ABSTRACT

Title of Document: PRE-SERVICE TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING: A COMPARISON OF TWO UNIVERSITY MATHEMATICS COURSES

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One enduring problem in the field of mathematics education is preparing teachers to present mathematics in sufficiently deep and meaningful ways to their students. A focus of this preparation is developing in practitioners sufficient knowledge of mathematics for teaching. Mathematical knowledge for teaching has been theorized widely and is currently the focus of many empirical investigations in the field. This study positions itself within this literature and seeks to connect the research to undergraduate, pre-service elementary school teachers (PSTs), and the content courses which comprise the bulk of their mathematical preparation within a typical university teacher education program.

Little is known about the impact that these courses have on teacher knowledge and still less has been studied about the efficacy of different pedagogical—or mathematical—approaches in these courses among PSTs. In order to test claims

made in situated learning theory and respond to prevalent political rhetoric about mathematics teacher education, this project compared mathematics courses designed for PSTs in two different universities along three dimensions: (1) Differences in pedagogical and mathematical approaches to developing content knowledge for teaching in PSTs; (2) Resulting differences in PST performance on mathematical knowledge for teaching instruments (3) Resulting differences among PSTs' attitudes about mathematics, teaching, and their perception of the course's relevance to their anticipated work as elementary school teachers. Data from multiple data sources reveals that, though differences were small, PSTs' mathematical knowledge for teaching was substantively different between the two campuses. In addition, the data indicate that PSTs developed different attitudes about mathematics and teaching. Finally, PSTs' evaluated their course's relevance for teaching practice differently.

This study suggests that when designing content courses for pre-service teachers, teacher educators should pay close attention to the interaction between mathematical approaches and pedagogical perspectives.

PRE-SERVICE TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING: A COMPARISON OF TWO UNIVERSITY MATHEMATICS COURSES

By

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Dedication

The work that follows is dedicated to my wife Theresa, whose unwavering love and support made this journey possible and worthwhile.

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This research project came to fruition because of the generosity of many people. In particular, I am in the debt of those at Hilada and Rio Universities who allowed me to explore the work that they do in their content courses. Department administrators, course coordinators, instructors, and students were very gracious in their efforts to help me learn more about how teachers learn mathematics. I hope that this work will make a constructive contribution to the field which will benefit the work that they do.

I am in debt to the community which is the Mid-Atlantic Center for Mathematics Teaching and Learning: with the financial and intellectual support of this group, I was able to take advantage of the opportunity to engage in two years of full-time graduate study in mathematics education, as well as multiple other years of part-time study, which often required of them much patience and guidance. Among this group of amazing people, I learned what it means to be a contributing member of a scholarly community. In particular, Jim Fey, Anna Graeber, Pat Campbell, and Anne Morris have been generous mentors who also happened to provide heavy-duty intellectual challenges for me at key points along the way. Along with these people, Dan Chazan has been a thoughtful and caring advisor: he consistently served a well-balanced blend of cheerleading, advice, focus, and scholarly debate which has helped me to clarify my thinking throughout this process and toward the future. If I can contribute a small fraction of the goodwill and productive insight each of these people has brought to the field of mathematics education, I will be very proud.

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Any intellectual milestone that I reach is a direct consequence of the love and support I have received from my wonderful family. I thank my parents, who have tenderly fostered my curiosity and love of learning all my life; and my brother and sister, who at once connect me to wonderful memories of childhood and offer me the best friendships I have as an adult. I thank my daughter Kathryn who is developing into a wonderful little girl as I work on this project. One of the most important aspects of my growth in these years has been an appreciation for the vigor and inherent logic in children's thinking and its simultaneous fragility in the face of adult influence. I hope that I will be able to nurture her blossoming curiosity into a love of learning about the world around her. Finally, and most importantly, I thank my wife Theresa, whose immeasurable patience, resolute support, loving sacrifices, and consistent optimism enabled me to take every step of this scholastic journey. I hope that the completion of this process can in some small way better enable me to give her endeavors the same caring support that she has given mine.

Table of Contents

List of Tables

List of Figures

Chapter 1: The Problem of Teaching Mathematics to Teachers

Introduction

How do we ensure that all teachers of mathematics know the mathematics and pedagogy essential for teaching the subject?

Skip Fennell, National Council of Teachers of Mathematics President, NCTM News Bulletin (July 2007)

The question by Fennell (2007) above implicitly represents a twofold problem, both pieces of which pose persistent challenges in mathematics education: what should teachers know and how do they come to know it in such a way that it fosters effective teaching practice? This is an issue I have encountered as a community college mathematics instructor. One of my responsibilities is to teach mathematics courses for pre-service elementary school teachers (PSTs). In the early versions of the courses I taught, I concentrated on giving undergraduates a behind-the-scenes look at mathematics that they took for granted, or worse, never learned. In order to help them connect our discussions in class to classroom teaching, I required the PSTs to complete a service learning project which focused on helping children to learn mathematics in school settings. These projects often took the form of after-school tutoring programs or working with a teacher at a local elementary school.

From my perspective, the results of the project were mixed at best. In more than a few cases, students had to navigate confusing bureaucracy just to set foot in a classroom. When they had accomplished this, they were often sent to do menial tasks for their cooperating teachers such as making copies or running school errands. Even when the

PSTs did not encounter these difficulties, I found that their reflections about the experience had little or no mathematical component. I had hoped that their encounters with students and classroom mathematics would present them with mathematical issues that intersected richly with the topics we discussed in class. I expected that children's questions, difficulties, and intuitions would bring the PSTs closer to understanding why they were enrolled in a mathematics course focused on elementary school topics. In their reflections, I had hoped to read for example, that they would use a student's difficulty to understand multi-digit multiplication as an opportunity to discuss the importance of place value, single-digit multiplication, and multiplication's links with other operations. As they took notes, wrote in their journals, and finally worked to summarize their experiences, I exhorted PSTs to pay attention to the mathematics that they saw and did with students and seek to draw connections between their experiences, but these requests generally went unfulfilled.

Meanwhile, the PSTs saw the assignment in a completely different light. Sometimes, the PSTs discussed the fact that they saw students working on similar mathematical ideas that we had during class, and that in the course of helping children, they discovered confidence in their knowledge they hadn't known before. The PSTs tended focus on classroom issues that were not specific to mathematics. They tended to write in general terms about "differentiated instruction" and choosing "fun" activities that would naturally and easily keep *all* children "on task" and (miraculously) foster understanding. I was deeply skeptical that the PSTs understood the challenges implicit in their writing about these things; their writing often suggested that the simple introduction of manipulatives or games would create understanding, set up a constructive learning environment, and make mathematics fun for students all at once. Despite this skepticism, I realized that I had underestimated the PSTs' desire to use the language and ideas of their other courses, those likely often used by their undergraduate professors, and probably by cooperating teachers in these experiences. They were discovering a new role for themselves as teachers and sought to apply the language, roles, and norms of that community, even though they were not yet a part of it.

Ultimately, I dropped the assignment from my courses, feeling that it had not accomplished my primary goal of pushing PSTs to confront children's thinking about mathematics, and recognize the experience as a key component of understanding how to teach but also of understanding mathematics. Despite the assignment's positive outcome by some measures, I felt that PSTs' time and energy could be better spent on activities that could focus their attention on that which they so often needed the most assistance: mathematics. It is important to note that nearly all of my students related the experience as being a positive one. In their reflective papers and later in course evaluations, they consistently commented that it was one of the best components of the course; they felt that they had learned much from participating in it. Although I felt that the assignment had failed to help them think carefully about the relevant mathematical ideas, it had overwhelmingly succeeded in giving PSTs an opportunity to participate in classroom (or classroom-like) activities with an unfamiliar if not altogether new perspective: a teacher's.

I sought to offer the PSTs other positive experiences like the one they had with the service learning assignment while challenging them to deepen their understanding of mathematics. But how could they come to know it in such a way that it would be available to them in the classroom environment, in which numerous split-second decisions must be made on a regular basis? The service learning assignment appealed to PSTs' desire to play the role of teacher even at this early stage in their preparation, but they clearly needed a less volatile environment in which to work, and which would allow them to concentrate on the mathematical ideas that they had yet to learn. What would such an environment look like, and could it be incorporated into a mathematics course in order to deepen PSTs' mathematical knowledge?

The Problem

Though working with service learning was an important learning experience for me as a teacher educator, the questions posed by Fennel remained: what should teachers know and how do they come to know it in such a way that it fosters effective teaching practice? What teachers should know is rightly tied to what children should know, but that is a judgment that continues to be debated fiercely throughout the country. The question of *should* is relentlessly elusive, due in large part to the high stakes attached to mathematics education in the United States, and the influence wielded by many interested—and often conflicting—groups. On the other hand, the answer to the latter question about how teachers come to know what they should feels different: it has an empirical quality to it that suggests that it is a testable question.

Until recently, there was little evidence about whether teacher knowledge had an impact on student achievement mathematics classrooms (Hill, Rowan, and Ball, 2005). This positive correlation is unnerving given the evidence presented over the last 25 years have demonstrated a startling lack of mathematical knowledge among teachers (e.g., Cooney, 1985; Tirosh & Graeber, 1989; Stein, Baxter, & Leinhardt, 1990; Zazkis &

Campbell, 1996; Ma, 1999). What's more, we know that teachers continue to use decades-old methods while students continue to perform at disappointing levels (Fey, 1979; Stigler & Hiebert, 1999). This attention to teacher knowledge of mathematics has occurred contemporaneously with a widespread push for reform in school mathematics, the rhetoric of which places new responsibilities on the part of teachers (Lampert & Ball, 1999). For example, *Principles and Standards for School Mathematics* (NCTM, 2000) suggests a vision of mathematics teaching that increases and deepens the knowledge demands on teachers. In this document, teachers are called upon to make connections among diverse branches of mathematics, incorporate ways of working that imitate those of mathematicians—such as encouraging the process of exploration, conjecture, and proof—and use intuitive and often idiosyncratic ideas generated by some students to foster the mathematical competence of all. In another influential policy document, The Conference Board of Mathematical Sciences (CBMS, 2001) writes:

> *…to make intelligent curricular decisions for their students and to teach current school curricula, future teachers need to know more and somewhat different mathematics than mathematics departments have previously provided to teachers. Because they are being urged to teach in different ways, prospective teachers also need to experience learning mathematics in those ways themselves (p. 122).*

These documents are grounded in traditional understandings of teacher knowledge and yet broaden it to include deeper knowledge of mathematics as a discipline and as students experience it.

The result of such rhetoric in Maryland has been to create new course requirements that include more subject-matter preparation for teachers in mathematics. Because there is very little research about undergraduate content courses in teacher education (Ambrose & Vincent, 2003; Wilson, Floden, Ferrini-Mundi, 2001), these changes have been implemented at institutions across the state without explicit reference to empirical results that may help inform their design or facilitate greater coherence within teacher education programs. What's more, courses and programs across the nation are operating with a great lack of understanding how content courses can contribute positively to teacher education programs and ultimately, to better teaching.

This call for a deepening of teachers' mathematical knowledge necessarily implicates teacher learning, an area of scholarship that has received increasing attention in recent decades. Situated and sociocultural learning theories have influenced much of this work (Putnam & Borko, 2000). These perspectives privilege the activities of teaching practice as central components of knowledge and claim that learning is a fundamental consequence of participating in these activities (e.g. Lave & Wenger, 1991; Greeno, 1997). Communities of practice—people working together in a shared enterprise such that every participant makes important contributions—have become central components of many teacher education efforts (Garet, Porter, Desimone, Birman, Suk Yoon, 2001) and form a foundation for framing the problem of teacher development over time (Cochran-Smith & Lytle, 1999; Hammerness, Darling-Hammond, Bransford, Berliner, Cochran-Smith, McDonald, & Zeichner, 2005). Teachers' ongoing identification of themselves with teaching and its associated tasks are a central component of this perspective. In addition, there is a growing body of research that focuses on specialized knowledge that teachers must have in order to be effective. In the two decades since Shulman (1986) proposed the existence of pedagogical content knowledge (PCK), others (notably Ma, 1999 and Hill, Rowan, & Ball, 2005) have demonstrated and described a corollary to Shulman's construct: mathematical knowledge for teaching (MKT). The notion of knowledge held by teachers as part of and in service to their roles and responsibilities in classrooms has made the use of situated theories of learning and knowing all the more compelling.

Undergraduate teacher education is a natural choice for putting these theories into practice for the purpose of improving teacher knowledge. However, teacher education is often considered inadequate to the task of preparing teachers for a variety of reasons. First, some argue that the field lacks a common knowledge base (e.g., Shulman, 1998) that is employed consistently across preparation programs. Second, teaching and teacher education has long suffered a lack of intellectual and scholarly respect (Clifford & Guthrie, 1988; Herbst, 1989; Labaree, 2004). In addition, there is a widely documented tension between pre-service teachers' expectations and teacher educators' academic agendas (Ducharme, 1993). At least as far back as Dewey (1904), teacher education has struggled with balancing theory and practice. These tensions often translate into a perception that both pre-service and practicing teachers maintain: their academic preparation is inadequate for confronting the realities of teaching (Britzman, 1986; Eisenhart, Behm, & Romagnano, 1991; Zeichner and Tabachnik, 1981; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992). The resulting ethos is that teaching is a practice best learned "on the job," as opposed to the traditional, undergraduate education program. It seems likely that if teachers dismiss out of hand their academic preparation as irrelevant, there is little that teacher educators can do to improve PSTs' knowledge.

Several teacher development environments have taken advantage of situated and sociocultural learning theories in an effort to address this problem and provide opportunities to think deeply about mathematics, teaching, and their connections (Cohen, 2004; Harrington, 1995; Lampert & Ball, 1998). These approaches rely on the use of artifacts of teaching in order to engage PSTs and teachers in mathematical and pedagogical ideas. These artifacts include items such as student work, teacher notes, classroom video, curricular materials, and written cases. Where these opportunities have been available, they typically have been offered for practicing teachers or with PSTs in methods courses, late in their academic preparation. There is evidence that these experiences boost teachers' confidence in mathematics they teach (Cohen, 2004) and provide an early opportunity to think carefully about the complex relationships between mathematical ideas and the pedagogical choices teachers make (Lampert & Ball, 1998). They offer as well the opportunity to achieve what may of the PSTs appreciated about my service learning assignment—the chance to interact with children and teaching practices—without all the intricacies and pressure of a real-time classroom environment.

Despite positive results from this work, there has been little application of this perspective to content courses that PSTs take early in their undergraduate programs, and not much is known about how well these approaches foster the development of greater mathematical understanding that population. Many questions thus remain about teacher preparation in mathematics. How can it address the lack of mathematical knowledge that teachers often demonstrate? What approaches foster strong mathematical understanding while promoting a perspective on teaching that accounts for its complexities? Can using activities and artifacts of teaching really bridge the gap between theory and practice while further developing mathematical knowledge in PSTs? In order to address these issues, I designed and carried out a research project that focuses on the following questions:

(1) What mathematics do prospective teachers learn by engaging in activities of teaching practice such as examining curriculum, student work, and classroom video?
    a. Do PSTs who regularly engage in such activities display evidence of different mathematical proficiency than PSTs who participate in more traditional course work?
    b. Do PSTs engaging in such activities display different mathematical knowledge for teaching (MKT) than PSTs participating in more traditional course work?
    c. Do PSTs engaging in such activities develop different attitudes about mathematics and teaching than PSTs participating in more traditional coursework?

(2) To what extent do prospective teachers see their mathematics course work as relevant to their future work?
    a. Do different course approaches set up differing perspectives among PSTs on the contribution of the course to their future work?
    b. Do different course approaches set up differing views among PSTs about their confidence and abilities in mathematics?

These questions represent an opportunity to challenge empirically the notion that teachers' knowledge is embedded in activities of practice and that learning mathematical content for teaching is inherent to participation in these activities. More broadly, these questions also address the question offered by Fennel above. Potential answers to the first set of research questions are that there is no difference between the knowledge in the two groups of teachers, or that concentrating on teaching issues inhibits the development of strong mathematical knowledge. However, I anticipated at the outset of this project that instead, evidence would suggest a third alternative: that PSTs who engage in tasks that make explicit the role of teaching will develop more sophisticated understanding that coincides with the vision of mathematics teaching espoused by the NCTM (2000). The questions also address the persistent lack of fit between prospective teachers' expectations of teacher education and the importance of theory to teacher educators as they seek to disrupt teachers' apprenticeship of observation. If the use of teaching artifacts in content courses does in fact address this gap, then this suggests important implications for the continuing debate about how to integrate theory and practice in undergraduate teacher education.

Chapter 2: Theoretical Lenses

How do Teachers Learn to Teach Mathematics?

Decision-Making in Content Courses

As a teacher educator at a community college, many of my choices are constrained by articulation agreements with state universities. Yet, there is still much latitude in seeking experiences for students to learn the mathematics which I responsible for teaching to them. These choices have often been dictated by some tacit assumptions about how teachers come to know and develop their craft. Though my courses were located within a mathematics department, and my task was to teach them mathematics, they were designed and intended for PSTs. As such, their future in teaching was never far from their minds—or from mine. I introduced the service learning assignment because I felt that encouraging PSTs to experience a classroom or tutoring situation from a teacher's perspective was a unique opportunity to integrate issues of mathematics and teaching. I hoped that it would bring important ideas about content and pedagogy to the fore, ideas that I otherwise would have had difficulty highlighting.

Empirical outcomes did not drive this decision-making process. I did not access the literature, synthesize the results and plan a course of action accordingly. Yet, even had I sought such literature and results to apply directly to my situation, I would have found little on which to draw. The fact is that most of what we know about how teachers learn to teach mathematics comes from research on practicing teachers, and to a lesser extent, from student teachers and interns. In turn, this work depends for its foundations upon a long tradition of generalized theories of learning, and related results from teacher development. In recent years, teacher education research has focused in large part on a situated perspective on learning to understand more about how teachers develop their practice (Putnam and Borko, 2000). In this chapter, I outline the foundations of situated learning theory, its application to teacher education in general, and in mathematics teacher education in particular.

Situated Learning Theory

Theories of learning have occupied the attention of psychologists and educators for at least the last century. Broadly speaking, this work can be broken into three categories: behaviorism, cognitivism, and situated learning theory. While behaviorism addresses learning as a system of psuedo-mechanistic responses to external stimuli that can be strengthened with repetition, cognitive learning theory approaches it as the acquisition and development of representations that order and structure ideas in the mind of the knower. Like behaviorism, cognitive learning theory objectifies knowledge: it can be acquired, stored, and retrieved by individuals. Situated learning theory on the other hand asserts that knowledge is located within and among communities of people, focusing on the activity and interactions they share. There is no consensus about the relationship between situated and cognitive learning theories: while situated learning theory is sometimes contrasted with the cognitive perspective (Anderson, Reder, & Simon, 1996; Greeno, 1997; Anderson, Reder & Simon, 1997; Cobb & Bowers, 1999; Moore, 1999; Anderson, Greeno, Reder, & Simon, 2000), others suggest that the situated perspective developed as a reaction to limitations of cognitive learning theory and represents a “second wave of the cognitive revolution” (Decorte, Greer, & Vershaffel 1996). Greeno (1997) and Cobb and Bowers (1999) argue that the two theories operate under fundamentally different assumptions, though Moore (1999) asserts that rather than operating at some deeper level, the differences between the two are merely a shift in vocabulary.[1] Simon (2009) claims that the variety of theories is useful because they serve to address different kinds of questions. The debate regarding the distinctions between cognitive and situated learning theories has created a need for theorists in the younger perspective—situated learning theory—to refine and clarify the arguments and assumptions that give it explanatory power. Because salient aspects of the situated perspective are brought to light in this conversation, I note in some detail the critiques of situated learning by theorists in the cognitive tradition.

For some situated theorists, its power lies in its ability to explain the uniqueness of school learning (and knowledge) from other types of knowing generated in other circumstances. Brown, Collins, and Duguid (1989) use such an approach in calling for “cognitive apprenticeship” in school. They describe a divide between the tools of disciplines and the culture that created them which interferes with a deep understanding of these tools and thus accounts for the difficulties that students have learning in school. For example:

> *Old-fashioned pocket knives… have a device for removing stones from horses' hooves. People with this device may know its use and be able to talk wisely about horses, hooves, and stones. But they may never betray or even recognize that they would not begin to know how to use this implement on a horse. Similarly, students can often manipulate algorithms, routines, and definitions they have acquired with apparent competence and yet not reveal, to their teachers or*

[1] Moore argues that situated learning theorists have co-opted tenets of cognitive learning and applied them exclusively to groups instead of individuals.

> *themselves, that they would have no idea what to do if they came upon the domain equivalent of a limping horse.* (p. 33)

Bridging this divide involves creating a "cognitive apprenticeship" for students in which tasks and activities are "authentic." Many school-based activities are considered inauthentic in this view because they insufficiently match school work with activities that occur outside of school, which are not simple and discrete, but often complicated and multidisciplinary.[2] Brown, Collins, and Duguid argue that schooling as currently conceived inherently stifles interactions inherent to authentic activity; consequently, "we have ended up with wholly inappropriate methods of teaching" (p. 41).

A weakness of this approach to the theory is that it implies that some activity is situated (or properly situated) and other activity is not. A primary criticism on this point is made by referencing research in which learners used knowledge in contexts that differ substantially from that in which they were learned. Anderson, Reder, and Simon (1996) review claims such as those made by Brown, Duguid, and Collins as misguided and overstated. They argue that the notion of authenticity fails to acknowledge that learning occurs in discrete stages: "what is important is what cognitive processes a problem evokes, and not what real-world trappings it might have. Often, real-world problems involve a great deal of busy work and offer little opportunity to learn the target competencies" (p. 9). The process of focusing on the "target competencies" involves isolating the relevant skills and learning each in turn before applying them together in a more complex problem situation. Anderson, et. al. assert that it is unreasonable to learn

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[2] Scholars such as Saxe (1985) and Carraher (1985) are often linked with situated learning because they generate similar conclusions in their research. I do not include them here because within these particular pieces, they do not associate themselves explicitly with this theoretical perspective.

all skills of a complex practice simultaneously and that research in cognitive psychology shows that "part training is often more effective when the part component is independent, or nearly so, of the larger task" (p. 9). Anderson, Reder, and Simon further argue that while transfer—the ability of an individual to employ knowledge acquired in one context in another, different context—is often tenuous, it can be strengthened by encouraging learners to reflect on potential avenues for transfer during their initial encounter of concepts. One could imagine an example of an algebra teacher pointing out that solving simultaneous linear equations can be applied to linear programming problems, which the class may address when the other component parts of the problem have been learned.[3]

Situated learning theory seeks to address the problems of complex problem situations and transfer through the concept of participation. This is perhaps best described by Lave and Wenger's seminal text, *Situated Learning: Legitimate Peripheral Participation* (1991); it is routinely associated with the foundations of situated learning theory (Greeno, 1998). Yet the authors make explicit their dissatisfaction with the perspective as it was then theorized, and offer instead the construct of "legitimate peripheral participation" as a transformation of situated learning theory (p.122). In the years since its publication, situated theorists' writing reflects Lave and Wenger's understanding, which builds on work like Brown, Collins, and Duguid (1989), but clarifies and refines the position.

In an implicit criticism of Brown, et., al., Lave and Wenger disassociate themselves with conventional notions of apprenticeship that privilege informal, experience-based learning; such perspectives label only particular activities as situated.

[3] Other component skills for solving linear programming problems might be: solving systems of linear inequalities, constructing objective functions and constraints, etc.

For them, situated cognition offered explicit recognition that all activity is situated. The authors discount the assertion that learning is situated in practice, "as if it were some independently reifiable process that just happened to be located somewhere" (p. 35). Instead, they assert that learning is a critical, contemporaneous component of practice. That is, "learning is an integral part of generative social practice in the lived-in world" (p. 35). As people engage with one another in communities to which they belong, learning is an inextricable element of that interaction. The community of practice is the medium in which legitimate peripheral participation operates. Within these communities are new-comers and old-timers—sometimes novices and experts, respectively—who are active in continuously reproducing the community as new-comers contribute more fully and old-timers end their participation. New-comers' "legitimacy" refers to the fact that their membership in the community has purpose and substantially contributes to the community, even when that contribution may be labeled small. Hence, Lave and Wenger contrast operating on the "periphery" with irrelevance. As a result, though peripheral participation means that new-comers play a strictly supporting role in the community, it is nonetheless one that serves to maintain the existence of the community.

Lave and Wenger describe five studies in which the theory of legitimate peripheral participation is readily illustrated. Through these examples, they outline features of communities of practice that foster effective learning among new-comers. First, they note that the relationships between apprentices and masters (or new-comers and old-timers) are neither fixed nor consistent across practices. In some cases, the relationship is quite hierarchical while others are egalitarian. This means that there is no single defining feature of the relationship between novices and experts, and that the relationship may be structured differently within communities of teachers than within communities of welders or lawyers. More important is the way resources are structured to enable novices to participate in ways that foster their learning and increase their participation. For example, they highlight the fact that in many cases, "apprentices learn mostly in relation with other apprentices" (p. 93). In addition, newcomers must have access to the practices, language, and structure of full practitioners. Though their status as peripheral participants means that they are not responsible for the entire range of relevant activities, the ones with which they do work nonetheless simulate the central practices of the community. The tasks are low-stakes and typically highlight specific aspects of practice. This may be analogous to Anderson et. al.'s conception of learning "component skills," but it is most certainly distinct from it, since novices do not engage in skill-building apart from their practice. Rather, new-comers focus only on particular aspects of the practice, which at once acknowledges the complexity of full participation and yet attempts to reduce that complexity for novices.

Acquiring a practical language is part of this, and represents an especially important component of novices' participation in the community. This language acquisition occurs largely through narrative discourse about particularly difficult and ill-defined problems within the practice. Newcomers learn most effectively as participants in discourse centered on issues of practice rather than strictly as passive observers of others who talk about practice.

Participation in the community is neither exclusive of observation nor limited to working in groups. For example, observation of tasks performed by others in the community can be an integral part of participation. However, it is not a substitute for more active contribution: "for newcomers…the purpose is not to learn *from* talk as a substitute for peripheral participation; it is to learn *to* talk as a key to legitimate peripheral participation" (p. 109). This means that not only are novices watching practice or hearing about it, they contribute meaningfully to the production (and reproduction) of the community in such a way that their talk is about *their* practice—however peripheral—rather than others'. Likewise, engaging in solitary work requires the practitioner (novice or otherwise) to think and act as a member of the community, even if she is not physically surrounded by colleagues. The particular physical and social settings in which the learner finds herself are indeed important in the situated perspective, but through this lens, transfer applies not to the knower's ability to transport knowledge from one location or set of circumstances to another. In the situated perspective, transfer refers to the consistency of activity patterns across situations, so that it is not necessarily the knower that takes knowledge from one place to the next, but salient features of the situation itself that are transferred (Greeno, 1997). The knower is able to attune properly to the features of activity that are germane to both settings depending on the learner's level of participation in the activities and her contribution to the resulting interaction (Greeno, 1998).

Thus, participation in the community is co-creative with the participant's identity: as members participate, they create an identity as contributors to the community. Similarly, as their identities develop, participants are able to contribute ever more fully to the community. This component of the theory addresses the problem of complexity: situated learning theory does not interpret all participation as equally complicated, but emphasizes the increasing participation of learners within communities. Though the larger context of their participation may be multifaceted and complex, learning is portrayed in the situated perspective as the process of making ever larger contributions to the community, meaning that they do not take on the fullness of responsibility or complexity in their earliest encounters, but their practice nonetheless builds in complexity as they increase their participation.

The identity formation of learners is critical. In school settings, the "didactic caretaker" makes changing the identity of the learners the central aspect of teaching. Lave and Wenger contrast this with successful apprenticeships, where the tasks of practice are the central and explicit objects of change: "As opportunities for understanding how well or how poorly ones' efforts to contribute are evident in practice, legitimate participation of a peripheral kind provides an immediate ground for self-evaluation," and therefore, re-formation. Thus, identity re-formation occurs through participation rather than by fiat. Though the goal of learning is to change the learner's identity on some level, the means by which this change occurs most effectively is fundamentally different than currently found in situations where didactic practices are dominant (pp. 111-112).[4] In such situations, the result is the commoditization of learning, in which knowledge has an exchange value established by testing; learning to display knowledge takes priority over learning for understanding.

> *The latter type of learning is a disposition that one would expect to find a practitioner adopting, for his ability to participate in the community is inextricably linked to his learning and development of knowledge. Situated learning theory—in its incarnation as legitimate peripheral participation—makes this link between activity, communities of practice, and knowledge the focus of analysis. Instead of conceptual emphasis on the exercise of conceptual skill, situated learning theory "provides activity structures in which those aspects of…knowing are meaningful and functional"* (Greeno, 1998, p. 19).

[4] Brown, Collins, and Duguid (1989) state that schools are examples of this type of situation.

Brown, Collins, and Duguid's notion of "cognitive apprenticeship" is echoed here: learning takes place as a part of participating with and among practitioners. One does not learn in order to demonstrate knowledge in some artificial way, but in order to participate more fully in the community.

How does this perspective apply to teacher education and the process of learning how to teach? What are the implications of situated learning theory in teacher education?

Learning to Teach

If one approaches learning (and by extension, knowledge) as being an intrinsic feature of participation with others in a community of practice, then teachers' knowledge should be located in and acquired through activities in which they engage: planning lessons, working with students, assessing student work, and collaborating with colleagues, among other things. Legitimate peripheral participation requires that new-comers be inducted over a period of time and that they be given—by old-timers—opportunities to develop the language and work of the practice by working through various sub-domains of practice in low-stakes environments. This means that PSTs must begin to develop their practice long before the time where they are given sole responsibility of a class full of students, such as an internship experience.

There were elements of this perspective embedded in my service learning assignments, which resulted as situated learning predicts they would: PSTs made clear efforts to adopt the language of practitioners and project their observation experiences forward into their own future classrooms. They used terms like "differentiated instruction" and "making mathematics meaningful," even when their articulation of these ideas did not include any evidence that they understood the complexities and challenges involved with carrying them out. Moreover, the PSTs in that experience usually were not paying attention to all salient features of the classrooms they observed. In particular, in most cases ignored the most important mathematical features of their situations. This ignorance may be attributed to the incongruence between the two contexts: our college classroom was too different from the elementary school settings in which they were working.

If one accepts the cognitive learning perspective in the area of teacher knowledge, then one must address the problem of transfer between necessarily very different environments: the university classroom and the elementary school classroom. Teacher educators must find ways to help teachers bridge the divide between these environments. The situated learning theoretical stance places a difference spin on the problem. It claims that the knower is not the one responsible for making this transition, but the environment: the university classroom should more closely emulate the experience of the elementary school classroom.[5]

The situated learning perspective has developed contemporaneously with terms such as pedagogical content knowledge (PCK) and mathematical knowledge for teaching (MKT) (Shulman, 1986; Ball, 2002). Both constructs highlight that there is knowledge that special to the work of teaching, knowledge that is maintained within the context of teaching practice and employed inherently and exclusively in service of important tasks of teaching practice.

[5] Many argue that the converse is also true: elementary school classrooms should look more like university classrooms—or at least they should more resemble the parent disciplines in academia. Lampert (1993) and Cuoco (2001) are among those who take this perspective in mathematics. In the work I propose here, I am playing out one possible way to bridge the two experiences. I am not arguing against the value of bringing the disciplines into elementary or secondary classrooms with more integrity.

Emerging conceptual frameworks for teaching and learning complement this scholarship. Cochran-Smith and Lytle (1999) describe the importance of "knowledge of practice"[6] which privileges the systematic inquiry of real issues that emerge from teaching, rather than perpetuating the divide between theory and practice. This approach unites the work of practicing and prospective teachers as they engage in focused case study that requires teachers to develop skills for attending to contextual details while also drawing more general inferences about teaching from them. Teacher knowledge which is conceptualized as being integrally related to practice reflects the same perspective on knowledge taken by situated learning theory.

In addition, research and rhetoric on teacher development has adopted the novice-expert (new-comer/old-timer) model of interaction within communities. Hammerness, Darling-Hammond, Bransford, Berliner, Cochran-Smith, McDonald, and Zeichner (2005) concentrate on the idea of adaptive experts described by Bransford, Derry, Berliner, Hammerness, and Beckett (2005) and the process by which teachers develop it. Hammerness, et., al. assert the importance of expertise, which for the authors consists of strong capabilities along dimensions of efficiency and innovation: expert teachers notice and address subtleties of typical classroom activity in ways that novices cannot. Moreover, experts can assess non-routine situations with skill and use them to help students make progress. The authors turn to research on learning to observe three guiding principles for teacher development. Teacher education should: (1) make teachers' "apprenticeship of observation" explicit and offer alternative perspectives that build upon—and in some cases challenge—this experience; (2) enable teachers to acquire

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[6] The authors contrast this with knowledge "for" and "in" practice. Each of these is made distinct by the relationships formed between (prospective or practicing) teachers and researchers, but all of them are conceived as being located within a context of teaching practice.

a deep foundation of knowledge and techniques for giving this knowledge a conscious structure; (3) provide access to meta-cognitive tools for teachers to continue to develop their knowledge in and through their practice.

Teaching is a very "complex and demanding task" (Hammerness, et. al., 2005, p. 37). In order to manage this complexity, teachers must be skilled on-the-job-learners. Here again, learning on the job requires more than casual observation but systematic inquiry and carefully reasoned interventions that can be applied to a (unique) situation. This is not necessarily something that arises naturally among teachers. Thus, teacher education is responsible for developing teachers' meta-cognitive skills that help them monitor their thinking as they work with students and design their instruction. The authors suggest that this can be accomplished through a collaborative approach which seeks to develop a strong, active knowledge of teaching rather than "book" learning which is often inert. Hammerness, et. al. further outline a framework that relies on teachers developing within a community of colleagues. Within the community, teachers should develop (1) a vision for teaching; (2) a repertoire of teaching practices; (3) a set of conceptual and practical tools for classroom use; (4) habits of thinking and acting as a teacher; and (5) deep conceptual understanding of content. There are strong echoes in this framework of Lave and Wenger's notion of legitimate peripheral participation; novices learn to become experts by engaging with experts in solving problems that arise in work of teaching. The distinction between the two groups implies that they work through problems differently and contribute different things to the community, but the success and continuation of the community requires that everyone participate.

Members of this community are practicing teachers, of course, but also included are PSTs and university faculty: they participate in the community in different ways, but are no less central to the perpetuation of the practice. In Hammerness, et. al.'s description, university faculty mediate the interactions between them.

Theory and Practice

The distinction and separation between novices and experts, teachers and university faculty, is an important and valuable one. But this distinction underscores a larger issue of teacher education. Integrating education theory with education practice has been an ongoing debate and tension in American teacher education throughout U.S. history (Clifford & Guthrie, 1988; Borrowman, 1965). Practicing and prospective teachers alike report that their undergraduate preparation is too theoretical and disconnected from their teaching practice (e.g., Ducharme, 1993; Britzman, 1991). Whether or not it is well-founded, this attitude could potentially impede PSTs' ability to develop important teaching skills. This ethos also perpetuates the belief that learning to teach only occurs by teaching, though some have argued (e.g. Cochran-Smith & Lytle, 2005; Dewey, 1965) that the full responsibility of practice is fraught with complexity and is too challenging an environment in which to learn.

Rosenthal (2003) calls it "field-based teacher education's dirty little secret," a secret which teachers know all too well: being a practicing teacher is very different from the preparation one receives in schools of education. Britzman (1986) outlines a few causes of this gap between teacher education and teaching. First, prospective teachers bring with them preconceived notions of the role of teacher. The so called "apprenticeship of observation" (Lortie, 1975) is a powerful influence for these students who view teaching as a series of methods used to maintain control, authority, and expertise in the classroom. Moreover, there is a lingering perspective of teaching as relatively unskilled labor that reinforces these views. Labaree (2000) points to an expectation of the general public—and likely PSTs—that anyone can teach

> *because the kinds of skills and knowledge that it transmits to students become generic in the population at large. Therefore, unlike college professors who are expected to be experts at a level well beyond the understanding of ordinary citizens, schoolteachers are seen as masters of what most adults already know.* (pg. 232)

Teachers' special domain of expertise therefore is often seen as focused on behavior management and is childcare-centric. Therefore, "education course work which does not immediately address 'know-how' or how to 'make do' with the way things are, appears impractical and idealistic" (Britzman, 1986, pg. 446).

This gap also becomes apparent to teachers during their student teaching experience. Theory often appears to play a small role in school settings; teachers (especially student teachers) are in a position of survival rather than reflection, and they are often evaluated not on their pedagogy or content knowledge, but on their skill in managing the classroom effectively. Britzman refers to interviews she conducted with student teachers: "…education courses were not considered as real experience. Instead, in the minds of these student teachers, their education courses failed to demonstrate the value of theory, or even to shed light on their pragmatic fields" (pg. 447).

Thus, the theoretical focus of teacher education fits neither with common views of teachers' role in the classroom, nor with the encounters PSTs have with the reality of schools as they complete their student teaching and begin their practice. Situated learning, as conceived through legitimate peripheral participation, calls for rehearsal of teaching tasks in a low-stakes, low-pressure environment (such as content and methods courses). This kind of induction process may strengthen teachers' ability to use what they learn as undergraduates in the classroom. This means that PSTs should be engaging in practices of teaching even before they have responsibility to children, parents, and administrators. It calls for a kind of transformation of the university classroom into a more authentic analog of the school classroom. An "artifacts approach" [7] to teaching PSTs may connect them more closely to their professional goals, instead of being another hoop through which teachers must jump on their long journey toward certification. Connecting undergraduate preparation and professional development to teaching in this way challenges the popular misconception that one *only* learns to teach when "on the job," while yet acknowledging the kernel of truth at its heart. PSTs can be "on the job" even when they are not planning for or implementing a lesson with real children for whom they are responsible.

Learning to Teach Mathematics

By and large, little of the work described above specifies what mathematics that teachers are supposed to be learning. Though there is not a consensus about what characterizes effective teaching generally, research has produced some accepted results. Early studies of teaching conducted in the process-product framework generally neglected teacher knowledge in favor of the effect of teacher behaviors on student achievement, such as timed studies of content instruction versus classroom management (Hill, Rowan, & Ball, 2005). Other work has also shown the limitations of coursework

[7] Recall that artifacts of teaching include—but are not limited to—student work, video of classroom interaction, teachers' notes, curricular materials, and written cases.

as a proxy for teacher knowledge and effectiveness (National Mathematics Advisory Panel, 2008).

In response to much of this work, Shulman (1987) suggested categories of knowledge that effective teachers require. In addition to knowledge that transcends subject matter (e.g., general pedagogical knowledge, knowledge of students) and knowledge of the subject itself, Shulman argued that there is a specialized content knowledge: "curriculum knowledge" and "pedagogical content knowledge" (p. 8). In the decades that followed the introduction of this compelling idea, educational researchers have embraced the notion of PCK as a critical feature of teacher education and development (Ball, Thames, & Phelps, 2008).

In an effort to refine and clarify Shulman's compelling idea of PCK, researchers in mathematics education have turned to mathematical knowledge for teaching (MKT) in an effort to explain at least some of the impact that teacher knowledge has in classrooms. It is closely related to the roots of situated cognition theory in that it describes a knowledge that teachers can build as part of their practice and that it exists inseparably from it. MKT involves an understanding of typical mathematics that may be common to many educated people, but it also involves flexible and robust knowledge of mathematical representations and how to select and interpret examples. It also involves knowledge of common student misconceptions, how to identify them and remediate them, and the ability to notice connections to more sophisticated mathematical ideas.

Some components attributed to MKT are common to other groups and practices, while others are particular to the work and activities of teaching. This idea that knowledge can be situated within a practice is directly linked to situated cognition theory.

The literature reveals that teachers who engage in development activities designed with this perspective report deeper understandings of mathematical content among teachers who also demonstrate greater skill for and commitments to attending to learning opportunities in their practice (Cohen, 2004; Steinberg, Empson, Carpenter, 2004; Kazemi & Franke, 2004; Chamberlin, 2005; Manouchehri, 2002). These development activities not only involved creating, fostering, and sustaining a community of teachers but engaging them in activities that focus on real issues of practice. They immersed participants in artifacts of classrooms: video from class interactions, copies of written student responses to teacher prompts, and teacher notes and reflections (Featherstone, Smith, Beasely, Corbin, & Shank, 1995; Hammer and Schifter 2001). A similar approach, called lesson study, engages teachers in the planning, teaching, and revision of a lesson; it is a collaborative, iterative process which requires close attention to issues of both content and pedagogy (Lewis and Tsuchida, 1998; Hiebert, Morris, & Glass, 2003).

Whether or not these approaches, when applied to undergraduate content course work, will have similar influence on pre-service teachers remains an open-question (Putnam & Borko, 2000; Wilson, Floden, Ferrini-Mundi, 2001). However, some researchers have investigated the use of artifacts of teaching in methods courses (e.g., Lampert & Ball, 1998; Harrington, 1995). The evidence supports the conclusions generated by work among practicing teachers: that undergraduates demonstrate more sophisticated and nuanced thinking about teaching and have experience with new, important tools for further development and learning as they enter the profession.

These results are not without caveat, however. First, the focus of much of this research has been on the increased confidence of teachers and their greater facility in extracting and making use of student ideas. Little as yet is known about whether or not teachers demonstrate stronger mathematical knowledge, though recent studies with in-service teachers show promising results (e.g. Hill & Ball, 2004; Hill, Rowan, & Ball, 2005). Also, in many cases, the researchers report an important preliminary hurdle: teachers and undergraduate teacher candidates alike lack experience in using these artifacts to make actionable judgments in the classroom. Learning to do this takes time and significant mentorship; teacher education would therefore be well-served by incorporating these approaches throughout its programmatic requirements, from year one courses through internship experiences.

Finally, the introduction of classroom practices in undergraduate courses is not a panacea; teachers are not necessarily more fully-formed as they complete their preparation and certification under these conditions. Yet, as Hiebert, Morris, and Glass (2003) argue, the goal of teacher education can shift from producing fully-fledged teachers to providing "prospective teachers with the tools they need to become increasingly effective mathematics teachers as they enter the classroom" (p. 202).

Studying Undergraduate Mathematics Courses

There are principally four formal opportunities for prospective teachers to learn mathematics: (1) their own primary and secondary mathematical course work; (2) undergraduate content courses; (3) methods courses; (4) and student teaching and field experiences. Thus, mathematics educators have four contexts in which to study and potentially influence prospective teachers' mathematical development. The first option is already widely studied but is not informative about teachers because of the relatively small (and unpredictable) proportion of young students who choose to become teachers and the wide variety of primary and secondary school environments in which they learn. The last alternative—field experience—is a volatile context for studying the mathematical education of teachers because so many more immediate and pragmatic factors overshadow the potential for developing mathematical knowledge (Dewey, 1965; Britzman, 1991; Lampert and Ball, 1998). Because methods courses are generally the particular domain of mathematics educators, researchers often concentrate on making use of these courses to influence and study prospective teachers' content knowledge. Indeed, these contexts have been shown to impact the mathematical choices made by novice teachers (Borko, H., Peressini, D., Romagnano, L., Knuth, E., Yorker, C., Wooley, C., Hovermill, J., & Masarik, K., 2000). They have also been used as sites for investigating the use of case studies and other artifacts of practice (Harrington, 1995; Lampert & Ball, 1998). The temporal proximity of methods courses to teachers' first work experience makes them well-suited to engaging prospective teachers in problems of practice. Despite this convenience, learning to work effectively with these records is not an easy task; researchers report that teaching teachers to make systematic and evidence-based inquiries takes a significant amount of time and energy (Featherstone, Smith, Beasely, Corbin, & Shank, 1995; Lampert & Ball, 1998; Cohen, 2004). Thus, content courses already designated for prospective teachers and taken early in education programs may be an avenue for introducing a stance of inquiry. Moreover, situated learning theory suggests that it may positively impact teachers' mathematical knowledge for use in their future practice.

However, it is notable once again that research about teacher learning in these content courses is under-represented in the field (Wilson, Floden, Ferrini-Mundi, 2001).

Philip, Ambrose, Lamb, Sowder, Schappelle, Sowder, Thanheiser, and Chauvot (2007) have arguably conducted the most comprehensive look at the effects of different treatments on how PSTs learn in their mathematics courses. They describe a comparison study in which PSTs enrolled in a first semester mathematics course for PSTs were assigned different lab experiences connected with their content course.[8] The PSTs were assigned to one of five different groups: the first group was placed into elementary school classrooms to observe teachers and their work. This group of teachers was labeled "reform-oriented," which meant that they had participated "enthusiastically" in "reform-based professional development efforts." A second group observed teachers who conveniently located near the PSTs, and were not necessarily "reform-oriented." The third group of PSTs was assigned to meet regularly to watch and discuss video of children working on mathematics problems. A fourth group was assigned to watch the same videos but also conducted live interviews of children in which they tried to draw out and understand children's thinking. Finally, the last group was considered a control group and did not participate in any extra-curricular work associated with the project. All PST participants completed instruments designed to test PSTs knowledge and measure their beliefs about mathematics. PSTs completed these instruments at the beginning and at the end of the semester course in which they were enrolled. The authors situate this research against the spectrum of laboratory vs. apprenticeship experiences.

Philip, Ambrose, et., al. report on the knowledge of PSTs in a first-semester content course for teachers who were placed in elementary school classrooms as a component of their courses, and others who were not. The authors state that PSTs who visited classrooms noticed similarities between the mathematical ideas in the elementary

[8] This assignment was not random, but depended on PSTs availability and class schedules.

classroom and their undergraduate course, and that PSTs reported feeling a strong connection with the role of teaching and this classroom experience. This echoes my anecdotal experience with service learning and further supports the use of a situated learning perspective: PSTs value the ability to "picture" themselves as teachers and practice filling that role. Ambrose and Vincent (2003) call this an example of "curriculum authentication," which recalls the notion of authenticity envisioned by Brown, Collins, and Duguid (1989). That is, PSTs see these kinds of activities as authentic because they associate the classroom activities with those they expect to encounter as teachers. Generally, the PSTs who visited elementary school classrooms reported a stronger identification with teachers and teaching and an authentication of what they discussed in their content courses.

However, the data suggest that making this emotional connection was the strongest result of visiting classrooms:

> *PSTs wrote about the importance of teaching mathematics in a variety of ways to meet the needs of all students but had little appreciation for what doing so would entail. We posit that [PSTs who visited classrooms], lacking an occasion to discuss their observations, failed to appreciate any phenomena that entailed children's mathematical thinking. We concluded that a structured laboratory environment was more likely to support the beliefs change we hoped to cultivate than a loosely organized apprenticeship structure.* (p. 467 – 468)

What's more, the PSTs that visited classrooms that focused on drill and practice developed beliefs about mathematics teaching that were antithetical to the goals of the university faculty. Ambrose and Vincent conclude that:

> *for bridging the university classroom and the elementary school classroom to enhance the mathematical learning of the PSTs, alternatives to unsupervised visits to elementary school mathematics classes should be considered…The elementary school classroom is a complex environment, and it is not designed to optimize the learning*

> *of PSTs. It is perhaps not the best place in which to situate PST's learning about mathematics.* (p. 30)

The data suggest that the PSTs who concentrated solely on children's mathematical thinking (those watched video alone and those who also interviewed children) scored ¼ of a standard deviation better on the knowledge instrument than their peers who did not engage in those experiences. This is a much smaller effect size than the authors anticipated, but it is a significant finding nonetheless.

Thus, there is empirical evidence that such activities can span the gap between theory and practice, and at least do not negatively impact PSTs' mathematical proficiency. Yet, accomplishing this goal by means of classroom visitation has its drawbacks, and the authors suggest that other environments might be better suited for helping PSTs learn mathematics. The results, while encouraging, are not sufficient for developing a complete picture of PSTs' growth in content courses. Most importantly, the study did not account for the structure of the course in which all participating PSTs were enrolled. It was considered was a constant factor which enabled the researchers to isolate the variable which they intended to study: the extent to which PSTs engaged with children and their thinking. However, the course itself plays a role in the PSTs' development, and this influence is not addressed in the research.

Even if one were able to replicate these PST experiences outside of their traditional coursework, there is no evidence that suggests one could achieve similar results because of the hidden interaction between the course and the extracurricular activity. Moreover, even if the effect sizes were significantly larger in magnitude, what would the implications of such a result be? Should teacher education programs design extracurricular lab experiences for PSTs throughout their mathematics content sequence?

This may be hard to justify in the context of programmatic requirements that seem to increase regularly. If such results were robust across disciplines, PSTs might be placed into such activities for all disciplinary domains, and the problem of simplifying practice by reducing the complexity for PSTs re-emerges: how could they focus on salient aspects of content with so many things going on at once in real-time? What's more, this research does not shed any light on theory vs. practice issues: did PSTs find certain extra-curricular activities particularly motivating or relevant? The idea that we can simultaneously motivate students to explore while teaching them important content-based ideas is a fundamental principle of much of the reform efforts in education over the last decades, but it does not necessarily represent an explicit goal of teacher education. Teacher educators should look for ways to teach important content-based ideas, while motivating and equipping PSTs to learn more independently. One result of Philip, et., al.'s work is that there is much more to explore about teaching PSTs mathematics in content courses. In particular, the field should be looking for ways that help to bridge the theory vs. practice divide, particularly from PSTs' perspective. In addition, an important question that remains unanswered is: which aspects of the courses themselves are influential for PSTs' mathematical knowledge for teaching?

Curricular Materials & Epistemological Issues

One way to learn about what is being taught in these courses is to analyze the textbooks and curriculum materials that are used. McCrory (2006) argues that many recently published mathematics textbooks that are used for PST content courses attempt to cast mathematics as a discipline that is not arbitrary and that is connected in rich and sensible ways. The texts emphasize rigor, definition, and understanding, but "these very characteristics create problems that may be inherent in trying to teach a complex, sophisticated subject to naïve learners" (p. 28). Though texts may provide clear and mathematically correct explanations, "there is no single 'correct' version of this mathematics, and we do not know what confusion is generated over time by the small but significant differences in what teachers are taught" (p. 28).

This tension is also visibly present in the differences between disciplinary mathematics and school mathematics. Moreira and Davis (2008) assert that teaching mathematics in schools can be in direct conflict with mathematics as viewed from a set of definitions and axioms: "To create the real number system from nothing, that is, by postulating its existence as 'anything' satisfying the complete ordered field axioms, ends up in an inversion of what is done in school…academic mathematical knowledge may not be 'naturally' a helpful instrument for the teacher in school practice" (pp. 37-38). Here again, situated learning theory suggests an explanation: school mathematics and mathematics in the academy are fundamentally different things because they take place within fundamentally different communities of practice, which employ different relationships, different structures, different organizations, and ultimately, different epistemologies. This is another component of the argument given by many mathematics educators in making a case for MKT, the specialized knowledge to which teachers—and not most others—have access, by virtue of their direct participation in school teaching.

That school mathematics and academic mathematics are occasionally in conflict echoes some of the tensions seen between the two disciplines in universities across the country: mathematics is a founding discipline of post-secondary education and represents some of the deepest-seated research traditions in academia. Education, on the other hand is a relative newcomer to universities and for many reasons, it still exists on the margins of the academic community (Labaree, 2004). The nature of these relationships makes the study of school mathematics in universities inherently problematic. How is one to reconcile the gap between mathematics as practiced in the discipline and as practiced in schools? How do schools of education acknowledge the importance of maintaining integrity between school subjects and academic disciplines while acknowledging the fundamental differences between them?

Content courses for PSTs are a microcosm of all these tensions: they exist in the intersection of education programs and the disciplinary departments that rightfully feel responsible for the content embedded in them. Content courses must walk a fine line: they are often designed exclusively for PSTs, and yet often they are not designated as education courses; they often serve as PSTs' final experience—after 12+ years of math in schools—of mathematics content course work, but in many respects they recapitulate the earliest of those experiences from elementary school; these courses represent and serve different content constituencies in mathematicians and mathematics educators; they are gatekeepers of mathematical knowledge *and* mathematical knowledge for teaching.

All of these things make mathematics content courses for PSTs fertile ground for study. The tensions that put the courses in a peculiar position between school mathematics and mathematics in the discipline is one good reason for focusing on them. Another is the growing need to understand whether or not the teacher education strategies that show such promise among practicing and student teachers can be applied to earlier experiences in undergraduate teacher education such as these. Philip, et., al. demonstrate evidence that these approaches do indeed have an impact, and yet the field needs to know more about the courses themselves, and how incorporating teaching practice into the courses might play out and what impact it could have on PSTs' knowledge of mathematics for teaching.

Chapter 3: A New Investigation of the Problem

An Overview of the Project

A review of the literature produces questions that remain unanswered. In particular, not enough is known about what PSTs learn in their content courses and whether and how activities that privilege teaching practices have an influence on the PSTs' development of MKT. In addition, undergraduate teacher education is plagued by the persistent notion that it is irrelevant to the tasks of teaching. Again, content courses are implicated: Ball, Thames, and Phelps (2008) argue that

> *…subject matter courses in teacher preparation programs tend to be academic in the best and worst sense of the word, scholarly and irrelevant, either way remote from classroom teaching. Disciplinary knowledge has the tendency to be oriented in directions other than teaching, toward the discipline…* (p. 404)

Situated learning theory offers a means by which this can be explained and suggests that the mathematical knowledge that PSTs develop in their content courses is intertwined with the nature of the activities in which they participate. With this perspective in mind, I have asked the following questions:

(1) What mathematics do prospective teachers learn by engaging in activities of teaching practice such as examining curriculum, student work, and classroom video?
   a. Do PSTs who regularly engage in such activities display evidence of different mathematical proficiency than PSTs who participate in more traditional course work?
   b. Do PSTs engaging in such activities display different mathematical knowledge for teaching (MKT) than PSTs participating in more traditional course work?
   c. Do PSTs engaging in such activities develop different attitudes about mathematics and teaching than PSTs participating in more traditional coursework?

(2) To what extent do prospective teachers see their mathematics course work as relevant to their future work?
    a. Do different course approaches set up differing perspectives among PSTs on the contribution of the course to their future work?
    b. Do different course approaches set up differing views among PSTs about their confidence and abilities in mathematics?

These are empirical questions that require suitable circumstances for data collection and analysis in order to answer. My own teaching is a potential site for investigating these questions. I could set up sections of content courses for PSTs incorporating these varied artifacts of teaching practice and compare these with other sections that do not incorporate them at all. This is unfortunately not ideal, as I do not teach more than two or three sections of these courses in any given not to mention the attending challenges of studying one's own teaching. Similarly, I could design and implement such courses campus-wide at my college, but the resources necessary for designing the courses appropriately and supervising instructors for faithful execution of the design is prohibitive for a project of this scale. It is therefore necessary to seek courses in existence that might present a contrast along the lines I have described. Though at best, this option would offer a quasi-experimental design, it has the advantage of involving courses in existence instead of conjured out of thin air. They would be courses whose reality demonstrates a measure of viability and pragmatism. That is to say, whatever conclusions generated by the project under this scenario are not the antiseptic results of a laboratory, but rather something of a field test.

Finding the context in which collect the data is only the beginning, however. What data is necessary in order to answer the questions I've asked? I believe that a variety of data collection techniques is necessary given the difficulty of measuring knowledge and attitudes. For obvious reasons, knowledge is often measured using multiple choice instruments, though mathematics education has often relied on more qualitative data collection techniques when seeking to understand teachers' knowledge (e.g, Cooney, 1985; Tirosh & Graeber, 1989; Stein, Baxter, & Leinhardt, 1990; Zazkis & Campbell, 1996; Ma, 1999). In addition, the questions assume the existence of differences between two courses, necessitating sources of data that would enable me to compare the courses in question.

The Importance of Setting

The Universities

This study was conducted across two universities in the Mid-Atlantic United States in the spring semester of 2008. Both Hilada University and Rio University[9] are former "land-grant" colleges:[10] large and research-oriented with large proportions of graduate students.[11] Both draw largely from native student populations in their respective states, but also boast of enrollment and retention from all over the globe. Each university houses a school of education in which teachers are prepared for teaching from kindergarten through eighth grade,[12] and both support mathematics educators and their research, offering courses in mathematics education for PSTs, local school teachers, and graduate students.

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[9] Both are pseudonyms.
[10] This terms refers to the Morrill Acts in 1862 and 1890 in which states designated colleges and universities to receive federal funding. Both universities in this study were designated land grant universities following the 1862 Act.
[11] Both universities have enrollments of over 20,000 total students, of which 20% are graduate students at one institution and nearly 1/3 are graduate students at the other.
[12] At Hildada, the College of Education prepares PSTs for certification through 12th grade

The Courses

Both universities offer a sequence of three courses to PSTs in order to fulfill the bulk of their undergraduate mathematics requirements. The first course in this sequence at both institutions considers closely issues of number: ways of representing numbers, relationships between numbers, numeration (number systems), the four fundamental arithmetic operations, and related topics that foster computational fluency.[13] Hilada offers this sequence through the mathematics department, and the courses are often taught by graduate students in mathematics. Here, I call it MATH 281: Fundamentals of Number and Operation.[14] The stated purpose of MATH 281 is to explore and explain why typical algorithms applied in school mathematics "work." Preliminary observations suggested that this course as implemented at the time of study made relatively little use of teaching artifacts. PSTs interacted only occasionally with student work, video of classroom interaction, teachers' notes, curricular materials, or other windows into legitimate teaching practice. At Rio University an analogous course to MATH 281 is offered as a mathematics course, but it is taught in the University's school of education (MATH 291). Graduate students and tenured faculty in mathematics education teach the course. The mathematics education unit at Rio has organized research projects around the use of classroom artifacts in MATH 291 and its partners in the sequence. Though the subject matter is decidedly mathematical, mathematics educators in charge of the course are using the tool of lesson study[15] throughout the teacher education program to examine teacher learning and the changes in PSTs' analysis of teaching situations. Each

[13] There are clear differences at the two institutions, but the bulk of each course is devoted to these topics, broadly described by the NCTM Principles and Standards for School Mathematics (2000) in the content standard of "Number and Operation."
[14] The course numbers and names given here are pseudonyms.
[15] In the spirit of Lewis and Tsuchida (1998).

university typically offers three to six sections of the course each semester. PSTs who enroll in the courses, by virtue of the typical trajectory though their undergraduate coursework, take the first course in the sequence in greater numbers in the fall. Spring enrollments are often lower, and often include students who have not successfully completed the course in the previous semester.

Though the location of these courses at the different universities (one in a mathematics department, one in the school of education) is important, the fact that they appear to take different approaches to similar course material is the primary focus in this study. Within these two existing courses is an opportunity to test situated learning theory and its implications for mathematics teacher education. Both are large universities, drawing students primarily from the Mid-Atlantic region, but these courses are housed in different university units and are typically taught by graduate students in related, but distinct fields of study.

The Instructors and Students

In the semester in which this study was conducted, there were four sections of MATH 281 at Hilada University enrolling nearly 100 students. MATH 291, at Rio University, enrolled about 75 students in three sections. My observations in these courses for pre-service elementary school teachers supports the conventional wisdom and prevailing demographic research: PSTs enrolled in these sections were overwhelmingly white and female[16]. The instructors at Hilada University were graduate students in mathematics who fulfilled their responsibilities to their graduate assistantships with these instructional positions. They met occasionally throughout the semester to discuss

[16] For example, please see http://nces.ed.gov/programs/coe/2007/section4/indicator33.asp for recent national data on teacher characteristics.

relevant issues and were coordinated directly by a full-time lecturer at the university with a PhD in mathematics education. This faculty member wrote course notes for the MATH 281 instructors, and otherwise advised the instructors about the material as it approached during the course. Each instructor taught two sections of MATH 281. At Rio University, there were two instructors responsible for three sections, an adjunct instructor who was recently a teacher in local elementary and middle schools who was supported by the other instructor, a full-time tenure-stream faculty member with a PhD in mathematics education and who has a research interest in the courses. The full-time faculty member taught two of the three sections of MATH 291 offered at Rio.

Strand One: Are the Courses Different?

I have hypothesized that different course approaches will result in differing outcomes for students. In order to make such an assertion, I have to demonstrate that the courses were indeed different. In order to show this, I designed data collection to enable such differences to emerge. Using different methods, I gathered information that would characterize each course: I conducted 34 class observations over 21 different class days and collected syllabi, assignments, exams, and other handouts throughout the semester. Together, I believe that these data give some insight into each course, their goals, and the techniques used by the instructors to achieve these goals.

Observation

At Hilada University, I took notes during 19 observations across 12 different days spread throughout the spring semester and among all four sections of MATH 281. This yielded 55 pages of field notes, which included lists of assigned problems from the textbook. At Rio University, I observed MATH 291 15 times over nine different class days throughout the semester and across all three sections. These observations yielded 45 pages of field notes regarding class discussions, homework assignments, and other related information.

Document Analysis

During my observations, I collected all major course documents such as syllabi, course projects, papers, and exams, in addition to a sample of quizzes and homework assignments given during the class meetings for which I was present. Finally, the notes used by instructors show intentions for the course by instructors and course designers. Since the project was limited in the number of observations that could be carried out during the semester, the extent to which the observations match the course notes will indicate whether or not one can reasonably interpolate classroom trends and activities without direct observation. Together with my observations, these documents offer a set of data that allows me to describe differences between the courses as they pertain to the use of artifacts of teaching practice. I hypothesize that the differences in the courses will result in measurably different outcomes on other instruments that form the basis of the data collection.

Strand Two: Do the PSTs Develop Different Mathematical Knowledge for Teaching?

Teacher knowledge is hypothesized to be a critical component of student learning. The focus of this study is to learn more about what influences teacher knowledge of mathematics for teaching. In order to determine differences in student knowledge of mathematics for teaching between the two institutions, I administered two rounds of instruments at each university, once at the beginning of the semester, and once at the end. The purpose for this was to discriminate between differences that existed between PSTs that were attributable to the courses themselves, and any relevant pre-existing differences between the cohorts.

Each round involved a set of identical multiple-choice items developed by the Learning Mathematics for Teaching (LMT) project at the University of Michigan. This project has created a large group of items designed to measure mathematical knowledge for teaching in elementary grades (Hill & Ball, 2004). Because elementary mathematics encompasses a wide variety of topics, and the courses in this study concentrated on number and operation in particular, I chose a narrow subset of items from the LMT collection. These items were chosen using three important criteria: (1) each item had to connect to the curriculum at both universities; (2) a collection of items that had a wide range of difficulty, as determined in pilot testing at the LMT project; and (3) a collection of items with a strong reliability score, based on statistical analysis offered by that project. With those criteria, 31 items were chosen to form an instrument that I am calling the Mathematical Knowledge for Teaching Instrument (MKTI).[17]

These items do not include mathematics related to rational numbers represented as fractions, for this was not explicitly discussed in MATH 291. Items involving finite decimal representations of rational numbers were included as this was territory covered by both courses. In addition, questions about arithmetic operations, appropriate representations for quantities and arithmetic sentences, alternative algorithms, and viable explanations for mathematical conventions figure prominently in the MKTI, as they were

[17] The instrument cannot be reproduced here as a result of restrictions placed on my use of the items. Released items from the Learning Mathematics for Teaching project are available at the following site: http://sitemaker.umich.edu/lmt/files/LMT_sample_items.pdf

important aspects of both courses. The mathematics identified by most items was directly connected to each course, although a few were peripherally so. Some of these items were included to test the boundaries of PSTs' knowledge, and others were included with an assumption that they would be discussed, when in some cases, it appears that they were not. An example of the latter is the first set of items, which focus on order of operations; I have no evidence that this was a topic of consideration in MATH 291, and I did not witness the lesson in which it was a topic in MATH 281. Finally, some items were included because they were part of a collection whose statistical integrity demanded that they remain together. A few items were chosen indirectly for this reason.

Each PST's response to the instrument was recorded at the beginning of the semester and again at the end of the semester. The collection of responses at each point was given a raw score based on how many of the 31 items were correctly answered. The family of items from which the MKTI items were drawn were designed to be very difficult, and the LMT project reports that their collections of items (which include a broader selection of topics than I incorporated into the MKTI) are designed so that mean scores will hover near 50% correct. Raw scores are not reflections of any standard of knowledge and are helpful primarily in comparisons between and among people. For this reason, I do not report the raw scores here, as they—in themselves—may be misleading about PST knowledge for teaching. This project is concerned with differential achievement between the courses and thus, my focus is on the growth that can be measured among PSTs in these courses. The raw scores themselves are thus unimportant. Rather, the changes in these scores across time and institutions are the measures I will be using.

Despite their utility on many levels, multiple-choice instruments do not give a complete view of anyone's knowledge in any field. In order to augment the picture given by the MKTI, I used interviews with selected students in each course near the conclusion of their work that semester. These interviews involved asking PSTs open-ended questions about mathematics teaching situations: questions designed to elicit explanations from PSTs about how they would employ their knowledge in a teaching scenario. These interviews were conducted with two purposes in mind, only one of which was to learn more about PSTs' mathematical knowledge for teaching. I chose three interview prompts designed to elicit PSTs' thinking about mathematical ideas contained in the courses. One item is very similar to items found on that multiple-choice instrument. Its inclusion was intended to gather more information about PSTs' thinking about such items. The middle item was chosen to learn about the extent to which PSTs attended to children's thinking, while the final item in the interview protocol was generated in response to observations during the course of the semester.

Interviewees were chosen by compartmentalizing scores on the MKTI at the beginning of the semester. One PST was recruited from each of the following segments: scores within one standard deviation of the mean raw score, scores more than one standard deviation below the raw score, and scores more than one standard deviation above the raw score. This approach was designed to gather data on PSTs that could represent the spectrum of MKT as measured by the multiple-choice instrument.

Strand Three: Do the PSTs Develop Different Attitudes about Mathematics and Teaching?

Situated learning theory suggests that becoming a part of a community requires that one begin to identify with the community into which she is to be initiated. Teacher education literature is replete with evidence that undergraduate courses are not effective in helping teachers connect to the teaching community of which they seek to become a part. This means that in addition to the challenge of helping PSTs develop mathematical knowledge, undergraduate teacher education must confront the problem of presenting PSTs with experiences that they recognize as authentic; a form of legitimate peripheral participation in the teaching practice.

In order to determine the efficacy of each course in its effort to connect PSTs to teaching, I administered two rounds of a survey which asked PSTs about their attitudes related to mathematics, teaching mathematics, and how well their course helped them to prepare for their teaching practice. The survey responses were formatted on a Likert scale from one to five. The items were drawn from a variety of sources, but primarily from Zambo's beliefs instrument (1994), which focuses in large part on problem-solving and elementary mathematics. These items were helpful in contrast to many other beliefs and attitudes surveys because they focused on ideas about practice rather than referring to the respondent's actions in practice, to which the PSTs as yet would not be able to respond. These items are described in more detail in Chapter Five, and are located in their entirety in Appendix A. In pilot surveys, the instrument was found to have a Cronbach alpha reliability coefficient of .8, which was a threshold met during each administration of the survey in this study. Changes in these attitudes could be tracked by finding differences in mean response scores from the entire instrument.

The survey also sought to identify the extent to which PSTs recognized their courses as authentic activity in their developing teaching practice. There were four items written for this purpose, and were separated from the rest of the items in the survey to signal a change in the focus of the questions. These four Likert-type items were augmented by open-ended items designed to elicit ideas from PSTs about how they would summarize and describe the course, and what aspects of the course were most memorable (See Table 11 on page 174). I expected to learn more about what particular assignments and activities impacted PSTs most during these experiences. The clinical interviews also were designed to shed light on this question, as prompts designed for the interviews were aimed toward an evaluation of the course in terms of how PSTs felt they were prepared for the tasks of teaching.

Analysis

Because I propose a wide range of data collection methods, the methods of analysis will necessarily be diverse. Statistical investigations of the mathematics content assessment data and Likert-based survey responses are supplemented by a qualitative analysis of MKTI item responses, the open-ended survey items, classroom observations, and interviews with PSTs.

Recall that the classroom observation data is intended to provide a description of the typical interactions, discussions, and activities during class. The observation instrument is not intended to provide a "thick description" of each class in the sense of an ethnographic approach. Rather, its purpose is to establish differences between the two classes in terms of activity and interaction.[18] Field notes and course documents were collected and synthesized to provide a description of the course that, while not complete, captured the essence and typical ways of working in each course. Descriptions of the courses that follow were sent to coordinators of each course for comments, which were subsequently incorporated into the description.

The MKTI scores were the basis for a statistical comparison of class means which enable me to determine what difference(s) between and within institutions existed at the beginning and end of the semester respectively. These comparisons have been accomplished using ANOVAs. Before doing ANOVA, several assumptions were addressed: the cases must be independent, the scores in the population must normally distributed, and variances in the populations must homogeneous. The first assumption is addressed by design – PSTs complete the assessments independently. The condition of normality can be justified by calculating skewness and kurtosis statistics. The homogeneity of variance can be tested using Levene's statistic. For the Likert-scale survey data, ANOVAs are similarly appropriate, if the same conditions are met. Reports on these conditions will follow, with the accounts of the data. In either case, when conditions are not satisfied for ANOVA, the Mann-Whitney test is often used as a non-parametric substitute (Wackerly, Mendenhall, & Scheaffer, 1996).

In analyzing the responses to open-ended survey items and interview prompts, a qualitative approach is necessary. One analytical stance I take with these data is adapted from Fennema, Carpenter, Franke, Levi, Jacobs, and Empson (1996). There, the authors describe a four-tiered scheme in which they analyze teachers' ability to acknowledge and

[18] Should the results of this study demonstrate a dramatic need for such a description, other methodologies could be employed in future projects.

incorporate children’s thinking into their teaching. This is a major component of mathematical knowledge for teaching, and as such plays a role in the items I used in the data collection. The categorization of these responses was viewed in part using the following categories, adapted from Fennema, et., al. (1996):

(1) Does not believe children can solve problems without instruction

(2) Struggling with the belief that children can solve problems without instruction

(3) Believes that children can solve problems without instruction, in a limited way that students’ thinking can be used to make instructional decisions.

(4) Believes that children can solve problems without instruction across mathematical content domains.

However, a more robust scheme for analyzing this qualitative data may be given by Ball, Thames, and Phelps (2008), who offer a map of the landscape of MKT. They argue that PCK—as originally described by Shulman, and as it relates to mathematics—is a sub-domain of MKT. In the figure below, the three domains on the left-side (Common Content Knowledge, Horizon Content Knowledge, and Specialized Content Knowledge) represent subject matter knowledge, while the three right domains (Knowledge of Content and Students, Knowledge of Content and Teaching, and Knowledge of Content and Curriculum) represent PCK as Shulman described it.

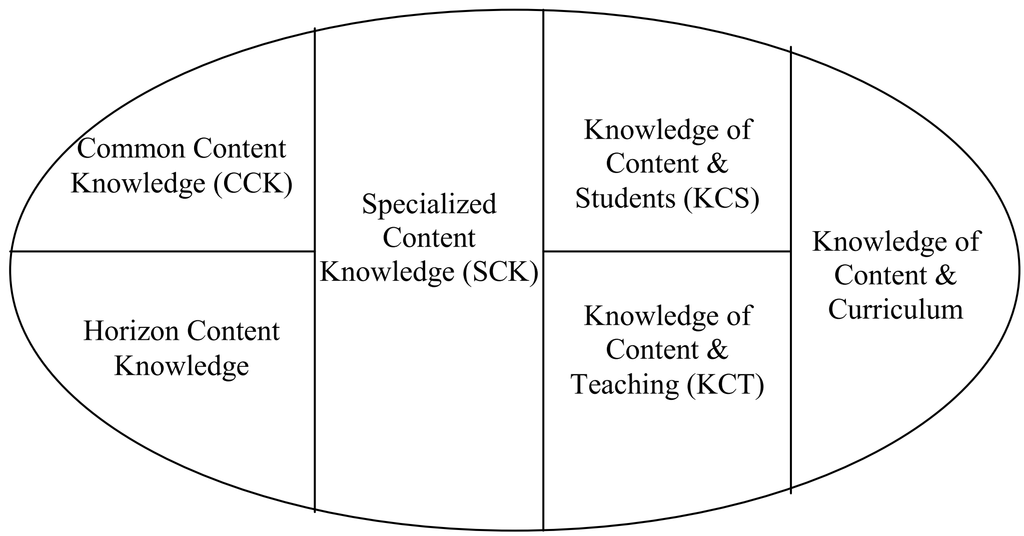

Figure 1: A Descriptive Map of MKT

Common content knowledge (CCK) is described as mathematical knowledge to which any educated person would have access, and is not specific to teaching: we would certainly expect that teachers could reliably and efficiently perform arithmetic computations, but we would expect many other people to be able to do this. For example, the ability to compute the product of 14 x 37 would be considered CCK. Moreover, if someone were to make a mistake in this computation, the ability to identify that a mistake had occurred would also be part of this category of knowledge, even if (or maybe especially when) the evaluation process is relatively unsophisticated: "If my answer differs from someone else's, and I am confident in my answer, then the other person has probably made a mistake." Computation is only one facet of elementary mathematics, and so CCK also extends to knowledge of basic mathematical facts like "even numbers are divisible by two;" or "the integers are composed of whole numbers and their opposites."

Specialized content knowledge (SCK) might be described in this example as one knowing that the standard algorithm (shown below) involves a decomposition of the numerals into tens and ones and corresponds to an application of the distributive property which can be represented as (10 + 4) x (30 + 7).

$$\begin{array}{r} 14 \\ \times\ 37 \\ \hline 98 \\ 420 \\ \hline 518 \end{array}$$

These ideas are important and relevant for understanding the standard multiplication algorithm, ideas that are not necessary for computing the product accurately. In this way, SCK represents something of a departure from CCK. Morris, Hiebert, and Spitzer (2009) write that SCK is

> *content knowledge of a particular kind. It is implicated in common teaching tasks such as choosing representations of mathematical ideas that reveal key subconcepts of the ideas, evaluating whether student responses show an understanding of key subconcepts, and justifying why arithmetic algorithms work. It involves unpacking or decompressing mathematical knowledge in order to make particular aspects of it visible for students or to identify the source of students' difficulties.* (p. 494)

Ma (1999) has referred to an ability to "unpack" knowledge, which is the kind of ability that SCK is intended to capture. Specialized content knowledge does not require any particular knowledge of students or teaching, and this property makes it a useful target of content courses for PSTs (Morris, Hiebert, & Spitzer, 2009).

Horizon knowledge refers to knowledge of how a topic or procedure relates to other, more sophisticated (or more general) applications, which are likely to occur on a student's "horizon." In this example, it might refer to knowledge of the fact that this problem has a correspondence with the procedures for operating with polynomials in algebra, such as $(2x - 6)(3x + 7)$. Knowledge of content and students (KCS) refers to the ability of teachers to determine how children are likely to think about a particular topic, including common misconceptions and which of these her students are most likely to develop. For example, a teacher may need to be able to determine what logic is behind the following mistake:

$$\begin{array}{r} 1\;4 \\ \times\; 3\;7 \\ \hline 2\;1\;8 \\ 6\;2\;0 \\ \hline 8\;3\;8 \end{array}$$

Continuing with this example, knowledge of content and teaching (KCT) might enable the teacher to determine what examples, representations, or scaffolding procedures (and in what order to present them) might help the student understand the difference between his fallacious method and the standard algorithm. Ball and her colleagues (2008) have shown empirically that these divisions in people's knowledge exist. One prominent example of their results is that they have found that mathematicians' knowledge about elementary mathematics is largely confined to CCK. Thus, the researchers conclude that these other sub-domains are primarily the territory of teachers.

This map of MKT will be the basis for much of the analysis of the data that follows: what kinds of knowledge are being developed in the courses? What kinds of knowledge do the PSTs demonstrate in the interviews? Are there differential responses on the MKTI depending on the type of knowledge (CCK, SCK, KCS, KCT, etc.) targeted by particular items?

Limitations

The Effects of Teaching

First and foremost, though the study emulates an experimental design, it is not. Though this study is not conceived in the tradition of Gage (1963, as reported in Floden, 2001; and Hamilton & McWilliam, 2001), I believe that the question and associated methodology can be associated with an “effects of teaching” tradition (Floden, 2001). The history of such research in education has long been criticized for its lack of scientific rigor. This study is unlikely to speak to those seeking the conclusions generated by large-scale, experimental research. The PST participants were not randomly assigned to “control” or “treatment” groups; in fact, there is no “treatment” in this classic sense. The comparison in this study is not whether or not a particular intervention is better than doing nothing, but a comparison between two differing perspectives on how PSTs learn mathematics for teaching. The study cannot and does not seek to reveal how the PSTs in one course would have done or what they would have learned in the other course. However, given the similarities between the universities, the general homogeneity in the demographics of the population of PSTs, and the differences manifest between the courses, there is nonetheless an opportunity to learn about potential effects. I believe that these features together with the breadth of data collected enable me to draw at least tentative conclusions about the efficacy of these courses.

Thus, generalizability of any results in this project are rightfully questionable; these courses are not necessarily representative of other courses offered at other institutions around the country. The lack of coherence among teacher education programs generally is widely criticized, and among content courses early in those programs, there may be greater cause for concern regarding consistency. This project only tests claims implied by situated learning theory in the context of these courses at these universities during this given semester, and does not claim to offer best practices in mathematics teacher education. On the other hand, to the extent that the courses are successful in their stated missions, they may be viewed as exemplars for other institutions to emulate. In cases where there is evidence that one course produces differing results than the other, I leave to others judgments about which is a "better" outcome.

The "Test-Retest" Effect

A potential weakness of the project is the fact that PSTs completed identical instruments at the beginning and end of the semester. The same MKTI items were used at the beginning of the semester and at the end, and the vast majority of items on the attitudes survey were identical at both endpoints of the semester. There is arguably a test-retest effect present in the scores on the MKTI instrument, though the LMT project reports that over a span of months, this effect is minimal (G. Phelps, personal communication November 21, 2007). In addition, this test-retest effect would presumably apply equally to both groups of PSTs, and therefore should not bear upon the relative results from each course.

The Use of Likert-type Surveys to Measure Attitudes

Philip, Clement, Thanheiser, Schappelle, and Sowder (2003) claim that Likert-scale surveys are fundamentally flawed because they lack context and deny participants an opportunity to explain or justify their choices. In addition, "Likert items do not carry with them good ways for assessing the depth with which one holds a belief. One may respond in a way that indicates the existence of a belief that is not central to the respondent" (p. 5). However, no one instrument can fully address all of these criticisms. Still, the use of a Likert scale survey does provide some advantages in this study: first, it enables an efficient collection of data about PSTs' perceptions about the course and its ability to help them in their preparation. Undergraduates can be difficult to recruit for participation in extracurricular research, so a simple method of collecting this data is important.[19] In addition, the Likert format lends itself to analyses that are not possible in a more qualitative context. I have designed a study that investigates two primary things: (1) PSTs' mathematical knowledge for teaching and (2) PSTs' perceptions about the relevance of certain approaches to teacher education—during content courses in particular. I believe that my research design incorporates numerous data sources for answering each question, which provides a sort of triangulation and strength to the conclusions that I can draw from it.

Determining the Impact of Content Courses on PSTs' Teaching

Finally, one might critique the use of interviews as a measure of PSTs' ability to use their knowledge in teaching situations. The adage "only time will tell" applies here; we cannot know how well these undergraduates will perform in their own classrooms until they actually set foot in one as the teacher of record, with all attendant authority and responsibility. Hill and Ball (2004) have called measuring mathematical knowledge for teaching in this way a "secondary measure," and assert that other tools to bridge the gap between knowledge and classroom action are not only useful, but necessary. However, the research questions which I have made the focus of this project are not meant to address this issue, and thus such concerns are beyond the scope of the project. Still, one

[19] Philip, et. al.'s project worked with much greater funding with which to draw participants than did the project reported here.

might respond to them by noting that the study *does* test questions raised by the rhetoric surrounding "best practices" research, and thus might indirectly bring into question the suggestions made by this literature.

Hypotheses from a Situated Learning Perspective

At this point, looking back at the project through the theoretical lens I have employed is appropriate. I have designed it as a test of situated learning theory in the context of teacher education and it is reasonable to consider what kinds of things one might expect from the circumstances given.

Situated learning theory argues that all learning is situated in some context, and the context in which that learning takes place is an integral part of what is learned. I have argued that this perspective has informed much of research and reform in teacher education, particularly among in-service teachers. This research shows that teachers learn important mathematics and gain valuable confidence in their knowledge when they learn mathematics that is placed in the context of their work as teachers. This enables them to connect with mathematics in ways that are intimately related to their day-to-day work, but also provides an avenue for becoming students of their own work, and that they will find new opportunities to learn within the contexts of their classroom, as opposed to approximations of it. Though PSTs are unlikely to be able to attend to all of the important details in a real classroom, they can begin to take salient features of classroom situations and work that highlights subsets of the knowledge they need to develop as they enter teaching. In other words, asking PSTs (especially early in their undergraduate programs) to competently observe a real classroom may be unrealistic, but giving them access to artifacts of classrooms, stripped of many of the classroom management concerns, whole-school responsibilities, student dynamics, and others of the most complex interactions, PSTs can begin to tackle important mathematical ideas that arise in classrooms. Research has demonstrated that PSTs and practicing teachers alike favor practical, on-the-job experience to their undergraduate course work, citing the former as more influential experiences than the latter on their teaching.

Given two undergraduate mathematics courses that are in stark contrast to one another along these dimensions, with a situated learning theoretic perspective, one could reasonably make the following hypotheses:

1. PSTs in a course that more closely identifies with practices of teaching will perform better on measures of Mathematics Knowledge for Teaching, than PSTs in a course that does not.
2. PSTs in a course that more closely identifies with practices of teaching will develop different attitudes about mathematics and teaching than PSTs in a course that does not.
3. PSTs in a course that more closely identifies with teaching will be more likely to reflect on the experience as a valuable one than PSTs in a course that does not.

These hypotheses correspond *roughly* to the research questions I outlined in Chapter One:

(1) What mathematics do prospective teachers learn by engaging in activities of teaching practice such as examining curriculum, student work, and classroom video?
   a. Do PSTs who regularly engage in such activities display evidence of different mathematical proficiency than PSTs who participate in more traditional course work?
   b. Do PSTs engaging in such activities display different mathematical knowledge for teaching (MKT) than PSTs participating in more traditional course work?
   c. Do PSTs engaging in such activities develop different attitudes about mathematics and teaching than PSTs participating in more traditional coursework?

(2) To what extent do prospective teachers see their mathematics course work as relevant to their future work?

a. Do different course approaches set up differing perspectives among PSTs on the contribution of the course to their future work?
b. Do different course approaches set up differing views among PSTs about their confidence and abilities in mathematics?

As the data will show, the differences between the two courses—especially with respect to their relative immersion in the practices and artifacts of teaching—are nuanced and not as clear cut as the hypotheses (and research questions) above assume. As I will describe below, the courses, though different, were certainly not unrecognizable to one another: they discussed much the same mathematics, employed many of the same tools, and suggested a coherent approach to mathematics for PSTs to experience. However, those differences might yet explain differences in the data, even if the courses are not different along the dimensions I have described in the extreme.

However, the hypotheses generated by an extreme example may point in the same direction as the context reported here, though the magnitude of the differences is substantially smaller. And yet, another possibility is that the data shows that the hypotheses are not borne out, and potentially that they are misdirected altogether: maybe PSTs who concentrate exclusively on mathematical issues devoid of their teaching context would demonstrate measurably better scores on MKT measures than their counterparts, etc. This would require a re-examination of situated learning theory as a tool for understanding teacher learning of mathematics. Still, even in this scenario, the data collected as I described above should provide evidence of such an outcome, even if it is manifest in relatively small differences. If the courses are indeed different, the learning theory predicts that the PSTs would have learned measurably different things about both mathematics *and* teaching. In addition, we might expect that a course that "feels" closer to the experience of teaching will be embraced more closely by PSTs than one that does not. Yet again, the methodology has not pre-determined such an outcome: the data collected as I described above can help determine whether or not such an expectation is borne out empirically.

Moreover, the theory shines a bright light on the contexts in which these courses are set: context is critical to understanding learning from the situated perspective. MATH 281 is located in a mathematics department at a large, research-oriented university, designed by working mathematicians, and taught by graduate students in mathematics; MATH 291 in the school of education at a large, research oriented university, designed by mathematics educators and taught by mathematics educators, and mathematics education graduate students. These different settings put those in charge of the courses (departments, faculty, instructors) in different places, and may give them different perspectives on not only what elementary school mathematics looks like, but what *mathematics* is about.

Elementary school mathematics can be viewed with all the coherence, beauty, and power that is contained in the forms and representations invented for rigorous mathematical reasoning. Fundamental arithmetic can be understood by investigating the properties of numbers and the operations associated with them, as in abstract algebra. This was arguably the motivation of important mathematics education movements set in motion by Bourbaki and the new math in the mid-twentieth century. However, that is not the only perspective on school mathematics. One might also view these elementary ideas from a very different starting point: how children build and develop ideas about arithmetic through working with broader sets of numbers. At the time of the "new math," there was little to no research or understanding about how children built these ideas from imprecise, but ultimately useful intuition. However, in the last 20-30 years, a great deal of progress has been made within mathematics education, which has looked closely at the learning processes of children in mathematics. Both perspectives can be marshaled to present clear, coherent, and—most of all—meaningful ways to think about fundamental mathematical ideas that attempt to maintain the integrity of mathematics as it is practiced in the discipline. Yet, they are also fundamentally different perspectives and thus, using a situated learning stance, one would expect that PSTs would learn different things as part of these different environments. Again, these descriptions are unlikely to describe MATH 281 and MATH 291 with precision, but their different contexts may demonstrate elements of these different perspectives, and again, the data collected should provide evidence that helps to illuminate the consequences. Next, I will turn to an examination of the data, and what it reveals about the courses and the PSTs who completed them.

Chapter 4: A Tale of Two Courses

Two sections of the same course at the same university are never precisely the same, and are often quite different. It is no surprise then that two different courses offered at two universities—while they play similar roles in the certification of elementary school teachers—are substantially and meaningfully different. A glance of the syllabi for each course makes it clear that they diverge even in terms of mathematical topics that form the scope of each course. With these underlying variations, the usual distinguishing features of instructors, student personalities, setting, and other related class characteristics will by necessity create very distinct environments for learning mathematics. On the other hand, the two courses in this study were designed for identical purposes and focused in particular on number and operation in elementary school mathematics. They enrolled similar populations of students who were learning to become elementary school teachers and who were beginning to take on the mantle of the profession, even if they were on the wide periphery of this community. When comparing students in these courses along the strands I have identified (mathematical proficiency for teaching and attitudes about mathematics and teaching), it may be possible to connect these distinctions with the different courses they took. The purpose of this research is to determine whether or not any of these connections exist empirically.

Course Notes

At both Hilada and Rio Universities, instructors of MATH 281 and 291 respectively had extensive notes as a resource for each class meeting. The notes outlined what the class was supposed to do during each class meeting, including salient examples, illustrative activities, and homework assignments. At Rio, the notes were written by a committee that both designs MATH 291 and uses it as a site of ongoing research through lesson study. At Hilada, the notes were primarily a product of the course coordinator, who also met regularly with instructors of the course in a given semester to receive and give feedback on using them. The value of these notes at each university was quite high: despite the occasional departure, and accounting for typical set-backs and differences in teaching style among instructors, there was a consistent correlation between the trajectories of activities suggested by the notes what happened during each class I observed.

There are numerous examples of this at each university, but I will highlight two here. In MATH 281 at Hilada University, the notes for Chapter Five cover approximately five days of activity. On the fifth day, the class is supposed to turn its attention to alternative algorithms for multiplying 16 and 24. The notes begin,

> *[Go to the student] Packet, Page 37: Begin by having students work in groups to analyze the student-invented algorithms pictured. Ask them to identify the properties used…Student #3 is probably the most difficult to understand. Apparently they began by writing the column of 4 24's in the center of the work. Can you work it out from there?*

One day in early April, the instructor began class by asking students to open their packets to page 37, and asked the class to discuss in groups how each student solved the problems shown there. As the students begin to work, the instructor, apparently prompted by the course notes said the following: "In #3, they did the middle 24s first." In the section immediately preceding this class (which this same instructor also taught), the instructor began a discussion of #3 by asking, "What about #3? What if I said they worked from the middle outward?" The instructor had not only introduced precisely the activity suggested by the notes, but had provided the hint described by them. Much of the rest of the class that day was devoted to writing coherent number sentences that made explicit each arithmetic property implicitly used by the students in these alternative algorithms, which is also highlighted in the notes:

> *In particular, it is common for students to abuse the "equals" sign, treating it as they do the equals key on the calculator: as a signal to calculate what they've got so far. It is important for students to learn early, however, that the equals sign signifies both sides are truly equal: this is an essential concept in algebra. With this in mind, consider how to notate the student work on the following pages.*

Similarly, at Rio University, instructors in MATH 291 closely adhered to their course notes. Lesson 11 describes the two- to three-class arc related to exploring the meanings of and connections between multiplication and division. There, toward the end of the lesson, the instructor should give the students three sets of division number sentences, asking them to construct three word problems for each, for a total of nine word problems. Each number sentence should be given a context in which the students think children will solve the problem using a repeated subtraction model of division, a partitioning model of division, and finally a problem in which children will solve the problem using multiplication. In addition to constructing the word problems, students are supposed to draw diagrams that represent the part-whole relationship expressed by the problem. This lesson took place at Rio within a week of the lesson I described at Hilada above. That day in class, the instructor introduced to the students the terms "repeated subtraction" and "partitioning" and their associated part-whole representations, just as notes suggest. The instructor then asked students to work in groups to complete the activity precisely as described above.

These are but examples in a catalog of ways in which the classes unfolded largely according to the notes written for each course. The general arc of each observation can be traced to the instructor's course notes, and in many cases the specific examples given in those notes were used in class. While it was rarely, if ever, the case that instructors were reading directly from notes in their presentations to students, the instructors generally had sets of notes to which they occasionally referred during class meetings, and they were otherwise familiar with the trajectory of the course, presumably through the use of the course notes. This adherence to the course notes was consistent throughout the semester, across sections and universities. For this reason, I have confidence inferring the occurrence of events in each course that I did not directly witness by way of these notes. Certainly, the presence of an activity in these documents does not guarantee that it took place in any given class. Likewise, the absence of something in the notes does not ensure that it did not occur. However, the collection of notes together says many things about the material as it is presented to students, and the mathematical and pedagogical values that are brought to the fore in a semester-long experience.

When comparing the courses at the different universities, I make claims along differing data dimensions: course notes are an important source of this data as well as syllabi and exams, class observations, and comments from students regarding messages they received about what topics and techniques were important in the course.

Similarities

In broad terms, the courses are very similar, which is an important feature that makes them valuable for comparison. Both MATH 281 and 291 focus on elementary number and operation: both devote substantial effort to understanding the meanings of addition, subtraction, multiplication, and division on whole numbers, integers, and decimal representations of rational numbers. In MATH 281, more than half[20] of the course schedule was devoted to addition, subtraction, multiplication and division on the syllabus. The MATH 291 course calendar also set aside most of the semester for these topics. Both courses made explicit their goals for developing important mathematical knowledge for teaching in PSTs. The first page of the class activities supplement used in MATH 281 states:

> *In Math [281] and [282] you will be expected to be able to explain and explain why a problem is done a certain way, in addition to being expected to do the problem. As you work on problems in class and on homework, don't be satisfied with getting the correct answer; ask yourself why that method is logical, and how you could explain that logic to someone else.*

A similar statement is found on the MATH 281 syllabus:

> *Throughout this course…you will be asked to 'explain why or why not' or to 'justify your answer.' In other words, you will be expected to understand <u>why</u> the procedure you are using works or why the answer you give is correct…Seeking connections and meaning can be a very rewarding way to learn—and someday teach—these math ideas.* (emphasis in the original)

The latter statement is offered as the "philosophy" of MATH 281. Statements such as these made explicit the need for PSTs to begin developing skills for becoming expositors of mathematics. It is not enough simply to *know* the answer—the suggestion is that this course is a departure from typical mathematics courses—but PSTs must learn to explain *why* the answer is what it is, and why the method used to arrive at that answer works. The message is that these skills are *especially* important in a course for PSTs; their jobs as teachers will depend on them. The syllabus of MATH 291 at Rio University also points PSTs in the direction of teaching:

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[20] The course began on January 28 and the final exam date was May 15. These four operations account for class meetings spanning more than two months of this time.

> *MATH [291] may be different than any course you've had before…not the kind of mathematics you've studied before. In this course, you will learn the mathematics needed to become an effective teacher…It is mathematics that helps teachers understand how their students are thinking…helps teachers see how the different topics in elementary and middle school mathematics fit together…helps teachers to re-examine what they have learned before so they can understand the underlying concepts, and so they can effectively support their student' learning.*

MATH 291 was also a course in which explanation and justification are highly valued:

> *You will…examine patterns and structure; formulate generalizations and conjectures…and construct and evaluate mathematical arguments…be asked to explain your reasoning—how you were thinking while you were solving a problem…and why you think some methods for solving problems work better than others.*

Such statements are not surprising; these courses were designed specifically for this group of undergraduates, and as such, should feature learning goals that highlight skills and knowledge that teachers must build. Moreover, the goal of developing mathematical understandings that enable them to investigate, explain, and justify is neither unusual nor improper for such a course. They are the academic standards set out by mathematics departments across the country and form the basis for much of the NCTM Standards. But these courses are not necessarily intended to build PSTs' common content knowledge. Recall that the knowing that an answer is correct might be considered as part of this sub-domain of MKT; these courses set out to push PSTs beyond such understanding into other sub-domains. Both syllabi refer to understanding that is required for teaching children mathematics, and in this way, seek to orient PSTs to a different kind of knowledge.

In order to achieve the goals set out for PSTs, the courses were structured in similar ways: they met multiple times each week for lectures and small group activities, PSTs were expected to complete homework assignments, in-class quizzes, and common exams among sections. Exams were overwhelmingly large components of the course grade at both universities. In MATH 281, exam scores counted for nearly 75% of the final course grade; in MATH 291 this figure was 85%. The sections at each university were organized into similar sizes (20-30 PSTs), and PSTs were often divided up into groups for class activities. At Rio (MATH 291), the classrooms in which class was held were arranged into hexagonal tables, forcing PSTs into small-groups, even during whole-class discussions. At Hilada (MATH 281), the classrooms were set up with individual desks that were often moved around and reorganized during small group activities. Finally, both courses emphasized attendance and participation as keys to success in the course. This is notable not because it is particular to these courses, but in the manner in which this message was conveyed, at least in syllabus documents. Both syllabi highlight the importance of students collaborating in groups and that attendance and active participation are critical components of learning the material. This may be one of the important ways in which the syllabi intend to signal that these courses are not typical.

During the class activities, both courses employed the use of elementary classroom manipulatives to illustrate important concepts and to give PSTs experience in working with them, as they are likely to do as teachers. According to student survey, responses 33% of MATH 281 PSTs at Hilada recalled working with base-ten blocks as the most memorable activities of the semester. 41% of MATH 291 PSTs at Rio cited manipulatives such as base-ten blocks, and straws[21] as the most memorable activity in the course. With one exception, no other activity elicited as much feedback on the survey.

[21] In MATH 291, straws were often used in bundling activities that highlighted the key ideas of place value in base-ten, and other bases.

46% of MATH 291 PSTs recalled watching video of children doing mathematics as the most memorable activity or assignment they did during the course. This result is one to which I will return later, because it, as well as the use of classroom manipulatives, gets to the heart of the purposes for the research described here.

Finally, according to instructors' notes, both courses explored student solutions to typical, multi-digit arithmetic problems in order to investigate alternative algorithms, discussed number systems in bases other than ten, and watched video of children doing mathematics, though the data demonstrates that this last kind of activity was overwhelmingly more common in MATH 291 than in MATH 281.

Differences

The similarities I described above demonstrate that the two courses were alike enough so as to make a comparison reasonable. Of course, there was a limit to extent of the similarity between MATH 281 and MATH 291. There were substantial and substantive differences between them that might account for differential performances on MKT measures, and distinctions between their responses to items related to attitudes the PSTs expressed about mathematics and teaching. An important difference was one of mathematical approach: MATH 281 was based upon Beckmann's (2005) focus on operations (personal communication with the course coordinator), while MATH 291 was predicated upon the concept of place value. This difference in organizing concept sent the courses onto different paths in their day-to-day work, and impacted how the courses communicated with students about what was mathematically important. In turn, this variation may have influenced the manner and extent to which the practices and artifacts of teaching infiltrated each course. In order to clarify these differences in approach, I turn again to the data, and offer several vignettes as illustrative examples.

Organizing Concept

In MATH 291, place value was a central, unifying concept. It took nearly of month of class meetings before the course turned its attention explicitly to base-ten numbers and arithmetic operations. Up until this point, the course focused on establishing defining features of number systems, in which PSTs explored ancient number systems, constructed their own base-six system in an activity called Alphabetia (Bassarear, 2007), and were introduced to the concept of the Basic Measuring Unit (BMU), which featured prominently throughout the course. The BMU establishes the size of "one" so that all subsequent groupings or divisions of that unit depend upon its definition. These activities were aimed at setting up the features of Hindu-Arabic numeration and arithmetic that would form the bulk of the course. One of the instructors in MATH 291 introduced the Alphabetia assignment this way:

> *Treat it as an awareness exercise. You get to re-experience what children go through when they are trying to acquire an understanding of the Hindu-Arabic system...This will help you to learn to make ideas explicit for you something you know, but which is hard to explain. Place value is key to understanding decimals in 4th through 6th grades…This lays the groundwork for younger kids, <u>and</u> you might decide to teach older kids… This is not easy, but don't give up too soon…describe how things are different if you can't get a system with all of our properties.*

In a different section of the course at Rio, taught by a different instructor a couple of weeks later, there was a continuing discussion of counting in different bases. This discussion took place within the context of two entire lessons[22] spelled out in the course notes labeled "Place Value." Here, the notion of the BMU (as well as the idea of "Measuring Units" (MUs) more generally, which are the units or quantities that are associated with each place value) features prominently:

> ***I***: *For homework last time I asked you to think about 124_5 and how to represent the area of that numeral if the BMU is equal to a small square like* $\square$.
> ***S***: *There is one group of 25.*
> ***I***: *What we call 25?*
> ***S***: *Yes.*
> ***I***: *What is that in base five?*
> ***S***: *One hundred?*
> ***I***: *Be careful, in base five, we don't say one hundred.*
> ***S***: *One, zero, zero?*

The instructor continued throughout the lesson to highlight this linguistic distinction, but asked the PSTs to focus on the importance of area in interpreting the diagrams.[23] The class discussed how to represent larger and smaller measuring units in different bases noting that for each place one moves left within a numeral (or in other cases, the associated picture, or bundle of straws) the measuring unit associated with a particular place value increases by a factor of the base. In other words, in a given base *b*, *b* copies of the measuring unit are associated with the next larger measuring unit, or place value. Likewise, one can determine the size of a place value to the right by partitioning a measuring unit into *b* equal parts. This class meeting featured nearly thirty minutes work on finite "decimals" in other bases, and different representations that would result by choice of the representation for the BMU, as above. The homework assignment

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[22] In the MATH 291 course notes, there are 20 Lessons, which correlates to an average of just more than one week per lesson. Thus, two lessons in MATH 291 are likely to encompass as much as two weeks of class meetings.

[23] One might use a unit of length as a BMU, and then subsequent "measuring units" would also then be referenced in terms of length.

associated with this lesson and activities in subsequent lessons asked students to interpret given MUs and generate measuring units of their in numeral systems with different bases.

Choosing and using measuring units was a consistent theme throughout MATH 291. Later in the semester, while discussing models for addition and subtraction, PSTs were asked to speculate how children would solve additions and subtraction problems using finite decimal representations such as 3.4 and 1.8 using tools such as snap cubes or graph paper. One of the lesson goals given in the course notes states: "Students will flexibly and appropriately select basic measuring units." PSTs worked in groups discussing potential informal strategies that children might use with these manipulatives, and then were asked to present their ideas and solutions to the rest of the class. Indeed, during these presentations, much of the discussion focused on the choice of the BMU. PSTs made clear at the beginning of their explanations what their choice of the BMU was, and how this choice affected how they grouped cubes, or sticks, or blocks (depending on the representation of the numeral they chose). The instructor asked groups to repeat their explanations and often pointed out ways that the group showed the meaning of "one." Such interactions occurred across sections and these issues continued to be explicit and in focus through lessons on multiplication and division.

One quiz featured a single multiplication problem (1.2 x 0.9 = ?) which featured five questions: "What are your measuring units? How did you interpret the meaning of the number sentence? How did you represent the different quantities? How exactly did you use your diagram to determine the final answer? What is the final answer?" The first question asks explicitly what measuring units the PST chose—presumably beginning with the BMU—and half of the remaining questions relate to the implications of choosing those measuring units on the process of solving the problem. On a final exam review sheet given out to the PSTs at the end of the semester, four of ten suggested problems are direct questions about place value. An example of one of the questions is the following: "A child correctly uses the standard subtraction algorithm to solve the problem below. Which of the following statements justifies the 8 being written where it is the solution? (a) 5 – 7 = 8; (b) 15 – 7 = 8; (c) 50 – 70 = 80; (d) 150 – 70 = 80; (e) None of the above."

In contrast, in MATH 281, place value was a more peripheral concept, highlighted for a brief time early in the semester and then referred to occasionally as a way to convince children about the viability of algorithms. Instead, the organizing concept of MATH 281 at Hilada University was that of arithmetic operation. Like at Rio, the first month of MATH 281 was designed to lay a foundation for the key ideas of the latter portion of the semester. During this time, PSTs discussed the fundamental theorem of arithmetic and divisibility rules (listed in the syllabus as "number theory"), and explored ways of representing, comparing, and simplifying decimals and fractions. In the course notes, there were class meetings devoted to discussing ancient number systems, and representing numbers in different bases. Later, these last few concepts, as well as place value did not command much attention either in the course notes or during class meetings. Place value receives explicit attention only on a single day as outlined in the notes, and it shares that time with a discussion of divisibility rules. During class, when introducing a base-eight system, one MATH 281 instructor attempted to justify the utility of learning to use numbers in bases other than ten: "[This is] a little weird, and not standard kid fare, but this is good for an advanced kid to work on. You may not like math, but you will have kids who do and you'll need something in your pocket for them." After showing examples of how to count in various bases and convert from one base into another, the following exchange took place:

> ***S***: *What grade would kids be in to think like this?*
> ***I***: *This is an advanced thing…you don't want to do this until they are very confident with base-ten. I don't know specifically, but it's appropriate for ages 18,19,20,21,…*

Such exchanges—possibly prompted by small attention devoted specifically to it in the course notes—marginalized the purpose of working in other bases, which is to develop a deeper understanding of, and appreciation for, place value.

In MATH 281, all of these interactions laid the groundwork for the later material, which took each arithmetic operation in turn, investigating it carefully within typical number sets found in elementary school: whole numbers, integers, and rational numbers (as represented by fractions and decimals). Addressed first was addition, then subtraction, multiplication, and finally division. The reasons for exploring number theory and working with fractions and decimals early in the semester is not explicitly documented in the syllabus, the supplementary Class Activities Manual, or the course notes. Presumably, the number theory topics were important for discussing fractions: in order to simplify and operate on fractions, understanding the value of greatest common factor and least common multiple are often pre-requisite knowledge. Having a shared experience in working with fractions was important as the course transitioned into understanding arithmetic operations on different sets of numbers, including rational numbers.

One element of this focus on operation was to elucidate the importance and power of some properties of operating in the real number system. Understanding and using the commutative properties of addition and multiplication, associative properties of addition and multiplication, and the distributive property of multiplication over addition and subtraction were high priorities in MATH 281. On Exam #3 given in MATH 281, three of nine questions require PSTs to reference the properties in order to draw an illustrative diagram, or justify a calculation. The course notes outline three days of class meetings to discuss the commutative and distributive properties of multiplication alone. These properties received explicit, if less intense attention during the chapter on addition. The course notes state, "Practically speaking, [the commutative property of addition] is often taught as a way to make the 'counting on' method of addition more efficient: if students need to add 2 + 7, it's easier to start with the seven and count on two more." Indeed, these properties are critical components of understanding elementary school mathematics: many intuitive strategies, standard algorithms, mental calculations, and common misconceptions stem from these important properties.

Putting the Organizing Concept to Use

I argue that the different organizing concepts in MATH 281 and MATH 291 resulted in an expansion of differences between the courses. In MATH 281, the focus on arithmetic operation was coincident with valuing rigor, justification, and mathematical correctness. There was a feeling of top-down progression: it is possible to justify elementary arithmetic operations by using the tools of upper-level mathematics. On the other hand, MATH 291's concern for place value attempted to accomplish similar goals from the opposite direction: the elementary school student's perspective.

One day, early in the semester, one MATH 281 instructor was reviewing an assignment that the PSTs had just received back with comments. On the board, the instructor wrote "*Patterns* $\neq$ *Reasons,*" and said, "A pattern is never an explanation for something. It may help you see something but it is not a reason." Later that same class, the instructor reiterated this point: "Why does $2\frac{1}{3}=\frac{7}{3}$? Because 2 x 3 + 1 = 7 is not a 'why.' We want to stick with something that is actually *math*. Don't rely on rules or patterns."

These comments echo the perspective taken by the textbook used in MATH 281 at Hilada University, written by Beckmann, and analyzed by McCrory (2006). McCrory argues that, though its mathematical rigor is not the same as in mathematical journals, in Beckmann's text, and others like it,

> *they pay attention to definitions, logical development of topics, making connections across topics, and mathematical reasoning…in ways that some other books, written by nonmathematicians, are not…they are often (though not always) explicit in trying to teach the prospective teachers about the importance of rigor and clarity in mathematics, portraying mathematics as an endeavor in which care and accuracy are both important.* (p. 23)

The central construct of MATH 281 was the idea of operation and one of the overwhelming messages given to students about this idea focused on clarity and mathematical rigor. PSTs appeared to have received this message. Consider the following typical responses to an item on the end-of-semester survey asking PSTs to describe the course as if to a friend:

> *This class helps you to understand how to solve patterns but more importantly, the reasons why the problems are correct and misconceptions. If anything, you should be ready to explain yourself and your answer. You will learn how to explain to students the reason why you do certain things in math and also you will learn how to draw correct visual aids.*

This class covers basic mathematics and the ideas behind them so they can be easily taught and understood by young students. She should be ready for learning the actual meaning behind simple operations rather than just processes.

In an interview, one PST in MATH 281, Carla,[24] expressed similar sentiments, if not in exactly the way envisioned by course documents. She felt that the course was too narrow-minded in its view of what constituted a correct answer:

> *…another criticism I had of the class is that they wanted, like, the answers in a very, very, very specific way. Like the diagrams, and the pictures had to be drawn very specific ways. And I don't know if that was for ease of grading, or if it was—I don't know, I don't what that was for, but I know in the real world, you're going to see things in a bunch of different ways, you know? And a kid isn't going to—some kids are going to make big bubbles, and some kids are going to—and while I think it's good to try to, try to teach in a way that their drawings or their diagrams are clear, I don't think it's practical, or even good to try to make it all so uniform.*

Another interviewee, Eliot, reacted similarly:

> *I thought it was, 'She doesn't have this word, so that's minus two points…And I was told that that was the rubric for our exams. And that really bothered me, because if I understand it, and I'm just not saying it the way you want me to say it, our class, the whole class is supposed to be about teaching it to different learning styles. If I'm not learning it the way you're teaching it, but I'm learning it, what does it matter?*

In these two quotes, as in the survey responses above, Hildada PSTs expressed the fact that in MATH 281, rigor, clarity, and specific formats were important in class. A classroom example of this stance or rigor and clarity took place about a month into the semester. The class was working on topics in Chapter Four of Beckmann's book, which focuses on addition and subtraction concepts. After discussing a page containing two-digit subtraction problems solved by children using invented and unconventional

[24] This, and all other names given here are pseudonyms.

strategies, the instructor showed the students another piece of student work on an addition problem:

$$136 + 50 = 186 + 7 = 193$$

"How could we write this correctly?" the instructor asked. Many PSTs did not seem to understand what the instructor meant by this: indeed, may PSTs had been writing number statements much like this. One PST suggested that the child should subtract seven from all sides of the equation, as though solving for an unknown in algebra. After one PST volunteered that the statement 136 + 50 is not actually equivalent to 186 + 7, the instructor led the class in a discussion of how to notate these calculations properly, while highlighting their use of various properties. For example, in order to solve the problem 123 – 58, the instructor suggested the following work:

$$\begin{aligned} 123 - 58 &= (120 + 3) - (60 - 2) \\ &= (120 + 3) - 60 + 2 \\ &= (120 - 60) + 3 + 2 \\ &= 60 + 3 + 2 \\ &= 63 + 2 \\ &= 65 \end{aligned}$$

In order to explain the transition from – (60 – 2) to – 60 + 2 (from the second to third steps), the instructor said, "If they understand subtracting a negative, you can write it just like this, but if not, just go straight to – 60 + 2." The implication here is that this is how teachers would (or should) notate such calculations with children. After working a few of these examples, the instructor told the class, referring to the use of equals signs, "If you can't write it clearly, you should not write it." Moreover, this lesson was not atypical of interactions over the course of the semester. A month later, when discussing alternative algorithms for multiplication, the primary avenue for justifying non-standard calculations was to write out carefully and correctly the steps and properties used by the child. One child calculated 16 x 24 by describing verbally that he found 12 x 10 (120), found 12 x 6 (72), and added them together, and then doubled the result. On the board, the instructor wrote and matched the following notation to the child's description: 16 x 12 = (10 + 6) x 12 = (10 x 12) + (6 x 12) = 120 + 72 = 192. The instructor asked why the doubling part was important and a PST responded that the doubling transformed the problem 16 x 12 into 16 x 24, at which point the instructor put the following equalities on the board: 16 x 24 = 16 x (12 x 2) = (16 x 12) x 2 = 16 x 12 + 16 x 12. This emphasis on correct notation was not a special project of this particular instructor: the course notes state when outlining these days' activities that a "challenge to address when encouraging students to use their own invented methods is notating their thinking in a way that is faithful to both their insight and the conventions of mathematics."

MATH 291 at Rio University also valued mathematical justifications over inductive reasoning when exploring elementary mathematics topics, but its central construct—place value—led it in a different direction in addressing these same topics. In MATH 291, the importance of place value went hand-in-hand with a concern for student thinking, less emphasis on mathematical conventions, and a more idiosyncratic development of mathematical ideas.

On day one, the first thing one instructor of MATH 291 said to the class after finishing the business of going over the syllabus, structure of the course, and office hours information was to appeal to their status as future teachers: "To teach math well, you must develop knowledge of kids and how they think, knowledge of pedagogy—that's in the methods course—and knowledge of mathematics." Understanding how children think and acknowledging the sense-making in it, even when flawed, was a common theme throughout MATH 291. When responding to an item on the survey about how to describe the course to a friend, several MATH 291 PSTs explicitly mentioned that the course was about learning how children think about mathematics.

> *In this class you will learn how to teach kids mathematics. You will learn and understand how they think and why they solve problems the way they do. You will learn multiple strategies for each problem.*

This was a typical sentiment expressed on many of the surveys. The first and last statements, that the course was about learning "how to teach" mathematics and that PSTs were supposed to learn multiple strategies, were quite common responses (occurring in 33% of all surveys). However, the second statement, that PSTs are supposed to learn how children think about mathematics (or ones like it) is found on 8 out of 41 completed surveys from MATH 291 (~20%).[25]

In one episode in the middle of the semester, PSTs were writing story problems to match addition number sentences. One group of students asked the instructor about the wording of their problem, which concluded with "how many ounces were consumed altogether?"

> ***S:*** *Should I use words that are too big for kids if these problems are for kids? (referring to 'consumed')*
> ***I:*** *What's wrong with this wording?*
> ***S:*** *Kids have trouble reading words not related to math.*
> ***I:*** *That's very thoughtful of you.*

This may seem like an expression of the character of the PST who suggested this; that she was somehow going above and beyond what would have been required of her as a teacher, but it was thoughtfulness in this very sense that was an explicit goal of the course. In another exchange in a different section, PSTs watched video of students

[25] By way of comparison, only 2 out of 61 (~3%) of respondents at Hilada cited this feature of the course.

solving addition and subtraction word problems. The fourth video in a sequence of five featured a child who solved a problem that children in previous video clips had failed to solve: *Sarah had some trucks. She gave 6 to Jeff. Now she has 9 trucks left. How many trucks did she have to start with?* The instructor asked PSTs to suggest differences in this video from the others they had watched so far, eventually turning their attention to this problem in particular, referred to as the "missing whole" problem.

> ***I:*** *Any differences here?*
> ***S****:* *She used her fingers.*
> ***I****:* *Did she use the same approach as the others?*
> ***S****:* *No, she counted up instead of guessing and recounting.*
> ***I****:* *This is a big move: from counting all to counting on. What other differences were there? How did she solve the missing whole problem?*
> ***S****:* *She counted separate piles and counted all.*
> ***I****:* *This is the opposite of the action in the problem…There's no action in this one, nothing to model.*

The main goals of this lesson, as stated in the course notes, are that PSTs should:

> *…recognize that children can use features of the problem situation to guide their choice of solution strategies… understand that the multiple strategies children use to solve addition and subtraction problems can be reconciled through the part-whole structure…recognize why certain types of word problems can be difficult for children.*

Here again, the instructor was highlighting an issue of language and the need to pay attention to small details that may not seem strictly mathematical, yet they impact how children might think of mathematics nonetheless. Understanding why certain things are difficult for children—developing a sense of (mathematical) empathy—is an explicit purpose of the lesson.

Finally, whereas in MATH 281, the course maintained relatively strict adherence to conventional mathematics/arithmetic notations, in MATH 291, instructors and PSTs generally confined their work to representations used in elementary school. Consider the following exchange that took place in a similar context as that I described above in MATH 281. The problem was 36 x 17:

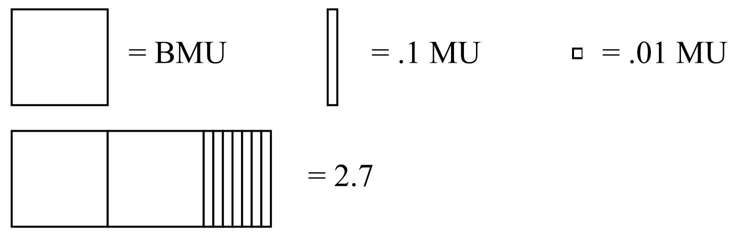
> ***I***: *This is an invented algorithm. What did he do?*
> ***S***: *I saw it differently than the worksheet, but he breaks 36 into 30 and 6 so 30 is ten and ten and ten. He does 10 x 17 three times, but he waits until the end to put together.*
> ***(I** restates the PST's explanation)*
> ***I***: *Then?*
> ***S***: *He decides to decompose 17 into ten and seven and does ten times six and seven times six, then he adds quantities together and gets six hundred twelve.*
> ***I***: *What types of strategies are here?*
> ***S***: *Distribution?*
> ***I***: *Before that, he…*
> ***S***: *…Decomposes.*
> ***I***: *This is an invented strategy. You should tap into their prior knowledge and lead them toward the standard algorithm, using the intermediate algorithm.*
>
> *(**I** writes on the board): Intermediate Algorithm:*

$$\begin{array}{rcl} 36 & & \\ \times\ 17 & & \\ \hline 42 & \longrightarrow & 7\times 6 \\ 210 & \longrightarrow & 7\times 30 \\ 60 & \longrightarrow & 10\times 6 \\ 300 & \longrightarrow & 30\times 10 \\ \hline 612 & & \end{array}$$

It is important to say here that such conversations also occurred in MATH 281, though they were shorter, less frequent, and these alternative notations did not drive conversations in the same way as they did in MATH 291. Another episode demonstrates how different the emphasis was on notation and representations. During a discussion on decimal multiplication, the class worked for fifteen minutes on how to represent the problem 3.2 x 1.4, and how this representation helped show the appropriate solution to the problem:

*I***:** *Let's figure out the number sentence that goes with the diagram. This is an important skill to have when looking at student work. (See the drawing below)…Now look at #1. ____ will walk us through it.*

S: *For 3.4 x 2.7:*



I: *Why show the two-point-seven?*
S: *Because it's the second number.*
I: *This shouldn't be a procedure…<u>why</u> do we start with two-point-seven? <u>Not</u> because it's the second number. This should have <u>meaning</u>.*
S: *Because it means "groups of."*

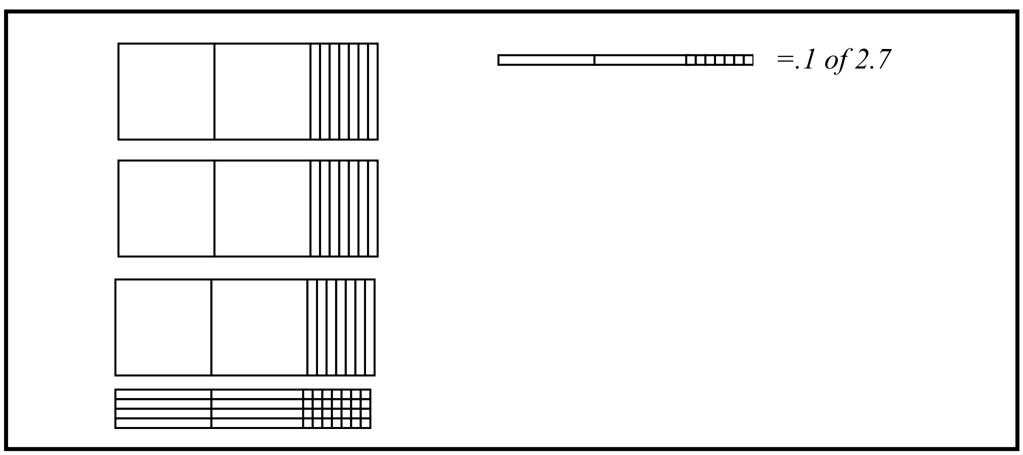


Figure 2: This shows the PST's drawing, representing three groups of 2.7 and four groups of one-tenth of 2.7

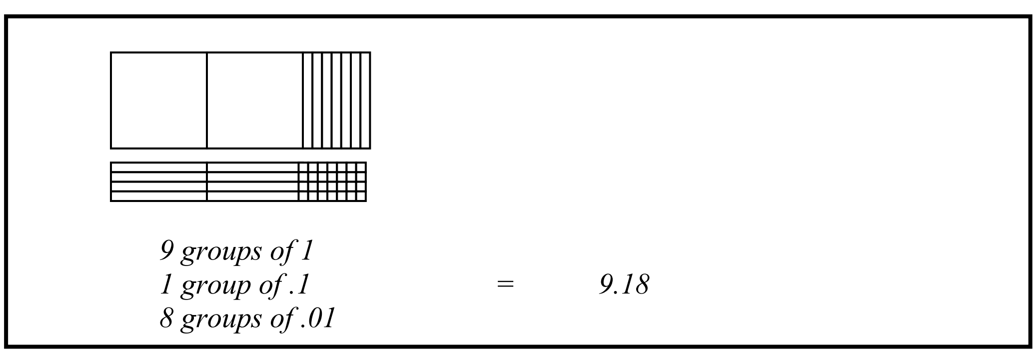


Figure 3: To find the numerical value of the product, the PST regrouped by labeling the BMUs 1,2,3,…,6 and circled collections of ten .1 MUs, etc., writing the final product as 9.18

> ***I***: *Is there anything you would add to make this diagram more clear? Pretend that this is student work.*
> ***S***: *Show the 2.7 cut into ten pieces to <u>show</u> one tenth of 2.7.*
> ***I*** : *Good suggestion. Also, your diagram doesn't show the 7, 8, and 9 groups of the BMU, though you explained it nicely in words…if you showed that in the picture, it would be more clear.*

Use of pictorial representations for multiplying decimals also appeared in MATH 281, but again, the fidelity to it as a reason and meaning for the procedures to follow was unique to MATH 291. In MATH 281, the course notes describe using diagrams to explain decimal multiplication, but *after* using multiplicative inverses of powers of ten to justify a connection with fraction multiplication. In other words, in MATH 281, the use of the diagram is a way of illustrating a concept which is justified with a set of symbolic manipulations, rather than motivating the concept itself. The primary difference here is not which symbols and pictures were used in which course: many of the same pictures and symbols were used in both settings. The fundamental difference is how they were used: in one course, the algebraic symbols were used to justify mathematical choices, while pictures were a representation of the logic inherent in the symbols. In the other course, it was precisely the opposite: pictures and concrete representations formed the basis for mathematical reasoning, and formal symbols were used to represent the ideas developed in the pictures, rather than to justify them.

In the map of MKT suggested by Ball, Thames, and Phelps (2008), a facility with formal mathematical procedures probably falls in the gray area between common content knowledge and specialized content knowledge. Many people among different practices must be familiar with the manipulation of formal mathematical symbols, but it may be that teachers have access to and use for certain kinds of manipulation which does not overlap with other groups. The pictorial representations and special understanding of children fall under specialized content knowledge in that these representation are those used in elementary school exclusively, and teachers must have access to it because those representations are designed to provide children insight into the primary concept of place value. This data demonstrates that the two courses were operating in different, if intersecting, areas within the MKT sphere.

Artifacts and Practices of Teaching

I have argued that content courses for PSTs may be more successful in preparing teachers if they construct courses as a way to begin participating in the practices of teaching. This can happen in many different ways, some of which were evident at Rio and Hilada Universities, though not to the same extent in each location. In this research, the degree to which these courses used artifacts and practices of teaching is the variable of greatest interest: the research questions for the study hypothesize implicitly that courses which differed along this dimension would result in measurably different outcomes among PSTs. Thus, documenting the ways in which MATH 281 and MATH 291 differed along this dimension is central to the study.

Video

In MATH 291, watching video of children was a frequent class activity and once, near the end of the semester, became a homework assignment.[26] At Rio, on the first day of classes, course notes state that PSTs should watch two videos of children doing arithmetic. One video shows a child misusing the standard subtraction algorithm, and

[26] Watching video at home was a marked change from the course notes, which suggested using the video as an in-class activity. Instructors commented that doing this at the end of the semester saved time and gave PSTs multiple opportunities to attend the complexities of the student thinking in each episode.

another who misuses the standard addition algorithm. The course notes suggest that these videos should be a motivation for the doing the work PSTs are about to undertake: “Pre-service teachers will begin the process of treating lessons as experiments by developing and improving their ability to generate hypotheses about children’s conceptual understanding and procedural skills.” Seven of the twenty lessons involved watching video as a primary activity.

As I described above, one of the key goals for using such materials was to develop an understanding of children’s thinking, and the issues that arise in learning elementary mathematics. In one class meeting, PSTs watched video of children solving addition and subtraction word problems (e.g., “Sally had 13 marbles. If she gave four marbles to Tony, how many does she have now?”). The instructor, introducing the activity, said, “…We’re going to watch kids….kids use a lot of strategies, though eventually earlier strategies tend to die out. Notice how the kids stick close to the story in their modeling of the problem, even though we might see them as addition and subtraction problems.” The purpose of this video in MATH 291 was to connect the types of story problems they had been discussing to the strategies that children use to solve typical problems.

Implicit in these activities was that the problems that teachers pose for children matter: the way that children construct mathematical knowledge depends on how they model problems put before them, and their modeling is closely related to the format in which the problem is offered. Thus, teachers affect the mathematics that children learn simply by the problems they choose! This is a profound statement to make about teaching, but it is typical of those that are implicit in much of the research in mathematics learning, and in learning mathematics for teaching. Ball, Thames, and Phelps (2008) argue one of the characteristics that distinguish knowledge of content and teaching from other kinds of knowledge is precisely this: understanding how to choose appropriate examples for a given learning situation. Here, different children used different strategies to solve the same problem, and PSTs had an opportunity to see how these strategies change as children get older, and develop more sophisticated understanding of mathematics. In addition, such activity targeted specialized content knowledge because the various strategies made visible for PSTs many of the skills children need for understanding these operations.

Watching video in MATH 291 had deeper goals than simply observing that children use different strategies. Facility in identifying and *naming* those strategies was an important objective of watching these videos of classroom interactions. Late in the semester, this became an explicit focus of the course, as lessons turned entirely on understanding children's invented algorithms in solving two- and three-digit arithmetic problems. The course notes for Lesson 15 include a handout given to PSTs as they watch a video of children adding and subtracting multi-digit numerals. As the children on the screen solve the problems, PSTs were asked to identify and label the strategy that each child was using:

Problem	**Jaycee**			
87 + 24	80 and 20 is 100. 6 and 4 is 10, so 7 and 4 is 11. So the answer is 111.			
Strategy Type?				
Problem	**Gary**	**Elizabeth**	**Stephen**	**Chris**
26 -17	20 – 10 = 10 10 – 7 = 3 3 + 6 = 9	20 – 10 = 10 6 – 6 = 0 10 – 1 = 9	20 – 10 = 10 6 – 7 = 1 10 + 1 = 11	20 – 10 =10 10 + 6 = 16 16 – 7 = 9
Strategy Type?				
Problem	**Chris**	**Marie**	**Chihol**	**Brent**
9 +6	Make the 9 a 10 and the 6 a 5. Then 10 + 5 = 15.	Take 1 away from the 6 and add it to the 9, and that would make 10 and 5 which is 15.	9 + 7 = 16 16 – 1 = 15	Take 3 off the 9 and that makes the 9 a 6. 6 and 6 is 12 and then add on the 3 which would be 15.
Strategy Type?				

Table 1: This is an excerpt of the handout given to PSTs in this lesson. In the handout, there are a total of eight problems, worked out by 18 children

These invented strategies were then connected to the standard algorithm in explicit ways for PSTs. In a later lesson on children's invented division strategies, the instructor showed how invented strategies could lead to an understanding of the standard long division algorithm:

> *A lot of teachers use invented strategies to develop an intermediate algorithm and then go to the standard algorithm. This is also called the scaffold algorithm. I've even seen teachers in high school use this because kids never got the standard algorithm. The way it works is that I make guesses and we hope we get better at guessing…*

This is how videos were used throughout MATH 291: PSTs watched how children solved problems, attempted to identify underlying mathematics for each strategy exemplified in the video, and then connected these ideas to standard algorithms and techniques traditionally found in elementary school.

This is a key component of what Ma (1999) referred to in describing “profound understanding of fundamental mathematics.” When PSTs become teachers, they will need to “diagnose” children’s thinking and act upon that thinking appropriately by introducing appropriate examples. Specialized content knowledge is required for this diagnosis, while knowledge of content and curriculum that enables teachers to choose remediating examples. Neither course gave careful attention to what decisions teachers should make under particular circumstances, or even what research claims can be made about what choices teachers have. These issues are typically reserved for methods courses, later in PSTs undergraduate preparation, and often in conjunction with intensive observation experiences or during student-teaching. However, before a teacher can make choices about what to do in a particular situation, one must first be able to identify the mathematical ideas that are germane to the given situation. MATH 291 made this diagnostic skill an explicit and important part of learning mathematics for teaching.

MATH 281 also made use of video to consider student thinking, though the evidence suggests that it was not a central resource for the course. Consider the testimony of Carla:

> ***C:*** *…this one video we saw was great, I mean, and I think it—I don’t know if--*
> ***M:*** *It was just one, right? The whole semester?*
> ***C:*** *It was just one, yeah…*

In survey responses, among MATH 281 PSTs, only Carla’s classmates (in her section) mentioned watching video in class, and apparently it only happened once during the semester.[27] MATH 281 course notes suggest once that video may be a part of class activities, but it is offered as an optional activity, and given time constraints of a 15-week semester, it is not surprising that instructors chose not to incorporate discretionary activities in favor of those that seemed more central to the goals of the course.

Other Artifacts

This is not to say that practices and artifacts of elementary school mathematics teaching were absent in MATH 281; video is not the only medium through which PSTs can begin to relate to, understand, and participate in practices of teachers. Exploring student work, using classroom manipulatives, and invoking educational research are others. In particular, typical elementary mathematics classroom manipulatives were a prevalent component of both courses. Examples of these manipulatives are base-ten blocks, straws, or rulers. In many cases in class, these manipulatives were used as models for explanation or justification rather than physically being present in the room. It is hard to determine how often these manipulatives were actually present. I did not witness PSTs in MATH 281 use base-ten blocks or rulers, though they were cited often during my observations, and mentioned frequently in end-of-course surveys. As I mentioned above, nearly one-third of PSTs in MATH 281 recalled using manipulatives as the most memorable classroom activity. A similar proportion of PSTs in MATH 291 cited the use of manipulatives as the most memorable activity. In one section of MATH 291, I observed an activity involving the use of straws, used to illustrate a base-three numeration system. Three PSTs were asked to go to the front of the room and line up. The right-most PST acted as a ones-place, the center PST served as a place for groups of

[27] Four of the fourteen participants from this section mentioned watching video as the most memorable class activity of the semester. No one else at Hilada mentioned watching video.

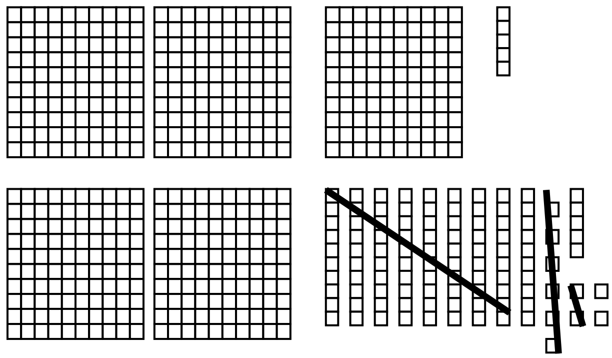
three, while the left-most PST was there when the straws spilled over into groups of nine. The instructor continued to give straws to the right-most PST in the ones-place, while she (and the rest of the class) kept track of how many straws she was "allowed" to hold before giving them to her neighbor, who bundled groups of three to represent her place. When the threes-place PST had amassed three groups of bundled straws, she in turn gave those bundles to her neighbor on the other side who again bundled them into groups of nine. Straws continued to be tools for doing calculations, particularly in other bases. Here again, this subtle difference has meaning. The use of straws is a robust representation of counting in any base because straws can be bundled and taken apart as numbers are in arithmetic operations. Though the use of money is often cited as a resource for teaching children, with it come difficulties that do not carry over into Hindu-Arabic arithmetic (Ball, Thames, & Phelps, 2008;).

However, the vast majority of instances in which I observed the "use of manipulatives," those objects were not physically present but served as a construct on which ideas could be built. Consider the following exchange in MATH 281 in which the class was discussing a homework assignment in preparation for an exam. The instructor showed how to use base-ten blocks to illustrate the calculation 305 – 88.

I: *This is like on the quiz. It shows up on every exam. 305 – 88. I like to start with the top number and do a take-away:*

(1)

(2)



S*:* *You have to show 305 but not 88?*
I*:* *Right. You could show both, but I'm just taking away from 305.*

Figure 4: Illustrating the calculation of 305 – 88 in MATH 281 at Hilada

This figure represents what was drawn on the board, not objects that were physically present in the room. Similarly, though I did observe PSTs' use of straws on a handful of occasions in MATH 291, generally, the activities dealt with drawing pictures, of straws or, more often, pictorial representations of BMUs and their relative measuring unit counterparts.

Another avenue for working with teaching practices is investigating student work. This is also something that was present in both courses, though again to different degrees. Consider the comment made by one instructor of MATH 281 as PSTs filtered into the room before class began: "This is the most fun lesson of the year…it's an exciting day." The primary activity for the class meeting—and the lesson to which the instructor referred—was to investigate student-invented algorithms for subtraction, and justify them using properties of whole numbers (commutative, associative, distributive). The algorithms were printed in the supplemental class activities manual; PSTs looked at the written work done by each child and discussed what the child was doing in order to solve the problem. Though this was not the only time PSTs in MATH 281 worked with student-generated algorithms,[28] the comment by the instructor signaled it as unusual. In MATH 291, PSTS engaged in similar work, though they did so generally through the medium of video, rather than written examples. In one class, PSTs watched video of children solving $159 \div 13$. As with the addition and subtraction algorithms activity I described above, the PSTs were asked to identify the strategies used by each child in the video, named and defined by the instructor in a previous class meeting. These activities are designed to help PSTs develop a sense of how children think and make sense of elementary mathematics. Idiosyncratic algorithms for addition, subtraction, multiplication, and division are among the most widely studied area of children's learning and mathematical development. It is an obvious place to begin encouraging PSTs to work on learning to understand children's thinking. In MATH 281, alternative algorithms were the most visible and explicit place for PSTs to examine children's thinking up close. In MATH 291, in addition to video and written examples of student's work, there were further opportunities for this, offered in different contexts. In MATH 281 children were peripheral objects, frequently mentioned, but rarely actually incorporated into the course. In MATH 291 however, children were often the focus of study, exploring how they reason and what mathematics they are likely to encounter.

<u>Addressing Addition and Subtraction: A Portrait of the Two Courses</u>

Together, the differences between the courses in treatment of the primary topics, their differing use of artifacts of teaching, and disparate emphasis on mathematical issues related to teaching provide a picture of courses that look very similar in a course

[28] Later in the semester, PSTs also explored student-invented algorithms for multiplication.

catalogue description or on a syllabus, but in fact were very different in practice. I suggest that these differences led the courses in different mathematical directions: these differences were subtle but important to the mathematics that PSTs had opportunities to develop through their participation. In order to illustrate this in a more focused manner, I offer here a picture of how MATH 281 and MATH 291 treated the central topics of addition and subtraction. The trajectory of these topics in each course provides insight into the different mathematics that PSTs encountered in their courses.

As they did throughout the semester, there was much content in one course that was mirrored by the other. Both courses introduced PSTs to various models for addition and subtraction, highlighting the part-whole relationship between quantities. Both courses made explicit the fact that the nature of this relationship could be illustrated with different actions. For example, the problem "Pablo had three trucks; how many more does he need to have seven trucks?" is modeled as an addition problem. On the other hand, "Tasha has eight jelly beans. Three of them are cherry flavored, and the rest are grape flavored; how many jelly beans are grape flavored?" is modeled as a subtraction problem.[29] Both courses also addressed the ways in which students compute using these operations on multi-digit numerals (see the discussion above), and both sets of PSTs operated on whole numbers, integers, and rational numbers with finite decimal representations. MATH 291 and MATH 281 similarly asked PSTs to model addition and

[29] The addition problem is often described as modeling a "joining" action of two parts to form a whole, though the second part is missing, or unknown. The subtraction problem similarly involves two parts and a whole, although in this case, the action modeled is often take-away, implying subtraction. The distinction is not whether or not addition or subtraction is the "right" operation to use, but simply what operation is implied by the situation as presented. Such distinctions have been made explicit primarily in the research associated with Cognitively Guided Instruction (CGI), which originated at the University of Wisconsin. See, for example, Carpenter, Hiebert, & Moser 1983 or Carpenter, Fennema, Peterson, & Carey, 1988.

subtraction problems using base-ten blocks[30] and to write their own word problems involving these operations.

However, there were substantial differences among and between these similarities which distinguished the two courses and show the different mathematical concepts that PSTs encountered and the knowledge they had opportunities to develop. There were two primary differences between the treatment of these operations between the two cohorts: (1) in MATH 281, addition and subtraction were applied to all integers and rational numbers (represented as finite decimals and fractions), while MATH 291 worked only with finite decimal representations;[31] (2) MATH 281 spent less than half the instructional time on these operations over the course of the semester than did MATH 291.[32] The first difference meant that MATH 281 was challenged to unify the algorithms for addition and subtraction across different representations of rational numbers, whereas MATH 291's limitation to rational numbers in decimal representation afforded more coherence to the treatment of algorithms for addition and subtraction (as well as multiplication and division). The second difference meant that PSTs in MATH 291 had more opportunity to explore children's mathematical thinking, while MATH 281 PSTs were more focused on working with and understanding algorithms for these operations.

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[30] As described above, in many cases the blocks themselves were not present in class, but PSTs were expected to draw the blocks that would be necessary for a computation.

[31] MATH 291 PSTs did encounter rational numbers with infinite decimal expansions, but only did so in the context of other bases. For example, 1/3 has an infinite decimal expansion, but it can be represented as .1 in base three. For rational numbers such as these, only these finite representations were used—the data imply that fraction representations were never explicitly part of MATH 291 at Rio University. It is notable that the term "decimal" only properly applies in the context of base-ten numerals, though for simplicity, I refer to all numerals using the symbol '.' as decimals.

[32] This large gap in instructional time is true of multiplication and division as well, though to a lesser extent. A primary cause of this was the extent to which MATH 281 addressed working with fractions generally and number theory concepts like divisibility.

Consider first the fact that MATH 281 treated the addition and subtraction of rational numbers using fraction representations, while MATH 291 did not. Addition and subtraction of fractions was introduced in MATH 281 by highlighting for PSTs some typical misconceptions that arise when trying to generalize addition and subtraction across number sets. For example, children often add numerators and denominators as if they were separate entities. The instructors of the course emphasized that in such problems, the size of one whole must be the same. On one visit I made to that class, the instructor began by asking PSTs how to add ½ and 1/3:

> **I**: How do we do ½ + 1/3? What do we need to do?
> **S**: Change the denominator.
> **I** (writes): $\frac{1}{2}\cdot\frac{3}{3}+\frac{1}{3}\cdot\frac{2}{2}$
> $\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$
> **I**: This is a hard thing for students, maybe it's a hard thing for you. If you have a good concept of what this <u>means</u>, it can be obvious. What does 3/6 <u>mean</u>?
> **S**: Three equal pieces out of six.
> **I**: The denominator tells you how big the pieces are as it relates to one whole. This is why you need to change them to sixes and why you don't add the sixes together….Why do we change them to sixths?
> **S**: To get same size pieces.

This exchange between the instructor and the PST illustrates a potential difficulty of treating fractions before discussing multiplication and division concepts: nothing has been explicitly stated about *why* the three pieces must be of equal size and how this relates to concepts of division. Moreover, the act of multiplying fractions which is necessary for properly changing the denominators so that they agree had not been supported by the same fundamental concepts of multiplication that the PSTs were then encountering with addition and subtraction. This happened principally because these ideas about multiplication and division were part of discussions that were set to take place a few weeks *after* this conversation. It's not that MATH 281 neglected to address these issues, but instead that it hadn't done so yet. Such considerations may not be necessary if one is aiming specifically to develop common content knowledge and the requisite ability to compute fluently, but these connections are part of a specialized content knowledge that was not addressed in MATH 281: how does understanding about operating on different numbers develop and what sub-concepts are necessary in developing this understanding?

As a result, the concepts, strategies, representations, and algorithms PSTs discussed related to whole number arithmetic lacked strong connection to those used for operating on fractions. Recall that one advantage of addressing each operation in turn is the ability to show the coherence with which these operations can be viewed. This lack of connection inhibited the development of this coherence, because the same operations on different kinds of numerals appeared in fact to be distinct from one another. One indication of this is that later in that same class meeting, PSTs were asked to write word problems that could be solved by modeling operations on fractions. Nearly every group working on the task generated a situation that could be modeled with a take-away strategy (subtraction) and as the instructor roamed the classroom talking to groups, no mention was made of the previous models that were named for whole number problems (e.g. take-away, comparison, etc.), despite the fact that this was a highlighted aspect of subtraction with whole numbers.

In contrast, the fact that MATH 291 did not address fraction arithmetic[33] afforded instructors to highlight the consistency with which representations, strategies, and algorithms can be used for whole numbers, negative integers, and finitely-represented decimal numbers alike. At the beginning of one class meeting, PSTs began by presenting word problems they had written in an earlier meeting that involved decimal computations, and for which they were expected to illustrate different types of models (take-away, comparison, join, etc.). In that same class meeting, the instructor asked PSTs to consider how children might model these kinds of calculations using blocks and line segment lengths: the same representational tools (these were used in MATH 291 to represent the choice of the BMU) were used to show how one can compute with whole numbers and decimals alike. This juxtaposition of whole number arithmetic, the PSTs' work with children's modeling strategies, and decimal numbers highlighted the coherence of the ideas involved: the same approaches and the same tools apply to all sets of numbers. This is precisely the sort of connection that was not evident in MATH 281, and an illustration of where PSTs had different opportunities to learn specialized content knowledge.

The second major difference manifest within these treatments of addition and subtraction was the fact that one course devoted substantially more instructional time to it than the other. This fact does not mean that MATH 291 PSTs spent an extra amount of time discussing one particular mathematical topic instead of another when compared to MATH 281 PSTs. Instead, this extra time was occupied principally with PSTs working closely with video of children working on addition and subtraction problems, discussing

[33] Understanding operations on rational numbers as represented with fractions is a primary focus of MATH 292, the second course in the sequence of three content courses at Rio University. MATH 291 however made no explicit mention of fractions and focused entirely on numbers with finite decimal representations.

written documents of student-invented algorithms for multi-digit computations, and investigating how some of these algorithms can be used as bridges between common intuitive approaches and "the" standard addition and subtraction algorithms. In contrast, in MATH 281, the extent of treatment of these kinds of activities comprised less than a single class meeting. Recall the Hilada instructor who claimed that "today is a special day" when describing how they were going to discuss student-generated algorithms for subtraction. That same class meeting also included extensive discussion of how to properly use equals signs to signify a string of computations as well as properly representing symbolic justifications for combining numbers in particular ways using commutative, associative, and distributive properties of the arithmetic operations.

This provided PSTs in MATH 291 opportunities to develop specialized content knowledge related to these topics that PSTs in MATH 281 did not have. In MATH 291, PSTs were referred explicitly to children's thinking in three different ways over the course of numerous days as it related to addition and subtraction. The first time was early on in the discussion of the meaning of the two operations. The purpose of this encounter was to familiarize PSTs with the intuitive approaches children use to solve addition and subtraction problems without using algorithms. Three of the five learning goals associated with this activity—as stated in the teaching notes—are:

> 3. [PSTs] will recognize that children have a rich variety of informal material counting strategies (based on the use of concrete manipulatives) and verbal counting strategies (based on forward or backward counting) for successfully solving addition and subtraction problems.
> 4. [PSTs] will understand that the different strategies children use to solve addition and subtraction problems can be reconciled through the part-whole structure of the problems (i.e., that there exists a mathematical consistency among the different solution strategies and that it is possible to establish that consistency).
> 5. [PSTs] will recognize why certain types of story problems can be difficult for children.

Note how these goals stress the importance of the structure of the problem and how this informs how children are likely to think about it. Such encounters are likely to facilitate PSTs' SCK, because it forces them to confront a wide variety of thinking strategies and they are thus presented with the dilemma of evaluating them and determining their mathematical value.

The second main encounter PSTs had with children adding and subtracting was similar, but as they watched video of children computing, they were asked to predict how children would solve various word problems, given its structure. For example, "Jill has three toys. Her mother gave her five more. How many does she have altogether?" is a joining problem for which the two parts are known and the value of the whole is unknown. In this activity, PSTs are asked to observe that children typically will solve such problems by physically modeling the action described in the problem. For the example above, children often will gather three blocks together, then five, and then count the two groups together in a single group. One value of this activity is to recognize that, as described in the teaching notes, "[t]his is an important ability in teaching. You should be able to predict children's strategies, children's thinking on different kinds of problems, and children's difficulties. Then you can use this information when you plan lessons and when you teach them." This quote describes an important component of specialized content knowledge.

The third encounter MATH 291 students had with children's thinking about addition and subtraction came significantly later in the semester than the first two, and *after* PSTs had done similar activities as described above for multiplication and division.

This encounter involved becoming familiar with alternative algorithms for computing with multi-digit whole numbers, and using this familiarity to develop ideas about which algorithms could be thought of as "intermediate" algorithms, processes that make certain aspects of the computation explicit which are hidden in more conventional algorithms. Again, this is a component of specialized content knowledge: an understanding of the constituent skills and knowledge related to a fundamental goal of elementary mathematics instruction such as multi-digit addition and subtraction.

On the other hand, in MATH 281, PSTs spent much more time adding and subtracting fractions, as if this were the primary skill set on which they needed to work. This may in fact be the case generally—it is widely believed that PSTs are weakest arithmetically when it comes to operating with fractions. On the other hand, the practical implications of this difference was a shift of focus away from strictly elementary school topic of operating on integers and decimals and the specialized knowledge that teachers need to in order to unpack the sub-skills that are part of learning these things. The attention on fractional representations of rational numbers obscured the inherent connections between the various techniques for computing with the different number sets and instead focused attention on the fundamental rules and properties of the operations and how they applied to different number sets. The advantage of working only with finite-decimal representations was that there was a consistency with which PSTs could apply the ideas they were discussing, while in MATH 281, there seemed to be different sets of rules for rational numbers, depending on how they were represented. One big reason for this disparity was the fact that Hilada PSTs were encountering operations with fractions before they had discussed multiplication and division in depth.

These same course similarities and—just as importantly—the course differences were mirrored throughout the semester under study. MATH 281 was a course that, by design and by consequence of the choice to study operations across many number representations, focused PSTs' attention more on procedures and algorithms than on the underlying concepts and fundamental ideas that are often hidden in such algorithms and that can be made visible in the work of children. On the other hand, MATH 291 was designed to highlight such skills for PSTs, and by virtue of the fact that the course concentrated on fewer kinds of representations, had the ability to introduce children's thinking through the use of video and written work. In this way the two cohorts, despite covering very similar ideas at one level, were actually addressing different components of MKT: Hilada PSTs had more opportunities to attend to computational fluency associated with common content knowledge (CCK) and Rio PSTs had more opportunities to develop understanding related to evaluating strategies based on stronger ability to recognize the structures and fundamental sub-skills that support a strong understanding of the operations.

Attending to Impacts of These Differences: A Return to the Research Questions

Recapitulating the Comparison

In this chapter, I have argued that MATH 281 at Hilada University and MATH 291 at Rio University addressed primarily the same mathematical topics, and that there are many aspects of their structure which are similar. Both courses concentrate on the fundamental concepts of number and operation: numeration, addition, subtraction, multiplication and division. A look at the respective syllabi for each course reveals that they discussed many of the same topics. Moreover, Rio and Hilada organized sections of these courses into similar sizes, expected PSTs to complete assignments, quizzes, and exams, and met multiple times each week to engage in lecture and small group activities.

On the other hand, there were substantive differences between the courses that may have implications for what PSTs learned as part of their course experience. MATH 291 at Rio did not explicitly address the notion of rational numbers as represented by fractions. Instead, PSTs in that class operated on rational numbers represented as finite decimals. MATH 281 at Hilada wove fraction operations throughout the course, often using them as a way to justify actions taken with decimal representations. The courses differed as well in the approach they took to teaching common content: MATH 281 organized the course around operations and justifying algorithms used for those operations using formal mathematical arguments. MATH 291, by contrast built primarily upon the idea of place value, using the concept of the Basic Measuring Unit and its associated measuring units. Algorithms were justified in MATH 291 using different representations of these BMUs and MUs. Interpreted with a situated learning theoretic perspective, this difference in organizing concept—operation and formal mathematical argument versus place value and representation—should result in PSTs learning different mathematics, despite the fact that all are learning about addition, subtraction, multiplication, and division. Still other differences between the courses may also have led to different learning among PSTs between the two institutions: while both courses employed the use of artifacts of teaching in order to motivate PSTs and examine important mathematical ideas, this was evident in MATH 291 more frequently than in MATH 281. Videos of children solving problems, analyzing word problems and how children solve them, and a closer fidelity to the development of mathematics in elementary school appeared to embed PSTs at Rio University more deeply into the practice of teaching, the practice into which PSTs are being inducted, and for which these courses are designed to prepare them.

Reviewing the Purpose of the Comparison

I have argued in this chapter that the differences between these courses create a useful context for testing situated learning theory. The aim of the project is to determine whether or not the theory can answer the following questions:

(1) What mathematics do prospective teachers learn by engaging in activities of teaching practice such as examining curriculum, student work, and classroom video?
    a. Do PSTs who regularly engage in such activities display evidence of different mathematical proficiency than PSTs who participate in more traditional course work?
    b. Do PSTs engaging in such activities display different mathematical knowledge for teaching (MKT) than PSTs participating in more traditional course work?
    c. Do PSTs engaging in such activities develop different attitudes about mathematics and teaching than PSTs participating in more traditional coursework?

(2) To what extent do prospective teachers see their mathematics course work as relevant to their future work?
    a. Do different course approaches set up differing perspectives among PSTs on the contribution of the course to their future work?
    b. Do different course approaches set up differing views among PSTs about their confidence and abilities in mathematics?

The theory may help explain how PSTs learn mathematics for teaching, which may have broad implications for undergraduate teacher education across disciplines. Did PSTs at the different universities learn mathematics differently? Did they develop different attitudes about mathematics and teaching? Did they view their courses as having different relevance to their preparation and future practice as teachers? These are the essential questions I have endeavored to answer, and for which I have collected data. While the evidence suggests that the courses met the initial criteria that they be different, it remains to be seen whether these differences can explain any gaps in outcomes measures between the PSTs at the different universities. It is to this data that I will turn in the next chapter.

Chapter 5: Results

MATH 281 and MATH 291 corresponded with one another in many respects: they each discussed nearly identical mathematical topics, enrolled similar numbers of PSTs, structured PSTs' time alike, and made similar claims on the goals they had for PSTs' knowledge and skills. On the other hand, the two courses organized their respective mathematical work differently, and these different approaches meant that PSTs did different things. The central question of this study is to learn what effects these differences had on PSTs' MKT, their attitudes about mathematics and teaching, and whether or not they felt that MATH 281 and MATH 291 were similarly relevant for their future careers. In order to discern the answers to these questions, I collected data about PSTs' MKT via a 31-item multiple choice instrument (the MKTI) and interviews with six selected PSTs. I also gathered responses to survey instruments designed to give insight about PSTs' attitudes about mathematics and teaching, and through the survey instrument and interview prompts sought information about PSTs' perceptions about their courses relevance.

Recall the hypotheses I described at the end of Chapter Three, which correspond to three primary research questions of this project:

1. PSTs in a course that more closely identifies with practices of teaching will perform better on measures of Mathematics Knowledge for Teaching, than PSTs in a course that does not.
2. PSTs in a course that more closely identifies with practices of teaching will develop different attitudes about mathematics and teaching than PSTs in a course that does not.

3. PSTs in a course that more closely identifies with teaching will be more likely to reflect on the experience as a valuable one than PSTs in a course that does not.

What does it mean to "more closely identify with teaching?" Although common content knowledge is an integral component of MKT, a mathematics course that "closely identifies with teaching" is one which concentrates on sub-domains of MKT that are particular to teaching rather than those which overlap with other practices and disciplines. In particular, the analysis in the last chapter indicates that MATH 291 concentrated on PSTs' SCK in ways that MATH 281 did not. MATH 281 focused greater attention on issues of CCK such as computational fluency and understanding of standard algorithms..

However, the differences between the two courses are nuanced; these courses were *not* at polar ends of a spectrum, but rather both could be located somewhere in the middle. Neither course claimed to be a kind of immersion experience in teaching and neither aspired to be a course which is isolated from the teaching profession. Still the evidence demonstrates that MATH 291 incorporated more explicit attention to a sub-domain of MKT that is not shared with other practices and disciplines. As such, situated learning theory predicts different outcomes along the dimensions I have outlined.

From the perspective of the theory, one might predict that the differences I have described between MATH 281 and MATH 291 would result in PSTs at Rio scoring higher on the MKTI, demonstrating different attitudes about mathematics and teaching, and claiming stronger affinity for what they learned in their math course than did Hilada PSTs. The data I report on in this chapter support these hypotheses, but the full range of data is necessary to reveal it, as a single data stream inadequately describes the outcomes. I therefore address the focus of each research question in turn, using all relevant sources to discern what the data reveal. First, I analyze the data with respect to the question about PSTs' relative MKT, turning to the results of the MKTI and the interviews. Next, I undertake the question of PSTs' attitudes about mathematics and teaching, using the data from the survey instrument. Finally, the survey instrument and the interviews form the basis of an analysis of PSTs' perception of their course's relevance for their teaching practice. Throughout these descriptions, I will explicate methods particular to the analysis as they arise in the data. These methods were described in Chapter Three, though new details are revealed here as they became necessary during the analysis itself.

PSTs' Relative MKT: Did the PSTs in MATH 281 Develop Different MKT than PSTs in MATH 291?

One might assume that if most of the PSTs did not fail their course, then their instructors certified that they had in fact learned mathematics over the duration of the semester. There is nothing revelatory about this statement. However, while course grades can be effective measures for assessing individual progress, they are ineffective ways to describe a group of students and are undesirable measures for comparing different groups of students in different courses, as this study intends to do. Moreover, given the descriptions above, it may be hard to decide the extent to which the course grades reflect PSTs' mathematical knowledge for teaching, as the assessments within and among the two courses were differentially geared toward this goal.[34] Thus, this research has employed the use of interviews and a multiple-choice instrument for measuring mathematical knowledge for teaching (MKTI).

[34] This is true both between and within courses, and understandably, the variety of assessments during a semester lend themselves to different foci among the regions of these different kinds of knowledge.

The data generated by these tools suggests that PSTs in both courses learned important mathematics from the beginning of the semester to the end, though teasing apart the differences between the PSTs at Hilada and Rio on this dimension is not trivial. Below I outline results from statistical analyses of the MKTI data that compare mean scores on the MKTI and what these results say about the knowledge PSTs developed at each institution. Subsequent analysis of differential achievement on individual items and of the interview data paints a slightly different picture, giving insight into PSTs' mathematical knowledge that might not be visible through the quantitative lens.

Results of the MKTI

Recall that PSTs completed a 31-item instrument designed to assess their mathematical knowledge for teaching, what I am calling the Mathematical Knowledge for Teaching Instrument (MKTI). From a pool of hundreds of items written by the Learning Mathematics for Teaching (LMT) project at the University of Michigan, the items were narrowed first to reflect the content that was common to both courses, focusing in particular on number and operation concepts. Next, the items were chosen to reflect a variety of difficulty levels and to maximize reliability.[35] PSTs completed the instrument at the beginning of the semester and then again, approximately three months later, as they neared the end of the semester. At each campus, there were a handful of PSTs who completed the first round of the MKTI that did not participate in the second administration. These were eliminated from the analysis described below; only PSTs who completed both the pre- and post-assessments were included.

[35] Analyses of items resulting from pilot studies in the LMT project provided baseline reliability statistics.

	MATH 281	MATH 291
Total # of Students Enrolled	93	76
# of PSTs completing Round One	65	47
Response Rate	70%	62%
#of PSTs Completing Round Two	60	41
Retention Rate	92%	87%
Total Response Rate	65%	54%

Table 2: A summary of PST participants

Analyses of Variance

Within the set of PSTs who completed both rounds of instruments, descriptive statistics were computed for each campus, including mean and median raw scores (number of correct responses), standard deviation, variance, skewness, and kurtosis. These descriptive characteristics of the data indicate that the assumption of normality necessary for the ANOVA is a reasonable one. Mean and median values[36] for each administration of the MKTI at each campus were similar[37] and histograms of the aggregate data and for each administration of MKTI at each university reveal symmetric shapes not unlike a normally distributed data set. Below are some examples of these histograms. Related Kolmogorov-Smirnov and Shapiro-Wilk tests also suggest that the data can be approximated using a normal distribution.[38] Tests for homogeneity of

[36] A condition of my use of the items from the LMT Project at Michigan was that reports of the data would not include raw scores on the instrument. Scores that result from responses on the items should not be interpreted as identifying some benchmark or level of knowledge. Rather, the scores are useful for comparative purposes, which is one of primary the reasons I employed these items in this project.
[37] The difference between mean and median at each campus was less than ¼ of a standard deviation, which means that these two measures of center were never further apart than the value of a single correct answer on the 31-item instrument.
[38] The test statistics for both the K-S and Shapiro-Wilk tests for all administrations of the MKTI have p-values larger than .10, which means that one cannot reject the null hypothesis, which is that a normal

variance also supported the assumption necessary to maintain integrity in the analysis of variance computations.

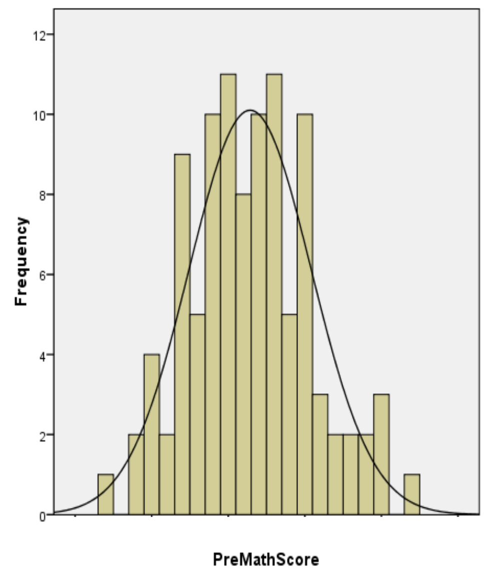

Figure 5: Aggregate Beginning-of-Semester MKTI Scores

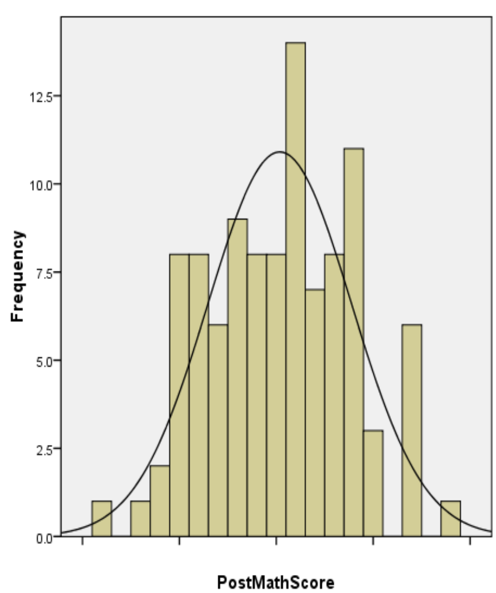

Figure 6: Aggregate End-of-Semester MKTI Scores

distribution can be fit to the data. The Shapiro-Wilk test is the more appropriate procedure of the two for samples of the size I gathered.

The analysis of variance performed on the mean MKTI scores at each location shows that there was no statistical distinction between PSTs at the two campuses at the beginning of their respective courses.[39] This is important, because it indicates the early homogeneity—along this dimension—of the two groups of PSTs in the independent locations. Had the two campuses demonstrated significant differences at the outset, this fact may have complicated the subsequent analysis of the influence the courses had on PSTs. This result indicates that, at least by this metric, the PSTs began their respective courses with similar mathematical knowledge for teaching.

		Sum of Squares	df	Mean Square	F	Sig.
PreMathScore	Between Groups	4.488	1	4.488	.280	.598
	Within Groups	1586.205	99	16.022		
	Total	1590.693	100			
PostMathScore	Between Groups	2.430	1	2.430	.177	.675
	Within Groups	1362.362	99	13.761		
	Total	1364.792	100			

Table 3: ANOVA Results Comparing Mean MKTI Scores between Hilada and Rio Universities at the beginning of the semester and then again at the end.

An ANOVA performed on end-of-the-semester MKTI scores again shows no statistical distinction between MATH 281 and MATH 291.[40] In other words, these data do not indicate that PSTs at either institution developed any more (or less) mathematical knowledge for teaching relative to PSTs at the other.

However, whether partitioned by campus or aggregated across campuses, the mean MKTI scores at both Hilada and Rio Universities rose by approximately one

[39] $p < .6$
[40] $p < .68$

standard deviation from the beginning- to end-of-course administrations of the MKTI instrument. This result is comparable to answering four additional items (~13% of 31 total items) correctly at the May administration of the MKTI than at the February MKTI administration. This increase was statistically significant in all cases.[41] Broadly speaking, this indicates that both courses had a similarly positive effect on PSTs' mathematical knowledge for teaching.

Multiple Regression Analysis: Which Variable(s) Influenced MKT Scores Most?

While the ANOVAs indicate that the campus from which the PSTs came had no effect on their end-of-semester MKTI scores, there may have been other relevant variables that affect—and can thus predict—these scores. In order to discern which variables might be able to accomplish this, I constructed a multiple regression model by entering following data: institution,[42] attitudes survey score change, average response to items related to course relevance, and the MKTI pre-test scores. Using a stepwise regression analysis,[43] the model excluded the institutional variable, citing it as not significant. In fact, the only one of those variables I listed above which made it into the final regression model was the early semester score on the MKTI. This pre-test score explained 52% of the variance in post-test scores. In another model that forced all of

[41] $p < .001$
[42] PSTs in MATH 281 were coded with a "1" while those in MATH 291 were coded with a "2." Thus, any positive correlation between this variable and other(s) shows that a higher institution score (MATH 291) is associated with a higher dependent variable score, which in this case in the MKTI score. Conversely, a negative correlation would suggest that a lower institutional affiliation (MATH 281) is associated with a higher score on the dependent variable.
[43] The stepwise regression method begins with the variable with the highest correlation to the dependent variable, and adds variables into the model only as long as they independently contribute significantly to the sum-of-squares calculation.

these variables to be included,[44] these other variables together accounted for only an extra 3% of the variance.

	Variables Included	Variables Excluded	Variance in MKTI Scores explained by the model
Model One	MKTI Pre-Test Scores	Institution, attitudes survey score change, average response to items related to course relevance, whether or not students were repeating the course	52%
Model Two	Institution, attitudes survey score change, average response to items related to course relevance, MKTI pre-test scores, whether or not students were repeating the course	none	55%

Table 4: A summary of the two regression models used to predict end-of-semester MKTI scores

This means that of the data collected in this study, the only variable that is likely to shed much light on how PSTs will perform at the end of the semester is how well they performed at the beginning of the semester. PSTs who earned higher scores at the beginning of the semester were likely to produce the higher scores at the end of the semester, and the PSTs with lower scores early in the semester were likely to remain relatively low-scoring at the end. More specifically, this model predicts that a change of one standard deviation on the beginning-of-semester MKTI score results in a nearly ¾ standard deviation change in the end-of-semester MKTI. In other words, two PSTs separated by a single standard deviation at the beginning of the semester are still likely to

[44] This is known as the "enter" method.

be separated by nearly that much at the end—though both scores are likely to be significantly higher.

This result is revealing, because it suggests that neither of the two interventions represented by these different courses are better than the other in overcoming the effects of the knowledge with which the PSTs entered their courses. Neither course was able to trump the influence of PSTs' prior knowledge and understanding so that all or most PSTs demonstrated similar achievement. This is not a surprising result, insofar as it would seem unlikely for a semester course to nullify 12 or more years of formal education in mathematics. On the other hand, it also suggests and reinforces the fact that both courses improved the MKT of all PSTs by similar amounts; the courses thus conform to the standard of "raising all boats" similarly. The "typical" PST answered 13% more items correctly at the end of the semester than at the beginning: the PSTs appear to have made measureable progress in developing MKT.

Among the handful of other variables in the second model described above, one variable in particular appears to have a disproportionate influence on the variance in MKTI scores. The amount of change exhibited in PSTs' attitudes about mathematics and teaching between beginning and end of the semester is weakly, but positively correlated with the end-of-semester MKTI score. This indicates that succeeding in changing (increasing) PSTs' scores on the attitudes survey instrument leads to increased scores on the MKTI. There are many reasons why it is hard to be conclusive about this last result, because it may well be that those who scored higher were more prepared to undergo such attitudinal changes. I will return to this topic later when discussing the data on PSTs' attitudes.

Differences in PSTs' Common Content Knowledge and Specialized Content Knowledge: Sub-scale Analysis

The broad result of the MKTI analysis reveals that PSTs' at both institutions improved their MKT, but it does not address whether or not PSTs learned different mathematics as the theory predicts they would. In order to determine this, I turn to a different analysis of the MKTI data, in which items are pooled together in subscales according to their assessment of common content knowledge (CCK) or specialized content knowledge (SCK). With the assistance of a mathematics educator with expertise in teacher education and MKT in particular, I partitioned the 31 items into two subsets. Recall that common content knowledge addresses knowledge that any well-educated person would be expected to know. This includes basic fluency with computation, ability to identify errors in calculations, and recollection of basic mathematical facts. Specialized content knowledge is knowledge that is particular to the work of teaching. It includes being able to evaluate non-standard approaches to calculations and to determine the validity of such approaches in other contexts. In addition, SCK is characterized by an ability to identify appropriate representations for key ideas and recognizing that particular skills can be decomposed into constituent sub-skills.

Using this scheme, 15 of 31 items were categorized as CCK while the remaining 16 items were labeled SCK. ANOVA run on each set of data (beginning- and end-of semester common content knowledge scores and beginning- and end-of-semester specialized content knowledge scores) demonstrates that PSTs were not statistically different from one another on either measure at the beginning[45] of the semester. Both

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[45] For CCK, $p < .62$ and for SCK, $p < .68$

cohorts of PSTs scored similarly on both types of items on the February administration of the MKTI.

		Sum of Squares	df	Mean Square	F	Sig.
PreTestCCK	Between Groups	1.303	1	1.303	.254	.615
	Within Groups	506.935	99	5.121		
	Total	508.238	100			
PreTestSCK	Between Groups	1.088	1	1.088	.182	.671
	Within Groups	591.724	99	5.977		
	Total	592.812	100			

Table 5: ANOVA Results Comparing Mean CCK and SCK sub-scale scores between Hilada and Rio Universities at the beginning of the semester.

However, at the end of the semester, there is evidence that the PSTs at Hilada had developed more CCK than their counterparts.[46] At the same time, Rio PSTs opened a similarly sized gap between themselves and Hilada PSTs in terms of SCK.[47]

		Sum of Squares	df	Mean Square	F	Sig.
PostTestCCK	Between Groups	12.116	1	12.116	3.216	.076
	Within Groups	372.974	99	3.767		
	Total	385.089	100			
PostTestSCK	Between Groups	19.163	1	19.163	3.481	.065
	Within Groups	545.055	99	5.506		
	Total	564.218	100			

Table 6: ANOVA Results Comparing Mean CCK and SCK sub-scale scores between Hilada and Rio Universities at the end of the semester.

[46] $p < .08$
[47] $p < .07$

This means that while all PSTs increased their measured MKT, PSTs at Hilada developed different aspects of their MKT than did their counterparts at Rio. Furthermore, these differences manifest themselves along the same dimensions which were foremost the focus in each course, namely common content knowledge (at Hilada University) and specialized content knowledge (at Rio University). It is notable that while the differences are statistically significant, they were not large in magnitude: in the May administration of the MKTI, Hilada PSTs outscored Rio PSTs on common content knowledge items by a single item. This means that on average, Hilada PSTs answered one more CCK item correctly than did their Rio counterparts. The magnitude of the differences between the two cohorts in terms of SCK items was also approximately one item.

This result points in the direction predicted by the learning theory and occurred over a relatively short span of instruction. Though the size of differences was relatively small, there are other data that support the conclusion that the two courses learned different mathematics for teaching.

Differences in PSTs' Common Content Knowledge and Specialized Content Knowledge: MKTI Item Analysis

On the scale of the entire MKTI, the PSTs at the different universities did not demonstrate different MKT. However, partitioning the instrument into the two categories on which it assesses reveals that there were differences between the two institutions. A closer look at individual items reveals a similar avenue of interpretation. One of the conditions of my use of the items generated by the LMT project at the University of Michigan was to report only relative results as opposed to raw scores. Therefore, I have chosen to compare responses on these items in terms of the magnitude of the difference between the percentages of PSTs at each school who answered the item correctly. For example, on a particular item, if 47% of Rio PSTs answered correctly, while 52% of Hilada PSTs answered correctly, I report the difference between them as 5%. For purposes of the analysis, I chose a relatively arbitrary difference to consider significant enough to warrant further attention: 10%. I chose this value because on many items, the difference in achievement was less than five percent so that a double-digit magnitude stood out among the 31 items on the instrument. Nearly half (15 of the 31) of the items were answered by PSTs at both universities within five percentage points. Seven items had differences of between five and ten percent. The nine remaining items which differed by greater than ten percent are prominent, and are thus reported here.

With these criteria in place, I explored the results of each item at each campus for each administration of the MKTI. As I argued in Chapter Four, the differences between the courses stemmed primarily from emphasizing different sub-domains of MKT. One would expect that at the end of the semester, Hilada PSTs would perform better on items focusing on CCK while Rio PSTs should perform better on SCK items.

Looking at the results of performance on individual items, there were clear differences between the two cohorts. Consider item #2: from the beginning- to the end-of-semester MKTI, Hilada PSTs increased their percentage correct by 22 points, while Rio PSTs increased by 14 percentage points. On item #4, Rio students increased their percentage correct by nearly 50 points, while Hilada PSTs increased by only 11 points. On item #8a, Hilada PSTs increased the 8% gap that existed at the beginning of the semester to 22%.[48] On #9a, the same thing happened, but roles were reversed. On item #10, Hilada PSTs increased their percentage of correct responses 34% from beginning to

[48] Rio students actually answered this item correctly in smaller numbers than at the beginning of the semester.

end, and Rio PSTs improved only 14%. On item #12, Rio PSTs gained nearly 60 points while Hilada PSTs gained only 12 points. On item #14, Rio PSTs gained 28 points while Hilada PSTs gained only 16%.

Item	Increase of Correct Responses at Hilada	Increase Of Correct Responses at Rio
2	22%	14%
4	11%	49%
6(d)	12%	40%
8(a)	9%	-6%
9(a)	3%	20%
9(b)	-8%	-1%
10	35%	14%
12	12%	60%
13(e)	32%	0%
14	17%	28%

Table 7: Changes in percentages of correct responses on selected items by PSTs at each university

All of these examples were considered noteworthy because they were unusual in the following sense: the proportion of correct answers on the end-of-course administration of the MKTI was at least ten percentage points higher on one campus than the other. In many of these cases, one campus dramatically improved the percentage of PSTs answering correctly, while the other campus gains were more modest. While the analysis of the differences between the two courses I offered in Chapter Four can explain many of these results, it does not account for all.

Consider first the item with the largest gap between the two campuses: at the end of the semester, item #4 was answered correctly by a proportion of Rio PSTs 43 percentage points higher than at Hilada.[49] Yet, at the beginning of the semester, the two

[49] Note that in Table 7, the difference between the gains on item #4 at the two campuses is 38%. Since the proportion of Rio PSTs answering this item correctly was 5% higher at the beginning of the semester, and the corresponding improvement was 38%, this is how I arrive at the 43% figure.

groups of PSTs were similar: MATH 281 PSTs answered correctly within about 5% of MATH 291 PSTs. Item #4 involves an analysis of a child's use of an alternative algorithm for a two digit subtraction, which features a non-standard regrouping process. The regrouping process in the item is illustrated in the example 364 – 79 below:[50]

$$\begin{array}{rccc} & & 16 & 14 \\ & 3 & \cancel{6} & \cancel{4} \\ & 1 & 8 & \\ - & & \cancel{7} & 9 \\ \hline & 2 & 8 & 5 \end{array}$$

Why would Rio PSTs perform so much better? One explanation might be that only MATH 291 PSTs worked on such an algorithm in class, and therefore, the MATH 281 PSTs were at a disadvantage in trying to solve it. This is not the case: both courses addressed precisely this algorithm. MATH 281 discussed it during the alternative algorithms activity that PSTs worked on in groups out of the supplemental activities manual. The course notes suggest that instructors highlight this very algorithm for the problem 364 – 79:

> *The most challenging student to figure out is #4. His method is actually a standard algorithm in some countries; very possibly the student was taught this approach rather than inventing it him- or herself. It uses the same concept as #7. For example, when the student changes the 4 in the ones column of the top number to 14, that has what affect on the value of 364? (adds ten). In order to preserve the difference between these two numbers, he then needs to add ten to the 79. Instead of doing that in the ones column, however, he added a ten in the tens column, making the 7 into an 8. Similarly, when he changes the 6 in the tens column of the top number to a 16, that changes the value—how much? (pause here—some students will says only ten is being added, not recognizing that ten tens or one hundred has been added; it is critical that this misconception be*

[50] This example is not meant to illustrate the item itself, but simply is an example of the regrouping process in that item. Item #4 does not involve two regrouping steps as this example does, but this is the problem used in MATH 281's course notes, described below.

> *addressed) Since 100 has been added to the top number, 100 needs to be added to the bottom number. See the little diagonal mark below the 3? That is a 1 in the hundreds column of the bottom number. Now the subtraction can be carried out, knowing that the difference between 3 hundreds + 16 tens + 14 ones and 1 hundred + 8 tens + 9 ones is the same as the difference between 364 and 79. Try this algorithm on another pair of numbers (e.g., 203 – 57 from last class).*

A related, but distinct interpretation is that Rio PSTs were better equipped through their course to analyze such work. In Ball, et. al.'s (2008) mapping of MKT, this item targets PSTs' specialized content knowledge: it asks PSTs to deconstruct the numerals into their constituent place values and an analyze of how those values can be manipulated differently from the standard algorithm. The analysis of the two courses shows that MATH 291 focused more on this component of MKT than did MATH 281, and thus PSTs in this course were ready to address the issue on this item in greater proportion than MATH 281 PSTs. Thus, Rio PSTs' knowledge was activated by this item, while Hilada PSTs' knowledge lay inert.

Alternatively, the most lopsided item in Hilada's favor was item #8a, in which PSTs were responded to a question involving the number of fractions between zero and one. This item was included in the instrument despite the fact that fractions were not commonly addressed in both courses. Items such as this were designed by the LMT project and grouped with other items that *were* commonly addressed by both courses, and in order to maintain their statistical integrity, these groupings, where they occurred, were retained. Fractions were frequently discussed in MATH 281, and though the knowledge required for this item did not appear to be an explicit goal of the course, it is a reasonable corollary to the discussion described in the course notes about locating fractions on the number line:

> *Number lines also give a way to visualize some important facts about fractions. These should remind you of analogous facts about decimals studied in chapter 2.*
>
> 1. *Rounding: Frequently "unusual" fractions like 17/30 are mentally replaced with a familiar fraction that is close in size, such as 1/2.*
>
> 2. *Fractions Between Fractions: Number lines also allow us to "zoom in" and find fractions lurking between two given fractions. For example, name one fraction between 17/30 and 1/2. Between 17/30 and 18/30. Name two fractions between 2/3 and 3/4.*

This does appear to be a case of one class—but not the other—addressing the topic. MATH 291 PSTs simply did not have an opportunity to learn about the ideas in this item over the course of the semester, while MATH 281 PSTs did.

Of the four items on which Hilada PSTs performed substantially better (> ten percentage points) than Rio PSTs, three are related to fractions and were classified as CCK with respect to fraction concepts. On the other hand, five of the six items on which Rio PSTs performed substantially better were classified as SCK. This split in performance maps directly back to the split I described between the respective foci of the courses, and likely influenced the statistical results I reported above. This is further evidence that the differences between the two courses influenced the kinds of mathematical understanding that PSTs developed. It appears that these differences arose from a simple difference in opportunities to learn, and in important ways, this is true. However, these opportunities were direct consequences of the mathematical and pedagogical design choices made for each course. MATH 291's focus on children's thinking and unpacking of elementary school ideas generated a different perspective on the same algorithms and ideas as addressed by MATH 281's focus on operation and notation that sought to explain and justify using more sophisticated mathematical approaches.

One item in particular forms a potential counterpoint to the analysis above. Item #2, in which PSTs were asked to choose the best representation a student's description of her method for computing 12×14, which is an application of the distributive property of multiplication across addition. At the beginning of the semester, Hilada PSTs answered this correctly in greater proportion by 5%. During the semester, while both courses discussed how to deconstruct such statements, MATH 281 focused on the use of algebraic properties such as distribution, while MATH 291 concentrated on place value arguments. In May, the gap on this item had increased to 14% in favor of Hilada PSTs. This is an SCK item which both courses addressed but in different ways. Generally speaking, PSTs from Rio answered these kinds of items in greater proportion than did Hildada PSTs, and it is unclear why it was not the case of item #2. It may be that this item bore resemblance to the emphasis MATH 281 placed on rewriting mathematical statements using various algebraic properties of the given sets (in this case the distributive property).

Another item for which the description of the courses does not provide explanation is item #9b. It was the only item for which both groups of PSTs answered incorrectly in greater proportion in May compared to February. This item requires PSTs to interpret an example of an alternative method for multi-digit multiplication and decide if it is a method that will "work" for all such problems. This item was classified as SCK. As such, Rio PSTs attended to this kind of mathematical knowledge more often than did Hilada, and they indeed performed better on this item than their Hilada counterparts, but both groups answered correctly less often compared to the beginning-of-semester MKTI. This is another result for which I do not yet have an explanation, especially as this item was part of a group of items in which this result was not repeated.

In addition to exploring the differences in percentage of PSTs responding correctly, I also noted two items in which the polarity of the gap was reversed. For example, at the beginning of the semester, the proportion of MATH 281 PSTs who answered #6d correctly was nearly *twice* the proportion of MATH 291 PSTs who answered that question correctly. Item #6 requires PSTs to choose appropriate collections of base-ten bocks to represent a number. This item groups multiple representations together and asks PSTs if they are correct: in other words, do the relationships between the blocks represent the relationship between the place values expressed by the numeral? Some choices are incorrect, one would be a typical, correct choice of blocks, and another would be a correct, if unconventional, choice.

In February, Hilada PSTs appeared to understand #6d in this collection in much greater numbers than did their counterparts at Rio University. At the end of the semester, the Rio PSTs in MATH 291 outscored the Hilada PSTs in MATH 281 by eight percentage points. Although more Hilada PSTs answered this question correctly at the end than at the beginning of the semester, the improvement was substantially smaller than that displayed by MATH 291 PSTs at Rio. This kind of dramatic reversal from beginning of the semester to the end happened only one other time, and as such, warrants further investigation. Why would the Rio PSTs' improvement on this item so greatly dwarf that of their Hilada counterparts?

On the other hand, at the beginning of the semester, MATH 291 PSTs answered item #13e correctly 22 percentage points higher than their MATH 281 counterparts, while in May, MATH 281 PSTs answered 15 percentage points higher than MATH 291 PSTs.[51] Item #13e involves PSTs' assessment of a particular "rule of thumb" related to division. The rule of thumb is false, though it is true when the number system being used is whole numbers. Counter-examples exist among integers and fractions, and fractions are the archetypal counter example to the rule of thumb.

These are the only two cases of such extraordinary turnarounds in the percentage differences between Hilada and Rio. This data, along with the sub-scale analysis suggest that indeed, the two cohorts of PSTs learned different aspects of MKT. In the case of #6d, the unorthodox choice of manipulatives, is special among the other items in the group because they are more straightforward uses of the blocks and thus they would not necessarily show differences between PSTs' MKT. Item #6d requires a greater facility with and understanding of place value and representations for it, which I've argued that Rio PSTs had more opportunities to develop. It arguably also reveals something about PSTs' knowledge of content and teaching, which requires teachers to choose appropriate representations for instruction. Here too, I have argued that Rio PSTs had more opportunity to develop this aspect of their MKT. It is not surprising then that the Rio PSTs would show significantly more improvement than MATH 281 PSTs.

In the case of #13e, it may be that this turn-around is due to the fact that Hilada PSTs' had greater experience with dividing rational numbers—fractions in particular—on which the counter-examples to this statement are often based. Recall that MATH 291 did

[51] Rio University (MATH 291) PSTs answered this item correctly in nearly the same proportion from beginning to end of the semester, while the percentage of correct responses rose 36% among MATH 281 PSTs.

not address operating with fractions explicitly,[52] while this was a regular theme in MATH 281. A statement like #13e was classified as common content knowledge, which was the type of mathematical knowledge that was more prominent in MATH 281 than in MATH 291, and again, it should not be surprising that MATH 281 PSTs performed better than their counterparts on this item.

At the same time, some differences were smoothed out from beginning to end of the semester. There were eight items for which one campus answered correctly at least ten percentage points more than the other at the beginning of the semester.[53] Of these eight, only items #6d and 13e (discussed above) had such a large gap at the end of the semester. One explanation for this "smoothing effect" is the broad result of the statistical analysis of MKTI data: the two courses were both effective in teaching PSTs important mathematical ideas for teaching. On the other hand, like the other items I've highlighted above, each of them focused on a particular sub-domain of MKT, sub-domains that would seem to privilege one group of PSTs over another, and interestingly, this appears to account for much of the "smoothing" effect. Consider the fact that four of the five items clustered in #6, which assesses PSTs' understanding of place value and relates to their SCK, were answered correctly in greater proportions by Hilada PSTs than Rio PSTs at the beginning of the semester. At the end of the semester, this specialized content knowledge that Rio PSTs had more opportunities to develop in MATH 291 closed the gap to within five points.

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[52] While it is true that a statement like $.6_7 \times .3_7$ can be translated into a statement like $\frac{6}{7}\times\frac{3}{7}$, this connection was not part of MATH 291, and I do not presume that PSTs would make this connection on their own. Making connections such as these appears to be the purpose of MATH 292, the next course in the sequence at Rio.

[53] 16 of the items on the beginning-of-semester MKTI resulted in gaps of less than 5%. The other seven items had gaps of between five and ten percent.

The use of situated learning theory to interpret this data suggests that PSTs at the different universities developed measurably different MKT, but it would be incorrect to assert that PSTs in one setting did not develop MKT associated with the other. For example, on item #9c (related to alternative algorithms for multi-digit multiplication which is associated with specialized content knowledge, Hilada closed a large gap that existed at the beginning of the semester and at the end answered this correctly in greater proportion than did Rio. Conversely, Rio closed a 14 percentage-point gap to four points on item #13d, which states what is often referred to as the multiplication property of equality; this item was classified as CCK.

The thrust of this item analysis is therefore to support the sub-scale analysis which shows that Rio PSTs and Hilada PSTs gained differential understanding of MKT: while Hilada PSTs' learned more common content knowledge than Rio PSTs, the Rio PSTs learned more specialized content knowledge than Hilada PSTs. The statistical analyses show that both courses seem to have helped the PSTs make significant progress in developing MKT and that neither course stands out against the other in terms of *how much* MKT PSTs developed. The item analysis does not contradict this conclusion: the number of items with lop-sided percentage differences was relatively equally distributed. However, the differences in *which* items PSTs answered correctly demonstrate that the different foci of the courses may indeed have developed MKT in measurably different ways.

Interview Analysis: MKT

If the statistical analysis of the MKTI data is not definitive about the differences between PSTs' mathematical knowledge for teaching, the analysis of individual items suggests that a complete description of the effects of these courses may require deeper analysis. The interview data were collected for precisely this purpose, and I argue below that they provide more evidence that there were substantive distinctions among the MKT that PSTs developed at the two universities. One of the limitations of a multiple choice assessment like the MKTI instrument is that the PSTs had no opportunity to explain why they answered the way that they did. Interviews can lay bare more of PSTs' understanding about how children think, and whether and how they might apply their knowledge in teaching situations. Below, I describe my analysis of the interviews as it pertains to the PSTs' knowledge of mathematics for teaching. This component of the analysis offers a subtly different view of the results contained in the statistical analysis of the MKTI, and supports the results of the subscale analysis as well as the individual item analysis.

Recall that three PSTs from each course were selected to be interviewed. Those selected were chosen on the basis of their pre-test scores on the MKTI. One PST from each class was chosen that scored more than a standard deviation below the mean for their course sample, one PST who scored within a standard deviation of the course mean, and one who scored more than a standard deviation above the course mean. The intent for this was to draw interviewees from a wide cross-section of each course in terms of mathematical knowledge for teaching. PSTs were grouped according to these criteria and selected randomly to participate. When two PSTs who were selected were unavailable to participate in this component of the project, another PST in that group was randomly selected and recruited for the interview.

Although many individual statements uttered by PSTs at one university can be found in the transcripts of PSTs at the other, data stemming from the interviews suggest that the courses are in fact distinguishable along the MKT dimension. Nothing demonstrates this more clearly than the answers PSTs gave to the final prompt in the interview. Carla was a PST enrolled in MATH 281 at Hilada University, and Ann was enrolled at Rio University in MATH 291. Both PSTs scored in the middle third of their respective courses. As I described above, this middle-third pair of PSTs scored within a single standard deviation of their respective course means, though both scored higher on the pre- and post-test than the course means and medians. Carla's post-test score was an improvement by about ½ a standard deviation, while Ann's post-test score was nearly two standard deviations above her pre-test score. Ann's pre-test score was lower than Carla's, but her post-test score was substantially higher than Carla's.[54]

All interviewees were asked to discuss three prompts that were intended to elicit information about their ability to apply their mathematical knowledge to teaching situations. The final item among the three asked PSTs to evaluate the division statement $9 \div .6$, without context or a suggestion to use a particular model or algorithm. PSTs were asked to solve the problem as if it were a question they needed to answer for themselves in a personal situation: the way *they* would prefer to perform the computation. Later, interviewees were asked how they might help a struggling student to understand the technique they used for division, and what kind of story could be modeled by the number sentence given.

[54] Carla's survey means were higher than Ann's at the beginning and the end of the semester.

Both Carla and Ann used the standard long division algorithm to solve the problem, shifting the decimal point on the divisor rightward one place and then doing the same to the dividend. The resulting division problem then became $90 \div 6$, which could then be evaluated with a straightforward application of the long division algorithm. Carla appeared to have a strong procedural understanding of the algorithm and had confidence in her use of it:

> **C:** *(laughs) So, there you go, so it's, uh, fifteen and you don't need the decimal there [to the right of the ones place], but…*
> **M:** *Ok. Um, why do you move the decimal like that?*
> **C:** *Um, because the divisor—no, divid—yeah, divisor, dividend, right?*
> **M:** *I think so.*
> **C:** *Yeah (laughs). Uh, the divisor can't be a fraction. It has to be a whole. So, you move, you have to move that, and since you're moving this, a, tenth, right? This is the tenths place, then you have to do the same to the dividend.*
> **M:** *Ok.*
> **C:** *And then you do your long division.*

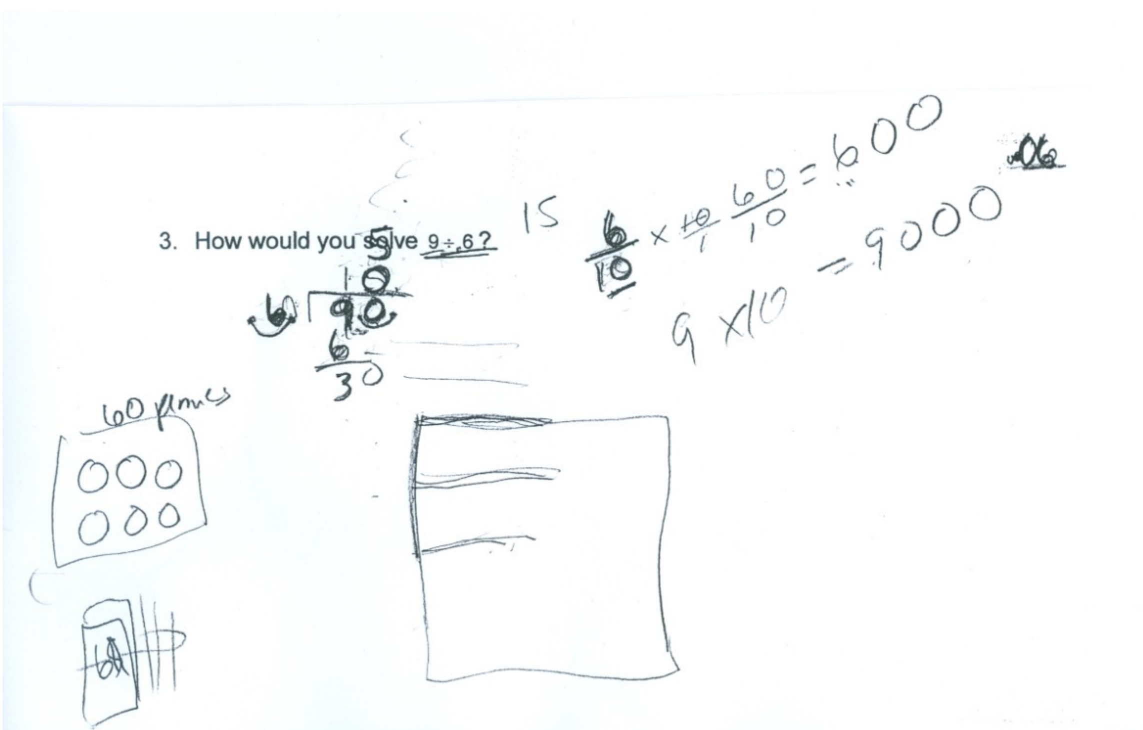

Figure 7: Carla's work on interview prompt #3, evaluating the problem $9 \div .6$. Note in particular her use of the standard long division algorithm, and the use of fractions (though the statements are not true, these and the pictures were written during a different part of the prompt.

Though Carla did not demonstrate mastery of the terminology, she expressed no reservation about how to perform the algorithm and showed no doubt that her computation had resulted in the correct value. Contrast those facts with Ann's response to the prompt. Ann also used the standard algorithm for its efficiency, yet she was not confident that her solution was correct:

> ***A:*** *I just got down here [to 30] (laughs)…Um, I don't know if I'm going to get the right answer. I hate division. (laughs). Um……… is that the right answer? (laughs). No?*
> ***M:*** *Well, I'm going to—what I want to ask you next is…what were you doing there that you just erased?*
> ***A:*** *Oh, I was seeing if it worked (laughs).*
> ***M:*** *Did it work? How were you doing it to see if it was the answer?*
> ***A:*** *Multiplying fifteen by point-six.*
> ***M:*** *Ok. And, did it work?*
> ***A:*** *Yeah. Right? (more writing)…Yeah.*

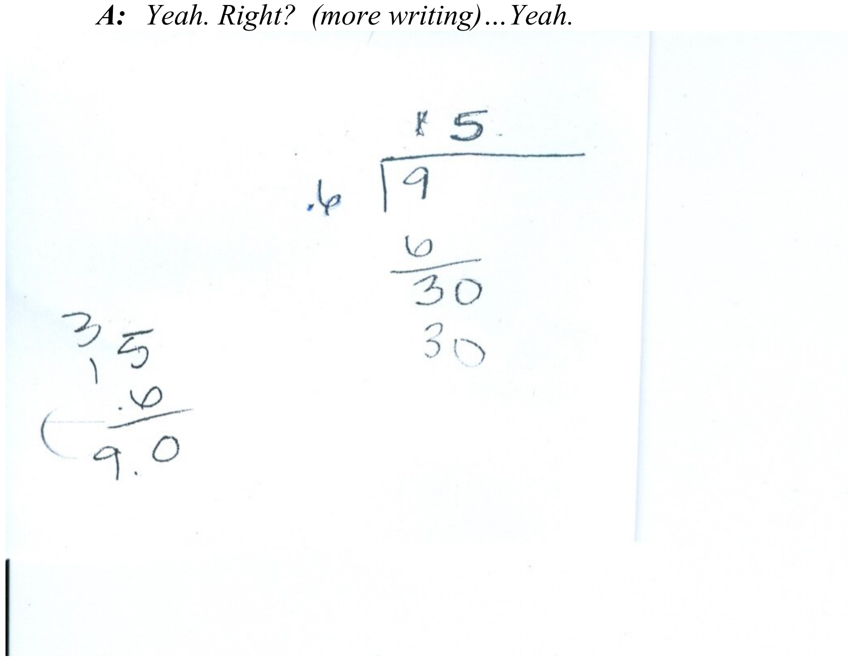

Figure 8: Ann's work on interview prompt #3. Again, Ann uses the standard long division algorithm, but also uses multiplication to check her work.

Ann used the standard algorithm for its efficiency, and yet lacked confidence that her solution was correct. Carla struggled a bit with vocabulary but she showed no hesitation in explaining how the shift of decimal places led to the correct number. On the other hand, Ann's first utterance after working on the problem demonstrated her discomfort. Ann performed the initial calculation using an algorithm which she knew to be efficient, but one which she could not trust to be effective, because she did not understand the algorithm very well. In talking through the next part of this task, in which I asked PSTs how they would represent the problem to a child who struggled to understand the method they used to compute the answer, Ann revealed more about her uneasiness with the standard algorithm, and her preference for an alternative approach:

> ***M:*** *What could you do to help me? What would you want me to understand that might help me understand this thing better?*
> ***A:*** *I don't know. I remember being taught how to do this[long division]…(laughs)…Um…*
> ***M:*** *You've talked about these kinds of problems in class, right?*
> ***A:*** *Yeah, um…I feel like when we did them in class, we'd always like, draw them. And we don't, like, I don't know. We'd never actually, like, write them out.*
> ***M:*** *Ok.*
> ***A:*** *So when we do this…*
> ***M:*** *So, you chose to write it…Yeah…*
> ***A:*** *Yeah…*
> ***M:*** *…how come you chose to do that? If, I mean…*
> ***A:*** *'Cause it's easier (laughs).*
> ***M:*** *I suspect that you probably don't do a whole lot of these in your daily life, right?*
> ***A:*** *Yeah.*
> ***M:*** *Probably most of the ones you've done like this, you've done in class.*
> ***A:*** *Mhmm.*
> ***M:*** *Right? So, if you draw those out in class, what made you do that now?*
> ***A:*** *'Cause it's faster (laughs).*
> ***M:*** *Ok.*
> ***A:*** *Um, I mean, I don't know how to explain it that well. Like, it [the division statement] makes more sense to me to explain it with the picture (laughs). Um…*

Ann and Carla, though they had both used the same algorithm to generate the same answer to the problem, showed different levels of confidence in their knowledge, but this does not necessarily mean that they had developed different knowledge. Indeed, the fact that they both chose the same approach indicates that their knowledge of problems like these was similar. At this level, there is support for the statistical result of the MKTI comparison: there was no measurable difference in PSTs' knowledge. However, knowledge as viewed in situated learning theory is predicated upon a context in which problems occur. Though this "naked" problem has a context of its own, that context has no special place in teaching practice. Being able perform this computation is simply Common Content Knowledge to which many people have access. For this reason, the task expanded beyond this initial prompt to get PSTs' thoughts about how to help a struggling child with the very same problem.

In particular, I asked PSTs to talk about what representations, tools, or ideas they would use to help a student who was struggling to compute the answer using the standard algorithm. Because both MATH 281 and MATH 291 emphasized the use of pictorial representations, I guided PSTs toward a use of pictures to represent the problem. It was at this juncture that Ann and Carla demonstrated a substantive difference in their knowledge; evidence of a difference in MKT. Ann had developed a representation and subsequent understanding of division that could support the connections she may be called upon to make as a teacher:

> ***M:*** *So, could you use a picture to explain this? I mean, if I'm your student, and we've talked about this in class…*
> ***A:*** *I think it'd be easier to explain it with the picture.*
> ***M:*** *So, could you, I mean…just pretend with me for a minute.*
> ***A:*** *(laughs)*
> ***M:*** *Can you go through with me how you might do that? How you might explain this with a picture?*
> ***A:*** *Um, get, like graph paper, and make, like, one, and draw nine of them and then cut them into five—like, point-six, whatever…Like a wall or whatever…Um, and then go over there and count them all up. And I think it better illustrates what it means, rather than be like…'Cause like, when I'm*

> *thinking about this, I'm like, 'Oh, ignore the decimal and six goes into nine one time, and…'*

Ann's previous hedging was no longer evident as she described how she would represent the problem for a child and explained that a pictorial representation (See Figure 9) would have greater explanatory power for her than the standard algorithm.

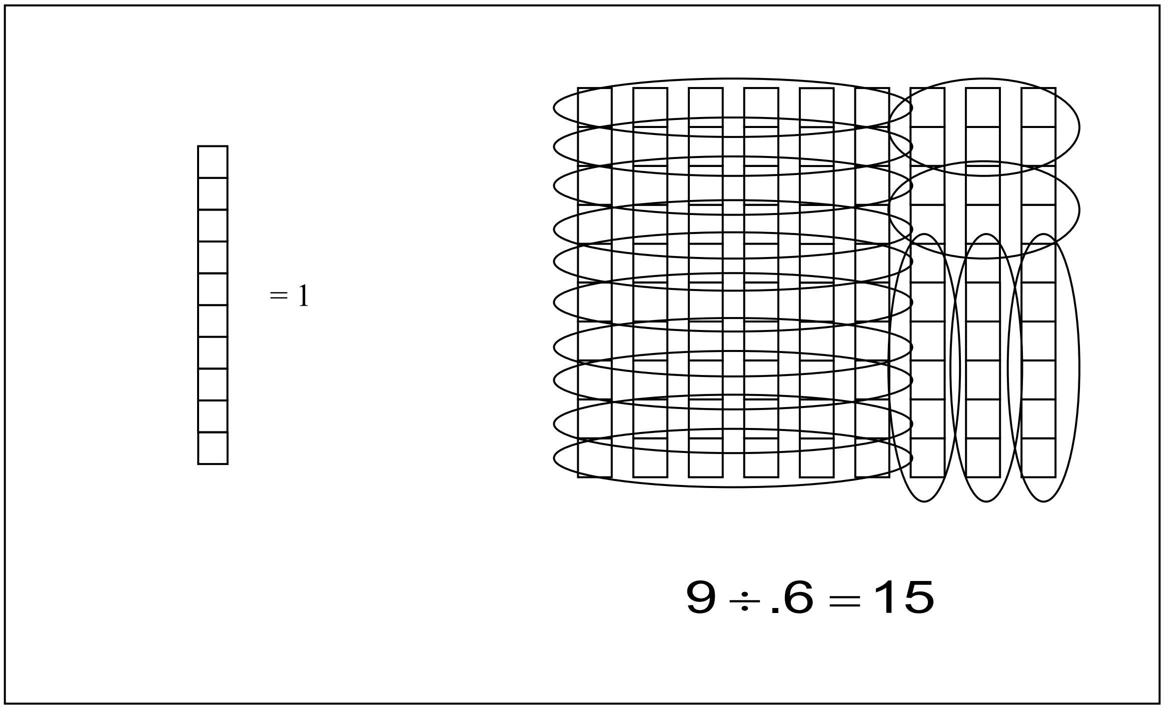

Figure 9: A representation of what Ann was likely describing in her explanation of how to solve $9 \div .6$ using a picture. She says to draw "a wall or whatever," which I have interpreted as the vertical column representing one. Note that there are 15 groups of .6 represented. Ann did not create such a drawing during the interview, but drawings like this—to which she referred—were prevalent in class.

On the other hand, when Carla was asked to explain her use of the standard division algorithm, Carla drew on her knowledge of fractions. Recall that operations with fractions were a large component of Carla's work in MATH

281, in addition to the pictorial representations similar to those that Ann described. Carla's choice was to argue from rules about operating on fractions:

> ***C:*** *I probably would look at saying that a decimal is like a fraction…*
> ***M:*** *Uh huh.*
> ***C:*** *…and that this [.6] is six tenths. Um, and that nine, nine is a whole number, um, and so if we're going to—ok, so that's six tenths—and I'm going to make that a…into just six, I have to make it—I have to multiply it by…do I have to multiply by ten? Yeah. Multiply by ten, right? Is that right?*
> ***M:*** *Seems right.*
> ***C:*** *So then it becomes sixty tenths, does that make sense? Sixty tenths is six, yes it does.*
> ***M:*** *Right.*
> ***C:*** *Um, so that's how I'm—but this child is just learning…I don't know. I actually don't know. If a child that's learning how to do this, can—I would assume that if you're doing, if you're doing, um, decimals, ten you also have to understand the concept of fractions.*
> ***M:*** *Ok.*
> ***C:*** *I would think. Because if they're going to understand that that's, that's…that this is a tenth, then they have to understand that this is what it means.*
> ***M:*** *Right, ok.*
> ***C:*** *That it's six portions of ten.*
> ***M:*** *Right.*
> ***C:*** *And, um, so to make it a whole number, they would have to multiply it by ten…*
> ***M:*** *Right.*
> ***C:*** *…to get the numbers to six. And so, so if I got this to be six, then to make this, to make this...I can't just leave this as a nine. Then I would have to multiply the nine times ten as well, because whatever I do to this one, I have to do to this one. So if I multiply the nine times ten, I get ninety. So then it makes the problem actually, six, um, ninety divided by six, which is fifteen. And, and it works because…it would work regardless. I guess I could make this sixty and make this nine hundred and it would still be the same thing, or, six hundred and this nine thousand, and it would still be the same thing. As long as we're increasing by tens, by hundreds I guess…by hundreds, um, both numbers, then it's ok.*

The choice that Carla made may have been one of necessity. The other primary model for interpreting these problems in MATH 281 was to draw pictures, and Carla was not able to draw upon this work during the interview:

> ***M:*** *I know that a lot of problems that you guys did together, pictures were a big deal.*
> ***C:*** *Right. And that's what I'm trying to come up with…*
> ***M:*** *I'm wondering if there's a picture that…*
> ***C:*** *…a picture.*
> ***M:*** *…that would…*
> ***C:*** *Right.*
> ***M:*** *…make that work, that there would be a way to draw that picture.*
> ***C:*** *Mmm. Um, I mean I guess you could have…gosh I really can't remember how we did the pictures for these, and it was not that long ago.*

Both PSTs used the long division algorithm to compute the answer to the problem, and Carla showed more confidence with this algorithm than Ann did, though when each PST confronted representing the problem to a struggling student, Ann showed more confidence than Carla in generating a representation that illustrates the relationships between the place values which are critical to evaluating the statement. The use of fractions to show how to do the problem is certainly legitimate, though children who are learning long division are unlikely to have gained much facility with fraction operations, and as such, fractions may be may be an inadequate mechanism for the task. In addition, this representation is further removed from typical representations for dividing with whole numbers, while the picture to which Ann referred is a common one among pictures of division with whole numbers. Certainly, both PSTs have an incomplete understanding of division and how to teach it, but this is evidence that Ann has a tool that Carla does not.[55]

The final question related to this prompt asked PSTs to come up with a word problem, a situation that might be best for illustrating this problem—and the concept of division—to children. Here is where the differences between Carla and Ann stand out most.

> ***M:*** *Could you think of how you might devise a word problem that might—that would get the class started with a problem like that?*
> ***A:*** *Um………I always use kids and candy (laughs).*
> ***M:*** *(laughs)*
> ***A:*** *Um, it can be like, there are nine pounds of candy, and you put point-six pounds in a goody bag, how many people get candy, or how many kids get candy, something like that.*
> ***M:*** *Ok.*
> ***A:*** *Is that…good (laughs)?*
> ***M:*** *Alright, let me...*
> ***A:*** *Or, something like that (laughs)!*
> ***M:*** *…nine pounds of candy, you can put point-six pounds in a goody bag, how many kids will get candy?*
> ***A:*** *Or, 'how many goody bags will there be' would make more sense…*
> ***M:*** *Alright.*
> ***A:*** *But, yeah.*
> ***M:*** *Alright…how does that problem match that do you think? Like what about your problem…*
> ***A:*** *And like, what it means?*
> ***M:*** *Yeah, like, what about your problem is going to generate talking about this, this math problem?*
> ***A:*** *I guess just explaining that the meaning of how many times will point-six fit into nine.*

Note Ann's use of the phrase "I always use," which signifies that this kind of problem is something with which she has experience. In fact, writing story problems that could be modeled with number sentences like these was a primary component of the work MATH 291 PSTs did at Rio. Ann modified the language of her initial problem ("…'how many

[55] In fact, Ann's interview took place in the second-to-last week of classes in Rio's Spring semester. Her class was had not completed its discussion of the long division algorithm. At the time of Carla's interview, her class was to meet one more time to review for the final exam.

good bags will there be' would make more sense…"), but otherwise did not show the hesitation that she demonstrated with the long division algorithm. Her last statement also indicates her interpretation of the number sentence as being modeled with repeated subtraction as opposed to associating it with a partitioning model of division.[56] Although it is possible to construct a problem around the statement $9 \div .6$, such stories are often convoluted and awkward. Repeated subtraction is typically a more natural fit. In other words, Ann not only appears to understand the meaning of division here, but also has developed something of a catalog of stories that can be matched to division problems. Contrast this with the struggle that Carla has in turning these numbers into a story problem:

> ***C:*** *Then, I would probably do something a little more practical, like getting a bunch of pennies. Like, getting six pennies, and I could take—'here's six pennies. How many six pennies, um, does it take, to, to um…nine dollars?' Right? Yeah. Nine dollars.*
> ***M:*** *How many six pennies does it take to make nine dollars?*
> ***C:*** *How many groups of six pennies, would it take to make nine dollars? And then I would say, 'Well, another way that you could look at that that would make that a lot easier, is how many six dollar bills are in ninety? Except that there's no such thing as six dollar bills.*
> ***M:*** *(laughs).*
> ***C:*** *How many piles of sixes…*
> ***M:*** *Right, you could, you could…*
> ***C:*** *...how many piles of sixes…*
> ***M:*** *…tweak it a little bit.*
> ***C:*** *Right. How many stacks of six dollars?*
> ***M:*** *Ok. So, I can see why six dollars—stacks of six dollars would go into ninety dollars…there would be fifteen of those stacks…*
> ***C:*** *Mhmm. There would be fifteen of those stacks.*

[56] Division is often modeled in one of two ways. Both models associate the dividend with the total number of objects in the problem (e.g., Suppose there are 45 toys). Repeated subtraction associates the divisor with the number of groups into which the total number of objects is to be divided (the toys are divided into equally-sized groups of five) and the number of groups is unknown (how many groups will there be?) Partitioning associates the divisor with the number of equal-sized groups into which the total is to be partitioned (the toys are divided into five equally-sized groups) and the size of each groups into which that total is to be divided is unknown (how many toys will there be in each group?).

M: *But if I do six pennies…*
C: *Mhmm.*
M: *There are going to be a lot more than 15 groups of six pennies in nine dollars, aren't there?*
C: *Oh, this would be sixty pennies. But that would make it even suckier. Yeah, because, I'm sorry, you could have…*
M: *You could have…*
C: *…a hundred and fifty pennies…*
M: *So…*
C: *Yeah.*

Here, Carla also seems to be trying to use a repeated subtraction model for the division—she is trying to figure out how many groups of a particular size fit into a total—but she demonstrates great difficulty in articulating a story that would require such a calculation. In particular, when attempting to use money to illustrate the numerals, she does not have a firm grasp on the connection between the relationships between pennies and dollars, the decimals in the numerals representing their monetary value, and the place values which they represent. Carla wanted to associate the decimals in the naked statement with the decimals that are prominent in using money, but she could not connect the physical representation of those decimals with the numerals themselves as Ann had done.

This is one of the key components that Ma (1999) highlights in her description of profound understanding of fundamental mathematics, and is the primary element of SCK. Ann had already developed a collection of stories that fit models for division, and Carla, while making progress as she talked out loud, appeared to be confronting the problem as if for the first time. Ball claims that one important piece of MKT is ability to choose appropriate representations and examples to illustrate key mathematical ideas. In the exchanges above, Carla showed some discomfort in choosing a representation for the number sentence $9 \div .6 = 15$ and struggled even more when trying to come up with a story in which the procedure might be applied. In contrast, Ann confidently gave a quick description of how she would draw a picture to show the relationships between the numbers in this situation. These relationships form the basic sub-concepts which make up the larger idea(s) of division. She said "I always use…" as if she had much experience designing problems around division number sentences. In fact, she did have some experience doing this, because it was a focus of the MATH 291 course which she had nearly completed. MATH 281, on the other hand, did not emphasize these connections as strongly and so it is not surprising that this difference between Carla and Ann arose.

I claim that these differences are explained by the differences arising from the two courses themselves. Carla's knowledge was procedurally sound and made an important connection between representing rational numbers both as fractions and decimals. It was a reflection of the focus in her course on procedural fluency, and justification for those procedures across these different representations. Based on this evidence, there is reason to expect that Carla could operate on rational numbers with relative ease. When Carla was asked to generate a context for these calculations, she struggled. On the other hand, Ann admitted a lack of confidence in working with rational numbers (even when in decimal form) and her experience in MATH 291 was unlikely to help her make the kind of connection between decimal and fraction representations that Carla made. However, MATH 291 clearly had an influence on Ann's use of representation when thinking about a context in which she was a teacher of young children. Despite her uneasiness with the problem itself, Ann showed no corresponding fear of describing a picture for solving the problem and a related situation for which it could serve as a mathematical model. Though she may not have been able to use it for operations on fractions, Ann's representation is robust enough to support such a conceptual move. Carla, unable to bring such a representation to bear on the problem, struggled greatly to come up with a context in which the problem could be set. Carla's knowledge about this appears to be confined to the Common Content Knowledge and Specialized Content Knowledge sub-domains of MKT, while Ann shows evidence of Knowledge of Content and Teaching. These differences echo the differences between the courses each PST took and, I believe, are explained by them.

The different course experiences can explain these different responses, if not completely, then at least in large measure. If we were to consider Carla and Ann isolated cases, it may be easy to dismiss them as anomalous. After all, Ann appeared to have made much greater progress over the semester than Carla did on the MKTI scores. There is evidence however, that Carla and Ann are not accidents, but part of a trend.

Consider other evidence that shows parallel results between Eliot and Maeby, who earned among the highest scores of their respective courses on both pre- and post-tests of MKT. Eliot was a student in MATH 281 at Hilada, while Maeby was enrolled at Rio in MATH 291. Eliot was the only PST who completed the interview significantly *after* her course was over.[57] Interestingly, neither Eliot nor Maeby made use of the standard long division algorithm to solve the problem. Eliot immediately invoked division of fractions, multiplied nine times ten-sixths, and simplified. Instead, Maeby used a guess-and-check strategy through multiplication. Like Ann, Maeby showed a lack of confidence in her first instinct:

[57]All Rio PSTs were interviewed in the second-to-last week of classes, while two Hilada PSTs were interviewed during the last week of classes. Eliot's interview took place nearly a month after classes at Hilada had concluded.

MA: *…and then another, so that [refers to six] would be one. So, twelve, fourteen, fifteen.*
M: *Ok.*
MA: *A little drawn out, but, (laughs).*
M: *But, if that's how you do it, that's…Ok, so fifteen, and would be pretty confident with that answer? Like if somebody said, 'Alright, I'll give you one hundred bucks if you're right and nothing if you are wrong…'*
MA: *(laughs).*
M: *Would you feel pretty confident about that?*
MA: *Yeah.*
M: *Yeah? Ok. Well, you just made a face like, 'Oh, maybe not.'*
MA: *(laughs). Um, I'd be about eighty-five percent confident (laughs).*

Maeby's figure of 85% seemed high to me at the time; her demeanor and voice indicated that she had little confidence that she was right. This time, it was not so much that she did not believe in her method for evaluating the statement (guessing and checking is a relatively inefficient, idiosyncratic approach which it seems unlikely she would have used without understanding) but rather than she felt sure that she had made a mistake along the way. If Maeby showed similarities with Ann in her lack of confidence of the initial computation, she showed similar comfort with creating a representation of the problem that could help her evaluate it:

M: *What could you do to make yourself more confident, other than getting out a calculator?*
MA: *Other than getting out a calculator? I honestly would get out my graph paper and do like, what we did in class, like make…*
M: *Can you give me a sketch of what that might involve?*
MA: *Um, yeah. So, the BMU would probably be like, ten blocks. So I'd draw out those….like, this is ten.*
M: *Ok.*
MA: *(whispers) one, two, three, four, five…*
M: *And so each one of those has ten little blocks…*
MA: *Yeah, ten little blocks, so it'd be ninety all together…*
M: *Alright.*
MA: *…and then um…no wait, now I'm trying to think…this is bad because the final's going to be on this (laughs)…*
M: *(laughs).*

MA: *Um, and then, so then point-six would be six blocks, and so you'd see how many groups of six blocks fit into that ninety.*
M: *Ok, so then you'd try and...would you count, would you circle, what would you do to figure that out?*
MA: *Um…yeah, I would probably count, like every six, and block it off, block it off, block off every six.*
M: *Ok.*
MA: *And then when I got that done, I'd just count how many blocks…*
M: *Right.*
MA: *…I'd done.*
M: *How many blocks do you think you'd get?*
MA: *(sighs)……so I guess you'd be doing—that, at that point, it would just be ninety divided by six, because you have ninety blocks and you're dividing it into like, sixes.*
M: *Oh, ok, uh huh.*
MA: *Um so……Um, maybe I've—I don't know, now I think I'm wrong (laughs).*
M: *Well, what were you going to write down?*
MA: *I was…*
M: *So you say there's ninety blocks…*
MA: *Yeah, ninety blocks…*
M: *…divided by six little bitty blocks. What were…*
MA: *Well, it's divided into groups of six blocks.*
M: *Oh, groups of six blocks, ok.*
MA: *Um, so I guess that would just figure out like…um, it would be, so it'd be ten is sixty blocks and we need ninety. Yeah…thirty…yeah, it'd be fifteen groups of six blocks. Right? Yeah.*
M: *Which is what you got here.*
MA: *Yeah. Yes…Yeah! (laughs)*
M: *Alright. So, are you more confident now that it's fifteen?*
MA: *Yes.*
M: *Ok. Um, ok, very interesting. So…*
MA: *Do you know the real answer? Am I right (laughs)?*

Maeby could not shake the lingering doubts raised by her first computation. Still, she quickly chose a BMU ("So, the BMU would probably be like, ten blocks…"), described it ("So I'd draw out those….like, this is ten…"), and then outlined how that choice would lead her to an answer to the problem ("…so then point-six would be six blocks, and so you'd see how many groups of six blocks fit into that ninety…").

Eliot, a Hilada student, immediately and confidently used a fraction representation to answer the question but, like Carla, had difficulty coming up with an alternative representation or generating a word problem that might require such a computation:

> ***M:*** *But how you would actually solve that?*
> ***E:*** *(Writes). (Mumbles). Yeah.*
> ***M:*** *Ok.*
> ***E:*** *That's [.6 is] an easy number—an easy decimal for me, because it's easily put into a fraction. Um, so I put it into a fraction, and you multiply—you um, multiply by the reciprocal, so then you have fifteen.*
> ***M:*** *Ok. And if I'm a kid your class, and you've taught me how to do this kind of problem like this, I might come up to you after class, or I raise my hand during class, and I say, 'You know, I just don't get this, dividing by this and then you flip it over…can you…?'*
> ***E:*** *Yeah.*
> ***M:*** *How would you do that? Like, how am I supposed to do that? What's going on there?*
> ***E:*** *Um, it's hard to have point-six of something, isn't it? Um…*
> ***M:*** *Well, let me ask you this: is there something about the [281] class that you can draw on? Is there something that you feel like you learned in that class…*
> ***E:*** *Yeah, I'm just not remembering it (laughs).*
> ***M:*** *Well, ok, that's important. So, even if you just described…*
> ***E:*** *Yeah.*
> ***M:*** *…a little bit about what that might be.*
> ***E:*** *When we learned how to divide it was, and I don't remember which way it went but you had two divided by—uh, four divided by two, meaning that you have…I don't know which way it went. But you have two—you have four items that you put into two groups…*
> ***M:*** *Ok.*
> ***E:*** *I just can't remember the order.*
> ***M:*** *Ok.*
> ***E:*** *Which comes first. So you have four ítems and you ask how many items can go into each group? And so I guess it's this way: if you have point-six items, which is hard to explain to kids…*
> ***M:*** *Mhmm.*
> ***E:*** *That you have six-tenths of—probably of, maybe like a pie.*
> ***M:*** *Ok.*
> ***E:*** *Mmm, no, a cake, that's square.*
> ***M:*** *Ok.*
> ***E:*** *So then you turn—so then you have six pieces of cake…No, you have nine pieces of cake, that need to be divided into point-six groups, which is really hard to—I have no clue.*
> ***M:*** *Ok.*

> ***E:*** *I mean, I have a clue: I learned it, I can't remember it.*

Eliot, like Carla, showed confidence and comfort with her procedure for computing the value. Eliot chose to evaluate the expression using fraction representations exclusively, while Carla's approach was to use the standard algorithm. However, both PSTs from Hilada demonstrated facility with the computational demand of the problem: neither PST expressed any concern about evaluating the statement, nor any reservation about the answer and whether it was correct. Eliot went so far as to call the problem "easy" ("That's an easy number—an easy decimal for me, because it's easily put into a fraction…"). Eliot's comment that, "I don't remember which way it went" suggests that there is a "right" way to interpret the problem and is evidence that she was unaware that division can be modeled in two ways. Again, though repeated subtraction is a more natural model to use, there is nothing about the number sentence that precludes the use of partitioning as a model. An ignorance of both division models could have played a role in Eliot's struggle to fit the problem into a partitioning model of division. Recall however, that in MATH 281, referring to these models often disappeared after working with whole numbers and operating on fractions instead. Likewise, Eliot's MATH 281 did make use of drawings and discussed word problems for these kinds of mathematical statements, though these activities did not appear to make a strong impact on Eliot because she was unable to draw upon them in order to respond to the prompts.

This transcript excerpt is not simply a demonstration of Eliot's lack of MKT. Rather, it shows that her MKT was possibly more concentrated in common content knowledge instead of other sub-domains. Moreover, this excerpt contains evidence that Eliot had in fact developed knowledge of content and teaching; she acknowledged that a rectangular object such as a cake would be more amenable to fraction representations than a circular pie. This knowledge that some representations make more sense in instruction than others (it is much harder in a classroom situation to divide circles accurately into equal-sized groups than rectangles) is a key component of KCT. Still, the ability to deploy this culinary image in a coherent story problem remained elusive for her.

On the other hand Maeby, who lacked confidence in her calculation, spoke with some authority on the different models for division and how making a decision about which model to use has an impact on the difficulty of the problem.

> ***M:*** *…And it's interesting that you say that the, you would at least start out by thinking about it like this…*
> ***MA:*** *Because that way helps me...realize, I guess, what this means. Um, because the way I usually start out thinking about it is: you can't have point-six groups…*
> ***M:*** *Ok.*
> ***MA:*** *...so that must mean that you are dividing it groups of point-six.*
> ***M:*** *Right, ok.*
> ***MA:*** *Because it can go either way with division.*
> ***M:*** *Uh huh.*
> ***MA:*** *So then that immediately makes it a lot easier to think about, because you're like what's point-six of a group? You know what I mean?*
> ***M:*** *Right.*
> ***MA:*** *So that immediately—at least it gets you on the right track; you start thinking about in a certain way that'll cause less problems.*

Like Ann, Maeby used the phrase "The way I usually start out thinking about it is…," which suggests that despite her computational concerns, she has experience evaluating these problems. Moreover, Maeby's original approach to the problem gave her insight into the meaning of division; her strategy turned the problem into a multiplication problem, building the number nine using multiples of .6, which in turn illuminated the fact that repeated subtraction was the "easier" way to think of the problem.

In this group of PST interviewees, two students in the high performance category struggled to address this issue completely, but Maeby's struggle was ameliorated with a representation in mind, a tool by which she could deduce (and explain) the division to a child. Eliot, on the other hand, struggled in a much more fundamental way: what does the division actually mean? This difference again can be interpreted through the distinct approaches taken by the courses that these PSTs had completed. Maeby had more opportunities to develop these particular tools than did Eliot, whose MKT advantage lay in her computational facility.

Finally, consider Lindsay and Laverne, who were two PSTs representing their peers who scored more than one standard deviation below their respective class means on the beginning-of-semester MKTI. Laverne doubled the number of correct answers she gave from February to May (more than two standard deviations), while Lindsay's increase was more closely aligned to the effect size for both courses (approximately one standard deviation). Their mean scores on the survey instruments were nearly identical at both points in the semester, and were lower than their respective course averages. Lindsay was enrolled in MATH 291 at Rio, while Laverne was enrolled at Hilada in MATH 281. These PSTs, in contrast the previous interviewees I have described, both answered the division problem in question incorrectly though both used the standard division algorithm. At first, they each responded that the solution was 1.5, instead of 15. Laverne eventually corrected her mistake. Lindsay checked her answer by multiplying, but since she proceeded to compound the error by using the wrong numerals, the multiplication (6×1.5 instead of $.6 \times 1.5$) confirmed for her that she arrived at the correct answer. Here, the models used in their respective courses had not made a strong impact on either PST; neither could recall how to apply models they had discussed in class to the problem at hand, though they could speak in general terms about what they were "supposed" to do. It was as if the pictorial representation and the fraction justifications were just as routinized—and just as poorly understood—for these PSTs as the standard division algorithm was:

> ***LA:*** *We went over this in class the other day…and I really don't remember.*
> ***M:*** *Ok.*
> ***LA:*** *Because you move the decimal point over—[the instructor] told us why you move it—but I don't remember why.*
> ***M:*** *Ok.*
> ***LA:*** *So, I mean…I would have to, like, look at some notes or something to tell you that. But I would tell him why we move the decimal place over one…*
> ***M:*** *Ok. Why is that?*
> ***LA:*** *I don't remember.*
> ***M:*** *Oh, ok.*
> ***LA:*** *I've got to look at some notes.*
> ***M:*** *Ok.*
> ***LA:*** *I don't remember why.*
> ***M:*** *Ok.*
> ***LA:*** *I just know that I'm supposed to do it. Let me think. Actually, let me think about it.*
> ***M:*** *I mean, would drawing a picture help?*
> ***LA:*** *It has something to do with fractions. I don't remember.*

Laverne's comments suggest that she lacks the computational proficiency demonstrated by her Hilada classmates, Carla and Eliot. This may explain her lower-than-average scores on the MKTI. My guidance to use a picture elicited no meaningful response from Laverne, which may be because the pictures discussed in her class were not effective in helping her to understand the fundamental concepts. But again, this was not a particularity of Laverne, but a trend among all the Hilada PST interviewees: pictorial representations of numerals did not serve as useful tools; mathematical notation did. This was why Laverne, even when I suggested drawing a picture, insisted that the key to understanding the problem lay in fraction representations.

Lindsay showed a similar lack of confidence as Laverne did, claiming that she did not really understand how the process worked, and was similarly non-committal about the details of the approach:

> ***LI:*** *Because like, I'm not really sure myself, so I don't know, like how I would really explain it to a kid that it would make sense. Because yeah, it makes sense to me, but I don't know how to explain it to them about the zero, and moving the decimal…*
> ***M:*** *Mhmm.*
> ***LI:*** *…and like bringing it up and bringing it down, and subtracting and checking your work. Like, I don't know—I don't really know how I would explain it in a way that they would understand.*
> ***M:*** *How—were you doing this? Because I didn't get to see what you guys did with these, but were you doing it like this? How were you doing it in class?*
> ***LI:*** *We were doing it more like, you would have to draw your BMU. And like, doing it with partitioning division, and like, repeated subtraction…*
> ***M:*** *Ok.*
> ***LI:*** *…and like, do it that way. And we would have to show it. Like we would have, like, if you're doing partitioning and like, how you go, one to this group, one to this group, one to this group. And then one to this group, one to this group, one to this group. You know, like that way?*
> ***M:*** *Uh huh.*
> ***LI:*** *So we weren't even touching any of that. It was, you do it this way, and count it up.*
> ***M:*** *Ok.*
> ***LI:*** *But like, I'm shaky on that (laughs), so like, I don't know how I would explain to a kid…Yeah, it's easy to be like, 'Ok, just keep going until you can't give one to these groups anymore,' but…*
> ***M:*** *Uh huh.*
> ***LI:*** *…but I don't know.*

Lindsay used the language of the different division models and referred to the BMU as a way to evaluate the problem, but only after I prompted her to think about her course. This is evidence that she did not make the connections on her own: the BMU and the two models of division are ways to unpack this problem and the process of evaluating it for children—and understand them for ones' self. Indeed, Lindsay said as much: despite the work she had done in class and her other years of mathematics preparation, division was simply something she could replicate (albeit, at times incorrectly) not something she understood:

> ***M:*** *What kinds of things do you think are important to understand about division for either of these ways of approaching it to make sense?*
> ***LI:*** *You have to understand that you're putting a number—you're seeing how much of another number can go into a number. So like, you have to think of it that way. I mean, division to me is division. Division is division, it's just something I do. I don't really know why I do it. Like, sometimes, it's because I want to know how much of something I need or need to get, but I mean, I couldn't tell you the meaning of division. I couldn't tell you like, why or whatever.*
> ***M:*** *Ok.*
> ***LI:*** *I just do it because I know I need to do it. I mean, it's a way of getting an answer I need to get. But, I couldn't tell you…that sounds really bad (laughs). I'm in college and I can't tell you what division is!*

Lindsay demonstrated her awareness of two division models, but the distinction did not reveal anything to her about performing computations or about how children might come to understand them. Division remained a relatively meaningless computation that Lindsay had learned to follow directions in order to complete, but did not understand well enough to deconstruct for children.

In this last group of PSTs, the differences that were apparent between the other pairs are not as clear, since they each had a more fundamental difficulty with the problem, the meaning of division, and they still—at the end of their course—lacked models that would support their own understanding. What may be most germane to this analysis is the fact that each PST spoke about solving the problem in similar ways as their classmates did. The MATH 281 PSTs clung resolutely to the use of fractions to compute and justify a division involving decimals, and they either could not recall a pictorial representation or did not mention it as relevant to the problem. On the other hand, the MATH 291 PSTs inevitably sought the concept of the BMU as represented by pictures used in their class. These behaviors echo the work and focus of their respective courses, and indicate that indeed, the way they talked about mathematics, and its proximity to elementary school mathematics influenced their responses during the interview.

Consider as well the second mathematical prompt in the interview, which supposed a second grade child trying to figure out the problem 5 x 2 using counters (or chips), and who got out a set of two counters and then a set of five counters. PSTs were asked to describe what they thought the child misunderstood about multiplication, and whether or not the child's idea could be "salvaged." That is, could this be the starting point for a correct answer to the problem? This prompt is aimed at PSTs' SCK as well as their knowledge of content and teaching. The prompt requires PSTs to discern what a student is thinking given incomplete information and use a concrete representation to support a child's developing understanding. In both courses, counters were used for thinking about multiplication in a discrete model (such as repeated addition) and the operation was defined so that the first numeral in the expression represented the number of groups repeatedly added, while the second numeral represented the size of each group. Again, the MATH 281 devoted a substantial amount of time and energy demonstrating and justifying the consistency with which multiplication could be applied across number sets (whole numbers, integers, rational numbers) and the appropriate notation for expressing these computations. In contrast, MATH 291, though it defined multiplication in precisely the same way, explored non-integer numbers (expressed as finite decimals) through its continued use of the BMU.

Four of the six interviewees expressed some confusion about the precise relationship between the two numerals, namely which numeral represented the number of groups and which numeral represented the size of the groups. Two of the three PSTs from Hilada changed their initial answers, indicating that for them, the definition was still unclear. Consider the example of Eliot, whose score on the MKTI was more than a standard deviation higher than her peers at Hilada:

> ***E:*** *Oh, oh, ok. Well…ok, so she has the right idea. Like, that she needs one row of five, and—or like, rows of five. But how many rows of five does she need? And so then she has two, so if we did...If I made her—if I somehow showed her that, if we flipped those, instead of making them horizontal, make them vertical…So, you have one, two. And like show that you can have two rows of five. Or, five rows of two…Right now, she's seeing five and two and she's not understanding that it's five groups of two. Or, if it was the other way, then I guess it would be—it wouldn't be this one, because the way that we teach it is five groups of two.*

Eliot's last statement amended her earlier remarks that 5 x 2 could be represented as two groups of five. Carla similarly equivocated when it came to creating a representation for 5 x 2:

> ***M:*** *This seems to be really important to know which [number] is groups and which is how many…*
> ***C:*** *I know, and I'm not quite sure either. So, say, ok, here we go, right? Say like the way they want to be able to do it is to draw a diagram that can show that and that. So this one, this, shows, um…the…I'm sorry, it's actually two groups of five—I had it mixed up—and this would show five groups of twos.*

Eliot and Carla were the only PSTs who changed their minds about this issue. Their corrections led them to the accepted definition of multiplication in both courses. The other four interviewees did not hesitate about the correct representation for this problem. Two of them gave a representation contrary to that offered by their instructors, saying that a proper representation would be two groups of five. The two who not only arrived at the representation consistent with both courses but did so without changing their initial response were Maeby and Ann, both of Rio University. Maeby and Ann were counterparts to Eliot and Carla respectively, in terms of their beginning-of-semester MKTI scores. The four highest scoring interviewees eventually expressed 5 x 2 using five groups of two, but the Rio PSTs did not express any confusion or uncertainty about the representation, while their counterparts at Hilada did. Here again, Rio PSTs demonstrated more facility in concretely representing mathematical statements, a reflection of the emphasis placed in their course on specialized content knowledge. Hilada PSTs, working with the same definition of multiplication, referred constantly to and made use of the commutative and distributive properties of multiplication, and concentrated less on generating multiplication story problems. Rio PSTs did not focus on the formal mathematical arguments like their counterparts did and instead referred frequently to scenarios in which the definition of multiplication (# of groups multiplied by the size of each group) was a necessary feature of the work they did. Thus, the MATH 291 PSTs had a stronger sense of this definition than did their MATH 281 counterparts.

Another reason I chose this prompt was because it provided an opportunity to discern whether and how PSTs could build on student thinking toward a conventional understanding of multiplication. This is associated with a knowledge of content and teaching. When I have asked PSTs this question in the past in my own teaching, two typical responses are (1) Starting Over: that the child is likely thinking of addition, and must start over by pushing the counters aside and then creating five groups of two; and

(2) Salvaging: that while the child may be thinking of addition in behaving this way, a teacher could show the child that the five could represent places to signify each group of two, and that the child could arrange two counters under each of the five counters to express ten. Both of these answers are problematic for different reasons. A "starting over" response ignores the idiosyncratic and developing nature of children's thinking and dismisses the idea as wrong and something that must be disposed of. The "salvaging" response credits the student with sense-making and acknowledges the fact that addition and multiplication are related, but the presence of 15 counters (five counters to signify each group, and then the ten counters that display the answer) instead of ten might be confusing to the child and therefore be counterproductive. I wanted to see which response PSTs would give and whether or not they would identify these potential problems. In fact, nearly every interviewee chose a third kind of response, which might be considered a hybrid of the two responses I described above. Most PSTs claimed that the child could use the counters she had gotten originally, but that they should be arranged to show the five groups of two (or in the cases of Lindsay and Laverne, two groups of five). Ann articulated this position most clearly among her peers at both universities:

> ***M:*** *Like, can you say, 'Alright, let's start here, and then I can take you to where you want to go'?*
>
> ***A:*** *I guess you can like put them all in, like, a pile. I don't know. Like, you wouldn't get the right answer, but then just, like have a pile and start counting out two and she'd run out because she wouldn't have enough. But, then you know, like, you form groups of two so that you have five of them and then you can count them all, but she didn't, 'cause like, you don't know how many to start out with… you could just start counting out twos, and when she runs out start counting another pile, I mean…and then start grabbing from the pile until you have five.*

Here, Ann describes using the initial response by the child to turn the discussion toward repeated addition, using the group of two as a starting point, and decomposing the group of five into groups of two, adding in new counters as necessary until one arrives at five groups of two. Other than the fact the two of the three Rio PSTs initially responded to this task with an interpretation of multiplication consistent with their course work while two of the three Hilada PSTs had to change their answers, it did not result in any discernable patterns of differences across campuses.

There were three interview prompts directly targeting PSTs' MKT: I have described PSTs' responses to two of them. The remaining prompt was actually the first mathematically oriented task in the interview; it required PSTs to examine student-generated work on the subtraction statement 75 – 48, respond by explaining what the child was doing for each, whether or not the method worked in the case presented, and if so, whether it would always work. This item was intended to echo similar items on the MKTI and get more information about how the PSTs interpreted such problems and what mathematical ideas they could extract from them.

Three Children Work out 75 – 48:

(a)
$$\begin{array}{r} 75 \\ -\ \ 48 \\ \hline 3 \\ 30 \\ \hline 27 \end{array}$$

(b)
$$\begin{array}{r} 75 \\ -\ \ 48 \\ \hline 5+2=7 \\ 7-5=2 \\ \hline 27 \end{array}$$

(c)
$$\begin{array}{r} 75 \\ -\ \ 48 \\ \hline 33 \end{array}$$

Figure 10: Prompt for the first task in the interview

In item (a), the child subtracts place values, and then adds the results of these two computations: 75 – 48 = (70 + 5) – (40 + 8) = 70 – 40 + 5 – 8. Item (b) was especially difficult for the PSTs who addressed it. The format of the writing obscures the fact that the procedure is to round the subtrahend (48) into the next multiple of ten (50) and add the number required in that rounding (two) to the minuend, in order to maintain the "distance" between the numbers. The third line represents the ones calculation and the fourth line represents the resulting tens calculation. During the first two interviews at Hilada, I chose to move on to other tasks instead of dwelling too long on it. Item (c) is simply incorrect, though the child is similar to that in (a) by working within place values instead of regrouping across them. This is a commonly reported mistake that children make, subtracting five from eight instead of the other way around, which is required by the problem.

This task was difficult to analyze across universities as I have done for the other tasks, because in two of the interviews time constraints prevented me from probing deeply about all three scenarios. In particular, Carla and Laverne (both Hilada students) did not address all three tasks; neither of them discussed part (b) in any detail, except to express their lack of understanding of it. These PSTs were the first interviewees of the six. As I was concerned about dwelling too long on the first mathematical task in the interview, when discussions of items (a) and (c) did not end until past the 30-minute mark in a scheduled 45-60 minute interview, I chose to move on and try to acquire data from the other prompts in the interview protocol. As a result, I did not get a complete set of responses from all PST interviewees. For this reason, I will not describe PSTs' responses to (b) in any detail; they do not shed any information that can be adequately compared with the others'.

Part of the reason for this difficulty during the interviews is likely that the items in this task—particularly (b)—were flawed. I had intentionally written the items to mimic those found in the MKTI, and I obscured some of the steps taken by the hypothetical students. For example, I did not write a negative symbol in front of the three in the third-to-last line of item (a). I intended to find out how well PSTs could discern the mathematical ideas in the limited information written by students "A," "B," and "C." For item (b), in retrospect, it seems that an alternative format might have made it more useful. This might mean writing out what a child might say to explain himself in a hypothetical transcript related to this item. All PSTs struggled with the different items, in their own ways, and to that extent, their struggles may invite opportunities for understanding their MKT. On the other hand, I do not think that the data from this task provide particularly strong evidence for drawing conclusions about the PSTs' MKT relative to one another.

All PSTs recognized that (c) was flawed and pointed out that it was likely that the child was subtracting five from eight instead of the other way around. All but one PST struggled to see in (a) that the student might be subtracting three from 30 (or adding -3 to 30) to arrive at 27, and when, in each interview I suggested that the child would do this, PSTs generally had difficulty understanding why. Consider Eliot's reaction to Child A's work. She said,

> *For A, um, I don't really know how this works out. But, uh, what I think they're doing is they're subtracting the tens places, and then—I mean the ones and then the tens, so I think they did it again, where eight—five—eight minus five is three but seventy minus forty is thirty…and then they subtracted those two [three from 30]. I don't really know, you know?*

The other PSTs across campuses responded in similar ways. Only Maeby identified that the child was subtracting three from thirty without prompting. Not a single interviewee discerned that the child would have subtracted the three from 30 because it represented a negative number. When I suggested that an adult might write the problem differently by putting a negative symbol in front of the three, some differences emerged. Carla responded this way:

> ***M:*** *So if I put—If I changed the way I wrote that problem by just putting a negative in front [of three], would that…*
> ***C:*** *Mhmm.*
> ***M:*** *…give you a sense of whether or not this was going to work?*
> ***C:*** *Yeah. Absolutely. But I wonder if a child who's just learning how to do this [subtract multi-digit numerals], can really understand the idea of negatives, you know? This does work…*
> ***M:*** *Right.*
> ***C:*** *…but is the child really understanding that in order to subtract, the number that you're subtracting from has to be larger? You know?*
> ***M:*** *Mhmm.*
> ***C:*** *And they are getting the right answer, but are they understanding what subtraction really is?*

Carla appeared to believe that this algorithm would work generally (though there is not direct evidence of this in the transcript: she responded only to the question of whether or not the negative gave her a "sense of whether" it would work). Her primary concern was with the child's understanding and a fundamental doubt that a child would be capable of such reasoning: "I wonder if a child who's just learning how to do this can really understand the idea of negatives…" is evidence of this doubt. Laverne, Eliot and Carla's classmate at Hilada was not sure how to evaluate the method generally, apart from trying multiple cases:

> ***LA:*** *Oh. I would probably make another problem ad have him do it the exact same way just a couple of times, to see if it worked, and if it worked…*

> ***M:*** *Uh huh.*
> ***LA:*** *…I might be like, 'Ok,' but it depends on where I am. If it's just me and this child and we're like, like this, by ourselves, we're off somewhere…*
> ***M:*** *Mhmm.*
> ***LA:*** *I don't really know what I would do. But if we were like in a tutoring session where there's like a math specialist around or something, I'd go to them. Right away.*
> ***M:*** *Uh huh.*
> ***LA:*** *No hesitation.*
> ***M:*** *Right.*
> ***LA:*** *We probably wouldn't have even gotten to this step. I'd have been like, 'What is that?'*

There was little consensus among the responses about (a) by PSTs at Hilada.

At Rio, Lindsay expressed her own doubts, but unlike Carla, she implicitly conceded that the child *could* do this. Her concern was whether or not the child's work gave evidence of such knowledge. For her, the presence or absence of the negative symbol was key to whether that evidence of the child's understanding was present:

> ***M:*** *…what if Kid A is really thinking about this like this?' And I put a negative there [next to the three]?*
> ***LI:*** *Well then, Kid A is right. Because then like, if Kid A thinks of it as a negative, and they're showing—he showed me that he thought of it as a negative, knowing full well that he'd have to take that [the three] away from thirty, it's ok then, because I know that they understand that they can't do eight minus—five minus eight.*
> ***M:*** *Without that negative…*
> ***LI:*** *Yeah…*
> ***M:*** *…you wouldn't feel very confident.*
> ***LI:*** *No.*
> ***M:*** *No.*
> ***LI:*** *Because I'd feel like they wouldn't understand the concept of five minus eight.*
> ***M:*** *Ok.*
> ***LI:*** *It can't be done. Well, it can, but…that negative makes a whole bunch of difference.*

Lindsay did not express the reservation that Carla did that children were incapable of thinking in this way, but instead she felt that given the child's work—and lack of a negative symbol—she could not infer the child's understanding of the approach. Ann expressed a similar sentiment, noting that the negative would be an important part of knowing whether or not the child understood what the three meant:

> ***A:*** *What if they understand that this is—like you're taking this number[eight] really from this [five], then it would work. But they have to know[the three is] a negative number.*
> ***M:*** *So, so you're thinking that in order for somebody to really use this properly they should really know what role that three is playing?*
> ***A:*** *Mhmm.*

Ann implies that there is not enough evidence of this fact to believe that the child can reliably apply this technique to multi-digit subtraction problems. Thus, at Rio, this issue of looking for evidence of children's understanding of the algorithm came up multiple times. Maeby, Ann and Lindsay's classmate at Rio, had a different, but related concern:

> ***M:*** *But what if I put in that [a negative symbol]? Does that make it make more sense?*
> ***MA:*** *Yeah. 'Cause then, um, well…'cause then you get twenty seven—it's like you're treating it as an addition problem, the thirty and the negative three. So, you're saying thirty plus negative three.*
> ***M:*** *Mhmm.*
> ***MA:*** *Um, (laughs)…but because you're not, exchanging, carrying it over, it's—technically, negative three is the answer to five minus eight…*
> ***M:*** *Uh huh.*
> ***MA:*** *…um, so, yeah, it…*
> ***M:*** *So, it prevents the exchanging…*
> ***MA:*** *Yeah.*
> ***M:*** *…but it also means that you have to do negative…*
> ***MA:*** *Yeah, you have to realize, then you have to switch it—you have to get out of subtraction mode, kind of, and make it…*
> ***M:*** *Right.*
> ***MA:*** *…like, addition mode.*
> ***M:*** *Right.*
> ***MA:*** *Because thirty minus negative three would be thirty three.*

Instead of thinking about whether or not the child really understood the value of the three, Maeby was concerned with whether or not there was evidence that the child understood the subtle shift from subtraction to addition. Indeed, this kind of shift could rely on the kind of representation used frequently in MATH 281 (75 – 48 = (70 + 5) – (40 + 8) = 70 – 40 + 5 – 8), but it is identified—if not expressed this way—by a PST from MATH 291 instead.

There is some commonality among Rio PSTs, in the sense that they each expressed concern for the evidence they can find of a child's knowledge given the limited information in each item. This is an important pedagogical stance to take, but again, it does not necessarily reveal any difference in their MKT with respect to the Hilada PSTs.

Summing up Results of the MKT Analysis

Because of the flawed execution of the last prompt during the interview, it generated little data about the relative MKT developed by the PSTs at the different campuses over the course of the semester, though other evidence from the MKTI and the interviews does point in a particular direction: the two sets of PSTs learned developed different MKT. All of the abilities PSTs showed in the interviews—connecting fractions to decimals or generating story problems from number sentences, for example—are clearly goals of teacher education in mathematics, and are primary components of MKT as defined by Ball, et., al. (2008). However, these differences are evidence that the PSTs learned differently in their course experiences, and as such, may have different knowledge to bring to bear when they arrive in classrooms. Ma (1999) describes that a critical component of profound understanding of fundamental mathematics is the ability to "unpack" knowledge and make it accessible to children. The evidence in this collection of data indicates that MATH 291 PSTs were better prepared for this responsibility than were MATH 281 PSTs.

Though MATH 281 and MATH 291 shared many important characteristics and mathematical ideas, they also showed differences that ran deep in each course. Though the distinctions were not manifest in the broad statistical analyses of the MKTI, they did reveal that both courses helped to develop PSTs' MKT. Sub-scale analysis of items classified as CCK and SCK—as well as individual item analysis—revealed that some of the differences between the courses were echoed in the responses PSTs gave on that instrument. This data suggests that the differences between MATH 281 and MATH 291 might have influenced PSTs answers.

Interview data was similarly revealing and yet not definitive. The fact that disparities arose in the answers that PSTs gave in the interviews, often paralleling the differing foci of the two courses indicates that they did indeed matter; PSTs' knowledge was a creation—at least in part—of the context in which they learned. Situated learning theory suggests that teachers will be able to apply the knowledge they learn as undergraduate PSTs best when the contexts of their undergraduate preparation are most closely matched with the teaching practice they will confront in classrooms. The interview data give a more complete picture of whether and how PSTs could deploy their knowledge in the service of teaching tasks, but that picture is still unfinished. The limited number of interviews makes it difficult to determine irrefutably that the results would be replicated among the other PSTs in each cohort, much less in other cohorts.

In short, the MKTI and interview data are inconclusive, and yet they point in the same direction that situated learning theory predicts they would. One may speculate about more extreme conditions that would have produced less ambiguous results, but the advantage of these circumstances was that they were living, breathing courses, and not theoretical constructs whose fidelity could be questioned upon implementation. Despite the ambiguity, I have argued that these data show that how one approaches content courses for elementary teachers—as opposed to simply concentrating on what content is covered—influences what PSTs learn as a result.

In MATH 281, PSTs were directed more frequently to aspects of elementary mathematics in the common content knowledge while MATH 291 PSTs' work with video and student thinking led them toward specialized content knowledge instead. Hilada PSTs' proficiency with numbers was reflected in their apparent facility with fractions and connections to decimals. One might expect this flexibility across number sets to be robust, as it was a major focus of the course. Rio PSTs' work appeared to sensitize them to children's thinking and the use of pedagogically appropriate representations of mathematical concepts. These representations were not only valuable teaching tools, but in many cases formed the foundations on which the PSTs understood the concepts themselves.

Thus, both sets of PSTs learned important mathematics for teaching, but they apparently developed *different* mathematical knowledge for teaching. Though this is a primary concern of the project, it had other foci as well. In the next section, I describe the data that are germane to the PSTs' relative attitudes about mathematics and teaching.

PSTs' Relative Attitudes about Mathematics and Teaching

The attitudes survey administered to PSTs coincided with their completion of the MKTI instrument. Thus, there are early- and late-semester data about PSTs' attitudes regarding mathematics and teaching. With these data, we can learn something about how the PSTs' attitudes changed over the course of the semester. While Likert-type surveys of the kind I administered are not particularly revelatory about individual attitudes (Vincent, et. al., 2003), they can be useful for thinking about groups of people, allowing a comparison between courses.

Nearly all items on the survey were paired into positive and negative statements of the same sentiment. For example: "Before class ends, a teacher needs to clarify those wrong answers, incorrect methods or misstatements that may have been made by students" was paired with "Having students determine and discuss their solution methods is a good use of class time, even if the discussion and questions about those methods takes more than one class period." Another example of these pairs is: "The idea of teaching math scares me" paired with "I am looking forward to teaching children about mathematics concepts." PSTs responded to each item using a five-point Likert scale, ranging from strongly agree to strongly disagree. The data were then coded to reflect the extent to which the PSTs demonstrated a comfort with mathematics and teaching that might be said to align with goals of the NCTM Principles and Standards for School Mathematics (2000). For example, the statement "Seeing/hearing different ways to solve the same problem confuses children" does not conform to the NCTM vision of problem solving in school:

> *Different strategies are necessary as students experience a wider variety of problems. Students must become aware of these strategies as the need for them arises, and as they are modeled during classroom activities, the teacher should encourage students to take note of them. For example, after a student has shared a solution and how it was obtained, the teacher may identify the strategy by saying, "It sounds like you made an organized list to find the solution. Did anyone solve the problem a different way? (p. 53).*

PSTs who responded "Strongly Agree" to the statement above was coded with a one and those who answered "Strongly Disagree" were coded with a five. Similarly, for the counterpart item, "Students should hear methods that other students use to solve problems," PSTs who answered "Strongly Agree" were coded with a five, while those who strongly disagreed were coded with a one. Each response was given a numerical value in the range 1-5 in this manner, with higher numbers representing a more "adaptive" set of attitudes and lower numbers "maladaptive" (Bassarear, 2007).[58] The twenty-one items on this portion of the survey form the "attitudes" portion of the instrument. Recall that in Chapter three, I reported that the collection of these items, taken together, were piloted with different PSTs previous to the data collection period and were shown to have a reliability coefficient (Cronbach's alpha) of .785. This is generally considered to be an acceptable value which indicates that the instrument in question is reliable.

The items in the "attitudes" component were generic in the sense that they did not apply to any given situation or context, but were general statements about teaching, mathematics, and teaching mathematics. In addition to this component, each survey included items that related directly to the course in which PST participants were enrolled. At the beginning of the semester, these items focused on PSTs' understanding of the importance of their course to help them in their goal to become teachers: "In this class, I expect to learn more about what it is like to teach mathematics." At the end of the semester, a similar set of questions were asked, though they were from a reflective standpoint: "In this class, I learned more about what it is like to teach mathematics." The responses to these items were similarly coded as those in the rest of the instrument, with

[58] Bassarear identifies the NCTM vision as promoting adaptive beliefs.

higher numbers reflecting a sense that the course should (or did) have strong relevance for PSTs' work toward teaching certification. The coded responses were averaged for each PST, producing two scores for each administration: a mean score for responses to items regarding PSTs' attitudes about mathematics and teaching, and a mean score for PSTs' beliefs related to the relevance of their course for their goals as future teachers. Thus, each PST had beginning- and end-of-semester scores on the attitudes instrument, as well as corresponding scores for the relevance items on the survey. Later, I will discuss the exploration of the relevance items. Here, I turn to an analysis of the attitudes items on the survey.

Like the MKTI data, the survey data allow for statistical comparisons. Since the distribution of mean scores on this instrument did not meet assumptions necessary for performing ANOVAs, Mann-Whitney tests[59] performed on the attitudes portion of the survey indicate significant differences between PSTs attitudes at the end of the semester, even though there is no statistical evidence that they were different at the beginning. While there is data on this question in the interviews, the interviews were designed to focus more on the questions of MKT and course relevance. As such, the survey provides the most robust data for answering the research question related to PSTs' attitudes about mathematics and teaching, and is the focus of this portion of the data analysis.

Attitudes Survey: Quantitative Analysis

Again, only PSTs who completed both rounds of instruments were included in the analysis, and again descriptive statistics were computed for each campus, including mean

[59] The Mann-Whitney statistic is a non-parametric computation that can is analogous to the t-statistic used in ANOVAs in cases when the assumptions required for ANOVA are not met, or when the scale of measurement is arbitrary, such as a Likert-scale (Wackerly, Mendenhall, & Scheaffer, 1996)

and median raw scores (number of correct responses), standard deviation, variance, skewness, and kurtosis. These descriptive characteristics of the data, in addition to the normality tests I described above suggest that this set is not normally distributed and therefore ANOVA is not an appropriate tool for testing claims about the mean survey scores. Below is a table of the mean values at each campus for each administration of the survey.

	Mean Score on Attitudes Items (February)	Standard Deviation (February)	Mean Score on Attitudes Items (May)	Standard Deviation (May)
MATH 281	3.35	.32	3.34	.34
MATH 291	3.41	.28	3.47	.36

Table 8: Mean scores and standard deviations associated with PSTs attitudes about mathematics and teaching

These values indicate that each group of PSTs can be described as leaning toward the adaptive end of the spectrum but that these attitudes were not deeply entrenched in either setting. Note also that while the score of Rio PSTs jumped a bit, the mean score at Hilada fell only slightly. A value of three on the items in this survey indicate a response of "No Opinion," so that PSTs in both courses generally agreed with adaptive statements such as, "Having students determine and discuss their solution methods is a good use of class time, even if the discussion and questions about those methods takes more than one class period," and disagreed with statements such as "Before class ends, a teacher needs to clarify those wrong answers, incorrect methods or mis-statements that may have been made by students." However, the fact that the mean scores are close to three indicates that these attitudes are not strongly held by PSTs as a group.

Since the assumptions required for ANOVA were not met, the Mann-Whitney test can serve as a non-parametric substitute (Wackerly, Mendenhall, & Scheaffer, 1996).

The test, performed for each administration of the survey, indicates that while there was no statistical distinction between the universities' mean scores at the beginning of the semester, at the end of the semester, a measurable difference can be detected. In addition, a comparison of the mean change in attitudes scores from February to May also is statistically different. The results show that MATH 291 mean survey score is significantly higher at the end of the semester than MATH 281 mean survey score. This means that MATH 291 PSTs appear to have responded more adaptively to the items on the survey instrument at the end of the semester than their MATH 281 peers. Likewise, the analysis suggests that the change in mean survey scores noted in the table above is also statistically significant.[60]

	Institution	N	Mean Rank	Sum of Ranks
SurveyPreTestAverage	1	60	49.35	2961.00
	2	41	53.41	2190.00
	Total	101		
SurveyPostTestAverage	1	60	44.93	2696.00
	2	41	59.88	2455.00
	Total	101		
SurveyChange	1	60	44.92	2695.00
	2	41	59.90	2456.00
	Total	101		

Table 9: Summary of ranks associated with scores at Hilada and Rio at both points in the semester

[60] For the end-of-semester mean scores $p < .012$, while in the analysis for the mean change in survey score $p < .011$

	SurveyPreTest Average	SurveyPostTest Average	SurveyChange
Mann-Whitney U	1131.000	866.000	865.000
Wilcoxon W	2961.000	2696.000	2695.000
Z	-.686	-2.520	-2.533
Asymp. Sig. (2-tailed)	.493	.012	.011

(Grouping Variable: Institution)

Table 10: Summary of results of the Mann-Whitney test on mean survey scores between Rio and Hilada at both points of the semester

Though these changes in attitudes were statistically significant, they are indeed small. Attributing these results to the differences I described between the courses may be unwise. However, the coincidence of these phenomena—the fact that PSTs from the course focusing on aspects of MKT that relate most closely to the practice of teaching responded more adaptively to items on the attitudes survey—suggests that there is more to learn about the link between attitudes about mathematics and teaching and developing MKT.

Though the interviews were not designed to elicit data on the question of how the courses might have influenced PSTs' attitudes about mathematics and teaching, there is some evidence that suggests a link between MKT and attitudes about mathematics and teaching. Consider these comments made by Laverne, who scored more than a standard deviation below her classmates at Hilada on the MKTI:

> ***M:*** *…did this class have any influence on that?*
> ***L:*** *I mean, I learned some things, and it did help improve my skills and the way I look at things now…*
> ***M:*** *Mhmm.*
> ***L:*** *But, I still have the same attitude towards math, like, I don't really like it…If ever I become a math teacher, I'm definitely going to probably have a tutor. Cause I would never go to teach this and not know it.*
> ***M:*** *Uh huh.*

> ***L:*** *I would rather be like, 'We're in math class one day—Laverne, this is what we're doing…'*
> ***M:*** *(laughs)*
> ***L:*** *I will find someone who does—I don't know—somehow, we'd work it out. But I wouldn't sit there and try to teach this when obviously I don't understand it.*

Laverne showed serious reservations about being able to teach mathematics to children, and did not feel prepared—mathematically—to take on this responsibility. In contrast, Eliot who scored more than a standard deviation above the mean MKTI score of her classmates at Hilada, expressed her confidence in mathematics:

> ***M:*** *Ok. Was your original impression right?*
> ***E:*** *It wasn't exactly. At first it started out, um, I thought that they were re-teaching me the math, and sometimes throughout the course it felt like that and um…I don't know, I like to think that I'm good at math and so re-learning things that I learned in sixth grade was counterproductive for me.*
> ***M:*** *Mhmm.*
> ***E:*** *But that's not the case for everyone. I understand that. So, the majority of the time it wasn't just re-teaching…it wasn't just re-learning the math, it was learning how to teach it.*

Unlike Laverne, Eliot's experience reinforced her thought that teaching mathematics to children was a challenge of which she was capable. It is reasonable to infer that the different levels of confidence and the perspectives shown by these PSTs would be mutually influential with their MKT: the more one knows, the better one is equipped to handle the challenges of practice, and thus the more adaptive ones' attitudes are likely to be.

The analysis of the survey data suggests again that at the end of the semester, there were differences between PSTs at one university and the other, differences that did not exist at the beginning of the semester. This time, the differences relate to the attitudes about mathematics and teaching which PSTs expressed through the survey instrument. It would be difficult, if not impossible, to map this result back to specific differences between the courses, since the attitudes expressed in the statements given in the survey do not match up well with the analysis I offered above that described the two courses in this study. On the other hand, this result again fits with the hypotheses I suggested at the end of Chapter Three. PSTs in the course which focused most carefully on practices of mathematics teaching did develop different attitudes about mathematics and teaching than those PSTs whose work was not focused in the same way. In fact, these results suggest that MATH 291 PSTs developed slightly more "adaptive" attitudes than did their MATH 281 counterparts.

PSTs' Perception of Course Relevance

So far, the data have supported the hypotheses I stated in Chapter Three, hypotheses that arose from the theory and educational research based upon situated learning theory. It now appears that PSTs who engaged more closely in practices of teaching did in fact learn different mathematics for teaching than PSTs who focused on other things. In addition, they developed different attitudes about mathematics and teaching. The effect sizes attributable to the course PSTs took have been small—but significant—in most cases. In this section, I turn to analyze the data as they relate to another primary research question guiding this project: To what extent do prospective teachers see their mathematics course work as relevant to their future work?

The literature reports that in-service and pre-service teachers alike perceive a lack of fit between their undergraduate preparation and their eventual teaching practice. There is an ethos among practitioners that teaching is best learned on the job, and while the theoretical principles discussed in teacher preparation programs are well-intentioned, they do not account for the realities of daily classroom interaction. Teacher educators, who often are also educational researchers—obviously disagree with this perspective, as theory continues to drive the work that they do. Bridging the gap between theory and practice is a perpetual problem in education, but situated learning theory and the teacher development activities that have been associated with it in recent decades show promise in helping teachers to learn relevant mathematics for teaching. The data I have presented from this project have supported these claims and may yet also be able to narrow the gap between what teachers perceive as well-meaning, but ultimately irrelevant university advice and the more pressing day-to-day demands of classroom life.

In order to find out if the MATH 281 and MATH 291 were perceived differently in terms of relevance, I included items at the end of the survey instrument (not included in the analysis of attitudes above) that asked PSTs to rate the value of their mathematics course to their future teaching. At the beginning of the semester, these questions focused on the expectations the PSTs had for their course. There were four Likert-scale items, shown below as they appeared at the end of the survey:

1. The **activities** we do in this class are supposed to help me make significant progress in my goal of becoming a teacher.	SA	A	N	D	SD
2. The **assignments** we do in this class are supposed to help me make significant progress in my goal of becoming a teacher.	SA	A	N	D	SD
3. The **exams we take** in this class are supposed to help me make significant progress in my goal of becoming a teacher.	SA	A	N	D	SD
4. In this class, I expect to learn more about what it is like to teach mathematics.	SA	A	N	D	SD

Table 11: The four Likert-scale “relevance” items on the survey.

At the end of the semester, the first four items were re-worded to reflect not PSTs' expectations but the reality of their view of the course. For example, the first item read, "The activities we did in this class helped me make significant progress in my goal of becoming a teacher." Responses to these items were coded on a scale from one to five, where "five" indicated that the student expected the course to be relevant to the PST's teaching practice and "one" no expectation of relevance. For the end-of-semester survey responses, the same codes were used except that they now signified de facto evaluations of the course with respect to its relevance for each PST's future practice. In addition, the May survey instrument contained three open-ended items. These items were designed to indirectly gather data about the nature of their experience in the course. These data were used to describe how the courses differed in Chapter Four, but they also can give insight into what connections PSTs saw between the course and their anticipated careers. What did they remember most about the course? How would they describe the course to a friend? Shown below are the items as they appeared on the May version of the survey:

1. The most memorable assignment in this course was:	
2. The most memorable activity in this course was:	
3. Suppose a friend of yours was thinking of taking this class and wanted to know more about it. How would you describe the class, what would you say to your friend about what she should be ready for, and what would she learn?	

Table 12: The three open-ended items, contained only on the end-of-semester survey.

Statistical analysis shows again that there were differences between MATH 281 PSTs and MATH 291 PSTs in the way they evaluated their courses in terms of its relevance for teaching. These results are supported by interview data and statements made on the open-ended items shown above.

Statistical Analysis

At Hilada, the mean score on the early semester items was 4.19, while at Rio, the mean score was 4.40. These scores mean that PSTs at Rio had slightly higher expectations for how relevant their course was going to be for their teaching career. At the end of the semester, both scores were lower, indicating that their expectations had not been fully met. Yet, the drop was more precipitous at Hilada: there, the mean score on these items was 3.29, while at Rio, the mean score was 3.97. At Hilada, the PSTs' responses dropped by almost a full point. At the beginning of the semester, their average response indicated that they agreed or strongly agreed that their course *should* be relevant to learning how to teach mathematics. At the end of the semester, PSTs' responses were nearly neutral (a score of three indicated a "no opinion" response) about whether or not the course *was* relevant to learning how to teach mathematics. At Rio, this score also dropped, but not nearly as much, and this mean score indicated that PSTs were still in agreement that the course had accomplished the goals *they* had for the course.

	Mean Score on Course Relevance Items (February)	Standard Deviation (February)	Mean Score on Course Relevance Items (May)	Standard Deviation (May)
MATH 281	4.19	.55	3.29	.95
MATH 291	4.40	.49	3.97	.62

Table 13: Means and Standard Deviation statistics for scores on the four relevance items on the survey

Are these results statistically significant, or could they have occurred simply by chance? The data, like the rest of the survey data, did not meet the criteria for performing ANOVA[61] making Mann-Whitney tests more appropriate for discerning differences between the cohorts. This analysis shows that while the PSTs' mean response scores did differ significantly ($p < .06$) at the beginning of the semester, and then again in May, the mean response scores were even more clearly different ($p < .001$). Though the Rio PSTs had higher expectations for their course's relevance than did their Hilada counterparts, at the end of the semester, those higher expectations were met better than those at Hilada.

	Institution	N	Mean Rank	Sum of Ranks
PreFourItemSurvey	1	60	46.49	2789.50
	2	41	57.60	2361.50
	Total	101		
PostFourItemSurvey	1	60	41.42	2485.50
	2	41	65.01	2665.50
	Total	101		

Table 14: Summary of ranks associated with mean scores on relevance items

	PreFourItemSurvey	PostFourItemSurvey
Mann-Whitney U	959.500	655.500
Wilcoxon W	2789.500	2485.500
Z	-1.922	-4.009
Asymp. Sig. (2-tailed)	.055	.000

Grouping Variable: Institution

Table 15: Summary of results of the Mann-Whitney test on mean relevance scores

[61] The test statistics for both the K-S and Shapiro-Wilk tests for all mean scores on the relevance items have p-values less than .001, which means that one must reject the null hypothesis that the distribution can be approximated with a normal distribution.

Other data sources suggest the reasons for the difference in relevance scores at the end of the courses, but these quantitative data shed some light on the issue. Correlations computed between all variables show that end-of-semester relevance scores are positively correlated with beginning-of-semester relevance scores,[62] end-of-semester MKTI scores[63], as well as institutional affiliation.[64] [65] Though these correlations are not particularly strong, they are significant. This means that higher the mean relevance scores at the end of the semester were associated with higher scores on the MKTI. It is reasonable to speculate that PSTs who find more connections with teaching will invest more heavily in the work of the course and therefore take more from it; it is a premise on which this project is predicated. Though the relative difference between the cohorts' MKT was not definitive in the quantitative analysis, but to the extent that PSTs across campuses felt their course was relevant to their understanding of teaching, the evidence suggests it is positively correlated with MKT. But this makes the fact that the institutions did differ along this dimension intriguing; why did the relevance scores differ?

Relevance: Survey Data

At the end of the semester, nearly 1/5 of all PST participants (19 out of 101) wrote that they would describe the course to a friend as helping them learn to teach mathematics to children. "We learn how to teach math to students and since we already understand standard math, they challenge us by using different bases," is a representative of this kind of response. Interestingly, notwithstanding the statistical analysis above,

[62] Pearson coefficient = .321, $p < .001$
[63] Pearson coefficient = .189, $p < .03$
[64] Pearson coefficient = .385, $p < .0005$
[65] Recall that PSTs in MATH 281 were coded with a "1" while those in MATH 291 were coded with a "2." This positive correlation shows that a higher institution score (MATH 291) is associated with a higher dependent variable score. In this case, that means that MATH 291 PSTs were associated more strongly with a higher relevance score.

fewer MATH 291 PSTs responded this way than did MATH 281 PSTs (seven compared to 12). Contrast this with a related, but distinct response that did not explicitly mention teaching children, but instead described learning how other people think about mathematics: "In this class we learn how children think mathematically and what, as teachers, we can do to promote their knowledge growth." In this case, seven PSTs at Rio responded that they learned how other people (children in particular) think about mathematics, but only three Hilada PSTs did. Statements that mention "learning how to teach" and "learning how other people think" are the only ones that mention teaching explicitly, either in an active sense or in a more passive sense—when assessing children's thinking. The vast majority of responses to this item related that the course was about looking at mathematics in a different way, or understanding the reasons behind procedures rather than simply the procedures themselves.

	MATH 281 (Hilada)	MATH 291 (Rio)
This course is about learning how to teach mathematics	20%	17%
This course is about learning how other people/children think about mathematics	5%	17%
This course is about understanding why procedures work	43%	49%
This course was difficult	7%	13%

Table 16: Summary of common responses to the survey item asking PSTs to describe to a friend what to expect to learn from the course.

Since 29 PSTs mentioned teaching in this way, only about 1/3 of all PSTs in the study would describe the course to a friend as being about teaching mathematics. From the perspective of course designers, this may be a good thing, as the courses—as described on the syllabus—are NOT intended to teach PSTs *how* to teach mathematics. Among teacher education programs, this particular goal is the traditional domain of methods courses and student teaching experiences. On the other hand, it demonstrates the strong desire of PSTs to play out these roles early in their program instead of near the end of it. Interestingly, despite the fact that MATH 281 tended to use more advanced mathematical symbols and arguments than did MATH 291, a fewer proportion of PSTs in that course would warn a friend that it was difficult. Responses to this item do not reveal major differences between the campuses however. It appears that relatively similar proportions of each cohort answered with similar sentiments regarding what they were supposed to learn in the course. The biggest gap is the one related to learning how children think; Rio PSTs used this description substantially more often than did their Hilada counterparts.

The other items on this portion of the survey asked PSTs to recall the most memorable aspects of the course, particularly the assignments and activities. These questions were designed to gather data about what “stuck” with PSTs. The table below summarizes the kinds of responses that were given to prompts dealing with memorable assignments and activities, indicating how the courses differed, but also might provide insight into the results of the statistical analysis.

	MATH 281 (Hilada)	MATH 291 (Rio)
Most Memorable Activity or Assignment: **Watching Video**	7%	46%
Most Memorable Activity of Assignment: **Working with Classroom Manipulatives**	36%	41%
Most Memorable Activity or Assignment: **Writing Word Problems based on Given Number Sentences**	10%	0%
Most Memorable Activity or Assignment: **Working in other Bases (e.g., base-five arithmetic)**	2%	61%
Most Memorable Activity or Assignment: **Using Graph Paper to Solve Problems**	0%	13%
Most Memorable Activity or Assignment: **Written Reflections and Projects (e.g., the "Star Project")**	64%	0%

Table 17: Responses from PSTs on open-ended items of the survey. Percentages add to more than 100% because some PSTs mentioned multiple activities and assignments as being most memorable.

A glance at the column associated with Rio PSTs shows that they heavily cited activities that I have earlier described as being closely associated with artifacts of teaching. The largest category of response, related to PSTs' work in other bases, is directly related to the focus in MATH 291 on place value, which is itself designed to match closely a fundamental concept in elementary school mathematics. Another reason for this discrepancy might be that a large number of the PSTs recalling the work in other bases were describing the Alphabetia project I highlighted in Chapter Four. This was an activity that was not replicated at Hilada.

Likewise, one of the categories which Hilada recalled in significantly greater proportions than did Rio PSTs was the "written reflections and projects" and "writing word problems from given number sentences." The first of these categories is also one that was particular to MATH 281: this was not an experience that occurred in both locations. In MATH 281 the written reflections and projects refer to specific assignments given to PSTs that were of an extended nature. Most nightly assignments given during class were short sets of problems taken from the textbook, and the written reflections and projects were a departure from this. Writing Assignment #2 involves writing and solving a word problem related to the sentence $2\frac{1}{4} \div \frac{1}{2}$, and might in fact be the memorable "word problems" assignment to which many of the PSTs referred on the survey. Writing Assignment #1 was a problem-solving exercise related to prime factorization, and Project #1 was an assignment related to GCF and LCM, and resembles an assignment in an introductory abstract algebra course.

Thus, PSTs overwhelmingly reported extended assignments as most memorable. There were relatively few of these in each course, and it is not surprising that they would thus stay with PSTs at the end of the semester. Though not all PSTs gave reasons why they reported these assignments as memorable, the reasons can give insight into why Rio had higher relevance scores on the survey instrument. HIlada responses can be represented by the following comments:

> *We had a take home project where we had to figure out some nonsense dealing with lockers. The point is that it took me a long time to do it and I felt really good when I finally figured it out. I got an F on that project.*
>
> *The star problem because it took a very long time to figure out, but one I did I felt like I fully understood the concept and was very comfortable with it.*
>
> *The locker problem. This really required critical thinking. Also, it could be solved by various methods.*
>
> *The written reflections were memorable because they were more in depth than assignments in class.*

The most memorable activity was the locker activity. It was very fun trying to figure out which lockers would be left open, after you eliminated certain lockers due to prime factors, etc.

The first comment was Laverne's, who scored more than a standard deviation below her peers at Hilada on the February MKT. What is notable about these comments is that they do not include any mention of these assignments' relevance to teaching, or to the larger goals of the course. Contrast these reasons with those given by Rio PSTs for why working in other bases (likely the Alphabetian project) was their most memorable assignment (recall that 61% of Rio PSTs cited this as most memorable):

Working with different bases to solve ordinary addition problems because it puts you in the mindset of an elementary level student.

The most memorable assignment was making up our own number system because it basically introduced the concepts of other number systems and helped to understand a lot of the course.

When we were asked to create our own numeration system using only 6 digits. This really helped me to understand the concepts behind number systems.

Though MATH 291 PSTs reasons do not necessarily cite the activity's connection to teaching, they are consistent in connecting the activity to the overall goals of the course, which my analysis has suggested was more closely in touch with these practices than was MATH 281.

The general thrust of this survey data is that MATH 291 PSTs recalled more opportunities to connect to teaching practices, artifacts, and fundamental elementary school mathematical topics; they report such activities as being memorable in considerably larger proportions than did their MATH 281 counterparts. It is likely that this was an important reason why Rio students responded more emphatically that their course was relevant to their teaching practice than Hilada PSTs did.

Relevance: Interview Data

The interview data support this inference. Note that at Hilada, only thirteen of the 61 PSTs participated in watching video, and these thirteen PSTs—according to interview data—only had one opportunity to do so. Nearly 1/3 of them—four out of the 13—mentioned video on the survey as the most memorable activity of the entire semester. This is an important result: a single viewing of children doing mathematics was the most memorable activity for a large proportion of the PSTs in that class. Carla (at Hilada) indicated what advantages she thought video had over other activities:

> ***M:*** *Some of the time I went to class, there was talk about a…you know there was a page in the book or a word—there were problems in the book that had, 'Suppose kids solved problems this way…' How would you compare those kinds of things to seeing the video?*
> ***C:*** *Seeing the video is much more practical because, you know, 'Suppose the child does this, this, and this.' Even though some of those were really interesting—and there were a few of those—there weren't many of those either. Um, I think seeing it, and actually watching a child work through the process…*
> ***M:*** *Mhmm.*
> ***C:*** *And the person that was with her, the adult that was with her, you know, really just kind of let her work through, and might have pointed out one thing or another, but…*
> ***M:*** *Right.*
> ***C:*** *…really let her work through it; it gave us just such—it was just so visual and so like, 'There it is,' you know, so concrete to understand how she went through it. And I don't think that just reading a hypothetical is as valuable.*

Carla thus placed a high premium on children's presence in the math classroom (even if that presence was virtual, via video recording) as being a "concrete" experience of how children think and an opportunity to confront what decisions teachers must make in real time. Though Carla thought the interviewer in the video might have done things differently, the video was a unique way to present her with this dilemma: the real child and teacher in the video forced her to speculate how *she* would respond to a child's thinking, laid out in front of her. Maeby (at Rio) responded similarly, recalling that video presented a more immediate and real example of how children work and think:

> ***MA:*** *Um, I think that for things like this, the videos have helped the most, because for that we had like, each video would have like eight kids, and they'd each do the problem differently. Um, and so—and they usually do it on a blackboard, so you can see like, step by step what they're thinking about.*
> ***M:*** *Uh huh.*
> ***MA:*** *Um, so, I mean, that's—like for B, that's how I was kind of familiar with the simplifying…*
> ***M:*** *Right.*
> ***MA:*** *...portion of it, because we've seen examples of that… …Because it allows you to see—like instead of saying, 'A kid might do this,' like, it actually—it allows you to see like the entire thought process, from like, first step to last step.*
> *… sometimes they're confusing, because I mean, they're kids. Like, a lot of things get lost in translation…*
> ***M:*** *Yeah.*
> ***MA:*** *...um, so like, today, she gave us written out what she did and it was easier to understand. Um, but I think overall it's—even though it's kind of hard to understand like, throughout, but like, I guess when you see it all done, it starts like, clicking. Like, 'Oh, ok, this is what they're thinking.'*

Maeby connected her analysis of the first interview prompt (75 – 48) to this work, noting that she could follow the logic underneath unusual procedures because of her experience watching the video. Both Carla and Maeby referred to discussing video as useful activities for learning mathematics for teaching. Unlike Carla and Maeby, Anne (at Rio) did not mention the videos in the relevance items of the survey instrument, and she did not raise the issue through nearly the entire interview. However, when she was asked which components of the course she thought should be included in an ideal version of MATH 291, Anne made similar observations as Carla and Maeby did:

> ***M:*** *Anything else you'd want to keep [in an ideal version of this course]?*
> ***A:*** *I think the videos helped too, like watching…or not even just the videos, like when we have the pieces of paper with all the different word problems—not word problems, like, algorithms worked out where the kids try to do all their little invented algorithms and everything. Um, 'cause we saw, like, five hundred ways to do one problem that we would have never thought of.*
> ***M:*** *Ok.*
> ***A:*** *Um, and then watching kids try to do it in the video was just…it made you think there's not one way to do it (laughs), that they don't think like the standard algorithm and that they kind of create their own thing. (inaudible)*
> ***M:*** *So the videos and the worksheets where you get the invented algorithms…*
> ***A:*** *Mhmm.*
> ***M:*** *That helped you get a sense that there were other ways of doing it that are out there…*
> ***A:*** *Mhmm.*
> ***M:*** *Um, was there anything else those helped you to do?*
> ***A:*** *Um…*
> ***M:*** *Those…I mean…*
> ***A:*** *I guess they kind of helped me to explain why things work or don't work, but I'm still working on that.*

Ann demonstrated less enthusiasm for the video over other activities, like examining student work. But, like Maeby and Carla, she acknowledged that they gave her an opportunity to understand the range of ideas that she might confront as a teacher in a classroom of 25-30 children. These PSTs watched the video *not* as students in a mathematics course, but as *teachers*. Though none are yet certified and none yet have responsibility for a classroom of children, the videos gave them a new experience of what it means to teach mathematics.

This is not to say that it is their only experience of what it is like to teach mathematics,[66] but video is a unique context for the reasons I laid out in Chapter Two, and the PSTs echoed these sentiments during the interviews: it is a low-stakes setting which strip away some of the complexity of real classroom teaching that PSTs can use to practice taking apart the relevant mathematical ideas and consider the implications of students' understanding for the knowledge and work of the teacher. Lindsay did not mention videos until I prompted her, but then implied this very concern in discussing the value of activities like watching video:

> ***M:*** *Is there anything you can think of—again, putting yourself in the position where you can—you can do whatever you want. You can make your dream math class for teachers. Is there anything you'd want to do brand new that you wouldn't keep or take out, but you'd plug in.?*
>
> ***LI:*** *Mmm, maybe hands on. Like, actually working with some kids. 'Cause, like…but, uh…No, actually scratch that. Because like, I wouldn't feel comfortable explaining something to a kid that I don't really understand…*
>
> ***M:*** *Uh huh.*
>
> ***LI:*** *…so, no I don't think I'd add anything. I think it's fine the way it is. I think it's the testing that I'd (inaudible). That's it.*
>
> ***M:*** *It's interesting though you mentioned working with kids.*
>
> ***LI:*** *I mean, maybe at the end. At the end...like, very end, like [MATH 293]…*
>
> ***M:*** *Right.*
>
> ***LI:*** *'Cause then by that time you should understand everything, but, I don't know…*
>
> ***M:*** *Well, the reason I think that's interesting is because one of the things I saw you guys do today even, was watching videos of kids. Um, and I know that in some of the assignments there have been questions like this…*
>
> ***LI:*** *Mhmm.*
>
> ***M:*** *…or, you know. And…is that a good—or even so-so substitute for working with kids?*
>
> ***LI:*** *Yeah, it's good, 'cause you get to see the kids. It's just sometimes, you know, the videos are hard to understand like, the kids talking. So yeah, it's like, a little hard. But I*

[66] Teacher education research contains many examples in which the "apprenticeship of observation" (Lortie, 1975) features prominently.

mean, they narrate for us so we understand…yeah, it's good.

M: *I mean, do you feel like that was a helpful…*

LI: *Yeah, it was helpful.*

M: *How would you compare that to the manipulatives?*

LI: *It, well, they're the same. They're both helpful, because like one, you're actually doing it and the other you're actually seeing it. So…*

For Lindsay, the use of video was not more valuable than working with manipulatives, but both put her in direct contact with children in different ways. With manipulatives, PSTs are working with objects that children work with to develop understanding ("…you're actually doing it…"), while video gives PSTs an experience of how children think ("…you're actually seeing it…"). Though Lindsay did not elevate videos over other materials or activities in value to her development as a teacher, she acknowledged that working with living, breathing children would present a complexity for which she was not prepared; indeed, deciphering video was challenging enough, and it provided her with enough complexity to begin developing her skill extracting relevant information for her practice. The same sentiment appears in all of these comments: working with these materials demonstrates to PSTs and gives them some experience of what knowledge children bring to classrooms, and the practical, mathematical decisions that teachers must make about what and how to engage with that knowledge. It is not surprising then that PSTs who had more frequent opportunities with such tasks viewed them as directly relevant to their future work.

Beyond the use of videos, PSTs cited the use of manipulatives, and trying to understand alternative algorithms as the most memorable aspects of their course. These are all components of the courses that made explicit their position as PSTs, and acknowledged, even at this early stage, their desire to enter into the teaching community, and don a teaching persona. Eliot, a student at Hilada—and who did not watch any video—cited manipulative materials as key components of her learning in the course:

> ***E:*** *I would keep the activities, with—and the real-life activities, with base ten blocks, even though some people thought they were kind of stupid, and childish, and…*
>
> ***M:*** *Mhmm.*
>
> ***E:*** *You know, I now can explain why you can't subtract that [referring to five minus eight in the first interview task] because of base ten, because I saw it. Because, you know, I worked it out. Um, I think using things that kids will actually use in class like that, is really important, because then you start thinking like kids, in a positive sense where you're on the same level as them and I can understand—at first I couldn't—but I can understand why there's a three there, like a basic check. I think that's huge, using things that kids would be using in class. Um, for the most part, I think I would teach the class pretty similarly. I would keep the focus on how to teach it, and how to reach all sort of kids. I mean, we had one class all about common misconceptions that children have when dealing with math problems, and I think that's huge. And I maybe would have even done another class on that, with problems like this. This whole, 'Why are they doing it this way?' 'How can you show them that it's not the way to do it?'*

In this response, Eliot expressed a desire to learn more about how children think about mathematics and common mistakes they make. She acknowledged that there were elements of this in her MATH 281 course, but she also wanted to learn more about how to teach children math.

This was a common sentiment among PSTs' responses on the survey: a large number of them said that they would describe the course to a friend as "learning how to teach" mathematics. 17% (17/101) of PSTs wrote that they would tell a friend that the course was about how to teach mathematics to children. While this may not be the explicit goal of the courses or the instructors, the use of video, manipulatives, student work, studying cases, and other artifacts of teaching practice may give PSTs the impression that this is in fact what they are doing. Indeed, situated cognition theory suggests that, to the extent these courses mimic the mathematics done in elementary school, PSTs are in fact learning how to teach mathematics. Again, it should not be surprising that when PSTs engage in these activities, they assigned a special relevance to them that they were less likely to attach to other kinds of assignments.

Part of the reason may be that without these connections to teaching, PSTs may feel like they are being treated like the children with whom they intend to work one day. Consider Eliot's comments that compare MATH 281 her concurrent experience in MATH 282, which focuses on geometry:

> ***E:*** *... make the focus about how to teach it and not necessarily—I mean, it's frustrating when you felt like…for me, it's frustrating to feel like you're being taught third grade stuff, because most of these people are freshmen and sophomores in college and…I mean, I'm a junior in college and—it didn't happen as much in this class, which is good. But it's frustrating when you know the material. You don't want to be taught—you don't want to be taught like you don't know it.*
> ***M:*** *Right.*
> ***E:*** *So—and I think that [MATH 281] did a good job of not doing that, so…*

Here, Eliot expressed frustration with courses that did not acknowledge their developing role as teachers. Carla expressed frustration that her other education courses, even those directly associated with disciplinary studies, did a better job of this than MATH 281:

> ***C:*** *... I think I've learned, actually, more or come up with and worked with more interesting math lesson plans, in say, you know, like my music for education class, or my, even art for education. I mean, we did really cool math things. You know in art, we came up with lesson on how to, how to do geometric shapes ad giving kids specific angles that they would have to include in their art pieces and for them to add up all the angles that they had in their art piece…*
> ***M:*** *Uh huh.*

> ***C:*** *In, uh…Oh, in music, they have to count their measures, and then they—you know, that's simple addition obviously but—then they could do multiplication as well, and we came up with little songs to remember formulas, and um…Yeah, I just feel like, that, in other education courses, that's [constructing interdisciplinary lessons] a big focus.*

Carla described the fact that she saw important connections to teaching mathematics in her other courses and expected to see more connections like those in her course about mathematics. These comments were preceded by a concern that the mathematics was too often a review of how to do mathematics:

> ***C:*** *And, I felt like some of those things we were doing, even though conceptually it [concentrating on explaining mathematical procedures]—I guess it helped us in the long run, kind of trying to understand math better, I don't think it did much to show us how to teach math better, you know? I guess, I guess in the long run, hopefully we'll have a better understanding of it ourselves, ultimately, we'll be able to translate that to the kids more…*

Eliot's comment that "you don't want to be taught like you don't know it" is an important expression of the sentiment that is at the heart of this project: one way to treat undergraduates like the adults they are and the teachers they will soon be, is to show them what elementary school mathematics is like (and how it is done) from a teacher's perspective. Without this component, teacher educators run the risk of PSTs feeling like they are being taught elementary mathematics all over again.[67] This is an important component of my argument as it relates to situated learning theory. Even though these PSTs are mathematics students and *not* teachers, it is possible to engage them as teachers by developing opportunities to think carefully about students' thinking and how to

[67] Some teacher educators might argue that this is necessary: PSTs are notoriously immature and weak mathematically. The counterargument is this: if they did not—as students—learn what they should have in the previous twelve or more years, a different approach may be necessary instead.

unpack the knowledge that is commonly held by many. The goal to learn why procedures work is an important piece in this puzzle, but it is not necessarily the special domain of teachers: a goal of teaching reform efforts for many years across disciplines has been to help students understand more than just facts, but the reasoning and processes that are used in disciplines which generated them. I have argued here that the two different courses oriented them differently with respect to their roles as future teachers, which created differences in how they viewed their courses' relevance.

It is important to note Eliot's final comment, that she felt like MATH 281 "did a good job." The final prompt in the interview was to describe how an ideal version of their course would be similar to and different from the course they took. All PSTs claimed that they would want to keep most aspects of their courses the same. Complaints and changes usually centered on issues of assessment: the testing and grading structures in both courses were frustrating for many of the PSTs (consider Laverne's comment from above about receiving an 'F' on an assignment she thought she understood) and were not exclusive to one campus or the other. Such sentiments underscore the fact that PSTs in both courses learned mathematics, they were challenged to think in ways they hadn't before, and many—if not most—PSTs could point to ways in which they connected to the teaching profession.

Summary

In summarizing the results of the analysis above, I return to the research questions I posed early in the project:

(1) What mathematics do prospective teachers learn by engaging in activities of teaching practice such as examining curriculum, student work, and classroom video?

a. Do PSTs who regularly engage in such activities display evidence of different mathematical proficiency than PSTs who participate in more traditional course work?
b. Do PSTs engaging in such activities display different mathematical knowledge for teaching (MKT) than PSTs participating in more traditional course work?
c. Do PSTs engaging in such activities develop different attitudes about mathematics and teaching than PSTs participating in more traditional coursework?

(2) To what extent do prospective teachers see their mathematics course work as relevant to their future work?
a. Do different course approaches set up differing perspectives among PSTs on the contribution of the course to their future work?
b. Do different course approaches set up differing views among PSTs about their confidence and abilities in mathematics?

Using responses to the MKTI and survey at two distinct points in the semester, as well as interviews from a cross-section of each cohort of PSTs, I have employed both statistical and qualitative analyses to determine what answers to these questions are revealed by the data. This variety of data collection methods enabled me to learn more about the PSTs than I could have using a single approach. Both the interviews and MKTI data lent insight into PSTs' mathematical knowledge for teaching, while the survey instrument and interviews resulted in information about PSTs' attitudes about mathematics and teaching. Thus, all three data collection methods helped to shed light on question #1 broadly. Finally, I learned more about whether ad how PSTs saw connections between their courses' structure and their future teaching practice through data generated by the survey instrument and interviews.

With respect to question #1a, there was little data in the corpus that could help answer it either way. Certainly, two of three MATH 281 PSTs who participated in interviews showed confidence and facility with the standard division algorithm that was not present among the MATH 291 PSTs. To the extent that the MKTI assesses PSTs' mathematical proficiency, there is no evidence that PSTs at one university developed greater performance than the other. The statistical analysis of the MKTI revealed that the two cohorts were statistically indistinguishable at both beginning and end of the semester.

This means that one cannot conclude from this data set that one cohort learned more than another. However, the item analysis revealed that while this claim is unsubstantiated, they indeed appeared to perform best on different sub-domains of MKT. MATH 281 PSTs were strongest on items that focused on common content knowledge (this might also bolster a claim that this group developed stronger mathematical proficiency; see question #2c). On the other hand, when MATH 291 PSTs showed stronger performance on individual items, they tended to be in the area of specialized content knowledge. This sub-domain of MKT is described by Ball and colleagues (2008) as being particular to teaching practice, while the other might be shared knowledge among other communities. The responses given by PSTs across the six interviews support the item analysis and show the depth of these differences. Thus, the short answer to question #1b is *yes*.

As for question #1c, the data again indicate that there were differences between PSTs at Hilada and at Rio. Though these differences were small, they were nonetheless significant statistically, and suggest that MATH 291 PSTs developed more adaptive attitudes toward mathematics and a greater comfort with teaching mathematics than did MATH 281 PSTs. This might be offered as an answer to question #2c as well. The reason for these differences remains unclear, as correlation statistics and interview data suggest that the greatest factor in determining attitudes scores on the survey was the strength of PSTs mathematical knowledge for teaching.

Finally, there is the question of relevance: did PSTs at one university view their course as more relevant than PSTs at the other? Here again, the hypothesis that generated the question (PSTs who engage in practices of teaching during their mathematics course work will view these courses as more relevant than those who do not) was borne out. Rio PSTs reported feeling a greater connection between their course and their future practice than did Hilada PSTs,

I have offered as explanation the differences between the two courses and their foci. While Hilada endeavored to place elementary mathematics on a solid logical foundation, concentrated on justifying and connecting arithmetic operations across number sets, and encouraged conventional mathematical notation, Rio looked to the notion of place value as a foundational principle in elementary school mathematics, developed robust new representations of numbers and operations, and pushed PSTs to confront the idiosyncratic nature of children's mathematical thinking through consistent use of videos and student work. The two courses made different connections to elementary teaching, both in number and kind; those differences had an impact on how PSTs viewed working in their course, the attitudes they developed about mathematics and teaching, as well as the mathematical knowledge for teaching they demonstrated.

The results of this study are not likely to reshape the landscape of mathematics education. The documentable effects of the differences I outlined between the courses are relatively small and yet they nevertheless have implications for the field, especially as it attempts to learn more about how teachers learn the mathematics they need to know in order to improve student understanding. With this study, I sought to answer questions that I had generated informally as a community college teacher which evolved into more formal research questions. The answers provided by the project can inform my teaching and that of many others in the field, while new questions have arisen in their place. It is these issues and questions to which I will turn in the next chapter.

Chapter 6: Conclusions and Further Research

My disappointment with service learning gave rise to an effort to find ways to acknowledge and take advantage of PSTs' desire (and need) to begin developing their craft early in their preparation. A key component of this is for PSTs to develop strong mathematical knowledge for teaching which, as the phrase indicates, presupposes that teachers' content knowledge is tied closely to their practice. Careful data collection confirmed what preliminary observation suggested: MATH 281 at Hilada University and MATH 291 at Rio University discussed analogous mathematical ideas, but were different courses that situated themselves differently with respect to teaching. I have argued that as a consequence of this difference, the PSTs learned different mathematics, developed different attitudes about mathematics and teaching, and recalled the relevance of their courses with respect to their preparation as teachers differently. All of these are predicted results of situated learning theory, which fundamentally and inextricably links the learner's environment to her development as a practitioner. The results of the project do not offer easy solutions to problems of teacher education in mathematics, yet they do suggest that PSTs indeed benefit from examining mathematical ideas through the lens of teaching practice.

In returning to the research questions at the end of the last chapter, I highlighted the results of the data collected as they related to each question. Here, I expand on these results a bit more, to argue what the implications of these findings are.

Question 1: What mathematics do prospective teachers learn by engaging in activities of teaching practice such as examining curriculum, student work, and classroom video?

a. Do PSTs who regularly engage in such activities display evidence of different mathematical proficiency than PSTs who participate in more traditional course work?
b. Do PSTs engaging in such activities display different mathematical knowledge for teaching (MKT) than PSTs participating in more traditional course work?
c. Do PSTs engaging in such activities develop different attitudes about mathematics and teaching than PSTs participating in more traditional coursework?

The MKTI data could be said to address questions #1a and #1b. The answer, with respect to this project is a qualified "yes." Though the general statistical analyses did not reveal any differences between PSTs' MKT, the subscale analysis together with the item analysis indicates that the differences in the courses may have played a role in the differential achievement on CCK items and SCK items. On one hand, it appears that these differences--where they exist—can be attributed simply to the different opportunities each cohort had to learn these things. In a way, this is precisely what happened.

However, it is not enough simply to say that the PSTs learned different mathematics because they were taught different mathematics. First, by many measures they were taught the *same* mathematics: both courses focused primarily on the four fundamental operations and understanding how (and why) the traditional algorithms and procedures associated with these concepts worked. Second, the choice to implement a concentration on student thinking (for which video played a key role in MATH 291) is inextricably tied to a mathematical approach that seeks an authentically elementary view. If indeed the PSTs did learn different mathematics for teaching, one must investigate carefully the circumstances of that learning: mathematical and pedagogical situative components are not distinct in this perspective.

Consider a scenario in which a teacher educator wants to incorporate student thinking and development into a mathematics course. This teacher educator can think of this implementation in two ways: (1) begin with student thinking and determine what mathematics are needed to sustain this focus; or (2) begin with a mathematical approach, and find examples of student thinking and classroom practice that illustrate the relevant mathematical ideas. For option (1), in order to lay bare children's thinking about mathematics, the teacher educator must develop in PSTs an understanding of the mathematics which the child(ren) confront in their classrooms. PSTs will not have an opportunity to understand this thinking without making explicit the mathematical ideas which form the foundation of elementary arithmetic, such as place value. As for option (2), if the focus is to find examples of practice that bring insight to an a priori mathematical approach, the mathematical ideas with which one begins must reflect those in schools. In either case, the mathematics that forms the focus of the course must have a close and direct relationship to the work of teaching for which the course is designed to prepare PSTs. Simultaneously, the work of teaching that is of import is not devoid of a disciplinary context, but is dependent on the mathematics at hand.

Thus, I argue that it is not a coincidence that artifacts like video were not as preferentially incorporated into MATH 281 as they were in MATH 291: the mathematics which they were discussing did not always lend itself to learning anything interesting about what children do or how they think. If a child misapplies an algorithm, it is probably not because he has misunderstood the property of distribution, or has incorrectly used mathematical symbols to express the value of the statement. Instead, it is likely to stem from some failure to understand or apply aspects of the base-ten number system and the consequent relationships between place values. These last ideas are key components of elementary school mathematics, and while the ideas highlighted by MATH 281 are neither incorrect nor irrelevant to PSTs, they were less directed toward PSTs ability to unpack mathematical concepts with the intuitive understandings that we well know children possess. Instead, MATH 281 worked more on areas beyond elementary school, when curricula tend to concentrate more on generalizing operations beyond whole numbers and integers.

It may be that MATH 281 PSTs possessed a stronger sense of the mathematical horizon than did MATH 291 PSTs; this was another sub-domain of MKT described by Ball and colleagues (2008) which I did not explore in this project. My hypothesis is that this would be the case. As it is, I argue that despite their relatively small magnitude, the differences between PSTs' MKT at each campus did exist, and is a result of the mathematical and pedagogical design of the courses. One of the main advantages situated learning theory offers teacher education is that it brings to the fore the links between mathematics and pedagogy: teachers understand the mathematics they teach through some practice. MATH 291 PSTs' activity was aligned more closely with elementary teaching practice than MATH 281 PSTs were, and thus, there were different manifestations of knowledge in the data.

Without this perspective, the data are relatively meaningless; one cannot determine whether or not it was the artifacts, which were used more often in MATH 291 than MATH 281 that created these differences. One might argue alternatively that the differences may have originated in the overall mathematical approach, or other characteristics of the course which I have described (or not). The design of the study was not sensitive enough to discern between these causes, in part because it was predicated on the theory that these are constituent, inextricably connected elements of the same situation. The mathematical approach influences the nature and extent of the use of teaching artifacts and vice versa.

And yet, the approaches espoused in MATH 291 are not magic potions; they neither substitute for nor supersede factors like the knowledge PSTs bring to the course. The data showed that this prior knowledge was the most significant component of the knowledge they showed at the end of the semester. Both MATH 281 and MATH 291 presented to PSTs coherent mathematical frameworks on which to build their knowledge for teaching. Those frameworks provided some basis for developing ideas about the mathematics that children are supposed to learn, how children think, and how to provide opportunities for developing PSTs' *own* mathematical knowledge.

As for Question #1c, again, I believe that the different nature of the courses has something to do with the differences in attitudes about mathematics and teaching revealed by the survey. However, I do not think that the data reinforce this claim. The survey items did not adequately match the differences between the courses, and it is not apparent to me yet how I may have generated more useful data along those lines. Even if it were possible to prove such a result shaping teachers' beliefs and attitudes has not been shown conclusively to lead to measurably different practices in the classroom. Such a result would strengthen the theory's claim that identity re-formation occurs through participation rather than by fiat. Though the goal of learning is to change the learner's identity, the means by which this change occurs most effectively is fundamentally different than situations where didactic practices are dominant. In other words, the theory posits that you can't really *tell* a teacher what to believe about children's thinking or what mathematical challenges and complexities students will present to them; only in confronting them directly as they develop their practice will teachers (and by extension, PSTs) have a chance to develop the perspective that teacher educators want to reveal.

<u>Question 2: To what extent do prospective teachers see their mathematics course work as relevant to their future work?</u>

a. Do different course approaches set up differing perspectives among PSTs on the contribution of the course to their future work?
b. Do different course approaches set up differing views among PSTs about their confidence and abilities in mathematics?

The answer to #2b, like the answers to Question #1, is a qualified "yes." It turns out not to be a question about quantity as much as one of type: MATH 281 PSTs developed confidence in different areas of mathematics than did MATH 291 PSTs, owing to the different foci of their courses. The data is not conclusive about this but indicates that the confidences of PSTs lay in different areas.

Question #2a is one of the more intriguing results of this project. The data indicate that MATH 291 PSTs felt that their course was significantly more relevant to their future teaching practice than their MATH 281 counterparts. Here it appears that the use of video in the course was a primary factor in this evaluation, suggesting that PSTs may be more motivated by courses that explicitly acknowledge their developing role as teachers and implicitly move their status as students toward the background. Recall

Eliot's statement "you don't want to be taught like you don't know it," which referred to a course which treats PSTs as though they are learning the mathematics for the first time. While it is true that PSTs (as well as teachers and mathematicians) can always learn new mathematics and new perspectives on well-worn ideas, one can accomplish this by inviting teachers into mathematical understanding for teaching: suddenly, "when are we going to use this" can become "we are going to need a deeper understanding of mathematics in order to decipher what children are saying and decide how to facilitate their learning." In my experience, PSTs are often studying to become teachers because of their self-professed "love of kids." Teacher education should take advantage of this affectionate stance by making children a central part of their study.

Implications for Teacher Education

Artifacts of Teaching Practice are Not a Panacea

A naïve hypothesis in this project was that the use of video and a concentration on student thinking are somehow a panacea for teaching mathematics to PSTs. In some ways, I replaced my early faith in service learning with the use of video cases, believing that the simple implementation of the idea would create an environment in which PSTs somehow would learn relevant mathematics by default. This study demonstrates the simplicity of that hypothesis; PSTs who watched video substantially more often than those who did not did not perform better (or worse) on the MKTI instrument. On other measures, the differences were relatively small, and are unlikely to motivate a sea change in undergraduate pre-service teacher education. Like the use of service learning, the benefits are probably limited in a setting like a semester-long course at a university.

Yet, the fact that videos, cases, and student work are somewhat less complex versions of service learning (or student teaching) is important. Use of such materials has the advantage of allowing teacher educators to strip away obfuscating characteristics of classroom activity without removing the important mathematical challenges faced regularly by classroom teachers. It allows teacher educators to focus PSTs' attention on particular (e.g., mathematical) aspects of the teaching situation. The interview data suggests that PSTs who engaged in this video analysis and concentrated largely on student thinking had tools and concept images that they could use not only to further their own understanding, but that of their future students. In fact, experience with video was cited directly by PSTs' for changes they saw in their abilities along these lines. On the other hand, though MATH 281 PSTs arguably were more mathematically proficient—the data suggests at least that their strengths were in the realm of common content knowledge—there was little evidence that they were as well prepared as the other cohort to articulate this knowledge to children or create opportunities to develop in them the same capacity. For instance, it seems hard to imagine that an explanation of the division algorithm which appeals to fraction operations will be useful to children who are learning to divide without much experience using fractions. What's more, the analysis suggests that PSTs develop more adaptive attitudes to mathematics and teaching when they engage with these materials. This may stem in part from the fact that they believe the work they are doing is directly relevant to their future practice; this was also a result of the analysis I offered above.

Listening to Pre-Service Teachers

It is not known whether or not one sub-domain of MKT is more important than others when it comes to teaching effectiveness. This means that it would short-sighted to conclude that one course did a "better" job helping PSTs develop MKT than the other. The evidence does not support such a claim. On the other hand, there are reasons why undergraduate content courses for PSTs might want to aim for particular sub-domains rather than others. First, the very nature of common content knowledge (as defined by Ball, Thames, & Phelps, 2008) is that it is not confined to the world of teaching and learning. This is an area of mathematical knowledge that is shared by other groups of people and is arguably the very sub-domain of knowledge that PSTs should already possess as they begin their undergraduate preparation. The situated nature of this knowledge (if one chooses this lens) makes it especially well-suited to PSTs and teaching, as many undergraduates—and adults in general—develop whatever mathematical knowledge they have in the classroom context.

However, the roles that people play as students in these classroom contexts are very different from the roles they will play as teachers. The evidence indicates that one can increase PSTs' knowledge while also addressing their desire to participate in the community of teachers. Moreover, the question of transfer, as theorized by many education researchers is at stake. What maximizes the chances that PSTs will be capable of applying what they did in their content courses to their teaching practice? Situated learning theory argues that engaging PSTs in elements of that practice will increase the probability of transfer. Recall that from the situated perspective, transfer refers to the consistency of activity patterns across situations, so that it is not necessarily the knower that takes knowledge from one place to the next, but salient features of the situation itself that are transferred (Greeno, 1997). The knower is able to attune properly to the features of activity that are germane to both settings depending on the learner's level of participation in the activities and her contribution to the resulting interaction (Greeno, 1998). I argue that it is the work of MATH 291 that more readily transfers from the university classroom to the elementary school classroom.

This is not to say that mathematics courses for PSTs should not address common content knowledge issues. Knowing the differences between mean, median, and mode or knowing that $\frac{0}{4} = 0$ while $\frac{4}{0}$ is undefined are clearly necessary for teachers to understand. Ball, Thames, and Phelps note that in their field observations, "when a teacher mispronounced terms, or made calculation errors, or got stuck solving a problem at the board, instruction suffered and instructional time was lost" (2008, p. 399). But this study offers evidence that concentrating on issues of teaching provides opportunities to develop, refine, and strengthen this knowledge while PSTs learn to confront the special knowledge and situations they will face as teachers.

Thus a major implication of this project is that PSTs' desire to connect somehow—even indirectly—with the children they will teach should not be ignored by teacher educators. When making decisions about how to design courses, PSTs preferences may not be the only consideration, but these results suggest that teacher education may be wise to look for more ways to unite PSTs with the teaching community directly or indirectly.

Implications of Course Design

However, the results of this study do not compel teacher educators (among whom many are mathematicians, whether they identify as such or not) to shape their practice around the teaching community simply to make PSTs feel better about what they are doing or to give them a voice in their own development.[68] The data suggest that such approaches not only impact PSTs attitudes and beliefs about how their academic work connects to teaching, but this influence extends to the opportunities PSTs have to learn mathematics. The differences between the *mathematical* frameworks of these courses were interdependent on the ways in which the courses connected with teaching. MATH 291 concentrated on place value and its relationship to the fundamental arithmetic operations. PSTs developed skills with choosing and representing BMUs, which in turn provided them with a basis for understanding and justifying common algorithms. Concentrating on place value also enabled MATH 291 to simulate the trajectory of elementary school mathematics by first building a system of counting in groups of ten and then learning to manipulate these numerals by taking advantage of the characteristics of the system. MATH 281's focus on operation across so many number sets did not preclude it from incorporating children's work or classroom video, but it did often require a mathematical trajectory that conflicts with a typical elementary school curriculum. The difficulty with organizing elementary mathematics material around the concept of operation in the way that MATH 281 did is that one must establish certain rational number concepts in some depth in order to operate on them. In fact, this is what MATH 281 did by introducing some number theoretic ideas (divisibility, GCF, and LCM, for example) in order to address rational number issues like simplifying fractions. This

[68] There may be powerful arguments to be made in favor of both of these perspectives.

means that the instructor of the course must address issues of multiplication and division on whole numbers (which is what gives rise to rational numbers in the first place) without first addressing what it means to add and subtract, let alone multiply and divide. But this inverts a conventional trajectory of elementary school mathematics: children learn how to add and subtract long before a serious discussion of multiplication or division is addressed, in large part because multiplication and division are deeply rooted in the other two arithmetic operations. Rational numbers—as fractions—are not a robust part of typical school curricula until the fundamentals of operating on whole numbers are laid as a foundation. In this much more subtle way then, MATH 291 represents a more authentic experience of teaching elementary school mathematics, and the data indicate that this may have contributed to MATH 291 PSTs showing a greater capacity to "unpack" the mathematical concepts about which they were learning.

Moreover, this authenticity did not necessarily sacrifice the mathematical generality sought by MATH 281's approach, nor did it "dumb down" MATH 291's perspective.[69] In a strict sense, fractions and ratio and proportion concepts were not explicitly part of MATH 291 at Rio. On the other hand, PSTs, working in the context of decimals and BMUs, explored a construct that supports a discussion of such concepts and representations. The diagrams drawn and much of the language used—as it pertained to operating on "decimal" numerals in other bases—are directly applicable to the set of rational numbers, even though PSTs would likely balk at any fraction question given to them on a MATH 291 exam!

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[69] Recall that a greater proportion of PSTs at Rio would describe their course as "challenging" or "hard" to a friend.

I am interpreting the differences between the two courses as boiling down to the following: while both courses presented a coherent view of mathematics, one course presented a view with greater authenticity with respect to elementary school mathematics teaching. Here I do not use the term "authentic" to mean some fidelity to one or another philosophy of mathematics, but instead I mean to highlight the compatibility of what mathematics was in each course with how mathematics is practiced in elementary school. The fact that this may have influenced PSTs' perception of the relevance of the course is an important component of what opportunities PSTs had to learn mathematics. Aside from this "motivational factor," the framework used by MATH 281 necessarily appealed to concepts that, while mathematically coherent, did not lend themselves to the kinds of knowledge unpacking required of teachers in the MKT scheme and in the rhetoric of teaching reform. Alternatively, MATH 291's approach, though somewhat idiosyncratic, provided PSTs with opportunities to develop MKT that was less explicit in MATH 281.

It is worth restating Moreira and Davis' (2008) assertion that teaching mathematics in schools can be in direct conflict with mathematics as viewed from its disciplinary referent: "To create the real number system from nothing, that is, by postulating its existence as 'anything' satisfying the complete ordered field axioms, ends up in an inversion of what is done in school…academic mathematical knowledge may not be 'naturally' a helpful instrument for the teacher in school practice" (pp. 37-38). School mathematics and mathematics in the academy are fundamentally different things because they take place within fundamentally different communities, which employ different relationships, different structures, different organizations, and ultimately, different epistemologies. One of the challenges of teacher education is to find ways to bridge this gap: teachers must understand and work within both worlds as they seek to show children the important characteristics of one while being constrained by the realities of the other. This means that in order to develop coherent models of teacher education in mathematics, there must be ongoing and constructive dialogue between the fields of mathematics and mathematics education (not simply between individual mathematicians and mathematics educators).

Exploring Differences Beyond MATH 281 and MATH 291

The results reported here are a replication—of sorts—of those described by the Philip, et., al. (2008). In that project, the authors note that PSTs who engaged with children via video (or in person) showed only small—though discernibly different—mathematical knowledge than peers who did not participate in such activities.[70] PSTs who engaged in focused discussions of children's thinking reported significant changes in their beliefs, while others showed smaller differences, if any. There are analogous results in the analyses I have provided above, but instead of focusing on extra-curricular activity, it examined the work of the content course itself, attempting to discern whether and how this factor (held constant in Philip, et., al.'s work) might affect mathematical knowledge for teaching, attitudes, and also PSTs' perceived relevance. This kind of replication in educational research is not common but it is important; such research is rarely large-scale or truly experimental, and so understanding how similar phenomena play out in numerous contexts is necessary. The two projects together appear to provide a compelling argument that focusing PSTs on issues of teaching—by way of examining children's mathematical thinking—does indeed change their beliefs and their ownership

[70] The PSTs who discussed children's thinking in formal ways also scored better than those who simply observed classroom teachers.

of the mathematics as important to their practice. The relatively small differences between the cohorts' MKT here—and in Philip, et. al.'s (2007) work—likely means one of two things: (1) focusing PSTs on children's thinking does not strongly influence PSTs' MKT in the short term (2) the influence has not been detected by these projects. One question that emerges is whether or not these results are robust across a wider variety of circumstances. Though I believe that evidence is mounting in favor of explanation (1), I do not believe that there is enough research to argue this claim conclusively.

Even if it is true that the differences between these single-semester courses did not strongly impact PSTs' MKT, another question emanating from this work is: what happens to these PSTs beyond this first content course? Would there be more obvious differences between PSTs' MKT at the end of their three-course sequence? Would three courses which focus on children's thinking and observing their mathematical abilities have a growing cumulative effect on MKT, or would any gains made in one course get washed out by events taking place in subsequent courses? Additionally, the field lacks research on what influence a coherent undergraduate program taking on these approaches might have on PSTs' knowledge, attitudes, and beliefs. Still more, what would such differences—if they exist—imply for PSTs' teaching at the conclusion of their programs, and would these differences—if they exist—have any impact on children's mathematical knowledge? Some of these questions are already the focus of research programs[71] but by necessity, these offer only small pieces of the puzzle. Additional, longitudinal data of PSTs and their mathematical development are necessary to answer many of these questions.

[71] Some of the projects underway at the Mid-Atlantic Center for Mathematics Teaching and Learning come to mind.

Finally, it is noteworthy that our understanding of how secondary PSTs in mathematics learn mathematics for teaching—or even what MKT looks like in secondary mathematics teaching—is even more nascent than our understanding of elementary PSTs. Teachers who work with more advanced students have potentially different mathematical goals and needs than their elementary school counterparts. What knowledge teachers in these situations need and how they come to know it are not questions that have received much attention. There may be parallels between the elementary level and secondary level, but there may also be significant differences, and these issues also should be explored empirically.

Final Remarks

Hammerness, et., al. (2005) point out that a key component of teacher education should be to make the subtleties of school experience more explicit for PSTs, to sensitize them to issues of which students cannot be aware, but teachers must. They write that teacher education should: (1) make teachers' "apprenticeship of observation" explicit and offer alternative perspectives that build upon—and in some cases challenge—this experience; (2) enable teachers to acquire a deep foundation of knowledge and techniques for giving this knowledge a conscious structure; (3) provide access to meta-cognitive tools for teachers to continue to develop their knowledge in and through their practice.

Both MATH 281 and MATH 291 accomplished these goals in subtly different ways, albeit for the short duration of the semester. PSTs at Rio and Hilada frequently mentioned on the surveys that they learned to think of and about mathematics in new ways. PSTs in interviews talked about the challenges they faced in completing the courses: learning why procedures work that were long taken for granted or misunderstood, how children think about elementary mathematics, and in some cases, what it's like to be in the shoes of a child learning to do such "basic" mathematics. Consider Maeby's comment about working in different bases:

> *[The instructor] turned it around on us…when she, she always makes us do problems, like multiplication problems, in different bases, which—they're annoying at the time, but they really help, because, I mean, if you tell me I have to do five times three, like I know how to do that, but if I had to do it in base eight, that's a whole different concept, and like, I have to—I have to think about it, like (sighs)…it's just, you have to think about place value and stuff like, things that you would normally…or you would automatically say the answer, you wouldn't think about the place value stuff like that . So it's kind of putting us in a child's position…*

These are opportunities that are not the special province of content courses for teachers; teacher educators have endeavored to create these kinds of opportunities for PSTs within methods courses and student teaching for some time. However, content courses are not independent of these other experiences that often occur late in PSTs' undergraduate preparation. This study assumed the truth of the emerging consensus that there is special knowledge that teachers must develop in order to enhance their ability to improve student learning, and supports the speculation that situating PSTs' learning within the contexts and tasks of teaching influences them. While it is unclear the relative effect this approach has on PSTs' mathematical knowledge for teaching, this project suggests that there is more to investigate and leaves open the possibility that one potential avenue for improving undergraduate teacher education is to re-cast content courses in ways that address the community of teachers of which PSTs will one day be a part.

As a teacher educator, this study has influenced my understanding of how PSTs learn, and what they value in undergraduate mathematics courses. As a researcher and a member of the mathematics education community, this study demonstrates to me that there is more work to be done, and more questions to be answered, but that there is building evidence that understanding PSTs' developing MKT with a situated perspective and designing instruction with this understanding moves the field in a constructive direction. In particular, I believe that this project contributes to the field by confirming some conclusions reached in other research but also contributing something new: this work provides information about how PSTs perceive their content course experiences and what kinds of participation draws them toward learning the mathematics we wish for them to learn.

Fennell's question, with which I opened this essay—how do we ensure that all teachers of mathematics know the mathematics and pedagogy essential for teaching the subject—ultimately is not answerable. We cannot ensure that *all* teachers know anything in particular, but teacher education has a responsibility to design coherent learning opportunities for PSTs in which they can participate in a community of teachers, a community that is continuously created and nurtured as they deepen and refine their contributions.

Appendices

Appendix A: Survey Items Administered to PSTs at the Beginning and End of the Semester

[February 2008]
Your Beliefs about Mathematics, Math Teaching, and This Math Course

	Strongly Agree	Agree	No Opinion	Disagree	Strongly Disagree
1. One of a teacher's major responsibilities is to show students how to solve problems and then to give them similar problems to practice.	SA	A	N	D	SD
2. Some people have mathematical minds and some don't, and neither good teaching nor student effort can overcome that.	SA	A	N	D	SD
3. Children can develop their problem solving skills by working in small groups and hearing the ideas of other students.	SA	A	N	D	SD
4. A good math test is one that consists of a variety of items that are that are just like the problems students completed in class or in homework.	SA	A	N	D	SD
5. Having students determine and discuss their solution methods is a good use of class time, even if the discussion and questions about those methods takes more than one class period.	SA	A	N	D	SD
6. If students are expected to solve mathematics problems before the teacher has explained the problem and solution, the students will become frustrated.	SA	A	N	D	SD
7. Some students may have more aptitude for mathematics than others, but all students can learn to understand mathematics.	SA	A	N	D	SD
8. I think I will be just as comfortable teaching the mathematics content taught to children in kindergarten through second grade as the mathematics content taught to children in the fourth or fifth grade.	SA	A	N	D	SD
9. It is more important for children to compute quickly and accurately than to solve word problems.	SA	A	N	D	SD
10. The best way to become good at mathematics is to solve a lot of new problems, thinking about the ideas and strategies used to solve prior problems.	SA	A	N	D	SD

11. Getting the correct answer is the most important goal in math class.	SA	A	N	D	SD
12. Students should work on mathematics problems before the teacher introduces the skills and vocabulary traditionally used to solve those problems.	SA	A	N	D	SD
13. Seeing/hearing different ways to solve the same problem confuses children.	SA	A	N	D	SD
14. A good math test is one that contains some challenging yet attainable problems that are not like problems worked in class.	SA	A	N	D	SD
15. The idea of teaching math scares me.	SA	A	N	D	SD
16. Before class ends, a teacher needs to clarify those wrong answers, incorrect methods or mis-statements that may have been made by students.	SA	A	N	D	SD
17. Students will become engaged in mathematics if they are expected to figure out the solutions to questions.	SA	A	N	D	SD
18. Students should hear methods that other students use to solve problems.	SA	A	N	D	SD
19. Discussing wrong answers is likely to confuse children about the right way to work on problems.	SA	A	N	D	SD
20. I am looking forward to teaching children about mathematics concepts.	SA	A	N	D	SD
21. In mathematics class, a teacher's role is to show students how to complete tasks and to solve problems, helping students who get stuck.	SA	A	N	D	SD

ABOUT YOUR MATH COURSE

5. The **activities** we do in this class are supposed to help me make significant progress in my goal of becoming a teacher	SA	A	N	D	SD
6. The **assignments** we do in this class are supposed to help me make significant progress in my goal of becoming a teacher.	SA	A	N	D	SD
7. The **exams** we take in this class are supposed to help me make significant progress in my goal of becoming a teacher.	SA	A	N	D	SD
8. In this class, I expect to learn more about what it is like to teach mathematics.	SA	A	N	D	SD

[MAY 2008]
Your Beliefs about Mathematics, Math Teaching, and This Math Course

	Strongly Agree	Agree	No Opinion	Disagree	Strongly Disagree
22. One of a teacher's major responsibilities is to show students how to solve problems and then to give them similar problems to practice.	SA	A	N	D	SD
23. Some people have mathematical minds and some don't, and neither good teaching nor student effort can overcome that.	SA	A	N	D	SD
24. Children can develop their problem solving skills by working in small groups and hearing the ideas of other students.	SA	A	N	D	SD
25. A good math test is one that consists of a variety of items that are that are just like the problems students completed in class or in homework.	SA	A	N	D	SD
26. Having students determine and discuss their solution methods is a good use of class time, even if the discussion and questions about those methods takes more than one class period.	SA	A	N	D	SD
27. If students are expected to solve mathematics problems before the teacher has explained the problem and solution, the students will become frustrated.	SA	A	N	D	SD
28. Some students may have more aptitude for mathematics than others, but all students can learn to understand mathematics.	SA	A	N	D	SD
29. I think I will be just as comfortable teaching the mathematics content taught to children in kindergarten through second grade as the mathematics content taught to children in the fourth or fifth grade.	SA	A	N	D	SD
30. It is more important for children to compute quickly and accurately than to solve word problems.	SA	A	N	D	SD
31. The best way to become good at mathematics is to solve a lot of new problems, thinking about the ideas and strategies used to solve prior problems.	SA	A	N	D	SD
32. Getting the correct answer is the most important goal in math class.	SA	A	N	D	SD
33. Students should work on mathematics problems before the teacher introduces the skills and vocabulary traditionally used to solve those problems.	SA	A	N	D	SD
34. Seeing/hearing different ways to solve the same problem confuses children.	SA	A	N	D	SD
35. A good math test is one that contains some challenging yet attainable problems that are not like problems worked in class.	SA	A	N	D	SD

36. The idea of teaching math scares me.	SA	A	N	D	SD
37. Before class ends, a teacher needs to clarify those wrong answers, incorrect methods or mis-statements that may have been made by students.	SA	A	N	D	SD
38. Students will become engaged in mathematics if they are expected to figure out the solutions to questions.	SA	A	N	D	SD
39. Students should hear methods that other students use to solve problems.	SA	A	N	D	SD
40. Discussing wrong answers is likely to confuse children about the right way to work on problems.	SA	A	N	D	SD
41. I am looking forward to teaching children about mathematics concepts.	SA	A	N	D	SD
42. In mathematics class, a teacher's role is to show students how to complete tasks and to solve problems, helping students who get stuck.	SA	A	N	D	SD

ABOUT YOUR MATH COURSE

9. The **activities** we did in this class helped me make significant progress in my goal of becoming a teacher	SA	A	N	D	SD
10. The **assignments** we did in this class helped me make significant progress in my goal of becoming a teacher.	SA	A	N	D	SD
11. The **exams** we took in this class helped me make significant progress in my goal of becoming a teacher.	SA	A	N	D	SD
12. In this class, I learned more about what it is like to teach mathematics.	SA	A	N	D	SD
13. The most memorable assignment in this course was:					
14. The most memorable activity in this course was:					
15. Suppose a friend of yours was thinking about taking this class and wanted to know more about it. How would you describe the class, what would you say to your friend about what she should be ready for, and what she would learn?					

Appendix B: Excerpt from Chapter Five Notes (MATH 281 at Hilada University)

Packet p. 37: Begin by having students work in groups to analyze the student-invented algorithms pictured. Ask them to identify the properties used.

Student #3 is probably the most difficult to understand. Apparently they began by writing the column of 4 24's in the center of the work. Can you work it out from there?

One of the challenges of having students use invented approaches is helping them to notate what they've done in efficient, mathematically sensible ways. In particular, it is common for students to abuse the "equals" sign, treating it as they do the equals key on the calculator: as a signal to calculate what they've got so far. It is important for students to learn early, however, that the equals sign signifies both sides are truly equal: this is an essential concept in algebra. With this in mind, consider how to notate the student work on the following pages. (Perhaps assign one problem to each group and have them put it on the chalkboard and explain.)

Class Activity 5U (p. 126) #1, 2, 3

Also identify the properties used in each.

Homework:
Read text 5.8
Do p. 201 #2, 3a, 3f, 5

Appendix C: Excerpt from Lesson 11 Notes (MATH 291 at Rio University)

Time: 75 min.
Activity Flow – Part 4 - Homework

Rationale
This activity extends the multiplication story problem activity completed in class to division story problems. The activity is designed to develop the preservice teachers' ability to write both partitioning and repeated subtraction division problems, and to help them connect the meanings of multiplication and division. The preservice teachers will learn to represent multiplication story problems, and repeated subtraction and partitioning division story problems with part-whole diagrams, and understand the connections among these different part-whole models. This will help them understand the missing factor interpretation of the ÷ sign when it is introduced in the next lesson.

Activity
As a teacher, you will create word problems for students to help develop their conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning. For the operation or action of division, you will want to pose problems that involve both interpretations of division in order to help children create meaning for this operation or action. In the next activity, you will practice writing multiplication and division word problems. You are writing problems for both operations because many fourth grade teachers complain that their students cannot see the connection between multiplication and division. As you write both types of word problems, see if you can make this connection.

Hand out L11_MultMean3DivMean1_HW.doc

Instructor Notes
1. Homework Assignment [L11_MultMean3DivMean1_HW.doc]

Part 1
Questions 1-3 list three numbers that can be combined into multiplication and division number sentences. Use two of these numbers (the other number will be the answer) and the given quantities to write three word problems:

a. one where students will use repeated subtraction division to find the answer;
b. one where students will use partitioning division to find the answer;
c. one where students will use multiplication to find the answer.

Assign the same numbers to the same quantities in all three word problems. For example, if you write a repeated subtraction problem for part (a) in which you make 12 the number of apples altogether, 4 the number of apples in a tree, and 3 the number of trees, do the same thing in parts (b) and (c). That is, keep 12 the number of apples altogether, 4 the number of apples in a tree, and 3 the number of trees in parts (b) and (c) too.

Now try to make a part-whole diagram for each word problem that you write. What is the difference between the part-whole diagrams for repeated subtraction division problems, partitioning division problems, and multiplication problems?

1. Numbers: 12, 3, 4
   Quantities: trees, apples
2. Numbers: 4, 5, 20
   Quantities: children, ounces of lemonade
3. Numbers: 3, 1.8, 0.6
   Quantities: miles of road, days.

Part 2
Try to do what you did in the first three questions for these numbers: 0.9, 1.5, and 0.6. Pick your own quantities.

Appendix D: Written Reflection Assignments at Hilada University

Written Reflection #1 **(10 points)**

due date: ______________

Lots of Lockers

One hundred bored students decided to pass the time with the following activity: They lined up in front of a line of lockers numbered 1 to 100. The first student opened every locker door. The second student closed the door of lockers numbered 2, 4, 6, 8, etc. (i.e., all the even-numbered lockers). The third student changed the door position of the lockers numbered 3, 6, 9, 12, etc. (i.e., every third locker). If the door had been open he closed it; if it had been closed he opened it. Similarly, the fourth student changed the door position of every fourth locker, the fifth student changed the door position of every fifth locker, and so on, until the hundredth student changed the door position of locker #100.

Which locker doors were standing open at the end of this activity? Why? Be complete.

Written Reflection #2

due date: _____________

Instructions: Your response to this is to be typed, double-spaced. Please answer completely, in well-written paragraphs. A diagram may be hand-drawn in to accompany your response.

Write a word problem that would be correctly modeled by the division problem
2 1/4 ÷ 1/2.

Draw a diagram to illustrate this division in the context of your word problem. Explain and show how your diagram illustrates the solution.

Give a detailed numeric solution path and final answer. Include words on your numbers and show each step logically.

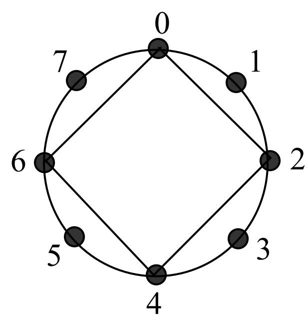
Identify what division concept your word problem illustrates ("How big is each group?" (partitioning), "How many groups?" (repeated subtraction), or missing factor), and explain how you know.

Rubric (10 points total)

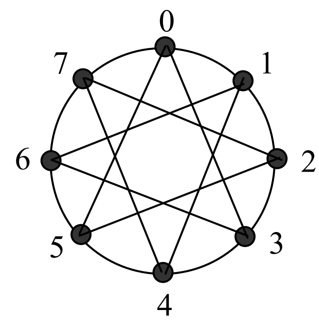
	0 points	1 point	2 points
word problem	incorrect operation	correct operation but incorrect numbers	clear and correct
diagram	missing or incorrect	unclear or partly correct	clear and correct
numeric solution	missing or incorrect	unclear or partly correct	clear and correct
solution shown in diagram	missing or incorrect	unclear or partly correct	clear and correct
division concept	missing or incorrect	correct but not explained or unclear explanation	correct and clearly explained

Project #1: Star Patterns

The following two figures illustrate star patterns. The left figure, Star (8, 2), was drawn by starting at 0 and "skipping" two spaces clockwise until returning to 0. In the right figure, each "skip" moves three spaces clockwise.



Star (8, 2)



Star (8, 3)

Both of these stars belong to the "8-family" because they have 8 dots evenly spaced around the circle. Notice that the number "8" does not appear, however; only the numbers 0 through 7 are used as labels.

Many patterns can be seen in these star figures. Collect data for the 8-family, the 7-family, the 9-family, and the 12-family on the following pages. After collecting this data, answer the questions below. For each, give the most general outcome, not just particular or special case(s).

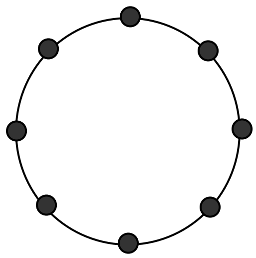
1. When will two stars in a family be identical? In other words, when will star (a, b) look exactly like star (a, c)? Include an example to illustrate.

2. For star (a, b), what happens when b divides evenly into a? Include an example to illustrate.

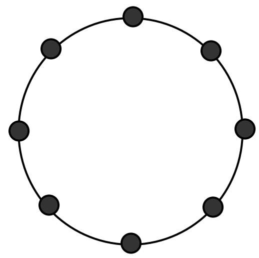
3. When will star (a,b) touch every number? Write a rule in English and then write a mathematical formula in terms of a and b. Give an example and a non-example.

4. a. In general, how many points will star (a, b) touch? Give a rule in English and then write a mathematical formula in terms of a and b. Give an example.
   b. Explain why your formula in 4a works. (Hint: think about LCM).

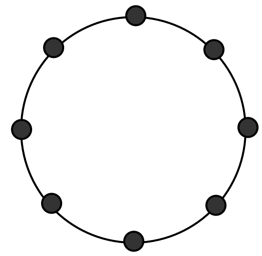
5. *(Optional)* Star (8, 3) above makes three clockwise circuits of the star in the process of connecting all the dots. How many circuits does star (a, b) make? Give a rule in English and then write a mathematical formula in terms of a and b. Give an example.



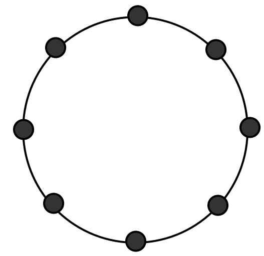
Star (8, 1)



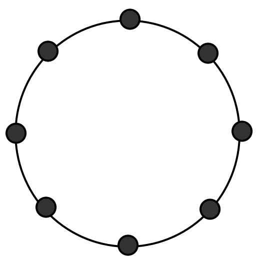
Star (8, 2)



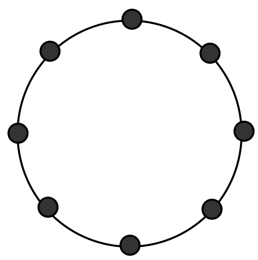
Star (8, 3)



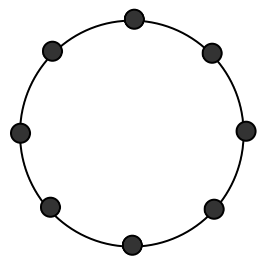
Star (8, 4)



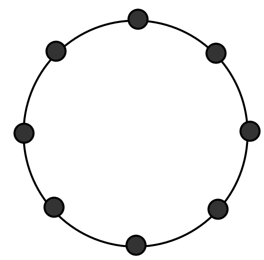
Star (8, 5)



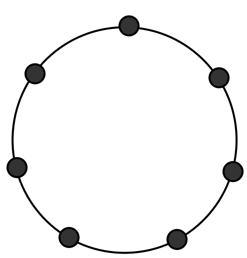
Star (8, 6)



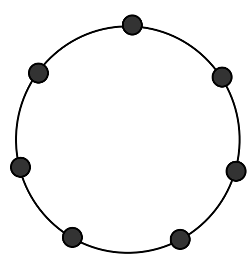
Star (8, 7)



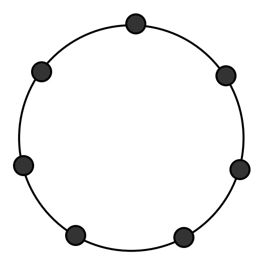
Star (8, 8)



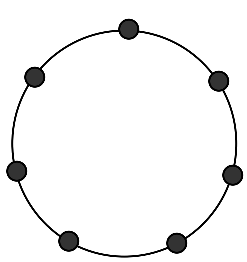
Star (7, 1)



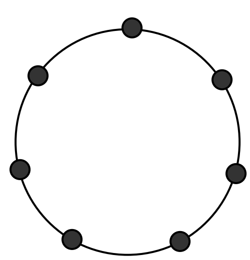
Star (7, 2)



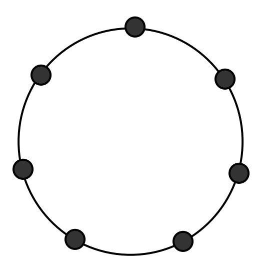
Star (7, 3)



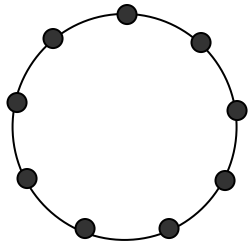
Star (7, 4)

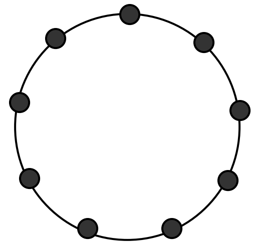


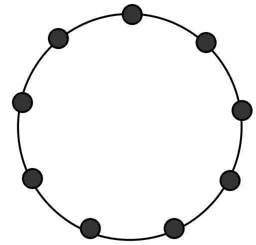
Star (7, 5)



Star (7, 6)


Star (9, 1)


Star (9, 2)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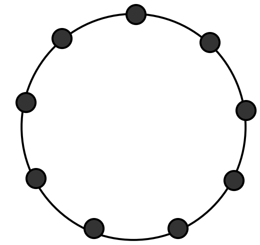
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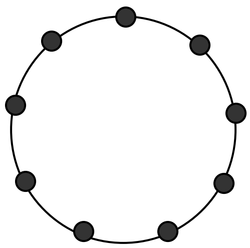
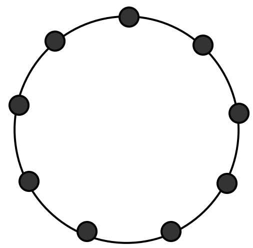
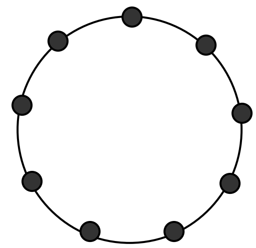
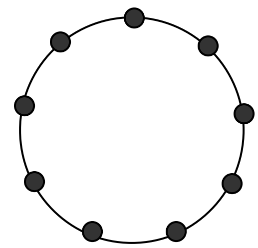
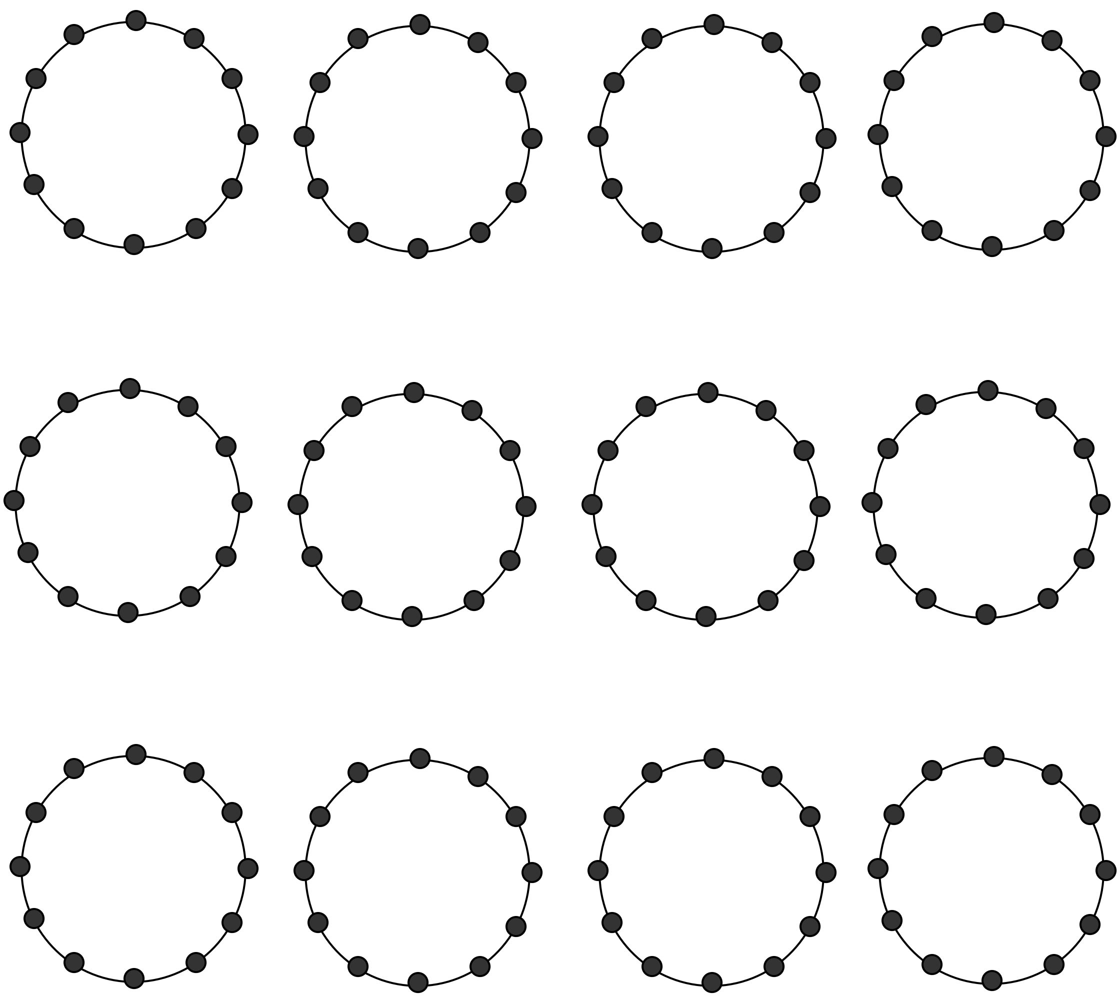

Star (9, 5)


Star (9, 6)


Star (9, 7)


Star (9, 8)



Appendix F: Project Assignment #1 from MATH 281 at Hilada University

[Refer to Beckmann, 2007]
Project #2 (20 points)
due date: ____________

Follow the instructions in the text for the following problems:

p. 270 #3

p. 270 #4

p. 271 #12

p. 272 #17

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