ABSTRACT

Title of dissertation: DISTRIBUTED ESTIMATION OVER NETWORKS WITH COMMUNICATION COSTS

Gabriel-Mihai Lipsa, 2010

Dissertation directed by: Professor Nuno C. Martins
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We analyze how distributed or decentralized estimation can be performed over networks, when there is a price to be paid whenever nodes in the network communicate with each other. The work here has application especially in the network control systems. Assume that different nodes in the network can track perfectly or with imperfectly some stochastic processes, while other nodes in the network need to estimate these stochastic processes. The nodes which can observe the stochastic processes can send information directly to the nodes which need to estimate the processes, or information can be sent to intermediate nodes. When each transmission is performed a cost for communication is paid. The goal of the network is to optimize jointly a cost which consists both of a function of the estimation error and a function of the transmission cost. We show here that for some simple topologies the decision to send information over the network is a threshold policy, while the estimators are linear estimators which resemble with the Kalman-filter. For the result dealing with simple topologies we have proved the results using majorization theory.

It is also shown here both analytically and numerically that things can immediately

become quite complicated. If we take into consideration multidimensional problems or problems with multiple agents and/or transmission noise, the optimal strategies can no longer be found analytically and it can be quite difficult to compute numerically the optimal strategies.

DISTRIBUTED ESTIMATION OVER NETWORKS WITH COMMUNICATION COSTS

by

Gabriel-Mihai Lipsa

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Advisory Committee:
Professor Nuno C. Martins, Chair/Advisor
Professor Prakash Narayan
Professor Andre Tits
Professor P.S. Krishnaprasad
Professor Radu Balan

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Chapter 1

Introduction

1.1 Motivation

Decentralized systems appear in a wide number of application, such as Internet, sensor networks, MANET (mobile as-hoc networks), robotics, multi-core CPUs, telecommunications, surveillance networks, control of autonomous aerial or underwater vehicles etc. Decentralized systems are made from multiple components, where each component has total or partial information about the state of the system. In centralized systems, if these systems are not fully observable, the control has a dual aspect, the actual control which alters the state of the system and the estimation, i.e. the control that alters the future information about the state of the system. In decentralized systems, the control has one other function, the communication, i.e. the control that alters the future information that other components or agents have about the state of the system. Hence, in decentralized systems we can talk about the triple aspect of control: actual control, estimation and communication. The term triple aspect of control was introduced by P. Varayia. Even in the case where in the network there are components which have perfect information about the state of the system, there is still the issue of what information must send those agents to other components. It theory at least, if the communication is noiseless, they can send the entire information, but since in practice the communication channels are noisy and the components have constrained computation capabilities, it is useful to analyze what information is sufficient for the components with partial observation of the system in order to achieve the common goal of the network.

We analyze how distributed or decentralized estimation can be performed over networks, when there is a price to be paid whenever nodes in the network communicate with each other. The work here has application especially in the network control systems. Assume that different nodes in the network can track perfectly or with imperfectly some stochastic processes, while other nodes in the network need to estimate these stochastic processes. The nodes which can observe the stochastic processes can send information directly to the nodes which need to estimate the processes, or information can be sent to intermediate nodes. When each transmission is performed a cost for communication is paid. The goal of the network is to optimize jointly a cost which consists both of a function of the estimation error and a function of the transmission cost. We show here that for some simple topologies the decision to send information over the network is a threshold policy, while the estimators are linear estimators which resemble with the Kalman-filter. For the result dealing with simple topologies we have proved the results using majorization theory.

When the topologies become slightly more complicated, the optimal policies become more complex and it becomes more difficult to analyze analytically or to compute numerically these optimal strategies.

1.2 Literature review

Multi-agent systems can be classified based on the objective of the agents as teams or on the information available to the agents as static or dynamic systems. Dynamic systems can be further decomposed in sequential and non-sequential. Sequential systems can be decomposed in systems with classical information structures and non classical information structures. The multi-agent systems as teams were studied first by Radner in [20], Marschak and Radner in [21], later in control systems by Witsenhausen in [22] and [24], and Ho in [27] and [28], and others. The distinction between sequential and non-sequential systems was given by Witsenhausen in [23] and [25]. Witsenhausen studied also the optimal design of non-sequential systems in [24]. Properties of non-sequential systems were studied by Andersland [31], Andersland and Teneketzis [32] and [33], Teneketzis [34] and Teneketzis and Andersland [35]. The importance of information structures was first highlighted by Witsenhausen in [22]. The role of information structures in specific teams problems was studied by Ho and Chu in [29], Chu in [30], Yoshikawa in [36] and others.

Previous work has been done in the field of distributed estimation and in filtering. We mention here the work of Hajek [1], which explores the optimization of paging and registration policies in cellular networks. Motion is modeled as a discrete-time Markov process, and minimization of the discounted, infinite-horizon average cost is addressed. Majorization theory and Rieszs rearrangement inequality are used to show that jointly optimal paging and registration policies are given for symmetric or Gaussian random walk models by the nearest-location-first paging policy and distance threshold registration policies. An iterative algorithm is proposed and investigated, which alternates between paging policy optimization and registration policy optimization. This paper [1] refers only to random walk or Gaussian random walk, while we are looking at linear systems driven by Gaussian noise and we are using the square of the estimation error in computing the cost.

In [7], the authors consider a nonlinear filtering problem of a diffusion process, when several sensors are available. A nonlinear filter can use any number of these sensors at each time, with each of the sensor having an associated cost. The problem considered in [7] is the optimal selection of a schedule of these sensors from the available set, so as to optimally estimate a function of the state at the final time. This problem is more general than what we are solving, but it is very difficult to compute the optimal policies in practice.

In [5], the authors consider a sequential estimation problem with two decision makers, one agent makes sequential observation about the state of a stochastic process and decides whether to send information to the other agent, which will estimate the state of the underlying stochastic process. These agents have a common objective of minimizing a performance criterion, with the constraint that the observer agent can send information to the estimator agent only a limited number of times. In [5], the authors assume that the decision policies are threshold policies, while in our paper we prove the optimality of the threshold policies for similar problems.

The work in [8] is motivated by large-scale sensor network where data collection from all sensors is prohibitive. These sensors are part of a network control system in which a controller can observe the sensors. The observations are not fixed, the controller can choose which sensors to observe and each choice has a cost associated with it. The work in [8] looks mainly at the linear quadratic Gaussian problem and also looks at a problem similar to Problem 2.1, for which the authors found numerically that the optimal policy is a threshold policy.

In [9], the author presents an optimization problem dealing with selecting one measurement from many sensors, where each measurement has an associated cost. In [9] it is shown that the problem of selecting the optimal strategy can be transformed into a deterministic control problem. The computation of the measurement policy takes place offline and the optimal strategy is adopted. In contrast to our result, the decisions analyzed in [9] are taken in an off-line fashion. In [10], the paper considers a class of problems known as measurement adaptive problems, in which the control is available not only to the plant but also the measurement subsystem. In the special case of linear systems, quadratic cost, and Gaussian random processes, the authors showed that the optimization of plant control can be carried out independently of the measurement control optimization. Moreover the optimization of the measurement control can be done apriori, hence the optimization of the measurement subsystem is done off-line.

In [6], the control and the estimation are separated and the estimation problem is exactly the same problem that we address in this paper. The authors assume that estimator policy is a linear estimator and show using dynamic programming that the decision to send a sample depends on the estimation error. The problem analyzed in [6] deals also with the multidimensional case, which we handle handle in Chapter 4. In contrast to the work in [6] we proved analytically that the state estimator is linear for the scalar case and that there exists a threshold policy which is an optimal sampling decision policy.

In [17], [42], similar problems are discussed, in which estimation is performed with a single sensor and a single measurement and the question is when to take send a measurement. The author could not prove the optimality of the threshold sampling, he proved that the scheme is better than a deterministic scheme. Another problem discussed in [17] is similar to the one discussed in Chapter 2, i.e. the level-triggered sampling scheme. Again the author did not prove the optimality of such a scheme, while we prove its optimality in Chapter 2.

In [15], [16], the optimal design of multi-agent sequential teams is investigated, and a methodology is presented to convert the search of a multistage design into a sequence of nested optimization problems. This conversion is called sequential decomposition and it drastically simplifies the search of optimal solution for both finite and infinite horizon problems.

In [18], it is considered a stochastic dynamic decision problem where at each step two decision must be taken, the first one is what information about the signal should be sent, while the second one what control must be adopted. For a finite horizon first order ARMA model, with Gaussian statistics and quadratic cost criterion, the authors showed that the optimal measurement strategy consists of transmitting the innovation linearly, which will imply that the optimal control law is also linear. The authors show that for higher order ARMA models, there exist a nonlinear design that outperforms the optimal affine design.

In [19], it is discussed the optimal controllers for linear-quadratic stochastic systems, where the measurement channels are no longer fixed, but they will be a part of the overall design. The authors show that, for the scalar case, the optimal measurement channel is linear and the optimal controller is also linear. In the vector version however, it is possible to find nonlinear design which outperforms the optimal linear design.

In [40, 44], the authors are looking at distributed estimation problems and place the Witsenhausen couterexample [26] within a broad class of dynamic decision problems with nonclassical information. In [51], a vector version of the Witsenhausen counterexample is presented. Moreover, it was reported in [43] that the discretized version of Witsenhausen's counter-example is NP-complete. This fact has motivated the numerical studies in [45, 46, 47].

The work in [29, 30], considered the case where a linear information pattern is defined by a directed graph. Using the notion of partially nested information structure, the authors of [29, 30] characterize when the optimal solution can be found, while bounds are derived when the optimal is unknown. In [48], it is shown that if Witsenhausen Counterexample is modified using an induced norm then the optimal control is linear. In [50], the authors show that linear sensing policies over Gaussian channels might not be optimal in a distributed multi-sensor, single controller scenario, for the minimization of a quadratic cost function. This is in contrast with the corresponding single-sensor problem, which does admit an optimal linear solution. The work in [49] addresses one follow-up question listed in the paper by Witsenhausen, more specifically, [49] discusses the connections between partially nested structures, for which linear controllers are known to be optimal, and quadratically invariant structures, for which the optimal linear control is known to be convex.

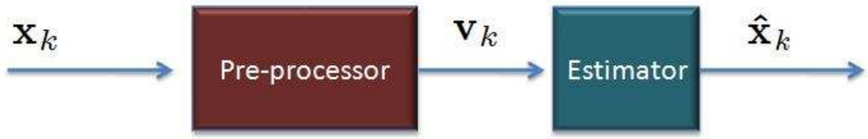

Figure 1.1: The Two Blocks

1.3 Thesis Structure

In Chapter 2 we solve a distributed estimation problem which consists from a pre-processor or encoder and an estimator or a decoder, shown in Figure 1.1. The pre-processor has perfect knowledge about a stochastic process and the decoder has access only to the information which it receives from the decoder. Each time the encoder sends information to the decoder it must pay a cost for communication. The encoder and the decoder must jointly optimize a common cost, which consists from the estimation error and the communication cost. The problem which arises is when and what information must be sent to the estimator. Using theory of majorization [4], it was shown that the optimal policy to send sample to the estimator is a threshold policy.

In Chapter 3, we present some applications of the problem presented in Chapter 2, from which we include general costs and noise distributions, noisy observation at the pre-processor side, a quadratic control problem, a problem where we consider packet drop with acknowledgement, infinite time horizon problems(the discounted cost and the average cost) and the tandem problem.

In Chapter 4, we show that if we tackle the problem described in Chapter 2, but we look at the multidimensional case, things can get quite complicated. First, the method used for proving the linearity of the estimator fails. If we consider a linear estimator (or equivalent, a symmetric policy at the pre-processor), it is difficult to prove properties of the decision sets for time horizons bigger or equal to three. Moreover, for the time horizon two, or at the penultimate stage, we found numerically that the decision sets need not be convex.

In Chapter 5, we present a problem with multiple agents and noisy transmission links. In this case, we show that simple affine strategies are not optimal, despite the fact that the problem has quadratic costs and Gaussian noise. We show numerically that signalling strategies perform actually better.

In Chapters 2 and 3 we show how to solve the two blocks problem, while in Chapters 4 and 5, we show the limitations of the methods used in Chapters 2 and 3.

In Chapter 6, we present future research directions, i.e. how the problems studied in Chapters 2, 3, 4 and 5 can be applied to general network topologies or to control problem with communication costs.

Chapter 2

The Two Blocks Problem: A Majorization Theory Approach

2.1 Problem Formulation

We address the design of a finite horizon optimal state estimation system featuring two causal operators; a pre-processor $\mathcal{P}_{0,T}$ and a remote estimator $\mathcal{E}$, where T denotes the time-horizon. At each time instant, the pre-processor outputs either an erasure symbol or a real number, based on causal measurements of the state of a first order linear time-invariant system driven by process noise. The estimator has causal access to the output of the pre-processor and its output is denoted as state estimate. We consider an optimization problem characterized by cost functions that combine the state estimation error and a communication cost. In our formulation, the communication cost depends on the output of the pre-processor, where we ascribe zero cost to the erasure symbol and a pre-specified positive constant otherwise. The state process, denoted as $\mathbf{X}_k$, is given and the two causal operators $\mathcal{P}_{0,T}$ and $\mathcal{E}$ are to be jointly designed so as to minimize the given cost function.

Most of this Section is dedicated to precisely formulating such an optimal estimation problem. In subsection 2.1.1 we give a description of the information structure of our framework, followed by subsections 2.1.2, where we give the problem formulation. In Section 2.2, we state the optimal solution of the problem studied in this chapter, without proof, while Section 2.4 is dedicated to presenting notions from majorization theory and to setting up the proof the optimality of the scheme presented in Section 2.2. In Sec-

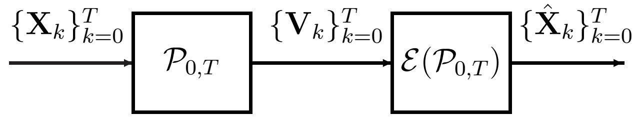

Figure 2.1: Schematic representation of the distributed estimation system considered in this chapter. It depicts the pre-processor $\mathcal{P}_{0,T}$ and the corresponding optimal estimator $\mathcal{E}(\mathcal{P}_{0,T})$, which produces the minimum mean squared error estimate of the process $\{\mathbf{X}_k\}_{k=0}^T$ given in (2.5).

tion 2.5, we prove the optimality scheme presented in Section 2.2. In Section 2.6, we present a simulation example, where we show how the optimal solution of the main problem from this chapter works. We also need to mention that in Appendices A.1 and A.2 we state and prove lemmas that are supporting results used throughout the chapter.

2.1.1 Preliminary Definitions and Information Pattern Description

We start by describing the three stochastic processes and the two classes of causal operators (pre-processor and estimator) that constitute our problem formulation.

Definition 2.1 ***(State Process)*** *Given a real constant a, and a positive real constant σ_W^2, consider the following first order, linear time-invariant discrete-time scalar system driven by process noise:*

$$\mathbf{X}_0 \overset{def}{=} x_0 \tag{2.1}$$

$$\mathbf{X}_{k+1} \overset{def}{=} a\mathbf{X}_k + \mathbf{W}_k,\ k \geq 0 \tag{2.2}$$

where $\{\mathbf{W}_k\}_{k=0}^T$ is an independent identically distributed (i.i.d.) Gaussian zero mean stochastic process with variance σ_W^2 and x_0 is a real number. The filtration generated by

$\{\mathbf{X}_k\}_{k=0}^{T}$ *is denoted as:*

$$\mathcal{X}_k \stackrel{def}{=} \sigma\left(\mathbf{X}_t; 0 \leq t \leq k\right) \tag{2.3}$$

where $\sigma\left(\mathbf{X}_t; 0 \leq t \leq k\right)$ *is the smallest sigma algebra generated by* $\{\mathbf{X}_t, 0 \leq t \leq k\}$*, for all integers* k*.*

Definition 2.2 ***(Pre-processor and remote link process)*** *Consider an erasure symbol denoted as* $\mathfrak{E}$ *and a causal map* $\mathcal{P}_{0,T} : (x_0, \ldots, x_k) \mapsto v_k$*, defined for* $k \in \{0, \ldots, T\}$ *and* $v_k \in \mathbb{R} \cup \{\mathfrak{E}\}$*. Hence, at each time instant* k*, the preprocessor* $\mathcal{P}_{0,T}$ *outputs either a real number or the erasure symbol, based on past observations of the state process.* $\mathcal{P}_{0,T}$ *generates a stochastic process* $\{\mathbf{V}_k\}_{k=0}^{T}$ *via the application of the operator* $\mathcal{P}_{0,T}$ *to the process* $\{\mathbf{X}_k\}_{k=0}^{T}$ *(See Figure 2.1). The map* $\mathcal{P}_{0,T}$ *is a valid pre-processor if the following two conditions hold: (1) The pre-processor transmits the initial state* x_0 *at time zero, i.e.,* $\mathbf{V}_0 = x_0$*. (2) The pre-processor is measurable in the sense that the process* $\{\mathbf{V}_k\}_{k=0}^{T}$ *is adapted to* $\mathcal{X}_k$*.*

The filtration generated by $\{\mathbf{V}_k\}_{k=0}^{T}$ *is denoted as* $\{\mathcal{B}_k\}_{k=0}^{T}$ *and it is obtained as:*

$$\mathcal{B}_k \stackrel{def}{=} \sigma\left(\mathbf{V}_t; 0 \leq t \leq k\right) \tag{2.4}$$

Remark 2.1 *Notice that any finite vector of reals can be encoded into a single real number via a suitable invertible transformation. Hence, without loss of generality, we can also assume that the pre-processor can transmit either a vector of real numbers or the erasure symbol.*

Definition 2.3 ***(Optimal estimate and optimal estimator)*** *Given a pre-processor* $\mathcal{P}_{0,T}$*, we consider the optimal estimator in the expected squared sense whose optimal estimate at* *time k is denoted as $\hat{\mathbf{X}}_k$ and takes values:*

$$\hat{x}_k \stackrel{def}{=} \begin{cases} E\left[\mathbf{X}_k | \{v_t\}_{t=0}^{k}\right] & \text{if } k \geq 1 \\ x_0 & \text{if } k = 0 \end{cases} \tag{2.5}$$

*where $E\left[\mathbf{X}_k | \{v_t\}_{t=0}^{k}\right]$ represents the expectation of the state $\mathbf{X}_k$ conditioned on the observed current and past outputs of the pre-processor $\{v_t\}_{t=0}^{k}$ (see Figure 1). We use $\mathcal{E}(\mathcal{P}_{0,T})$ to denote the **optimal estimator** associated with a given pre-processor policy $\mathcal{P}_{0,T}$.*

Notice that from Definition 2.2 we assume that the pre-processor always transmits the initial state x_0. Hence, the initial estimate is set to satisfy $\hat{x}_0 = v_0 = x_0$. Such an assumption is a key element that will allow us to prove the optimality of a certain scheme, via an inductive method. This will be discussed later on in Section 2.5.

Remark 2.2 *It is important to note that all the information available at the estimator $\mathcal{E}(\mathcal{P}_{0,T})$ is also available at the pre-processor $\mathcal{P}_{0,T}$. Hence, the pre-processor $\mathcal{P}_{0,T}$ can construct the state estimate $\hat{\mathbf{X}}_k$ by reproducing the estimation algorithm executed at the optimal estimator.*

2.1.2 The Two Blocks Problem and Main Results

In this subsection, we define the optimal estimation paradigm that is central to this chapter. We start by specifying the cost, which is used as a merit criterion throughout the chapter, followed by the problem definition.

Definition 2.4 ***(Finite time horizon cost function)*** *Given a valid pre-processor $\mathcal{P}_{0,T}$ (Definition 2.2), a real constant a, a positive integer T, a positive real number d less* *than one and positive real constants σ_W^2 and c, we define:*

$$\mathcal{J}_{0,T}\left(a,\sigma_W^2,c,\mathcal{P}_{0,T}\right) \stackrel{def}{=} \sum_{k=1}^{T} d^{k-1} E\left[\left(\mathbf{X}_k - \hat{\mathbf{X}}_k\right)^2 + \underbrace{c\mathbf{R}_k}_{\textit{communication cost}}\right] \tag{2.6}$$

where $\mathbf{X}_k$ is the state of the system defined in (2.1)-(2.2), $\hat{\mathbf{X}}_k$ is the optimal estimate specified in Definition 2.3, and $\mathbf{R}_k$ is the following indicator function:

$$\mathbf{R}_k \stackrel{def}{=} \begin{cases} 0 & \textit{if } \mathbf{V}_k = \mathfrak{E} \\ 1 & \textit{otherwise} \end{cases}, \qquad k \geq 1 \tag{2.7}$$

Remark 2.3 ***(Cost does not depend on** $\mathbf{X}_0$) Notice that because the plant (2.1)-(2.2) is linear, the fact that $\hat{x}_0 = x_0$ holds (see Definition 2.3) implies, in view of Remark 2.2, in particular, a is known at the estimator, that the homogenous part of the state can be reproduced at the estimator. Hence, the optimal estimator will incorporate such an homogeneous term, thus subtracting it out from the estimation error $\mathbf{X}_k - \hat{\mathbf{X}}_k$, for $k \geq 0$. This also implies that the cost (2.6) does not depend on the homogeneous term nor on the initial condition $\mathbf{X}_0$.*

The following is the main problem addressed in this chapter.

Problem 2.1 *Let a real constant a, the variance of the process noise σ_W^2 and the initial condition x_0 be given. In addition, consider that a positive real c, a positive real number d less then one and a positive integer T are given, specifying the cost as in Definition 2.4. Find:*

$$\mathcal{P}_{0,T}^* \in \textit{arg}\min_{\mathcal{P}_{0,T}} \mathcal{J}_{0,T}(a,\sigma_W^2,c,\mathcal{P}_{0,T}) \tag{2.8}$$

2.2 Optimal Solution to the Two Block Problem

In this section, we start by defining a particular choice of estimator (section 2.2.1) and pre-processor (section 2.2.3), which we denote as Kalman-like and symmetric threshold policy, respectively. As we argue later on, in Theorem 2.1, such estimator and pre-processor are optimal for Problem 2.1.

2.2.1 A Kalman-like estimator

Definition 2.5 *(**Kalman-like estimator**) Given the process defined in (2.1)-(2.2) and a pre-processor $\mathcal{P}_{0,T}$, define the map $\mathcal{Z} : (v_0, \ldots, v_k) \mapsto z_k$, for k in the set $\{0, \ldots, T\}$, where z_k is computed as follows:*

$$z_0 \overset{def}{=} x_0 \tag{2.9}$$

$$z_k \overset{def}{=} \begin{cases} az_{k-1} & \text{if } v_k = \mathfrak{E} \\ v_k & \text{otherwise} \end{cases}, \text{ with } k \geq 1 \tag{2.10}$$

Remark 2.4 *The Kalman-like filter generates the process $\{\mathbf{Z}_k\}_{k=0}^{T}$ via the operator $\mathcal{Z}$ applied to the process $\{\mathbf{V}_k\}_{k=0}^{T}$. Notice that the pre-processor has access to the estimate $\mathbf{Z}_k$ because it has access and full control of the input applied to $\mathcal{Z}$.*

2.2.2 The Set $\mathbb{P}_T$ - of Admissible Pre-Processors

We proceed by defining a class of admissible pre-processors, which is amenable to the use of recursive methods for performance analysis. We argue in Remark 2.6 that there always exist an admissible pre-processor that is an optimal solution to Problem 2.1. This implies that we incur no loss of generality in constraining our analysis to admissible pre-processors.

The following Remark provides an equivalent characterization of the class of admissible pre-processors.

Remark 2.5 *Let $T \in \mathbb{N}$ and let $\mathcal{P}_{0,T}$ be given. Then $\mathcal{P}_{0,T}$ is admissible if and only if for each $m \in \{0, \dots, T\}$ there exists a map $\mathcal{P}_{m,T} : (x_m, \dots, x_k) \mapsto v_k$ and a binary process $\{r_j\}_{j=0}^T$:*

$$r_m = 1 \implies \mathcal{P}_{q,T}(x_q, \dots, x_k) = \mathcal{P}_{m,T}(x_m, \dots, x_k), \qquad x_q, \dots, x_k \in \mathbb{R}, k \geq m \geq q \geq 0 \tag{2.11}$$

Given an admissible pre-processor $\mathcal{P}_{0,T}$, later on we will also refer to the time-restricted pre-processors $\{\mathcal{P}_{m,T}\}_{m=1}^T$ according to Definition 2.6, or equivalently as implied by (2.11).

Definition 2.6 ***(Admissible pre-processor)*** *Let a horizon T larger than zero and a pre-processor policy $\mathcal{P}_{0,T}$ be given. The pre-processor $\mathcal{P}_{0,T}$ is admissible if there exist maps $\mathcal{P}_{m,T} : (x_m, \dots, x_k) \mapsto v_k$, with $0 \leq m \leq T$ and $k \geq m$, that satisfies the following recursion:*

***Algorithm** $\mathcal{P}_{m,T}$*

- ***(Initial step)*** *Set $k = m$, $r_m = 1$ and transmit the current state, i.e., $v_m = x_m$.*

- ***(Step A)*** *Set $k = k + 1$. If $k > T$ then terminate, otherwise execute Step B.*

- ***(Step B)*** *Obtain the pre-processor output at time* k *by computing* $\mathcal{P}_{m,T}(x_m, \ldots, x_k)$*. If* $\mathcal{P}_{m,T}(x_m, \ldots, x_k) = \mathfrak{E}$ *then set* $r_k = 0$ *and* $v_k = \mathfrak{E}$ *(i.e. send the erasure symbol) and go back to Step A. Otherwise execute algorithm* $\mathcal{P}_{k,T}$*.*

———————— ***End of Algorithm for*** $\mathcal{P}_{m,T}$ ————————

The class of <u>*all admissible*</u> *pre-processors is denoted as* $\mathbb{P}_T$ *.*

Remark 2.6 *Given a positive time-horizon* T*, there is no loss of generality in restricting our search for an optimal pre-processor to the set* $\mathbb{P}_T$*. Indeed, let an optimal pre-processor policy* $\mathcal{P}^*_{0,T}$ *be given. If a transmission takes place at some time* m *(*$r_m = 1$ *holds) then the optimal output at the pre-processor is* $v_k = x_k$*, since, given that a real number is transmitted, the choice* $v_k = x_k$ *must be optimal because it leads to a perfect estimate* $\hat{x}_m = x_m$*. Hence, given that* $r_m = 1$*, by Markovianity we conclude that the current and future output produced by the pre-processor* $\{\mathbf{V}_k\}_{k=m}^{T}$ *will* <u>*not*</u> *depend on the state* $\mathbf{X}_k$ *for times* k *prior to* m*. Consequently,* $\mathcal{P}^*_{0,T}$ *satisfies (2.11), and hence it is admissible.*

2.2.3 Symmetric threshold pre-processor

Definition 2.7 *In order to simplify our notation, we define the following process:*

$$\mathbf{Y}_k \stackrel{def}{=} \mathbf{X}_k - a\mathbf{Z}_{k-1} \tag{2.12}$$

Using Definitions 2.1 and 2.5, we find that $\{\mathbf{Y}_k\}_{k=0}^T$ can be rewritten as:

$$\mathbf{Y}_0 = 0 \tag{2.13}$$

$$\mathbf{Y}_{k+1} = \begin{cases} a\mathbf{Y}_k + \mathbf{W}_k & \text{if } \mathbf{R}_k = 0 \\ \\ \mathbf{W}_k & \text{if } \mathbf{R}_k = 1 \end{cases} \tag{2.14}$$

Remark 2.7 *$\mathbf{Y}_k$ has an even probability density function, since $\mathbf{W}_k$ has an even probability function. This fact makes $\{\mathbf{Y}_k\}_{k=0}^T$ a more convenient process to work with, in comparison to $\{\mathbf{X}_k\}_{k=0}^T$. This motivates its use in our analysis hereon, whenever possible. No loss of generality is incurred because $\{\mathbf{Y}_k\}_{k=0}^T$ can be recovered from $\{\mathbf{X}_k\}_{k=0}^T$, and vice-versa, via the use of $\{\mathbf{Z}_k\}_{k=0}^T$, which is common information at the pre-processor and estimator (See Remark 2.4). In addition, notice that the cost (2.6) can be re-written in terms of $\{\mathbf{Y}_k\}_{k=0}^T$ as follows:*

$$\mathcal{J}_{0,T}\left(a, \sigma_W^2, c, \mathcal{P}_{0,T}\right) \stackrel{def}{=} \sum_{k=1}^{T} d^{k-1} E\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}_k\right)^2 + c\mathbf{R}_k\right] \tag{2.15}$$

where $\hat{\mathbf{Y}}_k \stackrel{def}{=} E\left[\mathbf{Y}_k | \{\mathbf{V}_t\}_{t=0}^k\right]$. A key fact here is that $\hat{\mathbf{Y}}_k = \hat{\mathbf{X}}_k - a\mathbf{Z}_{k-1}$ holds, leading to the validity of the identity $\mathbf{Y}_k - \hat{\mathbf{Y}}_k = \mathbf{X}_k - \hat{\mathbf{X}}_k$.

Definition 2.8 *Given a positive integer horizon T and an arbitrary sequence of positive real numbers (thresholds) $\tau = \{\tau_k\}_{k=1}^T$, for each m in the set $\{0, \dots, T\}$, we define the following algorithm for $k \geq m$, which we denote as $\mathcal{S}_{m,T}$:*

***Algorithm** $\mathcal{S}_{m,T}$*

- ***(Initial step)** Set $k = m$, $r_m = 1$ and transmit the current state, i.e., $v_m = x_m$ or equivalently set $y_m = 0$.*

- ***(Step A)*** *Increase the time counter k by one. If $k > T$ holds then terminate, otherwise execute Step B.*

- ***(Step B)*** *If $|y_k| < \tau_k$ holds then set $r_k = 0$, transmit the erasure symbol, i.e., $v_k = \mathfrak{E}$, and return to Step A. If $|y_k| \geq \tau_k$ holds then set $m = k$ and execute $\mathcal{S}_{m,T}$.*

______________ *End of Algorithm* $\mathcal{S}_{m,T}$ ______________

Definition 2.9 ***(Symmetric threshold policy)*** *The algorithm $\mathcal{S}_{0,T}$, as in Definition 2.8, is denoted as symmetric threshold pre-processor and the class of all symmetric threshold policies is denoted as $\mathbb{S}_T$.*

The following is the main result of this chapter.

Theorem 2.1 *Let the variance of the process noise σ_W^2, the system's dynamic constant a, the communication cost c, the discount factor d and the time horizon T be given. There exists a sequence of positive real numbers $\tau^* = \{\tau_k^*\}_{k=1}^T$, such that the corresponding symmetric threshold policy $\mathcal{S}_{0,T}^*$ is an optimal solution to (2.8) and the corresponding optimal estimator $\mathcal{E}(\mathcal{S}_{0,T}^*)$ is $\mathcal{Z}$. Here $\mathcal{S}_{0,T}^*$ and $\mathcal{Z}$ follow Definitions 2.9 and 2.5, respectively.*

Note: The proof of Theorem 2.1 is given in Section 2.5.

2.3 Auxiliary optimality results

We start by defining the following class of path-dependent pre-processor policies, which is an extension of Definition 2.9 so as to allow time-varying thresholds that depend on past decisions. Such a class of admissible pre-processors will be used later in Section 2.5, where we provide a proof for Theorem 2.1.

Definition 2.10 *(**Algorithm** $\mathcal{D}_{m,T}$) Given a horizon T, consider that a sequence of (threshold) functions $\mathcal{T} \stackrel{def}{=} \{\mathcal{T}_{m,k} | m < k \leq T, 1 \leq m \leq T\}$, with $\mathcal{T}_{m,k} : \{0,1\}^{m-k} \rightarrow \mathbb{R}$, is given. For every m in the set $\{1, \ldots, T\}$, we define the following algorithm, which we denote as $\mathcal{D}_{m,T}$:*

———————— ***Algorithm*** $\mathcal{D}_{m,T}$ ————————

- ***(Initial step)*** *Set $k = m$, $r_m = 1$ and transmit the current state, i.e., $v_m = x_m$ or equivalently set $y_m = 0$.*
- ***(Step A)*** *Increase the time counter k by one. If $k > T$ holds then terminate, otherwise execute Step B.*
- ***(Step B)*** *If $|y_k| < \mathcal{T}_{m,k}(r_m, \ldots, r_{k-1})$ holds then set $r_k = 0$, transmit the erasure symbol, i.e., $v_k = \mathfrak{E}$, and return to Step A. If $|y_k| \geq \mathcal{T}_{m,k}(r_m, \ldots, r_{k-1})$ holds then execute $\mathcal{D}_{k,T}$.*

———————— ***End of Algorithm*** $\mathcal{D}_{m,T}$ ————————

Recall that r_0 through r_{k-1} represent past decisions by the pre-processor, where $r_k = 1$ indicates that the state is transmitted to the estimator at time k, while $r_k = 0$ implies that an erasure symbol was sent.

Definition 2.11 ***(Path-dependent symmetric threshold policy)*** *Given a horizon T, consider that a sequence of (threshold) functions $\mathcal{T} \stackrel{def}{=} \{\mathcal{T}_{m,k} | m < k \le T, 1 \le m \le T\}$, with $\mathcal{T}_{m,k} : \{0,1\}^{m-k} \to \mathbb{R}$, is given. The path-dependent symmetric threshold pre-processor associated with $\mathcal{T}$ is implemented via the execution of the algorithm $\mathcal{D}_{0,T}$, as specified in Definition 2.10. Typically, we denote such an admissible pre-processor as $\mathcal{D}_{0,T}$. We use $\mathbb{D}_{0,T}$ to denote the entire class of path-dependent symmetric threshold pre-processors with time horizon T.*

The goal of this section is to provide the following two results that are crucial in the proof of Theorem 2.1: In Proposition 2.1, we prove that if $\mathcal{D}_{0,T}$ is any given path-dependent symmetric threshold pre-processor policy then the associated optimal estimator $\mathcal{E}(\mathcal{D}_{0,T})$ is $\mathcal{Z}$. In Lemma 2.1 we prove that if we optimize within the class of path-dependent policies then the optimum is of the path-independent type, as specified in Definition 2.9. This fact might raise the question of whether Definition 2.11 is needed. The answer is *yes* because we adopt a constructive argument in the proof of Theorem 2.1 in Section 2.5, which uses Definition 2.11.

Proposition 2.1 *Let $\mathcal{D}_{0,T}$ be a pre-selected path-dependent symmetric threshold policy (Definition 2.11), it holds that the optimal estimator $\mathcal{E}(\mathcal{D}_{0,T})$ is $\mathcal{Z}$, as described in Definition 2.5.*

Remark 2.8 *Proposition 2.1 could be recast by stating that $\hat{\mathbf{X}}_k = \mathbf{Z}_k$ holds in the presence of path-dependent symmetric threshold pre-processors.*

Proof: (of Proposition 2.1) In order to simplify the proof, we define $\{\tilde{\mathbf{X}}_k\}_{k=0}^{T}$ as the process quantifying the error incurred by adopting a Kalman-like estimator $\mathcal{Z}$ (See

Definition 2.5), i.e., $\tilde{\mathbf{X}}_k \overset{def}{=} \mathbf{X}_k - \mathbf{Z}_k$. More specifically, $\{\tilde{\mathbf{X}}_k\}_{k=0}^{T}$ can be equivalently expressed as follows:

$$\tilde{\mathbf{X}}_0 = 0 \tag{2.16}$$

$$\tilde{\mathbf{X}}_{k+1} = \begin{cases} a\tilde{\mathbf{X}}_k + \mathbf{W}_k & \text{if } \mathbf{R}_k = 0 \\ 0 & \text{if } \mathbf{R}_k = 1 \end{cases}, \qquad 0 \leq k \leq T-1 \tag{2.17}$$

The proof follows from the symmetry of all probability density functions involving $\tilde{\mathbf{X}}_k$ and $\mathbf{V}_k$. More specifically, under symmetric path-dependent threshold policies the probability density function of $\tilde{\mathbf{X}}_k$, given the past and current observations $\{\mathbf{V}_t\}_{t=0}^{k}$, is even. Hence, we conclude that $E[\tilde{\mathbf{X}}_k|\{\mathbf{V}_t\}_{t=0}^{k}] = 0$, which implies that $\hat{\mathbf{X}}_k \overset{def}{=} E[\mathbf{X}_k|\{\mathbf{V}_t\}_{t=0}^{k}] = \mathbf{Z}_k$. ■

2.3.1 Optimizing within the class $\mathbb{D}_T$

Remark 2.9 *If $\mathcal{D}_{0,T}$ is a symmetric path-dependent threshold pre-processor (see Definition 2.11) then $\hat{\mathbf{Y}}_k = 0$ holds, leading to the following equality:*

$$\mathcal{J}_{0,T}\left(a, \sigma_W^2, c, \mathcal{D}_{0,T}\right) = \sum_{k=1}^{T} d^{k-1} E\left[\mathbf{Y}_k^2 + c\mathbf{R}_k\right], \qquad \mathcal{D}_{0,T} \in \mathbb{D}_T \tag{2.18}$$

The process defined in (2.14) is a Markov Decision Process (MDP) whose state and control are $\mathbf{Y}_k$ and $\mathbf{R}_k$, respectively. Hence the minimization of (2.18) with respect to pre-processor policies $\mathcal{D}_{0,T}$ in the class $\mathbb{D}_T$ can be cast as a dynamic program [13]. To do so, we define the sequence of functions $\mathcal{V}_{t,T} : \mathbb{R} \to \mathbb{R}, t \in \{1, \ldots, T+1\}$ which represent the cost-to-go as observed by the pre-processor. Here T represents the horizon, while t denotes the time at which the decision was taken, and the argument of the function is the MDP state $\mathbf{Y}_t$. In order to simplify our notation, we adopt the convention that

$\mathcal{V}_{T+1,T}(y_{T+1}) \stackrel{def}{=} 0$, $y_{T+1} \in \mathbb{R}$. Using dynamic programming, we can find the following recursive equations for $\mathcal{V}_{t,T}(y_t),\ t \in \{1, \ldots, T\}$:

$$\mathcal{V}_{t,T}(y_t) \stackrel{def}{=} \min_{r_t \in \{0,1\}} \mathcal{C}_{t,T}(y_t, r_t),\ t \in \{1, \ldots, T\} \tag{2.19}$$

where $\mathcal{C}_{t,T} : \mathbb{R} \times \{0,1\} \to \mathbb{R}$ is defined as:

$$\mathcal{C}_{t,T}(y_t, r_t) \stackrel{def}{=} \begin{cases} c + dE\left[\mathcal{V}_{t+1,T}(\mathbf{W}_t)\right] & \text{if } r_t = 1 \\ \\ y_t^2 + dE\left[\mathcal{V}_{t+1,T}\left(ay_t + \mathbf{W}_t\right)\right] & \text{if } r_t = 0 \end{cases} \tag{2.20}$$

From (2.20) it immediately follows that an optimal decision policy r_t^* at any time t is given by:

$$r_t^* = \begin{cases} 1 & \text{if } \mathcal{C}_{t,T}(y_t, 1) \leq \mathcal{C}_{t,T}(y_t, 0) \\ \\ 0 & \text{if } \mathcal{C}_{t,T}(y_t, 0) < \mathcal{C}_{t,T}(y_t, 1) \end{cases} \tag{2.21}$$

Using the MDP given in Definition 2.7 and the value functions from equation (2.19), we prove the following Lemma, which states that, *within the class of symmetric path-dependent pre-processors $\mathbb{D}_T$ (Definition 2.11)*, there exists an optimal path-*independent* symmetric threshold policy $\mathcal{S}^*_{0,T}$ (Definition 2.9) for Problem 2.1.

Lemma 2.1 *Let the parameters specifying Problem 2.1 be given, i.e., the variance of the process noise σ_W^2, the system's dynamic constant a, the communication cost c, the discount factor d and the time horizon T are pre-selected. Consider Problem 2.1 with the additional constraint that the pre-processor must be of the symmetric path-dependent type $\mathbb{D}_T$ specified in Definition 2.11. There exists an optimal path-*independent *symmetric threshold policy $\mathcal{S}^*_{0,T}$, as given in Definition 2.9, whose associated threshold selection*

$\{\tau_k^*\}_{k=1}^T$ *is given by a solution to the following equations:*

$$\mathcal{C}_{t,T}(\tau_t^*, 0) = \mathcal{C}_{t,T}(\tau_t^*, 1),\ t \in \{1, \ldots, T\} \tag{2.22}$$

Proof: From (2.21), we conclude that in order to prove this Lemma we only need to show that there exist thresholds $\{\tau_k^*\}_{k=1}^T$ for which the following equivalences hold:

$$|y_t| \geq \tau_t^* \Longleftrightarrow \mathcal{C}_{t,T}(y_t, 1) \leq \mathcal{C}_{t,T}(y_t, 0), \qquad t \in \{1, \ldots, T\} \tag{2.23}$$

Indeed, if (2.23) holds then the optimal strategy in (2.21) can be implemented via a threshold policy. In order to prove that there exist thresholds $\{\tau_k^*\}_{k=1}^T$ such that (2.23) holds, we will use the following facts (A.1 thorugh A.4):

- (**Fact A.1**): For every t in the set $\{1, \ldots, T\}$, $\mathcal{C}_{t,T}(y_t, 1)$ depends only on t, i.e., it is a time-dependent constant independent of y_t.
- (**Fact A.2**): It holds that $\mathcal{C}_{t,T}(0, 0) < \mathcal{C}_{t,T}(y_t, 1)$ for $y_t \in \mathbb{R}$.
- (**Fact A.3**): For every t in the set $\{1, \ldots, T\}$ there exists a positive constant u_t such that $\mathcal{C}_{t,T}(y_t, 0) > \mathcal{C}_{t,T}(y_t, 1)$ and $\mathcal{C}_{t,T}(-y_t, 0) > \mathcal{C}_{t,T}(-y_t, 1)$ hold for every y_t satisfying $|y_t| > u_t$.
- (**Fact A.4**): It holds that $\mathcal{C}_{t,T}(y_t, 0)$ is a continuous, even, quasi-convex and unbounded function of y_t, for every t in the set $\{1, \ldots, T\}$.

Facts A.1 and A.2 follow directly from (2.20), while Fact A.3 follows from Fact A.4, which requires a proof that we defer to a later stage. At this point we assume that Fact A.4 is valid, and we proceed by noticing that continuity of $\mathcal{C}_{t,T}(y_t, 0)$ with respect to y_t, as well as Facts A.2 and A.3, imply that the equations in (2.22) have at least one

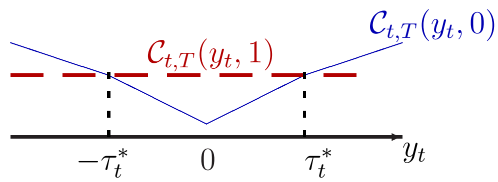

Figure 2.2: Illustration suggesting that Facts A.1 through A.4. imply the existence of thresholds for which equation (2.23) holds.

solution $\{\tau_k^*\}_{k=1}^T$. Moreover, from Facts A.1 through A.4 we can conclude that such a solution $\{\tau_k^*\}_{k=1}^T$ guarantees that (2.23) is true (See Figure 2.2).

(Proof of Fact 4) Since y_t^2 is an even, convex, unbounded and continuous function of y_t, from (2.20) we conclude that it suffices to prove by induction that $\mathcal{V}_{t,T}(y_t)$ is even, quasiconvex, bounded and continuous for each t in the set $\{1, \ldots, T\}$.

Since $\mathcal{V}_{T+1,T}(y_{T+1}) = 0$ holds by convention, the following is true:

$$\mathcal{V}_{T,T}(y_T) = \min\left(c, y_T^2\right), \qquad y_T \in \mathbb{R}$$

Hence $\mathcal{V}_{T,T}(y_T)$ is an even, quasiconvex, bounded and continuous function of y_T. Using Lemma A.10 in Appendix A.2, we conclude that $E\left[\mathcal{V}_{T,T}(ay_{T-1} + \mathbf{W}_{T-1})\right]$ is also an even, quasiconvex, bounded and continuous function of y_{T-1}, which implies that so is $\mathcal{V}_{T-1,T}(y_{T-1})$. By induction it follows that $\mathcal{V}_{t,T}(y_t)$ is an even, quasiconvex, bounded and continuous of y_t, for each t in the set $\{1, \ldots, T\}$.$\blacksquare$

Remark 2.10 *Lemma 2.1 shows that the optimal policy, which solves Problem 2.1 under the additional constraint that the pre-processor must be of the symmetric path-dependent type, is in fact a symmetric path-independent policy. We want to compute the optimal thresholds* $\{\tau_k^*\}_{k=1}^T$ *and for that we need the value functions* $\{\mathcal{V}_{k,T}\}_{k=1}^T$. *The value func-*

tions can be computed recursively using equations (2.19) and (2.20) and the fact that $\mathcal{V}_{T+1,T}(y) = 0$ for all $y \in \mathbb{R}$. From facts $\mathbf{A.1}$ and $\mathbf{A.4}$ in the proof of Lemma 2.1 and the fact that $\mathcal{C}_{t,T}(y_t, 0)$ is strictly increasing for $y_t > 0$ and strictly decreasing for $y_t < 0$ for all $t \in \{1, \ldots, T\}$, it follows that the optimal thresholds are given by the solution of the equations:

$$y^2 + dE\left[\mathcal{V}_{t+1,T}\left(ay + \mathbf{W}_t\right)\right] = c + dE\left[\mathcal{V}_{t+1,T}(\mathbf{W}_t)\right], \qquad t \in \{1, \ldots, T\} \tag{2.24}$$

Since the functions $\{\mathcal{V}_{k,T}\}_{k=1}^{T}$ are even, quasiconvex, bounded and continuous, it follows that the solution of the system of equation (2.24) is unique, hence the optimal thresholds $\{\tau_k^*\}_{k=1}^{T}$ are unique.

2.4 Notation, Definitions and Basic Results for the Proof of Theorem 2.1

This section is dedicated to introducing notation, definitions and basic results in majorization theory that will streamline our proof of Theorem 2.1. The proof of Theorem 2.1 is given in Section 2.5. In Subsection 2.4.1, we introduce basic majorization theory and state a few Lemmas, which are supporting results for the proof of Theorem 2.1. In Subsection 2.4.2, we introduce notation and we derive recursive equations for the time update of certain conditional probability density functions of interest.

2.4.1 Basic Results, Notation and Definitions from Theory of Majorization

In [1], the authors define what a neat probability mass functions is. We will adapt this definition for probability density functions on $\mathbb{R}$.

Definition 2.12 ***(Neat pdf)*** *Let $f : \mathbb{R} \to \mathbb{R}$ be a probability density function. We say that f is neat if f is quasiconcave and there exists a real number b such that f is non-decreasing on the interval $(-\infty, b]$ and non-increasing on $[b, \infty)$.*

Remark 2.11 *Throughout the chapter, we will use the useful fact that the convolution of two neat and even probability density functions is also neat and even. The complete proof of this fact is given in Lemma A.1 in Appendix A.1.*

Hajek gives in [1] the definition of symmetric non-increasing function on $\mathbb{R}^n$. Since we work only on the real line, it suffices to notice that a probability density function $f : \mathbb{R} \to \mathbb{R}$ is symmetric non-increasing if and only if it is neat and even. Hence, without loss of generality, in this chapter only use *symmetric non-increasing* to qualify certain probability density functions throughout the chapter.

Let $\mathbb{A}$ be a given Borel measurable subset of $\mathbb{R}$, we denote its Lebesgue measure by $\mathcal{L}(\mathbb{A})$. If the Lebesgue measure of $\mathbb{A}$ is finite then the symmetric rearrangement of $\mathbb{A}$, denoted by $\mathbb{A}^\sigma$, is a symmetric closed interval centered around the origin with Lebesgue measure $\mathcal{L}(\mathbb{A})$:

$$\mathbb{A}^\sigma = \left\{ x \in \mathbb{R} : |x| \leq \frac{\mathcal{L}(\mathbb{A})}{2} \right\}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a given non-negative function, we define f^σ, the symmetric non-decreasing rearrangement of f, as follows:

$$f^\sigma(x) \stackrel{def}{=} \int_0^\infty \mathcal{I}_{\{z\in\mathbb{R}:f(z)>\rho\}^\sigma}(x)d\rho \tag{2.25}$$

where $\mathcal{I}_{\{z\in\mathbb{R}:f(z)>\rho\}^\sigma} : \mathbb{R} \to \{0,1\}$ is the following indicator function:

$$\mathcal{I}_{\{z\in\mathbb{R}:f(z)>\rho\}^\sigma}(x) \stackrel{def}{=} \begin{cases} 1 & \text{if } x \in \{z\in\mathbb{R} : f(z) > \rho\}^\sigma \\ 0 & \text{otherwise} \end{cases}, \qquad x \in \mathbb{R}$$

If f and g are two probability density functions on $\mathbb{R}$, then we say that f majorizes g, which we denote as $f \succ g$, provided that the following holds:

$$\int_{|x|\le\rho} g^\sigma(x)dx \le \int_{|x|\le\rho} f^\sigma(x)dx, \text{ for all } \rho \ge 0 \tag{2.26}$$

One interpretation of the inequality in (2.26) is that, f majorizes g, if and only if for any Borel set $\mathbb{F}' \subset \mathbb{R}$ with finite Lebesgue measure, there exists another Borel set $\mathbb{F} \subset \mathbb{R}$ satisfying $\mathcal{L}(\mathbb{F}') = \mathcal{L}(\mathbb{F})$ and such that the following holds:

$$\int_{\mathbb{F}'} g(x)dx \le \int_{\mathbb{F}} f(x)dx$$

Given a probability density function $f : \mathbb{R} \to \mathbb{R}$ and a Borel set $\mathbb{K}$, such that $\int_{\mathbb{K}} f(x)dx > 0$, we define the restriction of f to $\mathbb{K}$ as follows:

$$f_{\mathbb{K}}(x) \stackrel{def}{=} \begin{cases} \frac{f(x)}{\int_{\mathbb{K}} f(x)dx} & \text{if } x \in \mathbb{K} \\ 0 & \text{otherwise} \end{cases} \tag{2.27}$$

It is clear that $f_{\mathbb{K}}$ is also a probability density function.

The following Lemma is a supporting result for the proof of Theorem 2.1 given in Section 2.5.

Lemma 2.2 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two probability density functions, such that f is neat and even and $f \succ g$. Let κ be a real number in the interval $\kappa \in (0, 1)$, and let $\mathbb{A} = [-\tau, \tau]$ be the symmetric closed interval such that $\int_{-\tau}^{\tau} f(x)dx = 1 - \kappa$. For any function $h : \mathbb{R} \to [0, 1]$ satisfying $\int_{\mathbb{R}} g(x)h(x)dx = 1 - \kappa$, the following holds:*

$$f_{\mathbb{A}} \succ \frac{g \cdot h}{1 - \kappa} \tag{2.28}$$

where $g \cdot h : \mathbb{R} \to \mathbb{R}$ is defined as $g \cdot h(x) \stackrel{def}{=} g(x)h(x)$, for $x \in \mathbb{R}$.

Proof: From Lemma A.6 given in Appendix A.1, we know that for any function $h : \mathbb{R} \to [0, 1]$ satisfying $\int_{\mathbb{R}} g(x)h(x)dx = 1 - \kappa$, there exists a set $\mathbb{A}' \subset \mathbb{R}$, satisfying $\int_{\mathbb{A}'} g(x)dx = 1 - \kappa$, such that the following holds:

$$g_{\mathbb{A}'} \succ \frac{g \cdot h}{1 - \kappa} \tag{2.29}$$

From Lemma A.5 given in Appendix A.1, we know that $f_{\mathbb{A}} \succ g_{\mathbb{A}'}$. From equation (2.29) and the fact that $f_{\mathbb{A}} \succ g_{\mathbb{A}'}$ holds, equation (2.28) follows. ■

The following Lemma, which we state without proof, can be found in [1]:

Lemma 2.3 *[1, Lemma 6.7] Let f and g be two probability density functions on $\mathbb{R}$, with f symmetric non-increasing and $f \succ g$. For a symmetric non-increasing probability density function h the following holds:*

$$f * h \succ g * h \tag{2.30}$$

Lemma 2.4 *Let f be a neat and even probability density function on the real line. Let g be a probability density function on the real line satisfying $g \prec f$. The following holds:*

$$\int_{\mathbb{R}} x^2 f(x)dx \leq \int_{\mathbb{R}} (x - y)^2 g(x)dx, \qquad y \in \mathbb{R} \tag{2.31}$$

Proof: The result follows by selecting $h(x) = x^2$ in Lemma A.9 found in Appendix A.■

Remark 2.12 *Consider the conditions of Lemma 2.4. The fact that the probability density function f is even implies that $\int_{\mathbb{R}} xf(x)dx = 0$. Hence, if we select $y = \int_{\mathbb{R}} xg(x)dx$ then it follows from equation (2.31) that the variance of f is less than or equal to the variance of g.*

2.4.2 Conditional probabilities and conditional probability density functions

Before proving Theorem 2.1, in this subsection we need to make a few remarks and introduce more notation, which will streamline our proof. This subsection contains two parts: We start by introducing the notation for certain conditional probability density functions of interest, while in the second part we will derive recursive equations for the time update of the conditional densities, and we will also obtain a recursive expansion for the cost associated with any given admissible pre-processor policy $\mathcal{P}_{0,T}$.

Definition 2.13 *Let a pre-processor $\mathcal{P}_{0,T}$, implementing a decision policy as in Definition 2.2, be given. We define the following notation for conditional probability densities, which will streamline our proof of Theorem 2.1:*

1. *Define the conditional probability density function of $\mathbf{Y}_k$ given that only erasure symbols were transmitted up until time k as follows:*

$$\gamma_{k|k}(y) \stackrel{def}{=} f_{\mathbf{Y}_k|\mathbf{R}_1=0,\ldots,\mathbf{R}_k=0}(y), \qquad y \in \mathbb{R}$$

2. *Define the conditional probability density function of $\mathbf{Y}_k$ given that only erasure symbols were transmitted up until time $k-1$ as follows:*

$$\gamma_{k|k-1}(y) \stackrel{def}{=} f_{\mathbf{Y}_k|\mathbf{R}_1=0,\ldots,\mathbf{R}_{k-1}=0}(y), \qquad y \in \mathbb{R}$$

Definition 2.14 *We define the following streamlined notation for certain conditional probabilities of interest:*

1. *Define the probability that, under policy $\mathcal{P}_{0,T}$, only erasure symbols have been transmitted up until time k:*

$$\varsigma_k \stackrel{def}{=} \begin{cases} P(\mathbf{R}_1 = 0, \ldots, \mathbf{R}_k = 0) & \textit{if } k \geq 1 \\ 1 & \textit{if } k = 0 \end{cases}$$

2. *Define the conditional probability that, under policy $\mathcal{P}_{0,T}$, the pre-processor transmits the erasure symbol at time k, given that only erasure symbols have been transmitted up until time $k-1$.*

$$\varsigma_{k|k-1} \stackrel{def}{=} \begin{cases} P(\mathbf{R}_k = 0|\mathbf{R}_1 = 0, \ldots, \mathbf{R}_{k-1} = 0) & \textit{if } k > 1 \\ \varsigma_1 & \textit{if } k = 1 \end{cases}$$

Definition 2.15 *Let $\mathcal{P}_{0,T}$ be a decision policy given as in Definition 2.2. Let k be a positive integer and y be a real number. For a positive integer k, define the function $\rho_k : \mathbb{R} \to [0,1]$ as follows:*

$$\rho_k(y) \stackrel{def}{=} P(\mathbf{R}_k = 0|\mathbf{Y}_k = y, \mathbf{R}_1 = 0, \ldots, \mathbf{R}_{k-1} = 0), \qquad x \in \mathbb{R} \tag{2.32}$$

which is the probability that, at time k, the erasure symbol is transmitted, given that $\mathbf{Y}_k = y$, where y is any real number, and the fact that only erasure symbols have been transmitted up until time $k-1$.

Notation: For a random variable $\mathbf{Y}$ described by a probability density function f and a real function h, we denote by $E_f[h(\mathbf{Y})]$, the expected value of the random variable $h(\mathbf{Y})$ under the probability density function f.

2.4.3 Time Evolution

Now, we describe how the conditional probability density functions presented in subsection 2.4.2 evolve in time, for a given policy $\mathcal{P}_{0,T}$. For a real number a, below we define the conditional probability density function of $a\mathbf{Y}_k$ given that no observation was received up until time k:

$$\gamma^a_{k|k}(y) \stackrel{def}{=} f_{a\mathbf{Y}_k|\mathbf{R}_1=0,\ldots,\mathbf{R}_k=0}(y)$$

We denote by $\mathcal{N}_{\sigma^2_W}$ the probability density function of $\mathbf{W}_k$, for all k, i.e., the Gaussian zero mean probability density with variance σ^2_W, or more concretely $\mathcal{N}_{\sigma^2_W}(x) = \frac{1}{\sqrt{2\pi\sigma^2_W}}e^{-\frac{x^2}{2\sigma^2_W}}$. Since the sequence $\{\mathbf{W}_k\}_{k=0}^{T}$ is i.i.d., $\mathbf{W}_{k-1}$ is also independent of $\{Y_l\}_{l=0}^{k-1}$, which implies that the following holds:

$$\gamma_{k|k-1} = \gamma^a_{k-1|k-1} * \mathcal{N}_{\sigma^2_w} \tag{2.33}$$

Proposition 2.2 *The conditional densities $\gamma_{k|k-1}$ and $\gamma_{k|k}$ are related via the following time-recursion:*

$$\gamma_{k|k}(y) = \frac{\gamma_{k|k-1}(y)\rho_k(y)}{\varsigma_{k|k-1}}, \qquad \varsigma_{k|k-1} \neq 0, k \geq 1 \tag{2.34}$$

Proof: In order to arrive at (2.34), we use Baye's rule to write:

$$f_{\mathbf{Y}_k|\mathbf{R}_1=\mathfrak{e},\ldots,\mathbf{R}_k=\mathfrak{e}}\left(y\right)=\frac{P\left(\mathbf{R}_k=0|\mathbf{Y}_k=y,\mathbf{R}_1=0,\ldots,\mathbf{R}_{k-1}=0\right)}{P\left(\mathbf{R}_k=0|\mathbf{R}_1=0,\ldots,\mathbf{R}_{k-1}=0\right)}f_{\mathbf{Y}_k|\mathbf{R}_1=0,\ldots,\mathbf{R}_{k-1}=0}\left(y\right) \tag{2.35}$$

The recursion (2.34) follows from (2.35) by rewriting it according to Definitions 2.13, 2.14 and 2.15. Equation (2.35) holds only if $P\left(\mathbf{R}_k=0|\mathbf{R}_1=0,\ldots,\mathbf{R}_{k-1}=0\right)=\varsigma_{k|k-1}\neq 0$. If $\varsigma_{k|k-1}=0$ then the conditional density function $f_{\mathbf{Y}_k|\mathbf{R}_1=0,\ldots,\mathbf{R}_k=0}\left(y\right)$ is no longer defined.■

Definition 2.16 *Given an admissible pre-processor $\mathcal{P}_{0,T}$ and an integer $m\in\{0,\ldots,T\}$, we adopt the following definition for the partial cost computed for the horizon $\{m+1,\ldots,T\}$ under the assumption that $r_m=1$:*

$$\mathcal{J}_{m,T}\left(a,\sigma_W^2,c,\mathcal{P}_{m,T}\right)\overset{def}{=}\begin{cases}\sum_{k=m+1}^{T}d^{k-m-1}E\left[\left(\mathbf{Y}_k-\hat{\mathbf{Y}}_k\right)^2+c\mathbf{R}_k\right] & \textit{if } 0\leq m<T\\ 0 & \textit{if } m=T\end{cases} \tag{2.36}$$

Remark 2.13 *Given an integer m, we notice that the cost in (2.36) will not depend on the value of the state at time m. This is so because, according to Definition 2.6, since $\mathcal{P}_{0,T}$ is admissible it holds that the current and future* output *of $\mathcal{P}_{m,T}$ will* <u>not</u> *depend on the current and past state observations. This Remark is an extension of Remark 2.3, which considered the case for $m=0$.*

Proposition 2.3 *Given an arbitrarily selected admissible pre-processor $\mathcal{P}_{0,T}$, the finite* *horizon cost (2.6) can be expanded as:*

$$\begin{aligned}&\mathcal{J}_{0,T}\left(a,\sigma_W^2,c,\mathcal{P}_{0,T}\right)\\&=\sum_{k=1}^{T}d^{k-1}\left(E_{\gamma_{k|k}}\left[\left(\mathbf{Y}_k-\hat{\mathbf{Y}}_k\right)^2\right]\varsigma_k+\left(c+\mathcal{J}_{k,T}\left(a,\sigma_W^2,c,\mathcal{P}_{k,T}\right)\right)\varsigma_{k-1}(1-\varsigma_{k|k-1})\right)\end{aligned} \tag{2.37}$$

Here we use the notation $E_{\gamma_{k|k}}\left[\left(\mathbf{Y}_k-\hat{\mathbf{Y}}_k\right)^2\right]\stackrel{def}{=}E\left[\left(\mathbf{Y}_k-\hat{\mathbf{Y}}_k\right)^2|\mathbf{R}_1=0,\ldots,\mathbf{R}_k=0\right]$*, where* $\gamma_{k|k}$ *is given in Definition 2.13.*

Proof: We start by noticing that, by the total probability law, we can expand the cost as:

$$\begin{aligned}&\mathcal{J}_{0,T}\left(a,\sigma_W^2,c,\mathcal{P}_{0,T}\right)\\&=\sum_{k=1}^{T}d^{k-1}\Bigg(E\left[\left(\mathbf{Y}_k-\hat{\mathbf{Y}}_k\right)^2|\mathbf{R}_1=0,\ldots,\mathbf{R}_k=0\right]P\left(\mathbf{R}_1=0,\ldots,\mathbf{R}_k=0\right)+\\&\quad+\left(c+E\left[\mathcal{J}_{k,T}\left(a,\sigma_W^2,c,\mathcal{P}_{k,T}\right)|\mathbf{R}_k=1,\mathbf{R}_1=0,\ldots,\mathbf{R}_{k-1}=0\right]\right)\times\\&\quad P\left(\mathbf{R}_k=1,\mathbf{R}_1=0,\ldots,\mathbf{R}_{k-1}=0\right)\Bigg)\end{aligned} \tag{2.38}$$

We proceed by obtaining the following identities:

$$\begin{aligned}&P\left(\mathbf{R}_k=1,\mathbf{R}_1=0,\ldots,\mathbf{R}_{k-1}=0\right)=P\left(\mathbf{R}_1=0,\ldots,\mathbf{R}_{k-1}=0\right)-\\&-P\left(\mathbf{R}_1=0,\ldots,\mathbf{R}_k=0\right)=P\left(\mathbf{R}_1=0,\ldots,\mathbf{R}_{k-1}=0\right)-\\&-P\left(\mathbf{R}_k=0|\mathbf{R}_1=0,\ldots,\mathbf{R}_{k-1}=0\right)P\left(\mathbf{R}_1=0,\ldots,\mathbf{R}_{k-1}=0\right)=\\&=\varsigma_{k-1}(1-\varsigma_{k|k-1}),\qquad k\geq 1\end{aligned} \tag{2.39}$$

Notice that, using standard probability theory, from $\{\varsigma_k\}_{k=1}^T$ we can compute $\left\{\varsigma_{k|k-1}\right\}_{k=1}^T$ and vice versa. Here, equation (2.39) is still valid for $k=1$, since we defined $\varsigma_0=1$ and

$\varsigma_{1|0} = \varsigma_1$. Finally, notice that from Remark 2.13, we conclude the following:

$$E\left[\mathcal{J}_{k,T}\left(a, \sigma_W^2, c, \mathcal{P}_{k,T}\right) | \mathbf{R}_k = 1, \mathbf{R}_1 = 0, \ldots, \mathbf{R}_{k-1} = 0\right] = \mathcal{J}_{k,T}\left(a, \sigma_W^2, c, \mathcal{P}_{k,T}\right) \tag{2.40}$$

The proof of this Proposition is complete once we substitute (2.39) and (2.40) into (2.38).■

Definition 2.17 *The following is a convenient definition for the optimal cost:*

$$\mathcal{J}^*_{m,T}\left(a, \sigma_W^2, c\right) \stackrel{def}{=} \begin{cases} \min_{\mathcal{P}_{m,T} \in \mathbb{P}_{T-m}} \mathcal{J}_{m,T}\left(a, \sigma_W^2, c, \mathcal{P}_{m,T}\right), & T \geq 1 \\ 0, & T = 0 \end{cases} \tag{2.41}$$

From Proposition 2.3, we can immediately state the following Corollary:

Corollary 2.1 *The following inequality holds for every admissible pre-processor $\mathcal{P}_{0,T}$:*

$$\mathcal{J}_{0,T}\left(a, \sigma_W^2, c, \mathcal{P}_{0,T}\right) \geq$$
$$\sum_{k=1}^{T} d^{k-1}\left(E_{\gamma_{k|k}}\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}_k\right)^2\right]\varsigma_k + \left(c + \mathcal{J}^*_{k,T}\left(a, \sigma_W^2, c\right)\right)(1 - \varsigma_{k|k-1})\varsigma_{k-1}\right) \tag{2.42}$$

2.5 Proof of Theorem 2.1

Our **strategy** to prove Theorem 2.1 is to show that for every admissible pre-processor policy $\mathcal{P}_{0,T}$, there exists a path-dependent symmetric threshold policy $\mathcal{D}^o_{0,T}$ which does not underperform $\mathcal{P}_{0,T}$. This fact, which we denote as **Fact B.1**, leads to the following conclusions:

- (**Fact B.2**): Lemma 2.1 (Section 2.3.1), in conjunction with Fact B.1, implies that an optimum $\mathcal{S}^*_{0,T}$ for Problem 2.1 exists and that it is of the symmetric threshold type $\mathbb{S}_T$ (Definition 2.9).

- (**Fact B.3**): From Fact B.2 and Proposition 2.1 (Section 2.3), we conclude there exists a symmetric threshold policy $\mathcal{S}^*_{0,T}$ and a Kalman-like estimator $\mathcal{Z}$ (Definition 2.5) that are jointly optimal for Problem 2.1.

Proof: (of Theorem 2.1) Facts B.2 and B.3 constitute a proof for Theorem 2.1. It remains to prove the validity of Fact B.1.

(**Proof of Fact B.1**): Here we will use an inductive approach that is analogous to the one used in [1, Lemma 6.5]. Our proof for Fact B.1 is organized in two parts. In **Part I**, we will prove Fact B.1 for the case when the time-horizon T is one, while in **Part II**, we prove the general induction step.

Notation: According to the definitions of Section 2.4.2 , any given pre-processor has associated with it conditional probability density functions $\left\{\gamma_{k|k}\right\}_{k=1}^{T}$ and $\left\{\gamma_{k|k-1}\right\}_{k=1}^{T}$, as well as conditional probabilities $\{\varsigma_k\}_{k=1}^{T}$ and $\left\{\varsigma_{k|k-1}\right\}_{k=1}^{T}$. Hence, we assume that the path-dependent symmetric threshold policy $\mathcal{D}^o_{0,T}$ - to be constructed as part of this proof - defines conditional probability density functions $\left\{\gamma^o_{k|k}\right\}_{k=1}^{T}$ and $\left\{\gamma^o_{k|k-1}\right\}_{k=1}^{T}$ as well as conditional probabilities $\{\varsigma^o_k\}_{k=1}^{T}$ and $\left\{\varsigma^o_{k|k-1}\right\}_{k=1}^{T}$.

Part I: Here we will prove Fact B.1 for $T = 1$. We will do so by constructing a policy $\mathcal{D}^o_{0,1}$ as follows:

$$r^o_1 \overset{def}{=} \begin{cases} 1 & \text{if } |y_1| > \tau_1 \\ \\ 0 & \text{otherwise} \end{cases} \tag{2.43}$$

where τ_1 is a threshold that we will select appropriately. Hence, if the absolute value of y_1 is less than or equal to τ_1 then the pre-processor transmits the erasure symbol, otherwise it sends x_1. Consider that a policy $\mathcal{P}_{0,1}$ is given. We start by noticing that for $\mathcal{P}_{0,1}$ and

$\mathcal{D}^o_{0,1}$ it holds that $\gamma_{1|0} = \gamma^o_{1|0} = N_{\sigma^2_W}$, while the cost associated with policy $\mathcal{P}_{0,1}$ is:

$$\mathcal{J}_{0,1}\left(a, \sigma^2_W, c, \mathcal{P}_{0,1}\right) = E_{\gamma_{1|1}}\left[\left(\mathbf{Y}_1 - \hat{\mathbf{Y}}_1\right)^2\right]\varsigma_1 + c(1-\varsigma_1) \tag{2.44}$$

where $\hat{\mathbf{Y}}_1 = E_{\gamma_{1|1}}[\mathbf{Y}_1]$. We construct a desirable $\mathcal{D}^o_{0,1}$ by selecting τ_1 such that $\varsigma^o_1 = \varsigma_1$, which from (2.43) leads to a probability density function $\gamma^o_{1|1}$ that is neat and even. Furthermore, Lemma 2.2 implies that $\gamma_{1|1} \prec \gamma^o_{1|1}$ holds. From Lemma 2.4 we arrive at the following inequality:

$$E_{\gamma^o_{1|1}}\left[\left(\mathbf{Y}_1 - \hat{\mathbf{Y}}^o_1\right)^2\right] \leq E_{\gamma_{1|1}}\left[\left(\mathbf{Y}_1 - \hat{\mathbf{Y}}_1\right)^2\right] \tag{2.45}$$

The cost associated with the policy $\mathcal{D}^o_{0,1}$ is given by:

$$\mathcal{J}_{0,1}\left(a, \sigma^2_W, c, \mathcal{D}^o_{0,1}\right) = E_{\gamma^o_{1|1}}\left[\left(\mathbf{Y}_1 - \hat{\mathbf{Y}}^o_1\right)^2\right]\varsigma_1 + c(1-\varsigma_1) \tag{2.46}$$

Finally, we conclude from (2.44), (2.45) and (2.46) that:

$$\mathcal{J}_{0,1}\left(a, \sigma^2_W, c, \mathcal{P}_{0,1}\right) \geq \mathcal{J}_{0,1}\left(a, \sigma^2_W, c, \mathcal{D}^o_{0,1}\right) \tag{2.47}$$

which leads to the desired conclusion that $\mathcal{D}^o_{0,1}$ does not underperform $\mathcal{P}_{0,1}$.

Part II: (General induction step) Let T^I be a given horizon that is strictly larger than one. Assume the **inductive hypothesis** that Fact B.1 is valid for any horizon T less than T^I.

We start by noticing that the validity of our inductive hypothesis implies the following facts:

- (**Fact B.4**): The inductive hypothesis in conjunction with Lemma 2.1 implies that Problem 2.1 has an optimum for every horizon T less than T^I.

- (**Fact B.5**): The inductive hypothesis also implies that Problem 2.1 admits an optimal pre-processor policy of the symmetric threshold type (Definition 2.9), for every horizon T less than T^I.

Hence, Fact B.5 implies that there exist $\mathcal{S}^*_{1,T^I}$ through $\mathcal{S}^*_{T^I,T^I}$ that satisfy the following:

$$\mathcal{J}_{m,T^I}(a,\sigma_W^2,c,\mathcal{S}^*_{m,T^I}) = \min_{\tilde{\mathcal{P}}_{m,T^I}\in\mathbb{P}_{T^I-m}} \mathcal{J}_{m,T^I}(a,\sigma_W^2,c,\tilde{\mathcal{P}}_{m,T^I}) \underset{(a)}{=} \mathcal{J}^*_{m,T^I}(a,\sigma_W^2,c) \qquad 1\leq m\leq T^I \tag{2.48}$$

where $\mathcal{S}^*_{m,T^I}$ is of the symmetric threshold type $\mathbb{S}_{T^I-m}$ and (a) above follows by definition from (2.41).

Now we proceed to showing that the general induction step holds. In order to do so, we show that for any admissible policy $\mathcal{P}_{0,T^I}$, we can construct a path-dependent symmetric threshold policy $\mathcal{D}^o_{0,T^I}$ that does not underperform $\mathcal{P}_{0,T^I}$. Henceforth, assume that $\mathcal{P}_{0,T^I}$ is an arbitrarily chosen admissible policy.

The following is our algorithm for $\mathcal{D}^o_{0,T^I}$:

Description of Algorithm for $\mathcal{D}^o_{0,T^I}$

- **(Initial step)** Set $k=0$ and transmit the current state, i.e., $v_0=x_0$ or equivalently set $y_0=0$.

- **(Step A)** Increase the time counter k by one. If $k>T^I$ holds then terminate, otherwise execute Step B.

- **(Step B)** If $|y_k|<\tau^o_k$ holds then set $r_k=0$, transmit the erasure symbol, i.e., $v_k=\mathfrak{E}$, and return to Step A. If $|y_k|\geq\tau^o_k$ holds then execute $\mathcal{S}^*_{k,T^I}$, as defined in (2.48).

where $\{\tau_k^o\}_{k=1}^{T^I}$ are appropriately chosen thresholds, as described next.

End of description of Algorithm for $\mathcal{D}_{0,T^I}^o$

Notice that $\mathcal{D}_{0,T^I}^o$ is a path-dependent symmetric threshold strategy (Definition 2.10), for which we can also conclude that $\mathcal{D}_{m,T^I}^o = \mathcal{S}_{m,T^I}^*$ holds for $1 \leq m \leq T^I$.

In order to complete the specification of $\mathcal{D}_{0,T^I}^o$ so that it does not underform $\mathcal{P}_{0,T^I}$, we proceed by appropriately selecting the thresholds $\{\tau_k^o\}_{k=1}^{T^I}$.

(**Selection of thresholds** $\{\tau_k^o\}_{k=1}^{T^I}$) We proceed to describing how to choose the threshold sequence $\{\tau_k^o\}_{k=1}^{T}$ and what this choice implies. Notice that $\gamma_{1|0}^o = \mathcal{N}_{\sigma_W^2}$ and that the Gaussian probability density function is neat and symmetric. Choose τ_1^o such that $\varsigma_1^o = \varsigma_1$, it follows that the probability density function $\gamma_{1|1}^o$ is neat and even. From equation (2.33), which describes how the conditional probability density functions evolve in time, it holds that $\gamma_{2|1}^o$ is neat and even. By further selecting τ_2^o such that $\varsigma_{2|1}^o = \varsigma_{2|1}$, it also follows that $\gamma_{2|2}^o$ and $\gamma_{3|2}^o$ are neat and even. By repeated execution of this selection process, we can choose all the thresholds τ_k^o such that $\varsigma_{k|k-1}^o = \varsigma_{k|k-1}$ for all k in $\{1, \ldots, T^I\}$. These choices also imply that $\gamma_{k|k}^o$ and $\gamma_{k|k-1}^o$ are neat and even for all k in $\{1, \ldots, T^I\}$. Since $\varsigma_{k|k-1}^o = \varsigma_{k|k-1}$ holds for all k in $\{1, \ldots, T^I\}$, it follows that $\varsigma_k^o = \varsigma_k$ is satisfied for all k in $\{1, \ldots, T^I\}$.

At this point, we know that $\gamma_{1|0} = \gamma_{1|0}^o = \mathcal{N}_{\sigma_W^2}$ and that the Gaussian probability density function $\mathcal{N}_{\sigma_W^2}$ is neat and even. Hence, then from Lemma 2.2, we conclude that $\gamma_{1|1} \prec \gamma_{1|1}^o$. It also follows from Lemma A.7 in the Appendix A.1 and Lemma 2.3 that $\gamma_{2|1} \prec \gamma_{2|1}^o$ holds. From the repeated application of this idea, it follows that $\gamma_{k|k} \prec \gamma_{k|k}^o$ for all k in $\{1, \ldots, T^I\}$ and, in addition, since $\gamma_{k|k}^o$ is neat and even, it holds that $\hat{\mathbf{Y}}_k^o =$

$E_{\gamma^o_{k|k}}[\mathbf{Y}_k] = 0$ for all k in $\{1, \ldots, T^I\}$. Since $\gamma_{k|k} \prec \gamma^o_{k|k}$ holds and $\gamma^o_{k|k}$ is neat and even, Lemma 2.4 implies that the following is true:

$$E_{\gamma^o_{k|k}}\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}^o_k\right)^2\right] \leq E_{\gamma_{k|k}}\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}_k\right)^2\right], \qquad k \in \{1, \ldots, T^I\} \tag{2.49}$$

The cost obtained by applying the pre-processor policy $\mathcal{P}^o$ can be expressed using (2.37) as follows:

$$\begin{aligned}\mathcal{J}_{0,T^I}\left(a, \sigma_W^2, c, \mathcal{D}^o_{0,T^I}\right) = \sum_{k=1}^{T^I} d^{k-1} \Bigg(& E_{\gamma^o_{k|k}}\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}^o_k\right)^2\right] \varsigma_k + \\ & \left(c + \mathcal{J}_{k,T^I}\left(a, \sigma_W^2, c, \mathcal{D}^o_{0,T^I}\right)\right)(1 - \varsigma_{k|k-1})\varsigma_{k-1} \Bigg)\end{aligned} \tag{2.50}$$

Using (2.48), we can re-write (2.50) as follows:

$$\begin{aligned}\mathcal{J}_{0,T^I}\left(a, \sigma_W^2, c, \mathcal{D}^o_{0,T^I}\right) = \sum_{k=1}^{T^I} d^{k-1} \Bigg(& E_{\gamma^o_{k|k}}\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}^o_k\right)^2\right] \varsigma_k + \\ & \left(c + \mathcal{J}^*_{k,T^I}\left(a, \sigma_W^2, c\right)\right)(1 - \varsigma_{k|k-1})\varsigma_{k-1} \Bigg)\end{aligned} \tag{2.51}$$

From inequality (2.42), which lower bounds the cost associated with any pre-processor policy, equation (2.51) and equation (2.49), we conclude that:

$$\mathcal{J}_{0,T^I}\left(a, \sigma_W^2, c, \mathcal{D}^o_{0,T^I}\right) \leq \mathcal{J}_{0,T^I}\left(a, \sigma_W^2, c, \mathcal{P}_{0,T^I}\right) \tag{2.52}$$

That we were able to construct $\mathcal{D}^o_{0,T^I}$ satisfying (2.52) for an arbitrarily chosen admissible pre-processor $\mathcal{P}_{0,T^I}$ constitutes a proof for Fact B.1.■

2.6 Simulation Example

In this section, we will show the results of simulation for Problem 2.1, if we adopt the optimal pre-processor and the optimal estimator. We consider $a = 1.5$, $\sigma_W^2 = 2$,

$c = 3$ and $T = 220$. With green line, we depict the process $\mathbf{Y}_k$, with red line we depict the process $\tilde{\mathbf{Y}}_k$, which we define to be $\tilde{\mathbf{Y}}_k = \mathbf{X}_k - \hat{\mathbf{X}}_k$, while with blue line we show the thresholds τ_k. First we notice that τ_k converges, fact that will be discussed in Chapter 3 in Problem 3.6, which is the infinite time horizon counterpart of Problem 2.1. We notice that as long as $\mathbf{Y}_k$ is within the blue lines, i.e. $|\mathbf{Y}_k| \leq \tau_k$ then the estimation error is less than the threshold and $\mathbf{Y}_k = \tilde{\mathbf{Y}}_k$. On the other hand, if $|\mathbf{Y}_k| > \tau_k$ then the estimation error is bigger than the threshold, hence the preprocessor sends the true value of the system to the estimator, which implies that $\tilde{\mathbf{Y}}_k = 0$.

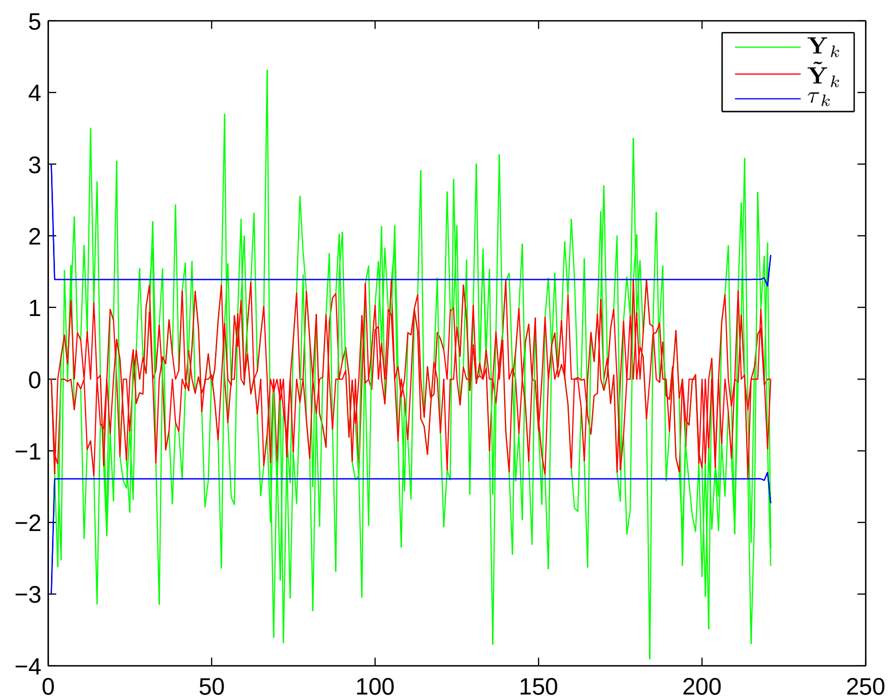

Figure 2.3: Simulation results

Chapter 3

Applications of Problem 2.1

3.1 Introduction

In this chapter, we will solve a few problems on which we will use the results and the proofs from Chapter 2. We will show how to extend the results for Problem 2.1. First, we will extend Problem 2.1 to more general costs, more general noise distributions. We will then deal with the problem where the pre-processor has noisy observations. Problem 2.1 is an estimation problem, we will show then how to solve a quadratic control problem with communication costs. We will solve then a similar problem with Problem 2.1, in which we will allow packet drop, i.e. the information sent from the pre-processor to the estimator can be lost. We will extend Problem 2.1 to its infinite horizon counterpart, where we will deal with the infinite horizon discounted cost and average cost. In the end, we will solve a tandem problem, where the information will be sent over multiple pre-processors.

In this chapter we will use similar notions for the pre-processor, estimator and the processes from Chapter 2. We will use in general the definitions from Chapter 2, but we will also give new definitions, if needed.

3.2 General Costs and General Noise Distributions

We define the state process:

Definition 3.1 *(**State Process**) Given a real constant a, consider the following first order, linear time-invariant discrete-time system driven by process noise:*

$$\mathbf{X}_0 \overset{def}{=} x_0 \tag{3.1}$$

$$\mathbf{X}_{k+1} \overset{def}{=} a\mathbf{X}_k + \mathbf{W}_k,\ k \geq 0 \tag{3.2}$$

where $\{\mathbf{W}_k\}_{k=0}^{T}$ is an independent identically distributed (i.i.d.) zero mean stochastic process with an even and quasiconcave probability density function h_W and x_0 is a real number.

We use the Definitions 2.2 and 2.3 for the pre-processor and estimator. In Definition 3.1, we relaxed the assumption from Definition 2.1 that the process noise $\{\mathbf{W}_k\}_{k=0}^{T}$ is an i.i.d. process, Gaussian, zero mean with variance σ_W^2. We consider that the process noise $\{\mathbf{W}_k\}_{k=0}^{T}$ is an i.i.d. process, zero mean with an even and quasiconcave probability density function.

Consider the set of functions $\{h_i\}_{i=1}^{T}$, $h_i : \mathbb{R} \to \mathbb{R}$, for all $i \in \{1, \ldots, T\}$, such that the functions $\{h_i\}_{i=1}^{T}$ are continuous, even and quasiconvex.

Consider the following cost:

Definition 3.2 ***(Finite time horizon cost function for general cost and general function)*** *Given a valid pre-processor $\mathcal{P}_{0,T}$ (Definition 2.2), a real constant a, a positive integer T, a positive real number d less than one, the probability density function h_W, the set of* *functions* $\{h_i\}_{i=1}^T$ *and the positive constant* c*, we define:*

$$\mathcal{J}_{0,T}\left(a, \sigma_W^2, c, \mathcal{P}_{0,T}\right) \stackrel{def}{=} \sum_{k=1}^{T} E\left[h_k\left(\mathbf{X}_k - \hat{\mathbf{X}}_k\right) + \underbrace{c\mathbf{R}_k}_{\textit{communication cost}}\right] \tag{3.3}$$

where $\mathbf{X}_k$ *is the state of the system defined in (3.1)-(3.2),* $\hat{\mathbf{X}}_k$ *is the optimal estimate specified in Definition 2.3, and* $\mathbf{R}_k$ *is the following indicator function:*

$$\mathbf{R}_k \stackrel{def}{=} \begin{cases} 0 & \textit{if } \mathbf{V}_k = \mathfrak{E} \\ 1 & \textit{otherwise} \end{cases}, \qquad k \geq 1 \tag{3.4}$$

We define the following problem:

Problem 3.1 *Let a real constant* a*, the probability density function* h_W *and the initial condition* x_0 *be given. In addition, consider that a positive real* c*, the set of functions* $\{h_i\}_{i=1}^T$ *and a positive integer* T *are given, specifying the cost as in Definition 3.2. Find:*

$$\mathcal{P}_{0,T}^* \in \textit{arg}\min_{\mathcal{P}_{0,T}} \mathcal{J}_{0,T}(a, \sigma_W^2, c, \mathcal{P}_{0,T}) \tag{3.5}$$

The following is the <u>main result of this section</u>.

Theorem 3.1 *Let the parameters specifying Problem 3.1 be given, i.e., the real constant* a*, the probability density function* h_W*, the communication cost* c*, the set of functions* $\{h_i\}_{i=1}^T$ *and the time horizon* T *are pre-selected. There exists a sequence of positive real numbers* $\tau^* = \{\tau_k^*\}_{k=1}^T$*, such that corresponding symmetric threshold policy* $\mathcal{S}_{0,T}^*$ *is an optimal solution to (3.5) and the corresponding optimal estimator* $\mathcal{E}(\mathcal{S}_{0,T}^*)$ *is* $\mathcal{Z}$*. Here* $\mathcal{S}_{0,T}^*$ *and* $\mathcal{Z}$ *follow Definitions 2.9 and 2.5 from Chapter 2, respectively.*

Proof: We note that in the proof of Theorem 2.1 we just needed that the process noise $\mathbf{W}_k$ have an even and quasiconcave probability density function and the cost need not be quadratic but an even, continuous and quasiconvex function. ■

In order to define the optimal thresholds, we need to define the value functions $\mathcal{V}_{t,T} : \mathbb{R} \to \mathbb{R}$, for $t \in \{1, T+1\}$:

$$\begin{aligned}
&\mathcal{V}_{T+1,T}\left(y_{T+1}\right) = 0, \forall y_{T+1} \in \mathbb{R} \\
&\mathcal{V}_{t,T}\left(y_t\right) = \min\left(c + E\left[\mathcal{V}_{t+1,T}\left(\mathbf{W}_t\right)\right], h_t\left(y_t\right) + E\left[\mathcal{V}_{t+1,T}\left(a y_t + \mathbf{W}_t\right)\right]\right)
\end{aligned} \tag{3.6}$$

An immediate application of Problem 3.1 is to choose the functions $\{h_i\}_{i=1}^{T}$ to be quadratic functions as follows:

$$h_i\left(x\right) = b_i x^2, \text{ for all } i \in \{1, \ldots, T\} \tag{3.7}$$

where b_i are strictly positive real numbers for all $i \in \{1, \ldots, T\}$. Hence the cost defined in equation (3.3) becomes:

$$\mathcal{J}_{0,T}\left(a, \sigma_W^2, c, \mathcal{P}_{0,T}\right) \stackrel{def}{=} \sum_{k=1}^{T} E\left[b_k\left(\mathbf{X}_k - \hat{\mathbf{X}}_k\right)^2 + c\mathbf{R}_k\right] \tag{3.8}$$

Moreover, we can select the process noise $\{\mathbf{W}_k\}_{k=0}^{T}$ to be white, zero-mean and Gaussian, but the variance of $\mathbf{W}_k$ need not be the same for all k. For this noise and the cost defined in equation (3.8) the optimal policy is a symmetric threshold policy, as stated in Theorem 3.1 with the optimal thresholds defined in equations (3.6), by adopting $h_i\left(x\right) = b_i x^2$ for all $i \in \{1, \ldots, T\}$ and by taking the apprpriate statistics for the process noise $\{\mathbf{W}_k\}_{k=0}^{T}$.

3.3 Distributed Estimation with Observation Noise

The next application of Problem 2.1 is the situation where we consider that the preprocessor has noisy observation of the state process. We will define the state process $\mathbf{X}_k$,

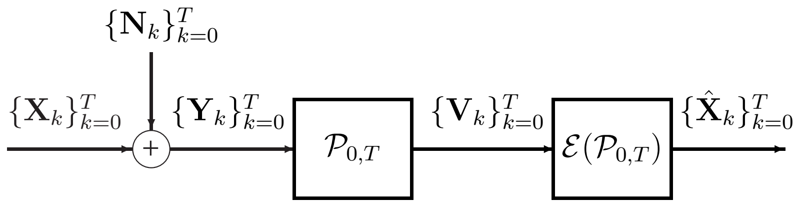

Figure 3.1: Schematic representation of the distributed estimation system considered in Problem 3.2, where we consider observation noise at the pre-processor side.

which will be the same as in Chapter 2, the observation process $\mathbf{Y}_k$ and the pre-processor. The estimator will the the same as in Chapter 2 (Definition 2.3).

Definition 3.3 ***(State Process for Estimation with Observation Noise)*** *Given a real constant a, positive real constants σ_W^2 and σ_N^2, a real number x_0, consider the first order, linear, time-invariant and discrete-time system driven by process noise:*

$$\mathbf{X}_0 \stackrel{def}{=} x_0 \tag{3.9}$$

$$\mathbf{X}_{k+1} \stackrel{def}{=} a\mathbf{X}_k + \mathbf{W}_k, \qquad k \geq 0 \tag{3.10}$$

$$\mathbf{Y}_k \stackrel{def}{=} \mathbf{X}_k + \mathbf{N}_k \tag{3.11}$$

where the process noise $\{\mathbf{W}_k\}_{k=0}^{\infty}$ is an i.i.d., Gaussian and zero mean stochastic process with variance σ_W^2 and the observation noise $\{\mathbf{N}_k\}_{k=0}^{\infty}$ is an i.i.d., Gaussian and zero mean stochastic process with variance σ_N^2.

The filtration generated by $\left\{\{\mathbf{X}_t\}_{t=0}^k\right\}$ is denoted as:

$$\mathcal{X}_k \stackrel{def}{=} \sigma\left(\mathbf{X}_t; 0 \leq t \leq k\right) \tag{3.12}$$

where $\sigma\left(\mathbf{X}_t; 0 \leq t \leq k\right)$ is the smallest sigma algebra generated by the random variables $\left\{\{\mathbf{X}_t\}_{t=0}^k\right\}$, for all integers $k \in \{0, \ldots, T\}$.

The filtration generated by $\left\{\{\mathbf{Y}_t\}_{t=0}^{k}\right\}$ is denoted as:

$$\mathcal{Y}_k \stackrel{def}{=} \sigma\left(\mathbf{Y}_t; 0 \leq t \leq k\right) \tag{3.13}$$

where $\sigma\left(\mathbf{Y}_t; 0 \leq t \leq k\right)$ is the smallest sigma algebra generated by the random variables $\left\{\{\mathbf{Y}_t\}_{t=0}^{k}\right\}$, for all integers $k \in \{0, \ldots, T\}$.

Definition 3.4 ***(Pre-processor and remote link process)*** *Consider an erasure symbol denoted as $\mathfrak{E}$ and a causal map $\mathcal{P}_{0,T} : (x_0, \ldots, x_k) \mapsto v_k$, defined for $k \in \{0, \ldots, T\}$ and $v_k \in \mathbb{R} \cup \{\mathfrak{E}\}$. Hence, at each time instant k, $\mathcal{P}_{0,T}$ outputs a real number or the erasure symbol, based on past observations of the observation process.$\mathcal{P}_{0,T}$ generates a stochastic process $\{\mathbf{V}_k\}_{k=0}^{T}$ via the application of the operator $\mathcal{P}_{0,T}$ to the process $\{\mathbf{X}_k\}_{k=0}^{T}$ (See Figure 3.1). The map $\mathcal{P}_{0,T}$ is a valid pre-processor if the following two conditions hold: (1) The pre-processor transmits the initial state x_0 at time zero, i.e., $v_0 = x_0$ (Put in other words the estimator knows x_0). (2) The pre-processor is measurable in the sense that the process $\{\mathbf{V}_k\}_{k=0}^{T}$ is adapted to $\mathcal{Y}_k$.*

The filtration generated by $\{\mathbf{V}_k\}_{k=0}^{T}$ is denoted as $\{\mathcal{B}_k\}_{k=0}^{T}$ and it is obtained as:

$$\mathcal{B}_k \stackrel{def}{=} \sigma\left(\mathbf{V}_t; 0 \leq t \leq k\right) \tag{3.14}$$

where $\sigma\left(\mathbf{V}_t; 0 \leq t \leq k\right)$ is the smallest sigma algebra generated by $\{\mathbf{V}_t, 0 \leq t \leq k\}$, for all non-negative integers k.

We define the cost just like in Definition 2.4, which repeat here for clarity purposes.

Definition 3.5 ***(Finite time horizon cost function with observation noise)*** *Given a valid pre-processor $\mathcal{P}_{0,T}$ (Definition 3.4), a real constant a, a positive integer T, a positive real* *number d less than one and positive real constants σ_W^2, σ_V^2 and c, we define:*

$$\mathcal{J}_{0,T}\left(a, \sigma_W^2, \sigma_N^2, c, \mathcal{P}_{0,T}\right) \stackrel{def}{=} \sum_{k=1}^{T} E\left[\left(\mathbf{X}_k - \hat{\mathbf{X}}_k\right)^2 + c\mathbf{R}_k\right] \tag{3.15}$$

where $\mathbf{X}_k$ is the state of the system defined in (3.9)-(3.10), $\hat{\mathbf{X}}_k$ is the optimal estimate specified in Definition 2.3 (Chapter 2), and $\mathbf{R}_k$ is the following indicator function:

$$\mathbf{R}_k \stackrel{def}{=} \begin{cases} 0 & \textit{if } \mathbf{V}_k = \mathfrak{E} \\ 1 & \textit{otherwise} \end{cases}, \qquad k \geq 1 \tag{3.16}$$

We will state now the main problem of this section and then we will give the optimal solution.

Problem 3.2 *Let a real constant a, the variance of the process noise σ_W^2, the variance of the observation noise σ_N^2 and the initial condition x_0 be given. In addition, consider that a positive real c, a positive real number d less then one and and a positive integer T are given, specifying the cost as in (3.15). Find:*

$$\mathcal{P}^* \in \textit{arg} \min_{\mathcal{P}} \mathcal{J}(a, \sigma_W^2, \sigma_N^2, c, \mathcal{P}) \tag{3.17}$$

We define the optimal cost for the infinite horizon cost:

$$\mathcal{J}^*\left(a, \sigma_W^2, \sigma_N^2, c\right) \stackrel{def}{=} \inf_{\mathcal{P}} \mathcal{J}(a, \sigma_W^2, \sigma_N^2, c, \mathcal{P})$$

We state now the theorem, which solves Problem 3.2:

Theorem 3.2 *Let the parameters specifying Problem 3.2 be given, i.e., the variance of the process noise σ_W^2, the variance of the observation noise σ_N^2, the system's dynamic* *constant a, the communication cost c, the discount factor d and the time horizon T are pre-selected. There exists a sequence of positive real numbers $\tau^* = \{\tau_k^*\}_{k=1}^T$, such that the corresponding symmetric threshold policy $\mathcal{S}_{0,T}^*$ is an optimal solution to (3.17) and the corresponding optimal estimator $\mathcal{E}(\mathcal{S}_{0,T}^*)$ is $\mathcal{Z}$. Here $\mathcal{S}_{0,T}^*$ and $\mathcal{Z}$ follow Definitions 2.9 and 2.5, respectively.*

Proof: In order to prove Theorem 3.2, we notice that the pre-processor $\mathcal{P}$ can compute the state estimate $\tilde{\mathbf{X}}_k$ as a function of the observations $\{\mathbf{Y}_j\}_{j=1}^k$. Due to the linearity of the process and of the observation, $\tilde{\mathbf{X}}_k$ is given by the usual Kalman filter. We notice that $\hat{\mathbf{X}}_k$ computed at the estimator side are functions of $\{\mathbf{V}_j\}_{j=1}^k$. The variables $\mathbf{V}_k$ are functions of the observation noise $\{\mathbf{Y}_j\}_{j=1}^k$. It follows that $\hat{\mathbf{X}}_k$ are functions of the observation noise $\{\mathbf{Y}_j\}_{j=1}^k$. We can re-write the cost 3.15 as follows:

$$\begin{aligned}
\mathcal{J}_{0,T}\left(a, \sigma_W^2, \sigma_N^2, c, \mathcal{P}_{0,T}\right) &= \sum_{k=1}^T d^{k-1} E\left[\left(\mathbf{X}_k - \hat{\mathbf{X}}_k\right)^2 + c\mathbf{R}_k\right] \\
&= \sum_{k=1}^T d^{k-1} E\left[\left(\mathbf{X}_k - \tilde{\mathbf{X}}_k + \tilde{\mathbf{X}}_k - \hat{\mathbf{X}}_k\right)^2 + c\mathbf{R}_k\right] \\
&= \sum_{k=1}^T d^{k-1} E\left[\left(\mathbf{X}_k - \tilde{\mathbf{X}}_k\right)^2 + c\mathbf{R}_k\right] + d^{k-1} E\left[\left(\tilde{\mathbf{X}}_k - \hat{\mathbf{X}}_k\right)^2\right] \\
&+ 2d^{k-1} E\left[\left(\tilde{\mathbf{X}}_k - \hat{\mathbf{X}}_k\right)\left(\mathbf{X}_k - \tilde{\mathbf{X}}_k\right)\right] \\
&= \sum_{k=1}^T d^{k-1} E\left[\left(\tilde{\mathbf{X}}_k - \hat{\mathbf{X}}_k\right)^2 + c\mathbf{R}_k\right] + d^{k-1} E\left[\left(\mathbf{X}_k - \tilde{\mathbf{X}}_k\right)^2\right]
\end{aligned}$$

The cross term $E\left[\left(\tilde{\mathbf{X}}_k - \hat{\mathbf{X}}_k\right)\left(\mathbf{X}_k - \tilde{\mathbf{X}}_k\right)\right]$ disappears due to the orthogonality principle. The term $E\left[\left(\mathbf{X}_k - \tilde{\mathbf{X}}_k\right)^2\right]$ cannot be affected in any way, hence we need to optimize the cost:

$$\sum_{k=1}^T d^{k-1} E\left[\left(\tilde{\mathbf{X}}_k - \hat{\mathbf{X}}_k\right)^2 + c\mathbf{R}_k\right]$$

From the standard Kalman filtering, the process $\tilde{\mathbf{X}}_k$ follows the dynamics:

$$\tilde{\mathbf{X}}_{k+1} = a\tilde{\mathbf{X}}_k + \tilde{\mathbf{W}}_k$$

where $\tilde{\mathbf{W}}_k$ is the innovation process from the standard Kalman filtering. We note the that innovation process is independent, zero-mean and Gaussian, but it is not i.i.d.. Combining this with the results from Problem 3.1, the result in Theorem 3.2 follows. Note that the factor d^{k-1} can be replaced by any strictly positive real number. ■

3.4 Control Problem with Communication Costs

Before we start actually to present the control problem, we will solve a simple estimation problem. We consider a process and a cost similar to the ones in Problem 2.1 from Chapter 2.

Definition 3.6 ***(State Process)*** *Given a real constant a, and a positive real constant σ_W^2, consider the following first order, linear time-invariant discrete-time system driven by process noise:*

$$\mathbf{X}_0 \stackrel{def}{=} x_0 \tag{3.18}$$

$$\mathbf{X}_{k+1} \stackrel{def}{=} a\mathbf{X}_k + \mathbf{U}_k + \mathbf{W}_k,\ k \geq 0 \tag{3.19}$$

where $\{\mathbf{W}_k\}_{k=0}^T$ is an independent identically distributed (i.i.d.) Gaussian zero mean stochastic process with variance σ_W^2 and x_0 is a real number. The filtration generated by $\{\mathbf{X}_k\}_{k=0}^T$ is denoted as:

$$\mathcal{X}_k \stackrel{def}{=} \sigma\left(\mathbf{X}_t; 0 \leq t \leq k\right) \tag{3.20}$$

where $\sigma(\mathbf{X}_t; 0 \leq t \leq k)$ is the smallest sigma algebra generated by $\{\mathbf{X}_t, 0 \leq t \leq k\}$, for all integers k. For now, we just say that the random variables $\mathbf{U}_k$ are measurable with respect to $\mathcal{X}_k$.

We note that since the random variables $\mathbf{U}_k$ are measurable with respect to $\mathcal{X}_k$, the sigma algebras $\{\mathcal{X}_k\}_{k=0}^{\infty}$ are well defined. We will give a precise definition of the process $\{\mathbf{U}_k\}_{k=0}^{\infty}$ later. We define the pre-processor and the estimator like in the Definitions 2.2 and 2.3 from Chapter 2.

Let $\mathcal{H} : (v_0, \ldots, v_k) \mapsto u_k$ where $u_k \in \mathbb{R}$ and $v_j \in \mathbb{R} \cup \{\mathfrak{E}\}$. Hence $\mathcal{H}$ is a deterministic map which takes the output of the pre-processor (what is received by the estimator) and maps it into a real number.

We define the process $\{\mathbf{U}_k\}_{k=0}^{\infty}$ to be the process generated by the map $\mathcal{H}$ applied to the process $\{\mathbf{V}_k\}_{k=0}^{\infty}$. We notice that this is consistent with the Definition 3.6. Moreover, we note that the process $\{\mathbf{U}_k\}_{k=0}^{\infty}$ is known both at the estimator and the pre-processor side.

Definition 3.7 ***(Finite time horizon cost function)*** *Given a valid pre-processor $\mathcal{P}_{0,T}$ (Definition 2.2), a real constant a, the mapping $\mathcal{H}$, a positive integer T, a positive real number d less than one and positive real constants σ_W^2 and c, we define:*

$$\mathcal{J}_{0,T}\left(a, \sigma_W^2, c, \mathcal{P}_{0,T}\right) \stackrel{def}{=} \sum_{k=1}^{T} d^{k-1} E\left[\left(\mathbf{X}_k - \hat{\mathbf{X}}_k\right)^2 + c\mathbf{R}_k\right] \tag{3.21}$$

where $\mathbf{X}_k$ is the state of the system defined in (3.18)-(3.19), $\hat{\mathbf{X}}_k$ is the optimal estimate

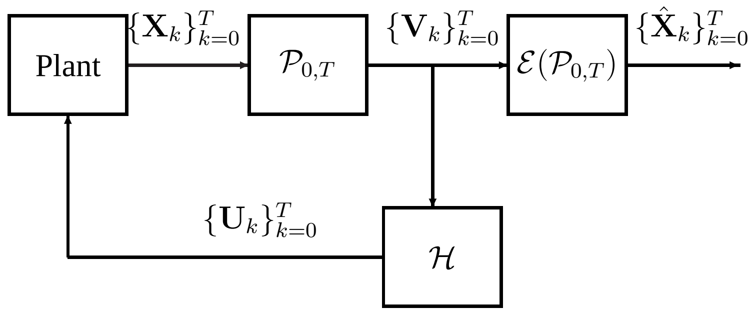

Figure 3.2: Schematic representation of the distributed estimation system considered in Problem 3.3, where the estimator can influence the process $\mathbf{X}_k$.

specified in Definition 2.3, and $\mathbf{R}_k$ is the following indicator function:

$$\mathbf{R}_k \stackrel{def}{=} \begin{cases} 0 & \textit{if } \mathbf{V}_k = \mathfrak{E} \\ 1 & \textit{otherwise} \end{cases}, \qquad k \geq 1 \tag{3.22}$$

The following is the problem addressed in this section.

Problem 3.3 *Let a real constant a, the variance of the process noise σ_W^2, the mapping $\mathcal{H}$ and the initial condition x_0 be given. In addition, consider that a positive real c, a positive real number d less then one and a positive integer T are given, specifying the cost as in Definition 3.7. We want to find an optimal solution $\mathcal{P}_{0,T}^*$ to the following optimization problem:*

$$\mathcal{P}_{0,T}^* \in \textit{arg} \min_{\mathcal{P}_{0,T}} \mathcal{J}_{0,T}(a, \sigma_W^2, c, \mathcal{P}_{0,T}) \tag{3.23}$$

We notice that this problem has just a slight modification in comparison to Problem 2.1.

Similar to Definition 2.5, we define the following estimator:

Definition 3.8 ***(Kalman-like estimator)*** *Given the process defined in (3.18)-(3.19) and a pre-processor $\mathcal{P}_{0,T}$, define the map $\mathcal{Z} : (v_0, \ldots, v_k) \mapsto z_k$, for k in the set $\{0, \ldots, T\}$, where z_k is computed as follows:*

$$z_0 \stackrel{def}{=} x_0 \tag{3.24}$$

$$z_k \stackrel{def}{=} \begin{cases} az_{k-1} + u_{k-1} & \textit{if } v_k = \mathfrak{E} \\ & \\ v_k & \textit{otherwise} \end{cases}, \textit{ with } k \geq 1 \tag{3.25}$$

where $u_k = \mathcal{H}(v_0, \ldots, v_k)$.

We define a process similar to the process $\{\mathbf{Y}_k\}_{k=0}^{\infty}$ as follows:

Definition 3.9 *We define the following process:*

$$\mathbf{Y}_k \stackrel{def}{=} \mathbf{X}_k - (a\mathbf{Z}_{k-1} + \mathbf{U}_{k-1}) \tag{3.26}$$

Using Definitions 3.6 and 3.8, we find that $\{\mathbf{Y}_k\}_{k=0}^{T}$ can be rewritten as:

$$\mathbf{Y}_0 = 0 \tag{3.27}$$

$$\mathbf{Y}_{k+1} = \begin{cases} a\mathbf{Y}_k + \mathbf{W}_k & \text{if } \mathbf{R}_k = 0 \\ & \\ \mathbf{W}_k & \text{if } \mathbf{R}_k = 1 \end{cases} \tag{3.28}$$

We notice that the equations (3.28) is exactly as the equation (2.14).

We can state now the optimal solution of Problem 3.3, which will be just a corollary of Theorem 2.1.

Corollary 3.1 *Let the variance of the process noise σ_W^2, the mapping $\mathcal{H}$, the system's dynamic constant a, the communication cost c, the discount factor d and the time horizon*

T be given. There exists a sequence of positive real numbers $\tau^* = \{\tau_k^*\}_{k=1}^T$, such that the corresponding symmetric threshold policy $\mathcal{S}_{0,T}^*$ is an optimal solution to (3.23) and the corresponding optimal estimator $\mathcal{E}(\mathcal{S}_{0,T}^*)$ is $\mathcal{Z}$. Here $\mathcal{S}_{0,T}^*$ and $\mathcal{Z}$ follow Definitions 2.9 and 3.8, respectively.

Proof: It follows from Remark 2.3, which states that initial condition x_0 does not influence the total cost since it can be subtracted at the estimator side. The same arguments hold for the process $\{\mathbf{U}_k\}_{k=0}^\infty$, since it is known both at the estimator and the pre-processor. ■

We will proceed now to define a quadratic control problem with communication costs. We keep the Definition 3.6 for the state process and the definition for the pre-processor to be Definition 2.2. We need to define a controller, which will generate the process $\{\mathbf{U}_k\}_{k=0}^T$.

Definition 3.10 ***(Controller and the Control Process)*** *Given a pre-processor $\mathcal{P}_{0,T}$, consider the mapping $\mathcal{C}_{0,T} : \left(\{\mathbf{v}_t\}_{t=0}^k\right) \mapsto u_k$, which we call controller. The controller generates the stochastic process $\{\mathbf{U}_k\}_{k=0}^T$ via the operator $\mathcal{C}_{0,T}$ applied to the process $\{\mathbf{V}_k\}_{k=0}^T$. Hence, the process $\{\mathbf{U}_k\}_{k=0}^T$ is adapted to the filtration $\{\mathcal{X}_k\}_{k=0}^T$ and it represents the output of the controller.*

Remark 3.1 *Just like in Remark 2.2, the pre-processor has all the information which the controller has. Hence, the pre-processor $\mathcal{P}_{0,T}$ can construct the control $\mathbf{U}_k$ by reproducing the control algorithm executed at the controller.*

We will define the performance criterion and the main problem from this section.

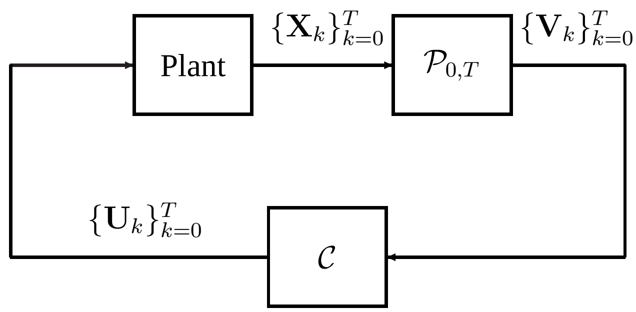

Figure 3.3: Schematic representation of the distributed estimation system considered in Problem 3.4, where we have a quadratic control problem with communication costs.

Definition 3.11 ***(Finite time horizon control cost)*** *Given a positive integer T, a measurable pre-processor $\mathcal{P}_{0,T}$ (Definition 2.2), a controller $\mathcal{C}_{0,T}$ (Definition 3.10), a real constant a and the positive real constant σ_W^2 and c, we define:*

$$\mathcal{J}_T\left(a, \sigma_W^2, c, \mathcal{P}_{0,T}, \mathcal{C}_{0,T}\right) \stackrel{def}{=} \sum_{k=0}^{T} E\left[\mathbf{X}_{k+1}^2 + \mathbf{U}_k{}^2 + c\mathbf{R}_k\right] \tag{3.29}$$

where $\mathbf{X}_k$ indicates the state of the process from Definition 3.6, $\mathbf{U}_k$ denotes the input provided by the controller $\mathcal{C}_{0,T}$, E indicates expectation and $\mathbf{R}_k$ is defined as follows:

$$\mathbf{R}_k \stackrel{def}{=} \begin{cases} 1, \mathbf{V} \neq \mathfrak{E}, & k \geq 0 \\ 0, \mathbf{V}_k = \mathfrak{E}, & k \geq 0 \end{cases} \tag{3.30}$$

Problem 3.4 *Let a real constant a, the variance of the process noise σ_W^2 and the initial condition x_0 be given. In addition, consider that a positive real c is given. We want to find an optimal solution $\left(\mathcal{P}_{0,T}{}^*, \mathcal{C}_{0,T}^*\right)$ for the following optimization problem:*

$$\left(\mathcal{P}_{0,T}^*, \mathcal{C}_{0,T}^*\right) \in \textit{arg} \min_{\left(\mathcal{P}_{0,T}, \mathcal{C}_{0,T}\right)} \mathcal{J}_T\left(a, \sigma_W^2, c, \mathcal{P}_{0,T}, \mathcal{C}_{0,T}\right) \tag{3.31}$$

Here the pre-processor $\mathcal{P}_{0,T}$ and the estimator $\mathcal{C}_{0,T}$ must be optimized jointly so as to minimize the cost function.

Before, we state the main result of this section, we need to define a certain controller.

Definition 3.12 *(Scalar Discrete Riccati Equations) Given a real constant a, define the sequence of real numbers:*

$$\begin{aligned} p_{T+1} &\stackrel{def}{=} 1 \\ p_t &\stackrel{def}{=} 1 + a^2 p_{t+1} - \frac{a^2 p_{t+1}^2}{1 + p_{t+1}}, t \in \{0, \ldots, T\} \end{aligned} \tag{3.32}$$

We notice that Definition 3.12 is the scalar version of the Riccati equation.

Definition 3.13 *Given a real constant a, define the map $\mathcal{C}_{0,T}^{(a)}$, as follows:*

$$\mathcal{C}_{0,T}^{(a)}(v_0, \ldots, v_k) \stackrel{def}{=} u_k, \quad \{v_t\}_{t=0}^{k} \in (\mathbb{R} \cup \{\mathfrak{E}\}) \quad k \geq 0 \tag{3.33}$$

and u_k is constructed using a supporting variable z_k which will have the role of state estimate. We define $u_{-1} \stackrel{def}{=} 0$ and $z_{-1} \stackrel{def}{=} 0$, then z_k and u_k are defined as follows

$$u_k \stackrel{def}{=} -\frac{a p_k}{1 + p_k} z_k, \ k \in \{0, \ldots, T\} \tag{3.34}$$

where z_k follows the dynamics from Definition 3.8 and $\{p_t\}_{t=0}^{T+1}$ was defined in equation (3.32).

We notice that in Definition 3.13 together with Definition 3.8, u_k is a function of z_k and z_k is a function of v_k, u_{k-1} and z_{k-1}, recursively it follows that both z_k and u_k are functions of the values $\{v_t\}_{t=0}^{k}$, hence u_k is well defined.

We are ready now to state the main result from this section.

Theorem 3.3 *Let the parameters specifying Problem 3.4 be given, i.e., the variance of the process noise σ_W^2, the system's dynamic constant a, the communication cost c, and* *the time horiazon T are pre-selected. There exist a sequence of positive real numbers $\tau = \{\tau_k\}_{k=0}^{T}$, such that the associated $S_{0,T}^{*}$ and $\mathcal{C}_{0,T}^{(a)}$ are an optimal solution to (3.31).*

Proof: We will make some manipulation of the cost in equation (3.29).

$$
\begin{aligned}
\mathcal{J}_T\left(a, \sigma_W^2, c, \mathcal{P}_{0,T}, \mathcal{C}_{0,T}\right) &= \sum_{k=1}^{T} E\left[\mathbf{X}_{k+1}^2 + \mathbf{U}_k^2 + c\mathbf{R}_k\right] \\
&= \sum_{k=0}^{T} E\left[\mathbf{X}_{k+1}^2 + \mathbf{U}_k^2 + p_k\mathbf{X}_k^2 - p_k\mathbf{X}_k^2 + c\mathbf{R}_k\right] \\
&= p_k\mathbf{X}_0^2 + \sum_{k=0}^{T} E\left[p_{k+1}\mathbf{X}_{k+1}^2 + \mathbf{U}_k^2 + \mathbf{X}_k^2 - p_k\mathbf{X}_k^2 + c\mathbf{R}_k\right] \\
&= p_k\mathbf{X}_0^2 + \sum_{k=0}^{T} E\left[p_{k+1}a^2\mathbf{X}_k^2 + p_{k+1}\mathbf{U}_k^2 + \mathbf{U}_k^2 + \mathbf{X}_k^2 - p_k\mathbf{X}_k^2 + c\mathbf{R}_k\right] \\
&+ \sum_{k=0}^{T} E\left[2p_{k+1}a\mathbf{X}_k\mathbf{U}_k + 2p_{k+1}a\mathbf{X}_k\mathbf{W}_k + 2p_{k+1}\mathbf{U}_k\mathbf{W}_k + p_{k+1}\mathbf{W}_k^2\right] \\
&= \sum_{k=0}^{T} E\left[\frac{p_{k+1}^2a^2}{1+p_{k+1}}\mathbf{X}_k^2 + \left(p_{k+1}+1\right)\mathbf{U}_k^2 + 2p_{k+1}a\mathbf{X}_k\mathbf{U}_k + c\mathbf{R}_k\right] \\
&+ p_k\mathbf{X}_0^2 + \sum_{k=0}^{T} p_{k+1}\sigma_W^2 \\
&= \sum_{k=0}^{T} E\left[\frac{p_{k+1}^2a^2}{1+p_{k+1}}\hat{\mathbf{X}}_k^2 + \left(p_{k+1}+1\right)\mathbf{U}_k^2 + 2p_{k+1}a\hat{\mathbf{X}}_k\mathbf{U}_k\right] \\
&+ \sum_{k=0}^{T} E\left[\frac{p_{k+1}^2a^2}{1+p_{k+1}}\left(\mathbf{X}_k - \hat{\mathbf{X}}_k\right)^2 + c\mathbf{R}_k\right] \\
&+ \sum_{k=0}^{T} E\left[2\frac{p_{k+1}^2a^2}{1+p_{k+1}}\left(\mathbf{X}_k^2 - \hat{\mathbf{X}}_k^2\right)\mathbf{X}_k^2 + 2p_{k+1}a\left(\mathbf{X}_k - \hat{\mathbf{X}}_k\right)\mathbf{U}_k\right] \\
&+ p_k\mathbf{X}_0^2 + \sum_{k=0}^{T} p_{k+1}\sigma_W^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{p_{k+1}+1}\sum_{k=0}^{T} E\left[\left(p_{k+1}a\hat{\mathbf{X}}_k + (p_{k+1}+1)\,\mathbf{U}_k\right)^2\right] \\
&+ \sum_{k=0}^{T} E\left[\frac{p_{k+1}^2 a^2}{1+p_{k+1}}\left(\mathbf{X}_k - \hat{\mathbf{X}}_k\right)^2 + c\mathbf{R}_k\right] \\
&+ p_k \mathbf{X}_0^2 + \sum_{k=0}^{T} p_{k+1}\sigma_W^2
\end{aligned}
$$

In the equations above, the last two terms are constant, the first term (which can be only positive) can be made equal to zero by selecting $\mathbf{U}_k = -\frac{p_{k+1}a}{p_{k+1}+1}\hat{\mathbf{X}}_k$, while the second term can be minimized according to Corollary 3.1 and the result of Theorem 3.3 follows.■

3.5 Distributed Estimation with Communication Costs and Packet Drops

Just like in Problem 2.1, we address the design of an optimal state estimation system featuring two blocks; a pre-processor $\mathcal{P}_{0,T}$ and a remote estimator $\mathcal{E}$. The pre-processor has causal access to the state of a first order, linear and time-invariant system driven by Gaussian zero mean, white process noise and, at each time instant, it outputs either an erasure symbol or a real number into a communication channel. The communication channel acts as an erasure link and with some probability it can drop the packets received from the pre-processor. The estimator has access to the output of the channel and its output is denoted as the state estimate. When the pre-processor transmits a real number, the channel can drop this real number and the estimator will receive an erasure symbol. If the channel sends to the estimator the erasure symbol instead of a real number, it will give an acknowledgement to the pre-processor that it dropped the real number. Whenever the pre-processor will transmit the erasure symbol, the estimator will receive this erasure symbol. The estimator cannot make the difference between an erasure symbol received from the pre-processor or because the channel lost he packet.

We define first the channel process.

Definition 3.14 *Let p be a positive real number less then one. We define the channel process to be $\{\mathbf{C}_k\}_{k=0}^{\infty}$, as follows:*

- $\mathbf{C}_0 = 1$*;*
- $\{\mathbf{C}_k\}_{k=1}^{\infty}$ *is a Bernoulli process with parameter p (i.e. $P(C_k = 1) = p$, for all integeres k bigger or equal to one.)*

We will define for clarity purposes the process, which will have a definition like in Chapter 2.

Definition 3.15 ***(State Process for Estimation)*** *Given a real constant a, a positive real constant σ_W^2, a real number x_0, consider the following first order, linear, time-invariant and discrete-time system driven by process noise:*

$$\mathbf{X}_0 \stackrel{def}{=} x_0 \tag{3.35}$$

$$\mathbf{X}_{k+1} \stackrel{def}{=} a\mathbf{X}_k + \mathbf{W}_k, \qquad k \geq 0 \tag{3.36}$$

where the process noise $\{\mathbf{W}_k\}_{k=0}^{\infty}$ is an i.i.d., Gaussian and zero mean stochastic process with variance σ_W^2.

The filtration generated by $\left\{\{\mathbf{X}_t\}_{t=0}^{k}, \{\mathbf{C}_t\}_{t=0}^{k}\right\}$ is denoted as:

$$\mathcal{X}_k^{\mathcal{E}} \stackrel{def}{=} \sigma\left(\mathbf{X}_t; 0 \leq t \leq k; \mathbf{C}_t; 0 \leq t \leq k;\right) \tag{3.37}$$

where $\sigma\left(\mathbf{X}_t; 0 \le t \le k; \mathbf{C}_t; 0 \le t \le k;\right)$ *is the smallest sigma algebra generated by the random variables* $\left\{\{\mathbf{X}_t\}_{t=0}^{k}, \{\mathbf{C}_t\}_{t=0}^{k}\right\}$*, for all integers* $k \in \{0, \dots, T\}$*.*

The filtration generated by $\left\{\{\mathbf{X}_t\}_{t=0}^{k}, \{\mathbf{C}_t\}_{t=0}^{k-1}\right\}$ *is denoted by* $\mathcal{X}_k^{\mathcal{P}}$*:*

$$\mathcal{X}_k^{\mathcal{P}} \stackrel{def}{=} \sigma\left(\mathbf{X}_t; 0 \le t \le k; \mathbf{C}_t; 0 \le t \le k-1;\right), \; k \ge 0 \tag{3.38}$$

where $\sigma\left(\mathbf{X}_t; 0 \le t \le k; \mathbf{C}_t; 0 \le t \le k;\right)$ *is the smallest sigma algebra generated by random variables* $\left\{\{\mathbf{X}_t\}_{t=0}^{k}, \{\mathbf{C}_t\}_{t=0}^{k-1}\right\}$*, for all integers* $k \in \{0, \dots, T\}$*.*

Definition 3.16 ***(Estimation Pre-processor)*** *Consider an erasure symbol denoted as* $\mathfrak{E}$ *and a causal map* $\mathcal{P}_{0,T} : (x_0, \dots, x_k, c_0, \dots, c_{k-1}) \mapsto \hat{v}_k$*, defined for* $k \in \{1, \dots, T\}$*,* $x_k \in \mathbb{R}$*,* $c_k \in \{0, 1\}$ *and* $\hat{v}_k \in \mathbb{R} \cup \{\mathfrak{E}\}$*. Hence, at each time instant* k*,* $\mathcal{P}_{0,T}$ *outputs a real number or an erasure symbol, based on past observations of the process* $\{\mathbf{X}_j\}_{j=0}^{k}$ *and and* $\{\mathbf{C}_j\}_{j=0}^{k-1}$*.* $\mathcal{P}_{0,T}$ *generates a stochastic process* $\{\hat{\mathbf{V}}_k\}_{k=0}^{T}$ *via the application of the operator* $\mathcal{P}_{0,T}$ *to the processes* $\{\mathbf{X}_k\}_{k=0}^{T}$ *and* $\{\mathbf{C}_j\}_{j=0}^{T-1}$ *and we note that the random variable* $\hat{\mathbf{V}}_k$ *is measurable with respect to the* σ*-algebra* $\mathcal{X}_k^{\mathcal{P}}$*. The pre-processor* $\mathcal{P}_{0,T}$ *is valid if at time zero,* $\hat{v}_0 = x_0$

Definition 3.17 ***(Remote link process)*** *The remote link process is denoted as* $\{\mathbf{V}_k\}_{k=0}^{T}$ *and it takes values in* $\mathbb{R} \bigcup \{\mathfrak{E}\}$*, where* $\mathfrak{E}$ *signifies erasure (See Definition 3.16). Given a real constant* a*, the positive real constants* σ_W^2*,* x_0 *and a pre-processor* $\mathcal{P}_{0,T}$*, the process* $\{\mathbf{V}_k\}_{k=0}^{T}$ *is adapted to* $\{\mathcal{X}_k^{\mathcal{E}}\}_{k=0}^{T}$ *and it represents the input received by the estimator* $\mathcal{E}$ *via the following relationship:*

$$\mathbf{V}_k \stackrel{def}{=} \begin{cases} \hat{\mathbf{V}}_k, \; \mathbf{C}_k = 1, \quad k \ge 0 \\ \mathfrak{E}, \; \mathbf{C}_k = 0, \quad k \ge 0 \end{cases} \tag{3.39}$$

Hence, at each time instant k, the pre-processor outputs a real number or an erasure symbol, based on the past observations of the state process and the channel process. If the channel process $\mathbf{C}_k = 1$ at the current time time k then the output of the preprocessor will be received by the estimator, otherwise the estimator will receive an erasure symbol. We notice that the estimator receives the initial condition $\mathbf{X}_0$ since we set $\mathbf{C}_0 = 1$. The filtration generated by $\{\mathbf{V}_k\}_{k=0}^{T}$ is denoted as $\{\mathcal{V}_k\}_{k=0}^{T}$ and it is obtained as:

$$\mathcal{V}_k \stackrel{def}{=} \sigma\left(\mathbf{V}_t; 0 \leq t \leq k\right) \tag{3.40}$$

where $\sigma\left(\mathbf{V}_t; 0 \leq t \leq k\right)$ is the smallest sigma algebra generated by $\{\mathbf{V}_t, 0 \leq t \leq k\}$, for all integers $k \in \{0, \ldots, T\}$.

Definition 3.18 ***(Optimal estimate and optimal estimator)*** *Given a pre-processor $\mathcal{P}_{0,T}$, we consider optimal estimators in the expected squared sense whose optimal estimate at time k is denoted as $\hat{\mathbf{X}}_k$ and is expressed as follows:*

$$\hat{\mathbf{X}}_k \stackrel{def}{=} \begin{cases} E\left[\mathbf{X}_k | \{\mathbf{V}_t\}_{t=0}^{k}\right] & \text{if } k \geq 1 \\ x_0 & \text{if } k = 0 \end{cases} \tag{3.41}$$

where $E\left[\mathbf{X}_k | \{\mathbf{V}_t\}_{t=0}^{k}\right]$ represents the expectation of the state $\mathbf{X}_k$ conditioned on current and past information received by the estimator $\{\mathbf{V}_t\}_{t=0}^{k}$. We use $\mathcal{E}(\mathcal{P}_{0,T})$ to denote the ***optimal estimator*** *for the given pre-processor policy $\mathcal{P}_{0,T}$*

Note: The estimator given in Definition 3.18 is the same with the one given in Definition 2.3.

Definition 3.19 ***(Finite time horizon cost function)*** *Given a positive integer T, a measurable pre-processor $\mathcal{P}_{0,T}$ (Definition 3.16), a real constant a, a positive real constant p*

less than or equal to one and positive real constants σ_W^2 and c, we define:

$$\mathcal{J}_{0,T}\left(a, \sigma_W^2, p, c, \mathcal{P}_{0,T}\right) \stackrel{def}{=} \sum_{k=1}^{T} E\left[\left(\mathbf{X}_k - \hat{\mathbf{X}}_k\right)^2 + c\mathbf{R}_k\right] \tag{3.42}$$

where $\mathbf{X}_k$ is the state of the system defined in (3.35)-(3.36), $\hat{\mathbf{X}}_k$ is the optimal estimate specified in Definition 2.3, and $\mathbf{R}_k$ is the communication cost defined in (3.43).

$$\mathbf{R}_k \stackrel{def}{=} \begin{cases} 0 & \textit{if}\, \hat{\mathbf{V}}_k = \mathfrak{E} \\ \\ 1 & \textit{if}\, \hat{\mathbf{V}}_k \neq \mathfrak{E} \end{cases}, \qquad k \geq 1 \tag{3.43}$$

We define the process $\{\mathbf{L}_k\}_{k=0}^{T}$ as follows:

$$\mathbf{L}_k = \mathbf{R}_k \mathbf{C}_k \tag{3.44}$$

It follows that $\mathbf{L}_k = 0$ either if the pre-processor sends the erasure symbol, or if the channel drops the packet and $\mathbf{L}_k = 1$ if the pre-processor sends a real number and the channel does not drop the packet. Hence the process $\mathbf{L}_k$ is zero if the estimator receives an erasure symbol and is equal to one if the estimator receives a real number. Since $\mathbf{C}_0 = 1$ and $\hat{\mathbf{V}}_0 = \mathbf{X}_0$, it follows that $\mathbf{L}_0 = 1$, hence the estimator knows the initial condition of the system described in (3.35)-(3.36)

Remark 3.2 ***(Cost does not depend on $\mathbf{X}_0$).*** *Just like in the Remark 2.3, notice that because the plant (3.35)-(3.36) is linear, the fact that $\hat{x}_0 = x_0$ holds (see Definition 3.18) implies that the homogenous part of the state can be reproduced at the estimator. Hence, the optimal estimator will include such an homogeneous term, thus subtracting it out from the estimation error $\mathbf{X}_k - \hat{\mathbf{X}}_k$, for $k \geq 0$. This also implies that the cost (3.42) does not depend on the homogeneous term nor on the initial condition $\mathbf{X}_0$.*

Problem 3.5 *Let a real constant a, the variance of the process noise σ_W^2, the initial condition x_0 and the parameter p of the channel process be given. In addition, consider that a positive real c is given. We want to find an optimal solution $\mathcal{P}_{0,T}^*$ for the following optimization problem:*

$$\mathcal{P}_{0,T}^* \in \textit{arg} \min_{\mathcal{P}_{0,T}} \mathcal{J}_{0,T}\left(a, \sigma_W^2, p, c, \mathcal{P}_{0,T}\right) \tag{3.45}$$

3.5.1 A Kalman-like filter

Definition 3.20 ***(Kalman-like estimator)*** *Given the process defined in (3.35)-(3.36) and a pre-processor $\mathcal{P}_{0,T}$ define the map $\mathcal{Z} : (v_0, \ldots, v_k) \mapsto z_k$, for k in the set $\{0, \ldots, T\}$, where z_k is computed as follows:*

$$z_0 \stackrel{def}{=} x_0 \tag{3.46}$$

$$z_k \stackrel{def}{=} \begin{cases} az_{k-1} & \textit{if } v_k = \mathfrak{E} \\ \\ v_k & \textit{if } v_k \neq \mathfrak{E} \end{cases}, \textit{ with } k \geq 1 \tag{3.47}$$

Remark 3.3 *Notice that the pre-processor has access to the estimate $\mathbf{Z}_k$ because it has access and full control of the input applied to $\mathcal{Z}$.*

Remark 3.4 *Notice that Definition 3.20 is identical with Definition 2.5, but we must point that v_k has different meanings in these two definitions. In Definition 2.5, v_k is the output of the pre-processor , while in Definition 3.20, $v_k$is the output of the channel.*

3.5.2 The Set $\mathbb{P}_T$ - of Admissible Pre-Processors

We proceed by defining a class of pre-processors, which is amenable to the use of recursive methods for perfomance analysis. If a pre-processor belongs to such a class then we denote it as admissible, and we argue in Remark 3.6 that there always exist an admissble pre-processor that is an optimal solution to Problem 3.5. This implies that we incur no loss of generality in constraining our analysis to admissible pre-processors.

Definition 3.21 ***(Admissible pre-processor)*** *Let a horizon T larger than zero, a pre-processor policy $\mathcal{P}_{0,T}$ and a map $\mathcal{F}_{0,T}$ be given. The pre-processor $\mathcal{P}_{0,T}$ is admissible if there exists maps $\mathcal{P}_{m,T} : (x_m, \ldots, x_k, c_m, \ldots, c_{k-1}) \mapsto \hat{v}_k$, with $0 \leq m \leq T$ and $k \geq m$, such that $\mathcal{P}_{0,T}$ can be specified recursively as follows:*

Algorithm for $\mathcal{P}_{m,T}$

- ***(Initial step)*** *Set $k = m$, $l_m = 1$ and transmit the current state, i.e., $\hat{v}_m = x_m$ and the channel will deliver the packet, i.e. $c_m = 1$, which implies that $l_m = 1$.*

- ***(Step A)*** *Increase the counter k by one. If $k > T$ holds then terminate, otherwise execute Step B.*

- ***(Step B)*** *Obtain the pre-processor output at time k via $\hat{v}_k = \mathcal{P}_{m,T}(x_m, \ldots, x_k, c_m, \ldots, c_{k-1})$. If $\hat{v}_k = \mathfrak{E}$ then set $r_k = 0$ which implies $l_k = 0$ and go back to Step A. If $\hat{v}_k \neq \mathfrak{E}$ and if $c_k = 0$, go to step A, if $\hat{v}_k \neq \mathfrak{E}$ and if $c_k = 1$ then execute algorithm $\mathcal{P}_{k,T}$.*

End of Algorithm for $\mathcal{P}_{m,T}$

The class of <u>all admissible</u> pre-processors is denoted as $\mathbb{P}_T$.

The following Remark provides an equivalent characterization of the class of admissible pre-processors.

Remark 3.5 *Let a horizon T larger than zero and a pre-processor policy $\mathcal{P}_{0,T}$ be given. The pre-processor $\mathcal{P}_{0,T}$ is admissible if and only if for each $m \in \{1, \ldots, T\}$ there exists a map $\mathcal{P}_{m,T} : (x_m, \ldots, x_k, c_m, \ldots, c_{k-1}) \mapsto \hat{v}_k$ such that the following holds:*

$$\begin{aligned} & l_m = 1 \implies \mathcal{P}_{q,T}(x_q, \ldots, x_k, c_q, \ldots, c_{k-1}) = \\ & \qquad \mathcal{P}_{m,T}(x_m, \ldots, x_k, c_m, \ldots, c_{k-1}), \\ & x_q, \ldots, x_k \in \mathbb{R}, c_m, \ldots, c_{k-1} \in \{0, 1\}, \\ & \qquad k > m \geq q \geq 0 \end{aligned} \tag{3.48}$$

Given an admissible pre-processor $\mathcal{P}_{0,T}$, later on we will also refer to the time-restricted pre-processors $\{\mathcal{P}_{m,T}\}_{m=1}^{T}$ according to Definition 3.21, or equivalently as implied by (3.48).

Remark 3.6 *Given a positive time-horizon T, there is no loss of generality in constraining our search for optimal an pre-processor to the set $\mathbb{P}_T$. Indeed, let an optimal pre-processor policy $\mathcal{P}^*_{0,T}$ be given. If a transmission takes place at some time m ($r_m = 1$ holds) then the optimal output at the pre-processor is $\hat{v}_k = x_k$. If the transmission is successful (i.e. $c_k = 1$), it holds that $v_k = \hat{v}_k = x_k$. Since, given that a real number is transmitted, the choice $\hat{v}_k = x_k$ must be optimal because it leads to a perfect estimate $\hat{x}_m = x_m$. Hence, given that $l_m = 1$ (i.e. $r_{m=1}$, $c_m = 1$), by Markovianity we conclude that the current and future values produced by the pre-processor $\{\hat{\mathbf{V}}_k\}_{k=m}^{T}$ will <u>not</u> depend on observations prior to m. Consequently, $\mathcal{P}^*_{0,T}$ satisfies (3.48), and hence it is admissible.*

3.5.3 Symmetric threshold pre-processor

Definition 3.22 *In order to simplify our notation, we define the following process:*

$$\mathbf{Y}_k \overset{def}{=} \mathbf{X}_k - a\mathbf{Z}_{k-1} \tag{3.49}$$

Using Definitions 3.15 and 3.20, we find that $\{\mathbf{Y}_k\}_{k=0}^T$ can be rewritten as:

$$\mathbf{Y}_0 = 0 \tag{3.50}$$

$$\mathbf{Y}_{k+1} = \begin{cases} a\mathbf{Y}_k + \mathbf{W}_k & \text{if } \mathbf{L}_k = 0 \\ \mathbf{W}_k & \text{if } \mathbf{L}_k = 1 \end{cases} \tag{3.51}$$

Remark 3.7 *We notice that the process $\{\mathbf{Y}_k\}$ is defined in a similar way in (3.28) or (2.14). This remark is the same as Remark 2.7 and we repeat it here for clarity purposes. $\mathbf{Y}_k$ has an even probability density function. This fact makes $\{\mathbf{Y}_k\}_{k=0}^T$ a more convenient process to work with, in comparison to $\{\mathbf{X}_k\}_{k=0}^T$, which motivates its use in our analysis hereon, whenever possible. No loss of generality is incurred because $\{\mathbf{Y}_k\}_{k=0}^T$ can be recovered from $\{\mathbf{X}_k\}_{k=0}^T$, and vice-versa, via the use of $\{\mathbf{Z}_k\}_{k=0}^T$, which is common information at the pre-processor and estimator (See Remark 2.4 or Remark 3.3). In addition, notice that the cost (3.42) can be re-written in terms of $\{\mathbf{Y}_k\}_{k=0}^T$ as follows:*

$$\mathcal{J}_{0,T}\left(a, \sigma_W^2, p, c, \mathcal{P}_{0,T}\right) \overset{def}{=} \sum_{k=1}^{T} E\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}_k\right)^2 + c\mathbf{R}_k\right] \tag{3.52}$$

where $\hat{\mathbf{Y}}_k \overset{def}{=} E\left[\mathbf{Y}_k | \{\mathbf{V}_t\}_{t=0}^k\right]$. A key fact here is that $\hat{\mathbf{Y}}_k = \hat{\mathbf{X}}_k - a\mathbf{Z}_{k-1}$ holds, leading to the validity of the identity $\mathbf{Y}_k - \hat{\mathbf{Y}}_k = \mathbf{X}_k - \hat{\mathbf{X}}_k$.

Definition 3.23 *Given positive integer horizon T and an arbitrary sequence of positive real numbers (thresholds) $\tau = \{\tau_k\}_{k=1}^T$, for each m in the set $\{0, \dots, T\}$, we define the following algorithm for $k \geq m$, which we denote as $\mathcal{S}_{m,T}$:*

Algorithm $\mathcal{S}_{m,T}$

- ***(Initial step)*** *Set* $k = m$, $l_m = 1$ *(i.e.* $r_m = 1$ *and* $c_m = 1$*) and transmit the current state, i.e.,* $\hat{v}_m = x_m$ *or equivalently set* $y_m = 0$.

- ***(Step A)*** *Increase the time counter* k *by one. If* $k > T$ *holds then terminate, otherwise execute Step B.*

- ***(Step B)*** *If* $|y_k| < \tau_k$ *holds then set* $r_k = 0$*, transmit the erasure symbol, i.e.,* $\hat{v}_k = \mathfrak{E} = v_k$*, and return to Step A. If* $|y_k| \geq \tau_k$ *holds and if* $c_k = 0$ *return to Step A, if* $|y_k| \geq \tau_k$ *holds and if* $c_k = 1$ *then set* $m = k$ *and execute* $\mathcal{S}_{m,T}$.

End of Algorithm $\mathcal{S}_{m,T}$

Definition 3.24 ***(Symmetric threshold policy)*** *The algorithm* $\mathcal{S}_{0,T}$*, as in Definition 3.23, is denoted as* <u>*symmetric threshold pre-processor*</u>*. The pre-processor* $\mathcal{S}_{0,T}$ *is admissible and the class of* <u>*all symmetric threshold policies*</u> *is denoted as* $\mathbb{S}_T$.

Theorem 3.4 *Let the parameters specifying Problem 3.5 be given, i.e., the variance of the process noise* σ_W^2*, the system's dynamic constant* a*, the communication cost* c*, the parameter* p *of the channel process, and the time horizon* T *are pre-selected. There exists a sequence of positive real numbers* $\tau^* = \{\tau_k^*\}_{k=1}^T$*, such that the corresponding symmetric threshold policy* $\mathcal{S}_{0,T}^*$ *is an optimal solution to (3.45) and the corresponding optimal estimator* $\mathcal{E}(\mathcal{S}_{0,T}^*)$ *is* $\mathcal{Z}$*. Here* $\mathcal{S}_{0,T}^*$ *and* $\mathcal{Z}$ *follow Definitions 3.24 and 3.20, respectively.*

3.5.4 Optimizing within the class $\mathbb{D}_T$

We start by defining the following class of path-dependent pre-processor policies, which is an extension of Definition 3.24 so as to allow time-varying thresholds that depend on past decisions. Such a class of pre-processors will be used later when we provide a proof for Theorem 3.4.

Definition 3.25 *(**Algorithm** $\mathcal{D}_{m,T}$) Given a horizon T, consider that a sequence of (threshold) functions $\mathcal{T} \stackrel{def}{=} \{\mathcal{T}_{m,k} | m \leq k \leq T, 1 \leq m \leq T\}$, with $\mathcal{T}_{m,k} : \{0,1\}^{m-k} \to \mathbb{R}$, is given. Given a selection of the threshold functions $\mathcal{T}$, for every m in the set $\{1, \ldots, T\}$, we define the following algorithm for $k \geq m$, which we denote as $\mathcal{D}_{m,T}$:*

Algorithm $\mathcal{D}_{m,T}$

- ***(Initial step)*** *Set $k = m$, $l_m = 1$ (i.e. $r_m = 1$ and $c_m = 1$) and transmit the current state which will be received by the estimator, i.e., $\hat{v}_m = v_m = x_m$ or equivalently set $y_m = 0$.*

- ***(Step A)*** *Increase the time counter k by one. If $k > T$ holds then terminate, otherwise execute Step B.*

- ***(Step B)*** *If $|y_k| < \mathcal{T}_{m,k}(l_m, \ldots, l_{k-1})$ holds then set $r_k = 0$, transmit the erasure symbol, i.e., $\hat{v}_k = v_k = \mathfrak{E}$, and return to Step A. If $|y_k| \geq \mathcal{T}_{m,k}(l_m, \ldots, l_{k-1})$ and if $c_k = 0$, then $v_k = \mathfrak{E}$ and return to Step A, if $|y_k| \geq \mathcal{T}_{m,k}(l_m, \ldots, l_{k-1})$ and if $c_k = 1$ hold then execute $\mathcal{D}_{k,T}$.*

End of Algorithm $\mathcal{D}_{m,T}$

Recall that l_0 through l_{k-1} represent past information received by the estimator, where $l_k = 1$ indicates that the state is received at the estimator at time k, while $l_k = 0$ implies that an erasure was received.

Definition 3.26 ***(Path-dependent symmetric threshold policy)*** *Given an horizon T, consider that a sequence of (threshold) functions $\mathcal{T} \stackrel{def}{=} \{\mathcal{T}_{m,k} | m \leq k \leq T, 1 \leq m \leq T\}$, with $\mathcal{T}_{m,k} : \{0,1\}^{m-k} \to \mathbb{R}$, is given. The path-dependent symmetric threshold pre-processor associated with $\mathcal{T}$ is implemented via the execution of the algorithm $\mathcal{D}_{0,T}$, as specified in Definition 3.25. We denote such an admissible pre-processor as $\mathcal{D}_{0,T}$. We use $\mathbb{D}_{0,T}$ to denote the entire class of path-dependent symmetric threshold pre-processors with time horizon T.*

The goal of this section is to provide the following two results that are crutial in the proof of Theorem 3.4: In Proposition 3.1, we prove that if $\mathcal{P}_{0,T}$ is any given path-dependent symmetric threshold pre-processor policy then the associated optimal estimator $\mathcal{E}(\mathcal{P}_{0,T})$ is $\mathcal{Z}$. In Lemma 3.1 we prove that if we optimize within the class of path-dependent policies then the optimum is of the path-independent type specified in Definition 3.24. This fact might raise the question of whether Definition 3.26 is needed. The answer is *yes* because we adopt a constructive argument in the proof of Theorem 3.4 in Subsection 3.5.7, which will make use of Definition 3.26.

Proposition 3.1 *Let $\mathcal{P}_{0,T}$ be a pre-selected path-dependent symmetric threshold policy (Definition 3.26), it holds that the optimal estimator $\mathcal{E}(\mathcal{P}_{0,T})$ is $\mathcal{Z}$, as described in Definition 3.20.*

Remark 3.8 *Proposition 3.1 could be recast by stating that* $\hat{\mathbf{X}}_k = \mathbf{Z}_k$ *holds in the presence of path-dependent symmetric threshold pre-processors.*

Proof (of Proposition 3.1) In order to simplify the proof, we define $\{\tilde{\mathbf{X}}_k\}_{k=0}^T$ as the process quantifying the error incurred by adopting a Kalman-like estimator (See Definition 3.20), i.e., $\tilde{\mathbf{X}}_k \overset{def}{=} \mathbf{X}_k - \mathbf{Z}_k$. More specifically, $\{\tilde{\mathbf{X}}_k\}_{k=0}^T$ can be equivalently expressed as follows:

$$\tilde{\mathbf{X}}_0 = 0 \tag{3.53}$$

$$\tilde{\mathbf{X}}_{t+1} = \begin{cases} a\tilde{\mathbf{X}}_t + \mathbf{W}_t & \text{if } \mathbf{L}_t = 0 \\ \\ 0 & \text{if } \mathbf{L}_t = 1 \end{cases} \tag{3.54}$$

The proof follows from the symmetry of all probability density functions involving $\tilde{\mathbf{X}}_k$ and $\mathbf{V}_k$. More specifically, under symmetric path-dependent threshold policies the probability density function of $\tilde{\mathbf{X}}_k$, given the past and current observations $\{\mathbf{V}_k\}_{k=0}^t$, is even. Hence, we conclude that $E[\tilde{\mathbf{X}}_t | \{\mathbf{V}_k\}_{k=0}^t] = 0$, which implies that $\hat{\mathbf{X}}_t \overset{def}{=} E[\mathbf{X}_t | \{\mathbf{V}_k\}_{k=0}^t] = \mathbf{Z}_t$.■

Remark 3.9 *If* $\mathcal{D}_{0,T}$ *is a symmetric path-dependent threshold pre-processor (see Definition 3.26) then* $\hat{\mathbf{Y}}_k = 0$ *holds, leading to the following equality:*

$$\mathcal{J}_{0,T}\left(a, \sigma_W^2, p, c, \mathcal{D}_{0,T}\right) = \sum_{k=1}^{T} E\left[{\mathbf{Y}_k}^2 + c\mathbf{R}_k\right], \qquad \mathcal{D}_{0,T} \in \mathbb{D}_T \tag{3.55}$$

The process defined in (3.51) is a Markov Decision Process (MDP) whose state and control are $\mathbf{Y}_k$ and $\mathbf{R}_k$, respectively. Hence the minimization of (3.55) with respect to pre-processor policies $\mathcal{D}_{0,T}$ in the class $\mathbb{D}_T$ can be cast as a dynamic program [13]. To do so, we define the sequence of functions $\mathcal{V}_{t,T} : \mathbb{R} \to \mathbb{R}, t \in \{1, 1, \ldots, T+1\}$ which represent the cost-to-go as observed by the pre-processor. Here T represents the horizon, while t denotes that the decision at time t was taken, and the argument of the function is the MDP state $\mathbf{Y}_t$, as seen by the pre-processor. In order to simplify our notation, we adopt the convention that $\mathcal{V}_{T+1,T}(y_{T+1}) \stackrel{def}{=} 0$, $y_{T+1} \in \mathbb{R}$. Using dynamic programming, we can find the following recursive equations for $\mathcal{V}_{t,T}(y_t),\ t \in \{1, \ldots, T\}$:

$$\mathcal{V}_{t,T}(y_t) \stackrel{def}{=} \min_{r_t \in \{0,1\}} \mathcal{C}_{t,T}(y_t, r_t),\ t \in \{1, \ldots, T\} \tag{3.56}$$

where $\mathcal{C}_{t,T} : \mathbb{R} \times \{0, 1\} \to \mathbb{R}$ is defined as:

$$\begin{aligned}
\mathcal{C}_{t,T}(y_t, 1) &\stackrel{def}{=} c + pE\left[\mathcal{V}_{t+1,T}(\mathbf{W}_t)\right] \\
&\quad + (1-p)\left({y_t}^2 + E\left[\mathcal{V}_{t+1,T}\left(ay_t + \mathbf{W}_t\right)\right]\right) \qquad (3.57) \\
\mathcal{C}_{t,T}(y_t, 0) &\stackrel{def}{=} {y_t}^2 + E\left[\mathcal{V}_{t+1,T}\left(ay_t + \mathbf{W}_t\right)\right]
\end{aligned}$$

From (3.57) it immediately follows that an optimal decision policy r_t^* at any time t is given by:

$$r_t^* = \begin{cases} 1 & \text{if } \mathcal{C}_{t,T}(y_t, 1) \le \mathcal{C}_{t,T}(y_t, 0) \\ 0 & \text{if } \mathcal{C}_{t,T}(y_t, 0) < \mathcal{C}_{t,T}(y_t, 1) \end{cases} \tag{3.58}$$

We state the next Proposition, which will state properties of the functions $\mathcal{V}_{t,T}$ for $t \in \{1, \ldots, T+1\}$:

Proposition 3.2 *There exist functions* $\tilde{\mathcal{V}}_{t,T} : \mathbb{R} \to \mathbb{R},\ t \in \{1, \ldots, T+1\}$ *and positive real numbers* $\left\{m_t^T\right\}_{t=1}^{T+1}$ *such that:*

$$\mathcal{V}_{t,T}(y_t) = m_t^T y_t^2 + \tilde{\mathcal{V}}_{t,T}(y_t) \tag{3.59}$$

where $\mathcal{V}_{t,T}(y_t),\ t \in \{1,\ldots,T+1\}$ *are defined in equation 3.56. Moreover, the functions* $\tilde{\mathcal{V}}_{t,T}:\mathbb{R}\to\mathbb{R},\ t\in\{1,\ldots,T+1\}$ *are even, quasi-convex, continuous and bounded.*

Proof: Define the sequence of positive numbers $\left\{m_t^T\right\}_{t=1}^{T+1}$ as follows:

$$\begin{aligned} m_{T+1}^T &\overset{def}{=} 0 \\ m_t^T &\overset{def}{=} (1-p)\left(1+a^2 m_{t+1}^T\right),\ t\in\{1,\ldots,T\} \end{aligned} \tag{3.60}$$

Define the functions $\tilde{\mathcal{V}}_{t,T}:\mathbb{R}\to\mathbb{R},\ t\in\{1,\ldots,T+1\}$ as follows:

$$\begin{aligned} &\tilde{\mathcal{V}}_{T+1,T}(y_{T+1}) \overset{def}{=} 0 \\ &\quad \tilde{\mathcal{V}}_{t,T}(y_t) \overset{def}{=} m_{t+1}^T\sigma_W^2 + (1-p)E\left[\tilde{\mathcal{V}}_{t+1,T}\left(ay_t+\mathbf{W}_t\right)\right] \\ &\qquad\qquad + \min\left(c+pE\left[\tilde{\mathcal{V}}_{t+1,T}\left(\mathbf{W}_t\right)\right], +p(1+a^2m_{t+1}^T)y_t^2 + pE\left[\tilde{\mathcal{V}}_{t+1,T}\left(ay_t+\mathbf{W}_t\right)\right]\right), \\ &\qquad\qquad\qquad t\in\{1,\ldots,T\} \end{aligned} \tag{3.61}$$

We will show that the functions $\tilde{\mathcal{V}}_{t,T}:\mathbb{R}\to\mathbb{R},\ t\in\{1,\ldots,T+1\}$ defined in equations (3.61) and the sequence of positive numbers $\left\{m_t^T\right\}_{t=1}^{T+1}$ defined in equations (3.60) satisfy equations (3.59).

The function $\tilde{\mathcal{V}}_{T-1,T}(y_{T-1}) = \min\left(c, py_{T-1}^2\right)$ is bounded. By induction, it follows that the functions $\tilde{\mathcal{V}}_{t,T},\ t\in\{1,\ldots,T+1\}$ are bounded.

We prove equation (3.59) by induction:

$$\begin{aligned} \mathcal{V}_{T,T}(y_T) &= \min\left(c+(1-p)y_T^2, y_T^2\right) \\ &= (1-p){y_T}^2 + \min\left(c, py_T^2\right) \\ &= m_T^T y_T^2 + \tilde{\mathcal{V}}_{T,T}(y_T) \end{aligned}$$

We can compute the function $E\left[\mathcal{V}_{T,T}\left(ay_{T-1}+\mathbf{W}_{T-1}\right)\right]$:

$$E\left[\mathcal{V}_{T,T}\left(ay_T+\mathbf{W}_T\right)\right] = (1-p)a^2y_T^2 + (1-p)\sigma_W^2 + E\left[\tilde{\mathcal{V}}_{T,T}\left(ay_T+\mathbf{W}_T\right)\right]$$

We note that the function $E\left[\mathcal{V}_{T,T}\left(ay_T + \mathbf{W}_T\right)\right]$ is well defined for all real numbers y_T, because $\tilde{\mathcal{V}}_{T,T}$ is bounded hence $E\left[\tilde{\mathcal{V}}_{T,T}\left(ay_T + \mathbf{W}_T\right)\right]$ exists for all real numbers y_T.

Assume that $\mathcal{V}_{k,T}\left(y_k\right)$ satisfies equation (3.59) for all $k \in \{t+1, \ldots, T+1\}$, which implies that the functions $E\left[\mathcal{V}_{k+1,T}\left(ay_k + \mathbf{W}_k\right)\right],\ k \in \{t, \ldots, T\}$ are well defined for all real numbers y_k, since:

$$E\left[\mathcal{V}_{k+1,T}\left(ay_k + \mathbf{W}_k\right)\right] = m_{k+1}^T a^2 y_k^2 + m_{k+1}^T \sigma_W^2 + E\left[\tilde{\mathcal{V}}_{k+1,T}\left(ay_k + \mathbf{W}_k\right)\right],\ k \in \{t, \ldots, T\}$$

In order to prove equation (3.59) for t, we use equations (3.56) and (3.57) :

$$\begin{aligned}
\mathcal{V}_{t,T}(y_t) = \min\Big(&c + pE\left[\mathcal{V}_{t+1,T}(\mathbf{W}_t)\right] + (1-p)\left(y_t^2 + E\left[\mathcal{V}_{t+1,T}\left(ay_t + \mathbf{W}_t\right)\right]\right), \\
&y_t^2 + E\left[\mathcal{V}_{t+1,T}\left(ay_t + \mathbf{W}_t\right)\right]\Big) \\
= &(1-p)\left(y_t^2 + E\left[\mathcal{V}_{t+1,T}\left(ay_t + \mathbf{W}_t\right)\right]\right) \\
&+ \min\left(c + pE\left[\mathcal{V}_{t+1,T}(\mathbf{W}_t)\right], p\left(y_t^2 + E\left[\mathcal{V}_{t+1,T}\left(ay_t + \mathbf{W}_t\right)\right]\right)\right) \\
= &(1-p)y_t^2 + (1-p)a^2 m_{t+1}^T {y_t}^2 + (1-p)\sigma_W^2 + (1-p)E\left[\tilde{\mathcal{V}}_{t+1,T}\left(ay_t + \mathbf{W}_t\right)\right] \\
&+ \min\Big(c + pm_{t+1}^T\sigma_W^2 + pE\left[\tilde{\mathcal{V}}_{t+1,T}\left(\mathbf{W}_t\right)\right], \\
&\quad py_t^2 + pm_{t+1}^T a^2 y_t^2 + pm_{t+1}^T\sigma_W^2 + pE\left[\tilde{\mathcal{V}}_{t+1,T}\left(ay_t + \mathbf{W}_t\right)\right]\Big) \\
= &(1-p)\left(1 + a^2 m_{t+1}^T\right) y_t^2 + m_{t+1}^T\sigma_W^2 + (1-p)E\left[\tilde{\mathcal{V}}_{t+1,T}\left(ay_t + \mathbf{W}_t\right)\right] + \\
&\min\left(c + pE\left[\tilde{\mathcal{V}}_{t+1,T}\left(\mathbf{W}_t\right)\right], py_t^2 + pm_{t+1}^T a^2 y_t^2 + pE\left[\tilde{\mathcal{V}}_{t+1,T}\left(ay_t + \mathbf{W}_t\right)\right]\right) \\
= &m_t^T y_t^2 + \tilde{\mathcal{V}}_{t,T}(y_t) \qquad \blacksquare
\end{aligned}$$

Using the MDP given in Definition 3.22 and the value functions from equation (3.56), we prove the following Lemma, which states that, *within the class of symmetric path-dependent pre-processors $\mathbb{D}_T$ (Definition 3.26)*, there exists an optimal path-*independent* symmetric threshold policy $\mathcal{S}_{0,T}^*$ (Definition 3.24) for Problem 3.5.

Lemma 3.1 *Let the parameters specifying Problem 3.5 be given, i.e., the variance of the process noise σ_W^2, the system's dynamic constant a, the communication cost c, the channel process parameter p and the time horizon T are pre-selected. Consider Problem 3.5 with the additional constraint that the pre-processor must be of the symmetric path-dependent type $\mathbb{D}_T$ specified in Definition 3.26. There exists an optimal path-*independent *symmetric threshold policy $\mathcal{S}_{0,T}^*$, as given in Definition 3.24, whose associated threshold selection $\{\tau_k^*\}_{k=1}^T$ is given by a solution to the following equations:*

$$\mathcal{C}_{t,T}(\tau_t^*, 0) = \mathcal{C}_{t,T}(\tau_t^*, 1),\ t \in \{1, \ldots, T\} \tag{3.62}$$

Proof: From (3.58), we conclude that in order to prove this Lemma we only need to show that there exist thresholds $\{\tau_k^*\}_{k=1}^T$ such that the following equivalences hold:

$$|y_t| \geq \tau_t^* \Longleftrightarrow \mathcal{C}_{t,T}(y_t, 1) \leq \mathcal{C}_{t,T}(y_t, 0),\ t \in \{1, \ldots, T\} \tag{3.63}$$

Indeed, if (3.63) holds then the optimal strategy in (3.58) can be implemented via a threshold policy. In order to prove that there exist thresholds $\{\tau_k^*\}_{k=1}^T$ such that (3.63) holds, we will use the following facts (A.1 thorugh A.4):

- (**Fact A.1**): For every t in the set $\{1, \ldots, T\}$, $\mathcal{C}_{t,T}(y_t, 1)$ depends only on t, i.e., it is time-dependent constant independent of y_t.

- (**Fact A.2**): It holds that $\mathcal{C}_{t,T}(0, 0) < \mathcal{C}_{t,T}(y_t, 1)$ for $y_t \in \mathbb{R}$.

- (**Fact A.3**): For every t in the set $\{1, \ldots, T\}$ there exists a positive constant u_t such that $\mathcal{C}_{t,T}(y_t, 0) > \mathcal{C}_{t,T}(y_t, 1)$ and $\mathcal{C}_{t,T}(-y_t, 0) > \mathcal{C}_{t,T}(-y_t, 1)$ hold for every y_t satisfying $|y_t| > u_t$.

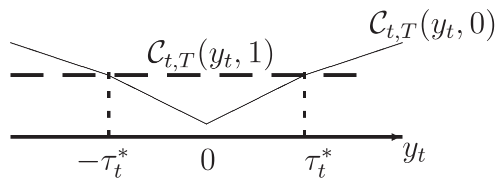

Figure 3.4: Illustration suggesting that Facts A.1 through A.4. imply the existence of thresholds for Problem 3.5, where we allow packet drop with ackowledgement.

- (**Fact A.4**): It holds that $\mathcal{C}_{t,T}(y_t, 0)$ is a continuous, even, quasi-convex and unbounded function of y_t, for every t in the set $\{1, \ldots, T\}$.

Facts A.1 and A.3 follow immediately from Fact A.4, which requires a proof that we defer to a later stage. Fact A.2 also follows from Fact A.4 and from (2.20), which implies that $\mathcal{C}_{t,T}(0,0) < \mathcal{C}_{t,T}(0,1)$. At this point we assume that Fact A.4 is valid, and we proceed by noticing that continuity of $\mathcal{C}_{t,T}(y_t, 0)$ with respect to y_t, as well as Facts A.2 and A.3, imply that the equations in (2.22) have at least one solution. Moreover, from Facts A.1 through A.4 we can conclude that such a solution guarantees that (3.63) is true (See Figure 3.4).

(Proof of Fact 4) It follows from Proposition 3.2. ■

3.5.5 Conditional probabilities and conditional probability density functions

Before proving Theorem 3.4, in this subsection we need to make a few remarks and introduce more notation, which will streamline our proof. This subsection contains two parts: We start by introducing the notation for certain conditional probability density functions of interest, while in the second part we will derive recursive equations for the time update of the conditional densities, and we will also obtain a recursive expansion for the cost associated with any given admissible pre-processor policy $\mathcal{P}_{0,T}$.

Definition 3.27 *Let a pre-processor $\mathcal{P}_{0,T}$, implementing a decision policy as in Definition 3.16, be given. We define the following notation for conditional probability densities, which will streamline our proof of Theorem 3.4:*

1. *Define the conditional probability density function of $\mathbf{Y}_k$ given that only erasure symbols were received by the estimator up until time k as follows:*

$$\gamma_{k|k}\left(y\right) \stackrel{def}{=} f_{\mathbf{Y}_k|\mathbf{L}_1=0,\ldots,\mathbf{L}_k=0}\left(y\right), \qquad y \in \mathbb{R}$$

2. *Define the conditional probability density function of $\mathbf{Y}_k$ given that only erasure symbols were received up until time $k-1$ as follows:*

$$\gamma_{k|k-1}\left(y\right) \stackrel{def}{=} f_{\mathbf{Y}_k|\mathbf{L}_1=0,\ldots,\mathbf{L}_{k-1}=0}\left(y\right), \qquad y \in \mathbb{R}$$

Definition 3.28 *We define the following streamlined notation for certain conditional probabilities of interest:*

1. *Define the probability that, under policy $\mathcal{P}_{0,T}$, only erasure symbols have been received up until time k:*

$$\varsigma_k \stackrel{def}{=} \begin{cases} P\left(\mathbf{L}_1 = 0, \ldots, \mathbf{L}_k = 0\right) & \textit{if } k \geq 1 \\ 1 & \textit{if } k = 0 \end{cases}$$

2. *Define the conditional probability that, under policy $\mathcal{P}_{0,T}$, the pre-processor transmits the erasure symbol at time k, given that only erasure symbols have been received up until time $k-1$.*

$$\varsigma_{k|k-1} \stackrel{def}{=} \begin{cases} P\left(\mathbf{R}_1 = 0\right), \text{ if } k = 1 \\ P\left(\mathbf{R}_k = 0 | \mathbf{L}_1 = 0, \ldots, \mathbf{L}_{k-1} = 0\right), \text{ ow} \end{cases}$$

Definition 3.29 *Let $\mathcal{P}_{0,T}$ be a decision policy given as in Definition 3.16. Let k be a positive integer and y be a real number. For a positive integer k, define the function $\rho_k : \mathbb{R} \to [0,1]$ as follows:*

$$\rho_k\left(y\right) \stackrel{def}{=} P\left(\mathbf{R}_k = 0 | \mathbf{Y}_k = y, \mathbf{L}_1 = 0, \ldots, \mathbf{L}_{k-1} = 0\right), \tag{3.64}$$

where $y \in \mathbb{R}$. The function $\rho_k\left(y\right)$ is the probability that, at time k, the erasure symbol is transmitted, given that $\mathbf{Y}_k = y$, where y is any real number, and the fact that only erasure symbols have been received up until time $k-1$.

3.5.6 Time Evolution

Now, we describe how the conditional probability density functions presented in subsection 3.5.5 evolve in time, for a given policy $\mathcal{P}_{0,T}$. For a real number a, define the conditional probability density function of $a\mathbf{Y}_k$ given that no observation was received up until time k under the decision policy $\mathcal{P}_{0,T}$:

$$\gamma^a_{k|k}(y) \stackrel{def}{=} f_{a\mathbf{Y}_k|\mathbf{L}_1=0,\ldots,\mathbf{L}_k=0}\left(y\right)$$

We denote by $\mathcal{N}_{\sigma^2_W}$ the probability density function of $\mathbf{W}_k$, for all k, i.e., the Gaussian zero mean probability density with variance σ^2_W, or more concretely $N_{\sigma^2_W}(x) =$

$\frac{1}{\sqrt{2\pi\sigma_W^2}}e^{-\frac{x^2}{2\sigma_W^2}}$. Since the sequence $\{\mathbf{W}_k\}_{k=0}^T$ is i.i.d., $\mathbf{W}_k$ is also independent of $\{Y_l\}_{l=0}^k$, which implies that the following holds:

$$\gamma_{k|k-1} = \gamma_{k-1|k-1}^a * \mathcal{N}_{\sigma_W^2} \tag{3.65}$$

Proposition 3.3 *The conditional densities $\gamma_{k|k-1}$ and $\gamma_{k|k}$ are related via the following time-recursion:*

$$\begin{aligned}\gamma_{k|k}(y) &= \frac{\gamma_{k|k-1}(y)\rho_k(x) + (1-p)(1-\rho_k(y))\gamma_{k|k-1}(y)}{\varsigma_{k|k-1} + (1-\varsigma_{k|k-1})(1-p)},\\ &\varsigma_{k|k-1} + (1-\varsigma_{k|k-1})(1-p) \neq 0,\ k \geq 1\end{aligned} \tag{3.66}$$

Proof: In order to arrive at (3.66), we use Baye's rule to write:

$$f_{\mathbf{Y}_k|\mathbf{L}_0=0,\ldots,\mathbf{L}_k=0}(y) = \frac{P\left(\mathbf{L}_k = 0|\mathbf{Y}_k = y, \mathbf{L}_0 = 0, \ldots, \mathbf{L}_{k-1} = 0\right)}{P\left(\mathbf{L}_k = 0|\mathbf{L}_0 = 0, \ldots, \mathbf{L}_{k-1} = 0\right)} f_{\mathbf{Y}_k|\mathbf{L}_0=0,\ldots,\mathbf{L}_{k-1}=0}(y) \tag{3.67}$$

The recursion (3.66) follows from (3.67) and by rewriting it according to Definitions 3.27, 3.28 and 3.29. Equation (3.67) holds only if $\varsigma_{k|k-1} + (1-\varsigma_{k|k-1})(1-p) \neq 0$, otherwise the conditional density function $f_{\mathbf{Y}_k|\mathbf{L}_1=0,\ldots,\mathbf{L}_k=0}(y)$ is no longer defined.■

Definition 3.30 *Given an admissible pre-processor $\mathcal{P}_{0,T}$ and an integer $m \in \{0, T\}$, we adopt the following definition for the partial cost computed for the horizon $\{m, \ldots, T\}$ under the assumption that $l_m = 1$:*

$$\mathcal{J}_{m,T}\left(a, \sigma_W^2, p, c, \mathcal{P}_{m,T}\right) \stackrel{def}{=} \sum_{k=m+1}^{T} E\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}_k\right)^2 + c\mathbf{R}_k\right], \textit{if } 0 \leq m < T \tag{3.68}$$

If $m < 0$ or $m \geq T$ we define $\mathcal{J}_{m,T}\left(a, \sigma_W^2, p, c, \mathcal{P}_{m,T}\right)$ to be equal to zero.

Remark 3.10 *Given an integer m, we notice that the cost in (3.68) will not depend on the value of the state at time m. This is so beause, according to Definition 3.21, since $\mathcal{P}_{0,T}$*

is admissible it holds that the current and future output *of* $\mathcal{P}_{m,T}$ *will* <u>*not*</u> *depend on the current and past state observations. This Remark is an extension of Remark 3.2, which considered the case for* $m = 0$.

Proposition 3.4 *Given an arbitrarily selected admissible pre-processor* $\mathcal{P}_{0,T}$*, the finite horizon cost (3.42) can be expanded as:*

$$\begin{aligned}
&\mathcal{J}_{0,T}\left(a, \sigma_W^2, p, c, \mathcal{P}_{0,T}\right) \\
&= \sum_{k=1}^{T}\Bigg(\left(E_{\gamma_{k|k}}\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}_k\right)^2\right] + \frac{c(1-p)\varsigma_{k|k-1}}{\varsigma_{k|k-1} + (1-\varsigma_{k|k-1})(1-p)}\right)\varsigma_{k|k} \\
&+ \left(c + \mathcal{J}_{k,T}\left(a, \sigma_W^2, p, c, \mathcal{P}_{k,T}\right)\right)\left(\varsigma_{k-1} - \left(\varsigma_{k|k-1} + (1-\varsigma_{k|k-1})(1-p)\right)\varsigma_{k-1}\right)\Bigg)
\end{aligned} \tag{3.69}$$

Here we use the notation $E_{\gamma_{k|k}}\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}_k\right)^2\right] \overset{def}{=} E\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}_k\right)^2 |\mathbf{L}_1 = 0, \ldots, \mathbf{L}_k = 0\right]$, *where* $\gamma_{k|k}$ *is given in Definition 3.27.*

Proof: We start by noticing that, by the total probability law, we can expand the cost as:

$$\begin{aligned}
&\mathcal{J}_{0,T}\left(a, \sigma_W^2, c, \mathcal{P}_{0,T}\right) \\
&= \sum_{k=1}^{T}\Bigg(\Bigg(E\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}_k\right)^2 |\mathbf{L}_1 = 0, \ldots, \mathbf{L}_k = 0\right] \\
&+ cP\left(\mathbf{R}_k = 1 | \mathbf{L}_1 = 0, \ldots, \mathbf{L}_k = 0\right)\Bigg) P\left(\mathbf{L}_0 = 0, \ldots, \mathbf{L}_k = 0\right) \\
&+ cP\left(\mathbf{L}_k = 1, \mathbf{L}_1 = 0, \ldots, \mathbf{L}_{k-1} = 0\right) \\
&+ E\left[\mathcal{J}_{k,T}\left(a, \sigma_W^2, p, c, \mathcal{P}_{0,T}\right) |\mathbf{L}_k = 1, \mathbf{L}_1 = 0, \ldots, \mathbf{L}_{k-1} = 0\right] \\
&\cdot P\left(\mathbf{L}_k = 1, \mathbf{L}_1 = 0, \ldots, \mathbf{L}_{k-1} = 0\right)\Bigg)
\end{aligned} \tag{3.70}$$

We proceed by obtaining the following identities: We are interested in computing the following conditional probabilities:

$$\begin{aligned}
&P\left(\mathbf{L}=1,\mathbf{L}_0=0,\dots,\mathbf{L}_{k-1}=0\right)=\\
&=P\left(\mathbf{L}_1=0,\dots,\mathbf{L}_{k-1}=0\right)-P\left(\mathbf{L}_1=0,\dots,\mathbf{L}_k=0\right)\\
&=P\left(\mathbf{L}_1=0,\dots,\mathbf{L}_{k-1}=0\right) \qquad (3.71)\\
&-P\left(\mathbf{L}_k=0|\mathbf{L}_1=0,\dots,\mathbf{L}_{k-1}=0\right)P\left(\mathbf{L}_1=0,\dots,\mathbf{L}_{k-1}=0\right)\\
&=\varsigma_{k-1}-\left(\varsigma_{k|k-1}+(1-\varsigma_{k|k-1})(1-p)\right)\varsigma_{k-1}
\end{aligned}$$

$$\begin{aligned}
P\left(\mathbf{R}_k=1|\mathbf{L}_k=0,\dots,\mathbf{L}_1=0\right)&\\
&=P\left(\mathbf{R}_k=1|\mathbf{L}_{k-1}=0,\dots,\mathbf{L}_1=0\right)\\
&\quad\cdot\frac{P\left(\mathbf{L}_k=0|\mathbf{R}_k=1,\mathbf{L}_{k-1}=0,\dots,\mathbf{L}_1=0\right)}{P\left(\mathbf{L}_k=0|\mathbf{L}_{k-1}=0,\dots,\mathbf{L}_1=0\right)} \qquad (3.72)\\
&=\frac{(1-p)\cdot\varsigma_{k|k-1}}{\varsigma_{k|k-1}+(1-\varsigma_{k|k-1})(1-p)}
\end{aligned}$$

Notice that, using standard probability theory, from $\left\{\varsigma_k\right\}_{k=1}^{T}$ we can compute $\left\{\varsigma_{k|k-1}\right\}_{k=1}^{T}$ and vice versa. Here, equations (3.71) and (3.72) are still valid for $k=1$, since we defined $\varsigma_0=1$ and $\varsigma_{1|0}=\varsigma_1$. Finally, notice that from Remark 3.10, we conclude the following:

$$E\left[\mathcal{J}_{k,T}\left(a,\sigma_W^2,p,c,\mathcal{P}_{k,T}\right)|\mathbf{L}_k=1,\mathbf{L}_1=0,\dots,\mathbf{L}_{k-1}=0\right]=\mathcal{J}_{k,T}\left(a,\sigma_W^2,p,c,\mathcal{P}_{k,T}\right) \qquad (3.73)$$

The proof of this Proposition is complete once we substitute (3.71), (3.72) and (3.73) into (3.70). ■

Definition 3.31 *The following is a convenient definition for optimal cost:*

$$\mathcal{J}^*_{m,T}\left(a,\sigma_W^2,p,c\right)\overset{def}{=}\min_{\mathcal{P}_{m,T}}\mathcal{J}_{m,T}\left(a,\sigma_W^2,p,c,\mathcal{P}_{m,T}\right), \qquad (3.74)$$

where $T\geq 1$*. If* $T=0$*, we set* $\mathcal{J}^*_{m,T}\left(a,\sigma_W^2,p,c\right)\overset{def}{=}0$.

From Proposition 3.4, we can immediately state the following Corollary:

Corollary 3.2 *The following inequality holds for every admissible pre-processor $\mathcal{P}_{0,T}$:*

$$
\begin{aligned}
&\mathcal{J}_{0,T}\left(a, \sigma_W^2, p, c, \mathcal{P}_{0,T}\right) \\
&\geq \sum_{k=1}^{T}\left(\left(E_{\gamma_{k|k}}\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}_k\right)^2\right] + \frac{c(1-p)\varsigma_{k|k-1}}{\varsigma_{k|k-1} + (1-\varsigma_{k|k-1})(1-p)}\right)\varsigma_{k|k}\right. \\
&\left. + \left(c + \mathcal{J}^*_{k,T}\left(a, \sigma_W^2, p, c\right)\right)\left(\varsigma_{k-1} - \left(\varsigma_{k|k-1} + (1-\varsigma_{k|k-1})(1-p)\right)\varsigma_{k-1}\right)\right)
\end{aligned}
\tag{3.75}
$$

3.5.7 Proof of Theorem 3.4

Before proceeding with the actual proof of Theorem 3.4, we state Lemma 3.2, which is a supporting result for the proof Theorem 3.4 and extends existing results from majorization theory (See Section 2.4.1). Before stating Lemma 3.2, we need the following definition.

Definition 3.32 *Given a probability density function $f : \mathbb{R} \to \mathbb{R}$ and a Borel set $\mathbb{K}$, such that $\int_{\mathbb{K}} f(x)dx > 0$, and a positive real constant $p \leq 1$ we define the probability density function $f_{\mathbb{K}^p}$ as follows:*

$$
f_{\mathbb{K}}^p(x) \stackrel{def}{=} \begin{cases} \frac{f(x)}{\int_{\mathbb{K}} f(x)dx + (1-p)\int_{\mathbb{R}\setminus\mathbb{K}} f(x)dx}, x \in \mathbb{K} \\ \frac{(1-p)f(x)}{\int_{\mathbb{K}} f(x)dx + (1-p)\int_{\mathbb{R}\setminus\mathbb{K}} f(x)dx}, x \notin \mathbb{K} \end{cases}
$$

It is clear that $f_{\mathbb{K}}^p$ is also a probability density function.

Lemma 3.2 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two probability density functions, such that f is neat and even and $f \succ g$. Let κ be a real number in the interval $\kappa \in (0,1)$ and let p be a real number such that $p \in (0,1]$. Let $\mathbb{A} = [-\tau, \tau]$ be the symmetric interval, such that*

$\int_{-\tau}^{\tau} f(x)dx = 1-\kappa$. *For any function* $h : \mathbb{R} \to [0,1]$ *satisfying* $\int_{\mathbb{R}} g(x)h(x)dx = 1-\kappa$, *the following holds:*

$$f_{\mathbb{A}}^{p} \succ \frac{g \cdot h + (1-p)(1-h) \cdot g}{1-\kappa+(1-p)\kappa} \tag{3.76}$$

Proof: Let $\mathbb{F} \in \mathbb{R}$ be a Borel set with $\mathcal{L}(\mathbb{F}) < \infty$. Since f is even and quasiconcave, it holds that, $f(x) \leq f\left(\frac{\mathcal{L}(\mathbb{F})}{2}\right),\ x \in \mathbb{F} \setminus \left[-\frac{\mathcal{L}(\mathbb{F})}{2}, \frac{\mathcal{L}(\mathbb{F})}{2}\right]$ and $f(x) \geq f\left(\frac{\mathcal{L}(\mathbb{F})}{2}\right),\ x \in \left[-\frac{\mathcal{L}(\mathbb{F})}{2}, \frac{\mathcal{L}(\mathbb{F})}{2}\right] \setminus \mathbb{F}$. It follows that:

$$\int_{\mathbb{F}} f(x)dx \leq \int_{-\frac{\mathcal{L}(\mathbb{F})}{2}}^{\frac{\mathcal{L}(\mathbb{F})}{2}} f(x)dx \tag{3.77}$$

Since $f \succ g$, and let $\mathbb{F} \in \mathbb{R}$ be a Borel set with $\mathcal{L}(\mathbb{F}) < \infty$ we obtain that:

$$\int_{\mathbb{F}} g(x)dx \leq \int_{-\frac{\mathcal{L}(\mathbb{F})}{2}}^{\frac{\mathcal{L}(\mathbb{F})}{2}} f(x)dx \tag{3.78}$$

We need to analyze two cases. The first case is when $\mathcal{L}(\mathbb{F}) \leq \mathcal{L}(\mathbb{A}) = 2\tau$, which implies that $\frac{\mathcal{L}(\mathbb{F})}{2} \leq \tau$.

$$\begin{aligned}
&\int_{-\frac{\mathcal{L}(\mathbb{F})}{2}}^{\frac{\mathcal{L}(\mathbb{F})}{2}} f(x)dx \geq \int_{\mathbb{F}} g(x)dx \\
&= \int_{\mathbb{F}} h(x)g(x)dx + \int_{\mathbb{F}} (1-h(x))\, g(x)dx \\
&\geq \int_{\mathbb{F}} h(x)g(x)dx + \int_{\mathbb{F}} (1-p)\,(1-h(x))\, g(x)dx
\end{aligned} \tag{3.79}$$

The second case is when $\mathcal{L}(\mathbb{F}) \geq \mathcal{L}(\mathbb{A}) = 2\tau$, which implies that $\frac{\mathcal{L}(\mathbb{F})}{2} \geq \tau$.

$$\begin{aligned}
&\int_{-\frac{\mathcal{L}(\mathbb{F})}{2}}^{\frac{\mathcal{L}(\mathbb{F})}{2}} f(x)dx = \int_{-\tau}^{\tau} f(x)dx + \int_{-\frac{\mathcal{L}(\mathbb{F})}{2}}^{-\tau} f(x)dx \\
&+ \int_{\tau}^{\frac{\mathcal{L}(\mathbb{F})}{2}} f(x)dx \geq \int_{\mathbb{F}} h(x)g(x)dx + \int_{\mathbb{F}} (1-h(x))\, g(x)dx
\end{aligned}$$

We know that $\int_{-\tau}^{\tau} f(x)dx = 1-\kappa \geq \int_{\mathbb{F}} h(x)g(x)dx$. If we have that:

$$\int_{-\frac{\mathcal{L}(\mathbb{F})}{2}}^{-\tau} f(x)dx + \int_{\tau}^{\frac{\mathcal{L}(\mathbb{F})}{2}} f(x)dx \geq \int_{\mathbb{F}} (1-h(x))\, g(x)dx$$

It holds then that:

$$\begin{aligned}\int_{-\tau}^{\tau} f(x)dx + (1-p)\left(\int_{-\frac{\mathcal{L}(\mathbb{F})}{2}}^{-\tau} f(x)dx + \int_{\tau}^{\frac{\mathcal{L}(\mathbb{F})}{2}} f(x)dx\right) \\ \geq \int_{\mathbb{F}} h(x)g(x)dx + (1-p)\int_{\mathbb{F}} (1-h(x))\, g(x)dx\end{aligned} \tag{3.80}$$

If we have that:

$$\int_{-\frac{\mathcal{L}(\mathbb{F})}{2}}^{-\tau} f(x)dx + \int_{\tau}^{\frac{\mathcal{L}(\mathbb{F})}{2}} f(x)dx \leq \int_{\mathbb{F}} (1-h(x))\, g(x)dx$$

Then the following holds:

$$\begin{aligned}\int_{-\tau}^{\tau} f(x)dx - \int_{\mathbb{F}} h(x)g(x)dx &\geq \int_{\mathbb{F}} (1-h(x))\, g(x)dx \\ &- \left(\int_{-\frac{\mathcal{L}(\mathbb{F})}{2}}^{-\tau} f(x)dx + \int_{\tau}^{\frac{\mathcal{L}(\mathbb{F})}{2}} f(x)dx\right) \\ &\geq (1-p)\int_{\mathbb{F}} (1-h(x))\, g(x)dx \\ &- (1-p)\left(\int_{-\frac{\mathcal{L}(\mathbb{F})}{2}}^{-\tau} f(x)dx + \int_{\tau}^{\frac{\mathcal{L}(\mathbb{F})}{2}} f(x)dx\right)\end{aligned}$$

It follows that:

$$\begin{aligned}\int_{-\tau}^{\tau} f(x)dx + (1-p)\left(\int_{-\frac{\mathcal{L}(\mathbb{F})}{2}}^{-\tau} f(x)dx + \int_{\tau}^{\frac{\mathcal{L}(\mathbb{F})}{2}} f(x)dx\right) \\ \geq \int_{\mathbb{F}} h(x)g(x)dx + (1-p)\int_{\mathbb{F}} (1-h(x))\, g(x)dx\end{aligned} \tag{3.81}$$

Multiplying the inequalities (3.79), (3.81) and (3.81) by $\frac{1}{1-\kappa+(1-p)\kappa}$ and using Definition 3.32, we obtain that for any Borel set $\mathbb{F}$ with $\mathcal{L}(\mathbb{F})$, there exists a set, i.e. the interval $[-\frac{\mathcal{L}(\mathbb{F})}{2}, \frac{\mathcal{L}(\mathbb{F})}{2}]$, such that:

$$\int_{-\frac{\mathcal{L}(\mathbb{F})}{2}}^{\frac{\mathcal{L}(\mathbb{F})}{2}} f_{\mathbb{A}}^{p}(x) \geq \int_{F} \frac{g(x)h(x) + (1-p)(1-h(x))g(x)}{1-\kappa+(1-p)\kappa} dx \tag{3.82}$$

Equation (3.82) implies that:

$$f_{\mathbb{A}}^{p} \succ \frac{g \cdot h + (1-p)(1-h)\cdot g}{1-\kappa+(1-p)\kappa}$$

■

Our **proof strategy** is to show that for every pre-processor policy $\mathcal{P}_{0,T}$, there exists a symmetric path-dependent threshold policy $\mathcal{D}^o_{0,T}$ which does not underperform $\mathcal{P}_{0,T}$, when evaluated according to Problem 3.5. This fact, which we denote as **Fact B.1**, leads to the following conclusions:

- (**Fact B.2**): Lemma 3.1, in conjunction with Fact B.1, implies that an optimum for Problem 3.5 exists and that it is of the symmetric threshold type (Definition 3.24).
- (**Fact B.3**): From Fact B.2 and Proposition 3.1, we conclude that symmetric threshold policies (Definition 3.24) and Kalman-like estimators (Definition 3.20) are jointly optimal for Problem 3.5.

Proof of Theorem 3.4:

We prove Theorem 3.4 by induction. Let $\mathcal{P}_{0,T}$ be an arbitrary admissible policy, given in Definition 3.21, for this policy we will construct a symmetric path-dependent policy $\mathcal{D}^o_{0,T}$ given in Definition 3.25, which does not underperform $\mathcal{P}_{0,T}$. Hence it is enough to search the optimum admissible policy $\mathcal{P}^*_{0,T}$ for Problem 3.5 only on the set of symmetric path-dependent policies. The optimal symmetric path-dependent pre-processor policy is given in Lemma 3.1 and it is actually a symmetric path-independent policy.

Let T, the time horizon be equal to one. The pre-processor policy $\mathcal{P}_{0,1}$ defines the conditional probabilities $\varsigma_{1|0}$, $\varsigma_{1|1}$ from Definiton 3.28 and the conditional probability density functions $\gamma_{1|0}(y)$ and $\gamma_{1|1}(y)$ from Definition 3.27. The cost associated with the pre-processor policy $\mathcal{P}_{0,1}$ is given in Proposition 3.69:

$$\begin{aligned}&\mathcal{J}_{0,1}\left(a,\sigma_W^2,p,c,\mathcal{P}_{0,1}\right)\\&=\left(E_{\gamma_{1|1}}\left[\left(\mathbf{Y}_1-\hat{\mathbf{Y}}_1\right)^2\right]+\frac{c(1-p)\varsigma_{1|0}}{\varsigma_{1|0}+(1-\varsigma_{1|0})(1-p)}\right)\varsigma_{1|1}\\&+c\left(\varsigma_0-\left(\varsigma_{1|0}+(1-\varsigma_{1|0})(1-p)\right)\varsigma_0\right)\end{aligned}\tag{3.83}$$

We need to construct a path-depedent pre-processor policy, but since the time horizon T is equal to one, we just need to select one threshold. Hence, $\mathcal{D}^o_{0,1}$ is given in Definition 3.25, i.e. if $|y_1|<\mathcal{T}_{0,1}(l_0)$ transmit the erasure symbol, otherwise transmit the true value of the system $\mathbf{X}_1$. Notice that l_0 was set from the beginning to be equal to one, i.e. the estimator knows the value of x_0. The pre-processor policy $\mathcal{D}^o_{0,T}$ has associated the conditional probability density functions $\gamma^D_{1|0}(y)$ and $\gamma^D_{1|1}(y)$ from Definition 3.27 and the conditional probabilities $\varsigma^D_{1|0}$, $\varsigma^D_{1|1}$ from Definiton 3.28. Choose the threshold $\mathcal{T}_{0,1}(1)$ such that $\varsigma^D_{1|0}=\varsigma_{1|0}$. It follows immediately that $\varsigma^D_{1|1}=\varsigma_{1|1}$. The cost associated with $\mathcal{D}^o_{0,1}$ is given in Proposition 3.69:

$$\begin{aligned}&\mathcal{J}_{0,1}\left(a,\sigma_W^2,p,c,\mathcal{D}^o_{0,1}\right)\\&=\left(E_{\gamma^D_{1|1}}\left[\left(\mathbf{Y}_1-\hat{\mathbf{Y}}_1\right)^2\right]+\frac{c(1-p)\varsigma_{1|0}}{\varsigma_{1|0}+(1-\varsigma_{1|0})(1-p)}\right)\varsigma_{1|1}\\&+c\left(\varsigma_0-\left(\varsigma_{1|0}+(1-\varsigma_{1|0})(1-p)\right)\varsigma_0\right)\end{aligned}\tag{3.84}$$

We have used the fact that $\varsigma^D_{1|0}=\varsigma_{1|0}$ and $\varsigma^D_{1|1}=\varsigma_{1|1}$.

We notice that $\gamma^D_{1|0}(y)=\gamma_{1|0}(y)=\mathcal{N}_{\sigma_W^2}(y)$. From Lemma 3.2 it follows that $\gamma^D_{1|1}$ is neat and even and that $\gamma^D_{1|1}\succ\gamma_{1|1}$. Since $\gamma^D_{1|0}$ is neat and even, it follows that $\hat{\mathbf{Y}}_1=E_{\gamma^D_{1|1}}\left[\mathbf{Y}_1\right]=0$, and from Lemma 2.4 it follows that $E_{\gamma_{1|1}}\left[\left(\mathbf{Y}_1-\hat{\mathbf{Y}}_1\right)^2\right]\geq E_{\gamma^D_{1|1}}\left[\left(\mathbf{Y}_1-\hat{\mathbf{Y}}_1\right)^2\right]$, hence:

$$\mathcal{J}_{0,1}\left(a,\sigma_W^2,p,c,\mathcal{P}_{0,1}\right)\geq\mathcal{J}_{0,1}\left(a,\sigma_W^2,p,c,\mathcal{D}^o_{0,1}\right)$$

Hence for $T = 1$, for any pre-processor policy $\mathcal{P}_{0,1}$ there exists a symmetric path-dependent threshold policy $\mathcal{D}^o_{0,1}$, which does not underperform $\mathcal{P}_{0,1}$. It follows from Lemma 3.1 that there exists an optimal policy, which is a symmetric path-independent threshold policy.

Assume that for all time horizons in the set $\{1, \ldots, T-1\}$, the claim of Theorem 3.4 is true, i.e. Problem 3.5 has an optimal policy, which is a symmetric, path-independent threshold policy. We need to show the claim for the time horizon equal to T. Let $\mathcal{P}_{0,T}$ be an arbitrary policy, this policy defines the conditional probabilities $\varsigma_{k|k-1}$, $\varsigma_{k|k}$ from Definiton 3.28 and the conditional probability density functions $\gamma_{k|k-1}(y)$ and $\gamma_{k|k}(y)$ from Definition 3.27, for $k \in \{1, \ldots, T\}$. We need to construct the symmetric path-dependent threshold policy $\mathcal{D}^o_{0,T}$. First we choose $\{\mathcal{T}_k\}_{k=1}^T$ positive real numbers, and then we construct $\mathcal{D}^o_{0,T}$ as follows:

—— Description of Algorithm $\mathcal{D}^o_{m,T}$ ——

- **(Initial step)** Set $k = 0$, $l_0 = 1$ (i.e. $r_0 = 1$ and $c_0 = 1$) and transmit the current state which will be received by the estimator, i.e., $\hat{v}_0 = v_0 = x_0$ or equivalently set $y_0 = 0$.

- **(Step A)** Increase the time counter k by one. If $k > T$ holds then terminate, otherwise execute Step B.

- **(Step B)** If $|y_k| < \mathcal{T}_k$ holds then set $r_k = 0$, transmit the erasure symbol, i.e., $\hat{v}_k = v_k = \mathfrak{E}$, and return to Step A. If $|y_k| \geq \mathcal{T}_k$ and if $c_k = 0$, then $v_k = \mathfrak{E}$ and return to Step A, if $|y_k| \geq \mathcal{T}_k$ and if $c_k = 1$ hold then execute $\mathcal{S}^*_{k,T}$.

—— End of description of Algorithm $\mathcal{D}^o_{m,T}$ ——

We remind to the reader that $\mathcal{S}^*_{k,T}$ is the optimal symmetric threshold policy, if the initial time is k and the final time is T, or better said the time horizon is $T-k$. According to Lemma 3.1, $\mathcal{S}^*_{k,T}$ is path-independent, and by the induction step, this policy is optimal among all admissible policies. Hence the policy $\mathcal{D}^o_{0,T}$ can be described as follows, if $|\mathbf{Y}_k| \leq \mathcal{T}_k$ transmit the erasure symbol, if $|\mathbf{Y}_k| \geq \mathcal{T}_k$ send the true state of the process, if the channel sends the true state safely then adopt the optimal policy from that point on. The policy $\mathcal{D}^o_{0,T}$ defines the conditional probabilities $\varsigma^D_{k|k-1}$, $\varsigma^D_{k|k}$ from Definiton 3.28 and the conditional probability density functions $\gamma^D_{k|k-1}(y)$ and $\gamma^D_{k|k}(y)$ from Definition 3.27, for $k \in \{1,\dots,T\}$. Choose the threshold $\{\mathcal{T}_k\}_{k=1}^T$ such that $\varsigma^D_{k|k-1} = \varsigma_{k|k-1}$ for all $k \in \{1,\dots,T\}$, which implies that $\varsigma^D_{k|k} = \varsigma_{k|k}$ for all $k \in \{1,\dots,T\}$.

The cost associated with the policy $\mathcal{D}^o_{0,T}$ is given by Proposition 3.69:

$$\begin{aligned}
&\mathcal{J}_{0,T}\left(a,\sigma^2_W,p,c,\mathcal{D}^o_{0,T}\right)\\
&= \sum_{k=1}^{T}\left(\left(E_{\gamma^D_{k|k}}\left[\left(\mathbf{Y}_k-\hat{\mathbf{Y}}_k\right)^2\right] + \frac{c(1-p)\varsigma_{k|k-1}}{\varsigma_{k|k-1}+(1-\varsigma_{k|k-1})(1-p)}\right)\varsigma_{k|k}\right.\\
&\left.+\left(c+\mathcal{J}_{k,T}\left(a,\sigma^2_W,p,c,\mathcal{D}^o_{k,T}\right)\right)\left(\varsigma_{k-1}-\left(\varsigma_{k|k-1}+(1-\varsigma_{k|k-1})(1-p)\right)\varsigma_{k-1}\right)\right)\\
&= \sum_{k=1}^{T}\left(\left(E_{\gamma^D_{k|k}}\left[\left(\mathbf{Y}_k-\hat{\mathbf{Y}}_k\right)^2\right] + \frac{c(1-p)\varsigma_{k|k-1}}{\varsigma_{k|k-1}+(1-\varsigma_{k|k-1})(1-p)}\right)\varsigma_{k|k}\right.\\
&\left.+\left(c+\mathcal{J}^*_{k,T}\left(a,\sigma^2_W,p,c\right)\right)\left(\varsigma_{k-1}-\left(\varsigma_{k|k-1}+(1-\varsigma_{k|k-1})(1-p)\right)\varsigma_{k-1}\right)\right)
\end{aligned}$$

We notice that $\gamma^D_{1|0}(y) = \gamma_{1|0}(y) = \mathcal{N}_{\sigma^2_W}(y)$. From Lemma 3.2 it follows that $\gamma^D_{1|1}$ is neat and even and that $\gamma^D_{1|1} \succ \gamma_{1|1}$. From equation (3.65) and Lemma 2.3, we have that $\gamma^D_{2|1} \succ \gamma_{2|1}$, again using Lemma 3.2 it follows that $\gamma^D_{2|2}$ is neat and even and that $\gamma^D_{2|2} \succ \gamma_{2|2}$. By an induction argument we conclude that $\gamma^D_{k|k}$ is neat and even for all $k \in$

$\{1, \ldots, T\}$ and that $\gamma^D_{k|k} \succ \gamma_{k|k}$ for all $k \in \{1, \ldots, T\}$. It follows that $\hat{\mathbf{Y}}_k = E_{\gamma^D_{k|k}}[\mathbf{Y}_k] = 0$ for all $k \in \{1, \ldots, T\}$, and from Lemma 2.4 it follows that $E_{\gamma_{k|k}}\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}_k\right)^2\right] \geq E_{\gamma^D_{k|k}}\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}_k\right)^2\right]$ for all $k \in \{1, \ldots, T\}$. Using Corollary 3.2 we obtain that:

$$\mathcal{J}_{0,T}\left(a, \sigma^2_W, p, c, \mathcal{D}^o_{0,T}\right) \leq \mathcal{J}_{0,T}\left(a, \sigma^2_W, p, c, \mathcal{P}_{0,T}\right)$$

■

3.6 Infinite Horizon - Discounted Cost Problem

We will look now at the infinite horizon counterpart of Problem 2.1. For this we first extend naturally the definitions for the process $\mathbf{X}_k$ (Definition 3.1), the pre-processor (Definition 2.2), the pre-processor algorithms (Definitions 2.6, 2.8 and 2.10) by letting the time-horizon T go to infinity.

We define the following cost:

$$\mathcal{J}\left(a, \sigma^2_W, c, \mathcal{P}\right) \stackrel{def}{=} \sum_{k=1}^{\infty} d^{k-1} E\left[\left(\mathbf{X}_k - \hat{\mathbf{X}}_k\right)^2 + \underbrace{c\mathbf{R}_k}_{\text{communication cost}}\right] \tag{3.85}$$

We state now the infinite horizon counter part of Problem 2.1:

Problem 3.6 *Let a real constant a, the variance of the process noise σ^2_W and the initial condition x_0 be given. In addition, consider that a positive real c and a positive real number d less then one specifying the cost as in (3.85). We want to find an optimal solution $\mathcal{P}^*$ to the following optimization problem:*

$$\mathcal{P}^* = \textit{arg}\min_{\mathcal{P}} \mathcal{J}(a, \sigma^2_W, c, \mathcal{P}) \tag{3.86}$$

We define the optimal cost for the infinite horizon cost:

$$\mathcal{J}^* \left(a, \sigma_W^2, c\right) \stackrel{def}{=} \inf_{\mathcal{P}} \mathcal{J}(a, \sigma_W^2, c, \mathcal{P})$$

We state now the theorem, which solves Problem 3.6:

Theorem 3.5 *Let the parameters specifying Problem 3.6 be given, i.e., the variance of the process noise σ_W^2, the system's dynamic constant a, the communication cost c, the discount factor d and the time horizon T are pre-selected. There exists a positive real number τ and the sequence of positive real numbers $\tau^* = \{\tau_k^*\}_{k=1}^{\infty}$, with $\tau_k = \tau$ for all integers k, such that the corresponding symmetric threshold policy $\mathcal{S}_{0,\infty}^*$ is an optimal solution to (3.86) and the corresponding optimal estimator $\mathcal{E}(\mathcal{S}_{0,\infty}^*)$ is $\mathcal{Z}$. Here $\mathcal{S}_{0,T}^*$ and $\mathcal{Z}$ follow Definitions 2.9 and 2.5, respectively.*

Proof: For the infinite time horizon, choose a horizon T and adopt the following policy:

- if the current time t is less than T choose the optimal policy for time horizon T;
- if the current time t is greater than T choose to transmit the current state $\mathbf{X}_k$.

We note that for the infinite horizon case, this policy might not be optimal, hence we obtain the following inequalities:

$$\mathcal{J}_{0,T}^*(a, \sigma_W^2, c) \leq \mathcal{J}^*(a, \sigma_W^2, c) \leq \mathcal{J}_{0,T}^*(a, \sigma_W^2, c) + c\frac{d^{T-1}}{1-d} \tag{3.87}$$

Noting that $d < 1$, taking the limit as T goes to infinity it follows that:

$$\lim_{T \to \infty} \mathcal{J}_{0,T}^*(a, \sigma_W^2, c) = \mathcal{J}^*(a, \sigma_W^2, c) \tag{3.88}$$

Since there exists a symmetric threshold policy, which is optimal for the finite horizon case, it follows that the policy chosen to prove equations (3.87) and (3.88) is a threshold policy. Hence, for every positive number ϵ there exists a threshold policy applied to the estimation error which gives a cost less then $\mathcal{J}_d^*(a, \sigma_W^2, c) + \epsilon$. It is enough to show that it exists an optimal threshold policy for the infinite horizon case.

Just like in the finite horizon case, for the infinite horizon case, we can restrict the estimator to be linear estimator $\mathcal{E}$ given in Definition 2.5. The pre-processor can observe the estimation error $\mathbf{X}_k - a\mathbf{Z}_{k-1}$, which follows the dynamics given in Definition 2.7. The infinite horizon cost can be rewritten in terms of $\mathbf{Y}_k$ as follows:

$$\mathcal{J}\left(a, \sigma_W^2, c, \mathcal{P}\right) = \sum_{k=1}^{\infty} d^{k-1}\left(E\left[\mathbf{Y}_k^2 + c\mathbf{R}_k\right]\right) \tag{3.89}$$

Just like in the finite horizon case, we need to solve a Markov Decision Process problem, with the dynamics given in equation (2.14). We need to define the value function for this problem. We define the value function $\mathcal{V} : \mathbb{R} \to \mathbb{R}$, where for a real number y, $\mathcal{V}(y)$ is the cost-to-go when the initial estimation error is equal to y. It holds that:

$$\mathcal{V}_{0,T}(y) \leq \mathcal{V}(y) \leq \mathcal{V}_{0,T}(y) + c\frac{d^{T-1}}{1-d} \tag{3.90}$$

similar as in equation (3.87). It follows from equation (3.90), that:

$$\mathcal{V}(y) = \lim_{T\to\infty} \mathcal{V}_{0,T}(y) \tag{3.91}$$

Moreover the limit in equation (3.91) in uniform on y, which follows from equation (3.90). Hence the properties of $\mathcal{V}_{0,T}$ are inherited by $\mathcal{V}$, i.e. $\mathcal{V}$ is an even, bounded, continuous and quasiconcave function, and $\mathcal{V}$ satisfies the optimality equation:

$$\mathcal{V}(y) = \min\left(c + dE\left[\mathcal{V}\left(\mathbf{W}\right)\right], y^2 + E\left[\mathcal{V}\left(ay + \mathbf{W}\right)\right]\right) \tag{3.92}$$

From equation (3.92), it follows that there exist a unique threshold τ, which gives the optimal policy and τ is the solution of the equation:

$$y^2 + E\left[\mathcal{V}\left(ay + \mathbf{W}\right)\right] = c + dE\left[\mathcal{V}\left(\mathbf{W}\right)\right] \tag{3.93}$$

■

3.7 Infinite Horizon - Average Cost Problem

We will look now at the infinite horizon average counterpart of Problem 2.1. For this we first extend naturally the definitions for the process $\mathbf{X}_k$ (Definition 3.1), the pre-processor (Definition 2.2), the pre-processor algorithms (Definitions 2.6, 2.8 and 2.10) by letting the time-horizon T go to infinity.

For this we define the following cost:

$$\mathcal{J}_{avg}\left(a, \sigma_W^2, c, \mathcal{P}\right) \stackrel{def}{=} \limsup_{T\to\infty} \frac{\sum_{k=1}^{T} E\left[\left(\mathbf{X}_k - \hat{\mathbf{X}}_k\right)^2 + c\mathbf{R}_k\right]}{T} \tag{3.94}$$

We state now the infinite horizon average cost counter part of Problem 2.1:

Problem 3.7 *Let a real constant a, the variance of the process noise σ_W^2 and the initial condition x_0 be given. In addition, consider that a positive real c specifying the cost as in (3.94). We want to find an optimal solution $\mathcal{P}^*$ to the following optimization problem:*

$$\mathcal{P}^* = \textit{arg}\min_{\mathcal{P}} \mathcal{J}_{avg}(a, \sigma_W^2, c, \mathcal{P}) \tag{3.95}$$

We define the optimal cost for the infinite horizon average cost:

$$\mathcal{J}_{avg}^*\left(a, \sigma_W^2, c\right) \stackrel{def}{=} \inf_{\mathcal{P}} \mathcal{J}_{avg}(a, \sigma_W^2, c, \mathcal{P})$$

We state now the theorem, which solves Problem 3.7:

Theorem 3.6 *Let the parameters specifying Problem 3.7 be given, i.e., the variance of the process noise σ_W^2, the system's dynamic constant a, the communication cost c and the time horizon T are pre-selected. There exists a positive real number τ and the sequence of positive real numbers $\tau^* = \{\tau_k^*\}_{k=1}^{\infty}$, with $\tau_k = \tau$ for all integers k, such that the corresponding symmetric threshold policy $\mathcal{S}_{0,\infty}^*$ is an optimal solution to (3.95) and the corresponding optimal estimator $\mathcal{E}(\mathcal{S}_{0,\infty}^*)$ is $\mathcal{Z}$. Here $\mathcal{S}_{0,T}^*$ and $\mathcal{Z}$ follow Definitions 2.9 and 2.5, respectively.*

Proof: We can show that the optimal estimator is linear, using the same technique like in the proof of Theorem 2.1. Just like in the finite horizon case, we can restrict the estimator to be linear estimator $\mathcal{E}$ given in Definition 2.5. The pre-processor can observe the estimation error $\mathbf{X}_k - a\mathbf{Z}_{k-1}$, which follows the dynamics given in Definition 2.7. The infinite horizon cost can be rewritten in terms of $\mathbf{Y}_k$ as follows:

$$\mathcal{J}_{avg}(a, \sigma_W^2, c, \mathcal{P}) = \limsup_{T\to\infty} \frac{\sum_{k=1}^{T} \left(E\left[\mathbf{Y}_k^2 + c\mathbf{R}_k\right]\right)}{T} \tag{3.96}$$

Both in the finite horizon case and infinite horizon case (the discounted cost) we notice that, we always send a real number if $\mathbf{Y}_k^2 \geq c$. This follows from the dynamic programming equations. After the real number is sent by the pre-processor the process given in Definition 2.7 is reset to zero. Just analyzing the cost in (3.96), we notice that, the pre-processor needs to send if $\mathbf{Y}_k^2 > c$. We can restrict ourselves to the policies, for which the pre-processor will transmit the state of the process, if $\mathbf{Y}_k^2 > c$. Hence, at each time, if the pre-processor sends if $\mathbf{Y}_k^2 > c$, there exists $p > 0$ such that with a probability greater or equal to p, the pre-processor sends a real number the next time.

There are two cases to be analyzed. First case is when $-\sqrt{c} \leq \mathbf{Y}_k \leq \sqrt{c}$, then based on the policy adopted by the pre-processor the next state will be either $a\mathbf{Y}_k + \mathbf{W}_k$ or $\mathbf{W}_k$. We need to verify when $-\sqrt{c} \leq a\mathbf{Y}_k + \mathbf{W}_k \leq \sqrt{c}$. It follows that, if $\mathbf{W}_k \in (-\infty, -\sqrt{c}\,(1+|a|)) \cup (\sqrt{c}\,(1+|a|)\,, \infty)$ then $-\sqrt{c} \geq a\mathbf{Y}_k + \mathbf{W}_k$ or $a\mathbf{Y}_k + \mathbf{W}_k \geq \sqrt{c}$. Hence, the probability p can be taken to be $P\left(\mathbf{W}_k \in (-\infty, -\sqrt{c}\,(1+|a|)) \cup (\sqrt{c}\,(1+|a|)\,, \infty)\right)$.

The second case is when $|\mathbf{Y}_k| \geq \sqrt{c}$, then if $\mathbf{W}_k \in (-\infty, -\sqrt{c}) \cup (\sqrt{c}, \infty)$, $|\mathbf{Y}_{k+1}| \geq \sqrt{c}$, which implies that the pre-processor needs to send a real number. Hence we can take $p = P\left(\mathbf{W}_k \in (-\infty, -\sqrt{c}\,(1+|a|)) \cup (\sqrt{c}\,(1+|a|)\,, \infty)\right)$.

It follows that, the infinite horizon average cost problem has an optimal policy given by the dynamic programming inequality:

$$h(y) + J^*_{avg}(a, \sigma_W^2, c) \geq \min\left(c + E\left[h\left(\mathbf{W}\right)\right], y^2 + E\left[h\left(ay + \mathbf{W}\right)\right]\right) \tag{3.97}$$

where $h : \mathbb{R} \to \mathbb{R}$ is the value function, $\mathbf{W}$ is a generic random variable, Gaussian, zero mean with variance σ_W^2. Moreover, there exists an increasing subsequence $\{d_k\}_{k=1}^{\infty}$, such that, $0 < d_k < 1$, for all integers k and:

$$\lim_{k\to\infty} d_k = 1 \tag{3.98}$$

Let $\mathcal{V}_k$ be the value function of the infinite time horizon problem with discounted cost, given in Problem 3.6 and Theorem 3.5, with the dynamic's constant a, the communication cost c, the variance of the process noise σ_W^2 and discount factor d_k. Then the value function for the average cost problem is given by the following limit:

$$h(y) = \lim_{k\to\infty} (1 - d_k)\left(\mathcal{V}_k(y) - \mathcal{V}_k(0)\right) \tag{3.99}$$

From equation (3.99), we notice that the function $h(y)$ is even and quasiconvex, hence there exists a threshold τ such that:

$$y^2 + E\left[h\left(ay + \mathbf{W}\right)\right] \leq c + E\left[h\left(\mathbf{W}\right)\right] \iff |y| \leq \tau \tag{3.100}$$

■

3.8 Tandem Networks

We address the design of an optimal state estimation system featuring three blocks; two pre-processor $\mathcal{P}^1_{0,T}$ and $\mathcal{P}^2_{0,T}$ and an estimator $\mathcal{E}$. The pre-processor $\mathcal{P}^1_{0,T}$ has causal access to the state of a first order, linear and time-invariant system driven by Gaussian zero mean, white process noise and, at each time instant, it outputs an erasure symbol or a real number. The pre-processor $\mathcal{P}^2_{0,T}$ has causal access to the output of the pre-processor $\mathcal{P}^1_{0,T}$, and it outputs a real number or an erasure symbol. The estimator has causal access to the output of the pre-processor $\mathcal{P}^2_{0,T}$ and its output is denoted as state estimate. We consider an optimization problem characterized by cost functions that depends on both the state estimation error and the communication cost. In our formulation, the communication cost is a function of the output of the pre-processors, if erasure symbols are sent is assigned a zero cost and a pre-specified positive constants otherwise. In our formulation, the state processes, denoted as $\mathbf{X}_k$ is given and the three causal operators $\mathcal{P}^1_{0,T}$, $\mathcal{P}^2_{0,T}$ and $\mathcal{E}$ are to be jointly designed so as to minimize the given cost function.

Definition 3.33 ***(State Process)*** *Given a real constant a, a real number x_0 and a positive real constant σ^2_W, consider the following first order, linear, time-invariant and discrete-time system driven by process noise:*

$$\mathbf{X}_0 \stackrel{def}{=} x_0 \tag{3.101}$$

$$\mathbf{X}_{k+1} \stackrel{def}{=} a\mathbf{X}_k + \mathbf{W}_k,\ k \geq 0 \tag{3.102}$$

where $\{\mathbf{W}_k\}_{k=0}^{\infty}$ *is white, Gaussian and zero mean stochastic processes with variances* σ_W^2*. The filtration generated by* $\{\mathbf{X}_k\}_{k=0}^{\infty}$ *is denoted as*

$$\mathcal{X}_k \stackrel{def}{=} \sigma\left(\mathbf{X}_t; 0 \leq t \leq k\right) \tag{3.103}$$

where $\sigma\left(\mathbf{X}_t; 0 \leq t \leq k\right)$ *is the smallest sigma algebra generated by* $\mathbf{X}_t$ *for all integers* t*.*

Definition 3.34 ***(First pre-processor and first remote link process)*** *Consider an erasure symbol denoted as* $\mathfrak{E}$ *and a causal pre-processor* $\mathcal{P}_{0,T} : (x_0, \dots, x_k) \mapsto v_k^1$*, defined for* $k \in \{0, \dots, T\}$ *and* $v_k^1 \in \mathbb{R} \cup \{\mathfrak{E}\}$*. Hence, at each time instant* k*, the preprocessor outputs a real number or the erasure symbol, based on past observations of the state process. Notice that a pre-processor generates a stochastic process* $\{\mathbf{V}_k^1\}_{k=0}^{T}$ *via the application of the operator* $\mathcal{P}_{0,T}$ *to the process* $\{\mathbf{X}_k\}_{k=0}^{T}$*. The map* $\mathcal{P}_{0,T}$ *is a valid pre-processor if the following two conditions hold: (1) The pre-processor transmits the initial state* x_0 *at time zero, i.e.,* $v_0^1 = x_0$*. (2) The pre-processor is measurable in the sense that the process* $\{\mathbf{V}_k^1\}_{k=0}^{T}$ *is adapted to* $\mathcal{X}_k$*.*

The filtration generated by $\{\mathbf{V}_k^1\}_{k=0}^{T}$ *is denoted as* $\{\mathcal{B}_k^1\}_{k=0}^{T}$ *and it is obtained as:*

$$\mathcal{B}_k^1 \stackrel{def}{=} \sigma\left(\mathbf{V}_t^1; 0 \leq t \leq k\right) \tag{3.104}$$

where $\sigma\left(\mathbf{V}_t^1; 0 \leq t \leq k\right)$ *is the smallest sigma algebra generated by* $\{\mathbf{V}_t^1, 0 \leq t \leq k\}$*, for all non-negative integers* k*.*

Note: Definition 3.34 is the same as Definition 2.2.

Definition 3.35 ***(Second pre-processor and second remote link process)*** *Consider an erasure symbol denoted as $\mathfrak{E}$ and a causal map $\mathcal{P}^2_{0,T} : (v^1_0, \ldots, v^1_k) \mapsto v^2_k$, defined for $k \in \{0, \ldots, T\}$ and $v^1_i \in \mathbb{R} \cup \{\mathfrak{E}\}$ for $i \in \{0, \ldots, k\}$ and $v^2_k \in \mathbb{R} \cup \{\mathfrak{E}\}$. Hence, at each time instant k, $\mathcal{P}^2_{0,T}$ outputs a real number or the erasure symbol, based on past observations of the state process. $\mathcal{P}^2_{0,T}$ generates a stochastic process $\{\mathbf{V}^2_k\}^T_{k=0}$ via the application of the operator $\mathcal{P}^2_{0,T}$ to the process $\{\mathbf{V}^1_k\}^T_{k=0}$. The map $\mathcal{P}^2_{0,T}$ is a valid pre-processor if the following two conditions hold: (1) The pre-processor transmits the initial state v^1_0 at time zero, i.e., $v^2_0 = v^1_0$. (2) The pre-processor is measurable in the sense that the process $\{\mathbf{V}^2_k\}^T_{k=0}$ is adapted to $\mathcal{B}^1_k$.*

The filtration generated by $\{\mathbf{V}^2_k\}^T_{k=0}$ is denoted as $\{\mathcal{B}^2_k\}^T_{k=0}$ and it is obtained as:

$$\mathcal{B}^2_k \overset{def}{=} \sigma\left(\mathbf{V}^2_t; 0 \le t \le k\right) \tag{3.105}$$

where $\sigma\left(\mathbf{V}^2_t; 0 \le t \le k\right)$ is the smallest sigma algebra generated by $\{\mathbf{V}^2_t, 0 \le t \le k\}$, for all non-negative integers k.

Definition 3.36 ***(Optimal estimate and optimal estimator)*** *Given the pre-processors $\mathcal{P}^1_{0,T}$ and $\mathcal{P}^2_{0,T}$, we consider optimal estimators in the expected squared sense whose optimal estimate at time k is denoted as $\hat{\mathbf{X}}_k$ and is expressed as follows:*

$$\hat{x}_k \overset{def}{=} \begin{cases} E\left[\mathbf{X}_k | \{v^2_t\}^k_{t=0}\right] & \text{if } k \ge 1 \\ x_0 & \text{if } k = 0 \end{cases} \tag{3.106}$$

where $E\left[\mathbf{X}_k | \{v_t\}^k_{t=0}\right]$ represents the expectation of the state $\mathbf{X}_k$ conditioned on the observed current and past outputs of the second pre-processor $\{v^2_t\}^k_{t=0}$. We use $\mathcal{E}(\mathcal{P}^1_{0,T}, \mathcal{P}^2_{0,T})$

to denote the ***optimal estimator*** *associated with a given pre-processor policies* $\mathcal{P}^1_{0,T}$ *and* $\mathcal{P}^2_{0,T}$.

Notice that from Definitions 3.34 and 3.35 we assume that the pre-processors always transmits the initial state x_0*. Hence, the initial estimate is set to satisfy* $\hat{x}_0 = v^2_0 = v^1_0 = x_0$.

Remark 3.11 *It is important to note that the pre-processor* $\mathcal{P}^1_{0,T}$ *has more information than the estimator and the pre-processor* $\mathcal{P}^2_{0,T}$*, which implies that the pre-processor* $\mathcal{P}^1_{0,T}$ *can reproduce all computation performed at the estimator* $\mathcal{E}$ *and the pre-processor* $\mathcal{P}^2_{0,T}$.

We define the following cost:

Definition 3.37 ***(Cost function fininte time horizon)*** *Given measurable pre-processors (Definition 3.34 and 3.35) , an estimator (Definition 3.36), a real constant* a*, a positive real number* d *less than one, a positive real constant* σ^2_W*, a positive integer* T *and positive real numbers* c_1 *and* c_2*, we define:*

$$\mathcal{J}_{0,T}\left(a, \sigma^2_W, c_1, c_2, \mathcal{P}^1_{0,T}, \mathcal{P}^2_{0,T}\right) \stackrel{def}{=} \sum_{k=1}^{T} d^{k-1} E\left[\left(\mathbf{X}_k - \hat{\mathbf{X}}_k\right)^2 + c_1\mathbf{R}_k + c_2\mathbf{P}_k\right] \quad (3.107)$$

where $\mathbf{R}_k$ *and* $\mathbf{P}_k$ *are defined as follows:*

$$\mathbf{R}_k \stackrel{def}{=} \begin{cases} 0 & \textit{if } \mathbf{V}^1_k = \mathfrak{E} \\ 1 & \textit{otherwise} \end{cases}, \ k \geq 0 \quad (3.108)$$

$$\mathbf{P}_k \stackrel{def}{=} \begin{cases} 0 & \textit{if } \mathbf{V}^2_k = \mathfrak{E} \\ 1 & \textit{otherwise} \end{cases}, \ k \geq 0 \quad (3.109)$$

Problem 3.8 *Let a real constant a and the variance of the process noise σ_W^2, that completely specify the state process $\{\mathbf{X}_k\}_{k=0}^{\infty}$, be given. In addition, consider that positive reals c_1 and c_2, the integer T and a positive real number $d < 1$ are given. We want to find an optimal solution $\left(\mathcal{P}_{0,T}^{1*}, \mathcal{P}_{0,T}^{2*}\right)$ to the following optimization problem:*

$$\min_{\left(\mathcal{P}_{0,T}^{1}, \mathcal{P}_{0,T}^{2}\right)} \mathcal{J}_{0,T}\left(a, \sigma_W^2, c_1, c_2, \mathcal{P}_{0,T}^{1}, \mathcal{P}_{0,T}^{2}\right) \tag{3.110}$$

Definition 3.38 *Consider the map $\hat{\mathcal{P}}_{0,T}^{2} : (v_0^1, \ldots, v_k^1) \mapsto v_k^2$, with given by:*

$$\hat{\mathcal{P}}_{0,T}^{2} : (v_0^1, \ldots, v_k^1) \overset{def}{=} v_k^1 \tag{3.111}$$

Theorem 3.7 *Let the parameters specifying Problem 3.8 be given, i.e., the variance of the process noise σ_W^2, the system's dynamic constant a, the integer T, the communication costs c_1 and c_2, and the discount factor d are pre-selected. There exists a sequence of positive real numbers $\tau^* = \{\tau_k^*\}_{k=1}^{T}$, such that the corresponding symmetric threshold policy $\mathcal{S}_{0,T}^{*}$ and the $\hat{\mathcal{P}}_{0,T}^{2}$ is an optimal solution to (3.110) and the corresponding optimal estimator $\mathcal{E}(\mathcal{S}_{0,T}^{*}, \hat{\mathcal{P}}_{0,T}^{2})$ is $\mathcal{Z}$. Here $\mathcal{S}_{0,T}^{*}$, $\hat{\mathcal{P}}_{0,T}^{2}$ and $\mathcal{Z}$ follow Definitions 2.9, 3.38 and 2.5, respectively.*

Remark 3.12 *The second pre-processor just passes the information from the first pre-processor to the estimator and then Problem 3.8 reduces to Problem 2.1 with the communication cost $c_1 + c_2$.*

Proof: We will show that Remark 3.12 is true, by proving that for each pair of pre-processor policies $\left(\mathcal{P}_{0,T}^{1}, \mathcal{P}_{0,T}^{2}\right)$ there exists another pair $\left(\mathcal{S}_{0,T}^{1}, \hat{\mathcal{P}}_{0,T}^{2}\right)$ such that:

$$\mathcal{J}_{0,T}\left(a, \sigma_W^2, c_1, c_2, \mathcal{P}_{0,T}^{1}, \mathcal{P}_{0,T}^{2}\right) \geq \mathcal{J}_{0,T}\left(a, \sigma_W^2, c_1, c_2, \mathcal{S}_{0,T}^{1}, \hat{\mathcal{P}}_{0,T}^{2}\right) \tag{3.112}$$

First consider an arbitrary pair $\left(\mathcal{P}^1_{0,T}, \mathcal{P}^2_{0,T}\right)$. We can assume, using Remark 2.1, that whenever each of the pre-processor transmits a real number, they can transmit the entire history of the process $\mathbf{X}_k$ (for $\mathcal{P}^1_{0,T}$) or the process $\mathbf{V}^1_k$ (for $\mathcal{P}^2_{0,T}$).

We perform analysis on sample paths. Assume that there are cases when $\mathcal{P}^2_{0,T}$ transmits twice a real number (or more that twice), before $\mathcal{P}^1_{0,T}$ transmits a real number. Consider the policies, $\tilde{\mathcal{P}}^1_{0,T}$ and $\tilde{\mathcal{P}}^2_{0,T}$ and for the estimator we pick $\mathcal{E}(\mathcal{P}^1_{0,T}, \mathcal{P}^2_{0,T})$ (we do not pick $\mathcal{E}(\tilde{\mathcal{P}}^1_{0,T}, \tilde{\mathcal{P}}^2_{0,T})$, which is optimal for $(\tilde{\mathcal{P}}^1_{0,T}, \tilde{\mathcal{P}}^2_{0,T})$). We define $\tilde{\mathcal{P}}^2_{0,T}$ to perform exactly like $\mathcal{P}^2_{0,T}$, except for the cases where it transmits more consecutive numbers before consecutive numbers before $\mathcal{P}^1_{0,T}$ transmits. We restrict $\tilde{\mathcal{P}}^2_{0,T}$ to send only the first time $\mathcal{P}^2_{0,T}$ used to send. Notice from Remark 3.11 that $\mathcal{P}^1_{0,T}$ has all the information available to $\mathcal{P}^2_{0,T}$, hence the decision of $\mathcal{P}^1_{0,T}$ depends only on the process $\mathbf{X}_k$. We also adopt $\tilde{\mathcal{P}}^1_{0,T} = \mathcal{P}^1_{0,T}$. It is clear that in this case the output of the estimators in these two cases are the same, hence the estimation cost is the same, but the communication cost for the choice $(\tilde{\mathcal{P}}^1_{0,T}, \tilde{\mathcal{P}}^2_{0,T})$ is smaller, hence:

$$\mathcal{J}_{0,T}\left(a, \sigma^2_W, c_1, c_2, \mathcal{P}^1_{0,T}, \mathcal{P}^2_{0,T}\right) \geq \mathcal{J}_{0,T}\left(a, \sigma^2_W, c_1, c_2, \tilde{\mathcal{P}}^1_{0,T}, \tilde{\mathcal{P}}^2_{0,T}\right)$$

Hence, we can assume that $\mathcal{P}^2_{0,T}$ does not transmit, twice (or more) a real number before $\mathcal{P}^1_{0,T}$ transmits a real number.

Assume that there are cases when $\mathcal{P}^1_{0,T}$ transmits twice a real number (or more that twice), before $\mathcal{P}^2_{0,T}$ transmits a real number. Consider the policies, $\tilde{\mathcal{P}}^1_{0,T}$ and $\tilde{\mathcal{P}}^2_{0,T}$ and for the estimator we pick $\mathcal{E}(\mathcal{P}^1_{0,T}, \mathcal{P}^2_{0,T})$ (we do not pick $\mathcal{E}(\tilde{\mathcal{P}}^1_{0,T}, \tilde{\mathcal{P}}^2_{0,T})$, which is optimal for $(\tilde{\mathcal{P}}^1_{0,T}, \tilde{\mathcal{P}}^2_{0,T})$). We define $\tilde{\mathcal{P}}^1_{0,T}$ to perform exactly like $\mathcal{P}^1_{0,T}$, except for the cases where it transmits more consecutive numbers before $\mathcal{P}^2_{0,T}$ transmits. We allow $\tilde{\mathcal{P}}^1_{0,T}$ to transmit only the last time $\mathcal{P}^1_{0,T}$ was supposed to transmit. Notice from Remark 3.11 that $\mathcal{P}^1_{0,T}$ has all the information available to $\mathcal{P}^2_{0,T}$, hence $\mathcal{P}^1_{0,T}$ knows what $\mathcal{P}^2_{0,T}$ has to do, and we need to define $\tilde{\mathcal{P}}^2_{0,T}$ so it transmits at the same times as $\mathcal{P}^2_{0,T}$. In this case the output of the estimators are the same, hence the estimation cost is the same, but the communication cost for the choice $(\tilde{\mathcal{P}}^1_{0,T}, \tilde{\mathcal{P}}^2_{0,T})$ is smaller, hence:

$$\mathcal{J}_{0,T}\left(a, \sigma_W^2, c_1, c_2, \mathcal{P}^1_{0,T}, \mathcal{P}^2_{0,T}\right) \geq \mathcal{J}_{0,T}\left(a, \sigma_W^2, c_1, c_2, \tilde{\mathcal{P}}^1_{0,T}, \tilde{\mathcal{P}}^2_{0,T}\right)$$

We established the face that for each transmission of $\mathcal{P}^1_{0,T}$ there is a transmission from $\mathcal{P}^2_{0,T}$. We only need to establish the fact that they transmit in the same time. Clearly, because of the cases discussed above, $\mathcal{P}^1_{0,T}$ transmits before $\mathcal{P}^2_{0,T}$. Assume that at time k_1, $\mathcal{P}^1_{0,T}$ transmits, and the corresponding transmission at $\mathcal{P}^1_{0,T}$ takes place at $k_2 > k_1$. Let $\tilde{\mathcal{P}}^2_{0,T} = \hat{\mathcal{P}}^2_{0,T}$. Clearly, the communication costs are the same, but in the latter case the estimation error is smaller, hence:

$$\mathcal{J}_{0,T}\left(a, \sigma_W^2, c_1, c_2, \mathcal{P}^1_{0,T}, \mathcal{P}^2_{0,T}\right) \geq \mathcal{J}_{0,T}\left(a, \sigma_W^2, c_1, c_2, \mathcal{P}^1_{0,T}, \hat{\mathcal{P}}^2_{0,T}\right)$$

The result of Theorem 3.7 follows then from Theorem 2.1. ■

Chapter 4

Multidimensional Counterpart of Problem 2.1

4.1 Introduction

We address the design of an optimal state estimation system featuring two blocks; a pre-processor $\mathcal{P}$ and a remote estimator $\mathcal{E}$. The pre-processor has causal access to observation of the state of a first order, linear and time-invariant system driven by Gaussian zero mean, white process noise and, at each time instant, it outputs either an erasure symbol or a real finite dimensional vector. The estimator has causal access to the output of the pre-processor and its output is denoted as state estimate. We consider an optimization problem characterized by cost functions that depends on both the state estimation error and the communication cost. In our formulation, the communication cost is a function of the output of the pre-processor, where to the erasure symbol is assigned zero cost and a pre-specified positive constant otherwise. In our formulation, the state process, denoted as $\mathbf{X}_k$ and the two causal operators $\mathcal{P}$ and $\mathcal{E}$ are to be jointly designed so as to minimize the given cost function.

Remark 4.1 *We note that the problem described in this chapter is the multidimensional counterpart of Problem 2.1 presented in Chapter 2. Most of the definitions in this chapter are similar to the definitions from Chapter 2, but we will repeat them for clarity purposes.*

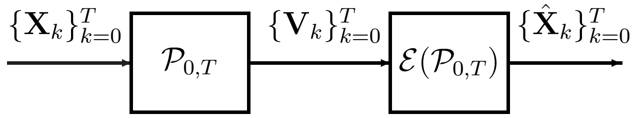

Figure 4.1: Schematic representation of the distributed estimation system considered in Problem 4.1, which is the multidimensional counterpart of Problem 2.1 in Chapter 2.

4.1.1 Preliminary Definitions and Information Pattern Description

We start by describing the three stochastic processes and the two classes of causal operators (pre-processor and estimator) that constitute our problem formulation.

Definition 4.1 ***(State Process)*** *Given a positive integer n greater or equal to two, a real square matrix $\mathbb{A}$ of dimension $n \times n$, and a positive definite matrix Σ_W of dimension $n \times n$, consider the following first order, linear time-invariant discrete-time system driven by process noise:*

$$\mathbf{X}_0 \overset{def}{=} x_0 \tag{4.1}$$

$$\mathbf{X}_{k+1} \overset{def}{=} \mathbb{A}\mathbf{X}_k + \mathbf{W}_k, \ k \geq 0 \tag{4.2}$$

where $\{\mathbf{W}_k\}_{k=0}^T$ is an independent identically distributed (i.i.d.) Gaussian zero mean stochastic process with variance Σ_W and x_0 is a real vector of dimension n. The filtration generated by $\{\mathbf{X}_k\}_{k=0}^T$ is denoted as:

$$\mathcal{X}_k \overset{def}{=} \sigma\left(\mathbf{X}_t; 0 \leq t \leq k\right) \tag{4.3}$$

where $\sigma\left(\mathbf{X}_t; 0 \leq t \leq k\right)$ is the smallest sigma algebra generated by $\{\mathbf{X}_t, 0 \leq t \leq k\}$, for all integers k.

Definition 4.2 *(**Pre-processor and remote link process**) Consider an erasure symbol denoted as $\mathfrak{E}$ and a causal map $\mathcal{P}_{0,T} : (x_0, \ldots, x_k) \mapsto v_k$ (pre-processor), defined for $k \in \{0, \ldots, T\}$ and $v_k \in \mathbb{R}^n \cup \{\mathfrak{E}\}$. Hence, at each time instant k, $\mathcal{P}_{0,T}$ outputs either a real number or the erasure symbol, based on past observations of the state process. The map $\mathcal{P}_{0,T}$ generates a stochastic process $\{\mathbf{V}_k\}_{k=0}^{T}$ via the application of the operator $\mathcal{P}_{0,T}$ to the process $\{\mathbf{X}_k\}_{k=0}^{T}$ (See Figure 4.1). The map $\mathcal{P}_{0,T}$ is a valid pre-processor if the following two conditions hold: (1) The pre-processor transmits the initial state x_0 at time zero, i.e., $\mathbf{V}_0 = x_0$. (2) $\mathcal{P}_{0,T}$ is measurable in the sense that the process $\{\mathbf{V}_k\}_{k=0}^{T}$ is adapted to $\mathcal{X}_k$.*

The filtration generated by $\{\mathbf{V}_k\}_{k=0}^{T}$ is denoted as $\{\mathcal{B}_k\}_{k=0}^{T}$ and it is obtained as:

$$\mathcal{B}_k \overset{def}{=} \sigma\left(\mathbf{V}_t; 0 \leq t \leq k\right) \tag{4.4}$$

Remark 4.2 *Notice that any finite vector of real numbers can be encoded into a single real vector of dimension n via a suitable invertible transformation. Hence, without loss of generality, we can also assume that the pre-processor can transmit either a vector of real numbers of dimension n or the erasure symbol.*

Definition 4.3 *(**Optimal estimate and optimal estimator**) Given a valid pre-processor $\mathcal{P}_{0,T}$, we consider optimal estimator in the expected squared sense whose optimal estimate at time k is denoted as $\hat{\mathbf{X}}_k$ and is expressed as follows:*

$$\hat{x}_k \overset{def}{=} \begin{cases} E\left[\mathbf{X}_k | \{v_t\}_{t=0}^{k}\right] & \textit{if } k \geq 1 \\ x_0 & \textit{if } k = 0 \end{cases} \tag{4.5}$$

where $E\left[\mathbf{X}_k | \{v_t\}_{t=0}^{k}\right]$ represents the expectation of the state $\mathbf{X}_k$ conditioned on the observed current and past outputs of the pre-processor $\{v_t\}_{t=0}^{k}$ (see Figure 4.1). We use $\mathcal{E}(\mathcal{P}_{0,T})$

*to denote the **optimal estimator** associated with a given pre-processor policy $\mathcal{P}_{0,T}$.*

Notice that from Definition 4.2 we assume that the pre-processor always transmits the initial state x_0. Hence, the initial estimate is set to satisfy $\hat{x}_0 = v_0 = x_0$. Such an assumption is a key element that will allow us to prove the optimality of a certain scheme, via an inductive method.

Remark 4.3 *Remark 2.2 is repeated here for emphasis. All the information available at the estimator $\mathcal{E}(\mathcal{P}_{0,T})$ is also available at the pre-processor $\mathcal{P}_{0,T}$. Hence, the pre-processor $\mathcal{P}_{0,T}$ can construct the state estimate $\hat{\mathbf{X}}_k$ by reproducing the estimation algorithm executed at the optimal estimator.*

Remark 4.4 *The definitions in this chapter are very similar to the definitions from Chapter 2. This is natural since in these definitions we made little or no use of the dimension of the system defined in equations 4.1 and 4.2.*

4.1.2 The Two Blocks Problem - The Multi Dimensional Case

In this subsection, we define the estimation paradigm that is central to this chapter. We start by specifying the cost, which is used as a merit criterion throughout the chapter, followed by the problem definition.

Definition 4.4 ***(Finite time horizon cost function)*** *Given a valid pre-processor $\mathcal{P}_{0,T}$ (Definition 4.2), a real square matrix $\mathbb{A}$ of dimension $n \times n$, a positive integer T, a positive real number $d < 1$ and positive definite real matrix Σ_W and a nonnegative real* *number c, we define:*

$$\mathcal{J}_{0,T}\left(\mathbb{A},\Sigma_W,c,\mathcal{P}_{0,T}\right)\overset{def}{=}\sum_{k=1}^{T}d^{k-1}E\left[\left(\mathbf{X}_k-\hat{\mathbf{X}}_k\right)^T\left(\mathbf{X}_k-\hat{\mathbf{X}}_k\right)+\underbrace{c\mathbf{R}_k}_{\textit{communication cost}}\right] \tag{4.6}$$

where $\mathbf{X}_k$ is the state of the system defined in (4.1)-(4.2), $\hat{\mathbf{X}}_k$ is the optimal estimate specified in Definition 4.3, and $\mathbf{R}_k$ is the following indicator function:

$$\mathbf{R}_k \overset{def}{=} \begin{cases} 0 & \textit{if } \mathbf{V}_k = \mathfrak{E} \\ 1 & \textit{otherwise} \end{cases}, \qquad k \geq 1 \tag{4.7}$$

Remark 4.5 ***(Cost does not depend on*** $\mathbf{X}_0$***)*** *This remark is similar with the Remark 2.3 from Chapter 2. Notice that because the plant (4.1)-(4.2) is linear, the equality $\hat{x}_0 = x_0$ (see Definition 4.3), implies, in view of Remark 4.3, in particular $\mathbb{A}$ is known at the estimator, that the homogenous part of the state can be reproduced at the estimator. Hence, the optimal estimator will incorporate such an homogeneous term, thus subtracting it out from the estimation error $\mathbf{X}_k - \hat{\mathbf{X}}_k$, for $k \geq 0$. This also implies that the cost (4.6) does not depend on the homogeneous term nor on the initial condition $\mathbf{X}_0$.*

The following is the main problem addressed in this chapter.

Problem 4.1 *Let be an integer n greater than one, real square matrix $\mathbb{A}$ of dimension $n \times n$, the variance of the process noise Σ_W and the initial condition x_0 be given. In addition, consider that a positive real c, a positive real number d less then one and a positive integer T are given, specifying the cost as in Definition 4.4. Find:*

$$\mathcal{P}^*_{0,T} \in \textit{arg}\min_{\mathcal{P}_{0,T}} \mathcal{J}_{0,T}(a,\sigma_W^2,c,\mathcal{P}_{0,T}) \tag{4.8}$$

4.2 Optimal Symmetric Solution to the Two Blocks Problem

In this section, we start by defining a particular choice of estimator (section 4.2.1) and pre-processor (section 4.2.3), which we denote as Kalman-like and symmetric policy, respectively. As we argue later on, in Conjecture 4.1, such estimator and pre-processor are optimal for Problem 4.1, if we restrict ourselves to the class of policies, to be defined in Section 4.2.3.

4.2.1 A Kalman-like estimator

Definition 4.5 ***(Kalman-like estimator)*** *Given the process defined in (4.1)-(4.2) and a pre-processor $\mathcal{P}_{0,T}$, define the map $\mathcal{Z} : (v_0, \ldots, v_k) \mapsto z_k$, for k in the set $\{0, \ldots, T\}$, where z_k is computed as follows:*

$$z_0 \overset{def}{=} x_0 \tag{4.9}$$

$$z_k \overset{def}{=} \begin{cases} \mathbb{A} z_{k-1} & \textit{if } v_k = \mathfrak{E} \\ \\ v_k & \textit{otherwise} \end{cases}, \textit{ with } k \geq 1 \tag{4.10}$$

Remark 4.6 *The Kalman-like filter generates the process $\{\mathbf{Z}_k\}_{k=0}^T$ via the operator $\mathcal{Z}$ applied to the process $\{\mathbf{V}_k\}_{k=0}^T$. Notice that the pre-processor has access to the estimate $\mathbf{Z}_k$ because it has access and full control of the input applied to $\mathcal{Z}$.*

4.2.2 The Set $\mathbb{P}_T$ - of Admissible Pre-Processors

We proceed by defining a class of admissible pre-processors, which is amenable to the use of recursive methods for performance analysis. We argue in Remark 4.8 that there always exist an admissible pre-processor that is an optimal solution to Problem 4.1. This implies that we incur no loss of generality in constraining our analysis to admissible pre-processors.

The following Remark provides an equivalent characterization of the class of admissible pre-processors.

Remark 4.7 *Let $T \in \mathbb{N}$ and let $\mathcal{P}_{0,T}$ be given. Then $\mathcal{P}_{0,T}$ is admissible if and only if for each $m \in \{0, \ldots, T\}$ there exists a map $\mathcal{P}_{m,T} : (x_m, \ldots, x_k) \mapsto v_k$ and a binary process $\{r_j\}_{j=m}^{k}$ such that the following holds:*

$$r_m = 1 \implies \mathcal{P}_{q,T}(x_q, \ldots, x_k) = \mathcal{P}_{m,T}(x_m, \ldots, x_k), \qquad x_q, \ldots, x_k \in \mathbb{R}^n, k > m \geq q \geq 0 \tag{4.11}$$

Given an admissible pre-processor $\mathcal{P}_{0,T}$, later on we will also refer to the time-restricted pre-processors $\{\mathcal{P}_{m,T}\}_{m=1}^{T}$ according to Definition 4.6, or equivalently as implied by (4.11).

Definition 4.6 ***(Admissible pre-processor)*** *Let a horizon T larger than zero and a pre-processor policy $\mathcal{P}_{0,T}$ be given. The pre-processor $\mathcal{P}_{0,T}$ is admissible if there exist maps $\mathcal{P}_{m,T} : (x_m, \ldots, x_k) \mapsto v_k$, with $0 \leq m \leq T$ and $k \geq m$ that satisfies the following recursion:*

$\mathcal{P}_{m,T}$

- ***(Initial step)*** *Set $k = m$, $r_m = 1$ and transmit the current state, i.e., $v_m = x_m$.*

- ***(Step A)*** *Set $k = k + 1$. If $k > T$ holds then terminate, otherwise execute Step B.*

- ***(Step B)*** *Obtain the pre-processor output at time k by computing $\mathcal{P}_{m,T}(x_m, \ldots, x_k)$. If $\mathcal{P}_{m,T}(x_m, \ldots, x_k) = \mathfrak{E}$ then set $r_k = 0$ and $v_k = \mathfrak{E}$ and go back to Step A. If $\mathcal{P}_{m,T}(x_m, \ldots, x_k) \neq \mathfrak{E}$ then execute algorithm $\mathcal{P}_{k,T}$.*

—————— *End of Algorithm for $\mathcal{P}_{m,T}$* ——————

The class of <u>all admissible</u> pre-processors is denoted as $\mathbb{P}_T$.

Remark 4.8 *Given a positive time-horizon T, there is no loss of generality in restricting our search for an optimal pre-processor to the set $\mathbb{P}_T$. Indeed, let an optimal pre-processor policy $\mathcal{P}^*_{0,T}$ be given. If a transmission takes place at some time m ($r_m = 1$ holds) then the optimal output at the pre-processor is $v_k = x_k$. Since, given that a real number is transmitted, the choice $v_k = x_k$ must be optimal because it leads to a perfect estimate $\hat{x}_m = x_m$. Hence, given that $r_m = 1$, by Markovianity we conclude that the current and future output produced by the pre-processor $\{\mathbf{V}_k\}_{k=m}^T$ will <u>not</u> depend on the state $\mathbf{X}_k$ for times k prior to m. Consequently, $\mathcal{P}^*_{0,T}$ satisfies (4.11), and hence it is admissible.*

4.2.3 Symmetric threshold pre-processor

Definition 4.7 *In order to simplify our notation, we define the following process:*

$$\mathbf{Y}_k \stackrel{def}{=} \mathbf{X}_k - \mathbb{A}\mathbf{Z}_{k-1} \tag{4.12}$$

Using Definitions 4.1 and 4.5, we find that $\{\mathbf{Y}_k\}_{k=0}^T$ can be rewritten as:

$$\mathbf{Y}_0 = 0 \tag{4.13}$$

$$\mathbf{Y}_{k+1} = \begin{cases} \mathbb{A}\mathbf{Y}_k + \mathbf{W}_k & \text{if } \mathbf{R}_k = 0 \\ \\ \mathbf{W}_k & \text{if } \mathbf{R}_k = 1 \end{cases} \tag{4.14}$$

Remark 4.9 *$\mathbf{Y}_k$ has an even probability density function, since $\mathbf{W}_k$ has an even probability density function. This fact makes $\{\mathbf{Y}_k\}_{k=0}^T$ a more convenient process to work with, in comparison to $\{\mathbf{X}_k\}_{k=0}^T$. This motivates its use in our analysis hereon, whenever possible. No loss of generality is incurred because $\{\mathbf{Y}_k\}_{k=0}^T$ can be recovered from $\{\mathbf{X}_k\}_{k=0}^T$, and vice-versa, via the use of $\{\mathbf{Z}_k\}_{k=0}^T$, which is common information at the pre-processor and estimator (See Remark 4.6). In addition, notice that the cost (4.6) can be re-written in terms of $\{\mathbf{Y}_k\}_{k=0}^T$ as follows:*

$$\mathcal{J}_{0,T}\left(\mathbb{A}, \Sigma_W, c, \mathcal{P}_{0,T}\right) \stackrel{def}{=} \sum_{k=1}^{T} d^{k-1} E\left[\left(\mathbf{Y}_k - \hat{\mathbf{Y}}_k\right)^T \left(\mathbf{Y}_k - \hat{\mathbf{Y}}_k\right) + c\mathbf{R}_k\right] \tag{4.15}$$

where $\hat{\mathbf{Y}}_k \stackrel{def}{=} E\left[\mathbf{Y}_k | \{\mathbf{V}_t\}_{t=0}^k\right]$. A key fact here is that $\hat{\mathbf{Y}}_k = \hat{\mathbf{X}}_k - \mathbb{A}\mathbf{Z}_{k-1}$ holds, leading to the validity of the identity $\mathbf{Y}_k - \hat{\mathbf{Y}}_k = \mathbf{X}_k - \hat{\mathbf{X}}_k$.

We found that solving Problem 4.1 is quite difficult, hence we will restrict the search for the optimal policies, only to a class of policies, which we will name symmetric policies. Towards defining the symmetric policies and the optimal solution within this class of policies we first define a class of sets and functions. We start by defining the star sets.

Definition 4.8 *Let $\mathbb{K} \subset \mathbb{R}^n$ be Lebesgue measurable. We say that $\mathbb{K}$ is a symmetric star set if the following hold:*

- *The set $\mathbb{K}$ is symmetric, i.e. if $x \in \mathbb{K}$, then $-x \in \mathbb{K}$;*
- *The set $\mathbb{K}$ is a star set, i.e. if $x \in \mathbb{K}$, then $\alpha x \in \mathbb{K}$, for every real number $\alpha \in [0, 1]$.*

Definition 4.9 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an even nonnegative function. We say that f is a star function if the level sets of f are star sets (Definition 4.8).*

Next, we extend the definition of central convex unimodal distribution from [38] to nonnegative functions.

Definition 4.10 *Let M be a positive real number. Let:*

$$\mathcal{C}_M \stackrel{def}{=} \{\alpha \mathbb{I}_{\mathbb{K}}, \alpha > 0, \mathbb{K} \subset \mathbb{R}^n, \textit{symmetric, compact and convex}, \textit{ such that } \alpha \mathcal{L}(\mathbb{K}) \leq M\} \tag{4.16}$$

where $\mathcal{L}(\mathbb{K})$ is the Lebegue measure $\mathbb{K}$. We say that a nonnegative function $f : \mathbb{R}^n \to \mathbb{R}$ is central convex unimodal if there exists $M > 0$, such that $f \in$ Co $\{\mathcal{C}_M\}$, where Co $\{\mathcal{C}_M\}$ is the closure ($\mathcal{L}(\mathbb{R}^n)$ topology) of the convex hull generated by $\mathcal{C}_M$.

We denote by $\mathcal{C}$ the set of central convex unimodal functions.

Remark 4.10 *We note that $\mathcal{C}_M$ in Definition 4.10 is the set of indicator functions of symmetric, compact and convex sets, scaled by positive real numbers such that if $f \in \mathcal{C}_M$, then it holds that $\int_{\mathbb{R}^n} f(x)dx \leq M$. It follows that if $f \in$ Co $\{\mathcal{C}_M\}$, then f can be approximated (in $\mathcal{L}(\mathbb{R}^n)$ sense) by linear positive combinations of indicator functions of symmetric, compact and convex sets. Moreover, it holds that $\int_{\mathbb{R}^n} f(x)dx \leq M$.*

Remark 4.11 *The central convex unimodal functions are star functions (see Definition 4.9).*

The following lemma from [38] will give some insight about the main problem from this chapter.

Note: By $f * g$ we mean the convolution between f and g.

Lemma 4.1 *Let f and g be two central convex unimodal functions, then $f * g$ is also central convex unimodal.*

Proof: Since f and g are central convex unimodal, it follows from Remark 4.10, that there exist M_1 and M_2 such that $\int_{\mathbb{R}^n} f(x)dx \leq M_1$ and $\int_{\mathbb{R}^n} g(x)dx \leq M_2$, then it follows that $f * g$ is well defined and $\int_{\mathbb{R}^n} f * g(x)dx \leq M_1 M_2$. Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be two symmetric, compact and convex sets. Let's assume that $f = \mathbb{I}_{\mathbb{K}_1}$ and $g = \mathbb{I}_{\mathbb{K}_2}$. It follows that $f * g$ is even and quasiconcave from [37]. Since an integrable, even and quasiconcave function can be approximated (in $\mathcal{L}(\mathbb{R}^n)$ sense) by a positive linear combination of convex, compact symmetric sets, the lemma is proved for this particular case.

Now let f and g be arbitrary central convex unimodal functions. Then f and g can be approximated by positive linear combinations of indicator functions of symmetric, compact and convex sets, then $f * g$ is a linear combination of integrable, even and quasiconvex functions, hence it is central convex unimodal. It follows that for general central convex unimodal functions f and g, $f * g$ is central convex unimodal.■

Next, we extend the definition of monotone unimodal distribution from [38] to non-negative functions.

Definition 4.11 *Let M be a positive real number. We define $\mathcal{CM}_M$ to be the set of functions as follows:*

$$\mathcal{CM}_M = \left\{ f : \mathbb{R}^n \to \mathbb{R},\ \textit{s.t. } f(x) \geq 0,\ \forall x \in \mathbb{R}^n,\ \int_{\mathbb{R}^n} f(x)dx \leq M \right\} \tag{4.17}$$

such that $\int_{\mathbb{D}+\alpha x} f(y)dy$ is non-increasing as a function of α, for $\alpha \in [0,\infty)$, for all $x \in \mathbb{R}^n$ and all sets $\mathbb{D} \subset \mathbb{R}^n$, where $\mathbb{D}$ is compact, convex and symmetric. If $f \in \mathcal{CM}_M$, for some real number M, then we say that f is monotone unimodal.

We denote by $\mathcal{CM}$ the set of monotone unimodal functions. The following result is from [38], which will be used later.

Lemma 4.2 *Let f be a monotone unimodal function and g be a central convex unimodal, then $f * g$ is a monotone unimodal function.*

We define yet another class of functions related to Definitions 4.10 and 4.11.

Definition 4.12 *We define $\mathcal{CL}$ to be the set of functions as follows:*

$$\mathcal{CL} = \{f : \mathbb{R}^n \to \mathbb{R},\ \textit{s.t.}\ f(x) \geq 0,\ \forall x \in \mathbb{R}^n\} \tag{4.18}$$

such that $\int_{-\tau}^{\tau} f(\alpha x + \beta y)d\beta$ is non-increasing as a function of α, for $\alpha \in [0,\infty)$, for all $\tau > 0$ and for all $x,\ y \in \mathbb{R}^n$.

Lemma 4.3 *It holds that $\mathcal{C} \subset \mathcal{CM} \subset CL$*

Proof See [38]. ■

We now define a class of symmetric pre-processor policies. We restrict the search to the optimal pre-processor policy to the class of policies given in Definition 4.14 .

Definition 4.13 ***(Algorithm** $\mathcal{D}_{m,T}$**)*** *Given a horizon T, consider a sequence of set functions $\mathcal{T} \overset{def}{=} \{\mathcal{T}_{m,k} | m < k \leq T, 1 \leq m \leq T\}$, with $\mathcal{T}_{m,k} : \{0,1\}^{m-k} \to \mathcal{B}(\mathbb{R}^n)$ such that $\mathcal{T}_{m,k}(r_m, \ldots, r_{k-1})$ is a symmetric set in $\mathbb{R}^n$ for all $r_m, \ldots, r_{k-1} \in \{0,1\}$, is given, where*

$\mathcal{B}(\mathbb{R}^n)$ denotes the Borel σ-algebra generated by $\mathbb{R}^n$. For every m in the set $\{1, \ldots, T\}$, we define the following algorithm, which we denote as $\mathcal{D}_{m,T}$:

———————— ***Algorithm*** $\mathcal{D}_{m,T}$ ————————

- ***(Initial step)*** *Set $k = m$, $r_m = 1$ and transmit the current state, i.e., $v_m = x_m$ or equivalently set $y_m = 0$.*

- ***(Step A)*** *Increase the time counter k by one. If $k > T$ holds then terminate, otherwise execute Step B.*

- ***(Step B)*** *If $y_k \in \mathcal{T}_{m,k}(r_m, \ldots, r_{k-1})$ (also $-y_k \in \mathcal{T}_{m,k}(r_m, \ldots, r_{k-1})$ from symmetry) holds then set $r_k = 0$, transmit the erasure symbol, i.e., $v_k = \mathfrak{E}$, and return to Step A. Otherwise execute $\mathcal{D}_{k,T}$.*

———————— ***End of Algorithm*** $\mathcal{D}_{m,T}$ ————————

Recall that r_0 through r_{k-1} represent past decisions by the pre-processor, where $r_k = 1$ indicates that the state is transmitted to the estimator at time k, while $r_k = 0$ implies that an erasure symbol was sent.

Definition 4.14 ***(Symmetric policy)*** *Given a horizon T, consider that a sequence of functions $\mathcal{T} \stackrel{def}{=} \{\mathcal{T}_{m,k} | m < k \leq T, 1 \leq m \leq T\}$, with $\mathcal{T}_{m,k} : \{0,1\}^{m-k} \to \mathcal{B}(\mathbb{R}^n)$, is given. The symmetric pre-processor associated with $\mathcal{T}$ is implemented via the <u>execution of the algorithm $\mathcal{D}_{0,T}$</u>, as specified in Definition 4.13. We denote such an admissible pre-processor as $\mathcal{D}_{0,T}$. We use $\mathbb{D}_{0,T}$ to denote the <u>entire class</u> of symmetric pre-processors with time horizon T.*

We will define a special class of the symmetric pre-processor policy, namely, the ones which are path-independent.

Definition 4.15 *Given a positive integer horizon T and an arbitrary sequence of symmetric star sets $\tau = \{\mathbb{T}_k\}_{k=1}^{T}, \ \mathbb{T}_k \in \mathcal{B}(\mathbb{R}^n)$, for each m in the set $\{0, \ldots, T\}$, we define the following algorithm for $k \geq m$, which we denote as $\mathcal{S}_{m,T}$:*

***Algorithm** $\mathcal{S}_{m,T}$*

- ***(Initial step)*** *Set $k = m$, $r_m = 1$ and transmit the current state, i.e., $v_m = x_m$ or equivalently set $y_m = 0$.*
- ***(Step A)*** *Increase the time counter k by one. If $k > T$ holds then terminate, otherwise execute Step B.*
- ***(Step B)*** *If $y_k \in \mathbb{T}_k$(also $-y_k \in \mathbb{T}_k$) holds then set $r_k = 0$, transmit the erasure symbol, i.e., $v_k = \mathfrak{E}$, and return to Step A. If $y_k \notin \mathbb{T}_k$ holds then set $m = k$ and execute $\mathcal{S}_{m,T}$.*

***End of Algorithm** $\mathcal{S}_{m,T}$*

We note from Definitions 4.13 and 4.15 that the differences between the Algorithms $\mathcal{D}_{m,T}$ and $\mathcal{S}_{m,T}$ are the facts that in Definition 4.15, the sets $\mathbb{T}_k$ do not depend on the past decisions and are symmetric star sets.

Definition 4.16 ***(Symmetric threshold policy)*** *The algorithm $\mathcal{S}_{0,T}$, as in Definition 4.15, is denoted as symmetric threshold pre-processor and the class of all symmetric threshold policies is denoted as $\mathbb{S}_T$.*

The Problem 4.1 proved to be quite difficult to solve even if we restrict the search of the optimal policies to the symmetric policies. since we do not know how to prove the general result, we just state a conjecture, which we believe to be correct. We were able to prove the conjecture only for $T \in \{1, 2\}$

Conjecture 4.1 *Let the dimension n, the variance of the process noise Σ_W, the system's dynamic matrix $\mathbb{A}$, the communication cost c, the discount factor d and the time horizon T be given. There exists a sequence of star sets $\tau^* = \{\mathbb{T}_k^*\}_{k=1}^T$, such that the corresponding symmetric threshold policy $\mathcal{S}_{0,T}^*$ is an optimal solution to:*

$$\mathcal{S}_{0,T}^* \in \textit{arg} \min_{\mathcal{P}_{0,T} \in \mathbb{D}_{0,T}} \mathcal{J}_{0,T}(\mathbb{A}, \Sigma_W, c, \mathcal{P}_{0,T}) \tag{4.19}$$

and the corresponding optimal estimator $\mathcal{E}(\mathcal{S}_{0,T}^*)$ is $\mathcal{Z}$. Here $\mathcal{S}_{0,T}^*$ and $\mathcal{Z}$ follow Definitions 4.15 and 4.5, respectively.

4.3 Auxiliary optimality results

Proposition 4.1 *Let $\mathcal{D}_{0,T}$ be a pre-selected path-dependent symmetric threshold policy (Definition 4.14), it holds that the optimal estimator $\mathcal{E}(\mathcal{D}_{0,T})$ is $\mathcal{Z}$, as described in Definition 4.5.*

Remark 4.12 *Proposition 4.1 could be recast by stating that $\hat{\mathbf{X}}_k = \mathbf{Z}_k$ holds in the presence of path-dependent symmetric threshold pre-processors.*

Proof: (of Proposition 4.1) In order to simplify the proof, we define $\{\tilde{\mathbf{X}}_k\}_{k=0}^T$ as the process quantifying the error incurred by adopting a Kalman-like estimator $\mathcal{Z}$ (See

Definition 4.5), i.e., $\tilde{\mathbf{X}}_k \stackrel{def}{=} \mathbf{X}_k - \mathbf{Z}_k$. More specifically, $\{\tilde{\mathbf{X}}_k\}_{k=0}^{T}$ can be equivalently expressed as follows:

$$\tilde{\mathbf{X}}_0 = 0 \tag{4.20}$$

$$\tilde{\mathbf{X}}_{k+1} = \begin{cases} \mathbb{A}\tilde{\mathbf{X}}_k + \mathbf{W}_k & \text{if } \mathbf{R}_k = 0 \\ \\ 0 & \text{if } \mathbf{R}_k = 1 \end{cases}, \qquad 0 \le k \le T-1 \tag{4.21}$$

The proof follows from the symmetry of all probability density functions involving $\tilde{\mathbf{X}}_k$ and $\mathbf{V}_k$. More specifically, under symmetric policies the probability density function of $\tilde{\mathbf{X}}_k$, given the past and current observations $\{\mathbf{V}_t\}_{t=0}^{k}$, is even. Hence, we conclude that $E[\tilde{\mathbf{X}}_k|\{\mathbf{V}_t\}_{t=0}^{k}] = 0$, which implies that $\hat{\mathbf{X}}_k \stackrel{def}{=} E[\mathbf{X}_k|\{\mathbf{V}_t\}_{t=0}^{k}] = \mathbf{Z}_k$. ■

4.3.1 Optimizing within the class $\mathbb{D}_T$

Remark 4.13 *If $\mathcal{D}_{0,T}$ is a symmetric path-dependent threshold pre-processor (see Definition 4.14) then $\hat{\mathbf{Y}}_k = 0$ holds, leading to the following equality:*

$$\mathcal{J}_{0,T}\left(\mathbb{A}, \Sigma_W, c, \mathcal{D}_{0,T}\right) = \sum_{k=1}^{T} d^{k-1} E\left[\mathbf{Y}_k^T \mathbf{Y}_k + c\mathbf{R}_k\right], \qquad \mathcal{D}_{0,T} \in \mathbb{D}_T \tag{4.22}$$

The process defined in (4.14) is a Markov Decision Process (MDP) whose state and control are $\mathbf{Y}_k$ and $\mathbf{R}_k$, respectively. Hence the minimization of (4.22) with respect to pre-processor policies $\mathcal{D}_{0,T}$ in the class $\mathbb{D}_T$ can be cast as a dynamic program [13]. To do so, we define the sequence of functions $\mathcal{V}_{t,T} : \mathbb{R}^n \to \mathbb{R}, t \in \{1, \ldots, T+1\}$ which represent the cost-to-go as observed by the pre-processor. Here T represents the horizon, while t denotes the time at which the decision was taken, and the argument of the function is the MDP state $\mathbf{Y}_t$. In order to simplify our notation, we adopt the convention that

$\mathcal{V}_{T+1,T}(y_{T+1}) \stackrel{def}{=} 0$, $y_{T+1} \in \mathbb{R}^n$. Using dynamic programming, we can find the following recursive equations for $\mathcal{V}_{t,T}(y_t),\ t \in \{1, \ldots, T\}$:

$$\mathcal{V}_{t,T}(y_t) \stackrel{def}{=} \min_{r_t \in \{0,1\}} \mathcal{C}_{t,T}(y_t, r_t),\ t \in \{1, \ldots, T\} \tag{4.23}$$

where $\mathcal{C}_{t,T} : \mathbb{R}^n \times \{0, 1\} \to \mathbb{R}$ is defined as:

$$\mathcal{C}_{t,T}(y_t, r_t) \stackrel{def}{=} \begin{cases} c + dE\left[\mathcal{V}_{t+1,T}(\mathbf{W}_t)\right] & \text{if } r_t = 1 \\ \\ y_t^T y_t + dE\left[\mathcal{V}_{t+1,T}\left(\mathbb{A}y_t + \mathbf{W}_t\right)\right] & \text{if } r_t = 0 \end{cases} \tag{4.24}$$

From (4.24) it immediately follows that an optimal decision policy r_t^* at any time t is given by:

$$r_t^* = \begin{cases} 1 & \text{if } \mathcal{C}_{t,T}(y_t, 1) \leq \mathcal{C}_{t,T}(y_t, 0) \\ \\ 0 & \text{if } \mathcal{C}_{t,T}(y_t, 0) < \mathcal{C}_{t,T}(y_t, 1) \end{cases} \tag{4.25}$$

Using the MDP given in Definition 4.7 and the value functions from equation (4.23), we discuss the following Remark, which states that if Conjecture 4.1 is true, *within the class of symmetric pre-processors* $\mathbb{D}_T$ *(Definition 4.14)*, there exists an optimal path-*independent* symmetric threshold policy $\mathcal{S}_{0,T}^*$ (Definition 4.16) for Problem 4.1.

Remark 4.14 *Let the dimension of the system* n*, the variance of the process noise* Σ_W*, the system's dynamic matrix* $\mathbb{A}$*, the communication cost* c*, the discount factor* d *and the time horizon* T *be given. Consider Problem 4.1 with the additional constraint that the pre-processor must be of the symmetric type* $\mathbb{D}_T$ *specified in Definition 4.14. If Conjecture 4.1 is correct, then there exists an optimal path*-independent *symmetric threshold policy* $\mathcal{S}_{0,T}^*$*, as given in Definition 4.16, and the associated star sets* $\{\mathbb{T}_k^*\}_{k=1}^T$ *have the boundaries* *given by the solution to the following equations:*

$$\mathcal{C}_{t,T}(y_t, 0) = \mathcal{C}_{t,T}(y_t, 1),\ t \in \{1, \dots, T\} \tag{4.26}$$

From (4.25), we conclude that in order to show that Conjecture 4.1 is true, we only need to show that there exist symmetric star sets $\{\mathbb{T}_k^*\}_{k=1}^T$ for which the following equivalences hold:

$$y_t \notin \mathbb{T}_t^* \Longleftrightarrow \mathcal{C}_{t,T}(y_t, 1) \leq \mathcal{C}_{t,T}(y_t, 0), \qquad t \in \{1, \dots, T\} \tag{4.27}$$

Indeed, if (4.27) holds then the optimal strategy in (4.25) can be implemented via a threshold policy. Similar to the scalar case from Chapter 2, we will use the following facts (A.1 thorugh A.4):

- **(Fact A.1)**: For every t in the set $\{1, \dots, T\}$, $\mathcal{C}_{t,T}(y_t, 1)$ depends only on t, i.e., it is a time-dependent constant independent of y_t.
- **(Fact A.2)**: It holds that $\mathcal{C}_{t,T}(0, 0) < \mathcal{C}_{t,T}(y_t, 1)$ for $y_t \in \mathbb{R}^n$.
- **(Fact A.3)**: For every t in the set $\{1, \dots, T\}$ there exists a symmetric star set $\mathbb{U}_t$ such that $\mathcal{C}_{t,T}(y_t, 0) > \mathcal{C}_{t,T}(y_t, 1)$ and $\mathcal{C}_{t,T}(-y_t, 0) > \mathcal{C}_{t,T}(-y_t, 1)$ hold for every y_t satisfying $y_t \notin \mathbb{U}_t$.
- **(Fact A.4)**: It holds that for every positive constant M, the function $M - \min\left(M, \mathcal{C}_{t,T}(y_t, 0)\right)$ is a continuous, and belongs to the function set $\mathcal{CL}$, given in Definition 4.12 , for every set t in the set $\{1, \dots, T\}$.

Facts A.1 and A.2 follow directly from (4.24), while Fact A.3 follows from Fact A.4. The only difficulty is to prove Fact A.4, which we will discuss later. At this point we assume that Fact A.4 is valid, and we proceed by noticing that continuity of $\mathcal{C}_{t,T}(y_t, 0)$ with respect to y_t, as well as Facts A.2 and A.3, imply that the equations in (4.27) have at least one solution $\{\mathbb{T}_k^*\}_{k=1}^T$. Moreover, from Facts A.1 through A.4 we can conclude that such a solution $\{\mathbb{T}_k^*\}_{k=1}^T$ guarantees that (4.27) is true.

(Discussion of Fact 4) Since $y_t^T y_y$ is an even, convex, unbounded and continuous function of y_t, from (2.20) we conclude that it suffices to prove by induction that $\mathcal{V}_{t,T}(y_t)$ is even bounded, continuous and belong to the set of functions $\mathcal{CL}$, for each t in the set $\{1, \ldots, T\}$.

Since $\mathcal{V}_{T+1,T}(y_{T+1}) = 0$ holds by convention, the following is true:

$$\mathcal{V}_{T,T}(y_T) = \min\left(c, y_T^T y_T\right), \qquad y_T \in \mathbb{R}$$

From equation (4.24), it follows that $\mathcal{C}_{T,T} = y_T^T y_T$. Hence Fact A.4 holds trivially. It follows then, that $\mathcal{V}_{T,T}(y_T)$ is an even, quasiconvex, bounded and continuous function of y_T. It follows that the function:

$$g\left(y_T\right) = c - \mathcal{V}_{T,T}(y_T)$$

is an even, continuous, bounded and quasiconcave function and has a compact support (which implies that it is integrable). It follows from Lemma 4.1 that the function:

$$E\left[g\left(\mathbb{A}y_{T-1} + \mathbf{W}_{T-1}\right)\right] = c - E\left[\mathcal{V}_{T,T}\left(\mathbb{A}y_{T-1} + \mathbf{W}_{T-1}\right)\right]$$

is an even, continuous, bounded and monotone unimodal function. It follows that the decision set for:

$$\min\left(c + E\left[\mathcal{V}_{T,T}\left(\mathbf{W}_{T-1}\right)\right], y_{T-1}^T y_{T-1} + E\left[\mathcal{V}_{T,T}\left(\mathbb{A}y_{T-1} + \mathbf{W}_{T-1}\right)\right]\right) \qquad (4.28)$$

is a unique star set. We notice that the expression in equation (4.28) is in fact $\mathcal{V}_{T-1,T}\left(y_{T-1}\right)$. The question is, whether $\mathcal{V}_{T-1,T}\left(y_{T-1}\right)$ is monotone unimodal. If this is the case then we can conclude that the decision set:

$$\min\left(c + E\left[\mathcal{V}_{T-1,T}\left(\mathbf{W}_{T-1}\right)\right], y_{T-2}^{T}y_{T-2} + E\left[\mathcal{V}_{T-1,T}\left(\mathbb{A}y_{T-2} + \mathbf{W}_{T-1}\right)\right]\right) \tag{4.29}$$

is a symmetric star set.

We were able to prove that the functions $\mathcal{V}_{t,T}$, for $t \in \{1, T+1\}$ are monotone unimodal if it holds that, for any monotone unimodal function f:

$$g(x) = \max(f(x), C) - C \tag{4.30}$$

is monotone unimodal, for any positive real number C. This latter question, however, is open.

4.4 Decision Sets Need NOT Be Convex

The results from Chapter 2 tell that for the scalar problem, the decision sets are symmetric intervals. One immediate thought would be to check if the decision sets in the multidimensional case are symmetric convex sets. In [41], the author investigates the policies associated to Problem 4.1, by looking only at symmetric convex sets. In this section, we present a simple numerical example where we show that the symmetric convex sets are not optimal for Problem 4.1, even if we restrict ourselves only to symmetric policies.

Let $\Sigma_W = \mathbb{I}$, $\mathbb{A} = diag(1, 7)$, $T = 2$ and $d = 0.99$.

Notice that in the provious section, we have already proved the for $T = 2$, the decision sets are star sets. We will show here, in the figures below the value function $\mathcal{V}_{1,2}$ and the decision set for $T = 2$.

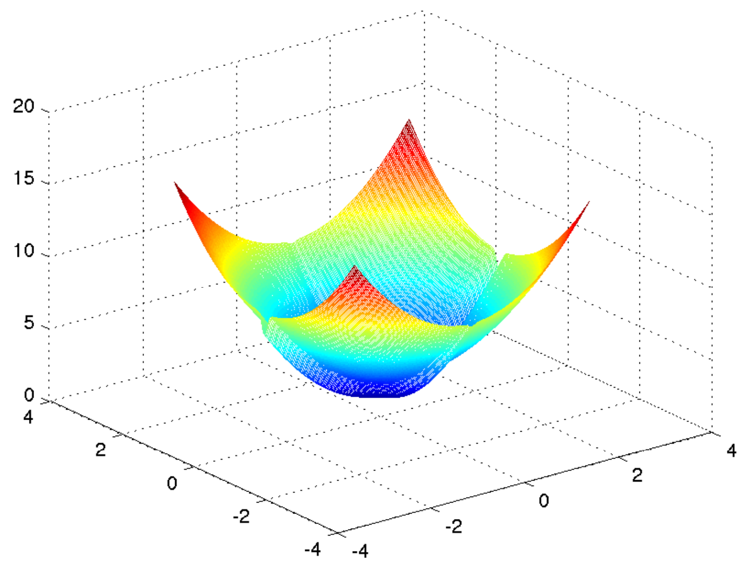

Figure 4.2: The value function $\mathcal{V}_{1,2}$ on the set $[-2.5, 2.5] \times [-2.5, 2.5]$

In the figures below, we show the decision set at time $t = 1$.

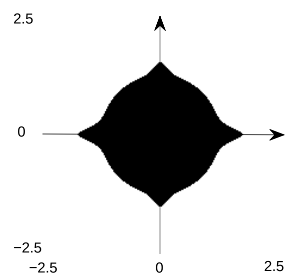

Figure 4.3: The set $\mathbb{T}_1$ on $[-2.5, 2.5] \times [-2.5, 2.5]$

It can be clearly seen, especially in Figure 4.3 that the set $\mathbb{T}_1$ is not a convex set.

Chapter 5

Basic Network Topologies with Noisy Transmission Links

5.1 Introduction

In this chapter, we will approach an estimation problem with communication cost, but the cost will be different than in the previous chapters. In chapters 2, 3 and 4, the cost is taken to be a positive constant, while in this chapter, the communication costs will have the meaning of transmission power and will be the second moment of the random variables, which represent the signal send over a communication link. Moreover, we will study systems, which consists from more than two agents and we will analyze how this fact will affect the optimal policies and we will also look at transmission noise which will affect the communication. In this chapter, we investigate control strategies for a scalar, one-step delay system in discrete time, i.e., the state of the system is the input delayed by one time unit. In contrast with classical approaches, here the control action must be a memoryless function of the output of the plant, which consists the current state corrupted by measurement noise. We adopt a first order state-space representation for the delay system, where the initial state is a Gaussian random variable. In addition, we assume that the measurement noise is drawn from a white and Gaussian process with zero mean and constant variance. Performance evaluation is carried out via a finite-time quadratic cost that combines the second moment of the control signal, and the second moment of the difference between the initial state and the state at the final time. We show that if the time-horizon is one or two then the optimal control is a linear function of the plant's output, while for a sufficiently large horizon a control taking on only two values will outperform the optimal affine solution.

Consider the following discrete-time delay system:

$$X(k+1) = U(k), \qquad k \geq 0 \tag{5.1}$$

$$Y(k) = X(k) + V(k), \qquad k \geq 0 \tag{5.2}$$

where $V(k)$, $U(k)$, $X(k)$, and $Y(k)$ take values on the reals, and they represent the measurement noise, input, state, and output of the plant, respectively. In addition, we assume that the initial state $X(0)$ is a Gaussian random variable, with zero mean and variance σ_0^2. The measurement noise $\{V(k)\}_{k=0}^{\infty}$ is white, Gaussian, zero mean and with constant variance given by σ_V^2. We also assume that the noise $\{V(k)\}_{k=0}^{\infty}$ and $X(0)$ are mutually independent. In this chapter, we will investigate the following problem:

Problem 5.1 *Let σ_0^2 and σ_V^2 be pre-selected positive constants representing the variance of $X(0)$ and $V(k)$, for all $k \in \{0, \ldots, m-1\}$ and m be a given integer denoting the length of an optimization horizon. Consider that the system described by (5.1)-(5.2) accepts a control strategy of the following form:*

$$U(k) = \mathcal{F}_k(Y(k)),\ k \in \{0, \ldots, m-1\} \tag{5.3}$$

where, for each k in the set $\{0, \ldots, m-1\}$, $\mathcal{F}_k : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function. Given a positive real parameter ϱ, we wish to determine Lebesgue measurable *functions* $\{\mathcal{F}_k\}_{k=0}^{m-1}$ *that minimize the following cost:*

$$\begin{aligned} \mathcal{J}(\{\mathcal{F}_k\}_{k=0}^{m-1}, \varrho, \sigma_0^2, \sigma_V^2) \overset{def}{=} & \ E[(X(m) - X(0))^2] \\ & + \varrho \sum_{k=0}^{m-2} E[U(k)^2] \end{aligned} \tag{5.4}$$

In Figure 5.1, we present a graphic interpretation of Problem 5.1. Notice that Problem 5.1 can be viewed as an optimal control problem aimed at the design of a memory element capable of storing $X(0)$. The memory element must be constructed using a one-step delay and memoryless components $\{\mathcal{F}_k\}_{k=0}^{m-1}$, which are used in a feedback configuration. In addition, the memoryless control has access to noisy measurements of the delay's state. Minimizing the cost function defined in (5.4) amounts to finding the minimal energy memoryless control that leads to the optimal recovery of $X(0)$ from $Y(m-1)$, in a mean square sense.

The following is the organization of this chapter (introduction not included):

- In **Section 5.2**, we derive the optimal solution to Problem 5.1, subject to the constraint that the feedback maps $\{\mathcal{F}_k\}_{k=0}^{m-1}$ are affine. We also show that if m is one or two then affine solutions are optimal over all feedback maps.

- In **Section 5.3**, we adopt a class of functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ that take on only two values for each step k. Given σ_0^2 and σ_V^2, we show that there exists m for which the two valued strategy outperforms the optimal affine control and we provide numerical examples, and in the end we discuss conclusions and open issues.

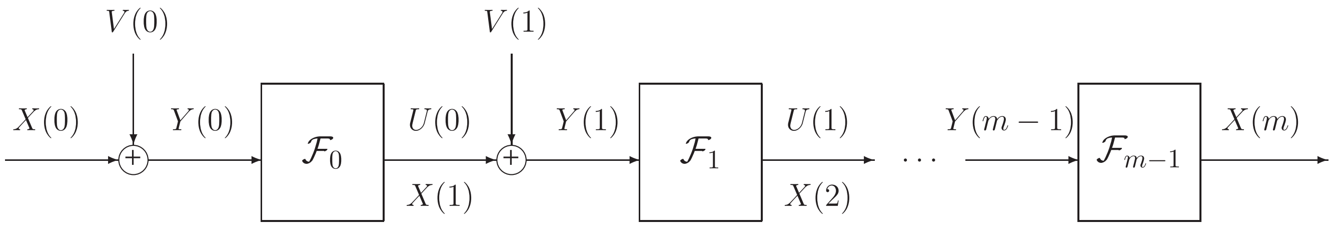

Figure 5.1: Graphical interpretation to Problem 5.1.

5.2 Optimal affine memoryless control

In this section, we solve Problem 5.1 under the constraint that the functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ are affine. In particular, we adopt the following steps:

- We start this section by defining an auxiliary problem (Problem 5.2), in which we adopt the cost $E\left[(X(0) - X(m))^2\right]$ subject to an upper bound constraint on $\sum_{k=0}^{m-2} E[U(k)^2]$, where $X(k)$ and $U(k)$ are as defined in Problem 5.1;
- Proposition 5.1 below solves Problem 5.2, for two stages ($m = 2$), for the special case where the initial noise is set to zero ($V(0) = 0$);
- In Lemma 5.1 below, we find the optimal solution to Problem 5.2, subject to affine memoryless control strategies;
- In Proposition 5.2 below, we give the optimal solution of Problem 5.2, for two stages $(m = 2)$, and we show that the optimal memoryless policy is affine;
- The main result of the section is given in Theorem 5.1, in which the optimal cost of Problem 5.1 is computed subject to affine memoryless control strategies.

Problem 5.2 *Let σ_0^2 and σ_V^2 be pre-selected positive constants representing the variance of $X(0)$ and $V(k)$, for all $k \in \{0, \ldots, m-1\}$ and m be a given integer denoting the* *length of an optimization horizon. Consider that the system described by (5.1)-(5.2) accepts a control strategy of the following form:*

$$U(k) = \mathcal{F}_k(Y(k)),\ k \in \{0, \dots, m-1\} \tag{5.5}$$

where, for each k in the set $\{0, \dots, m-1\}$, $\mathcal{F}_k : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function. Given a positive real parameter γ, we wish to determine Lebesgue measurable functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ that minimize the following cost:

$$\mathcal{C}(\{\mathcal{F}_k\}_{k=0}^{m-1}, \sigma_0^2, \sigma_V^2) \stackrel{def}{=} E[(X(m) - X(0))^2] \tag{5.6}$$

$$\textit{s.t.} \sum_{k=0}^{m-2} E[U(k)^2] \leq (m-1)\sigma_V^2 \gamma \tag{5.7}$$

Using standard Lagrangian relaxation [39], it is readily verified that there exists a positive real number ϱ, such that the optimal solution of Problem 5.2, is also an optimal solution of the problem.

$$\min_{\{\mathcal{F}_k\}_{k=0}^{m-1}} E[(X(m) - X(0))^2] + \varrho \sum_{k=0}^{m-2} E[U(k)^2]$$

with $X(0)$, $X(m)$ and $U(k)$ defined as in Problem 5.2, where ϱ is the Lagrange multiplier associated with the constraint $\sum_{k=0}^{m-2} E[U(k)^2] \leq (m-1)\sigma_V^2\gamma$. Hence, using Lagrangian relaxation we can recover Problem 5.1. We will show later in Theorem 5.1, that, subject to affine memoryless control and under some additional conditions, Problem 5.1 and Problem 5.2 share an optimal solution. We introduce Problem 5.2 because it will aid in the solution of Problem 5.1, subject to affine memoryless control.

The following proposition is an important supporting result for this section. It provides a solution to Problem 5.2, for the particular case, where m is two and the initial noise is set to zero ($V(0) = 0$). Our proof uses a result in [40], where a similar problem was analyzed. In Figure 5.2, we present an alternative interpretation of Proposition 5.1.

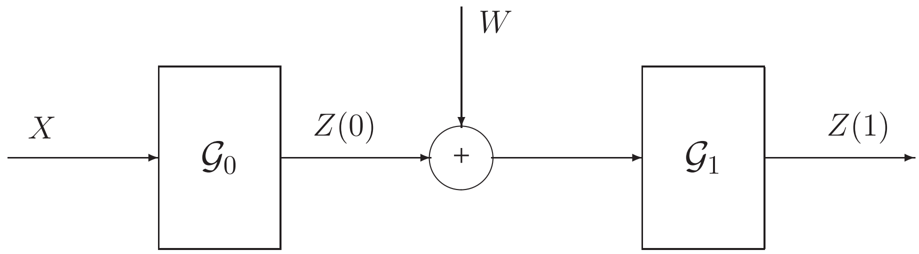

Figure 5.2: Graphical interpretation to Proposition 5.1.

Proposition 5.1 *Given strictly positive real numbers σ_X^2 and σ_W^2, let X and W be zero mean Gaussian independent random variables with variance σ_X^2 and σ_W^2, respectively. For a positive real number σ^2, define the optimal cost:*

$$J^* \stackrel{def}{=} \min_{\mathcal{G}_0,\mathcal{G}_1} E\left[(X - Z(1))^2\right] \tag{5.8}$$

$$s.t. E\left[Z(0)^2\right] \leq \sigma^2 \tag{5.9}$$

where $Z(0)$ and $Z(1)$ are random variables defined as $Z(0) \stackrel{def}{=} \mathcal{G}_0(X)$ and $Z(1) \stackrel{def}{=} \mathcal{G}_1(Z(0) + W)$ respectively, and $\mathcal{G}_0$ and $\mathcal{G}_1$ are Lebesgue measurable functions. The following holds:

$$J^* = \sigma_X^2 \left(1 - \frac{\sigma^2}{\sigma^2 + \sigma_W^2}\right)$$

and the functions $\mathcal{G}_0^*$ and $\mathcal{G}_1^*$, which minimize the cost, are linear and given by:

$$\mathcal{G}_0^*(x) = \pm\frac{\sigma}{\sigma_X}x \ \text{ and } \ \mathcal{G}_1^*(x) = \pm\frac{\sigma_X \cdot \sigma}{\sigma^2 + \sigma_W^2}x$$

Proof: Using standard optimization techniques (e.g. page 243 in [39]), we can verify that there exists a positive real number μ, such that the problem in the statement of the proposition, shares an optimal solution with the problem.

$$\min_{\mathcal{G}_0,\mathcal{G}_1} E\left[(X - Z(1))^2\right] + \mu E\left[Z(0)^2\right] \tag{5.10}$$

where X, $Z(0)$, $Z(1)$ and $\mathcal{G}_0$ and $\mathcal{G}_1$ are defined in the statement of the Proposition and the positive number μ is the Lagrange multiplier associated with the constraint $E\left[Z(0)^2\right] \leq \sigma^2$. Basar and Bansal proved in [40], that an optimal solution of (5.10) is given by linear $\mathcal{G}_0^*$ and $\mathcal{G}_1^*$. This implies that there exists a linear optimal solution for (5.8)-(5.9), hence it suffices to consider $\mathcal{G}_0^*(x) = ax$ and $\mathcal{G}_1^*(x) = bx$, where a and b are real numbers. Equivalently, we can consider that $Z(0) = aX$ and $Z(1) = abX + bW$. The problem in the statement of the Proposition becomes:

$$\begin{aligned} &\min_{a,b} E\left[((1-ab)X - bW)^2\right] \\ &\text{s.t. } \; E\left[a^2X^2\right] \leq \sigma^2 \end{aligned}$$

Knowing that the variance of X is σ_X^2, the variance of W is σ_W^2 and that X and W are independent, the problem becomes:

$$\begin{aligned} &\min_{a,b} (1-ab)^2\sigma_X^2 + b^2\sigma_W^2 \\ &\text{s.t. } \; a^2\sigma_X^2 \leq \sigma^2 \end{aligned} \tag{5.11}$$

By the first order necessary condition (page 243 in [39]), for the optimal a^* and b^*, there exist a nonnegative real number λ such that:

$$-2b^*(1-a^*b^*)\sigma_X^2 + 2\lambda a^*\sigma_X^2 = 0 \tag{5.12}$$

$$-2a^*(1-a^*b^*)\sigma_X^2 + 2b^*\sigma_W^2 = 0 \tag{5.13}$$

Let us assume that $\lambda = 0$. If λ takes the value zero, then, from (5.12), it follows that $b^* = 0$ or $1 - a^*b^* = 0$. If $b^* = 0$, from (5.13), it follows that $a^* = 0$. If $1 - a^*b^* = 0$, then it follows that $b^* = 0$, which contradicts the condition that $1 - a^*b^* = 0$. This implies that, if the optimal λ is equal to zero, then $a^* = b^* = 0$ and the optimal cost is $J^* = \sigma_X^2$.

Let us assume that $\lambda \neq 0$, then the constraint from equations (5.11) is active, which implies that $a^* = \frac{\sigma}{\sigma_X}$ or $a^* = -\frac{\sigma}{\sigma_X}$. Let $a^* = \frac{\sigma}{\sigma_X}$, then, from equation (5.13), $b^* = \frac{\sigma_X \cdot \sigma}{\sigma^2 + \sigma_W^2}$. In the same way, we show that, if $a^* = -\frac{\sigma}{\sigma_X}$, then $b^* = -\frac{\sigma_X \cdot \sigma}{\sigma^2 + \sigma_W^2}$. In both these cases $J^* = \sigma_X^2 \left(1 - \frac{\sigma^2}{\sigma^2 + \sigma_W^2}\right) < \sigma_X^2$. The value of the cost when $a = b = 0$ is σ_X^2, hence, the constraint in (5.11) is active and the optimal solution is given by $a^* = \frac{\sigma}{\sigma_X}$ and $b^* = \frac{\sigma_X \cdot \sigma}{\sigma^2 + \sigma_W^2}$ or by $a^* = -\frac{\sigma}{\sigma_X}$ and $b^* = -\frac{\sigma_X \cdot \sigma}{\sigma^2 + \sigma_W^2}$.

Therefore the functions:

$$\mathcal{G}_0^*(x) = \pm \frac{\sigma}{\sigma_X} x \quad \text{and} \quad \mathcal{G}_1^*(x) = \pm \frac{\sigma_X \cdot \sigma}{\sigma^2 + \sigma_W^2} x$$

are the optimal solution to (5.8)-(5.9) and the optimal cost is:

$$J^* = \sigma_X^2 \left(1 - \frac{\sigma^2}{\sigma^2 + \sigma_W^2}\right) \quad \blacksquare$$

Lemma 5.1 describes the solution of Problem 5.2, subject to affine memoryless policies. Before stating Lemma 5.1, we define a class of affine memoryless strategies of interest.

Definition 5.1 *Let all parameters defining Problem 5.2 be given. Let the real numbers* $\{\lambda(k)\}_{k=0}^{m-1}$ *and* $\{\beta(k)\}_{k=0}^{m-1}$ *be given. Define the class of affine memoryless strategies as follows:*

$$U(k) \stackrel{def}{=} \lambda(k)Y(k) + \beta(k),\ k \in \{0, \dots, m-1\} \tag{5.14}$$

$$\tag{5.15}$$

In view of equation (5.1) a direct consequence of (5.14) is equation (5.16) below (set $k = m-1$).

$$X(m) \stackrel{def}{=} \lambda(m-1)Y(m-1) + \beta(m-1) \tag{5.16}$$

Consider the following cost:

$$\mathcal{C}_A\left(\{\lambda(k)\}_{k=0}^{m-1}, \{\beta(k)\}_{k=0}^{m-1}, \sigma_0^2, \sigma_V^2\right) \stackrel{def}{=} E[(X(m) - X(0))^2] \tag{5.17}$$

which must be computed with the control (5.14) applied to (5.1)-(5.2).

Definition 5.2 *Given real a positive constant γ, define the following optimal cost:*

$$\mathcal{C}_A^*\left(m, \gamma, \sigma_0^2, \sigma_V^2\right) \stackrel{def}{=} \min_{\{(\lambda(k),\beta(k))\}_{k=0}^{m-1}} \mathcal{C}_A\left(\{\lambda(k)\}_{k=0}^{m-1}, \{\beta(k)\}_{k=0}^{m-1}, \sigma_0^2, \sigma_V^2\right) \tag{5.18a}$$

$$\textit{s.t.} \quad \sum_{k=0}^{m-2} E[U(k)^2] \leq (m-1)\gamma\sigma_V^2 \tag{5.18b}$$

where $U(k),\ k \in \{0, \ldots, m-2\}$ are defined in equation (5.14).

Lemma 5.1 *Let all parameters defining Problem 5.2 be given and let γ be a positive real number. Adopt an affine memoryless control strategy, as given in equation (5.14). The following holds:*

$$\mathcal{C}_A^*\left(m, \gamma, \sigma_0^2, \sigma_V^2\right) = \sigma_0^2 \left(1 - \frac{\sigma_0^2}{\sigma_0^2 + \sigma_V^2} \frac{\gamma^{m-1}}{(1+\gamma)^{m-1}}\right) \tag{5.19}$$

and the optimum is reached by selecting the following affine functions:

$$\beta(k) = 0, k \in \{0 \ldots m-1\} \tag{5.20}$$

$$\lambda(0) = \sqrt{\frac{\gamma\sigma_V^2}{\sigma_V^2 + \sigma_0^2}} \tag{5.21}$$

$$\lambda(k) = \sqrt{\frac{\gamma}{\gamma+1}}, \quad k \in \{1 \ldots m-2\} \tag{5.22}$$

$$\lambda(m-1) = \sqrt{\frac{\sigma_0^2}{\sigma_0^2 + \sigma_V^2}} \left(\frac{\gamma}{\gamma+1}\right)^{\frac{m-1}{2}} \sqrt{\frac{\sigma_0^2}{\gamma\sigma_V^2 + \sigma_V^2}} \tag{5.23}$$

Before we prove Lemma 5.1, we need to state and prove two supporting results.

Lemma 5.2 *Let all parameters and cost function defining Lemma 5.1 be given. Given the positive numbers $\{\sigma_i^2\}_{i=1}^{m-1}$, define the optimal cost:*

$$\mathcal{C}_\sigma^*\left(m, \sigma_0^2, \sigma_V^2, \{\sigma_i^2\}_{i=1}^{m-1}\right) \stackrel{def}{=} \min_{\{(\lambda(k),\beta(k))\}_{k=0}^{m-1}} \mathcal{C}_A\left(\{\lambda(k)\}_{k=0}^{m-1}, \{\beta(k)\}_{k=0}^{m-1}, \sigma_0^2, \sigma_V^2\right) \tag{5.24}$$

$$\textit{s.t.} \quad E[U(k)^2] = \sigma_{k+1}^2, \quad k \in \{0 \dots m-2\} \tag{5.25}$$

Then the following holds:

$$\mathcal{C}_\sigma^*\left(m, \sigma_0^2, \sigma_V^2, \{\sigma_i^2\}_{i=1}^{m-1}\right) = \sigma_0^2\left(1 - \prod_{i=0}^{m-1}\frac{\sigma_i^2}{\sigma_i^2+\sigma_V^2}\right) \tag{5.26}$$

$$E[X(m)^2] = \sigma_0^2\prod_{i=0}^{m-1}\frac{\sigma_i^2}{\sigma_i^2+\sigma_V^2}$$

and the optimum is reached by selecting the following affine functions:

$$\beta(k) = 0, k \in \{0 \dots m-1\} \tag{5.27}$$

$$\lambda(k) = \sqrt{\frac{\sigma_{k+1}^2}{\sigma_k^2+\sigma_V^2}}, \quad k \in \{0 \dots m-2\} \tag{5.28}$$

$$\lambda(m-1) = \prod_{i=0}^{m-1}\sqrt{\frac{\sigma_i^2}{\sigma_i^2+\sigma_V^2}} \cdot \sqrt{\frac{\sigma_0^2}{\sigma_{m-1}^2+\sigma_V^2}} \tag{5.29}$$

Proof: We notice that the affine functions at each step with $k \in \{0, \dots m-2\}$ act only as scale factors. Because of linearity the values of the $\lambda(k)$'s $k \in \{0, \dots m-2\}$ appear immediately the way they are written in equation (5.28), $\lambda(m-1)$ can be computed using the fact that $Y(m-1)$ is Gaussian, hence $X(m) = E[X(0)|Y(m-1)]$, and with the values of $\beta(k) = 0, \forall k$. Note that the values of $\lambda(k)$ are not unique. It is straightforward to show that if we take a even number of parameters $\lambda(k)$, when $\beta(k) = 0$ and flip their sign the value of the cost, given in the statement of the theorem, remains the same. However, the values for $\beta(k)$ are unique, i.e. if there exists at least one index $k \in \{0, \ldots, m-1\}$ such that $b(k) \neq 0$, then the cost will be larger than the one from equation (5.26).

First we will show that the optimal $\lambda(k) \neq 0$, for all $k \in \{0, \ldots, m-1\}$, then we will prove by induction that the cost $\mathcal{C}_A\left(\{\lambda(k)\}_{k=0}^{m-1}, \{\beta(k)\}_{k=0}^{m-1}, \sigma_0^2, \sigma_V^2\right)$ is lower bounded by $\sigma_0^2\left(1 - \prod_{i=0}^{m-1} \frac{\sigma_i^2}{\sigma_i^2+\sigma_V^2}\right)$ from equation (5.26) for all $\lambda(k)$ and $\beta(k)$, $k \in \{0, \ldots, m-1\}$ which satisfy the constraint from equation (5.25). Then, we show that the cost from equation (5.26) can be reached by selecting the values for $\lambda(k)$ and $\beta(k)$ from equations (5.28), (5.29) and (5.27). Finally, we will show that the optimal values for $\beta(k)$, for all $k \in \{0, \ldots, m-1\}$ are always zero. We process now through these three steps.

We first show that for a general m, there is no $k \in \{0, \ldots, m-1\}$ for which $\lambda(k) = 0$. Assume that exists such a k, then $U(k) = \beta(k)$, which will be just a constant, and all the $Y(l), l \geq k$ will be independent of $X(0)$, which will make $X(m)$ independent of $X(0)$. Hence, the cost function becomes:

$$E[(X(0) - X(m))^2] = E[X(0)^2] + E[X(m)^2] \geq \sigma_0$$

but the values of $\lambda(k)$ and $\beta(k)$ from equations (5.28), (5.29) and (5.27) satisfy the constraints from equation (5.25) and have the associated cost from equation (5.26) which less then σ_0^2, hence we conclude that the optimal $\lambda(k)$, $k \in \{0, \ldots, m-1\}$ are always non zero. Since we showed that the optimal values for $\lambda(k)$ are non-zero we will consider from this point on that $\lambda(k) \neq 0$ for all $k \in \{0, \ldots, m-1\}$.

Second, we show that the cost from equation (5.26) can be reached by selecting the values for $\lambda(k)$ and $\beta(k)$ from equations (5.28), (5.29) and (5.27). It is a standard computation to show that the lemma hold for $m = 1$ and for $m = 2$. For $m = 1$, $X(1) = E[X(0)|Y(0)]$, due to the Gaussianity of $X(0)$ and the noise, and the result is immediate. The results for $m = 2$ are found also in the proof for Proposition 5.2. Assume that the claim holds for $m \geq 2$. We need to prove that it holds also for $m+1$. Let it be the $m+1$ stage problem. Let $\widetilde{X}(m)$ the best affine estimator of $X(0)$ given $Y(m-1)$. By the properties of the affine estimators $\widetilde{X}(m)$ is an affine function of $Y(m-1)$ and $E[\widetilde{X}(m)] = E[X(0)] = 0$. Since all the $\lambda(k) \neq 0, k \in \{0 \dots m-2\}$ it follows that $\widetilde{X}(m)$ is an invertible affine function of $Y(m-1)$. This means that $X(m)$, being an affine function of $Y(m-1)$, is an affine function of $\widetilde{X}(m)$. Using the orthogonality principle we can write the cost:

$$
\begin{aligned}
&E[(X(0) - X(m+1))^2] \\
&= E[(X(0) - \widetilde{X}(m) + \widetilde{X}(m) - X(m+1))^2] \\
&= E[(X(0) - \widetilde{X}(m))^2] + E\left[(\widetilde{X}(m) - X(m+1))^2\right] \\
&\quad + 2E[(X(0) - \widetilde{X}(m))(\widetilde{X}(m) - X(m+1))] \\
&= E[(X(0) - \widetilde{X}(m))^2] + E[(\widetilde{X}(m) - X(m+1))^2]
\end{aligned}
$$

The value $E[(\widetilde{X}(m) - X(m+1))^2]$ can be bounded from below using Proposition 5.1, since $X(m)$ is an affine function of $\widetilde{X}(m)$ and $E[X(m)^2] = \sigma_m^2$. We know that $\widetilde{X}(m)$ is the best affine estimator of $X(0)$ given $Y(m-1)$. Then using the orthogonality principle we obtain:

$$E[X(0)^2] = E[(X(0) - \widetilde{X}(m) + \widetilde{X}(m))^2]$$
$$= E[(X(0) - \widetilde{X}(m))^2] + E[\widetilde{X}(m)^2] + 2E[(X(0) - \widetilde{X}(m))\widetilde{X}(m)]$$
$$= E[(X(0) - \widetilde{X}(m))^2] + E[\widetilde{X}(m)^2]$$

Looking back at the initial cost:

$$E[(X(0) - X(m+1))^2]$$
$$= E[(X(0) - \widetilde{X}(m))^2] + E[(\widetilde{X}(m) - X(m+1))^2]$$
$$\geq E[(X(0) - \widetilde{X}(m))^2] + E[\widetilde{X}(m)^2]\left(1 - \frac{\sigma_m^2}{\sigma_m^2 + \sigma_V^2}\right)$$

$$= E[X(0)^2]\left(1 - \frac{\sigma_m^2}{\sigma_m^2 + \sigma_V^2}\right) + E[(X(0) - \widetilde{X}(m))^2]$$
$$- E[(X(0) - \widetilde{X}(m))^2]\left(1 - \frac{\sigma_m^2}{\sigma_m^2 + \sigma_V^2}\right)$$
$$= E[X(0)^2]\left(1 - \frac{\sigma_m^2}{\sigma_m^2 + \sigma_V^2}\right) + E[(X(0) - \widetilde{X}(m))^2]\frac{\sigma_m^2}{\sigma_m^2 + \sigma_V^2}$$
$$\geq \sigma_0^2\left(1 - \prod_{i=0}^{m-1}\frac{\sigma_i^2}{\sigma_i^2 + \sigma_V^2}\right)\frac{\sigma_m^2}{\sigma_m^2 + \sigma_V^2} + \sigma_0^2\left(1 - \frac{\sigma_m^2}{\sigma_m^2 + \sigma_V^2}\right)$$
$$= \sigma_0^2\left(1 - \prod_{i=0}^{m}\frac{\sigma_i^2}{\sigma_i^2 + \sigma_V^2}\right)$$

The first inequality takes place due to the fact that $X(m)$ is an affine function of $\widetilde{X}(m)$ and $E[X^2(m)] = \sigma_m^2$, so the second term can be lower bounded using 5.1 and the second inequality appears due to the induction. Both inequalities can be reached with equality by selecting the parameters $\lambda(k), k \in \{0, \ldots m-2\}$ and $\beta(k), k \in \{0, \ldots m-2\}$ for the m stage problem and the values for $\lambda(m-1)$, $\lambda(m)$, $\beta(m-1)$, $\beta(m)$ and $E[X(m+1)^2]$ follow from 5.1.

Third and finally, we show that the optimal values $b(k)$, for all $k \in \{0, \ldots, m-1\}$ are always zero. We will rewrite for the reader convenience the equation which govern the system when we adopt affine control strategies:

$$\begin{aligned} X(k+1) &= U(k), && k \geq 0 \\ Y(k) &= X(k) + V(k), && k \geq 0 \\ U(k) &= \lambda(k)Y(k) + \beta(k), && k \geq 0 \end{aligned}$$

Adopt $\lambda(k)$ and $\beta(k)$, for $k \in \{0, \ldots, m-2\}$ such that the constraints from equation (5.25) are satisfied. Since we are trying to minimize the cost function $E\left[(X(0) - X(m))^2\right]$, we let $X(m)$ to be:

$$X(m) = E\left[X(0)|Y(m-1)\right]$$

Since $X(0)$ and $V(k)$, $k \in \{0, \ldots, m-1\}$ are Gaussian random variables and are mutually independent, it follow that $X(m)$ is an affine function of $Y(m-1)$. We can write $U(k)$ as follows:

$$U(k) = \prod_{i=0}^{k} \lambda(i)X(0) + \sum_{i=0}^{k} V(k) \prod_{j=i}^{k} \lambda(j) + \sum_{i=0}^{k-1} b(i) \prod_{j=i}^{k-1} \lambda(j) + \beta(k) \tag{5.30}$$

The real numbers $\lambda(k)$ and $\beta(k)$, for $k \in \{0, \ldots, m-2\}$ are chose such that the constraint from equation (5.25) is satisfied, hence it holds that:

$$E\left[U^2(k)\right] = \sigma_{k+1}^2, \qquad k \in \{0, \ldots, m-2\}$$

Define the positive real numbers $\{\tilde{\sigma}_k^2\}_{k=1}^{m-1}$ as follows:

$$\tilde{\sigma}_k^2 \stackrel{def}{=} E\left[\left(\prod_{i=0}^{k} \lambda(i)X(0) + \sum_{i=0}^{k} V(k) \prod_{j=i}^{k} \lambda(j)\right)^2\right], \qquad k \in \{1, \ldots, m-1\} \tag{5.31}$$

Since $X(0)$ and $V(k)$, for all $k \in \{0, m-2\}$ are zero mean random variables, it follows that $\tilde{\sigma}_k^2 \leq \sigma_k^2$ for all $k \in \{1, \ldots, m-1\}$. We find immediately that:

$$\begin{aligned} \lambda(0)^2 &= \frac{\tilde{\sigma}_1^2}{\sigma_0^2 + \sigma_V^2} \\ \lambda(k)^2 &= \frac{\tilde{\sigma}_{k+1}^2}{\tilde{\sigma}_k^2 + \sigma_V^2}, \qquad k \in \{1, \ldots, m-2\} \end{aligned} \tag{5.32}$$

Since we are computing $X(m) = E\left[X(0)|Y(m-1)\right]$, from standard estimation theory we obtain:

$$\begin{aligned} \lambda(m-1) &= \sigma_0^2 \frac{\prod_{i=0}^{m-2} \lambda(i)}{\tilde{\sigma}_{m-1}^2 + \sigma_V^2} \\ \beta(m-1) &= \sigma_0^2 \frac{\prod_{i=0}^{m-2} \lambda(i)}{\tilde{\sigma}_{m-1}^2 + \sigma_V^2} \left(\sum_{i=0}^{m-3} b(i) \prod_{j=i}^{k-1} \lambda(j) + \beta(m-2) \right) \\ E\left[(X(0) - X(m))^2 \right] &= \sigma_0^2 - \sigma_0^4 \frac{\prod_{i=0}^{m-2} \lambda(i)^2}{\tilde{\sigma}_{m-1}^2 + \sigma_V^2} \end{aligned}$$

Using equation (5.32) we obtain:

$$\begin{aligned} E\left[(X(0) - X(m))^2 \right] &= \sigma_0^2 \left(1 - \prod_{i=0}^{m-1} \frac{\tilde{\sigma}_i^2}{\tilde{\sigma}_i^2 + \sigma_V^2} \right) \\ &\leq \sigma_0^2 \left(1 - \prod_{i=0}^{m-1} \frac{\sigma_i^2}{\sigma_i^2 + \sigma_V^2} \right) \end{aligned} \tag{5.33}$$

where the inequality appears due to the fact that $\tilde{\sigma}_k^2 \leq \sigma_k^2$, for all $k \in \{1, \ldots, m-1\}$ and because the function $\frac{x}{1+x}$ is strictly increasing for positive real numbers x. Let $\tilde{k} = \inf\{k \leq m-2 : b(k) \neq 0\}$, it follows that $\tilde{\sigma}_{\tilde{k}+1}^2 < \sigma_{\tilde{k}+1}^2$, then the inequality from equation (5.33) will become strict, hence $b(k) = 0$ for all $k \in \{0, \ldots, m-2\}$, which implies that $b(m-1) = 0$. ■

The Lemma 5.3 below is a supporting result for Lemma 5.1

Lemma 5.3 *Let $\{\alpha_i\}_{i=1}^n$ be positive real numbers. Consider the following cost function:*

$$C\left(\{\alpha_i\}_{i=1}^n\right) \stackrel{def}{=} \prod_{i=1}^{n} \frac{\alpha_i}{1 + \alpha_i}$$

Given a positive real number P, define the following optimal cost:

$$\mathcal{C}^* \stackrel{def}{=} \max_{\{\alpha_i\}_{i=1}^n} C\left(\{\alpha_i\}_{i=1}^n\right)$$

$$\textit{s.t.} \quad \begin{array}{l} \sum_{i=1}^n \alpha_i \leq P \\ \alpha_i \geq 0, i \in \{1, \ldots n\} \end{array}$$

Then the following hold:

$$\mathcal{C}^* = \frac{\left(\frac{P}{n}\right)^n}{\left(1 + \frac{P}{n}\right)^n}$$

$$\alpha_i^* = \frac{P}{n}, i \in \{1, \ldots, n\}$$

where $\{\alpha_i^*\}_{i=1}^n$ *are the optimal values of* $\{\alpha_i\}_{i=1}^n$ *for which the problem is solved.*

Proof: First we show that the optimization problem is equivalent to the following problem:

$$\max_{\{\alpha_i\}_{i=1}^n} C\left(\{\alpha_i\}_{i=1}^n\right)$$

$$\text{s.t.} \quad \begin{array}{l} \sum_{i=1}^n \alpha_i \leq P \\ \alpha_i \geq \epsilon, i \in \{1, \ldots n\} \end{array}$$

for some $\epsilon > 0$.

The cost function is positive for any choice of positive $\alpha_i \geq 0$ and is zero if exist an integer i s.t. $\alpha_i = 0$. Choose any $\alpha_i > 0$ such that $\sum_{i=1}^n \alpha_i \leq P$. For this choice, let $\prod_{i=1}^n \frac{\alpha_i}{\alpha_i+1} = \bar{\epsilon} > 0$. Then for any $k \in \{1, \ldots, n\}$, $\displaystyle\prod_{i=1}^n \frac{\alpha_i}{\alpha_i + 1} \leq \frac{\alpha_k}{\alpha_k + 1} \leq \alpha_k$ Choose $\epsilon = \frac{\bar{\epsilon}}{2}$, then if $\alpha_k \leq \epsilon$, then $\prod_{i=1}^n \frac{\alpha_i}{\alpha_i+1} < \bar{\epsilon}$ no matter of the values of the other $\alpha_i,\ i \in \{1, \ldots, k-1, k+1, \ldots, n\}$. This shows that the first problem and the second problem are equivalent. Moreover it shows that for the second problem, the inequality constraints $\alpha_i, i \in \{1,\dots n\}$ are inactive. Then the second problem can be solved by solving the equivalent problem:

$$\max_{\{\alpha_i\}_{i=1}^n} \log \prod_{i=1}^n \frac{\alpha_i}{\alpha_i+1}$$

$$\text{s.t.} \quad \begin{array}{l} \sum_{i=1}^n \alpha_i \leq P \\ \alpha_i \geq \epsilon, i \in \{1,\dots n\} \end{array}$$

which is the same with:

$$\max_{\{\alpha_i\}_{i=1}^n} \sum_{i=1}^n \log \frac{\alpha_i}{\alpha_i+1}$$

$$\text{s.t.} \quad \begin{array}{l} \sum_{i=1}^n \alpha_i \leq P \\ \alpha_i \geq \epsilon, i \in \{1,\dots n\} \end{array}$$

We note that the optimization function is strictly concave on the maximization domain and the inequality constraints are affine functions, which means that values for $\{\alpha_i\}_{i=1}^n$ which reach the maximum are unique. From the argument of the previous problem the inequality constraints $\alpha_i \geq \epsilon, i \in \{1,\dots n\}$ are inactive, so the Lagrange multipliers associated with these constraints are 0. Let μ be the Lagrange multiplier associated with the remaining inequality constraint. Then for the optimization problem the first order optimality conditions can be written:

$$\frac{\partial \sum_{i=1}^n \log \frac{\alpha_i}{\alpha_i+1}}{\partial \alpha_k} + \mu = 0, k \in \{1,\dots n\}$$

which is after differentiation:

$$\frac{1}{\alpha_k} - \frac{1}{\alpha_k+1} + \mu = 0, k \in \{1,\dots n\}$$

First we note that $\mu < 0$ and that the inequality constraint is active. Then α_k can be written as a function of μ: $\alpha_k = \dfrac{-1+\sqrt{1-\frac{4}{\mu}}}{2}$ We obtain that the $\alpha_k, k \in \{1,\dots n\}$ are equal and, $\alpha_k = \dfrac{P}{n}$ and the result follows.■

Proof of Lemma 5.1 The initial optimization problem:

$$\mathcal{C}_A^*\left(m, \gamma, \sigma_0^2, \sigma_V^2\right) \stackrel{def}{=} \min_{\{(\lambda(k),\beta(k))\}_{k=0}^{m-1}} \mathcal{C}_A\left(\{\lambda(k)\}_{k=0}^{m-1}, \{\beta(k)\}_{k=0}^{m-1}, \sigma_0^2, \sigma_V^2\right)$$
$$\text{s.t.} \sum_{k=0}^{m-2} E[U(k)^2] \leq (m-1)\gamma\sigma_V^2$$

is equivalent to the following optimization problem:

$$\mathcal{C}_A^*\left(m, \gamma, \sigma_0^2, \sigma_V^2\right) \stackrel{def}{=} \min_{\left\{\sigma_i^2\right\}_{i=1}^{m-1}} \mathcal{C}_\sigma^*\left(m, \sigma_0^2, \sigma_V^2, \left\{\sigma_i^2\right\}_{i=1}^{n}\right)$$
$$\text{s.t.} \sum_{i=1}^{m-1} \sigma_i^2 \leq (m-1)\gamma\sigma_V^2$$

Taking into consideration that σ_i^2's are the variances of some random variables, hence they must be positive, the results of Lemma 5.1 follow directly from Lemma 5.2 and Lemma 5.3.■

The following proposition, in conjunction with Lemma 5.1, shows that affine strategies are optimal for the two stage version of Problem 5.2.

Proposition 5.2 *Let all the parameters defining Problem 5.2 be given and assume that $m = 2$. Given a positive real constant γ, let $\mathcal{F}_0$ be a Lebesgue measurable function satisfying $E[U(0)^2] \leq \gamma\sigma_V^2$. The following holds:*

$$E[(X(2) - X(0))^2] \geq \mathcal{C}_A^*\left(2, \gamma, \sigma_0^2, \sigma_V^2\right) \tag{5.34}$$

where $\mathcal{C}_A^*(2, \gamma, \sigma_0^2, \sigma_V^2)$ is given by (5.19).

Proof: Let $\widetilde{X}(1) = E[X(0)|Y(0)] = \dfrac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}Y(0)$. The cost can be written as follows:

$$\begin{aligned}
&E[(X(2)-X(0))^2]\\
&= E[(X(2)-\widetilde{X}(1)+\widetilde{X}(1)-X(0))^2]\\
&= E[(X(2)-\widetilde{X}(1))^2] + E[(\widetilde{X}(1)-X(0))^2]\\
&+ 2E[(X(2)-\widetilde{X}(1))(\widetilde{X}(1)-X(0))]
\end{aligned} \tag{5.35a}$$

$$\begin{aligned}
&= E[(X(2)-\widetilde{X}(1))^2] + E[(\widetilde{X}(1)-X(0))^2]\\
&= E[(X(2)-\widetilde{X}(1))^2] + \frac{\sigma_0^2\sigma_V^2}{\sigma_0^2+\sigma_V^2}
\end{aligned} \tag{5.35b}$$

We note that $\widetilde{X}(1)$ is a linear function of $Y(0)$, which means that $Y(0)$ can be written as a linear function of $\widetilde{X}(1)$ and also $X(1) = U(0)$ is a function of $\widetilde{X}(1)$. The variable $X(2)$ is a function of $Y(0)$ and $V(1)$, hence, it follows that $X(2)$ is a function of $\widetilde{X}(1)$ and $V(1)$. The noise $V(1)$ is independent of $X(0)$ and $V(0)$, hence it follows that the equality between equations (5.35a) and (5.35b) is valid, because the cross term is zero due to the orthogonality principle. Moreover, we can use Proposition 5.1, by letting $\widetilde{X}(1)$ take the place of X, $X(1)$ the place of $Z(0)$, $X(2)$ the place of $Z(1)$ and $V(1)$ the place of W, leading to the following lower bound on $E[(X(2)-\widetilde{X}(1))^2]$:

$$\begin{aligned}
&E[(X(2)-X(0))^2] = E[(X(2)-\widetilde{X}(1))^2] + \frac{\sigma_0^2\sigma_V^2}{\sigma_0^2+\sigma_V^2}\\
&\geq \frac{\sigma_0^4}{\sigma_V^2+\sigma_0^2}\left(1-\frac{\sigma_V^2\gamma}{\sigma_V^2+\sigma_V^2\gamma}\right) + \frac{\sigma_0^2\sigma_V^2}{\sigma_0^2+\sigma_V^2}\\
&= \sigma_0^2\left(1-\frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}\frac{\gamma}{1+\gamma}\right) = \mathcal{C}_A^*\left(2,\gamma,\sigma_0^2,\sigma_V^2\right) \blacksquare
\end{aligned}$$

Remark 5.1 *As shown in Lemma 5.1, inequality (5.34) can become an equality by adopting* $U(0) = \sqrt{\frac{\gamma\sigma_V^2}{\sigma_0^2+\sigma_V^2}}Y(0)$ *and* $X(2) = \sqrt{\frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}\frac{\gamma}{\gamma+1}}\sqrt{\frac{\sigma_0^2}{\gamma\sigma_V^2+\sigma_V^2}}Y(1)$. *Hence, optimal* *feedback strategies for Problem 5.2, with $m = 2$, are given by:*

$$\mathcal{F}_0^*(x) = \pm\sqrt{\frac{\gamma\sigma_V^2}{\sigma_0^2 + \sigma_V^2}}x$$
$$\mathcal{F}_1^*(x) = \pm\sqrt{\frac{\sigma_0^2}{\sigma_0^2 + \sigma_V^2}\frac{\gamma}{\gamma + 1}}\sqrt{\frac{\sigma_0^2}{\gamma\sigma_V^2 + \sigma_V^2}}x$$

The optimal strategies $\mathcal{F}_0^*$ and $\mathcal{F}_1^*$ can be derived also from Proposition 5.1.

The following Theorem gives the optimal solution of Problem 5.1 subject to affine memoryless control.

Theorem 5.1 Part I. *Let all parameters defining Problem 5.1 be given, with m larger than or equal to two. We denote by $\mathcal{J}_A^*(m, \varrho, \sigma_0^2, \sigma_V^2)$ the optimal cost of Problem 5.1 subject to affine strategies of the form (5.14). The following equality holds:*

$$\mathcal{J}_A^*\left(m, \varrho, \sigma_0^2, \sigma_V^2\right) = \min_{\gamma\geq 0}\left[\mathcal{C}_A^*\left(m, \gamma, \sigma_0^2, \sigma_V^2\right) + (m-1)\varrho\gamma\sigma_V^2\right] \tag{5.36}$$

where $\mathcal{C}_A^*(m, \gamma, \sigma_0^2, \sigma_V^2)$ is given by (5.19).

Part II. *Consider the following conditions: (a) $\varrho = \frac{\sigma_0^4}{\left(\sigma_0^2+\sigma_V^2\right)\sigma_V^2}\frac{\gamma^{m-2}}{(1+\gamma)^m}$ and (b) $m < \gamma + 2$. Given ϱ and m, if there exists a positive real number γ for which the conditions (a) and (b) hold, then γ is an optimal solution of (5.36). If no such γ exists, then $\gamma = 0$ is an optimal solution of (5.36).*

Remark 5.2 *For a fixed value of ϱ and for large enough m, the optimal solution of (5.36) is γ equal to zero. That is, if the number of stages is large enough, then the optimal affine solution is to adopt $\mathcal{F}_k = 0$ for $k \in \{0, \dots, m-2\}$ and then it follows that the optimal $\mathcal{F}_{m-1}$ is also zero.*

Proof: Using Lagrange relaxation [39], there exists a postive real number γ, such that Problem 5.1 subject to affine strategies of the form (5.14) shares an optimal solution with the problem defined in Lemma 5.1, for this particular γ. Using the results from Lemma 5.1, **Part I** of Theorem 5.1 follows.

In order to prove **Part II** of the Theorem, we need to define the following function:

$$f(\gamma) \stackrel{def}{=} \mathcal{C}_A^*\left(m, \gamma, \sigma_0^2, \sigma_V^2\right) + \varrho(m-1)\gamma\sigma_V^2$$

The function $f(\gamma)$ is the function to be minimized in equation (5.36). We will show that there are at most two values γ for which condition (a) is satisfied and the larger of the two is a point of local minima. We will show that if conditions (a) and (b) hold, then the local minima identified by condition (a) is in fact a point of global minima. If either condition (a) or condition (b) fails for every positive γ then zero is the global optimum.

We proceed with case $m \geq 3$, while the treatment for the case where $m = 2$ is left at the end of the proof. Let m, the number of stages be greater than or equal to three. We will show that for a fixed m and ϱ, there are at most two points which can satisfy condition (a).

In order to find the minimum of $f(\gamma)$, we take the derivative of $f(\gamma)$ with respect to γ and, using equation (5.19), we obtain:

$$\frac{\partial f}{\partial \gamma}(\gamma) = -(m-1)\frac{\sigma_0^4}{(\sigma_0^2+\sigma_V^2)}\frac{\gamma^{m-2}}{(1+\gamma)^m} + \varrho(m-1)\sigma_V^2$$

The fact that the derivative of f with respect to γ satisfies condition (a) is equivalent to $\frac{\partial f}{\partial \gamma}(\gamma) = 0$. The function $\frac{\gamma^{m-2}}{(1+\gamma)^m}$ has a single stationary point, which is a point of maximum at $\frac{m-2}{2}$, for $\gamma > 0$. This implies that $\frac{\partial f}{\partial \gamma}(\gamma)$ has a single point of minimum at $\frac{m-2}{2}$ and moreover $\frac{\partial f}{\partial \gamma}$ is strictly decreasing for $\gamma \leq \frac{m-2}{2}$ and strictly increasing for $\gamma \geq \frac{m-2}{2}$.

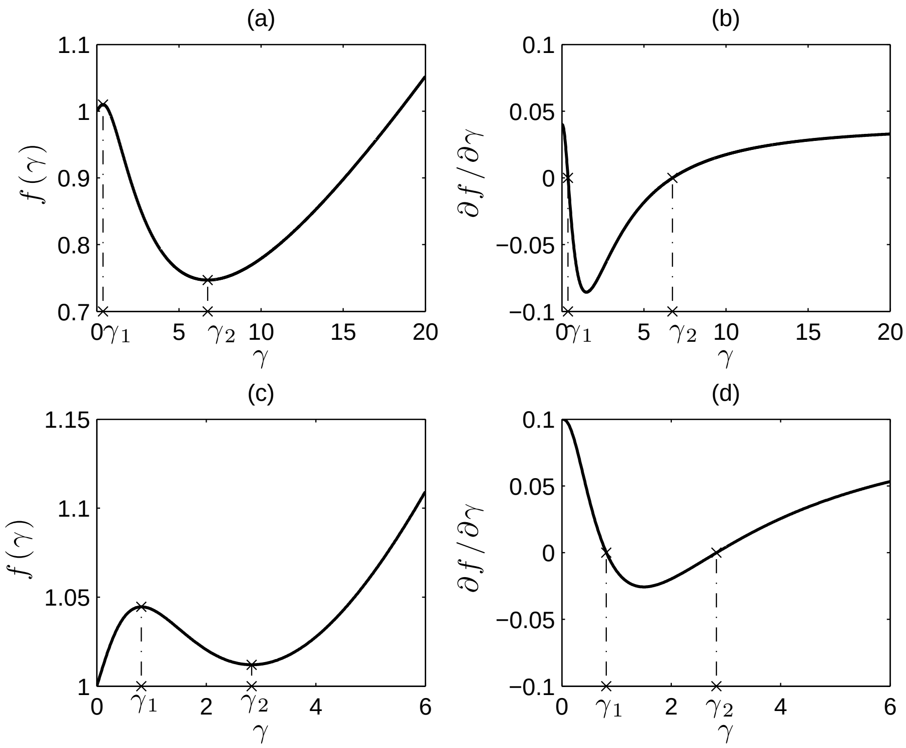

Figure 5.3: (a) The function $f(\gamma)$ satisfies conditions (a) and (b); (b) The derivative of the function from (a); (c) The function $f(\gamma)$ satisfies condition (a) but does not satisfies (b); (d) The derivative of the function from (c)

We will show next, that there are at most two values γ for which condition (a) is satisfied. We notice that $\dfrac{\partial f}{\partial \gamma}(0) = \lim\limits_{\gamma \to \infty} \dfrac{\partial f}{\partial \gamma}(\gamma) = \varrho(m-1)\sigma_V^2$ and that $\frac{\partial f}{\partial \gamma}(\gamma)$ is continuous as a function of γ. There are three cases to be analyzed. The first case is $\frac{\partial f}{\partial \gamma}\left(\frac{m-2}{2}\right) > 0$; in this case, $\frac{\partial f}{\partial \gamma}(\gamma) > 0$ for all postive real numbers γ. The second case is $\frac{\partial f}{\partial \gamma}\left(\frac{m-2}{2}\right) = 0$ and this takes place if $\varrho = \frac{4\sigma_0^4(m-2)^{m-2}}{\sigma_V^2\left(\sigma_0^2+\sigma_V^2\right)m^m}$. In this case $\gamma = \frac{m-2}{2}$ is the unique point which satisfies condition (a). The third case is $\frac{\partial f}{\partial \gamma}\left(\frac{m-2}{2}\right) < 0$; in this case, there exist two real numbers $\gamma_1 < \frac{m-2}{2} < \gamma_2$ such that $\frac{\partial f}{\partial \gamma}(\gamma_1) = \frac{\partial f}{\partial \gamma}(\gamma_2) = 0$. If γ is in the interval $[0, \gamma_1]$, the function $f(\gamma)$ is increasing, since its derivative is positive, on the interval $[\gamma_1, \gamma_2]$, f is decreasing, and on the interval $[\gamma_2, \infty)$, f is increasing. This means that the function $f(\gamma)$ has two points of local minimum, one at $\gamma = 0$ and the second one at $\gamma = \gamma_2$, hence, in order to compute the minimum, one has to compute $f(0)$ and $f(\gamma_2)$ and take the minimum between these two.

Assume that condition (a) is not satisfied, then this implies that the equation $\frac{\partial f}{\partial \gamma}(\gamma) = 0$, has no solution, which corresponds to the case $\frac{\partial f}{\partial \gamma}\left(\frac{m-2}{2}\right) > 0$. We know that $\frac{\partial f}{\partial \gamma}(0)$ is strictly positive and $\frac{\partial f}{\partial \gamma}(\gamma)$ is continuous in γ. It follows then, that $\frac{\partial f}{\partial \gamma}(0) > 0$, for all $\gamma \geq 0$, which implies that f is increasing for $\gamma \geq 0$ and $f(\gamma) \geq f(0)$, for all positive γ.

Assume that the condition (a) is satisfied, in this case we need to analyze $\frac{\partial f}{\partial \gamma}\left(\frac{m-2}{2}\right) = 0$ and $\frac{\partial f}{\partial \gamma}\left(\frac{m-2}{2}\right) > 0$. Let $\frac{\partial f}{\partial \gamma}\left(\frac{m-2}{2}\right) = 0$, it follows that $\frac{\partial f}{\partial \gamma}(\gamma) \geq 0$ for all $\gamma \geq 0$, which implies that $f(\gamma) \geq f(0)$ for all $\gamma \geq 0$ and that zero is a global minimizer. Since $\gamma = \frac{m-2}{2}$ is the unique positive real number, which satisfies condition (a), we notice in this case that condition (b) cannot be satisfied for $m \geq 3$.

We just need to discuss the case when condition (a) is satisfied with $\frac{\partial f}{\partial \gamma}\left(\frac{m-2}{2}\right) < 0$. By the analysis above, it follows that condition (a) is satisfied and that there exist γ_1 and γ_2, the solutions of the equation $\frac{\partial f(\gamma)}{\partial \gamma} = 0$, such that $\gamma_1 < \frac{m-2}{2} < \gamma_2$. Moreover, condition (a) implies that $\gamma = 0$ and $\gamma = \gamma_2$ are points of local minimum for the function f, when $\gamma \geq 0$, and one of these points is actually a global minimum. It is immediate that $f(0) = \sigma_0^2$, while $f(\gamma_2)$ is given below:

$$f(\gamma_2) = \sigma_0^2 \left(1 - \frac{\sigma_0^2}{\sigma_0^2 + \sigma_V^2} \left(\frac{\gamma_2}{\gamma_2 + 1}\right)^{m-1} \left(1 - \frac{m-1}{\gamma_2 + 1}\right)\right)$$

Since $\gamma_1 < \frac{m-2}{2} < m-2$, only γ_2 can satisfy both condition (a) and (b). Assume that, besides condition (a), condition (b) is also satisfied, then $m < \gamma_2 + 2$. Condition (b) implies that $f(\gamma_2) < f(0)$, hence $\gamma = \gamma_2$ is a global minimizer for f.

If condition (a) is satisfied, but condition (b) is not satisfied, then $m \geq \gamma + 2$, which implies that $f(\gamma_2) \geq f(0)$, hence $\gamma = 0$ is a global minimizer for f. In Figure 5.3 we provide a few plots that clarify the analysis above. In Figure 5.3(a), the function $f(\gamma)$ satisfies both conditions (a) and (b), while in Figure 5.3(c), the function $f(\gamma)$ satisfies condition (a) but does not satisfy condition (b). In Figure 5.3(b) and (d), there are the derivatives of the functions from Figure 5.3(a) and (c).

Let $m = 2$, then the derivative of f, with respect to γ, is:

$$\frac{\partial f}{\partial \gamma}(\gamma) = -\frac{\sigma_0^4}{(\sigma_0^2 + \sigma_V^2)} \frac{1}{(1+\gamma)^2} + \varrho \sigma_V^2$$

We kow that $\lim_{\gamma \to \infty} \frac{\partial f}{\partial \gamma}(\gamma) = \varrho \sigma_V^2$. The function $\frac{1}{(1+\gamma)^2}$ is decreasing for $\gamma \geq 0$, hence if $\frac{\partial f}{\partial \gamma}(0) \geq 0$, then the derivative of f with respect to γ is always positive, which implies that f is minimized when $\gamma = 0$ and also condition (a) is never satisfied. If $\frac{\partial f}{\partial \gamma}(0) < 0$, then there exists a unique positive γ such that, $\frac{\partial f}{\partial \gamma}(\gamma) = 0$, which implies that the following holds:

$$\varrho = \frac{\sigma_0^4}{(\sigma_0^2 + \sigma_V^2)\,\sigma_V^2} \frac{\gamma^{m-2}}{(1+\gamma)^m}$$

notice that this corresponds to condition (a). Since for $\gamma = 0$, the derivative is negative, it is clear that the γ, which satisfies $\frac{\partial f(\gamma)}{\partial \gamma} = 0$ is a point of minimum. The condition (b) is always satisfied, since we are studying the case $m = 2$.■

Remark 5.3 *The Lagrange multiplier of the constrained problem in Lemma 5.1, for a* *fixed* γ*, has the value* $\varrho = \frac{\sigma_0^4}{(\sigma_0^2+\sigma_V^2)\sigma_V^2}\frac{\gamma^{m-2}}{(1+\gamma)^m}$*. This is consistent with conditions (a) and (b) from* **Part II** *of Theorem 5.1.*

5.3 Two valued memoryless control

In this section, we show that, in general, affine functions are not optimal for Problem 5.1. The main result in this section is Theorem 5.2, where we show that two valued control reaches a cost that is lower than what would be the cost for the optimal affine control. The section ends with numerical results, illustrating that two valued control can be better than the optimal affine strategy.

We proceed by defining the class of two-valued control strategies, along with its associated cost.

Definition 5.3 *Given positive real numbers* $\{\sigma_i^2\}_{i=1}^m$*, define the class of functions* $\mathcal{F}_i^B : \mathbb{R} \to \{-1, 1\}\,,\ i \in \{0, 1, \ldots, m-1\}$ *as follows:*

$$\mathcal{F}_i^B(x) = \sigma_{i+1} sgn(x), i \in \{0, \ldots, m-1\} \tag{5.37}$$

where the function $sgn : \mathbb{R} \to \{-1, 1\}$ *is the standard sign function.*

Definition 5.4 *Given positive real numbers* $\{\sigma_i^2\}_{i=1}^m$*, assume that the control strategies for Problem 5.1 are obtained via the functions* $\left\{\mathcal{F}_i^B\right\}_{i=0}^{m-1}$*, given in (5.37), as is follows:*

$$U(k) = \mathcal{F}_k^B\left(Y(k)\right),\ k \in \{0, \ldots m-1\} \tag{5.38}$$

Consider the following cost:

$$\mathcal{C}_B(\{\sigma_k^2\}_{k=1}^m, \sigma_0^2, \sigma_V^2) \overset{def}{=} E\left[(X(m) - X(0))^2\right] \tag{5.39}$$

obtained under the control law (5.38).

Lemma 5.4 *Let the parameters in Problem 5.1 and the positive real numbers* $\{\sigma_i^2\}_{i=1}^m$ *be given. Adopt the two valued control strategies from Definition 5.4. The following holds:*

$$\begin{aligned}&\mathcal{C}_B(\{\sigma_k^2\}_{k=1}^m,\sigma_0^2,\sigma_V^2)=\sigma_0^2+\sigma_m^2\\&-4\frac{\sigma_m\sigma_0^2}{\sqrt{2\pi(\sigma_0^2+\sigma_V^2)}}\prod_{i=1}^{m-1}\left(2P(V(i)\leq\sigma_i)-1\right)\end{aligned}\tag{5.40}$$

Proof: Before proving the claim in Lemma 5.4, one needs to prove the following.

$$P\left(U(k)=\sigma_{k+1}\right)=\frac{1}{2},\quad k\in\{0,\dots m-1\}$$

The proof of the claim above is done by induction. For $k=0$, $P\left(U(0)=\sigma_1\right)=P\left(Y(0)>0\right)=\frac{1}{2}$. Assume that the claim holds for $0\leq k<m-1$.

$$\begin{aligned}&P\left(U(k+1)=\sigma_{k+2}\right)\\&=P\left(Y(k+1)>0\right)=P\left(U(k)+V(k+1)>0\right)\\&=\frac{1}{2}P\left(U(k)+V(k+1)>0|U(k)=\sigma_{k+1}\right)\\&+\frac{1}{2}P\left(U(k)+V(k+1)>0|U(k)=-\sigma_{k+1}\right)\\&=\frac{1}{2}P\left(V(k+1)>-\sigma_{k+1}\right)+\frac{1}{2}P\left(V(k+1)>\sigma_{k+1}\right)=\frac{1}{2}\end{aligned}$$

We remind to the reader that $U(m-1)=X(m)$. We need to prove that:

$$\begin{aligned}E\left[X(m)|Y(m-k-1)<0\right]&=-\sigma_m\prod_{i=m-k}^{m-1}\left(2P\left(V(i)\leq\sigma_i\right)-1\right)\\E\left[X(m)|Y(m-k-1)>0\right]&=\sigma_m\prod_{i=m-k}^{m-1}\left(2P\left(V(i)\leq\sigma_i\right)-1\right)\end{aligned}$$

for $1 \le k \le m$. We prove this by induction. For $k = 1$:

$$E\left[X(m)|Y(m-2)<0\right]$$

$$= \sigma_m P\left(X(m) = \sigma_m | Y(m-2) < 0\right)$$

$$- \sigma_m P\left(X(m) = -\sigma_m | Y(m-2) < 0\right)$$

$$= \sigma_m P\left(V(m-1) > \sigma_{m-1}\right) - \sigma_m P\left(V(m-1) < \sigma_{m-1}\right)$$

$$= -\sigma_m \left(2P\left(V(m-1) < \sigma_{m-1}\right) - 1\right)$$

In the same way we show that:

$$E\left[X(m)|Y(m-2)>0\right] = \sigma_m \left(2P\left(V(m-1) < \sigma_{m-1}\right) - 1\right)$$

Assume that the claim holds for all i, $1 \le i \le k$. We need to prove it for $k+1$.

$$E\left[X(m)|Y(m-k-2)<0\right]$$

$$= E\left[X(m)|Y(m-k-1)<0, Y(m-k-2)<0\right]$$

$$\cdot P\left(Y(m-k-1)<0|Y(m-k-2)<0\right)$$

$$+ E\left[X(m)|Y(m-k-1)>0, Y(m-k-2)<0\right]$$

$$\cdot P\left(Y(m-k-1)>0|Y(m-k-2)<0\right)$$

$$= E\left[X(m)|Y(m-k-1)<0\right] P\left(V(m-k-1) < \sigma_{m-k-1}\right)$$

$$+ E\left[X(m)|Y(m-k-1)>0\right] P\left(V(m-k-1) > \sigma_{m-k-1}\right)$$

$$= -\sigma_m \prod_{i=m-k}^{m-1} \left(2P\left(V(i) \le \sigma_i\right) - 1\right) P\left(V(m-k-1) < \sigma_{m-k-1}\right)$$

$$+ \sigma_m \prod_{i=m-k}^{m-1} \left(2P\left(V(i) \le \sigma_i\right) - 1\right) P\left(V(m-k-1) > \sigma_{m-k-1}\right)$$

$$= -\sigma_m \prod_{i=m-k-1}^{m-1} \left(2P\left(V(i) \le \sigma_i\right) - 1\right)$$

In the same way, we show the induction step for $E\left[X(m)|Y(m-k-2)>0\right]$.By the way the functions $\mathcal{F}_i^B,\ i\in\{0,\ldots,m-1\}$ are defined, the following equalities are true:

$$
\begin{aligned}
E\left[X(m)|Y(k)<0\right] =& E\left[X(m)|Y(k)=-\alpha\right] \\
=& E\left[X(m)|U(k+1)=-\sigma_{k+1}\right] \\
E\left[X(m)|Y(k)>0\right] =& E\left[X(m)|Y(k)=\beta\right] \\
=& E\left[X(m)|U(k+1)=\sigma_{k+1}\right]
\end{aligned}
$$

where $0\leq k\leq m-2$ and α and β are any postive real numbers. These equalities are immediate since $\mathcal{F}_k^B(x)=\sigma_k sgn(x)$.

The cost function defined in the lemma is:

$$
\begin{aligned}
&E\left[(X(0)-X(m))^2\right] = \sigma_0^2+\sigma_m^2-2E\left[X(0)X(m)\right] \\
&E\left[X(0)X(m)\right] = E\left[E\left[X(0)X(m)|X(0),V(0)\right]\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[X(0)X(m)|X(0)=x_0,V(0)=v_0\right] \\
&\qquad \cdot \frac{1}{2\pi\sqrt{\sigma_V^2\sigma_0^2}} e^{-\left(\frac{x_0^2}{2\sigma_0^2}+\frac{v_0^2}{2\sigma_V^2}\right)} dx_0 dv_0 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{-v_0} x_0 E\left[X(m)|X(0)=x_0,V(0)=v_0\right]
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \frac{1}{2\pi\sqrt{\sigma_V^2\sigma_0^2}} e^{-\left(\frac{x_0^2}{2\sigma_0^2}+\frac{v_0^2}{2\sigma_V^2}\right)} dx_0 dv_0 \\
&+ \int_{-\infty}^{\infty}\int_{-v_0}^{\infty} x_0 E\left[X(m)|X(0)=x_0, V(0)=v_0\right] \\
&\cdot \frac{1}{2\pi\sqrt{\sigma_V^2\sigma_0^2}} e^{-\left(\frac{x_0^2}{2\sigma_0^2}+\frac{v_0^2}{2\sigma_V^2}\right)} dx_0 dv_0 \\
&= -\sigma_m \prod_{i=1}^{m-1}\left(2P\left(V(i)\le\sigma_i\right)-1\right) \\
&\int_{-\infty}^{\infty}\int_{-\infty}^{-v_0} x_0 \frac{1}{2\pi\sqrt{\sigma_V^2\sigma_0^2}} e^{-\left(\frac{x_0^2}{2\sigma_0^2}+\frac{v_0^2}{2\sigma_V^2}\right)} dx_0 dv_0 \\
&+ \sigma_m \prod_{i=1}^{m-1}\left(2P\left(V(i)\le\sigma_i\right)-1\right) \\
&\int_{-\infty}^{\infty}\int_{-v_0}^{\infty} x_0 \frac{1}{2\pi\sqrt{\sigma_V^2\sigma_0^2}} e^{-\left(\frac{x_0^2}{2\sigma_0^2}+\frac{v_0^2}{2\sigma_V^2}\right)} dx_0 dv_0 \\
&= 2\sigma_m \prod_{i=1}^{m-1}\left(2P\left(V(i)\le\sigma_i\right)-1\right)\frac{\sigma_0^2}{\sqrt{2\pi(\sigma_0^2+\sigma_V^2)}}
\end{aligned}
$$

It follows that:

$$
\begin{aligned}
&E\left[(X(0)-X(m))^2\right] = \sigma_0^2+\sigma_m^2 \\
&-4\sigma_m \prod_{i=1}^{m-1}\left(2P\left(V(i)\le\sqrt{\sigma_i^2}\right)-1\right)\frac{\sigma_0^2}{\sqrt{2\pi(\sigma_0^2+\sigma_V^2)}} \blacksquare
\end{aligned}
$$

The cost (5.40) in Lemma 5.4 can be minimized with respect to σ_m as follows:

$$
\mathcal{C}_B^*(\{\sigma_k^2\}_{k=1}^{m-1}, \sigma_0^2, \sigma_V^2) \stackrel{def}{=} \min_{\sigma_m} \mathcal{C}_B\left(\{\sigma_k^2\}_{k=1}^{m}, \sigma_0^2, \sigma_V^2\right) \tag{5.41}
$$

Minimization of (5.40) with respect to σ_m leads to:

$$
\mathcal{C}_B^*\left(\{\sigma_k^2\}_{k=1}^{m-1}, \sigma_0^2, \sigma_V^2\right) = \sigma_0^2 - \frac{4}{2\pi}\sigma_0^2 \prod_{i=1}^{m-1}\left(2P(V(i)\le\sigma_i)-1\right)^2 \cdot \frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}
$$

Theorem 5.2 *Let all the parameters defining Problem 5.1 be given. There exists a positive real number ϱ, an integer m and measurable nonlinear functions $\{\mathcal{F}_i\}_{i=0}^{m-1}$ such that:*

$$
\mathcal{J}(\{\mathcal{F}_k\}_{k=0}^{m-1}, \varrho, \sigma_0^2, \sigma_V^2) < \mathcal{J}_A^*\left(m, \varrho, \sigma_0^2, \sigma_V^2\right) \tag{5.42}
$$

where $\mathcal{J}_A^*(m, \varrho, \sigma_0^2, \sigma_V^2)$, as defined in Theorem 5.1, is the optimal cost of Problem 5.1 subject to affine strategies of the form (5.14).

Proof: In order to prove Theorem 5.2, we will choose a positive real number γ and an integer m such that the following conditions hold: **(i)** $\frac{\left(2\Phi\left(\sqrt{\gamma}\right)-1\right)^{2(m-1)}}{\left(\frac{\gamma}{1+\gamma}\right)^{m-1}} > \frac{2\pi}{4}$ and **(ii)** $m < \gamma + 2$. We denote by $\Phi(x)$ the cumulative distribution function of a normal random variable with zero mean and unit variance.

For the chosen pair of parameters (γ, m) we will show that $\mathcal{C}_A^*(m, \gamma, \sigma_0^2, \sigma_V^2) > \mathcal{C}_B^*\left(\{\gamma\sigma_V^2\}_{k=1}^{m-1}, \sigma_0^2, \sigma_V^2\right)$. We will choose $\varrho = \frac{\sigma_0^4}{\left(\sigma_0^2+\sigma_V^2\right)\sigma_V^2}\frac{\gamma^{m-2}}{(1+\gamma)^m}$, and the functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ from the class of function given in Definition 5.4, for which we select $\sigma_k = \sqrt{\gamma}\sigma_V,\ k \in \{1, \ldots, m-1\}$ and choose σ_m in order to minimize the cost defined in equation (5.41). For this choice of nonlinear functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ we will prove that $\mathcal{J}(\{\mathcal{F}_k\}_{k=0}^{m-1}, \varrho, \sigma_0^2, \sigma_V^2) < \mathcal{J}_A^*(m, \varrho, \sigma_0^2, \sigma_V^2)$.

We need to prove that there exists a pair of parameters (γ, m) which satisfies the conditions **(i)** and **(ii)**. We notice that $\lim_{\gamma\to\infty}\left(\frac{\gamma}{1+\gamma}\right)^{\gamma+1} = e^{-1}$, $\lim_{\gamma\to\infty}\left(2\Phi(\sqrt{\gamma})-1\right)^{2(\gamma+1)} = 1$ and $e > \frac{2\pi}{4}$. Adopt $m = \lfloor \gamma + 1 \rfloor$ and then choose γ large enough, and it follows that both conditions **(i)** and **(ii)** are satisfied for the pair (γ, m).

Choose $\mathcal{F}_k(x) = \sqrt{\gamma\sigma_V^2}sgn(x), k \in \{0, \ldots m-2\}$, choose $\mathcal{F}_{m-1}(x) = \sigma_m sgn(x)$ and let σ_m be the minimizer of the cost defined in (5.41) for which $\sigma_k^2 = \gamma\sigma_V^2, k \in \{1, \ldots m-1\}$. It is clear that by this choice of functions, it holds that $\sum_{k=0}^{m-2} E\left[U(k)^2\right] =$

$(m-1)\gamma\sigma_V^2$, while the cost $\mathcal{C}_B^*\left(\{\gamma\sigma_V^2\}_{k=1}^{m-1}, \sigma_0^2, \sigma_V^2\right)$ becomes:

$$
\begin{aligned}
&\mathcal{C}_B^*\left(\{\gamma\sigma_V^2\}_{k=1}^{m-1}, \sigma_0^2, \sigma_V^2\right) \\
&= \sigma_0^2 - \frac{4}{2\pi}\sigma_0^2 \prod_{i=1}^{m-1} \left(2P\left(V(i) \le \sqrt{\gamma}\sigma_V\right) - 1\right)^2 \frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2} \\
&= \sigma_0^2 - \frac{4}{2\pi} \prod_{i=1}^{m-1} \left(2P\left(\frac{V(i)}{\sigma_V} \le \sqrt{\gamma}\right) - 1\right)^2 \frac{\sigma_0^4}{\sigma_0^2+\sigma_V^2} \\
&= \sigma_0^2 - \frac{4}{2\pi}\left(2\Phi\left(\sqrt{\gamma}\right) - 1\right)^{2(m-1)} \frac{\sigma_0^4}{\sigma_0^2+\sigma_V^2}
\end{aligned}
$$

Since the pair (γ, m) satisfies conditions **(i)** and **(ii)** it follows that:

$$
\begin{aligned}
&\sigma_0^2\left(1 - \frac{4}{2\pi}\left(2\Phi\left(\sqrt{\gamma}\right) - 1\right)^{2(m-1)} \frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}\right) \\
&< \sigma_0^2\left(1 - \frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}\frac{\gamma^{m-1}}{(1+\gamma)^{m-1}}\right) = \mathcal{C}_A^*(m, \gamma, \sigma_0^2, \sigma_V^2)
\end{aligned}
$$

Adopt $\varrho = \frac{\sigma_0^4}{\left(\sigma_0^2+\sigma_V^2\right)\sigma_V^2}\frac{\gamma^{m-2}}{(1+\gamma)^m}$. We note that with the ϱ and m chosen above, the conditions (a) and (b) of Theorem 5.1 are satisfied. Hence the cost of Problem 5.1 subject to affine strategies of the form (5.14) is given by equation (5.36), with the optimum γ being non-zero. Moreover, the γ chosen above, which together with m satisfies conditions **(i)** and **(ii)**, is the optimal solution of the minimization problem from equation (5.36), for the selected ϱ and m. It follows that:

$$
\begin{aligned}
\mathcal{J}_A^*\left(m, \varrho, \sigma_0^2, \sigma_V^2\right) &= \mathcal{C}_A^*\left(m, \gamma, \sigma_0^2, \sigma_V^2\right) + \varrho(m-1)\gamma\sigma_V^2 \\
&> \sigma_0^2 - \frac{4}{2\pi}\sigma_0^2\left(2\Phi\left(\sqrt{\gamma}\right) - 1\right)^{2(m-1)} \frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2} \\
&+ \varrho(m-1)\gamma\sigma_V^2 = \mathcal{J}(\{\mathcal{F}_k\}_{k=0}^{m-1}, \varrho, \sigma_0^2, \sigma_V^2)
\end{aligned}
$$

This shows that the cost obtained by nonlinear controls is less than the cost obtained by affine controls, hence the optimum of Problem 5.1 is reached in general by nonlinear functions rather than affine functions. ■

5.3.1 Numerical Results

The following cost will be used throughout this subsection:

$$\begin{aligned}\mathcal{J}_N\left(\gamma, m, \varrho, \sigma_0^2, \sigma_V^2\right) &\stackrel{def}{=} \varrho(m-1)\gamma\sigma_V^2 \\ &+ \sigma_0^2 - \frac{4}{2\pi}\left(2\Phi\left(\sqrt{\gamma}\right)-1\right)^{2(m-1)}\frac{\sigma_0^4}{\sigma_0^2+\sigma_V^2},\ \gamma \geq 0\end{aligned} \tag{5.43}$$

We notice that $\mathcal{J}_N\left(\gamma, m, \varrho, \sigma_0^2, \sigma_V^2\right)$ is the cost associated with the two-valued control strategy given in Definition 5.3, for which $\sigma_i^2 = \gamma\sigma_V^2$, for $k \in \{1, \dots, m-1\}$ and σ_m is chosen to minimize (5.41). Let γ_{opt}^N be an optimal solution for $\min_{\gamma\geq 0} \mathcal{J}_N\left(\gamma, m, \varrho, \sigma_0^2, \sigma_V^2\right)$, and γ_{opt}^L be an optimal solution for (5.36).

In the proof of Theorem 5.2, we compared the optimal cost of Problem 5.1 subject to affine strategies, i.e. $\mathcal{J}_A^*\left(m, \varrho, \sigma_0^2, \sigma_V^2\right)$ with $\mathcal{J}_N\left(\gamma_{opt}^L, m, \varrho, \sigma_0^2, \sigma_V^2\right)$. We notice that by adopting $\sigma_i^2 = \gamma_{opt}^N\sigma_V^2$ for $k \in \{1, \dots, m-1\}$ in Definition 5.3 with the appropriate σ_m from equation (5.41), we arrive at $\mathcal{J}_N\left(\gamma_{opt}^N, m, \varrho, \sigma_0^2, \sigma_V^2\right) \leq \mathcal{J}_N\left(\gamma_{opt}^L, m, \varrho, \sigma_0^2, \sigma_V^2\right)$.

We now proceed to discussing the numerical results in Table 5.1. Subsequently, we set the parameters σ_0^2 and σ_V^2 to 1 and 0.1, respectively. The numerical results from Table 5.1 are structured as follows, the first two columns denote the parameters m, i.e. the number of stages and ϱ, the third column gives the optimal value for γ_{opt}^L, the fourth column gives the optimal cost for affine functions $\mathcal{J}_A^*\left(m, \varrho, \sigma_0^2, \sigma_V^2\right)$, the fifth column denotes γ_{opt}^N, while the sixth column is the cost $\mathcal{J}\left(\gamma_{opt}^N, m, \varrho, \sigma_0^2, \sigma_V^2\right)$.

For $m = 2$, we have proved analytically that affine functions achieve the optimal cost, hence we chose not to include any corresponding numerical data in the table. For $m = 3$ or $m = 4$, all our numerical experiments showed that the affine strategies are better than nonlinear strategies, but we could not prove it analytically. Hence, for $m = 3$ and $m = 4$, we do not know whether the optimal solution is affine. For $m \geq 5$, for some values of ρ, we were able to find nonlinear strategies that achieve smaller cost when compared to the optimal affine.

This chapter investigates the design of a sequential linear quadratic Gaussian estimation system comprising of multiple decision stages. Our paradigm can also be cast as the optimal control of a unit delay system in discrete-time driven by white Gaussian noise, and subject to memoryless strategies over a finite time-horizon. We conclude from our analysis given that, for certain expected squared error measures, optimal strategies are linear for up to two stages and nonlinear for a sufficiently large number of stages. Since our framework features a non-nested information pattern for two or more stages, the existence of optimal linear strategies for our problem cannot be predicted via other existing methods. Several problems remain open, such as determining if linear strategies can be optimal for three or four stages, and devising systematic methods for designing high performance strategies for the cases where linear solutions are not optimal.

m	ϱ	γ_{opt}^{L}	$cost_{Lin}$	γ_{opt}^{N}	$cost_{Nonlin}$	Opt
3	0.42	3.03	0.74	3.17	0.84	L
3	0.21	5.00	0.58	4.47	0.68	L
3	0.06	10.77	0.37	6.72	0.52	L
4	0.32	3.00	0.90	3.45	0.94	L
4	0.14	5.89	0.68	5.12	0.71	L
4	0.06	10.21	0.50	6.69	0.57	L
5	0.24	0.00	1.00	3.85	0.98	NL
5	0.19	3.92	0.93	4.39	0.90	NL
5	0.10	6.75	0.75	5.69	0.72	NL
5	0.06	9.61	0.62	6.65	0.63	L
10	0.09	0.00	1.00	0.00	1.00	NL-L
10	0.06	0.00	1.00	6.44	0.87	NL
10	0.04	8.81	0.97	7.27	0.75	NL
10	0.02	15.63	0.76	8.61	0.61	NL
10	0.005	37.36	0.45	11.20	0.48	L

Table 5.1: Comparison between the optimal cost with affine functions and the cost with discrete value functions. Here, $cost_{Lin}$ and $cost_{Nonlin}$ refer to $\mathcal{J}_A^*\left(m, \varrho, \sigma_0^2, \sigma_V^2\right)$ and $\mathcal{J}\left(\gamma_{opt}^N, m, \varrho, \sigma_0^2, \sigma_V^2\right)$, respectively.

Chapter 6

Conclusions and Future Directions: Extension to More General Network Topologies

6.1 General Network Topologies

We have showen in Chapter 2 we solve a distributed estimation problem which consists from a pre-processor or encoder and an estimator or a decoder, shown in Figure 1.1. The preprocessor has perfect knowledge about a stochastic process and the decoder has access only to the information which it receives from the decoder. Each time the encoder sends information to the decoder it must pay a cost for communication. The encoder and the decoder must jointly optimize a common cost, which consists from the estimation error and the communication cost. The problem which arises is when and what information must be sent to the estimator. It was shown that the optimal policy to send sample to the estimator is a threshold policy. In Chapter 3, we present some applications of the problem presented in Chapter 2, from which we include general costs and noise distributions, noisy observation at the pre-processor side, a quadratic control problem, a problem where we consider packet drop with acknowledgement, infinite time horizon (the discounted cost and the average cost) and the tandem problem. In Chapter 4, we show that if we tackle the problem described in Chapter 2, but we look at the multidimensional case, things can get quite complicated. First, the method used for proving the linearity of the estimator fails.

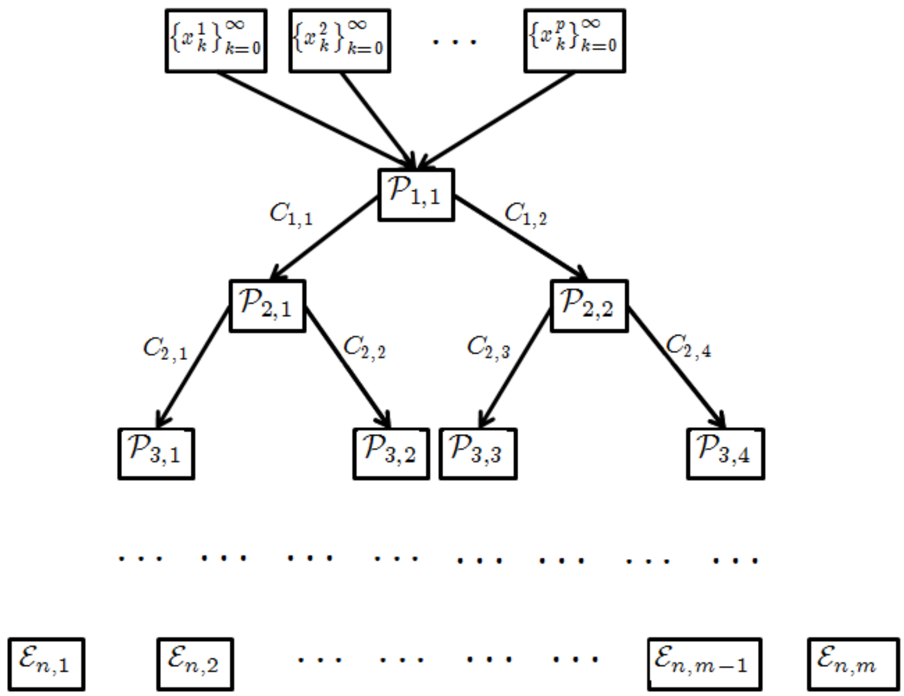

Figure 6.1: Tree Topology

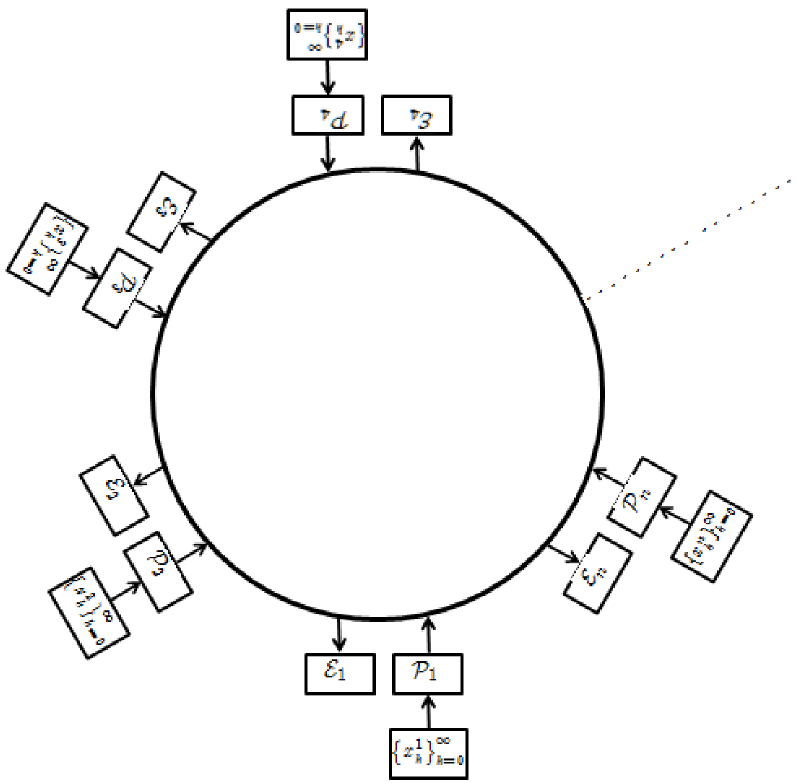

Figure 6.2: Ring Topology

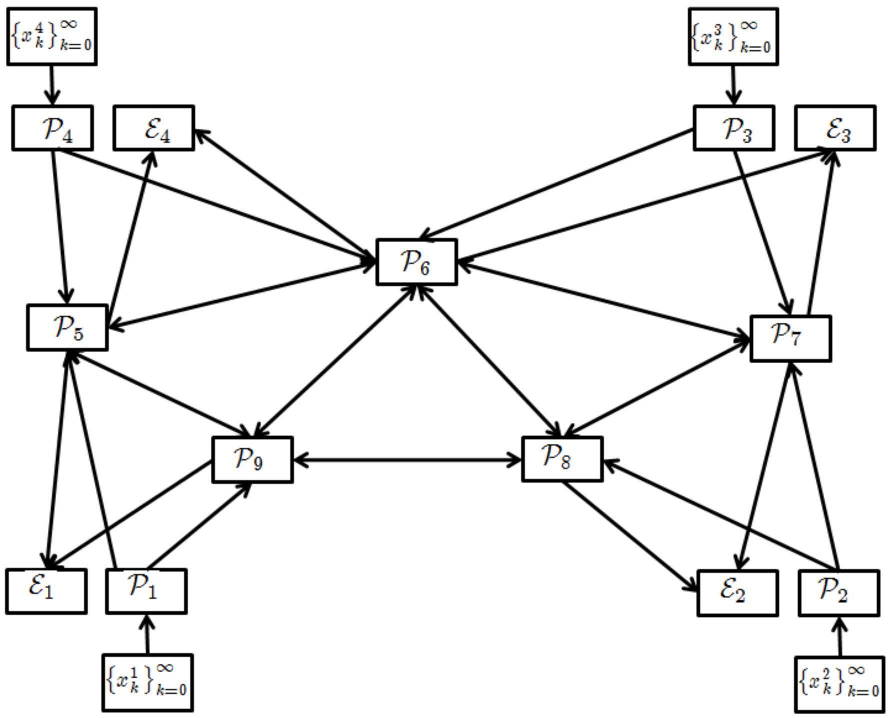

Figure 6.3: Network

If we consider a linear estimator (or equivalent, a symmetric policy at the pre-processor), it is difficult to prove properties of the decision sets for time horizons bigger or equal to three. Moreover, for the time horizon two, or at the penultimate stage, we found numerically that the decision sets need not be convex. In Chapter 5, we present a problem with multiple agents and noisy transmission links. In this case, we show that simple affine strategies are not optimal, despite the fact that the problem has quadratic costs and Gaussian noise. We show numerically that signalling strategies perform actually better. moreover, we cannot compute the optimal strategies.

In Chapters 2 and 3 we show how to solve the two blocks problem, while in Chapters 4 and 5, we show the limitations of the methods used in Chapters 2 and 3. The future directions of these work are to look at general network topologies and performing distributed estimation and control over networks. In Fig. 6.1, we present a network with a tree topology, where a pre-processor tracks a number of stochastic processes. The pre-processor has to send information about these processes to the intermediate pre-processes and will pay a communication cost. The intermediate pre-processors have to route these information eventually to the estimators, which represent the leafs of the tree topology. The estimators have to estimate the stochastic processes tracked by the root pre-processor. The pre-processors and the estimators must jointly optimize a cost function, which consists both from the communication costs and the estimation error. In Fig. 6.2, we present a ring topology, where each node tracks a stochastic process and has an estimator, which will try to estimate processes from other nodes. Just like in the previous case, although not depicted in the figure, for each transmission there is a communication cost, and the entire network must jointly optimize a cost consisting from functions on the estimation error and functions on the communication costs. The goal is to end up with general topologies as in Fig. 6.3, where some of the nodes can just pass information through the network, like $\mathcal{P}_5$ through $\mathcal{P}_9$, or can be nodes that either track some process or estimate other processes like $\mathcal{P}_1$ through $\mathcal{P}_4$ and $\mathcal{E}_1$ through $\mathcal{E}_4$. For all these networks the transmission links can be noisy links, i.e. the signal can be affected by transmission noise, like a Gaussian addtive noise, or the information send through the links can be lost, as in the packet drop cases. The goal is to analyze all these networks having as base the results obtained in the chapters 2, 3, 4 and 5.

Appendix A

Appendix

A.1 Majorization Theory

Lemma A.1 *If f and h are neat and even probability density functions, then $f * h$ is also neat and even, where by $f * h$ we mean the convolution between f and h.*

Proof: Since b is a distribution function, it implies that is also measurable. Let $g : \mathbb{R} \to \mathbb{R}$ be defined as:

$$g(x) = \begin{cases} 1, x \in [-\alpha, \alpha] \\ 0, x \notin [-\alpha, \alpha] \end{cases}$$

where α is a positive real number. We notice that g is an indicator function. We claim that $f * g$ is neat and even.

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt = \int_{-\alpha}^{\alpha} f(x-t)dt = \int_{-\alpha-x}^{\alpha-x} f(y)dy \tag{A.1}$$

Since the function f is neat and even, it is clear that $f * g$ is neat and even from equation (A.1). The function $f * g$ is neat and even also for the case when $g(x) = 1$ on a symmetric open interval $(-\alpha, \alpha)$.

We need to prove the main claim of Lemma A.1. We do this by approximating the function b with a sum of functions of the type of function g. Since b is neat and even it follows that $b(0) \geq b(x)$, for any real number x. For a positive integer number n, and positive integer $k \leq n$, define the function b_n as follows:

$$b_n(x) = b(0)\frac{k}{n}, \quad b(0)\frac{k}{n} \leq b(x) < b(0)\frac{k+1}{n} \tag{A.2}$$

It follows that $b_n(x) \leq b_{n+1}(x)$ for every real number x and that $b_n \to b$. Moreover, from the monotone convergence theorem [14], it follows that $f * b_n \to f * b$.

Since b is neat and even it follows that for every integer n and integer $k \leq n$, there exists a positive α_k^n such that $b(x) \geq b(0)\frac{k}{n}$ on the interval $\mathbb{I}_k^n = [-\alpha_k^n, \alpha_k^n]$ or $\mathbb{I}_k^n = (-\alpha_k^n, \alpha_k^n)$ and $b(x) < b(0)\frac{k}{n}$ outside $\mathbb{I}_k^n$. The function b_n can be written as follows:

$$b_n(x) = b(0)\frac{1}{n}\sum_{k=0}^{n} \mathcal{I}_{\mathbb{I}_k^n}(x)$$

where by $\mathcal{I}_{\mathbb{I}_k^n}$ we denote the indicator function of the interval $\mathbb{I}_k^n$.

$$f * b_n = b(0)\frac{1}{n}\sum_{k=0}^{n} f * \mathcal{I}_{\mathbb{I}_k^n}$$

It follows that $f * b_n$ is neat and even, hence $f * b$ is neat and even. ■

Remark A.1 *From the proof of Lemma A.1, it follows that the claim of Lemma A.1 holds if f and b are any nonnegative, even, quasiconcave and integrable functions.*

We will state now two important inequalities, which are useful for this paper. The first one is the Riesz's rearrangement inequality:

Lemma A.2 (Riesz's Rearrangement inequality [2]) *If f, g and h are nonnegative functions on $\mathbb{R}^n$, then:*

$$\int_{\mathbb{R}^n} f(x)\,(g * h)\,(x)dx \leq \int_{\mathbb{R}^n} f^\sigma(x)\,(g^\sigma * h^\sigma)\,(x)dx \tag{A.3}$$

The second important inequality, which we need is the Hardy-Littlewood inequality [3].

Lemma A.3 (Hardy-Littlewood inequality [3]) *Let f and g be two nonnegative measurable functions defined on the real line, which vanish at infinity, then the following holds:*

$$\int f(x)g(x)dx \leq \int f^{\sigma}(x)g^{\sigma}(x)dx \tag{A.4}$$

We state and prove the following Lemmas, which are a supporting results for Lemma 2.2 in Subsection 2.4.1.

Lemma A.4 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a symmetric and nonincreasing probability distribution function. Then for any positive $\kappa \leq 1$, there exists a symmetric convex set $\mathbb{K}$ centered around zero such that:*

$$\int_{\mathbb{K}} f(x)dx = 1 - \kappa$$

and for any other set $K' \subset \mathbb{R}^n$, for which:

$$\int_{\mathbb{K}'} f(x)dx = 1 - \kappa$$

the following holds:

$$f_{\mathbb{K}} \succ f_{\mathbb{K}'} \tag{A.5}$$

Proof: Assume that there exists ρ such that $\int_{\{x\in\mathbb{R}^n : f(x)>\rho\}} f(x)dx = 1 - \kappa$, then let $\mathbb{K} = \{x \in \mathbb{R}^n : f(x) > \rho\}$. Since, f is symmetric and nonincreasing, it follows that K is a symmetric set. Let any other set $\mathbb{K}'$ such that $\int_{\mathbb{K}'} f(x)dx = 1 - \kappa$. Choose any set $\mathbb{F}' \subset \mathbb{K}'$, if $\mathcal{L}^n(\mathbb{F}') \geq \mathcal{L}^n(\mathbb{K})$, let $\mathbb{F} \subset \mathbb{R}^n$ be any measurable set, such that $\mathcal{L}^n(\mathbb{F}) = \mathcal{L}(\mathbb{F}')$ and $\mathbb{K} \subset \mathbb{F}$, it follows that :

$$\int_{\mathbb{F}} f_{\mathbb{K}}(x)dx = 1 \geq \int_{\mathbb{F}'} f_{\mathbb{K}'}(x)dx$$

since both $f_{\mathbb{K}}$ and $f_{\mathbb{K}'}$ are probability distribution functions. If $\mathcal{L}^n(\mathbb{F}') \leq \mathcal{L}^n(\mathbb{K})$, then choose any set $\mathbb{F} \subset \mathbb{K}$, such that $\mathcal{L}^n(\mathbb{F}) = \mathcal{L}^n(\mathbb{F}')$. Let $\mathbb{F}_1 = \mathbb{F} \cap \mathbb{F}'$, then by the way the set $\mathbb{K}$ is defined, for any real number $x \in \mathbb{F}' \setminus \mathbb{F}_1$ it holds that $f(x) \leq \rho$, while on the set $\mathbb{F} \setminus \mathbb{F}_1$, $f(x) \geq \rho$.

$$\begin{aligned}
\int_{\mathbb{F}} f_{\mathbb{K}}(x)dx = \frac{1}{1-\kappa}\int_{\mathbb{F}} f(x)dx &= \frac{1}{1-\kappa}\left(\int_{\mathbb{F}_1} f(x)dx + \int_{\mathbb{F}\setminus\mathbb{F}_1} f(x)dx\right) \\
&\geq \frac{1}{1-\kappa}\left(\int_{\mathbb{F}_1} f(x)dx + \int_{\mathbb{F}\setminus\mathbb{F}_1} \rho dx\right) \\
&\geq \frac{1}{1-\kappa}\left(\int_{\mathbb{F}_1} f(x)dx + \int_{\mathbb{F}'\setminus\mathbb{F}_1} f(x)dx\right) \\
&= \frac{1}{1-\kappa}\int_{\mathbb{F}'} f(x)dx = \int_{\mathbb{F}'} f_{\mathbb{K}'}(x)dx
\end{aligned}$$

The second ineqaulity is due to the fact that $\mathbb{F} \setminus \mathbb{F}_1$ and $\mathbb{F}' \setminus \mathbb{F}_1$ have the same measure.

Assume that, there is no such ρ, such that $\int_{\{x\in\mathbb{R}^n : f(x)>\rho\}} f(x)dx = 1-\kappa$. The integral $\int_{\{x\in\mathbb{R}^n : f(x)>\rho\}} f(x)dx$ is decreasing as a function of ρ and is also bounded. It follows than that, there exist a ρ such that $\int_{\{x\in\mathbb{R}^n : f(x)>\rho\}} f(x)dx < 1-\kappa$ and $\int_{\{x\in\mathbb{R}^n : f(x)\geq\rho\}} f(x)dx \geq 1-\kappa$. Both the sets $\{x \in \mathbb{R}^n : f(x) > \rho\}$ and $\{x \in \mathbb{R}^n : f(x) \geq \rho\}$ are symmetric and convex and $\{x \in \mathbb{R}^n : f(x) > \rho\} \subset \{x \in \mathbb{R}^n : f(x) \geq \rho\}$. Then we can find a $\mathbb{K} \subset \{f(x) \geq \rho\}$ symmetric around the origin and convex such that $\int_{\mathbb{K}} f(x)dx = 1-\kappa$. Using the same type of arguments like in the first case we get that $f_{\mathbb{K}} \succ f_{\mathbb{K}'}$ for any $\mathbb{K}' \subset \mathbb{R}^n$ such that $\int_{\mathbb{K}'} f(x)dx = 1-\kappa$ ■

An immediate consequence of Lemma A.4 is the fact that if the probability distribution function f is defined on the real line, then the convex set $\mathbb{K}$ is a symmetric interval centered around zero.

Lemma A.5 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two probability distribution functions, such that f is* *neat and symmetric and $f \succ g$. Let κ be a real number such that $0 < \kappa < 1$. Let $\mathbb{K}$ be the symmetric interval given by Lemma A.4 for the distribution f and the number κ. Then for any set $\mathbb{K}' \subset \mathbb{R}$ such that $\int_{\mathbb{K}'} g(x)dx = 1 - \kappa$ the following holds:*

$$f_{\mathbb{K}} \succ g_{\mathbb{K}'} \tag{A.6}$$

Proof: Fix $\mathbb{K}' \in \mathbb{R}$. Choose a set $\mathbb{F}' \in \mathbb{K}'$ with strictly positive Lebesgue measure. If $\mathcal{L}(\mathbb{F}') \geq \mathcal{L}(\mathbb{K})$, choose $\mathbb{F}$ any set with $\mathcal{L}(\mathbb{F}) = \mathcal{L}(\mathbb{F}')$, such that $\mathbb{K} \subset \mathbb{F}$. It is clear in this case that $\int_{\mathbb{F}} f_{\mathbb{K}}(x)dx = 1 \geq \int_{\mathbb{F}'} g_{\mathbb{K}'}(x)dx$. If $\mathcal{L}(\mathbb{F}') \leq \mathcal{L}(\mathbb{K})$, then because $f \succ g$, there exists a set $\mathbb{F}'' \in \mathbb{R}$, such that $\mathcal{L}(\mathbb{F}'') = \mathcal{L}(\mathbb{F}')$ and $\int_{\mathbb{F}''} f(x)dx \geq \int_{\mathbb{F}'} g(x)dx$. Choose $\mathbb{K}''$ a set which contains $\mathbb{F}''$ and $\int_{\mathbb{K}''} f(x)dx = 1 - \kappa$. By Lemma A.4, $f_{\mathbb{K}''} \prec f_{\mathbb{K}}$, so it follows that there exists a set $\mathbb{F} \subset \mathbb{K}$, with the same Lebesgue measure as $\mathbb{F}''$ such that $\int_{\mathbb{F}} f(x)dx \geq \int_{\mathbb{F}''} f(x)dx \geq \int_{\mathbb{F}'} g(x)dx$ ■

Lemma A.6 *Let $f : \mathbb{R} \to \mathbb{R}$ be a probability distribution function and let κ be a positive real number, less then one. Let $\lambda : \mathbb{R} \to [0, 1]$ be a measurable positive function such that $\int_{\mathbb{R}} \lambda(x)f(x)dx = 1 - \kappa$. Then there exists a set $\mathbb{K} \in \mathbb{R}$ such that, $\int_{\mathbb{K}} f(x)dx = 1 - \kappa$ and $f_{\mathbb{K}} \succ \frac{\lambda \cdot f}{1-\kappa}$.*

Remark A.2 *Note that by the way they are defined $f_{\mathbb{K}}$ and $\frac{\lambda \cdot f}{1-\kappa}$ are probability distribution functions. Lemma A.6 states that for any probabilistic trimming can be majorized by a deterministic trimming.*

Proof: If exists ρ such that $\int_{\{x \in \mathbb{R} : f(x) > \rho\}} f(x)dx = 1-\kappa$, then let $\mathbb{K} = \{x \in \mathbb{R} : f(x) > \rho\}$.

If no such ρ exists, just like in the proof of Lemma A.4, there exists a ρ such that:

$$\int_{\{x\in\mathbb{R}:f(x)>\rho\}} f(x)dx < 1-\kappa, \quad \text{and} \quad \int_{\{x\in\mathbb{R}:f(x)\geq\rho\}} f(x)dx \geq 1-\kappa$$

i.e., there exists a set of Lebesgue measure strictly positive, such that $f(x)=\rho$. Choose a set $\mathbb{K}'=\{x\in\mathbb{R}:f(x)>\rho\}$. From the set $\{x\in\mathbb{R}:f(x)=\rho\}$, choose a subset $\mathbb{K}''$ of measure $\frac{1-\kappa-\int_{\{x\in\mathbb{R}:f(x)>\rho\}} f(x)dx}{\rho}$. Let $\mathbb{K}=\mathbb{K}'\cup\mathbb{K}''$. It follows that $\int_{\mathbb{K}} f(x)dx = 1-\kappa$ and by the way the set $\mathbb{K}$ is defined, it holds that $f(x)\geq\rho$, for all $x\in\mathbb{K}$. Let $\mathbb{F}'$ be a set in $\mathbb{R}$. If $\mathcal{L}(\mathbb{F}')\geq\mathcal{L}(\mathbb{K})$, choose $\mathbb{F}$ such that $\mathcal{L}(\mathbb{F}')=\mathcal{L}(\mathbb{F})$ and $\mathbb{K}\subset\mathbb{F}$. Then the following holds:

$$\int_{\mathbb{F}} f_{\mathbb{K}}(x)dx = 1 \geq \int_{\mathbb{F}'} \frac{f(x)}{1-\kappa}\lambda(x)dx$$

If $\mathcal{L}(\mathbb{F}')\leq\mathcal{L}(\mathbb{K})$, let $\mathbb{F}_1=\mathbb{F}'\cap\mathbb{K}$ and let $\mathbb{F}_2\subset\mathbb{K}\setminus\mathbb{F}_1$ such that $\mathcal{L}(\mathbb{F}_1\cup\mathbb{F}_2)=\mathcal{L}(\mathbb{F}')$. If $x\in\mathbb{F}_1$, $f(x)\geq\lambda(x)f(x)$, and if $x\in\mathbb{F}_2$, $f(x)\geq\rho$, and if $x\in\mathbb{F}'\setminus\mathbb{F}_1$, $\lambda(x)f(x)\leq f(x)\leq\rho$. It follows then:

$$\int_{\mathbb{F}_1\cup\mathbb{F}_2} f_{\mathbb{K}}(x)dx \geq \int_{\mathbb{F}'} \lambda(x)\frac{f(x)}{1-\kappa}dx$$

■

Lemma A.7 *Let $f,g:\mathbb{R}\to\mathbb{R}$ be two probability distribution functions such that $f\succ g$ on $\mathbb{R}$. Then, for any non zero constant a, define the following probability distribution functions:*

$$\tilde{f}(x) = \frac{1}{|a|}f\left(\frac{x}{a}\right)$$

$$\tilde{g}(x) = \frac{1}{|a|}g\left(\frac{x}{a}\right)$$

The following holds:

$$\tilde{f} \succ \tilde{g} \tag{A.7}$$

Remark A.3 *We notice that Lemma A.7 is well posed since $\tilde{f}$ and $\tilde{g}$ are also probability distribution functions. If f is the probability distribution function of a random variable X, then $\tilde{f}$ is the probability distribution function of the random variable aX.*

Proof: For a set $\mathbb{A} \subset \mathbb{R}$ and for a non zero constant α, define the set $\alpha\mathbb{A} = \left\{x \in \mathbb{R} : \frac{1}{\alpha}x \in \mathbb{A}\right\}$. Assume a to be positive and let $\mathbb{F}'$ be a set of positive and finite Lebesgue measure.

$$\int_{\mathbb{F}'} \tilde{g}(x)dx = \int_{\frac{1}{a}\mathbb{F}'} g(x)adx$$

since $f \succ g$, there exists a set $\mathbb{F}''$ with the same Lebesgue measure as $\frac{1}{a}\mathbb{F}'$ such that:

$$\int_{\frac{1}{a}\mathbb{F}'} g(x)adx \leq \int_{\mathbb{F}''} f(x)adx = \int_{a\mathbb{F}''} \tilde{f}(x)dx$$

Pick $\mathbb{F} = a\mathbb{F}''$. Clearly, $\mathbb{F}$ and $\mathbb{F}'$ have the same Lebesgue measure, then it follows that:

$$\int_{\mathbb{F}'} \tilde{g}(x)dx \leq \int_{\mathbb{F}} \tilde{f}(x)dx$$

which implies that $\tilde{g} \prec \tilde{f}$. Same arguments hold for a negative.■

From the Riesz's rearrangement inequality, Hajek states and proves in [1] the following result:

Lemma A.8 *[1, Page 619] Let f and g be a probability distribution function defined on the real line, such that, f is neat and symmetric, and $f \succ g$. Let h be a nonnegative, symmetric and nonincreasing function. The following holds:*

$$\int h(x)g(x)dx \leq \int h(x)f(x)dx \tag{A.8}$$

In order to prove Lemma 2.4, we state the following Lemma.

Lemma A.9 *Let f be a neat and even probability density function on the real line, Let g, be a probability density function on the real line, such that $g \prec f$. Let h be a positive, even and quasiconvex function. Then the following holds:*

$$\int_{\mathbb{R}} h(x)f(x)dx \leq \int_{\mathbb{R}} h(x-y)g(x)dx \tag{A.9}$$

where y is any real number.

Proof: Let c be a positive real number and define the functions:

$$h_c(x) = c - \min\left(c, h(x)\right)$$

$$h_c(x,y) = c - \min\left(c, h(x-y)\right)$$

for any real number y. We notice that the function h_c is symmetric and non-increasing, it is then immediate, that $h_c = h_c^{\sigma}$ and $h_c = h_c^{\sigma}(\cdot, y)$ for all real numbers y. The following inequalities are true:

$$\int_{\mathbb{R}} h_c(x,y)g(x)dx \leq \int_{\mathbb{R}} h_c(x)g^{\sigma}(x)dx \leq \int_{\mathbb{R}} h_c(x)f(x)dx$$

for any $y \in \mathbb{R}$. The first inequality follows from the Hardy-Littlewood inequality (A.3), while the second inequality follows from Lemma A.8. It follows that:

$$\int_{\mathbb{R}} h_c(x,y)g(x)dx \leq \int_{\mathbb{R}} h_c(x)f(x)dx \Rightarrow$$
$$\int_{\mathbb{R}} \left(c - \min\left(c, h(x-y)\right)\right) g(x)dx \leq \int_{\mathbb{R}} \left(c - \min\left(c, h(x)\right)\right) f(x)dx \Rightarrow$$
$$\int_{\mathbb{R}} \min\left(c, h(x-y)\right) g(x)dx \geq \int_{\mathbb{R}} \min\left(c, h(x)\right) f(x)dx$$

Taking the limit as c goes to infinity and using the monotone convergence theorem the result follows. ■

A.2 Quasiconvex Lemma

Lemma A.10 *Let $h : \mathbb{R} \to \mathbb{R}$, be a measurable, bounded, even and quasiconvex function. Let W be a random variable with an even and quasiconcave probability distribution function. Define $\bar{h} : \mathbb{R} \to \mathbb{R}$, such that $\bar{h} \stackrel{def}{=} E\left[h(x+W)\right]$, then $\bar{h}$ is a bounded, continuous, even and quasiconvex function. If the function h is also continuous then $\bar{h}$ is also continuous.*

Proof: Define $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

$$g(x, C) \stackrel{def}{=} E\left[C - \min\left(C, h(x+W)\right)\right]$$

We will show that the function $g(x, C)$ is continuous in C for every fixed real number x, and for every C the function $g(x, C)$ is even and quasiconcave in x. The function h is even and quasiconvex then, it follows that zero is a global minimizer of h. For any real number C and any real number x define the set:

$$D(x, C) \stackrel{def}{=} \{w \in \mathbb{R} : h(x+w) \leq C\}$$

Since h is even and quasiconvex then $D(0, C)$ is a convex set and is symmetric around zero, hence it is a symmetric interval it follows that:

$$D(0, C) = \begin{cases} \emptyset, & C \leq h(0) \\ [-\alpha(C), \alpha(C)] \text{ or } (-\alpha(C), \alpha(C)), & h(0) < C < \sup_x h(x) \\ (-\infty, \infty), & \sup_x h(x) \leq C \end{cases}$$

where by $\emptyset$ we denote the empty set. Note that for $h(0) < C < \sup_x h(x)$, the set $D(0, C)$ is a symmetric interval, which can be either closed or open.

$$D(x,C) = \begin{cases} \emptyset, & C \leq h(0) \\ [-\alpha(C) - x, \alpha(C) - x] \text{ or } (-\alpha(C) - x, \alpha(C) - x), & h(0) < C < \sup_x h(x) \\ (-\infty, \infty), & \sup_x h(x) \leq C \end{cases}$$

We will show that the function $g(x, C)$ is even and quasiconvex in x for any real number C. Let $f : \mathbb{R} \to \mathbb{R}$, be the probability distribution function of W. We can write $g(x, C)$:

$$g(x,C) = E\left[\min(C, h(x+W))\right] = C \int_{-\alpha(C)-x}^{\alpha(C)-x} f(w)dw$$
$$- \int_{-\alpha(C)-x}^{\alpha(C)-x} h(x+w)f(w)dw$$

For any positive real number δ, any real numbers C and x, it holds that:

$$E\left[|g(C+\delta, x+\mathbf{W})\right] - E\left[g(C, x+\mathbf{W})|\right] =$$

$$E\left[|\delta + \min(C+\delta, h(x+\mathbf{W})) - \min(C, h(x+\mathbf{W}))|\right] \leq 2\delta$$

It follows that for any real number x and any real number C, for any positive real number ϵ, choose $\delta = \frac{\epsilon}{2}$, then for any real number $\bar{C} \in (C - \delta, C + \delta)$, $|g(x, \bar{C}) - g(x, C)| < \epsilon$, hence the function $g(x, C)$ is a continuous function in C for every real number x.

Since the function h is even and quasiconvex, it follows that the function $C - \min(C, h(x))$ is even and quasiconcave, i.e. is neat and even. Moreover, from the definition of the set $D(0,C)$, we notice that the function $C-\min(C,h(x))$ is nonnegative, bounded and takes the value zero outside the set $D(x,C)$. If $C<\sup_x h(x)$, then the set $D(0,C)$ is the empty set or a finite interval (open or closed), it follows that, if $C<\sup_x h(x)$ the function $C-\min(C,h(x))$ is integrable. It holds that:

$$\begin{aligned} g(x,C)=E\left[C-\min(h(x+\mathbf{W},C)\right] &= \int_{-\infty}^{\infty}(C-\min(h(x+w,C))f(w)dw \\ &= \int_{-\infty}^{\infty}(C-\min(h(x+w,C))f(-w)dw \\ &= \int_{-\infty}^{\infty}(C-\min(h(x-\eta,C))f(\eta)d\eta \end{aligned}$$

The first equality comes from the fact that f is even, while the second inequality comes from the change of variable $\eta=-w$. It follows from Lemma A.1 and Remark A.1 that $g(x,C)$ is a neat and even function for every $C<\sup_x h(x)$. Since $g(x,C)$ is continuous in C it implies that $g(x,C)$ is neat and even for every real C and moreover the function $E\left[\min(C,h(x+\mathbf{W}))\right]$ is even and quasiconvex. From the monotone convergence theorem, it holds that:

$$\bar{h}(x)=\lim_{C\to\infty}E\left[\min(h(x+\mathbf{W}),C)\right]$$

and the properties of $E\left[\min(h(x+\mathbf{W}),C)\right]$ in x are kept for $\bar{h}$,i.e. $\bar{h}$ is even and quasiconvex.

Since h is bounded, it follows that $\bar{h}$ is bounded and we only need to prove the continuity of $\bar{h}$. We are given that h is even and quasiconvex, which implies that h is nondecreasing on $[0,\infty)$ and nonincreasing on $(-\infty,0]$. We are also given that h is bounded and continuous, which implies that h is uniform continuous on the interval $[0,\infty)$ and is also uniform continuous on the interval $(-\infty,0]$. It follows that the entire function h

is uniform continuous, i.e. for any real number x, for any positive real number ϵ, there exists a positive real number δ, which does not depend on x, such that for any real number $y \in (x-\delta, x+\delta)$, it holds that $|h(x) - h(y)| < \epsilon$. It follows that, for any real number x and for any real number $y \in (x-\delta, x+\delta)$, it holds that:

$$\begin{aligned}|E\left[h(x+\mathbf{w})\right] - E\left[h(y+\mathbf{w})\right]| &= |\int_{-\infty}^{\infty} h(x+w)f(w)dw - \int_{-\infty}^{\infty} h(y+w)f(w)dw| \\ &\leq \int_{-\infty}^{\infty} |h(x+w) - h(y+w)| f(w)dw \leq \epsilon\end{aligned}$$

This implies that $\bar{h}$ is continuous.■

Bibliography

[1] Bruce Hajek, Kevin Mitzel and Sichao Yang, *“Paging and Registration in Cellular Networks: Jointly Optimal Policies and an Iterative Algorithm,”* IEEE Transactions on Information Theory, vol. 54, no. 2, Feb 2008, pp 608-622

[2] F. Riesz, *”Sur une inegalite integrale,”* J. London Math. Soc., vol. 5, pp. 162-168, 1930.

[3] G. E. Hardy, J. E. Littlewood, and G. Polya, *”Inequalities.”* First / second edition, Cambridge University Press, London and New York, 1934 / 1952.

[4] Albert W. Marshall, Ingram Olkin, Barry Arnold, *”Inequalities: Theory of Majorization and Its Applications (Springer Series in Statistics),”* Academic Press, New York, 569 pp

[5] O. C. Imer and T. Basar. *“Optimal estimation with limited measurements,”* in Proc. IEEE CDC/ECC 2005, Seville, Spain submitted to IEEE Transactions on Automatic Control

[6] Y. Xu and J. Hespanha, *“Optimal Communication Logics for Networked Control Systems,”* Proc. of the 43rd Conf. on Decision and Contr., Dec. 2004

[7] J.S. Baras and A. Bensoussan, *“Optimal Sensor Scheduling in Nonlinear Filtering of Diffusion Processes,”* SIAM Journal on Control and Optimization, Vol. 27, No. 2, pp. 786-814, July 1989.

[8] Wei Wu and Ari Arapostathis, *“Optimal Sensor Querying: General Markovian and LQG Models With Controlled Observations,”* IEEE Transactions on Automatic Control, vol. 53, no. 6, July 2008, pp 1392-1405.

[9] Michael Athans, *“On the determination of optimal costly measurement strategies for linear stochastic systems,”* Automatica, vol. 8, 1972, pp 397-412.

[10] L. Meier, J. Peschon, and R. M. Dressler, *”Optimal control of measurement subsystems,”* IEEE Trans. Automat. Control, vol. AC-12, no. 5, pp. 528536, Oct. 1967.

[11] P. Billingsley, *”Probability and measure,”* Wiley-Interscience, 3rd edition, 1995

[12] D. Bertsekas, *”Dynamic Programming and Optimal Control,”* Athena Scientific, 3rd edition, Vol. I 2005, Vol. II 2007

[13] D. Bertsekas and S. E. Shreve, *"Stochastic Optimal Control: The Discrete Time Case,"* Athena Scientific, 1996

[14] H. Royden, *"Real Analysis,"* Prentice-Hall, 3rd edition, 1988

[15] Aditya Mahajan, *"Sequential decomposition of sequential teams: applications to real-time communication and networked control systems,"* University of Michigan, September 2008, PhD Thesis

[16] A. Mahajan, D. Teneketzis, *"Optimal Design of Sequential Real-Time Communication Systems,"* IEEE Transactions on Information Theory, Vol 55, No 11, November 2009

[17] Maben Rabi , *"Packet based Inference and Control,"* September 2007, Department of Electrical and Computer Engineering, University of Maryland, College Park, PhD Thesis

[18] R. Bansal and T. Basar , *"Simultaneous design of measurement and control strategies for stochastic systems with feedback,"* Automatica, Vol. 25, No. 5, pp. 679-694, 1989

[19] T. Basar and R. Bansal , *"Optimum design of measurement channels and control policies for linear-quadratic stochastic systems,"* European Journal of Operational Research, Vol. 73, No 2, pp. 226-236, 1994

[20] Radner, R., *"Team decision problems"*, Annals of Mathematical Statistics, 1962, Vol. 33, pp. 857-881

[21] Marschak, J. and Radner, R., *"Economic Theory of Teams"* Yale University Press, 1972, New Haven

[22] Witsenhausen, H.S., *"Separation of estimation and control for discrete time systems"*, Proceedings of the IEEE, 1971, Vol. 59, No. 11, pp. 1557-1566

[23] Witsenhausen, H.S., *"On information structures, feedback and causality "*, SIAM Journal of Control, 1971, Vol. 9, No. 2, pp. 149-160

[24] Witsenhausen, H.S., *"A standard form for sequential stochastic control"*, Mathematical Systems Theory, 1973, Vol. 7, No. 1, pp. 5-11

[25] Witsenhausen, H.S., *"The intrinsec model for stochastic control: Some open problems"*, Lecture notes in Economics and Mathematical Systems, 1973, pp. 322-335, Springer Verlag

[26] H. S. Witsenhausen, *“A counterexample in stochastic optimum control,”* SIAM J. Control, Vol 6, pp. 131-147, 1968.

[27] Ho, Y-C., Kastner, M. and Wong, E., *”Teams, signaling and information theory”,* IEEE Transactions on Automatic Control, 1978, Vol. 23, No. 2, pp. 305-312

[28] Ho, Y-C., Kastner, M. and Wong, E., *”Team decision theory and information structures”,* Proceedings of the IEEE, 1980, Vol. 68, No. 6, pp. 644-654

[29] Ho, Y-C. and Chu, K.C., *”Team decision theory and information structures in optimal control-part 1”,* IEEE Transactions on Automatic Control, 1972, Vol. 17, No. 1, pp. 15-22

[30] Chu, K.C., *”Team decision theory and information structures in optimal control-part 2”,* IEEE Transactions on Automatic Control, 1972, Vol. 17, No. 1, pp. 22-28

[31] Andersland, M.S., *”Decoupling non-sequential control problems”,* Systems and Control Letters, 1991, Vol. 16, No. 1, pp. 65-69

[32] Andersland, M.S. and Teneketzis, D., *”Information structures, causality, and non-sequential stochastic control I: Design-independent properties”,* SIAM Journal on Control and Optimization, 1992, Vol. 30, No. 6, pp. 1447-1475

[33] Andersland, M.S. and Teneketzis, D., *”Information structures, causality, and non-sequential stochastic control II: Design-independent properties”,* SIAM Journal on Control and Optimization, 1994, Vol. 32, No. 6, pp. 1726-1751

[34] Teneketzis, D., *”Communication in Decentralized Control”,* 1996, PhD Thesis, Dept. of EECS, MIT Cambridge, MA

[35] Teneketzis, D. and Andersland, M.S., *”On partial order characterization of information structures”,* Mathematics of Control, Signals and Systems, 2000, Vol. 13, pp. 277-292

[36] Yoshikawa, T., *”Decomposition of dynamic team decision problems”,* IEEE Transactions on Automatic Control, 1978, Vol. 23, No. 4, pp. 627-632

[37] Sherman, S., *”A Theorem on Convex Sets with Applications”,* Ann. Math. Statist., 1955, 26, pp. 763-767

[38] S. W. Dharmadhikari, S. W. and Kumar Jogdeo, *”Multivariate Unimodality ”,* Ann. Math. Statist., Vol. 4, No. 3 (May, 1976), pp. 607-613

[39] S. Boyd and L. Vandenberghe, *"Convex Optimization,"* Cambridge University Press

[40] R. Bansal and T. Basar, *"Stochastic teams with nonclassical information revisited: When is an affine law optimal?,"* IEEE Transaction on Automatic Control, Vol 32, pp. 554-559, 1987

[41] R. Cogill, *"Event-Based Control Using Quadratic Approximate Value Functions,"* in Proc. 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference, December 16-18, 2009, Shanghai, P.R. China

[42] M. Rabi, G.V. Moustakides and J. S. Baras, *"Multiple Sampling for Real-time Estimation on a Finite Horizon,"* Proc. of the 45th Conf. on Decision and Contr., Dec. 2006, pp 1351-1357

[43] C. H. Papadimitriou and J. N. Tsitsiklis, *" Intractable problems in control theory,"* SIAM J. Control Optim., vol 24, pp. 639-654, 1986.

[44] T. Basar, *"Variations on the Theme of the Witsenhausen Counterexample,"* 47th IEEE Conference on Decision and Control, Cancun, Mexico

[45] M. Deng and Y. C. Ho, *"An ordinal optimization approach to optimal control problems,"* Automatica, Vol 35, pp. 331-338, 1999

[46] J. T. Lee, E. Lau and Y. C. Ho, *"The Witsenhausen counterexample: A hierarchical search approach for nonconvex optimization problems,"* IEEE Transactions on Automatic Control, Vol 46, pp. 382-397, 2001

[47] M. Baglietto, T. Parisini and R. Zoppoli, *"Numerical solutions to the Witsenhausen counterexample by approximating networks,"* IEEE Transaction on Automatic Control, Vol 46, No 9, pp. 1471-1477, 2001

[48] M. Rotkowitz, *"Linear Controllers are Uniformly Optimal for the Witsenhausen Counterexample,"* Proceedings of the IEEE Conference on Decision and Control, pp. 553-558, December 2006

[49] M. Rotkowitz, *"On Information Structures, Convexity, and Linear Optimality,"* 47th IEEE Conference on Decision and Control, Cancun, Mexico, pp. 1642-1647, December 2008

[50] S. Yuksel and S. Tatikonda, *"A Counterexample in Distributed Optimal Sensing and Control,"* IEEE Trans. Automatic Control, to appear

[51] P. Grover and A. Sahai, *"A Vector Version of Witsenhausens Counterexample: A Convergence of Control, Communication and Computation*," 47th IEEE Conference on Decision and Control, Cancun, Mexico, pp. 1636-1641, December 2008