2023. 19(1). 56–63

СТРОИТЕЛЬНАЯ МЕХАНИКА ИНЖЕНЕРНЫХ КОНСТРУКЦИЙ И СООРУЖЕНИЙ

STRUCTURAL MECHANICS OF ENGINEERING CONSTRUCTIONS AND BUILDINGS

ISSN 1815-5235 (Print), 2587-8700 (Online)

HTTP://JOURNALS.RUDN.RU/STRUCTURAL-MECHANICS

Аналитические и численные методы расчета конструкций
Analytical and numerical methods of analysis of structures

DOI 10.22363/1815-5235-2023-19-1-56-63
EDN: DXBQYB
UDC 531.2

RESEARCH ARTICLE / НАУЧНАЯ СТАТЬЯ

Calculation of radially inhomogeneous ring loaded with normal and tangential loads

Vladimir I. Andreev

Moscow State University of Civil Engineering (National Research University), Moscow, Russian Federation
asv@mgsu.ru

Article history
Received: November 19, 2022
Revised: January 17, 2023
Accepted: January 20, 2023

For citation
Andreev V.I. Calculation of radially inhomogeneous ring loaded with normal and tangential loads. *Structural Mechanics of Engineering Constructions and Buildings.* 2023;19(1): 56–63. http://doi.org/10.22363/1815-5235-2023-19-1-56-63

Abstract. The aim of the study is to solve the problem of the stress-strain state of a thin ring under radial and ring loads, factoring in the radial inhomogeneity of the ring. Also, the task is to compare the two calculation methods to the example of solving the problem of uneven load distribution along the outer surface of the ring with one-dimensional inhomogenuity. Analytical or numerical-analytical solutions are used in the two-dimensional plane problem of the theory of elasticity in polar coordinates for an inhomogeneous body. Most of these problems consider centrally symmetric circular bodies. As a rule, this is possible when all unknown functions depend on the radius. These tasks correspond with the majority of ring and cylindrical structures. Pipes are suitable for creating pipeline systems and civil engineering, they are used for gas pipelines, in heating networks and water supply systems. The key feature of the work lies in the consideration of uneven radial and ring loads distribution along the outer surface of the ring. Comparison of the calculation results obtained by two methods makes it possible to determine the stressed and deformed states with sufficient accuracy, an indicator of which is the obtaining of the ring stresses.

Keywords: thin ring, plane task, radial loads, ring loads, radial inhomogeneity, analytical methods, numerical-analytical methods

Расчет радиально неоднородного кольца, нагруженного нормальными и касательными нагрузками

В.И. Андреев

Национальный исследовательский Московский государственный строительный университет, Москва, Российская Федерация
asv@mgsu.ru

История статьи
Поступила в редакцию: 19 ноября 2022 г.
Доработана: 17 января 2023 г.
Принята к публикации: 20 января 2023 г.

Аннотация. Исследование нацелено на решение задачи напряженно-деформированного состоянии тонкого кольца при радиальных и кольцевых нагрузках с учетом радиальной неоднородности кольца, а также на сравнение двух методов расчета на примере решения задачи, когда нагрузки неравно-

Vladimir I. Andreev, Doctor of Technical Sciences, Professor of the Department of Strength of Materials, Moscow State University of Civil Engineering (National Research University), 26 Yaroslavskoye Shosse, Moscow, 129337, Russian Federation; ORCID: 0000-0002-1057-4329, Scopus Author ID: 57198780961, eLIBRARY SPIN-code: 9906-7214; asv@mgsu.ru

Андреев Владимир Игоревич, доктор технических наук, профессор кафедры сопротивления материалов, Национальный исследовательский Московский государственный строительный университет, Российская Федерация, 129337, Москва, Ярославское шоссе, д. 26; ORCID: 0000-0002-1057-4329, Scopus Author ID: 57198780961, eLIBRARY SPIN-код: 9906-7214; asv@mgsu.ru

мерно распределены вдоль внешней поверхности кольца при одномерной неоднородности. В двумерной плоской задаче теории упругости в полярных координатах для неоднородного тела используются аналитические или численно-аналитические решения. В большинстве таких задач рассматриваются центрально симметричные круглые тела. Как правило, это возможно, когда все неизвестные функции зависят от радиуса. Эти задачи соответствуют большинству кольцевых и цилиндрических сооружений. Трубы пригодны для трубопроводных систем и строительства, применяются для газопроводов, в тепловых сетях и водопроводах. Особенность работы в рассмотрении задачи, когда радиальные и кольцевые нагрузки неравномерно распределены вдоль внешней поверхности кольца. Сравнение результатов расчета, полученных двумя методами, позволяет с достаточной точностью определить напряженное и деформированное состояния, показателем чего является получение кольцевых напряжений.

Ключевые слова: тонкое кольцо, плоская задача, радиальные нагрузки, кольцевые нагрузки, радиальная неоднородность, аналитические методы, численно-аналитические методы

Для цитирования
Андреев В.И. Расчет радиально неоднородного кольца, нагруженного нормальными и касательными нагрузками // Строительная механика инженерных конструкций и сооружений. 2023. Т. 19. № 1. С. 56–63. http://doi.org/10.22363/1815-5235-2023-19-1-56-63

Introduction

The article considers a plane problem of the theory of elasticity in polar coordinates for a radially inhomogeneous disk under the action of variable normal and tangential loads, factoring in the radial inhomogeneity. The solution of the problem makes it possible to calculate the strength and deformability of pipes under the action of internal and external pressure, as well as torsion. The feature of this article is the heterogeneity of the structure, which significantly expands the scope of solving problems of the mechanics of deformation in solids.

The first articles on the formulation and solution of problems in the mechanics of inhomogeneous bodies began in the 50s and 60s, when the first computers appeared. A significant contribution to the development of the mechanics of inhomogeneous bodies was made by Russian scientists: S.G. Mikhlin [1], G.B. Lekhnitsky [2], V.A. Lomakin [3], G.B. Kolchin [4; 5]. The works of Polish scientists, primarily V. Olshak and his students [6; 7] should also be mentioned. The author of this article began to pursue this field of mechanics in 1974 [8–11] and continues to do so with his students and colleagues to the present day [12–15].

The proposed article is dedicated to solving the problem of the stress-strain state of a thin ring under radial and ring loads factoring in the radial inhomogeneity of the ring.

The problem is a two-dimensional one with one-dimensional inhomogeneity. The author uses the method of separation of variables, which is based on the development of the generalized solution of J. Michell for the plane problem in polar coordinates, which was written about in [16]. Two solutions to the problem are given: an analytical one and a numerical-analytical one, and comparison of the two calculations' results is shown.

Formulation of the problem

We consider the problem of the equilibrium of a thin ring when forces are applied to its outer surface:

$$p = p_0(1+\cos 2\theta)/2; \quad q = p_0 \sin 2\theta/2, \tag{1}$$

and the inner surface is load-free (Figure 1).

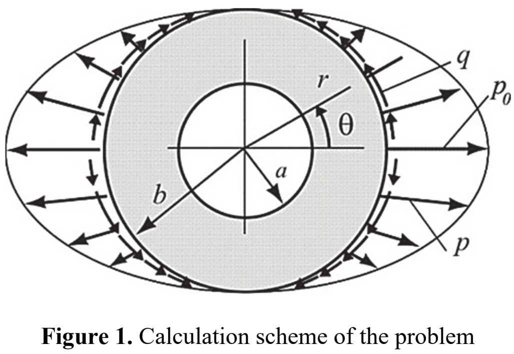

Figure 1. Calculation scheme of the problem

The modulus of elasticity of the ring material changes according to the power law:

$$E(r) = E_0\left(\frac{r}{a}\right)^{\delta}, \tag{2}$$

Poisson's ratio ν = const.

Analytical solution

The analytical solution is made on the basis of the method [17], derived from the development of the generalized J.H. Michell solution [16], for calculating a plane problem in polar coordinates for inhomogeneous structures. In this method, the solution is obtained in the form:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} + \begin{pmatrix} \varphi_1 \\ \psi_1 \end{pmatrix}\theta + \begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix}\theta\sin\theta + \begin{pmatrix} \varphi_3 \\ \psi_3 \end{pmatrix}\theta\cos\theta +$$
$$+\begin{pmatrix} \varphi_4 \\ \psi_4 \end{pmatrix}\begin{pmatrix} \varphi_3 \\ \psi_3 \end{pmatrix}\sin\theta + \begin{pmatrix} \varphi_5 \\ \psi_5 \end{pmatrix}\cos\theta + \sum_{n=2}^{\infty}\left[\begin{pmatrix} \varphi_{sn} \\ \psi_{sn} \end{pmatrix}\sin n\theta + \begin{pmatrix} \varphi_{cn} \\ \psi_{cn} \end{pmatrix}\cos n\theta\right], \tag{3}$$

in which each summand consists of the product of unknown functions (φ_i, ψ_i), multiplied by a certain trigonometric function. In (3), the indices s and c mean that the corresponding functions are multiplied by sin$n\theta$ or cos$n\theta$.

The solution is reduced to an infinite partially decomposing system of second-order equations. In the problem under consideration, the corresponding equations are selected from this system depending on the boundary conditions and formulas for stresses.

According to (1) the boundary conditions have the form

$$r = a;\quad \sigma_r = \tau_{r\theta} = 0;\quad r = b;\quad \sigma_r = p(\theta);\quad \tau_{r\theta} = -q(\theta). \tag{4}$$

Comparing the boundary conditions and stresses, it can be assumed that the expressions for stresses should contain the functions φ_0, ψ_1, φ_{s2} and ψ_{s2}. From the system of differential equations mentioned above, it is necessary to consider the following four equations to determine those functions:

$$\left(\lambda^* + 2\mu\right)\left(\varphi_0'' + \frac{\varphi_0'}{r} - \frac{\varphi_0}{r^2}\right) + \frac{\lambda^* + \mu}{r}\psi_1' - \frac{\lambda^* + 3\mu}{r^2}\psi_1 + \left[(\lambda^*)' + 2\mu'\right]\varphi_0' + \frac{(\lambda^*)'}{r}\left(\varphi_0 + \psi_1\right) = 0; \tag{5}$$

$$\mu\left(\psi_1'' + \frac{\psi_1'}{r} - \frac{\psi_1}{r^2}\right) + \mu'\left(\psi_1' - \frac{\psi_1}{r}\right) = 0; \tag{6}$$

$$-2\frac{\lambda^* + \mu}{r}\varphi_{c2}' - 2\frac{\lambda^* + 3\mu}{r^2}\varphi_{c2} + \mu\left(\psi_{s2}'' + \frac{\psi_{s2}'}{r} - \frac{\psi_{s2}}{r^2}\right) - 4\frac{\lambda^* + 2\mu}{r^2}\psi_{s2} - \frac{\mu'}{r}\left(2\varphi_{c2} - r\psi_{s2}' + \psi_{s2}\right) = 0; \tag{7}$$

$$\left(\lambda^* + 2\mu\right)\left(\varphi_{c2}'' + \frac{\varphi_{c2}'}{r} - \frac{\varphi_{c2}}{r^2}\right) - \frac{4\mu}{r^2}\varphi_{c2} + 2\frac{\lambda^* + \mu}{r}\psi_{s2}' - 2\frac{\lambda^* + 3\mu}{r^2}\psi_{s2} + \left[(\lambda^*)' + 2\mu'\right]\varphi_{c2}' + \frac{(\lambda^*)'}{r}\left(\varphi_{c2} + 2\psi_{s2}\right) = 0. \tag{8}$$

A transition is made in the above equations, from a plane deformed state to a plane stressed state by replacing the parameter λ with $\lambda^* = \frac{E\nu}{1-\nu^2}$.

Considering that the solution of this problem factoring in the unambiguity condition, should not have terms containing θ, it follows from (3) that $\psi_1' - \frac{\psi_1}{r} = 0$. Given this equality, equation (6) takes the form $\psi_1'' = 0$. The integral of this equation is a linear function that becomes a constant at *r = a, b*. Since the boundary conditions (18) for $\tau_{r\theta}$ considering (1) do not contain a constant, we should set $\psi_1 = 0$. Then equation (5) is simplified:

$$(\lambda^* + 2\mu)\left(\varphi_0'' + \frac{\varphi_0'}{r} - \frac{\varphi_0}{r^2}\right) + \left[(\lambda^*)' + 2\mu'\right]\varphi_0' + \frac{(\lambda^*)'}{r}\varphi_0 = 0, \tag{9}$$

and can serve to determine the function of φ_0.

Thus, the problem can be divided into two parts: to determine the axisymmetric component of the solution using (9) and to use (7), (8) for determination of the component that depends on θ. Substituting formula (2) into the these equations leads them to the form

$$\varphi_0'' + (1+\delta)\frac{\varphi_0'}{r} + (\delta\nu - 1)\frac{\varphi_0}{r^2} = 0; \tag{10}$$

$$\psi_{s2}'' + (1+\delta)\frac{\psi_{s2}'}{r} - \left(\frac{9-\nu}{1-\nu} + \delta\right)\frac{\psi_{s2}}{r^2} - \frac{2(1+\nu)}{1-\nu}\frac{\varphi_{c2}'}{r} - \left(\frac{6-2\nu}{1-\nu} + 2\delta\right)\frac{\varphi_{c2}}{r^2} = 0; \tag{11}$$

$$\varphi_{c2}'' + (1+\delta)\frac{\varphi_{c2}'}{r} + (\delta\nu - 3 + 2\nu)\frac{\varphi_{c2}}{r^2} + (1+\nu)\frac{\psi_{s2}'}{r} - (3 - \nu - 2\delta\nu)\frac{\psi_{s2}}{r^2} = 0. \tag{12}$$

The solution of equation (10) is the following function:

$$\varphi_0 = C_1 r^{\frac{1-\alpha+\beta}{2}} + C_2 r^{\frac{1-\alpha-\beta}{2}}, \tag{13}$$

where $\alpha = (1+\delta)$, $\beta = \sqrt{(1-\alpha)^2 - 4(\delta\nu - 1)}$.

The constants of integration that are used in (13) can be found from the boundary conditions for the axisymmetric component of the external load:

$$r = a;\quad \sigma_r = 0;\quad r = b;\quad \sigma_r = p_0/2.$$

The system of two ordinary differential equations (11), (12) can be reduced to one fourth-order equation as follows. From equation (11) we express ψ_{s2}' as a function of ψ_{s2}, φ_{c2} and its derivatives:

$$\psi_{s2}' = f_1(\psi_{s2}, \varphi_{c2}'', \varphi_{c2}', \varphi_{c2}). \tag{a}$$

Using the differentiation of this expression by *r* and substituting equality (a) into it, we obtain

$$\psi_{s2}'' = f_2(\varphi_{c2}''', \varphi_{c2}'', \varphi_{c2}', \varphi_{c2}, \psi_{s2}). \tag{b}$$

Substituting (a) and (b) into (5.40), we find the expression for ψ_{s2} :

$$\psi_{s2} = f_3\left(\varphi'''_{c2}, \varphi''_{c2}, \varphi'_{c2}, \varphi_{c2}\right). \tag{c}$$

By differentiating the last equation once with by r, we equate the obtained expression with (a) by substituting (c) in it. As a result, we obtain a fourth-order equation with respect to the function φ_{c2} :

$$r^4\varphi_{c2}^{IV} + (6+2\delta)r^3\varphi'''_{c2} + (5\delta+\delta^2+\nu\delta-3)r^2\varphi''_{c2} + (\nu\delta^2+\nu\delta-9\delta-9)r\varphi'_{c2} + (3\nu\delta^2+3\nu\delta-3\delta+9)\varphi_{c2} = 0. \tag{14}$$

The resulting equation can be reduced to a differential equation with constant coefficients by introducing a variable t with the dependence $r = e^t$:

$$\frac{d^4\varphi_{c2}}{dt^4} + 2\delta\frac{d^3\varphi_{c2}}{dt^3} + (\delta^2+\nu\delta-\delta-10)\frac{d^2\varphi_{c2}}{dt^2} + (\nu\delta^2-\delta^2-10\delta)\frac{d\varphi_{c2}}{dt} + (3\nu\delta^2+3\nu\delta-3\delta+9)\varphi_{c2} = 0.$$

The characteristic equation corresponding to the resulting equation will be

$$l^4 + 2\delta l^3 + (\delta^2+\nu\delta--10)l^2 + (\nu\delta^2-\delta^2-10\delta)l + 3\nu\delta^2 + 3\nu\delta - 3\delta + 9 = 0.$$

Using the substitution $\eta = l^2 + \delta l$, this equation can be reduced to a quadratic one:

$$\eta^2 + (\nu\delta-\delta-10)\eta + 3\nu\delta^2 + 3\nu\delta - 3\delta + 9 = 0.$$

The final solution of equation (14) has the form

$$\varphi_{c2} = \sum_{i=1}^{4} D_i e^{lt},$$

its constants D_i are determined from the boundary conditions for the non-axisymmetric component:

$$r = a;\quad \sigma_r = \tau_{r\theta} = 0;\quad r = b;\quad \sigma_r = p_0\cos 2\theta/2;\quad \tau_{r\theta} = -p_0\sin 2\theta/2.$$

The function ψ_{s2} can be found from equality (c).

Below is an example of a calculation performed for the following initial data: $\delta = -1$; $b/a = 2$; $\nu = 1/3$; $E = 2\cdot 10^4$ MPa.

Figure 2 shows the stress diagrams along the three radius directions. It can be concluded from the graphs shown above that the consideration of heterogeneity in this case does not lead to a qualitative change in the character of the diagrams. Numerical differences in some cases amount to approximately 20%. It is logical to assume that with more substantial heterogeneity, the differences in the results for homogeneous and heterogeneous materials may be more significant.

In this case, it is simple enough to trace the dependence of displacements on the inhomogeneity of the material. Figure 3 shows the diagrams of the displacements of the points along the angular coordinate of the inner contour of the ring. It can be noticed that the displacements in the inhomogeneous ring are larger than in the homogeneous ring. This fact is explained by the fact that when $\delta = -1$ then the modulus of elasticity decreases from the inner contour to the outer contour twice, and this leads to a decrease in the stiffness of the ring as a whole.

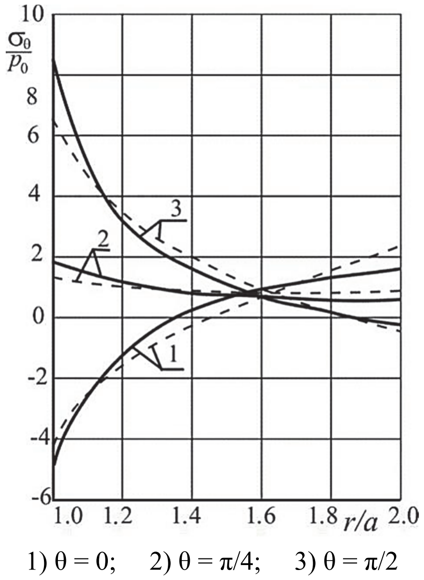

Figure 2. Stresses σ_θ in the ring:
– – inhomogeneous material; – – – - homogeneous material

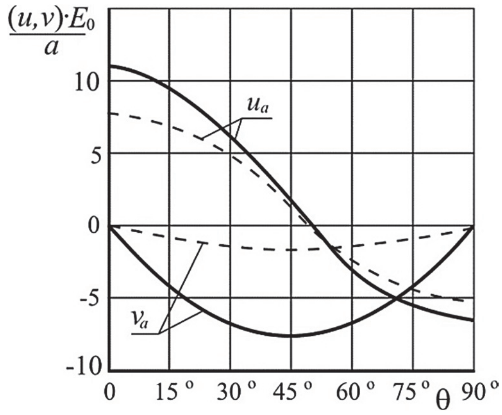

Figure 3. Displacements of points on the inner contour of the ring:
– – inhomogeneous material; - – – – homogeneous material

Solution using numerical-analytical method

In this section we consider the problem, the analytical solution of which was obtained in item 3. By the example of this problem the application of numerical-analytical method will be demonstrated, including numerical solution of systems of ordinary differential equations [6; 7]. At the same time it is possible to determine the accuracy of the method under consideration.

As it was shown above, in order to satisfy the boundary conditions (4) considering (1), it is sufficient to restrict formulas (3) to the summands φ_0, $\varphi_{c2}\cos 2\theta$ and $\psi_{s2}\sin 2\theta$. The expressions for the stresses σ_r and $\tau_{r\theta}$ in the boundary conditions will have the form

$$\sigma_r = \frac{E}{(1+\nu)(1-\nu)}\left[\left(\varphi' + \nu\frac{\varphi_0}{r}\right) + \left(\varphi'_{c2} + \nu\frac{\varphi_{c2} + 2\Psi_{s2}}{r}\right)\cos 2\theta\right];$$

$$\tau_{r\theta} = \frac{E}{2(1+\nu)}\left(\Psi'_{s2} - \frac{\Psi_{s2} + 2\varphi_{c2}}{r}\sin 2\theta\right). \quad (15)$$

Equations (10)–(12) are valid for the functions φ_0, φ_{c2} and ψ_{s2} for the law of changing for the modulus of elasticity (2).

Equation (10) is reduced to a system of two first-order equations, introducing the following notations $y_1 = \varphi_0, y_2 = \varphi'_0$:

$$y'_1 = y_2; \quad y'_2 = -(1+\delta)\frac{y_2}{r} - (\delta\nu - 1)\frac{y_1}{r^2}. \quad (16)$$

Boundary conditions (4) for the *axisymmetric* component of stresses will take the form:

$$r = a,\ \ y_2 + \nu\frac{y_1}{r} = 0; \qquad r = b,\ y_2 + \nu\frac{y_1}{r} = -\frac{p_0(1+\nu)(1-\nu)}{2E(b)}. \quad (17)$$

Equations (11), (12) with introduction of notations $y_1 = \varphi_{c2}$, $y_2 = \varphi'_{c2}$, $y_3 = \psi_{s2}$, $y_4 = \psi'_{s2}$ are transformed to a system of four equations of the first order. Adding the boundary conditions (4) for the non-axisymmetric component to these equations, we obtain the boundary task for functions φ_{c2} and ψ_{s2}.

The calculation was carried out on the interval (*a, b*) with an equal division into 100 steps for the same initial data as in item 3. Table shows comparative values of stresses in the inhomogeneous ring at θ = 45°, obtained as a result of analytical and numerical calculations.

Stresses in the ring

r/a	**Analytical solution**		**Numerical solution**	
	σ_r	$\tau_{r\theta}$	σ_r	$\tau_{r\theta}$
1.0	0	0	0	0
1.2	0.252	–1.266	0.263	–1.296
1.4	0.377	–1.338	0.378	–1.363
1.6	0.444	–1.108	0.444	–1.121
1.8	0.480	–0.807	0.480	–0.813
2.0	2.000	–0.500	2.000	–0.500

Comparison of the results obtained by the two methods allows us to conclude that the accuracy of the numerical-analytical method is sufficiently high.

If we consider a ring with sufficiently large ratio of outer and inner radii, then we can obtain the solution of the problem of tension-compression of a plate with a small circular hole using the calculation method shown above. The solutions for tension of an inhomogeneous plate with a hole in one direction, tension-compression in two directions, and shearing of a plate with a hole were obtained in [8]. It was demonstrated on the basis of numerical-analytical calculation of a homogeneous plate and comparison of the results with the solution of the Kirsch problem [4] that satisfactory accuracy can be achieved when the ratio of the plate dimensions to the hole radius is more than 10.

Conclusion

The problem considered in the paper is an example of using a generalized method for calculating a plane two-dimensional problem for a radially inhomogeneous ring. The possibility of obtaining an analytical solution to such problems largely depends on the inhomogeneity of the material, i.e., first of all, on the dependence of the modulus of elasticity of the material on the radius. The degree dependence of the modulus of elasticity on the radius selected in the paper is the simplest one.

The second solution obtained by the numerical method shows good compliance with the analytical solution and may be used for calculations of two-dimensional planar problems with radial inhomogeneity at any *continuous* dependence of the modulus of elasticity on the radius.

References

1. Mikhlin S.G. The plane problem of elasticity theory. *Proceedings of Seism. Institute of the USSR Academy of Sciences.* 1935;65:81–82. (In Russ.)

2. Lehnitsky S.G. Radial stress distribution in a wedge and a half-plane with a variable modulus of elasticity. *Applied Mathematics and Mechanics.* 1962;26(1):146–151. (In Russ.)

3. Lomakin V.A. *Theory of elasticity of inhomogeneous bodies.* Moscow: URSS Publ.; 2014. (In Russ.)

4. Kolchin G.B. On the applicability of the iterative method in problems of the theory of elasticity of inhomogeneous bodies. *Applied Mathematics and Programming* (issue 2). Chisinau: AN MSSR; 1969. (In Russ.)

5. Kolchin G.B. *Planar problems of the theory of elasticity of inhomogeneous bodies*. Chisinau: Stiinza Publ.; 1977. (In Russ.)

6. Olszak W., Urbanowski W., Rychlewski J. Elastic-plastic thick-walled inhomogeneous cylinder under the influence of internal pressure and longitudinal force. *Archiwum Mechaniki Stosowanej.* 1955;VII:315–336.

7. Olszak W., Rychlewski J. Nichthomogenitats-Probleme in elastischen und vorplastischen Bereich. *Osterreich. Ingen.* 1961;15:61–76.

8. Andreev V.I., Shishchits I.Yu. Investigation of stresses around holes in space taking into account the compressibility of the material. *Collection of works of MISI.* 1974;(118):59–62. (In Russ.)

9. Andreev V.I., Dubrovsky A.V. Taking into account the heterogeneity of the material when calculating the dry protection of the reactor. *Issues of Atomic Science and Technology. Design and Construction*. 1982;(3):3–8. (In Russ.)

10. Andreev V.I. About one method of solving of plane problem of the theory of elasticity for radial inhomogeneous body. *Applied Mechanics.* 1987;23(4):16–23. (In Russ.)

11. Andreev V.I. A method for solving a certain class of three-dimensional problems for an elastic radially inhomogeneous cylinder. *News of Universities. Construction and Architecture*. 1985;(8):27–31. (In Russ.)

12. Andreev V.I. *Some problems and methods of mechanics of inhomogeneous bodies*. Moscow: ASV Publ.; 2002. (In Russ.)

13. Andreev V.I., Tsybin N.Yu. Generalization of Michel's solution to plane problem theory of elasticity in polar coordinates in the event of a radially inhomogeneous body. *WIT Transactions on Modelling and Simulation.* 2014;57:215–227.

14. Polyakova L.S., Andreev V.I. Solution of the axisymmetric problem of thermoelasticity of a radially inhomogeneous cylindrical shell by numerical-analytical method and the finite element method. *Structural Mechanics of Engineering Constructions and Buildings.* 2019;15(4):323–326. (In Russ.) http://doi.org/10.22363/1815-5235-2019-15-4-323-326

15. Andreev V., Maksimov M. Elastic – plastic equilibrium of a hollow ball made of inhomogeneous ideal-plastic material. In: Akimov P., Vatin N. (eds.) *Proceedings of FORM 2021. Lecture Notes in Civil Engineering*. 2021;170:177–188. https://doi.org/10.1007/978-3-030-79983-0_16

16. Timoshenko S., Goodier J.N. *Theory of elasticity.* New York, Toronto, London: McGraw-Hill Book Company; 1951.

17. Kukudzhanov V.N. *Numerical methods in continuum mechanics.* Moscow: MATI Publ.; 2006. (In Russ.)