large n, encoding the fact that $b'_k = c'_1$ adds less than $4\log(2k+R)$ to $\mathcal{L}(b'd_1c'd_2)$. On the other hand, we have that there is a uniform constant $K_5 > 0$ such that $\mu(A_{k,k+|d_1|}|b') \leq K_5\lambda^{-U}$, by Lemma 4.2.28 (4). Thus, there exists n_3 and a uniform constant K_6 such that for $n \geq n_3$, we have

$$\mathcal{L}(b'd_1c'd_2) + \log \mu(b'd_1c'd_2) \leq (|\mathcal{I}| + 1)(4\log(2k + R)) + NK_6 - (m + U + |J|)\log\lambda, \tag{4.5.14}$$

with $|\mathcal{I}| \leq 2k/U + 2$, $N \leq |\mathcal{I}|$, and $|J| \geq 2k - j - 1$. Then by Lemma 4.5.2 there is a polynomial $p_4(x)$ such that for $n \geq n_3$,

$$\mu(\{b'd_1c'd_2 : (b,c) \in S^j_{n,k}\}) \leq p_4(k)^{k/U}\lambda^{-(m+U+2k-j)}. \tag{4.5.15}$$

Note that the number of pairs (b, c) associated to the a path $b'd_1c'd_2$ is at most k^2, and hence

$$\mu \times \mu(S^j_{n,k}) \leq k^2 p_4(k)^{k/U}\lambda^{-(m+U+2k-j)}. \tag{4.5.16}$$

Now let $p_3(x) = x^2 p_4(x)$, and the proof is complete. □

4.5.2 Entropy

Recall that if G is a graph, then β_G is the random variable such that $\beta_G(\omega)$ is the spectral radius of the adjacency matrix of $G(\omega)$.

Theorem 4.5.13. *Let (G_n) be a sequence of graphs that satisfies the Standing Assumptions 4.2.21 and such that (G_n) has local uniqueness of paths, small diameter, and bounded distortion of weights (conditions (C4), (C5), and (C7) in 4.2.23). Then for $1/\lambda < \alpha \leq 1$ and $\epsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}_\alpha(|\beta_{G_n} - \alpha\lambda| \geq \epsilon) = 0,$$

and the convergence to the limit is exponential in $m(G_n)$.

Remark 4.5.14. If we assume that X is irreducible in the statement of Theorem 4.1.3, then Theorem 4.1.3 is a direct corollary of Theorem 4.5.13, obtained by choosing (G_n) to be the sequence of n-block graphs of an irreducible SFT with positive entropy (and using the fact that such a sequence satisfies the hypotheses of Theorem 4.5.13 by Proposition 4.2.29). In the case when X is reducible, X has a finite number of irreducible components of positive entropy, $X_1, \ldots, X_r$, and there exist i such that $\mathbf{h}(X_i) = \mathbf{h}(X)$. For all large n, we have that $B_n(X_i) \cap B_n(X_j) = \emptyset$ for $i \neq j$, which means that the entropies of the random subshifts appearing inside each of these components are mutually independent. Applying Theorem 4.5.13 to each of these components, we obtain Theorem 4.1.3 for reducible X.

Proof of Theorem 4.5.13. Let α be in $(1/\lambda, 1]$. Let $m = m(G_n)$ and $U = U(G_n)$. Let b be a path in $G_n = (V_n, E_n)$. Let $\xi_b : \Omega_n \to \mathbb{R}$ be the random variable defined by

$$\xi_b(\omega) = \begin{cases} 1, & \text{if } b \text{ is allowed in } G_n(\omega) \\ 0, & \text{else.} \end{cases}$$

Now let

$$\phi_{n,k} = \sum_{b \in B_k(G_n)} \xi_b, \qquad \text{and} \qquad \psi_{n,k} = \frac{1}{|V_n|} \sum_{b \in \mathrm{Per}_k(G_n)} \xi_b.$$

For each n and k, we have that $\psi_{n,k} \leq \beta_n^k \leq \phi_{n,k}$. Indeed, $\psi_{n,k}$ is the average number of loops of length k based at a vertex in G_n. Thus there is at least one vertex v with at least $\psi_{n,k}$ loops of length k based at v, and it follows that $k^{-1} \log \psi_{n,k} \leq \log \beta_n$ since these loops may be concatenated freely. Also, it follows from subadditivity that $\log \beta_n = \lim_k k^{-1} \log \phi_{n,k} = \inf_k k^{-1} \log \phi_{n,k}$, which implies that $\beta_n^k \leq \phi_{n,k}$ for all n and k.

Fix $0 < \nu < 1$, and let $k = \lceil m^{1+\nu} \rceil + i$, where i is chosen such that $0 \leq i \leq \mathrm{per}(G_1) - 1$ and $\mathrm{per}(G_1)$ divides k. Recall that if (G_n) is the sequence of n-block graphs of a fixed graph G, then by Proposition 4.2.29 we have that m and n differ by at most a uniform constant, and thus $k \sim n^{1+\nu}$. We will show below that as n tends to infinity,

(I) $\left(\mathbb{E}_\alpha \phi_{n,k}\right)^{1/k}$ tends to $\alpha\lambda$;

(II) $\left(\mathbb{E}_\alpha \psi_{n,k}\right)^{1/k}$ tends to $\alpha\lambda$;

(III) there exists $K_1 > 0$ and $\rho_1 > 0$ such that $\frac{\mathrm{Var}(\phi_{n,k})}{(\mathbb{E}_\alpha \phi_{n,k})^2} \leq K_1 e^{-\rho_1 m}$;

(IV) there exists $K_2 > 0$ and $\rho_2 > 0$ such that $\frac{\mathrm{Var}(\psi_{n,k})}{(\mathbb{E}_\alpha \psi_{n,k})^2} \leq K_2 e^{-\rho_2 m}$.

Recall Definitions 4.5.5 - 4.5.8, as well as the modification of these definitions using “hats.” Notice that

$$\mathbb{E}_\alpha \phi_{n,k} = \sum_{b \in B_k(G_n)} \mathbb{E}_\alpha \xi_b = \sum_{b \in B_k(G_n)} \alpha^{|E_n(b)|} = \sum_{j=1}^{k} \alpha^j |\hat{N}^j_{n,k}|.$$

Also,

$$|V_n| \mathbb{E}_\alpha \psi_{n,k} = \sum_{b \in \mathrm{Per}_k(G_n)} \mathbb{E}_\alpha \xi_b = \sum_{b \in \mathrm{Per}_k(G_n)} \alpha^{|E_n(b)|} = \sum_{j=1}^{k} \alpha^j |\hat{Q}^j_{n,k}|.$$

Regarding variances, we have

$$\mathrm{Var}(\phi_{n,k}) = \sum_{(b,c) \in B_k(G_n)^2} \alpha^{|E_n(b) \cup E_n(c)|} (1 - \alpha^{|E_n(b) \cap E_n(c)|}) \leq \sum_{j=1}^{2k-1} \alpha^j |\hat{D}^j_{n,k}|,$$

and

$$|V_n|^2 \,\mathrm{Var}(\psi_{n,k}) = \sum_{(b,c) \in \mathrm{Per}_k(G_n)^2} \alpha^{|E_n(b) \cup E_n(c)|} (1 - \alpha^{|E_n(b) \cap E_n(c)|}) \leq \sum_{j=1}^{2k-1} \alpha^j |\hat{S}^j_{n,k}|.$$

For the remainder of this proof, we use the following notation: if (x_n) and (y_n) are two sequences, then $x_n \sim y_n$ means that the limit of the ratio of x_n and y_n tends to 1 as n tends to infinity.

Proof of (I). By Lemma 4.2.28 (1), there exists a uniform constant $K_1 > 0$ such that

$$\mathbb{E}_\alpha \phi_{n,k} = \sum_{j=1}^{k} \alpha^j |\hat{N}^j_{n,k}| \geq \alpha^k \sum_{j=1}^{k} |\hat{N}^j_{n,k}| = \alpha^k |B_k(G_n)| \geq K_1 \alpha^k \lambda^{m+k}. \qquad (4.5.17)$$

Taking k-th roots, letting n tend to infinity, and using that $m/k \sim m^{-\nu}$ tends to 0, we obtain that $\liminf_n \big(\mathbb{E}_\alpha \phi_{n,k}\big)^{1/k} \geq \alpha\lambda$.

By Lemma 4.2.28 (1) and Lemma 4.5.9, we have that there exists n_0, a polynomial $p_0(x)$, and a uniform constant $K_2 > 0$ such that for $n \geq n_0$,

$$\begin{aligned}
\mathbb{E}_\alpha \phi_{n,k} &= \sum_{j=1}^{k} \alpha^j |\hat{N}^j_{n,k}| \\
&\leq \sum_{j=1}^{k-1} \alpha^j |N^j_{n,k}| + \alpha^k |B_k(G_n)| \\
&\leq \big(p_0(k)\big)^{k/U} \left(\sum_{j=1}^{k-1} (\alpha\lambda)^j \right) + K_2 \alpha^k \lambda^{k+m} \\
&\leq (\alpha\lambda)^k \lambda^m \left(\frac{1}{\alpha\lambda - 1} p_0(k)^{k/U} \lambda^{-m} + K_2 \right).
\end{aligned}$$

By condition (C4) and the fact that $k \sim m^{1+\nu}$, we have that

- m tends to infinity as n tends to infinity by the Standing Assumptions 4.2.21;
- $m/k \sim m^{-\nu}$, which tends to zero as n tends to infinity;
- $U \geq m - C$, which tends to infinity as n tends to infinity.

Taking k-th roots and letting n tend to infinity, we have that $\limsup_n \big(\mathbb{E}_\alpha \phi_{n,k}\big)^{1/k} \leq \alpha\lambda$, which concludes the proof of (I).

Proof of (II). Let $p = \mathrm{per}(G_1) = \mathrm{per}(G_n)$. Note that since p divides k, there exists a uniform constant $K_3 > 0$ such that $|\operatorname{Per}_k(G_n)| \geq K_3 \lambda^k$ for large enough k. We choose n large enough so that this inequality is satisfied. Then we have that

$$\begin{aligned}
\mathbb{E}_\alpha \psi_{n,k} &= |V_n|^{-1} \sum_{j=1}^{k} \alpha^j |\hat{Q}^j_{n,k}| \\
&\geq |V_n|^{-1} \alpha^k \sum_{j=1}^{k} |\hat{Q}^j_{n,k}| \\
&= |V_n|^{-1} \alpha^k |\operatorname{Per}_k(G_n)| \\
&\geq K_3 \lambda^{-m} \alpha^k \lambda^k.
\end{aligned}$$

Taking k-th roots, letting n tend to infinity, and using that $m/k \sim m^{-\nu}$ tends to 0, we get that $\liminf_n \big(\mathbb{E}_\alpha \psi_{n,k}\big)^{1/k} \geq \alpha\lambda$. Recall that $0 \leq \psi_{n,k} \leq \phi_{n,k}$. Therefore it follows from (I) that $\limsup_n \big(\mathbb{E}_\alpha \psi_{n,k}\big)^{1/k} \leq \alpha\lambda$. Thus we have shown (II).

Proof of (III). For $j \leq 2k-1$, Lemma 4.5.10 implies that there is n_1 and a polynomial p_1 such that $|D^j_{n,k}| \leq p_1(k)^{k/U}\lambda^{j+m}$ and $|D^{2k-1}_{n,k}| \leq p_1(k)\lambda^{2k+m}$ for $n \geq n_1$. Now using that $\mathbb{E}_\alpha \phi_{n,k} \geq K_1 \alpha^k \lambda^{m+k}$ (see Equation (4.5.17)), we obtain that there exists a uniform constant $K_5 > 0$ such that

$$\begin{aligned}
\frac{\operatorname{Var}\phi_{n,k}}{(\mathbb{E}_\alpha \phi_{n,k})^2} &\leq \frac{\sum_{j=1}^{2k-1} \alpha^j |\hat{D}^j_{n,k}|}{K_1^2 \alpha^{2k} \lambda^{2m+2k}} \\
&= \frac{\sum_{j=1}^{2k-m-1} \alpha^j |\hat{D}^j_{n,k}| + \sum_{j=2k-m}^{2k-1} \alpha^j |\hat{D}^j_{n,k}|}{K_1^2 \alpha^{2k} \lambda^{2m+2k}} \\
&\leq \frac{\sum_{j=1}^{2k-m-1} \alpha^j |D^j_{n,k}| + \alpha^{2k-m} \sum_{j=2k-m}^{2k-1} |\hat{D}^j_{n,k}|}{K_1^2 \alpha^{2k} \lambda^{2m+2k}} \\
&\leq \frac{p_1(k)^{k/U} \lambda^m \sum_{j=1}^{2k-m-1} (\alpha\lambda)^j + \alpha^{2k-m} |D^{2k-1}_{n,k}|}{K_1^2 \alpha^{2k} \lambda^{2m+2k}} \\
&\leq \frac{K_5 p_1(k)^{k/U} \lambda^m (\alpha\lambda)^{2k-m} + \alpha^{2k-m} p_1(k) \lambda^{2k+m}}{K_1^2 \alpha^{2k} \lambda^{2m+2k}} \\
&\leq \frac{K_5}{K_1^2} \frac{p_1(k)^{k/U}}{(\alpha\lambda)^m \lambda^m} + \frac{p_1(k)}{K_1^2 (\alpha\lambda)^m} \\
&\leq \frac{K_5}{K_1^2} \left(\frac{p_1(k)^{k/Um}}{(\alpha\lambda)} \right)^m + \frac{p_1(k)}{K_1^2 (\alpha\lambda)^m}.
\end{aligned}$$

Using the facts that $U \geq m - C$ and $k \sim m^{1+\nu}$, we have that k/Um is asymptotically bounded above by $2m^{\nu-1}$. Since $\nu - 1 < 0$, it holds that $p_1(k)^{k/Um}$ tends to 1. Thus we obtain that for any $0 < \rho_1 < \ln \alpha\lambda$, there exists $K_6 > 0$ and n_2 such that for $n \geq n_2$, it holds that $\operatorname{Var}\phi_{n,k}(\mathbb{E}_\alpha \phi_{n,k})^{-2} \leq K_6 e^{-\rho_1 m}$, which proves (III).

Proof of (IV). For $j \leq 2k-1$, Lemma 4.5.12 together with (C4) implies that there is n_3 and a polynomial p_3 such that $|S^j_{n,k}| \leq p_3(k)^{k/U}\lambda^j$ for $n \geq n_3$. Also, Lemma 4.5.11 implies that there is n_4 and a polynomial p_2 such that $|S^{2k-1}_{n,k}| \leq p_2(k)\lambda^{2k-U}$ for $n \geq n_4$. Now using that $|V_n|\mathbb{E}_\alpha\psi_{n,k} \geq K_3\alpha^k\lambda^k$, we obtain that there exists $K_7 > 0$ such that, with $K := K_3$,

$$\begin{aligned}
\frac{\operatorname{Var}\psi_{n,k}}{(\mathbb{E}_\alpha\psi_{n,k})^2} &\leq \frac{\sum_{j=1}^{2k-1}\alpha^j|\hat{S}^j_{n,k}|}{K^2\alpha^{2k}\lambda^{2k}} \\
&= \frac{\sum_{j=1}^{2k-U-1}\alpha^j|\hat{S}^j_{n,k}| + \sum_{j=2k-U}^{2k-1}\alpha^j|\hat{S}^j_{n,k}|}{K^2\alpha^{2k}\lambda^{2k}} \\
&\leq \frac{\sum_{j=1}^{2k-U-1}\alpha^j|S^j_{n,k}| + \alpha^{2k-U}\sum_{j=2k-U}^{2k-1}|\hat{S}^j_{n,k}|}{K^2\alpha^{2k}\lambda^{2k}} \\
&\leq \frac{p_3(k)^{k/U}\sum_{j=1}^{2k-U-1}(\alpha\lambda)^j + \alpha^{2k-U}|S^{2k-1}_{n,k}|}{K^2\alpha^{2k}\lambda^{2k}} \\
&\leq \frac{K_7p_3(k)^{k/U}(\alpha\lambda)^{2k-U} + \alpha^{2k-U}p_2(k)\lambda^{2k-U}}{K^2\alpha^{2k}\lambda^{2k}} \\
&\leq \frac{K_7}{K^2}\frac{p_3(k)^{k/U}}{(\alpha\lambda)^U} + \frac{p_2(k)}{K^2(\alpha\lambda)^U} \\
&\leq \frac{K_7}{K^2}\left(\frac{p_3(k)^{k/U^2}}{(\alpha\lambda)}\right)^U + \frac{p_2(k)}{K^2(\alpha\lambda)^U}.
\end{aligned}$$

Using the facts that $U \geq m - C$ and $k \sim m^{1+\nu}$, we have that k/U^2 is asymptotically bounded above by $2m^{\nu-1}$. Since $\nu - 1 < 0$, it holds that $p_3(k)^{k/U^2}$ tends to 1. Thus we obtain that for any $0 < \rho_2 < \log\alpha\lambda$, there exists $K_8 > 0$ and n_5 such that for $n \geq n_5$,

$$\frac{\operatorname{Var}\phi_{n,k}}{(\mathbb{E}_\alpha\phi_{n,k})^2} \leq K_8e^{-\rho_2 m},$$

which proves (IV).

Proof of Theorem 4.5.13 using (I)-(IV). Recall that $\psi_{n,k} \leq \beta^k_n \leq \phi_{n,k}$. Let $\epsilon > 0$. Since $\alpha\lambda > 1$, we may assume without loss of generality that $\alpha\lambda - \epsilon > 1$.

Then

$$\mathbb{P}_\alpha\Big(|\beta_n - \alpha\lambda| \geq \epsilon\Big) = \mathbb{P}_\alpha\Big(\beta_n \geq \alpha\lambda + \epsilon\Big) + \mathbb{P}_\alpha\Big(\beta_n \leq \alpha\lambda - \epsilon\Big) \tag{4.5.18}$$

$$= \mathbb{P}_\alpha\Big(\beta_n^k \geq (\alpha\lambda + \epsilon)^k\Big) + \mathbb{P}_\alpha\Big(\beta_n^k \leq (\alpha\lambda - \epsilon)^k\Big) \tag{4.5.19}$$

$$\leq \mathbb{P}_\alpha\Big(\phi_{n,k} \geq (\alpha\lambda + \epsilon)^k\Big) + \mathbb{P}_\alpha\Big(\psi_{n,k} \leq (\alpha\lambda - \epsilon)^k\Big). \tag{4.5.20}$$

We will bound each of the two terms in Equation (4.5.20). Notice that

$$\begin{aligned}\mathbb{P}_\alpha\Big(\phi_{n,k} \geq (\alpha\lambda + \epsilon)^k\Big) &= \mathbb{P}_\alpha\Big(\phi_{n,k} - \mathbb{E}_\alpha\phi_{n,k} \geq (\alpha\lambda + \epsilon)^k - \mathbb{E}_\alpha\phi_{n,k}\Big) \\ &= \mathbb{P}_\alpha\left(\phi_{n,k} - \mathbb{E}_\alpha\phi_{n,k} \geq \mathbb{E}_\alpha\phi_{n,k}\left(\Big(\frac{\alpha\lambda + \epsilon}{(\mathbb{E}_\alpha\phi_{n,k})^{1/k}}\Big)^k - 1\right)\right).\end{aligned}$$

Let $d_{n,k}^1 = (\mathrm{Var}(\phi_{n,k}))^{1/2}/\mathbb{E}_\alpha\phi_{n,k}$. Then by Chebychev's Inequality,

$$\mathbb{P}_\alpha\Big(\phi_{n,k} \geq (\alpha\lambda + \epsilon)^k\Big) = \tag{4.5.21}$$

$$= \mathbb{P}_\alpha\left(\phi_{n,k} - \mathbb{E}_\alpha\phi_{n,k} \geq (\mathrm{Var}(\phi_{n,k}))^{1/2}\frac{1}{d_{n,k}^1}\left(\Big(\frac{\alpha\lambda + \epsilon}{(\mathbb{E}_\alpha\phi_{n,k})^{1/k}}\Big)^k - 1\right)\right) \tag{4.5.22}$$

$$\leq \left(\frac{d_{n,k}^1}{(\frac{\alpha\lambda+\epsilon}{(\mathbb{E}_\alpha\phi_{n,k})^{1/k}})^k - 1}\right)^2. \tag{4.5.23}$$

The denominator in the right-hand side of (4.5.23) might be 0 for finitely many n, but by properties (I) and (III), there exists $K_9 > 0$ such that for large enough n,

$$\mathbb{P}_\alpha\Big(\phi_{n,k} \geq (\alpha\lambda + \epsilon)^k\Big) \leq \left(\frac{d_{n,k}^1}{(\frac{\alpha\lambda+\epsilon}{(\mathbb{E}_\alpha\phi_{n,k})^{1/k}})^k - 1}\right)^2 \leq K_9 e^{-\rho_1 m}.$$

Similarly, we let $d_{n,k}^2 = (\mathrm{Var}(\psi_{n,k}))^{1/2}/\mathbb{E}_\alpha\psi_{n,k}$, and then Chebychev's Inequality gives that

$$\mathbb{P}_\alpha\Big(\psi_{n,k} \le (\alpha\lambda - \epsilon)^k\Big) = \tag{4.5.24}$$

$$= \mathbb{P}_\alpha\left(\psi_{n,k} - \mathbb{E}_\alpha\psi_{n,k} \le (\mathrm{Var}(\psi_{n,k}))^{1/2}\frac{1}{d_{n,k}^2}\Big(\Big(\frac{\alpha\lambda-\epsilon}{(\mathbb{E}_\alpha\psi_{n,k})^{1/k}}\Big)^k - 1\Big)\right) \tag{4.5.25}$$

$$\le \left(\frac{d_{n,k}^2}{(\frac{\alpha\lambda-\epsilon}{(\mathbb{E}_\alpha\psi_{n,k})^{1/k}})^k - 1}\right)^2 \tag{4.5.26}$$

Again, the denominator in the right-hand side might be 0 for finitely many n, but by properties (II) and (IV), there exists $K_{10} > 0$ such that for large enough n,

$$\mathbb{P}_\alpha\Big(\psi_{n,k} \le (\alpha\lambda - \epsilon)^k\Big) \le \left(\frac{d_{n,k}^2}{(\frac{\alpha\lambda-\epsilon}{(\mathbb{E}_\alpha\psi_{n,k})^{1/k}})^k - 1}\right)^2 \le K_{10}e^{-\rho_2 m}.$$

In conclusion, we obtain that there exists $K_{11} > 0$ such that for large enough n,

$$\mathbb{P}_\alpha(|\beta_n - \alpha\lambda| \ge \epsilon) \le K_{11}e^{-\min(\rho_1,\rho_2)m}.$$

□

4.5.3 Irreducible components of positive entropy

Theorem 4.5.15. *Let (G_n) be a sequence of graphs that satisfies the Standing Assumptions 4.2.21, with $p = per(G_1) = per(G_n)$, and such that*

- *(G_n) has bounded degrees (condition (C1) in 4.2.23),*
- *(G_n) has fast separation of periodic points (condition (C3) in 4.2.23),*
- *and (G_n) has uniform forward and backward expansion (condition (C8) in 4.2.23).*

Let $\mathcal{U}_{G_n}$ be the event in Ω_{G_n} that $G_n(\omega)$ contains a unique irreducible component C of positive entropy. Also, let $\mathcal{W}_{G_n}$ be the event (contained in $\mathcal{U}_{G_n}$) that the induced edge shift on C has period p. Then there exists $c > 0$ such that for $1 - c < \alpha \leq 1$,

$$\lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{U}_{G_n}) = 1, \quad \textit{and} \quad \lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{W}_{G_n}) = 1,$$

and the convergence to these limits is exponential in $m(G_n)$.

Remark 4.5.16. Theorem 4.1.4 is a corollary of Theorem 4.5.15: if X is an irreducible SFT of positive entropy, then the sequence of n-block graphs for X satisfies the hypotheses of Theorem 4.5.15 by Proposition 4.2.29. In fact, if X is a reducible SFT, we may apply Theorem 4.1.4 to each irreducible component independently, which allows us to conclude the following. Let X be a reducible SFT with irreducible components $X_1, \ldots, X_r$ such that $p_i = \mathrm{per}(X_i)$ for each i. Let $\mathcal{W}_n$ be the event in Ω_n that X_ω has exactly r irreducible components with periods $p_1, \ldots, p_r$. Then there exists $c > 0$ such that for $\alpha \in (1 - c, 1]$, we have that $\lim_n \mathbb{P}_\alpha(\mathcal{W}_n) = 1$, with exponential (in n) convergence to the limit.

Definition 4.5.17. Let G be a directed graph. For each vertex v in G, and for each ω in Ω_G, let $\Gamma^+_\omega(v)$ be the union of $\{v\}$ and the set of vertices u in G such that there is an allowed path from v to u in $G(\omega)$. Similarly, for each vertex v in G and each ω in Ω_G, let $\Gamma^-_\omega(v)$ be the union of $\{v\}$ and the set of vertices u in G such that there is an allowed path from u to v in $G(\omega)$. Also, let $I_\omega(v) = \Gamma^+_\omega(v) \cap \Gamma^-_\omega(v)$, which is the vertex set of the irreducible component containing v in $G(\omega)$.

The proof of the following proposition is an adaptation of the proof of Lemma 2.2 in [3].

Proposition 4.5.18. *Let (G_n) be a sequence of graphs satisfying the Standing Assumptions 4.2.21 and such that (G_n) has bounded degrees and uniform forward and backward expansion (conditions (C1) and (C8) in 4.2.23). Let r_n be a sequence of integers such that $r_n \geq am(G_n)$, for some $a > 0$, for all large n. Let $C^+_{G_n}$ be the event in Ω_{G_n} consisting of all ω such that there exists a vertex v in G_n with $r_n \leq \Gamma^+_\omega(v) \leq |V_n|/2$. Then there exists $c > 0$ such that for $\alpha > 1 - c$,*

$$\lim_{n\to\infty} \mathbb{P}_\alpha(C^+_{G_n}) = 0, \tag{4.5.27}$$

and the convergence of this limit is exponential in $m(G_n)$. Furthermore, the same statement holds with "+" replaced by "−".

Proof. Let $m = m(G_n)$. Let $b > 0$ be such that both (G_n) and (G_n^T) are b-expander sequences (where the existence of such a b is guaranteed by condition (C8)). We use the notation in Definition 4.5.17. For any v in V_n and any ω in Ω_{G_n}, the set $\Gamma^+_\omega(v)$ has the property that all edges in $E_n(\Gamma^+_\omega(v), \overline{\Gamma^+_\omega(v)})$ are forbidden (by ω). Then the fact that G_n is a b-expander implies that for a particular subset S of V_n, the probability that $S = \Gamma^+_\omega(v)$ for some v is bounded above by $(1-\alpha)^{b|S|}$. The number of subsets S of V_n with $|S| = r$ that could appear as $\Gamma^+_\omega(v)$ for some v is bounded above by $(\Delta e)^r$, where e is the base of the natural logarithm [3, Lemma 2.2] (see also [2, Lemma 2.1] or [60, p. 396, Exercise 11]). Then for α such that

$\Delta e(1-\alpha) < 1$, we have that for any $0 \leq r_n \leq |V_n|/2$,

$$\mathbb{P}_\alpha(C^+_{G_n}) = \mathbb{P}_\alpha(\exists v \text{ such that } r_n \leq |\Gamma^+_\omega(v)| \leq \frac{|V_n|}{2}) \tag{4.5.28}$$

$$\leq \sum_{r=r_n}^{\frac{|V_n|}{2}} |V_n|(\Delta e)^r(1-\alpha)^{br} \tag{4.5.29}$$

$$\leq |V_n|(\Delta e(1-\alpha)^b)^{r_n}\frac{1}{1-\Delta e(1-\alpha)} \tag{4.5.30}$$

$$\leq (\lambda^{1/a}\Delta e(1-\alpha)^b)^{am}\frac{1}{1-\Delta e(1-\alpha)}. \tag{4.5.31}$$

Thus there is a $c > 0$ (depending only on a, b, λ, and Δ) such that if $\alpha > 1-c$, then the right-hand side of the inequality in (4.5.31) tends to zero exponentially in $m(G_n)$ as n tends to infinity. In particular, we may take

$$c = \left(\frac{1}{\lambda}\right)^{1/ab}\left(\frac{1}{\Delta e}\right)^{1/b}.$$

Since (G_n^T) is also a uniform b-expander, the same estimates hold with $C^-_{G_n}$ in place of $C^+_{G_n}$. □

Proof of Theorem 4.5.15. Let (G_n) be as in the statement of Theorem 4.5.15. Let $m = m(G_n)$, $z = z(G_n)$, and $p = \text{per}(G_1) = \text{per}(G_n)$. We use the notation in Definition 4.5.17. Consider the following events:

$$F_n^+ = \{\omega \in \Omega_n : \forall v \in V_n, \Gamma^+_\omega(v) \leq z(G_n) - 2p \text{ or } \Gamma^+_\omega(v) > |V_n|/2\}$$

$$F_n^- = \{\omega \in \Omega_n : \forall v \in V_n, \Gamma^-_\omega(v) \leq z(G_n) - 2p \text{ or } \Gamma^-_\omega(v) > |V_n|/2\}$$

$$F_n = F_n^+ \cap F_n^-.$$

Recall that condition (C3) gives $a > 0$ such that $z \geq am$. Note that Proposition 4.5.18 gives $c > 0$ such that for $1-c < \alpha \leq 1$, there exists $K_1, K_2 > 0$ and $\rho_1, \rho_2 > 0$

such that for large n,

$$\mathbb{P}_\alpha(\Omega_n \setminus F_n^+) \leq K_1 e^{-\rho_1 m} \qquad \text{and } \mathbb{P}_\alpha(\Omega_n \setminus F_n^-) \leq K_2 e^{-\rho_2 m}.$$

Fix such an α, and note that for all large enough n, we have the following estimate: $\mathbb{P}_\alpha(\Omega_n \setminus F_n) \leq 2\max(K_1, K_2)e^{-\min(\rho_1,\rho_2)m}$.

Consider ω in F_n. Suppose that there exists v_1 and v_2 in V_n such that $|I_\omega(v_1)| > z - 2p$ and $|I_\omega(v_2)| > z - 2p$. Then by definition of F_n, we must have that $\Gamma_\omega^+(v_1) \cap \Gamma_\omega^-(v_2) \neq \emptyset$ and $\Gamma_\omega^-(v_1) \cap \Gamma_\omega^+(v_2) \neq \emptyset$. It follows that there is a path from v_1 to v_2 in $G_n(\omega)$, and there is a path from v_2 to v_1 in $G_n(\omega)$. Thus $I_\omega(v_1) = I_\omega(v_2)$. We have shown that for ω in F_n, there is at most one irreducible component of cardinality greater than $z - 2p$. Note that this argument implies that for ω in F_n, all allowed periodic orbits γ such that $|V_n(\gamma)| > z - 2p$ must lie in the same irreducible component.

By definition of z, if I_ω is an irreducible component of $G_n(\omega)$ with positive entropy, then $|I_\omega| > z$ (since it must contain at least two periodic orbits with overlapping vertex sets). We deduce that for ω in F_n, there is at most one irreducible component of $G_n(\omega)$ with positive entropy.

We now show that there exists an irreducible component of positive entropy with probability tending exponentially to 1. Let $z_1 = z - i$, where i is chosen (for each n) such that $0 \leq i \leq p - 1$ and p divides z_1. Then let $z_2 = z_1 - p$. Consider the following sequences of random variables:

$$f_n = \sum_{b \in \mathrm{Per}_{z_1}(G_n)} \xi_b, \qquad \text{and} \qquad g_n = \sum_{b \in \mathrm{Per}_{z_2}(G_n)} \xi_b. \tag{4.5.32}$$

Note that by the definition of z and Lemma 4.3.3, we have that $|E_n(b)| = |b|$ for any periodic path b with period less than or equal to z. Furthermore, any two such paths are disjoint. Therefore the random variables $\{\xi_b\}_{b\in \operatorname{Per}_{z_1}(G_n)}$ are jointly independent, and the random variables $\{\xi_b\}_{b\in \operatorname{Per}_{z_2}(G_n)}$ are also jointly independent. Thus

$$\begin{aligned}
\mathbb{E}_\alpha f_n &= \sum_{b\in \operatorname{Per}_z(G_n)} \alpha^{z_1} = \alpha^{z_1} |\operatorname{Per}_{z_1}(G_n)|,\\
\mathbb{E}_\alpha g_n &= \sum_{b\in \operatorname{Per}_{z_2}(G_n)} \alpha^{z_2} = \alpha^{z_2} |\operatorname{Per}_{z_2}(G_n)|,\\
\operatorname{Var}(f_n) &= \sum_{b\in \operatorname{Per}_{z_1}(G_n)} \alpha^{z_1}(1-\alpha^{z_1}) = \alpha^{z_1}(1-\alpha^{z_1}) |\operatorname{Per}_{z_1}(G_n)|\\
\operatorname{Var}(g_n) &= \sum_{b\in \operatorname{Per}_{z_2}(G_n)} \alpha^{z_2}(1-\alpha^{z_2}) = \alpha^{z_2}(1-\alpha^{z_2}) |\operatorname{Per}_{z_2}(G_n)|.
\end{aligned}$$

As n tends to infinity, z tends to infinity since $z \geq am$ and m tends to infinity. Then by the Standing Assumptions 4.2.21 (in particular, we use that $\operatorname{Sp}_\times(G_n) = \operatorname{Sp}_\times(G_1)$) and the fact that p divides z_1 and z_2, we have that each of of the sequences $\lambda^{-z_1}|\operatorname{Per}_{z_1}(G_n)|$ and $\lambda^{-z_2}|\operatorname{Per}_{z_2}(G_n)|$ tends to a finite, non-zero limit as n tends to infinity (and in fact the limit is p). For two sequences x_n and y_n of positive real numbers, let $x_n \sim y_n$ denote the statement that their ratio tends to a finite, non-zero limit as n tends to infinity. Then we have that $\mathbb{E}_\alpha f_n \sim (\alpha\lambda)^{z_1} \sim \operatorname{Var}(f_n)$ and $\mathbb{E}_\alpha g_n \sim (\alpha\lambda)^{z_2} \sim \operatorname{Var}(g_n)$. Note that $\mathbb{E}_\alpha f_n \geq \operatorname{Var}(f_n)$ and $\mathbb{E}_\alpha g_n \geq \operatorname{Var}(g_n)$. A simple application of Chebychev's inequality implies that

$$\begin{aligned}
\mathbb{P}_\alpha(f_n \leq 0) &\leq \mathbb{P}_\alpha\Big(f_n - \mathbb{E}_\alpha f_n \leq -\operatorname{Var}(f_n)\Big)\\
&\leq \left(\frac{1}{\operatorname{Var}(f_n)^{1/2}}\right)^2 \sim \left(\frac{1}{\alpha\lambda}\right)^{z_1} \leq \left(\frac{1}{\alpha\lambda}\right)^{am-i},
\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}_\alpha(g_n \leq 0) &\leq \mathbb{P}_\alpha\Big(g_n - \mathbb{E}_\alpha g_n \leq -\operatorname{Var}(g_n)\Big)\\ &\leq \left(\frac{1}{\operatorname{Var}(g_n)^{1/2}}\right)^2 \sim \left(\frac{1}{\alpha\lambda}\right)^{z_2} \leq \left(\frac{1}{\alpha\lambda}\right)^{am-i-p}.\end{aligned}$$

We have shown that the probability that there is no periodic orbit of period z_1 tends to 0 exponentially in m as n tends to infinity, and the probability that there exists no periodic orbit of period z_2 tends to 0 exponentially in m as n tends to infinity.

In summary, we have shown that the following events occur with probability tending to 1 exponentially in m as n tends to infinity:

- there exists a periodic point of period $z - i$;
- there exists a periodic point of period $z - i - p$;
- any two periodic points of period greater than $z-2p$ lie in the same irreducible component (of necessarily positive entropy);
- there is at most one irreducible component of positive entropy.

We conclude that with probability tending to 1 exponentially in m as n tends to infinity, there exists a unique irreducible component of positive entropy, and the induced edge shift on that component has period p. □

4.6 Remarks

Remark 4.6.1. The proofs of Theorems 4.3.1 and 4.4.2 do not require all of the Standing Assumptions 4.2.21. In fact, these proofs only use that $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$ for each n and that $z(G_n)$ tends to infinity as n tends to infinity.

Remark 4.6.2. Theorem 4.3.1 states that at the critical threshold $\alpha = 1/\lambda$, the probability of emptiness tends to zero. Using the fact that entropy is a monotone increasing random variable (as defined in Section 4.2.3), one may deduce from Theorem 4.5.13 that for $\alpha = 1/\lambda$, the probability that the random SFT has zero entropy tends to 1. It might be interesting to know more about the behavior of typical random SFTs at the critical threshold.

Remark 4.6.3. We have considered only random $\mathbb{Z}$-SFTs, but one may also consider random $\mathbb{Z}^d$-SFTs for any d in $\mathbb{N}$ by adapting the construction of Ω_n and $\mathbb{P}_\alpha$ in the obvious way. It appears that most of the proofs presented above may not be immediately adapted for $d > 1$, but there is one exception, which we state below. Let X be a non-empty $\mathbb{Z}^d$-SFT. For $d > 1$, there are various zeta functions for X (for a definition distinct from ours, see [65]); we consider

$$\zeta_X(t) = \exp\left(\sum_{p=1}^{\infty} \frac{N_p}{p} t^p\right),$$

where N_p is the number of periodic points x in X such that the number of points in the orbit of x divides p. The function ζ_X has radius of convergence $1/\rho$, where $\log \rho = \limsup_p p^{-1} \log(N_p)$. For example, for a full $\mathbb{Z}^d$ shift on a symbols, $\rho = a$, regardless of d. Using exactly the same proof as presented in Section 4.3, we obtain that

$$\limsup_{n\to\infty} \mathbb{P}_\alpha(\mathcal{E}_n) \leq \begin{cases} (\zeta_X(\alpha))^{-1}, & \text{if } \alpha \in [0, 1/\rho) \\ 0, & \text{if } \alpha \in [1/\rho, 1]. \end{cases}$$

For $\alpha \geq 1/\rho$, this bound implies that the limiting probability of emptiness is 0. In this context, we note that there is no algorithm, which, given a $\mathbb{Z}^d$-SFT X defined by a finite list of finite forbidden configurations, will decide whether X is empty [9]. Nonetheless, we may be able to compute the limiting probability of emptiness. For example, if X is a full shift on a symbols, then for $\alpha \geq 1/a$, we have that the limiting probability of emptiness is 0.

Remark 4.6.4. One may also consider more general random subshifts. Recall that a set $X \subset \mathcal{A}^{\mathbb{Z}}$ is a subshift if it is closed and shift-invariant. For a non-empty subshift X and a natural number n, we may consider the (finite) set of subshifts obtained by forbidding words of length n from X. After defining a probability measure $\mathbb{P}_\alpha$ on this space as in Section 4.2, we obtain random subshifts of X. Now we may investigate the asymptotic probability of properties of these random subshifts. Recall that any subshift X can be written as $\cap X_n$, where (X_n) is a sequence of SFTs (called the Markov approximations of X) and $\lim_n h(X_n) = h(X)$. A subshift X is called *almost sofic* [79] if there exists a sequence (X_n) of irreducible SFTs such that $X_n \subset X$ and $\lim_n h(X_n) = h(X)$. Using this inner and outer approximation by SFTs, the conclusion of Theorem 4.1.3 still holds if the system X is only assumed to be an almost sofic subshift.

Remark 4.6.5. Theorem 4.5.15 asserts the existence of a constant $c > 0$, but we are left with several questions about this constant. Fix a sequence (G_n) satisfying the hypotheses of Theorem 4.5.15. Let $\alpha_* = \inf\{\alpha > 0 : \lim_n \mathbb{P}_\alpha(\mathcal{U}_n) = 1\}$. What is α_*? What is α_* in the case that (G_n) is the sequence of n-block graphs of a mixing SFT of positive entropy (or even a full shift)?

Appendix A

Additional facts

A.1 The Realization Theorem of Downarowicz and Serafin

For general references on the ergodic theory of topological dynamical systems, see [46, 78, 87]. For a topological dynamical system (X, T), we write $M(X, T)$ to denote the space of Borel probability measures on X which are invariant under T. We give $M(X, T)$ the weak* topology. It is well known that in this setting $M(X, T)$ is a metrizable, compact, convex subset of a locally convex topological vector space (see, for example, [46, 78]). The set of extreme points of $M(X, T)$ is the set of ergodic measures, $M_{\text{erg}}(X, T)$. Furthermore, the fact that each measure μ in $M(X, T)$ has a unique ergodic decomposition (see [46, 78]) translates to the fact that $M(X, T)$ is a Choquet simplex. Since we are only interested in simplices arising from dynamical systems, we consider only metrizable Choquet simplices. It was shown in [34] that every metrizable Choquet simplex K can be obtained as the space of invariant Borel probability measures for a dynamical system.

We write $h : M(X, T) \to [0, \infty)$ to denote the function that assigns to each measure μ in $M(X, T)$ its metric entropy. For any dynamical system (X, T), Boyle and Downarowicz defined a reference candidate sequence $\mathcal{H}_{ref}(X, T)$ on $M(X, T)$ that is u.s.c.d. and harmonic. Further, Downarowicz defined an **entropy structure** on $M(X, T)$ to be any candidate sequence on $M(X, T)$ that is uniformly equivalent to $\mathcal{H}_{ref}$ (see Section 2.1.5 for definitions). Almost all known methods of defining or computing entropy can be adapted to form an entropy structure [35]. The work of Downarowicz and Serafin [38] implies the following realization theorem:

Theorem A.1.1 ([35, 38]). *Let $\mathcal{H}$ be a candidate sequence on a Choquet simplex K that is uniformly equivalent to a harmonic candidate sequence with u.s.c. differences. Then $\mathcal{H}$ is (up to affine homeomorphism) an entropy structure for a minimal homeomorphism of the Cantor set.*

The importance of Theorem A.1.1 lies in the fact that it allows one to translate questions in the theory of entropy structures and dynamical systems into the terms of functional analysis.

A.2 Proof of Fact 2.2.24

The following fact was given as Fact 2.5 in [36], where there is a sketch of the proof. In this appendix we fill in some details of this proof.

Fact (Fact 2.2.24). *Let K be a metrizable Choquet simplex, and let $f : K \to [0, \infty)$ be convex and u.s.c. Then $(f|_{\mathrm{ex}(K)})^{har}$ is u.s.c.*

Proof. Let $f : K \to [0, \infty)$ be convex and u.s.c. Let $g : \mathcal{M}(K) \to [0, \infty)$ be defined for each μ in $\mathcal{M}(K)$ as

$$g(\mu) = \int f d\mu.$$

Now let $G : K \to [0, \infty)$ be given by $G(x) = \sup\{g(\mu) : \mathrm{bar}(\mu) = x\}$ for all x in K. We have that g is u.s.c. because f is u.s.c., and G is u.s.c. because g is u.s.c. (Remark 2.1.16 (iii)).

Now we claim that $f(x) \leq \int f d\mu$ for any μ such that $\text{bar}(\mu) = x$. To see this, fix x and μ such that $\text{bar}(\mu) = x$. Let f_m be a decreasing sequence of continuous functions, $f_m : K \to [0, \infty)$, whose limit is f. Let $\delta > 0$. Partition the support of μ into a finite number of sets S_j of diameter smaller than δ. For each j, if $\mu(S_j) > 0$, let $z_j = \text{bar}(\mu_{S_j})$, where μ_{S_j} is the measure μ conditioned on the set S_j. Then let $\nu = \sum_j \mu(S_j)\epsilon_{z_j}$. Note that $\text{bar}(\nu) = \text{bar}(\mu) = x$, and ν tends to μ in $\mathcal{M}(K)$ as δ tends to zero. We have shown that there exists a sequence of measures ν_k such that each ν_k is a finite convex combination of point measures, ν_k converges to μ in $\mathcal{M}(K)$, and $\text{bar}(\nu_k) = x$ for each k. Now choose such a sequence ν_k, and note that for any m, any $\epsilon > 0$, and any large enough k (depending on ϵ and m), by the convexity of f,

$$f(x) \leq \int f \, d\nu_k \leq \int f_m \, d\nu_k \leq \int f_m \, d\mu + \epsilon.$$

Letting m tend to infinity, the Dominated Convergence Theorem implies that $f(x) \leq \int f d\mu + \epsilon$. Since ϵ was arbitrary, we see that $f(x) \leq \int f d\mu$, which implies in particular that $f(x) \leq \int f d\mathcal{P}_x$.

Then for any μ with $\text{bar}(\mu) = x$,

$$\int f d\mu \leq \int \left(\int f d\mathcal{P}_y \right) d\mu(y) = \int f \, d\mathcal{P}_x,$$

where the equality of the last two expressions follows from the fact that $x \mapsto \int f \, d\mathcal{P}_x$ defines a harmonic function on K (Remark 2.2.22).

Thus $G(x) = \int f d\mathcal{P}_x$, which shows that $G = (f|_{\text{ex}(K)})^{har}$. Since G is u.s.c., the proof is complete. $\square$

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