ABSTRACT

Title of dissertation: EQUIVARIANT GIAMBELLI FORMULAE FOR GRASSMANNIANS

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In this thesis we use Young's raising operators to define and study polynomials which represent the Schubert classes in the equivariant cohomology ring of Grassmannians. For the type A and maximal isotropic Grassmannians, we show that our expressions coincide with the factorial Schur S, P, and Q functions. We define factorial theta polynomials, and conjecture that these represent the Schubert classes in the equivariant cohomology of non-maximal symplectic Grassmannians. We prove that the factorial theta polynomials satisfy the equivariant Chevalley formula, and that they agree with the type C double Schubert polynomials of [IMN] in some cases.

Equivariant Giambelli Formulae for Grassmannians

by

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Chapter 1

Introduction

1.1 Classical Schubert Calculus

Schubert Calculus was invented in the late nineteenth century by Hermann Schubert in order to solve various counting problems in projective geometry.

We let the Grassmannian $G(k,n)$ be the set of k-dimensional subspaces of some fixed n-dimensional complex vector space V. The goal of Schubert Calculus in this setting is to describe the intersection theory, or equivalently, the cohomology ring of the Grassmannian as a projective complex manifold.

We fix a complete flag $F_\bullet = 0 \subset F_1 \subset \cdots \subset F_n = V$ where $\dim_{\mathbb{C}}(F_j) = j$. For the Grassmannian $G(k,n)$ it is well known that the Schubert cells/varieties/classes are indexed by partitions λ whose Young diagrams are contained in a k by $n-k$ rectangle. The Schubert cell, X_λ° corresponding to a partition λ is given by the following formula:

$$X_\lambda^\circ = \{\Sigma \in G(k,n) | \dim(\Sigma \cap F_{n-k+i-\lambda_i}) = i \text{ for } i \leq k\}.$$

These cells are affine spaces of dimension $k(n-k) - |\lambda|$, where $|\lambda| = \sum_i \lambda_i$. The Schubert variety corresponding to the partition λ is the closure of this Schubert cell, and the class of this variety in the cohomology ring, denoted σ_λ, is independent of the choice of flag $F_\bullet$.

The Grassmannian described above is the standard Lie type A Grassmannian. We can similarly define symplectic/orthogonal Grassmannians for the other Lie types as follows. Let V be a complex vector space of dimension $2n$ for types C and D, or of dimension $2n+1$ for type B. Let $\langle\, , \rangle$ be an non-degenerate bilinear form on V, where for type C it is skew symmetric, and for types B and D it is symmetric. Then we define the symplectic/orthogonal Grassmannians as follows. For type C we have isotropic Grassmannians, $IG(n-k, 2n) = \{\Sigma^{n-k} \subset V | \forall v, w \in \Sigma^{n-k} \langle v, w\rangle = 0\}$, where $0 \leq k \leq n$. Where if $k = 0$ then we obtain the Lagrangian Grassmannian $LG(n, 2n)$. For type B (resp. type D) we have orthogonal Grassmannians, denoted $OG(n-k, 2n+1)$ (resp. $OG(n-k, 2n)$), which are defined in the same way.

We can similarly define Schubert cells/varieties/classes in these types by fixing a complete flag and providing incidence conditions for an isotropic subspace with respect to this flag. The indexing set for Schubert varieties will be discussed in further detail later in the thesis.

Two main problems arise in the Schubert calculus of the Grassmannian. The first problem is the Giambelli problem. In the Grassmannian the Schubert classes corresponding to partitions with one part are called special classes, and are the Chern classes of the tautological quotient vector bundle over $G(k, n)$. It is known that these special classes are generators for the cohomology ring over $\mathbb{Z}$. The Giambelli problem is to find a formula which gives any Schubert class in terms of these special classes. The answer, due to Giambelli, is well known for $G(k, n)$ and is usually given as the determinant below

$$\sigma_\lambda = det[\sigma_{\lambda_i+j-i}]_{1\leq i,j\leq k}.$$

The other main problem in classical Schubert calculus is determining what happens when we multiply an arbitrary Schubert class σ_λ by a special class σ_p. This is called the Pieri rule. We note that the Pieri rule equivalently tells one how the Schubert variety X_λ intersects the variety X_p. In all of the classical settings if one solves the Giambelli problem then the Pieri rule follows formally, and visa versa.

1.2 Equivariant Cohomology

In equivariant cohomology, we are considering both the intersection theory of some homogeneous space and the action of a torus. The general setup is as follows. Let G be a classical Lie group, fix a Borel subgroup B, let $P \supseteq B$ be a parabolic subgroup, and let $T \subset B$ be a maximal torus. There is an action of T on the homogeneous space G/P. We find a contractible space ET on which T acts freely and then form

$$ET \times^T G/P := (ET \times G/P)/[(e \cdot t, x) \sim (e, t \cdot x)].$$

This is otherwise known as the quotient stack $[T\backslash G/P]$. The equivariant cohomology $H_T^*(G/P) = H^*(ET \times^T G/P)$. Since $ET \times G/P$ is homotopy equivalent to G/P we are in effect taking the cohomology of G/P modulo the action of T.

For the Grassmannians G/P, where P is usually a maximal parabolic subgroup, the equivariant cohomology ring is similarly generated by special classes corresponding to partitions of length one. The difference is now they generate the equivariant cohomology ring of the Grassmannian over $\mathbb{Z}[t_1, \ldots t_n]$, where n is the rank of G and the t_i's are Chern classes corresponding to characters of the torus.

1.3 Equivariant Schubert Calculus

Similarly to the classical situation, equivariant Schubert calculus studies the intersection theory of the equivariant Schubert varieties. A main problem of equivariant Schubert calculus is to represent the equivariant Schubert classes by polynomials whose multiplicative structure coincides with the intersection theory of the Schubert varieties.

Equivariant Giambelli formulas for the usual type A Grassmannian have been obtained by various authors (cf. [MS], [F2], [FP], etc.). Most of these authors express a Schubert class as a Schur determinant and note that the classes can be represented by factorial Schur S-functions where the variables are Chern classes. There are also nice polynomial representatives for the equivariant Schubert classes of Grassmannians of maximal isotropic subspaces in types B,C, and D. In these cases the equivariant Schubert class is usually represented as a Pfaffian and can be recognized as factorial Schur Q- or$P$- functions. One reference which includes both formulas is chapter 3 and chapter 7 of [FP]. We will describe these polynomials in chapters 3 and 4 of the thesis. In all cases the polynomials representing the equivariant Schubert classes will coincide with the double Schubert polynomials of Lascoux and Schützenberger for type A, and the double Schubert polynomials of Ikeda, Mihalcea, and Naruse for types B, C, and D. To recognize these polynomials as representatives of the equivariant Schubert classes for the Grassmannian we consider the projection of the complete flag variety into the Grassmannian, and use the induced injection of the cohomology rings. It is well known that the factorial

Schur S functions coincide with the double Schubert polynomials of Lascoux and Schützenberger. Also Ikeda, Mihalcea, and Naruse show that their double Schubert polynomials for types B, C, and D coincide with the factorial Schur P functions for types B and D and factorial Schur Q functions for type C whenever the indexing permutation of their double Schubert polynomial corresponds to a strict partition.

In the equivariant setting the Pieri rule is significantly more complicated. Thus one first tries to describe the product of a general equivariant Schubert class σ_λ with the equivariant class of a Schubert divisor σ_1. This is known as the equivariant Chevalley Formula.

1.4 Raising Operators as a Tool for Schubert Calculus

Recently Buch, Kresch, and Tamvakis [BKT] used Young's raising operators to solve the Giambelli problem for all Grassmannians in classical Lie types. As mentioned in the first section, the solution to the Giambelli problem for the standard Lie type A Grassmannian is a determinant. In the case of the maximal isotropic Grassmannian in Lie type C, otherwise known as the Lagrangian Grassmannian, the Giambelli problem was solved using a Pfaffian by Pragacz [FP]. The raising operators give a beautiful way to interpolate between the two solutions and in the process give a solution for general isotropic Grassmannians which was not known before.

1.5 Outline of Thesis

In my thesis I aim to use Young's raising operators to give a polynomial representation for an equivariant Schubert class in the equivariant cohomology ring of a Grassmannian of any classical Lie type. In chapter 3, this is done for the type A Grassmannian $G(k,n)$, where I give a raising operator expression which coincides with the factorial Schur S functions. I show that my expression does indeed coincide with the corresponding Schur S function, and then prove the Chevalley formula for my expression independent of previous results. In chapter 4, I give expressions which coincide with the Schur Q functions for the Lagrangian Grassmannian, along with the Schur P functions for the maximal orthogonal Grassmannians. Again I prove the Chevalley formula independently. Finally, in chapter 5, I give a conjecture for a Giambelli type formula for a Schubert class in a general symplectic isotropic Grassmannian, prove the corresponding Chevalley formula, and prove that it represents a Schubert class in cases where the indexing partition is sufficiently small.

Chapter 2

Preliminaries

2.1 Schubert Cells, Varieties, and Classes

Let G be a complex reductive algebraic group. Fix a Borel subgroup B, and a maximal torus $T \subset B$. Then if we consider a flag variety G/P for a parabolic subgroup $P \supset B$, we can index the Schubert varieties of this flag variety by a particular subset of the Weyl group $W = N_G(T)/T$. The Schubert cells of this flag variety are the B-orbits of G/P, and so are indexed by W/W_P, where W_P is the subgroup of W which fixes P. Let w_0 be the element of longest length in W. Each coset vW_P has a unique element of shortest length in W. Let W^P be the set of minimal length coset representatives, and let w_0 be the element of longest length in W. Then the Schubert varieties are indexed by the set $\{w_0 v \in W : v \in W^P\}$. We denote the Schubert cell as $X_w^\circ = Bw_0wP/P$, and its closure, the Schubert variety, as X_w. The class $[X_w]$ is the Schubert class in the classical cohomology of G/P. Each Schubert cell contains exactly one torus fixed point which we will denote as $e_w \in X_w^\circ$.

In the equivariant setting we need to also consider the action of a maximal torus. Back in section 1.2 we described the quotient stack $[T\backslash G/P]$. We can similarly define $ET \times^T X_w$ this will be the equivariant Schubert variety, and its class in the cohomology ring will be its equivariant Schubert class, which is usually denoted as

σ_w^T. Since the rest of the thesis will be about equivariant Schubert classes, we will shorten the notation to σ_w for an equivariant Schubert class corresponding to a Weyl group element w.

For Weyl group elements u, v, w let c_{uv}^w be such that $\sigma_u \cdot \sigma_v = \sum_w c_{uv}^w \sigma_w$. Also the restriction of σ_w to the torus fixed point e_v will be denoted $\sigma_w\big|_v$. For a given classical Lie type, these restrictions will be stable as the rank of G increases. In other words, let n and m be the ranks of classical Lie groups G_n and G_m of a fixed Lie type with Weyl groups W_n and W_m respectively. Then if $n < m$ we have $W_n \subset W_m$ and for any $v, w \in W_n$, we have that $\sigma_w\big|_v \in H^*_{T_m}(e_v)$ is constant as m varies, where T_m is the corresponding maximal torus for G_m. These descriptions of the Schubert classes are referred to in [IMN, §1.1] as stable Schubert classes. As in [IMN] we will let H_∞ be the span of the stable Schubert classes. We will describe the restriction map in greater detail for the Grassmannians in the later chapters of the thesis.

2.2 Equivariant Cohomology Rings for Complete Flags in Types A and C

We note that in every case we can consider the projection $G/B \to G/P$ of a complete flag variety G/B onto a Grassmannian G/P. This projection induces an inclusion of the cohomology ring of the Grassmannian into the cohomology ring of the complete flag. In my thesis all of the results, including the conjecture made in chapter 5, match the descriptions of double Schubert polynomials which describe the equivariant Schubert classes for a complete flag; the reader can refer to chapter one of [FP] for the type A double Schubert polynomials, and to chapter eight of [IMN] for the double Schubert polynomials in other types. I will be using the same conventions as these sources.

2.2.1 The Presentations in Terms of Chern Classes

Let $F_\bullet = 0 = F_0 \subset F_1 \subset \cdots \subset F_r = V$ be a flag representing a point in G/B for a given classical Lie group G of rank n and a Borel subgroup B. We note that if G is type A then $r = n$, if G is type B then $r = 2n+1$ and if G is types C or D then $r = 2n$. Also if G is types B,C, or D, then F_i is isotropic for $i \leq n$ and $F_i = F_{r-i+1}^{\perp}$. Given a Lie type we can describe the tautological sub- and quotient bundles $\{S_i\}_{i=1}^r$ and $\{Q_i\}_{i=1}^r$ of the corresponding flag variety. The fiber above the point $F_\bullet$ for S_i is F_i and the fiber for Q_i is V/F_i. We can make all of these bundles equivariant by instead considering the point $ET \times^T F_\bullet$ inside of $[T\backslash G/B]$. Let $e_1, \ldots, e_r$ be the standard set of basis vectors for V, then we will also fix a flag $E_\bullet = 0 \subset \langle e_r \rangle \subset \langle e_r, e_{r-1} \rangle \subset \cdots \subset \langle e_r, \ldots, e_1 \rangle = V$. Then we can define a new set of vector bundles $\{\mathcal{T}_i\}_{i=1}^r$ where the fiber above every point of G/B for $\mathcal{T}_i$ is E_i.

For all Lie types, let $t_i = c_1(\mathcal{T}_i)$; we note that these will correspond to characters of the torus. In the following we let $\Lambda_T = \mathbb{Z}[t_1, \ldots, t_n]$.

For type A, the variable $x_i = c_1(ker(Q_i \to Q_{i-1}))$. Then for the equivariant cohomology ring we consider the projection

$$p : G/B \to G(k, n).$$

This map induces an injection of the equivariant cohomology rings so that we can realize $H_T^*(G(k,n);\mathbb{Z})$ as a subring of

$$H_T^*(G/B;\mathbb{Z}) \cong \Lambda_T[x_1,\ldots,x_n]/\langle e_i(x)-e_i(t)\rangle,$$

using the Borel presentation of the equivariant cohomology of the complete flag variety as described in [F2] and [LS]. In fact, for the Grassmannian $G(k,n)$, we will only need the variables $x_1,\ldots,x_{n-k}$ and we will denote this $(n-k)$-tuple as simply x, as in case of the ordinary cohomology when the Schubert classes are realized as Schur polynomials in [F3, 10.6]. In the equivariant case it has been shown [cf. F] that the double Schubert polynomial $\mathfrak{S}_{w_\lambda}(x,t)$ of [LS] which corresponds to an equivariant Schubert class in the Grassmannian is the factorial Schur function $s_\lambda(x|t)$, which we define in chapter 3.

For types B, C, and D, x will denote a countably infinite set of *formal* variables $x_1, x_2, \ldots$. Also in types B, C, and D we will need an additional set of variables $z_i = c_1(S_{n-i+1}/S_{n-i})$ where $1 \le i \le n$. Let $\mathcal{T} = \oplus_{i=1}^n \mathcal{T}_i^*$ where $\mathcal{T}_i^*$ is the dual bundle to $\mathcal{T}_i$ and recall that for vector bundles $\mathcal{A}$ and $\mathcal{B}$ the total Chern class of the formal difference $\mathcal{A}-\mathcal{B}$ is defined by

$$c(\mathcal{A}-\mathcal{B}) = c(\mathcal{A})/c(\mathcal{B}) = \frac{1+c_1(\mathcal{A})+c_2(\mathcal{A})+\ldots}{1+c_1(\mathcal{B})+c_2(\mathcal{B})+\ldots}$$

so that

$$\sum_{i=0}^{\infty} c_i(\mathcal{A}-\mathcal{B})u^i = \frac{\sum_{i=0}^{\infty} c_i(\mathcal{A})u^i}{\sum_{i=0}^{\infty} c_i(\mathcal{B})u^i}.$$

As in [IMN, §10] we define $\beta_i = c_i(S_n^* - \mathcal{T})$ where S_n^* is the dual bundle to the tautological subbundle S_n. We note that $\mathcal{T}$ and S_n^* have Chern roots $-t_1,\ldots,-t_n$

and $-z_1, \ldots, -z_n$ respectively so that

$$\sum_{i=0}^{\infty} \beta_i u^i = \frac{\prod_{i=1}^{n} 1 - z_i u}{\prod_{i=1}^{n} 1 - t_i u}.$$

As noted in [IMN, §1.1] we are considering a limit as $n \to \infty$, so for the stable Schubert classes of the complete flag we will have infinitely many z variables to apply the restriction map to. Recall from §2.1 that H_∞ is the span of the stable Schubert classes. In [IMN, §5.2, 6.1] the authors show that H_∞ injects into $\varprojlim H^*_{T_n}(G_n/B_n; \mathbb{Z})$ and, in type C, produce an isomorphism from H_∞ to $\Lambda_T[z_1, z_2, \ldots] \otimes_{\mathbb{Z}} \mathbb{Z}[Q_1(x), Q_2(x), \ldots]$ where $Q_i(x)$ is the i^{th} Schur Q-function which is defined in chapter 4. In [IMN, §10] the authors produce a ring homomorphism

$$\pi_n : \Lambda_T[z_1, z_2, \ldots] \otimes_{\mathbb{Z}} \mathbb{Z}[Q_1(x), Q_2(x), \ldots] \to H^*_{T_n}(G_n/B_n; \mathbb{Z})$$

such that $\pi_n(Q_i(x)) = \beta_i$ where $\pi_n(t_i) = t_i$, and $\pi_n(z_i) = z_i$ for $i \leq n$ while $\pi_n(z_i) = 0$ for $i > n$. For any Schubert class $\sigma_w \in H^*_{T_n}(G_n/B_n; \mathbb{Z})$, the authors of [IMN] define the type C double Schubert polynomial $\mathfrak{C}_w(x, z, t) \in \Lambda_T[z_1, z_2, \ldots] \otimes_{\mathbb{Z}} \mathbb{Z}[Q_1(x), Q_2(x), \ldots]$ and the type B and type D double Schubert polynomials $\mathfrak{B}_w(x, z, t), \mathfrak{D}_w(x, z, t) \in \Lambda_T[z_1, z_2, \ldots] \otimes_{\mathbb{Z}} \mathbb{Z}[P_1(x), P_2(x), \ldots]$ where $P_i(x)$ is the Schur P-function which is defined in chapter 4. These double Schubert polynomials have the property that $\pi_n(\mathfrak{S}_w(x, z, t)) = \sigma_w$ where $\mathfrak{S}_w(x, z, t) = \mathfrak{B}_w(x, z, t)$ if $G_n = \mathrm{SO}_{2n+1}(\mathbb{C})$, $\mathfrak{S}_w(x, z, t) = \mathfrak{C}_w(x, z, t)$ if $G_n = \mathrm{Sp}_{2n}(\mathbb{C})$, and $\mathfrak{S}_w(x, z, t) = \mathfrak{D}_w(x, z, t)$ if $G_n = \mathrm{SO}_{2n}(\mathbb{C})$.

Then for the equivariant cohomology ring of the isotropic Grassmannian we consider the projection $p : G_n/B_n \to IG(n-k, 2n)$. Again this induces an injection of the equivariant cohomology rings, so that we can realize $H^*_{T_n}(IG(n-k, 2n); \mathbb{Z})$

as a subring of $H^*_{T_n}(G_n/B_n;\mathbb{Z})$. The Borel presentation of the complete flag in type C [cf. FP] is given by

$$H^*_{T_n}(G_n/B_n;\mathbb{Z}) = \Lambda_{T_n}[z_1, z_2, \ldots, z_n]/\langle e_i(z_1^2, z_2^2, \ldots, z_n^2) - e_i(t_1^2, t_2^2, \ldots, t_n^2)\rangle.$$

Then for a Schubert class $\sigma_\lambda \in H^*_{T_n}(IG(n-k, 2n);\mathbb{Z})$ we have that

$$p^*(\sigma_\lambda) = \sigma_{w_\lambda} = \pi_n(\mathfrak{C}_{w_\lambda}(x, z, t)).$$

Our goal is to define a raising operator expression which will coincide with these double Schubert polynomials.

2.3 The Vanishing Theorem and Ramifications

A major ingredient in proving that a polynomial expression represents a Schubert class is a vanishing theorem for the restriction of a Schubert class to the fixed points of the torus (cf. [MS] for type A, and [IMN] for other Lie types).

Theorem 1 (Vanishing)**.** $\sigma_w\big|_v = 0$ *unless* $w \leq v$.

One notes that we expect this since we should only expect $\sigma_w\big|_v \neq 0$ if $e_v \in X_w$, which only happens if $\sigma_v \subseteq \sigma_w$ which implies $w \leq v$ in the Bruhat order.

If the vanishing theorem holds then we immediately get several nice equations.

$$c^w_{ww} = \sigma_w\big|_w \tag{2.1}$$

This equation holds since if look at $\sigma_w \cdot \sigma_w = \sum c^v_{ww}\sigma_v$ and restrict both sides to the fixed point corresponding to w we get that the only non-zero term on the right will be when $v = w$, thus canceling $\sigma_w\big|_w$ from each side we get the desired equation.

Similarly we obtain

$$c_{wv}^{v} = \sigma_w\big|_v. \tag{2.2}$$

From this we can use that our expressions given in chapters 3, 4 and 5 are homogeneous polynomials in each separate set of variables to show that we have Schubert classes.

It is well known that the equivariant Schubert classes form a basis for the equivariant cohomology ring for Grassmannians of all types, where in each type the indexing set for the Schubert classes in the Grassmannian can be recognized as a set of partitions. In my thesis I give a raising operator expression in each Lie type for a general equivariant Schubert class. In each type, my raising operator expression specializes to a known solution to the classical Giambelli formula when each t_i is set to 0 (cf. [F3] for type A, [BKT] for other classical Lie types). Therefore using that my expressions are homogeneous polynomials, which are indexed by the same set as the Schubert classes, one can show that in each case my set of expressions will form a basis for the given cohomology ring.

To show that the raising operator expression represents an equivariant Schubert class we just need to show that it is the same basis as the Schubert basis for the equivariant cohomology ring over Λ_T. Let my basis be denoted as $\{T_\lambda\}$ and let the Schubert basis be denoted $\{\sigma_\lambda\}$ which we can realize as a set of polynomials by considering the projection of the complete flag onto the Grassmannian in each case, and using the induced map on the equivariant cohomology rings to express each Schubert class as a double Schubert polynomial. Then we expand my basis in terms of the Schubert basis as follows:

$$T_\lambda = \sum_\mu a_{\lambda\mu}\sigma_\mu.$$

If we assume the vanishing theorem for both bases, then we note that if $a_{\lambda\mu} \neq 0$ then $\lambda \subseteq \mu$ since otherwise we could restrict to μ and get 0 on the left hand side and a non-zero value on the right hand side. Since both bases are homogenous polynomials, we know that $|\mu| \leq |\lambda|$. Therefore $a_{\lambda\mu} = 0$ unless $\lambda = \mu$. Hence both bases are the same.

2.4 The Chevalley Formula

For the Grassmannian the Chevalley Formula is a special case of the Pieri rule. In general the Chevalley formula on the flag variety G/B tells us how to multiply any Schubert class by the class of a divisor, denoted by σ_{s_i}, where s_i is a simple reflection which is a Weyl group element of length one. Each simple reflection corresponds to a simple root. The set of all positive roots will be the set of all linear combinations of simple roots where the coefficients on the simple roots are all non-negative.

For any positive root α set $c_{\alpha,s_i} = (w_i, \alpha^\vee)$ where $(\ ,\ )$ is the standard inner product for V, w_i is the i^{th} fundamental weight corresponding to s_i and $\alpha^\vee$ is the coroot of α.

Let R^+ be the a set of positive roots. The general equivariant Chevalley Formula (cf. [IMN, Lemma 6.8]) is

$$\sigma_w \cdot \sigma_{s_i} = \sigma_{s_i}\Big|_w \sigma_w + \sum_{\alpha \in R^+, \ell(ws_\alpha)=1+\ell(w)} c_{\alpha,s_i} \sigma_{ws_\alpha}.$$

In order to prove the Chevalley formula, one simply notes that we are only multiplying the general class by a polynomial of degree one. One knows that the classical terms will still appear (i.e. the σ_{ws_α}) since when all t_i are set to zero we have the classical expression. Hence we only need to find the coefficient of σ_w, and this was shown to be $\sigma_{s_i}\big|_w$ already using the vanishing theorem.

In each of the following chapters of the thesis we will prove using a raising operator approach that the given raising operator expression satisfies the Chevalley formula, and note that this gives strong evidence that the given expression is a solution to the Giambelli problem. The author hopes that in the future this information can be used to prove the conjecture stated in chapter 5.

Chapter 3

The Type A Equivariant Giambelli Rule

3.1 Preliminaries for Type A Grassmannians

Consider the Grassmannian $G(k,n)$ of k-planes in n-dimensional complex space $\mathbb{C}^n$ (i.e. $G(k,n) \cong GL_n(\mathbb{C})/P$, where P is the parabolic subgroup of $GL_n(\mathbb{C})$ which consists of block matrices where there are zeros in the bottom left k by $n-k$ entries and the rest of the entries can be arbitrary so long as the resulting matrix is invertible).

3.1.1 The Classical Schubert Classes and Schur S functions

In section 9.4 of [F3], Fulton describes how the Schubert classes of $G(k,n)$ are indexed by partitions λ whose Young diagram fit in a k by $(n-k)$ rectangle. As discussed in the introduction, once we fix a complete flag in type A $F_\bullet = 0 \subset F_1 \subset \cdots \subset F_n = V$ where the $\dim_{\mathbb{C}}(F_j) = j$, then the Schubert cell corresponding to λ is

$$X_\lambda^\circ = X_\lambda^\circ(F_\bullet) = \{\Sigma \in G(k,n) | \dim(\Sigma \cap F_{n-k+i-\lambda_i}) = i \text{ for } i \leq k\}.$$

Also the Schubert variety corresponding to λ is the closure of the cell, so it is

$$X_\lambda = X_\lambda(F_\bullet) = \{\Sigma \in G(k,n) | \dim(\Sigma \cap F_{n-k+i-\lambda_i}) \geq i \text{ for } i \leq k\}.$$

Also in 9.4 of [F3], Fulton describes how one can obtain a cohomology class σ_λ from the above variety using Poincaré duality. The Schubert class, σ_λ, does not depend on the choice of flag $F_\bullet$.

Fulton also describes in [F3] how one can represent the Schubert class σ_λ as the Schur S polynomial $s_\lambda(x)$. In Chapter 6 of [F3], Fulton gives the Jacobi-Trudy identity for the Schur S functions below.

$$s_\lambda(x) = det[e_{\lambda_i+j-i}(x)]_{1\leq i,j\leq k} = det[h_{\lambda_i'+j-i}(x)]_{1\leq i,j\leq n-k}$$

where $e_r(x)$ and $h_r(x)$ are the elementary and complete symmetric polynomials and $x = (x_1, \ldots x_{n-k})$ are the variables described in §2.2.1. For an infinite list of variables $a = (a_1, a_2, \ldots)$ the generating functions for these polynomials are

$$E(a,u) = \prod_{i=1}^{\infty}(1+a_iu) = \sum_{i=0}^{\infty} e_i(a)u^i \qquad H(a,u) = \prod_{i=1}^{\infty}(1-a_it)^{-1} = \sum_{i=0}^{\infty} h_i(a)u^i$$

We note that if x is a finite set of k variables then $e_i(x) = 0$ for $i > k$. The elementary and complete symmetric polynomials also have the following very nice property which we will use later.

Let $a = (a_1, a_2, \ldots)$ and define $a^{(j)}$ to be $(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots)$ so that a_j is removed. Then

$$e_i(a) = e_i(a^{(j)}) + a_je_{i-1}(a^{(j)}) \text{ and } h_i(a) = h_i(a^{(j)}) + a_jh_{i-1}(a). \qquad (3.1)$$

3.1.2 Torus Fixed Points and the Equivariant Cohomology

Let

$$E_\bullet = 0 \subset \langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \cdots \subset \langle e_1, \ldots, e_n\rangle$$

be the complete flag given by the standard basis vectors. Let T be the usual maximal torus of diagonal matrices. Then we note that a subspace Σ of dimension k will be fixed by the action of T only when exactly k standard basis vectors give a basis for the subspace. In other words we have $\Sigma = \langle e_{i_1}, \ldots, e_{i_k} \rangle$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. One notices that each Schubert cell in $G(k, n)$ will contain exactly one torus fixed point. For a partition λ whose Young diagram is contained in the k by $n-k$ rectangle, the torus fixed point is

$$e_\lambda = \langle e_{n-k-\lambda_1+1}, e_{n-k-\lambda_2+2}, \ldots, e_{n-\lambda_k} \rangle \in X^\circ_\lambda(E_\bullet).$$

Let $H^*_T(G(k, n); \mathbb{Z})$ be the equivariant cohomology of this Grassmannian with respect to the maximal torus $T \cong (\mathbb{C}^*)^n$ of diagonal matrices in $GL_n(\mathbb{C})$. It is well known that the equivariant Schubert classes form a basis of this ring over $\Lambda_T = \mathbb{Z}[t_1, \ldots, t_n]$ where the t_i's are first equivariant Chern classes of line bundles over the Grassmannian (namely t_i is the first Chern class of the line bundle whose fiber is the complex line generated by the ith standard basis vector as described in §2.2.1). To get a presentation of the ring in terms of the variables described we consider the projection $p : Gl_n(\mathbb{C})/B \to G(k, n)$. As described in §2.2.1, this map induces an injection of the equivariant cohomology rings so that we can realize $H^*_T(G(k, n); \mathbb{Z})$ as a subring of

$$H^*_T(Gl_n(\mathbb{C})/B; \mathbb{Z}) \cong \Lambda_T[x_1, \ldots, x_n]/\langle e_i(x) - e_i(t) \rangle_{1 \leq i \leq n} \tag{3.2}$$

using the Borel presentation of the equivariant cohomology of the complete flag variety. The goal of this section is to use raising operators to express the equivariant Schubert classes as a polynomial in the presentation (3.2).

There is a known representation of Schubert classes as polynomials in these variables given by factorial Schur S-polynomials. The factorial Schur S-polynomial $s_\lambda(x|t)$ is defined as one of 2 expressions in Equations (3.3) and (3.4) below:

$$s_\lambda(x|t) = \frac{det[(x_j|t)^{n-k-i+\lambda'_i}]_{1\leq i,j\leq n-k}}{\prod_{i<j}(x_i - x_j)} \tag{3.3}$$

where $(x_j|t)^m = \prod_{r=1}^{m}(x_j - t_r)$ and λ' is the conjugate partition to λ (i.e. λ' has λ's rows as it's columns and λ's columns as it's rows); or

$$s_\lambda(x|t) = det[e^{n-k+i-\lambda_i}_{\lambda_i+j-i}]_{1\leq i,j\leq k} \tag{3.4}$$

where

$$e^q_p = e_p(x_1,\ldots,x_{n-k}|t_1,\ldots,t_q) = \sum_{i=0}^{p}(-1)^i e_{p-i}(x)h_i(t).$$

We note that the usual definition of $s_\lambda(x|t)$ has the conjugate (or transpose) of the partitions indicated in these equations. We use the transpose so that we can apply these results directly for the classes indexed by "small" partitions in Chapter 5. We note that the two definitions above where shown to be equivalent in [M, I.3]; see also [MS] and [FP,F].

3.1.3 The Type A Grassmann Permutation for a Partition λ

Let λ be a partition whose Young diagram fits in a k by l box where $l = n-k$. We recall that all such partitions will index the Schubert classes in the Grassmannian $G(k,n)$. As in section 7 of [F] we denote the set of jumping numbers for λ to be $I(\lambda) = l+1-\lambda_1, l+2-\lambda_2, \ldots, l+k-\lambda_k$. Also we similarly set $J(\lambda)$ to be the complement of $I(\lambda)$ in the set $\{1, \dots, n\}$.

The Weyl group element corresponding to λ is w_λ where $w_\lambda(i) = \lambda'_{l-i+1} + i$ for $i \le l$ where λ' is the conjugate partition for λ and is a strictly increasing sequence for the remaining $l+1 \le i \le n$.

Notice that w_λ has the property that $w_\lambda(i) < w_\lambda(i+1)$ for all $i \ne l$.

Proposition 1. *We have* $\{w_\lambda(1), \dots, w_\lambda(l)\} = J(\lambda)$.

Proof. Assume for a contradiction that $w_\lambda(l-i+1) \in I(\lambda)$ for some $i \le l$. Then there exists a $j \le k$ such that $l + j - \lambda_j = \lambda'_i + l - i + 1$. If this were true then we would have

$$0 = \lambda_j - j + \lambda'_i - i + 1. \tag{3.5}$$

Consider cases.

1. If there is a box in the ij^{th} place of the Young diagram of λ then we know that $\lambda_j \ge i$ and $\lambda'_i \ge j$. Therefore the right hand side of Equation (3.5) is positive and we have a contradiction.

2. If there is not a box in the ij^{th} place of the Young diagram of λ then we know that $\lambda_j < i$ and $\lambda'_i < j$. Therefore the right hand side of Equation (3.5) is negative and we have a contradiction.

Hence $\{w_\lambda(1), \dots, w_\lambda(l)\} = J(\lambda)$ as desired. □

3.1.4 Restriction to Torus Fixed Points in Type A

We have shown that the fixed points of the torus action on $G(k,n)$ will correspond to partitions, and we call the permutation w_λ corresponding to the partition λ a Grassmann permutation. In order to get the restriction map we look at the inclusion $\iota : pt \hookrightarrow G(k,n)$, where pt denotes a torus fixed point. This induces a ring homomorphism on the equivariant cohomology rings $\iota^* : H_T^*(G(k,n)) \to H_T^*(pt) = \mathbb{Z}[t_1, \ldots, t_n]$. This homomorphism takes the equivariant Chern class x_i to the equivariant Chern class $t_{w_\lambda(i)}$ if the point we are restricting to corresponds to λ.

So when restricting to a fixed point e_λ we have the following:

$$x_i \to t_{w_\lambda(i)}$$

and as l-tuples, where $t_{J(\lambda)} = \{t_j\}_{j \in J(\lambda)}$, we have

$$x \to t_{J(\lambda)}.$$

When we restrict a Schubert class σ_λ to the fixed point e_μ we will denote the image as $\sigma_\lambda\big|_\mu$. In particular the above gives us that

$$\sigma_1\big|_\lambda = \sum_{j \in J(\lambda)} t_j - \sum_{i=1}^{l} t_i \tag{3.6}$$

since in $G(k,n)$, σ_1 is represented by the polynomial $\sum_{i=1}^{l}(x_i - t_i)$. We will use Equation (3.6) in §3.4.

3.2 The Raising Operator Expression

The raising operator, R_{ij} for $i < j$, as defined by A. Young, acts on a tuple of integers by raising the i^{th} entry by 1, and lowering the j^{th} entry by 1. So for example $R_{2,4}(5,4,3,2,1) = (5,5,3,1,1)$. For integer sequences λ and γ, we define the monomial $m_{\lambda;\gamma} = \prod_{i=1}^{\ell} e_{\lambda_i}^{\gamma_i}$ where ℓ is the length of the sequence λ (i.e. $\ell = max\{i : \lambda_i \neq 0\}$). In this monomial we agree that if any number in either λ or γ is negative then $m_{\lambda;\gamma} = 0$. We define $R_{ij} m_{\lambda;\gamma} = m_{R_{ij}(\lambda);\gamma}$. From here on out all of our integer sequences will have finite length, however we may still regard the integer sequence as an infinite sequence with only a finite number of nonzero entries.

Let

$$T_{\lambda;\gamma} = \prod_{1 \leq i < j \leq \ell} (1 - R_{ij}) m_{\lambda;\gamma} \tag{3.7}$$

For a partition λ we define $\gamma(\lambda)$ to be the integer sequence such that $\gamma(\lambda)_i = n - k + i - \lambda_i$ and define $T_\lambda = T_{\lambda;\gamma(\lambda)}$.

Theorem 2. *The equivariant Schubert class σ_λ is represented by T_λ.*

We will prove the above theorem by showing that the raising operator expression (3.7) satisfies the vanishing theorem.

3.3 The Vanishing Theorem

We show below that our raising expression is equivalent to the expression given in (3.4).

Let $w \in S_n$ be a permutation. Let $I_w = \{(i,j) : i < j, w(i) > w(j)\}$ be the set of inversions of w. Define

$$R^w := \prod_{(i,j)\in I_w} R_{ij}.$$

Then we claim that the following raising operators are equivalent.

$$\prod_{1\leq i<j\leq n}(1-R_{ij}) = \sum_{w\in S_n}(-1)^{\ell(w)}R_w. \tag{3.8}$$

Note that we can think of R_{ij} as acting on a polynomial in the variables $x_1,\ldots,x_n$ by raising the power of the x_i by one and lowering the power of x_j by one. In this setting the raising operator R_{ij} is simply multiplication by x_i/x_j. Let

$$A = \begin{pmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{pmatrix}.$$

Then

$$det(A) = \sum_{w\in S_n}(-1)^{\ell(w)}x_1^{w(1)-1}\cdots x_n^{w(n)-1} = \left(\prod_{j=1}^{n} x_i^{i-1}\right)\sum_{w\in S_n}(-1)^{\ell(w)}x_1^{w(1)-1}\cdots x_n^{w(n)-n}$$

and from Vandermonde we also have

$$det(A) = \prod_{1\leq i<j}(x_j-x_i) = \left(\prod_{j=1}^{n} x_i^{i-1}\right)\prod_{1\leq i<j}(1-\frac{x_i}{x_j}).$$

Then we simply note that $\prod_{i=1}^n x_i^{w(i)-i} = \prod_{(i,j)\in I_w}\frac{x_i}{x_j}$ which can be proven by induction on $\ell(w)$. So Equation (3.8) amounts to two different calculations of the Vandermonde determinant.

Theorem 3. $T_\lambda = s_\lambda(x|t)$

Proof.

$$\begin{aligned}
T_\lambda &= \prod_{1\le i<j\le \ell}(1-R_{ij})m_{\lambda;\gamma(\lambda)} \\
&= \sum_{w\in S_\ell}(-1)^{l(w)}R_w\prod_{i=1}^{\ell}e_{\lambda_i}^{n-k+i-\lambda_i} \\
&= \sum_{w\in S_\ell}(-1)^{l(w)}\prod_{i=1}^{\ell}e_{\lambda_i+(w(i)-i)}^{n-k+i-\lambda_i} \\
&= det[e_{\lambda_i+(j-i)}^{n-k+i-\lambda_i}]_{i,j\le\ell} \\
&= s_\lambda(x|t) \text{ by Equation (3.4)}
\end{aligned}$$

□

Below we include a proof of the vanishing theorem similar to the one for Theorem 2.1 of [MS] for completeness.

Theorem 4. *[MS] $s_\lambda(t_\mu|t) = 0$ unless $\lambda \subseteq \mu$, where t_μ is the image of the x variables under the localization map $\iota_\mu^* : H_T^*(G(k,n)) \to H_T^*(e_\mu)$ where $\iota_\mu^*(x_i) = t_{w_\mu(i)}$.*

Proof. In this setting we will use Equation (3.3), so recall

$$s_\lambda(x|t) = \frac{det[(x_j|t)^{n-k-i+\lambda_i'}]_{1\le i,j\le n-k}}{\prod_{i<j}(x_i-x_j)}.$$

Then the numerator of the expression when restricted to μ will be $det[(t_{w_\mu(j)}|t)^{n-k+i-\lambda_i}]_{1\le i,j\le n-k}$. If λ is not contained in μ this means that there exists an index $r\le k$ such that $\lambda_r > \mu_r$. Similarly there is a part in their conjugate partitions $s \le n-k$ such that $\lambda_s' > \mu_s'$. Then $x_{n-k-s+1}$ is mapped to $t_{\mu_s'+n-k-s+1}$. Then since $\lambda_s' > \mu_s'$ we have that $\lambda_s' + n-k-s \ge \mu_s'+n-k-s+1$. For $i\le s\le j$

we have the following inequality:

$$1 \leq \mu'_j + n - k - j + 1 \leq \mu'_s + n - k - s + 1 \leq \lambda'_s + n - k - s \leq \lambda'_i + n - k - i.$$

Note that the matrix given by $A = [(t_{w_\mu(j)}|t)^{n-k+i-\lambda_i}]_{1 \leq i,j \leq n-k} = [a_{ij}]$ will have the property that

$$a_{ij} = (t_{\mu'_j+n-k-j+1} - t_1) \cdots (t_{\mu'_j+n-k-j+1} - t_{\lambda'_i+n-k-i})$$

so $a_{ij} = 0$ for all $i \leq s \leq j$. This gives us a block of zeros which implies that $det(A) = 0$, and thus proves the theorem. □

Combining Theorems 3 and 4 we have the following Corollary.

Corollary 1. *T_λ satisfies the vanishing theorem so that if $T_\lambda\big|_\mu \neq 0$ then $\lambda \subset \mu$.*

Proof. This is a consequence of the above theorems and Macdonald's proof of the equivalence of Equations (3.4) and (3.3). □

3.4 The Chevalley Formula

We will show that our raising operator expression when multiplied by σ_1 has the same outcome as when we multiply the corresponding equivariant Schubert class by σ_1. We note that this is true as a consequence of §3.3, but we will give an independent raising operator proof.

Recall that the equivariant Chevalley Formula for $G(k, n)$ is

$$\sigma_\lambda \cdot \sigma_1 = \sigma_1\big|_\lambda \sigma_\lambda + \sum_{\lambda^+} \sigma_{\lambda^+}$$

where $\sigma_1\big|_\lambda$ is the restriction of the divisor class to a fixed point of the torus, and where λ^+ is a partition containing λ such that $|\lambda^+| = |\lambda| + 1$.

To prove the analogue of this statement for our T_λ expressions we will use a series of lemmas. The first lemma gives a relation between two different T expressions.

Lemma 1. *For any integer sequences λ, μ, and γ, and integers r and s we have:*

$$T_{(\lambda_1,\ldots,\lambda_{j-1},r,s,\mu);\gamma} = -T_{(\lambda_1,\ldots,\lambda_{j-1},s-1,r+1,\mu);s_j\gamma}.$$

where s_j is the transposition which switches the j^{th} and $(j+1)^{th}$ entries.

Proof. We prove this statement via an induction argument similar to that of Buch, Kresch and Tamvakis in [BKT, Lemma 1.1] for isotropic Grassmannians.

Note that

$$T_{(\lambda_1,\ldots,\lambda_{j-1},r);\gamma} = \sum_{\alpha\in\{0,1\}^{j-1}} (-1)^{|\alpha|} T_{\lambda+\alpha;\gamma} T_{r-|\alpha|;\gamma_j} \tag{3.9}$$

where $T_{r-|\alpha|;\gamma_j} = e^{\gamma_j}_{r-|\alpha|}$

Using equation 3.9 once we prove the statement is true for $\mu = \emptyset$ we will have our result by induction on $\ell(\mu)$. For the base case we assume that the μ in our lemma is empty and prove the statement $T_{(\lambda_1,\ldots,\lambda_{j-1},r,s);\gamma} = -T_{(\lambda_1,\ldots,\lambda_{j-1},s-1,r+1);s_j\gamma}$. Here we apply the above recursion twice to get

$$T_{(\lambda_1,\ldots,\lambda_{j-1},r,s);\gamma} = \sum_{\alpha,\beta\in\mathbb{Z}_{\{0,1\}^{j-1}}} (-1)^{|\alpha|+|\beta|} T_{\lambda+\alpha+\beta;\gamma}(e^{\gamma_j}_{r-|\alpha|}e^{\gamma_{j+1}}_{s-|\beta|} - e^{\gamma_j}_{r+1-|\alpha|}e^{\gamma_{j+1}}_{s-1-|\beta|}).$$

Similarly we get

$$T_{(\lambda_1,\ldots,\lambda_{j-1},s-1,r+1);s_j\gamma} = \sum_{\alpha,\beta\in\mathbb{Z}_{\{0,1\}^{j-1}}} (-1)^{|\alpha|+|\beta|} T_{\lambda+\alpha+\beta;\gamma}(e^{\gamma_{j+1}}_{s-1-|\beta|}e^{\gamma_j}_{r+1-|\alpha|} - e^{\gamma_{j+1}}_{s-|\beta|}e^{\gamma_j}_{r-|\alpha|}).$$

Hence $T_{(\lambda_1,\ldots,\lambda_{j-1},r,s);\gamma} = -T_{(\lambda_1,\ldots,\lambda_{j-1},s-1,r+1);s_j\gamma}$ and so the base case is proven. Then we use Equation (3.9) to perform the inductive step and we have proven the lemma.

$\square$

Corollary 2. *Let λ be a partition such that $\lambda = (\mu, r, r, \nu)$. Also let γ be such that*

$$\gamma_i = \begin{cases} \gamma(\lambda)_i & \text{if } i \neq l(\mu)+1 \\ \gamma(\lambda)_i + 1 & \text{if } i = l(\mu)+1 \end{cases}$$

Then $T_{(\mu,r+1,r,\nu);\gamma} = 0$.

Proof. Note that $\gamma(\lambda)_{l(\mu)+1} = n-k-r+l(\mu)+1$ and $\gamma(\lambda)_{l(\mu)+2} = n-k-r+l(\mu)+2$. Hence $\gamma = s_{l(\mu)+1}\gamma$ and by Lemma 1 we get the expression $T_{(\mu,r+1,r,\nu);\gamma} = -T_{(\mu,r+1,r,\nu);\gamma}$ so $T_{(\mu,r+1,r,\nu);\gamma} = 0$. $\square$

The next lemma is a general relation between factorial elementary symmetric polynomials which is necessary later in showing that the T_λ's produce the correct structure constants.

Lemma 2. $e^{r+1}_{p+1} = e^r_{p+1} - t_{r+1} \cdot e^{r+1}_p$

Proof.

$$\begin{aligned} e^{r+1}_{p+1} &= \sum_{i=0}^{p} (-1)^i e_{p+1-i}(x) h_i(t_1,\ldots,t_{r+1}) \\ &= \sum_{i=0}^{p} (-1)^i e_{p+1-i}(x)(h_i(t_1,\ldots,t_r) + t_{r+1}h_{i-1}(t_1,\ldots,t_{r+1})) \\ &= e^r_{p+1} - e^{r+1}_p(t_{r+1}) \end{aligned}$$

via a shifting of indices.

$\square$

The next lemma gives us the Chevalley formula for multiplying a T_λ by T_1. We note that

$$(e_1(x) - h_1(t))\big|_\lambda = \sum_{j \in J(\lambda)} t_j - \sum_{i=1}^{n-k} t_i = \sigma_1\big|_\lambda$$

so that the expression below does indeed coincide with the Chevalley formula discussed in §3.1.4 Equation (3.6).

We recall that the J-set, $J(\lambda)$, for a partition λ is the set of numbers less than or equal to n which are not jumping numbers for the partition (i.e. not equal to $n-k+i-\lambda_i$ for some i).

Lemma 3.

$$T_\lambda \cdot T_1 = \sum_{\lambda^+} T_{\lambda^+} + \left(\sum_{j \in J(\lambda)} t_j - \sum_{i=1}^{n-k} t_i \right) T_\lambda.$$

where λ^+ is such that $|\lambda^+| = \lambda + 1$ and $\lambda \subset \lambda^+$.

Proof. Note that $T_1 = e_1^{n-k} = e_1^{n-k+r} + \sum_{i=1}^{r} t_{n-k+i}$, and λ^+ must be a partition of length at most one more than the length of λ. We enable raising operator expressions of length $\ell+1$ by multiplying the expression by $\prod_{i=1}^{\ell} \frac{1-Ri,\ell+1}{1-Ri,\ell+1}$. We note that since there is only one box added to the diagram of λ, so $(1-R_{i,\ell+1})^{-1} = 1 + R_{i,\ell+1} + R_{i,\ell+1}^2 + R_{i,\ell+1}^3 \dots$ will act the same as simply $(1+R_{i,\ell+1})$ since $R_{i,\ell+1}^r$ will act on the monomial and result in an expression which is zero. Therefore

$$
\begin{aligned}
&T_\lambda \cdot T_1 \\
&= \prod_{1\le i<j\le \ell} (1-R_{ij}) m_{\lambda;\gamma(\lambda)} \cdot e_1^{n-k} \\
&= \prod_{1\le i<j\le \ell} (1-R_{ij}) \left(1+\sum_{r=1}^{\ell} R_{r,\ell+1}\right) \left(\prod_{i=1}^{\ell} e_{\lambda_i}^{\gamma(\lambda)_i}\right) \cdot e_1^{n-k+1+\ell-1} + \left(\sum_{i=1}^{\ell} t_{n-k+i}\right) T_{\lambda;\gamma(\lambda)} \\
&= \prod_{1\le i<j\le \ell+1} (1-R_{ij}) \left(m_{\lambda,1;\gamma(\lambda,1)} + \sum_{r=1}^{\ell} \left(\prod_{i=1}^{r-1} e_{\lambda_i}^{n-k-\lambda_i+i} \cdot e_{\lambda_r+1}^{n-k-\lambda_r+r} \cdot \prod_{i=r+1}^{\ell} e_{\lambda_i}^{n-k-\lambda_i+i}\right)\right) \\
&+ \left(\sum_{i=1}^{\ell} t_{n-k+i}\right) T_{\lambda;\gamma(\lambda)}
\end{aligned}
$$

by the previous lemma $e_{\lambda_r+1}^{n-k-\lambda_r+r} = e_{\lambda_r+1}^{n-k-\lambda_r-1+r} + e_{\lambda_r}^{n-k-\lambda_r+r}(-t_{n-k+r-\lambda_r})$ so the above

$$
\begin{aligned}
&= \prod_{1\le i<j\le \ell+1} (1-R_{ij}) \left(m_{\lambda,1;\rho} + \sum_{r=1}^{\ell} \left(\prod_{i=1}^{r-1} e_{\lambda_i}^{\gamma(\lambda)_i} \cdot e_{\lambda_r+1}^{\gamma(\lambda)-1} \cdot \prod_{i=r+1}^{\ell} e_{\lambda_i}^{\gamma(\lambda)_i}\right)\right) \\
&+ \left(\sum_{i=1}^{\ell} (-t_{n-k+i-\lambda_i} + t_{n-k+i})\right) T_{\lambda;\gamma(\lambda)} \\
&= \sum_{\nu} T_{\nu;\gamma(\nu)} + \left(\sum_{i=1}^{\ell} (t_{n-k+i} - t_{n-k+i-\lambda_i})\right) T_\lambda
\end{aligned}
$$

where ν is obtained by adding one box to any row of λ

$$
= \sum_{\lambda^+} T_{\lambda^+} + \left(\sum_{j\in J(\lambda)} t_j - \sum_{i=1}^{n-k} t_i\right) T_\lambda
$$

since Lemma 1 implies that $T_\nu = 0$ for any ν which is not a partition and since $\{n-k+1,\ldots,n\} \smallsetminus I(\lambda) = I(\lambda)^c \smallsetminus \{1,\ldots,n-k\} = J(\lambda) \smallsetminus \{1,\ldots,n-k\}$.

$\square$

Hence we have shown the given raising operator expression satisfies the Chevalley formula independent of the previous results for factorial Schur S functions. In the following chapters we will look at the other Lie types.

Chapter 4

The Giambelli Formula for the Maximal Isotropic Grassmannians

4.1 Preliminaries for the Lagrangian Grassmannian

Let V be a $2n$-dimensional complex vector space and $\langle\, ,\, \rangle$ be a skew-symmetric bilinear form on V. Consider the Lagrangian Grassmannian

$$LG(n, 2n) = \{\Sigma \subset V |\, \dim(\Sigma) = n, \forall v, w \in \Sigma\, \langle v, w\rangle = 0\}.$$

We call a subspace W of V isotropic if $\forall v, w \in W, \langle v, w\rangle = 0$.

Notice that $LG(n, 2n)$ is the set of maximal isotropic subspaces of V.

Let $F_\bullet = 0 \subset F_1 \subset \cdots \subset F_{2n} = V$ be a type C complete flag so that for $i \leq n, F_i$ is an isotropic subspace of V, and $F_{2n-i} = F_i^{\perp}$. Then Schubert cells and varieties are defined similarly to the type A case. The Schubert varieties for LG are indexed by strict partitions λ with $\lambda_1 \leq n$, where a partition is called strict if $\lambda_i > \lambda_{i+1}$ for all i such that $\lambda_i > 0$. For such a strict partition λ the Schubert variety corresponding to λ is given by

$$X_\lambda = X_\lambda(F_\bullet) = \{\Sigma \in LG(n, 2n) |\, \dim(\Sigma \cap F_{n+1-\lambda_j}) \geq j \text{ for } j \leq l(\lambda)\}.$$

The corresponding Schubert cell is

$$X_\lambda^\circ = \{\Sigma \in LG(n, 2n) |\, \dim(\Sigma \cap F_{n+1-\lambda_j}) = j \text{ for } j \leq l(\lambda)\}.$$

Like the type A case, from any Schubert variety we can obtain a class in the cohomology ring which does not depend on the choice of flag.

For $LG(n, 2n)$ we similarly have that the torus fixed points are n dimensional subspaces generated by exactly n basis vectors. We recall that for a vector space V^{2n} the standard basis vectors are $e_1, \dots, e_{2n}$ with the standard antidiagonal symplectic form $\langle\, ,\, \rangle$ where $\langle e_i, e_{2n-i+1}\rangle = 1$ and $\langle e_i, e_j\rangle = 0$ for all $j \neq 2n - i + 1$. Let T be the maximal torus with respect to this basis. Let $E_\bullet = 0 \subset \langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \cdots \subset \langle e_1, \dots, e_{2n}\rangle$. Similarly, any Schubert cell will contain exactly one torus fixed point. For any strict partition λ of length $\ell \leq n$, the torus fixed point is

$$e_\lambda = \langle e_{n+1-\lambda_1}, \dots, e_{n+1-\lambda_\ell}, e_{n+i_1}, \dots e_{n+i_{n-\ell}}\rangle \in X_\lambda^\circ(E_\bullet)$$

where $i_1 < i_2 < \cdots < i_{n-\ell}$ is such that $\lambda_j \neq i_r$ for any j, r.

It is well known that the classical Schubert classes on $LG(n, 2n)$ can be represented by Schur Q functions. Let $x = (x_1, x_2, \dots)$. Then the functions $q_i(x)$ are defined by the generating function:

$$\prod_{i=1}^{\infty} \frac{1 - x_i u}{1 + x_i u} = \sum_{i=0}^{\infty} q_i(x) u^i.$$

For a pair of positive integers r, s, we set

$$Q_{r,s}(x) = q_r(x)q_s(x) + 2\sum_{i\geq 1}(-1)^i q_{r+i}(x)q_{s-i}(x) =: \frac{1 - R_{1,2}}{1 + R_{1,2}} q_r(x)q_s(x).$$

For any strict partition λ of we can regard it has having even length ℓ by setting the last part of λ be zero if the actual length of λ is odd, and leaving it unchanged otherwise. Let

$$Q_\lambda(x) = Pf[Q_{\lambda_i,\lambda_j}]_{i<j\leq\ell}.$$

It is well known that

$$Q_\lambda(x) = \prod_{1\leq i<j\leq n} \frac{1 - R_{ij}}{1 + R_{ij}} m_\lambda$$

where $m_\lambda = \prod_i q_{\lambda_i}(x)$. In the above the operator $\prod_{1\le i<j\le n} \frac{1-R_{ij}}{1+R_{ij}}$ acts on a monomial m_λ just as in Chapter 3 where we expand each factor $\frac{1-R_{ij}}{1+R_{ij}} := 1 - 2R_{ij} + 2R_{ij}^2 - 2R_{ij}^3 + \dots$ as a formal sum.

4.2 The Raising Operator Expression

We define the factorial Schur Q-functions in terms of the usual Schur Q functions and elementary symmetric polynomials as follows in a similar manner as in section 4 of [IMN]. For a positive integer k, the one part factorial Schur Q function is

$$Q_k^r(x|t) := \sum_{i=0}^{k-1} (-1)^i Q_{k-i}(x_1, x_2, \dots) e_i(t_1, \dots, t_r).$$

Then the 2 part factorial Schur Q function is

$$Q_{k,l}(x|t) := Q_k^{k-1}(x|t) Q_l^{l-1}(x|t) + 2\sum_{i=1}^{\infty} (-1)^i Q_{k+i}^{k-1}(x|t) Q_{l-i}^{l-1}(x|t).$$

Finally the factorial Schur Q function for a strict partition λ of even length ℓ is defined as

$$Q_\lambda(x|t) = Pf(Q_{\lambda_i,\lambda_j}(x|t))_{i<j\le\ell}. \tag{4.1}$$

Proposition 2. *[IMN, Prop. 4.2] For a strict partition λ, $Q_\lambda(x|t)$ satisfies the vanishing theorem.*

We set the following notation:

$$\begin{aligned} Q_k^r[j] &:= \sum_{i=0}^{k-1} (-1)^i Q_{k+j-i}(x_1, x_2 \dots) e_i(t_1, \dots, t_r) \\ &= 2\sum_{i=1}^{\infty} x_i^{k-r+j}(x_i - t_1)\cdots(x_i - t_r). \end{aligned}$$

For integer sequences λ, γ, ρ where the length of λ is ℓ we let $m_{\lambda,\gamma,\rho} = \prod_{i=1}^{\ell} Q_{\lambda_i}^{\gamma_i}[\rho_i]$. Then a raising operator R acts on $m_{\lambda,\gamma,\rho}$ by acting on only ρ. We notice

$$Q_{k,l}(x|t) = \frac{1-R_{1,2}}{1+R_{1,2}} Q_k^{k-1}[0] Q_l^{l-1}[0]. \tag{4.2}$$

We define

$$T_{\lambda;\gamma;\rho} := \prod_{1 \leq i < j \leq \ell} \frac{1-R_{ij}}{1+R_{ij}} m_{\lambda,\gamma,\rho}.$$

In $LG(n, 2n)$ we define $\gamma(\lambda)$ such that $\gamma(\lambda)_i = \lambda_i - 1$. For the remainder of the thesis when we call an integer sequence $\mathbf{0}$ we mean the infinite sequence of zeros.

Define $T_\lambda = T_{\lambda;\gamma(\lambda);\mathbf{0}}$. We wish to show that T_λ is the same as the factorial Schur-Q polynomials described in section 4 of [IMN]. We note that since $\gamma(\lambda)_i = \lambda_i - 1$ we have that $Q_{\lambda_i}^{\gamma(\lambda)_i}[j] = Q_{\lambda_i+j}^{\lambda_i-1}[0]$ since for $j < 0$ this is true for any γ and for $j > 0$ we will have that $e_r(t_1, \ldots, t_{\lambda_i-1}) = 0$ for $r > \lambda_i - 1$ so that

$$\begin{aligned}
Q_{\lambda_i}^{\lambda_i-1}[j] &= \sum_{i=1}^{\lambda_i-1} (-1)^i Q_{\lambda_i+j-i+1}(x_1, x_2 \ldots) e_i(t_1, \ldots, t_{\lambda_i-1}) \\
&= \sum_{i=1}^{\lambda_i+j-1} (-1)^i Q_{\lambda_i+j-i+1}(x_1, x_2 \ldots) e_i(t_1, \ldots, t_{\lambda_i-1}) \\
&= Q_{\lambda_i+j}^{\lambda_i-1}[0].
\end{aligned}$$

Hence for any shift in the index ρ is equivalent to the same shift in the index λ. From this we can assume that our raising operator is acting on the sequence λ rather than the sequence ρ in the special case of T_λ.

Note that

$$\prod_{1 \leq i < j \leq \ell} \left(\frac{1-R_{ij}}{1+R_{ij}} \right) = Pf \left(\frac{1-R_{ij}}{1+R_{ij}} \right)_{i<j\leq \ell}$$

due to an identity of Schur.

We have

$$
\begin{aligned}
T_\lambda &= \prod_{1\le i<j\le \ell}\left(\frac{1-R_{ij}}{1+R_{ij}}\right)\prod_{k=1}^{\ell} Q_{\lambda_k}^{\lambda_k-1}\\
&= Pf\left(\frac{1-R_{ij}}{1+R_{ij}}\right)\prod_{k=1}^{\ell} Q_{\lambda_k}^{\lambda_k-1}\\
&= \frac{1}{2^{\ell/2}(\ell/2)!}\sum_{\sigma\in S_\ell}(-1)^{l(\sigma)}\prod_{i=1}^{\ell/2}\left(\frac{1-R_{\sigma(2i-1),\sigma(2i)}}{1+R_{\sigma(2i-1),\sigma(2i)}}\right)\prod_{k=1}^{\ell} Q_{\lambda_k}^{\lambda_k-1}\\
&= \frac{1}{2^{\ell/2}(\ell/2)!}\sum_{\sigma\in S_\ell}(-1)^{l(\sigma)}\prod_{i=1}^{\ell/2} Q_{\lambda_{\sigma(2i-1)},\lambda_{\sigma(2i)}}\\
&= Pf(Q_{\lambda_i,\lambda_j})_{i<j\le\ell}\\
&= Q_\lambda
\end{aligned}
$$

From this and Proposition 2 we have that T_λ will satisfy the same vanishing theorem as in [IMN] and §2.3, and thus represents the equivariant Schubert class σ_λ.

4.3 The Chevalley Formula

The equivariant Chevalley formula for $LG(n,2n)$ states

$$
\sigma_\lambda\cdot\sigma_1 = \sigma_{\lambda,1} + 2\sum_{\lambda^+}\sigma_{\lambda^+} + 2\sum_{i=1}^{\ell} t_{\lambda_i}\sigma_\lambda
$$

where again λ^+ is a strict partition obtained from adding a box to λ and $\sigma_1\big|_\lambda = \sum_i 2t_{\lambda_i}$ as in [I, §4.5]. We show that the T_λ satisfy the equivariant Chevalley formula for $LG(n,2n)$ independently. First we will prove a lemma similar to Lemma 1.

Lemma 4. *For any integer sequences λ, μ, and γ, where the length of λ is $j-1$ and any integers r and s we have that $T_{(\lambda_1,\ldots,\lambda_{j-1},r,s,\mu);\gamma;\mathbf{0}} = -T_{(\lambda_1,\ldots,\lambda_{j-1},s,r,\mu);s_j\gamma;\mathbf{0}}$.*

Proof. We will prove this using the following recursion: For any partition λ of length ℓ

$$T_{\lambda,r;\gamma} = \sum_{\alpha\in\mathbb{Z}^{\ell}_{\geq 0}} (-1)^{|\alpha|} 2^{\#\{i:\alpha_i>0\}} T_{\lambda;\gamma;\alpha} Q_r^{\gamma_{\ell+1}}[-|\alpha|].$$

Then similarly to the proof of Lemma 1 we proceed by induction on the length of μ and using the recursion above we need only prove the base case where μ is empty. So applying the above recursion twice we get

$$T_{(\lambda_1,\ldots,\lambda_{j-1},r,s);\gamma;0} = \sum_{\alpha,\beta\in\mathbb{Z}^{j}_{\geq 0}} (-1)^{|\alpha|+|\beta|} T_{\lambda;\gamma;\alpha+\beta} \times$$

$$\left(Q_r^{\gamma_j}[-|\alpha|] Q_s^{\gamma_{j+1}}[-|\beta|] + 2 \sum_{i\in\mathbb{Z}_{\geq 0}} (-1)^i Q_r^{\gamma_j}[i-|\alpha|] Q_s^{\gamma_{j+1}}[-i-|\beta|] \right).$$

We similarly get

$$T_{(\lambda_1,\ldots,\lambda_{j-1},s,r);s_j\gamma,0} = \sum_{\alpha,\beta\in\mathbb{Z}^{j}_{\geq 0}} (-1)^{|\alpha|+|\beta|} T_{\lambda;\gamma;\alpha+\beta} \times$$

$$\left(Q_s^{\gamma_{j+1}}[-|\beta|] Q_r^{\gamma_j}[-|\alpha|] + 2 \sum_{i\in\mathbb{Z}_{\geq 0}} (-1)^i Q_s^{\gamma_j}[i-|\beta|] Q_r^{\gamma_{j+1}}[-i-|\alpha|] \right).$$

The fact that $Q_{r,s}(x) = -Q_{s,r}(x)$ is a well known relation of Q functions. We notice in the above that

$$
\begin{aligned}
&\left(Q_r^{\gamma_j}[-|\alpha|]Q_s^{\gamma_{j+1}}[-|\beta|] + 2\sum_{i\in\mathbb{Z}_{\geq 0}}(-1)^i Q_r^{\gamma_j}[i-|\alpha|]Q_s^{\gamma j+1}[-i-|\beta|]\right) \\
&= \left(\sum_{k=0}^{r-1}(-1)^k Q_{r-|\alpha|-k}(x)e_k(t_1,\ldots,t_{\gamma_j})\right)\left(\sum_{k=0}^{s-1}(-1)^k Q_{s-|\beta|-k}(x)e_k(t_1,\ldots,t_{\gamma_{j+1}})\right) \\
&+ 2\sum_{i\in\mathbb{Z}_{\geq 0}}(-1)^i\left(\sum_{k=0}^{r-1}(-1)^k Q_{r-|\alpha|-k+i}(x)e_k(t_1,\ldots,t_{\gamma_j})\right)\cdot \\
&\left(\sum_{k=0}^{s-1}(-1)^k Q_{s-|\beta|-k-i}(x)e_k(t_1,\ldots,t_{\gamma_{j+1}})\right) \\
&= \sum_{p<r;q<s}(-1)^{p+q}e_p(t_1,\ldots,t_{\gamma_j})e_q(t_1,\ldots,t_{\gamma_{j+1}})\cdot \\
&\left(Q_{r-|\alpha|-p}(x)Q_{s-|\beta|-q}(x) + 2\sum_{i\in\mathbb{Z}_{\geq 0}}(-1)^i Q_{r-|\alpha|-p+i}(x)Q_{s-|\beta|-q-i}(x)\right) \\
&= \sum_{p<r;q<s}(-1)^{p+q}e_p(t_1,\ldots,t_{\gamma_j})e_q(t_1,\ldots,t_{\gamma_{j+1}})Q_{a,b}(x)
\end{aligned}
$$

where $a+p = r-|\alpha|$ and $b+q = s-|\beta|$. Similarly

$$
\begin{aligned}
&\left(Q_s^{\gamma_{j+1}}(-|\beta|)Q_r^{\gamma_j}(-|\alpha|) - 2\sum_{i\in\mathbb{Z}_{\geq 0}} Q_s^{\gamma_j}(i-|\beta|)Q_r^{\gamma j+1}(-i-|\alpha|)\right) \\
&\qquad = \sum_{p<r;q<s}(-1)^{p+q}e_p(t_1,\ldots,t_{\gamma_j})e_q(t_1,\ldots,t_{\gamma_{j+1}})Q_{b,a}(x)
\end{aligned}
$$

where $a+p = r-|\alpha|$ and $b+q = s-|\beta|$. Hence using the relation $Q_{r,s}(x) = -Q_{s,r}(x)$ we get that $T_{\lambda_1,\ldots,\lambda_{j-1},r,s;\gamma} = -T_{\lambda_1,\ldots,\lambda_{j-1},s,r;s_j\gamma}$ and the lemma follows. $\square$

Corollary 3. *If* $\lambda = (\mu, r, r, \nu)$ *then* $T_\lambda = 0$.

Proof. This is immediate from the last lemma since $\gamma(\lambda)_j = \gamma(\lambda)_{j+1}$ if the r is the j^{th} and $(j+1)^{th}$ row of λ. $\square$

Now we will give a relation amongst the factorial Schur Q polynomials which will be useful in proving the Chevalley formula.

Lemma 5. *For integers p and r, $Q^r_{p+1}[0] = Q^{r+1}_{p+1}[0] - t_{r+1}Q^r_p[0]$.*

Proof.

$$\begin{aligned}
Q^{r+1}_{p+1}[0] &= \sum_{i=0}^{p}(-1)^i Q_{p+1-i}(x)e_i(t_1,\ldots,t_{r+1}) \\
&= \sum_{i=0}^{p}(-1)^i Q_{p+1-i}(x)(e_i(t_1,\ldots,t_r) + t_{r+1}e_{i-1}(t_1,\ldots,t_r)) \\
&= Q^r_{p+1}[0] - t_{r+1}Q^r_p[0] \text{ once we shift indices.}
\end{aligned}$$

□

Theorem 5. *For a strict partition λ of length ℓ, we have that T_λ satisfies the Chevalley formula for $LG(n, 2n)$.*

Proof. Note that $T_1 = Q_1^0[0]$ so we have the following:

$$
\begin{aligned}
&T_\lambda \cdot T_1 \\
&= \prod_{1\le i<j\le \ell} \left(\frac{1-R_{ij}}{1+R_{ij}}\right) \prod_{i=1}^{\ell} Q_{\lambda_i}^{\lambda_i-1}[0] \cdot Q_1^0[0] \\
&= \prod_{1\le i<j\le \ell+1=l} \left(\frac{1-R_{ij}}{1+R_{ij}}\right) (1+2\sum_{r=1}^{\ell} R_{i,l}) \prod_{i=1}^{\ell} Q_{\lambda_i}^{\lambda_i-1}[0] \cdot Q_1^0[0] \\
&= T_{\lambda,1;\gamma(\lambda,1);\mathbf{0}} + 2\sum_{r=1}^{\ell} \prod_{1\le i<j\le \ell} \left(\frac{1-R_{ij}}{1+R_{ij}}\right) \prod_{i=1}^{r-1} Q_{\lambda_i}^{\lambda_i-1}[0] \cdot Q_{\lambda_r}^{\lambda_r-1}[1] \cdot \prod_{i=r+1}^{\ell} Q_{\lambda_i}^{\lambda_i-1}[0] \\
&= T_{\lambda,1;\gamma(\lambda,1);\mathbf{0}} + 2\sum_{r=1}^{\ell} \prod_{1\le i<j\le \ell} \left(\frac{1-R_{ij}}{1+R_{ij}}\right) \prod_{i=1}^{r-1} Q_{\lambda_i}^{\lambda_i-1}[0] \cdot Q_{\lambda_r+1}^{\lambda_r-1}[0] \cdot \prod_{i=r+1}^{\ell} Q_{\lambda_i}^{\lambda_i-1}[0] \\
&= T_{\lambda,1} + 2\sum_{r=1}^{\ell} \prod_{1\le i<j\le \ell} \left(\frac{1-R_{ij}}{1+R_{ij}}\right) \prod_{i=1}^{r-1} Q_{\lambda_i}^{\lambda_i-1}[0] \cdot (Q_{\lambda_r+1}^{\lambda_r}[0] + t_{\lambda_r} Q_{\lambda_r}^{\lambda_r-1})[0] \cdot \prod_{i=r+1}^{\ell} Q_{\lambda_i}^{\lambda_i-1}[0] \\
&= T_{\lambda,1} + 2\sum_{\nu} T_{\nu;\gamma(\nu)} + 2\sum_{r=1}^{\ell} t_{\lambda_r} T_\lambda.
\end{aligned}
$$

Where ν is obtained from λ by adding one box. By Lemma 4 the only ν where $T_{\nu;\gamma(\nu)}$ is non-zero will be strict partitions. Hence we have proven the Chevalley formula for $LG(n, 2n)$.

$\square$

4.4 Giambelli Revisited for Maximal Orthogonal Grassmannians

We will now consider the maximal orthogonal Grassmannians $OG(n, 2n)$ and $OG(n, 2n+1)$. These are isomorphic as varieties, but the Lie groups which act on them are different. The Lagrangian Grassmanian and the maximal orthogonal Grassmannians are defined similarly; the difference is that the Lagrangian Grassmannian is a set of subspaces which are isotropic with respect to a skew-symmetric form, while the orthogonal Grassmannians are sets of subspaces which are isotropic with respect to a symmetric form. Given the similarity between the two spaces it is not surprising that the Schubert classes will have similar representatives.

Here we define factorial Schur P functions as section 4 of in [IMN].

$$\begin{aligned} P_k^r(j) &= P_{k+j}(x|t_1,\dots,t_r) \\ &= \sum_{i=0}^{k-1}(-1)^i P_{k+j-i}(x_1,x_2\dots)e_i(t_1,\dots,t_r) \\ &= \sum_{i=1}^{\infty} x_i^{k-r+j}(x_i-t_1)\cdots(x_i-t_r). \end{aligned}$$

As in the case of the Lagrangian Grassmannian, the 2-part factorial Schur P functions are defined as:

$$P_{k,l}(x|t) = P_k(x|t)P_l(x|t)+2\sum_{i\geq 1}(-1)^i P_{k+i}(x|t)P_{l-i}(x|t) = \left(\frac{1-R_{1,2}}{1+R_{1,2}}\right) P_k^{k-1}[0]P_l^{l-1}[0]$$

as in [IMN]. In fact the reader will notice that the only difference between the P functions and the Q functions is a factor of two. Also similar to the Lagrangian case, for any strict partition λ of even length ℓ, where again we set every strict partition to have even length by making the last part 0 if the strict partition is actually of odd length, we have

$$P_\lambda(x|t) = Pf[P_{\lambda_i,\lambda_j}]_{1\leq i,j\leq \ell}.$$

Thus we can take our Giambelli Formula for the Lagrangian Grassmannian, multiply it by the appropriate factor of $\frac{1}{2}$, and appropriately shift the t variables as in [IMN, Theorem 6.6] for whether we are in the even or odd orthogonal case to obtain an equivariant Giambelli formula for the orthogonal Grassmannians.

Theorem 6. *Set*

$$T_\lambda = 2^{-\ell} \prod_{1\le i<j\le \ell} \left(\frac{1-R_{ij}}{1+R_{ij}}\right) \prod_{i=1}^{\ell} Q_{\lambda_i}^{\lambda_i}.$$

Then T_λ represents the Schubert class corresponding to λ in a maximal orthogonal Grassmannian, where we fix $t_1 = 0$ in the odd orthogonal case.

We can show that these formulas coincide with the formulas in [IMN, Theorem 6.6] similarly to the case of $LG(n, 2n)$ where we note that we are using the factorial Q functions with the appropriate factor of 2 removed instead of using the factorial P functions for the theorem. This is to ensure that the equivariant correction term $T_1\big|_\lambda$ in the Chevalley formula will not have the factor of 2 which is inherited from the expansion of the raising operator to include $R_{i,\ell+1}$.

Chapter 5

The Giambelli Formula for General Isotropic Grassmannians

5.1 Preliminaries for $IG(n-k, 2n)$

We now consider the non-maximal symplectic Grassmannian $IG(n-k, 2n)$ where $k \leq n$. $IG(n-k, 2n) = \{\Sigma^{n-k} \subset V | \forall v, w \in \Sigma^{n-k} \, \langle v, w \rangle = 0\}$, where $\langle \; , \; \rangle$ is a symplectic form on $V \simeq \mathbb{C}^{2n}$.

In [BKT2], the authors describe how any Schubert variety in the symplectic Grassmannian, $IG(n-k, 2n)$, is indexed by a k-strict partition whose Young diagram fits inside a $(n-k)$ by $(n+k)$ rectangle. A partition λ is called k-strict if for all i such that $\lambda_i > k$ we have that $\lambda_i > \lambda_{i+1}$. Fix a complete flag in type C, $F_\bullet = 0 \subset F_1 \subset \cdots \subset F_{2n} = V$ so that for $i \leq n, F_i$ is an isotropic subspace of V, and $F_{2n-i} = F_i^\perp$. Then Schubert cells and varieties are defined similarly to the type A case. For any k-strict partition λ, the Schubert variety is described below as it is in section 1 of [BKT2]

$$X_\lambda = X_\lambda(F_\bullet) = \{\Sigma \in IG(n-k, 2n) | \dim(\Sigma \cap F_{p_j(\lambda)}) \geq j \text{ for } 1 \leq j \leq \ell(\lambda)\}$$

where for any k-strict partition λ,

$$p_j(\lambda) = n + k + 1 - \lambda_j + \#\{i < j : \lambda_i + \lambda_j \leq 2k + j - i\}. \tag{5.1}$$

The corresponding Schubert cell is

$$X_\lambda^\circ = \{\Sigma \in IG(n-k, 2n) | \dim(\Sigma \cap F_{p_j(\lambda)}) = j \text{ for } 1 \leq j \leq \ell(\lambda)\}.$$

Again from any Schubert variety we can obtain a class in the cohomology ring of $IG(n-k, 2n)$ which does not depend on the choice of flag $F_\bullet$. When $k = 0$, we denote $IG(n, 2n)$ by $LG(n, 2n)$, as it is the Lagrangian Grassmannian of maximal isotropic subspaces described in Chapter 4.

For $IG(n-k, 2n)$ we similarly have that all torus fixed points for the standard maximal torus are $n-k$ dimensional subspaces generated by exactly $n-k$ standard basis vectors. We recall that for a vector space V^{2n} with an antidiagonal skew symmetric non-degenerate bilinear form $\langle\, ,\, \rangle$, the standard basis vectors are $e_1, \ldots, e_{2n}$ where $\langle e_i, e_{2n-i+1}\rangle = 1$ and $\langle e_i, e_j\rangle = 0$ for all $j \neq 2n-i+1$. Let $E_\bullet = 0 \subset \langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \cdots \subset \langle e_1, \ldots, e_{2n}\rangle$. Similarly, we have that any Schubert cell, $X^\circ_\lambda(E_\bullet)$, will contain exactly one torus fixed point. For any k-strict partition λ, the torus fixed point is given by:

$$e_\lambda = \langle e_{p_1(\lambda)}, \ldots, e_{p_{n-k}(\lambda)}\rangle \in X^\circ_\lambda(E_\bullet)$$

where $p_j(\lambda)$ is as defined in Equation (5.1).

5.2 The Theta Polynomials of [BKT] and the Classical Chevalley Formula

We begin by recalling a few definitions from [BKT].

Definition 1. [BKT2] Let λ be a Young diagram. Then the box $[r, c]$ and the box $[r', c']$are called k-related if $c + c' = 2k + 2 + r - r'$ where $c \leq k < c'$. For example in Figure 5.1 below the two marked boxes are k-related.

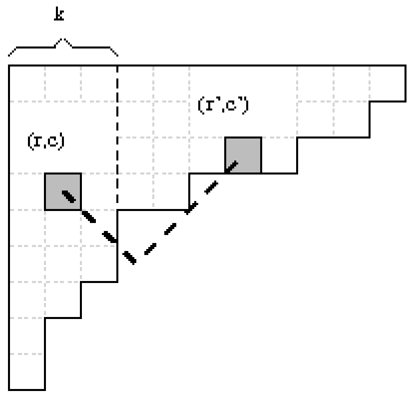


Figure 5.1: k-related: In this example the box $[r,c]$ is k-related to $[r',c']$.

For an integer sequence λ, we define $\mathcal{C}(\lambda) = \{(i,j)|i < j \leq \ell(\lambda), \lambda_i + \lambda_j > 2k + j - i\}$ as in [BKT], and we let $\Delta^\circ = \{(i,j) \in \mathbb{N} \times \mathbb{N}|1 \leq i < j\}$. We define a partial order on this set by $(i,j) \leq (i',j')$ if $i \leq i'$ and $j \leq j'$. We call a finite subset of Δ° valid if it is an order ideal. We recall from [BKT] that a subset is valid iff it is equal to $\mathcal{C}(\lambda)$ for some k-strict partition λ.

Definition 2. For any valid set of pairs D the raising operator R^D is defined as

$$R^D = \prod_{i<j}(1 - R_{ij}) \prod_{(i,j)\in D}(1 + R_{ij})^{-1}.$$

We set $R^{\mathcal{C}(\lambda)} = R^\lambda$, and notice that

$$R^\lambda = \prod_{1\leq i<j\leq \ell}(1 - R_{ij}) \prod_{(i,j)\in \mathcal{C}(\lambda)}(1 + R_{ij})^{-1}$$

where ℓ is the length of the partition λ.

Definition 3. [BKT, §5.1] Let $x = (x_1, x_2, \dots)$ and $z = (z_1, \dots, z_k)$ be the variables defined for type C in section §2.2.1. For a positive integer r, the standard Theta polynomial $\theta_r(x,z) := \sum_{i=0}^r q_{r-i}(x)e(z)$ where $q_i(x)$ is the usual Schur q function of degree i. For an integer sequence α, set $\theta_\alpha = \prod_{i=1}^\ell \theta_{\alpha_i}$.

Then for a k-strict partition λ, the theta polynomial Θ_λ is defined by

$$\Theta_\lambda = R^\lambda \theta_\lambda.$$

In [BKT], the authors prove that the above theta polynomials satisfy the Pieri rule for $IG(n-k, 2n)$, where n is sufficiently large. Below we specialize their proof to the Chevalley formula.

For any two k-strict partitions λ and μ, we write $\lambda \to \mu$ if μ may be obtained by adding a box to λ, or removing r boxes from a single column of the first k columns of λ and adding $r+1$ boxes in a single row of the result, so that the removed boxes and the bottom box of μ in that column are each k-related to one of the added boxes. If $\lambda \to \mu$, let $e_{\lambda\mu} = 2$ if $\mu \supset \lambda$ and the added box in μ is not k-related to a bottom box in one of the first k columns of λ, and otherwise set $e_{\lambda\mu} = 1$. The Chevalley rule for IG states that

$$\sigma_1 \cdot \sigma_\lambda = \sum_{\lambda \to \mu} e_{\lambda\mu}\, \sigma_\mu. \tag{5.2}$$

For any valid set D and any integer sequence α we define $T(D, \alpha) = R^D \theta_\alpha$ so that $\Theta_\lambda = T(\mathcal{C}(\lambda), \lambda)$.

Theorem 7.

$$\theta_1 \cdot T(\mathcal{C}(\lambda), \lambda) = \sum_{\lambda \to \mu} e_{\lambda\mu}\, T(\mathcal{C}(\mu), \mu).$$

In [BKT] this is proven via a series of Lemmas and a substitution rule. The following Lemmas are taken from [BKT].

Lemma 6. *[BKT, Lemma 1.2] Let $\lambda = (\lambda_1, \dots, \lambda_{j-1})$ and $\mu = (\mu_{j+2}, \dots, \mu_\ell)$ be integer vectors. Assume that $(j, j+1) \notin D$ and that for each $h < j$, $(h, j) \notin D$ iff*

$(h, j+1) \notin D$. Then for any integers r and s, where $D = \mathcal{C}(\lambda, r, s, \mu)$ we have

$$T(D, (\lambda, r, s, \mu)) = -T(D, (\lambda, s-1, r+1, \mu))\,.$$

In particular, $T(D, (\lambda, r, r+1, \mu)) = 0$.

Lemma 7. *[BKT, Lemma 1.3] Let $\lambda = (\lambda_1, \ldots, \lambda_{j-1})$ and $\mu = (\mu_{j+2}, \ldots, \mu_\ell)$ be integer vectors, assume $(j, j+1) \in D$, and that for each $h > j+1$, $(j, h) \in D$ iff $(j+1, h) \in D$. If $r, s \in \mathbb{Z}$ are such that $r + s > 2k$, where $D = \mathcal{C}(\lambda, r, s, \mu)$, then we have*

$$T(D, (\lambda, r, s, \mu)) = -T(D, (\lambda, s, r, \mu))\,.$$

In particular, $T(D, \lambda, r, r, \mu) = 0$ for any $r > k$.

Below we give a simplified version of the substitution rule of [BKT] which suffices in proving the Chevalley formula above holds.

5.2.1 The Substitution Rule

We fix the k-strict partition λ of length ℓ, let $\mathcal{C} = \mathcal{C}(\lambda)$, and choose n sufficiently large so that all Pieri terms that can possibly appear in the Chevalley formula do not vanish. For any $d \geq 1$ define the raising operator R_d^λ by

$$R_d^\lambda = \prod_{1 \leq i < j \leq d} (1 - R_{ij}) \prod_{i<j\,:\,(i,j) \in \mathcal{C}} (1 + R_{ij})^{-1}.$$

We compute that

$$\sigma_1 \cdot \Theta_\lambda = \sigma_1 \cdot R_\ell^\lambda\, \theta_\lambda = R_{\ell+1}^\lambda \cdot \prod_{i=1}^{\ell} (1 - R_{i,\ell+1})^{-1}\, \theta_{\lambda,1} = R_{\ell+1}^\lambda \cdot \prod_{i=1}^{\ell} (1 + R_{i,\ell+1})\, \theta_{\lambda,1}$$

and therefore

$$\sigma_1 \cdot T(\mathcal{C}, \lambda) = \sum_{j=1}^{\ell+1} T(\mathcal{C}, \lambda^{(j)}), \tag{5.3}$$

where $\lambda^{(j)}$ is the integer sequence obtained from λ such that $\lambda_i^{(j)} = \lambda_i$ for all $i \neq j$ and $\lambda_j^{(j)} = \lambda_j + 1$. We aim to show that the right hand side of Equation (5.3) is equal to the right hand side of the Chevalley rule.

Let $m \geq 1$ be minimal such that $\lambda_m \leq k$; we call m the *middle* row of λ.

Definition 4. A *valid triple* is a triple $\psi = (D, \mu, S)$, such that D is a valid set of pairs containing $\mathcal{C}$, all pairs (i,j) in D satisfy $i \leq m$ and $j \leq \ell + 1$, μ is an integer sequence of length at most $\ell + 1$, and S is a subset of $D \smallsetminus \mathcal{C}$. The evaluation of ψ is defined by $\mathrm{ev}(\psi) = T(D, \mu) \in H^*(\mathrm{IG}, \mathbb{Z})$.

All valid triples encountered here will also satisfy that $D \subset \mathcal{C} \cup \partial\mathcal{C}$, but for technical reasons we do not require this in the definition. We will represent the set Δ° as the positions above the main diagonal of a matrix, and the various sets of pairs in D as sets of entries in this matrix, as in [BKT].

For $1 \leq h \leq m$, we let $b_h = \min\{j \geq m \mid (h,j) \notin \mathcal{C}\}$ and $g_h = b_{h-1}$ (by convention we set $g_1 = \ell + 1$); the next result follows from [BKT, Lemma 3.5].

Lemma 8. *If $2 \leq h \leq m$ then we have $\lambda_{h-1} - \lambda_h \geq g_h - b_h + 1$.*

Let $(i,j) \in \Delta^\circ$ be arbitrary. We define a weight condition $\mathrm{W}(i,j)$ on an integer sequence μ as follows.

$$\mathrm{W}(i,j) \ : \ \mu_i + \mu_j > 2k + j - i \, .$$

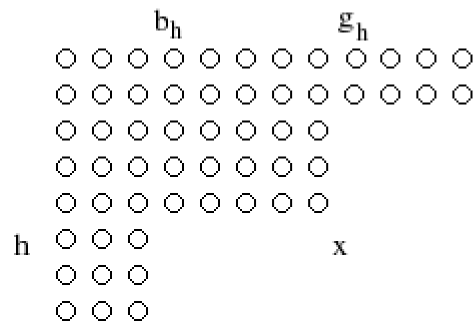


Figure 5.2: The set $\mathcal{C}$ near the pair $\mathbf{x} = (h, g_h) \in \partial\mathcal{C}$

Also we will call (i, j) an outer corner of a valid set D if $D \cup (i, j)$ is also a valid set. We note that if there is no outer corner in column j then this means that $(i, j-1) \notin D$ iff $(i, j) \notin D$, and similarly if there is no outer corner in row i then this means that $(i-1, j) \in D$ iff $(i, j) \in D$. This gives us a new way to view the hypotheses of Lemmas 6 and 7.

The following *substitution rule* will be applied iteratively to rewrite the right hand side of (5.3). It may be applied to any valid triple and will result in either a REPLACE statement, indicating that the triple should be replaced by one or two new triples, or a STOP statement, indicating that the triple should not be replaced.

Substitution Rule

Let (D, μ, S) be a valid triple. Let $h \leq \ell + 1$ be largest such that one of the following four conditions is true (if none hold for any h, then STOP).

(i) $(h-1, h) \notin D$ and there is an outer corner (i, h) of D with $i \leq m$ such that W(i, h) holds;

(ii) $(h-1, h) \notin D$, D has no outer corner in column h, and $\mu_h = \mu_{h-1} + 1$;

(iii) $(h-1,h) \in D$ and there is an outer corner (h,j) of D with $j \leq \ell+1$ such that $\mathrm{W}(h,j)$ holds;

(iv) $(h-1,h) \in D$ and $\mu_h = \mu_{h-1}$.

If condition **(i)** holds, then REPLACE (D,μ,S) with $(D \cup (i,h), \mu, S)$ and $(D \cup (i,h), R_{ih}\mu, S \cup (i,h))$. If **(iii)** holds, then REPLACE (D,μ,S) with $(D \cup (h,j), R_{hj}\mu, S \cup (h,j))$ if $\mu_j > \mu_{j-1}$, or REPLACE (D,μ,S) with $(D \cup (h,j), R_{hj}\mu, S \cup (h,j))$ and $(D \cup (h,j), \mu, S)$ if $\mu_h \leq \mu_{h-1}$. If **(ii)** or **(iv)** holds, then STOP.

Definition 5. [BKT, Definition 3.9] Let **(x)** be one of the conditions **(i)**–**(iv)** of the Substitution Rule. We say that a valid triple ψ *meets* condition **(x)** if ψ reaches condition **(x)** in the Substitution Rule, and condition **(x)** is satisfied. The corresponding integer $h \geq 1$ is called the *level* of (D,μ,S). Whenever the Substitution Rule REPLACES ψ by one or two triples ψ_i, we refer to ψ as the *parent* term and the ψ_i are its *children*.

Initially we let $\Psi = \{(\mathcal{C}, \lambda^{(j)}, \emptyset) : 1 \leq j \leq \ell+1\}$ so that $\sum_{\psi \in \Psi} \mathrm{ev}(\psi)$ agrees with the right hand side of Equation (5.3). We then apply the above substitution rule which will change this set by replacing some triples with one or two new valid triples. Whenever the substitution rule results in a REPLACE statement, then the set is changed accordingly; otherwise the substitution rule results in a STOP statement, in which case the triple (D,μ,S) is left unchanged. These substitutions are iterated until no further elements can be REPLACED, i.e., until the substitution rule results in a STOP statement when applied to any remaining triple. Since the set of pairs D is not allowed to grow beyond column $\ell+1$, this algorithm will terminate after a finite number of steps.

Suppose that the triple $\psi = (D, \mu, S)$ occurs in the algorithm. If ψ is REPLACED by two triples ψ_1 and ψ_2, we deduce from the identity

$$1 - R_{ij} = \frac{1 - R_{ij}}{1 + R_{ij}} + \frac{1 - R_{ij}}{1 + R_{ij}} R_{ij}$$

that $\mathrm{ev}(\psi) = \mathrm{ev}(\psi_1) + \mathrm{ev}(\psi_2)$. Moreover, if ψ meets **(iii)** and is REPLACED by $\psi' = (D \cup (h, j), R_{hj}\mu, S \cup (h, j))$, one can show using Lemma 6 that $\mathrm{ev}(\psi) = \mathrm{ev}(\psi')$ since we must have that $\mu_{j-1} = \mu_j - 1$ and that $D \cup (h, j)$ has no outer corner in column j, so $\mathrm{ev}(D \cup (h, j), \mu) = 0$.

When the algorithm terminates, let Ψ_0 denote the collection of all triples (D, μ, S) in the final set such that **(i)**-**(iv)** fail for all h, and Ψ_1 the remaining triples (i.e. the ones which satisfy conditions **(ii)** or **(iv)**). We say that a triple ψ *survives the algorithm* if at least one of its successors lies in Ψ_0 as in [BKT]. The above analysis implies that

$$\sum_{j=1}^{\ell+1} T(\mathcal{C}, \lambda^{(j)}) = \sum_{\psi \in \Psi_0} \mathrm{ev}(\psi) + \sum_{\psi \in \Psi_1} \mathrm{ev}(\psi).$$

We remark that the triples $\psi \in \Psi_0$ with $\mu_{\ell+1} < 0$ evaluate to zero trivially.

Claim 1. For any $\psi \in \Psi_1$ we have $\mathrm{ev}(\psi) = 0$.

Proof. If $(h-1, h) \notin D$, then ψ meets **(ii)** and $\mu_{h-1} = \mu_h - 1$. Since there is no outer corner in column h by Lemma 6 we have that $\mathrm{ev}(\psi) = 0$. If ψ meets **(iv)**, where $(h-1, h) \in D$ and $\mu_{h-1} = \mu_h$, then Lemma 8 implies that D has no outer corner in row h since in this case we must have that $g_h = b_h$. Hence, Lemma 7 shows that $\mathrm{ev}(\psi) = 0$. The next assertion will therefore prove that the Chevalley rule holds. □

Claim 2. For each triple $\psi = (D, \mu, S)$ in Ψ_0 with $\mu_{\ell+1} \geq 0$, μ is a k-strict partition with $\lambda \to \mu$ and $\mathrm{ev}(\psi) = T(\mathcal{C}(\mu), \mu)$. Furthermore, for each such partition μ, there are exactly $e_{\lambda\mu}$ such triples ψ, in agreement with the Chevalley rule.

In the appendix we provide an example of the equivariant substitution rule. One can also view the example as an example of the classical substitution rule by making the adjustments described in Remark 2.

Recall the fixed choices of λ, ℓ, $\mathcal{C}$, and m. Let $\psi = (D, \mu, S)$ denote a triple which occurs at some step in the algorithm. The symbols D, μ, and S will refer to components of the triple ψ. Throughout the algorithm, observe that μ is obtained from the initial composition ν by removing boxes from rows weakly below the middle row m of λ and adding them to rows weakly above the middle row. The set D is initially equal to $\mathcal{C}$ and grows when REPLACE statements are encountered. All pairs added to D come from the outer rim $\partial\mathcal{C}$:

Lemma 9. *If $(j-1, j) \notin D$ and ψ does not meet* **(i)** *or* **(ii)** *at any level $h \geq j$, then $\mu_j \leq \lambda_{j-1}$.*

Proof. Assume that $\mu_j > \lambda_{j-1}$. Since ψ does not meet **(ii)**, at level j, it follows that D has an outer corner (i, j) in column j. Since $(i, j-1) \in \mathcal{C}$ we obtain $\mu_j + \mu_i > \lambda_{j-1} + \lambda_i > 2k + (j-1) - i$. Hence ψ satisfies $\mathrm{W}(i, j)$ and meets **(i)** at level j, which is a contradiction. $\square$

Proposition 3. *Let $\psi = (D, \mu, S)$ be a valid triple and suppose that $(i, j), (i', j') \in D \smallsetminus \mathcal{C}$. Then either $i = i'$ or $j = j'$.*

The above proposition is special to the Chevalley formula, as it tells us that not only is $D \subset \mathcal{C} \cup \partial\mathcal{C}$, but that the set difference $D \setminus \mathcal{C}$ will only be a row or a column.

Proof. Let ψ be as in the statement of the proposition. Recall that initially $\Psi = \{(\mathcal{C}, \lambda^{(j)}, \emptyset) : 1 \leq j \leq \ell\}$. Let r be such that ψ is a descendent of $(\mathcal{C}, \lambda^{(r)}, \emptyset) := \psi_0$. This proof will trace through the family tree of ψ_0 to ψ. We first note that if $\psi = \psi_0$ then $D \setminus \mathcal{C}$ is empty. So we consider cases for the children of ψ_0. Let h be the level of ψ_0. Since $\lambda^{(r)}$ is obtained by only adding a box to the r^{th} row of λ we must have that any change in the pairs which satisfy the weight condition must contain r. We also notice that all pairs in Ψ which have non-trivial descendants will satisfy condition **(i)**, since if $(i,j) \notin \mathcal{C}$ but $W(i,j)$ is satisfied for $i < j$, then we must have $(j-1, j) \notin \mathcal{C}$.

1. If $(r-1, r) \in \mathcal{C}$, then $h > r$ since ψ_0 satisfies condition **(i)**, and we have $W(r,h)$ holds. Hence ψ_0 has children $\psi_1 = (\mathcal{C} \cup (r,h), \lambda^{(r)}, \emptyset)$ and $\psi_2 = (\mathcal{C} \cup (r,h), R_{rh}\lambda^{(r)}, \{(r,h)\})$.

   We note ψ_1 cannot have any children since condition **(i)** cannot be satisfied since h is the largest such that $\lambda_r + 1 + \lambda_h > 2k + h - r$, and (r,h) is in D for ψ_1. Also condition **(iii)** cannot be satisfied for ψ_1 since r would have to be the level and again h is the largest such that $\lambda_r + 1 + \lambda_h > 2k + h - r$, and (r,h) is in D for ψ_1. If $\psi = \psi_1$ then $D \setminus \mathcal{C}$ is only a singleton.

   We notice that ψ_2 may have children. The sequence μ for ψ_2 has the property that $\mu_i = \lambda_i$ for $i \neq h, r$, where $\mu_r = \lambda_r + 2$ and $\mu_h = \lambda_h - 1$.

Thus any change in the pairs which satisfy the weight condition must again contain r, since the r^{th} position is the only position which was increased. We note ψ_2 cannot satisfy condition **(i)** since $(r,h) \in D$ and so the level of ψ_2 is less than h, as we have progressed past h in the algorithm. If $\psi = \psi_2$ then $D \smallsetminus \mathcal{C}$ is again only a singleton. If ψ_2 satisfies condition **(iii)** then $W(r,h+1)$ is satisfied. We note this happens when $\lambda_h = \lambda_{h+1}$. Then ψ_2 has children $\psi_3 = (\mathcal{C} \cup (r,h) \cup (r,h+1), R_{rh}\lambda^{(r)}, \{(r,h)\})$ and $\psi_4 = (\mathcal{C} \cup (r,h) \cup (r,h+1), R_{r,h}R_{r,h+1}\lambda^{(r)}, \{(r,h),(r,h+1)\})$. Similarly to ψ_1, ψ_3 cannot have any children, and similarly ψ_4 can have children as ψ_2 did if ψ_4 satisfies condition **(iii)**. We note that if ψ_4, or its descendants have children then their D sets will only increase along row r since in each case r is the only position where μ is increased, and hence the only position which can take part in a change in pairs which satisfy the weight condition.

Thus in the case where $(r-1,r) \in \mathcal{C}$ we have that ψ, as a descendent of ψ_0, will have the property that if $(i,j),(i',j') \in D \smallsetminus \mathcal{C}$, then $i = i' = r$.

2. If $(r-1,r) \notin \mathcal{C}$, then $h = r$, and there exists an $i \leq m$ such that (i,r) is an outer corner of $\mathcal{C}$ and $W(i,r)$ holds. Hence ψ_0 has children $\psi_1 = (\mathcal{C} \cup (i,r), \lambda^{(r)}, \emptyset)$ and $\psi_2 = (\mathcal{C} \cup (i,r), \lambda^{(i)}, \{(i,r)\})$.

   We note that in this case ψ_2 cannot have any children since condition **(i)** cannot be satisfied since r is the largest such that $\lambda_r + 1 + \lambda_i > 2k + r - i$, and (i,r) is in D for ψ_2. Also condition **(iii)** cannot be satisfied for ψ_2 since i would have to be the level and again r is the largest such that $\lambda_r + 1 + \lambda_i > 2k + r - i$, and (i, r) is in D for ψ_2. If $\psi = \psi_2$ then $D \setminus \mathcal{C}$ is only a singleton.

We notice that ψ_1 could possibly have children since if $\lambda_i = \lambda_{i+1} + 1$ then ψ_1 satisfies condition **(i)** again with level r. If this were the case then ψ_1 would have children $\psi_3 = (\mathcal{C} \cup (i, r) \cup (i+1, r), \lambda^{(r)}, \emptyset)$ and $\psi_4 = (\mathcal{C} \cup (i, r) \cup (i+1, r), \lambda^{(i+1)}, \{(i, r), (i+1, r)\})$. Similarly we have that ψ_4 cannot have children, while ψ_3 can have descendants if $\lambda_i = \lambda_{i+2} + 2$.

Thus in the case where $(r-1, r) \notin \mathcal{C}$ we have that ψ, as a descendent of ψ_0, will have the property that if $(i, j), (i', j') \in D \setminus \mathcal{C}$, then $j = j' = r$.

From the above 2 cases the proposition is proved.

□

Proposition 4. *Any (i, j) which is an outer corner of D such that $W(i, j)$ holds for μ, has the property that $\mu_i + \mu_j = 2k + j - i + 1$.*

Proof. If $\mu = \lambda^{(i)}$ for some i then this is clear, since $\mathcal{C} \subseteq D$ and each row of μ only differs from λ by one box. Otherwise, assume not. Then $\mu_i + \mu_j > 2k + j - i + 1$, while $\lambda_i + \lambda_j \leq 2k + j - i$. This can only happen if S is non-empty and contains at least one pair (i, r) where $r < j$. In this case we must have that $D \setminus \mathcal{C}$ is a row. Let r be minimal such that $(i, r) \in D \setminus \mathcal{C}$. Then we must have that $\lambda_r = \lambda_{r+1} = \cdots = \lambda_j$ since from the proof of the previous proposition this is the only way to make $D \setminus \mathcal{C}$ be a row. The ancestor of such a progression will be $(\mathcal{C}, \nu, \emptyset)$ where $\nu = \lambda^{(i)}$. This ν has the property that $\nu_r + \nu_i > 2k + r - i$ while we know $\lambda_r + \lambda_i \leq 2k + r - i$. Therefore $\nu_r + \nu_i = 2k + r - i + 1$. The triple $(\mathcal{C}, \nu, \emptyset)$ satisfies condition **(i)** for $h = r$ and so it has children. As we continue along row i we note that we are increasing part i of each descendent by looking at the child, which will be a product of condition **(iii)**, that adds (i,h) to S whenever (i,h) is an outer corner at that point in the algorithm. Thus at most $\mu_i = \nu_i + j - r$. We also note that μ_j will remain unchanged during this process, so that $\mu_j = \lambda_j = \lambda_r = \nu_r$. Therefore $2k+j-i+1 < \mu_i + \mu_j \le \nu_i + \nu_r + j - r$. This would imply that $\nu_i + \nu_r > 2k + r - i + 1$ which is a contradiction. So any (i,j) which is an outer corner of D such that $W(i,j)$ holds for μ, has the property that $\mu_i + \mu_j = 2k + j - i + 1$. □

We next study the triples $\psi = (D, \mu, S) \in \Psi_0$ with $\mu_{\ell+1} \ge 0$.

Proposition 5. *Suppose that $\psi = (D, \mu, S) \in \Psi_0$ and $\mu_{\ell+1} \ge 0$. Then μ is a k-strict partition with $|\mu| = |\lambda| + 1$, satisfying $\lambda_j - 1 \le \mu_j \le \lambda_{j-1}$ for every $j \ge 1$, and $\lambda_j \le \mu_j$ when $\lambda_j > k$. Furthermore, we have $D = \mathcal{C}(\mu)$ and $\lambda \to \mu$.*

Proof. First note that $\psi = (D, \mu, S) \in \Psi_0$ means that ψ does not meet conditions **(ii)** or **(iv)**. Also if ψ_0 is part of the initial set of triples Ψ then we know $\psi_0 = (\mathcal{C}, \lambda^{(j)}, \emptyset)$ for some j where $|\lambda^{(j)}| = |\lambda| + 1$, and so since every REPLACE step of the substitution rule does not change the weight of the integer sequence we have that every $\psi = (D, \mu, S) \in \Psi_0$ has the property that $|\mu| = |\lambda| + 1$. Also since $D \smallsetminus \mathcal{C}$ is either a row or a column we only have a couple possibilities for a descendent $\psi \in \Psi_0$ of $\psi_0 \in \Psi$. Let $\psi = (D, \mu, S)$ be a descendent of $\psi_0 = (\mathcal{C}, \lambda^{(j)}, \emptyset)$. Consider cases:

1. If $D \smallsetminus \mathcal{C} = \emptyset$ then ψ_0 does not meet conditions **(i) - (iv)**. So $\psi = \psi_0$ and we claim that $\lambda^{(j)}$ is a k-strict partition since λ is. Indeed, the only way $\lambda^{(j)}$ will not be a k-strict partition is if either $k \ge \lambda_{j-1} = \lambda_j$ in which case since ψ_0

does not satisfy condition **(i)** it would satisfy condition **(ii)** which would be a contradiction, or if $k < \lambda_{j-1} = \lambda_j + 1$ in which case since does not satisfy condition **(iii)** it would satisfy condition **(iv)** which would be a contradiction. Also since ψ_0 does not meet conditions **(i) - (iv)**, there are no outer corners of $\mathcal{C}$ such that the weight condition holds so $\mathcal{C} = \mathcal{C}(\lambda^{(j)})$ in this case. It is clear that in this case if we set $\mu = \lambda^{(j)}$ then $\lambda_j - 1 \leq \mu_j \leq \lambda_{j-1}$ for every $j \geq 1$, $\lambda_j \leq \mu_j$ when $\lambda_j > k$, and $\lambda \to \mu$.

2. If $D \smallsetminus \mathcal{C}$ is in the h^{th} column, then we have either $j = h$ or $j \neq h$.

   If $j = h$ then we must have $j > m$. In this case ψ_0 must meet condition **(i)**. Since $D \smallsetminus \mathcal{C}$ is the j^{th} column we must have that in the family tree of ψ_0, ψ is always a descendent of the terms whose integer sequence $\lambda^{(j)}$ remains unchanged. Let (r, j) be the last new addition to D, so that ψ's parent is $(D \smallsetminus (r, j), \lambda^{(j)}, \emptyset)$. Then $\psi = (D, \mu, S)$ is either $(D, \lambda^{(j)}, \emptyset)$ or $(D, \lambda^{(r)}, \{(r, j)\})$. In both case μ is k-strict partition with $|\mu| = |\lambda| + 1$, satisfying $\lambda_j \leq \mu_j$ and $\lambda \to \mu$. To see that $D = \mathcal{C}(\mu)$ note that D is a valid set such that the weight condition is only met for pairs in D, and that every pair of D meets the weight condition for μ since $\lambda \subset \mu$ and the only pairs which were added to $\mathcal{C}$ to form D satisfied the weight condition for μ.

   If $j \neq h$ then we must have that ψ_0 meets condition **(i)** at position h since we have only changed λ in the j^{th} row and so the fact that only the h^{th} column is changed means that (j, h) must be an outer corner of $\mathcal{C}$. In this case ψ_0 must be the parent of ψ since $D \smallsetminus \mathcal{C}$ is a column. So $\psi = (D, \mu, S) = (\mathcal{C} \cup (j, h), \lambda^{(j)}, \emptyset)$

or $\psi = (D, \mu, S) = (\mathcal{C} \cup (j,h), R_{jh}\lambda^{(j)}, (\{(j,h)\})$. Since ψ does not meet conditions **(iii)** or **(iv)** we must have that μ is a k-strict partition satisfying the desired properties. We note that in the case $\mu = R_{jh}\lambda^{(j)}$ we have $\lambda \to \mu$ since we would have removed a box from the λ_h^{th} column and added two boxes to the j^{th} row.

3. If $D \smallsetminus \mathcal{C}$ is in the h^{th} row, then again we have either $j = h$ or $j \neq h$.

   If $j = h$ then we must have $j \leq m$. Let c be the smallest such that $(j,c) \in D \smallsetminus \mathcal{C}$. Then ψ_0 satisfies condition **(i)** at position c. If $|D \smallsetminus \mathcal{C}| > 1$ then we must have that ψ is a descendent of $(\mathcal{C} \cup (j,c), R_{jc}\lambda^{(j)}, \{(j,c)\})$. We remark that it is necessary for a triple to satisfy condition **(iii)** in order to add to row j of $\mathcal{C} \cup (j,c)$ for descendants of ψ_0 and the triple $(\mathcal{C} \cup (j,c), \lambda^{(j)}, \emptyset)$ will not satisfy condition **(iii)** since $\lambda_{c+1}^{(j)} = \lambda_{c+1} \leq \lambda_c = \lambda_c^{(j)}$. Then if d is the greatest such that $(j, c+d) \in D \smallsetminus \mathcal{C}$ we have that $\psi = (\mathcal{C} \cup_{i=0}^{d} (j, c+i), \prod_{i=0}^{d}(R_{j,c+i})\lambda^{(j)}, \{(j, c+i)\}_{i=0}^{d})$. We note that if $\psi = (\mathcal{C} \cup_{i=0}^{d} (j, c+i), \prod_{i=0}^{d-1}(R_{j,c+i})\lambda^{(j)}, \{(j, c+i)\}_{i=0}^{d-1})$ then this would have to vanish by Lemma 6. The fact that condition **(iv)** is not satisfied guarantees that μ is k-strict. We have that $\lambda \to \mu$ since in order to meet condition **(iii)** at each branch in the family tree of ψ_0 we must have had that $\lambda_c = \lambda_{c+1} = \cdots = \lambda_{c+d}$ so that μ is obtained from λ by removing $d+1$ boxes from column λ_c of λ and adding $d+2$ boxes to row j. We similarly argue that $\mathcal{C}(\mu) = D$ in this case.

   If $j \neq h$ then we must have (h, j) is an outer corner of $\mathcal{C}$ and that ψ_0 satisfies condition **(i)** at position j. We note that λ is a k-strict partition so it is not possible that $\lambda_j^{(j)} = \lambda_{j+1}^{(j)}$ hence $|D \smallsetminus \mathcal{C}| = 1$ and we have that μ is either $\lambda^{(j)}$ or $\lambda^{(h)}$ and since none of the conditions hold for ψ this will guarantee that μ and D have the desired properties.

$\square$

Proposition 5 tells us that if $\psi = (D, \mu, S)$ is any triple in Ψ_0 with $\mu_{\ell+1} \geq 0$, then μ is a k-strict partition with $\lambda \to \mu$, $D = \mathcal{C}(\mu)$ is uniquely determined by μ, and $\mathrm{ev}(\psi) = T(\mathcal{C}(\mu), \mu)$ is a term appearing in the Chevalley rule.

To account for the multiplicities, fix an arbitrary k-strict partition μ such that $\lambda \to \mu$, and suppose that $e_{\lambda\mu} = 2$. Then the unique box $B = [i, c]$ of $\mu \smallsetminus \lambda$ is not k-related to a bottom box in one of the first k columns of λ. We associate a pair (i, j) to B as follows: i is equal to the row number of B and j such that B is k-related to the box $[j-1, d-1]$ in the first k columns of λ and $[j, d] \notin \lambda$; if there is no such box then let $j = \ell + 1$. We then have that the two triples in Ψ_0 which contribute to $T(\mathcal{C}(\mu), \mu)$ are $(\mathcal{C}(\mu), \mu, \emptyset)$ and $((\mathcal{C}(\mu), \mu, \{(i, j)\})$.

This concludes the summary of the argument that the Theta polynomials satisfy the classical Chevalley formula. In §5.5 we define factorial Theta polynomials and attempt to show that they satisfy the equivariant Chevalley formula. In order to understand the equivariant Chevalley formula, we first need to understand Grassmann permutations and the restriction map to a torus fixed point.

5.3 The k-Grassmannian Signed Permutation of a Partition λ

We let λ be a k-strict partition which fits inside the $n-k$ by $n+k$ rectangle. We let λ' be the partition whose rows are the columns of λ. Note λ' is not going to be k-strict.

The length of a k-related diagonal for λ is given by the length of a diagonal, where first we draw a line from the each of bottom boxes of the first k columns of λ to the k-line and then find the length of the diagonal going up from the vertical k-line to the first place where it intersects λ. The lengths of these diagonals are given by

$$n_j(\lambda) = \text{ \# of } \{(i,j') | j' > \max(j, \lambda_j); \lambda_i < j'; (\lambda'_j, j) \text{ is } k\text{-related to } (i,j')\}$$

for $1 \leq j \leq k$.

Then we define the signed Grassmann permutation corresponding to λ as in [BKT2]. Let m be the middle row of λ as defined in §5.2.1. The signed permutation corresponding to λ is w_λ where w_λ has the property that $w_\lambda(i) < w_\lambda(i+1)$ for all $i \neq k$, $w_\lambda(i) > 0$ for all $i \geq k+m$, and

$$w_\lambda(i) = \begin{cases} n_{k-i+1}(\lambda) & \text{if } 1 \leq i \leq k \\ k - \lambda_{i-k} & \text{if } k+1 \leq i < k+m \end{cases}$$

This is perhaps easier to see in a picture, taken from [BKT2]. Let $n = 7, k = 3$ and let λ be the partition (8,5,2,1). Below the diagram for λ drawn with the appropriate k-related diagonals filled with dots.

We note that the $w_\lambda(i)$ will be the length of a k-related diagonal. So in Figure

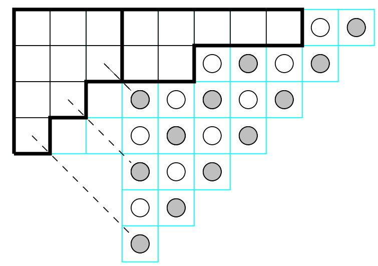


Figure 5.3: The diagram of λ with marked k-related diagonals.

5.3 we have that the first k-related diagonal is given by $n_k(\lambda)$ where in this case since $k = 3$ we have $\lambda'_3 = 2$ and we notice that the boxes 3-related to the $(2,3)$ box are $(3,4),(2,5)$, and $(1,6)$, but since $\lambda_1 \geq 6$ and $\lambda_2 \geq 5$ we see that $n_3(\lambda) = 1$ and we see in the figure that the smallest k-related diagonal has length 1. Similarly the other k-related diagonals have length 4 and 7 respectively. Also we note that $k - \lambda_1 = -5$ and $k - \lambda_2 = -2$ but $k - \lambda_i \geq 0$ for $i > 2$ so in the above example, for $n = 7, k = 3$ and $\lambda = (8,5,2,1)$ we have $w_\lambda = (1,4,7,-5,-2,3,6)$.

5.4 Restriction to Torus Fixed Points in Type C

We have an inclusion $\iota : pt \hookrightarrow IG(n-k,2n)$. This induces a ring homomorphism on the equivariant cohomology rings $\iota^* : H_T^*(IG(n-k,2n)) \to H_T^*(pt) = \mathbb{Z}[t_1,\ldots,t_n]$. When the point we are restricting to corresponds to λ, this homomorphism takes the equivariant Chern class z_i defined in §2.2.1 to the equivariant Chern class $t_{w_\lambda(i)}$ and takes x_i to t_{λ_i-k} or 0 if $\lambda_i - k \leq 0$, where we cite [IMN] for the geometric details for the localization map.

Remark 1. We note that as a tuple the z variables are mapped to the tuple of

t variables indexed by the lengths of the related k-diagonals, and that for the x variables x_i is mapped to the t variable indexed by the absolute value of $w_\lambda(k+i)$ if $w_\lambda(k+i) < 0$ and is mapped to zero otherwise. We also note that if $w_\lambda \in S_n$ then all x variables are mapped to zero!

5.5 Factorial Theta Polynomials

We will first define the factorial Theta polynomials indexed by partitions of length one, use these to define Θ_λ for an arbitrary k-strict partition λ, and then we will give a conjectural Giambelli formula for the general Schubert classes. Note that it is necessary that when we consider the non-equivariant cohomology for $IG(n-k, 2n)$ it should coincide with the known results of Buch, Kresch, and Tamvakis. Thus we have the following preliminary definition:

Definition 6. For $p > k$

$$\theta_p(x, z|t) = \sum_{i=k+1}^{p} (-1)^{p-i} \theta_i(x_1, x_2, \ldots | z_1, \ldots, z_k) e_{p-i}(t_1, \ldots, t_{p-k-1})$$

and for $p \leq k$

$$\theta_p(x, z|t) = \sum_{i=1}^{p} (-1)^i \theta_{p-i}(x_1, x_2, \ldots | z_1, \ldots, z_k) h_i(t_1, \ldots, t_{k-p+1})$$

It should be noted that the above factorial theta polynomials coincide with the double Schubert functions described in [IMN]. We prove this in a proposition below.

Proposition 6. *The $\theta_p(x, z|t)$ defined above represent the special Schubert classes for the partition p.*

Proof. Let $\mathfrak{C}_w(x,z|t)$ for a Weyl group element w denote the type C double Schubert polynomial described in [IMN], let $\mathfrak{C}_v(x,z)$ represent the type C Billey-Haiman Schubert polynomial for a Weyl group element v, which are known to represent the Schubert class corresponding to v in a presentation of the classical cohomology of the complete flag variety in type C, and let $\mathfrak{S}_u(t)$ be the classical type A Schubert polynomial for a permutation u which is known to represent the Schubert class corresponding to u in the classical cohomology of the complete flag variety in type A. Then by [IMN, Cor. 8.10] we have

$$\mathfrak{C}_w(x,z|t) = \sum_{u,v} \mathfrak{S}_{u^{-1}}(-t)\mathfrak{C}_v(x,z)$$

where $l(w) = l(u) + l(v), uv = w$, and $v, w \in W$, u a permutation.

Then we note in type C, if the rank of the group is n, W is generated by simple reflections $\{s_i\}_{i=0}^{n}$, where $s_i = (i, i+1)$ for $i \geq 1$, and $s_0(i) = -i$ for all i. Then we can write the Weyl group elements corresponding to the special classes as $s_{k-p+1} \dots s_k$ for σ_p when $p \leq k$, and $s_{p-k-1} \dots s_0 s_1 \dots s_k$ for σ_p when $n+k \geq p > k$.

In general we call such a listing of simple reflections whose product is a given Weyl group element a word. Observe that u is a permutation only when we can write it as a combination of simple reflections which do not contain s_0. We notice that for $p \in \{1 \dots k\}$, any u which is a subword of w_p, will be a permutation. However for $p \in \{k+1, \dots, n+k\}$, we notice that only a small subset of subwords do not contain the reflection s_0. To get our result we only need to look at cases.

Consider the case when $p \leq k$. Then we get all possibilities of u, v such that $uv = w$ since s_0 is not part of the word for w_p. Then we note that in each case

u^{-1} is the permutation corresponding to the partition 1^r in type A, where r is the length of the subword. Therefore the Schubert polynomial in type A for u^{-1} will be $h_r(t_1, \ldots, t_{k-p+1})$. We also notice that for each u the remaining v will be the Weyl group element for $q \leq p$. Then we get the desired term by summing all possibilities, and noting that the Billey-Haiman type C Schubert polynomials coincide with the theta polynomials in this case.

Now consider the case when $p > k$. Then we notice that the only possible subwords u are those that don't contain s_0. Therefore we are left with u^{-1} which correspond to a one part partition of length $r < p - k - 1$ in type A, for which the Schubert polynomial will be $e_r(t_1, \ldots, t_{p-k-1})$. Again the remaining v will be the Weyl group element for $k+1 \leq q \leq p$. And again we get the desired term by summing all possibilities. $\square$

Definition 7. For any integers p, r, j we define

$$\theta_p^r[j] = \begin{cases} \sum_{i=0}^{p-k-1} (-1)^i \theta_{p+j-i}(x_1, x_2, \ldots | z_1, \ldots, z_k) e_i(t_1, \ldots, t_r) \text{ if } k < p \\ \sum_{i=0}^{p+j} (-1)^i \theta_{p+j-i}(x_1, x_2, \ldots | z_1, \ldots, z_k) h_i(t_1, \ldots, t_r) \text{ if } k \geq p \end{cases}$$

where if either $p + j$ or r are negative we take the corresponding factorial theta polynomial to be zero and if $r = 0$ then we recover the standard theta polynomial $\theta_{p+j}(x, z)$ of Definition 3. We also note that if $p > k$, $r = p - k - 1$, and $j > 0$ then $\theta_p^r[j] = \theta_{p+j}^r[0]$ and similarly if $p \leq k$ and $p + j \leq k$ for an integer j then $\theta_p^r[j] = \theta_{p+j}^r[0]$.

Definition 8. For a valid set D define the following two functions:

1. $a_j(D) = \#\{i | i < j; (i, j) \notin D\}$

2. $c_j(D) = \#\{i|i < j; (i,j) \in D\} = j - 1 - a_j(D)$

For simplicity we define $a_j(\lambda) := a_j(\mathcal{C}(\lambda))$ and $c_j(\lambda) := c_j(\mathcal{C}(\lambda))$.

Definition 9. For integer sequences λ, γ, ρ let $\theta_{\lambda;\gamma;\rho} = \prod_{i=1}^{\ell} \theta_{\lambda_i}^{\gamma_i}[\rho_i]$.

For any k-strict partition λ, set $\gamma(\lambda)$ to be the ℓ-tuple such that $\gamma(\lambda)_i = |k - \lambda_i + 1 + a_i(\lambda)|$.

Definition 10. Let ν, ρ, and γ be any integer sequences, and let D be any valid set. Define

$$\Theta^D_{\nu;\gamma;\rho} = R^D \theta_{\nu;\gamma;\rho}$$

where for any $i < j$, $R_{ij}\theta_{\nu;\gamma;\rho} := \theta_{\nu;\gamma;R_{ij}(\rho)}$.

Proposition 7. *If $p, q > k$ then*

$$\left(\frac{1 - R_{1,2}}{1 + R_{1,2}}\right) \theta_p^r[0]\theta_q^s[0] = -\left(\frac{1 - R_{1,2}}{1 + R_{1,2}}\right) \theta_q^s[0]\theta_p^r[0]$$

for any non-negative integers r, s.

Proof. Let $t_{(r)} = (t_1, \ldots, t_r)$ and similarly let $t_{(s)} = (t_1, \ldots, t_s)$. Then

$$\begin{aligned}
&\left(\frac{1 - R_{1,2}}{1 + R_{1,2}}\right) \theta_p^r[0]\theta_q^s[0] \\
&= \theta_p^r\theta_q^s + 2\sum_{j=1}^{\infty}(-1)^j \theta_p^r[j]\theta_q^s[-j] \\
&= \left(\sum_{i=0}^{p-k-1} (-1)^i \theta_{p-i}(x,z)e_i(t_{(r)})\right)\left(\sum_{i=0}^{q-k-1} (-1)^i \theta_{q-i}(x,z)e_i(t_{(s)})\right) \\
&+ 2\sum_{j=1}^{\infty}(-1)^j \left(\sum_{i=0}^{p-k-1} (-1)^i \theta_{p+j-i}(x,z)e_i(t_{(r)})\right)\left(\sum_{i=0}^{q-k-1} (-1)^i \theta_{q-j-i}(x,z)e_i(t_{(s)})\right) \\
&= \sum_{a=k+1}^{p}\sum_{b=k+1}^{q} (-1)^{p+q-a-b} e_{p-a}(t_{(r)})e_{q-b}(t_{(s)})\Theta_{a,b}
\end{aligned}$$

Then by Lemma 7 $\Theta_{a,b}(x,z) = -\Theta_{b,a}(x,z)$ since $a+b > 2k$. Note that by a similar argument

$$\left(\frac{1-R_{1,2}}{1+R_{1,2}}\right)\theta_q^s[0]\theta_p^r[0] = \sum_{a=k+1}^{p}\sum_{b=k+1}^{q}(-1)^{p+q-a-b}e_{p-a}(t_{(r)})e_{q-b}(t_{(s)})\Theta_{b,a}.$$

Therefore

$$\left(\frac{1-R_{1,2}}{1+R_{1,2}}\right)\theta_p^r[0]\theta_q^s[0] = -\left(\frac{1-R_{1,2}}{1+R_{1,2}}\right)\theta_q^s[0]\theta_p^r[0].$$

□

Definition 11. For a k-strict partition λ, define the factorial Theta polynomial corresponding to λ as

$$\Theta_\lambda = \Theta^{C(\lambda)}_{\lambda;\gamma(\lambda);\mathbf{0}} = R^\lambda\theta_{\lambda;\gamma(\lambda);\mathbf{0}}.$$

Conjecture 1. Let λ be a k-strict partition. Then the Schubert class corresponding to λ in the equivariant cohomology of $IG(n-k,2n)$ is represented by Θ_λ.

We'll now prove a series of lemmas similar to Lemmas 6 and 7 which the factorial Theta polynomials satisfy.

Lemma 10. *Let $\lambda = (\lambda_1,\dots,\lambda_{j-2})$ and $\mu = (\mu_{j+1},\dots,\mu_\ell)$ be integer vectors. Assume that $(j-1,j)\notin D$ and that for each $h<j$, $(h,j-1)\notin D$ iff $(h,j)\notin D$. Then for any integers r and s, and any integer sequences ρ and $\hat\rho$ such that $\rho_{j-1}+1 = \hat\rho_j$ and $\hat\rho_{j-1}+1 = \rho_j$ we have*

$$\Theta^D_{(\lambda,r,s,\mu);\gamma;\rho} = -\Theta^D_{(\lambda,s,r,\mu);s_j(\gamma);\hat\rho}.$$

To prove this we will use a similar recursion and induction as in the type A case. As in [BKT; 1.2] we set for a sequence of nonnegative integers α and a positive integer ℓ we set $m(D,\alpha,\ell) = \#\{i \leq c_\ell(D) : \alpha_i > 0\}$ and call α (D,ℓ)-compatible if $\alpha_i \in \{0,1\}$ for $i > c_\ell(D)$. Note that for any integer sequences λ, ρ of length $\ell - 1$ we have

$$\Theta^D_{(\lambda,r);\gamma;(\rho,s)}$$

$$= \sum_\alpha (-1)^{|\alpha|} 2^{m(D,\alpha,\ell)} \Theta^D_{\lambda;\gamma;\alpha+\rho} \theta_r^{\gamma_{\ell+1}}[s - |\alpha|]$$

where α is (D,ℓ)-compatible.

Proof. From the above recursion we can assume that μ is empty. Then applying the recursion twice to $\Theta^D_{(\lambda,r,s);\gamma;\rho}$ we have:

$$\sum_{\alpha,\beta} (-1)^{|\alpha|+|\beta|} 2^{m(D,\alpha,j)+m(D,\beta,j-1)} (\Theta^D_{\lambda;\gamma;\alpha+\beta+\rho'}) \times$$

$$(\theta_r^{\gamma_{j-1}}[\rho_{j-1} - |\alpha|] \theta_s^{\gamma_j}[\rho_j - |\beta|] - \theta_r^{\gamma_{j-1}}[\rho_{j-1} + 1 - |\alpha|] \theta_s^{\gamma_j}[\rho_j - 1 - |\beta|])$$

where $\rho' = \rho_1, \ldots, \rho_{j-2}$, α is (D,j)-compatible, and β is $(D,j-1)$-compatible. Similarly applying the recursion twice to $\Theta_{(\lambda,s,r);s_j(\gamma);\hat{\rho}}$ we have:

$$\sum_{\alpha,\beta} (-1)^{|\alpha|+|\beta|} 2^{m(D,\alpha,j-1)+m(D,\beta,j)} (\Theta^D_{\lambda;\gamma;\alpha+\beta+\hat{\rho}'}) \times$$

$$(\theta_s^{\gamma_j}[\hat{\rho}_{j-1} - |\beta|] \theta_r^{\gamma_{j-1}}[\hat{\rho}_j - |\alpha|] - \theta_s^{\gamma_j}[\hat{\rho}_{j-1} + 1 - |\alpha|] \theta_r^{\gamma_{j-1}}[\hat{\rho}_j - 1 - |\beta|])$$

where $\hat{\rho}' = \hat{\rho}_1, \ldots, \hat{\rho}_{j-2}$, α is $(D,j-1)$-compatible, and β is (D,j)-compatible. We note that since for each $h < j$, $(h,j-1) \notin D$ iff $(h,j) \notin D$ the set of sequences which are $(D,j-1)$-compatible is the same as those which are (D,j)-compatible, so that $\Theta_{(\lambda,s,r);s_j(\gamma);\hat{\rho}}$ is the negative of the expression for $\Theta_{D,(\lambda,r,s);\gamma;\rho}$ since we have $\rho_{j-1} + 1 = \hat{\rho}_j$ and $\hat{\rho}_{j-1} + 1 = \rho_j$ so the lemma is proven. $\square$

Corollary 4. *Suppose that we have the same situation as in the previous lemma where $\lambda_{j-1} = \lambda_j \leq k$ and for each $h < j$, $(h, j-1) \notin \mathcal{C}(\lambda)$ iff $(h,j) \notin \mathcal{C}(\lambda)$. Then we have the following*

$$R^\lambda R_{j,\ell+1}\theta_{(\lambda,1);\gamma(\lambda);\mathbf{0}} = -t_{k-\lambda_j+1+a_j(\lambda)}\Theta_\lambda.$$

Proof. Let $\lambda^{(j)}$ be the result of adding one box to the j^{th} row of λ (note that this might no longer be a partition), and let $\gamma = \gamma(\lambda)$ for this proof. Note that since for each $h < j$, $(h,j) \notin \mathcal{C}(\lambda)$ iff $(h, j+1) \notin \mathcal{C}(\lambda)$ we have that $\mathcal{C}(\lambda) = \mathcal{C}(\lambda^{(j)})$.

$$\begin{aligned}
&R^\lambda R_{j,\ell+1}\theta_{(\lambda,1);\gamma;\mathbf{0}} \\
&= R^\lambda \prod_{i=1}^{j-1} \theta_{\lambda_i}^{|k+1-\lambda_i+a_i(\lambda)|}[0]\theta_{\lambda_j}^{k+1-\lambda_j+a_j(\lambda)}[1] \prod_{i=j+1}^{\ell} \theta_{\lambda_i}^{k+1-\lambda_i+a_i(\lambda)}[0] \\
&= R^\lambda \prod_{i=1}^{j-1} \theta_{\lambda_i}^{|k+1-\lambda_i+a_i(\lambda)|}[0]\theta_{\lambda_j}^{k-\lambda_j+a_j(\lambda)}[1] \prod_{i=j+1}^{\ell} \theta_{\lambda_i}^{k+1-\lambda_i+a_i(\lambda)}[0] \\
&+ R^\lambda \prod_{i=1}^{j-1} \theta_{\lambda_i}^{|k+1-\lambda_i+a_i(\lambda)|}[0](-t_{k-\lambda_j+1+a_j(\lambda)})\theta_{\lambda_j}^{k+1-\lambda_j+a_j(\lambda)}[0] \prod_{i=j+1}^{\ell} \theta_{\lambda_i}^{k+1-\lambda_i+a_i(\lambda)}[0] \\
&\text{by equation (3.1) for complete symmetric polynomials} \\
&= \Theta^{\mathcal{C}(\lambda)}_{\lambda;\gamma^{(j-)};\epsilon_j} - t_{k-\lambda_j+1+a_j(\lambda)}\Theta_{\lambda;\gamma} \\
&= -t_{k-\lambda_j+1+a_j(\lambda)}\Theta_\lambda
\end{aligned}$$

Where here we set $\gamma^{(j-)} = \gamma$ everywhere except the j^{th} spot where it is one less. So since $a_{j-1}(\lambda) + 1 = a_j(\lambda)$ since for each $h < j$, $(h,j) \notin \mathcal{C}(\lambda)$ iff $(h, j+1) \notin \mathcal{C}(\lambda)$, we have $s_j(\gamma^{(j-)}) = \gamma^{(j-)}$. Hence by the previous lemma $\Theta^{\mathcal{C}(\lambda)}_{\lambda;\gamma^{(j-)};\epsilon_j} = -\Theta^{\mathcal{C}(\lambda)}_{\lambda;\gamma^{(j-)};\epsilon_j}$ where ϵ_j is the sequence which is zero everywhere except the j^{th} position where it is 1. This means it must be zero, and we are left with the desired result. □

Lemma 11. *Let $\lambda = (\lambda_1, \ldots, \lambda_{j-1})$ and $\mu = (\mu_{j+2}, \ldots, \mu_\ell)$ be integer vectors, assume $(j, j+1) \in D$, and that for each $h > j+1$, $(j, h) \in D$ iff $(j+1, h) \in D$. If $r, s \in \mathbb{Z}$ are such that $r, s > k$ and $D = \mathcal{C}(\lambda, r, s, \mu)$, then we have*

$$\Theta^D_{(\lambda,r,s,\mu);\gamma;\mathbf{0}} = -\Theta^D_{(\lambda,s,r,\mu);s_j(\gamma);\mathbf{0}}\,.$$

Proof. We will again proceed by using our recursion formula to assume that μ is empty. Following the proof of Lemma 7 in [BKT] we set $\ell = j+1$, and note that $(h, h') \in D$ for all $h < h' \leq \ell$. If $m > 0$ is the least integer such that $2m \geq \ell$, we claim that $\Theta^D_{\alpha;\gamma;\mathbf{0}} = \Theta^D_{(\lambda,r,s);\gamma;\mathbf{0}}$ satisfies the relation

$$\Theta^D_{\alpha;\gamma;\mathbf{0}} = \sum_{i=2}^{2m} (-1)^i \, \Theta^D_{(\alpha_1,\alpha_i);(\gamma_1,\gamma_i);\mathbf{0}} \, \Theta^D_{(\alpha_2,\ldots,\widehat{\alpha_i},\ldots,\alpha_{2m});(\gamma_2,\ldots,\widehat{\gamma_i},\ldots,\gamma_{2m});\mathbf{0}}. \tag{5.4}$$

Equation (5.4) follows from the formal identity of raising operators

$$\prod_{1\leq h<h'\leq 2m} \frac{1-R_{hh'}}{1+R_{hh'}} = \sum_{i=2}^{2m} (-1)^i \frac{1-R_{1i}}{1+R_{1i}} \prod_{\substack{2\leq h<h'\leq 2m \\ h\neq i\neq h'}} \frac{1-R_{hh'}}{1+R_{hh'}},$$

which is equivalent to the classical formula

$$\prod_{1\leq h<h'\leq 2m} \frac{x_h - x_{h'}}{x_h + x_{h'}} = \text{Pfaffian}\left(\frac{x_h - x_{h'}}{x_h + x_{h'}}\right)_{1\leq h,h'\leq 2m}$$

due to Schur. The proof is completed using induction, starting from the base case of $j = 1$, which was obtained in Proposition 7.

∎

Corollary 5. *If λ is a k-strict partition, $\lambda_{j-1} > \lambda_j > k$ and $\lambda_j + 1 = \lambda_{j-1}$ then if for each $h > j+1$, $(j, h) \in \mathcal{C}(\lambda)$ iff $(j+1, h) \in \mathcal{C}(\lambda)$ we have*

$$R^\lambda R_{j,\ell+1} \theta_{(\lambda,1);\gamma(\lambda);\mathbf{0}} = t_{\lambda_j - k} \Theta_{\lambda;\gamma(\lambda);\mathbf{0}}$$

Proof. For this proof we let $\gamma^{(j+)} = \gamma(\lambda)$ everywhere except the j^{th} place where it is one more. Then since $\lambda_j + 1 = \lambda_{j-1} > k+1$ we have $s_j(\gamma^{(j+)}) = \gamma^{(j+)}$. We also note that since $\lambda_j > k$ we have that $\theta_{\lambda_j}^{\lambda_j-k-1}[1] = \theta_{\lambda_j+1}^{\lambda_j-k-1}[0]$. Hence we have:

$$
\begin{aligned}
& R^{\lambda} R_{j,\ell+1}\theta_{(\lambda,1);\gamma;\mathbf{0}} \\
&= R^{\lambda}\prod_{i=1}^{j-1}\theta_{\lambda_i}^{(\lambda_i-k-1)}[0]\theta_{\lambda_j}^{(\lambda_j-k-1)}[1]\prod_{i=j+1}^{\ell}\theta_{\lambda_i}^{|k+1-\lambda_i+a_i(\lambda)|}[0] \\
&= R^{\lambda}\prod_{i=1}^{j-1}\theta_{\lambda_i}^{(\lambda_i-k-1)}[0]\theta_{\lambda_j+1}^{(\lambda_j-k)}[0]\prod_{i=j+1}^{\ell}\theta_{\lambda_i}^{|k+1-\lambda_i+a_i(\lambda)|}[0] \\
&+ R^{\lambda}\prod_{i=1}^{j-1}\theta_{\lambda_i}^{(\lambda_i-k-1)}[0](t_{\lambda_j-k})\theta_{\lambda_j}^{(\lambda_j-k-1)}[0]\prod_{i=j+1}^{\ell}\theta_{\lambda_i}^{|k+1-\lambda_i+a_i(\lambda)|}[0] \\
& \text{by equation (3.1) for elementary symmetric polynomials} \\
&= \Theta_{\lambda^{(j)};\gamma^{(j+)};\mathbf{0}}^{\mathcal{C}(\lambda)} + t_{\lambda_j-k}\Theta_{\lambda} \\
&= t_{\lambda_j-k}\Theta_{\lambda}.
\end{aligned}
$$

Since $\mathcal{C}(\lambda) = \mathcal{C}(\lambda^{(j)})$ since for each $h > j+1$, $(j,h) \in \mathcal{C}(\lambda)$ iff $(j+1,h) \in \mathcal{C}(\lambda)$ and since $\Theta_{\lambda^{(j)};\gamma^{(j+)};\mathbf{0}}^{\mathcal{C}(\lambda)} = 0$ by the above lemma.

$\square$

5.6 The Equivariant Chevalley Formula

We will recall the equivariant Chevalley formula for $IG(n-k,2n)$ will be indexed by the same partitions as the classical Chevalley formula, and will have an additional equivariant correction given by $\sigma_1\big|_{\lambda}\sigma_{\lambda}$. Therefore the equivariant

Chevalley formula is given by

$$\sigma_1 \cdot \sigma_\lambda = \sigma_1\big|_\lambda \sigma_\lambda + \sum_{\lambda \to \mu} e_{\lambda\mu}\, \sigma_\mu \tag{5.5}$$

where $e_{\lambda\mu}$ is as it is defined in the classical case. Also we note that when we consider the special Schubert class σ_1 as a Theta polynomial we have the following coefficient on our equivariant correction term:

$$\sigma_1\big|_\lambda = \theta_1^k\big|_\lambda = 2\sum_{i=1}^{\ell} t_{\lambda_i - k} + \sum_{j=1}^{k} t_{w_\lambda(j)} - \sum_{l=1}^{k} t_l$$

where we set $t_i = 0$ whenever $i \leq 0$.

The way we prove the equivariant Chevalley formula will be very similar to how the classical Chevalley formula is proven. The difference here will be that in our substitution algorithm we need to also keep track of any equivariant corrections.

5.7 The Equivariant Substitution Rule

We fix a k-strict partition λ of length ℓ, set m to be the middle row of λ defined in §5.2.1, let $\mathcal{C} = \mathcal{C}(\lambda)$, and choose n sufficiently large so that all Pieri terms that can possibly appear in the Chevalley formula do not vanish just as we did in the classical case. For the equivariant substitution rule we will need to keep track of the changes necessary to transform $\gamma(\lambda)$ into $\gamma(\mu)$ as well as the usual transforming of $\mathcal{C}$ into $\mathcal{C}(\mu)$ and ρ to $\mathbf{0}$. To account for this we will need to make our triple into a 4-tuple.

Definition 12. For any integer sequence ρ the 4-tuple (D, μ, S, ρ) will be called valid if the triple $(D, \mu + \rho, S)$ was valid in the classical case. A 4-tuple will correspond to a Schubert class if it is of the form $(\mathcal{C}(\mu), \mu, S, \mathbf{0})$ for some k-strict partition μ. For any valid 4-tuple $\psi = (D, \mu, S, \rho)$ we let

$$ev(\psi) = R^D \theta_{\mu;\gamma(D,\mu,\rho);\rho}$$

where $\gamma(D, \mu, \rho)_i = |k + 1 - \rho_i - \mu_i + a_i(D)|$. We note that if $D = \mathcal{C}(\mu)$ and $\rho = \mathbf{0}$ then $ev(\psi) = \Theta_\mu$ since $\gamma(\mathcal{C}(\mu), \mu, \mathbf{0}) = \gamma(\mu)$.

For a 4-tuple $\psi = (D, \mu, S, \rho)$ we define a new weight condition. We say the 4-tuple ψ satisfies the weight condition $W(i, j)$ for $j > i$ if $\mu_i+\rho_i+\mu_j+\rho_j > 2k+j-i$.

Let $\gamma = \gamma(\lambda, 1)$ which is the same as $\gamma(\lambda)$ in the first ℓ components, let $\lambda^{(j)}$ be such that $\lambda^{(j)}_i = \lambda_i$ for all $i \neq j$ and $\lambda^{(j)}_j = \lambda_j + 1$, and let $\gamma^{(j+)} = \gamma + \epsilon_j$ and $\gamma^{(j-)} = \gamma - \epsilon_j$, where we recall that ϵ_j is the sequence which has a 1 in the j^{th} position and zeros elsewhere.

We compute that

$$\Theta_\lambda \cdot \Theta_1 = R^\lambda \theta_{(\lambda,1);\gamma;\mathbf{0}} + \sum_{i=1}^{a_{\ell+1}(\lambda,1)} t_{k+i}\Theta_\lambda$$

since $\Theta_1 = \Theta^{\emptyset}_{1;k;\mathbf{0}}$ and since $\gamma(\lambda, 1)_{\ell+1} = k + a_{\ell+1}(\lambda, 1)$. Then the above can be expanded as in the classical case to

$$R^\lambda_{\ell+1} \cdot \prod_{i=1}^{\ell} (1 + R_{i,\ell+1})\, \theta_{(\lambda,1);\gamma;\mathbf{0}} + \sum_{i=1}^{a_{\ell+1}(\lambda,1)} t_{k+i}\Theta_\lambda.$$

At this point we notice this is the same as

$$\Theta_\lambda \cdot \Theta_1 = \sum_{j=1}^{\ell+1} \Theta^{\mathcal{C}}_{(\lambda;\gamma(\lambda);\epsilon_j)} + \sum_{i=1}^{a_{\ell+1}(\lambda,1)} t_{k+i}\Theta_\lambda, \tag{5.6}$$

where we note $\theta^r_0[1] = \theta^r_1[0]$.

We aim to show that the right hand side of Equation (5.6) is equal to the right hand side of the equivariant Chevalley rule. We will do this using the an equivariant version of the substitution rule from the classical case.

Thus in the above the 4-tuples corresponding to terms on the right hand side of Equation (5.6) will be $\{(\mathcal{C}, \lambda, \emptyset, \epsilon_j) : 1 \leq j \leq \ell + 1\} := \Psi$.

The following *substitution rule* will be applied iteratively to rewrite the right hand side of (5.6). It may be applied to any valid 4-tuple and will result in either a REPLACE statement, indicating that the 4-tuple should be replaced by one or two new 4-tuples, or a STOP statement, indicating that the 4-tuple should not be replaced.

Equivariant Substitution Rule

Let (D, μ, S, ρ) be a valid 4-tuple. Let $h \leq \ell + 1$ be largest such that one of the following five conditions is true (if none hold for any h, then STOP).

(i) $(h-1, h) \notin D$ and there is an outer corner (i, h) of D with $i \leq m$ such that $W(i, h)$ holds;

(ii) $(h-1, h) \notin D$, D has no outer corner in column h, and $\mu_h + \rho_h = \lambda_{h-1} + 1$;

(iii) $(h-1, h) \in D$ and there is an outer corner (h, j) of D with $j \leq \ell + 1$ such that $W(h, j)$ holds;

(iv) $(h-1,h) \in D$ and $\mu_h + \rho_h = \mu_{h-1} + \rho_{h-1}$.

(v) $\rho \neq \mathbf{0}$.

If condition **(i)** holds, then REPLACE (D, μ, S, ρ) with $(D \cup (i,h), \mu, S, \rho)$ and $(D \cup (i,h), \mu, S \cup (i,h), R_{ih}\rho)$. If **(iii)** holds, then REPLACE (D, μ, S, ρ) with $(D \cup (h,j), \mu, S \cup (h,j), R_{hj}\rho)$ if $\mu_h > \mu_{h-1}$, or REPLACE (D, μ, S, ρ) with $(D \cup (h,j), \mu, S \cup (h,j), R_{hj}\rho)$ and $(D \cup (h,j), \mu, S, \rho)$. If **(v)** holds then REPLACE (D, μ, S, ρ) with $(D, \mu + \rho, S, \mathbf{0})$ If **(ii)** or **(iv)** holds, then STOP.

We begin by performing the Substitution rule on terms at level $h = \ell+1$. First note that $(\ell, \ell+1) \notin \mathcal{C}$ and of the first 4 conditions, only condition **(i)** has any chance of holding. If condition **(i)** holds then we note that it must hold for $i = 1$, and in fact will hold for $i = 1, 2, \ldots, c_{\ell+1}(\lambda, 1)$. Thus we will have $c_{\ell+1}(\lambda, 1)$ additional terms, corresponding to the valid 4-tuples $(\mathcal{C}, \lambda, \{(j, \ell+1)\}, \epsilon_j)$ for $j = 1, 2, \ldots, c_{\ell+1}(\lambda, 1)$, and this will correspond to the following in our expansion in the Chevalley formula for the factorial Theta polynomials.

$$
\begin{aligned}
&\Theta_\lambda \cdot \Theta_1 \\
&= R^{\lambda,1} \prod_{i=1}^{c_{\ell+1}(\lambda,1)} (1+2R_{i,\ell+1})\theta_{(\lambda,1);\gamma} \prod_{i=c_{\ell+1}(\lambda,1)+1}^{\ell} (1+R_{i,\ell+1})\theta_{\lambda,1;\gamma} + \sum_{i=1}^{a_{\ell+1}(\lambda,1)} t_{k+i}\Theta_{\lambda;\gamma} \\
&= \Theta_{\lambda,1} + 2\sum_{j\le c_{\ell+1}(\lambda,1)} R^\lambda \theta_{\lambda;\gamma;\epsilon_j} + \sum_{j>c_{\ell+1}(\lambda,1)} R^\lambda \theta_{\lambda;\gamma;\epsilon_j} + \sum_{i=1}^{a_{\ell+1}(\lambda,1)} t_{k+i}\Theta_{\lambda;\gamma;\mathbf{0}} \\
&= \Theta_{\lambda,1} + 2\sum_{j\le c_{\ell+1}(\lambda,1)} R^\lambda \theta_{\lambda;\gamma^{(j+)};\epsilon_j} + \sum_{j>c_{\ell+1}(\lambda,1),j<m} R^\lambda \theta_{\lambda;\gamma^{(j+)};\epsilon_j} \\
&+ \sum_{m\ge j>c_{\ell+1}(\lambda,1)} R^\lambda \theta_{\lambda;\gamma^{(j-)};\epsilon_j} \\
&+ \left(\sum_{j\le c_{l\ell+1}(\lambda,1)} 2t_{\lambda_j-k} + \sum_{j>c_{\ell+1}(\lambda,1)} (t_{\lambda_j-k} - t_{k-\lambda_j+1+a_j(\lambda)}) + \sum_{i=1}^{a_{l+1}(\lambda,1)} t_{k+i} \right) \Theta_\lambda
\end{aligned}
$$

where we note that if the index on t is negative, then we take it to be zero. The equivariant correction terms on the last line are gathered using the same relations amongst elementary and complete symmetric functions as in the proofs of Corollaries 4 and 5. Thus our new goals are to show that the coefficient on Θ_λ is $\Theta_1\big|_\lambda$, and that the descendants of the 4-tuples:

1. $(\mathcal{C}, \lambda, \{(j, \ell+1)\}, \epsilon_j)$ for $j \le c_{\ell+1}(\lambda, 1)$

2. $(\mathcal{C}, \lambda, \emptyset, \epsilon_j)$ for $j < m$

3. $(\mathcal{C}, \lambda, \emptyset, \epsilon_j)$ for $j \ge m$

which evaluate to the right hand side of the above equation correspond to Schubert classes appearing the right hand side of Equation (5.5).

To do this we will apply the equivariant substitution rule, and show the algorithm will give us the correct results.

First note that in input for the algorithm, our 4-tuples correspond to the triples in the classical case in that our equivariant algorithm acts on (D, μ, S, ρ) the same way as the classical algorithm acts on $(D, \mu+\rho, S)$, where we use Lemma 10 in place of Lemma 6 and Lemma 11 in place of Lemma 7. Hence we need only show that after the algorithm has terminated, the remaining $(\mathcal{C}(\mu), \mu, S, \rho)$ have the property that that $\rho = 0$. We also must show that our replace steps match up algebraically with correct manipulations of the factorial theta polynomials.

We work through this algorithm with the example $\lambda = (7, 4, 3, 2, 1, 1)$ for $IG(9-2, 18)$ in the appendix.

Recall from the classical setting that the REPLACE step for conditions **(i)** and **(iii)** corresponds to equality of the raising operators below

$$1 - R_{ij} = \frac{1 - R_{ij}}{1 + R_{ij}} + \frac{1 - R_{ij}}{1 + R_{ij}} R_{ij}. \tag{5.7}$$

We have $\mathrm{ev}(D, \mu, S, \rho) = R^D \theta_{\mu;\gamma(D,\mu,\rho);\rho}$. For simplicity let $\gamma = \gamma(D, \mu, \rho)$. Then when we apply Equation (5.7) we would a priori have

$$\mathrm{ev}(D, \mu, S, \rho) = R^{\{D \cup (i,j)\}} \theta_{\mu;\gamma;\rho} + R^{\{D \cup (i,j)\}} \theta_{\mu;\gamma;R_{ij}\rho}.$$

Then we note that in the algorithm if we perform this REPLACE step then we were in one of two possible cases, where both cases will have the property that $W(i, j)$ holds for $\mu+\rho$ but (i, j) is an outer corner of D. This means that $\gamma_i = \mu_i + \rho_i - k - 1$ while $\gamma_j = k + 1 - \mu_j - \rho_j + \#\{r : r < j, (r, j) \notin D\}$. So $(\gamma^{(i+)})_i = \mu_i + \rho_i - k$ and

$(\gamma^{(j-)})_j = k - \mu_j - \rho_j + \#\{r : r < j, (r,j) \notin D\}$. Since (i,j) is an outer corner, we have that

$$\#\{r : r < j, (r,j) \notin D\} = j - i.$$

$W(i,j)$ being satisfied by $\mu+\rho$ means that $\mu_i+\rho_i+\mu_j+\rho_j > 2k+j-i$. Combining these relations we must have that

$$(\gamma^{(i+)})_i - \gamma_j = \mu_i\rho_i + \mu_j + \rho_j - 2k - j + i - 1.$$

Then by Proposition 4 we know $\mu_j+\rho_j+\mu_i+\rho_i = 2k+j-i+1$ so we have $(\gamma^{(i+)})_i = \gamma_j$. We notice that $\gamma^{(i+)} = \gamma(D\cup(i,j), \mu, R_{ij}\rho)$ and $\gamma^{(j-)} = \gamma(D\cup(i,j), \mu, R_{ij}\rho)$.

Hence our REPLACE step corresponds to the usual application of Equation 5.7 along with the relations

$$\theta_{\mu;\gamma;\rho} = \theta_{\mu;\gamma^{(j-)};\rho} - t_{\gamma_j}\theta_{\mu;\gamma;\rho-\epsilon_j}$$

from Equation (3.1) for the complete symmetric functions, and

$$\theta_{\mu;\gamma;R_{ij}\rho} = \theta_{\mu;\gamma^{(i+)};R_{ij}\rho} + t_{(\gamma^{(i+)})_i}\theta_{R_{ij}\mu;\gamma;\rho-e_i}$$

from Equation (3.1) for the elementary symmetric functions. Note that $\rho - \epsilon_j = R_{ij}\rho - e_i$ so these correction terms will cancel by the above argument. Hence we have

$$\mathrm{ev}(D,\mu,S,\rho) = ev(D\cup(i,j),\mu,S,\rho) + ev(D\cup(i,j),\mu,S\cup(i,j),R_{ij}\rho)$$

and our REPLACE step for conditions **(i)** and **(iii)** correspond to correct manipulation of the Theta polynomials.

For condition **(v)** we note that for a 4-tuple $\psi = (D, \mu, S, \rho)$ if R_{ij} is applied to ρ in a REPLACE step then $i \leq m$ and $j > m$, then as long as $\rho_i \geq 0$ for $i \leq m$ and $\rho_i \leq 0$ for $i > m$ we have that $ev(\psi) = ev(\psi')$ where $\psi' = (D, \mu + \rho, S, \mathbf{0})$. Thus we need only check what happens for our initial 4-tuples $\psi = (\mathcal{C}, \lambda, S, \epsilon_j)$ for $j > m$. Then we note that if $\lambda_j < k$ then $\lambda_j + 1 \leq k$ so again we have $ev(\psi) = ev(\psi')$ where $\psi' = (\mathcal{C}, \lambda^{(j)}, S, \mathbf{0})$. So at this point we know that as long as $\lambda_j \neq k$ then we have that the REPLACE from condition **(v)** corresponds to correct manipulations of the indices.

Assume $\lambda_j = k$ and consider the 4-tuple $\psi = (\mathcal{C}, \lambda, S, \epsilon_j)$. We consider the following cases:

1. If $\lambda_{j-1} > k + 1$, then $a_j(\mathcal{C}) = 0$ so that $\gamma(\mathcal{C}, \lambda, \epsilon_j)_j = 0$. So we note that $\theta_k^0[1] = \theta_{k+1}(x, z) = \theta_{k+1}^0[0]$ and so applying the REPLACE step for condition **(v)** is an algebraically correct operation.

2. If $\lambda_{j-1} = k + 1$, then $(\mathcal{C}, \lambda, S, \epsilon_j)$ satisfies condition **(i)** so it has children $\psi_1 = (\mathcal{C} \cup (j-1, j), \lambda, S \cup \{(j-1, j)\}, e_{j-1})$ and $\psi_2 = (\mathcal{C} \cup (j-1, j), \lambda, S \cup \{(j-1, j)\}, e_j)$. Then ψ_1 has the property that $\rho_i = 0$ except for when $i = j - 1$ and there we have that $\lambda_{j-1} > k$ so that $ev(\psi_1) = ev(\psi_1')$ where $\psi_1' = (\mathcal{C} \cup (j-1, j), \lambda + e_{j-1}, S, \mathbf{0})$ which amounts to applying the REPLACE step for condition **(v)**. We then consider ψ_2. If $\gamma(\lambda)_j = 2$ then $\gamma(\mathcal{C} \cup (j-1, j), \lambda, \epsilon_j)_j = 0$ and we note that since $\theta_k^0[1] = \theta_{k+1}(x, z) = \theta_{k+1}^0[0]$ and $\gamma(\lambda)_{j-1} = 0$ so that $ev(\psi_2)$ is zero by Lemma 11. If $\gamma(\mathcal{C} \cup (j-1, j), \lambda, \epsilon_j)_j \neq 0$ then we must have that $\lambda_{j-r} = k + r$ for some $r > 1$, in which case ψ_2 satisfies condition **(i)** and

has children $\psi_{21} = (\mathcal{C} \cup \{(j-1,j),(j-2,j)\}, \lambda, S \cup \{(j-2,j)\}, e_{j-2})$ and $\psi_{22} = (\mathcal{C} \cup \{(j-1,j),(j-2,j)\}, \lambda, S \cup \{(j-1,j)\}, e_j)$ where here ψ_{21} can be evaluated similarly to ψ_1 by applying the REPLACE step for condition **(v)** and ψ_{22} is either zero by Lemma 11 or satisfies condition **(i)**. Continuing in this fashion we are able to make all terms vanish except those where $\rho = 0$.

3. If $\lambda_{j-1} = k$, then we notice that ψ will either satisfy condition **(i)** or condition **(ii)**. If it satisfies condition **(ii)** then it will vanish by Lemma 10. If it satisfies condition **(i)** then it will have a descendant which will satisfy condition **(ii)** and all other descendants will have the property that $\rho_i > 0$ only when $i \leq m$. In such a case we know that condition **(v)** will be satisfied and the REPLACE step is a correct manipulation of indices.

Then the fact that all of the surviving 4-tuples are terms in the right hand side of the equivariant Chevalley formula follows from the classical case.

Hence to prove the Chevalley formula it remains to show that our initial correction of

$$\left(\sum_{j \leq c_{\ell+1}(\lambda,1)} 2t_{\lambda_j - k} + \sum_{\ell \geq j > c_{\ell+1}(\lambda,1)} (t_{\lambda_j - k} - t_{k-\lambda_j+1+a_j(\lambda)}) + \sum_{i=1}^{a_{\ell+1}(\lambda,1)} t_{k+i} \right)$$

is the correct coefficient for σ_λ. We know the correct coefficient of σ_λ is

$$\sigma_1|_\lambda = 2\sum_{j=1}^{\ell} t_{\lambda_j - k} + \sum_{i=1}^{k} (t_{w_\lambda(i)} - t_i)$$

where w_λ is the signed permutation corresponding to λ.

We notice that these are equivalent coefficients only if

$$\sum_{i=1}^{k+a_{\ell+1}(\lambda,1)} t_i = \sum_{j > c_{\ell+1}(\lambda,1)} (t_{\lambda_j - k} + t_{k-\lambda_j+1+a_j(\lambda)}) + \sum_{i=1}^{k} t_{w_\lambda(i)}.$$

Observe first that if the indices on the right hand side are distinct then there are precisely $a_{\ell+1}(\lambda, 1)$ of them. This is because the set $\{\ell \geq j > c_{\ell+1}(\lambda, 1)\}$ has cardinality $\ell - c_{\ell+1}(\lambda, 1) = a_{l+1}(\lambda, 1)$, and only one of $\lambda_j - k$ or $k - \lambda_j + 1 + a_j(\lambda)$ will be positive. Then we also note that so long as $j \leq \ell$ we have that

$$k - \lambda_j + 1 + a_j(\lambda) \leq k + a_j(\lambda) < k + a_{\ell+1}(\lambda, 1).$$

Similarly so long as $j > c_{\ell+1}(\lambda, 1)$ we know that $\lambda_j \leq 2k + \ell - j$ so

$$\lambda_j - k \leq k + \ell - j < k + a_{\ell+1}(\lambda, 1).$$

It is known that w_λ will have a unique jump after $i = k$ and is increasing before then, so we calculate $w_\lambda(k)$. This will be equal to $\ell + k - c_{\ell+1}(\lambda, 1) = k + a_{\ell+1}(\lambda, 1)$. This is because if we look at the length of the k related diagonal coming from the first column of λ it will be $\ell + k - \#\{j|\lambda_j > 2k + \ell - j\}$, since we note that the k related diagonal will "hit" the j^{th} row only if $\lambda_j > k + \ell - j$. Therefore $w_\lambda(k) = k + a_{\ell+1}(\lambda, 1)$, and all other terms on the right hand side are less than it.

Thus to show we have the correct coefficient we need only show that the terms

$$\sum_{j > c_{l+1}(\lambda,1)} (t_{\lambda_j - k} + t_{k - \lambda_j + 1 + a_j(\lambda)}) + \sum_{i=1}^{k} t_{w_\lambda(i)}$$

are distinct. To do this we simply notice that the indices are all the absolute value of $w_\lambda(i)$ for some $1 \leq i \leq n$. This is obvious for the $t_{w_\lambda(i)}$. Then we also know that for all j such that $\lambda_j - k > 0$, $w_\lambda(k + j) = -(\lambda_j - k)$ so $|w_\lambda(k + j)| = \lambda_j - k$. Lastly we know that if we have $\lambda_j - k \leq 0$, then $w_\lambda(k + j)$ corresponds to the length of a non-related diagonal, which is $k - \lambda_j + 1 + a_j(\lambda)$ as explained in §5.3. So these indices all correspond to w_λ and are thus distinct.

Hence we have the correct coefficient for σ_λ and have proven the Chevalley Formula for our raising operator expression.

5.8 The Vanishing Theorem

Now in order to prove the Conjecture 1 we still need to show that the raising operator expressions satisfies the vanishing theorem described in §2.3.

Currently I have only proven this in the case that the Schubert class is indexed by a k-strict partition λ, for which $\lambda_i \leq k$ for all i. From here until the end of the chapter we will use the word "small" to describe any such partition λ

Theorem 8. *Let λ, μ be small partitions such that $\lambda \not\subseteq \mu$. Then $\Theta_\lambda\big|_\mu = 0$.*

Proof. We will use the localization map for restricting to the torus fixed point e_μ described in §5.4. We note that in this map the variable x_i will go to $t_{\mu_i - k}$ which in the case of μ being small will always be 0. Also the localization map sends z_i to $t_{w_\mu(i)}$. Thus we have the following:

$$\begin{aligned}\Theta_\lambda\big|_\mu &= R^\lambda \theta_{\lambda;\gamma(\lambda);\mathbf{0}}\big|_\mu \\ &= \prod_{1\leq i<j\leq \ell}(1-R_{ij})\prod_{i=1}^{\ell}\theta_{\lambda_i}^{k-\lambda_i+1+a_i(\lambda)}(0)\big|_\mu\end{aligned}$$

since λ is small. We also note that in general $w_\mu(i)$ for $i \leq k$ will be the k related diagonals as described in §5.3. Let μ' be the partition given by the columns of μ. Then the k related diagonals are exactly $\mu'_{k-i+1} + i$. We note that in this case

w_μ in Lie type C is the same as w_μ in Lie type A when we are considering the Grassmannian $G(n-k,n)$. Thus we note that $t_{w_\mu}(i)$ is the image of both x_i when considering $G(n-k,n)$ and z_i when considering $IG(n-k,2n)$. Also since λ is small we have that $a_i(\lambda)=i-1$ for all i. Then note that

$$\begin{aligned}\theta_{\lambda_i}^{(k-\lambda_i+i)}(0)\Big|_\mu &= \sum_{\substack{r,s\\ r+s\leq\lambda_i}} (-1)^s q_{\lambda_i-r-s}(x)e_r(z)h_s(t)\Big|_\mu \\ &= \sum_{j=1}^{\lambda_i}(-1)^j e_{\lambda_i-j}(t_\mu)h_j(t).\end{aligned}$$

Hence

$$\Theta_\lambda\Big|_\mu = s_\lambda(t_\mu|t).$$

Thus the Vanishing theorem in this case follows from the type A result. $\square$

5.9 Generalizations and Further Work

There are several directions I can go from here. The most obvious of which is to prove my conjecture. Once I am able to prove this conjecture one could also generalize these results to the Quantum Equivariant Cohomology ring of a Grassmannian. I may also try to generalize to the Quantum Equivariant Cohomology ring in order to prove my conjecture, since a recent result of Mihalcea in [Mi] shows that satisfying the equivariant quantum Chevalley formula is enough to prove the Giambelli formula in this setting.

Chapter A

Example of the Substitution Rule

A.1 Setup

Let us consider the 2-strict partition $\lambda = (7,4,3,2,1,1)$ indexing a Schubert cell in $IG(9-2,18)$. The Chevalley formula corresponding to this partition is

$$\sigma_\lambda \cdot \sigma_1 =$$

$$2\sigma_{(8,4,3,2,1,1)} + \sigma_{(10,4,3,2)} + 2\sigma_{(7,5,3,2,1,1)} + \sigma_{(7,6,3,1,1,1)} + \sigma_{(7,4,3,2,2,1)} + \sigma_{(7,4,3,2,1,1,1)}$$

$$+(2t_5 + t_2 + t_1 + t_4 + t_8)\sigma_\lambda.$$

In this example $\mathcal{C} = \mathcal{C}(\lambda) = \{(1,2),(1,3),(1,4),(2,3)\}$. Our initial set of 4-tuples is $\Psi_0 = \{(\mathcal{C},\lambda,\emptyset,\epsilon_1),(\mathcal{C},\lambda,\emptyset,\epsilon_2),(\mathcal{C},\lambda,\emptyset,\epsilon_3),(\mathcal{C},\lambda,\emptyset,\epsilon_4),(\mathcal{C},\lambda,\emptyset,\epsilon_5),(\mathcal{C},\lambda,\emptyset,\epsilon_6),$ $(\mathcal{C},\lambda,\emptyset,\epsilon_7)\}$. We note that for this example $\gamma(\lambda) = (4,1,0,3,6,7)$. Below we will follow the substitution algorithm for each 4-tuple (D,μ,S,ρ), including the changes in the integer sequences ρ and μ in the Young diagram and the set D in the figure of circles just underneath each diagram. In the pictures below the gray boxes will represent the positive parts of the integer sequence ρ and the dashed boxes are the negative parts of the integer sequence ρ. The survivors of the algorithm will match up with the Chevalley formula given above.

Remark 2. We note that the solid outline of the resulting diagram is $\mu + \rho$ and thus will match up with the steps of the classical substitution rule.

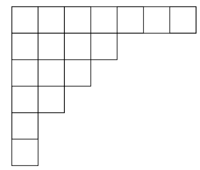
$\lambda =$ 

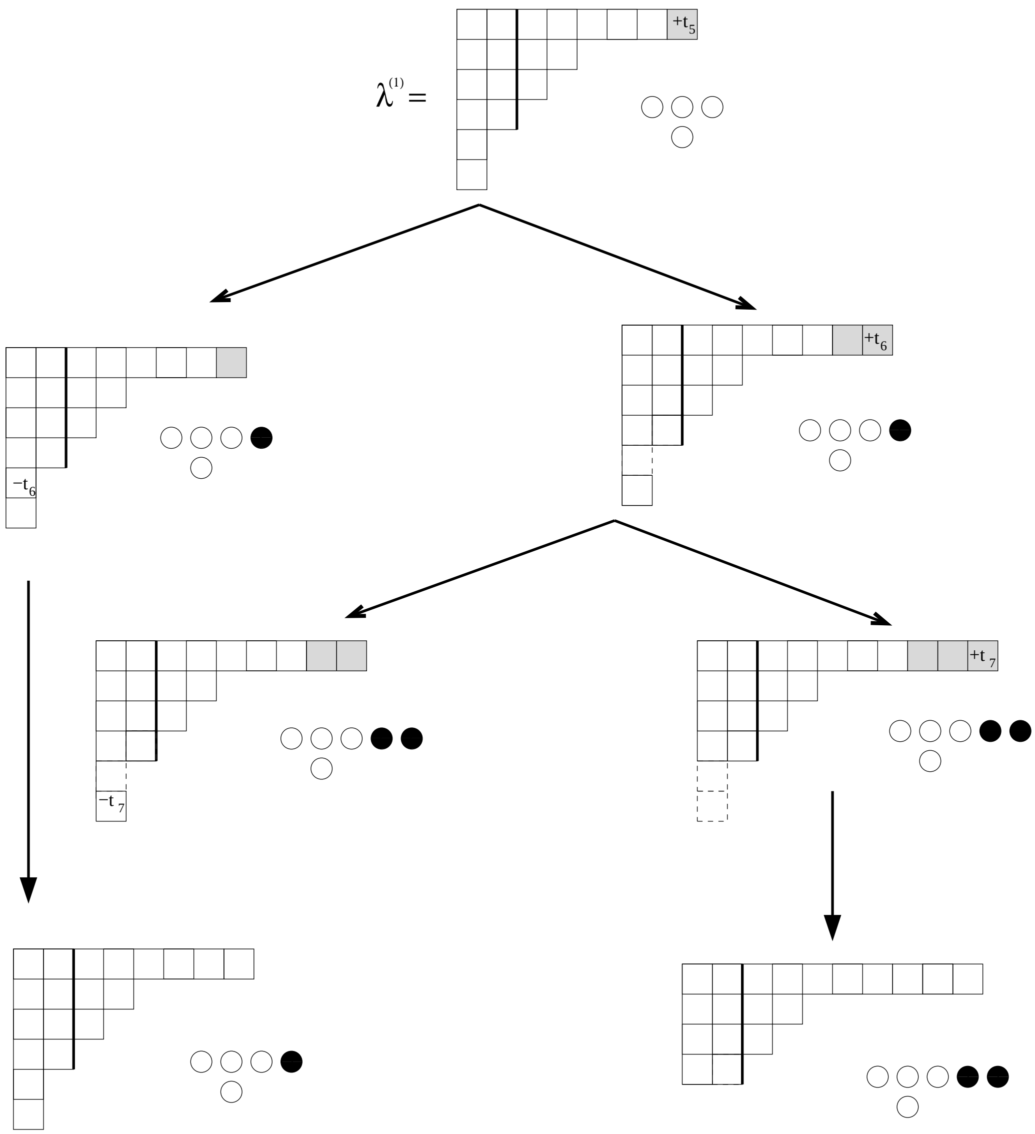
Figure A.1: The Young diagram of λ.

In the figures below the REPLACE steps for conditions **(i)** and **(iii)** will result in a new generation of 2 children (indicated by a double arrow in the figure) while we will resolve the REPLACE step for condition **(v)** (indicated by a single downward arrow) within the description for the parent generation.

We also include the equivariant corrections in the figure. A $\pm t_i$ in the last box of the j^{th} row of the diagram indicates that for the corresponding 4-tuple (D, μ, S, ρ), ρ makes it necessary to include or remove t_i from the j^{th} part of the monomial $\theta_{\mu,\gamma(D,\mu,\rho),\rho}$ while it was or wasn't included in the previous generation's monomial. We perform this correction using equation (3.1) as described in §5.5.

A.2 Algorithm With Corrections

Figure A.2: The Substitution Rule applied to $(\mathcal{C}, \lambda, \emptyset, \epsilon_1)$



First generation:

$\bullet(\mathcal{C}, \lambda, \emptyset, \epsilon_1)$ satisfies condition **(i)** at level $h = 5$ since $7+1+1 > 4+5-1$. We note that $\gamma(\mathcal{C}, \lambda, \epsilon_1) = (5, 1, 0, 3, 6, 7) = \gamma + \epsilon_1$ so we must add $t_5 \cdot \mathrm{ev}(\mathcal{C}, \lambda, \emptyset, \epsilon_1 - \epsilon_1) = t_5 \Theta_\lambda$ in order for $\mathrm{ev}(\mathcal{C}, \lambda, \emptyset, \epsilon_1)$ to be correct.

Second generation:

$\bullet(\mathcal{C}\cup(1,5),\lambda,\emptyset,\epsilon_1)$ meets condition **(v)** and is replaced with $(\mathcal{C}\cup(1,5),\lambda^{(1)},\emptyset,\mathbf{0})$ which does not meet any conditions and survives the algorithm. We note that $\gamma(\mathcal{C}\cup(1,5),\lambda,\epsilon_1)=(5,1,0,3,5,7)=\gamma(\mathcal{C},\lambda,\epsilon_1)-\epsilon_5$ so we must subtract $t_6\cdot\mathrm{ev}(\mathcal{C}\cup(1,5),\lambda,\emptyset,\epsilon_1-\epsilon_5)$.

$\bullet(\mathcal{C}\cup(1,5),\lambda,\{(1,5)\},2\epsilon_1-\epsilon_5)$ satisfies condition **(iii)** at level $h=1$ since $7+2+1>4+6-1$. We note that $\gamma(\mathcal{C}\cup(1,5),\lambda,2\epsilon_1-\epsilon_5)=(6,1,0,3,6,7)=\gamma(\mathcal{C},\lambda,\epsilon_1)+\epsilon_1$ so we must add $t_6\cdot\mathrm{ev}(\mathcal{C}\cup(1,5),\lambda,\{(1,5)\},2\epsilon_1-\epsilon_5-\epsilon_1)$. We note that this term will cancel with the correction term from this 4-tuple's second generation brother.

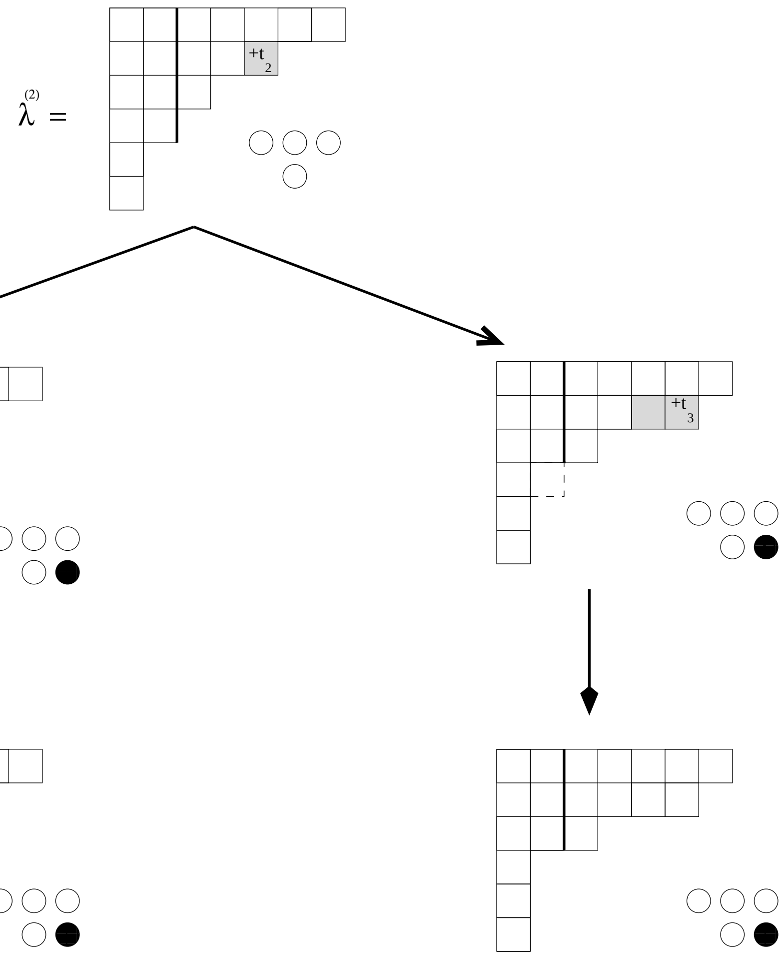
Third generation:

$\bullet(\mathcal{C}\cup(1,5)\cup(1,6),\lambda,\{(1,5)\},2\epsilon_1-\epsilon_5)$ satisfies condition **(ii)** at level $h=6$ and its evaluation vanishes by Lemma 10. We note that $\gamma(\mathcal{C}\cup(1,5)\cup(1,6),\lambda,2\epsilon_1-\epsilon_5)=(6,1,0,3,6,6)=\gamma(\mathcal{C},\lambda,2\epsilon_1-\epsilon_5)-\epsilon_6$ so we must subtract $t_7\cdot\mathrm{ev}(\mathcal{C}\cup(1,5)\cup(1,6),\lambda,\{(1,5)\},2\epsilon_1-\epsilon_5-\epsilon_6)$.

$\bullet(\mathcal{C}\cup(1,5)\cup(1,6),\lambda,\{(1,5),(1,6)\},3\epsilon_1-\epsilon_5-\epsilon_6)$ meets condition **(v)** and is replaced with $(\mathcal{C}\cup(1,5),(10,4,3,2),\emptyset,\mathbf{0})$ which does not meet any conditions and survives the algorithm. We note that $\gamma(\mathcal{C}\cup(1,5)\cup(1,6),\lambda,3\epsilon_1-\epsilon_5-\epsilon_6)=(7,1,0,3,6,7)=\gamma(\mathcal{C},\lambda,2\epsilon_1-\epsilon_5)+\epsilon_1$ so we must add $t_7\cdot\mathrm{ev}(\mathcal{C}\cup(1,5)\cup(1,6),\lambda,\{(1,5),(1,6)\},3\epsilon_1-\epsilon_5-\epsilon_6-\epsilon_1)$. We note that this term will cancel with the correction term from this 4-tuple's third generation brother.

We notice that in the end the descendants (D,μ,S) of $(\mathcal{C},\lambda^{(1)},\emptyset)$ have the property that $D\setminus\mathcal{C}$ is a row.

Figure A.3: The Substitution Rule applied to $(\mathcal{C}, \lambda, \emptyset, \epsilon_2)$



First generation:

$\bullet(\mathcal{C}, \lambda, \emptyset, \epsilon_2)$ satisfies condition **(i)** at level $h = 4$ since $4+1+2 > 4+4-2$. We note that $\gamma(\mathcal{C}, \lambda, \epsilon_2) = (4, 2, 0, 3, 6, 7) = \gamma(\lambda) + \epsilon_2)$ so we must add $t_2 \cdot \mathrm{ev}(\mathcal{C}, \lambda, \emptyset, \epsilon_2 - \epsilon_2) = t_2\Theta_\lambda$.

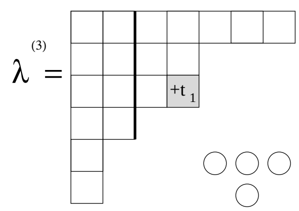
Second generation:

$\bullet(\mathcal{C} \cup (2, 4), \lambda, \emptyset, \epsilon_2)$ meets condition **(v)** and is replaced with $(\mathcal{C} \cup (2, 4), \lambda^{(2)}, \emptyset, \mathbf{0})$ which does not meet any conditions and survives the algorithm. We note that $\gamma(\mathcal{C} \cup (2, 4), \lambda, \epsilon_2) = (4, 2, 0, 2, 6, 7) = \gamma(\mathcal{C}, \lambda, \epsilon_2) - \epsilon_4)$ so we must subtract $t_3 \cdot \mathrm{ev}(\mathcal{C} \cup$

$(2,4), \lambda, \emptyset, \epsilon_2 - \epsilon_4)$.

$\bullet(\mathcal{C} \cup (2,4), \lambda, \{(2,4)\}, 2\epsilon_2 - \epsilon_4)$ meets condition **(v)** and is replaced with $(\mathcal{C} \cup (2,4), (7,6,3,1,1,1), \{(2,4)\}, \mathbf{0})$ which does not meet any conditions and survives the algorithm. We note that $\gamma(\mathcal{C} \cup (2,4), \lambda, 2\epsilon_2 - \epsilon_4) = (4,3,0,3,6,7) = \gamma(\mathcal{C}, \lambda, \epsilon_2) + \epsilon_2)$ so we must add $t_3 \cdot \mathrm{ev}(\mathcal{C} \cup (2,4), \lambda, \{(2,4)\}, 2\epsilon_2 - \epsilon_4 - \epsilon_2)$. We note that this term will cancel with the correction term from this 4-tuple's second generation brother.

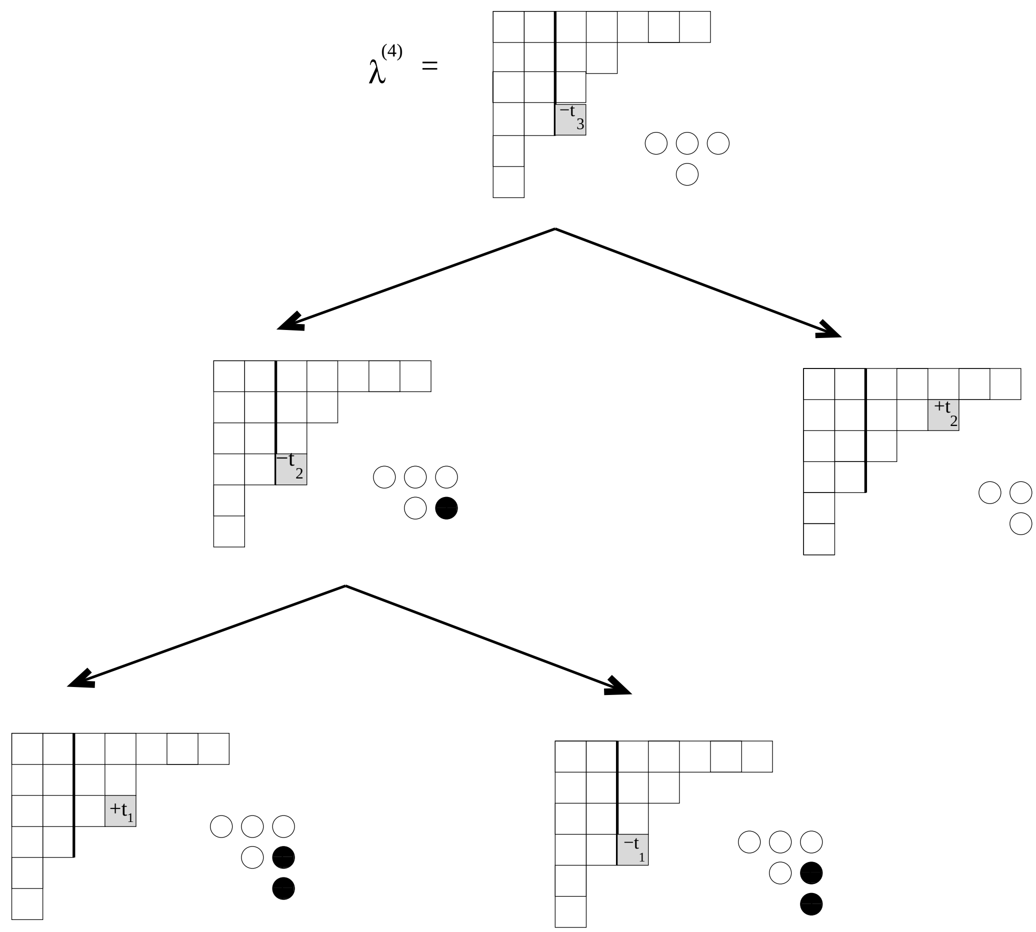
Figure A.4: The Substitution Rule applied to $(\mathcal{C}, \lambda, \emptyset, \epsilon_3)$



First generation:

$\bullet(\mathcal{C}, \lambda, \emptyset, \epsilon_3)$ satisfies condition **(iv)** at level $h = 3$ and its evaluation vanishes by Lemma 11. We note that $\gamma(\mathcal{C}, \lambda, \epsilon_3) = (4,1,1,3,6,7) = \gamma(\lambda) + \epsilon_3)$ so we must add $t_1 \cdot \mathrm{ev}(\mathcal{C}, \lambda, \emptyset, \epsilon_3 - \epsilon_3) = t_1 \Theta_\lambda$.

Figure A.5: The Substitution Rule applied to $(\mathcal{C}, \lambda, \emptyset, \epsilon_4)$



First generation:

•$(\mathcal{C}, \lambda, \emptyset, \epsilon_4)$ satisfies condition **(i)** at level $h = 4$ since $4+2+1 > 4+4-2$. We note that $\gamma(\mathcal{C}, \lambda, \epsilon_4) = (4,1,0,2,6,7) = \gamma(\lambda) - \epsilon_4)$ so we must subtract $t_3 \cdot \mathrm{ev}(\mathcal{C}, \lambda, \emptyset, \epsilon_4 - \epsilon_4) = -t_3\Theta_\lambda$.

Second generation:

•$(\mathcal{C} \cup (2,4), \lambda, \emptyset, \epsilon_4)$ satisfies condition **(i)** at level $h = 4$ since $3+2+1 > 4+4-3$. We note that $\gamma(\mathcal{C} \cup (2,4), \lambda, \epsilon_4) = (4,2,0,1,6,7) = \gamma(\mathcal{C}, \lambda, \epsilon_4) - \epsilon_4)$ so we must subtract $t_2 \cdot \mathrm{ev}(\mathcal{C} \cup (2,4), \lambda, \emptyset, \epsilon_4 - \epsilon_4)$.

$\bullet(\mathcal{C} \cup (2,4), \lambda, \{(2,4)\}, \epsilon_2)$ satisfies condition **(v)** and is replaced with $(\mathcal{C} \cup (2,4), \lambda^{(2)}, \{(2,4)\}, \mathbf{0})$ which does not meet any conditions and survives the algorithm. We note that $\gamma(\mathcal{C} \cup (2,4), \lambda, \epsilon_2) = (4,3,0,2,6,7) = \gamma(\mathcal{C}, \lambda, \epsilon_4) + \epsilon_2)$ so we must add $t_2 \cdot \mathrm{ev}(\mathcal{C} \cup (2,4), \lambda, \emptyset, \epsilon_2 - \epsilon_2)$. We note that this term will cancel with the correction term from this 4-tuple's second generation brother.

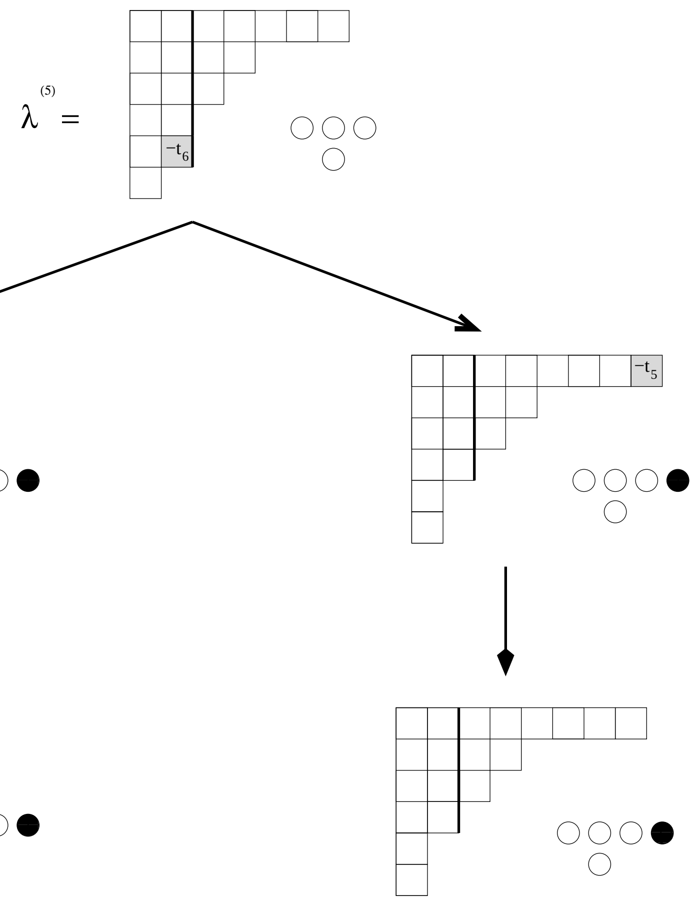
Third generation:

$\bullet(\mathcal{C} \cup (2,4) \cup (3,4), \lambda, \emptyset, \epsilon_4)$ satisfies condition **(iv)** at level $h = 4$ and its evaluation vanishes by Lemma 11. We note that $\gamma(\mathcal{C} \cup (2,4) \cup (3,4), \lambda, \epsilon_4) = (4,2,0,0,6,7) = \gamma(\mathcal{C} \cup (2,4), \lambda, \epsilon_4) - \epsilon_4)$ so we must subtract $t_1 \cdot \mathrm{ev}(\mathcal{C} \cup (2,4) \cup (3,4), \lambda, \emptyset, \epsilon_4 - \epsilon_4)$.

$\bullet(\mathcal{C} \cup (2,4) \cup (3,4), \lambda, \{(3,4)\}, \epsilon_3)$ satisfies condition **(iv)** at level $h = 3$ and its evaluation vanishes by Lemma 11. We note that $\gamma(\mathcal{C} \cup (2,4) \cup (3,4), \lambda, \epsilon_3) = (4,2,1,1,6,7) = \gamma(\mathcal{C} \cup (2,4), \lambda, \epsilon_4) + \epsilon_3)$ so we must add $t_1 \cdot \mathrm{ev}(\mathcal{C} \cup (2,4) \cup (3,4), \lambda, \emptyset, \epsilon_3 - \epsilon_3)$. We note that this term will cancel with the correction term from this 4-tuple's third generation brother.

We notice that in the end the descendants (D, μ, S) of $(\mathcal{C}, \lambda^{(4)}, \emptyset)$ have the property that $D \setminus \mathcal{C}$ is a column.

Figure A.6: The Substitution Rule applied to $(\mathcal{C}, \lambda, \emptyset, \epsilon_5)$



First generation:

$\bullet(\mathcal{C}, \lambda, \emptyset, \epsilon_5)$ satisfies condition **(i)** at level $h = 5$ since $7 + 1 + 1 > 4 + 5 - 1$. We note that $\gamma(\mathcal{C}, \lambda, \epsilon_5) = (4, 1, 0, 3, 5, 7) = \gamma(\lambda) - \epsilon_5)$ so we must subtract $t_6 \cdot \mathrm{ev}(\mathcal{C}, \lambda, \emptyset, \epsilon_5 - \epsilon_5) = -t_6\Theta_\lambda$.

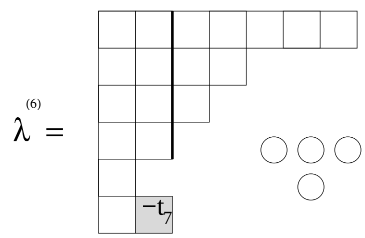
Second generation:

$\bullet(\mathcal{C} \cup (1, 5), \lambda, \emptyset, \epsilon_5)$ meets condition **(v)** and is replaced with $(\mathcal{C} \cup (1, 5), \lambda^{(5)}, \emptyset, \mathbf{0})$ which does not meet any conditions and survives the algorithm. We note that $\gamma(\mathcal{C} \cup (1, 5), \lambda, \epsilon_5) = (4, 1, 0, 3, 4, 7) = \gamma(\mathcal{C}, \lambda, \epsilon_5) - \epsilon_5)$ so we must subtract $t_5 \cdot \mathrm{ev}(\mathcal{C} \cup$

$(1,5), \lambda, \emptyset, \epsilon_5 - \epsilon_5)$.

$\bullet(\mathcal{C} \cup (1,5), \lambda, \emptyset, \epsilon_1)$ meets condition **(v)** and is replaced with $(\mathcal{C} \cup (1,5), \lambda^{(1)}, \emptyset, \mathbf{0})$ which does not meet any conditions and survives the algorithm. We note that $\gamma(\mathcal{C} \cup (1,5), \lambda, \epsilon_1) = (5,1,0,3,5,7) = \gamma(\mathcal{C}, \lambda, \epsilon_5) - \epsilon_5)$ so we must add $t_5 \cdot \mathrm{ev}(\mathcal{C} \cup (1,5), \lambda, \emptyset, \epsilon_1 - \epsilon_1)$. We note that this term will cancel with the correction term from this 4-tuple's second generation brother.

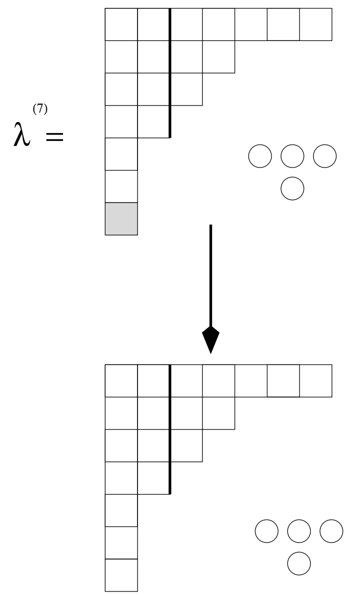
Figure A.7: The Substitution Rule applied to $(\mathcal{C}, \lambda, \emptyset, \epsilon_6)$



First generation:

$\bullet(\mathcal{C}, \lambda, \emptyset, \epsilon_6)$ satisfies condition **(ii)** at level $h = 6$ and its evaluation vanishes by Lemma 10. We note that $\gamma(\mathcal{C}, \lambda, \epsilon_6) = (4,1,0,3,6,6) = \gamma(\lambda) - \epsilon_6)$ so we must subtract $t_7 \cdot \mathrm{ev}(\mathcal{C}, \lambda, \emptyset, \epsilon_6 - \epsilon_6) = -t_7 \Theta_\lambda$.

Figure A.8: The Substitution Rule applied to $(\mathcal{C}, \lambda, \emptyset, \epsilon_7)$



First generation:

$\bullet(\mathcal{C}, \lambda, \emptyset, \epsilon_7)$ meets condition **(v)** and is replaced with $(\mathcal{C}, \lambda^{(7)}, \emptyset, \mathbf{0})$ which does not meet any conditions and survives the algorithm. We note that $\gamma(\mathcal{C}, \lambda, \epsilon_7) = (4, 1, 0, 3, 6, 7, 8)$ and that in the Chevalley product, $\Theta_1 = \Theta_1(x, z) - t_1 - t_2$ so we must add $(\sum_{i=3}^{8} t_i)\Theta_\lambda$ which is not included as a correction on our figure.

Remark 3. In the end when we apply the evaluation map we notice the survivors match with the terms in the equivariant Chevalley formula for λ given at the beginning of the appendix.

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