ABSTRACT

Title of dissertation:	K-THEORETIC ASPECTS OF STRING THEORY DUALITIES
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String theory is a a physical field theory in which point particles are replaced by 1-manifolds propagating in time, called strings. The 2-manifold representing the time evolution of a string is called the string worldsheet. Strings can be either closed (meaning their worldsheets are closed surfaces) or open (meaning their worldsheets have boundary). A D-brane is a submanifold of the spacetime manifold on which string endpoints are constrained to lie.

There are five different string theories that have supersymmetry, and they are all related by various dualities. This dissertation will review how D-branes are classified by K-theory. We will then explore the K-theoretic aspects of a hypothesized duality between the type I theory compactified on a 4-torus and the type IIA theory compactified on a $K3$ surface, by looking at a certain blow down of the singular limit of $K3$. This dissertation concludes by classifying D-branes on the type II orientifold $\mathbb{T}^n/\mathbb{Z}_2$ when the $\mathbb{Z}_2$ action is multiplication by -1 and the H-flux is trivial. We find that classifying D-branes on the singular limit of $K3$, $\mathbb{T}^4/\mathbb{Z}_2$ by equivariant K-theory agrees with the classification of D-branes on a smooth $K3$ surface by

ordinary K-theory.

K-THEORETIC ASPECTS OF STRING THEORY DUALITIES

by

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Dedication

This dissertation is dedicated to my grandfather, Maurice Holtzman.

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It is impossible to remember all, and I apologize to those I've inadvertently left out.

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Chapter 1

Introduction

String theory is a physical field theory in which point particles are replaced by 1-manifolds called strings. The 2-manifold representing the time evolution of a string is called the string worldsheet. Strings can either be closed, meaning that their worldsheets are closed surfaces, or open, meaning that their worldsheets have boundary. The simplest string theory is known as the bosonic string theory. This theory, however, is not realistic since it contains only bosons (particles with integer spin) and not fermions (particles with half-integer spin). To obtain a physically relevant theory, we must include fermions in the theory. String theories with fermions and supersymmetry (which is a symmetry relating fermions and bosons) are called superstring theories. There are five superstring theories: type I, type IIA, type IIB, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic. While the focus of this dissertation will be on the five superstring theories and the relations between them, it is useful to first look at the bosonic string theory to define the bosonic fields and then add in the fermionic fields.

This chapter will begin by giving a brief overview of string theory, assuming the reader has some knowledge of quantum mechanics and general relativity. For more detail on the subject, the reader is referred to [7, 60, 61, 94, 39]. Sections 1.1 and 1.2 will mainly follow [7]. This chapter will then go on to describe some of the known dualities in string theory. There will then be a brief overview of K-theory and the chapter will conclude by describing how K-theory is relevant to string theory and its dualities.

Chapters 2 and 3 contain the original content in this dissertation. In chapter 2 we will explore the K-theoretic aspects of a duality between the type I superstring theory on $\mathbb{T}^4$ and the type IIA superstring theory on a $K3$ surface. Chapter 3 will focus on classifying stable D-branes on orientifolds and how the classifications relate to string dualities.

1.1 Bosonic String Theory

Let us parameterize the worldsheet by $\sigma^0 = \tau$ and $\sigma^1 = \sigma$ such that σ parameterizes the spatial dimension of the string and τ parameterizes the time dimension of the worldsheet. Furthermore, let us choose our parameterization so that the entire length of the string is traversed as σ goes from 0 to π. Let the functions $X^\mu(\tau, \sigma)$ where $\mu = 0, 1, \ldots, d$ be embeddings of the worldsheet into the spacetime manifold with total spacetime dimension $D = d + 1$. Note that if a string is closed then $X^\mu(\tau, 0) = X^\mu(\tau, \pi)$.

Let $g_{\mu\nu}$ be the metric on the spacetime manifold. The worldsheet then inherits a metric,

$$h_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu, \tag{1.1}$$

where $a, b = 0, 1$ and $\partial_a = \frac{\partial}{\partial\sigma^a}$. We can determine the equations of motion for the string using the variational principle.

1.1.1 The Bosonic String Action

Relativistic point particles move along geodesics, so the action of a relativistic point particle is proportional to the invariant length of the particle's trajectory. This way, when we minimize the action we are stating that particles travel along paths of least distance, geodesics. Similarly for strings, their motion will be such that the area of their worldsheet is minimized. Therefore, in units where $\hbar = c = 1$, the string action takes the form

$$S_{NG} = -T \int d\mathcal{A}, \tag{1.2}$$

where T is the string tension and $d\mathcal{A}= \sqrt{-h} d\tau d\sigma$, with $h = \det h_{ab}$. This is known as the *Nambu-Goto* action and is the simplest string action. Note that in order for the action to be dimensionless, T must have dimensions of $\text{length}^{-2} = \frac{\text{mass}}{\text{length}}$. The presence of the square root makes the Nambu-Goto action difficult to quantize. Therefore we will instead use the *string sigma model action*, which is classically equivalent to the Nambu-Goto action, but easier to quantize. The string sigma model action is given by

$$S_\sigma = -\frac{1}{2} T \int d\tau d\sigma \sqrt{-h} h^{ab} \partial_a X \cdot \partial_b X, \tag{1.3}$$

where $h^{ab} \equiv (h_{ab})^{-1}$ and $Y \cdot Z \equiv g_{\mu\nu} Y^\mu Z^\nu$.

In Minkowski spacetime S_σ is invariant under Poincaré transformations:

$$\delta X^\mu = a^\mu_\nu X^\nu + b^\mu,\ a_{\mu\nu} = -a_{\nu\mu} \quad \text{and} \quad \delta h^{ab} = 0, \tag{1.4}$$

reparametrizations:

$$\sigma^a \to f^a(\sigma) \tag{1.5}$$

and Weyl transformations:

$$h_{ab} \to e^{\phi(\tau,\sigma)} h_{ab} \quad \text{and} \quad \delta X^\mu = 0. \tag{1.6}$$

The induced worldsheet metric h_{ab} is symmetric, so it has only three independent components. Plugging a reparametrization transformation (1.5) into equation (1.1), we see that h_{ab} transforms as

$$h_{ab}(\sigma) = \frac{\partial f^c}{\partial \sigma^a} \frac{\partial f^d}{\partial \sigma^b} h_{cd}(f(\sigma))$$

under reparametrizations. Therefore, we may choose two of the components of h_{ab} because of reparametrization invariance. Invariance of the action under Weyl transformations allows us to fix the one remaining independent component of h_{ab}. Since we can completely gauge fix h_{ab}, we can choose

$$h_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.7}$$

as long as there are no topological obstructions (meaning the Euler characteristic of the worldsheet is 0). Since no kinetic term of h_{ab} occurs in the action 1.3, the equation of motion for h^{ab} implies the worldsheet energy-momentum tensor,

$$T_{ab} = -\frac{2}{T\sqrt{-h}} \frac{\delta L_\sigma}{\delta h^{ab}}, \text{ where } S_\sigma = \int d^2\sigma L_\sigma, \tag{1.8}$$

vanishes. Once the worldsheet metric has been gauge fixed, we can write S_σ as

$$S = \frac{T}{2} \int d^2\sigma (\dot{X}^2 - X'^2), \tag{1.9}$$

where $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$ and $X'^\mu = \frac{\partial X^\mu}{\partial \sigma}$. Since we have gauge fixed h_{ab}, we must now add its equation of motion, $T_{ab} = 0$, as a constraint.

1.1.2 Equations of Motion and Mode Expansions

Solving for the equations of motion for X^μ from the action (1.9), we find that X^μ is governed by the wave equation

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu = 0. \tag{1.10}$$

Adding in the constraint $T_{ab} = 0$, we obtain

$$\begin{aligned} T_{01} = T_{10} &= \dot{X} \cdot X' = 0 \\ T_{00} = T_{11} &= \frac{1}{2}(\dot{X}^2 + X'^2) = 0. \end{aligned} \tag{1.11}$$

When we vary the action with respect to X^μ we obtain a boundary term

$$-T \int d\tau (X'_\mu \delta X^\mu|_{\sigma=\pi} - X'_\mu \delta X^\mu|_{\sigma=0}). \tag{1.12}$$

The boundary term must vanish in order for the variation of the action to be zero. For closed strings, the boundary term automatically vanishes, since a string being closed implies the embedding functions are periodic, $X^\mu(\tau, \sigma + \pi) = X^\mu(\tau, \sigma)$. For open strings there are two ways to make the boundary term vanish.

The first is to impose what are called *Neumann boundary conditions*:

$$X'_\mu|_{\sigma=0,\pi} = 0. \tag{1.13}$$

This means that no momentum travels through the string endpoints, since the momentum normal to the worldsheet boundary is zero. Under Neumann boundary conditions the string endpoints can move freely in directions tangential to the boundary. Neumann boundary conditions are Poincaré invariant in D-dimensional spacetime.

The second way to make the boundary term vanish is to impose *Dirichlet boundary conditions*:

$$\delta X^\mu|_{\sigma=0,\pi} = 0. \tag{1.14}$$

Under Dirichlet boundary conditions the string endpoint is fixed. We could have any combination of Dirichlet and Neumann boundary conditions, but Dirichlet boundary conditions are not Poincaré invariant. If we have Dirichlet boundary conditions for $\mu = 1, \ldots, D-p-1$ and Neumann boundary conditions for the other dimensions, X^μ for $\mu = 1, \ldots, D-p-1$ will be constant at the end points. The other p spatial dimensions that obey Neumann boundary conditions define submanifolds of the spacetime manifold with p-spatial dimensions, called D_p*-branes*, located at $X^\mu|_{\sigma=0}$ and $X^\mu|_{\sigma=\pi}$, on which the string endpoints are free to move. Note that as a D_p-brane moves in time it defines a $(p+1)$-dimensional worldvolume. D_p-branes are an important part of this dissertation that we will return to in more detail in sections 1.3 and 1.4, but we will start by assuming Neumann boundary conditions for all coordinates in order to preserve Poincaré invariance. We will see in section 1.3 that Dirichlet boundary conditions (hence D_p-branes) will arise naturally in the theory.

To solve the equations of motion and constraint equations it is useful to use worldsheet *light-cone coordinates*,

$$\sigma^\pm = \tau \pm \sigma. \tag{1.15}$$

In these coordinates,

$$\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \quad \text{and} \quad \begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.16}$$

The wave equation for X^μ in light-cone coordinates is

$$\partial_+\partial_- X^\mu = 0. \tag{1.17}$$

The most general solution to equation (1.17) can be written as the sum of right-movers and left-movers,

$$X^\mu(\tau, \sigma) = X^\mu_R(\tau - \sigma) + X^\mu_L(\tau + \sigma). \tag{1.18}$$

We are looking for solutions for equation (1.17) that are real and obey the constraint equations,

$$(\partial_- X_R)^2 = (\partial_+ X_L)^2 = 0, \tag{1.19}$$

coming from the constraint $T_{ab} = 0$.

The most general of such solutions satisfying the closed string boundary condition is

$$\begin{aligned} X^\mu_R &= \frac{1}{2}x^\mu + \frac{1}{2}l_S^2 p^\mu(\tau - \sigma) + \frac{i}{2}l_S \sum_{n\neq 0} \frac{1}{n}\alpha^\mu_n e^{-2in(\tau-\sigma)} \\ X^\mu_L &= \frac{1}{2}x^\mu + \frac{1}{2}l_S^2 p^\mu(\tau + \sigma) + \frac{i}{2}l_S \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}^\mu_n e^{-2in(\tau+\sigma)}, \end{aligned} \tag{1.20}$$

where x^μ is the center of mass position and p^μ is the total string momentum. l_s is called the *string length scale* and is related to the string tension by

$$T = \frac{1}{\pi l_S^2}.$$

The string excitation modes are represented by the exponential terms. The constraint that X^μ be real implies that x^μ and p^μ are real and the positive and negative modes are conjugate to each other,

$$(\alpha_n^\mu)^\star = \alpha_{-n}^\mu \quad \text{and} \quad (\tilde{\alpha}_n^\mu)^\star = \tilde{\alpha}_{-n}^\mu.$$

The left and right-moving modes combine to form standing waves in the general solution to equation (1.17) for an open string obeying Neumann boundary conditions. The general solution under the constraints T_{ab}=0 is

$$X^\mu(\tau,\sigma) = x^\mu + l_S^2 p^\mu \tau + i l_S \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma). \tag{1.21}$$

The canonical momentum conjugate to X^μ is

$$P^\mu(\tau,\sigma) = \frac{\delta L}{\delta \dot{X}_\mu} = T\dot{X}^\mu, \tag{1.22}$$

where $S = \int d^2\sigma L$. Plugging the mode expansions into the classical Poisson brackets for canonical coordinates, we can solve for the Poisson brackets of the modes. After quantization, the Poisson brackets are replaced by i times the commutator. We would find that the modes obey the following commutation relations:

$$\begin{gathered} [\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0} \\ [\alpha_m^\mu, \tilde{\alpha}_n^\nu] = 0. \end{gathered} \tag{1.23}$$

We can define modes

$$a_m^\mu = \frac{1}{\sqrt{m}}\alpha_m^\mu \quad \text{and} \quad a_m^{\mu\dagger} = \frac{1}{\sqrt{m}}\alpha_{-m}^\mu \quad \text{for} \quad m > 0, \tag{1.24}$$

that satisfy the algebra defined by

$$[a_m^\mu, a_n^{\mu\dagger}] = [\tilde{a}_m^\mu, \tilde{a}_n^{\mu\dagger}] = \eta^{\mu\nu}\delta_{m,n} \quad \text{for} \quad m,n > 0. \tag{1.25}$$

This is almost the algebra of raising and lowering operators for the quantum mechanical harmonic oscillator, except the commutators of time components are negative. Therefore, we can treat $a_m^{\mu\dagger}$ as raising operators and a_m^{μ} as lowering operators for $m > 0$. The spectrum is built by applying raising operators to the ground state, $|0\rangle$, which by definition is killed by all lowering operators. When writing a general state $|\phi\rangle$, we can also specify its eigenvalue of the momentum operator, k^μ, the momentum of the state. Therefore a general state can be written as

$$|\phi\rangle = a_{m_1}^{\mu_1\dagger} a_{m_2}^{\mu_2\dagger} \cdots a_{m_n}^{\mu_n\dagger} |0; k\rangle. \tag{1.26}$$

The presence of the negative sign in the time component of equation (1.25) gives rise to negative norm states (any state with an odd number of time-component raising operators will have negative norm). We cannot have negative norm states in the physical spectrum because it would break unitarity. It can be shown (see section 2.4 of [7]) that the negative norm states decouple from all physical processes and all physical states have positive norm if the spacetime manifold has total dimension 26.

1.2 Superstring Theory

The bosonic string theory described in the previous section is unrealistic, since it does not contain fermions. String theories that contain fermions are more manageable when they include supersymmetry. The are two ways to formulate supersymmetric string theory. The *Ramond-Neveu-Schwarz* (RNS) *formulation* was developed from the 1971 papers [64] and [54]. The RNS formalism has supersymmetry on the string worldsheet. The other approach to including supersymmetry in string theory is the *Green-Schwarz* (GS) *formalism*. It has supersymmetry in 10-dimensional Minkowski spacetime and can be generalized to other spacetime geometries. In 10-dimensional Minkowski spacetime, the two formulations are equivalent. The GS formalism was developed by Green and Schwarz between 1979 and 1984. We will begin with the RNS formulation, since with it, many of the features we are interested in are easier to derive. We will discuss the GS formulation later when we want to generalize to other background spacetimes.

For the RNS formulation, in addition to the bosonic fields, $X^\mu(\tau,\sigma)$, discussed in the previous section, we add fermionic partners $\psi^\mu(\tau,\sigma)$. The fermionic fields are spinors on the 2-dimensional worldsheet, but transform as vectors under Lorentz transformations of the D-dimensional spacetime. Therefore, they should be Majorana fermions that belong to the vector representation of the Lorentz group $\mathrm{SO}(D-1,1)$[7]. In addition to the action (1.3) for the bosonic fields, we include the standard Dirac action for free massless fermions to the total action. Therefore, in the conformal gauge, the action is

$$S = -\frac{T}{2}\int d^2\sigma(\partial_a X^\mu \partial^a X^\mu + \bar{\psi}^\mu \rho^a \partial_a \psi_\mu). \tag{1.27}$$

The ρ^a are the generators of what is known in the mathematics literature as the 2-dimensional *Clifford algebra*, and in the physics literature as the 2-dimensional *Dirac algebra*. They obey the anticommutation relations

$$\{\rho^a, \rho^b\} = 2\eta^{ab}. \tag{1.28}$$

We can choose a basis so that

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.29}$$

The spinor fields, ψ^μ, have two components,

$$\psi^\mu = \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix}. \tag{1.30}$$

The conjugate of a spinor is defined to be

$$\bar{\psi} = i\psi^\dagger \rho^0. \tag{1.31}$$

Classically, the fermionic fields are made of real Grassmann numbers in this representation of the Dirac algebra. Therefore, they obey the relations

$$\begin{gathered} \{\psi^\mu, \psi^\nu\} = 0, \\ \psi_+^\star = \psi_+ \quad \text{and} \quad \psi_-^\star = \psi_-. \end{gathered} \tag{1.32}$$

In the worldsheet light-cone coordinates, the fermionic part of the action (1.27) can be written as

$$S_f = iT \int d^2\sigma \; \eta_{\mu\nu}(\psi_-^\mu \partial_+ \psi_-^\nu + \psi_+^\mu \partial_- \psi_+^\nu). \tag{1.33}$$

The equations of motion for ψ_- and ψ_+ are

$$\partial_+ \psi_-^\mu = 0 \quad \text{and} \quad \partial_- \psi_+^\mu = 0. \tag{1.34}$$

This is the Dirac equation and describes left-moving and right-moving waves. Equation (1.34) is also the Weyl condition for spinors in two dimensions. Therefore, $\psi_\pm$ are Majorana-Weyl spinors, which to mathematicians means "they belong to two inequivalent real one-dimensional spinor representations of the two-dimensional Lorentz group $Spin(1,1)$"[7]. Since $\psi^\mu_\pm$ are solutions to the Dirac equation, after quantization the fermionic fields will obey the canonical anticommutation relations

$$\{\psi^\mu_A(\tau,\sigma),\psi^\nu_B(\tau,\sigma')\} = i\eta^{\mu\nu}\delta_{AB}\delta(\sigma-\sigma'), \tag{1.35}$$

where $A, B = \pm$.

The action (1.27) is invariant under the transformations

$$\delta X^\mu = \bar{\varepsilon}\psi^\mu \quad \text{and} \quad \delta\psi^\mu = \rho^a\partial_a X^\mu\varepsilon, \tag{1.36}$$

where ε is an infinitesimal Majorana spinor. These symmetry transformations mix the worldsheet fermionic and bosonic fields and are known as worldsheet supersymmetry. This is a global symmetry of the theory because ε does not depend on τ or σ. When the transformations are written out in components, the symmetry is not manifest. To make the symmetry manifest and write the transformations in a non-component form, we must reformulate the action (1.27) using superfields. *Superfields* are fields defined on *superspace*, an extension of spacetime including additional anticommuting Grassmann coordinates. For a full description on how this is done see [7].

We still have the vanishing of the energy momentum tensor as a constraint obtained by varying the action with respect to the worldsheet metric. In worldsheet light-cone coordinates, the energy-momentum tensor is

$$\begin{aligned} T_{++} &= \partial_+X_\mu\partial_+X^\mu + \tfrac{i}{2}\psi^\mu_+\partial_+\psi_{+\mu}, \\ T_{--} &= \partial_-X_\mu\partial_-X^\mu + \tfrac{i}{2}\psi^\mu_-\partial_-\psi_{-\mu}, \\ T_{+-} &= T-+ = 0. \end{aligned} \tag{1.37}$$

We can also obtain the *worldsheet supercurrent*, the conserved current associated to the worldsheet supersymmetry, using Noether's method. The worldsheet supercurrent is

$$J^a_A = -\frac{1}{2}(\rho^b \rho^a \psi_\mu)_A \partial_b X^\mu. \tag{1.38}$$

Superconformal symmetry implies the vanishing of the supercurrent. For details see section 4.3.4 of [33].

1.2.1 Mode Expansions and Spectrum

The mode expansions for the bosonic fields are the same as for the bosonic string theory described in the previous section. Varying the fermionic part of the action (1.33) with respect to $\psi^\mu_\pm$ gives a boundary term proportional to

$$\int d\tau \, \{(\psi_{+\mu}\delta\psi^\mu_+ - \psi_{-\mu}\delta\psi^\mu_-)|_{\sigma=\pi} - (\psi_{+\mu}\delta\psi^\mu_+ - \psi_{-\mu}\delta\psi^\mu_-)|_{\sigma=0}\}. \tag{1.39}$$

The action is minimized when the equations of motion (1.34) are satisfied and the boundary term vanishes.

For open strings, the term $(\psi_{+\mu}\delta\psi^\mu_+ - \psi_{-\mu}\delta\psi^\mu_-)$ must vanish at each endpoint separately. This means that at each endpoint,

$$\psi^\mu_+ = \pm\psi^\mu_-.$$

The overall sign of ψ^μ is a matter of convention, so we can choose

$$\psi^\mu_+|_{\sigma=0} = \psi^\mu_-|_{\sigma=0}. \tag{1.40}$$

Then the relative sign at the other endpoint becomes important.

The choice

$$\psi_+^\mu|_{\sigma=\pi} = \psi_-^\mu|_{\sigma=\pi} \tag{1.41}$$

is known as *Ramond* (R) *boundary conditions*. Fermionic fields with R boundary conditions are said to be in the R sector and have the form

$$\psi_-^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau-\sigma)}, \tag{1.42}$$

$$\psi_+^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau+\sigma)}. \tag{1.43}$$

Since $\psi_\pm^\mu$ are real, $d_n^{\mu\dagger} = d_{-n}^\mu$.

The other choice

$$\psi_+^\mu|_{\sigma=\pi} = -\psi_-^\mu|_{\sigma=\pi} \tag{1.44}$$

is known as *Neveu-Schwarz* (NS) *boundary conditions*. The mode expansion for fermionic fields in the NS sector is

$$\psi_-^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-ir(\tau-\sigma)}, \tag{1.45}$$

$$\psi_+^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-ir(\tau+\sigma)}. \tag{1.46}$$

Again, $b_n^{\mu\dagger} = b_{-n}^\mu$.

For closed strings, the left and right-moving modes are independent. The boundary term (1.39) will vanish if the left and right-moving fields obey periodic (R) or antiperiodic (NS) boundary conditions,

$$\psi_\pm^\mu(\sigma) = \pm\psi_\pm^\mu(\sigma+\pi). \tag{1.47}$$

The periodicity conditions for the right and left-movers can be chosen independently.

Therefore, the mode expansion for the right movers can be either

$$\psi_-^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in(\tau-\sigma)}, \quad \text{for R boundary conditions,} \tag{1.48}$$

or

$$\psi_-^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-ir(\tau-\sigma)}, \quad \text{for NS boundary conditions.} \tag{1.49}$$

And the mode expansions for the left-movers can be either

$$\psi_+^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-in(\tau+\sigma)}, \quad \text{for R boundary conditions,} \tag{1.50}$$

or

$$\psi_+^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+1/2} \tilde{b}_r^\mu e^{-ir(\tau+\sigma)}, \quad \text{for NS boundary conditions.} \tag{1.51}$$

We can have any combination of R and NS boundary conditions with left and right-movers, so there are four individual sectors in the closed string spectrum. They are the NS-NS, R-R, NS-R and R-NS sectors.

After quantization, the anticommutation relations for the modes d_n^μ and d_r^μ can be determined from the canonical anticommutation relations for the fermionic fields (1.35). The fermionic modes satisfy

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0} \quad \text{and} \quad \{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0}. \tag{1.52}$$

The bosonic modes satisfy the same commutation relations as for the bosonic string given in equation (1.23). We now look at the spectrum for open strings first, since we can build the spectrum for closed strings by combing our results for open strings with both left and right-movers.

The ground states in the two sectors $|0\rangle_R$ and $|0\rangle_{NS}$ are killed by the lowering operators in their respective sectors, that is

$$\alpha_n^\mu|0\rangle_R = d_n^\mu|0\rangle_R = 0 \quad \text{for} \quad n > 0 \tag{1.53}$$

and

$$\alpha_n^\mu|0\rangle_{NS} = b_r^\mu|0\rangle_{NS} = 0 \quad \text{for} \quad n, r > 0. \tag{1.54}$$

The spectrum is built by acting on the ground state with raising operators (negative modes). Raising operators increase the mass of the state.

In the R sector, the operators d_0^μ are neither raising nor lowering operators. They can act on a state without changing the mass, as we will see later. Therefore, the ground state is degenerate. By equation (1.52), we know the d_0^μ satisfy the algebra

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}. \tag{1.55}$$

Except for a factor of 2, this is the same as the Dirac algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}. \tag{1.56}$$

Therefore, the set of ground states must give a representation of the Dirac algebra. The degenerate ground state can be written as $|\hat{a}\rangle$, where $\hat{a}$ is a spinor index and

$$d_0^\mu|\hat{a}\rangle = \frac{1}{\sqrt{2}}\Gamma^\mu_{\hat{b}\hat{a}}|\hat{b}\rangle. \tag{1.57}$$

We see that the R sector ground state is a spacetime spinor, so it is a fermion. All of the modes are spacetime vectors, so all of the states in the R sector (obtained by acting on the ground state with negative modes) are also spacetime fermions.

In the NS sector there is no fermionic zero mode, so there is a unique ground state. The ground state is a spin 0 particle which corresponds to a scalar field. Therefore all of the states in the NS sector are spacetime vectors, hence bosons.

After quantization, the constraints coming from the vanishing of the energy momentum tensor and the worldsheet supercurrent must be altered. The modes of the energy momentum tensor are

$$L_n = \frac{1}{\pi}\int_{-\pi}^{\pi} d\sigma e^{in\sigma} T_{++} = L_n^{(b)} + L_n^{(f)}. \tag{1.58}$$

Classically,

$$L_m^{(b)} = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot\alpha_{m+n}, \tag{1.59}$$

where $\alpha_0^\mu = l_s p^\mu$. In the quantum theory, the $L_m^{(b)}$ are defined to be normal ordered,

$$L_m^{(b)} = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{-n}\cdot\alpha_{m+n}: . \tag{1.60}$$

The *normal ordering operation*, $:\cdot:$, is defined so that raising operators always appear on the left of lowering operators. Therefore,

$$:\alpha_{-n}\cdot\alpha_n:=:\alpha_n\cdot\alpha_{-n}:=\alpha_{-n}\cdot\alpha_n \quad \text{for} \quad n>0. \tag{1.61}$$

The only mode of the energy momentum tensor for which normal ordering is important is L_0;

$$L_0^{(b)} = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha_{-n}\cdot\alpha_n: = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n. \tag{1.62}$$

Because of the ambiguity in choosing the normal ordering, we must assume an arbitrary constant can be added to $L_0^{(b)}$. This alters the *Virasoro algebra* normally satisfied by the L_n to a Virasoro algebra with central extension equal to the total spacetime dimension, D (see equations (1.71-1.76) below). While we have quantized only $L_n^{(b)}$, the same prescription holds for quantizing $L_n^{(f)}$ and the modes of the supercurrent. Therefore, I will only state the results in the quantum theory. We will see that normal ordering only matters for the L_0 terms.

In the NS sector

$$L_n^{(f)} = \frac{1}{2} \sum_{r \in \mathbb{Z}+1/2} (r + \frac{n}{2}) : b_{-r} \cdot b_{n+r} : \qquad n \in \mathbb{Z}. \tag{1.63}$$

The modes of the supercurrent are

$$G_r = \frac{\sqrt{2}}{\pi} \int_{-\pi}^{\pi} d\sigma e^{ir\sigma} J_+ = \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot b_{r+n} \qquad r \in \mathbb{Z} + \frac{1}{2}. \tag{1.64}$$

The only operator for which normal ordering matters is L_0 and it can be written as

$$L_0 = \frac{1}{2}\alpha_0^2 + N_R, \tag{1.65}$$

where N_R is the number operator,

$$N_R = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{r=1/2}^{\infty} r b_{-r} \cdot b_r. \tag{1.66}$$

Again, note that an arbitrary constant can be added to this.

In the R sector

$$L_n^{(f)} = \frac{1}{2} \sum_{k \in \mathbb{Z}} (k + \frac{n}{2}) : d_{-k} \cdot d_{n+k} \qquad n \in \mathbb{Z}, \tag{1.67}$$

and the modes of the supercurrent are

$$F_n = \frac{\sqrt{2}}{\pi} \int_{-\pi}^{\pi} d\sigma e^{in\sigma} J_+ = \sum_{k \in \mathbb{Z}} \alpha_{-k} \cdot d_{n+k} \qquad n \in \mathbb{Z}. \tag{1.68}$$

L_0 is the only operator where normal ordering matters and it can be written as

$$L_0 = \frac{1}{2}\alpha_0^2 + N_{NS}, \tag{1.69}$$

where

$$N_{NS} = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + \sum_{n=1}^{\infty} n d_{-n} \cdot d_n. \tag{1.70}$$

The algebra satisfied by the modes of the energy momentum tensor and the supercurrent is called the *super Virasoro algebra*. It can be determined from equations (1.35) and (1.52) and the definitions of the modes given above. In the NS sector, the super Virasoro algebra is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{8}m(m^2-1)\delta_{m+n,0}, \tag{1.71}$$

$$[L_m, G_r] = (\frac{m}{2} - r)G_{m+r} \tag{1.72}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{D}{2}(r^2 - \frac{1}{4})\delta_{r+s,0}. \tag{1.73}$$

In the R sector it is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{8}m^3\delta_{m=n,0}, \tag{1.74}$$

$$[L_m, F_n] = (\frac{m}{2} - n)F_{m+n} \tag{1.75}$$

$$\{F_m, F_n\} = 2L_{m+n} + \frac{D}{2}m^2\delta_{m+n,0}. \tag{1.76}$$

Classically, the vanishing of the energy momentum tensor and supercurrent means the modes must vanish. That would be incompatible with the super Virasoro algebra. In the quantum theory, the constraint (which is the condition for a state to be physical) is altered to require only the positive modes to annihilate physical states. Therefore the physical state conditions in the NS sector are

$$G_r|\phi\rangle = 0 \qquad r > 0, \tag{1.77}$$

$$L_n|\phi\rangle = 0 \qquad n > 0, \tag{1.78}$$

$$(L_0 - a_{NS})|\phi\rangle = 0. \tag{1.79}$$

In the R sector, the physical state conditions are

$$F_n|\phi\rangle = 0 \qquad n \geq 0, \tag{1.80}$$

$$L_n|\phi\rangle = 0 \qquad n > 0, \tag{1.81}$$

$$(L_0 - a_R)|\phi\rangle = 0. \tag{1.82}$$

a_{NS} and a_R are constants due to the normal ordering ambiguity discussed earlier. Equations (1.79) and (1.82) are the mass shell conditions. The relativistic mass shell condition is $M^2 = -p_\mu p^\mu$. Using this as well as the definition of L_0 (equations (1.65) and (1.69)), equations (1.79) and (1.82) imply

$$\alpha' M^2 = N_\flat - a_\flat, \tag{1.83}$$

where $\flat = R, NS$. M is the mass of the state $|\phi\rangle$ and $N_\flat$ is its eigenvalue under the number operator. α' is the *Regge slope parameter* which is defined as

$$\alpha' = \frac{1}{2} l_s^2. \tag{1.84}$$

The negative sign on the right-hand side of the time component of equations (1.35) and (1.52) leads to states with negative norm, as was the case in the bosonic string theory. It can be shown that the negative norm states decouple from the physical states if $a_R = 0$, $a_{NS} = \frac{1}{2}$ and $D = 10$ (see section 4.5 of [7]). So we see that all realistic superstring theories have ten total spacetime dimensions.

As we will see later, the closed string spectrum contains one or two gravitinos. This implies that the theory must have spacetime supersymmetry and not just worldsheet supersymmetry as we currently have. Additionally, the NS sector ground state is a *tachyon* (has imaginary mass) due to the value of a_{NS} being $\frac{1}{2}$. To fix these problems we must perform what is known as *GSO projection*, named after Gliozzi, Scherk and Olive who introduced it in [32]. Before describing what the GSO projection is, it is useful to define the light-cone gauge.

In analogy with the worldsheet light-cone coordinates defined earlier, we can define *spacetime light-cone coordinates* as

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}). \tag{1.85}$$

The two null coordinates, $X^{\pm}$, and the $D-2$ transverse coordinates X^i make up the D total spacetime coordinates, X^{μ}. In these coordinates the flat metric has the nonzero components

$$g_{+-} = g_{-+} = -1 \quad \text{and} \quad g_{ii} = 1. \tag{1.86}$$

The Virasoro algebra appears because we have not fully gauge fixed the reparametrization symmetry. The action is still invariant under reparametrizations that are also Weyl rescalings. This additional symmetry allows us to choose a gauge in which

$$\alpha_n^+ = 0 \quad \text{for} \quad n \neq 0, \tag{1.87}$$

so

$$X^+(\tau, \sigma) = x^+ + l_S^2 p^+ \tau. \tag{1.88}$$

We can also choose

$$\psi^+(\tau, \sigma) = 0 \tag{1.89}$$

in the NS sector. In the R sector we need to keep the zero mode, which is a Dirac matrix. It can be shown that X^- and ψ^- are not independent degrees of freedom using the Virasoro constraints. Therefore, in this gauge (called the *light-cone gauge*), the independent physical excitations in the spectrum can be created by acting on the ground state with negative transverse bosonic and fermionic modes.

In order to define GSO projection we must introduce an operator, G, called *G-parity*. In the NS sector

$$G = (-1)^{F+1} = (-1)^{\sum_{r=1/2}^{\infty} b^i_{-r} b^i_r + 1}. \tag{1.90}$$

The eigenvalue of F is the number of b-oscillator excitations a state has. Therefore a state has positive or negative G-parity depending on if the state has an odd or even number of b-oscillator excitations. In the R sector

$$G = \Gamma_{11}(-1)^{\sum_{n=1}^{\infty} d^i_{-n} d^i_n}. \tag{1.91}$$

Γ_{11} is the product of the ten Dirac matrices in 10 dimensions,

$$\Gamma_{11} = \Gamma_0 \Gamma_1 \cdots \Gamma_9. \tag{1.92}$$

If a spinor satisfies

$$\Gamma_{11}\psi = \pm\psi, \tag{1.93}$$

it is said to have positive or negative chirality. Weyl spinors have a definite chirality.

In the NS sector, we keep only states with positive G-parity under the GSO projection, while eliminating the states with negative G-parity. So after GSO projection the NS sector consists of states with an odd number b-oscillator excitations. We can project onto states with positive or negative G-parity in the R sector depending on our choice for the chirality of the ground state, which is a matter of convention. We keep states with the same G-parity as the ground state.

After GSO projection, the old ground state in the NS sector, the tachyon, is eliminated from the spectrum and the new ground state is the lowest state that survives the GSO projection,

$$b^{\mu}_{-1/2}|0\rangle_{NS}. \tag{1.94}$$

The ground state in the NS sector is a massless vector boson with eight transverse degrees of freedom. The ground state in the R sector is a massless Majorana-Weyl spinor. The fact that there exist spinors that obey the Majorana and Weyl spinor conditions is due to Bott periodicity and the fact that we are have 10 total spacetime dimensions. A general spinor in 10 dimensions has 32 complex components. The Majorana condition implies that all of the components are real and the Weyl condition further cuts the number of components in half. So a Majorana-Weyl spinor has 16 real components. Since the ground state is a solution to the Dirac equation, this further cuts the number of components in half. Therefore the ground state in the R sector also has 8 degrees of freedom. There are the same number of on-shell bosonic and fermionic degrees of freedom in the massless spectrum. These form two inequivalent real representations of $Spin(8)$. While this is strong evidence for spacetime supersymmetry, it is not a proof. The full proof that the GSO projection gives spacetime supersymmetry was given by Green and Schwarz in [34].

Now let us look at the massless closed string spectrum. Massless states in the closed string spectrum are tensor products of massless left and right-moving states. The different ways we combine which sector the left and right-movers are in give four distinct closed string sectors: R-R, R-NS, NS-R and NS-NS.

As we saw above, the GSO projection in the R sector depends on the choice of chirality of the ground state. This allows us to define two different closed string theories. The left and right moving ground states in the R sector are chosen to have the same chirality in the *type IIB* theory. Since the choice of chirality is a matter of convention we will choose both the left and right-movers to have positive chirality. We will denote the positive chirality ground state in the R sector as $|+\rangle_R$, where we have suppressed a spinor index. Similarly, we will denote the negative chirality ground state by $|-\rangle_R$. In the *type IIA* theory the left and right-movers have opposite chirality. As we will see below, both type II string theories have two gravitinos, so they have $\mathcal{N} = 2$ supersymmetry.

1.2.2 Type IIB Superstring Theory

The massless closed string states in the type IIB theory are

$$|+\rangle_R \otimes |+\rangle_R, \tag{1.95}$$

$$|+\rangle_R \otimes b^i_{-1/2}|0\rangle_{NS}, \tag{1.96}$$

$$\tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes |+\rangle_R, \tag{1.97}$$

$$\tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes b^i_{-1/2}|0\rangle_{NS}. \tag{1.98}$$

$|+\rangle_R$ is an eight component spinor and $b^i_{-1/2}$ is an eight component vector, so there are a total of 64 states in each sector.

In the R-R sector, when the two Majorana-Weyl spinors are tensored together the two half integral spins combine to give a state with integral spin, a bosonic state. Since the two Majorana-Weyl spinors have the same chirality, the tensor product can be decomposed as the sum of a scalar gauge field (1 state), a 2-form gauge field (28 states) and a 4-form gauge field with a self dual 5-form field strength (35 states).

In both the NS-R and R-NS sectors the 64 states split as a spin $\frac{3}{2}$ fermion (56 states) called the *gravitino*, and a spin $\frac{1}{2}$ fermion (8 states) called the *dilatino*. For the type IIB theory, the two gravitinos have the same chirality.

In the NS-NS sector the spectrum decomposes as a scalar (1 state called the *dilaton*), an antisymmetric 2-form gauge field (28 states) and a symmetric traceless rank 2 tensor (35 states) called the *graviton*.

1.2.3 Type IIA Superstring Theory

In the type IIA theory the left and right-movers have opposite chirality, so the massless closed string states are

$$|-\rangle_R \otimes |+\rangle_R, \tag{1.99}$$

$$|-\rangle_R \otimes b^i_{-1/2}|0\rangle_{NS}, \tag{1.100}$$

$$\tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes |+\rangle_R, \tag{1.101}$$

$$\tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes b^i_{-1/2}|0\rangle_{NS}. \tag{1.102}$$

Just as in the type IIB theory there are 64 states in each sector. The spectrum in the NS-NS sector is exactly the same as in the type IIB theory. The spectrum in the NS-R and R-NS sectors is almost the same except the two gravitinos have opposite chirality in the type IIA theory. In the R-R sector of the type IIA theory, we are tensoring two Majorana-Weyl spinors with opposite chirality. This results in a vector gauge field (8 states) and a 3-form gauge field (56 states).

1.2.4 Type I and The Heterotic Superstring Theories

The *type I superstring theory* can be obtained as a projection of the type IIB theory. Strings in the type IIB theory are oriented, so their worldsheets must be orientable surfaces. Consider the *worldsheet parity transformation*

$$\Omega\colon\ \sigma \to -\sigma. \tag{1.103}$$

This transformation reverses the orientation of the worldsheet and interchanges the left and right-moving modes of the bosonic and fermionic fields. The worldsheet parity transformation is a symmetry of the type IIB theory because the left and right-moving fermions have the same chirality. In the type IIA theory, the left and right-moving fermions have opposite chirality, so worldsheet parity is not a symmetry of the theory. The type I theory is obtained from the type IIB theory by keeping only states that are even under the worldsheet parity transformation. So states in the type I theory are projections of states in the type IIB theory under the projection operator

$$P = \frac{1}{2}(1 + \Omega). \tag{1.104}$$

P projects onto the left-right symmetric part of a state, so closed strings in the type I theory are unoriented. As we saw in equation (1.98), the NS-NS massless closed string states in the type IIB theory are given by the tensor product of two vectors. The projection keeps only states that are symmetric in the two vectors. Therefore, only the dilaton and the graviton survive the projection, while the antisymmetric B-field is projected out.

Worldsheet parity interchanges $|+\rangle_R \otimes b^i_{-1/2}|0\rangle_{NS}$ and $\tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes |+\rangle_R$, so only their sum survives the projection. Therefore, the type I theory has only 1 gravitino and 1 dilatino, giving a total $56 + 8 = 64$ fermionic degrees of freedom. The fact that there is 1 gravitino means that the type I theory must have $\mathcal{N} = 1$ supersymmetry (half as much supersymmetry as the type IIB theory).

We can determine which R-R sector states survive the projection by requiring there be 64 total bosonic degrees of freedom to match the fermionic degrees of freedom under supersymmetry. We already saw that there are $35 + 1 = 36$ bosonic degrees of freedom in the NS-NS sector coming from the graviton and dilaton. This means we must have 28 bosonic degrees of freedom coming from the R-R sector. Under this requirement, we see that only the R-R 2-form gauge field survives the projection, while both the scalar and 4-form gauge fields are projected out.

We also get open string states in the type I theory coming from strings whose endpoints lie on the fixed points of $\sigma \to -\sigma$. Since they must also be even under the worldsheet parity transformation, open strings in the type I theory are unoriented as well. Since the endpoints of open strings must lie on D-branes, the presence of open strings in the type I spectrum shows that there must be spacetime filling D_9-branes in the theory. The type I theory itself is inconsistent. To make it consistent, it must be coupled to a super Yang-Mills theory with an SO(32) gauge group. We will see in section 1.3.3 how this gauge group arises by including additional degrees of freedom on the endpoints of strings called Chan-Paton degrees of freedom.

We will not go into great detail about the heterotic string theories as they will not be a big topic in this dissertation. In chapter 2 we will discuss a duality that passes through the heterotic theory, so for the sake of completeness we will give a brief definition of the heterotic theory. The two *heterotic superstring theories* are closed oriented superstring theories built by giving the left-movers the degrees of freedom of the 26-dimensional bosonic string theory and giving the right-movers the degrees of freedom of the 10 dimensional superstring. The right-moving currents of the string carry the supersymmetry charges, whereas the left-moving currents carry conserved charges of a Yang-Mills gauge symmetry. Quantum consistency requires that the gauge symmetry be locally either SO(32) or $E_8 \times E_8$. As we will see in section 1.5 the weakly coupled SO(32) heterotic is equivalent to the strongly coupled type I theory (and vice versa), so when working with the SO(32) heterotic string theory we will often consider the type I theory instead.

1.3 T-duality

As we have seen, there are five different superstring theories. This might seem discouraging at first when trying to determine a unified physical theory, but it turns out that the different theories are related to one another by dualities. When we say two theories are dual to one another, it means there is a transformation between the two theories that leaves the observable physics unchanged.

One of the most important and best understood dualities in string theory is known as *T-duality*. The simplest version of T-duality relates a spacetime with a dimension compactified on a circle of radius R to a dual spacetime with a dimension compactified on a circle of radius $\frac{\alpha'}{R}$. Multiple T-duality transformations can be performed on spacetimes with multiple compact dimensions. T-duality can be generalized to the case when the spacetime is fibered by circles but doesn't split as a product. We will not deal with this generalization until section 1.4.4.

Let us first look only at what happens to the bosonic fields under a T-duality transformation. Due to the worldsheet supersymmetry in the RNS string, the fermionic fields will have to transform in the same way as the bosonic fields. Returning to the conformal gauge, let's consider closed strings in 9-dimensional Minkowski space cross a circle of radius R ($\mathbb{R}^{8,1} \times S^1$). So if we give $\mathbb{R}^{8,1} \times S^1$ coordinates x^μ, $0 \leq \mu \leq 9$, and let x^9 be the coordinate in the S^1 direction, then

$$x^9 \sim x^9 + 2\pi R. \tag{1.105}$$

This introduces new closed string states. In a spacetime with no compact dimensions, all closed strings can be continuously deformed to zero size. In spacetimes with compact dimensions there exist closed strings that wrap around the compact dimension, so can't be continuously deformed to zero size. These additional string states can be differentiated by the number of times they wrap around the compact dimension (and in what direction if we are dealing with an oriented string theory, as both type II theories are).

The winding number of a closed string is best understood in the universal cover of the spacetime manifold. A closed string that wraps the compactified dimension lifts to an open string in the universal cover, where

$$X^9(\tau, \pi) = X^9(\tau, 0) + m(2\pi R). \tag{1.106}$$

m is called the *winding number*. This gives the periodic boundary condition in the spacetime manifold,

$$X^9(\tau,\sigma+\pi) = X^9(\tau,\sigma) + m(2\pi R). \tag{1.107}$$

This can sometimes seem confusing if the string wraps more than once around the compact dimension because then 2, or more, different points on the string occupy the same point in the spacetime manifold. The important thing to remember is that for a closed string, only the endpoints are identified. The interior of the string can intersect or wrap around itself, but each value of σ gives a different point of the string.

The quantum mechanical wave function contains a factor of $e^{ip^9x^9}$, and should be single valued on the circle. In order for the wave function to be invariant under a change of x^9 by a multiple of $2\pi R$, p^9 must be quantized as

$$p^9 = \frac{k}{R}, \qquad k \in \mathbb{Z}. \tag{1.108}$$

k is known as the *Kaluza-Klein excitation number.*

The components of the bosonic fields in the noncompact dimensions remain unchanged. In the compact direction, however, not only does the momentum become quantized but we must add a term linear in σ to account for the boundary conditions (1.107). The mode expansions for the bosonic field in the compact dimension are

$$X^9_R = \frac{1}{2}(x^9 - \tilde{x}^9) + (\alpha'\frac{k}{R} - mR)(\tau - \sigma) + \frac{i}{2}l_S \sum_{n\neq 0} \frac{1}{n}\alpha^9_n e^{-2in(\tau-\sigma)}, \tag{1.109}$$

$$X^9_L = \frac{1}{2}(x^9 + \tilde{x}^9) + (\alpha'\frac{k}{R} + mR)(\tau + \sigma) + \frac{i}{2}l_S \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}^9_n e^{-2in(\tau+\sigma)}, \tag{1.110}$$

where $\tilde{x}^9$ is a constant that cancels in the sum $X = X_R + X_L$. We can see that this is the same as for the bosonic field in noncompact dimensions except that the zero modes are now defined to be

$$\sqrt{2\alpha'}\alpha_0^9 = \alpha' \frac{k}{R} - mR, \tag{1.111}$$

$$\sqrt{2\alpha'}\tilde{\alpha}_0^9 = \alpha' \frac{k}{R} + mR. \tag{1.112}$$

All nonzero modes remain unchanged. The zero modes in the compact dimension add a contribution to the mass squared of

$$\left(\frac{k}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2. \tag{1.113}$$

It can also be shown from the mass-shell condition that

$$N_R - N_L = mk, \tag{1.114}$$

where $N_{R,L}$ are the number of right and left-moving excitations. Equations (1.113) and (1.114), and hence the spectrum, are invariant under the simultaneous interchange of m and k, and R and $\tilde{R} = \frac{\alpha'}{R}$. This symmetry between a spacetime with a dimension compactified on a circle of radius R and one compactified on a circle of radius $\tilde{R}$ and momentum and winding interchanged is known as T-duality.

From equation (1.111), we see that the zero modes in the compact direction transform as

$$\alpha_0 \to -\alpha_0 \quad \text{and} \quad \tilde{\alpha}_0 \to \tilde{\alpha}_0 \tag{1.115}$$

under a T-duality transformation. Note for later, that due to this duality in the $R \to 0$ limit the $n = 0$ states with all w form a continuum and the compact dimension reappears.

Before the T-duality transformation, a coordinate, x, parametrized the original circle with periodicity $2\pi R$. After the transformation, x is replaced by a coordinate,

$\tilde{x}$, that parameterizes the dual circle with periodicity $2\pi\tilde{R}$ and conjugate momentum, $\tilde{p} = mR/\alpha' = m/\tilde{R}$. From this it is clear that the right-moving part of the bosonic field must switch sign under a T-duality transformation, while the left-moving part remains the same. Therefore, the bosonic field in the compact direction in the dual theory is

$$\tilde{X}(\tau, \sigma) = \tilde{X}_L + \tilde{X}_R \tag{1.116}$$

$$= X_L - X_R. \tag{1.117}$$

The fermionic fields must transform in the same way as the bosonic fields due to the worldsheet supersymmetry. Therefore, the fermionic fields in the compact direction transform as

$$\psi^9_L \rightarrow \psi^9_L \quad \text{and} \quad \psi^9_R \rightarrow -\psi^9_R \tag{1.118}$$

under a T-duality transformation. A T-duality transformation switches the chirality of the right-moving R sector ground state. This changes the relative chirality between the left and right-moving ground states since the left-moving ground state remains unchanged. Since the type IIA and IIB theories are distinguished by the relative chirality of the left and right-moving R sector ground states, a T-duality transformation interchanges the type IIA and type IIB theories. If there was more than one compact direction we could perform T-duality transformations in each of the different compact directions. An odd number of T-duality transformations interchanges the type IIA and IIB theories, while an even number leaves the type of string theory invariant.

1.3.1 T-duality of Open Strings

When we compactify a spatial dimension in a theory containing open strings, the situation is a little different. We get no new open string states. Since the string end points aren't identified all open strings can be continuously deformed to zero size. That is, all open strings are homotopic to a point, so no matter how many times you wrap it around the compact dimension you can always unwrap it without tearing and continuously deform it to a string that does not wrap the compact dimension.

Since there are no new open string states, in the $R \to 0$ limit we do not get a continuum of new states, as in the closed string case, and the compactified dimension disappears leaving a $(D-1)$-dimensional system. This would seem to be a paradox since all open string theories must also contain closed strings, which we saw earlier live in a D-dimensional system after the $R \to 0$ limit, but can be explained by the fact that the endpoints of an open string are different than the interior. The interior of an open string is just like a closed string and lives in D-dimensions. The string endpoints are however confined to live on a $(D-1)$-dimensional hyperplane.

We saw the mode expansion for an open string with Neumann boundary conditions in equation (1.21). We can split this into left and right-moving parts to obtain

$$X_R(\tau-\sigma) = \frac{x-\tilde{x}}{2} + \alpha' p(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in(\tau-\sigma)}, \tag{1.119}$$

$$X_L(\tau+\sigma) = \frac{x+\tilde{x}}{2} + \alpha' p(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-in(\tau+\sigma)}. \tag{1.120}$$

Performing a T-duality transformation on the compact direction gives

$$X_R \to -X_R \quad \text{and} \quad X_L \to X_L. \tag{1.121}$$

Therefore, the dual coordinate in the compact direction is

$$\tilde{X}^9(\tau,\sigma) = X_L^9 - X_R^9 = \tilde{x}^9 + 2\alpha' p^9 \sigma + \sum_{n\neq 0} \frac{1}{n}\alpha_n^9 e^{-in\tau} \sin{(n\sigma)}. \tag{1.122}$$

Equation (1.122) is the mode expansion for an open string coordinate obeying Dirichlet boundary conditions. To see this, note that at $\sigma = 0, \pi$ the oscillatory terms vanish and since there is no term linear in τ, $\tilde{X}^9$ is fixed. Using $p^9 = \frac{k}{R}$, we see that at the endpoints

$$\tilde{X}(\tau,0) = \tilde{x} \quad \text{and} \quad \tilde{X}(\tau,\pi) = \tilde{x} + \frac{2\pi\alpha' k}{R} = \tilde{x} + 2\pi k \tilde{R}. \tag{1.123}$$

This string has no momentum in the compact direction since there is no term linear in τ in (1.122). This means the string only has oscillatory motion in the compact direction. The string does, however, wrap the dual circle nontrivially k times. The string's winding cannot be continuously deformed away without breaking the string because the string endpoints are fixed by the Dirichlet boundary conditions. The submanifold defined by $\tilde{X} = \tilde{x}$, is called a *D-brane*. A D-brane is defined to be a submanifold of the spacetime manifold on which open strings can end. As we will see, D-branes are dynamical, physical objects in the spectrum of string theory and not just a specified location. We have seen that T-duality changes a string with momentum, no winding and Neumann boundary conditions into a string with no momentum, nontrivial winding and Dirichlet boundary conditions. The converse is also true since T-duality is an order 2 symmetry.

A D-brane is normally specified by the number of spatial dimensions the submanifold has. A D_p-brane denotes a D-brane with p spatial dimensions and a $(p+1)$-dimensional worldvolume. Note that from equation (1.123), we see that the two endpoints end on the same D_8-brane in the case of a single T-dualized compact direction. The case of Neumann boundary conditions in all directions, as with the string we started with, is the case of a space filling D_9-brane.

1.3.2 T-Duality and Background Fields

So far we have left out many of the possible background fields. Three of the most important background fields are the metric, $g_{\mu\nu}(X)$, the NS-NS sector, antisymmetric 2-form gauge field, $B = B_{\mu\nu}(X)dx^\mu \wedge dx^\nu$, and the dilaton, $\Phi(X)$, also appearing in the NS-NS sector. The metric occurs as a background field in the part of the action we have considered so far,

$$S_g = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{h} h^{ab} g_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu. \tag{1.124}$$

So far we have only considered the case where $g_{\mu\nu} = \eta_{\mu\nu}$. This can easily be generalized to a more general spacetime metric in the obvious way.

The NS-NS sector antisymmetric 2-form gauge field, B, is called the *B-field*. The B-field is defined locally. It is not necessarily closed or globally defined. For most purposes, the fact that B is only defined locally doesn't matter, since strings are so small we only need to deal with things locally. When we need to look at global phenomena, we can use the *H-flux*, which is locally $H = dB$. H is a closed globally defined 3-form on X. The B-field contributes to the action a term proportional to the pullback of B along the spacetime embedding of the worldsheet integrated over the worldsheet. That means the contribution is

$$S_B = \frac{1}{4\pi\alpha'} \int d^2\sigma\ \varepsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu, \tag{1.125}$$

where ε^{ab} is the antisymmetric tensor density with components normalized to 1. $\epsilon^{ab} = \frac{1}{\sqrt{h}}\varepsilon^{ab}$ transforms as a tensor. This above term is only present in oriented string theories, such as both type II theories. If we project onto strings that are invariant under orientation reversal, due to its antisymmetry, the B-field vanishes from the spectrum, as we saw was the case for the type I theory in section 1.2.4.

For later use, note that if we denote the string world sheet by Σ, the spacetime manifold by X, the embedding of the worldsheet into the spacetime manifold by $\varphi : \Sigma \to X$ and let M be any 3-manifold in X bounded by $\varphi(\Sigma)$, then by Stokes' Theorem,

$$\begin{aligned} \int_\Sigma \varphi^* B &= \int_{\partial M} B \\ &= \int_M dB = \int_M H. \end{aligned} \tag{1.126}$$

In the above expression, $\varphi^* B$ is the pullback of B. We could redo the above formula using a different 3-manifold, M', bounded by $\varphi(\Sigma)$ and use the fact that e^{iS} should not depend on our choice of manifold bounded by $\varphi(\Sigma)$ to obtain

$$\frac{1}{4\pi\alpha'} \int_M H = \frac{1}{4\pi\alpha'} \int_{M'} H \pmod{2\pi\mathbb{Z}}. \tag{1.127}$$

This means that the H-flux corresponds to an integral cohomology class.

In a T-duality transformation, the metric and B-field transform via the *Buscher rules*, which were determined by Buscher in [19] and [20]. The Buscher rules mix the metric and B-fields. Equation (1.125) can be thought of as a generalization of the coupling between a 1-form Maxwell field and the worldline of a charged particle to the coupling of a 2-form field and the worldsheet of a charged string. So $B_{\mu\nu}$ defines a string charge in the same way a Maxwell field defines a particle charge. We will discuss this later in more detail.

The dilaton is related to the string coupling constant as

$$g_S = e^{\Phi}. \tag{1.128}$$

The dilaton adds a term to the action of

$$S_\Phi = \frac{1}{4\pi}\int d^2\sigma\, \sqrt{h}\Phi(X)R^{(2)}(h), \tag{1.129}$$

where $R^{(2)}(h)$ is the scalar curvature of the worldsheet. To determine how Φ transforms under a T-duality transformation it is easiest to look at how the string coupling constant transforms. Before doing that, let's first look at the significance of Φ.

If Φ is constant then equation (1.129) is a total derivative. Its value only depends on the worldsheet topology and does not contribute to the classical field equations. In this case equation (1.129) is

$$\Phi\chi(\Sigma), \tag{1.130}$$

where $\chi(\Sigma)$ is the Euler characteristic of the worldsheet Σ. The type II theories have only oriented closed strings, so worldsheets in the type II theories must be closed oriented Riemann surfaces. The Euler characteristic of a closed oriented Riemann surface is entirely determined by the genus of the surface, which corresponds to the number of string loops.

To determine how the string coupling constant changes under a T-duality transformation, it is easiest to look at the low energy effective action in the NS-NS sector. In the low energy limit, we can replace string theory by a supergravity theory that only describes the interactions of the massless modes. We can ignore the massive modes because they are too heavy to observe. The only result that we will need about the low energy effective actions is that for the type IIA theory compactified on a circle the low energy effective action in the NS-NS sector takes the form

$$\frac{2\pi R}{g_S^2}\int d^9x\,\mathcal{L}_{NS}, \tag{1.131}$$

and in the type IIB theory it takes the form

$$\frac{2\pi \tilde{R}}{\tilde{g}_S^2}\int d^9x\,\mathcal{L}_{NS}. \tag{1.132}$$

$\mathcal{L}_N S$ is the low energy effective action in the NS-NS sector. Since this is the only result we will need on this topic, we will not go into more detail about the explicit formula for $\mathcal{L}_{NS}$ and instead refer the reader to chapter 8 of [7]. Equations (1.131) and (1.131) should be equal by T-duality when $R\tilde{R} = \alpha'$. This implies that under a T-duality transformation

$$\tilde{g}_S = \frac{\sqrt{\alpha'}}{R} g_S. \tag{1.133}$$

1.3.3 Chan-Paton Factors and Wilson Lines

An additional non-dynamical degree of freedom can be added to the endpoints of the string since it is consistent with spacetime Poincaré invariance and worldsheet conformal invariance. The extra degree of freedom is known as a *Chan-Paton charge.*

Paton and Chan originally introduced these extra degrees of freedom in [57] to try to describe the global SU (2) isotopic spin symmetry between a quark and antiquark located at opposite ends of a string. It was later pointed out by Neveu and Scherk in [53] that Chan-Paton charges lead to a local gauge symmetry. In fact, we will see that including the Chan-Paton degrees of freedom will lead to a $\mathrm{U}(N)$ *Yang-Mills theory*. The modern interpretation of Chan-Paton charges was first described, 27 years after they were first introduced, by Witten in [91]. We now understand the Chan-Paton degrees of freedom to label which of N coincident D-branes a string endpoint lies on. As this is a crucial ingredient to using K-theory to describe string theory we will go through how this works now.

In addition to the usual Fock space label for a string state, we must also include a label, $i, j = 1, \ldots, N$, for the Chan-Paton charge at each end of the string. The Hamiltonian for the Chan-Paton charges vanishes, so they are non-dynamical. This means that once the Chan-Paton charge at a string endpoint is known, it will never change. Therefore we can decompose an arbitrary string state in terms of a basis $|\phi, k, ij\rangle$ with coefficients given by an $N \times N$ hermitian matrix, $\lambda = (\lambda_{ij})$, as

$$|\phi, k, \lambda\rangle = \sum_{i,j=1}^{N} |\phi, k, ij\rangle \lambda_{ij}. \tag{1.134}$$

The individual wave functions appearing on the right-hand side of the above equation are known as the *Chan-Paton factors*.

Consider an interaction between n oriented strings, whose tree level diagram is shown in figure 1.1. The scattering amplitude is obtained by summing over all

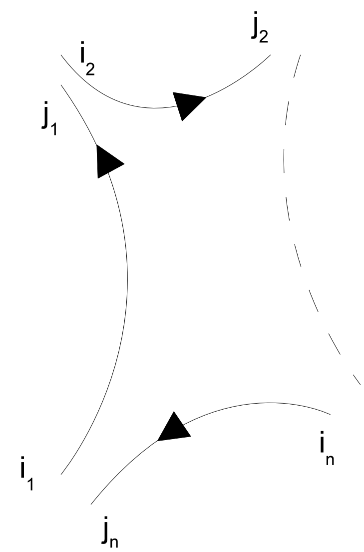

Figure 1.1: A scattering of n oriented open strings with Chan-Paton charges labeling each endpoint.

basis elements, so it picks up an overall factor of

$$\sum_{i_1,j_1=1}^{N} \cdots \sum_{i_n,j_n=1}^{N} \lambda^1_{i_1 j_1} \lambda^2_{i_2 j_2} \cdots \lambda^n_{i_n j_n}. \tag{1.135}$$

The $\sigma = \pi$ endpoint of the kth string must have the same Chan-Paton charge as the $\sigma = 0$ end of the $k+1 \pmod{n} th$ string, since the Chan-Paton charges are non-dynamical. Therefore we must include an additional factor of $\delta^{j_1 i_2} \cdots \delta^{j_{n-1} i_n} \delta^{j_n i_1}$ in the scattering amplitude. This makes the overall factor

$$\delta^{j_1 i_2} \cdots \delta^{j_{n-1} i_n} \delta^{j_n i_1} \lambda^1_{i_1 j_1} \lambda^2_{i_2 j_2} \cdots \lambda^n_{i_n j_n} = \mathrm{Tr}(\lambda^1 \cdots \lambda^n). \tag{1.136}$$

Such traces (and therefore all open string amplitudes) are invariant under the $\mathrm{U}(N)$ transformation

$$\lambda \to U \lambda U^{-1}. \tag{1.137}$$

Under such transformations, the $\sigma = 0$ endpoint transforms as the fundamental representation of $\mathrm{U}(N)$, $\mathbf{N}$, and the $\sigma = \pi$ endpoint transforms as the antifundamental (or conjugate) representation, $\bar{\mathbf{N}}$. From this and equation (1.134), we see that string states are matrices that transform in the adjoint representation of $\mathrm{U}(N)$. The Chan-Paton factors, $|\phi, k, ij\rangle$, transform with charge $+1$ under $\mathrm{U}(1)_i$ and charge -1 under $\mathrm{U}(1)_j$.

While this is only a local symmetry, it is raised to a global gauge symmetry of spacetime. Globally, the Chan-Paton factors define an N-dimensional bundle on the D-branes and the $\mathrm{U}(N)$ gauge field is a connection on the bundle.

Even if the gauge potential, A, is flat (has vanishing field strength, $F = dA + A \wedge A = 0$), it can have physical effects in a spacetime with compact dimensions. When the gauge potential in the compact direction has nonzero constant values it introduces a *holonomy* or *Wilson line*,

$$W = e^{i \int_0^{2\pi R} A_9 dx^9}. \tag{1.138}$$

A can be diagonalized by a constant gauge transformation, so that it may be written as

$$A = -\frac{1}{2\pi R} \mathrm{diag}(\theta_1, \theta_2, \ldots, \theta_N). \tag{1.139}$$

The Wilson line shifts the momentum of a state $|\phi, k, ij\rangle$ in the compact direction to

$$p^9 = \frac{k}{R} - \frac{\theta_i - \theta_j}{2\pi R}, \qquad k \in \mathbb{Z}. \tag{1.140}$$

After a T-duality transformation, the dual ij open string is expanded as

$$\tilde{X}^9_{ij} = \tilde{x}_0 + \theta_i \tilde{R} + 2\tilde{R}\sigma \left(k + \frac{\theta_j - \theta_i}{2\pi} \right) + \text{oscillator terms}. \tag{1.141}$$

Equation (1.141) shows us that the $\sigma = 0$ end of the string is located at $\tilde{x}_0 + \theta_i \tilde{R}$ and the $\sigma = \pi$ endpoint is located at $\tilde{x}_0 + \theta_j \tilde{R}$. In this way, we can interpret the Chan-Paton charges as labeling which of N coincident space filling D_9 branes the string endpoint lies on (similarly, there are $\bar{N}$ coincident space filling *anti-branes*, denoted as a $\bar{D}_9$-brane, with negative Chan-Paton charge interpreted as the Chan-Paton bundle having negative dimension). The N coincident D_9-branes transform to N D_8-branes with the ith brane located at the angular position θ_i on the dual circle. Therefore, the string wraps the dual circle an integral number of times only if $\theta_i = \theta_j$, otherwise it wraps the dual circle a fractional number of times.

The shift in the momentum caused by the inclusion of the Wilson line creates a corresponding shift in the mass spectrum. Therefore the only massless states when all of the θ_i's are different are the ones that represent strings that start and end on the same brane (so $\theta_j - \theta_i = 0$) and don't wrap the compact direction (so $k = 0$). This means that the diagonal states (states with $i = j$) with $k = 0$ define N different massless $\mathrm{U}(1)$ vectors when none of the branes coincide. This breaks the $\mathrm{U}(N)$ symmetry present when all of the branes were coincident to a $\mathrm{U}(1)^N$ symmetry. In general, if there are m coincident branes and l branes that do not coincide, with $m + l = N$, then the $\mathrm{U}(N)$ symmetry is broken into a $\mathrm{U}(m) \times \mathrm{U}(1)^l$ symmetry. The massless open string states create fluctuations in the geometry of the D-branes, showing that D-branes are actual dynamical objects [59].

For unoriented strings, such as type I strings, the gauge group changes. The Chan-Paton factors were originally extended to to the case of unoriented strings by Schwarz in [42] and Marcus and Sagnotti in [47]. For unoriented strings, the representation at the two endpoints must be the same. $N = \bar{N}$ implies that the fundamental representation of the symmetry group must be real. If the massless vectors are antisymmetric under orientation reversal then there are $\frac{N(N-1)}{2}$ such states and the symmetry group is SO(N). If the massless vectors are symmetric under orientation reversal, however, then there are $\frac{N(N+1)}{2}$ such states and the symmetry group is Sp(N). This group only exists for N even since symplectic matrices are even-dimensional.

1.4 K-Theory and D-brane Charges

We saw in the previous section that D-branes are actual dynamical objects. It turns out that they can carry charge. By charge and energy conservation a charged D-brane is stable. This section will begin by describing how D-branes carry charge. We will then give a brief overview of K-theory and describe how K-theory can be used to classify D-brane charges.

1.4.1 D-brane Charges

As we have seen, the different superstring theories contain numerous massless antisymmetric n-form gauge fields. We will denote a general n-form gauge field by

$$A_n = \frac{1}{n!} A_{\mu_1 \mu_2 \cdots \mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}. \tag{1.142}$$

Let us also define its $(n+1)$-form *field strength* to be

$$F_{n+1} = dA_n. \tag{1.143}$$

Since a D_p-brane, W, has a $(p+1)$-dimensional world volume it can couple electrically to a $(p+1)$-form gauge field, A_{p+1}. This is a generalization of the Maxwell field, A_1, that couples electrically to a point particle, which has a 1-dimensional worldline, to higher dimensions. Ordinary electromagnetism is described by Maxwell's equations

$$dF_2 = \star J_m \quad \text{and} \quad d \star F = \star J_e, \tag{1.144}$$

where $J = J_\mu dx^\mu$ is the 1-form magnetic (or electric) charge and current density and $\star$ denotes the Hodge dual. When we generalize this to higher dimensions the coupling contributes an interaction term to the action similar to the lower dimensional case given by

$$S_{int} = \mu_p \int_W A_{p+1}, \tag{1.145}$$

where μ_p is the charge of W. By Gauss' Law, μ_p can be calculated as

$$\mu_p = \int \star F_{p+2}, \tag{1.146}$$

The above integral is performed over a $(D-p-2)$-sphere in D total spacetime dimensions, because F, being a $p+2$-form, implies its Hodge dual is a $(D-p-2)$-form,

$$(\star F)^{\mu_1\mu_2\cdots\mu_{D-p-2}} = \frac{\varepsilon^{\mu_1\mu_2\cdots\mu_D}}{2\sqrt{-g}} F_{\mu_{D-p-1}\cdots\mu_D}. \tag{1.147}$$

There is a dual brane with magnetic charge given by

$$\int F_{p+2}, \tag{1.148}$$

where the integral is performed over $(p+2)$-sphere surrounding the brane. In D dimensions a $(p+2)$-sphere can surround a $(D-p-4)$-brane. Therefore in 10 dimensions the magnetic dual of a D_p-brane is a D_{6-p}-brane. The Dirac quantization condition generalizes to

$$\mu_p \mu_{6-p} \in 2\pi\mathbb{Z}. \tag{1.149}$$

Note that in the case of $D = 4$ dimensions and $p = 0$ we obtain ordinary electromagnetism.

Charged D-branes are stable due to energy and charge conservation. On the other hand, uncharged D-branes can decay, so only D-branes that couple either electrically or magnetically to a gauge field will be stable. We saw previously that in the R-R sector the Type IIA theory has n-form gauge fields for $n = 1, 3$. This means that in the type IIA theory there are stable, electrically charged D_p-branes for $p = 0, 2$ and magnetically charged D_p-branes for $p = 4, 6$. Under certain conditions it is also possible to have stable D_8-branes, so there are stable D_p branes for p even in the type IIA theory.

In the type IIB theory there are n-form gauge fields in the R-R sector for $n = 0, 2, 4$. So there are stable D_p-branes that are electrically charged for $p = -1, 1, 3$ and magnetically charged for $p = 3, 5, 7$. The D_{-1}-brane is called a *D-instanton* and is localized in both time and space. The 5-form field strength is self dual,

$$\star F_5 = F_5, \tag{1.150}$$

so the 4-form gauge field couples both electrically and magnetically to the same D_3-brane carrying an electric charge and its self-dual magnetic charge. There can also be space filling D_9-branes under certain conditions, so type IIB string theory has stable D_p-branes for p odd.

1.4.2 A Brief Overview of K-theory

The purpose of this section is to define K-theory and describe the properties we will need for the rest of this dissertation. A more detailed development of K-theory can be found in [36, 4, 3, 14]. We will state results in this section without proof. Proofs for any of the theorems in this section can be found in any of the above references.

While we have already used the term without giving a definition, we will be begin our discussion of K-theory by defining a vector bundle. Let X be a locally compact Hausdorff space. A *vector bundle over* X is described by a continuous open surjective map

$$\pi : E \to X \tag{1.151}$$

with the following conditions. E is a locally compact Hausdorff space with vector addition and scalar multiplication maps, $E \times_X E \to E$ and $\mathbb{C} \times E \to E$, that make $E_x = \pi^{-1}(x)$ into a vector space for each $x \in X$. E_x is called the *fiber of the bundle over* x. The above conditions define a *family of vector spaces over* X. Families of vector spaces over X define a category with morphisms given by commuting diagrams

$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi} & E_2\,, \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_2} & \\ & X & \end{array} \tag{1.152}$$

where φ is linear on the fibers. A vector bundle is a family of vector spaces over X with the added condition that there exists an open covering, $\{U_j\}$, of X such that $E|_{U_j} \cong U_j \times \mathbb{C}^n$ in the category of families of vector spaces over U_j for each

j, i.e., for all j there exists a homeomorphism, $\varphi_j : \pi^{-1}(U_j) \to U_j \times \mathbb{C}^n$, such that $p \circ \varphi(e) = \pi(e)$ for all $e \in \pi^{-1}(U_j)$, where p is the obvious projection of $U_j \times \mathbb{C}^n$ onto U_j. The homeomorphism φ_j along with the open set U_j is known as a *local trivialization of the vector bundle.* The rank of the fibers must be locally constant because of the local trivializations. Therefore, the rank of the fibers is constant on each connected component of X. If the rank is constant on all of X, it is called the *rank of the vector bundle.*

The local trivializations of a vector bundle must vary continuously. Let $\pi : E \to X$ be trivialized by the open covering $\{U_j\}$. Over the intersection of two sets U_i and U_j, the map

$$\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times \mathbb{C}^n) \to (U_i \cap U_j) \times \mathbb{C}^n \tag{1.153}$$

is well defined and obeys

$$\varphi_j \circ \varphi_i^{-1}(e, v) = (e, g_{ij}(e)v), \tag{1.154}$$

where $g_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C})$ is a continuous map. The g_{ij}'s are known as *transition functions* and satisfy the *cocycle identities*:

$$\begin{aligned} g_{ij}g_{ji} &= 1, \\ g_{ij}g_{jk}g_{ki} &= 1 \qquad \text{on } U_i \cap U_j \cap U_k. \end{aligned} \tag{1.155}$$

The transition functions determine how the different local trivializations are glued together and thus determines the topology of the *total space*, E. The *trivial bundle* is the bundle $X \times \mathbb{C}^n \to X$. If every transition function were of the form

$g_{ij} = h_i h_j^{-1}$ with $h_j : U_j \to \mathrm{GL}_n(\mathbb{C})$, then we could change the local trivialization functions, $\varphi_j : \pi^{-1}(U_j) \to U_j \times \mathbb{C}^n$ from $\varphi_j(e) = (u, v)$, with $\pi(e) = u$, to $\varphi_j(e) = (u, h_j v)$. Doing this would make all of the local trivializations match on their intersections, giving the trivial bundle. Conversely, the transition bundles for any bundle isomorphic to the trivial bundle can be written in the form $g_{ij} = h_i h_j^{-1}$. From this we see that isomorphism classes of vector bundles can be classified by cocycles $\{g_{ij}\}$ modulo coboundaries $\{g_{ij} : g_{ij} = h_i h_j^{-1}\}$; this is just $H^1(X; \underline{\mathrm{GL}_n(\mathbb{C})})$, the first cohomology group of X with coefficients in the sheaf of continuous functions with values in $\mathrm{GL}_n(\mathbb{C})$.

When discussing vector bundles, it is useful to define the pull back of a vector bundle over a continuous function. Let $E \xrightarrow{\pi} X$ be a vector bundle and $f : Y \to X$ be a continuous function. The *pull back along* f, $f^*(E)$, is defined by the commutative diagram

$$\begin{array}{ccc} f^*(E) & \overset{\hat{f}}{\dashrightarrow} & E \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X. \end{array} \tag{1.156}$$

In order for the above diagram to be commutative, we see that $f^*(E) = \{(y, e) \in Y \times E : f(y) = \pi(e)\}$, $f^*\pi$ is projection onto the first factor and $\hat{f}$ is projection onto the second factor. We can now use the pull back to define the direct sum and tensor product of two vector bundles.

Let $E_1 \xrightarrow{\pi_1} X$ and $E_2 \xrightarrow{\pi_2} X$ be vector bundles of rank n_1 and n_2 respectively. Then $\pi_1 \times \pi_2 : E_1 \times E_2 \to X \times X$ is a vector bundle of rank $n_1 + n_2$. The pull back of this bundle along the diagonal map $X \to X \times X$ gives a rank $n_1 + n_2$ bundle over X denoted by $E_1 \oplus E_2$ and called the *Whitney sum of* E_1 *and* E_2. Similarly, we can define the tensor product of two vector bundles, $E_1 \otimes E_2$, which is a rank $n_1 n_2$ vector bundle over X.

The direct sum operation makes the set of isomorphism classes of vector bundles over X (which we will denote by $\text{Vect}(X)$) into an abelian additive monoid with addition defined by $[E] + [F] = [E \oplus F]$, where $[E]$ denotes the isomorphism class of the vector bundle E. An *abelian monoid* is a set that obeys all of the axioms for an abelian group except for maybe the existence of inverses. $K(X)$ is the group completion of the monoid $\text{Vect}(X)$ for X compact. It is defined to be the set of formal differences of isomorphism classes of vector bundles over X,

$$[E] - [F], \tag{1.157}$$

under the equivalence relation

$$[E] - [F] = [E'] - [F'] \Leftrightarrow \exists H \in \text{Vect}(X) \text{ s.t. } E \oplus F' \oplus H \cong E' \oplus F \oplus H. \tag{1.158}$$

When the monoid is the set of isomorphism classes in an additive category, as is the case here, the group completion $K(X)$ is known as the *Grothendieck group*. $K(X)$ is an abelian group under addition and can be made into a commutative ring by defining

$$([E] - [F]) \cdot ([E'] - [F']) = [E \otimes E'] + [F \otimes F'] - [E \otimes F'] - [F \otimes E']. \tag{1.159}$$

$K(X)$ is a contravariant functor since a continuous map $f : Y \to X$ induces a homomorphism $f^* : K(X) \to K(Y)$ on K-theory. This comes from the pull back of f that we saw earlier and results in $f^* : \text{Vect}(X) \to \text{Vect}(Y)$.

We can extend K-theory to compact pairs (X, A), where X is compact and $A \subseteq X$ is closed. This done by defining

$$K(X, A) \equiv \tilde{K}(X/A) \equiv \operatorname{coker} q^* : K(\mathrm{pt}) \to K(X/A), \tag{1.160}$$

where $q : X/A \to \mathrm{pt}$ is the constant map. $\tilde{K}$ is known as *reduced K-theory*. By the definition of $\tilde{K}(X)$ and the fact that $K(\mathrm{pt}) = \mathbb{Z}$ (which we will see shortly) it can be shown that $K(X) = \tilde{K}(X) \oplus \mathbb{Z}$. $K(X, A)$ can be regarded as the Grothendieck group of virtual vector bundles over X of rank 0, with a fixed trivialization over A. From this perspective,

$$\begin{aligned} K(X, A) = \{[E, \xi_E] - [F, \xi_F] : E, F \in \mathrm{Vect}_n(X) \text{ and} \\ \xi_E : E|_A \xrightarrow{\cong} A \times \mathbb{C}^n, \xi_F : F|_A \xrightarrow{\cong} A \times \mathbb{C}^n \}, \end{aligned} \tag{1.161}$$

where $\mathrm{Vect}_n(X)$ is the set of isomorphism classes of rank n vector bundles over X. Note that $[E, \xi] \cong [E', \xi']$ if and only if there exists an isomorphism of vector bundles over X, $E \to E'$ that sends one trivialization to the other.

We can also extend the definition of $K(X)$ to spaces that are locally compact by defining the *K-theory with compact support* to be

$$K(X) \equiv \tilde{K}(X^+) = \ker i^* : K(X^+) \to K(\mathrm{pt}), \tag{1.162}$$

where X^+ is the one-point compactification of X and $i : \mathrm{pt} \to X^+$ is the inclusion of the point at infinity into X^+. Note that if X is compact then

$$K(X^+) = K(X \coprod \{\infty\}) = K(X) \oplus K(\{\infty\})$$

and $K(X) = \tilde{K}(X^+)$ agrees with our original definition of $K(X)$ for X compact.

We define $K^{-j}(X) \equiv K(X \times \mathbb{R}^j)$. We can make sense of $K^j(X)$ for j positive by *Bott periodicity.*

Theorem 1. (Bott Periodicity). *Let X be a locally compact space. Then there exists a natural isomorphism $K(X) \to K(X \times \mathbb{R}^2)$.*

Bott periodicity shows that $K^*(X)$ is $\mathbb{Z}_2$ graded, so there are only two different groups, $K^0(X) = K(X)$ and $K^1(X) = K^{-1}(X) = K(X \times \mathbb{R})$.

After extending K-theory to compact pairs, the functor $X \to K^*(X)$, where X is a compact space, obeys the *Eilenberg-Steenrod axioms* except for the dimension axiom.

Theorem 2. *Let X, Y be compact Hausdorff spaces and $A \subseteq X, B \subseteq Y$ be closed subspaces.*

(1) If $f, g : (X, A) \to (Y, B)$ are homotopic maps then they induce the same homomorphism on K-theory, $f^* = g^* : K^n(Y, B) \to K^n(X, A)$ for all n.

(2) If U is any subset of X whose closure is contained in the interior of A then the inclusion $i : (X - U, A - U) \to (X, A)$ induces an isomorphism $i^* : K^n(X, A) \to K^n(X - U, A - U)$ for all n.

(3) There exists a long exact sequence

$$\cdots \longrightarrow K^n(X, A) \xrightarrow{j^*} K^n(X) \xrightarrow{i^*} K^n(A) \xrightarrow{\delta} K^{n+1}(X, A) \longrightarrow \cdots,$$

where i and j are the inclusions $A \to X$ and $(X, \emptyset) \to (X, A)$ and δ is called the boundary map.

The above theorem means that the functor $X \to K^*(X)$ on the category of compact Hausdorff spaces is a generalized cohomology theory. This is also true in the category of locally compact spaces with all maps proper because of the extension of K-theory to locally compact spaces.

To describe the relationship between K-theory and ordinary cohomology, we must first define *Chern classes*. For the definition of a Chern class we will use the main theorem about Chern classes. The theorem contains all of the features of a Chern class, which can be taken as axioms.

Theorem 3. (Chern Classes). *There exists a unique sequence of functions* $c_i : \mathrm{Vect}(X) \to H^{2i}(X;\mathbb{Z})$ *such that, for any complex vector bundles* $E, F \to X$,

(1) $c_i(E) = c_i(F)$ *if* $[E] = [F]$.

(2) $c_i(f^*(E)) = f^*(c_i(E))$ *for any pullback* $f^*(E)$.

(3) $c_i(E) = 0$ *if* $i > \dim E$.

(4) $c(E \oplus F) = c(E) \cup c(F)$, *where* $\cup$ *is the cup product and* $c(E) = 1 + c_1(E) + c_2(E) + \cdots \in H^*(X;\mathbb{Z})$.

(5) $c_1(E)$ *is the usual generator of* $H^2(\mathbb{CP}^\infty;\mathbb{Z})$ *for the tautological line bundle* $E \to \mathbb{CP}^\infty$.

c_i is the ith Chern class and c is the total Chern class. Condition (1) simply states that c_i only depends on the isomorphism class of E. The *tautological line bundle over* $\mathbb{CP}^n$ is defined to be the bundle $E \xrightarrow{\gamma} \mathbb{CP}^n$ where $E = \{(x, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : v \in x\}$ and γ is the obvious projection. Condition (5) is a nontriviality and normalization condition. Without it $c_i = 0$ for all i would be possible and if c_i satisfied conditions $(1) - (4)$ so would $k^i c_i$ for a constant $k \in \mathbb{Z}$. Condition (3) ensures that $c = 1 + c_1 + c_2 + \cdots$ only has finitely many nonzero terms and so does in fact lie in $H^*(X;\mathbb{Z})$. We can determine the nth Chern class of $E \oplus F$ from condition (4) to be

$$c_n(E \oplus F) = \sum_{i+j=n} c_i(E) \cup c_j(F), \tag{1.163}$$

where $c_0 = 1$.

Chern classes behave well under direct sums but not under tensor products. For this reason, it is useful to introduce another function called the Chern character, but before this we need to describe vector bundles that can be decomposed as the direct sum of line bundles.

Lemma 1. (Splitting Principle). *Let X be a compact Hausdorff space and E be a rank n vector bundle over X. Then there exists a compact Hausdorff space $F(E)$ and a map $f : F(E) \to X$ such that the induced maps, f^*, on both K-theory and cohomology are injective and $f^*(E)$ splits as the direct sum of line bundles; $f^*(E) \cong L_1 \oplus \cdots \oplus L_n$.*

The space $F(E)$ together with the map $f : F(E) \to X$ is called the *flag bundle associated to the vector bundle $E \to X$*. The flag bundle is a *fiber bundle over X*, which is a generalization of a vector bundle where the fiber is no longer required to be a vector space. The definition of a fiber bundle is the same as for a vector bundle, if we replace the fiber $\mathbb{C}^n$ for a vector bundle with an arbitrary space F. The total space of the flag bundle, $F(E)$, is the set of all n-tuples of linearly independent lines in E_x for all $x \in X$. The map f sends an n-tuple of orthogonal lines in E_x to x. From this we see that the fiber is the *flag manifold*, $F(\mathbb{C}^n)$, which is the set of all n-tuples of linearly independent lines through the origin in $\mathbb{C}^n$. From condition (2) of theorem 3 we see that

$$c_i(L_1 \oplus \cdots \oplus L_n) = f^*(c_i(E)) \tag{1.164}$$

and by the splitting principle f^* is injective on cohomology so the Chern classes of E and $L_1 \oplus \cdots \oplus L_n$ are in one-to-one correspondence. This splitting of a rank n vector bundle into the direct sum of n line bundles is what broke the $\mathrm{U}(N)$ symmetry into the $\mathrm{U}(1)^N$ symmetry that we saw in section 1.3.3.

We can now define the Chern character; for a line bundle L, the *Chern character* is

$$\mathrm{Ch}(L) \equiv e^{c_1(L)} = 1 + c_1(L) + \frac{1}{2}c_1(L)^2 + \cdots . \tag{1.165}$$

Note that $\mathrm{Ch}(L) \in H^*(X;\mathbb{Q})$ because the power series has non-integer coefficients. For a direct sum of line bundles, the Chern character is defined to be

$$\mathrm{Ch}(L_1 \oplus \cdots \oplus L_n) \equiv \sum_{j=1}^{n} \mathrm{Ch}(L_j). \tag{1.166}$$

This along with the splitting principle defines the Chern character for a general rank n vector bundle. With this definition the Chern character satisfies

$$\mathrm{Ch}(E \oplus F) = \mathrm{Ch}(E) \oplus \mathrm{Ch}(F), \tag{1.167}$$

$$\mathrm{Ch}(E \otimes F) = \mathrm{Ch}(E)\,\mathrm{Ch}(F). \tag{1.168}$$

Due to the Chern character's behavior over direct sums and tensor products it extends to a ring homomorphism $\mathrm{Ch} : K(X) \to H^*(X;\mathbb{Q})$. A much stronger condition can be shown.

Theorem 4. $\mathrm{Ch} : K^*(X) \otimes \mathbb{Q} \to H^*(X;\mathbb{Q})$ *is an isomorphism of cohomology theories that sends products in* $K(X)$ *to cup products in* $H^*(X;\mathbb{Q})$*. It can be decomposed as*

$$\mathrm{Ch} : K(X) \otimes \mathbb{Q} \to H^{even}(X;\mathbb{Q}),$$

$$\mathrm{Ch} : K^1(X) \otimes \mathbb{Q} \to H^{odd}(X;\mathbb{Q}).$$

While this gives us a very powerful tool for relating K-theory to cohomology, the torsion in $K^*(X)$ can differ from the torsion in $H^*(X;\mathbb{Z})$. In practice we will often want to calculate $K^*(X)$, including the torsion, from $H^*(X;\mathbb{Z})$. To do this we will need the *Atiyah-Hirzebruch spectral sequence*

Theorem 5. (Atiyah-Hirzebruch). *There is a spectral sequence with*

$$E_2^{p,q} = H^p(X, K^q(pt))$$

($K^q(pt) = \mathbb{Z}$ for q even and 0 for q odd) that converges to $K^*(X)$. The first nonzero differential is $d_3 : H^p(X;\mathbb{Z}) \to H^{p+3}(X;\mathbb{Z})$ and is equal to the Steenrod operation Sq^3.

A spectral sequence is a sequence of bigraded abelian groups $\{E_r^{p,q}\}$ along with maps $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ (called differentials since $d_r d_r = 0$) such that E_{r+1} is the cohomology of E_r with respect to d_r. That is

$$E_{r+1}^{p,q} = \ker(d_r^{p,q}) / \operatorname{image}(d_r^{p-r,q+r-1}). \tag{1.169}$$

For the Atiyah-Hirzebruch spectral sequence $E_r^{p,q} = 0$ if q is odd, so only differentials d_r with r odd can be nonzero. This shows $E_3 = E_2, E_5 = E_4$, etc. If there exists

N such that $E_{r+1} = E_r$ for all $r > N$ then we call the stable value of E_r, E_∞. For the Atiyah-Hirzebruch spectral sequence, E_∞ gives $K^*(X)$ up to extensions of abelian groups. This provides us with a powerful tool to compute K-theory from cohomology that we will use throughout this dissertation. The next two corollaries of the Atiyah-Hirzebruch theorem simplify the results when X is a finite CW complex.

Corollary 1. *Let X be a finite CW complex. If $H^*(X;\mathbb{Z})$ is torsion-free, then so is $K^*(X)$, and $K(X) \cong H^{even}(X;\mathbb{Z})$ and $K^1(X) \cong H^{odd}(X;\mathbb{Z})$ as groups.*

Corollary 2. *Let X be a finite CW complex. The order of the torsion subgroup of $K^0(X)$ is less than or equal to the order of the direct sum of the torsion subgroups of the $H^{2i}(X;\mathbb{Z})$ and the order of the torsion subgroup of $K^1(X)$ is less than or equal to the order of the direct sum of the torsion subgroups of the $H^{2i+1}(X;\mathbb{Z})$. Moreover, if all of the torsion in $H^*(X;\mathbb{Z})$ is p-primary for some prime p, then all of the torsion in $K^*(X)$ is also p-primary.*

For later use, we will now introduce the *twisted K-theory of X*, $K^*(X,H)$, where $H \in H^3(X;\mathbb{Z})$. The defining characteristics of twisted K-theory are:

- if $H = 0$ then $K^*(X,H) = K^*(X)$.
- $K^*(X,H)$ is a module over $K^0(X)$.
- there is a cup product homomorphism

$$K^p(X,H) \otimes K^q(X,H') \to K^{p+q}(X,H+H').$$

Twisted K-theory is contravariant, so if $f : Y \to X$ is continuous then it induces a homomorphism

$$f^! : K^*(X,H) \to K^*(Y, f^*H).$$

Furthermore, if $f : X \to Y$ is a smooth K-oriented map between compact manifolds then there is a homomorphism

$$f_! : K^*(X, f^*H) \to K^{*+l}(Y,H),$$

where $l = \dim(X) - \dim(Y)$. We can still use the Atiyah-Hirzebruch spectral sequence to calculate $K^*(X,H)$, with the only change being that d_3 is changed so that $d_3(a) = \mathrm{Sq}^3(a) + H \cup a$.

1.4.3 Classifying D-brane charges by K-theory

Before describing how K-theory classifies D-brane charges, let us first look at the case of a single D_p-brane and a single $\bar{D}_p$-brane wrapping the same submanifold W of X (meaning they are coincident). Let us now label the Chan-Paton charges at the ends of a string by p if it ends on the D_p-brane and $\bar{p}$ if it ends on the anti-brane, so the charges live in a 2-dimensional quantum Hilbert space. We can consider the p state to be bosonic and the $\bar{p}$ state to be fermionic so that the GSO operator, $(-1)^F$, acts on the Chan-Paton factors by

$$(-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.170}$$

The Chan-Paton wave functions for p-p and $\bar{p}$-$\bar{p}$ open strings are even under $(-1)^F$, since they are diagonal, so we obtain the usual GSO projection on the oscillators. The p-$\bar{p}$ and $\bar{p}$-p open string states, however, are odd under $(-1)^F$ since the wave functions are off-diagonal. Therefore, the GSO projection is reversed, so the massless vector multiplet is projected away and the tachyon is kept. The tachyon represents a flow towards annihilation of the coincident brane and anti-brane. For a suitable expectation value the brane and anti-brane would annihilate each other and we would obtain the vacuum state without the pair.

We can extend this more generally to the case of N D_9-branes and N $\bar{D}_p$-branes all wrapping the same submanifold of X. Let E be any $\mathrm{U}(N)$ Chan-Paton gauge bundle on the D_p-branes. If the $\bar{D}_p$-branes have the same Chan-Paton bundle then the total system will have no D-brane charge and the branes and anti-branes will annihilate each other.

1.4.3.1 Classifying D-brane Charges in the Type IIB Theory

Let us first consider N space filling D_9-branes and $\bar{N}$ space filling $\bar{D}_9$-branes in the type IIB theory. The quantum theory requires complete cancellation of D-brane charges, so $N = \bar{N}$. There is a $\mathrm{U}(N)$ Chan-Paton gauge bundle, E, on the D-branes and a $\mathrm{U}(N)$ Chan-Paton Gauge bundle, F, on the $\bar{D}$-branes. As in the last section, we consider the bundle E to represent bosonic Chan-Paton states and F to represent fermionic Chan-Paton states. Note that the worldvolume of space filling branes and anti-branes is the entire spacetime manifold X, so E and F are bundles over X. This configuration of branes and anti-branes is determined by the pair (E, F). Two configurations of branes and anti-branes, (E, F) and (E', F'), should be equivalent to each other if they can be related by the creation or annihilation of M D_9-branes and M $\bar{D}_9$-branes that have the same $\mathrm{U}(M)$ Chan-Paton gauge bundle, G, on them. That means, $(E, F) \sim (E', F')$ if there exists a $\mathrm{U}(M)$ bundle, G, such that

$$(E', F') = (E \oplus G, F \oplus G), \tag{1.171}$$

or vice versa. But this is just $K(X)$. The requirement that $N = \bar{N}$ forces E and F to have the same rank, so brane anti-brane configurations that are consistent with the quantum theory modulo creation and annihilation of brane anti-brane pairs are classified by the reduced K-theory of X, $\tilde{K}(X)$. In most physical applications the total spacetime manifold will not be compact, so we will need to use K-theory with compact support. K-theory with compact support is indistinguishable from the reduced K-theory in this case. For this reason I will be purposefully vague about what type of K-theory is being used and just say that type IIB D-brane configurations are classified by $K(X)$ with the precise definition of K-theory needed depending on the situation.

We saw already that gauge fields living on the worldvolume of a single D-brane can couple electrically or magnetically to the D-brane, defining a charge. In the case of N coincident D-branes and N coincident $\bar{D}$-branes, the Chan-Paton bundles interact with the gauge fields. The interaction terms allows us to relate the gauge fields to the Chan-Paton bundles and determine D-brane charges in terms of the Chan-Paton bundles. We already stated in section 1.3.3 that the gauge fields are connections on the Chan-Paton bundles. Let us now be more precise. Let A be the $\mathrm{U}(N)$ gauge field living on the worldvolume of the D-branes and A' be the $\mathrm{U}(N)$ gauge field on the worldvolume of the anti-branes. A and A' are connections on E and F respectively. The tachyon field is a map

$$T\colon\ F \to E. \tag{1.172}$$

It can also be viewed as a section of $E \otimes F^*$, where F^* is the dual of the bundle F. Then all of the worldvolume field content can be put together into a *superconnection*

$$\mathcal{A} = \begin{pmatrix} A & T \\ \bar{T} & A' \end{pmatrix}. \tag{1.173}$$

For complete annihilation to occur the tachyon field should be such that the tachyon potential energy is a true minimum. The minimum value of the tachyon potential energy must be negative and completely cancel the branes' energy density.

In [51], Minasian and Moore showed that for a D-brane wrapped around W with embedding $f\colon\ W \hookrightarrow X$ and Chan-Paton bundle $E \to X$, its charge, Q, is given by

$$Q = \mathrm{ch}(f_! E)\sqrt{\hat{A}(TX)}, \tag{1.174}$$

where TX is the tangent bundle of X and

$$\mathrm{ch}(E) = \mathrm{Tr}_N e^{\frac{\mathcal{F}}{2\pi}}. \tag{1.175}$$

In the above expression $\mathcal{F} = F - f^*B$ where F is the Hermitian field strength of the $\mathrm{U}(N)$ gauge field that lives on the brane and f^*B is the pullback of the B-field along f.

The interaction between the Chan-Paton bundles and the gauge fields introduces an anomaly. An anomaly is an inconsistency in the theory that is usually caused by a topological invariant not vanishing. Therefore, anomaly cancellation usually involves some topological constraints. To describe the anomaly in the worldsheet path integral of type II superstring theory with D-branes we will follow [28].

Let Σ denote the worldsheet (which must be an oriented surface in the type II theory) and let $\varphi\colon\ \Sigma \to X$ be the embedding of the worldsheet into spacetime such that it it maps $\partial\Sigma$ to W, an oriented submanifold of X. So we are considering the type IIB string theory with a single D-brane, W. The worldsheet path integral contains the factors

$$\text{pfaff}(D) \cdot \exp(i \oint_{\partial\Sigma} A) \cdot \exp(i \int_{\Sigma} B), \tag{1.176}$$

where pfaff(D) is the pfaffian (square root of the determinant) of the worldsheet Dirac operator D and the second term is the holonomy of the brane gauge field A around the boundary of Σ. We will begin our discussion of the anomaly by assuming the B-field is trivial and ignore the last term in equation (1.176) until later.

The worldsheet left and right-movers have separate spin structures. Theorem 4.6 of [28] shows that our final result will not depend on the spin structures, so we may assume the left and right-movers have the same spin structure. Under this assumption, D, and thus pfaff(D), is real. The absolute value of pfaff(D) is well defined, but its sign is not.

To determine the ambiguity in the sign of pfaff(D), consider a one-parameter family of Σ's parameterized by the circle S^1. Our embedding into spacetime is now given by a map

$$\tilde{\varphi}\colon\ \Sigma \times S^1 \to X, \tag{1.177}$$

such that $\tilde{\varphi}(\partial\Sigma \times S^1) \subset W$. Freed and Witten showed that after going around the circle, pfaff(D) transforms as

$$\text{pfaff}(D) \to (-1)^{\alpha}\,\text{pfaff}(D), \tag{1.178}$$

where

$$\alpha = \int_{\partial\Sigma\times S^1} \tilde{\varphi}^* w_2(W). \tag{1.179}$$

In the above expression, $w_2(W)$ is the second Stiefel-Whitney class of W. If $w_2(W) \neq 0$ then pfaff(D) is not well defined. Note that $w_2(W) = 0$ implies that W is spin. We can reformulate the anomaly condition (equation (1.179)) in terms of the normal bundle ν to W in X.

X must be spin since we need to be able to define spinors on the spacetime manifold. Therefore $w_1(X) = w_2(X) = 0$. In the type II theory, W must be oriented, so $w_1(W) = 0$. The Whitney sum formula,

$$w_n(X) = \sum_{i+j=n} w_i(W) \cup w_j(\nu),$$

shows that $w_1(\nu) = 0$ and $w_2(\nu) = w_2(W)$. Therefore equation (1.179) can be written as

$$\alpha = \int_{\partial\Sigma\times S^1} \tilde{\varphi}^* w_2(\nu). \tag{1.180}$$

If pfaff(D) is not well defined, then the second term in equation (1.176) must change sign whenever pfaff(D) does in order for the string theory to be well defined. Therefore, A is not a globally defined U(1) gauge field, since its holonomy around a loop is not a well defined element of U(1). By looking at the Levi-Civita bundle on W, ω, we can show that A must be a Spin^c connection.

The structure group of ω is SO(n), where n is the dimension of W. The trace of the holonomy of ω in the spin representation of the double cover of SO(n), Spin(n), is only well defined up to sign because there are two ways to lift an element of $\mathrm{SO}(n)$ to $\mathrm{Spin}(n)$. It changes sign in the same way has the holonomy of A around $\partial\Sigma$ when going around a one-parameter family of loops parameterized by S^1. Therefore the trace of the product of the holonomy of ω and the holonomy of A is well defined, making the worldsheet measure well defined. The product of the holonomies is the holonomy from going around a loop $\partial\Sigma$ for spinors with charge 1 relative to A. Spinors on W can be understood as sections of the *spin bundle of* W, $S(W)$, the space of all spinors on W. Letting $\mathcal{L}$ denote the line bundle that A is a connection on, we see that neither $S(W)$ nor $\mathcal{L}$ are globally defined, but $S(W)\otimes\mathcal{L}$ is. The bundle $S(W)\otimes\mathcal{L}$ defines a Spin^c *structure* on W. The global anomaly is due to the fact that A is not a connection on a globally defined $\mathrm{U}(1)$ bundle, but $\omega + A$ is a connection on $S(W)\otimes\mathcal{L}$. The anomaly implies that W must be Spin^c and the Spin^c structure can be determined by the Levi-Civita connection and the gauge field A.

The fact that D-branes can only wrap submanifolds of X, W, if W admits a Spin^c structure allows us to show that the classification for stable D_9-brane configurations in the type IIB theory with trivial B-field by $K(X)$ includes stable D_p-branes for $p < 9$. A stable D-brane configuration that is wrapped on $W \subset X$ is classified by $K(W)$. It can be pushed forward to $K(X)$ along $f\colon W \hookrightarrow X$ by the Gysin map $f_!$ if W and X have Spin^c structures which f preserves.

We can rewrite the condition that W admits a Spin^c structure in terms of topological invariants. The short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{q} \mathbb{Z}_2 \longrightarrow 0, \tag{1.181}$$

induces the long exact sequence on cohomology,

$$\cdots \longrightarrow H^2(W;\mathbb{Z}) \xrightarrow{q^*} H^2(W;\mathbb{Z}_2) \xrightarrow{\beta} H^3(W;\mathbb{Z}) \longrightarrow \cdots . \tag{1.182}$$

β is the Bockstein map and $\beta(w_2(W)) = W_3(W) \in H^3(W;\mathbb{Z})$ is the third integral Stiefel-Whitney class of W. The second Stiefel-Whitney class $w_2(W)$ is the image under q^* of an element in $H^2(W;\mathbb{Z})$ if and only if $W_3(W) = 0$. There is only one Spin^c structure on W at each preimage of $w_2(W)$ under q^*. So W has a Spin^c structure if and only if $W_3(W) = 0$. Note that this is equivalent to ν having a Spin^c structure since $w_2(W) = w_2(\nu)$. In [92], Witten arrived at the same result by describing lower dimensional branes as bound states of coincident D_9 and $\bar{D}_9$-branes.

When we include a nontrivial H-flux, the requirement for a D-brane to be able to wrap a submanifold W is no longer $W_3(W) = 0$. The anomaly condition becomes

$$W_3(W) + H \cup W = 0. \tag{1.183}$$

In [92], Witten describes how the relation between $W_3(W)$ and $H \cup W$ implies that the gauge bundle on a D_9-brane in the type IIB theory can be described by transition functions on $U_i \cap U_j \cap U_k$ satisfying

$$g_{ij}g_{jk}g_{ki} = h_{ijk}, \tag{1.184}$$

where h_{ijk} is a cocycle with values in the nth roots of unity representing the lift of H to $H^2(W;\mathrm{U}(1))$ induced by the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathrm{U}(1) \longrightarrow 0.$$

n is the order of $H \in H^3(W;\mathbb{Z})$. This implies that in the presence of a nontrivial

H-flux, stable D-brane configurations in the type IIB theory are classified by the twisted K-theory, $K(X, H)$.

1.4.3.2 Classifying D-brane Charges in the Type I and Type IIA Theories

The classification of stable D-brane configurations in the type I superstring theory is very similar to the one in the type IIB theory. Strings in the type I theory are unoriented and must be invariant under the parity transformation. The parity transformation interchanges branes and anti-branes, so it sends Chan-Paton bundles to their conjugate. For this to be a symmetry, we see that the Chan-Paton bundles for open strings in the type I theory must be real. Furthermore, as we saw in section 1.3.3, Chan-Paton bundles in the type I theory are locally $\mathrm{SO}(N)$ gauge bundles. If E is an $\mathrm{SO}(N)$ bundle over the world volume of N coincident D-branes and F is an $\mathrm{SO}(N')$ bundle over the worldvolume of N' coincident anti-branes, tadpole cancellation in the type I theory requires $N' \equiv N \mod 32$. Therefore stable D-brane configurations are classified by a pair of real vector bundles (E, F), where E is an $\mathrm{SO}(N)$ bundle, F and $\mathrm{SO}(N')$ bundle and $N' \equiv N \mod 32$. Furthermore, a D-brane configuration classified by (E, F) is a equivalent to a D-brane configuration described by (E', F') if and only if there exists an $\mathrm{SO}(M)$ bundle G, for some $M \in \mathbb{Z}$, such that

$$(E', F') = (E \oplus G, F \oplus G), \tag{1.185}$$

or vice versa. This means stable D-brane configurations in the type I theory are classified by the K-theory of real vector bundles over X, $KO(X)$. KO is similar to K except it is defined in terms of real vector bundles. KO is also a generalized cohomology theory, but it has Bott periodicity of order 8. Note that there is no B-field in the type I theory because it is unoriented. Therefore, we do not need to worry about any twisting due to the H-flux.

Classifying stable D-brane configurations in the type IIA theory is a little more complicated than in the IIB theory since the worldvolume of a D-brane configuration has odd codimension in the type IIA theory. This problem is resolved by looking at bundles over $S^1 \times X$ rather than over X. If a brane wraps an odd dimensional submanifold, W, of X, we identify it with a submanifold $W' = z \times W$ of $S^1 \times X$, where z is any point in S^1. The worldvolume of W' has even codimension in $S^1 \times X$, so it can be shown that stable D-brane configurations in the type IIA theory are classified by elements of $K(S^1 \times X, H)$ that are trivial when restricted to X. That is, stable D-brane configurations in the type IIA theory are classified by $K^1(X, H)$.

1.4.4 A K-theoretic Description of T-duality

We saw earlier that the type IIA and IIB theories are related to each other by T-duality when the spacetime splits as a product $X \times S^1$. This can be extended to case when X is fibered by circles but doesn't split as a product by looking at the effect of T-duality on K-theory. Alvarez, Alvarez-Gaumé, Barbón and Lazano first looked at this generalization of T-duality in [1]. Their work was generalized by

Bouwknegt, Evslin and Mathai in [16, 15]. Since the physics in two spacetimes that are related by T-duality should be indistinguishable, stable D-brane configurations in one theory should map to a stable D-brane configuration in the other theory under a T-duality transformation. Therefore, the groups that classify stable D-brane configurations in the two different theories should be isomorphic. This along with the work of Bouwknegt, Evslin and Mathai led to the belief that there should be an axiomatic construction of the topological aspects of T-duality.

In [18], Bunke and Schick constructed a universal theory satisfying the following axioms:

(1) There is a suitable class of spacetimes X; each with a principal S^1-bundle $X \to Z$.

(2) On each X there can be any H-flux $H \in H^3(X;\mathbb{Z})$.

(3) There exists an involution $(X, H) \mapsto (\tilde{X}, \tilde{H})$ that keeps the base, Z, fixed.

(4) For $X = Z \times S^1$ and $H = 0$, $(\tilde{X}, \tilde{H})$ is topologically equivalent to $(Z \times S^1, 0)$ and for $Z = S^2$ and $(X, H) = (S^3, 0)$, $\tilde{X} = S^2 \times S^1$ and $\tilde{H}$ is the usual generator of $H^3(S^2 \times S^1; \mathbb{Z}) \cong \mathbb{Z}$.

(5) Let $\varphi\colon\ Z' \to Z$ and (X, H) be a pair over Z. Then for the pullback diagram

$$\begin{array}{ccc} \varphi^*(X) & \overset{\hat{\varphi}}{\dashrightarrow} & X \\ \downarrow & & \downarrow \\ Z' & \overset{\varphi}{\longrightarrow} & Z, \end{array} \tag{1.186}$$

the T-dual of $(\varphi^*(X), \hat{\varphi}^*(H))$ as a pair over Z' is the pull back of $(\tilde{X}, \tilde{H})$ constructed over Z.

(6) $K^*(X,H) \cong K^{*+1}(\tilde{X},\tilde{H})$.

In the above set of axioms $(\tilde{X},\tilde{H})$ denotes the T-dual of (X,H). In the Bunke-Schick construction, T-duality can be viewed as an isomorphism between the twisted K-groups that classify stable D-brane configurations in T-dual theories. The shift in dimension of the K-groups is due to the fact the T-duality maps the type IIA theory to the IIB theory and vice versa. Consider an oriented S^1-bundle $\pi\colon X \to Z$ with H-flux $H \in H^3(X;\mathbb{Z})$. It is characterized by its first Chern class $c_1(X) \in H^2(Z;\mathbb{Z})$. X gets mapped by T-duality to an oriented $\tilde{S}^1$-bundle[1] $\tilde{\pi}\colon \tilde{X} \to Z$ with H-flux $\tilde{H} \in H^3(\tilde{X};\mathbb{Z})$ such that

$$c_1(X) = \tilde{\pi}_*(\tilde{H}) \quad \text{and} \quad c_1(\tilde{X}) = \pi_*(H). \tag{1.187}$$

$\pi_*\colon H^k(X) \to H^{k-1}(Z)$ is the push forward map defined by integration of X along the fiber in terms of de Rham cohomology. This was generalized to higher dimensions, i.e., to spacetimes X with a principal $\mathbb{T}^n$-bundle $\pi\colon X \to Z$ by Mathai and Rosenberg in [49] and Bunke, Rumpf and Schick in [17].

Let us now look at circle bundles X over an oriented 2-manifold with genus g, Z, as an example. In this example the K-groups will be completely determined by the Atiyah-Hirzebruch spectral sequence and it is enough to look only at the differential $d_3 = \mathrm{Sq}^3 + H$. Sq^3 is trivial for this example, so d_3 is just the cup

[1]Here $\tilde{S}^1$ denotes the dual circle of radius $\frac{\alpha'}{R}$.

product with H. The twisted K-groups can be computed as:

$$K^0(X,H) = \frac{ker(H\cup : H^{even}(X) \to H^{odd}(X))}{H \cup H^{odd}(X)}, \tag{1.188}$$

$$K^1(X,H) = \frac{ker(H\cup : H^{odd}(X) \to H^{even}(X))}{H \cup H^{even}(X)}. \tag{1.189}$$

In the above and for the remainder of this section $H^*(X)$ denotes the cohomology of X with integer coefficients. E is completely classified by its first Chern class

$$c_1(E) = F \in H^2(Z) = \mathbf{Z},$$

where F is the cohomology class of the curvature of the bundle. So topologically, circle bundles are classified by an integer j, where $j = 0$ corresponds to the trivial bundle. If X is the trivial bundle, we can calculate its cohomology by the Künneth formula as:

$$\begin{aligned} H^0(X) &= \mathbf{Z}, & H^1(X) &= \mathbf{Z}^{2g+1}, \\ H^2(X) &= \mathbf{Z}^{2g+1}, & H^3(X) &= \mathbf{Z}. \end{aligned} \tag{1.190}$$

If the Chern class is $j \neq 0$ then the cohomology is:

$$\begin{aligned} H^0(X) &= \mathbf{Z}, & H^1(X) &= \mathbf{Z}^{2g}, \\ H^2(X) &= \mathbf{Z}^{2g} \oplus \mathbf{Z}_j, & H^3(X) &= \mathbf{Z}. \end{aligned} \tag{1.191}$$

The H-flux is classified by an integer k, since $H \in H^3(X) = \mathbb{Z}$. The cup product with H increases degree by 3, so is can be nonzero only with elements of $H^0(X)$ and is essentially multiplication by k. Therefore, we see from equation

(1.188) that when the H-flux is trivial the untwisted K-groups are

$$K^0(X) = H^0(X) \oplus H^2(X) = \begin{cases} \mathbf{Z}^{2g+2}, & j = 0 \\ \mathbf{Z}^{2g+1} \oplus \mathbf{Z}_j, & j \neq 0 \end{cases} \tag{1.192}$$

$$K^1(X) = H^1(X) \oplus H^3(X) = \begin{cases} \mathbf{Z}^{2g+2}, & j = 0 \\ \mathbf{Z}^{2g+1}, & j \neq 0. \end{cases} \tag{1.193}$$

When $H = k \neq 0$ times the usual generator of $H^3(E)$ then the twisted K-groups are:

$$K^0(X, H) = H^2(X) = \begin{cases} \mathbf{Z}^{2g+1}, & j = 0 \\ \mathbf{Z}^{2g} \oplus \mathbf{Z}_j, & j \neq 0 \end{cases} \tag{1.194}$$

$$K^1(X, H) = H^1(X) \oplus H^3(X)/kH^3(X) = \begin{cases} \mathbf{Z}^{2g+1} \oplus \mathbf{Z}_k, & j = 0 \\ \mathbf{Z}^{2g} \oplus \mathbf{Z}_k, & j \neq 0 \end{cases} \tag{1.195}$$

T-duality interchanges j and k, which results in the relevant twisted K-groups being interchanged.

1.5 Other Dualities in String Theory

While so far we have only discussed T-duality, there are many other dualities in string theory. A good general overview of dualities in string theory is [69]. Another important duality that we will need in this dissertation is called S-duality. *S-duality* relates a string theory with a strong string coupling constant to a weakly coupled string theory. S-duality is important to string theory because many calculations can only be done in the perturbative theory, where everything is expanded as a power series of the string coupling constant. The string coupling constant must be small for such power series to converge. This makes many computations impossible in a strongly coupled theory. Relating a strongly coupled theory to a weakly coupled one allows us to use perturbation theory to perform calculations in the weakly coupled theory and then relate them back to the strongly coupled theory via S-duality. This makes S-duality into a strong computational tool.

The best known example of S-duality is the S-duality between the type I theory and the SO(32) heterotic theory. That is, the weakly coupled type I superstring theory is dual to the strongly coupled SO(32) heterotic superstring theory [38, 90, 87, 21, 41, 62]. For this reason we will generally consider the type I theory in place of the SO(32) heterotic theory.

The focus of this dissertation will be on U-dualities between the type I superstring theory and the two type II theories. *U-duality* is a combination of S and T-dualities. The combination of the two dualities is more complicated than either of the individual dualities and the effect of U-duality on the K-theoretic description of stable D-brane configurations is unknown. In the next two chapters we will explore the K-theoretic aspects of U-dualities in the hope of defining some of the building blocks necessary to determine a K-theoretic description of U-dualities between the type I and type II theories analogous to the Bunke-Schick construction of T-duality.

Chapter 2

K-Theoretic Matching of Brane charges in a Type IIA/Type I Duality

As we have seen, the five superstring theories are all related through various dualities: T-duality, S-duality, and a combination of both, known as U-duality [41]. The main purpose of this dissertation is to develop the building blocks necessary to give a K-theoretic description of dualities between the type II string theories and the type I theory, as well as gain a greater understanding of the problems that arise and possible solutions. To this end, we will start with a known duality between the type IIA theory and the type I theory and try to give a K-theoretic description of the duality with the hope that we can then extend what we learn to more general examples. Note that since the type IIA and IIB theories are related by T-duality, it is enough to look only at type I/type IIA dualities when studying more general type I/type II dualities.

Sometimes the explicit dualities between the different superstring theories are unclear. By putting together the known duality between type I and the SO(32) heterotic string theories with the one between the SO(32) heterotic and type IIA string theories, one obtains an example of a U-duality between type I and type IIA string theories.

It is conjectured that type IIA string theory on $K3$ is dual to the SO(32) heterotic string on the 4-torus, $\mathbb{T}^4$ [40], [90, §4], [73], [8], [83], [7, p. 424]. The SO(32) heterotic string is believed to be equivalent to type I string theory via S-duality [38], [88], [21], so this gives a duality between type I string theory on $\mathbb{T}^4$ and type IIA string theory on $K3$. (This chain of equivalences is mentioned explicitly in [69, p. 258].)

We can determine a lot about possible dualities by looking at stable D-brane configurations. Stable D-branes in one theory should map to stable D-branes in any dual theories. So dual string theories should have equivalent D-brane configurations. (*Equivalent* here means, for instance, that it should be possible to match up the D-brane charges in the two theories, and thus these charges should live in isomorphic groups.) The stable D-brane configurations in a given theory depend only on the topology of the spacetime and can be classified by K-theory [52, 92, 93, 55]. We have seen such an isomorphism of the groups classifying D-brane charges with T-duality, and would now like to extend this idea.

Stable D-brane charges are classified by $KO(X)$ in type I string theory [92, 65], and by $\widetilde{K}(X)$ (resp., $K^{-1}(X)$) in type IIB theory (resp., type IIA theory) [92, 93, 7]. Since it is conjectured that type I string theory on $\mathbb{T}^4$ and type IIA string theory on $K3$ are dual to each other, they should have the same possible stable D-brane charges. Therefore we would expect to see an isomorphism between $KO^*(\mathbb{T}^4)$ and $K^{-1}(K3)$. The puzzle is that this is very far from being true, since $KO^*(\mathbb{T}^4)$ contains 2-torsion and $K^0(K3) \cong \mathbb{Z}^{24}$, while $K^{-1}(K3) \cong 0$, so that there wouldn't appear to be any stable D-brane charges at all in type-IIA theory compactified on $K3$! Even in IIB theory on $K3$, it appears there is no room for torsion brane charges! We show that if one first removes the sixteen isolated singularity points of an orbifold blow-down of $K3$, such an isomorphism is close to being achieved, albeit in a very nontrivial way. Thus this calculation provides an interesting test of S- and U-duality.

2.1 Classifying Stable D-brane configurations in Type I Compactified on $\mathbb{T}^4$

We work throughout with K-theory with compact support. Recall that for a locally compact space X which is not compact, $KO^*(X)$ is identified with $\widetilde{KO}^*(X^+)$, where $X^+ = X \cup \{\infty\}$ is the one-point compactification of X (e.g., $(\mathbb{R}^n)^+ = S^n$). As we saw in section 1.4.2, KO^* is a generalized cohomology theory. From the Eilenberg-Steenrod axioms (theorem 2), we can derive the formula

$$KO^k(X \times S^1) \cong KO^k(X) \oplus KO^{k-1}(X). \tag{2.1}$$

$KO^{-n}(\mathbb{T}^4)$ can be computed from

$$KO^{-i}(\mathrm{pt}) \cong \begin{cases} \mathbb{Z}, & i = 0 \pmod 4 \\ \mathbb{Z}_2, & i = 1, 2 \pmod 8 \\ 0, & \text{otherwise,} \end{cases}$$

by iterating equation (2.1). Thus we obtain:

$$
\begin{aligned}
KO^{-i}(\mathbb{T}^4) &\cong KO^{-i}(\mathbb{T}^3) \oplus KO^{-(i+1)}(\mathbb{T}^3) \\
&\cong KO^{-i}(\mathbb{T}^2) \oplus 2KO^{-(i+1)}(\mathbb{T}^2) \oplus KO^{-(i+2)}(\mathbb{T}^2) \\
&\cong KO^{-i}(\mathbb{T}) \oplus 3KO^{-(i+1)}(\mathbb{T}) \oplus 3KO^{-(i+2)}(\mathbb{T}) \oplus KO^{-(i+3)}(\mathbb{T}) \\
&\cong KO^{-i}(\mathrm{pt}) \oplus 4KO^{-(i+1)}(\mathrm{pt}) \oplus 6KO^{-(i+2)}(\mathrm{pt}) \oplus 4KO^{-(i+3)}(\mathrm{pt}) \\
&\qquad \oplus KO^{-(i+4)}(\mathrm{pt}).
\end{aligned}
$$

Since type I string theory is a ten-dimensional theory, the actual spacetime manifold for type I string theory compactified on $\mathbb{T}^4$ is $\mathbb{T}^4 \times \mathbb{R}^6$. Stable D-brane charges in type I string theory on $\mathbb{T}^4 \times \mathbb{R}^6$ are thus classified by

$$
KO^0(\mathbb{T}^4 \times \mathbb{R}^6) \cong KO^{-6}(\mathbb{T}^4) \cong 6\mathbb{Z} \oplus 5\mathbb{Z}_2. \tag{2.2}
$$

However, this may not be the end of the story. Let $\iota\colon Y^{p+1} \hookrightarrow \mathbb{T}^4 \times \mathbb{R}^6$ be the inclusion of a (proper) Dp-brane in $\mathbb{T}^4 \times \mathbb{R}^6$. Strings in the type I theory are unoriented, so the type I theory does not support a B-field. For anomaly cancellation in the type I theory, Y should be spin. The Gysin map in KO-theory gives a map $\iota_!\colon KO(Y) \to KO^{3-p}(\mathbb{T}^4)$ obtained as the following composite:

$$
KO(Y) \xrightarrow{\text{P.D.}} KO_{p+1}(Y) \xrightarrow{\iota_*} KO_{p+1}(\mathbb{T}^4\times\mathbb{R}^6) \xrightarrow{(\text{P.D.})^{-1}} KO^{10-(p+1)}(\mathbb{T}^4\times\mathbb{R}^6) \cong KO^{3-p}(\mathbb{T}^4).
$$

Here P.D. denotes the Poincaré duality isomorphism, $(\text{P.D.})^{-1}$ is its inverse and ι_* is the map induced by ι on KO-homology. A Chan-Paton bundle with orthogonal gauge group gives a class in $KO(Y)$, and thus via the Gysin map $\iota_!$ a D-brane charge in $KO^{3-p}(\mathbb{T}^4)$. This is $KO^{-6}(\mathbb{T}^4) \cong 6\mathbb{Z} \oplus 5\mathbb{Z}_2$ when $p = 9$ or 1. (Recall that real K-theory satisfies Bott periodicity with period 8.) Similarly, a Chan-Paton bundle with symplectic gauge group gives a class in $KSp(Y) \cong KO^4(Y)$ (since real and symplectic K-theory agree after a dimension shift by 4), and thus via the Gysin map $\iota_!$ a D-brane charge in $KO^{7-p}(\mathbb{T}^4)$. This can again be identified with $KO^{-6}(\mathbb{T}^4)$ when $p = 5$. The 9-branes and 1-branes with real Chan-Paton bundles, along with the 5-branes with symplectic Chan-Paton bundles, account for all the usual BPS-branes of type I superstring theory [7, p. 223]. But as pointed out by many authors,

p	bundle type	BPS?	Charge group
9	O	yes	$KO^{-6}(\mathbb{T}^4) \cong 6\mathbb{Z} \oplus 5\mathbb{Z}_2$
9	Sp	no	$KO^{-2}(\mathbb{T}^4) \cong 6\mathbb{Z} \oplus \mathbb{Z}_2$
8	O	no	$KO^{-5}(\mathbb{T}^4) \cong 4\mathbb{Z} \oplus \mathbb{Z}_2$
8	Sp	no	$KO^{-1}(\mathbb{T}^4) \cong 4\mathbb{Z} \oplus 5\mathbb{Z}_2$
7	O	no	$KO^{-4}(\mathbb{T}^4) \cong 2\mathbb{Z}$
7	Sp	no	$KO^{0}(\mathbb{T}^4) \cong 2\mathbb{Z} \oplus 10\mathbb{Z}_2$
6	O	no	$KO^{-3}(\mathbb{T}^4) \cong 4\mathbb{Z}$
6	Sp	no	$KO^{1}(\mathbb{T}^4) \cong 4\mathbb{Z} \oplus 10\mathbb{Z}_2$
5	O	no	$KO^{-2}(\mathbb{T}^4) \cong 6\mathbb{Z} \oplus \mathbb{Z}_2$
5	Sp	yes	$KO^{2}(\mathbb{T}^4) \cong 6\mathbb{Z} \oplus 5\mathbb{Z}_2$
4	O	no	$KO^{-1}(\mathbb{T}^4) \cong 4\mathbb{Z} \oplus 5\mathbb{Z}_2$
4	Sp	no	$KO^{3}(\mathbb{T}^4) \cong 4\mathbb{Z} \oplus \mathbb{Z}_2$
3	O	no	$KO^{0}(\mathbb{T}^4) \cong 2\mathbb{Z} \oplus 10\mathbb{Z}_2$
3	Sp	no	$KO^{4}(\mathbb{T}^4) \cong 2\mathbb{Z}$
2	O	no	$KO^{1}(\mathbb{T}^4) \cong 4\mathbb{Z} \oplus 10\mathbb{Z}_2$
2	Sp	no	$KO^{5}(\mathbb{T}^4) \cong 4\mathbb{Z}$
1	O	yes	$KO^{2}(\mathbb{T}^4) \cong 6\mathbb{Z} \oplus 5\mathbb{Z}_2$
1	Sp	no	$KO^{6}(\mathbb{T}^4) \cong 6\mathbb{Z} \oplus \mathbb{Z}_2$
0	O	no	$KO^{3}(\mathbb{T}^4) \cong 4\mathbb{Z} \oplus \mathbb{Z}_2$
0	Sp	no	$KO^{7}(\mathbb{T}^4) \cong 4\mathbb{Z} \oplus 5\mathbb{Z}_2$
-1	O	no	$KO^{4}(\mathbb{T}^4) \cong 2\mathbb{Z}$
-1	Sp	no	$KO^{8}(\mathbb{T}^4) \cong 2\mathbb{Z} \oplus 10\mathbb{Z}_2$

Table 2.1: Groups of Dp-brane charges for type I compactified on $\mathbb{T}^4$

e.g., [81, 82, 92, 9, 13, 2], there can be additional D-brane charges coming from non-supersymmetric, but still stable, branes with other values of p. Such charges (for type I superstring theory compactified on $\mathbb{T}^4$) are summarized in the following Table 2.1. The various kinds of branes are hypothetical; not all of them actually occur. Also note that after inverting 2, KO and KSp are the same, so the nature of the Chan-Paton gauge group only affects the 2-torsion.

2.2 $K3$ Surfaces

The string theories we have looked at so far do not make sense physically. Superstring theories on 10-dimensional Minkowski spacetime, while useful for for defining the fields involved, do not make sense since since we can only observe 4 spacetime dimensions. This problem is rectified by compactifying 6 of the dimensions on an *internal manifold*, whose size is small enough to have avoided detection so far. To be more precise, if the spacetime manifold is $X^6 \times \mathbb{R}^{3,1}$, where X is a 6-dimensional compact manifold of size l_X, then the compact dimensions will be unobservable at energies $E \ll 1/l_X$. Only at high energy will the compact dimensions become apparent and affect interactions; therefore, at low energies the 4 noncompact dimensions define our 4-dimensional spacetime for the purposes of particle physics.

We have already looked at superstring theories compactified on circles and tori. Compactifying any of the superstring theories on a 6-torus, to obtain an effective 4-dimensional theory, does not break any of the supersymmetry. Superstring theory compactified on $\mathbb{T}^6$ has either $\mathcal{N} = 4$ or $\mathcal{N} = 8$ in 4 dimensions. Extensions of the

$D = 4$ standard model that include supersymmetry must have the supersymmetry broken at some scale. Imposing supersymmetry introduces constraints on the theory that make it unrealistic for $\mathcal{N} \geq 2$. For $\mathcal{N} = 1$ the constraints make calculations easier, but do not make the theory unrealistic. Therefore, we would like to compactify string theory on an internal manifold that gives $\mathcal{N} = 1$ supersymmetry in 4 dimensions and we see that compactifications on a 6-torus are unrealistic.

Superstring theories are often compactified on *Calabi-Yau manifolds* since they do not have the isometries that a torus has and thus break some of the supersymmetry. Before we can define a Calabi-Yau manifold, we must first define a *Kähler manifold.*

Definition 1. Let X be a complex manifold with a hermitian metric $g_{i\bar{j}}$. We can define a $(1,1)$-form $\omega = \frac{i}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$ on X. X is called Kähler if ω is closed, $d\omega = 0$.

Definition 2. A Calabi-Yau n-fold, X, is an n-dimensional Kähler manifold satisfying one of the following equivalent conditions:

- The canonical bundle of X (the bundle of $(n,0)$-forms) is trivial.
- X has $\mathrm{SU}(n)$ holonomy.
- The first Chern class of X is 0 as an integral cohomology class.

There are many more equivalent definitions, but for our purposes, this will suffice. As a consequence of the definition, Calabi-Yau manifolds have vanishing Ricci curvature, which is another reason why they are so appealing for applications to physics.

Compactifying on a (simply connected) Calabi-Yau 3-fold breaks three fourths of the supersymmetry leaving $\mathcal{N} = 1$ supersymmetry for the heterotic and type I theories in 4 dimensions and $\mathcal{N} = 2$ for the type II theories. In fact, it can be shown that (see section 9.3 of [7]) the heterotic theory will only have $\mathcal{N} = 1$ supersymmetry if it is compactified on a Calabi-Yau 3-fold. While compactifications on Calabi-Yau 3-folds are the most physically realistic, it is informative to look first at compactifications on complex n-folds for $n < 3$. The only Calabi-Yau 1-folds are $\mathbb{C}$ and $\mathbb{T}^2$. We have already looked at these examples. There are only 2 compact Calabi-Yau 2-folds. They are $\mathbb{T}^4$ and $K3$ manifolds. Let f be any degree 4 homogenous polynomial in the coordinates $(z_1, z_2, z_3, z_4) \in \mathbb{C}^4$. A $K3$ surface is the submanifold of $\mathbb{CP}^3$ defined by

$$f(z_1, z_2, z_3, z_4) = 0, \tag{2.3}$$

assuming f is chosen so that the surface is smooth. Using a different quartic polynomial will give another $K3$ surface that is diffeomorphic to the manifold defined by f, but has a different complex structure.

Since all $K3$ surfaces are diffeomorphic they all have the same Betti numbers: $1, 0, 22, 0, 1$. Furthermore, the integral cohomology of $K3$ contains no torsion, so the Betti numbers completely define the cohomology of $K3$. Note that $H^3(K3) = 0$, so $K3$ surfaces cannot support a nontrivial H-flux. We can use Corollary 1 to calculate the K-theory of $K3$ to be

$$K^0(X) \cong H^{\text{even}}(X; \mathbb{Z}) \cong \mathbb{Z}^{24}, \tag{2.4}$$

$$K^1(X) \cong H^{\text{odd}}(X; \mathbb{Z}) \cong 0. \tag{2.5}$$

It becomes immediately obvious that we don't just want to use $K3$ as a possible dual topology to type-I theory on $\mathbb{T}^4$, because the complex K-theory of $K3$ contains no torsion, and $K^{-1}(K3) \cong 0$, so could not possibly match $KO^{-6}(\mathbb{T}^4)$.

2.2.1 K-Theory of a Desingularized $K3$

Instead of $K3$, a $\mathbb{Z}_2$ orbifold quotient of $\mathbb{T}^4$, which is a singular limit of $K3$, is often used in string theory because the Ricci-flat metric can be explicitly determined [7, §9.3]. For the purposes of string theory we can define an orbifold as follows. Let X be a smooth manifold with a discrete isometry group G. The G *orbifold quotient of* X is defined to be the quotient space X/G, where for $x, y \in X$, $x = y$ in X/G if and only if there exists $g \in G$ such that $x = gy$. Points in X with nontrivial isotropy under the action of G are called *singular*. If there are no singular points, the orbifold is itself a smooth manifold. Orbifold background geometries affect the physical spectrum. Orbifolds (and more general orientifolds) will be the topic of the next chapter. We need to define the term for what follows, but will delay further details until the next chapter.

Consider $\mathbb{T}^4$ as $\mathbb{C}^2$ under the equivalence relation

$$\begin{aligned} z_a &\sim z_a + 1, \\ z_a &\sim z_a + i, \end{aligned} \tag{2.6}$$

for $a = 1, 2$. Then $\mathbb{T}^4$ has a $\mathbb{Z}_2$ isometry group generated by

$$(z_1, z_2) \to (-z_1, -z_2). \tag{2.7}$$

The 16 points given by z_1 and z_2 taking one of the four values $0, \frac{1}{2}, \frac{i}{2}$ and $\frac{1+i}{2}$ are invariant under the $\mathbb{Z}_2$ isometry. Therefore, the orbifold $\mathbb{T}^4/\mathbb{Z}_2$ has 16 isolated singular points. To define what it means for $\mathbb{T}^4/\mathbb{Z}_2$ to be the singular limit of $K3$, we must first describe a process known as blowing up the singularities.

We can remove the 16 singular points by first removing 16 open balls in $\mathbb{T}^4$ surrounding each of the singular points. We then divide out by the $\mathbb{Z}_2$ action on $\mathbb{T}^4$ minus the 16 open balls to obtain a smooth manifold (with boundary), N. The boundary of N is 16 copies of $S^3/\mathbb{Z}_2 \cong \mathbb{RP}^3$. Since $K3$ is a Calabi-Yau manifold, we would like to replace the missing balls with a smooth Kähler manifold with vanishing Ricci curvature whose boundary is $\mathbb{RP}^3$. The unique manifold with these properties is called an *Eguchi-Hanson space*, which topologically is the unit disk bundle of the tangent bundle of $S^2 = \mathbb{CP}^1$. Glueing 16 copies of an Eguchi-Hanson space onto N along their common boundary creates a manifold with the same topology as $K3$. The metric on the Eguchi-Hanson space is

$$ds^2 = \left(1 - \left(\frac{a}{r}\right)^4\right)^{-1} dr^2 + \frac{1}{4}r^2 \left(1 - \left(\frac{a}{r}\right)^4\right)(d\psi + \cos\theta\, d\phi)^2 + \frac{1}{4}r^2 d\Omega_2^2, \qquad (2.8)$$

where $d\Omega_2^2$ is the area element of the 2-sphere, ψ has period 2π and the radial coordinate has a domain of $a \leq r \leq \infty$ for an arbitrary constant a. So blowing up the singularities gives a manifold with the same topology as $K3$ for $a > 0$, but in the $a \to 0$ limit we get back the orbifold $\mathbb{T}^4/\mathbb{Z}_2$. This is what is meant by saying $\mathbb{T}^4/\mathbb{Z}_2$ is the singular limit of $K3$.

In the physics literature, $\mathbb{T}^4/\mathbb{Z}_2$ with the singularities blown up is usually used instead of $K3$ because it is a manifold with the same topology as $K3$. An explicit form of the metric is known on the blow up of $\mathbb{T}^4/\mathbb{Z}_2$, but it must be smoothed to give a Calabi-Yau geometry. The blow up using Eguchi-Hanson spaces cannot be the correct manifold for our purposes of matching with the type I theory on $\mathbb{T}^4$ because again its cohomology (and thus its K-theory) contains no torsion. Since the original $K3$ has no singularities, we do not want to allow for any physical effects from the singularities[1]. Therefore, physically, we are only interested in fields that approach a constant value at the singularities, and it makes more sense simply to collapse the singularities and deal with the singular quotient space $(\mathbb{T}^4/\mathbb{Z}_2)/(\text{singularities}) \cong N/\partial N$. Since this space is the one-point compactification of the *interior* of N, we have $\widetilde{K}^*(N/\partial N) \cong K^*(N, \partial N)$, the relative K-theory of the manifold N rel its boundary. So this is what we shall compute.

To compute $K^*(N, \partial N)$ we will first need to compute the homology of N. Let M be $\mathbb{T}^4 - (16 \text{ open balls})$, which is the double cover of N. Since N is obtained from M by dividing out by a free $\mathbb{Z}_2$-action, there is a spectral sequence $H_p(\mathbb{Z}_2, H_q(M)) \Rightarrow H_{p+q}(N)$. (See for example [50, Theorem 8^{bis}.9].) So we must first determine the homology of M as a $\mathbb{Z}_2$-module. The homology of M is torsion free, so this will split as a direct sum of copies of two standard $\mathbb{Z}_2$-modules: the trivial module $\mathbb{Z}$, and $\mathbb{Z}$ with the non-trivial $\mathbb{Z}_2$-action (where the generator of the group acts by multiplication by -1). We call this latter module $\underline{\mathbb{Z}}$ to distinguish it from $\mathbb{Z}$ with the trivial $\mathbb{Z}_2$-action. First of all, note that the cohomology ring of $\mathbb{T}^4$ is an exterior algebra on 4 generators. Each of these generators is sent to its negative under

[1] We will see in the next chapter that orbifold singularities, like D-branes, are sources for R-R charges.

the $\mathbb{Z}_2$ action, so $\mathbb{Z}_2$ acts trivially on the even exterior powers and non-trivially on the odd exterior powers. So $H^1(\mathbb{T}^4) \cong H_1(\mathbb{T}^4) \cong H^3(\mathbb{T}^4) \cong H_3(\mathbb{T}^4) \cong \underline{\mathbb{Z}}^4$, while $H^2(\mathbb{T}^4) \cong H_2(\mathbb{T}^4) \cong \mathbb{Z}^6$. Now by a simple transversality argument, removing 16 balls from $\mathbb{T}^4$ does not change the fundamental group, so $\pi_1(M) \cong \pi_1(\mathbb{T}^4) \cong \mathbb{Z}^4$. Therefore $H_1(M) \cong \underline{\mathbb{Z}}^4$. To obtain $H_2(M)$ we can use the Mayer-Vietoris sequence:

$$\begin{array}{ccccccc}
H_2(M \cap 16B^4) & \longrightarrow & H_2(M) \oplus H_2(16B^4) & \longrightarrow & H_2(\mathbb{T}^4) & \longrightarrow & H_1(M \cap 16B^4) \\
\| & & \| & & \| & & \| \\
0 & \longrightarrow & H_2(M) & \longrightarrow & \mathbb{Z}^6 & \longrightarrow & 0
\end{array}$$

Here B^4 is the closed 4-ball, so $M \cap 16B^4 = 16S^3$. So we see $H_2(M) \cong \mathbb{Z}^6$. We can determine $H_3(M)$ from the long exact sequence of pairs, using the pair $(M, \partial M)$, where $\partial M = 16S^3$. The part of the long exact sequence we are interested in is

$$H_4(M) \to H_4(M, \partial M) \to H_3(16S^3) \to H_3(M) \to H_3(M, \partial M) \to H_2(16S^3).$$

$H_4(M) \cong 0$ since M has a nonempty boundary. By Poincaré duality $H_4(M, \partial M) \cong H^0(M) \cong H_0(M) \cong \mathbb{Z}$. And similarly, $H_3(M, \partial M) \cong H^1(M) \cong FH_1(M) \oplus TH_0(M) \cong \underline{\mathbb{Z}}^4$. Finally, $H_3(16S^3) \cong \mathbb{Z}^{16}$ (the $\mathbb{Z}_2$ action is trivial since it preserves orientation on S^3) and $H_2(16S^3) \cong 0$. Putting this all together, the long exact sequence becomes

$$0 \to \mathbb{Z} \to \mathbb{Z}^{16} \to H_3(M) \to \underline{\mathbb{Z}}^4 \to 0.$$

This shows us that $H_3(M) \cong \underline{\mathbb{Z}}^4 \oplus \mathbb{Z}^{15}$. Putting this all together we see

$$H_i(M) \cong \begin{cases} \mathbb{Z}, & i = 0 \\ \underline{\mathbb{Z}}^4, & i = 1 \\ \mathbb{Z}^6, & i = 2 \\ \mathbb{Z}^{15} \oplus \underline{\mathbb{Z}}^4, & i = 3 \\ 0, & \text{otherwise.} \end{cases}$$

We can now determine $H_*(N)$ from the spectral sequence with $E^2_{p,q} = H_p(\mathbb{Z}_2, H_q(M))$. Recall that $H_0(\mathbb{Z}_2, \mathbb{Z}) \cong \mathbb{Z}$ and $H_0(\mathbb{Z}_2, \underline{\mathbb{Z}}) \cong \mathbb{Z}_2$, so

$$E^2_{0,q} \cong H_0(\mathbb{Z}_2, H_q(M)) \cong \begin{cases} \mathbb{Z}, & q = 0 \\ {\mathbb{Z}_2}^4, & q = 1 \\ \mathbb{Z}^6, & q = 2 \\ \mathbb{Z}^{15} \oplus {\mathbb{Z}_2}^4, & q = 3 \\ 0, & \text{otherwise.} \end{cases}$$

For $p > 0$, $E^2_{p,q}$ is all torsion, so the free part of $H_q(N)$ is the same as for $E^2_{0,q}$. And we see that the Betti numbers of N are

$$\beta_i(N) = \begin{cases} 1, & i = 0 \\ 0, & i = 1 \\ 6, & i = 2 \\ 15, & i = 3 \\ 0, & \text{otherwise.} \end{cases}$$

First we know that $H_0(N) \cong \mathbb{Z}$ since N is connected. We also know $H_4(N) \cong 0$ because N has a nonempty boundary. Now $H_3(N) \cong H^1(N, \partial N) \cong FH_1(N, \partial N) \oplus$

$TH_0(N, \partial N)$, and $TH_0(N, \partial N) \cong 0$. So $H_3(N)$ is free, and thus isomorphic to $\mathbb{Z}^{15}$. We can use Mayer-Vietoris with N and 16 Eguchi-Hanson spaces, E, since $N \cup_{16\mathbb{RP}^3} 16E \cong K3$, to show that $H_2(N) \cong \mathbb{Z}^6$, since we have an exact sequence

$$\cdots \to H_{k+1}(K3) \to H_k(16\mathbb{RP}^3) \to H_k(N) \oplus H_k(16E) \to H_k(K3) \to \dots$$

Furthermore, E has the same homotopy type as S^2, since it is the unit disk bundle of the tangent bundle of S^2. The part of the Mayer-Vietoris sequence we are interested in is:

$$\begin{array}{ccccc} H_2(16\mathbb{RP}^3) & \longrightarrow & H_2(N) \oplus H_2(16E) & \longrightarrow & H_2(K3) \\ \| & & \| & & \| \\ 0 & \longrightarrow & H_2(N) \oplus \mathbb{Z}^{16} & \longrightarrow & \mathbb{Z}^{22}. \end{array}$$

From this we see that $H_2(N)$ injects into a free group and thus must be free. We have shown that only $H_1(N)$ can have any torsion.

We can calculate $H_1(N)$ from the spectral sequence $H_*(\mathbb{Z}_2, H_*(M)) \Rightarrow H_*(N)$.

$$E^2_{1,0} = H_1(\mathbb{Z}_2, H_0(M)) \cong \mathbb{Z}_2.$$

And no non-zero differential hits it or leaves it because we have a first quadrant spectral sequence.

$$E^2_{0,1} = H_0(\mathbb{Z}_2, H_1(M)) \cong {\mathbb{Z}_2}^4.$$

Again no differential hits it since $E^2_{2,0} = H_2(\mathbb{Z}_2, \mathbb{Z}) \cong 0$. Therefore $H_1(N)$ is an extension of $\mathbb{Z}_2$ by ${\mathbb{Z}_2}^4$. Also $H_1(N)$ is a quotient of ${\mathbb{Z}_2}^{16}$ as can be seen from Mayer-Vietoris:

$$\begin{array}{ccccc} H_1(16\mathbb{RP}^3) & \longrightarrow & H_1(N) \oplus H_1(16E) & \longrightarrow & H_1(K3) \\ \| & & \| & & \| \\ {\mathbb{Z}_2}^{16} & \longrightarrow & H_1(N) & \longrightarrow & 0, \end{array}$$

so all of its torsion is of order 2. Therefore the extension is trivial and $H_1(N) \cong {\mathbb{Z}_2}^5$. Putting this all together, we see that

$$H_i(N) \cong \begin{cases} \mathbb{Z}, & i = 0 \\ {\mathbb{Z}_2}^5, & i = 1 \\ \mathbb{Z}^6, & i = 2 \\ \mathbb{Z}^{15}, & i = 3 \\ 0, & \text{otherwise.} \end{cases}$$

By Poincaré duality for manifolds with boundary (also known as Alexander-Lefschetz duality), the cohomology of N relative to its boundary is thus

$$H^i(N, \partial N) \cong H_{4-i}(N) \cong \begin{cases} 0, & i = 0 \\ \mathbb{Z}^{15}, & i = 1 \\ \mathbb{Z}^6, & i = 2 \\ {\mathbb{Z}_2}^5, & i = 3 \\ \mathbb{Z}, & i = 4 \\ 0, & \text{otherwise.} \end{cases}$$

The K-theory is then computed from the Atiyah-Hirzebruch spectral sequence

$$H^p(N, \partial N; K^q(\mathrm{pt})) \Rightarrow K^{p+q}(N, \partial N),$$

but all differentials vanish since the first differential is the Steenrod operation Sq^3, which must vanish, and there is no room in this case for any higher differentials. Since the spectral sequence collapses at E_2, $K^0(N, \partial N) \cong H^2(N, \partial N) \oplus H^4(N, \partial N) \cong \mathbb{Z}^7$ and $K^{-1}(N, \partial N) \cong H^1(N, \partial N) \oplus H^3(N, \partial N) \cong \mathbb{Z}^{15} \oplus {\mathbb{Z}_2}^5$. Finally, the stable D-brane charges in type-IIA string theory on $(N/\partial N) \times \mathbb{R}^6$ are classified by $K^{-1}\big((N/\partial N)\times\mathbb{R}^6\big) \cong K^{-1}(N/\partial N) \cong K^{-1}(N,\partial N)\oplus K^{-1}(\text{pt}) \cong \mathbb{Z}^{15}\oplus{\mathbb{Z}_2}^5$ (since $K^{-1}(\text{pt}) \cong 0$).

2.3 Conclusions and Future Research

We have seen that the group that classifies the BPS D-branes in type I superstring theory compactified on $\mathbb{T}^4$, $KO(\mathbb{T}^4 \times \mathbb{R}^6)$, is isomorphic to $\mathbb{Z}^6 \oplus {\mathbb{Z}_2}^5$, which injects into $K^{-1}\big((N,\partial N) \times \mathbb{R}^6\big) \cong \mathbb{Z}^{15} \oplus {\mathbb{Z}_2}^5$ with an isomorphism on the torsion. The extra $\mathbb{Z}$ summands in the latter group could possibly correspond to other non-supersymmetric branes in type I on $\mathbb{T}^4$, since these would have K-theoretic charges living in the other groups listed in Table 2.1. For example, D_6-branes with real Chan-Paton bundles should have charges living in $KO^{-3}(\mathbb{T}^4) \cong 4\mathbb{Z}$. It is unknown which non-BPS branes in the type I theory transform to BPS branes in the type IIA theory. Further research into this phenomenon needs to be done before completing the classification. A correct classification of the BPS brane charges on the type IIA side is a powerful tool when studying non-BPS branes that map to BPS branes. Requiring the classification of all branes (both BPS and non-BPS) in the type I theory that map to BPS branes in the type IIA theory to match with $K^{-1}\big((N,\partial N) \times \mathbb{R}^6\big)$ gives constraints on which non-BPS branes are allowed. That is the classification for a non-BPS brane in the type I theory that maps to a BPS brane in the type IIA theory must be free of rank≤ 9.

While showing the stable D-brane configurations in the two theories match does not prove the two theories are dual, it does add further evidence to the already conjectured duality. This particular example also illustrates how first ensuring stable D-brane configurations match is a useful first step in checking a possible duality. By looking at the stable D-branes in this case, we saw immediately that we did not want to use $K3$, but rather a desingularized version of the orbifold blow-down, as matched with our physical intuition. Since N is only the same as $K3$ away from the singularities, it is possible that the including the contribution from the singularities can explain the difference of our two classifications. For this reason, we will look at classifying D-brane charges on orbifolds (taking into account the effect of the singularities rather than excising them) in the next chapter.

Chapter 3

Type II Superstring Theory Compactified on an Orientifold

As we saw in the previous chapter, the different superstring theories are often compactified on Calabi-Yau manifolds to reduce the total supersymmetry of the theory. Another way to reduce the supersymmetry is to compactify on an orientifold. An orientifold is a generalization of a orbifold that we will define shortly. Orbifolds are often used in string theory instead of Calabi-Yau manifolds because (as we saw with $K3$) the explicit form of the metric on most Calabi-Yau manifolds is unknown, but the class of orbifolds contains singular limits of Calabi-Yau manifolds on which the metric is known. Additionally, compactifying on orientifolds has proven promising for relating the type I theory to the type II theories. In fact, we have already seen in chapter 1 that the type I superstring theory can be defined as an orientifold projection of the type IIB theory. In this chapter we will focus on the orbifold limit of $K3$, $\mathbb{T}^4/\mathbb{Z}_2$, as well as the more general orientifold case of $\mathbb{T}^n/\mathbb{Z}_2$ for arbitrary n, where again the $\mathbb{Z}_2$ action is multiplication by -1. We will discuss the importance of this class of orientifolds to type I/type II dualities after giving a precise definition of an orientifold and describing how D-brane charges are classified on one.

3.1 Superstring Theory on an Orientifold

An *orientifold* is the quotient of an oriented orbifold by the action of an involution. Put more simply, an orientifold is an orbifold in which we allow the local isotropy group to reverse orientation. An orbifold is an orientifold with the trivial involution. Note than when n is even $\mathbb{T}^n$ can be described as $\mathbb{C}^{\frac{n}{2}}/\mathbb{Z}[i]^{\frac{n}{2}}$ and the action $(z_1, \ldots, z_{\frac{n}{2}}) \rightarrow (-z_1, \ldots, -z_{\frac{n}{2}})$ preserves orientation. If n is odd, however, $\mathbb{T}^n$ is described as $\mathbb{R}^n$ under the equivalence relation

$$x_{\dot{a}} \sim x_{\dot{a}} + 1, \tag{3.1}$$

where $\dot{a} = 1, \ldots, n$. Since there are an odd number of coordinates, the action $(x_1, \ldots, x_n) \rightarrow (-x_1, \ldots, -x_n)$ reverses orientation. Therefore $\mathbb{T}^n/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ action is multiplication by -1, is an orbifold for n even and an orientifold for n odd.

3.1.1 The String Spectrum

Performing an orientifold projection reduces the previous spectrum of allowable states, but also introduces new states. If the symmetry group reverses orientation (i.e., we take an orientifold projection rather than an orbifold projection) then when we mod out by the action we will obtain unoriented strings. Therefore we will concentrate only on the effect of performing an orbifold projection and keep in mind that we have the additional constraint that strings should be unoriented if the involution reverses orientation. There are two types of states that exist on an orbifold X/G. They are known as the twisted and untwisted states.

A state on the orbifold should be G-equivariant, so it transforms under the action by G as

$$g\Psi = \sigma(g)\Psi, \tag{3.2}$$

where $\sigma(g)$ is a character of G. *Untwisted states* are states Ψ whose transformation is given by the trivial character.

Untwisted states are states that exist on X and are invariant under the action by G. So a state Ψ in X is a state in X/G if and only if

$$g\Psi = \Psi \tag{3.3}$$

for all $g \in G$. If G is finite, we can construct a G-invariant state, Ψ_G, from any state Ψ in X as

$$\Psi_G = \frac{1}{|G|}\sum_{g\in G} g\Psi. \tag{3.4}$$

So all untwisted states in $\mathbb{T}^n/\mathbb{Z}_2$ can be written as

$$\frac{1}{2}(\Psi + g\Psi), \tag{3.5}$$

where Ψ is a string state on $\mathbb{T}^n$ and g acts as multiplication by -1 on the internal coordinates. Note that there are less states in the untwisted sector of X/G than in X.

Performing an orbifold projection introduces new closed string states. A *twisted state* is a state on the orbifold that transform as equation (3.2) where σ is a nontrivial character. Twisted states correspond to open strings in X that satisfy

$$X^\mu(\sigma + 2\pi) = gX^\mu(\sigma), \tag{3.6}$$

for some $g \in G$. Since $X^\mu = gX^\mu$ in X/G this describes new closed strings in the orbifold that do not exist in X. $g = 1$ gives the untwisted closed string states. There are actually various twisted sectors corresponding to the different conjugacy classes of G. This distinction only matters if G is nonabelian, which we will not deal with. In our example of $\mathbb{T}^n$, the twisted states correspond to strings in X satisfying

$$X^\mu(\sigma + 2\pi) = -X^\mu(\sigma). \tag{3.7}$$

Since twisted string states come from strings that connect a point to its image under G, after performing the orbifold projection the resulting closed string state must enclose a fixed of the G-action, that is an orbifold singularity.

3.1.2 Supersymmetry Breaking

To describe how compactifying on an orbifold breaks some of the supersymmetry we will use the example of $\mathbb{C}^n/\mathbb{Z}_k$ described in section 9.1 of [7]. This example is useful for us because all of the results hold true for the compact case $\mathbb{T}^{2n}/\mathbb{Z}_k$. To define the orbifold $\mathbb{C}^n/\mathbb{Z}_k$ consider $\mathbb{C}^n$ with coordinates $(z_1, \ldots, z_n)$ and let the generator of the $\mathbb{Z}_k$ action, α, be the isometry that performs a simultaneous rotation around each axis

$$\alpha\colon\ z_{\dot{a}} \to e^{i\phi_{\dot{a}}} z_{\dot{a}}, \tag{3.8}$$

where $\dot{a} = 1, \ldots, n$ and ϕ_a is an integer multiple of $\frac{2\pi}{k}$.

To describe the effect of this of this orbifold projection on supersymmetry we must first introduce the generator of the supersymmetry transformations. Since supersymmetry is only manifest in the GS formalism, the generator of supersymmetry can only be defined in the GS formalism of string theory that we alluded to earlier. Therefore, we will now give a brief overview of the features of the GS formalism we require here.

The GS formalism uses an extension of spacetime called *superspace* that includes additional anti-commuting Grassmann coordinates. So in the GS formalism we use super-worldsheet coordinates (σ^a, θ_A), where

$$\theta = \begin{pmatrix} \theta_- \\ \theta_+ \end{pmatrix} \tag{3.9}$$

is a Majorana spinor and the individual components, θ_A, $A = \pm$, are the anticommuting Grassmann coordinates. Fields defined on superspace are called *superfields*. The most general superfield can be written as a series expansion in θ as

$$Y^\mu(\sigma^a, \theta) = X^\mu(\sigma^a) + \bar{\theta}\psi^\mu(\sigma^a) + \frac{1}{2}\bar{\theta}\theta B^\mu(\sigma^a). \tag{3.10}$$

Any terms with more powers of θ would vanish as a result of the anticommutation relations satisfied by the Grassmann coordinates and a term linear in θ would be equivalent to the term linear in $\bar{\theta}$ since for Majorana spinors $\bar{\theta}\psi = \bar{\psi}\theta$. Introducing the *supercovariant derivative*,

$$D_A = \frac{\partial}{\partial\bar{\theta}^A} + (\rho^a\theta)_A\partial_a, \tag{3.11}$$

we can write the action in the GS formalism as

$$S = \frac{iT}{4}\int d^2\sigma d^2\theta \; \bar{D}Y^\mu DY_\mu. \tag{3.12}$$

Note that for Berezin integration over two Grassmann coordinates the only nonzero integral is

$$\int d^2\theta \; \bar{\theta}\theta. \tag{3.13}$$

Expanding the action (3.12) out in component form and performing the Grassmann integration gives

$$S = -\frac{T}{2}\int d^{\sigma}\;(\partial_a X_{\mu}\partial^a X^{\mu} + \bar{\psi}^{\mu}\rho^a\partial_a\psi_{\mu} - B_{\mu}B^{\mu}). \tag{3.14}$$

Solving the equations of motion for B^{μ} gives $B^{\mu} = 0$, under which the above action reduces to the same action we saw in RNS formalism (equation (1.27)). So we see that B^{μ} is an auxiliary field that does not affect the physics, but allows for manifest supersymmetry of the theory as we will see below. The above action also shows that X^{μ} and ψ^{μ} play the same role in the GS formalism as they did in the RNS formalism.

The generators of supersymmetry transformations are called *super charges* and are given by

$$Q_A = \frac{\partial}{\partial\bar{\theta}^A} + (\rho^a\theta)_A\partial_a. \tag{3.15}$$

They act on the superfield as

$$\delta Y^{\mu} = [\bar{\varepsilon}Q, Y^{\mu}] = \bar{\varepsilon}QY^{\mu}, \tag{3.16}$$

where ε is a constant Majorana spinor. Expanding this out in components, we find that the supersymmetry transformations are

$$\delta X^{\mu} = \bar{\varepsilon}\psi^{\mu}, \tag{3.17}$$

$$\delta\psi^{\mu} = \rho^a\partial_a X^{\mu}\varepsilon + B^{\mu}\varepsilon, \tag{3.18}$$

$$\delta B^{\mu} = \bar{\varepsilon}\rho^a\partial_a\psi^{\mu}. \tag{3.19}$$

The super charges and supercovariant derivatives anticommute, so DY^{μ} transforms the same way under a supersymmetry transformation as Y^{μ}. Therefore, the variation of the action under a supersymmetry transformation is

$$\delta S = \frac{iT}{4} \int d^2\sigma d^\theta \; \bar{\varepsilon} Q(\bar{D}Y^\mu DY_\mu). \tag{3.20}$$

There are two terms in the definition of Q (equation (3.15)). One gives a total derivative in terms of σ and the other a total derivative in terms of θ. There is no θ boundary term, but there can be nontrivial σ boundary terms. Therefore, with suitable σ boundary conditions there is worldsheet symmetry (equation (3.20) vanishes), but different σ boundary conditions can break the supersymmetry. Note that if include the field equation $B^\mu = 0$ then equations (3.17)-(3.19) reduce to the supersymmetry transformations in the RNS formalism, equation (1.36). So including the field equation $B^\mu = 0$ reduces the GS formalism to the RNS formalism and manifest supersymmetry is lost.

Let us now return to our example of $\mathbb{C}^n/\mathbb{Z}_k$ and look at what happens to the supersymmetry after performing the orbifold projection. The components of the supercharge Q_A on $\mathbb{C}^n$ that are invariant under the group action give the unbroken supersymmetries on the orbifold. Therefore, determining the unbroken supersymmetries is equivalent to determining how a spinor changes under the group action, which in this case is a rotation. Spinor representations of a rotation generator in $2n$ dimensions have weights of the form $(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2})$ with a total of 2^n states. This divides the exponent in equation (3.8) in half, which is due to the fact that a spinor reverses sign under a rotation by 2π. Irreducible spinor representations of Spin$(2n)$ have dimension 2^{n-1}. An even number of negative weights gives one representation while an odd number gives the other representation. Under the rotation given in equation (3.8), Q_A transforms as

$$\alpha\colon\ Q_A \to \exp\left(i\sum_{\dot{a}=1}^{n}\varepsilon_A^{\dot{a}}\phi^{\dot{a}}\right)Q_A, \tag{3.21}$$

where ε_A is a spinor weight. If we choose the $\phi^{\dot{a}}$ so that

$$\frac{1}{2\pi}\sum_{\dot{a}=1}^{n}\phi^{\dot{a}} = 0 \mod N \tag{3.22}$$

(which can always be done), then the components of Q_A that will be invariant under α are the components whose weights have the same sign for all n components, so that $\sum \varepsilon_A^{\dot{a}}\phi^{\dot{a}} = 0$. For each component of the supercharge that is not invariant under α the amount of unbroken supersymmetry is broken in half.

3.1.3 Classifying D-brane Charges on an Orientifold

As we have seen, D-brane charges are determined by their Chan-Paton bundles. Therefore, to classify stable D-brane charges on orientifolds we need to look at what happens to the Chan-Paton bundles under an orientifold projection. We will begin by looking at orbifold projections and will see, as originally proposed by Witten in [92], that stable D-brane charges on an orientifold are classified by equivariant K-theory. In this section, we will mainly be following [44], [31] and [35].

D-brane configurations in the orbifold X/G should come from D-brane configurations on X that are globally invariant under G. The pair of bundles (E, F) that classify a brane/antibrane configuration in X can be constructed in a G-invariant way, so we can consider G as acting on the pair (B, F). A D-brane configuration represented by the pair of bundles (E, E) can be created or annihilated if the tachyon field is G invariant, which means that G must act on both copies of E in the same way. To be more precise, let γ_g be the representation of $g \in G$ in the Chan-Paton gauge group (for type II string theories this is $\mathrm{U}(N)$) and let f be some fields with Chan-Paton factor λ. The action of G on the Chan-Paton factors is given by [55]

$$g \cdot f(\lambda) = f(\gamma_g \lambda). \tag{3.23}$$

The tachyon field transforms in the adjoint representation and the action of G is given by

$$g \cdot T(x) = \gamma_g T(g^{-1} \cdot x) \gamma_g^{-1}. \tag{3.24}$$

The tachyon field must be invariant under the above action to allow for creation and annihilation. D-brane charges on X/G are classified by pairs of vector bundles (B, F) with a G action modulo the relation

$$(B, F) = (B \oplus E, F \oplus E), \tag{3.25}$$

where E is any G-bundle on X. A *G-bundle on X* is a vector bundle on X, $E \xrightarrow{\pi} X$ with a G action such that

$$\pi(g \cdot e) = g \cdot \pi(e) \tag{3.26}$$

for all $e \in E$ and

$$g\colon\ E_x \rightarrow E_{g \cdot x} \tag{3.27}$$

is a homomorphism of vector spaces.

The equivalence classes of pairs of vector bundles (B, F) modulo the equivalence relation in equation (3.25) form the group $K_G(X)$ called the *G-equivariant*

K-theory of X. Therefore, stable D-brane charges on an orbifold X/G are classified by $K_G(X)$. To be more precise they are classified by $K_G(X)$ in the type IIB theory, $K^1_G(X)$ in the type IIA theory and by $KO_G(X)$ in the type I theory. Equivariant K-theory is a generalization of K-theory for which many of the basic properties, such as Bott periodicity, hold. A useful theorem for equivariant K-theory [68] shows that if G acts freely on X then

$$K_G(X) \cong K(X/G). \tag{3.28}$$

Another result from [68] that we will need later is that

$$K_G(\mathrm{pt}) = R(G), \tag{3.29}$$

where $R(G)$ is the representation or character ring of G. Since there is a natural map from X to a point, $K_G(X)$ is a $R(G)$ module for general X.

3.2 Classifying D-brane Charges in type II Superstring Theory on $\mathbb{T}^n/\mathbb{Z}_2$

Throughout this section let $G = \mathbb{Z}/2$ and identify $\mathbb{T}^n$ with $\mathbb{R}^n/\mathbb{Z}^n$ on which G acts by -1. We choose to look at this example because as Polchinski pointed out in [58], if you start with the type II theory compactified on a $(10-p)$-torus, where 10 is the total spacetime dimension, containing 16 D_p-branes in the noncompact dimensions and then take the $R \to 0$ limit, you obtain the type I string. Further evidence of the importance of this example to studying type I/type II dualities can be found in [22]. In their paper they classify type II orientifolds on $\mathbb{T}^{9-p} \times \mathbb{R}^{p+1}$ for

$p = 9, 8, 7, 6$ and give a partial description for $p = 5, 4$. Through their description there is further evidence that this class of Type II orientifolds should be able to be matched with some type I theory. The greatest piece of evidence is that in the case of $p = 9$, the only supersymmetric $\mathbb{Z}_2$ orientifold is type I on $\mathbb{R}^{10}$.

As we saw in the last section, classifying stable D-brane configuration in X/G comes down to calculating $K_G(\mathbb{T}^n)$. This has been computed as an abelian group in [29]. We will present a more general formulation where $K_G(\mathbb{T}^n)$ is computed as an $R(G)$-module.

3.2.1 $\mathbb{Z}_2$-Equivariant K-Theory of $\mathbb{T}^n$

For the remainder of this section, let $G = \mathbb{Z}_2$ and $R = R(G) = \mathbb{Z}[t]/(t^2 - 1)$, where t represents the nontrivial character of G. Let $I = (t - 1)$ and $J = (t + 1)$. These are prime ideals with $R/I \cong R/J \cong \mathbb{Z}$, and $R_{(I)} \cong R_{(J)} \cong \mathbb{Q}$.

A prime ideal, $\wp$, of R has support $\{1\}$ if and only if $\wp \supseteq I$. So I has support $\{1\}$ and J has support G. Given the identification of $\mathbb{T}^n$ with $\mathbb{R}^n/\mathbb{Z}^n$ on which G acts by -1, G has 2^n fixed points. By noting $K^*_G(\text{pt}) = R$, all in degree 0, by equivariant Bott periodicity and using the Segal localization theorem, we can obtain a preliminary result:

$$K^*_G(\mathbb{T}^n)_{(J)} = K^*_G((\mathbb{T}^n)^G)_{(J)} = K^*_G(2^n \text{ points})_{(J)} = 2^n R_{(J)}, \qquad (3.30)$$

all in degree 0. In order to prove a more general formula for $K^*_G(\mathbb{T}^n)$, without localization, we will need the following lemma.

Lemma 2. $\text{Hom}_R(J, R/J) = 0$, $\text{Ext}^1_R(R/J, R) = 0$, *and* $\text{Ext}^1_R(R/J, R/J) = 0$.

Proof. Note that $I \cdot J = I \cap J = 0$ and $I \cong R/J$, $J \cong R/I$ (as R-modules). A homomorphism $J \to R$ is determined by the image of $t+1$, which must lie in the ideal annihilated by $t-1$, which is J itself. So $\mathrm{Hom}(J, R) \cong J$. Now $R/J \cong I$, so a homomorphism $J \to R/J$ is the same thing as a homomorphism $J \to R$ with image in I. This is 0 since $I \cap J = 0$.

Next, consider the short exact sequence

$$0 \to J \to R \to R/J \to 0, \tag{3.31}$$

and apply the functor $\mathrm{Hom}_R(\_\_, R)$ to it. We get

$$0 \to \mathrm{Hom}_R(R/J, R) \to \mathrm{Hom}_R(R, R) \to \mathrm{Hom}_R(J, R) \to \mathrm{Ext}^1_R(R/J, R) \to \mathrm{Ext}^1_R(R, R) = 0. \tag{3.32}$$

We have $\mathrm{Hom}_R(R, R) = R$, and by the above, $\mathrm{Hom}_R(J, R) = J \cong R/I$. Similarly, $\mathrm{Hom}_R(R/J, R) = \mathrm{Hom}_R(I, R) = I$. So (3.32) becomes the sequence

$$0 \to I \to R \to R/I \to \mathrm{Ext}^1_R(R/J, R) \to 0, \tag{3.33}$$

and $\mathrm{Ext}^1_R(R/J, R) = 0$. Similarly, if we apply $\mathrm{Hom}_R(\_\_, R/J)$ to the short exact sequence (3.31), we get:

$$\begin{aligned} 0 \to \mathrm{Hom}_R(R/J, R/J) \to \mathrm{Hom}_R(R, R/J) \to \mathrm{Hom}_R(J, R/J) \\ \to \mathrm{Ext}^1_R(R/J, R/J) \to \mathrm{Ext}^1_R(R, R/J) = 0. \end{aligned} \tag{3.34}$$

But $\mathrm{Hom}_R(J, R/J) = \mathrm{Hom}_R(J, I) = I \cap J = 0$, so $\mathrm{Ext}^1_R(R/J, R/J) = 0$. □

Theorem 6. *Let G act on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ via multiplication by -1 on $\mathbb{R}^n$. Then $K^*_G(\mathbb{T}^n)$ is entirely concentrated in even degrees, and $K^0_G(\mathbb{T}^n) \cong 2^{n-1} \cdot R \oplus 2^{n-1} \cdot (R/J)$.*

Proof. Throughout this proof we will be using the result from [68]: If C is a closed G-invariant subspace of a locally compact G-space X then the sequence

$$K_G^0(X-C) \to K_G^0(X) \to K_G^0(C) \to K_G^1(X-C) \to K_G^1(X) \to K_G^1(C) \to K_G^0(X-C) \to \cdots \tag{3.35}$$

is exact. Here the last part was gotten using Bott periodicity, $K_G^2(X-C) = K_G^0(X-C)$. We will also use the result from [25]:

$$\begin{aligned} K_G^0(\text{pt}) = K_G^0(\mathbb{R}^k) &= R, \text{ if } k \text{ is even} \\ K_G^0(\mathbb{R}^k) &= R/J, \text{ if } k \text{ is odd} \\ K_G^1(\text{pt}) = K_G^1(\mathbb{R}^k) &= 0. \end{aligned} \tag{3.36}$$

A fundamental domain for $\mathbb{T}^n$ is $F = \{(x_1, \ldots, x_n) : |x_j| \le \frac{1}{2}\}/\sim$, where $-\frac{1}{2} \sim \frac{1}{2}$. Define:

$$Y_k = \bigcup_{i_{n-k}=i_{n-k-1}+1}^{n} \cdots \bigcup_{i_2=i_1+1}^{n-k+2} \bigcup_{i_1=1}^{n-k+1} \{(x_1, \ldots, x_n) : \\ x_{i_l} = \pm\frac{1}{2} \text{ for } 1 \le l \le (n-k) \text{ and } |x_j| \le \frac{1}{2} \text{ if } j \ne i_l\}/\sim. \tag{3.37}$$

So Y_k is the set of all n-tuples where at least $n-k$ coordinates are exactly $\pm\frac{1}{2}$. Note that Y_k is $\binom{n}{k}$ copies of $\mathbb{T}^k$ whose pairwise intersection is $\mathbb{T}^{k-1}$. The union of all the pairwise intersections is Y_{k-1}. Now by induction on k we will show that $K_G^0(Y_k)$ is given by

$$\binom{n}{k} R \oplus \binom{n}{k-1} R/J \oplus \binom{n}{k-2} R \oplus \cdots \oplus \binom{n}{1} R/J \oplus R,$$

if k is even, and is given by

$$\binom{n}{k} R/J \oplus \binom{n}{k-1} R \oplus \binom{n}{k-2} R/J \oplus \cdots \oplus \binom{n}{1} R/J \oplus R,$$

if k is odd. We will also show that $K^1_G(Y_k) = 0$ in both cases.

Note that $Y_0 = \text{pt}$, so $K^*_G(Y_0) = R$ all in degree 0. Let us now look at the case of $k = 1$. Y_1 is the one-point union of $\binom{n}{1} = n$ 1-tori. The point of intersection, $y = (\pm\frac{1}{2}, \pm\frac{1}{2}, \ldots, \pm\frac{1}{2})$, is a closed G-invariant subset of Y_1, so by (3.35) we get an exact sequence

$$K^0_G(Y_1\backslash\{y\}) \to K^0_G(Y_1) \to K^0_G(\text{pt}) \to K^1_G(Y_1\backslash\{y\}) \to K^1_G(Y_1) \to K^1_G(\text{pt})$$

$Y_1\backslash\{y\}$ is the disjoint union of n copies of $\mathbb{R}$. Using this and the above exact sequence we can see immediately that $K^1_G(Y_1) = 0$ since both $K^1_G(\mathbb{R})$ and $K^1_G(\text{pt})$ are 0 by (3.36). And we get a short exact sequence

$$0 \longrightarrow nR/J \longrightarrow K^0_G(Y_1) \longrightarrow R \longrightarrow 0,$$

which splits since R is free. So $K^0_G(Y_1) = nR/J \oplus R$.

Now let us look at the inductive step. Assume the above formula for $K^*_G(Y_k)$ holds for all $k < m$. Let us also assume m is even. Y_{m-1} is a closed G-invariant subset of Y_m. $Y_m\backslash Y_{m-1} \cong \binom{n}{m}\mathbb{R}^m$, since it is the set of all n-tuples with $n - m$ components exactly $\pm\frac{1}{2}$ and m components with absolute value strictly less than $\frac{1}{2}$. Therefore $K^*_G(Y_m\backslash Y_{m-1}) \cong \binom{n}{m}R$ all in degree 0, by (3.36) since m is even. Also, by the inductive assumption and since $m-1$ is odd, $K^*_G(Y_{m-1}) \cong \binom{n}{m-1}R/J \oplus \binom{n}{m-2}R \oplus \binom{n}{m-3}R/J \oplus \cdots \oplus \binom{n}{1}R/J \oplus R$ all in degree zero. By (3.35) we see that $K^1_G(Y_m) = 0$ and we get a short exact sequence:

$$0 \to K^0_G(Y_m\backslash Y_{m-1}) \to K^0_G(Y_m) \to K^0_G(Y_{m-1}) \to 0.$$

By lemma 2, the exact sequence splits and we see that

$$K_G^*(Y_m) \cong K_G^*(Y_m \backslash Y_{m-1}) \oplus K_G^*(Y_{m-1}) \cong \binom{n}{m} R \oplus \binom{n}{m-1} R/J \oplus \binom{n}{m-2} R \oplus \cdots \oplus \binom{n}{1} R/J \oplus R,$$

all in degree 0. The inductive step for m odd follows the same form. Note that Y_n is the entire space $\mathbb{T}^n$, so by the above inductive proof we have shown:

$$KO_G^0(\mathbb{T}^n) \cong \begin{cases} R \oplus \binom{n}{n-1} R/J \oplus \binom{n}{n-2} R \oplus \cdots \oplus \binom{n}{1} R/J \oplus R, & \text{if n is even} \\ R/J \oplus \binom{n}{n-1} R \oplus \binom{n}{n-2} R/J \oplus \cdots \oplus \binom{n}{1} R/J \oplus R, & \text{if n is odd.} \end{cases} \tag{3.38}$$

$$\cong \sum_{j \le n \text{ even}} \binom{n}{j} R \oplus \sum_{j \le n, \text{ odd}}^{n} \binom{n}{j} R/J$$

Note that $0 = (1-1)^n = \sum_{j \le n \text{ even}} \binom{n}{j} - \sum_{j \le n \text{ odd}} \binom{n}{j}$, which implies

$$\begin{aligned} 2^n &= (1+1)^n = \textstyle\sum_{j=0}^n \binom{n}{j} \\ &= \textstyle\sum_{j \le n \text{ even}} \binom{n}{j} + \sum_{j \le n \text{ odd}} \binom{n}{j} \\ &= 2 \textstyle\sum_{j \le n \text{ even}} \binom{n}{j}. \end{aligned}$$

Therefore we see that $\sum_{j \le n \text{ even}} \binom{n}{j} = \sum_{j \le n \text{ odd}} \binom{n}{j} = 2^{n-1}$. Putting this into (3.38) gives us our final result:

$$K_G^*(\mathbb{T}^n) \cong 2^{n-1} \cdot R \oplus 2^{n-1} \cdot (R/J),$$

all in degree zero. □

As we will discuss in the next section, when n is even, Theorem 6 classifies stable D-branes on the orbifold $\mathbb{T}^n/\mathbb{Z}_2$.

To observe that this is consistent with our previous result of the Segal localization theorem (3.30), observe:

Lemma 3. $(R/J)_{(J)} \cong R_{(J)}$.

Proof. Recall that $R_{(J)}$ is the result of inverting everything in R that is not in J. Thus we invert all prime numbers $p \in \mathbb{Z}$ as well as everything in I (except for 0). Since $I \cdot J = 0$, $J_{(J)} = 0$ and the quotient map $R \twoheadrightarrow R/J$ becomes an isomorphism after localization. $\square$

So we see that

$$K_G^*(\mathbb{T}^n)_{(J)} = 2^{n-1} \cdot R_{(J)} \oplus 2^{n-1} \cdot (R/J)_{(J)} = 2^{n-1} \cdot R_{(J)} \oplus 2^{n-1} \cdot R_{(J)} = 2^n \cdot R_{(J)},$$

all in degree zero, which agrees with (3.30).

3.3 Conclusions and Future Research

The K-groups given in Theorem 6 only make sense as a classification of stable D-brane configurations in the type II theory if n is even. When n is odd, the action of $\mathbb{Z}_2$ on $\mathbb{T}^n$ reverses orientation, as we saw in section 3.1, so we cannot define an oriented string theory on $\mathbb{T}^n/\mathbb{Z}_2$. In order to get a consistent string theory we must also mod out by the action of the worldsheet parity operator, to obtain unoriented strings. When this is done, the singularities are planes, called *orientifold planes*, that can couple to R-R gauge fields just like D-branes. Therefore stable D-brane configurations are classified by KR-theory, which is a generalization of K-theory that combines both KO-theory and K-theory, as described by Witten in [92]. While KR-theory is not a topic of this dissertation, it is an interesting topic for future research, since not only would it extend the classification of stable D-branes on $\mathbb{T}^n/\mathbb{Z}_2$ to the case of n odd, the end result of modding out by the action of the worldsheet parity operator will result in the type I theory. Many interesting K-theoretic aspects of the stable D-brane configuration in type I theories constructed as an orientifold projection of the type IIB theory are shown in [55]. While [55] describes how to construct the type I theory from the type IIB theory and the effect on K-theory, it does not describe actual dualities between type I and type II theories with indistinguishable spectra. It appears that for an unoriented orientifold, we only get a consistent string theory in the type I theory, which is built from the type IIB theory but not necessarily dual to a type II theory. When X/G is an oriented orbifold, however, it should be possible to define a type II theory on it.

Returning to the case of n even, $\mathbb{T}^n/\mathbb{Z}_2$ is an oriented orbifold, so Theorem 6 classifies stable D-brane configurations on $\mathbb{T}^n/\mathbb{Z}_2$ in the two type II theories when the H-flux is trivial. Note that the $\mathbb{Z}_2$ equivariant K-theory of $\mathbb{T}^4$, when viewed as an abelian group, is the same as the K-theory of $K3$. We believe that this is evidence that we can match stable D-brane configurations in the type II theory on $\mathbb{T}^4/\mathbb{Z}_2$ with D-brane configurations in the type I theory on $\mathbb{T}^4$ if we include a twisting by the H-flux. However, on an orientifold X/G, it is unclear what is meant by the H-flux. It does not make sense for H to live in $H^3(X;\mathbb{Z})$ since to make any sense on the orbifold, H would have to be G invariant. Recent work by Distler, Freed and Moore in [24] proposes using more exotic twistings involving equivariant cohomology, but the precise definition of the H-flux on an orientifold remains an open problem.

Once the H-flux on an orientifold is defined, it will be possible to completely classify stable D-brane configurations in the type II theories on the orbifold

$\mathbb{T}^n \times \mathbb{R}^{10-n}$, where the orbifold action is multiplication by -1. Afterwards we can determine any possible dual type I theories by looking for any isomorphisms between $K^i_{\mathbb{Z}_2}(\mathbb{T}^n \times \mathbb{R}^{10-n})$ and KO of some dual spacetime. Once a K-theoretic isomorphism is found, the spectra in the two theories need to be computed to see if there is an actual physical duality. After completing this classification, it would be interesting to extend to the case of $\mathbb{T}^4/\mathbb{Z}_2 \times \mathbb{R}^6$ with more general $\mathbb{Z}_2$ actions, to complete the classification of D-brane configurations on $\mathbb{Z}_2$ orientifolds of $\mathbb{T}^4 \times \mathbb{R}^6$ started in [22]. This is the simplest case in which every possible type of orientifold plane is apparent.

We are currently working on a project motivated by the fact that if X is locally compact then

$$KO(X \times (\mathbb{CP}^2 - \{\mathrm{pt}\})) \cong K(X). \tag{3.39}$$

If $\dim X = 6$, this suggests the possibility of a duality between the type I theory on $X \times (\mathbb{CP}^2 - \{\mathrm{pt}\})$ and the type IIB theory on $X \times \mathbb{R}^4 = X \times (S^4 - \{\mathrm{pt}\})$. Regarding $\mathbb{CP}^2$ as $S^5/\,\mathrm{U}(1)$, we see that $\mathbb{CP}^2$ can be viewed as $\mathrm{U}(1)$ orbifold of S^5, where the action of $\mathrm{U}(1)$ on S^5 has no fixed points so the orbifold is a smooth manifold. This could be a useful example to the research presented in this dissertation, since if it does describe a duality, any general rules determining the K-theoretic aspects of dualities between the type I theory and the type II theories on G orbifolds should include the above example when the action of G is constrained to be free.

To summarize, we saw that classifying stable D-brane configurations in the singular limit of $K3$, taking the effect of the singularities into account, successfully recreates the classification of stable D-brane configurations on a smooth $K3$ surface.

Furthermore, with further research, it should be possible to define a nontrivial H-flux on the orbifold limit, whereas it is not possible on a smooth $K3$ surface since $H^3(K3;\mathbb{Z}) = 0$. This, along with previous work by Polchinski describing the type I theory as an orientifold projection of the type IIB theory, shows that looking at the K-theoretic classification of D-brane configurations on orientifolds is a promising area to continue research into U-dualities between the type I and type II superstring theories.

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