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**Abstract**

In this paper, we provide convergence results for an Ant-Based Routing (ARA) Algorithm for wireline, packet-switched communication networks, that are acyclic. Such algorithms are inspired by the foraging behavior of ants in nature. We consider an ARA algorithm proposed by Bean and Costa [2]. The algorithm has the virtues of being adaptive and distributed, and can provide a multipath routing solution. We consider a scenario where there are multiple incoming data traffic streams that are to be routed to their respective destinations via the network. Ant packets, which are nothing but probe packets, are introduced to estimate the path delays in the network. The node routing tables, which consist of routing probabilities for the outgoing links, are updated based on these delay estimates. In contrast to the available analytical studies in the literature, the link delays in our model are stochastic, time-varying, and dependent on the link traffic. The evolution of the delay estimates and the routing probabilities are described by a set of stochastic iterative equations. In doing so, we take into account the distributed and asynchronous nature of the algorithm operation. Using methods from the theory of stochastic approximations, we show that the evolution of the delay estimates can be closely tracked by a deterministic ODE (Ordinary Differential Equation) system, when the step-size of the delay estimation scheme is small. We study the equilibrium behavior of the ODE system in order to obtain the equilibrium behavior of the routing algorithm. We also explore properties of the equilibrium routing probabilities, and provide illustrative simulation results.

**Keywords**: communication networks, ant routing algorithms, stochastic approximations and learning algorithms, queuing networks

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# 1 Introduction

"Ant algorithms" constitute a class of algorithms that have been proposed to solve a variety of problems arising in optimization and distributed control. They form a subset of the larger class of "Swarm Intelligence" algorithms, a topic which has received widespread attention recently; see, for example, the book of Bonabeau, Dorigo, and Theraulaz [7]. The central idea here is that a "swarm" of relatively simple agents can interact through simple mechanisms and collectively solve complex problems. Bonabeau, Dorigo, and Theraulaz [7] give examples of insect societies like those of ants, honey bees, and wasps, which accomplish fairly complex tasks of building intricate nests, finding food, responding to external threats etc., even though the individual insects themselves have limited capabilities. The abilities of ant colonies to collectively accomplish complex tasks have served as sources of inspiration for the design of "Ant algorithms".

Examples of "Ant algorithms" are the set of Ant-Based Routing algorithms (henceforth referred to simply as Ant Routing algorithms (ARA)) that have been proposed for communication networks. It was observed in an experiment conducted by biologists Deneubourg *et. al.* [12], called the double bridge experiment, that under certain conditions, a group of ants when presented with two paths to a source of food, is able to collectively converge to the shorter path. It was found that every ant lays a trail of a chemical substance called *pheromone* as it walks along a path. Subsequent ants follow paths with stronger pheromone trails, and in their turn reinforce the trails. Because ants take lesser time to traverse the shorter path, pheromone concentration increases more rapidly along this path. These "positive reinforcement" effects culminate in all ants following, and thus discovering, the shorter path. Various mathematical models have been proposed to describe the evolution in time of pheromone levels on trails; for a discussion see Dorigo, Stutzle [13].

Most of the ARA algorithms proposed in the literature are inspired by variations on the basic idea of creation and reinforcement of a pheromone trail on a path that serves as a measure of the quality of the path. These algorithms employ probe packets called ant packets (analogues of ants) that help create analogues of pheromone trails on paths. In the context of routing, these trails are based on measurements of path delays made by the ant packets. Routing tables at the nodes are updated based on the path pheromone trails. The update algorithms help direct data packets along outgoing links that lie on paths with lower delays.

In this paper, we consider a wireline, packet-switched network, and provide convergence results for an ARA algorithm proposed by Bean and Costa [2]. The algorithm retains some of the most attractive features of ARA algorithms. It is distributed, and the routing tables (for every node, this consists of routing probabilities for the outgoing links) are updated based on delay information collected by ant packets. This enables the algorithm to be adaptive. Furthermore, the scheme can provide a multipath routing solution — that is, the incoming traffic at a source is split between the multiple paths available to the destination. This enables efficient utilization of network resources. We now briefly dwell on the literature on ARA algorithms.

**Literature.** ARA algorithms have been proposed for all kinds of networks — circuit- and packet-switched wireline, as well as packet-switched wireless networks. We briefly discuss the algorithms for packet-switched networks, because they are more relevant; for a more comprehensive survey see Dorigo, Stutzle [13] and Bonabeau, Dorigo, and Theraulaz [7]. Most of the algorithms proposed and studied for packet-switched networks — for example, Gabber, Smith [14], Di Caro, Dorigo [10], Subramanian, Druschel, and Chen [23] (all the above are for wireline networks), and Baras, Mehta [1] (for wireless networks) — are variants of the Linear Reinforcement (LR) scheme considered in studies of stochastic learning automata (see Kaelbling, Littman, and Moore [17] and Thathachar, Sastry [24] [2]). In these works, variants of the LR scheme are used to adjust routing probabilities at the nodes based on path pheromone trails. Yoo, La, and Makowski [26] consider the scheme proposed by [23] for a network consisting of two nodes connected by $L$ parallel links. The link delays are deterministic. Ant packets are either routed uniformly at the nodes — called 'uniform routing' — or are routed based on the node routing tables — called 'regular routing'. A rigorous analysis then shows that the routing probabilities converge in distribution for the uniform routing case, and almost surely to a shortest path solution for the regular routing case. The LR scheme however, is not designed for applications where the delays are stochastic and time-varying, which is the case of main interest to us. ARA algorithms different from the LR scheme, are considered in [7], and in [2].

Though a large number of ARA algorithms (and in general, Ant algorithms) have been proposed, fewer analytical studies are available in the literature. Algorithms similar to those that aim to explain the observations in the double bridge experiment, have been rigorously studied in Makowski [21] and Das and Borkar [11]. Makowski considers the case where there are two paths of equal length to a food source, and a model where each ant chooses a path with a probability proportional to a power $\nu \geq 0$, of the number of ants that have previously traversed the path. Using stochastic

[2] The LR scheme has been proposed for various adaptive learning and control applications.

approximation and martingale techniques the paper provides convergence results, and shows that the asymptotic behavior can be quite complex (in particular, only for $\nu > 1$, is it true that all ants eventually choose one path). Das and Borkar consider a scenario where there are multiple disjoint paths between a source and a destination. There are three algorithms — a pheromone update algorithm that builds a pheromone trail based on the number of ants that have previously traversed the path and the path length, a utility estimate algorithm based on the pheromone trail on a path, and finally a routing probability update algorithm that uses the utility estimates. Using stochastic approximation methods, they show convergence to a shortest path solution if there is an 'initial bias', i.e., if initially there is a higher probability of choosing the shortest path. The paper also considers extensions to multi-stage problems. Gutjahr [15] considers a problem where ant-like agents help solve the combinatorial optimization problem of finding an optimal cycle on a graph, with no nodes being repeated except for the start node. The arc costs are deterministic. The agents sample walks based on routing probabilities, and reinforce pheromone trail levels on arcs, which in turn, influence the routing probabilities. The paper shows that asymptotically, with probability arbitrarily close to one, an optimal cycle can be found. Another analytical study is the paper [26] discussed above.

**Contributions and Related Work**. The above set of analytical studies have mostly concentrated on networks with deterministic link delays. We consider the Bean, Costa [2] scheme for wireline, packet-switched networks, that are acyclic [3]. In contrast to the studies above, we provide convergence results when the link delays are stochastic and time-varying, and are dependent on the link traffic. This is a more relevant and interesting case. In our work [22] we initiated study in this direction, by considering the Bean, Costa scheme for a simple routing scenario where there are $N$ parallel links between a source and a destination.

Bean and Costa [2] study their scheme using a combination of simulation and analysis. They employ a 'time-scale separation approximation' whereby average network delays are computed 'before' the routing probabilities are updated. Numerical iterations of an analytical model based on this approximation and simulations are shown to agree well. However, the time-scale separation is not justified, nor is any formal study of convergence provided.

We consider a stochastic model for the arrival processes and packet lengths of both the ant and the incoming data packet streams. The ARA scheme consists of a delay estimation algorithm and

[3] For a definition of such networks see Section 3.

a routing probability update algorithm, that utilizes the delay estimates. These algorithms run at every node of the network. The delay estimates are formed based on measurements of path delays (these delays are caused by queuing delays on the links) obtained by the ant packets. We describe the evolution of these algorithms by a set of (intricately coupled) discrete stochastic iterations. We consider constant step-size schemes, which can adapt to (track) long term changes in statistics of the delay processes. This feature of constant step-size schemes is well known in the literature on adaptive algorithms; see, for example, Benveniste, Metivier, Priouret [3]. Our formulation considers the distributed and asynchronous nature of the algorithm operation. We show, using methods from the theory of stochastic approximations, that the evolution of the delay estimates can be closely tracked by a deterministic ODE (Ordinary Differential Equation) system, when the step size of the delay estimation scheme is small. We then study the equilibrium behavior of the ODE system in order to obtain the equilibrium behavior of the routing algorithm. We explore properties of the equilibrium routing solution, and provide illustrative simulation results.

Our approach is most closely related to Borkar and Kumar [9], which studies an adaptive algorithm that converges to a form of routing equilibrium, known as a Wardrop equilibrium [25]. Our framework is similar to theirs — there is a delay estimation algorithm and a routing probability update algorithm which utilizes the delay estimates. Their routing probability update scheme moves on a slower "time scale" than the delay estimation scheme. In our case however, the routing probability update scheme is on the same "time scale" as the delay estimation scheme, and our method of analysis is consequently different. This could also be desirable in practice, because the algorithm convergence will be much faster.

The paper is organized as follows. In this paper we separately consider the two cases where ant packets are routed according to uniform and regular routing. There is a parallel development of the discussion related to these two forms of routing. In Section 2 we outline in detail the mechanism of operation of ARA algorithms, and discuss the Bean, Costa algorithm. Section 3 provides a formal discussion of our acyclic network model and assumptions, and a formulation of the routing problem. We analyse the routing algorithm in Section 4, and discuss our ODE approximation results and related computations. We also discuss the equilibrium behavior of the algorithm. In the next couple of sections, Section 5 and Section 6, we study in some detail two illustrative examples — an $N$ parallel links network and an acyclic network. Related simulation results are provided and discussed. The concluding section, Section 7, summarizes the paper and discusses a few directions for future research. In an appendix, Section 8, we outline a proof of convergence of the algorithm.

## 2 Ant Routing Algorithms: Mechanism of Operation

We provide in this section a brief description of the mechanism of operation of ant routing algorithms for a wireline communication network. Such a network can be represented by a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{L})$, with a set of nodes $\mathcal{N}$, and a set of directed links $\mathcal{L}$. Our formal description follows the framework in Di Caro and Dorigo [10], [13], which is general enough and adequate for our purposes. Alongside, we describe the Bean, Costa [2] scheme, that we analyse in this paper.

Every node $i$ in the network maintains two key data structures — a matrix of routing probabilities, the routing table $\mathcal{R}(i)$, and a matrix of various kinds of statistics used by the routing algorithm, called the network information table $\mathcal{I}(i)$. For a particular node $i$, let $N(i,k)$ denote the set of neighbors of $i$ (corresponding to the outgoing links $(i,j)$ from $i$) through which node $i$ routes packets towards destination node $k$. For the communication network consisting of $|\mathcal{N}|$ nodes, the matrix $\mathcal{R}(i)$ has $|\mathcal{N}| - 1$ columns, corresponding to the $|\mathcal{N}| - 1$ destinations towards which node $i$ could route data packets, and $|\mathcal{N}| - 1$ rows, corresponding to the maximum number of neighbor nodes of node $i$. The entries of $\mathcal{R}(i)$ are the probabilities $\phi_{ij}^k$. $\phi_{ij}^k$ denotes the probability of routing an incoming data packet at $i$ and bound for destination $k$ via the neighbor $j \in N(i,k)$. The matrix $\mathcal{I}(i)$ has the same dimensions as $\mathcal{R}(i)$, and its $(j,k)$-th entry contains various statistics pertaining to the route from $i$ to $k$ that goes via $j$ (denoted henceforth by $i \to j \to \cdots \to k$). Examples of such statistics could be mean delay and delay variance estimates of the route $i \to j \to \cdots \to k$. These statistics are updated based on the information the ant packets collect about the route. The matrix $\mathcal{I}(i)$ thus represents the characteristics of the network that are learned by the nodes through the ant packets. Based on the information collected in $\mathcal{I}(i)$, "local decision-making" — the update of the routing table $\mathcal{R}(i)$ — is done. The iterative algorithms that are used to update $\mathcal{I}(i)$ and $\mathcal{R}(i)$ will be referred to as the *learning algorithms*.

We now describe the mechanism of operation of ARA algorithms. For ease of exposition, we restrict attention to a particular fixed destination node, and consider the problem of routing from every other node to this node, which we label as $D$ (see Figure 1). The network information tables $\mathcal{I}(i)$ at the nodes contain only statistics related to estimates of mean delays.

**Forward ant generation and routing**. At certain intervals, forward ant (FA) packets are

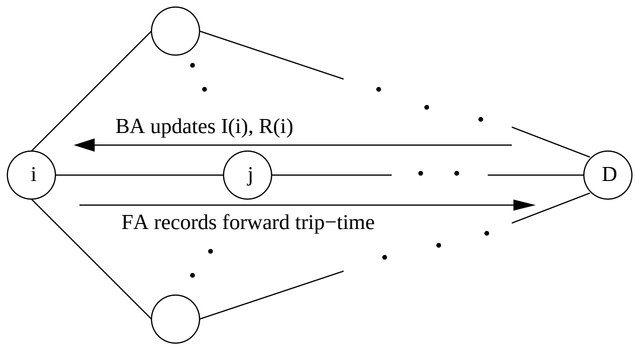


Figure 1: Forward Ant and Backward Ant packets

launched from a node $i$ towards destination $D$ to discover low delay paths to it. The FA packets sample walks on the graph $\mathcal{G}$ based either on the current routing probabilities at the nodes as in regular ant routing (regular ARA), or uniformly[4] as in uniform ant routing (uniform ARA). Uniform ant routing might be preferred in certain cases; for instance, when we want the ant packets to explore the network in a completely "unbiased" manner. FA packets share the same queues as data packets and so experience similar delay characteristics as data packets. Every FA packet maintains a stack of data structures containing the IDs of nodes in its path and the per hop delays encountered. The per hop delay measurements can be obtained through time stamping of the FA packets as they pass through the various nodes.

**Backward ant generation and routing**. Upon arrival of an FA at the destination $D$, a backward ant (BA) packet is generated. The FA packet transfers its stack to the BA. The BA packet then retraces back to the source $i$ the path traversed by the FA packet. BA packets travel back in high priority queues, so as to quickly get back to the nodes and minimize the effects of outdated or stale measurements. At each node that the BA packet traverses through, it transfers the delay information that was gathered by the FA packet. This information is used to update matrices $\mathcal{I}$ and $\mathcal{R}$ at the respective nodes. Thus the arrivals of BA packets at the nodes triggers the iterative learning algorithms.

We now describe the Bean, Costa [2] learning algorithm. Suppose that an FA packet measures the delay $\Delta_{ij}^{D}$ associated with a walk from $i$ to $D$ via the outgoing link $(i, j)$. This delay is more precisely the following. Let $\widetilde{J}_j^D$ denote a sample sum of the delays in the queues associated with the links, experienced by an FA packet moving from node $j$ to node $D$ (it is thus a sample of the

[4]routed with equal probability on each outgoing link

expected 'cost-to-go' from $j$ to $D$). Let $\widetilde{w}_{ij}$ denote a sample of the delay experienced by an FA packet traversing the link $(i,j)$. Then $\Delta_{ij}^D = \widetilde{w}_{ij} + \widetilde{J}_j^D$. When the corresponding BA packet comes back at node $i$ the delay information is used to update the estimate $X_{ij}^D$ of the mean delay using the simple exponential estimator

$$X_{ij}^D := X_{ij}^D + \epsilon(\Delta_{ij}^D - X_{ij}^D), \tag{1}$$

where $\epsilon \in (0,1)$ is a small constant. We also refer to $X_{ij}^D$ as *the mean delay estimate for the route* $i \to j \to \cdots \to D$. The mean delay estimates $X_{ik}^D$, corresponding to the other neighbors $k \in N(i,D)$, are left unchanged.

Simultaneously, the routing probabilities at $i$ are updated using the relation

$$\phi_{ij}^D = \frac{(X_{ij}^D)^{-\beta}}{\sum_{k \in N(i,D)} (X_{ik}^D)^{-\beta}}, \ \forall j \in N(i,D), \tag{2}$$

where $\beta$ is a constant positive integer. $\phi_{ij}^D$ is thus inversely proportional to $X_{ij}^D$. $\beta$ influences the extent to which outgoing links with lower delay estimates are favored compared to the ones with higher delay estimates.

We can interpret the quantity $(X_{ij}^D)^{-1}$ as analogous to a "pheromone trail or deposit" on the outgoing link $(i,j)$. This trail gets dynamically updated by the ant packets. The pheromone trail influences the routing tables through the relation (2). Equation (2) shows that the outgoing link $(i,j)$ is more desirable when $X_{ij}^D$, the delay through $j$, is smaller; in other words, when the pheromone deposit is higher, relative to the other routes.

## 3 Formulation of the Problem. The Acyclic Network Model

We consider the problem of routing from the various nodes $i$ of the network to a single destination node $D$. At every node $i$ there exist queues (buffers) $Q_{ij}$ associated with the outgoing links $(i,j)$; we assume these queues to be of infinite size. The service discipline in these queues is FIFO. The network can be thought of equivalently as a system of inter-connected queues (a queuing network). Every link $(i,j)$ has capacity $C_{ij}$. We assume that the queuing delays dominate the processing and propagation delays in the links. The latter delays can be accounted for with minimal changes in the discussion in the rest of the paper, but for simplicity, we assume that they are negligible.

We consider acyclic networks and define them following Bertsekas, Gallager [5]. A queue $Q_{ij}$ is said to be downstream with respect to a queue $Q_{kl}$ if some portion of the traffic through the latter queue flows through the former. An acyclic network is one for which it is not possible that simultaneously $Q_{ij}$ is downstream of $Q_{kl}$ and $Q_{kl}$ is downstream of $Q_{ij}$, for all $(i,j),(k,l)$. The set $N(i) = \{j : (i,j) \in \mathcal{L}\}$ denotes the set of downstream neighbors of $i$. An example of an acyclic network is given in Figure 5, pp. 31. We shall denote the routing probability entries of $\mathcal{R}(i)$ by $\phi_{ij}$ (i.e., without explicitly mentioning the destination). The mean delay estimate entries of $\mathcal{I}(i)$ are denoted by $X_{ij}$.

The general algorithm, as described in the previous Section 2, is asynchronous (and distributed). This is because the nodes launch the FA packets towards the destination in an unco-ordinated way. Moreover, there is a random delay as each FA-BA pair travels through the network. The learning algorithms at the nodes for updating $\mathcal{R}$ and $\mathcal{I}$ are thus triggered at random points of time (when BA packets come back). We consider a more simplified view of the algorithm operation, which is still asynchronous and distributed, retains the main characteristics and the essence of the algorithm, but is easier to analyze.

We assume that FA packets are generated according to a Poisson process of rate $\lambda_i^a > 0$ at node $i$ ($\lambda_D^a = 0$). We consider a model with the following assumptions on the algorithm operation.

($M1$) We assume that the BA packets take negligible time to travel back to the source nodes (from which the corresponding FA packets were launched) from destination $D$. Because BA packets are expected to travel back to the source through high priority queues, the delays might not be very significant, except for very large-sized networks with significant propagation delays. On the other hand, incorporating the effects of such delays into our model introduces additional complications related to asynchrony.

($M2$) Furthermore, we note that in the general algorithm operation, a BA packet updates the delay estimates at every node that it traverses on its way back to the source, besides the source itself. In what follows, we shall consider the more simplified algorithm operation, whereby only at the source node the delay estimates and the routing probabilities are updated.

We assume that data packets are generated according to a Poisson process of rate $\lambda_i^d \geq 0$ at node $i$; for some nodes it is possible that no data packets are generated, i.e., the rate is zero. For the destination, $\lambda_D^d = 0$.

Let $\{\alpha(m)\}_{m=1}^{\infty}$ denote the sequence of times at which FA packets are launched from the various nodes of the network. Let $\{\delta(n)\}_{n=1}^{\infty}$ denote the sequence of times at which FA packets arrive at the destination $D$ (we set $\alpha(0) = 0, \delta(0) = 0$). Because we have assumed that BA packets take negligible time to travel back to the source nodes, these are also the sequence of times at which BA packets come back to the source nodes. Consequently, these are the sequence of times at which algorithm updates are triggered at the various nodes. At time $\delta(n)$, let $X(n)$ and $\phi(n)$ denote, respectively, the vector of mean delay estimates and the vector of outgoing routing probabilities at the network nodes. The components of $X(n)$ and $\phi(n)$ are $X_{ij}(n), (i,j) \in \mathcal{L}$, and $\phi_{ij}(n), (i,j) \in \mathcal{L}$, respectively.

Thus, by time $\delta(n)$, overall $n$ BA packets will have come back to the network nodes. (At this point, it is useful to recall Assumption ($M2$)). Let $T(n)$ be the $\mathcal{N}$-valued random variable that indicates which node the $n$-th BA packet comes back to. Then $\xi_i(n) = \sum_{k=1}^{n} I_{\{T(k)=i\}}$ gives the number of BA packets that have come back at node $i$ by time $\delta(n)$[5]. Let $R_i(.)$ denote the routing decision variable for FA packets originating from node $i$. We say that the event $\{R_i(k) = j\}$ has occurred if the $k$-th FA packet that arrives at $D$ *and* that has been launched from $i$, has been routed via the outgoing link $(i,j)$. Let $\psi_{ij}(n) = \sum_{k=1}^{\xi_i(n)} I_{\{R_i(k)=j\}}$; $\psi_{ij}(n)$ gives the number of FA packets that arrive at node $D$ by time $\delta(n)$, having been launched from node $i$ and routed via $(i,j)$. By the zero delay assumption on the travel time of the BA packets and the assumption ($M2$) on algorithm operation, $\psi_{ij}(n)$ is also the number of BA packets that come back to $i$ via $j$, by time $\delta(n)$. Let $\{\Delta_{ij}(m)\}$ denote the sequence of delay measurements made by successive FA packets arriving at $D$ *and* that have been launched from node $i$, routed via the outgoing link $(i,j)$. This is also the sequence of delay measurements about the route $i \to j \to \cdots \to D$ made available to the source $i$ by the BA packets.

Lets suppose that at time $\delta(n)$ a BA packet comes back to node $i$. Furthermore, suppose that the corresponding FA packet was routed via the outgoing link $(i,j)$. When this BA packet comes back to node $i$, the delay estimate $X_{ij}$ is updated using an exponential estimator

$$X_{ij}(n) = X_{ij}(n-1) + \epsilon\Big(\Delta_{ij}(\psi_{ij}(n)) - X_{ij}(n-1)\Big), \tag{3}$$

with $\epsilon \in (0,1)$ being a small positive constant. The delay estimates $X_{ik}$ for the other routes $i \to k \to \cdots \to D$ $(k \in N(i), k \neq j)$ are left unchanged

$$X_{ik}(n) = X_{ik}(n-1). \tag{4}$$

Also, the delay estimates at the other network nodes do not change

$$X_{lp}(n) = X_{lp}(n-1), \quad \forall p \in N(l), \forall l \neq i. \tag{5}$$

[5] $I_A$ denotes the indicator random variable for the event $A$.

Also, as soon as the delay estimates are updated at node $i$, the outgoing routing probabilities are also updated

$$\phi_{ij}(n) = \frac{(X_{ij}(n))^{-\beta}}{\sum_{k \in N(i)} (X_{ik}(n))^{-\beta}}, \quad \forall j \in N(i). \tag{6}$$

The routing probabilities at the other nodes do not change.

In general thus the evolution of the delay estimates at the various nodes of the network can be described by the following set of stochastic iterative equations

$$\begin{aligned} X^{\epsilon}_{ij}(n) &= X^{\epsilon}_{ij}(n-1) + \epsilon \, I_{\{T^{\epsilon}(n)=i, R^{\epsilon}_i(\xi^{\epsilon}_i(n))=j\}} \Big( \Delta^{\epsilon}_{ij}(\psi^{\epsilon}_{ij}(n)) - X^{\epsilon}_{ij}(n-1) \Big), \\ & \qquad \forall (i,j) \in \mathcal{L}, n \geq 1, \end{aligned} \tag{7}$$

starting with the initial conditions $X^{\epsilon}_{ij}(0) = x_{ij}, \forall (i,j) \in \mathcal{L}$.

The routing probabilities are updated in the usual way

$$\phi^{\epsilon}_{ij}(n) = \frac{(X^{\epsilon}_{ij}(n))^{-\beta}}{\sum_{k \in N(i)} (X^{\epsilon}_{ik}(n))^{-\beta}}, \quad \forall (i,j) \in \mathcal{L}, n \geq 1, \tag{8}$$

starting with the initial conditions $\phi^{\epsilon}_{ij}(0) = \frac{(x_{ij})^{-\beta}}{\sum_{k \in N(i)} (x_{ik})^{-\beta}}$, $\forall (i,j) \in \mathcal{L}$. Though not explicitly mentioned, it is understood that there are no algorithm updates being made at $D$.

The $\epsilon$'s in the superscript in the algorithm update equations (7) and (8) above, recognize the dependence of the evolution of the quantities involved (for example, the delay estimates $X_{ij}$) on $\epsilon$. However, for most of the paper[6], we shall not use this notation; this enables the discussion to be less cumbersome. Also, we note that equations (7) and (8) describe the evolution of the delay estimates and the routing probabilities for the regular ARA as well as for the uniform ARA case.

We also introduce the following continuous time processes, $\{x(t), t \geq 0\}$ and $\{f(t), t \geq 0\}$, defined by the equations

$$\begin{aligned} x(t) &= X(n), \quad \text{for } \delta(n) \leq t < \delta(n+1), n = 0, 1, 2, \ldots, \\ f(t) &= \phi(n), \quad \text{for } \delta(n) \leq t < \delta(n+1), n = 0, 1, 2, \ldots. \end{aligned}$$

The components of $x(t)$ and $f(t)$ are denoted by $x_{ij}(t)$ and $f_{ij}(t)$, respectively.

In the case of regular ant routing, an ant (FA) packet as well as a data packet are routed at an intermediate node based on the current routing probabilities at the node. Thus, in view of the discussion in this section, a packet that arrives at node $i$ at time $t$, is routed according to the

[6] except when we are required to be more clear and precise

routing probabilities $f_{ij}(t), j \in N(i)$, and joins the corresponding queues. In the case of uniform ant routing, a data packet arriving at $i$ at time $t$, is routed according to the probabilities $f_{ij}(t), j \in N(i)$; an ant packet arriving at $t$ is routed uniformly (see Figure 2).

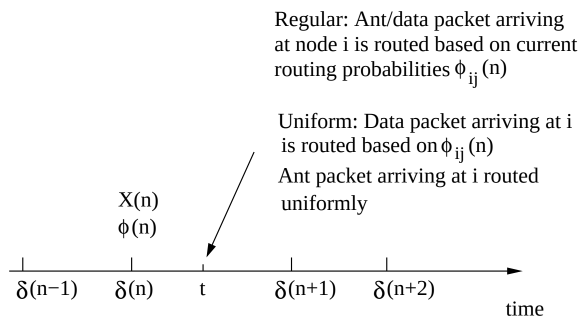


Figure 2: Routing of packet arrivals at a node at time $t$. Sequence $\{\delta(n)\}$ are the times at which algorithm updates are taking place.

# 4 Analysis of the Routing Algorithm

We view the routing algorithm, consisting of equations (7) and (8), as a set of discrete stochastic iterations of the type usually considered in the literature on stochastic approximation methods [18]. We provide below the main convergence result which states that, when $\epsilon$ is small enough, the evolution of the vector of delay estimates is closely tracked by a system of Ordinary Differential Equations (ODEs).

## 4.1 The ODE approximation

The key observation, which simplifies the analysis of the algorithm, is that there is a time-scale decomposition when $\epsilon > 0$ is small enough — the delay estimates $X_{ij}$ then evolve much more slowly compared to the delay processes $\Delta_{ij}$. The probabilities $\phi_{ij}$ also evolve at the same "time-scale" as the delay estimates (probabilities $\phi_{ij}$ are continuous functions of the delay estimates $X_{ij}$). Consequently, when $\epsilon$ is small enough, with the vector of delay estimates $X$ considered fixed at $z$ (equivalently the vector of routing probabilities fixed at $\phi$, the components of $\phi$ being $\phi_{ij} = \frac{(z_{ij})^{-\beta}}{\sum_{k \in N(i)} (z_{ik})^{-\beta}}$), the delay processes $\{\Delta_{ij}(.)\}$ converge to a stationary distribution, which is dependent on $z$. Given the routing probabilities $\phi_{ij}, (i,j) \in \mathcal{L}$, and a knowledge of the rates of incoming traffic streams into the queuing network, enable us to determine the total incoming arrival rates into each of the queues $Q_{ij}$. This can be done by simply solving the flow balance equations; see Bertsekas, Gallager [5], Mitrani [20]. We assume that the total arrival rate into each queue (assumed earlier to be of infinite size) is smaller than the service rate of packets in the queue. This assumption (a queue stability assumption) then ensures that, with the delay estimate vector $X$ considered fixed at $z$, the delay processes $\{\Delta_{ij}(.)\}$ converge to a stationary distribution, which depends on $z$. We denote the means under stationarity, for each $(i,j) \in \mathcal{L}$, by $D_{ij}(z)$ ($D^U_{ij}(z)$ for the uniform ant case), which is a finite quantity. We also make the following short note. $\{\Delta_{ij}(m)\}$ was defined to be the sequence of delay measurements made by successive FA packets *arriving at* $D$, *that have been launched from* $i$ *and routed via* $(i,j)$. When the delay estimate vector is considered fixed at $z$, its average under stationarity is denoted by $D_{ij}(z)$. Also, with the delay estimate vector considered fixed at $z$, the sequence, denoted by (say) $\{\Delta'_{ij}(m)\}$, of delay measurements made by successive FA packets *launched from* $i$ *and routed via* $(i,j)$, has the same stationary average $D_{ij}(z)$. This is because the latter sequence is just a rearrangement of the former, and hence the average is the same.

Also, when $X$ is considered fixed at $z$, let $\zeta_i(z), i \in \mathcal{N}$, ($\zeta^U_i(z)$ for the uniform ants case) denote, under stationarity, the long-term fraction of FA packets arriving at $D$ that have been launched from $i$. $\zeta_i(z)$ assumes values from the set $(0,1)$ ($\zeta_D(z) = 0, \zeta^U_D(z) = 0$).

Furthermore, when $\epsilon$ is small, the evolution of the vector of delay estimates can be tracked by an ODE system (an ODE approximation result). This result is shown in Section 8.1 of the Appendix. We now introduce some additional notation and state the assumptions under which this result holds. For any fixed $\epsilon \in (0,1)$, and for each $(i,j)$, consider the piecewise constant interpolation of $X^\epsilon_{ij}(n)$ given by

$$z^\epsilon_{ij}(t) = X^\epsilon_{ij}(n), \quad n\epsilon \le t < (n+1)\epsilon, \ n = 0,1,2,\ldots, \tag{9}$$

with the initial value $z^\epsilon_{ij}(0) = X^\epsilon_{ij}(0)$. Consider also the vector-valued piecewise constant process $z^\epsilon(t)$, for all $t \ge 0$, with components $z^\epsilon_{ij}(t)$, $(i,j) \in \mathcal{L}$. Let us now consider the increasing sequence of $\sigma$-fields $\{\mathcal{F}^\epsilon(n)\}$, where $\mathcal{F}^\epsilon(n)$ encapsulates the entire history of the algorithm for the time $t \le \delta(n)$. In particular, it contains the $\sigma$-field generated by the r.v.'s $X^\epsilon(0), X^\epsilon(1), \ldots, X^\epsilon(n)$. It also contains information regarding the arrival and packet service times, as well as information regarding the actual routing of packets. The ODE approximation result will be shown to hold under the following assumptions.

**Assumptions**:

$(A1)$ For every $(i,j) \in \mathcal{L}$, and for every $\epsilon \in (0,1)$, the sequence $\{\Delta^{\epsilon}_{ij}(m)\}$ is uniformly integrable; that is, $\sup_{m\geq 1} E[\Delta^{\epsilon}_{ij}(m) I_{\{\Delta^{\epsilon}_{ij}(m)\geq K\}}] \to 0$, as $K \to \infty$.

**Regular Ant case**.

$(A2)$ If $X(n)$ is held fixed at a value $z$ ($\phi(n)$ is then fixed at a value $\phi$; $\phi$ has components $\phi_{ij} = \frac{(z_{ij})^{-\beta}}{\sum_{k\in N(i)}(z_{ik})^{-\beta}}$) then, for every $l \geq 0$, and for every $(i,j) \in \mathcal{L}$, we have

$$\lim_{r\to\infty} \frac{\sum_{m=l+1}^{l+r} E[I_{\{T(m)=i, R_i(\xi_i(m))=j\}} \Delta_{ij}(\psi_{ij}(m))/\mathcal{F}(m-1)]}{r} = \zeta_i(z)\phi_{ij} D_{ij}(z) \quad \text{a.s.}, \tag{10}$$

$$\lim_{r\to\infty} \frac{\sum_{m=l+1}^{l+r} E[I_{\{T(m)=i, R_i(\xi_i(m))=j\}} /\mathcal{F}(m-1)]}{r} = \zeta_i(z)\phi_{ij} \quad \text{a.s.} \tag{11}$$

The quantities $T(n), R_i(n), \Delta_{ij}(n)$, as well as the sequence $\{\mathcal{F}(n)\}$ that appear in the equations above are defined in a similar way as for the case when the delay estimate vector $X$ is time-varying.

$(A3)$ We assume that the quantities $\zeta_i(z)\phi_{ij}D_{ij}(z)$ and $\zeta_i(z)\phi_{ij}$ are continuous functions of $z$.

**Uniform Ant case**.

$(A2')$ If $X(n)$ is held fixed at a value $z$ then, for every $l \geq 0$, and for every $(i,j) \in \mathcal{L}$, we have

$$\lim_{r\to\infty} \frac{\sum_{m=l+1}^{l+r} E[I_{\{T(m)=i, R_i(\xi_i(m))=j\}} \Delta_{ij}(\psi_{ij}(m))/\mathcal{F}(m-1)]}{r} = \frac{\zeta^U_i(z) D^U_{ij}(z)}{|N(i)|} \quad \text{a.s.}, \tag{12}$$

$$\lim_{r\to\infty} \frac{\sum_{m=l+1}^{l+r} E[I_{\{T(m)=i, R_i(\xi_i(m))=j\}} /\mathcal{F}(m-1)]}{r} = \frac{\zeta^U_i(z)}{|N(i)|} \quad \text{a.s..} \tag{13}$$

$(A3')$ We assume that the quantities $\zeta^U_i(z)D^U_{ij}(z)$ and $\zeta^U_i(z)$ are continuous functions of $z$.

Under the above assumptions, in Section 8.1 it is shown that the process $\{z^{\epsilon}(t), t \geq 0\}$ converges weakly to a (deterministic) process $\{z(t), t \geq 0\}$ as $\epsilon \downarrow 0$. For the regular ARA case, $z(t)$, whose components are $z_{ij}(t), (i,j) \in \mathcal{L}$, is a solution of the ODE system

$$\frac{dz_{ij}(t)}{dt} = \frac{\zeta_i(z(t))(z_{ij}(t))^{-\beta}\Big(D_{ij}(z(t)) - z_{ij}(t)\Big)}{\sum_{k\in N(i)}(z_{ik}(t))^{-\beta}}, \qquad \forall (i,j) \in \mathcal{L}, \quad t > 0, \tag{14}$$

with initial conditions given by $z_{ij}(0) = x_{ij}$, $\forall (i,j) \in \mathcal{L}$. We denote the right hand side of ODE (14) by the function $F_{ij}$; $F_{ij}(z(t)) = \frac{\zeta_i(z(t))(z_{ij}(t))^{-\beta}\Big(D_{ij}(z(t)) - z_{ij}(t)\Big)}{\sum_{k\in N(i)}(z_{ik}(t))^{-\beta}}$.

For the uniform ant case, $z(t)$, with components $z_{ij}(t), (i,j) \in \mathcal{L}$, is a solution of the ODE system

$$\frac{dz_{ij}(t)}{dt} = \frac{\zeta_i^U(z(t))\Big(D_{ij}^U(z(t)) - z_{ij}(t)\Big)}{|N(i)|}, \qquad \forall (i,j) \in \mathcal{L}, \quad t > 0, \tag{15}$$

with initial conditions given by $z_{ij}(0) = x_{ij}$, $\forall (i,j) \in \mathcal{L}$.

We now briefly discuss the assumptions. A sufficient condition under which $(A1)$ holds is $\sup_{n\geq 1} E\big[\big(\Delta_{ij}^{\epsilon}(n)\big)^{\gamma+1}\big] < \infty$, for some $\gamma > 0$. That is, some moment of the delay higher than the first moment is finite, which we assume. Assumptions $(A2)$ and $(A3)$ can be expected to hold, because they are forms of the strong law of large numbers (they are somewhat weaker because the terms involve conditional expectations). Similar remarks apply for Assumptions $(A2')$ and $(A3')$.

The dynamic behavior of the routing algorithm can be studied via the ODE approximation. Numerical solution of the ODE, starting from given initial conditions, requires computation of the means $D_{ij}(z)$ and the fractions $\zeta_i(z)$ (respectively, $D_{ij}^U(z)$ and $\zeta_i^U(z)$ for the uniform ants case), for given $z$. These computations depend upon the particular network under consideration. In the next subsection, we discuss how to compute these quantities under our assumptions on the statistics of the packet arrival processes and service times of the ant and data streams.

## 4.2 Computations related to the ODE approximation

We assume that, in every queue $Q_{ij}$ the successive service times of both ant (FA) and data packets are i.i.d. exponentially distributed with the same mean $\frac{1}{C_{ij}}$ [7]. Furthermore, the service times at each queue are also independendent of the service times at all other queues, and also independent of the arrival processes at the nodes. (We had assumed earlier that the arrival processes to the network are all Poisson.) These assumptions are the usual assumptions made for open Jackson networks, and enable us to remain within the realm of solvable models; see, for example, Bertsekas and Gallager [5] and Mitrani [20].

**Regular Ant case**. In this case, because ant and data packets are being routed in an identical fashion, we have a single class open Jackson network. Given $z$, we can compute the routing probabilities $\phi_{ij}, (i,j) \in \mathcal{L}$. The routing probabilities combined with a knowledge of the rates of the incoming streams (ant, data) into the network, enable us to determine the total arrival rate

[7] This amounts to assuming that the average length of a packet (ant or data) is one unit. This is not a restriction, and we can consider the general case by simply multiplying by the average length. However, both ant and data packets must have the same average length.

$A_{ij}(z)$ into each queue $Q_{ij}$. This can be done by simply solving the flow balance equations in the network. For each $(i,j) \in \mathcal{L}$, we assume that $A_{ij}(z) < C_{ij}$, the arrival rate is smaller than the service rate. Then, under our assumptions, there is a unique joint stationary distribution of the random variables denoting the total number of packets in the queues $Q_{ij}, (i,j) \in \mathcal{L}$. Moreover, this stationary distribution is of a product form. Also, we can compute various quantities of interest to us, like average stationary delays in the queues [5], [20]. Let $w_{ij}(z)$ denote the average stationary delay (sojourn time) in queue $Q_{ij}$, and let $J_j(z)$ denote the average stationary delay (expected 'cost-to-go') from node $j$ to the destination $D$, both experienced by an ant packet. $w_{ij}(z)$ is given by the formula, $w_{ij}(z) = \frac{1}{C_{ij} - A_{ij}(z)}$. The quantities $J_i(z), i \in \mathcal{N}$, satisfy the following equations

$$\begin{aligned} J_i(z) &= \sum_{j \in N(i)} \phi_{ij}\Big(w_{ij}(z) + J_j(z)\Big), \quad \forall i \in \mathcal{N}, i \neq D, \\ J_D(z) &= 0. \end{aligned} \tag{16}$$

Once these equations are solved for $J_i(z), i \in \mathcal{N}$, we can compute the quantities $D_{ij}(z), (i,j) \in \mathcal{L}$, using the relations

$$D_{ij}(z) = w_{ij}(z) + J_j(z). \tag{17}$$

Because ants are generated as a Poisson process with rates $\lambda_i^a$ at each node $i$, and because of Assumption $(M2)$, the fraction $\zeta_i(z) = \frac{\lambda_i^a}{\sum_{j \in \mathcal{N}} \lambda_j^a}$ (see Section 8.2 for a detailed argument).

**Uniform Ant case**. In this case, the FA packets and the data packets are routed differently. We thus have an open Jackson network with two classes of traffic, the first class consisting of the ant traffic and the second class of data traffic. Separate flow balance equations are set up for the two classes of traffic. These flow balance equations enable us to solve for the arrival rates $A_{ij}^a(z)$ and $A_{ij}^d(z)$ of ant and data packets into each queue $Q_{ij}$. The total arrival rate $A_{ij}(z)$ into $Q_{ij}$ is simply given by the sum $A_{ij}^a(z) + A_{ij}^d(z)$. The average stationary delay $w_{ij}^U(z)$ in $Q_{ij}$ is then given by $w_{ij}^U(z) = \frac{1}{C_{ij} - A_{ij}(z)}$. The rest of the computations which lead to the determination of the quantities $D_{ij}^U(z), (i,j) \in \mathcal{L}$, can be done in a similar manner (with modifications that are straightforward) as for the regular ants case. Again, because ant packets are generated as a Poisson process at all nodes, and because of Assumption $(M2)$, the fraction $\zeta_i^U(z) = \frac{\lambda_i^a}{\sum_{j \in \mathcal{N}} \lambda_j^a}$.

With the knowledge of the quantities $D_{ij}(z), (i,j) \in \mathcal{L}$, and $\zeta_i(z), i \in \mathcal{N}$ (respectively, $D_{ij}^U(z)$ and $\zeta_i^U(z)$ for the uniform ant case), we can numerically solve ODE (14) (respectively, (15) for the uniform ant case), starting from an initial condition: $z_{ij}(0), (i,j) \in \mathcal{L}$.

### 4.3 Equilibrium Behavior of the Routing Algorithm

We now study the equilibrium behavior of the routing algorithm. We denote the equilibrium values of the various quantities by attaching a $*$ to the superscript.

**Regular Ant case**. Consider the equilibrium points $z^*$ of the ODE system (14). Because the $\zeta_i(z^*)$ are all positive, the points $z^*$ with components $z^*_{ij}$ satisfy the equations

$$\frac{(z^*_{ij})^{-\beta}}{\sum_{k\in N(i)}(z^*_{ik})^{-\beta}}\cdot\Big(D_{ij}(z^*)-z^*_{ij}\Big)=0,\quad \forall(i,j)\in\mathcal{L}. \tag{18}$$

The interpolated delay estimate vector $z^\epsilon(t)$ approaches the set of equilibrium points $z^*$ asymptotically as $\epsilon\to 0$. More precisely, if $E$ denotes the set of equilibrium points and $N_\delta(E)$ denotes a small enough, $\delta$-neighborhood of $E$, then asymptotically (as $t\to\infty$), the fraction of time $z^\epsilon(t)$ spends in $N_\delta(E)$ goes to one in probability, as $\epsilon\to 0$ (see Kushner, Yin [18]). The vector of routing probabilities $\phi^\epsilon(n)$, being a continuous function of the delay estimate, asymptotically approaches the set of points $\phi^*$ with components $\phi^*_{ij}=\frac{(z^*_{ij})^{-\beta}}{\sum_{k\in N(i)}(z^*_{ik})^{-\beta}}$, $\forall(i,j)\in\mathcal{L}$ (the meaning of the term 'asymptotically approaches' is the same as described above for the delay estimate vector). In the discussion for the rest of this section, we shall refer to the quantity $z^*_{ij}$ as an equilibrium delay estimate, and $\phi^*_{ij}$ as an equilibrium routing probability, it being understood that the delay estimate $z^\epsilon_{ij}(t)$ and the routing probability $\phi^\epsilon_{ij}(n)$ are asymptotically very close to these quantities with probability close to one, when $\epsilon$ is chosen small enough.

Under our assumption that the total arrival rate into every queue is smaller than the packet service rate, the equilibrium delay estimates are finite, and so the equilibrium routing probabilities must be all positive. Consequently, the above equations (18) reduce to: $D_{ij}(z^*)=z^*_{ij},\forall(i,j)\in\mathcal{L}$. Now, denoting the functional dependence of the mean stationary delays on the routing probabilities also by $D_{ij}(\phi)$ (a slight abuse of notation), and noting that $\phi^*_{ij}=\frac{(z^*_{ij})^{-\beta}}{\sum_{k\in N(i)}(z^*_{ik})^{-\beta}}$, $\forall(i,j)\in\mathcal{L}$, we find that the equilibrium routing probabilities must satisfy the following fixed-point system of equations

$$\phi^*_{ij}=\frac{(D_{ij}(\phi^*))^{-\beta}}{\sum_{k\in N(i)}(D_{ij}(\phi^*))^{-\beta}},\quad \forall(i,j)\in\mathcal{L}. \tag{19}$$

We now check that, for a vector $\phi^*$, there is a unique vector with components $D_{ij}(\phi^*),(i,j)\in\mathcal{L}$. To that end, we first notice that, for every $(i,j)\in\mathcal{L}$,

$$D_{ij}(\phi^*)=w_{ij}(\phi^*)+J_j(\phi^*), \tag{20}$$

where $J_j(\phi^*)$ is the expected delay (expected 'cost-to-go') from node $j$ to destination $D$ experienced by an FA packet when the routing probability vector is $\phi^*$; $J_D(\phi^*)=0$. $w_{ij}(\phi^*)$ is the expected delay along link $(i,j)$ experienced by an FA packet when the routing probability vector is $\phi^*$; we assume for a given $\phi^*$, $w_{ij}(\phi^*)$ is unique [8]. The quantities $J_i(\phi^*), i \in \mathcal{N}$, satisfy the following equations

$$\begin{aligned} J_i(\phi^*) &= \sum_{j \in N(i)} \phi^*_{ij}\Big(w_{ij}(\phi^*) + J_j(\phi^*)\Big), \ \forall i \in \mathcal{N}, \quad i \neq D, \\ J_D(\phi^*) &= 0. \end{aligned} \tag{21}$$

Because our equilibrium probabilities $\phi^*_{ij}$ are all positive, there exists a path from every node $i$ to the destination $D$ consisting of a sequence of links $(i,k_1)$, …, $(k_n,D)$ for which $\phi^*_{ik_1} > 0$, …, $\phi^*_{k_nD} > 0$. Then, the above set of equations (21) have a unique solution (vector) $J(\phi^*)$, which has components $J_i(\phi^*), i \in \mathcal{N}$ (see Bertsekas and Tsitsiklis [6]). Taking note of this and relation (20), we see that for every vector $\phi^*$, there is a unique vector of delays $D_{ij}(\phi^*), (i,j) \in \mathcal{L}$.

Also, for any $(i,j) \in \mathcal{L}$, $D_{ij}(\phi^*)$ is a continuous function of the probabilities. (Furthermore, being at least equal to the average service time experienced by an FA packet in the queue $Q_{ij}$, it is lower bounded by a positive quantity.) Then, by an application of Brouwer's fixed-point theorem, there exists a vector of equilibrium routing probabilities $\phi^*$ satisfying the fixed-point system (19) (the right hand side of the fixed-point system maps a compact, convex set — a Cartesian product of probability simplices — to itself).

**Uniform Ant case**. For the uniform ant case, at equilibrium, the components $z^*_{ij}$ satisfy the following equations

$$\frac{\Big(D^U_{ij}(z^*) - z^*_{ij}\Big)}{|N(i)|} = 0, \quad \forall (i,j) \in \mathcal{L}. \tag{22}$$

We can show in a manner similar to the regular ant case, that the equilibrium routing probabilities must be all positive and satisfy the fixed-point system of equations

$$\phi^*_{ij} = \frac{\left(D^U_{ij}(\phi^*)\right)^{-\beta}}{\sum_{k \in N(i)} \left(D^U_{ij}(\phi^*)\right)^{-\beta}}, \quad \forall (i,j) \in \mathcal{L}. \tag{23}$$

Also, we can show that, for a vector of equilibrium routing probabilities $\phi^*$ there is a unique vector with components $D^U_{ij}(\phi^*), (i,j) \in \mathcal{L}$. Also there exists a solution to the set of fixed-point equations (23), by an application of Brouwer's fixed-point theorem.

[8] We have a similar abuse of notation for $w_{ij}$ and $J_j$ as we had for $D_{ij}$. In the previous Section 4.2, we had denoted by $w_{ij}(z)$ and $J_j(z)$ the average stationary delay in queue $Q_{ij}$, and the average stationary delay (expected 'cost-to-go') from node $j$ to destination $D$, both experienced by an FA packet, with delay estimate vector considered fixed at $z$.

# 5 Example: The N Parallel Links Case

In this section we consider the special case involving a simple routing scenario where arriving traffic at a single source node $S$ has to be routed to the single destination $D$. There are $N$ available parallel links between the source and the destination through which traffic can be routed. The network and its equivalent queuing theoretic model are shown in Figures 3 and 4 respectively. The queues represent the output buffers at the source and are associated with the $N$ outgoing links. We have more detailed results for this example that explore properties of the routing algorithm. In particular, we study the dependence of the equilibrium routing probabilities on capacities of the $N$ links and the effect of parameter $\beta$ on the equilibrium routing behavior. In this special case, packet service times are allowed to be generally distributed. In Section 5.1 we discuss in detail the regular ARA case, and in Section 5.2 we focus on the uniform ARA case.

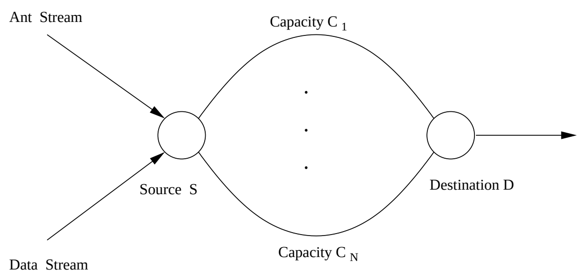


Figure 3: The network with $N$ parallel links

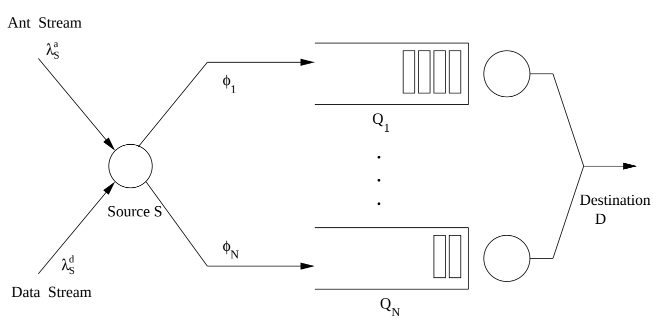


Figure 4: $N$ parallel links: The queueing theoretic model

### 5.1 The Regular Ant case

An ant and a data stream arrive at $S$ as Poisson processes with rates $\lambda_S^a > 0$ and $\lambda_S^d > 0$. At node $S$, every packet of the combined stream is routed according to the current routing probabilities towards queues $Q_1, \ldots, Q_N$. Samples of delays in the $N$ queues are collected by ant (FA) packets as they traverse through the queues along with data packets. The packet lengths of the combined stream constitute an i.i.d. sequence, which is also statistically independent of the arrival processes. The capacity of link $i$ is $C_i$ bits/sec ($i = 1, \ldots, N$). We assume that the lengths of an ant and a data packet are generally distributed with means $L_a$ and $L_d$ bits, respectively. If we denote the service times of an ant and a data packet in $Q_i$ by the generic random variables $S_i^a$ and $S_i^d$, then $S_i^a$ and $S_i^d$ are generally distributed (according to some c.d.f.'s, say $G_i^a$ and $G_i^d$) with means $E[S_i^a] = \frac{L_a}{C_i}$ and $E[S_i^d] = \frac{L_d}{C_i}$, respectively.

The delay estimation and routing probability update algorithms are special cases of the general update algorithms (7) and (8), and are hence not written down again here. The delay estimate vector $X(n)$ has components $X_1(n), \ldots, X_N(n)$ (corresponding to the $N$ links/queues), and similarly the routing probability vector $\phi(n)$ has components $\phi_1(n), \ldots, \phi_N(n)$.

#### 5.1.1 The ODE approximation

The ODE approximation result (14) specialized to the $N$ parallel links case reads as follows

$$\begin{aligned} \frac{dz_1(t)}{dt} &= \frac{(z_1(t))^{-\beta}\Big(D_1(z_1(t), \ldots, z_N(t)) - z_1(t)\Big)}{\sum_{k=1}^{N} (z_k(t))^{-\beta}}, \\ \vdots \quad & \quad \vdots \\ \frac{dz_N(t)}{dt} &= \frac{(z_N(t))^{-\beta}\Big(D_N(z_1(t), \ldots, z_N(t)) - z_N(t)\Big)}{\sum_{k=1}^{N} (z_k(t))^{-\beta}}, \end{aligned} \tag{24}$$

with the initial conditions $z_1(0) = X_1(0), \ldots, z_N(0) = X_N(0)$ [9]. Notice that clearly, $\zeta_S(z(t)) = 1$, and so this term is not explicitly mentioned in the ODE above. As usual, $D_i(z_1, \ldots, z_N), i = 1, \ldots, N$, are the mean delays (sojourn times) in the queues under stationarity as experienced by ant packets, with the delay estimates considered fixed at $z_1, \ldots, z_N$.

In order to numerically solve the ODE, we need to compute the quantities $D_i(z_1, \ldots, z_N)$ for our queuing system. With the delay estimates considered fixed at $z_1, \ldots, z_N$, the routing probabilities

[9] Instead of using the notation $z_{jD}(t), D_{jD}(z(t))$, we employ the simpler notation $z_j(t), D_j(z(t))$.

are given by $\phi_i = \frac{(z_i)^{-\beta}}{\sum_{k=1}^{N}(z_k)^{-\beta}}, i = 1, \ldots, N$. We now discuss how to compute the $D_i(z_1, \ldots, z_N)$'s given our assumptions on the statistics of the packet arrival processes and the packet lengths.

Every arrival at $S$ from either of the Poisson streams (ant or data) is routed independent of other arrivals with probability $\phi_i$ towards queue $Q_i$. Thus the incoming arrival process in $Q_i$ is a superposition of two independent Poisson processes with rates $\lambda_S^a \phi_i$ and $\lambda_S^d \phi_i$. Consequently, every incoming packet into $Q_i$ is, with probability $\frac{\lambda_S^a}{\lambda_S^a + \lambda_S^d}$ an ant packet, and with probability $\frac{\lambda_S^d}{\lambda_S^a + \lambda_S^d}$ a data packet. Also, under our assumptions, the queues evolve as independent $M/G/1$ queues. The cumulative incoming stream into $Q_i$ is Poisson with rate $(\lambda_S^a + \lambda_S^d)\phi_i$, and every incoming packet's service time is distributed according to the c.d.f. $G_i^a$ w.p. $\frac{\lambda_S^a}{\lambda_S^a + \lambda_S^d}$ and according to the c.d.f. $G_i^d$ w.p. $\frac{\lambda_S^d}{\lambda_S^a + \lambda_S^d}$. We assume that the input arrival rate is smaller than the service rate: $(\lambda_S^a + \lambda_S^d)\phi_i E[S_i] < 1, i = 1, \ldots, N$ (the queue stability condition). $E[S_i]$, the mean packet service time in $Q_i$, is given by $E[S_i] = \frac{\lambda_S^a E[S_i^a] + \lambda_S^d E[S_i^d]}{\lambda_S^a + \lambda_S^d}$. We further note that the average sojourn time experienced by an ant arrival to $Q_i$ is the same as the average sojourn time of packets in $Q_i$ by the PASTA (Poisson Arrival See Time Averages) property. Thus, using the Pollaczek-Khinchin formula for the average sojourn time, we obtain the following expression for $D_i(z_1, \ldots, z_N)$ $(i = 1, \ldots, N)$

$$D_i(z_1, \ldots, z_N) = E[S_i] + \frac{(\lambda_S^a + \lambda_S^d)\phi_i E[S_i^2]}{2\Big(1 - (\lambda_S^a + \lambda_S^d)\phi_i E[S_i]\Big)}, \tag{25}$$

where $E[S_i]$ and $E[S_i^2]$ are given by $E[S_i] = \frac{\lambda_S^a E[S_i^a] + \lambda_S^d E[S_i^d]}{\lambda_S^a + \lambda_S^d}$, $E[S_i^2] = \frac{\lambda_S^a E[(S_i^a)^2] + \lambda_S^d E[(S_i^d)^2]}{\lambda_S^a + \lambda_S^d}$, and $\phi_i = \frac{(z_i)^{-\beta}}{\sum_{k=1}^{N}(z_k)^{-\beta}}$.

Once the expressions for $D_i(z_1, \ldots, z_N)$ are available, we can numerically solve the ODE system (24), starting with initial conditions $z_1(0)$, ..., $z_N(0)$. We observe in our simulations that if we start the system with initial conditions such that the input arrival rate is smaller than the service rate for each queue, this condition is satisfied thereafter during the evolution of the system.

### 5.1.2 Equilibrium Behavior of the Routing Algorithm

For our present case, the fixed-point system of equations (19) reduce to

$$\phi_i^* = \frac{\big(D_i(\phi_i^*)\big)^{-\beta}}{\sum_{k=1}^{N}\big(D_k(\phi_k^*)\big)^{-\beta}}, \quad i = 1, \ldots, N. \tag{26}$$

The equilibrium routing probabilities are positive and are given by the solutions to the above equations (26). The equilibrium routing probabilities must also satisfy the conditions $(\lambda_S^a +$

$\lambda_S^d)\phi_i^* E[S_i] < 1, i = 1, \ldots, N$. We now show that the equations (26) form the necessary and sufficient optimality conditions for an optimization problem involving the minimization of a convex objective function of $(\phi_1, \ldots, \phi_N)$ subject to the above mentioned constraints. A consequence of this fact is that, if there exists a solution to equations (26) that also satisfies the mentioned constraints, then such a solution is unique.

Consider the optimization problem

$$\begin{gathered}\text{Minimize } F(\phi_1, \ldots, \phi_N) = \textstyle\sum_{i=1}^{N} \int_0^{\phi_i} x[D_i(x)]^\beta dx, \\ \text{subject to } \phi_1 + \cdots + \phi_N = 1, \\ 0 < \phi_1 < a_1, \\ \vdots \\ 0 < \phi_N < a_N,\end{gathered}$$

where $a_i = \frac{1}{(\lambda_S^a + \lambda_S^d) E[S_i]}, i = 1, \ldots, N$.

The cost function (a function of the delays) is a measure of congestion in the $N$ links. The feasible set $C \subset \mathbb{R}^N$ of the above optimization problem is convex. It is possible that the set $C$ is empty (for a given set of values of $\lambda_S^a$, $\lambda_S^d$, and $E[S_i^a], E[S_i^d]$,$i = 1, \ldots, N$), which means that there are no feasible solutions to the optimization problem in such a case. We assume in what follows that $C$ is nonempty.

We assume that functions $D_i(x)$ are positive, differentiable and monotonically increasing on their domains of definition. This is true in most cases of interest, because when the routing probability for an outgoing link increases, the amount of traffic flow into that link also increases, resulting in an increase of the delay. We then have the following easy proposition

**Proposition 1.** *Given the above assumption on delay functions $D_i(x), i = 1, \ldots, N$, a probability vector $\phi^*$ is a local minimum of $F$ over $C$ if and only if $\phi^*$ satisfies the fixed-point system* (26)*. $\phi^*$ is then also the unique global minimum of $F$ over $C$.*

**Proof:** The Hessian of $F$ is a diagonal matrix given by

$$\nabla^2 F(\phi_1, \ldots, \phi_N) = \text{diag}\Big([D_i(\phi_i)]^{\beta-1}\{D_i(\phi_i) + \beta\phi_i D_i'(\phi_i)\}\Big), \tag{27}$$

where $D_i'(.)$ denotes the derivative of $D_i(.)$. Under above assumptions on $D_i(x)$'s, $\nabla^2 F(\phi_1, \ldots, \phi_N)$ is positive definite over $C$, and so $F$ is a strictly convex function on $C$. Consequently, any local minimum of $F$ is also a global minimum of $F$ over $C$; furthermore, there is atmost one such global minimum [4].

If $\phi^* = (\phi_1^*, \ldots, \phi_N^*)$ is a local minimum of $F$ over $C$, we must have (Proposition 2.1.2 of Bertsekas [4])

$$\sum_{i=1}^{N} \frac{\partial F}{\partial \phi_i}(\phi^*)(\phi_i - \phi_i^*) \geq 0, \quad \forall \phi \in C. \tag{28}$$

Let us fix a pair of indices $i, j, i \neq j$. Then choose $\phi_i = \phi_i^* + \delta$ and $\phi_j = \phi_j^* - \delta$, and let $\phi_k = \phi_k^*, \forall k \neq i, j$. Now, choosing $\delta > 0$ small enough that the vector $\phi = (\phi_1, \ldots, \phi_N)$ is also in $C$, the above condition becomes

$$\begin{aligned} \Big(\frac{\partial F}{\partial \phi_i}(\phi^*) - \frac{\partial F}{\partial \phi_j}(\phi^*)\Big)\delta &\geq 0, \\ \text{or} \qquad \phi_i^*[D_i(\phi_i^*)]^\beta &\geq \phi_j^*[D_j(\phi_j^*)]^\beta. \end{aligned}$$

By a similar argument, we can show that $\phi_j^*[D_j(\phi_j^*)]^\beta \geq \phi_i^*[D_i(\phi_i^*)]^\beta$. Thus, the necessary conditions for $\phi^*$ to be a local minimum are

$$\phi_1^*[D_1(\phi_1^*)]^\beta = \ldots = \phi_N^*[D_N(\phi_N^*)]^\beta.$$

Combining this with the normalization condition, $\phi_1^* + \cdots + \phi_N^* = 1$, gives us the system of equations (26).

The necessary conditions above can also be written in the form

$$\frac{\partial F}{\partial \phi_1}(\phi^*) = \ldots = \frac{\partial F}{\partial \phi_N}(\phi^*).$$

We check that these conditions are also sufficient for $\phi^*$ to be a local minimum. Suppose $\phi^* \in C$ satisfies the above conditions. Then for every other vector $\phi \in C$, we have $\sum_{i=1}^{N}(\phi_i - \phi_i^*) = 0$. So, the quantity

$$\sum_{i=1}^{N} \frac{\partial F}{\partial \phi_i}(\phi^*)(\phi_i - \phi_i^*) = \frac{\partial F}{\partial \phi_1}(\phi^*) \sum_{i=1}^{N}(\phi_i - \phi_i^*) = 0.$$

Then, because $F$ is convex over $C$, by Proposition 2.1.2 of Bertsekas [4], $\phi^*$ is a local minimum. □

In our case, it is easy to check that functions $D_i(x)$, are positive, differentiable, and monotonically increasing. Thus, if there is an equilibrium routing probability vector satisfying fixed-point system (26), then such a vector is unique.

We have carried out a discrete event simulation of the queuing system. We present here a result with the number of parallel links $N = 3$. The step size $\epsilon = 0.002$ and $\beta = 1$. The ant and data arrival processes are Poisson with rates $\lambda_S^a = 1$ and $\lambda_S^d = 1$. For ant packets, service times in the queues are exponential with means $E[S_1^a] = 1/3.0$, $E[S_2^a] = 1/4.0$ and $E[S_3^a] = 1/5.0$. For data packets also, queue service times are exponential with the same means. Initial values of the delay estimates are set at $X_1(0) = 0.8$, $X_2(0) = 2.8$, and $X_3(0) = 5.6$. With the initial delay estimates set as above, the initial routing probabilities are $\phi_1(0) = 0.7$,$\phi_2(0) = 0.2$, and $\phi_3(0) = 0.1$. In general, we choose our initial delay estimates in a way that we satisfy the inequalities: $(\lambda_S^a + \lambda_S^d)\phi_i E[S_i] < 1, i = 1, \ldots, N$.

Let $\mu_i = \frac{1}{E[S_i]}$, be the service rate of packets in queue $Q_i, i = 1, 2, 3$. Delays $D_i(\phi_i)$, using relation (25), are then given by

$$D_i(\phi_i) = \frac{1}{\mu_i - \lambda\phi_i}, \tag{29}$$

where $\lambda = \lambda_S^a + \lambda_S^d$. Consequently, the fixed point equations (26) reduce to

$$\phi_i^* = \frac{\mu_i - \lambda\phi_i^*}{\sum_{j=1}^{3}\left(\mu_j - \lambda\phi_j^*\right)}, \quad i = 1, 2, 3,$$

which on simplification gives us

$$\phi_i^* = \frac{\mu_i}{\sum_{j=1}^{3}\mu_j}, \quad i = 1, 2, 3.$$

In this special case the equilibrium routing probabilities are directly proportional to the service rates in the queues. Figures 6a, 6b, and 6c provide plots of the interpolated delay estimates $z_i^\epsilon(t), i = 1, 2, 3$, in the three queues versus the components $z_i(t), i = 1, 2, 3$, of the ODE approximation (obtained by numerically solving (24)). The components of the ODE approximation track the delay estimates from the simulation well, for the mentioned value of $\epsilon$. Figures 7a, 7b, and 7c provide plots of the routing probabilities $\phi_1^\epsilon(n)$, $\phi_2^\epsilon(n)$, and $\phi_3^\epsilon(n)$, respectively. The equilibrium routing probabilities are $\phi_1^* = 3/12, \phi_2^* = 4/12, \phi_3^* = 5/12$ (note that $\mu_1 = 3, \mu_2 = 4, \mu_3 = 5$).

#### 5.1.3 Effect of the parameter $\beta$

We observed in Section 5.1.2 that, for a given $\beta$, the equilibrium routing probabilities satisfy the fixed point system (26). Lets denote the equilibrium routing probability vector by $\phi^*(\beta) = (\phi_1^*(\beta), \ldots, \phi_N^*(\beta))$ (a function of $\beta$). The delay function for the $i$-th queue is written as $D_i(\phi_i) = D(\phi_i, C_i)$, emphasizing its dependence on $\phi_i$ and on capacity $C_i$. We keep $\lambda_S^a, \lambda_S^d$ fixed throughout the discussion in this section. We assume that the delay function has the following properties: it is positive, is a strictly increasing function of $\phi_i$ when $C_i$ is held fixed, and a strictly decreasing function of $C_i$ when $\phi_i$ is held fixed. Also, let $\beta$ be a nonnegative real number (instead of being a positive integer).

Suppose $C_1 > C_2 = \cdots = C_N$. Then using the relations

$$\phi_1^*(\beta)\big[D(\phi_1^*(\beta), C_1)\big]^\beta = \cdots = \phi_N^*(\beta)\big[D(\phi_N^*(\beta), C_N)\big]^\beta,$$

it can be checked that [10]

$$\phi_1^*(\beta) > \phi_2^*(\beta) = \cdots = \phi_N^*(\beta),$$

and consequently that

$$D(\phi_1^*(\beta), C_1) < D(\phi_2^*(\beta), C_2) = \cdots = D(\phi_N^*(\beta), C_N). \tag{30}$$

We show that, as $\beta$ increases, the routing probability on the highest capacity path also increases. To arrive at a contradiction, lets suppose, for some small positive $\delta\beta$ that $\phi_1^*(\beta+\delta\beta) < \phi_1^*(\beta)$; then we also have $\phi_2^*(\beta+\delta\beta) > \phi_2^*(\beta)$. This implies that

$$\frac{\phi_1^*(\beta+\delta\beta)}{\phi_2^*(\beta+\delta\beta)} < \frac{\phi_1^*(\beta)}{\phi_2^*(\beta)}. \tag{31}$$

Using the relationships with the delays, we then have

$$\Big[\frac{D(\phi_2^*(\beta+\delta\beta), C_2)}{D(\phi_1^*(\beta+\delta\beta), C_1)}\Big]^{\beta+\delta\beta} < \Big[\frac{D(\phi_2^*(\beta), C_2)}{D(\phi_1^*(\beta), C_1)}\Big]^{\beta},$$

$$\text{or,} \quad \Big[\frac{D(\phi_2^*(\beta+\delta\beta), C_2)}{D(\phi_2^*(\beta), C_2)}.\frac{D(\phi_1^*(\beta), C_1)}{D(\phi_1^*(\beta+\delta\beta), C_1)}\Big]^{\beta+\delta\beta} < \Big[\frac{D(\phi_1^*(\beta), C_1)}{D(\phi_2^*(\beta), C_2)}\Big]^{\delta\beta}.$$

Using the hypothesis and the monotonicity property of the delay function with respect to the routing probability, it is easy to see that the left hand side of the above inequality is greater than one, which implies that

$$D(\phi_1^*(\beta), C_1) > D(\phi_2^*(\beta), C_2),$$

contradicting the relation (30). Thus, we must have $\phi_1^*(\beta+\delta\beta) > \phi_1^*(\beta)$ and $\phi_2^*(\beta+\delta\beta) < \phi_2^*(\beta)$.

We now consider an example studying what happens when $\beta \uparrow \infty$. The service times of ant and data packets are exponentially distributed and the means in a particular queue are the same ($E[S_i^a] = E[S_i^d]$). Then $E[S_i] = E[S_i^a]$; we let $\mu_i = \frac{1}{E[S_i]}$. The delays are then given by $D_i(\phi_i^*) = \frac{1}{\mu_i - \lambda\phi_i^*}$. Let the number of parallel links $N = 3$. The traffic parameters are $\lambda_S^a = 1, \lambda_S^d = 1, \mu_1 = 4, \mu_2 = 3, \mu_3 = 3$ [11]. The fixed point equations for the equilibrium routing probabilities then become

[10] More generally, if $C_1 > C_2 > \cdots > C_N$, it can be checked that $\phi_1^*(\beta) > \phi_2^*(\beta) > \cdots > \phi_N^*(\beta)$, so that the paths "are ranked" according to the capacities. Then, $D(\phi_1^*(\beta), C_1) < D(\phi_2^*(\beta), C_2) < \cdots < D(\phi_N^*(\beta), C_N)$.

[11] The service rates are proportional to the link capacities. We work with them instead of the capacities for convenience.

(with $\phi_2^* = \phi_3^*$)

$$\begin{aligned}
\phi_1^* &= \frac{\left(4 - 2\phi_1^*\right)^\beta}{\left(4 - 2\phi_1^*\right)^\beta + 2\,\left(3 - 2\phi_2^*\right)^\beta}, \\
\phi_2^* &= \frac{1 - \phi_1^*}{2}.
\end{aligned}$$

We solve the above fixed point system in Mathematica for increasing values of $\beta$. The equilibrium routing probabilities are close to $\phi_1^* = \frac{2}{3}, \phi_2^* = \frac{1}{6}, \phi_3^* = \frac{1}{6}$ for high values of $\beta$. It is not possible that $\phi_1^* \geq \frac{2}{3}$, because then $D_1(\phi_1^*) = \frac{1}{4-2\phi_1^*} \geq D_2(\phi_2^*) = \frac{1}{3-2\phi_2^*}$, which is impossible by (30). Thus it may be surmised in this case that when $\beta \uparrow \infty$, $\phi_1^*$ increases to 2/3 but never attains that value.

If we now increase the service rate in queue $Q_1$ to $\mu_1 = 6$, the equilibrium routing probabilities are close to $\phi_1^* = 1, \phi_2^* = 0, \phi_3^* = 0$ for high values of $\beta$; then all the incoming traffic is routed through $Q_1$ in steady state. It may be noted in this case, that for no $\phi_1^* \in [0, 1]$, is it possible that $D_1(\phi_1^*) \geq D_2(\phi_2^*)$.

Thus $\beta$ acts like a tuning parameter that can be used to modulate the fraction of flow on the outgoing links under equilibrium. Higher values of $\beta$ make the flows more concentrated on the outgoing links with more capacity — in the limiting case of $\beta \uparrow \infty$, as the example above shows we can even have all the incoming flow routed to the highest capacity path. Lower values of $\beta$ make the flows more evenly distributed on the outgoing links — in the limiting case of $\beta = 0$, the incoming flow is perfectly split: $\phi_i^* = \frac{1}{N}, i = 1, 2, \ldots, N$.

## 5.2 The Uniform ARA Case

We now turn our attention to the case when ant packets are routed uniformly. The discussion is brief because the same methods, as for the regular ant case, can be used to analyze this case. Specializing to the $N$ parallel links case the ODE approximation result (15), we have the following ODE system

$$\begin{aligned}
\frac{dz_1(t)}{dt} &= \frac{D_1^U\big(z_1(t), \ldots, z_N(t)\big) - z_1(t)}{N}, \\
&\vdots \\
\frac{dz_N(t)}{dt} &= \frac{D_N^U\big(z_1(t), \ldots, z_N(t)\big) - z_N(t)}{N},
\end{aligned} \tag{32}$$

with appropriate initial conditions. As for the regular ants case, $\zeta_S^U(t) = 1$ [12].

[12] Also, instead of using the notation $z_{jD}(t), D_{jD}^U(z(t)), j = 1, \ldots, N$, we employ the simpler notation $z_j(t), D_j^U(z(t)), j = 1, \ldots, N$.

The ODE above (32) can be numerically solved once the quantities $D_i^U(z_1, \ldots, z_N), i = 1, \ldots, N$, are available for a fixed $z = (z_1, \ldots, z_N)$. For our queuing system, with identical assumptions on the statistics of the arrival processes and the packet lengths of the ant and data streams as for the regular ant case, we can compute the quantities $D_i^U(z_1, \ldots, z_N)$ in an identical manner. The details are omitted. For each $i = 1, \ldots, N$, $D_i^U(z_1, \ldots, z_N)$ is given by the Pollaczek-Khinchin formula

$$D_i^U(z_1, \ldots, z_N) = E[S_i] + \frac{\left(\frac{\lambda_S^a}{N} + \lambda_S^d \phi_i\right) E[S_i^2]}{2\left(1 - (\frac{\lambda_S^a}{N} + \lambda_S^d \phi_i) E[S_i]\right)},$$

where $E[S_i] = \frac{\frac{\lambda_S^a}{N} E[S_i^a] + \lambda_S^d \phi_i E[S_i^d]}{\frac{\lambda_S^a}{N} + \lambda_S^d \phi_i}$, $E[S_i^2] = \frac{\frac{\lambda_S^a}{N} E[(S_i^a)^2] + \lambda_S^d \phi_i E[(S_i^d)^2]}{\frac{\lambda_S^a}{N} + \lambda_S^d \phi_i}$, and $\phi_i = \frac{(z_i)^{-\beta}}{\sum_{j=1}^N (z_j)^{-\beta}}$. We again require that the input arrival rate is smaller than the service rate for each queue: $(\frac{\lambda_S^a}{N} + \lambda_S^d \phi_i) E[S_i] < 1, i = 1, \ldots, N$.

The equilibrium routing probabilities must satisfy the fixed-point system of equations

$$\phi_i^* = \frac{\left(D_i^U(\phi_i^*)\right)^{-\beta}}{\sum_{k=1}^N \left(D_k^U(\phi_k^*)\right)^{-\beta}}, \tag{33}$$

and must be all positive. It can be shown, using methods similar to those in Section 5.1.2 that if there exists a solution to the equations (33) that also satisfies the conditions $(\frac{\lambda_S^a}{N} + \lambda_S^d \phi_i) E[S_i] < 1, i = 1, \ldots, N$, then such a solution is unique.

We also make the following observation comparing the equilibrium routing probabilities for the regular and the uniform ARA case for the $N$ parallel links network. We consider the two cases with $\beta = 1$, and with identical statistics on the arrival processes and service times of the packet streams. Service times in queue $Q_i$ of ant and data packets are exponential, with identical means $E[S_i^a] = E[S_i^d]$. The service rate in $Q_i$ is then $\mu_i = \frac{1}{E[S_i]} = \frac{1}{E[S_i^a]}$. For the regular ARA case, the equilibrium routing probabilities are given in Section 5.1.2 (they are directly proportional to the queue service rates). For the uniform ARA case, $\phi_i^*, i = 1, \ldots, N$, satisfy the equations

$$\phi_i^* = \frac{\mu_i - \left(\frac{\lambda_S^a}{N} + \lambda_S^d \phi_i^*\right)}{\sum_{j=1}^N \left[\mu_j - \left(\frac{\lambda_S^a}{N} + \lambda_S^d \phi_j^*\right]\right.}, \quad i = 1, \ldots, N. \tag{34}$$

These equations can be solved for the equilibrium routing probabilities

$$\phi_i^* = \frac{\mu_i - \frac{\lambda_S^a}{N}}{\sum_{j=1}^N \left[\mu_j - \frac{\lambda_S^a}{N}\right]}, \quad i = 1, \ldots, N. \tag{35}$$

For purposes of comparison, without much loss of generality, lets assume that $\mu_1 > \mu_2 > \cdots > \mu_N$. We denote the vectors of equilibrium routing probabilities in the uniform and in the regular case by $(\phi^*)^U$ and $(\phi^*)^R$, respectively. We can then check, by simply applying the definition, that $(\phi^*)^R$ is majorised by $(\phi^*)^U$, denoted by $(\phi^*)^R \prec (\phi^*)^U$ (Marshall and Olkin [19] is a reference on majorization theory). That is, the routing probabilities in the regular ARA case are "less spread out". This can be understood by observing that, when ant packets are routed uniformly, their contribution to the average delays in the $N$ queues are the same, and hence more uniform, than when they are routed as in the regular case. Thus, delays in queues with higher service rates are lower for the uniform than for the regular ARA case. The result can then be expected, because the routing probabilities are inversely proportional to the delays. Results in a similar spirit might be expected to hold when $\beta > 1$, but we haven't been able to show this.

**Stability of the ODE system**. We now show for the ODE system (32) that for almost all initial conditions, $z(t)$ converges to the set of equilibria of the ODE, which are solutions of the system of equations $z_i = D_i^U(z_1, \ldots, z_N), i = 1, \ldots, N$. We consider the special case when the lengths of an ant and a data packet are both exponentially distributed with the same mean. Then $E[S_i^a] = E[S_i^d]$ for each $i = 1, \ldots, N$, and we have the following expression for $D_i^U(z_1, \ldots, z_N)$

$$D_i^U(z_1, \ldots, z_N) = \frac{1}{\frac{1}{E[S_i^a]} - \left(\frac{\lambda_S^a}{N} + \frac{\lambda_S^d (z_i)^{-\beta}}{\sum_{j=1}^N (z_j)^{-\beta}}\right)}, \quad i = 1, \ldots, N. \tag{36}$$

Lets denote the right hand sides of the ODE system (32) by $F_i^U(z_1(t), \ldots, z_N(t)), i = 1, \ldots, N$. A straightforward computation shows that, for $j \neq i$,

$$\frac{\partial F_i^U(z)}{\partial z_j} = \frac{\beta \lambda_S^d (z_i)^\beta}{N T^2 P^2 (z_j)^{\beta+1}},$$

where $T = \frac{1}{E[S_i^a]} - \left(\frac{\lambda_S^a}{N} + \frac{\lambda_S^d (z_i)^{-\beta}}{\sum_{j=1}^N (z_j)^{-\beta}}\right)$ and $P = (z_i)^\beta \sum_{j=1}^N (z_j)^{-\beta}$ (note that $T > 0$, because the input rate is smaller than the service rate, for each $i$). Thus, for $j \neq i$,

$$\frac{\partial F_i^U(z)}{\partial z_j} \geq 0.$$

We thus have a cooperative ODE system. Such ODEs have been proposed and studied as models describing the behavior of a set of interacting agents (in our case a set of interacting queues). Hirsch, in a series of papers, extensively studied such ODEs [13]. We shall make use of results in one of the papers in the series [16].

We first note that, if the ODE starts in the convex open set $C = \{(z_1, \ldots, z_N) : z_1 > 0, \ldots, z_N > 0\}$, it remains in that set for all time $t > 0$. We can check this by noting that, from (36),

$$D_i^U(z_1, \ldots, z_N) \geq E[S_i^a] = n_i \quad \text{(say)}.$$

[13] In Borkar [8] it is shown to arise in other application contexts involving ODE approximations.

Then a trajectory $z(t) = (z_1(t), \ldots, z_N(t))$ of the ODE system (32) must satisfy the following relations

$$\begin{aligned} \frac{dz_1(t)}{dt} &\geq \frac{n_1 - z_1(t)}{N}, \\ &\vdots \quad \vdots \\ \frac{dz_N(t)}{dt} &\geq \frac{n_N - z_N(t)}{N}, \end{aligned}$$

and so all components shall remain away from zero, once the ODE starts in $C$.

Also, the trajectories of our ODE system remain bounded. To see this, we note from (36) that

$$D_i^U(z_1, \ldots, z_N) \leq \frac{1}{\frac{1}{E[S_i^a]} - \left(\frac{\lambda_S^a}{N} + \lambda_S^d\right)} = m_i \quad \text{(say)}.$$

Then a trajectory $z(t)$ of the ODE system satisfies

$$\begin{aligned} \frac{dz_1(t)}{dt} &\leq \frac{m_1 - z_1(t)}{N}, \\ &\vdots \quad \vdots \\ \frac{dz_N(t)}{dt} &\leq \frac{m_N - z_N(t)}{N}, \end{aligned}$$

and so must be bounded.

Consider a general ODE system $\dot{x}(t) = F(x(t))$, with initial condition $x(0) = x$, evolving on an open subset $W \subset \mathbb{R}^n$. The flow $\{\phi_t\}$ associated with the ODE is said to be strongly monotone [16], if for initial conditions $x, y$, with $x < y$, $\phi_t(x) < \phi_t(y)$, for all $t > 0$. (For vectors $p, q$, $p < q$ here means that componentwise $p_i < q_i, i = 1, \ldots, n$.)

Our vector field $F^U$ being cooperative and irreducible on the set $C$, by Theorem 1.5 of Hirsch [16] the flow $\{\phi_t\}$ is strongly monotone. Also, because the trajectories of our ODE system remain bounded, by Theorem 4.1 of [16], the forward trajectory starting from almost any initial condition in $C$ approaches the set of equilibria of the ODE system. Also, by Theorem 2.4 of Hirsch, the flow cannot have an attracting closed orbit (same as a periodic solution, since our ODE is autonomous).

# 6 Example: An Acyclic Network

In this section, we consider the acyclic network of Figure 5. The numbers beside the links indicate the link capacities ($C_{ij}$ units for link $(i, j)$). There are data packets coming in at nodes 1, 2 and

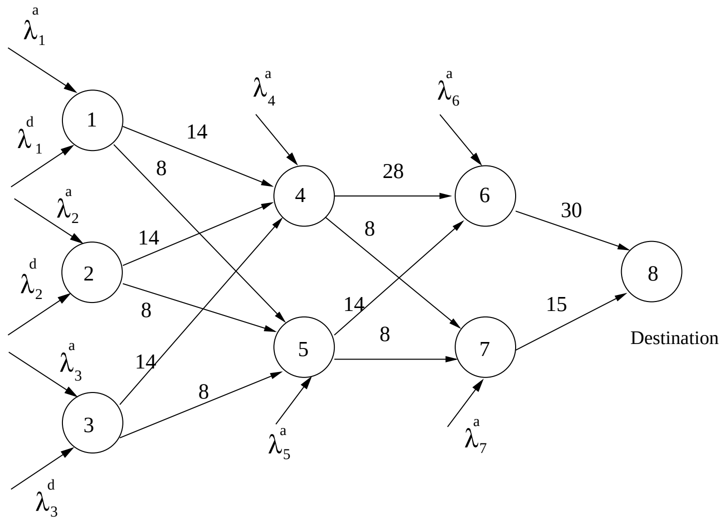


Figure 5: An Acyclic Network

3, as Poisson processes with rates $\lambda_1^d$, $\lambda_2^d$, and $\lambda_3^d$. Ant (FA) packets are coming in as a Poisson process at node $i$ with rate $\lambda_i^a$ $(i = 1, \ldots, 7)$.

We carry out a discrete event simulation of the network and present results for the regular ARA case. The arrival rates of the streams are as follows: $\lambda_i^a = 2, i = 1, \ldots, 7$, and $\lambda_1^d = 6$, $\lambda_2^d = 8$, and $\lambda_3^d = 6$. The parameter $\beta = 1$, and the step size $\epsilon = 0.002$. The ODE is numerically solved following the procedure described in Sections 4.1 and 4.2. Figures 8a, 8b, and 8c provide plots of the interpolated delay estimates $z_{14}^\epsilon(t)$, $z_{46}^\epsilon(t)$, and $z_{68}^\epsilon(t)$, and alongside plots of the corresponding components of the ODE system. The ODE approximation again tracks the interpolated delay estimates well. Figures 9a, 9b, and 9c provide plots of the routing probabilities $\phi_{14}^\epsilon(n)$, $\phi_{24}^\epsilon(n)$, and $\phi_{46}^\epsilon(n)$, respectively. We note that though we initially start with a routing probability $\phi_{14}^\epsilon(0) < 0.5$, the routing probability $\phi_{14}^\epsilon(n)$ converges to a value which is greater than 0.5. This is to be expected of a routing algorithm; the (equilibrium) routing probability on outgoing links that lie on paths with higher capacity links should be higher. In our case, this is a consequence of the fact that the equilibrium outgoing routing probability on an outgoing link is proportional to a decreasing function of the estimated delay to the destination along the link.

# 7 Concluding Remarks

**Extensions**. We can extend our results to the case when we have an acyclic network, with multiple destinations for the incoming data traffic into the network. As usual, at every node, ant (FA) packets would be sent out to explore delays in the paths towards each destination. The ant packets can be routed using either the regular or the uniform ARA algorithm. Suppose that there are $M$ destinations overall. With assumptions $(M1)$ and $(M2)$ regarding the algorithm operation in force, we can write down the stochastic iterative equations, describing the evolution of delay estimates and the routing probabilities, in a form similar to equations (7) and (8). We will have a set of equations corresponding to each of the $M$ destinations. Let us consider first the case when the queue $Q_{ij}$ associated with $(i, j)$, is shared by all ant and data packets that are bound for various destinations. The scheduling discipline is FIFO. In this case it can be checked that, we would again have an ODE approximation of the form (14) for the regular ARA case ((15) for the uniform ARA case). There is a set of equations for each of the $M$ destinations, and the equations considered together constitute a system of coupled ODEs. In order to compute the stationary means of delays — $D_{ij}(z)$, for a given $z$ — related to the ODE approximation, we can employ the same procedure as in Section 4.2, with appropriate modifications. In this regard we note that we have an open Jackson network, with $M$ classes for the regular ARA case, and with $M + 1$ classes for the uniform ARA case (data packets are routed according to routing probabilities at the nodes and ant packets are routed uniformly). Also, the equilibrium behavior of the routing algorithm can be described in a similar way as in Section 4.3.

The second, more general case is a per-destination queuing arrangement, which can be more appropriate in a routing context. In this case, for a link $(i, j)$, $M$ separate outgoing queues $Q^k_{ij}, k = 1, \dots, M$, are maintained, each corresponding to a particular destination. $Q^k_{ij}$ holds ant and data packets that are bound for destination $k$. The transmission capacity of link $(i, j)$ is then shared between the queues; the manner in which the sharing takes place is known as the link-scheduling discipline. In this case, the form of the update algorithms does not change, and we can arrive at an ODE approximation for the system as described above for the first case. However, in this case, it might not be always possible to compute analytically the stationary mean delays. Only for certain *symmetric* link scheduling disciplines like Processor-Sharing, which are analytically tractable (that is, have joint stationary product form distributions for the number of packets in the queues; see [20]), can we compute the stationary mean delays.

We also point out that the ODE approximation results hold whenever assumptions $(A1)$, $(A2)$, and $(A3)$ ($(A1)$, $(A2')$, and $(A3')$ for the uniform ARA case) hold. $(A2)$ and $(A3)$ are essentially law of large number-like conditions, which require that when the delay estimate vector $X$ is considered fixed at $z$, the queuing system converges to a stationary distribution. This might hold under more general conditions on the statistics of arrival processes and packet lengths of the packet streams, than we have considered. Under our assumptions though, we are explicitly able to compute the quantities $D_{ij}(z)$ and $\zeta_i(z)$ using results from the theory of queuing networks, and hence solve the ODE numerically. This then enables us to compare the theoretical ODE with the piecewise constant interpolation of the delay estimates, obtained through a discrete event simulation. In our framework, we can also consider a slightly more general dependence of outgoing routing probabilities on the delay estimates: $\phi_{ij} = \frac{g(X_{ij})}{\sum_{k \in N(i)} g(X_{ik})}$, where $g$ is a continuous function, that is positive real-valued, and nonincreasing. The analysis remains the same. An example of $g$ is $g(x) = e^{-\beta x}, x \geq 0$, where $\beta$ is a positive integer.

**Conclusions and Future Directions**. In summary, in this paper we have studied the convergence and the equilibrium behavior of an Ant Routing Algorithm for wireline packet-switched networks. We have considered acyclic network models, where there are multiple sources of incoming traffic whose packets are bound for specified destinations. We have considered stochastic models for the arrival processes and packet lengths for the ant and incoming data streams. The link delays are stochastic and time varying. We have shown that the evolution of the vector of delay estimates can be tracked by an ODE when the step-size of the estimation scheme is small. We then study the equilibrium routing behavior and properties of the equilibrium routing probabilities. We observe that, at equilibrium, the routing probabilities are higher for outgoing links that lie on paths with higher capacity links.

There are certain advantages of ARA algorithms, which are worth pointing out. ARA algorithms do not require explicit knowledge of the incoming traffic rates into the network, or a knowledge of the link capacities. Instead, ARA algorithms rely directly on online estimates of path delays in the network, which are collected by the ant packets. This enables the algorithm to adapt to changes in the incoming traffic rates, and/or changes in the network topology. On the other hand, because there is a learning process to ascertain the delays (based on which the routing probabilities are updated), the convergence of the algorithm can be slow. Further experimentation with the step-size $\epsilon$ is necessary, in order to enable the algorithm to be fast enough that it can react and adapt to changes.

In our work, we have considered models where are no cycles in the network. It remains to study convergence and equilibrium behavior of the algorithm when there are cycles. There are two issues that arise. First, cycles in the network affect adversely the process of estimation of the path delays by the ant packets. This is because the estimates can grow unbounded if there is a positive probability of an ant packet being routed in a cycle. Second, it might happen that we converge to an equilibrium routing solution which has *loops*. That is, for a given destination $k$, the equilibrium routing probabilities might be such that, for a sequence of links $(i_1, i_2), \ldots, (i_{n-1}, i_n), (i_n, i_1)$ that forms a cycle, $\phi^k_{i_1 i_2} > 0, \ldots, \phi^k_{i_{n-1} i_n} > 0, \phi^k_{i_n i_1} > 0$. There is no reason to believe that the scheme that we analyse in this paper can lead to a loop-free equilibrium solution. For the case when the network has cycles, we might need to modify the scheme so that it can converge to a loop-free routing solution which is always desirable.

# 8 Appendix

## 8.1 Proof of Convergence of the Ant Routing Algorithm

As discussed in Section 3, the evolution of the delay estimates and the routing probabilities is given by the equations ((7) and (8))

$$\begin{aligned} X^\epsilon_{ij}(n) &= X^\epsilon_{ij}(n-1) + \epsilon\, I_{\{T^\epsilon(n)=i, R^\epsilon_i(\xi^\epsilon_i(n))=j\}} \Big( \Delta^\epsilon_{ij}(\psi^\epsilon_{ij}(n)) - X^\epsilon_{ij}(n-1) \Big), \\ & \qquad \forall (i,j) \in \mathcal{L}, n \geq 1, \end{aligned}$$

$$\phi^\epsilon_{ij}(n) = \frac{(X^\epsilon_{ij}(n))^{-\beta}}{\sum_{k \in N(i)} (X^\epsilon_{ik}(n))^{-\beta}}, \quad \forall (i,j) \in \mathcal{L}, n \geq 1,$$

respectively, with the appropriate initial conditions.

We now consider the piecewise constant interpolation of $\{X^\epsilon_{ij}(n)\}$, which is the process $\{z^\epsilon_{ij}(t), t \geq 0\}$, defined by equation (9), Section 4.1. We also consider the vector-valued piecewise constant process $\{z^\epsilon(t), t \geq 0\}$. This process evolves on the path space $D^{|\mathcal{L}|}[0, \infty)$, consisting of right-continuous $\mathbb{R}^{|\mathcal{L}|}$-valued functions possessing left hand limits.

The stochastic iterations (7) (considered along with (8)) are an example of constant step-size stochastic approximation algorithms. For the proof of the ODE approximation for the algorithm, we follow the approach as given in the textbook of Kushner and Yin [18]. We provide the proof for the regular ant case; the proof for the uniform ant case can be similarly done. The main theorem is the following

**Theorem 1.** *Under assumptions (A1), (A2), and (A3), we have the following: there exists a subsequence* $\{\epsilon(k)\}$*, with* $\epsilon(k) \downarrow 0$ *as* $k \to \infty$*, such that the process* $\{z^{\epsilon(k)}(t)\}$ *converges weakly (as* $k \to \infty$*) to a solution* $\{z(t)\}$ *of the ODE approximation (14).*

**Proof:** A brief outline of the proof is as follows:

- We first show that the family of processes $\{z^\epsilon(t)\}, \epsilon \in (0,1)$, is tight. Then there exists a subsequence $\epsilon(k) \downarrow 0$ as $k \to \infty$, and a process $\{z(t)\}$ such that $\{z^{\epsilon(k)}(t)\}$ converges weakly to $\{z(t)\}$. The process $\{z(t)\}$ has Lipschitz continuous paths.
- The limit process $\{z(t)\}$ will be then shown to have the following property ($z(t)$ has components $z_{ij}(t), (i,j) \in \mathcal{L}$). Let $t, \tau > 0$ be arbitrary numbers, and let $0 \leq s_1, s_2, \ldots, s_p \leq t$ also be a set of arbitrary numbers. Then, for a bounded continuous function $h$, we show that

$$E\Big[h(z(s_1), z(s_2), \ldots, z(s_p))\Big(z_{ij}(t+\tau) - z_{ij}(t) - \int_t^{t+\tau} F_{ij}(z(u))du\Big)\Big] = 0, \tag{37}$$

  for each $(i,j) \in \mathcal{L}$. This fact then implies that $\{z_{ij}(t) - z_{ij}(0) - \int_0^t F_{ij}(z(u))du, t \geq 0\}$, is a martingale with respect to the filtration generated by the process $\{z(t)\}$. This martingale process, because it has Lipschitz continuous paths, can be shown to have zero quadratic variation [18]. It is hence a constant. Because the martingale is zero at $t = 0$, it is identically zero with probability one. We shall thus have the result.

The fact that (37) holds will be shown by showing that

$$E\Big[h(z^\epsilon(s_1), z^\epsilon(s_2), \ldots, z^\epsilon(s_p))\Big(z^\epsilon_{ij}(t+\tau) - z^\epsilon_{ij}(t) - \int_t^{t+\tau} F_{ij}(z^\epsilon(u))du\Big)\Big] = 0, \tag{38}$$

and using the fact that $\{z^\epsilon(t)\}$ converges weakly to $\{z(t)\}$ (we are actually going through the subsequence $\epsilon(k)$).

We now embark on the proof. We first show the tightness of the family $\{z^\epsilon(t)\}, \epsilon \in (0,1)$, using the uniform integrability assumption.

From equation (7) we can write

$$X^\epsilon_{ij}(n+1) = \Big(1 - \epsilon\, I_{\{T^\epsilon(n+1)=i, R^\epsilon_i(\xi^\epsilon_i(n+1))=j\}}\Big) X^\epsilon_{ij}(n) + \epsilon\, I_{\{T^\epsilon(n+1)=i, R^\epsilon_i(\xi^\epsilon_i(n+1))=j\}} \Delta^\epsilon_{ij}(\psi^\epsilon_{ij}(n+1)),$$

$$X^\epsilon_{ij}(n+1) \leq X^\epsilon_{ij}(n) + \epsilon\, I_{\{T^\epsilon(n+1)=i, R^\epsilon_i(\xi^\epsilon_i(n+1))=j\}} \Delta^\epsilon_{ij}(\psi^\epsilon_{ij}(n+1)).$$

Iterating we can see that for every positive integer $m$,

$$X^\epsilon_{ij}(n+m) \leq X^\epsilon_{ij}(n) \; + \epsilon\Big(\sum_{k=n+1}^{n+m} I_{\{T^\epsilon(k)=i, R^\epsilon_i(\xi^\epsilon_i(k))=j\}} \Delta^\epsilon_{ij}(\psi^\epsilon_{ij}(k))\Big).$$

Consequently, for any $L > 0$, we have

$$E\Big[|X_{ij}^{\epsilon}(n+m) - X_{ij}^{\epsilon}(n)|\Big] \leq \epsilon \sum_{k=n+1}^{n+m} E\Big[\Delta_{ij}^{\epsilon}(\psi_{ij}^{\epsilon}(k))\Big],$$

$$\leq \epsilon \sum_{k=n+1}^{n+m} E\Big[\Delta_{ij}^{\epsilon}(\psi_{ij}^{\epsilon}(k)) I_{\{\Delta_{ij}^{\epsilon}(\psi_{ij}^{\epsilon}(k)) \geq L\}} + \Delta_{ij}^{\epsilon}(\psi_{ij}^{\epsilon}(k)) I_{\{\Delta_{ij}^{\epsilon}(\psi_{ij}^{\epsilon}(k)) < L\}}\Big].$$

Thus, for any $n, n+m \in \{0, 1, 2, \ldots, \lfloor \frac{T}{\epsilon} \rfloor\}$ (for some fixed $0 < T < \infty$), we have

$$E\Big[|X_{ij}^{\epsilon}(n+m) - X_{ij}^{\epsilon}(n)|\Big] \leq m\epsilon\Big(L + \sup_{k \geq 1} E\Big[\Delta_{ij}^{\epsilon}(\psi_{ij}^{\epsilon}(k)) I_{\{\Delta_{ij}^{\epsilon}(\psi_{ij}^{\epsilon}(k)) \geq L\}}\Big]\Big).$$

If we now let $t = n\epsilon$ and $\tau = m\epsilon$, and noting that $z_{ij}^{\epsilon}(t) = X_{ij}^{\epsilon}(\lfloor \frac{t}{\epsilon} \rfloor)$, we have

$$\sup_{0 \leq t, t+\tau \leq T} E\Big[|z_{ij}^{\epsilon}(t+\tau) - z_{ij}^{\epsilon}(t)|\Big] \leq L\tau + \tau \sup_{k \geq 1} E\Big[\Delta_{ij}^{\epsilon}(\psi_{ij}^{\epsilon}(k)) I_{\{\Delta_{ij}^{\epsilon}(\psi_{ij}^{\epsilon}(k)) \geq L\}}\Big].$$

The uniform integrability of the sequence $\{\Delta_{ij}^{\epsilon}(m)\}$ allows us to choose $L$ large enough that the second term on the right hand side can be made as small as we like. Once $L$ is so chosen, we can choose $\tau$ small enough that the first term on the right can be made as small as we like. We then have the following result. For any $0 < T < \infty$,

$$\lim_{\tau \downarrow 0} \lim_{\epsilon \downarrow 0} \sup_{0 \leq t, t+\tau \leq T} E\Big[|z_{ij}^{\epsilon}(t+\tau) - z_{ij}^{\epsilon}(t)|\Big] = 0.$$

Now, because

$$E\Big[\| z^{\epsilon}(t+\tau) - z^{\epsilon}(t) \|\Big] \leq \sum_{(i,j) \in \mathcal{L}} E\Big[|z_{ij}^{\epsilon}(t+\tau) - z_{ij}^{\epsilon}(t)|\Big],$$

we have

$$\lim_{\tau \downarrow 0} \lim_{\epsilon \downarrow 0} \sup_{0 \leq t, t+\tau \leq T} E\Big[\| z^{\epsilon}(t+\tau) - z^{\epsilon}(t) \|\Big] = 0.$$

The fact that this holds for every $0 \leq T < \infty$ is sufficient for the family $\{z^{\epsilon}(t)\}, \epsilon \in (0, 1)$, to be tight.

We now show the validity of (38). We have the following expression for $X^\epsilon_{ij}(n)$

$$\begin{aligned}
X^\epsilon_{ij}(n) &= X^\epsilon_{ij}(0) + \epsilon \sum_{m=1}^{n} \Big( I_{\{T^\epsilon(m)=i, R^\epsilon_i(\xi^\epsilon_i(m))=j\}} \Delta^\epsilon_{ij}(\psi^\epsilon_{ij}(m)) \\
&\quad - E\Big[ I_{\{T^\epsilon(m)=i, R^\epsilon_i(\xi^\epsilon_i(m))=j\}} \Delta^\epsilon_{ij}(\psi^\epsilon_{ij}(m)) / \mathcal{F}^\epsilon(m-1) \Big] \Big) \\
&\quad + \epsilon \sum_{m=1}^{n} \Big( E\Big[ I_{\{T^\epsilon(m)=i, R^\epsilon_i(\xi^\epsilon_i(m))=j\}} \Delta^\epsilon_{ij}(\psi^\epsilon_{ij}(m)) / \mathcal{F}^\epsilon(m-1) \Big] \\
&\quad - \zeta_i(X^\epsilon(m-1)) \phi^\epsilon_{ij}(m-1) D_{ij}(X^\epsilon(m-1)) \Big) + \epsilon \sum_{m=1}^{n} \zeta_i(X^\epsilon(m-1)) \phi^\epsilon_{ij}(m-1) D_{ij}(X^\epsilon(m-1)) \\
&\quad - \epsilon \sum_{m=1}^{n} \Big( I_{\{T^\epsilon(m)=i, R^\epsilon_i(\xi^\epsilon_i(m))=j\}} X^\epsilon_{ij}(m-1) \\
&\quad - E\Big[ I_{\{T^\epsilon(m)=i, R^\epsilon_i(\xi^\epsilon_i(m))=j\}} X^\epsilon_{ij}(m-1) / \mathcal{F}^\epsilon(m-1) \Big] \Big) \\
&\quad - \epsilon \sum_{m=1}^{n} \Big( E\Big[ I_{\{T^\epsilon(m)=i, R^\epsilon_i(\xi^\epsilon_i(m))=j\}} X^\epsilon_{ij}(m-1) / \mathcal{F}^\epsilon(m-1) \Big] \\
&\quad - \zeta_i(X^\epsilon(m-1)) \phi^\epsilon_{ij}(m-1) X^\epsilon_{ij}(m-1) \Big) - \epsilon \sum_{m=1}^{n} \zeta_i(X^\epsilon(m-1)) \phi^\epsilon_{ij}(m-1) X^\epsilon_{ij}(m-1). \qquad (39)
\end{aligned}$$

We can then write, using the fact that $z^\epsilon_{ij}(t) = X^\epsilon_{ij}(\lfloor \frac{t}{\epsilon} \rfloor)$ for all $t \geq 0$,

$$\begin{aligned}
z^\epsilon_{ij}(t) &= z^\epsilon_{ij}(0) + \epsilon \sum_{m=1}^{\lfloor \frac{t}{\epsilon} \rfloor} M^\epsilon_{ij}(m) + \epsilon \sum_{m=1}^{\lfloor \frac{t}{\epsilon} \rfloor} N^\epsilon_{ij}(m) + \epsilon \sum_{m=1}^{\lfloor \frac{t}{\epsilon} \rfloor} \zeta_i(X^\epsilon(m-1)) \phi^\epsilon_{ij}(m-1) D_{ij}(X^\epsilon(m-1)) \\
&\quad - \epsilon \sum_{m=1}^{\lfloor \frac{t}{\epsilon} \rfloor} P^\epsilon_{ij}(m) - \epsilon \sum_{m=1}^{\lfloor \frac{t}{\epsilon} \rfloor} Q^\epsilon_{ij}(m) - \epsilon \sum_{m=1}^{\lfloor \frac{t}{\epsilon} \rfloor} \zeta_i(X^\epsilon(m-1)) \phi^\epsilon_{ij}(m-1) X^\epsilon_{ij}(m-1),
\end{aligned}$$

where $M^\epsilon_{ij}(m)$, $N^\epsilon_{ij}(m)$, $P^\epsilon_{ij}(m)$, and $Q^\epsilon_{ij}(m)$ refer to the corresponding quantities in the equation (39) above.

We now introduce the quantities: $G^\epsilon_1(t) = \epsilon \sum_{m=1}^{\lfloor \frac{t}{\epsilon} \rfloor} M^\epsilon_{ij}(m)$, $G^\epsilon_2(t) = \epsilon \sum_{m=1}^{\lfloor \frac{t}{\epsilon} \rfloor} N^\epsilon_{ij}(m)$, $G^\epsilon_3(t) = \epsilon \sum_{m=1}^{\lfloor \frac{t}{\epsilon} \rfloor} P^\epsilon_{ij}(m)$, and $G^\epsilon_4(t) = \epsilon \sum_{m=1}^{\lfloor \frac{t}{\epsilon} \rfloor} Q^\epsilon_{ij}(m)$. Now the term

$$\epsilon \sum_{m=1}^{\lfloor \frac{t}{\epsilon} \rfloor} \zeta_i(X^\epsilon(m-1)) \phi^\epsilon_{ij}(m-1) \Big( D_{ij}(X^\epsilon(m-1)) - X^\epsilon_{ij}(m-1) \Big) = \int_0^t F_{ij}(z^\epsilon(u)) du,$$

when $t$ is an integral multiple $n\epsilon$ of $\epsilon$, and is an approximation otherwise, the approximation error vanishing when $\epsilon \to 0$.

Hence, in order to show (38), from equation (39) and the developments above, we can see that all that we need to show is that

$$\lim_{\epsilon\to 0} E\Big[h(z^\epsilon(s_1), z^\epsilon(s_2), \ldots, z^\epsilon(s_p))\Big(G_1^\epsilon(t+\tau) - G_1^\epsilon(t) + G_2^\epsilon(t+\tau) - G_2^\epsilon(t) - \big[G_3^\epsilon(t+\tau) - G_3^\epsilon(t) + G_4^\epsilon(t+\tau) - G_4^\epsilon(t)\big]\Big)\Big] = 0.$$

We show that each of the summands in the expectation above tend to zero as $\epsilon \to 0$, i.e., for $i = 1, 2, 3, 4$,

$$\lim_{\epsilon\to 0} E\Big[h(z^\epsilon(s_1), z^\epsilon(s_2), \ldots, z^\epsilon(s_p))\Big(G_i^\epsilon(t+\tau) - G_i^\epsilon(t)\Big)\Big] = 0.$$

We start by showing that $\lim_{\epsilon\to 0} E\Big[h(z^\epsilon(s_1), \ldots, z^\epsilon(s_p))\ \Big(G_1^\epsilon(t+\tau) - G_1^\epsilon(t)\Big)\Big] = 0$. Now,

$$E\Big[h(z^\epsilon(s_1), z^\epsilon(s_2), \ldots, z^\epsilon(s_p))\Big(G_1^\epsilon(t+\tau) - G_1^\epsilon(t)\Big)\Big] = E\Big[h(z^\epsilon(s_1), z^\epsilon(s_2), \ldots, z^\epsilon(s_p))\Big(\epsilon \sum_{m=\lfloor \frac{t}{\epsilon}\rfloor+1}^{\lfloor \frac{t+\tau}{\epsilon}\rfloor} M_{ij}^\epsilon(m)\Big)\Big].$$

It can be checked that the sequence $\{M_{ij}^\epsilon(n)\}$ is a martingale difference sequence with respect to $\{\mathcal{F}^\epsilon(n)\}$ (that is, the sequence $T_{ij}^\epsilon(n) = \sum_{m=1}^n M_{ij}^\epsilon(m)$ is a martingale with respect to the same filtration). It then follows that

$$E[h(z^\epsilon(s_1), z^\epsilon(s_2), \ldots, z^\epsilon(s_p))\Big(G_1^\epsilon(t+\tau) - G_1^\epsilon(t)\Big)] = 0;$$

the result then also holds true when $\epsilon \to 0$.

Similarly, we can show that $\lim_{\epsilon\to 0} E[h(z^\epsilon(s_1), \ldots, z^\epsilon(s_p))\Big(G_3^\epsilon(t+\tau) - G_3^\epsilon(t)\Big)] = 0$.

The arguments for showing that $\lim_{\epsilon\to 0} E[h(z^\epsilon(s_1), \ldots, z^\epsilon(s_p))\ \Big(G_2^\epsilon(t+\tau) - G_2^\epsilon(t)\Big)] = 0$ and $\lim_{\epsilon\to 0} E[h(z^\epsilon(s_1), \ldots, z^\epsilon(s_p))\Big(G_4^\epsilon(t+\tau) - G_4^\epsilon(t)\Big)] = 0$, hold, are similar in nature. Consequently, we shall discuss in detail the steps for only one of them.

Lets show that $\lim_{\epsilon\to 0} E[h(z^\epsilon(s_1), \ldots, z^\epsilon(s_p))\Big(G_2^\epsilon(t+\tau) - G_2^\epsilon(t)\Big)] = 0$. We recall that

$$G_2^\epsilon(t+\tau) - G_2^\epsilon(t) = \epsilon \sum_{m=\lfloor \frac{t}{\epsilon}\rfloor+1}^{\lfloor \frac{t+\tau}{\epsilon}\rfloor} \Big(E\Big[I_{\{T^\epsilon(m)=i, R_i^\epsilon(\xi_i^\epsilon(m))=j\}}\ \Delta_{ij}^\epsilon(\psi_{ij}^\epsilon(m))/\mathcal{F}^\epsilon(m-1)\Big] - \zeta_i(X^\epsilon(m-1))\phi_{ij}^\epsilon(m-1)D_{ij}(X^\epsilon(m-1))\Big). \quad (40)$$

Now, for a scalar $\eta > \epsilon > 0$ (with $\eta < \tau$), the expression on the right hand side of (40) can be written as

$$\sum_{j=0}^{\lfloor \frac{\tau}{\eta} \rfloor} \eta \Bigg[ \frac{\epsilon}{\eta} \sum_{m=\lfloor \frac{t+j\eta}{\epsilon} \rfloor + 1}^{\lfloor \frac{t+(j+1)\eta}{\epsilon} \rfloor} \Big( E\Big[ I_{\{T^\epsilon(m)=i, R_i^\epsilon(\xi_i^\epsilon(m))=j\}} \ \Delta_{ij}^\epsilon(\psi_{ij}^\epsilon(m)) / \mathcal{F}^\epsilon(m-1) \Big]$$

$$- \zeta_i(X^\epsilon(m-1)) \phi_{ij}^\epsilon(m-1) D_{ij}(X^\epsilon(m-1)) \Big) \Bigg].$$

In the interval $\{\lfloor \frac{t+j\eta}{\epsilon} \rfloor + 1, \ldots, \lfloor \frac{t+(j+1)\eta}{\epsilon} \rfloor\}$, each of the summands above can be written as a sum of the terms

$$E\Big[ I_{\{T^\epsilon(m)=i, R_i^\epsilon(\xi_i^\epsilon(m))=j\}} \ \Delta_{ij}^\epsilon(\psi_{ij}^\epsilon(m)) / \mathcal{F}^\epsilon(m-1) \Big] - \zeta_i(X^\epsilon(\lfloor \frac{t+j\eta}{\epsilon} \rfloor)) \phi_{ij}^\epsilon(\lfloor \frac{t+j\eta}{\epsilon} \rfloor) D_{ij}(X^\epsilon(\lfloor \frac{t+j\eta}{\epsilon} \rfloor))$$

and

$$\zeta_i(X^\epsilon(\lfloor \frac{t+j\eta}{\epsilon} \rfloor)) \phi_{ij}^\epsilon(\lfloor \frac{t+j\eta}{\epsilon} \rfloor) D_{ij}(X^\epsilon(\lfloor \frac{t+j\eta}{\epsilon} \rfloor)) - \zeta_i(X^\epsilon(m-1)) \phi_{ij}^\epsilon(m-1) D_{ij}(X^\epsilon(m-1)).$$

Choosing an $\eta$ small enough, noting that $\zeta_i(x)\phi_{ij}D_{ij}(x)$ is assumed to be a continuous function of $x$ (Assumption $(A3)$), and the fact that $\{z^\epsilon(t)\}$ is tight, shows that the latter terms tend to zero in the mean as $\epsilon \to 0$. For the former terms, we note that for any small $\eta > 0$, the expression

$$\frac{\epsilon}{\eta} \sum_{m=\lfloor \frac{t+j\eta}{\epsilon} \rfloor + 1}^{\lfloor \frac{t+(j+1)\eta}{\epsilon} \rfloor} \Big( E\Big[ I_{\{T^\epsilon(m)=i, R_i^\epsilon(\xi_i^\epsilon(m))=j\}} \ \Delta_{ij}^\epsilon(\psi_{ij}^\epsilon(m)) / \mathcal{F}^\epsilon(m-1) \Big]$$

$$- \zeta_i(X^\epsilon(\lfloor \frac{t+j\eta}{\epsilon} \rfloor)) \phi_{ij}^\epsilon(\lfloor \frac{t+j\eta}{\epsilon} \rfloor) D_{ij}(X^\epsilon(\lfloor \frac{t+j\eta}{\epsilon} \rfloor)) \Big),$$

tends to zero as $\epsilon$ tends to zero, by Assumption $(A2)$. We will then have $\lim_{\epsilon \to 0} E[h(z^\epsilon(s_1), \ldots, z^\epsilon(s_p)) \Big( G_2^\epsilon(t+\tau) - G_2^\epsilon(t) \Big)] = 0.$ □

## 8.2 Expression for $\zeta_i(z)$

We show here that $\zeta_i(z) = \frac{\lambda_i^a}{\sum_{j \in \mathcal{N}} \lambda_j^a}$, for each $i \in \mathcal{N}$, for the regular ARA case. The same argument holds for the uniform ARA case. As discussed in Section 4.2, with the delay estimate vector $X$ considered fixed at $z$, we have a single class open Jackson network. For each queue $Q_{ij}$ with the arrival rate of packets $A_{ij}(z) < C_{ij}$, the queuing network converges to stationarity. Let $T_{ij}, (i,j) \in \mathcal{L}$, denote the total number of packets in the queues $Q_{ij}$ under stationarity. Let $\{R_n\}$ denote the sequence of times when $T_{ij}, (i,j) \in \mathcal{L}$, returns to the state consisting of all zeros. Thus, $\{B_n\}$, where $B_n = R_n - R_{n-1}$, constitutes the sequence of successive busy periods for the queuing network.

Under our assumptions on the statistics of the arrival processes and packet lengths of the various streams, $\{B_n\}$ is an i.i.d. sequence, with the mean $E[B_1] < \infty$. $\{R_n\}$ is a sequence of stopping times for the ant Poisson arrival processes at the nodes.

For each $i \in \mathcal{N}$, let $D_i(t)$ = Number of FA packets that arrive at destination $D$ by time $t$. Then

$$\zeta_i(z) = \lim_{t\to\infty} \frac{D_i(t)}{\sum_{j\in\mathcal{N}} D_j(t)}. \tag{41}$$

Furthermore, we have

$$\zeta_i(z) = \frac{E[D_i(B_1)]}{\sum_{j\in\mathcal{N}} E[D_j(B_1)]}. \tag{42}$$

This is intuitive, and can be established by using the Renewal Reward Theorem, with the inter-renewal times being the sequence $\{B_n\}$.

Now, because $D_i(B_1)$ = Number of ant Poisson arrivals at node $i$ in the interval $B_1$, the mean $E[D_i(B_1)] = \lambda_i^a E[B_1]$, and so

$$\zeta_i(z) = \frac{\lambda_i^a}{\sum_{j\in\mathcal{N}} \lambda_j^a}. \tag{43}$$

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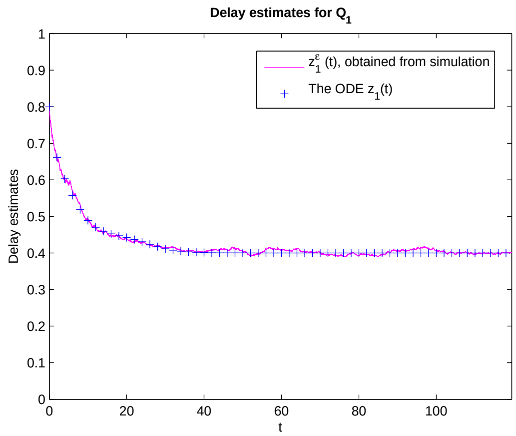
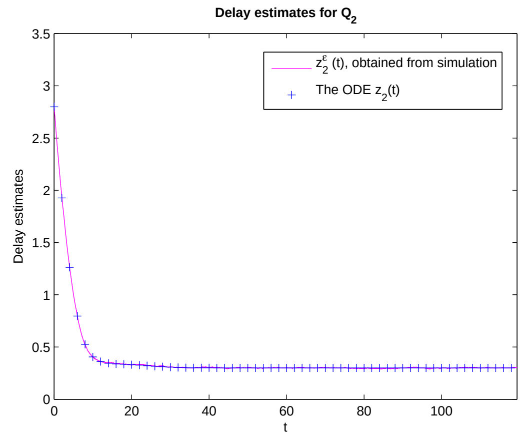
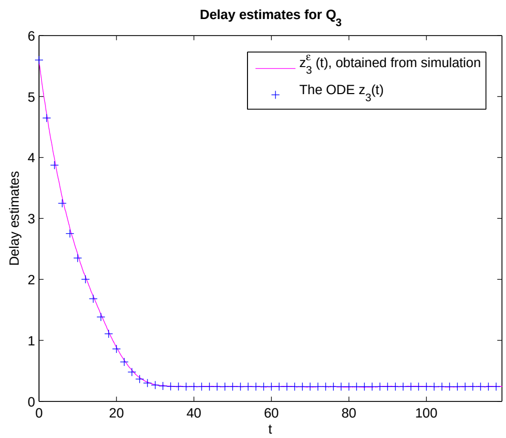



(a) The ODE approximation for $X_1(n)$

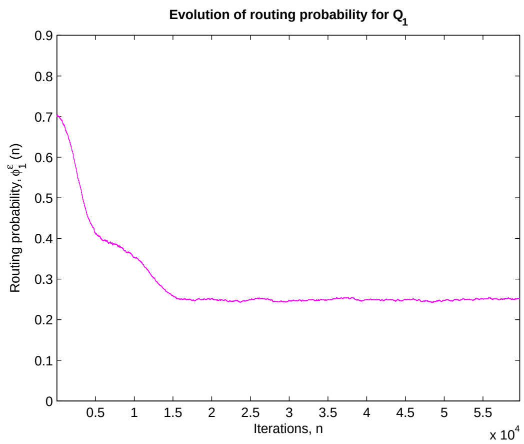



(b) The ODE approximation for $X_2(n)$

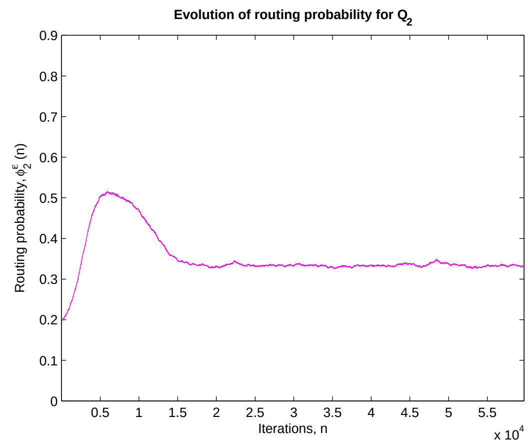



(c) The ODE approximation for $X_3(n)$

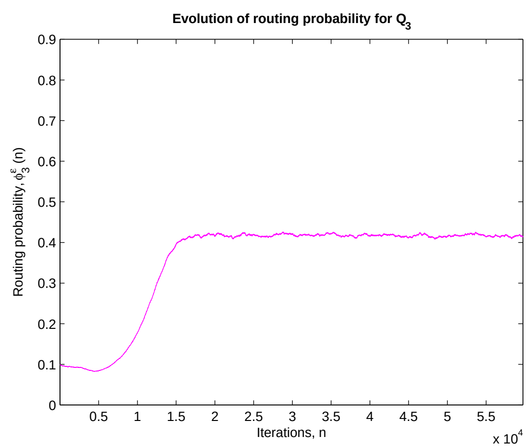
Figure 6: $N$ parallel links: The ODE approximations. Parameters: $\lambda_S^a = 1$, $\lambda_S^d = 1, E[S_1^a] = E[S_1^d] = 1/3.0, E[S_2^a] = E[S_2^d] = 1/4.0, E[S_3^a] = E[S_3^d] = 1/5.0, \beta = 1, \epsilon = 0.002$.




(a) The plot for $\phi_1^\epsilon(n)$

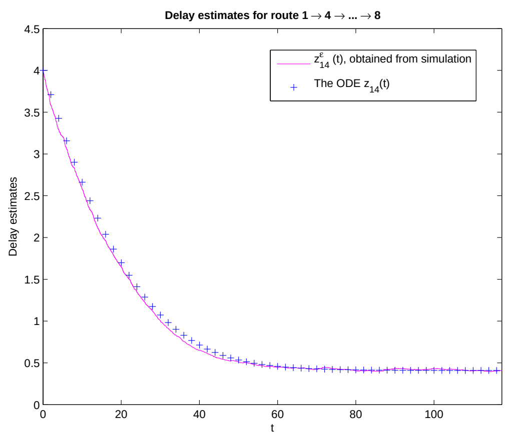



(b) The plot for $\phi_2^\epsilon(n)$

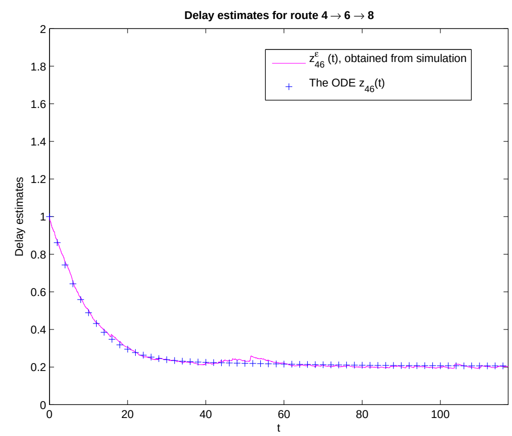



(c) The plot for $\phi_3^\epsilon(n)$

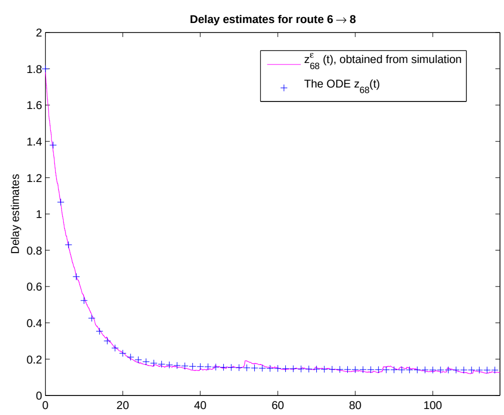
Figure 7: $N$ parallel links: Plots for routing probabilities. Parameters: $\lambda_S^a = 1,\ \lambda_S^d = 1, E[S_1^a] = E[S_1^d] = 1/3.0, E[S_2^a] = E[S_2^d] = 1/4.0, E[S_3^a] = E[S_3^d] = 1/5.0, \beta = 1, \epsilon = 0.002$.




(a) The ODE approximation for $X_{14}(n)$

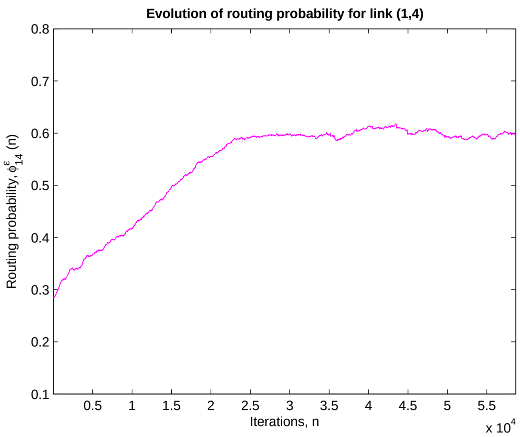



(b) The ODE approximation for $X_{46}(n)$

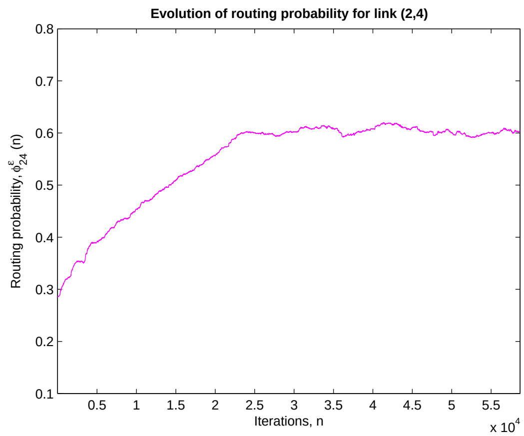



(c) The ODE approximation for $X_{68}(n)$

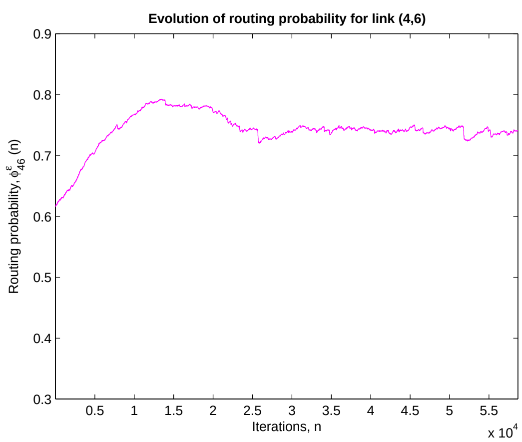
Figure 8: Acyclic network: The ODE approximations. Parameters: $\lambda_i^a = 2, i = 1, \ldots, 7$, $\lambda_1^d = 6, \lambda_2^d = 8, \lambda_3^d = 6, \beta = 1, \epsilon = 0.002$.




(a) The plot for $\phi^{\epsilon}_{14}(n)$




(b) The plot for $\phi^{\epsilon}_{24}(n)$




(c) The plot for $\phi^{\epsilon}_{46}(n)$

Figure 9: Acyclic network: Plots for routing probabilities. Parameters: $\lambda^a_i = 2, i = 1, \ldots, 7,\ \lambda^d_1 = 6, \lambda^d_2 = 8, \lambda^d_3 = 6, \beta = 1, \epsilon = 0.002$.