ABSTRACT

Title of dissertation: TRANSPORT COEFFICIENTS AND UNIVERSALITY IN HOT STRONGLY COUPLED GAUGE THEORIES

Aleksey Cherman, PhD 2010

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The gauge/gravity duality provides a valuable opportunity to study the behavior of relativistic fluids described by some strongly-interacting non-Abelian gauge theories. However, as yet no gravity duals are known for the field theories that are currently used to describe nature. Thus, it is particularly interesting to search for universal properties of theories with gravity duals. This dissertation discusses a broad class of theories with gravity duals, and it is shown that at high temperatures, the speed of sound squared is bounded from above by one-third of the speed of light squared. It is conjectured that this may be a universal property of theories with gravity duals. It is also shown that the temperature dependence of a number of transport coefficients takes a universal form in the high-temperature limit. In particular, in a high-temperature expansion, the power law of the leading correction away from the infinite temperature limit is universal for all of the transport coefficients, and is the same as that of the speed of sound squared.

TRANSPORT COEFFICIENTS AND UNIVERSALITY IN HOT STRONGLY COUPLED GAUGE THEORIES

by

Aleksey Cherman

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Dissertation Committee:
Associate Professor Paulo F. Bedaque
Associate Professor Zacharia Chacko
Professor Thomas D. Cohen, Chair/Advisor
Assistant Professor Carter Hall
Professor William Walters

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Chapter 1

QCD and Hydrodynamics

1.1 Introduction

General relativity and the Standard Model of particle physics are thought to provide a description of almost all known phenomena in nature[1]. The Standard Model is a relativistic quantum field theory that describes the properties and behavior of all of the known subatomic particles, while General Relativity is a classical (that is, non-quantum) theory that describes gravity. These basic physics theories underlie all of the rest of our scientific description of reality, in the sense that at least in principle the properties of nuclear physics and chemistry can be calculated from the Standard Model. Together with General Relativity, we can *in principle* use the Standard Model explain everything from where the atoms making up the earth came from, to the chemical properties of carbon, to the life cycles of living things, stars and galaxies.

In practice, of course, trying to calculate the properties of (say) benzene directly from the Standard Model is *completely* impractical, since the even a carbon nucleus is enormously complex on a subatomic level. It is not reasonable to expect to explain the properties of very complex objects directly in terms of the most fun-

[1]The big exceptions are dark matter and dark energy, which are not explained by the Standard Model.

damental degrees of freedom which is the concern of the Standard Model. However, it is nonetheless very useful to have successively deeper layers of understanding of nature. For benzene, while it may not be useful to try to explain its properties from the Standard Model of subatomic particles, it is very useful to understand the properties of benzene in terms of the the bonding of atoms into molecules. In turn, it is crucial for chemistry that there is a way to understand the bonding of atoms in terms of the properties of electrons in the presence of atomic nuclei, and so on.

Unfortunately, there is an embarrassing gap in this matryoshka doll series of descriptions, with each successive 'effective theory' (*e.g.* basic chemistry) having a useful and calculable description in terms of a deeper, more fundamental description (*e.g.*, quantum electrodynamics, which describes the behavior of electrons around atoms and thence atomic bonding.) Atomic nuclei are composed of protons and neutrons, which are examples of a broad class of subatomic particles called hadrons. Hadrons are in turn composed from particles called quarks and gluons. The theory that describes the behavior of quarks and gluons is called quantum chromodynamics (QCD), which is a part of the Standard Model. Unfortunately, while one can use QCD to calculate the behavior of quarks and gluons to great precision under some rather special circumstances (high energy collisions of various sorts), a detailed understanding of the properties of even the simplest hadrons in terms of QCD remains far out of reach without brute-force computer simulations.

The basic problem is that except in some special circumstances, QCD is a strongly-interacting theory, with a coupling constant of order unity. As a result, perturbation theory with the coupling as the expansion parameter, which is the calculational tool used throughout much of the rest of physics to great success, cannot be used in the most interesting situations in QCD.

Thus, one of the major challenges of theoretical particle physics is the search for calculational techniques that can be used to attack strongly coupled theories such as QCD. Unfortunately, it is still not known how to calculate observables in QCD without using perturbation theory (which is only possible in very special circumstances) or brute-force numerical methods. The numerical approach that has been successfully applied to QCD, lattice Monte Carlo simulations, can be used to calculate a wide variety of properties of QCD, but unfortunately cannot be used for a number of very interesting observables, in particular the ones that will be topic of this work. Even when lattice calculations can be used, by their nature as large-scale numerical methods, they do not give all of the insight into the strongly-interacting dynamics of QCD that we may want.

Despite decades of effort, an analytical method to reliably calculate quantities in QCD when it is strongly-coupled has not been found. However, a great deal of progress has been made in the last ten years towards the development of calculational tools for strongly coupled field theories which share a number of features with QCD, but also have a number of important differences. The progress came from a rather unexpected direction: string theory, which is a program to unify the Standard Model with General Relativity in a single framework.

Insights from string theory have led to the very startling realization that some gauge field theories (that are related to QCD, which is also a gauge theory, in a way that will be discussed later) that describe physics in a four dimensional world (three space dimensions and one time dimension) actually have an *equivalent* but very different description involving a quantum theory of gravity (string theory) in ten dimensions. Even more remarkably, when these gauge theories are strongly interacting, so that perturbation theory cannot be used, the equivalent 'dual' gravitational theory in ten dimensions becomes *weakly interacting* and classical, and describable by General Relativity. As a result, one can determine the properties of these strongly-interacting quantum fields theories by doing simple calculations in a higher dimensional classical theory of gravity. This set of ideas and techniques goes under the name of the gauge/string (or gauge/gravity) duality.

Unfortunately, the remarkable techniques of the gauge/gravity duality cannot be applied directly to QCD, because it is not known how to construct a string dual for QCD. Nonetheless, the gauge/gravity duality is an extremely rare case where we have a set of tools to understand the strong-coupling behavior of a large class of quantum field theories. Such calculable examples are very valuable for developing some theoretical intuition about the generic behavior of strongly-interacting field theories. If one is lucky, one might stumble across some universal properties of strongly-interacting field theories, which might conceivably hold even outside the class of theories in which one can currently use the gauge/gravity duality. The search for universal properties of strongly-interacting field theories is the focus of this thesis.

The regime we will be particularly interested in is the behavior of systems governed by QCD when they are heated to very high temperatures. At high temperatures, it is thought that hadrons fall apart into their constituent particles, quarks and gluons, producing a quark-gluon plasma. Temperatures this high are not easily reached, and they have only been accessed shortly after the Big Bang, since the universe was extremely hot when it was young, and in recent heavy atomic ion collision experiments at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory in the US. The transition between a gas of hadrons and quark-gluon plasma (QGP) is called a deconfinement transition.[2]

In QCD, recent observations imply that this plasma is strongly interacting at the temperatures where it can first be produced, but as the temperature is increased, the interactions are expected to become weaker. As a result, at extremely high temperatures, one can use perturbation theory to calculate the properties of the quark-gluon plasma. At lower temperatures, however, such as those reached at RHIC, the physics of the quark-gluon plasma is strongly interacting.

It turns out that the QGP produced at RHIC behaves like a fluid. But because it is a strongly-coupled relativistic fluid described by QCD, there are essentially no reliable tools to calculate its properties *ab initio*. Since we have essentially no other experience with such strongly-coupled relativistic fluids, we do not even have much theoretical intuition to build on in interpreting the experimental data.

As is mentioned above, the gauge/gravity duality can not be used to calculate

[2]The deconfinement transition may or may not be a phase transition. One can have first or second order transitions between the two 'phases' depending on how many flavors of quarks are active in the transitions (and their masses). In fact in real-world QCD with three colors, two light quark flavors (up and down quarks) and one moderately heavy quark flavor (the strange quarks), the deconfinement transition is thought to be a rapid crossover.

anything about the QGP seen in the real world, at RHIC, since that fluid is described by QCD and there is no known gravity dual for QCD. However, remarkably, the duality *does* provide us with a wide theoretical class of strongly-interacting relativistic fluids for which we can calculate a great deal, and it turns out that such strongly-coupled fluids have a number of apparently universal properties.

In particular, in this thesis it is shown that there is a universal bound on the speed of sound in such fluids described by a broad class of theories with gravity duals. Additionally, within this class of systems, it is shown that the temperature dependence of transport coefficients - quantities like viscosities and diffusion coefficients - takes a certain universal form. Both of these strong-coupling results stand in sharp contrast to what one sees in the same systems when they are weakly coupled, since these universal results do *not* hold apply in that case.

Since QCD is not within the class of systems to which our calculations apply, it is unclear whether the lessons one can draw from these interesting 'universal' behaviors apply to strongly-coupled fluids in the real world. Nonetheless, it is an interesting demonstration that at least in principle, some rather unexpected phenomena can occur at strong coupling that one would not have expected just from experience with weakly-coupled systems. Thus, it seems worthwhile to continue using the gauge/gravity duality to explore the physics of the strongly-coupled systems to which it applies.

In this chapter, we first review the most relevant features of QCD in Sec. 1.2. Next, in Sec. 1.3, we discuss the large N limit of QCD, and preview some of the issues that will arise in comparing QCD with theories with gravity duals. Finally, in Sec. 1.4, we review relativistic hydrodynamics.

Next, Chapter 2 of this thesis introduces the gauge/gravity duality, giving some technical details about how it can be used to do calculations in some strongly-coupled. Chapter 3 discusses the speed of sound in strongly-coupled systems with gravity duals, and it is shown that in such systems there is a universal bound on the speed of sound at high temperatures of $1/\sqrt{3}c$, where c is the speed of light. Chapter 4 applies the techniques that are developed in Chapter 3 to study other properties of relativistic fluids described by gravity duals, and it is shown that the temperature dependence of transport coefficients takes a certain universal form in such theories in the high-temperature limit. Chapter 5 serves as a summary of our results.

1.2 QCD

Quantum Chromodynamics is the quantum field theory that describes the strong nuclear force. It is a relativistic non-abelian gauge theory with gauge group $SU(3)$, coupled to $N_f = 6$ 'flavors' of fermions in the fundamental representation of the gauge group $SU(3)_c$. The rank of the gauge group is called the number of colors N_c of the theory. The fermions are referred to as quarks, while the quanta of the gauge field A are called gluons. The gauge field A is a Lorentz 4-vector, and can be written as a matrix-valued field $A_\mu(x^\nu) = A^a_\mu(x^\nu)t^a$, where the t^a with $a = 1, \ldots N_c^2 - 1$ are the generators of $SU(N_c)$ in the fundamental representation (defined in such a way that $\mathrm{tr} t^a t^b = \frac{1}{2}\delta^{a,b}$). In this notation A_μ is an $N_c \times N_c$ matrix.

The lagrangian density of QCD can be written as

$$\mathcal{L} = \frac{-1}{2g_{YM}^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{N_f}\bar{\psi}_i(\not{D} + m_i)\psi_i \tag{1.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the field strength associated with the the gauge field, $\not{D} \equiv \Gamma^\nu D_\mu$ (the Γ^ν are the Dirac gamma matrices), and $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative in the fundamental representation, $\bar{\psi} = \psi^\dagger\gamma^0$, i is a flavor index, and we have suppressed the color indices on the quarks, which run from $1, \ldots N_c$, so that the quarks look like color-vectors.

The quark masses m_i appearing in the Lagrangian are external parameters from the point of view of QCD, and their values are determined by a different sector of the Standard Model. There are six known quark flavors in nature, known as the up, down, strange, charm, bottom, and top quarks in order of increasing mass. The charm, bottom, and top quarks are very heavy compared to the up, down and strange quarks, and more importantly to the underlying QCD scale. As a result, the charm, bottom, and top quarks make very small contributions to the properties of the lightest hadrons. The up and down quarks are very light, with a mass of less than $10MeV$, while the strange quark is somewhat heavier, with a mass of order $100MeV$. For many purposes, it is actually a good approximation to treat the up and down quarks as massless (and sometimes the strange quark as well).

Apart from quark masses, QCD has only one obvious parameter, the Yang-Mills coupling constant g_{YM}. There is actually another parameter in QCD which is perhaps less obvious: the number of colors N_c, which is just 3 in QCD. As noted by 't Hooft[1], it turns out that some features of gauge theories like QCD simplify when $N_c \to \infty$, so that one can try to do calculations with $1/N_c$ as the expansion parameter. As noted by 't Hooft, when taking the large N_c limit, consistency requires one to keep the quantity

$$\lambda = g_{YM}^2 N_c \tag{1.2}$$

fixed to a value independent of N_c, and it is the 't Hooft coupling λ that plays the role of the coupling constant of the theory. For real-world QCD, the large N_c expansion parameter is 1/3, so that it is not obvious whether the expansion would always be phenomenologically useful. In fact, it turns out that the $1/N_c$ expansion *does* make useful phenomenological predictions for $N_c = 3$ QCD, for some (but not all) observables. This is discussed more in the next section.

The QCD coupling constant g_{YM} is dimensionless, and its value controls when the *classical* theory can be treated using perturbation theory. When g_{YM} is small, the theory can be treated perturbatively with g_{YM} as the expansion parameter, since for $g_{YM} = 0$ the theory is free and can be solved exactly. When g_{YM} is large, the theory is far from free, and the perturbative expansion in g_{YM} breaks down.

The situation is much more complicated when the theory is quantized. To get from the classical lagrangian above to the quantum theory, let us define the path integral Z as

$$Z \equiv \int DA_\mu D\bar{\psi} D\psi \exp\left(-i \int d^4x \mathcal{L}[A_\mu, \psi, \bar{\psi}]\right) \tag{1.3}$$

It is the path integral above that really defines QCD as a quantum field theory. The observables in a quantum theory are encoded in correlation functions, which can be obtained by taking functional determinants of Z with respect to sources[2].

For instance, suppose we promote g_{YM} to a function of spacetime as a (temporary) formal trick. Then the connected two-point correlation function of the field strength is given by

$$\langle \mathrm{tr}F^2(x)\mathrm{tr}F^2(0)\rangle \equiv \frac{\delta}{\delta g_{YM}^2(x)}\frac{\delta}{\delta g_{YM}^2(0)} \log Z[g_{YM}^2(x)]|_{g_{YM}^2(x)=g_{YM}^2} \tag{1.4}$$

$$= \frac{1}{Z}\int DA_\mu D\bar{\psi}D\psi \exp\left(-i\int d^4x\mathcal{L}[A_\mu,\psi,\bar{\psi}]\right)\mathrm{tr}F^2(x)\mathrm{tr}F^2(0) \tag{1.5}$$

The definition of the path integral is a very subtle business, and the definition of the path integral measure DA_μ necessarily involves some high-energy regulator, such as a discretization of spacetime or some cutoff on the momenta. Of course, for the theory to make sense it better be the case that the observables are independent of the details of our choice of regulator as the regulator is removed. For example, if the regulator was a discretization of spacetime, so that the path integral becomes a very high-dimensional normal integral, the correlation functions should become independent of the details of the discretization when the made more and more fine-grained. It turns out that for this to work, the parameters of the lagrangian density must themselves depend on the regulazation, with the regulator dependence canceling in the observables.

Since the value of g_{YM} is regulator-dependent, and thus not directly physical, the question of when one can use perturbation theory is much more subtle in the quantum theory than in the classical theory. It turns out that whether g_{YM} is small or not in QCD really depends on the questions one is trying to ask. For processes involving large momentum transfers (high energy collisions between quarks, for instance), it turns out that g_{YM} is small, while for lower energies g_{YM} is large. The fact that the coupling depends on the energy scale at which the theory is probed is referred to as the coupling 'running'.

If we view the regulator as being something like an energy cutoff at the energy μ, as μ is varied g_{YM} responds as

$$\mu \frac{dg^2_{YM}(\mu)}{d\mu} = \beta(g^2_{YM}(\mu) \tag{1.6}$$

When $g^2_{YM}(\mu)$ is small, one can show that

$$\beta(g^2_{YM}(\mu) = -\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)\frac{g^4_{YM}}{2\pi} \tag{1.7}$$

This means that when $g^2_{YM}(\mu)$ is small,

$$g^2_{YM} = \left[\frac{1}{2\pi}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)\log(\frac{\mu}{\Lambda_{QCD}})\right]^{-1} \tag{1.8}$$

Λ_{QCD} is at this point an integration constant, and its relation to observable quantities is regulator scheme dependent. It is nonetheless a very important parameter, since it tells us the scale at which QCD becomes strongly coupled. When $\mu \sim \Lambda_{QCD}$, the coupling blows up, while when $\mu \gg \Lambda_{QCD}$ the coupling gets very small. The property that a theory becomes weakly interacting at high energy scales is referred to asymptotic freedom, and it is one of the most important properties of QCD.

Another important property of QCD is that quarks and gluons are never observed in isolation: quarks and gluons are never found as free particles in the 'vacuum', at temperatures small compared to Λ_{QCD}. What is observed instead are the hadrons (for instance, the baryons, which are the constituents of atomic nuclei), which are bound states of quarks of quarks and gluons. This property of QCD is known as confinement. The scale of the masses of hadrons is set by Λ_{QCD}, with the typical hadronic scale being of order $\sim 700MeV$.

Confinement and asymptotic freedom lead to a rather embarrassing situation for QCD. QCD can be used to predict what happens in very high energy collisions involving hadrons, because the effective coupling constant in such collisions is small, and one can do calculations using perturbation theory. However, in many ways the most interesting situations are those involving low interaction energies, and perturbation theory is useless there. For instance, despite more than 30 years of intense effort, no one knows how to calculate the mass of any hadron from QCD analytically, and the low-energy interactions of hadrons, which are what is important for nuclear physics, remain out of theoretical reach.

As mentioned in the introduction, even when the temperature is high enough that the system goes through the deconfinement transition and a quark-plasma is formed (rather than a hadronic fluid), the QGP remains relatively strongly-interacting for $T \sim \Lambda_{QCD}$. In the strongly-coupled QGP (sQGP) phase, the hydrodynamic properties of the fluid are some of the basic observables of interest. Unfortunately, these properties of the sQGP are not calculable directly from QCD because of the inapplicability of perturbation theory and severe technical difficulties in numerical studies.

In fact, the *only* known general way to calculate non-perturbative quantities such as hadron masses directly from QCD is lattice QCD, a brute-force — if elegant — numerical method. For lattice QCD, one works with the path integral in Euclidean space (so that time is treated as an imaginary number), and regulates the path integral by discretizing spacetime on a lattice. This turns the evaluation of correlation functions into the problem of doing an enormous number of integrals numerically using Monte Carlo methods with the exponential of the classical Euclidean action as a statistical weight. Such calculations require state of the art supercomputers, and are very expensive. Also, since they are numerical techniques, they provide only a limited insight into the dynamics that gives the hadrons their mass.

A critical problem is that not all observables are calculable using lattice Monte Carlo methods. For instance, some observables cannot easily be extracted from Euclidean correlation functions, which is what can be calculated on the lattice, while for other observables (involving the properties of QCD at finite baryon density, for example) the Euclidean action becomes complex and Monte Carlo methods cease to be useful. Of particular relevance here is that transport coefficients, which are the observables relevant for the hydrodynamics of the sQGP, can only be reliably calculated from real-time correlation functions, and are extremely difficult to calculate using lattice Monte-Carlo methods.

The situation is not as grim in some theories that are theoretical cousins of QCD, thanks the to the gauge-gravity duality, which relates some field theories to gravity theories living in higher-dimensional spacetimes. Consider $\mathcal{N} = 4$ super Yang-Mills (SYM) theory, which is a gauge theory with gauge group $SU(N_c)$, four flavors of adjoint Majorana quarks, and six flavors of real adjoint scalars, with the interactions of the 'matter' in the theory uniquely determined by the symmetries of the theory. When $N_c \gg 1$ and $\lambda \equiv g_{YM}^2 N_c \gg 1$, so that the theory is strongly-coupled, $\mathcal{N} = 4$ SYM has a simple *weakly-coupled* gravity dual description. Using the gravity dual, it turns out to be possible to calculate the hydrodynamic transport coefficients for the $\mathcal{N} = 4$ SYM 'QGP'. The technology for doing this is discussed in the next chapter.

1.3 Large N QCD

In the previous section, we briefly mentioned that some aspects of QCD simplify in the large N_c limit. In the following it will be useful to mention some of the features of large N_c QCD, and to compare and contrast it with the properties of the theories for which gravity duals are known to exist.

First, so long as $N_c \gg N_f$, it is clear that QCD remains asymptotically free in the large N_c limit. Since asymptotic freedom (and corresponding infrared slavery) are at the heart of the difficulties of QCD, it may not be obvious whether the large N_c limit really buys one anything. In fact, despite decades of intense effort, large N_c QCD has not been solved, in the sense of (for instance) obtaining analytic expressions for the meson masses in terms of $\lambda = g_{YM}^2 N_c$.

However, QCD *does* become simpler in the large N_c limit. In the large N_c limit, meson decays and scattering amplitudes become suppressed by powers of N_c, while baryons become very heavy, with masses of order N_c. As a result, the light particles of the theory (the mesons) form a non-interacting gas of stable particles, with masses of order Λ_{QCD}. If one turns up the temperature of such a gas, there must be a phase transition at $T \sim \Lambda_{QCD}$ to a quark-gluon plasma. The deconfinement phase transition becomes first order in the large N_c limit. Since the mesons are colorless, the meson gas has an energy density of order N_c^0. On the other hand, the quark-gluon plasma has an energy density that scales as N_c^2, since the gluons have $\sim N_c^2$ color labels. Since the energy density must go from order N_c^0 to N_c^2 in a region of temperature that does not scale with N_c, a phase transition exists, and turns out to be first order, with a latent heat proportional to N_c^2.

While at lower temperatures QCD is a weakly interacting gas of hadrons in the large N_c limit, it is important to note that there is no reason to expect that the large N_c QGP will be weakly interacting at temperatures of the order of the phase transition, and it is expected that the quark-gluon plasma would remain strongly interacting at moderate temperatures in the large N_c limit.

This can be compared to the theories that have known gravity duals. All of these theories are gauge theories that share a number of features of QCD, with $N_c \to \infty$. The big difference is that all of the theories that have known duals also have a large *tunable* 't Hooft coupling λ. In some of these, this is a meaninful concept because these theories are conformal at the classical *and* quantum levels, so that λ does not run. In the others, conformal symmetry is broken and λ may run, but has a strong-coupling UV fixed point.

The tractability of the gravity dual theories *requires* the $N_c \to \infty$ and $\lambda \to \infty$ limits: moving away from these limits requires quantum gravity calculations in the dual theory. The dual theories only reduce to classical gravitational systems in the large N and strong coupling limits.

The large N limit has a direct counterpart in QCD. The theories with gravity duals do in fact share many of the properties of large N QCD: phase transitions in such theories are first order, with latent heats scaling with N_c^2; some of the theories have analogues of mesons that behave in the same way that mesons do in large N_c QCD, and so on. However, the large λ limit has no counterpart in QCD, where λ runs, and has a *weak-coupling* UV fixed point. This means that theories that have classical gravity duals are very different in at least one critical way than even large N_c QCD.

Nonetheless, thanks to the gauge/gravity duality we have a window that gives us a clear look at the strong-coupling behavior of a wide class of gauge theories. This is very valuable, since it can be used as a theoretical playground for exploring strong-coupling physics, and for trying to answer questions of principle about strongly-coupled gauge theory dynamics. In particular, one may be able to get important general insights about the behavior of strongly-coupled theories.

1.4 Relativistic Hydrodynamics

In this section, we briefly review relativistic hydrodynamics. The material in this section is standard, and can be found in a number of textbookd and review articles, for instance Refs. [3, 4]. Our development follows the exposition in Ref. [4].

Hydrodynamics is essentially an effective field theory describing the behavior of systems in local thermal equilibrium on length and time scales that are large compared to the microscopic scales of the system, for instance the mean free path of the particles making up the fluid. Hydrodynamics charaterizes fluid flow in terms of the evolution of the conserved quantities associated with the fluid.

Relativistic fluids always have a conserved stress-energy tensor $T^{\mu\nu}$, and may also have other conserved vector currents j_μ. Since hydrodynamics is supposed to describe the long-distance (*i.e.* low-wavelength) physics of a fluid, one makes a derivative expansion of $T^{\mu\nu}$. The hydrodynamics of the systems will then be determined by the conservation equations applied to the derivative expansion of the conserved quantities of the system.

If the fluid is in local thermal equilibrium, it can be characterized by the temperature $T(x)$ which can depend on position x, and the local four-velocity of the fluid u^μ, which satisfies $u^\mu u_\mu = -1$. Then at lowest order in the derivative expansion it can be shown that the stress-energy tensor takes the form

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - g^{\mu\nu} P \tag{1.9}$$

where ϵ is the energy density and P is the pressure exerted by the fluid in its local rest frame. When the fluid is at rest so that $u^\mu = (1, 0, 0, 0)$, this reduces to

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{1.10}$$

The motion of the fluid is determined by

$$\partial_\mu T^{\mu\nu} = 0 \tag{1.11}$$

with ∂_μ replaced by the appropriate covariant derivative in curved space-times. At this order, the fluid motion is adiabatic. To see this, consider the 'longitudinal' part of the above equation of motion, obtained by contracting it with u_ν:

$$u^\mu \partial_\mu \epsilon + (\epsilon + P)\partial_\mu u^\mu = 0. \tag{1.12}$$

Using the fact that $\epsilon + P = Ts$, $d\epsilon = Tds$ and $dP = sdT$, where T is the temperature and s is the entropy density, the above equation reduces to

$$\partial_\mu (su^u) = 0 \tag{1.13}$$

This shows that the entropy current

$$s^\mu = su^\mu \tag{1.14}$$

is conserved.

The other equations characterizing the fluid flow are contained in the projection of the equation of motion on the directions transverse to the fluid flow, using

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \tag{1.15}$$

as a projection operator. $P^{\mu\nu}$ satisfies $u_\mu P^{\mu\nu} = 0$ and $P^\mu_\mu = 0$. Applying this projection operator the the equation of motion yields the relativistic Euler equation:

$$P_{\rho\nu}\partial_\mu T^{\mu\nu} = -\partial_\rho P + u_\rho u^\mu \partial_\mu P + (\epsilon + P)u^\mu \partial_\mu u_\rho = 0 \tag{1.16}$$

At the next order in the derivative expansion, we add a new term that has extra derivatives of u^μ. One now writes

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - g^{\mu\nu} P - \sigma^{\mu\nu} \tag{1.17}$$

where $\sigma^{\mu\nu}$ is a new term which turns out to contribute to dissipation. The $\sigma^{\mu\nu}$ term has a trace-free and trace-full part:

$$\sigma^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta} \left[\eta \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta} \partial_\lambda u^\lambda \right) + \zeta g_{\alpha\beta} \partial_\lambda u^\lambda \right]. \tag{1.18}$$

The coefficients ζ and η are known as the shear and bulk viscosities respectively and are examples of hydrodynamic transport coefficients, and fluids with $\eta = 0$ and $\zeta = 0$ are known as 'ideal'. The shear viscosity essentially characterizes the 'friction' experienced by a fluid under shear, while the bulk viscosity ζ characterizes the dissipative effects in a fluid under compression. When a non-ideal fluid is compressed, the pressure temporarily rises by more than is predicted by the equation of state, and ζ characterizes the time scale over which the system remains out of equilibrium.

1.4.1 Hydrodynamic dispersion relations

To better understand the role of the viscosities, it is helpful to consider the normal modes of the hydrodynamic equations. Before proceeding to a discussion of the normal modes associated with the stress tensor, it will be useful to consider the simple dispersion relation that arises when the system of interest has a conserved current j^μ. This current will play a role in the hydrodynamic equations, with the equation of motion $\partial_\mu j^\mu = 0$. Its derivative expansion can be written to the leading dissipative order as

$$j^\mu = \rho u^\mu + D P^{\mu\alpha} \partial_\alpha \rho \tag{1.19}$$

where ρ is the charge density in the fluid rest frame, and D is a constant that turns out to be the diffusion constant.

To find the (quasi-)normal modes associated with j^μ and $T^{\mu\nu}$, suppose (without loss of generality) that the fluid moves in the z direction, so that the time and z dependence of the solutions of the equation of motion is $\exp(-i\omega t + kz)$. The equation of motion for j^μ reduces to

$$\partial_t \rho - D\nabla^2 \rho = 0 \tag{1.20}$$

giving the dispersion relation

$$\omega = -iDk^2. \tag{1.21}$$

This is the diffusion equation. Thus, as promised, D has the interpretation of a charge diffusion constant, and is associated with dissipation.

Now let us turn to the normal modes associated with the stress-energy tensor. It is not hard to see that there are two sets of normal modes, those associated with shear, and those associated with 'bulk' modes.

Let us first discuss the shear modes. These are obtained from fluctuations of T^{0a} and T^{za}, where $a = x, y$. Then Eq. 1.18 reduces to

$$T_{za} = -\eta \partial_z u^a T_{0a} = (\epsilon + P) u^a \tag{1.22}$$

so that

$$T_{za} = -\frac{\eta}{\epsilon + P} \partial_z T_{0a} \tag{1.23}$$

On the other hand, the equation of motion is $\partial^\mu T_{\mu\nu} = 0$, so

$$-\partial_t T_{0a} + \partial_z T_{za} = 0 \tag{1.24}$$

Taking into account the t, z dependence of T_{0a}, T_{za}, we finally learn that

$$\omega = -i\frac{\eta}{\epsilon + P} k^2 \tag{1.25}$$

for the shear mode. This is clearly a dissipative mode: note the i factor. This has the form of a diffusion dispersion relation for transverse momentum, so that one can define the transverse momentum diffusion coefficient

$$D_\eta = \frac{\eta}{\epsilon + P} \tag{1.26}$$

The other mode is called the sound mode, which involves the fluctuations of T_{00}, T_{zz}, and T_{0z}. The derivation of the dispersion relation for the sound mode is a bit more algebraically involved than for the shear mode, but the result is[4]

$$\omega = c_s k - \frac{i}{2}\frac{\frac{4}{3}\eta + \zeta}{\epsilon + P}k^2 \tag{1.27}$$

where c_s is the speed of sound of the fluid, defined as $c_s^2 = dP/d\epsilon$. Note again that the viscosities appear in the dissipative imaginary term in the dispersion relation, supporting our identification of the $\sigma^{\mu\nu}$ term in the hydrodynamic expansion as being associated with the appearance of dissipation.

1.4.2 Example: Weakly-interacting relativistic gas

A possibly counterintuitive feature of shear viscosity is that the shear viscosity of an ideal gas is *infinite*, not zero as one might have naively thought. Thus an ideal gas is *not* an ideal fluid. To see how this works, consider the shear viscosity in a dilute gas of weakly-interacting particles at a temperature T and density $n \sim T^3$, with a dimensionless coupling constant λ. The interparticle separation is $n^{1/3} \sim 1/T$, while the mean free path can be estimated as

$$l_{mfp} = (n\sigma v)^{-1} \tag{1.28}$$

where $v \sim 1$ is the particle velocity, and $\sigma \sim \lambda^2 T^{-2}$ is the interaction cross-section, so that in the $\lambda \to 0$ limit the mean free path of the particles is much larger than the interparticle separation:

$$\frac{1}{\lambda^2 T} \gg \frac{1}{T} \Rightarrow l_{mfp} \gg n^{1/3} \tag{1.29}$$

A simple way to estimate the shear viscosity is to consider the behavior of diffusion coefficients in dilute weakly-interacting gasses[5]. In this regime, diffusion can be thought of as a random walk, so that one can estimate the diffusion coefficient D as

$$D \sim l_{mfp} v \tag{1.30}$$

For simple charge diffusion, this yields the estimate

$$D \sim \frac{1}{\lambda^2 T} \tag{1.31}$$

However, for momentum diffusion, we have the relation

$$D_\eta = \frac{\eta}{\epsilon + P} \tag{1.32}$$

Thus we get an estimate for η:

$$\eta \sim l_{mfp}(\epsilon + P)v, \tag{1.33}$$

and since $\epsilon \sim T^4, P \sim T^4$, this means that

$$\eta \sim \frac{T^3}{\lambda^2}. \tag{1.34}$$

Clearly, this diverges as $\lambda \to 0$. The divergence is easy to trace to the divergence of the mean free path as the interaction strength goes to zero.

This behavior of the shear viscosity in the weak coupling limit may seem counterintuitive. However, it is important to recall that hydrodynamics - and thus η -

makes sense on distance scales much larger than the microscopic scales characterizing the system, which in this case are the interparticle separation and the mean free path. As the coupling is sent to zero, the mean free path diverges, and one must do experiments on larger and larger length scales to measure η to stay in the hydrodynamic limit. The shear viscosity can be viewed as a proportionality constant relating the amount of force dF experienced by a fluid area element dA due to a transverse velocity gradient dv/dy:

$$dF = \eta \frac{dv}{dy} dA \tag{1.35}$$

Suppose that we fix the force dF experienced by the area element dA, and imagine sending $\lambda \to 0$. To remain in the hydrodynamic limit, we must increase y as λ decreases. It is then not hard to convince oneself that this implies that η diverges as $\lambda \to 0$.

1.4.3 Transport coefficients from Green's functions

Suppose one has a microscopic theory describing a fluid in which we are interested. Since hydrodynamics describes the low-frequency physics of the theory, we should be able to connect the parameters entering the hydrodynamic equations to the correlation functions of the microscopic theory. Such a relation must obviously relate transport coefficients to the low-frequency behavior of the retarded Green's functions of the theory.

As sketched in Section 1.2 above, the correlation functions can be calculated by introducing a source $J(x)$ coupled to the operator $\mathcal{O}(x)$ of interest. The correlation functions are affected by the presence of the source, and the expectation values of $\mathcal{O}$ becomes non-zero, assuming that $\mathcal{O}$ has zero vacuum expectation value. Linear response theory then implies that if the sources are small, the expectation value of $\mathcal{O}$ shifts from its vacuum value to

$$\langle \mathcal{O}(x) \rangle_J = -\int d^4y G^R(x-y) J(y) \tag{1.36}$$

For the application to hydrodynamics, $\mathcal{O} = T^{\mu\nu}$, so we need to couple a source to $T^{\mu\nu}$. The source for the stress-energy tensor is simply the metric tensor $g^{\mu\nu}$, and turning on a small source just means introducing a small metric perturbation $h^{\mu\nu}$ around flat space $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$. We can use this to derive formulas for η and ζ in terms of the retarded correlation functions.

Following Ref. [4], let us take $h^{\mu\nu}$ to be of the form

$$g_{ij} = \delta_{ij} + h_{ij}(t) \tag{1.37}$$

$$g_{00} = 0, \; g_{0i} = 0. \tag{1.38}$$

and work in the rest frame of the fluid, so that $u^\mu = (1, 0, 0, 0)$. To analyze the effect of $h_{\mu\nu}$, we need the curved-space generalization of Eq. 1.18, which can be obtained simply by promoting the partial derivatives to covariant derivatives. With the form of $h^{\mu\nu}$ and u^μ we have chosen above, we get only contributions to the traceless part of $\sigma^{\mu\nu}$, coming from the Christoffel symbols in the covariant derivatives:

$$\sigma_{xy} = 2\eta\Gamma^0_{xy} = \eta\partial_0 h_{xy}(t) = i\eta\omega h_{xy} \tag{1.39}$$

where we took the time-depedence of the perturbation to be $h_{xy}(t) = e^{i\omega t} h_{xy}$. But from the linear-response relation above, this can be connected to the retarded

Green's function of T_{xy}, so that we can write

$$\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \text{Im} G^R_{xy,xy}(\omega, 0) \tag{1.40}$$

where

$$G^R_{xy,xy}(\omega, 0) \equiv \int dt dx e^{i\omega t} \theta(t) \langle [T_{xy}(t, x), T_{xy}(0, 0)] \rangle. \tag{1.41}$$

This is known as the Kubo formula for the shear viscosity. Very similar arguments (as shown for instance in Ref. [6]) can be used to derive a Kubo formula for the bulk viscosity:

$$\zeta = -\frac{4}{9} \lim_{\omega \to 0} \frac{1}{\omega} \text{Im} G^R_{tr}(\omega, 0) \tag{1.42}$$

where

$$G^R_{tr}(\omega, k) \equiv \int dt dx e^{i\omega t} \theta(t) \langle [\frac{1}{2} T^i_i(t, x), \frac{1}{2} T^i_i(0, 0)] \rangle \tag{1.43}$$

While the Kubo formulas give a simple relation between the microscopic physics in a fluid and its hydrodynamic properties, actual calculations of transport coefficients directly from the microscopic theory are very involved even when the microscopic theory is weakly coupled[5]. Thus, in a generic field theory it seems entirely hopeless to calculate the transport coefficients analytically from first principles. Moreover, since the Kubo formulas involve retarded correlation functions, the calculation of transport coefficients necessitates doing calculations in real time, rather than imaginary time (which suffices for many observables). Lattice Monte Carlo methods can only yield imaginary-time correlation functions, since Monte Carlo methods require a real probability measure, which only appears in path integrals after rotation to imaginary time. Thus, one cannot calculate transport coefficients using lattice Monte Carlo without attempting to analytically continue

Euclidean correlations functions back to Minkowski space. This turns out to be extremely difficult, because in lattice calculations one always works with a finite-volume Euclidean spacetime, which amounts to working with the theory at finite temperature, with Matsubara frequencies $\omega = 2\pi n/T$. Hydrodynamics comes from the $\omega \to 0$ limit of the correlation functions, and accessing this region on the lattice requires sending $T \sim 0$, which corresponds to having an arbitrarily large lattice. This is impractical, and so hydrodynamic transport coefficients for most field theories remain out of theoretical reach, resisting attacks by both analytic and numerical methods.

Remarkably, as we will see, for some very strongly coupled theories, one can calculate transport coefficients *analytically* using the gauge gravity duality. The strong-coupling calculations actually turn out to be much easier than calculating the transport coefficients in the same theories at weak coupling.

Chapter 2

Gauge/Gravity Duality

In this chapter we discuss the aspects of the gauge/gravity duality that we use in the rest of this dissertation. Gauge/gravity duality is a feature of string theory. String theory is a theory of relativistic strings and their interactions, and while it was originally invented in an attempt to describe the strong nuclear interaction, it was quickly recognized that it is really a more fundamental theory that includes quantum gravity instead. The requirement of self-consistency turns out to be extremely restrictive for a relativistic quantum theory of fundamental strings[7, 8, 9, 10], and string theory turns out to be consistent only in 10 spacetime dimensions. Furthermore, it is thought that one must make the strings theory supersymmetric to obtain a consistent theory. (Supersymmetry is a symmetry relating fermions such as quarks with bosons such as gluons.)

In this chapter, we first simply give a quick 'user's guide' to the duality, and discuss the dictionary relating gauge theory quantities to gravity theory quantities. We first show how the duality can be used to calculate Euclidean correlation functions of some strong-coupled field theories. We then briefly discuss how the duality can be used to calculate real-time correlation functions, which is essential for the calculation of hydrodynamic transport coefficients. Finally, we give a brief discussion of how the duality arises in string theory. This last discussion is not essential for following the rest of the dissertation.

2.1 The gauge/gravity duality dictionary

In this section we present the gauge/gravity duality dictionary. The simplest form of the duality is called the AdS/CFT correspondence, and the dictionary relating gauge theory quantities to gravity quantities was developed in this context. The AdS/CFT correspondence states that $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills (SYM) theory is equivalent to type IIB superstring theory on an $AdS_5 \times S^5$ background[11]. Let us unpack this statement.

On the 'CFT' side of the correspondence, we have the field theory: $\mathcal{N} = 4$ SYM theory, which we sometimes refer to as 'SYM' for short. This a Yang-Mills theory with gauge group SU(N), and the glue part of the theory is the same as glue sector of QCD. In addition to gluons, SYM theory has fermions that couple to the gluons, as does QCD. In contrast to QCD, in SYM the fermions transform in the adjoint representation of the gauge group, just like the gluons. SYM also has fundamental scalar fields coupling to the gluons, transforming in the adjoint representation of color; this feature has no analogue in QCD. In QCD, we can adjust the matter content of the theory (the number of quark flavors and their masses) without necessarily changing the qualitative features of the theory, confinement and asymptotic freedom. In contrast, SYM has very specific matter content: $N_f = 4$ flavors of masslesss Majorana adjoint fermions in SYM, and $N_f = 6$ flavors of massless adjoint real scalar fields.

The matter contents of SYM, as well as the interactions of the fermions and scalars in the theory, are fixed by the requirement of $\mathcal{N} = 4$ supersymmetry[9, 10]. For us, the important implication of the special symmetries of $\mathcal{N} = 4$ SYM gauge theory is that SYM theory is a *conformal field theory*, a CFT. This means that the theory does not have any mass scales, even after quantization. The matter content and interactions are chosen so that all of the contributions to the β function cancel exactly. Thus there is no Λ_{QCD} in SYM. The essential feature of SYM is that there is no intrinsic scale in the theory on which the dimensionless coupling can depend, the accordingly coupling g^2_{YM} in SYM *does not run*. The large amount of supersymmetry available in SYM theory allows one to show that the above statements remain true even non-perturbatively.

Thus, in contrast to QCD, the coupling g_{YM} of $\mathcal{N} = 4$ SYM theory an adjustable parameter of the theory, and we can tune it to be large or small. So $\mathcal{N} = 4$ SYM theory has *two* adjustable parameters: the number of colors N, and the 't Hooft coupling λ.

On the 'AdS' side of the correspondence, we have the gravity theory: type IIB superstring theory on an $AdS_5 \times S_5$ background. $AdS_5 \times S_5$ is a 10D spacetime that has the structure of 5D Anti-deSitter space AdS_5, and a five-sphere S^5. AdS_5 has the metric

$$ds^2 = \frac{r^2}{R^2}\left(dx_\mu dx^\mu + dr^2\right) \tag{2.1}$$

where $\mu = 0, 1, 2, 3$. The string theory has a rich spectrum, the details of which will not be important for what follows. There are two dimensionless parameters on the gravity side: the string coupling g_s and the ratio of the string length scale l_s and the radius of AdS_5 space R, R/l_s.

Having quickly defined the theories on the two sides of the correspondence, we now discuss the matching of parameters and observables between the two theories. The AdS/CFT correspondence is a strong/weak duality, with $R/l_s \sim \lambda^{1/4}$, and $g_s \sim 1/N$. When the field theory is weakly coupled and λ is small, allowing one to do perturbative calculations on the field theory side, the dual gravitational theory is sensitive to stringy physics, and cannot be described by Einstein gravity. In this regime one must use the full string theory. Furthermore, if N were small in the gauge theory, string loop corrections would be unsuppressed in the gravity theory, so that one would not be able to describe the string theory using string perturbation theory.

On the other hand, suppose that the field theory has $\lambda \to \infty$ and $N \to \infty$. On the string theory side, $g_s \to 0$, and $R/l_s \to \infty$, which implies that the string theory reduces to a *classical* two-derivative theory of gravity: general relativity with some specific matter content. This means that given a dictionary between the generating functionals of correlation functions of the two theories, one could calculate the correlation functions of observables in the *strongly-coupled* quantum field theory by doing an equivalent calculation in a classical *weakly-coupled* gravity theory.

Before we proceed to describe this dictionary., it is important to note that above we used $\mathcal{N} = 4$ SYM theory, a CFT, as an example, but in fact the discussion of the domain of validity of the correspondence applies to all of the known examples of the gauge/gravity correspondence, including non-conformal theories. In all known cases, field theories with gravity duals are in some kind of large N limit. In theories with gravity duals where the conformal symmetry is broken, and the coupling runs, there is still a *strongly-coupled* ultraviolet fixed point. The large strength of the coupling at this UV fixed point is used to provide a separation of scales on the gravity side between the curvature scale R of the backgrounds and the string scale l_s. There are no gravity duals known for asymptotically free theories like QCD, which have *weakly-coupled* ultraviolet fixed points. It is unknown whether a string dual for QCD exists; if it does, it is expected on general grounds that it would not have a large separation of scales between R and l_s because of the UV behavior of QCD.

2.1.1 Correlation functions

Consider a local gauge-invariant single-trace operator in the field theory, such as $\mathcal{O} = \frac{1}{N_c}\text{tr}F^2$. As we discussed previously in Sec. 1.2, to find the correlation functions in the field theory, one introduces a source ϕ_0 that couples to $\mathcal{O}$. The correlation functions of $\mathcal{O}$ can then be found by taking functional derivatives with respect to $\phi_0(x)$ of

$$\langle e^{\int d^4x\, \phi_0(x)\mathcal{O}(x)}\rangle_{CFT} \tag{2.2}$$

Gauge-invariant single-trace operators in a CFT are characterized by the representation of the Lorentz group they transform in (e.g., scalar, spinor, vector, etc). In addition to their symmetry properties under the Lorentz group, operators in a field theory have scaling dimensions, which give their behavior under scale transformations. For instance, classically the operator $\mathrm{tr}F^2$ has scaling dimension $\Delta = 4$, since it has two derivatives and two gauge fields. In the full theory, quantum corrections usually change the value of the scaling dimension away from the classical one. That is, operators can have anomalous dimensions.

The gauge/gravity duality dictionary associates a bulk field in the 10D gravity theory with every gauge-invariant local operator in the CFT. To state how these bulk fields are related to field theory quantities, we need a few more facts about the gravity theory. Recall that the metric of $AdS_5 \times S^5$ is

$$ds^2 = \frac{r^2}{R^2} dx_\mu dx^\mu + \frac{r^2}{L^2} dr^2 + L^2 d\Omega_5^2 \tag{2.3}$$

where $d\Omega_5^2$ is the metric on the S_5. We can make a choice of coordinates in the above metric $z = R^2/r$, which leads to an often more convenient expression for the $AdS_5 \times S^5$ metric,

$$ds^2 = \frac{R^2}{z^2} \left(dx_\mu dx^\mu + dz^2 \right) + L^2 d\Omega_5^2. \tag{2.4}$$

$\mu = 0, 1, 2, 3$ are the directions the field theory dual turns out to 'live' in, and z is an extra 'holographic' direction. z turns out to have the interpretation of something like an energy scale in the dual field theory, with small z corresponding to high energies and large z corresponding to low energies in the field theory. AdS_5 has a boundary located at $z \to 0$ in the coordinates used above, which is where the field theory can be said to 'live'. Note that because AdS_5 has a boundary, given a field in AdS_5, one must specify a boundary condition at $z = 0$ to obtain a solution to the equations of motion.

The isometry group of AdS_5 is $SO(4,2)$, which encodes the conformal symmetry of the dual CFT, which is described by the group of conformal transformation $SO(2,4)$. In generalizations of the AdS/CFT correspondence, the S^5 is replaced by some other 5D compact manifold X_5, while the AdS_5 is usually replaced by a space which becomes *asymptotically* AdS_5 in the limit $r \to \infty$. This corresponds to having a dual field theory with broken conformal symmetry at low energies ('the infrared', referred to as the IR), but with conformal symmetry reemerging at high energies ('the ultraviolet', which is referred to as the UV).

It turns out that many calculations, the S^5 part of the background is unimportant, and we suppress the dependence of the metric on the S^5 in what follows. This is because for many quantities, the details of X_5 affect the details of the matching of gravity theory quantities to field theory quantities, but not the general features of the results. In particular, the dependence of the bulk fields on the compact manifold will not be important for the quantities we will be calculating in the rest of this dissertation.[1].

Now we can state the mapping between bulk fields and field theory quantities. A scalar operator $\mathcal{O}$ with scaling dimension Δ in the field theory is associated with a scalar field $\phi(x,z)$ in AdS_5. More generally, a vector operator is associated with a vector field in the bulk, tensor operators with bulk tensor fields, and so on. The boundary value of $\phi(x,z)$ at $z=0$ is associated with the source $\phi_0(x)$ of $\mathcal{O}$ via the

[1]For some remarks on how our neglect of the X_5 affects our results, see the discussion in Chapter 3 regarding matching some of the parameters appearing in our results in a certain special case, with those of Ref. [12], in which the full 10D theory is analyzed.

relation

$$\phi(x, z \to 0) = z^{4-\Delta}\phi_0(x). \tag{2.5}$$

The factor of $z^{4-\Delta}$ is necessary to give $\phi_0(x)$ the correct mass dimension $[\phi_0] = 4-\Delta$, since a 5D scalar field in AdS is dimensionless. To see this, note that as befits a gravitational theory, the 5D action has an overall dimensionful factor involving the 5D Planck scale:

$$S_{5D} = \frac{1}{2\kappa^2}\int d^5x\sqrt{g}\left\{R + (\partial\phi)^2 + m^2\phi^2 + V(\phi)\right\} \tag{2.6}$$

where R is the Ricci scalar with $[R] = 2$, and $\kappa^2 = 8\pi G_N = 4\pi^2 R^3/N^2$ is the 5D Newton constant [2] with $[\kappa^2] = -3$.

The gauge/gravity conjecture then consists of the claim that

$$\langle e^{\int d^4x\, \phi_0(x)\mathcal{O}(x)}\rangle_{CFT} = \exp\left(S_{bulk}[\phi]\Big|_{\phi(z\to 0,x)=z^{4-\Delta}\phi_0(x)}\right) \tag{2.7}$$

On the left hand side of this equation is the generating functional of connected correlation functions of $\mathcal{O}$ in the strongly-coupled large N quantum field theory. On the right hand side, we would in general have the partition function of the string theory, which in the classical (due to large N) and low energy (due to strong coupling) limits reduces to that of a two-derivative theory of gravity coupled to matter fields ϕ. The gravitational action S_{bulk} is evaluated on bulk fields satisfying the bulk equations of motion, subject to the boundary condition in Eq. 2.5.

[2]The precise relation of κ with N requires that one know the full supergravity dual, including the behavior of the solution on X_5. Above we wrote the relation of κ with N that is appropriate to having $X_5 = S^5$, as happens with $\mathcal{N} = 4$ SYM.

The full (classical + quantum) scaling dimension of a scalar operator in the field theory determines the mass-squared m^2 of the associated bulk scalar field:

$$m^2R^2 = \Delta(\Delta - 4) \tag{2.8}$$

An example of a dimension-4 scalar operator in $\mathcal{N} = 4$ SYM is just the lagrangian density $\mathcal{L}$, which is sourced by the YM coupling constant. Since $\mathcal{N} = 4$ is a CFT, the coupling does not run, and the anomalous dimension of $\mathcal{L}$ zero. So the total scaling dimension of $\mathcal{L}$ remains $\Delta = 4$. The dual bulk field is a massless 5D scalar field. In the supergravity theory, this massless 5D scalar is the dilaton.

The correspondence also applies to higher-spin operators. For instance, the stress-energy tensor $T^{\mu\nu}$ of the field theory has scaling dimension $\Delta = 4$[3], and is sourced by the 4D metric in the field theory. The corresponding tensor bulk field is the 5D metric g_{MN}.

Using the dictionary, one can now calculate (for instance) the Euclidean two-point correlation function of a scalar operator with dimension $\Delta > 1$ in $\mathcal{N} = 4$ SYM. The calculation on the gravity side proceeds by introducing a bulk scalar field with the appropriate mass into the Wick-rotated action ($t \to it$):

$$S_{5D} = \frac{1}{2\kappa^2} \int d^5x \sqrt{g} \left\{ g^{MN} \partial_M \phi \partial_N \phi - \frac{1}{2} m^2 \phi^2 \right\} \tag{2.9}$$

$$= \frac{1}{2\kappa^2} \int d^5x \sqrt{g} \left\{ \frac{1}{2} \phi (\Box - m^2) \phi) + \partial_N (\phi \partial^N \phi) \right\} \tag{2.10}$$

where $\Box$ is the curved-space d'Alembertian operator and $m^2L^2 = \Delta(\Delta - 4)$, and the metric is that of AdS_5 space, Eq. 3.4, as is appropriate for strongly-coupled large

[3]The stress-energy tensor is conserved, so that $\partial_\mu T^{\mu\nu} = 0$ in flat space. The anomalous dimensions of conserved currents are exactly zero[2].

N $\mathcal{N} = 4$ SYM. Terms more than quadratic in ϕ do not affect the calculation of two-point functions, which is why we did stopped at the mass term in writing the potential for the scalar above. To find the two-point function, one must solve the equations of motion for the 5D scalar

$$\left(\partial_z^2 - \frac{3}{z}\partial_z - q^2 - \frac{m^2 R^2}{z^2}\right)\phi(z, q) = 0 \tag{2.11}$$

where we went to 4D Fourier space, subject to the boundary condition

$$\phi(z \to 0, q) = z^{4-\Delta}\phi_0(q) \tag{2.12}$$

where $\phi_0(x)$ is a prescribed source function. The solution of this equation which is regular in AdS_5 is

$$\phi(z, q) = f_q(z)\phi_0(q). \tag{2.13}$$

where

$$f_q(z) = \frac{z^{2-\Delta} J_{\Delta-2}(-iqz)}{\epsilon^{2-\Delta} J_{\Delta-2}(-iq\epsilon)} \tag{2.14}$$

where $J_n(y)$ is a Bessel function, normalized so that at the UV boundary $z = \epsilon \to 0$, we have $f_q(\epsilon) = 1$, and ϵ has been introduced as a UV cutoff. One must then evaluate the action on this solution. Clearly, only the boundary term remains once this is done:

$$S = \int \frac{d^4q}{(2\pi)^4}\phi_0(-q)\mathcal{F}(q, z)\phi(q, z)|_{z\to 0} \tag{2.15}$$

where

$$\mathcal{F}(q, z) = \frac{N^2}{8\pi^2}\frac{1}{z^3} f_{-q}(z)\partial_z f_q(z) \tag{2.16}$$

Now one simply takes two derivatives with respect to $\phi_0(x)$ to obtain the two-point

Euclidean correlation function:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_{CFT} = -2 \lim_{z\to 0} \mathcal{F}(p,z). \tag{2.17}$$

For $\Delta = 4$, for instance, this evaluates as[4].

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle^{\Delta=4}_{CFT} = \frac{N_c^2}{16\pi^2} q^4 \log(q^2\epsilon^2) \tag{2.18}$$

which is the behavior one expects from the field theory side.

It is natural at this point to ask how one really goes about matching bulk fields and gauge theory operators. The dictionary matches operators with given Lorentz quantum numbers and *total* scaling dimension with bulk fields with particular 5D masses. But the total scaling dimension of a gauge theory operator is in general not calculable in perturbation theory when λ is large, and generically one expects large quantum corrections to the classical scaling dimension.

This strong coupling problem means that given a generic bulk field with mass $m^2R^2 = \Delta(\Delta - 4)$, in general one cannot identify to which specific gauge theory operator it corresponds. One does not even know (without further information) whether there exists an operator in the gauge theory that has total scaling dimension Δ. In special cases (for instance with $\mathcal{N} = 4$ SYM), one can deduce the existence of such operators and their form in terms of the fundamental fields of the theory. For our purposes here, we can assume that such operators do exist for the bulk fields of interest. With this assumption, the gauge-gravity duality allows one to calculate

[4]We have thrown away terms that diverge as $z \to 0$. These divergences correspond to contact terms in the field theory, and the justification for discarding them in this case is the subject of holographic renormalization theory[13, 9]

the correlation functions of such operators.

However, despite the $\lambda \to \infty$ limit inherent to gauge/gravity duality, one can sometimes symmetries and associated non-renormalization theorems to identify the dual gauge theory operator. For instance, as we remarked above, the bulk metric is associated with the stress tensor of the field theory, and the total scaling dimension of the stress tensor is equal to its classical scaling dimension. Thus means that it is straightforward to identify the correlation functions of the stress tensor in terms of quantities on the dual gravity side.

2.2 Breaking of conformal symmetry

Let us again consider the relation between the scaling dimension of scalar gauge theory operators and the mass of the corresponding bulk fields:

$$m_5^2 = \Delta(\Delta - 4) \tag{2.19}$$

This formula tells us that irrelevant scalar operators correspond to bulk scalar fields with positive m^2, exactly marginal operators correspond to fields with $m^2 = 0$, and relevant operators correspond to fields with $m^2 < 0$. In flat space, fields with $m^2 < 0$ are a signal of instability, but this is not necessarily the case in a curved space like AdS_5: there is no instability for fields that have $m^2 R^2 > -3$ [14]. In the field theory, this corresponds to the restriction $\Delta > 1$, which is simply the unitarity bound on the dimension of a scalar operator in 4 dimensions[15].

Suppose that one wants to study systems with broken conformal symmetry in the dual field theory. A way to do this is to add an operator as a deformation. To be able to use the AdS/CFT dictionary, we need the field theory to remain conformal in the UV. This means that the deformations we will consider are relevant, with $\Delta < 4$. For instance, one can turn on a mass for the fermions, which at weak coupling corresponds to adding a $\Delta = 3$ operator to the theory. This introduces a scale, and breaks conformal symmetry.

As was noted in the preceding section, once conformal symmetry is broken in the field theory, the dual geometry will no longer be AdS_5. The bulk scalar field now has to be allowed to backreact on the geometry. The reemergence of conformal symmetry in the UV is reflected in the geometry becoming asymptotically AdS_5 as $z \to 0$. The effect of conformal symmetry breaking on hydrodynamic observables is the central subject of the next two chapters.

2.3 Finite temperature and real-time correlation functions in AdS/CFT

Turning on a finite temperature breaks conformal invariance in the field theory, since a non-zero temperature introduces a scale into the theory. On the gravity side, the claim of the gauge/gravity duality dictionary is that the background can no longer be AdS_5. It turns out that in the cases of interest in this dissertation, the dual background corresponding to a large N strongly coupled CFT with finite temperature T is that of a Schwarzschild black hole in AdS_5, often referred to later as AdS_5-Schwarzschild[16]:

$$ds^2 = \frac{r^2}{R^2}(-f(r)dt^2 + dx_i dx^i) + \frac{R^2}{r^2 f(r)} dr^2 \qquad (2.20)$$

where $f(r) = 1 - r_0^4/r^4$ and $i = 1, 2, 3$ are the spatial directions of the field theory. The temperature T of the field theory is identified with the Hawking temperature of the black hole

$$T = \frac{r_0}{\pi R^2}. \tag{2.21}$$

Note that the black hole horizon is a flat 3-brane in this solution, rather than the spherical horizon of a Schwarzschild black hole in 4D. The entropy of the dual field theory is associated with the Bekenstein-Hawking entropy $S = A/4G$, where A is the area of the horizon. This gives an entropy density for large N, strongly coupled $\mathcal{N} = 4$ SYM of

$$s = S/V = \frac{\pi^2}{2} N^2 T^3. \tag{2.22}$$

This turns out to be 3/4 of the entropy density of $\mathcal{N} = 4$ SYM theory[10, 9]. The N^2 dependence makes sense given that this is a theory with N^2 deconfined degrees of freedom.

2.3.1 Calculation of s and T for generic black brane geometries

It will be important later in this dissertation to be able to calculate s and T in generic backgrounds, so we give a discussion of how this is done here. Consider the following geometry:

$$ds^2 = a(r)^2 \left(-h(r)dt^2 + dx^2\right) + \frac{dr^2}{h(r)b(r)^2} \tag{2.23}$$

Here h has a simple zero at $r = r_h$, the location of the black hole horizon, and a, b are non-zero at $r = r_h$. The entropy density is very simple to obtain, as it can be read off directly from the geometry using the relation $S = A/4G$:

$$s = \frac{2\pi}{\kappa_5^2}|a(r_h)|^3, \tag{2.24}$$

where we chose to write the Newton constant in terms of the 5D Planck constant $\kappa_5^2 = 8\pi G$.

There are a number of ways to calculate the Hawking temperature, but the simplest way in practice is via 'analytic continuation'. In this method, one does a Wick rotation $t \to i\tau$. In field theory, the periodicity β of τ, $\tau + \beta \sim \tau$, is identified with the temperature $T = 1/\beta$. In the gravity theory, one makes the same identification. However, the metric must remain free of conical singularities for consistency. The demand that there be no conical singularity in the metric at $r = r_h$ relates T to the values of the metric coefficients at $r = r_h$.

To see how this works, let us consider the line element above for $r = r_h + \epsilon$, where $\epsilon \ll 1$. In this region, after the Wick rotation we get

$$ds^2 = a(r_h)^2 h'(r_h)\epsilon d\tau^2 + \frac{d\epsilon^2}{h'(r)\epsilon b(r_h)^2} \tag{2.25}$$

where we have dropped the dx^2 part of the metric, since it won't play a role in what follows. In terms of a new coordinate

$$u = \frac{2\epsilon^{1/2}}{b(r_h)h'(r_h)^{1/2}}, \tag{2.26}$$

the metric takes the form

$$ds^2 = a(r_h)^2 h'(r_h)^2 b(r_h)^2 \frac{u^2}{4} d\tau^2 + du^2. \tag{2.27}$$

This looks like the metric for a flat plane in polar coordinates, with τ as the angular variable $\theta \sim \theta + 2\pi$. If the metric is to have such an interpretation, we must have

(at $u = 1$)

$$2\pi = \int_0^{2\pi} d\theta = \int_0^{\beta} \frac{1}{2}|a(r_h)h'(r_h)b(r_h)|d\tau \qquad (2.28)$$
$$= \frac{1}{2}|a(r_h)h'(r_h)b(r_h)|\beta,$$

and since $\beta = 1/T$, we obtain

$$T = \frac{|a(r_h)h'(r_h)b(r_h)|}{4\pi} \qquad (2.29)$$

2.3.2 Real-time correlation functions

Now we can discuss the calculation of real-time correlation functions. The introduction of a black hole horizon presents some subtleties for the calculation of two point functions, since in addition to regularity in the UV, one must now also specify the boundary condition for solutions of the equations of motion at the horizon. The correct choice for calculating retarded two-point correlation functions turns out to be the imposition of incoming-wave boundary conditions at the horizon[17, 18, 19, 20, 21, 22, 23, 24].

The other subtlety one must deal with concerns the behavior of the boundary term in Eq. 2.9 when the background contains a black hole. Since there are now two boundaries, one at $z = 0$ and the other at the black hole horizon $z = z_h$, there are two contributions in the boundary term when the action is evaluated on solutions of the equations of motion subject to the incoming-wave boundary condition. The correct prescription was rigorously justified in Refs. [21, 20]. To obtain two-point retarded correlation functions, which is all we will need in this dissertation, the prescription is to simply *discard* the contribution from the horizon:

$$G^R(q) = -2 \lim_{z \to 0} \mathcal{F}(q, z) \tag{2.30}$$

where $\mathcal{F}(q, z)$ is defined analogously to the Euclidean case, but now one must solve the equations of motion in the black hole background, subject to incoming-wave boundary conditions at the horizon.

2.4 AdS/CFT duality in string theory

In this section we sketch the arguments that lead to the AdS/CFT conjecture in string theory.

As mentioned in the introduction, string theory is a theory of relativistic quantum strings, moving in 10 spacetime dimensions. There are two kinds of strings in string theory: open and closed strings. Closed strings do not have end-points, and their spectrum contains massless spin-two excitations, which are nothing other than gravitons; this is how it was first realized that string theory contains gravity. Open strings have end-points, and their spectrum generally contains various kinds of gauge fields.

String theory turns out to contain more than just strings. It also contains non-perturbative extended excitations called 'D-branes', that look like soliton-like hyperplanes in the 10D spacetime. These end points of open strings can be free to move around in spacetime, or they can end on a D-brane. (The 'D' in D-brane comes from the fact that strings that have an end that sits on a D-brane have Dirichlet boundary conditions at that end.) The open string spectrum contains massless vector excitations, which are associated with $U(1)$ gauge fields.

D-branes are extended objects, and a Dp brane has a $p+1$ dimensional world-volume. In this section we focus on D3 branes, which have $4D$ world-volume. Consider a system of N D3 branes lying on top of each other in a 10D spacetime described by type IIB superstring theory. The AdS/CFT conjecture is that this system has two equivalent descriptions: $\mathcal{N}=4$ SU(N) super-Yang-Mills theory, which is a CFT, and type IIB superstring theory on an $AdS_5 \times S^5$ background.

At low energy, the closed strings give rise to a 10D supergravity theory. Keeping this at the back of our minds, consider the open string sector associated with strings ending on the D3 branes, and follow the discussion of Ref. [9]. Let us first consider a *perturbative* description of the open-string dynamics of the D3 brane stack. In this description, one considers open strings with both ends attached to the D3 branes.

The perturbative low-energy spectrum of open strings in the presence of one D3 brane contains maximally supersymmetric $U(1)$ gauge theory living on the world-volume of the D3 brane, in four dimensions. The $U(1)$ gauge theory comes from open strings with both ends attached to the D3 brane. If one has N coincident D3 branes (that is, the branes are right on top of each other), the low-energy theory must capture the fact that each open string can start and end on N different branes, so that there are a total of N^2 different kinds of strings. The low energy theory turns out to be $\mathcal{N}=4$ YM SU(N) gauge theory ($\mathcal{N}=4$ SYM) living on the world-volume of the stack of N D3 branes. [5]

[5]One might think that the theory would have gauge group $U(N)$ instead, but it turns out that

As described in the previous section, $\mathcal{N} = 4$ SYM is a very special gauge theory: the contributions of the gauge bosons, fermions and scalars in the theory to the beta function cancel, and it can be shown that the theory is conformal. In particular, the coupling of the $\mathcal{N} = 4$ SYM theory *does not run*, in contrast to the behavior of the coupling in QCD. The YM gauge coupling g_{YM}^2 is related to the string coupling g_s via

$$g_{YM}^2 = 4\pi g_s \tag{2.31}$$

The above perturbative description of the open-string sector turns out to be valid when

$$\lambda = 2\pi g_s N \ll 1, \tag{2.32}$$

since string interactions cost powers of g_s, and there are N open-string 'species' to consider due to the presence of N D3 branes. This means that the perturbative open string description is valid when $\mathcal{N} = 4$ SYM is weakly coupled.

Next, one can show that in the perturbative (in the sense that $\lambda \ll 1$) low energy limit (compared to $1/l_s$), it turns out that the open strings living on the branes decouple from the closed strings living in the 10D 'bulk' spacetime. The reason for this is that it can be shown that the energy density of the D3 branes scales as N/g_s, while the Planck mass scales as $1/g_s^2$[9]. As a result, the closed string sector does not feel the presence of the D-branes, and the supergravity solution in this regime is just a flat 10D space. Thus we find that at low energies, the stack of N D3 branes can be described by *weakly-coupled* $\mathcal{N} = 4$ SYM theory and supergravity

the $U(1)$ factor decouples from the world-volume theory[9, 10]. In any case, the difference between $U(N)$ and $SU(N)$ is subleading in the large N limit.

fields living on a 10D flat background, which the two sectors decoupled from each other.

To recapitulate, in the $\lambda \ll 1$ limit, the string theory reduces to two decoupled systems in the low-energy limit':

- Flat 10D type IIB supergravity
- $\mathcal{N} = 4$ SYM .

Now consider the limit when $\lambda \gg 1$. Now we cannot trust the perturbative description of the D-branes in terms of open strings. However, in this limit the energy density of the D3 brane stack becomes non-negligible, and we have to into account the effects of the D3 branes on the supergravity solution.

Following Ref. [10], the relevant part of the low-energy (low compared to $1/l_s$) effective action describing the kind of string theory we will be focusing on here (type IIB string theory) is given by

$$S^{\text{string}} = \frac{1}{2\kappa^2} \int d^{10}x \, e^{-2\Phi} \left(R + 4\partial_M \Phi \partial^M \Phi - \frac{1}{4} F_5 \wedge \star F_5 \right) \tag{2.33}$$

where $M = 0, 1, \ldots 9$, Φ is a scalar field called the dilaton, R is the Ricci scalar associated with the metric G, F_5 is the field-strength 5-form associated with the 4-form field that is part of the massless string spectrum, and $\star$ is the Hodge star operator that maps p-forms to $10 - p$ forms. This action must be supplemented by a self-duality constraint $F = \star F$.

A stack of N coincident D3 branes is a solution to the equations of motion associated with Eq. 2.33 which takes the following form

$$ds^2 = H^{-1/2} dx^\mu dx_\mu + H^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right) \tag{2.34}$$
$$F_5 = N\frac{1}{2}(\omega_5 + \star\omega_5)$$

where $\mu = 0, \ldots 3$ are the coordinates along the world-volume of the D3 brane, $d\Omega_5^2$ is the metric on the five-sphere S^5,

$$H = 1 + \left(\frac{R}{r}\right)^4 \tag{2.35}$$

and ω_5 is an appropriately normalized volume form on S^5, obeying

$$\int_{S^5} \omega_5 = 1 \Rightarrow \int_{S^5} F_5 = N \tag{2.36}$$

The solution for the dilaton is related to the string coupling as

$$e^{2\Phi} = g_s^2, \tag{2.37}$$

while

$$\left(\frac{R}{2\pi l_s}\right)^4 = \frac{g_s N}{4\pi^3} \tag{2.38}$$

For this solution to be reliable, two conditions must be met. First, we must be able to neglect string loop corrections, which means that we must have $g_s \to 0$. Second, we have to make sure that the geometry does not have any regions where curvatures become comparable to the string scale, since then we would become sensitive to higher derivative terms that are not included in Eq. 2.33. It is natural to expect trouble with the latter issue when $R \to l_s$ in the D3 brane stack solution, and a calculation of curvature invariants for this geometry confirms that the solution is reliable only when $R/l_s \gg 1$. This corresponds to requiring $g_s N \gg 1$. Translating to YM theory quantities, we see that the supergravity solution above is reliable when

$$g_s = \frac{g_{YM}^2}{4\pi} \sim 1/N \ll 1 \tag{2.39}$$

$$g_s N = \frac{g_{YM}^2 N}{4\pi} \gg 1 \tag{2.40}$$

Now consider the low energy limit for excitations in the background described by Eq. 2.34. There turn out to be two kinds of *decoupled* low-energy excitations in this background: ones that live in the region $r \gg R$, where the spacetime is flat, and ones that live in the 'near-horizon' region where $r \ll R$[9]. The near-horizon limit has the metric

$$ds^2 = \frac{r^2}{R^2} dx_\mu dx^\mu + \frac{R^2}{r^2}\left(dr^2 + r^2 d\Omega_5^2\right) \tag{2.41}$$

which is simply $AdS_5 \times S^5$.

Thus in the $\lambda \gg 1$ limit, the string theory reduces to two decoupled systems in the low-energy limit:

- Flat 10D type IIB supergravity
- Type IIB supergravity on $AdS_5 \times S^5$

In *both* the $\lambda \ll 1$ and $\lambda \gg 1$ limits, the string theory reduces to two decoupled pieces. In both cases one of these pieces is described by supergravity on a flat 10D space. The AdS/CFT conjecture is that the remaining two pieces are equivalent to each other: $\mathcal{N} = 4$ SYM is equivalent to type IIB supergravity on $AdS_5 \times S^5$.

The AdS/CFT conjecture involves a strong/weak duality, and thus is very hard to prove, because the regimes in which one can do reliable calculate in the

CFT and in the gravity dual do not overlap. However, the conjecture has passed a large number of consistency tests. For instance, the coupling dependence of some special quantities (ones connected with anomalies, for instance) in $\mathcal{N} = 4$ SYM is protected by symmetries, and can be calculated in both the field theory and the dual. In all cases investigated so far, such calculations, which look very different on the two sides of the duality, give the same answer. As a result, it is generally believed in the community that the AdS/CFT conjecture is correct.

Chapter 3

Sound Bound

This chapter focuses on the behavior of the speed of sound in fluids described by theories with gravity duals, and is based on work done with Abhinav Nellore and Thomas Cohen which was published in Ref. [25]. The chapter follows the presentation of Ref. [25], and is organized as follows. After an introduction describing the search for universal quantities that are not sensitive to the details of the rather special theories that posess gravity duals, we describe a class of non-conformal field theories which have tractable gravity duals. We then calculate the speed of sound in this class of theories, and show that the speed of sound always approaches a third the speed of light from below. This is followed by a brief review of other speed of sound calculations in the literature that serves to place our results in context. We then make a conjecture that the speed of sound bound may continue to hold outside the class of theories for which we were able to prove it, and give some encouraging evidence for some variants of the conjecture. Finally, there is a technical appendix describing the calculations techniques.

3.1 Introduction

As was discussed in the previous two chapters, gauge/gravity duality can be used to get insights into the behavior of some strongly-coupled fluids described by large N_c gauge theories. The dual gravitational theory becomes tractable when the large N_c field theory is strongly-coupled, and reduces to general relativity coupled to various matter fields as discussed in the previous chapter. One can then use the duality to explore the properties of a strongly-coupled quantum field theory by doing *classical* calculations in the gravity dual. In particular, one can calculate transport coefficients, which are generally not theoretically accessible in strongly coupled gauge theories.

Unfortunately, there is no gravity dual known for QCD, and thus the gauge/gravity duality cannot be used to make reliable quantitative predictions for phenomenologically interesting theories. However, we might hope to be able to develop some qualitative insights into the behavior of generic strongly-coupled systems by looking for quantities that do not depend sensitively on the details of any particular gravity dual construction. With some luck, the lessons gleaned from such 'universal' properties may tells us something about theories that do not have known gravity duals.

The most well-known example of a universal quantity in the hydrodynamics of strongly coupled theories with gravity duals is the ratio η/s of the shear viscosity to the entropy density. This ratio takes the value $\eta/s = 1/4\pi$ (in units where $k_B = \hbar = 1$) in *all* theories with gravity duals, in the sense that the dual theories are in the $\lambda \to \infty, N_c \to \infty$ limits[23, 19, 17, 26, 4]. This result does not depend on any details of the gravity duals. For instance, it does not depend on the dimension of the spacetime the field theory lives in, or on the temperature and chemical potential. As was argued in Chapter 1, the viscosity of a weakly-coupled gas diverges with the strength of the coupling. In the opposite limit, with a strongly-coupled system, one might expect that η/s would approach zero, or perhaps saturate at some theory-dependent finite value of order unity. The fact that systems with gravity duals universally have $\eta/s = 1/4\pi$ may suggest that η/s indeed saturates at a finite value as the coupling becomes large, and was originally interpreted as implying that η/s approaches the *same* universal value $1/4\pi$ for all strongly-coupled theories.

The value of η/s in theories with gravity duals turns out to be lower than the value of η/s for any known fluid in nature. The only fluid competitive with the value of η/s seen in theories with gravity duals is the sQGP seen at RHIC. The fact that η/s as measured at RHIC is of the same order of magnitude[1] as $1/4\pi$ has been taken to be one of the reasons for believing that the QGP is strongly coupled in the conditions explored at RHIC.

The phenomenological implications of the universality of η/s in theories with gravity duals are not clear[27, 28]. Furthermore, the deviations from $\eta/s = 1/4\pi$ as one moves away from the $N_c \to \infty, \lambda \to \infty$ limit do not appear to be universal, as the sign of the deviations has been shown to depend on the details of the theory one is working with[29, 30]. However, it remains important to search for universal properties of theories with gravity duals for both theoretical and phenomenological reasons.

In this chapter, we consider the behavior of the speed of sound v_s in theories with gravity duals. In any consistent relativistic theory, the speed of sound must

[1]There are large uncertainties in the extraction of η/s from the experimental data on the sQGP, so arguably the best one can do is say that η/s in the sQGP is of the order of $1/4\pi$.

be less than the speed of light, so that $v_s^2 \leq 1$. We show that in the simplest (but broad) class of non-conformal 4D field theories with gravity duals, v_s^2 obeys a much stronger bound: v_s^2 is always bounded from above by $1/3$ at high temperatures. We work with natural units $c = \hbar = k_B = 1$ throughout; while our results can be easily generalized to theories in D dimensions, we focus on the specific case of $D = 4$, as it is the case of phenomenological interest.

The speed of sound is known not to be universal in the same sense as η/s, since the speed of sound depends on the detailed properties of a system both at weak and strong coupling. For instance, the speed of sound depends on the temperature T and chemical potential μ of a system. We work with $\mu = 0$ throughout, but we do not expect that turning on a finite chemical potential would qualitatively affect our results so long as $\mu \ll T$. In addition to showing that $v_s^2 \leq 1/3$ at high temperatures, we will show that the temperature dependence of v_s^2 takes a certain universal form at high temperature[2]. Moreover, since $v_s^2 > 1/3$ has so far never been observed in energetically stable configurations of theories with gravity duals, it is tempting to speculate that $v_s^2 = 1/3$ is a *universal upper bound* in a wide class of strongly coupled gauge theories, at least at high temperatures.

As was mentioned in Chapter 1, the speed of sound can be found from $v_s^2 = \partial p/\partial\epsilon$, where p is the pressure of a fluid and ϵ is its energy density, both measured in the local fluid rest frame. When $\mu = 0$, one can use standard thermodynamic

[2]By high temperatures we mean $T \gg \Lambda$, where Λ is any other energy scale in the system.

relations to rewrite v_s^2 in terms of the entropy density s as

$$v_s^2 = \frac{d \log T}{d \log s} \ . \tag{3.1}$$

If one turns on a temperature in a 4D conformal field theory, the temperature provides the only scale, so that by dimensional analysis, one must have $s \sim T^3$, and $v_s^2 = 1/3$. In non-conformal theories, on the other hand, there are by definition scales other than the temperature, and v_s^2 has a non-trivial dependence on T and other properties of the theories.

3.2 v_s^2 in strongly-coupled theories

We now turn to the calculation of the speed of sound in strongly-coupled large N gauge theories. Such calculations are only tractable in theories with gravity duals, which are best understood when the dual field theories are CFTs. However, since $v_s^2 = 1/3$ in CFTs at finite temperature, with a CFT there is nothing to calculate, and we must consider *non-conformal* theories. Thus we will study the behavior of the speed of sound in gravity duals of theories with broken conformal symmetry.

Suppose that we have a large N strongly coupled conformal field theory with an action S_{CFT} that has a gravity dual. To break conformal symmetry in the field theory, we can add a deformation term to the S_{CFT} with a gauge-invariant single-trace scalar operator $\mathcal{O}_\Delta$ of total scaling dimension Δ:

$$S_{QFT} = S_{CFT} + \int d^4x \, \Lambda^{4-\Delta} \mathcal{O}_\Delta, \tag{3.2}$$

where Λ is the mass scale introduced by the deformation. We restrict our attention to deformations by *relevant* operators, for which $\Delta < 4$, since in this case the theory is still conformal in the far UV. For instance, at weak coupling, an example of an operator deformation with $\Delta = 3$ is furnished by $\mathcal{O}_3 = \bar{\psi}\psi$, where ψ is a fermion field. Such a the deformation has the interpretation of turning on a mass for some fermions in the CFT, so that $\Lambda = m_f$. One can also imagine adding multiple relevant deformations of this sort, with several Λ_i. Turning on multiple deformations turns out not to affect our results in any deep way (in fact they turn out to be essentially unaltered), so for simplicity we will only consider deformations of the form in Eq. 3.2 above in what follows, and comment on the extension to multiple scalar deformations at the end.

Let us now consider what the above field theory setup looks like on the gravity dual side. The gravity dual theory is a 10D theory of gravity coupled to various matter fields. We assume that the geometry factorizes into a non-compact manifold times a compact manifold, as in the paradigmatic case of $\mathcal{N} = 4$ SYM, where the dual geometry at zero temperature is $AdS_5 \times S^5$. For a generic CFT that has a gravity dual (up to the caveats in Footnote 3), the compact space is in general some compact 5D manifold X_5, while the non-compact manifold remains AdS_5, since the isometries of anti-deSitter space encode the conformal symmetries of the CFT. The addition of the conformal-symmetry breaking scalar operator $\mathcal{O}_\Delta$ in the field theory corresponds to turning on a scalar field ϕ in the gravity dual. The scaling dimension Δ of $\mathcal{O}_\Delta$ is encoded on the gravity side in the mass term of the bulk scalar field, as described in Chapter 2.

The details of X_5 do not play a direct role in the thermodynamics and transport properties of the dual gauge theory, so we work with the non-compact 5D part of the dual geometry in what follows. More precisely, we assume that there is a consistent truncation of the 10D supergravity equations of motion where the scalar field ϕ has a trivial profile on X_5.[3] Then one can do the trivial integral over X_5 in the action, resulting in an effective 5D action, which is what we work in the rest of the chapter. The volume of the X_5 manifold then shows up in the relation between the 5D Newton constant and the 10D Newton constant, but does this does not affect calculations of thermodynamic or hydrodynamic observables.

The gravity theory which is dual to the class of non-conformal field theories we are considering has a 5D effective action given by

$$S = \frac{1}{2\kappa_5^2} \int d^5 x \sqrt{-g} \left[R + \frac{12}{L^2} + \frac{1}{2}(\partial\phi)^2 - V(\phi) \right] . \tag{3.3}$$

where $\kappa_5^2/(8\pi) \sim 1/N^2$ is the $5D$ gravitational constant, R is the Ricci scalar associated with the 5D metric g_{MN}, ϕ is a real scalar field which is associated with the operator $\mathcal{O}$ breaking the conformal symmetry in the dual field theory, and we assume that $V(\phi)$ is a smooth potential that is symmetric about an extremum at $\phi = 0$. When $\phi = 0$, the equations of motion associated with Eq. 3.3 are simply the Einstein equations with a positive cosmological constant $12/L^2$, and have the solution of an AdS_5 space with radius L:

$$ds^2 = \frac{L^2}{z^2}(dz^2 + dx_\mu dx^\mu) \tag{3.4}$$

[3]It is also possible to have a CFT which has a non-factorizable dual geometry[31] that takes the form of a 'warped product'; Ref. [32] gives a survey of some examples of field theories with duals of this sort. It would be an interesting problem for future work to understand the extension of the class of 5D effective theories we study here to cover deformations of such CFTs.

On the field theory side, this corresponds to the geometry dual to the the original CFT, with the deformation turned off. When $\phi \neq 0$ in the gravity dual, the conformal symmetry breaking deformation is turned on, and the dual geometry is no longer described by Eq. 3.4.

The specific form of the scalar potential $V(\phi)$ appropriate to various specific dual gauge theories would be determined when the action is embedded in a particular string dual. At the level of the 5D effective theory, different choices of $V(\phi)$ correspond to different dual gauge theories. Thus Eq. 3.3 describes the gravity dual of a whole *class* of field theories, rather than the dual to a particular specific gauge theory.

The AdS/CFT dictionary imposes one condition on $V(\phi)$, coming from the fact that ϕ is dual an operator of scaling dimension Δ. The scaling dimension Δ is encoded in the mass term for the ϕ:

$$\lim_{r\to\infty} V(\phi) = \frac{1}{2L^2}\Delta(\Delta-4)\phi^2 + \mathcal{O}(\phi^4)\ , \tag{3.5}$$

Our interest is in relevant deformations constrains, so that $\Delta < 4$, while the Breitenloher-Freedman bound mentioned in Chapter 2 implies that $m^2L^2 > -4$, which translates into the unitarity bound in field theories of $\Delta > 1$. Here we restrict our attention to $2 < \Delta < 4$, since there are some technical subtleties with extending our analysis to the region $1 < \Delta \leq 2$; however, it can be shown our results continue to hold throughout the region $1 < \Delta < 4$ [33]. We make no further assumptions about $V(\phi)$.

So far we have not discussed the introduction of a finite temperature T in the field theory. There are now two sources of conformal symmetry breaking in the field theory: the relevant deformation, and the non-zero temperature. On the gravity side, turning on a finite temperature translates to the appearance of a black hole in the dual geometry[4]. We are interested in describing 4D systems at finite temperature with translational invariance in the $(t, \vec{x})$ directions and $SO(3)$ invariance in the $\vec{x}$ directions. The most general metric ansatz in the gravity dual consistent with these symmetries is

$$ds^2 = a^2(-hdt^2 + d\vec{x}^2) + \frac{dr^2}{b^2 h} \,, \tag{3.6}$$

where a, b, and h are functions of the holographic coordinate r only, and $\phi = \phi(r)$. The function $h(r)$ is assumed to have a simple zero at $r = r_h$, where a black hole horizon occurs, while a, b and ϕ are all assumed to be regular at $r = r_h$.

The dictionary identifies the entropy density s and temperature T of the dual field theory with those the dual black hole. In terms of the ansatz above, s and T can be read off in the standard way:

$$s = \frac{2\pi}{\kappa_5^2} |a(r_h)|^3 \,, \qquad T = \frac{|a(r_h)b(r_h)h'(r_h)|}{4\pi} \,. \tag{3.7}$$

Single-scalar systems of the sort we are considering here have been previously explored in Refs. [34, 35], and it has been found that such systems have quite rich thermodynamics: the systems can undergo first- or second-order phase transitions, depending on the form of $V(\phi)$.

Let us now demonstrate that $v_s^2 \leq 1/3$ at high T in *all* theories of this class. The basic idea is that because at high temperatures the relevant deformation has

[4]We are focusing on the deconfined phase, for which the dual involves a black hole[16].

a small effect on the dual field theory, in the gravity dual the scalar field should be 'small' at high temperatures, so that $\phi(r) \ll 1$. By high T, we mean T much larger than all other energy scales in the system. Our derivation self-consistently shows that small $\phi_H \equiv \phi(r_h)$ corresponds to asymptotically high T, and at high T the gravity duals are sensitive *only* to the 'universal' part of $V(\phi)$ shown in Eq. 3.5. This universality can be traced simply to the fact that at high temperatures the scalar is small everywhere in the bulk.

Since the systems we are considering become approximately conformal in the UV, v_s^2 should certainly approach the conformal value of 1/3 as $T \to \infty$. The correction away from 1/3 must be a power law in Λ/T by dimensional analysis. However, the sign of the first non-zero correction as well as the power of Λ/T in the first correction in a high temperature expansion of v_s^2 is less obvious, and obtaining these is our goal in the rest of this section.

As $T \to \infty$, the temperature becomes the dominant source of conformal symmetry breaking, since the effects of the relevant deformation associated with the scale Λ become negligible. Thus we expect that the background geometry approximates an AdS-Schwarzschild black hole as $T \to \infty$. To determine the sign of the corrections to v_s^2, we solve the equations of motion resulting from Eq. 3.3 perturbatively around the AdS-Schwarzschild black hole solution. This is a high temperature expansion. The horizon value of ϕ, $\phi_H \ll 1$, can be viewed as the expansion parameter. By working to second order in ϕ_H, we obtained a closed-form expression for the backreaction of the scalar field on the geometry, which determines s and T from Eq. 3.7.

The perturbative expansion is somewhat messy, and we relegate a sketch of the derivation to the appendix; a fuller discussion of the derivation appears in Chapter 4. The result is simple to state:

$$\begin{aligned} v_s^2(\phi_H) &= 1/3 - C(\Delta)\phi_H^2 + \mathcal{O}(\phi_H^4), \text{ where} \\ C(\Delta) &= \frac{1}{576}(\Delta-4)^2\Delta\Big[16 + (\Delta-4)\Delta \\ &\times \int_1^\infty ds\, s\, {}_2F_1(2-\Delta/4; 1+\Delta/4; 2; 1-s)^2\Big] \\ &= \frac{1}{18\pi}(4-\Delta)(4-2\Delta)\tan(\pi\Delta/4)\ , \end{aligned} \tag{3.8}$$

where ${}_2F_1$ is a hypergeometric function, and the simplified form in the last line can be obtained by standard identities of Meijer-G functions. [5] Since $C(\Delta)$ is positive, $v_s^2 \leq 1/3$. This result was also obtained by different methods in Ref. [36].

We can rewrite our result in a more physical form in terms of T, since as follows from the derivation in the appendix and in Chapter 4,

$$\phi_H = (\pi T/\Lambda)^{\Delta-4}\frac{\Gamma(\Delta/4)^2}{\Gamma(\Delta/2-1)}. \tag{3.9}$$

This allows to write our result as

$$v_s^2(T) = 1/3 - \left[\frac{1}{9\pi}(4-\Delta)(2-\Delta)\tan(\pi\Delta/4)\,\frac{\Gamma(\Delta/4)^4}{\Gamma(\Delta/2-1)^2}\right]\left(\frac{\pi T}{\Lambda}\right)^{2(\Delta-4)} + \mathcal{O}\left[\left(\frac{\pi T}{\Lambda}\right)^{4(\Delta-4)}\right] \tag{3.10}$$

In Fig. 3.1, we show that the closed-form result for v_s^2 in Eqs. 3.10 matches a numerical solution for v_s^2 at large T.

Our expression for the speed of sound Eq. 3.10 holds *universally* at high temperatures in all non-conformal theories with gravity duals that can be viewed as

[5] We are grateful to Paul Hohler for showing us the way to do this.

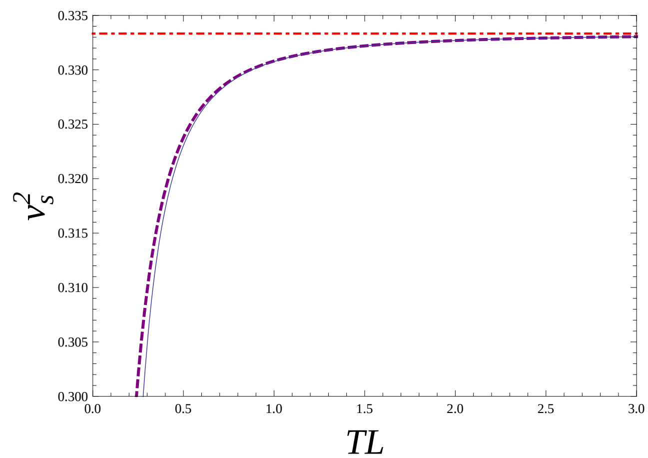

Figure 3.1: Plot [25] of the high-T approximation to v_s^2 in Eqs. 3.10 for $\Delta = 3$ (solid line) versus a numerical solution (dashed line) for v_s^2 found using the methods of Ref. [34]. The numerical solution is for $V(\phi) = -\frac{12}{L^2}\cosh(\frac{1}{2}\phi)$, which corresponds to $\Delta = 3$. The sound bound $v_s^2 = 1/3$ is shown as a horizontal dot-dashed line.

CFTs deformed by a relevant scalar operator. The result above makes clear that the *only* parameter of the dual gauge theory on which v_s^2 depends at high temperature is the scaling dimension Δ of the relevant operator deforming the CFT, apart from the obvious dependence on Λ/T. The speed of sound squared in this class of theories is universally bounded by 1/3 *from above*. We have thus proved that this class of strongly-coupled large N non-conformal gauge theories obeys a non-trivial bound on the speed of sound.

3.3 Comparison with other calculations of v_s^2

Given the apparent universality of our result for v_s^2, it is natural to wonder whether it may hold outside the class of theories that we considered here. In fact, it turns out all other calculations in the literature to date have found that $v_s^2 \leq 1/3$ at high temperature in 4D field theories.[6] To show the breadth of circumstances considered in the literature, we briefly survey some representative results on the speed of sound in non-conformal theories with gravity duals, with an eye to whether the sound bound $v_s^2 \leq 1/3$ is satisfied at high temperatures.

First, it is interesting to note that in all known cases, single-scalar models with potentials that do *not* fall into the class that we considered above also obey the sound bound at high temperatures[38]. The scalar potential in these theories is intended to represent a marginally relevant deformation of the dual field theory; this resembles the way classical conformal invariance is broken in massless QCD due to the running of the coupling.

The speed of sound of sound has also been calculated in the finite-temperature version of the Sakai-Sugimoto model[39, 40], which is dual to a strongly coupled 4D field theory with $N_f \ll N_c$ fundamental matter fields. It was found that $v_s^2 = 1/5$, which is less than 1/3 [41]. It has also been shown that $v_s^2 \leq 1/3$ at high T in the D3/D7 system [42], which is dual to another strongly-coupled large N_c 4D field theory with fundamental matter.

The speed of sound was also calculated in a 4D cascading gauge theory [43, 44]. This strongly coupled large N gauge theory has a rather exotic gauge group structure, since the effective rank of the gauge depends on the temperature. At high

[6] An apparent exception to this was found in Ref. [37], but the violation of the sound bound occurs in a somewhat exotic setup where the dual field theory is on an energetically unstable branch. On the energetically stable branch, the system obeys the bound.

T, the effective gauge group is $SU(K) \times SU(K+P)$, where $P \ll K$, and it was found that

$$v_s^2 = \frac{1}{3} - \frac{4}{9}\frac{P^2}{K} + \mathcal{O}\left(\frac{P^4}{K^2}\right) \tag{3.11}$$

As a final example, v_s^2 has also been calculated in $\mathcal{N} = 2^*$ gauge theory [45, 12]. This theory is $4D$ $\mathcal{N} = 4$ super Yang-Mills theory at finite T, which is deformed by turning on small masses for the bosons and fermions in two of the $\mathcal{N} = 1$ chiral multiplets that are part of the $\mathcal{N} = 4$ gauge theory. To leading order in $m_f/T, m_b/T$, it was found that

$$v_s^2 = \frac{1}{3} - \frac{2[\Gamma(3/4)]^4 m_f^2}{9\pi^4 T^2} + \mathcal{O}(m_b, m_f/T^4)\ , \tag{3.12}$$

which clearly satisfies the sound bound. We note that the 5D effective action describing this theory actually falls into the class of actions we considered above, except that it has two scalar fields, corresponding to bosonic and fermionic mass deformations. At high T, the scalar field corresponding to the fermionic deformations dominates over the scalar field associated with the bosonic deformation, and one obtains our single-scalar system with $\Delta = 3$. However, connecting the parameter Λ with m_f requires working with the full string dual.

3.4 Sound bound conjecture

As we have seen above, it seems that $v_s^2 \leq 1/3$ at high temperatures in a much broader class of systems than just the single-scalar models for which we were able to show a universal bound on the speed of sound at high temperatures. It is not clear just how broad the class of theories that satisfy the bound is. However, motivated by the examples in the previous section, we conjecture that $v_s^2 \leq 1/3$ at high temperatures in all $4D$ theories with gravity duals, at least for energetically stable systems [37] at zero chemical potential.

It is highly plausible that the sound bound holds more broadly than the context in which we have proved it. In all examples in the literature where v_s^2 has been calculated away from the high T limit, v_s^2 stays below 1/3 [46, 47]. Thus it seems likely that the sound bound may continue to hold away from the high T limit in a broad class of theories with gravity duals.

It is less obvious what class of systems the bound may apply to once one looks at theories that do *not* have gravity duals, especially away from the high-temperature limit. It is important to note that the bound cannot apply to *all* field theories in nature. For instance, in QCD at $T = 0$ and non-zero isospin chemical potential μ_I, v_s^2 can be accurately calculated in chiral perturbation theory, so long as one in the regime $\mu_I, m_\pi \ll \Lambda$, where Λ is a typical hadronic scale (such as the mass of the ρ meson), and m_π is the pion mass [48]. In the phase where $\mu_I > m_\pi$,

$$\frac{p}{\epsilon} = \frac{\mu_I^2 - m_\pi^2}{\mu_I^2 + 3m_\pi^2} , \tag{3.13}$$

so $v_s^2 \to 1$ as $m_\pi^2 \to 0$. Similar behavior for v_s^2 was seen earlier in ad hoc models, for instance in the Walecka model [49, 50].

These counterexamples show that the sound bound $v_s^2 \leq 1/3$ cannot apply generically to all systems with chemical potentials. It is not yet known whether the bound can be violated in systems with gravity duals at finite chemical potential. The closest investigation along these lines, Ref. [51], showed that $v_s^2 \leq 1/3$ in a

$D3/D7$ holographic model at finite isospin chemical potential in a regime where $\mu_I < m_\pi$. This is a different regime than the one considered in the QCD example above. It would be certainly be interesting to investigate the behavior of v_s^2 when $\mu_I > m_\pi$ in systems with gravity duals.

One can make a number of heuristic field-theoretic arguments that suggest that $v_s^2 \leq 1/3$ for theories that are weakly-coupled at high temperatures. For example [52], one can write the entropy in terms of the number of effective degrees of freedom $N_{\text{eff}}(T)$, so that the entropy density takes the form

$$s = \frac{16\pi^2}{45} N_{\text{eff}}(T) T^3 \tag{3.14}$$

Then one can easily find an expression for the speed of sound:

$$v_s^2 = \frac{1}{3 + T N'_{\text{eff}}(T)/N_{\text{eff}}(T)} \tag{3.15}$$

So long as one is working with an asymptotically free theory, which is well-defined in the UV, one expects that at high temperatures $N'_{\text{eff}}(T) \geq 0$, with $N'_{\text{eff}}(T) \to 0$ at large T. Then one sees that the speed of sound will approach 1/3 from below. In fact, this is exactly what happens in massless QCD at zero chemical potential [53, 54]. The conformal symmetry of the classical theory is broken by the running of the coupling, and one finds that

$$v_s^2 = \frac{1}{3} + \frac{5}{36\pi}\beta\left[\lambda(T)\right] + \mathcal{O}\left[\lambda(T)^4\right] \tag{3.16}$$

where $\beta[\lambda(T)] < 0$ is the β function of QCD given in Eq. 1.7, and $\lambda(T)$ is the 't Hooft coupling evaluated at the scale T.

The deviation from the conformal value is proportional to the beta function, so that one might wonder about the speed of sound in theories like quantum electrodynamics (QED), which are not asymptotically free and have a *positive* beta function. Such theories have a Landau pole in the UV, where the coupling becomes strong, rather than becoming weak. It is thought that this is a signal that these theories are not well-defined in the UV. However, the Landau pole occurs at very high energies, so it is legitimate to ask what happens to v_s^2 in, for instance, QED, when $T_l \gg T \gg m_e$, where T_l is the temperature associated with the Landau pole energy scale, and m_e is the electron mass. In this temperature regime, the theory is weakly coupled, and the conformal invariance of the classical theory is violated quantum mechanically by the running of the coupling. One can show that in QED in the regime $T_l \gg T \gg m_e$,[53]

$$v_s^2 = \frac{1}{3} + \frac{1}{162\pi}\alpha_{EM}^2 + \mathcal{O}\left[\alpha_{EM}(T)^3\right], \tag{3.17}$$

where $\alpha_{EM}(T) = e^2(T)/4\pi$. This is indeed larger than 1/3, violating the bound on the speed of sound. Note that where this formula is valid, v_s^2 is not *approaching* 1/3, but rather moving away from it, so that this behavior is qualitatively different what was seen in QCD at zero chemical potential, as well as in theories with gravity duals. Nevertheless, at weak coupling, it appears that the sign of $1/3 - v_s^2$ at high temperatures is controlled by the sign of the beta function of the theory.

Given the behavior of v_s^2 in weakly-coupled QED at high T, it is natural to ask whether anything can be said about the speed of sound in non-asymptotically-free *strongly-coupled* theories with gravity duals. Given what happens to the speed of sound at weak coupling, one might expect that $v_s^2 > 1/3$ at high T in such theories.

Calculations using the gauge/gravity duality for non-asymptotically-free theories are very technically demanding, and to our knowledge there is only one example of such a calculation [55]. Ref. [55] considered the D3/D7 system, but took into account the backreaction of the flavor fields. The field theory dual consists of $\mathcal{N} = 4$ SYM with N_f fundamental fields. When $N_f = 0$, the theory is conformal. Once one moves away from the $N_f \ll N_c$ limit, and takes into account the effects of the fundamental fields on the running of the coupling, the conformal symmetry breaks, and the theory gets a positive β function.

In a controlled regime (well away from the Landau pole), Ref. [55] used the gravity dual of the D3/D7 with 'unquenched' flavors that at strong 't Hooft coupling λ to show that the speed of sound is given by

$$v_s^2 = \frac{1}{3} - \frac{1}{18}\epsilon_h(T)^2 + \mathcal{O}\left[\epsilon(T)^3\right] \tag{3.18}$$

where $\epsilon(T) \ll 1$ is

$$\epsilon(T) \sim \lambda(T)\frac{N_f}{N_c}, \tag{3.19}$$

and $\lambda(T) \gg 1$ is the 't Hooft coupling of the theory at T.

Note that in this strongly-coupled theory with a *positive* beta functions, v_s^2 approaches 1/3 *from below*, in stark contrast to QED. Moreover, since the D3/D7 theory has a positive beta function at both strong and weak coupling, we see that something rather interesting must happen to v_s^2 as we increase λ while holding N_f/N_c and T fixed and large (but small compared to the break-down scale of the theory). At small λ, v_s^2 is greater than 1/3, while at large λ, v_s^2 is less than 1/3, so as a function λ, v_s^2 at fixed T and N_f/N_c must interpolate between the two limiting behaviors. If the system has no phase transitions as a function of λ, as seems reasonable to assume, then there must exist some intermediate $\lambda = \lambda_*$ for which $v_s^2(\lambda_*) = 1/3$.

3.5 Conclusions

In this chapter we have shown that $v_s^2 \leq 1/3$ at high T in theories with single-scalar gravity duals, corresponding to CFTs deformed by one relevant scalar operator. As we will see in the next chapter, our techniques can be extended to cover systems with multiple scalar deformations, and can be used to compute transport coefficients in addition to thermodynamic observables such as v_s. We have conjectured that the behavior of the sound bound may be universal in more than just the class of theories we considered.

3.6 Appendix: Computation of the dual geometry at high T

In this appendix we briefly describe the methods we used to compute v_s. A fuller description is given in the next chapter. Consider the action Eq. 3.3 with $V(\phi)$ as in Eq. 3.5 and the metric ansatz Eq. 3.6. There are three independent equations of motion: the scalar equation of motion and the tt and rr components of Einstein's equations. Since we are dealing with a theory of gravity, one must deal with diffeomorphism invariance. We find the gauge choice $a = r$ to be convenient. In this gauge, $r \to \infty$ is the UV boundary where the field theory lives.

When ϕ vanishes, the general solution to the equations of motion subject to the ansatz we have chosen is AdS_5-Schwarzschild. When ϕ is everywhere small, the scalar equation of motion in the black hole background gives it the profile

$$\phi(r) = \phi_0\, {}_2F_1(1 - \Delta/4; \Delta/4; 1; 1 - r^4/r_h^4) \;, \tag{3.20}$$

where ϕ_0 is an integration constant that measures the smallness of the scalar and will be related to the temperature below. Eq. (4.13) is one of the two solutions to the linearized scalar equation of motion in AdS_5-Schwarzschild. We have discarded a second solution that is not regular at $r = r_h$, the location of the black hole horizon.

Suppose we fix the entropy density s of AdS_5-Schwarzschild (when $\phi = 0$) at some large s_0 in the high-temperature regime. When the $\phi = 0$, one can read off the temperature in terms of s_0 from Eq. 3.7. We then need to find the temperature that corresponds to s_0 when ϕ is turned on, and the geometry is no longer AdS_5-Schwarzschild. To do this, we perform a perturbation expansion of the metric and the scalar in powers of $\phi_0 \ll 1$. Working up to $\mathcal{O}(\phi_0^n)$ is sufficient for computing corrections to v_s^2 up to $\mathcal{O}(\phi_0^n)$. The metric backreacts on the scalar at odd orders in ϕ_0, as was seen for $\mathcal{O}(\phi_0)$ above, and the scalar backreacts on the metric at even orders in ϕ_0.

Turning on a profile for ϕ corresponds to deforming the dual CFT lagrangian by the addition of the integral of $\Lambda^{4-\Delta}\mathcal{O}_\phi$ to the CFT action. On the gravity side, Λ is a new energy scale that also appears in the leading behavior of the scalar at the boundary[15]:

$$\phi(r \to \infty) \approx (\Lambda L)^{4-\Delta} r^{\Delta-4}. \tag{3.21}$$

We must keep Λ fixed in terms of L when computing equations of state, since we do not want to be varying the size of the deformation while doing the expansion. Thus we set $\Lambda L = 1$. Comparing the asymptotic form of Eq. (4.13) as $r \to \infty$ with Eq. (4.14) then gives a relationship connecting r_h and ϕ_0:

$$r_h^{\Delta-4}\Gamma(\Delta/4)^2 = \phi_0\Gamma(\Delta/2-1). \tag{3.22}$$

r_h can in turn be connected to the temperature via Eq. 3.7. Small ϕ_0 thus necessarily corresponds to large r_h, which means that the exact single-scalar background only approaches AdS_5-Schwarzschild in the high-temperature limit, as one would expect from general considerations. There are four boundary conditions that we impose at each order in ϕ_0: (1) maintain $\Lambda L = 1$ and hence also Eq. (3.22); (2) keep the horizon location $r = r_h$ so that s remains at s_0; (3) ensure that the solution is regular at $r = r_h$; and (4) preserve the boundary asymptotics $h \to 1$ and $b \to r/L$.

We do not present our results for b and h here; readers in search of a sleep aid may find useful the rather cumbersome and unilluminating explicit expressions for b and h, which are shown in the appendix of Chapter 4. However, we note that $\phi_H = \phi_0$ up to $\mathcal{O}(\phi_0^2)$. Once armed with the high T expansion for the geometry, it is easy to compute v_s^2 to $\mathcal{O}(\phi_H^2)$ by plugging the $\mathcal{O}(\phi_0^2)$ results for $b(r)$ and $h(r)$ into Eq. (3.7), using Eq. (3.22) to eliminate r_h, and finally using Eq. (4.19).

Chapter 4

Temperature Dependence Universality

In this chapter we show that the temperature dependence of transport coefficients becomes universal at high temperatures in a wide class of theories with gravity duals. The chapter is based on work done with Abhinav Nellore, and the results have been previously published in Ref. [56]. The chapter follows the presentation of Ref. [56], and the organization is as follows. To set the context, we start with a description of previous results on universality of transport coefficients, and then give a preview of our results. After a brief reminder of the class of theories we consider in this dissertation, we describe the high temperature expansion for the geometry of the gravity dual theories in detail. This is followed by calculations of the temperature dependence of the bulk viscosity, charge density diffusion coefficient, and the DC conductivity in the high temperature limit. It then follows that the temperature dependences of all of the transport coefficients calculated are shown to be identical in the hot strongly-coupled with gravity duals, which is our main result.

4.1 Introduction

In the previous chapter, we used the gauge/gravity duality to investigate the behavior of the speed of sound in strongly-coupled theories with gravity duals. The gauge/gravity duality is the only known theoretical tool that can give analytical insights into the behavior of strongly-coupled 4D gauge theories. Unfortunately, there are no known gravity duals for the gauge theories that are currently used to describe nature. The theories that have gravity duals are quite different from theories of direct phenomenological interest. As a result, at present the duality can only be reliably used as a source of calculable toy models that allow us to probe the qualitative features of strongly-coupled field theories.

Thus it makes sense to the search for universal properties of strongly-coupled theories with gravity duals. In Chapter 3, we used the duality to look at the high temperature limit of a wide class of strongly-coupled theories with broken conformal symmetry. The speed of sound in general depends sensitively on the details of a theory. However, at high temperatures, the behavior simplifies, and we found that the speed of sound always obeys the bound $v_s^2 \leq 1/3$. Our calculation, together with other evidence from the literature, suggests the conjecture that the sound bound may be a *universal* property of theories with gravity duals. In showing the existence of the sound bound, we saw in Chapter 3 that the temperature dependence of v_s^2 at high T turns out to depend only on the scaling dimension of the relevant operator perturbing the dual CFT, and not on any other details of the deformation. This turns out to foreshadow the results of the present chapter, as we will see shortly.

Given the apparent emergence of universality at high T for v_s^2, one may wonder whether something similar may happen for the hydrodynamic transport coefficients. The speed of sound v_s^2 is a thermodynamic observable, since it can be calculated from $v_s^2 = dp/d\epsilon$. In the gravity dual, v_s^2 can be extracted directly from the geometry of the dual, since the metric encodes the equilibrium pressure and energy density of the dual field theory. Transport coefficients are somewhat different, since they encode non-equilibrium fluid properties, and in general are not read off directly from the dual metric. Instead, as was mentioned in Chapter 1 and Chapter 2, to calculate transport coefficients using Kubo formulas, one must calculate the real-time correlation functions of the relevant conserved currents. In the dual, this reduces to solving the equations of motion for the associated bulk fields in a given geometrical background. Since the relevant bulk equations of motion are different for different transport coefficients, it is not *a priori* obvious that any of the universal features of speed of sound would have an analogue for transport coefficients.

The most famous example of a universal quantity in theories with gravity duals is the ratio of shear viscosity to the entropy density η/s, which takes the universal value $\eta/s = 1/4\pi$ in all such theories. In particular, the value of η/s does not depend on the temperature, and it is not affected by the breaking of conformal symmetry by relevant operators. However, other transport coefficients do depend on the detailed structure of the field theories with gravity duals. For instance, the ratio of the bulk viscosity to the entropy density ζ/s is known to be zero in conformal field theories, and becomes non-zero once conformal symmetry is broken by relevant deformations, when it becomes temperature-dependent.

In this chapter, we examine the behavior of several transport coefficients in the same class of non-conformal field theories with gravity duals. The class of field theories we consider consists of large N strongly-coupled CFTs deformed by n relevant scalar gauge-invariant single-trace operators $\mathcal{O}_i$ with scaling dimensions

$\Delta_i < 4$, which introduce the energy scales Λ_i:

$$S_{CFT} \to S_{CFT} + \int d^4x \sum_{i=1}^{n} \left(\Lambda_i^{4-\Delta_i} \mathcal{O}_i\right) \tag{4.1}$$

In this chapter, it is shown that at high T, the temperature dependence of all of the transport coefficients we studied becomes universal, and the power law behavior is identical to the temperature dependence of v_s^2. We study the bulk viscosity ζ, as well as the charge diffusion coefficient D, the charge susceptibility Ξ, and DC charge conductivity σ. We review the calculation of v_s^2, and extend it to treat multiple scalar deformations of the CFT. To discuss the temperature dependence, it is convenient to work with the appropriately normalized versions of these quantities, so that the normalized coefficients approach dimensionless constant values as $T \to \infty$. For the normalized set of transport coefficients[1] $\xi_i \in \{v_s^2, \zeta/s, 2\pi T D, \sigma/\pi T, \Xi/(2\pi^2 T^2)\}$ we show that at high T

$$\xi_i(T) = \xi_i^{\mathrm{CFT}} + \mathcal{C}_{\xi_i}(\Delta) \left(\frac{\Lambda}{T}\right)^{-2(4-\Delta)} + \mathcal{O}\left(\frac{\Lambda}{T}\right)^{-4(4-\Delta)} \tag{4.2}$$

where $\xi_i^{\mathrm{CFT}} = \xi_i(T \to \infty)$, $\Delta \equiv \max(\Delta_i)$, and Λ is the energy scale associated with the operator with the scaling dimension Δ [2]. As mentioned above, we follow the discussion in Ref. [56]. As a result, for technical reasons, in the case of $\xi = \zeta/s$, our analysis is limited to field theories with only one scalar deformation; for all of the other observables we examined, the methods we use here allow the treatment of n deformations. However, using a different approach, Yarom [33] has subsequently

[1]Here we abuse language slightly since we include v_s^2 in $\{\xi_i\}$, and v_s^2 is not a transport coefficient.

[2]In the degenerate case where there is more than one operator with the maximal value of Δ, Λ must instead be defined as in Eq. 4.44.

shown in an elegant paper that our result for ζ/s holds for multiple deformations as well.

As in Chapter 3, we work with units where $\hbar = c = k_B = 1$, and focus on 4D field theories. Below, we begin in Sec. 4.2 by describing the class of gravitational theories dual to theories of the form of Eq. 4.1. We then describe the high temperature expansion for the geometry in Sec. 4.3, focusing on the single-scalar system for clarity. This is followed by the calculation of the transport coefficients at high T in Sec. 4.4. The extension of the results to the case of multiple scalar deformations is given in Sec. 4.5, which also summarizes our results.

4.2 Gravity dual for deformed CFTs

We wish to describe the gravity dual of a strongly-coupled large N CFT that is deformed by n relevant scalar gauge-invariant single-trace operators $\mathcal{O}_i$ with scaling dimensions $\Delta_i < 4$. By the usual AdS/CFT dictionary, each scalar operator with scaling dimension Δ_i is associated with a bulk scalar field ϕ_i, with a mass term given by

$$m_i^2 L^2 = \Delta_i(\Delta_i - 4). \tag{4.3}$$

Since we wish to consider relevant deformations of the CFT, we have the restriction $\Delta_i < 4$. The unitarity bound in the field theory (which is the BF bound in the gravity dual) is that $\Delta_i > 1$. For simplicity of presentation, we work with $2 < \Delta_i < 4$ in the rest of this chapter; we expect our results to extend straightforwardly to $1 < \Delta \leq 2$, and this was shown to indeed be the case in Ref. [33].

The action for the class of gravity duals we will consider takes the form

$$S = \frac{1}{2\kappa_5^2} \int d^5 x \sqrt{-g} \left[R + \frac{12}{L^2} - \frac{1}{2} \sum_{i=1}^{n} (\partial \phi_i)^2 - V(\phi_1, \ldots, \phi_n) \right] , \tag{4.4}$$

where $\kappa_5^2/8\pi$ is the Newton constant, ϕ is a real scalar field, and V is analytic in ϕ_i near $\phi_i = 0$. We take $V(\phi_i)$ to be symmetric around $\phi_i = 0$. Different choices of $V(\phi_i)$ correspond to different dual gauge theories. Up to the restrictions above, the potentials we consider are arbitrary. The mass terms in the potentials are identified by the AdS/CFT dictionary with the scaling dimensions of the dual operators as in Eq. 4.3.

It is also important to note that in general one may have a modified kinetic term in the gravity dual, so that

$$\frac{1}{2} \sum_{i=1}^{n} (\partial \phi_i)^2 \rightarrow \frac{1}{2} \sum_{i,j=1}^{n} f^{ij}(\phi_1, \ldots, \phi_i) \partial \phi_i \partial \phi_j \tag{4.5}$$

for some function f^{ij}. In writing the action in the form of Eq. 4.4, we are considering theories in which $f^{ij} \rightarrow 1$ quickly enough as $\phi_i \rightarrow 0$, since our high temperature expansion turns out to only probe the behavior of the action in this regime[3].

Our interest in studying the dual field theory at finite temperature forces us to choose the metric ansatz

$$ds^2 = a^2(-h dt^2 + d\vec{x}^2) + \frac{dr^2}{b^2 h} , \tag{4.6}$$

where a, b and h are smooth functions of the holographic coordinate r only, and $\phi_i = \phi_i(r)$. This is the most general ansatz consistent with the symmetries we

[3]From the discussion in Sec. 4.3, it is not hard to convince oneself that if $f^{ij}(\phi_i \rightarrow 0) = \delta^{ij} + \mathcal{O}(\phi_i)$, the calculation of the background is not affected at the order to which we work here, $\mathcal{O}(\phi_i^2)$.

expect in the dual field theory. We assume that h has a simple zero at $r = r_h$, which is the location of the black hole horizon. The entropy density s and temperature T of the field theory are identified by the AdS/CFT dictionary with the entropy density and temperature of the black hole:

$$s = \frac{2\pi}{\kappa_5^2}|a(r_h)|^3\,, \qquad T = \frac{|a(r_h)b(r_h)h'(r_h)|}{4\pi}\,. \tag{4.7}$$

4.3 High-temperature expansion

In this section, we discuss the details of the high-temperature expansion for the backgrounds and scalar field profiles for gravity duals of the class of deformed CFTs discussed here and in Chapter 3. To streamline the presentation, we focus on the case of a single deformation in this section; the extension to multiple deformations is straightforward and is presented in Sec. 4.5. Ref. [45] developed a similar expansion for the study of $\mathcal{N} = 2^*$ SYM theory.

We choose to work in a gauge where $a = r$, so that the AdS_5 boundary is at $r = \infty$. There are three independent equations of motion that follow from Eq. 4.4 (with $n = 1$) and the ansatz Eq. 4.6:

$$0 = \frac{2V(\phi)}{b^2} + \frac{6(4h + rh')}{r^2} - h(\phi')^2 \tag{4.8}$$

$$0 = 6rb' + b(-6 + r^2(\phi')^2) \tag{4.9}$$

$$0 = -\frac{V'(\phi)}{b^2} + h'\phi' + h\left[\left(\frac{4}{r} + \frac{b'}{b}\right)\phi' + \phi''\right]. \tag{4.10}$$

The first two equations come from a convenient combination of the tt and rr components of the Einstein equations, while the last one is the scalar equation of motion.

To set up the high-temperature expansion, note that when $T \gg \Lambda$, the influence of the relevant operator $\mathcal{O}$ on the physics of the dual becomes negligible. This means that on the gravity side, the profile of the scalar field must vanish as $T \to \infty$. When $\phi(r) = 0$, the solution to Eq. 4.8 is just AdS_5-Schwarzschild spacetime, with metric coefficients given by

$$b = \frac{r}{L}, \qquad h = 1 - \left(\frac{r_0}{r}\right)^4, \tag{4.11}$$

where r_0 is the location of the black hole horizon. Using Eq. 3.7, we can compute the entropy density and temperature, obtaining

$$s = \frac{2\pi}{\kappa_5^2} r_0^3, \qquad T = \frac{r_0}{\pi L}. \tag{4.12}$$

From these expressions it is clear that $s \sim T^3$, as is appropriate for a CFT, and large r_0 corresponds to large entropies and temperatures.

The solution in Eq. 4.11 together with $\phi(r) = 0$ give the background and scalar profile at the 0^{th} order of the high-T expansion. To move to the first order in the expansion, suppose that $\phi(r)$ is everywhere small. Then it is consistent to compute the profile of $\phi(r)$ in the background of Eq. 4.11, with the result that

$$\phi(r) = \phi_0 \, {}_2F_1(1 - \Delta/4, \Delta/4; 1; 1 - r^4/r_h^4), \tag{4.13}$$

where ϕ_0 is an integration constant that characterizes the smallness of the scalar, and will be related to Λ/T below. In obtaining the expression above, we have discarded a solution to the equations of motion that diverges logarithmically as $r \to r_0$. Note that the solution above is only sensitive to the mass term in the scalar potential. Since the AdS/CFT dictionary relates the mass of the scalar to the scaling dimension of the dual relevant operator, the profile above is in universal in the sense that it depends on *only* the scaling dimension of the deformation Δ. This turns out to be crucial to the emergence of high-temperature universality that we describe later in this chapter.

Our aim is to compute the properties of the dual field theory at high temperature, when the scalar ϕ is small. Since the thermodynamic properties and transport coefficients of the field theory are encoded in the geometry of the gravity dual, to get the leading finite-temperature corrections to these observables we must work to the leading order in the high-temperature expansion at which the geometry is changed from its $T \to \infty$ form. To this end, we develop a perturbation expansion of the metric of the dual and the scalar profile in powers of ϕ_0.

Once the scalar field is non-zero, the dual is sensitive to the fact that we have added the term $\Lambda^{4-\Delta}\mathcal{O}$ to the field theory Lagrangian density. The energy scale Λ appears in the gravity dual in the coefficient of the $r^{\Delta-4}$ term of $\phi(r)$ as $r \to \infty$ [57]:

$$\phi(r) = (\Lambda L)^{4-\Delta} r^{\Delta-4} + \dots . \tag{4.14}$$

The size of ΛL should not change as we vary the temperature. Thus to choose the size of Λ in units $1/L$, we fix $\Lambda L = 1$. To relate Λ with ϕ_0, we expand Eq. 4.13 about $r = \infty$, obtaining

$$\phi(r) = \phi_0 \left(\frac{r}{r_h}\right)^{\Delta-4} \frac{\Gamma(\Delta/2-1)}{\Gamma(\Delta/4)^2} + \dots . \tag{4.15}$$

A comparison of Eq. 4.14 and Eq. 4.15 now yields a relationship connecting r_0 and ϕ_0:

$$r_h^{\Delta-4} = \phi_0 \frac{\Gamma(\Delta/2-1)}{\Gamma(\Delta/4)^2}. \tag{4.16}$$

Thus ϕ_0 is small when r_h is large; since T is large compared to Λ when r_h is large, this confirms that we are indeed performing an expansion valid at large s and T.

Since the differential order of the system of equations in Eq. 4.8 is four, we must impose four boundary conditions at each order in the ϕ_0 expansion. The first three boundary conditions are simple to state:

- preserve $\Lambda L = 1$ and consequently Eq. (4.16),
- maintain the boundary asymptotic $b \to r/L$, which has the consequence of keeping $h \to 1$, and
- ensure that $\phi(r)$ remains regular at $r = r_0$.

The fourth boundary condition is somewhat more subtle, as it encodes the physical meaning of ϕ_0. To see how the last boundary condition works, suppose we fix a temperature-dependent observable Ω of the undeformed CFT (so that $\phi(r) = 0$ in the gravity dual) at some given value Ω_0. Ω could be the temperature, the entropy, a transport coefficient, or the energy density. When the relevant deformation is turned on, and $\phi(r)$ becomes non-zero, our high-temperature expansion should give the form of the background in the gravity dual when $\Omega = \Omega_0$. The fourth boundary condition simply enforces that $\Omega = \Omega_0$ at every order in the ϕ_0 expansion.

For example, if we take $\Omega = T$, the fourth boundary condition requires that the temperature $T = |a(r_h)b(r_h)h'(r_h)|/4\pi$ remain at $r_0/\pi L$, its value in AdS_5-Schwarzschild. Eliminating r_0 in favor of T in Eq. 4.16 gives

$$\phi_0 = \frac{\Gamma(\Delta/4)^2}{\Gamma(\Delta/2-1)} \left(\pi L T\right)^{\Delta-4},$$

so the ϕ_0 expansion is essentially in the smallness of $1/LT$ when $\Omega = T$.

We find that the most convenient choice for explicit calculations is to take $\Omega = s$. Since $s = \frac{2\pi}{\kappa_5^2} r_0^3$, the fourth boundary condition above becomes the demand that the horizon of the black hole remain at $r_h = r_0$ at each order in the ϕ_0 expansion. From Eq. 4.12 and Eq. 4.16, we see that this expansion is in powers of

$$\phi_0 = \frac{\Gamma(\Delta/4)^2}{\Gamma(\Delta/2-1)} \left(\frac{s\kappa_5^2}{2\pi}\right)^{(\Delta-4)/3} . \tag{4.17}$$

Thus the expansion we pursue is essentially in the smallness of $1/s\kappa_5^2$; however, since large s corresponds to large T, we continue to refer to the expansion as a high T expansion.

Given this expansion scheme, it is possible to compute the geometry to any finite order in ϕ_0. At every odd order in the expansion, the geometry backreacts on the scalar profile; at every even order, the scalar field backreacts on the geometry. The transport coefficients are then determined from the appropriate solutions to the equations of motion of the relevant bulk field in the geometry derived using the high T expansion, as described in Chapter 2. Since the background is first corrected at $\mathcal{O}(\phi_0^2)$, we must work to $\mathcal{O}(\phi_0^2)$ to determine the leading deviations from the conformal values of the transport coefficients. To order $\mathcal{O}(\phi_0^2)$, the geometry is only sensitive to the mass term in the scalar potential, and hence is only sensitive to the scaling dimension Δ of the dual relevant operator. It turns out that the transport coefficients and the speed of sound squared can be written as

$$\xi_i = \xi_i^{CFT} + C_{\xi_i}(\Delta)\phi_0^2 + \mathcal{O}(\phi_0^4) . \tag{4.18}$$

when ϕ_0 is small, which corresponds to high entropy and temperature. This is the reason for the universality of the temperature dependence of the normalized transport coefficients universally at high temperatures. We have obtained closed-form solutions for $\phi(r)$, $b(r)$, and $h(r)$ to $\mathcal{O}(\phi_0^2)$, but the expressions are cumbersome and not especially illuminating; they can be found in the Appendix of this chapter.

4.4 Transport coefficients

4.4.1 Speed of sound

Let us briefly review the calculation of the speed of sound v_s from Chapter 3[25]. One can compute v_s^2 by using a Kubo formula; this comes down to examining the poles in the appropriate retarded Green's functions of the dual field theory (making use of Eq. 1.27) by relating them to the behavior of metric perturbations in the gravity dual. This method is rather involved, but has the virtue of also yielding the bulk viscosity. An easier method to obtain v_s^2 is to simply extract it from the equation of state of the dual field theory[58, 45, 42, 43, 44, 47]. Here, we describe the latter approach in the context of the high-temperature expansion presented in the previous section.

To find the speed of sound, we use the relation $v_s^2 = dp/d\epsilon$, where p is the pressure of a system and ϵ is its energy density, both measured in the local rest frame of the fluid, which is assumed to be in thermodynamic equilibrium. We are considering systems at zero chemical potential, and in this case one can get a formula for v_s^2 that involves just s and T, which are quantities that are particularly easy to read off from the geometry:

$$v_s^2 = \frac{d \log T}{d \log s} \ . \tag{4.19}$$

To determine v_s^2, we perform the high-temperature expansion to order $\mathcal{O}(\phi_0^2)$; at this order, the geometry receives its first non-trivial corrections. Now we evaluate Eq. 3.7, and applying Eq. 4.19, we find that

$$\begin{aligned} v_s^2(\phi_0) &= 1/3 - C_{v_s^2}(\Delta)\phi_0^2 + \mathcal{O}(\phi_0^3), \text{ where} \\ C_{v_s^2}(\Delta) &= \frac{1}{576}(\Delta-4)^2\Delta\Big[16 + (\Delta-4)\Delta \\ &\times \int_1^\infty ds\, s\, {}_2F_1(2-\Delta/4, 1+\Delta/4; 2; 1-s)^2\Big] \\ &= \frac{1}{9\pi}(4-\Delta)(2-\Delta)\tan(\pi\Delta/4) \ , \end{aligned} \tag{4.20}$$

where the simplified form in the last line can be found by making use of standard identities of Meijer G-functions.

This result has two important consequences. First, since $C_{v_s^2}(\Delta)$ is positive for $2 < \Delta < 4$, and in fact is also justifiable [33] and positive when $1 < \Delta \leq 2$, at high temperatures the speed of sound is always bounded from above by $1/3$ in this class of theories [25, 36]. Second, to the order to which we are working, it can be shown that

$$\phi_0 = \left(\frac{\Lambda}{\pi T}\right)^{4-\Delta} \frac{\Gamma(\Delta/4)^2}{\Gamma(\Delta/2-1)} \ . \tag{4.21}$$

This means that the temperature dependence of $1/3 - v_s^2$ takes the simple form $(1/3 - v_s^2) \sim (\Lambda/\pi T)^{2(4-\Delta)}$ as $T \to \infty$. Below, we will see that this result is *universal*, in the sense that this temperature dependence is shared by all of the transport coefficients we calculate. Of course, it is already clear that this result depends *only* on the 'universal' part of the scalar potential, which carries information only about the scaling dimension of the deformation operator; no other details about the information or the dual field theory affect v_s^2 at this order.

4.4.2 Conductivity

Let us now move on to investigate the behavior of transport coefficients. The easiest place to start is with the calculations of the transport coefficients associated with the presence of conserved charges in the dual field theory. Thus, suppose that the field theory has a $U(1)$ global symmetry, so that there is an associated conserved current J_μ, and a corresponding conserved charge. One can now consider some new transport coefficients associated with the conserved charges, for instance the charge diffusion coefficient D, the charge susceptibility Ξ, and the DC conductivity σ. In this section, we compute σ at high temperatures.

The AdS/CFT dictionary associates a bulk $U(1)$ gauge field A_M $(M = \mu, r)$ with the conserved current in the field theory. To calculate correlation functions of J_μ using the duality, we must add the term

$$-\int d^5x\,\sqrt{-g}\,\frac{1}{4g_5^2}F_{MN}F^{MN} \tag{4.22}$$

to the gravity dual action Eq. 4.4; here g_5^2 is the $5D$ gauge coupling and F_{MN} is the field strength. In terms of the parameters of the dual field theory, $g_5^2/L \sim N_c^{-2}$ [4]. The

[4] One can give a precise relation between g_5 and the parameters of the gauge theory when one works with a full string dual for a specific field theory. Since we are looking at class of 5D effective action that is dual to a class of gauge theories, the proportionality relation given in the text is the best we can do for general $V(\phi_i)$.

fact that transport coefficients can be calculated in linear response theory justifies treating the new term in the action we wrote above in the probe limit, where the bulk $U(1)$ gauge field does *not* backreact on the metric, in contrast to the scalar field ϕ. The Kubo formula for the DC conductivity σ is [59]

$$\sigma\delta^{ij} = \lim_{\omega\to 0} \frac{G_R^{ij}(\omega, \mathbf{0})}{i\omega}, \tag{4.23}$$

where $G_R^{ij}(\omega, \mathbf{k})$ is the retarded two-point Green's function of $J_i(\omega, \mathbf{k})$ ($i = x, y, z$), and ω is the frequency. Note that the σ is rotationally invariant as defined; one can also consider systems with different conductivities in different directions, but we will not do so here. The Green's function above can be calculated via the gauge/gravity duality by solving the equations of motion for A_M and following the prescription for real-time correlation functions described in Chapter 2. To find the high-temperature behavior of σ, one must solve the equations of motion for A_M in the high-temperature background. There is, however, a shortcut. Ref. [59] gives an elegant proof that when one does such a calculation in the class of backgrounds we are working with (Eq. 4.6), the final result for σ can always be written directly in terms of the metric coefficients evaluated at the black hole horizon r_h:

$$\sigma = \frac{1}{g_5^2}\frac{b(r)}{a(r)^3}\sqrt{-g(r)}\Bigg|_{r=r_h} . \tag{4.24}$$

This means that we can immediately write down σ using the high-temperature metric coefficients given in the Appendix, together with Eq. 4.21 and Eq. 4.24. At

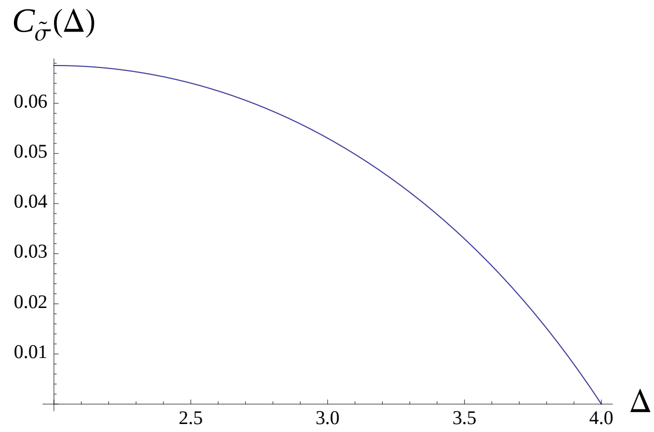

Figure 4.1: Plot of $C_{\tilde{\sigma}}$ versus Δ. [56]

high temperatures, we thus find that σ is given by

$$\begin{aligned} \tilde{\sigma}(T) &\equiv \frac{\sigma g_5^2}{\pi L T} = 1 - C_{\tilde{\sigma}}(\Delta)\phi_0^2\,, \text{ where} \\ C_{\tilde{\sigma}}(\Delta) &= \frac{1}{6\pi}(2-\Delta)\tan\left(\frac{\pi\Delta}{4}\right). \end{aligned} \tag{4.25}$$

A plot of $C_{\tilde{\sigma}}$ is shown in Fig. 4.1.

Like v_s^2, $\tilde{\sigma}$ is bounded from above in the high T limit, so that $\tilde{\sigma} < 1$ as $T \to \infty$. Also, $\tilde{\sigma}$ has an *identical* temperature dependence to v_s^2, in the sense that $1 - \tilde{\sigma} \sim (\Lambda/\pi T)^{2(4-\Delta)}$ in the high T limit, just as for v_s^2 we have $(1/3 - v_s^2) \sim (\Lambda/\pi T)^{2(4-\Delta)}$.

4.4.3 Diffusion coefficient and charge susceptibility

Once there is a conserved current in the field theory, in addition to the conductivity, one can also study the behavior of a small charge density perturbation. There are two transport coefficients associated with this: the charge diffusion coefficient D, and the charge susceptibility Ξ.

Let us start with D, which describes the relaxation to equilibrium of a small charge density perturbation in the field theory. Without loss of generality, suppose that the charge density perturbation is along the z direction. Then D appears as a pole in the two-point retarded Green's function of $J_z(\omega, k)$. (Recall the diffusion dispersion relation $\omega(k) = iDk^2$ from Chapter 2.) This two-point correlation function can be calculated using the duality, again by looking at the appropriate solutions to the equations of motion of the bulk gauge field in the curved background of the dual. Fortunately, Ref. [59] showed that the result of such a calculation again can be written in terms of the metric coefficients of the gravity dual background:

$$D = \sigma \int_{r_h}^{\infty} dr \frac{a(r)^2}{b(r)^2\sqrt{-g}} g_5^2 \ . \tag{4.26}$$

To find the high T behavior of D, we can simply read off D from the metric components of the high T geometry found in Sec. 4.3. To $\mathcal{O}(\phi_0^2)$,

$$\tilde{D}(T) \equiv 2\pi T D = 1 + C_{\tilde{D}}\phi_0^2 \,, \text{ where} \tag{4.27}$$

$$\begin{aligned} C_{\tilde{D}} = \frac{1}{96\pi}\Bigg(& 4\pi\Delta(\Delta-4) - 32(\Delta-2)\tan(\pi\Delta/4) + \\ & + \pi\Delta(\Delta-4)\int_1^{\infty} du\, u^5 {}_2F_1(2-\frac{\Delta}{4}, 1+\frac{\Delta}{4}; 2; 1-u^4)^2\Bigg). \end{aligned} \tag{4.28}$$

Unfortunately in this case we were unable to simplify $C_{\tilde{D}}$ further. We show a plot of $C_{\tilde{D}}$ in Fig. 4.2.

At high temperatures, we see that $2\pi TD \geq 1$. For us, the most important feature of our result for D is that the temperature dependence of $1 - \tilde{D}$ becomes the *same* as that of $1/3 - v_s^2$ and $1 - \tilde{D\sigma}$ as $T \to \infty$.

It turns out that in in the class of theories we work with here, it is known that D, Ξ and σ satisfy the Einstein relation $D\Xi = \sigma$ [59]. Using Eqs. 4.25 and Eq. 4.27,

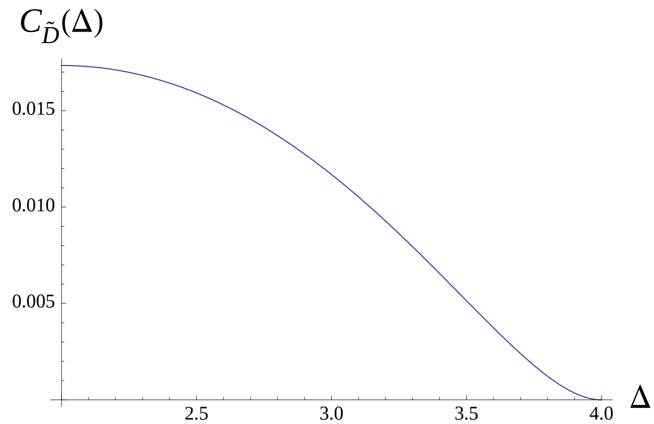

Figure 4.2: Plot of $C_{\tilde{D}}$ versus Δ. [56]

we find that

$$\tilde{\Xi} \equiv \frac{g_5^2 \Xi}{2L(\pi T)^2} = 1 - (C_{\tilde{\sigma}} + C_{\tilde{D}})\phi_0^2 \, . \tag{4.29}$$

Since $C_{\tilde{\sigma}}$ and $C_{\tilde{D}}$ are positive, $\tilde{\Xi} \leq 1$ at high temperatures. Obviously, $1 - \tilde{\Xi}$ again has the same temperature dependence as the other observables we have considered.

Kovtun and Ritz have proposed that there is a bound on $D = \sigma/\Xi$ in systems with gravity duals [60], given by

$$D \geq \frac{1}{2\pi T}. \tag{4.30}$$

Our results are consistent with this proposal.

4.4.4 Bulk viscosity

Finally, we turn to the bulk viscosity ζ. The bulk viscosity characterizes some aspects of the response of a non-ideal fluid to volume changes, as was discussed in Chapter 2. To compute ζ in field theory, one can use the Kubo formula Eq. 1.42. The Kubo formula relates ζ to the low-frequency behavior of an $SO(3)$-symmetric two-point retarded correlation function of the stress-energy tensor. Such correlation functions can be calculated using the gauge gravity duality by relating them to $SO(3)$-invariant time-dependent perturbations of the metric of the dual gravity theory [61, 62, 44, 63, 64, 43, 47, 12].

Calculating ζ using the duality is somewhat challenging as a practical matter, because the relevant metric perturbations mix with perturbations of other bulk fields, and one must solve a coupled set of a non-linear differential equations to find ζ. However, for field theories with single-scalar gravity duals, an elegant formalism was developed in Ref. [64, 65]. A key idea of the approach is to use a gauge where the bulk scalar ϕ is used as the holographic coordinate, so that $r = \phi$ in Eq. 4.6. The advantage of this gauge is that the metric and scalar perturbations decouple, which makes the calculation of ζ much easier in this gauge than in the gauge we use in the rest of this chapter.

Here we are interested in the high T behavior of the field theory. On the gravity side, perturbations around the $T \to \infty$ limit map to perturbations about AdS-Schwarzschild, for which $\phi = 0$ everywhere. Unfortunately, this makes the gauge choice $\phi = r$ inappropriate for computing the geometry at high temperatures. Thus to find a simple expression for the ζ, we use the $r = \phi$ gauge in the first part of this section to find a simple formula for ζ/s, and then switch back to the gauge we use in the rest of the paper (where $a(r) = r$) to make use of our results for the high T geometry.

As is shown in Ref. [64, 65], to compute ζ in the $r = \phi$ gauge one can consider the diagonal metric ansatz

$$\begin{aligned} ds^2 &= g_{00}dt^2 + g_{11}d\vec{x}^2 + g_{55}d\phi^2, \text{ where} \\ g_{00} &= -e^{2A}h(1+\frac{\lambda}{2}H_{00})^2 \ , \ g_{11} = e^{2A}(1+\frac{\lambda}{2}H_{11})^2 \\ g_{55} &= \frac{e^{2B}}{h}(1+\frac{\lambda}{2}H_{55})^2 \ . \end{aligned} \tag{4.31}$$

Here A, h, and B are functions of ϕ only, while H_{00}, H_{11}, and H_{55} are functions of t and ϕ, and λ is a formal expansion parameter. The functions H_{00}, H_{11} and H_{55} parameterize the $SO(3)$-invariant metric perturbations. When $\lambda = 0$, the metric perturbation is turned off, and the equations of motion of Eq. 4.4 with $n = 1$ determine the background with which we work. For us, this background is given by the high T geometry we computed above.

At first order in λ, the Einstein equations determine H_{00}, H_{11}, and H_{55}. By taking the ansatz $H_{11}(t,\phi) = e^{-i\omega t}h_{11}(\phi)$, the 11 component of the Einstein equations gives

$$\begin{aligned} h_{11}'' = \left(-\frac{1}{3A'} - 4A' + 3B' - \frac{h'}{h}\right) h_{11}' \\ + \left(-\frac{e^{-2A+2B}}{h^2}\omega^2 + \frac{h'}{6hA'} - \frac{h'B'}{h}\right) h_{11} \ , \end{aligned} \tag{4.32}$$

where the primes denote derivatives with respect to ϕ. By applying the recipe we discussed in Chapter 2 for computing retarded two-point functions and using the Kubo formula Eq. 1.42, Ref. [64, 65] showed that h_{11} is related to the bulk viscosity via

$$\frac{\zeta}{s} = \frac{1}{4\pi}h_{11}(\phi_H)^2\frac{V'(\phi_H)^2}{V(\phi_H)^2} \ , \tag{4.33}$$

where s is the entropy density and ϕ_H is the value of the scalar (which is the value of r as well in this gauge) at the horizon.

Having obtained an expression for ζ, we now do a change of coordinates in Eq. 4.32 (with $\omega = 0$) implementing the switch from the $r = \phi$ gauge back to the $a(r) = r$ gauge. Once this is done, we plug in the expressions for b, h, and ϕ up to second order in ϕ_0. It is convenient to define a new radial variable $u = r/r_h$. Then when we retain the leading term in ϕ_0 in the coefficients of h_{11}, dh_{11}/du, and d^2h_{11}/du^2, we find that

$$h''_{11} = \alpha(u,\Delta)h'_{11} + \beta(u,\Delta)h_{11} \ , \tag{4.34}$$

where the primes denote derivatives with respect to u, and $\alpha(u,\Delta)$ and $\beta(u,\Delta)$ are given in the Appendix. Crucially, $\alpha(u,\Delta)$ and $\beta(u,\Delta)$ do not depend on r_h, ϕ_0, or ϕ_H. Thus, Eq. 4.34 does not involve r_h, ϕ_0, or ϕ_H, which are the only parameters that depend on temperature.

This means that at high temperatures, $h_{11}(\phi = \phi_H) = h_{11}(u = 1)$ *only* depends on Δ. As a result, all of the temperature dependence of ζ/s lives in the $V'(\phi_H)^2/V(\phi_H)^2$ factor of Eq. 4.33. When one takes $r = r_h$ in Eq. 4.13, it is clear that $\phi_H = \phi_0$ up to $\mathcal{O}(\phi_0^2)$. Then it follows that

$$\frac{\zeta}{s} = C_{\zeta/s}(\Delta)\phi_0^2 \ , \tag{4.35}$$

where $C_{\zeta/s}(\Delta)$ is determined determined from the solution to Eq. 4.34.

From Eq. 4.35, it is immediately clear that the temperature dependence of ζ/s is $\zeta/s \sim (\Lambda/\pi T)^{2(4-\Delta)}$. Thus ζ/s has the same temperature dependence at high T as all of the other observables we examined. This shows that Eq. 4.2 holds, which is our main result in this chapter.

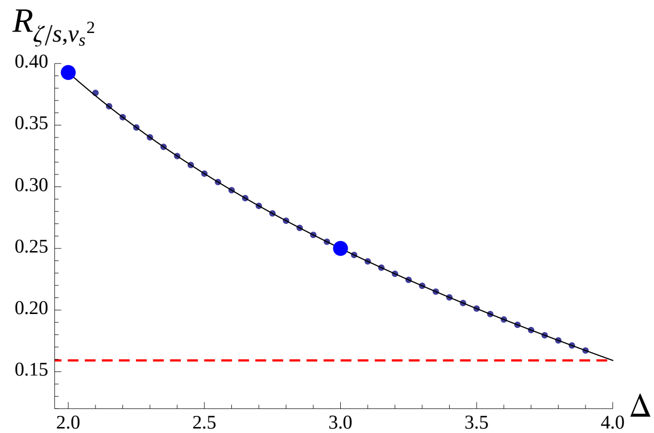

Figure 4.3: Plot of $R_{\zeta/s,v_s^2}(\Delta) = \frac{\partial(\zeta/s)/\partial \log T}{\partial(v_s^2)/\partial \log T}$ vs. Δ. [56] The small dots were obtained from a numerical solution of Eq. 4.34, which becomes unreliable near $\Delta = 2$. The solid curve represents a guess at an analytical form described in the text, and the two large dots are taken from the ζ/η results of Refs. [12, 61]. The dashed line is at $R_{\zeta/s,v_s^2} = 1/2\pi$, the bound suggested in Ref. [66].

For completeness, we note that we do not know of any obvious analytical solution to Eq. 4.34. To find ζ/s, we solved Eq. 4.34. In Fig. 4.3, we plot the quantity

$$R_{\zeta/s,v_s^2}(\Delta) \equiv \frac{\partial(\zeta/s)/\partial \log T}{\partial(v_s^2)/\partial \log T}. \tag{4.36}$$

which simply gives the leading deviation of ζ/s from its conformal value (0), divided by the deviation of v_s^2 from its conformal value (1/3).

The dots in Fig. 4.3 denote the results of the numerical solution. When this calculation was first described for publication in Ref. [56], it was observed that the numerical solution turns out to fit a simple analytical formula

$$C_{\zeta/s}(\Delta) = \frac{1}{9}(\Delta - 4)^2. \tag{4.37}$$

This analytic formula fits our numerics to great precision (except near $\Delta = 2$, where our numerics can no longer be trusted). For example, at $\Delta = 3$, the analytical formula predicts $C_{\zeta/s}(\Delta = 3) = 1/9$, and the numerical result is 0.1111111.

We also compared this conjectured analytic formula for ζ/s against the high-temperature calculations done for the $\mathcal{N} = 2^*$ theory in Refs. [41, 61]. $\mathcal{N} = 2^*$ theory is a deformation of $\mathcal{N} = 4$ SYM that is obtained when one turns on masses for the bosons and/or fermions in two of the $\mathcal{N} = 1$ chiral supermultiplets that are part of the $\mathcal{N} = 4$ gauge theory. In the notation we use here, these deformations correspond to looking at bulk scalars with $\Delta = 2$ or $\Delta = 3$. For $\Delta = 3$, corresponding to turning on masses for fermions, we find that $R_{\zeta/s, v_s^2}(\Delta = 3) = \pi$, while for $\Delta \to 2$, corresponding to turning on masses for the bosons, we get $R_{\zeta/s, v_s^2}(\Delta \to 2) = \pi^2/2$. These results precisely agree with the results of Ref. [61].

Given the remarkable agreement of Eq. 4.37 with our numerical solutions and with the results of Refs. [41, 61], we conjectured that Eq. 4.37 gives the correct analytic expression for $C_{\zeta/s}(\Delta)$. This conjecture was later proved by Yarom [33] using a different approach.

Finally, we note that Buchel [66] proposed the conjecture that

$$\frac{\zeta}{\eta} \geq 2\left(\frac{1}{3} - v_s^2\right) \tag{4.38}$$

in all theories with gravity duals. This translates to the statement that $R_{\zeta/s, v_s^2} \geq 1/2\pi$. As one can see from Fig. 4.3, the Buchel bound is always satisfied at high temperatures in the class of theories we consider for $2 < \Delta < 4$; the bound becomes saturated as $\Delta \to 4$.

4.5 Systems with multiple scalars

We have spent most of the previous few section discussing the class of CFTs that are deformed by one relevant operator. Here, we discuss the straightforward extension of our results to handle multiple relevant deformations. This extension is described for all of the observables we computed except ζ/s, since our treatment of ζ/s relies on a choice of gauge that makes the generalization difficult. It has since been shown by Yarom [33] that our results do indeed extend to the multiple deformation case for ζ/s, in the same way as for the other observables.

For clarity of exposition, we focus here on the case where we have two relevant deformations of the dual CFT. The generalization to $n > 2$ deformations will be obvious. When there are two relevant deformations of the CFT action, the action of the gravity dual has two bulk scalar fields:

$$S = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left[R + \frac{12}{L^2} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\chi)^2 - V(\phi,\chi)\right], \tag{4.39}$$

where ϕ and ξ are bulk scalar fields, and $V(\phi,\chi)$ is an analytic potential that is even in both ϕ and χ. Our interest in relevant deformations of the CFT translates on the gravity side to the demand that

$$\begin{aligned} \lim_{r\to\infty} V(\phi) &= \frac{1}{2L^2}\Delta_\phi(\Delta_\phi - 4)\phi^2 + \mathcal{O}(\phi^4) \\ &+ \frac{1}{2L^2}\Delta_\chi(\Delta_\chi - 4)\chi^2 + \mathcal{O}(\chi^4)\,, \end{aligned} \tag{4.40}$$

where Δ_ϕ and Δ_χ are the scaling dimensions of $\mathcal{O}_\phi$ and $\mathcal{O}_\chi$, respectively. Just as in the case of a single deformation, we restrict our attention to $2 < \Delta_\phi, \Delta_\chi < 4$. This result turns out to continue to hold for $1 < \Delta \leq 2$ [33].

The high temperature expansion for the background of the gravity dual goes through in almost completely the same way as in the single-scalar case discussed previous. The only substantive change comes from the fact that there now two expansion parameters, ϕ_0 and χ_0, which track the smallness of the two scalar fields. There are also two energy scales associated with the CFT deformations, Λ_ϕ and Λ_χ. Fixing the size of Λ_χ and Λ_ϕ in units of $1/L$ then gives relationships between r_h and ϕ_0, χ_0:

$$\begin{aligned} r_h^{\Delta_\phi - 4} &= \phi_0 \frac{\Gamma(\Delta_\phi/2 - 1)}{\Gamma(\Delta_\phi/4)^2} \\ r_h^{\Delta_\chi - 4} &= \chi_0 \left(\frac{\Lambda_\chi}{\Lambda_\phi}\right)^{\Delta_\chi - 4} \frac{\Gamma(\Delta_\chi/2 - 1)}{\Gamma(\Delta_\chi/4)^2} . \end{aligned} \tag{4.41}$$

To find the leading deviations from the conformal behavior of various observables in the field theory, we must work to order $\mathcal{O}(\phi_0^2, \chi_0^2)$ in the gravity dual. For any particular observable ξ_i,

$$\xi_i = \xi_i^{CFT} + C_{\xi_i,\phi}(\Delta_\phi)\phi_0^2 + C_{\xi_i,\chi}(\Delta_\chi)\chi_0^2 + \mathcal{O}(\phi_0^4, \chi_0^4, \phi_0^2\chi_0^2) .$$

Terms proportional to $\phi_0\chi_0$ cannot arise, because there are no terms proportional to $\phi\chi$ in a Taylor expansion of $V(\phi, \chi)$ about $(\phi, \chi) = (0, 0)$.

Now suppose that $\Delta_\phi \neq \Delta_\chi$, and take (without loss of generality) $\Delta \equiv \Delta_\phi > \Delta_\chi$. Then it is clear from the above discussion that $\phi_0^2 \gg \chi_0^2$ as $T \to \infty$. Thus, the leading deviation of ξ from ξ_{CFT} in the high T limit is driven by the least relevant operator, and

$$\xi = \xi_i^{CFT} + C_{\xi_i,\phi}(\Delta)\phi_0^2 \tag{4.42}$$

plus terms that vanish faster than ϕ_0^2 as $T \to 0$. This gives the same temperature dependence as we have seen in the single-deformation case.

Finally, consider the degenerate case when $\Delta_\phi = \Delta_\chi$. From Eq. 4.21, we find that

$$\xi = \xi_i^{CFT} + C_{\xi_i}(\Delta)\frac{\Gamma(\Delta/4)^4}{\Gamma(\Delta/2-1)^2}\left(\frac{\mathbf{\Lambda}}{\pi T}\right)^{2(4-\Delta)} , \tag{4.43}$$

where $\Delta = \Delta_\phi$, and

$$\mathbf{\Lambda}^{2(4-\Delta)} = \Lambda_\phi^{2(4-\Delta)} + \Lambda_\chi^{2(4-\Delta)} . \tag{4.44}$$

Again, the temperature dependence will be the same for all of the transport coefficients, in the same sense as before. The extension of the analysis above to the case of $n > 2$ deformations is obvious.

4.6 Summary

We now recap. The essential idea throughout our analysis is that in the high temperature limit, only the least relevant deformation matters as $T \to \infty$. Moreover, since the field theories we consider are CFTs deformed by relevant operators, at high temperatures the effects of the deformations become small. On the gravity side, this translates into the fact that the scalar fields associated with the deformations become 'small' at high temperatures. As a result, only the 'universal' part of the scalar potential in the dual is important for determining the high-temperature behavior of the theory, and the *only* property of a deformation that determines its effects on the physics at high temperatures is its scaling dimension.

Thus, as we have seen, our results in this chapter imply that in conformal strongly-coupled large N gauge theories with gravity duals that are deformed by

n relevant scalar operators, observables like the speed of sound and the transport coefficients, generically referred to as ξ_i, universally obey the relation

$$\xi_i(T) = \xi_i^{\mathrm{CFT}} + \mathcal{C}_{\xi_i}(\Delta)\left(\frac{\Lambda}{T}\right)^{2(4-\Delta)} + \mathcal{O}\left(\frac{\Lambda}{T}\right)^{4(4-\Delta)} \tag{4.45}$$

at high T in all the cases we have considered.

4.7 Appendix: High T metric coefficients

As we noted in the main text, we found it convenient to do the expansion at a fixed entropy. In the gravity dual, this corresponds to fixing the location of the black hole horizon at its AdS_5-Schwarzschild value, so that $r = r_h$ to all orders in the expansion. When there is one scalar operator deforming the CFT, one must compute the back-reaction of one scalar field on the geometry of the gravity dual. This gives the following results for $h(r)$ and $b(r)$ to $\mathcal{O}(\phi_0^2)$:

$$\begin{aligned}
b(r) &= \frac{r}{L} - \frac{\Delta^2(\Delta-4)^2\phi_0^2 r}{96 L r_h^8} \times && (4.46)\\
&\times \int_\infty^r dx\, x^7 \left({}_2F_1(2-\frac{\Delta}{4}, 1+\frac{\Delta}{4}; 2; 1-\frac{x^4}{r_h^4})\right)^2 \\
h(r) &= 1 - \frac{r_h^4}{r^4} - \frac{r_h\phi_0^2}{6r^4}\int_{r_h}^r dx\, x^2 f(x), \text{ where} && (4.47)\\
\frac{16 r_h^9}{\Delta(\Delta-4)x} f(x) &= 16 r_h^8\, {}_2F_1(1-\frac{\Delta}{4}, \frac{\Delta}{4}; 1; 1-\frac{x^4}{r_h^4}) \\
&+ x^4(r_h^2 - x^4)\Delta(\Delta-4)\, {}_2F_1(2-\frac{\Delta}{4}, 1+\frac{\Delta}{4}; 2; 1-\frac{x^4}{r_h^4}) \\
&- 8\Delta(\Delta-4)\int_\infty^x dy\, y^7 \left({}_2F_1(2-\frac{\Delta}{4}, 1+\frac{\Delta}{4}; 2; 1-\frac{y^4}{r_h^4})\right)^2.
\end{aligned}$$

Some functions that appear in the calculation of ζ/s in Eq. 4.34 are given by

$$\alpha(u,\Delta) = \frac{14u^4-9}{u-u^5} - \frac{u^3\,(\Delta-8)(4+\Delta)\,{}_2F_1(3-\frac{\Delta}{4},2+\frac{\Delta}{4};3;1-u^4)}{4\,{}_2F_1(2-\frac{\Delta}{4},1+\frac{\Delta}{4};2;1-u^4)} \tag{4.48}$$

$$\beta(u,\Delta) = \frac{32\,{}_2F_1(1-\frac{\Delta}{4},\frac{\Delta}{4};2;1-u^4)+(\Delta-8)(4+\Delta)\,{}_2F_1(1-\frac{\Delta}{4},\frac{\Delta}{4};3;1-u^4)}{2u^6(u^4-1)\,{}_2F_1(2-\frac{\Delta}{4},1+\frac{\Delta}{4};2;1-u^4)}.$$

Chapter 5

Conclusions

In this dissertation, we looked at the high-temperature properties of relativistic fluids described by a broad class of 4D large N strongly-coupled gauge theories with gravity duals. The gauge theories in this class can be viewed as conformal field theories deformed by the addition of n relevant gauge-invariant single-trace scalar operators. As we have stressed throughout, the theories that can be investigated using the gauge/gravity duality are not of direct phenomenological interest, as they are different in a number of important ways from theories that are known to describe the physical world. In particular, these theories are quite different from QCD, the theory that describes the strongly-coupled quark-gluon plasma, which is the perhaps the most famous recent example of a relativistic fluids described by a strongly-coupled non-Abelian gauge theory. Perhaps the biggest difference between theories with gravity duals and QCD is that theories with duals remain strongly coupled on all energy scales, while QCD becomes weakly coupled at high temperatures.

As a result, rather than looking at the quantitive properties of fluids described by theories with gravity duals, it makes sense to think of the gauge/gravity duality as a source of tractable toy models of strongly-coupled systems with a number of properties in common with QCD. In particular, it is important to try to understand any universal properties of theories with gravity duals, since it is conceivable that this may teach us some qualitative lessons that may be of use phenomenologically. Understanding which features of theories with gravity duals are universal may also give insights into how the duality can be extended to cover more interesting theories, perhaps by highlighting the current obstacles to such extensions.

To look for universal properties of theories with gravity duals, we considered a broad class of theories with gravity duals. These theories are 4D large N strongly-coupled conformal field theories that are deformed by n relevant operators. In looking at the high-temperature behavior of the speed of sound v_s, the bulk viscosity ζ, the DC conductivity σ, the charge diffusion constant D, and the charge susceptibility Ξ, we found two striking universal features:

- In Chapter 3, we showed that in the class of theories we examined, there is a bound on the speed of sound: at high temperatures, v_s^2 universally approaches 1/3 the speed of light squared from *below*. In fact, this sound bound appears to hold more generally in theories with gravity duals, and we conjectured that it may be a universal property of theories with gravity duals.

- In Chapter 4, we showed that for $\xi_i \in \{v_s^2, \zeta/s, , 2\pi TD, \sigma/\pi T, \Xi/(2\pi^2 T^2)\}$, at high T the temperature dependence of ξ_i takes a universal form:

$$\xi_i(T) = \xi_i^{\mathrm{CFT}} + \mathcal{C}_{\xi_i}(\Delta)\left(\frac{\Lambda}{T}\right)^{-2(4-\Delta)} + \mathcal{O}\left(\frac{\Lambda}{T}\right)^{-4(4-\Delta)}, \qquad (5.1)$$

  where ξ_i^{CFT} is the value of the ξ_i in the $T \to \infty$ limit, Δ is the scaling dimension of the deformation with scaling dimension closest to $\Delta = 4$, and Λ is the associated energy scale.

One might naively think that the latter result is a trivial consequence of dimensional analysis, as a consequence of the conformal symmetry in the $T \to \infty$ limit. Certainly, dimensional analysis tells us that the deviation from the conformal behavior must be a power law in Λ/T. However, the conformal symmetry of the $T \to \infty$ theory does *not* fix the exponent in the power law. To see this, note that if the whole result was a consequence of the $T \to \infty$ conformal symmetry, the temperature dependence of, for instance, v_s^2 and ζ/s would continue to be identical at weak coupling. However, at small λ, for conformal field theories deformed by a turning on a small mass m for a matter field in the theory, it is known that [54]

$$1/3 - v_s^2 \sim \frac{m^2}{T^2} \tag{5.2}$$

$$\zeta/s \sim \left(\frac{m^2}{T^2}\right)^2 \tag{5.3}$$

Thus at weak coupling, the temperature dependence of v_s^2 and ζ/s is *different* at high temperature, in stark contrast to what we saw in strongly-coupled theories, where the temperature dependence of these quite different observables is *identical*. This shows that the strong coupling limit is crucial for our results.

The universality of the high-temperature dependence of transport coefficients is a remarkable feature of theories with gravity duals. However, the fact that our results appear to rely so heavily on the fact that these theories remain strongly coupled at high temperatures makes it difficult to imagine that this can give much insight into the phenomenology of QCD fluids. Nonetheless, the universal aspects of the behavior of theories with gravity duals are interesting in their own right, at least as mathematical physics, and there are many interesting future research directions.

For instance, it would be interesting to investigate the effect of the leading $1/N$ and $1/\lambda$ corrections on the speed of sound and on the temperature dependence of transport coefficients. In the gravity duals, this would correspond to exploring the effect of higher-curvature corrections on the dynamics. Further, one can to go the next order in the hydrodynamic expansion, where one must introduce a number of additional transport coefficients. It would be interesting to understand whether these higher order transport coefficients continue to have the universal temperature dependence that we saw in the high temperature limit for first-order hydrodynamics.

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