ABSTRACT

Title of Dissertation: ROBUST REVENUE MANAGEMENT WITH LIMITED INFORMATION: THEORY AND EXPERIMENTS

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Revenue management (RM) problems with full probabilistic information are well studied. However, as RM practice spreads to new businesses and industries, there are more and more applications where no or only limited information is available. In that respect, it is highly desirable to develop models and methods that rely on less information, and make fewer assumptions about the underlying uncertainty. On the other hand, a decision maker may not only lack data and accurate forecasting in a new application, but he may have objectives (e.g. guarantees on worst-case profits) other than maximizing the average performance of a system.

This dissertation focuses on the multi-fare single resource (leg) RM problem with limited information. We only use lower and upper bounds (i.e. a parameter range), instead of any particular probability distribution or random process to characterize an uncertain parameter. We build models that guarantee a certain performance level under all possible realizations within the given bounds. Our methods are based on the regret criterion, where a decision maker compares his performance

to a perfect hindsight (offline) performance. We use competitive analysis of online algorithms to derive optimal static booking control policies that either (i) maximize the competitive ratio (equivalent to minimizing the maximum regret) or (ii) minimize the maximum absolute regret. Under either criterion, we obtain closed-form solutions and investigate the properties of optimal policies.

We first investigate the basic multi-fare model for booking control, assuming advance reservations are not cancelled and do not become no-shows. The uncertainty in this problem is in the demand for each fare class. We use information on lower and upper bounds of demand for each fare class. We determine optimal static booking policies whose booking limits remain constant throughout the whole booking horizon. We also show how dynamic policies, by adjusting the booking limits at any time based on the bookings already on hand, can be obtained. Then, we integrate overbooking decisions to the basic model. We consider two different models for overbooking. The first one uses limited information on no-shows; again the information being the lower and upper bound on the no-show rate. This is appropriate for situations where there is not enough historical data, e.g. in a new business. The second model differs from the first by assuming the no-show process can be fully characterized with a probabilistic model. If a decision-maker has uncensored historical data, which is often the case in reality, he/she can accurately estimate the probability distribution of no-shows. The overbooking and booking control decisions are made simultaneously in both extended models. We derive static overbooking and booking limits policies in either case.

Extensive computational experiments show that the proposed methods that use limited information are very effective and provide consistent and robust results. We also show that the policies produced by our models can be used in combination with traditional ones to enhance the system performance.

ROBUST REVENUE MANAGEMENT WITH LIMITED INFORMATION : THEORY AND EXPERIMENTS

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Dedication

I dedicate my dissertation to the following people in my life:

- My wife, Xiaoli (Serena) Z., whose utmost support made it possible at all for me to finish it in the most challenging time in my life, when our first baby came.
- To my mother-in-law, Wenfang Z., who stayed with us for half a year and helped us out in our most challenging time.
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Chapter 1

Introduction

Revenue management (RM) has become one of the most powerful management science and operations research (MS/OR) applications in the last decade. The success stories span many industries, including air travel, rail transportation, hospitality, cruise ships, and car rental. Common to these listed industries is the ability of a firm to sell its (perishable) inventory well in advance of the actual service time while changing the prices (or changing the availability to different customer segments/groups). There is a significant body of research in RM that proposes tactics, models, and methods to control these advance bookings in order to maximize revenues. The oldest of the RM tactics is *overbooking.* Overbooking is the practice of accepting more reservations than the firm's actual physical capacity to provide the service. This is done as a hedge against the uncertainty that accepted reservations may cancel prior to service time during the booking horizon or may become no-shows at the time of service. Overbooking is regarded as among the most economically important functions of revenue management. Smith et al. (1992) at American Airlines estimated that the benefit of overbooking at American in 1990 exceeded $225 million. According to a recent report, US Airways would have lost *$1 billion or more* of its $11.56 billion revenue in 2006 if the airline had not overbooked (Bailey, 2007). Current statistics on major airlines put the average no-show rate around 8% (Bailey, 2007, Matheu, 2007), but this rate varies significantly depending on the route and season of travel. Cancellations/no-shows are a serious problem for several other industries: The no-show rates for car rentals can be as high as 30% (Wade, 1996). The average no-show rate at the University of Maryland Golf Course was 25% in 2006 (Maynor, 2006).

The earliest papers on overbooking appeared more than 40 years ago (e.g., Beckmann, 1958, Thompson, 1961). Despite decades of research and practice, overbooking is still one of the most problematic when it comes to day-to-day operations. Any customer with advance reservations who has been denied boarding to a flight or has been turned down by the car rental company, or any employee that had to deal with disgruntled customers who have been denied service due to overbooking knows the true implications. Given the attractiveness of the opportunity to make additional revenues on the capacity that would otherwise remain idle, overbooking is a viable business practice. In that respect, *overselling is not an issue, but not knowing the exact number of no-shows is.*

Another critical problem in RM is capacity control, a practice which grew out of the deregulation of the U.S. airline industry in 1978. To compete against the low fares offered by new entrants, major airlines introduced a variety of discounted fares offered with advance-purchase, Saturday-night-stay, non-refundability and other restrictions. But to prevent potential revenue losses, airlines had to carefully control how many seats they allocated to these discounted fares. Thus, RM practice broadened in this post-deregulation period to incorporate capacity control methodology, which focuses on how to optimally allocate capacity to differentiated classes of demand (henceforth *fare-class allocation* or *seat inventory control*). Collectively, fare-class allocation and overbooking practices have produced dramatic improvements in revenues in the airline industry (see Weatherford and Bodily, 1992, Alstrup et al.,1989, and Smith et al., 1992). The hotel/hospitality industry (see Yeoman and Ingold, 1997) has widely adopted seat inventory control techniques as well. National Car Rental is another well-known success story in RM where effective use of forecasting, pricing, inventory control and overbooking resulted in dramatic changes (Geraghty and Johnson, 1997). We refer the reader to Talluri and van Ryzin (2004a) and Phillips (2005) for more information about the RM applications and the well-known research models and methods. Issues associated with application of RM in golf courses is discussed in Kimes and Schruben (2002).

The need for developing accurate forecasting systems first in a successful RM implementations, and challenges associated with lack of data or naive forecasts are emphasized by Lahoti (2002) and Lennon (2004). Even in industries such as airline and hospitality where RM has been effectively used for decades, lack of accurate forecasts is a persistent problem. Despite the need, research on robust methods and approaches that do not rely heavily on information for overbooking and fare-class allocation decisions has been scarce. Traditional research models and analysis rely on several assumptions about demand and cancellations/no-shows such as independence and stationarity, and assume knowledge of probability distributions or stochastic processes characterizing arrivals and/or cancellation/no-shows (see Talluri and van Ryzin, 2004a).

While the revenue management problems with fully characterized probabilistic information are well studied, there are more and more applications where there is only limited information, and it is highly desirable to develop revenue management models with only limited information. This dissertation focuses on the multi-fare single resource (leg) problems with limited information, characterized by only lower and upper bounds on uncertain parameters, instead of any particular probability distribution or random process. We use competitive analysis to determine robust policies for RM. We guarantee a certain performance level under *all* possible realizations within the given lower and upper bounds of the uncertain parameters.

The basic model for seat allocation without considering cancellations or no-shows is investigated in Chapter 3. The only information we use is the lower and upper bound of demand for each fare class. We focus on the best possible static policies whose booking limits remain constant throughout the booking horizon. We also show how dynamic policies, whose booking limits may be adjusted at any time based on the bookings of already on hand, can be obtained.

We then integrate overbooking decisions to the basic model. The problem and methods we investigate apply to a range of single-resource RM contexts such as golf courses and (in simplified form) rail transportation, car rentals, airlines and hotels. It is well known that overbooking and capacity control are two closely related sets of decisions. Our focus is on *joint decision-making* for overbooking and fare-class allocation under *limited information*. We consider two different models for overbooking in Chapters 4 and 5. The first one uses limited information of no-shows; again the information being the lower and upper bound on the no-show rate. This is appropriate for situations where there is not enough historical data, e.g. in a new business. The second model differs from the first by assuming no-show can be fully characterized with a probabilistic model. If a decision-maker has uncensored historical data, which is often the case for an existing business, he/she can accurately estimate the probability distribution of no-shows. The overbooking and booking control decisions are made simultaneously in both extended models.

Our extensive computational experiments in Chapter 6 show that the proposed methods that use limited information are very effective and provide consistent and robust results. It is also shown that the optimal overbooking level determined using the distribution free method can be used to enhance the performance of the traditional fare-class allocation methods that assume no cancellations. This is of practical importance given the challenges associated with estimating the no-show rates and determining appropriate overbooking levels that are the main inputs for seat inventory control optimization in typical revenue management applications. In practice, a virtual capacity (also known as pseudo capacity) - physical capacity plus an overbooking pad based on the no-show rate, cost of overbooking and/or service measures - is determined separately, and then used as the input to seat inventory control optimization. Phillips (2005) mentions this approach has practical advantages in terms of being able to mix-and-match different methods of overbooking and seat inventory control optimization. Our approach, with minimal computational requirements, produces overbooking levels that can be used as an input to any existing seat inventory control optimization model/module. The distribution-free overbooking levels tend to be conservative, leading to savings in overbooking costs and having additional intangible benefits, but with the potential to reduce revenues. However, we show in our computational experiments that the net revenues are not compromised using our overbooking levels when they are coupled with effective seat inventory control methods.

In summary, new approaches and models for the multi-fare single resource revenue management problem with limited information are studied in this dissertation and experiments are carried out to show the effectiveness of these new methods as compared to the traditional methods. Finally, we discuss future research directions in Chapter 7.

Chapter 2

Literature Review

In this chapter, we introduce the single-leg RM problem and provide a review of the research done on this problem.

2.1 Problem definition

The single-leg (-resource) RM problem is one of the most well known problems in RM literature. In the basic form of this problem, the provider has fixed amount of identical capacity available to sell to m distinct classes of customers at different prices in a fixed planning (booking) horizon. Customers come one by one, each customer belongs to a certain class and request for one unit, willing to pay the price assigned for that class. A request for a class (demand in that class), for which no units are available, is considered lost and any unsold units at the end of the selling horizon have no value. The objective is, of course, to maximize economic performance measures, i.e. maximize the revenues or profits.

In the RM context, products and their prices are differentiated on different sale conditions and restrictions, such as in the airline and hotel context. Of course, the underlying reason is the heterogeneity of the customers in the market. In traditional literature, the market segmentation is assumed to be perfect, so that there is a distinct product corresponding to a customer segment (class) and demand will never shift from one class to another. But in reality, no such ideal segmentation exists, instead, customers can often change their choice among the different classes and prices, especially when their primary choice is no longer available. Only in recent few years, the choice behavior of customers is explicitly modeled and integrated into RM decisions. In this dissertation, however, we focus on the traditional "independent demand model" where a customer belongs to a particular class, and leave exploration of choice behavior to future research.

Let us first consider a situation that once a unit is sold, it will never be returned and the sale can not be canceled. In such case, there is no warrant for selling more than the capacity, thus the decision is how to allocate the capacity to the various classes in order to maximize revenues from sales. This kind of decision is called fare-class allocation, originating from the airline industry where RM practice first flourished. At the operational level, the decision is quite simple to state: whether or not to make a unit available to an incoming request, knowing its class upon arrival (i.e. whether to accept or to reject a request), but without knowing what lies in the future. However, identifying effective fare-class allocation policies (or booking control policies) is not that simple. In practice, the types of policies that are used in booking control are limited. Policies with few parameters and ease of implementation are preferred. In the literature, the policies preferred and used by practitioners, with nice structural properties, are proved to be optimal under mild assumptions.

Booking limits are one of the most well known structures, which limit the amount of capacity to be sold to any particular class at a given point in time. For the purpose of illustration, let there be only two classes, where class-1 has higher price and class-2 lower. A booking limit of 11 on class-2 indicates that at most 11 units of capacity can be sold to customers in class-2, and beyond this point there will be nothing available for customers in class-2 and the class is said to be “closed”. Meanwhile, there could still be capacity available for class-1 customers, in which case we say class-1 is still “open” to its customers.

There are basically two types of booking limits: *partitioned* or *nested*. Partitioned booking limits divide the available capacity into m separate blocks (or buckets) in an exclusive way, such that each class can use only the capacity in its own bucket. In contrast, nested booking limits allow the capacity in one class to be sold not only to customers in that class, but to customers of other classes with different fares. Nested booking limits are easy to implement especially when the classes are ranked by their fares. Consider class-1 to have a higher fare than class-2. Under nesting, the booking limit for class-2 determines the maximum number of units to be sold to class-2, while any excess (when class-2 demand is lower than the booking limit) capacity can be used to satisfy the demand of class-1. Suppose the booking limits are 11 and 5 for classes 2 and 1, respectively, and there are 8 class-2 requests and 6 class-1 requests. With partitioned booking limits, only 5 out of 6 class-1 requests is accepted while all 8 class-2 requests are accepted, resulting in 3 unused units and 1 lost sale. This outcome is independent of the sequence of arrivals of requests. If we employ nested booking limits instead, all requests can be accepted, with no more than 11 units available for class-2 and up to 11+5=16 units available for class-1. The net result is higher customer satisfaction and resource utilization, as well as higher revenue in this example. Nested booking limits are commonly used in airline RM practice. There are examples of use of partitioned booking limits in railway applications. In Chapters 3 and 4 of this dissertation, we prove that nested booking limits are the optimal policy in our static problems.

Up to now we have assumed no returns or cancellations, but that is far from reality in most practical situations. Cancellations are reservations terminated by the customers strictly before the time of service, while no-shows occur when the customers simply do not show up at the time of service without any notice before hand. If a carrier does not allow for overbooking, i.e. selling more units than the physical capacity, estimations show that about 15% of all seats would go unsold. Hence overbooking in airline RM is desirable to avoid wasting of capacity (Smith et al. 1992). Overbooking increases the total sales volume beyond the total available capacity in anticipation of significant cancellations and no-shows. Clearly, too aggressive overbooking can lead to high service denials, when number of customers that show up exceeds the physical capacity. One must has to consider this trade-off in decreasing spoilage (waste in capacity) and increasing spill (denied service) in deciding exactly how much to overbook.

In summary, there are critical of decisions to be made in the single-leg RM problem: fare-class allocation and overbooking. In the next section we provide a brief review of the literature on these two topics.

2.2 Literature review

The recent book of Talluri and van Ryzin (2004a) provides a comprehensive overview of revenue management, including both seat inventory control and overbooking. Since the seminal work of Littlewood (1972) on the single-resource, multi-fare problem which assumes demand is monotonic in fares (i.e., lowest-fare is requested first) and its extension in Belobaba (1987,1989), the seat inventory control problem has attracted a lot of attention. Various formulations that differ in their assumptions primarily about the uncertainty in demand are proposed (e.g., Curry, 1990, Lee and Hersh, 1993, Brumelle and McGill, 1993, Robinson, 1995, Brumelle and Walzcak, 2003), including customer-choice based models (e.g., Talluri and van Ryzin, 2004b). We refer the reader to Talluri and van Ryzin (2004a) for the details about the extant literature on the single-resource seat inventory control problem. Interestingly, the bulk of the research on the single-resource problem disregards cancellations/no-shows; hence does not make any overbooking decisions.

Rothstein (1985) provides a very readable account of the history of overbooking in the airline industry. Similarly, Ratliff (1998) presents a survey and focuses on practical problems in overbooking decisions. The earliest overbooking models in the literature either take a cost-based approach (e.g. economic models that balance the cost of overselling with the opportunity cost of empty seats) and/or a service-level-based approach (e.g. a bound on the expected number of passengers denied service or the probability that a passenger is denied service because of overbooking). Beckmann (1958) provides a static, single period cost-based model, to determine an upper bound on the number of reservations to accept. Thompson (1961) shows a way to determine overselling probabilities for a static, single leg problem. His work is refined by Taylor (1962), and Rothstein and Stone (1967). Shlifer and Vardi (1975) provide static cost-based and service-level-based models for both a single-leg flight carrying two types of passengers and a two-leg flight. A multi-fare, single-resource model is analyzed in Coughlan (1999) assuming fare-class demand is Normal distributed. Several researchers have addressed dynamic models of overbooking for a single-resource (Chatwin, 1992, 1999, Rothstein, 1971, Subramanian et al., 1999, Alstrup et al., 1986). Models for the hotel industry are presented in Rothstein (1974), Bitran and Gilbert (1996) and Liberman and Yechiali (1978). Karaesmen and van Ryzin (2004), present a model that solves an overbooking problem where the resources are substitutable (e.g. hotel rooms, sequential flights on the same route, different types of car in a car-rental).

All of these models make assumptions about the nature of demand and cancellations/no-shows: Typically, probability distributions or stochastic processes are used to model uncertainty in demand and/or cancellations; independence across fare-classes are typically assumed, demand and cancellations are assumed to be Markovian, and so on. Recent research in revenue management questions the availability of information and some of the modeling assumptions, including the risk-neutrality of the decision-maker. There are several papers on robust pricing decisions and use of learning-based methods in pricing optimization (e.g., Rusmevichientong et al., 2006, Lim and Shanthikumar, 2006); cancellations are irrelevant in that context.

Here we mention the recent work on single-resource RM with no- or limited-information. Adaptive methods that assume no-information about the demand in each fare class have been proposed to compute the optimal nested booking limits (see van Ryzin and McGill, 2000, Huh and Rusmevichientong, 2006, Kunnumkal and Topaloglu, 2007). These methods update booking limits iteratively based on past observations, require arrivals that are monotonic in fares (we call this low-before-high, or LBH), learn or adapt from flight to flight but not during the booking control horizon of a particular flight. Eren and Maglaras (2006) also require no information on demand, and sequentially update booking control parameters and estimates of demand that are obtained using the maximum entropy method. Ball and Queyranne (2006) use competitive analysis of on-line algorithms and provide closed-form optimal booking limits using no information on demand. Perakis and Roels (2006) assume the demand in each fare class lies in a given interval, but make no further distributional assumptions. Birbil et al. (2006) assume there are inaccuracies associated with discrete probability distributions that characterize the demand in each fare class.

In the practice of airline RM, overbooking is carried separately from seat inventory control. Virtual capacities calculated externally become the main input to seat inventory control optimization (Phillips, 2005). Belobaba (2006) explains how practical rules are used to compute the virtual capacities when point estimates of no-show rates are given or when both point estimate and the standard error are known. In the former case, deterministic rules provide the virtual capacity, and in the latter, either the virtual capacity is computed based on measures of service levels, or a by solving a stochastic, cost-based model similar to a news-vendor formulation assuming no-show rate is Gaussian distributed.

In this dissertation, we propose a model for fare-class allocation in Chapter 3, and then extend this model for joint overbooking and fare-class allocation decisions in Chapters 4 and 5. Our models use limited information on demand and/or no-show rates. We use competitive analysis of on-line algorithms which benchmarks the performance of a policy to one that has hindsight information, extending the work of Ball and Queyranne (2006). We do not require a risk neutrality assumption in the analysis. Using computational experiments, we show robustness and effectiveness of our decisions. We also show the overbooking levels (virtual capacities) computed using our approach can be used as inputs to other seat inventory control optimization models and increase their effectiveness. This latter is of special practical importance.

Chapter 3

Fare-Class Allocation with Limited Information

In this chapter we study the fare-class allocation problem assuming no cancellations. Throughout this dissertation, let n denote the total capacity of the resource (seats, rooms, etc.) available and $m \geq 2$ the number of fare-classes. Let f_i denote the fare for class i, where $f_1 > f_2 > \cdots > f_m \geq 0$. There are no restrictions on the demand or arrival process except that requests arrive one-by-one and cancellations are not allowed. In our analysis, we assume each request demands only one unit but this can be generalized as we discuss below.

3.1 Problem Definition

We consider the single-leg RM problem from the perspective of *competitive analysis of on-line algorithms* (see the survey in Albers, 2003). This perspective evaluates the performance of a booking control policy relative to the performance of an offline algorithm that considers the entire input sequence simultaneously. An offline optimal solution is a solution obtained by an offline algorithm (with hindsight) that optimizes the objective function of interest. In competitive analysis, it is common to use *competitive ratio* (CR) as a measure of an algorithm's effectiveness. There are two other performance metrics of interest: *absolute regret*, which is the difference between the objective function values of the offline and on-line algorithms, and *relative regret*, which is the ratio of the absolute regret to the objective function value of the offline algorithm. Specifically, we are interested in having a guarantee on these measures under all possible demand scenarios (or 'input sequences' as we call them in our analysis).

CR is defined as the *minimum* of the ratio of revenues obtained by the on-line algorithm to the offline revenues. If we let Ω_Υ be the set of all possible input sequences to an on-line algorithm Υ and, for any $I \in \Omega_\Upsilon$, let $R(I;\Upsilon)$ be the objective value achieved by the on-line algorithm for input I and let $R^*(I)$ be the objective value achieved by an optimal offline algorithm. Then, CR is defined as:

$$\text{CR of } \Upsilon \quad = \quad \inf_{I \in \Omega_\Upsilon} \frac{R(I;\Upsilon)}{R^*(I)}.$$

Likewise, the maximum absolute regret (MAR) of the on-line algorithm is defined as

$$\text{MAR of } \Upsilon \quad = \quad \sup_{I \in \Omega_\Upsilon} |R^*(I) - R(I;\Upsilon)|.$$

By definition, CR is related to *maximum relative regret* (MRR): $CR = 1 - MRR$. Clearly, the algorithm/policy that maximizes the CR also minimizes the MRR. We use CR and MAR in our analysis. Note that these performance measures as defined above apply to deterministic algorithms, i.e., the algorithm applies the same decision rule and yields the same performance given the same input sequence as opposed to a "randomized" algorithm. While not as practical as deterministic ones, randomized algorithms are of special interest from a technical standpoint; discussion on this topic is deferred until Section 3.2.4. For now, we are interested in determining the best deterministic algorithm that maximizes the CR (or minimizes the MAR).

Based on the definition of robustness in Kouvelis and Yu (1997), maximizing the CR (minimizing the MAR) leads to decisions that are *relative-robust* (*deviation-robust*).

We are interested in determining nested booking limits $n \geq b_1 \geq b_2 \geq \cdot \geq b_m \geq 0$ that maximize the CR or minimize the MAR. Note that *nesting by revenue order* remains the core of booking control in many traditional RM implementations. Although Talluri and van Ryzin (2004b) discuss that these types of nested policies may not be optimal in general, we show later in the text that they are optimal relative to the CR and MAR criteria.

The booking limit b_i defines the maximum number of booking requests to be accepted in classes i to m. We use an additional variable called *bucket size* - denoted x_i - in our analysis; we define it as $x_m = b_m$ and $x_i = b_i - b_{i+1}$ for $i = 1, ..., m-1$. The notation b and x are used for vectors of variables. Note either of these vectors is sufficient to characterize a nested booking control policy. The *protection levels* that are commonly used in single-leg RM are easily derived from the booking limits (see Talluri and van Ryzin, 2004b). The number of seats protected for classes $j = 1, .., k$ from classes $k+1$ to m is equal to $n - b_{k+1}$, $k = 1, ..., m-1$. While our analysis is carried out with b and x, we interpret some of our results with respect to protection levels, as well.

In simple terms, competitive analysis of on-line algorithms is based on worst-case analysis: One can think there is an adversary in charge of generating booking requests. The adversary is aware of the algorithm (nested booking limits in our case) and chooses an input sequence (the number of requests and the arrival sequence) to minimize the algorithm performance (i.e., so that the algorithm achieves the lowest

CR or highest AR). Given the characterization of input sequences the adversary would choose, we determine the optimal parameters for the algorithm to maximize the CR (alternatively, to minimize the MAR).

By nature of the single-leg RM problem, the demand is discrete, although in many research papers, the analysis is done assuming continuity. In this dissertation, we distinguish between the *continuous problem* and *discrete problem*. In the *continuous problem*, the demands, and therefore the protection levels as well, may be any non-negative real numbers. In this case, any request in a fare-class is *partially accepted* (split). Although the continuous problem is less realistic, its analysis is simpler. The continuous case easily generalizes to multi-unit and multi-fare requests (as in batch arrivals) by allowing splitting of requests. Our analysis carries through as long as each request demands a non-negative and finite amount. Note that group bookings cannot be enforced/guaranteed because of splitting. In the *discrete problem*, the demands are integral and the protection levels are restricted to being integer. Our analysis of the discrete problem extends to multi-unit and multi-fare requests as long as requests are split into integral quantities. In the rest of the dissertation, we assume - without loss of generality - input sequences consist of requests that demand one unit of a fare-class each.

3.2 Optimal Static Policies for the Competitive Ratio Problem

In this section, we analyze the multi-fare single-leg problem and derive optimal booking control policies under the CR criterion assuming upper and lower bounds are available on demand in each fare class. That is, for each fare class i, we assume the input sequences, I, are restricted so that the total number of units demanded in class i falls between L_i and U_i, where L_i and U_i are integers with $0 \leq L_i \leq U_i$ for $i = 1, ..., m$.[1] We use the notation L and U to denote the respective vectors $(L_1, ..., L_m)$ and $(U_1, ..., U_m)$. Let $\Omega(L, U)$ be the set of all sequences where the total demand in each fare-class falls between the upper and lower bounds, $R^*(I)$ be the offline revenue obtained from sequence I, and $R(I; b)$ be the on-line revenue gained by a standard nested booking limit policy b. The following formulation, taking demand bounds into consideration, solves for the optimal booking control policy that maximizes the CR:

$$z^{CR} = \max_{b} \ z \ : \ z \leq \frac{R(I;b)}{R^*(I)}, \forall I \in \Omega(L, U) \tag{3.1}$$

where b is the vector of decision variables and z^{CR} is the optimal CR. There are some potential challenges in solving this maximization problem: 1) $R(I; b)$ might not have closed-form expression; 2) $|\Omega(L, U)|$ grows exponentially with m, prohibiting any serious direct attempt with even small problems. However, since there will be redundant constraints, which means the corresponding sequences are redundant as well, a sequence reduction approach seems natural, and, as we will show in the next two sections, this maximization problem can be rewritten in a very compact form. The solution to the above problem will result in a static control policy.

[1] It is typical in the robust optimization framework to assume uncertain parameters belong to a polyhedral set. For instance, Bertsimas and Sims (2004) assume parameters lie in a known interval centered at a nominal value. Similarly, we assume only "range forecasts" are given and demand lies in a given interval.

3.2.1 Sequence Reduction

A sequence consists of a finite stream of fare requests during the booking horizon. Since each request in a sequence demands one unit, we can characterize the sequences based on the fare-classes and units demanded: Let $I[j]$ be the total number of class j requests in sequence I. A *profile* of I is an m-dimensional vector of $[I] = (I[j] : j = 1, ..., m)$. Observe that offline optimal revenue realized in sequence I *only* depends on the profile, while on-line revenue (of a policy) also depends on the arrival order of requests. However, given a profile, we can ignore permutations of requests that do not yield the lowest on-line revenue (see Appendix A.1 of the On-line Addendum for the proof).

Proposition 1 *Relative to all input sequences with the same profile, a nested booking limit policy b generates the least revenue when applied to the unique LBH sequence with that profile. This is true for both the continuous and discrete problems.*

From the standpoint of the maximization problem in (3.1), all non-LBH sequences can be discarded based on Proposition 1. In the remaining sequences, there is a one-to-one relationship between sequences and profiles. This yields a substantial reduction in the size of the problem. Yet, the total number of profiles can be prohibitive. Therefore, we pursue further sequence reductions to define a tractable problem. We now introduce a categorization of input sequences, which is even a broader concept than profiles. The ultimate goal is to choose only one sequence in each category. For mathematical completeness, we now introduce a virtual fare-class $m + 1$ where $b_{m+1} = 0$, $f_{m+1} = 0$, and $L_{m+1}, U_{m+1} \geq 0$ are arbitrary, non-negative numbers.

Definition 1 *(Input Sequence Category) An input sequence category A_j^b is the set of sequences such that an input sequence I belongs to category A_j^b under policy b, if and only if j is the lowest-fare class whose booking limit is reached after executing b on the sequence I.*

To better understand the notion of a category, consider the following: First of all, the virtual class has $b_{m+1} = 0$, so at least the booking limit of class $m+1$ is always reached, and j is well defined. By construction, A_{m+1}^b includes all sequences for which no request is rejected when booking limit vector is b. Therefore, $\frac{R(I;b)}{R^*(I)} = 1$ for all $I \in A_{m+1}^b$. Second, some categories may be empty. For instance, in the trivial case of $b_i = n$ for all $i = 1, ..., m$ and $\sum_{i=1}^m U_i < n$, no booking limit (except for the virtual class) would be reached when policy b is applied to the sequences in $\Omega(L, U)$. In that case, all A_j^b for $j = 1, .., m$ would be empty, and the only category that would be non-empty is A_{m+1}^b. In fact, $A_{m+1}^b = \Omega(L, U)$ in this case. Finally, the categorization is complete, meaning $\cup_{j=1}^{m+1} A_j^b = \Omega(L, U)$ for all b.

Appending more requests in classes $k \geq j$ to a sequence in category j will not change the on-line revenue of policy b, since all such requests would be rejected. In addition, any permutation of the order of requests in a sequence will not change its category, which means the category is totally determined by the profile of the sequence. Note the notion of a category applies to both discrete and continuous problems, and so does our next result (see On-line Addendum, Appendix A.2 for the proof).

Definition 2 *(CAtegory-dominant-STream, or CAST) Given $j = 1, ..., m$, a LBH sequence I is called a category-dominant-stream (CAST) if I has the profile of $I[k] = U_k$ for $k \geq j$ and $I[k] = L_k$ for $k < j$.*

There are m CASTs in total. We denote each one by $CAST_j$, $j = 1, .., m$. By definition, $CAST_j[k] = U_k$ for $k \geq j$ and $CAST_j[k] = L_k$ for $k < j$.

Proposition 2 *(Dominance of CAST) Consider a nested booking limit policy b and all input sequences in some category A_j^b. $CAST_j \in A_j^b$ dominates the other sequences in category A_j^b, i.e., $\frac{R(I;b)}{R^*(I)} \geq \frac{R(CAST_j;b)}{R^*(CAST_j)}$ for all $I \in A_j^b$.*

For any nested policy, Proposition 2 effectively reduces the number of sequences to be considered, down to m, disregarding the virtual class $m+1$ (which is not critical since $\frac{R(I;b)}{R^*(I)} = 1$ for all $I \in A_{m+1}^b$). With these reductions, problem (3.1) can be reformulated as

$$z^{CR} = \max_b \quad z: \quad z \leq \frac{R(CAST_k;b)}{R^*(CAST_k)} \quad for \quad k = 1, ..., m. \tag{3.2}$$

We next show how the optimal solution to the reduced formulation can be obtained by defining an appropriate linear programming model. We first focus on the continuous problem.

3.2.2 A Linear Programming Formulation for the Continuous Problem

In determining the optimal booking control policy, we use the bucket sizes x_i, $i = 1, ..., m$, as the decision variables. We make two observations regarding the optimal policy below.

First, if $\sum_{i=1}^{m} U_i \leq n$, then the optimal bucket sizes are $x_i^* = U_i$, $\forall i$ and $z^{CR} = 1$. Second, if there is a class k such that $(n - \sum_{i=1}^{k} L_i) < 0$, then any reasonable policy would have $x_j = 0$ for all $j > k$, $k = 1, ..., m-1$, and the problem size can be reduced by ignoring classes $j > k$ in that case. This second observation relies on the following: Remember that the lower bounds imply that all sequences have at least L_i requests of class i. Consider the quantity $\sum_{i=1}^{k} L_i$; this represents the number of seats that are *guaranteed* to be sold to classes 1 to k. Consequently, any reasonable policy would protect at least that many seats for classes 1 to k. Hence, the remaining seats, i.e., $(n - \sum_{i=1}^{k} L_i)^+$, are the total number of seats that would be considered for sale for classes $k+1$ to m. Therefore, the m-fare problem is only interesting when $\sum_{i=1}^{m} U_i > n$ and $\sum_{j=1}^{m-1} L_j < n$. We impose these two conditions on the problem parameters to keep our problem statement general, the notation simpler and solutions to the problem non-trivial.

To express the term in the right hand side of the constraints in problem (3.2), we study properties of nested policies under the CASTs. Note that any reasonable policy would have $x_1 \leq U_1$ so as not to protect any seats for class-1 that would definitely remain unsold. Likewise, $x_i \leq U_i$ for all $i = 1, .., m$; otherwise there would be a class $k \geq 1$ whose bucket would have a "slack" (i.e., the booking limit of class k would not be reached). Naturally, we require that $\sum_{i=1}^{m} x_i \leq n$ so as not to exceed the capacity.

By definition, the profile of CASTs is known. We further define $\rho(k) = \min\{j : \sum_{i=1}^{j} CAST_k[i] > n,\ 1 \leq j \leq m\}$ which denotes the index of the lowest-fare class to be accepted in the offline optimal solution (requests of lower fares are rejected) when the input is $CAST_k$. Then, we can express $R^*(CAST_k)$, the offline optimal revenue for $CAST_k$, as

$$R^*(CAST_k) = \left(n - \sum_{j=1}^{\rho(k)-1} CAST_k[j]\right) f_{\rho(k)} + \sum_{j=1}^{\rho(k)-1} CAST_k[j] f_j.$$

Next, we focus on the on-line revenue obtained from the CASTs. For any non-trivial m-fare problem, the on-line revenue generated from $CAST_k$ can be considered in two parts: 1) from the L_i requests in classes $i < k$, with a subtotal revenue *at most* $\sum_{i=1}^{k-1} f_i L_i$ (because not all L_i may be accepted based on the bucket sizes), and 2) from the U_i requests in classes $i \geq k$, from which revenue is $\sum_{i=k}^{m} x_i f_i$ (note $x_i \leq U_i$, from our discussion above). Then, $R(CAST_k; b)$, the on-line revenue obtained by policy b and corresponding bucket sizes x satisfy $R(CAST_k; b) \leq \sum_{i=1}^{k-1} f_i L_i + \sum_{i=k}^{m} f_i x_i$. Therefore, we have an upper bound on the ratio of on-line revenues to that of offline, and this upper bound is linear in the bucket sizes x_j:

$$\frac{R(CAST_k; b)}{R^*(CAST_k)} \quad \leq \quad \frac{\sum_{i=1}^{k-1} f_i L_i + \sum_{j=k}^{m} f_j x_j}{R^*(CAST_k)}.$$

Combining these observations, we can formulate a linear program (LP) that finds a continuous, nested policy. We call this formulation the *General Bounded Model (GBM)*:

$$\begin{aligned} \textbf{GBM}: \quad \bar{z}^{CR} \quad = \quad & \max_x \quad z \\ & \text{s.t.} \quad R^*(CAST_j) z \leq \sum_{i=1}^{j-1} f_i L_i + \sum_{i=j}^{m} f_i x_i, \quad j = 1, ..., m \qquad (3.3) \\ & \qquad \sum_{i=1}^{m} x_i \leq n, \qquad (3.4) \\ & \qquad 0 \leq x_j \leq U_j, \quad j = 1, ..., m. \qquad (3.5) \end{aligned}$$

Although GBM has been developed based on a necessary set of constraints and relations that constitute an *upper bound* on the optimal CR, the solution to GBM provides the optimal nested policy for the CR problem introduced in (3.1). We formalize this statement and discuss how a closed-form solution to GBM can be derived in the next section.

Note that the actual CR problem expressed in (3.1) is not only reduced to a problem with a small number of constraints based on CAST, but its optimal policy in the continuous version is obtained by solving a LP with $2m+1$ constraints. The multi-fare continuous booking problem discussed in BQ can be represented using GBM. Theirs is a special case: They use no information about the demand, hence $L_i = 0$ and $U_i = \infty$ (or effectively $U_i \geq n$) for $i = 1, ..., m$ in their problem.

3.2.3 Optimal Policy for the Continuous Problem: Solution to GBM

Analysis of GBM relies heavily on the investigation of the intrinsic structure of the LP and the relationship among the parameters f_i, U_i, L_i for $i = 1, ..., m$, and n; details are provided in Appendix A.3 in the On-line Addendum. An important step in the analysis is to determine the 'critical' fare class $u \in \{1, ..., m\}$ such that classes $j > u$ would be closed (i.e., booking limit of $j > u$ would be zero) in the optimal solution. Once u is known, the solution to GBM can be determined by solving a set of linear equations that are composed of the binding constraints in the LP. We derive a closed-form solution to the GBM based on these observations. The proof of the next result is also available in the On-line Addendum, Appendix A.3.

Proposition 3 *(a) The optimal solution of GBM is*

$$\bar{z}^{CR} = \frac{R_u^+/f_u + N_u}{R^*(CAST_u)/f_u + \sum_{i=1}^{u-1} g_i} \tag{3.6}$$

$$x_j^{CR} = \begin{cases} g_j \bar{z}^{CR} + L_j & j < u \\ (R^*(CAST_u)\bar{z}^{CR} - R_u^+)/f_u & j = u \\ 0 & j > u \end{cases} \tag{3.7}$$

$$u = \max\{j \le m : \quad R_j^+ \sum_{i=1}^{j-1} g_i < N_j R^*(CAST_j)\} \tag{3.8}$$

where the index u denotes the critical fare-class such that all classes $k > u$ are closed, and g_i, N_j, R_j^+ are auxiliary parameters defined as

$$g_m = \frac{R^*(CAST_m)}{f_m}, \quad g_i = \frac{R^*(CAST_i) - R^*(CAST_{i+1})}{f_i}, \quad i = 1, \cdots, m-1, \tag{3.9}$$

$$N_j = n - \sum_{i=1}^{j-1} L_i, \quad R_j^+ = \sum_{i=1}^{j-1} f_i L_i \quad j = 1, .., m. \tag{3.10}$$

(b) The nested booking limits defined by

$$b_j^{CR} = \sum_{i=j}^{m} x_i^{CR} \text{ for } j = 1, \cdots, m \tag{3.11}$$

maximizes the CR in problem (3.1) and the optimal CR is $z^{CR} = \bar{z}^{CR}$.

Note that our analysis so far yields the optimal solution within the class of nested policies. We used reductions in the number of possible input sequences and formulated the GBM based on the properties of nesting. One question that remains to be answered is whether the nested policies are the best. The next result shows that, in fact, this is true (see the On-line Addendum, Appendix A.4 for the proof).

Theorem 1 *For the continuous m-fare problem with demand bounds, the nested booking control policy with booking limit vector b^{CR} defined by (3.11) has a CR of $\bar{z}^{CR}$ given by (3.6) and this is the best possible among deterministic policies.*

If all lower bounds are zero, the above result can be simplified, and the optimal protection levels can be expressed in terms of n and g_j for $j = 1, ..., m$. Although this is only a special case, we provide a formal proof of the next result in Appendix A.5 in the On-line Addendum.

Corollary 1 *For the continuous m-fare problem with all lower bounds equal to zero, the nested booking control policy with booking limits defined by:*

$$b_i^{CR} = n\frac{\sum_{j=i}^{m} g_i}{\sum_{j=1}^{m} g_j} \text{ for } i = 2, ..., m$$

has a CR of $n/(\sum_{j=1}^{m} g_j)$ and this is the best possible among deterministic policies.

Note that for the special case of $L_j = 0, U_j \geq n$ for all $j = 1, ..., m$, we have $R^*(CAST_j) = f_j n$, and $g_i = n(1 - f_{i+1}/f_i)$. This, in fact, defines the optimal nested policy obtained by BQ.

3.2.4 Randomized Policies

Our focus so far has been on deterministic algorithms. However, the performance of an on-line algorithm can sometimes be improved by allowing the use of randomization strategies. Specifically, a randomized algorithm makes random choices from a set of deterministic algorithms to process an input. To define a randomized algorithm Υ, we start by defining a set of deterministic policies $\{\Upsilon_1, \Upsilon_2, ...\}$. Prior to observing the input sequence, a deterministic policy is randomly chosen from the given set and used in processing the input. The same input sequence may result in different outputs when a randomized policy is executed. The random choice prior to each execution of the algorithm is a random variable that is independent and identically distributed with a known probability distribution. Let $\boldsymbol{\Upsilon}$ be a random variable defined over the set $\{\Upsilon_1, \Upsilon_2, ...\}$ given the probabilities $P(\boldsymbol{\Upsilon} = \Upsilon_j)$. The expected objective function value of the randomized algorithm Υ is then $E[R(I; \boldsymbol{\Upsilon})]$. Consequently, the CR of a randomized algorithm is then defined based on the *expected performance*:

$$\text{CR of } \Upsilon \quad = \quad \inf_{I \in \Omega_{\tilde{\Upsilon}}} \frac{E[R(I; \boldsymbol{\Upsilon})]}{R^*(I)}.$$

Randomized algorithms have received substantial attention within the computer science literature. We recognize that randomized policies may not be very desirable in practice (confusing the users and possibly the buyers). However, we believe our analysis is of theoretical interest and also provides further insight into the performance obtained by rounding the continuous policies.

Given the class of randomized policies for the single-leg RM problem, we can show that no policy can achieve a higher CR than $\bar{z}^{CR}$ given in (3.6). While this seems as a trivial extension of our previous results, analysis of the randomized policies requires a different approach in general and we use a different proof technique for the next result (see Appendix A.6 in the On-line Addendum).

Theorem 2 *For the continuous m-fare problem with demand bounds, no randomized booking policy has a CR larger than* $\bar{z}^{CR}$ *given in (3.6). Therefore, the deter-*

ministic nested booking control policy with booking limits defined by b^{CR} *in (3.11) is the best possible among* <u>*all*</u> *policies.*

As this result implies, *nesting by revenue order* provides the optimal policy for the single-leg RM problem where the objective is to maximize the CR under limited demand information.

3.2.5 Solution to the Discrete Problem

In most practical settings whole (or individual) units of a product are sold indicating that a realistic analysis should consider the discrete problem rather than the continuous one. On the other hand, we can view the continuous problem as a (possibly very close) approximation for the discrete one in that, rounding the booking limits up or down will have a relatively small impact on overall performance. We note that with integer upper and lower bounds rounding always produces feasible booking limits. To develop a randomized procedure for the discrete problem, we first consider the solution x^{CR} of GBM given in equation (3.7). If this solution is non-integral, we create deterministic policies by rounding up or down each x_i^{CR}, $i = 1, ..., m$. Based on the rounding scheme, we define probabilities associated with each policy. We provide the details on how the policies and probabilities are determined along with a proof of existence of a probability distribution associated with the randomized algorithm in the On-line Addendum, Appendix A.7. Based on the analysis in Appendix A.7, we can show that the expected performance of the randomized policy is exactly the same as the performance of the best deterministic policy for the continuous problem, which is our desired result:

Proposition 4 *For the discrete m-fare problem with demand bounds, an appropriately defined randomized policy achieves a CR of $\bar{z}^{CR}$ which is given in (3.6).*

While this result shows that a (randomized) policy exists for solving the discrete problem, it (possibly more importantly) provides justification for creating discrete policies by selectively rounding continuous policies.

3.3 Optimal Static Policies for the Absolute Regret Problem

The analysis in Section 3.2 easily extends to the problem with the absolute regret criterion where the objective is to minimize the MAR. Let us first express this problem in its general form:

$$z^{AR} = \min_{b} \; z \; : z \geq R^*(I) - R(I; b), \forall I \in \Omega(L, U) \tag{3.12}$$

where b is the vector of decision variables and z^{AR} is the optimal MAR. We can reformulate this problem by reducing the number of input sequences using Propositions 1 and 2. We only have to consider the CASTs as inputs. The following LP model (called GBM-AR) is designed to provide a *lower bound* on the minimum MAR in the continuous problem:

$$\begin{aligned} \textbf{GBM}-\textbf{AR}: \quad \bar{z}^{AR} = \min \quad & z \\ \text{s.t.} \quad & R^*(CAST_j) - z \leq \sum_{i=1}^{j-1} f_i L_i + \sum_{i=j}^{m} f_i x_i, \quad j = 1, \ldots, m \qquad (3.13) \\ & \text{and} \;\; (3.4), (3.5). \end{aligned}$$

This model is actually easier to analyze than the GBM model, and the closed-form solution is given below (proof is omitted):

$$\bar{z}^{AR} = R^*(CAST_{\tilde{u}}) - R^+_{\tilde{u}} - f_{\tilde{u}} x^{AR}_{\tilde{u}} \tag{3.14}$$

$$x^{AR}_j = \begin{cases} g_j + L_j & j < \tilde{u} \\ N_{\tilde{u}} - \sum_{i=1}^{\tilde{u}-1} g_i & j = \tilde{u} \\ 0 & j > \tilde{u} \end{cases} \tag{3.15}$$

$$\tilde{u} = \max\{j \leq m : \sum_{i=1}^{j-1} g_i < N_j\} \tag{3.16}$$

where $\tilde{u}$ is the critical fare class such that classes $k > \tilde{u}$ are closed, and the auxiliary parameters g_i, N_j, R^+_j for $j = 1, .., m$ are as defined in equations (3.9) and (3.10).

Following a similar argument for the CR problem, we can show that the optimal solution of GBM-AR provides the optimal MAR of the problem in (3.12)[2]. In addition, the optimal nested policy obtained by GBM-AR is the best possible under the MAR criteria (proof is omitted). Hence, nesting leads to both *deviation-robust* and *absolute-robust* decisions in the single-leg RM problem with limited demand information.

Theorem 3 *For the continuous m-fare problem with demand bounds, the nested booking control policy defined by the booking limits*

$$b^{AR}_j = \sum_{i=j}^{m} x^{AR}_i \text{ for } j = 1, ..., m \tag{3.17}$$

[2]Perakis and Roels (2006) develop a LP model, which is slightly different than ours, and obtain the same closed-form solution for the MAR problem. Their analysis of the MAR problem is significantly different, and they do not present a formal proof of optimality of nesting.

has minimum MAR of $\bar{z}^{AR}$ *(where* x_i^{AR} *and* $\bar{z}^{AR}$ *are given in (3.15) and (3.14)). No other policy, deterministic or randomized, has a lower MAR.*

A close look at the optimal policies of CR and MAR problems, given in equations (3.6)-(3.8) and (3.14)-(3.16), respectively, reveals that $u \geq \tilde{u}$. Therefore, MAR tends to deny bookings to a higher number of fare classes. Furthermore, $x_i^{CR} \leq x_i^{AR}$ for $i < \tilde{u} \leq u$ and $\sum_{i=1}^{u} x_i^{CR} = \sum_{i=1}^{\tilde{u}} x_i^{AR} = n$ at optimality. Our next result combines these observations:

Proposition 5 *In the continuous problem with demand bounds, the optimal nested booking limits obtained by (3.11) and (3.17) satisfy:* $b_i^{AR} \leq b_i^{CR}$ *for* $i = 1, ..., m$.

While competitive analysis of algorithms using CR and MAR criteria provides conservative solutions to guarantee worst-case performance, the above result shows that MAR criteria is more *aggressive in protecting seats,* i.e. seats available for lower-fare classes are fewer and higher number of seats are protected for higher fares.

Our analysis of the randomized policies and the discrete problem as presented in Sections 3.2.4 and 3.2.5, respectively, can be extended to the MAR problem. Details are omitted.

3.4 Dynamic Policies

In this section, we show how dynamically adjusting a static policy can improve the CR (or MAR) in our problem. Our analysis extends the discussion in BQ (who provide a direction for the CR problem only for $m = 2$) to multiple fare classes with demand bounds. Using the imaginary adversary paradigm, the CR (or MAR) of a policy effectively assumes that the adversary always sticks to an "optimal" strategy (i.e., adversary sends LBH inputs). On the other hand, if the adversary "makes a mistake", we can consider a new problem scenario based on the remaining capacity and requests accepted so far, create a new policy for the remaining requests in order to guarantee better overall performance.

Consider a dynamic scenario where a dynamic policy has been executing so that a partial input sequence has already been processed. Suppose h_i bookings have been accepted for fare class i for $i = 1, ..., m$ by processing the partial sequence I_0. This accumulates revenue of $\sum_{i=1}^{m} h_i f_i$ from the $\sum_{i=1}^{m} h_i$ sold seats. The question is whether the booking limits can be adjusted to improve the CR (MAR) achievable under a future input sequence $\hat{I}$, which yields the complete input sequence of $I = I_0\hat{I}$. The concatenation of I_0 and $\hat{I}$ into the complete input sequence I produces the following profile: $L_j \leq I[j] = I_0[j] + \hat{I}[j] \leq U_j, \quad j = 1, ..., m$. Since $I_0[j] \geq h_j, j = 1, ..., m$, it follows that $I[j] \geq h_j$ must hold, so that we can update lower and upper bounds for $\hat{I}$ as follows:

$$\hat{L}_j = \max(L_j, h_j) - h_j, \quad \hat{U}_j = U_j - h_j, \quad j = 1, ..., m.$$

Let $\hat{n} = n - \sum_{i=1}^{m} h_i$ denote the remaining number of available seats, $\hat{b}$ the nested booking limits for allocating the remaining $\hat{n}$ seats, and $\hat{x}$ the corresponding bucket sizes.

Given a partial policy $\hat{b}$ and partial input sequences $I_0, \hat{I}$, the overall CR can be improved by solving

$$\hat{z}^{CR} \quad = \quad \max_{\hat{b}} z : z \leq \frac{\sum_{i=1}^{m} h_i f_i + R(\hat{I};\hat{b})}{R^*(I_0\hat{I})}, \quad \forall \hat{I} \in \Omega(\hat{L},\hat{U}). \tag{3.18}$$

where $\hat{z}^{CR}$ is the new guarantee on the worst-case CR performance. Similarly, the improved minimum MAR of $\hat{z}^{AR}$ is determined by:

$$\hat{z}^{AR} \quad = \quad \min_{\hat{b}} z : R^*(I_0\hat{I}) - z \leq \sum_{i=1}^{m} h_i f_i + R(\hat{I};\hat{b}), \quad \forall \hat{I} \in \Omega(\hat{L},\hat{U}). \tag{3.19}$$

Applying Propositions 1 and 2 to the partial $\hat{I}$ sequences, we only have to focus on m sequences that dominate the others in $\Omega(\hat{L},\hat{U})$. These sequences have the same structure as CASTs. Following exactly the same steps in Sections 3.2.2 and 3.3, the optimal CR (MAR) can be computed solving a LP. This LP has the same structure as GBM (GMB-AR). Closed-form solutions are obtained by replacing appropriate parameters in (3.6)-(3.8) and (3.14)-(3.16). The closed-form solutions and discussion of special cases are provided in Appendix A.8 in the On-line Addendum.

These dynamic policies are easy to implement; specifically, the h_j should be updated each time a request is accepted, LPs are re-solved and the booking limits adjusted accordingly. These tasks require minimal computational resources as we have closed-form optimal solutions of the LPs. The idea of such dynamic adjustments is reminiscent of re-optimization in traditional RM where a static model approximates that is used in a rolling horizon fashion mimics the performance of a dynamic program. The major difference in our dynamic approach is that we consider the remaining capacity and the past performance (as represented by the revenue component $\sum_{i=1}^{m} h_i f_i$) from a partial input to guarantee a better worst-case performance for the *entire* booking horizon.

Chapter 4

Overbooking with Limited Information

Consider a seller with fixed inventory of a resource (e.g., a particular time-slot for the practice range of a golf course). Customers make advance reservations during a booking horizon at the end of which the service/good is to be provided. The seller offers his resource for sale by designing multiple fare-classes (i.e., products) that match differences in consumers willingness-to-pay for the product, as well as other characteristics that enable customer segmentation. In case of a golf-course, different fare-classes typically correspond to members (with multiple levels of membership based on affiliation etc.) vs. non-members. Advance reservations may be cancelled during the booking horizon or the customers may become no-shows at the time of the service. The seller makes use of overbooking and fare-class allocation to maximize his revenues net of overbooking costs at the end of the booking horizon.

We use the same notation as in Chapter 3: n denotes the total capacity of the resource available and $m \geq 2$ the number of fare-classes, f_i denotes the fare for class i, where $f_1 > f_2 > \cdots > f_m \geq 0$. The total number of requests in each class i falls with in a given range $[L_i, U_i]$ where L_i and U_i are lower and upper bounds, respectively, on the class-i demand. We focus on *static* decision rules on both overbooking and seat inventory control. In that respect, we do not require any information about the cancellations over time. However, we are concerned with accepted reservation requests, which are not honored when the customer either cancels in advance or is a last-minute “no-show”.

4.1 Problem Definition

We are interested in determining nested booking limits $b_1 \geq b_2 \geq \cdot \geq b_m \geq 0$ where b_j is the maximum number of requests to be accepted in classes j to m. We use an additional variable called *bucket size* - denoted x_i - in our analysis; we define it as $x_m = b_m$ and $x_i = b_i - b_{i+1}$ for $i = 1, 2, ..., m-1$. The notation b and x are used for the corresponding vectors of variables. Note that either of these vectors is sufficient to characterize a nested booking control policy. Given that the seller overbooks, the overbooking level (virtual capacity) is equal to $b_1 = \sum_{i=1}^{m} x_i \geq n$. While we first restrict ourselves to nested policies, we later prove that no other policy can provide better results in our problem. Throughout our analysis, we assume the decision variables take continuous values and any request can be split and partially accepted.

There are several practical approaches to making overbooking decisions based on estimates of no-show rates. One of the probabilistic models most commonly used in research and practice, is the following (Belobaba, 2006): Given that the total number of reservations is ξ, the number of no-shows is a random variable expressed as a function $\tilde{p}\xi$ where $\tilde{p}$ is the random no-show rate with a known probability distribution. Similar to this model, we express the no-shows using a proportional model: Given a no-show rate p, exactly $p\xi$ customers become no-shows and $(1 -$

$p)\xi$ customers show-up. However, p is unknown to the seller at the time of the overbooking decision in our study. The only information that is available is that the no-show rate p lies in the interval $[p_0, p_1]$ where $0 \leq p_0 \leq p_1 \leq 1$. This model is very simple and intuitive, and does not require specification of a probability distribution. The no-show rate is class-independent because overbooking is done at the resource level. Note that good estimates of upper and lower bounds on the no-show rate can easily be extracted from historical records on no-show rates at the resource-level, fare class-level, or even individual customer-level (see, for e.g., the cabin-level no-show model in Lawrence et al., 2003).

We assume there is a cost per denied service as is common in the economic models of overbooking (see Chapter 9 of Phillips, 2005). We denote this unit overbooking cost by V. In addition, we introduce a parameter β which is the fraction of the fare the service provider retains upon a no-show; $(1-\beta)$ is the refund for a no-show. In all the contexts when all reservations are refundable upon cancellation/no-show (e.g., hotel, car rental, golf course), we have $\beta = 0$. In case of non-refundable advance sales, $\beta = 1$. Our model is general enough to allow $0 \leq \beta \leq 1$. An immediate extension of this model includes *no-show fees*: our analysis and results hold with minor modifications for that case; see the discussion in Section 4.4. In Section 4.4, we also discuss what happens when no-show refund/penalty is class-dependent. Such class dependent refunds allow a better accounting of partial refunds or fees associated with advance cancellations which are typical of airline practice.

In this distribution-free environment, our focus is on the *worst-case* performance of the system. We again employ *competitive analysis of on-line algorithms* in our analysis. We use CR and MAR in our analysis. In this chapter, we are interested in determining the best deterministic policy that maximizes the CR (or minimizes the MAR).

4.2 Analysis of the Optimal Policy under the CR Criterion

Similar to our analysis in Chapter 3, we represent demand by input sequences. An input I includes information on the number and order of incoming booking requests. The set of feasible input sequences - an input sequence is characterized by the order and number of reservation requests - is $\Omega(L, U)$ where $L = (L_1, ..., L_m)$ and $U = (U_1, ..., U_m)$ are the vectors of lower and upper bounds, respectively, of the demand for fare-classes. $\Omega(L, U)$ includes all the input sequences that meet the lower and upper bound constraints of fare-class demand. A *scenario* in our problem is defined by a pair (I, p) where $I \in \Omega(L, U)$ is an input sequence and $p \in [p_0, p_1]$ is the no-show rate. The adversary chooses the scenario (I, p) while the decision-maker chooses the nested booking limit policy b.

Given a scenario (I, p), let $R^*(I, p)$ be the optimal offline revenue net of overbooking costs (hence forth referred to as the *net revenue*) and $R(I, p; b)$ be the on-line net revenue of a standard nested booking limit policy b, where $b_1 \geq n$ is the overbooking level. The following formulation, solves for the optimal booking control policy that maximizes the CR taking limited information into account:

$$\max_{b,\gamma} \quad \gamma : \quad \gamma \leq \frac{R(I, p; b)}{R^*(I, p)}, \forall I \in \Omega(L, U), p \in [p_0, p_1] \tag{4.1}$$

where γ is an auxiliary decision variable that represents the CR. There are clear challenges with this formulation: 1) $R(I, p; b)$ might not have closed-form expression; 2) If $p_1 > p_0$, there is an infinite number of constraints; 3) $|\Omega(L, U)|$ grows exponentially with m. While the number of constraints - equivalently the number of scenarios - is prohibitive even for small size problems, there will be redundant constraints. In other words, many scenarios will be dominated by others. The questions then become whether or not (i) the set of non-dominated scenarios can be identified and (ii) the optimization problem is tractable. As we show in the remainder of this section, the answer to both of these questions is yes. The maximization problem in (4.1) can be rewritten in a very compact form and its optimal solution is obtained in closed-form.

4.2.1 On-line and Offline Performance

An alternative way of thinking of our problem is to think of a game between an adversary and a decision maker. There is an adversary in charge of generating a sequence of booking requests and a no-show rate. By definition of our model, the adversary cannot choose who will become a no-show given an input sequence, but chooses a no-show rate that applies uniformly to all the reservation requests in an input sequence. The adversary is aware of the on-line algorithm (nested booking limits that allow for overbooking in our case) and all the problem parameters. The adversary chooses a scenario (I, p) so that the on-line algorithm achieves the worst (relative) performance. Given the scenario(s) the adversary would choose, the decision maker determines the optimal parameters for the algorithm to maximize the CR. Our analysis is focused on the characterization of adversary's actions first; hence characterization of the non-dominated scenarios. In order to understand and eliminate the dominated scenarios, we first study the structural properties of, and the relationship between, the on-line policy performance and the offline optimal given a scenario.

Let $q_i(I)$ denote the number of requests within fare class i in an input sequence I. Let $z_i(I, p)$ be the total number of reservations accepted in class i, after input I has been processed by *any* policy, either offline or on-line, prior to observing no-shows. Let $q(I)$ and $z(I, p)$ be the respective vectors. We use the term *booking profile* for the vector $z(I, p)$. The booking profile of the offline optimal policy is distinguished as $z^*(I, p)$ and that of a nested policy b is $z^b(I, p)$. Clearly, for any generic policy (offline or not) in this context, the number of accepted reservations cannot exceed the demand, i.e., $0 \leq z(I, p) \leq q(I)$. For shorthand notation, we use z, z^*, z^b and q to denote the vectors $z(I, p)$, $z^*(I, p)$, $z^b(I, p)$, and $q(I)$, respectively, when the context implies scenario (I, p).

Given the vector of accepted number of requests $w = (w_1, .., w_m)$ and a no-show rate p, the net revenues NR, are calculated as:

$$NR(w|p) \;=\; \sum_{i=1}^{m} w_i f_i - p(1-\beta)\sum_{i=1}^{m} w_i f_i - V[(1-p)\sum_{i=1}^{m} w_i - n]^+ \qquad (4.2)$$

where $[\cdot]^+ = max\{0, \cdot\}$. The first term on the right hand side of the above equation is the total revenues obtained from accepted reservations (prior to no-shows), the second term is the refund (if any) for the fraction of accepted reservations that become no-shows, and the third term is the cost of overbooking in case the number of show-ups exceeds the capacity. The offline optimal revenue given a scenario (I, p) can be determined by solving

$$R^*(I,p) = \max_w \{NR(w|p) : 0 \leq w \leq q(I)\} \tag{4.3}$$

where the decision vector w is in fact a booking profile. By definition

$$z^*(I,p) = argmax_w\{NR(w|p) : 0 \leq w \leq q(I)\}.$$

Before we proceed further, we point out to the following in our problem: Given a no-show rate p, the minimum net revenue from a reservation in class i is

$$f_i - p(1-\beta)f_i - (1-p)V = p\beta f_i + (1-p)(f_i - V).$$

If this minimum net revenue is positive, the seller prefers to accept as many reservations in class i regardless of how high the denied-service level would be. We call this irresponsible overbooking. The classes with fares high enough to induce irresponsible overbooking are excluded from our analysis, i.e., we assume

$$V > f_1(1 + \frac{p_1\beta}{1-p_1}) = \max\{f_i(1 + \frac{p\beta}{1-p}),\ i = 1, ..., m,\ p \in [p_0, p_1]\} \tag{4.4}$$

in our problem. This condition is hereafter referred to as the *responsible overbooking condition (ROC)*.[1] In light of ROC, the closed-form expressions for the optimal offline booking profile and offline optimal revenues are computed easily for a given scenario.

[1]Some airlines guarantee seat availability to the 'elite' members of their frequent flyer programs regardless of the time of a reservation. If the flight is sold-out, the fare quoted to these elite members (i.e., the fare of the class that is 'open' regardless of seat availability) far exceeds the unit cost of overbooking. This shows *irresponsible overbooking* is done in practice.

Proposition 6 *Given scenario* (I,p), *the offline optimal booking profile* $z^*(I,p)$ *and the corresponding net revenue* $R^*(I,p)$ *are*

$$z_i^*(I,p) = \min\left(\sum_{j=1}^{i} q_j(I), \frac{n}{1-p}\right) - \min\left(\sum_{j=1}^{i-1} q_j(I), \frac{n}{1-p}\right) \;\; \forall i = 1,...,m \quad (4.5)$$

$$R^*(I,p) = (1-p-p\beta)\sum_{i=1}^{m}(f_i - f_{i+1})\min\left(\sum_{j=1}^{i} q_j(I), \frac{n}{1-p}\right). \quad (4.6)$$

<u>Proof</u> Consider the optimization problem in (4.3). Any solution w that incurs overbooking penalties cannot be the optimal solution to this problem because of ROC. Therefore, the maximum number of reservations to be accepted is $\frac{n}{1-p}$. Consider the revised optimization problem with no overbooking penalties

$$Max\;\; [(1-p)+p\beta]\sum_{i=1}^{m} w_i f_i \;\; s.t. \;\; \sum_{i=1}^{m} w_i \leq \frac{n}{1-p}, \;\; 0 \leq w_i \leq q_i(I) \;\; \forall i = 1,..,m \quad (4.7)$$

where the decision vector $w = (w_1,...,w_m)$ is a booking profile. This is a continuous, knapsack problem with variable upper bounds where all variables have a coefficient of one in the constraint. Thus, the solution can be obtained in an iterative manner. The fares are monotonic and so are the objective function coefficients in problem (4.7). We start with the highest-fare class, and $z_1^* = min(q_1(I), \frac{n}{1-p})$. The remaining knapsack capacity is allocated to class-2: $z_2^* = min(q_2(I), \frac{n}{1-p} - z_1^*)$. Then, we move on to the next highest-fare class, and so on. The total number of reservations accepted is $\sum_{i=1}^{m} z_i^* = \min(\sum_{j=1}^{m} q_j(I), n/(1-p))$. Rewriting and rearranging the terms, the optimal booking profile $z^*(I,p)$ is expressed by the difference given in (4.5) and the corresponding offline optimal revenue is given by (4.6). •

With the understanding of the offline optimal profiles for a given scenario, we are ready to characterize the non-dominated scenarios the adversary would choose. In the next section, we show that the number of non-dominated scenarios is $m+1$.

4.2.2 Scenario Reduction

We first focus on the effect of no-show rates, given an input sequence I. Notice that no-show rate p affects the revenue retained after refunds and the overbooking cost but not the *gross revenue* (revenues accrued by accepted reservations prior to observing no-shows). The higher (lower) the no-show rate, the lower (higher) the revenue retained upon paying the refunds and the lower (higher) the cost of overbooking. Our next result shows that the minimum CR given a policy b and an input I is achieved when the no-show rate is either p_0 or p_1.

Proposition 7 *Given an input sequence I and a policy b with booking profile z^b, scenario (I,p) with $p \in (p_0,p_1)$ is dominated by either (I,p_0) or (I,p_1), that is,*

$$\frac{R(I,p;b)}{R^*(I,p)} \geq min\left\{\frac{R(I,p_0;b)}{R^*(I,p_0)}, \frac{R(I,p_1;b)}{R^*(I,p_1)}\right\}.$$

The proof is available in Appendix B.1. We provide a sketch of the proof here since the same argument is used to prove other results on scenario reduction. Given policy b and no-show rate p, consider the booking profile $z^b(I,p)$. The proof analyzes two cases: (i) If policy b does not incur any overbooking costs under (I,p) given its booking profile z^b, then it does not incur overbooking charges given (I,p_1), too. Scenario (I,p_1) has a higher no-show rate, hence a higher amount of refund; thus policy b net revenues are lower under (I,p_1) in this case. In this first case, the CR with scenario (I,p_1) is shown to be no-more than the CR with (I,p). (ii) If policy

b does incur overbooking costs under (I, p), then scenario (I, p_0) results in higher overbooking charges but also lower refunds. In this case, the CR in scenario (I, p_0) is shown to be no more than the CR in (I, p).

Proposition 7 reduces the choices of no-show rates to two alternatives, avoiding the complications associated with the continuum of constraints in problem (4.1). Next, we focus on the non-dominated input sequences given a no-show rate $p > 0$. The sequence reduction on inputs extends the work of Chapter 3 who identified the set of non-dominated sequences for the adversary assuming no overbooking/no-shows, i.e., $p_0 = p_1 = 0$. The first result studies the inputs focusing only on the order of requests.

Proposition 8 *Given a no-show rate p, a nested booking limit policy has the lowest revenue when requests come in LBH-order.*

Proof For the special case with $p = 0$, We have shown in Chapter 3 the following: Given a policy b, two inputs I and I^l where $q(I) = q(I^l)$ and I^l is LBH, (i) the gross revenues are lower under I^l, and (ii) $\sum_{i=1}^m z_i^b(I, p) = \sum_{i=1}^m z_i^b(I^l, p)$, although $z^b(I, p)$ and $z^b(I^l, p)$ need not be the same. Note that (i) and (ii) are true for any $p > 0$, too, because the profile of a policy z^b, hence the gross revenue $\sum_{i=1}^m z_i^b f_i$, are not affected by the no-show rate p. The refunds and overbooking penalties are only a function of the total number of reservations accepted in our model, which is the same under I and I^l due to property (ii). Hence the seller's net revenue is lower with an LBH input for any $p \geq 0$. •

Based on this result, the CR is lower with LBH inputs (because the adversary's performance is not affected by the order of requests), and any non-LBH input is dominated. We continue to reduce the number of input sequences by focusing on the amount of requests in each LBH input. We next use the terminology introduced in Chapter 3 to characterize the extreme sequences in this particular problem. Based on Definition 1 and 2 in Chapter 3, there are m CASTs in total, denoted $CAST_j$, $j = 1, .., m$. By definition, $q_k(CAST_j) = U_k$ for $k \geq j$ and $q_k(CAST_j) = L_k$ for $k < j$. Our next result shows that each of the CAST sequences is non-dominated in its category for a given p. The proof is in Appendix B.2.

Proposition 9 *Consider no-show rate p, nested booking limit policy b, and a category A_j^b for some $j = 1, ..., m$. Any sequence I in A_j^b is dominated by either $CAST_1$ or by $CAST_j$, i.e.,*

$$\frac{R(I,p;b)}{R^*(I,p)} \geq \min\left\{\frac{R(CAST_j,p;b)}{R^*(CAST_j,p)}, \frac{R(CAST_1,p;b))}{R^*(CAST_1,p)}\right\}.$$

Combining Propositions 7 and 9, the number of non-dominated scenarios are reduced to $2m$ with m distinct input sequences and 2 no-show rates. One more observation reduces the total number of non-dominated scenarios to $m + 1$; see Appendix B.3 for the proof.

Theorem 4 $(CAST_1, p_0), (CAST_1, p_1)$, $(CAST_2, p_1), \cdots, (CAST_m, p_1)$ *are the only non-dominated sequences.*

There is a special relation among these non-dominated sequences considering the overbooking penalties for a given policy b.

Corollary 2 *If scenario* $(CAST_k, p_1)$ *causes a nested policy* b *to incur overbooking penalties, then* $(CAST_k, p_1)$ *is dominated by* $(CAST_1, p_0)$ *for that policy, i.e.*

$$\frac{R(CAST_k, p_1; b)}{R^*(CAST_k, p_1)} \geq \frac{R(CAST_1, p_0; b)}{R^*(CAST_1, p_0)}. \tag{4.8}$$

<u>Proof</u> From the second case in the proof of Proposition 7, $(CAST_k, p_1)$ is dominated by $(CAST_k, p_0)$. As $p_0 < p_1$, scenario $(CAST_k, p_0)$ must result in higher overbooking penalties, so it is dominated by $(CAST_1, p_0)$. •

Here we give an example on extreme vs. dominated scenarios. Consider this example with $m = 2$, $f_1 = 200$, $f_2 = 100$, $V = 300$, $\beta = 0.2$, $p_0 = 0.1$, $p_1 = 0.2$, $C = 8$, $[L_2, U_2] = [7, 7]$, and $[L_1, U_1] = [4, 6]$. We calculate $\chi(p) = \frac{R(I,p;b)}{R^*(I,p)}$ of the nested booking limit $b = (b_1, b_2) = (10, 5)$ for different no-show rates and LBH inputs in Table 4.1 below.

Input	p		
	$p = 0.1$	$p = 0.15$	$p = 0.2$
I_1^*: $q_1(I_1^*) = 6$, $q_2(I_1^*) = 7$	.79	.86	.93
I_2^*: $q_1(I_2^*) = 4$, $q_2(I_2^*) = 7$	.98	.96	.92
I': $q_1(I') = 5$, $q_2(I') = 7$	.84	.92	1.0

Table 4.1: Ratio of on-line to offline net revenues of a booking limit policy for different scenarios

Note that in each row in Table 4.1, the value of $\chi(p)$ is smallest either for $p = 0.1$ or for $p = 0.2$, which is expected due to Proposition 7. Similarly, in each column of Table 4.1, $\chi(p)$ is smallest either for input I_1^* or I_2^*, which is expected by Proposition 9. We now use these results to represent the optimization problem (4.1) in a compact way.

4.2.3 Linear Programming Formulation

With the reduction in the number of scenarios, problem (4.1) can be expressed as

$$\gamma^{CR} = \max_{b,\gamma} \gamma : \gamma \le \frac{R(CAST_k, p_t; b)}{R^*(CAST_k, p_t)} \text{ for } (k,t) = (1,0), (1,1), \cdots, (m,1). \quad (4.9)$$

We next show how the optimal overbooking level and booking limits of this problem can be obtained by solving an appropriate linear programming (LP) model.

We use the bucket sizes x_i, as opposed to the booking limits b_i, $i = 1, ..., m$, as the decision variables in the LP. discuss Using the observation that any reasonable policy would have $x_i \le U_i$ for all $i = 1, .., m$ (otherwise the nested policy would result in unused capacity in at least one bucket), In Chapter 3 we have shown that the gross revenue of policy b given input $CAST_k$, $k = 1, ..., m$, is bounded above by a linear function of x_i, i.e.

$$\sum_{i=1}^{m} z_i^b(CAST_k) f_i \le \sum_{i=1}^{k-1} f_i L_i + \sum_{i=k}^{m} f_i x_i. \quad (4.10)$$

Due to Corollary 2, the overbooking penalties can be omitted for all the $(k, 1)$ scenarios, $k = 1, ..., m$, and denied service is only relevant for scenario $(1, 0)$. Let y be the (maximum) number of customers denied service. Given that the maximum number of accepted reservations is $\sum_{j=1}^{m} x_j$, we have $y = [(1 - p_0) \sum_{j=1}^{m} x_j - n]^+$. Given y, the overbooking penalty for scenario $(1, 0)$ is Vy.

Combining all this, we get an upper bound - as a linear function of x_i and y - on the net policy revenues for each of the non-dominated scenarios:

$$R(CAST_k, p_1; b) \leq (1 - p_1 + p_1\beta)[\sum_{i=1}^{k-1} f_i L_i + \sum_{i=k}^{m} f_i x_i], \quad k = 1, ..., m,$$
$$R(CAST_1, p_0; b) \leq (1 - p_0 + p_0\beta) \sum_{i=k}^{m} f_i x_i - Vy.$$

We let $f_j^t = (1 - p_t + p_t\beta) f_j$, $t = 0, 1$, to simplify the presentation below. Using the relations established thus far, we formulate an LP that finds the optimal booking limits and overbooking level. We call this model BON (Bounded No-show):

$$\textbf{BON}: \ \gamma^{BON} = \max_{x,y,\gamma} \quad \gamma$$
$$\text{s.t.} \quad R^*(CAST_j, p_1)\gamma \leq \sum_{i=1}^{j-1} f_i^1 L_i + \sum_{i=j}^{m} f_i^1 x_i, \ j = 1, ..., m \tag{4.11}$$
$$R^*(CAST_1, p_0)\gamma \leq \sum_{j=1}^{m} f_j^0 x_j - Vy \tag{4.12}$$
$$y \geq (1 - p_0) \sum_{j=1}^{m} x_j - n \tag{4.13}$$
$$y \geq 0, 0 \leq x_j \leq U_j, \quad j = 1, \cdots, m. \tag{4.14}$$

Note that the terms on the right-hand-side of constraints (4.11) and (4.12) constitute an *upper bound* on the net revenues. Hence, BON is designed to provide an upper bound on the optimal CR. However, the solution to BON provides the optimal solution for the CR problem introduced in (4.1). We formalize and prove this statement and present a closed-form solution to BON in the next section.

4.2.4 Closed-form Solution to BON

BON is an LP with $m+2$ variables and $m+2$ constraints. Given the size of a realistic single-resource RM problem, the optimal solution to BON can be obtained very easily using a standard LP solver. However, the structure of the constraints and the relationship among the coefficients of the constraints in BON permit a closed-form solution. The solution to BON can be determined by identifying the constraints that are binding in the optimal solution. Finding the *critical fare class* $u \in \{1, ..., m\}$ such that classes $j > u$ would be closed (i.e., booking limit of $j > u$ would be zero), provides the basis for such a solution. Once u is known, the optimal solution to BON is determined by solving a set of linear equations. The proof of the next result is available in Appendix B.4.

Theorem 5 *(a) The optimal solution to BON is*

$$\gamma^{BON} = \frac{R_u^+/f_u^1 + N_u}{R_u^*/f_u^1 + \sum_{i=0}^{u-1} g_i} \tag{4.15}$$

$$x_j^{BON} = \begin{cases} g_j\gamma^{BON} + L_j & j < u \\ (R_u^*\gamma^{BON} - R_u^+)/f_u^1 & j = u \\ 0 & j > u \end{cases} \tag{4.16}$$

$$y^{BON} = -(1-p_0)g_0\gamma^{BON} \tag{4.17}$$

$$u = \max\{j \leq m : R_j^+ \sum_{i=0}^{j-1} g_i < N_j R_j^*\} \tag{4.18}$$

where the index u denotes the critical fare-class such that all classes $k > u$ are *closed, and R_j^*, R_j^+, g_i, N_j are auxiliary parameters defined as*

$$R_j^* = R^*(CAST_j, p_1), \tag{4.19}$$

$$R_j^+ = \sum_{i=1}^{j-1} f_i^1 L_i, \tag{4.20}$$

$$g_0 = \frac{R^*(CAST_1, p_0)/f_1^0 - R_1^*/f_1^1}{(1-p_0)V/f_1^0}, \tag{4.21}$$

$$g_i = \frac{R_i^* - R_{i+1}^*}{f_i^1}, i = 1, ..., m-1, \tag{4.22}$$

$$N_j = \min\left(\sum_{i=1}^{m} U_i, \frac{n}{1-p_0}\right) - \sum_{i=1}^{j-1} L_i, j = 1, ..., m+1. \tag{4.23}$$

(b) The nested booking limits defined by

$$b_j^{BON} = \sum_{i=j}^{m} x_i^{BON} \text{ for } j = 1, ..., m \tag{4.24}$$

maximizes the CR in problem (4.1) and the optimal CR is $\gamma^{CR} = \gamma^{BON}$.

Our analysis gives the optimal solution to the CR problem within the class of nested policies. We next show that no other deterministic, static booking control and overbooking policy achieves a higher CR; hence nesting by revenue order is the best among the static policies. The proof is in the Appendix B.5.

Theorem 6 *The nested booking control policy with booking limit vector b^{BON} defined by (4.24) has a CR of γ^{BON} given by (4.15) and this is the best possible among all deterministic, static policies.*

4.3 Analysis of the Optimal Policy under the AR Criterion

Our analysis for the CR criterion easily extends to the absolute regret criterion where the objective is to minimize the MAR. The general formulation for that problem is:

$$\gamma^{MAR} = \min_{b,\gamma} \gamma \ : \gamma \geq R^*(I,p) - R(I,p;b), \forall I \in \Omega(L,U), p \in [p_0, p_1] \tag{4.25}$$

where b is the vector of decision variables and γ^{MAR} is the optimal MAR. All of our results on scenario reduction in Section 4.2 are valid for the MAR criteria. Therefore we only have to consider the same $m+1$ non-dominated scenarios to reformulate (4.25). The following LP model, called BON-MAR, is designed to provide a *lower bound* on the minimum MAR.

$$\mathbf{BON-MAR}:$$

$$\gamma^{MAR} = \min_{x,y,\gamma} \quad \gamma$$

$$\text{s.t.} \quad R^*(CAST_j, p_1) - \gamma \leq \sum_{i=1}^{j-1} f_i^1 L_i + \sum_{i=j}^{m} f_i^1 x_i, \ j = 1, ..., m \tag{4.26}$$

$$R^*(CAST_1, p_0) - \gamma \leq \sum_{j=1}^{m} f_j^0 x_j - Vy \tag{4.27}$$

$$y \geq (1-p_0) \sum_{j=1}^{m} x_j - n \tag{4.28}$$

$$y \geq 0, 0 \leq x_j \leq U_j, \ j = 1, ..., m. \tag{4.29}$$

The closed-form solution to BON-MAR is given below. The proof is omitted; technical details require the same type of analysis carried put in the proof of Theorem 5, part (a).

Theorem 7 *The optimal solution to BON-MAR is*

$$\gamma^{MAR} = R^*_{\tilde{u}} - R^+_{\tilde{u}} - x^{MAR}_{\tilde{u}} f^1_{\tilde{u}} \tag{4.30}$$

$$x^{MAR}_j = \begin{cases} g_j + L_j & j < u \\ N_{\tilde{u}} - \sum_{i=0}^{\tilde{u}} g_0 & j = u \\ 0 & j > u \end{cases} \tag{4.31}$$

$$y^{MAR} = -(1-p_0)g_0 \tag{4.32}$$

$$\tilde{u} = \max\{j \le m : \sum_{i=0}^{j-1} g_i < N_j\} \tag{4.33}$$

where $\tilde{u}$ is the critical fare class such that classes $k > \tilde{u}$ are closed, and the auxiliary parameters g_i, N_j, R^*_j, R^+_j for $j = 1,..,m$ are defined in equations (4.19), (4.20), (4.22), (4.23), respectively.

Similar to our analysis in Section 4.2, we can show that the optimal solution of BON-MAR provides the optimal MAR of the problem in (4.25). In addition, the optimal nested policy obtained by BON-MAR is the best possible under the MAR criteria. We formalize our result in the theorem below. The proof is omitted - the proof technique is similar to that used in the proof of Theorem 5, part (b) and Theorem 6.

Theorem 8 *The nested booking control policy with booking limits defined as*

$$b^{MAR}_j = \sum_{i=j}^{m} x^{MAR}_i \text{ for } j = 1, ..., m \tag{4.34}$$

has minimum MAR of γ^{MAR} (where x^{MAR}_i and γ^{MAR} are given in (4.31) and (4.30)). No other deterministic, static policy has a lower MAR.

When there are no cancellations/no-shows, in Chapter 3 we showed that the optimal booking limits obtained with the MAR criterion are more aggressive than the ones with CR, i.e., MAR protects more resources for higher fares. We make a similar observation: Comparing the optimal solutions of BON and BON-MAR in equations (4.15)-(4.18) and (4.30)-(4.33), respectively, we see that $u \geq \tilde{u}$, i.e. MAR makes resources available to fewer number of fare-classes. In addition, $x_i^{BON} \leq x_i^{MAR}$ for $i < \tilde{u} \leq u$ *and* $y^{BON} \leq y^{MAR}$ at optimality. Thus, MAR is not only more aggressive in protecting resources for higher fares, it also overbooks more aggressively.

4.4 Extensions of the Model

Our model assumes class-independent no-show rate p and class-independent refund rate β. This captures the situation where all reservations are refundable, non-refundable, or a fixed percentage of fares is retained upon no-shows. Examples of hotels, car-rentals and golf courses that do not charge customers in advance are easily represented in this model.

More recently, car rentals and golf courses started implementing policies that charge a fixed no-show fee. This fee is typically independent of the fare class. Suppose all fares are refundable but a fixed fee K is charged upon a no-show. In this case, we set $\beta = 0$ and add the term $pK \sum_{i=1}^{m} z_i^b(I, p)$ to the net revenues of policy b for scenario (I, p). Once the ROC in (4.4) is adjusted to take into account the fixed no-show fee, the offline optimal policy is characterized as in Proposition

6, number of non-dominated scenarios are reduced to $m+1$ using the analysis in Section 4.2.2, and a LP model similar to BON is developed to obtain the optimal booking limits. Similarly, if the no-show policy of the company refunds $(1-\beta)$ of the revenue for some $\beta > 0$ and charges a fixed K for no-shows, the ROC is updated accordingly, and an LP model similar to BON can be developed. These modeling extensions pose no difficulty in the analysis. The optimal booking limits can be obtained in closed-form in either case.

Another extension of our model would consider class-dependent no-show penalties, which is a better representation of airline practice where a mix of refundable and non-refundable fare classes are employed.[2] In this case, we have to introduce a vector $(\beta_1, ..., \beta_m)$ to better account for the revenues retained by the seller upon a no-show. Unfortunately, our analysis does not carry through when refunds are class-dependent. Consider a policy b which is nested by revenue order. LBH input sequences provide lowest net revenues in our model for a given p (Proposition 8). However, LBH inputs may be dominated when refunds are class-dependent: the classes may no longer be ordered based on the fares f_i, and one has to consider the adjusted fare $\tilde{f}_i(p) = (1-p+p\beta_i)f_i$ which varies with p. Consider an example with $m=2$, $f_1 = 1$, $f_2 = 0.7$, $\beta_1 = 0$, $\beta_2 = 1$, and $p \in [0, 0.5]$. When $p = 0$, we have $\tilde{f}_1(0) = 1$, $\tilde{f}_2(0) = 0.7$, thus $\tilde{f}_1(0) > \tilde{f}_2(0)$. When $p = 0.5$, we have $\tilde{f}_1(0.5) = 0.5$, $\tilde{f}_2(0.5) = 0.7$, which gives $\tilde{f}_1(0.5) < \tilde{f}_2(0.5)$. Notice that when p changes, the rank-

[2] Class-dependent refunds and cancellation/no-show rates are in general disregarded in the practice of airline overbooking; see the description of the cost-based model we used in our computational experiments.

ing of the fare classes based on the $\tilde{f}(p)$ values change, and so does the preferred sequence of arrivals. When $p = 0.5$, the lowest revenues are achieved when arrivals are ordered in opposite of LBH: sending in the high fares first. When $p = 0$, lowest revenues are achieved by LBH. This example suggests that (i) a static (constant) nesting by revenue-order may be a suboptimal booking control policy in this case, and (ii) we no longer can use our scenario reduction techniques when refunds are class-dependent. These two observations point to future research directions that are interesting from both a theoretical and a practical perspective.

The sub-optimality of static nesting of booking limits was reported in Subramanian et al. (1999) who provide one of the most detailed models of the multi-fare, single-resource RM problem with cancellations. Their Markov Decision Process (MDP) model captures cancellations over time, no-shows, and class-dependent refunds. Class-dependent refunds and the time trade-offs (e.g. early high fare booking may also have higher cancellation probabilities) no longer permit a static ranking of fare-classes in this dynamic model; hence a static nesting rule can no longer be optimal. We encounter a similar problem when the ordering of fare classes changes with the rate of no-show.

Notice that the challenges associated with class-dependent refunds also arise when one wants to incorporate class-dependent no-show rates to our model. However, overbooking is done at the resource level in practice, as opposed to class level, and incorporating class-dependent no-show information does not change the nature of the static overbooking decision.[3] Therefore the choice of disaggregate vs. ag-

[3]Resource-level overbooking is shown to be more effective than class-level overbooking for hotels

gregate information on no-show rates does not affect model realism, but is more a matter of the trade-off between accuracy/availability of information and model tractability.

in Hadjinicola and Panayi (1997).

Chapter 5

A Hybrid Overbooking and Fare-Class Allocation Approach

The model in Chapter 4 is practical for uses with or without accurate information on no-shows. In practice, one difficulty in the booking control RM problem is in estimating demand because historical data is censored (i.e. data does not reflect the true demand but only the sales, which is affected by the capacity and the booking control policy used). On the other hand, airlines and hotels have very good information on no-shows because cancellations are recorded. In this chapter, we consider the situation where a decision maker is confident about the probabilistic model underlying no-shows and has very good forecast, while there is only limited information about the demand uncertainty. We adopt a hybrid approach to integrate these two classes of information into decision-making in this chapter. This model can potentially yield policies that are more effective, as compared to the ones analyzed in Chapter 4.

5.1 Problem Definition

We have studied the overbooking problem in the previous chapter, with limited information concerning both demands for different fare-classes and no-shows. While good demand information is usually hard to obtain due to many reasons such as censored data, buy-ups and buy-downs, good no-show information can be obtained given an adequate number of observations. Unlike lost demand requests, which leave no trace in the system, no-show data are fully recorded in a computerized reservation system. Integrating good probabilistic no-show information with limited demand information in overbooking and seat allocation decision making is the subject matter in this chapter. We believe this well serves both theoretical and practical interests.

We keep the notation the same as before whenever appropriate. We define the summation over the elements of a vector Y as $\|Y\|_+ = \sum_{\forall i} Y_i$. Let e_i denote the unit vector with only the i^{th} element being one and all others being zero. n is the total capacity of the resource available and $m \geq 2$ the number of fare-classes. Let f_i denote the fare for class i, where $f_1 > f_2 > \cdots > f_m \geq 0$. Again I is the input sequence, $q(I)$ is its profile, a vector with $q_i(I)$ denoting the number of requests in fare-class i. The only restriction we put on an input stream I is this: $L \leq q(I) \leq U$, where L and U are the lower and upper bound vectors. Given a profile vector q, we use $\acute{q}$ to denote the unique low before high (LBH) input stream whose profile equals q. Thus, $\acute{q}(I)$ denotes the LBH re-arrangement of input I. Let $w = (w_1, \cdots, w_m)$ be a vector such that w_i is the number of accepted requests in class i when an input is processed by a particular booking control policy. We refer to vector w as the "outcome" of a given input and policy. The *gross revenue* of the outcome w is given by $\tilde{R}(w) = \sum_{i=1}^{m} w_i f_i$. That revenue does not take into account the cost of possible no-shows and service denials, which is our next consideration.

Let $Z(y)$ denote the random number of show-ups given the total number of bookings y. We assume that $Z(y)$ satisfies the semi-group property: $Z(x + y)$ is a random variable that has the same probability distribution as the sum of two random variables $Z(x)$ and $Z(y)$, i.e. $Z(x+y) \sim Z(x)+Z(y)$, where $\sim$ denotes equivalence in distribution. We also assume that $E[Z(y)] = \varphi y$, $0 \le \varphi \le 1$. Both assumptions hold regardless of the fare-class composition of x or y. So, for any class i, we have $E[Z(\|e_i\|_+)] = \varphi$. Notice that these two assumptions are satisfied by the no-show models commonly used in practice and research, including the Binomial distribution, where φ is the success probability and y is the total number of trials.

The parameter β is the same as in Chapter 4 and denotes the unit refund fee. Finally, $V(z)$ denotes the service denial cost given exactly z customers showing up. The net revenue would be a random variable depending on no-shows:

$$NR(w) = \sum_{i=1}^{m} f_i[Z(w_i) + \beta(w_i - Z(w_i))] - V(\sum_{i=1}^{m} Z(w_i)).$$

The expected net revenue from a booking outcome w is thus

$$E[NR(w)] = \sum_{i=1}^{m} f_i(\varphi + \beta(1-\varphi))w_i - E[V(Z(\|w\|_+))],$$

the term $E[V(Z(\|w\|_+))]$ follows from the semi-group property of random variables $Z(\cdot)$. Let $v(y) = E[V(Z(y))]$ denote the expected denial cost with y total bookings (when $y \le n$, there are no service denials and $v(y) = 0$), and $f_i' = f_i(\varphi + \beta(1-\varphi))$ for $i = 1, \cdots, m$, then we have

$$E[NR(w)] = \sum_{i=1}^{m} f_i' w_i - v(\|w\|_+).$$

The most considered form of $V(z)$ in the literature is linear in the number of denials (see Chapter 9 of Phillips, 2005), and in that case $v(y)$ is convex when $Z(y)$ is Binomial. Our analysis is not restricted to these special models and structures. In our analysis, we focus on the discrete problem, assuming that the requests are always of one unit, and that no fractional requests are allowed. However, the analysis can be easily extended to allow fractional requests.

We consider *static* decision policies for overbooking and fare-class allocation, and do not use any information about the cancellations over time. There are almost infinite possibilities about booking control policies, it is necessary to restrict our analysis on policies with special structures. As nested booking limits are shown to be optimal in our previous chapters, we will assume such a policy structure in this chapter as well. Unlike before, we do not provide any proof of the optimality of nested booking limits, but only give analysis of additional structural properties. The notation b and x are same as before, and denote the vector of nested booking limits and bucket sizes, respectively. We assume $0 \leq x_i \leq U_i$ for all $i = 1, \cdots, m$, to exclude trivially suboptimal policies from our analysis. When a policy x is applied to an input stream I, we use $w(I;x)$ to denote the outcome vector. Then the expected net revenue of an on-line policy x is $R(I;x) = E[NR(w(I;x))]$, and the gross revenue $\tilde{R}(I;x) = \tilde{R}(w(I;x))$.

The offline revenue is the optimal net revenue from the accepted requests with the decisions made when the entire input stream is already known. Let $R^*(I)$ denote the offline optimal, which is found by:

$$R^*(I) = \max E[NR(w)] : \text{ s.t. } 0 \leq w \leq q(I). \tag{5.1}$$

Let $w^*(I)$ be the offline optimal outcome, or $R^*(I) = E[NR(w^*(I))]$.

In our problem, there are two distinct types of uncertainty: full probabilistic characterization of no-shows on the one hand, and only limited, distribution-free information of demands on the other hand. Integrating them into a unified model is the key to successful decision making, and our approach is thus driven by these two uncertainty types. To utilize the limited information of demands, we use *competitive analysis of on-line algorithms* (see Albers, 2003). The full probabilistic characterization of no-shows is utilized and integrated into the competitive analysis when we define the on-line and offline revenues. The CR of an on-line policy x, is defined as:

$$\mathrm{CR}(x) \quad = \quad \min_{I \in \Omega(L,U)} \frac{R(I;x)}{R^*(I)} = \min_{I \in \Omega(L,U)} \frac{\sum_{i=1}^m f_i' w_i(I;x) - v(\|w(I;x)\|_+)}{\sum_{i=1}^m f_i' w_i^*(I) - v(\|w^*(I)\|_+)}. \quad (5.2)$$

Clearly, a policy that maximizes the CR, minimizes the maximum relative regret. Likewise, the maximum absolute regret (MAR) of an on-line policy x is defined as

$$\mathrm{MAR}(x) \quad = \quad \max_{I \in \Omega(L,U)} \{R^*(I) - R(I;x)\}.$$

We focus on CR analysis, but our results extend easily to MAR. In this chapter, we are interested in determining the best deterministic policy that maximizes the CR. At this point even evaluating the CR looks quite challenging as $|\Omega(L,U)|$ grows exponentially with m. However, we find that it is not necessary to consider all input streams in $\Omega(L,U)$, and in fact, we can reduce the set of necessary input streams to a manageable size. The profiles of those necessary input streams are called extreme profiles, and are studied in Section 3. In Section 4 structural properties of the optimal nested policy is discussed. And in Section 5, we propose a global optimization algorithm based on cross entropy (CE) and model reference adaptive search (MRAS) methods, to find the optimal on-line policy in a computationally tractable way.

5.2 Extreme Profiles

Competitive analysis can be viewed as a zero-sum two-player game. The two players are a policy maker and an adversary, who controls the input stream and tries to minimize the CR. Our primary goal in this section is to find dominating strategies, that is, input streams, for the adversary. Note that from the perspective of the adversary, an input stream I' dominates I if for any nested policy x,

$$R(I;x) - R(I';x) \geq R^*(I) - R^*(I'), \text{ and } R(I;x) - R(I';x) \geq 0. \tag{5.3}$$

It has been established in previous chapters that LBH input streams always have the lowest on-line revenues, and this result carries over naturally.

Theorem 9 *Among all input sequences with the same profile, a given nested policy x generates the least revenue when applied to the unique such LBH sequence.*

Proof We have already shown the following: given any input stream I and its LBH re-arrangement $\acute{q}(I)$, (i) the gross revenues have $\tilde{R}(I;x) \geq \tilde{R}(\acute{q}(I);x)$, and (ii) the total number of accepted requests remains the same, so $\|w(I;x)\|_+ = \|w(\acute{q}(I);x)\|_+$. Now because of (ii), the expected denial cost remains unchanged. Note that $R(w)$ can be rewritten as: $R(w) = (\varphi + \beta(1-\varphi))\tilde{R}(w) - v(\|w\|_+)$. Since the gross revenue decreases by (i), the net revenue also decreases with a LBH input. •

Based on this result, we only need to consider the LBH inputs in $\Omega(L,U)$. Although this significantly reduces the number of necessary inputs, the number of remaining LBH inputs is still high, and further reduction is necessary. Since a LBH input stream is uniquely determined by its profile, we introduce some definition on input profiles next.

Definition 3 *(Extreme Profile) A profile q is called extreme if there exists an integer $\eta \in [0, m]$ such that $q_j = L_j, j < \eta)$, $q_j = U_j, j > \eta$, and $L_j < q_j \leq U_j, j = \eta$.*

It is not hard to see that for each extreme profile q, there is a unique η satisfying the condition above. Further, we find that an extreme profile is completely determined by its total number of requests, and thus η is a function of $\|q\|_+$.

Proposition 10 *If q and q' are both extreme profiles, and $\|q\|_+ = \|q'\|_+$, then $q = q'$.*

Proof From Definition 3, clearly we only need to show that q and q' have the same η. It is also easy to show that if they don't, then $\|q\|_+ \neq \|q'\|_+$, a contradiction. •

Obviously, since $L \leq q \leq U$, we have $\|L\|_+ \leq \|q\|_+ \leq \|U\|_+$, so there can be at most $\|U\|_+ - \|L\|_+ + 1$ extreme profiles. On the other hand, for any integer $k \in [\|L\|_+, \|U\|_+]$, one can start with the profile $q = L$ and then add $k - \|L\|_+$ more requests one by one, each time adding one request of the lowest fare as long as the upper limit is not exceeded. From this observation, we can see that we have exactly $\|U\|_+ - \|L\|_+ + 1$ extreme profiles. In fact, we can easily arrange these extreme profiles into series. Let $q^k, k = \|L\|_+, \cdots, \|U\|_+$, recursively defined as: $q^{\|L\|_+} = L$, and for $\|L\|_+ < k \leq \|U\|_+$, let $\eta(k) = \max\{j : q_j^{k-1} < U_j\}$ and $q^k = q^{k-1} + e_{\eta(k)}$. Clearly, $\eta(k)$ is the next lowest fare request that can be added to profile q^{k-1}, and $k = \|q^k\|_+$. Given any extreme profile q, let $k = \|q\|_+$, then $q = q^k, \eta(k) = \min\{j : q_j^k > L_j\}$, which gives an equivalent way to express $\eta(k)$.

In the remainder of our analysis, we use the concept of category introduced in Chapter 3. Note that if an input stream with profile q belongs to the $c+1^{st}$ category according to a nested policy x, then $\sum_{i=j}^{c} q_i < \sum_{i=j}^{c} x_i$, for $j = 1, \cdots, c$. This set of category inequalities is particularly useful for the discussions that follow. Note that when $j = c$ we obtain $q_c < x_c$. Our next result proves that we only need to consider the LBH input streams $\acute{q}^k$ with extreme profiles.

Theorem 10 *Given a nested policy x, any input stream is dominated by one of the LBH streams with the extreme profiles.*

Proof We only need to consider LBH streams. Let the LBH input stream with profile q belongs to category $c+1$. Here we assume $q_i = U_i, i > c$, otherwise we can construct such one in the same category that dominates q by adding $U_i - q_i$ requests in each fare class $i > c$, which does not increase on-line revenue (because it is equivalent to append the added requests to the end of the input stream and then make adjustments to arrive at the desired LBH stream) while the corresponding offline revenue might increase. If q is already one of the extreme profiles, we are done; otherwise we construct a series of profiles that will end up with an extreme profile.

If q is not an extreme profile, then there is a $k < c$ such that $q_k > L_k$. Construct a new profile $q' = q + e_c - e_k$. As the LBH stream $\acute{q}$ belongs to category $c+1$, so $\sum_{i=j}^{c} q_i < \sum_{i=j}^{c} x_i, j \leq c$. Then we have $\sum_{i=j}^{c} q'_i \leq \sum_{i=j}^{c} x_i, j \leq c$, which means with profile q' none of the requests in fare classes $j = 1, \cdots, c$ will be rejected by x, exactly the same as what happens with q. So we have $\|w(\acute{q}; x)\|_+ = \|w(\acute{q}'; x)\|_+$, and $R(\acute{q}; x) - R(\acute{q}'; x) = f'_k - f'_c$. Now what happens with the offline revenues? Changing from q to q' can be seen as substituting one class-k request with a lower fare class-c request (since $q' = q + e_c - e_k$). Consider the offline optimal solution $w^*(\acute{q})$, which satisfies $w^*(\acute{q}) \leq q$. Clearly, $w' = w^*(\acute{q}) + e_c - e_k \leq q + e_c - e_k = q'$ is a feasible solution to (5.1) with $I = \acute{q}'$ if $w' \geq 0$. So we have a lower bound when $w' \geq 0$:

$$R^*(\acute{q}') \geq E[NR(w')] = R^*(\acute{q}) - (f'_k - f'_c). \tag{5.4}$$

If $w' \geq 0$ does not hold, then $w^*(\acute{q})$ is feasible solution to (5.1) with $I = \acute{q}'$, and thus $R^*(q') \geq R^*(q)$, and the lower bound on $R^*(q')$ in (5.4) is still valid as $f'_k - f'_c \geq 0$. So we have

$$R(\acute{q}; x) - R(\acute{q}'; x) = f'_k - f'_c \geq R^*(\acute{q}) - R^*(\acute{q}'),$$

thus $\acute{q}'$ dominates $\acute{q}$ by reason of (5.3).

Let $q^{(0)} = q$, and if $q^{(j)}, j \geq 0$ is not an extreme profile, we can recursively construct $q^{(j+1)}$ from $q^{(j)}$ in the same fashion as above, obtaining a series of profiles, each profile dominating the ones before it. Let $q^{(j)}$ belongs to category $c_j + 1$, the sum $\sum_{i=1}^{c_j - 1}(q_i^{(j)} - L_i)$ decreases by at least one as j increases by one. Clearly, the process will end within limited number of iterations until the sum is zero, where an extreme profile is found. •

We effectively reduced the set of necessary input streams to the LBH streams with the extreme profiles, and evaluating CR or MAR becomes computationally tractable, given that the function $v(\cdot)$ is readily computable. Before we start looking at suitable optimization procedures to find the optimal nested policies to either maximize the CR or minimize the MAR, it is worthwhile to mention the structural properties of optimal nested policies.

5.3 Structural Properties of Optimal Nested Policies

We have assumed nested policies in the first place, which confined our policy space and thus the policy we obtained might not be *the* optimal policy. So we say it is an *optimal nested policy* instead. The primary goal in this section is to understand what kind of nested policies are optimal in our problem. We investigate additional structural properties of the nested booking limits. We show that there always *exist* optimal nested policies with such structural properties, and we demonstrate that these policies will dominate those lacking such properties. Clearly, when there is a unique optimal nested policy, then that policy must have these structural properties.

Definition 4 *A nested policy x is called* lower bound aware (LBA) *if $x_i \geq L_i, \forall i < u(x)$ where $u(x) = \max\{i : x_i > 0\}$.*

The insight behind a LBA policy is this: if one accepts a given fare request, then one should never reject any of the requests with higher fares, the number of which is guaranteed to be at least equal to the lower bound. And indeed we have the following theorem:

Theorem 11 *There exists a LBA optimal nested policy.*

Proof Let x be an optimal nested policy. If x is LBA, we are done; otherwise we will show a way to construct a LBA optimal nested policy from x.

By the definition of LBA policy, if x is not LBA, then $x_i < L_i$ for some $i < u(x)$. Let $s = \max\{i < u(x) : x_i < L_i\}$ and $t = \min\{i > s : x_i > L_i \text{ or } i = u(x)\}$. Construct a new policy $x' = x + e_s - e_t$. We show that x' dominates x. Note that we have

$$x_i = L_i, \forall i : s < i < t \tag{5.5}$$

by the way s and t are found. Consider the corresponding booking limits b and b', we have $b'_i = b_i - 1, s < i \leq t$ and $b'_i = b_i$ otherwise. For a LBH input $\acute{q} \in \Omega(L, U)$, we consider two cases. The first case is when none of the limits $b_i, s < i \leq t$ were met. In this case it is clear that b' ends up accepting exactly the same set of requests, thus no change in on-line revenue. The second case is when some of the limits $b_i, s < i \leq t$ were met, let $\hat{j} = \max\{s < i \leq t : b_i \text{ is met}\}$, then all $b_i, s \leq i < \hat{j}$ are also met because of the equations in (5.5) and $x_s < L_s$. Let $w = w(\acute{q}; x)$, and $w' = w(\acute{q}; x')$, denoting the outcome vectors for x and x' respectively. We will compare w and w' as the fare requests in $\acute{q}$ arrives in LBH order. First note that $w_i = w'_i, i > t$, since $b_i = b'_i, i > t$; then for $\hat{j} < i \leq t$, since none of these limits of b is met, we still have $w_i = w'_i$. As for $i = \hat{j}$, because $b_{\hat{j}}$ is met, and $w_i = w'_i, i > \hat{j}$, it is obvious that b' would reject exactly one more class-$\hat{j}$ fare request than b would. After that, for i with $s < i < \hat{j}$, we clearly have $w_i = w'_i = x_i = x'_i = L_i$. So, when it comes to $i = s$, we have $w_i = x_i < L_i$ and $w'_i = x'_i \leq L_i$, so b' would accept exactly one more class-s fare request than b. For all the remaining fare classes, clearly we again have

$w_i = w'_i, i < s$. So the on-line revenue increased by $f'_s - f'_{\hat{j}}$ in this case. In both cases, the on-line revenue never decreases, so b' dominates b.

Let $x^{(0)} = x$, and if $x^{(k)}, k \geq 0$ is not a LBA policy, we can recursively construct $x^{(k+1)}$ from $x^{(k)}$ in the same fashion as above, obtaining a series of policies, each one dominating the ones before it. Consider the sum $\sum_{i=1}^{u(x^{(k)})-1}(L_i - x_i^{(k)})^+$ decreases by exactly one as k increases by one. Clearly, the process will end within limited number of iterations until the sum is zero, where a LBA policy is found, which is exactly what we are looking for. •

We point out that the dominance of LBA policies holds only for LBH streams, which is fine for competitive analysis, since we have shown that LBH streams are the worst cases. So, as we restrained the adversary to the LBH input streams, we can also narrow down the strategies of the seller to LBA policies. A natural questions here is: when the adversary is restrained to the extreme input streams $\acute{q}^k$, can we further narrow down the on-line policy space? Before we investigate this question, we introduce some notations. As we focus on the discrete problem, we assume $v(y)$ only takes discrete values. We define $\Delta v(y) = v(y) - v(y-1)$, and when $\Delta v(y)$ is non-decreasing in y, we say $v(y)$ is convex. Convexity of $v(y)$ is a desirable property but we only assume it when necessary. Let $s_k(x) = \|w(\acute{q}^k; x)\|_+$ be the total number of bookings when policy x is applied to extreme input $\acute{q}^k$. We have this definition below:

Definition 5 *A nested policy x is called* light loaded *if x is LBA and* $f'_{\eta(k)} >$

$\Delta v(s_k(x))$ *holds for all* k *such that* $\eta(k) \leq u(x)$.

Recall that $\eta(k)$ is the class of the additional fare request such that when it is added to q^{k-1}, we obtain q^k; or simply put: $q^k = q^{k-1} + e_{\eta(k)}$. The definition above shows that if the input stream changes from $\acute{q}^{k-1}$ to $\acute{q}^k$, and if the additional request in class $\eta(k)$ is accepted, then there is an increase of $f'_{\eta(k)}$, but on the other hand, the expected denial cost also increases by $\Delta v(s_k(x))$. As $f'_{\eta(k)} > \Delta v(s_k(x))$, the net revenue of a light loaded policy will never decrease, but will strictly increase when $s_k(x)$ increases.

Theorem 12 *If* $v(\cdot)$ *is convex, then there is a light loaded optimal nested policy.*

Proof Let x be an optimal LBA policy. If x is light loaded, we are done; otherwise we will construct a finite series of optimal LBA policies from x that would lead to a light loaded optimal nested policy.

We show that given an optimal LBA policy x that is not light loaded, we can always find an index $\hat{l}$ such that $x' = x - e_{\hat{l}}$ is also an optimal LBA policy. By Definition 5, if x is not light loaded, then

$$K(x) = \{k : \eta(k) \leq u(x), f'_{\eta(k)} \leq \Delta v(s_k(x))\} \neq \emptyset.$$

Let $k(x) = \min K(x)$. If $x_{\eta(k(x))} > L_{\eta(k(x))}$, then let $\hat{l} = \eta(k(x))$, otherwise let $\hat{l} = u(x)$. Since $\eta(k(x)) \leq u(x)$ by construction, we note that $\hat{l} \geq \eta(k(x))$. Clearly the derived new policy $x' = x - e_{\hat{l}}$ is also LBA. It remains to show that x' is also an optimal nested policy, where we simply compare the on-line revenues of x' with those of x for all extreme inputs. It should be clear that x' will not accept anything more than x, but rather x' could reject at most one more class $\hat{l}$ fare request. We show that such a rejection will not decrease the on-line revenue. For the inputs $\acute{q}^k$ with $k \geq k(x)$, as $s_{k(x)}(x) \leq s_k(x)$ and $\hat{l} \geq \eta(k(x))$, we have

$$f'_{\hat{l}} \leq f'_{\eta(k(x))} \leq \Delta v(s_{k(x)}(x)) \leq \Delta v(s_k(x))$$

by convexity of $v(\cdot)$ and construction of $k(x)$. So the rejection will not reduce the on-line revenue for any $\acute{q}^k$ with $k \geq k(x)$:

$$R(\acute{q}^k, x) \leq R(\acute{q}^k, x'), k \geq k(x).$$

Next, for the inputs $\acute{q}^k$ with $k < k(x)$, we consider two cases.

(i) $x_{\eta(k(x))} > L_{\eta(k(x))}$. Then we have $\hat{l} = \eta(k(x))$; we will also have $q_{\hat{l}}^{k(x)} \leq x_{\hat{l}}$, otherwise we have $s_{k(x)-1}(x) = s_{k(x)}(x)$ (since $q^{k(x)}$ is obtained from $q^{k(x)-1}$ by adding one more class-$\hat{l}$ request, and if $q_{\hat{l}}^{k(x)} > x_{\hat{l}}$, then that additional class-$\hat{l}$ request must have been rejected, and we end up with the same set of accepted requests), and that entails $k(x) - 1 \in K(x)$ (recall that $f_{\eta(k(x))} \geq f_{\eta(k(x)-1)}$), which is a contradiction as $k(x)$ is the minimal element in $K(x)$. Thus $q_{\hat{l}}^k \leq x_{\hat{l}} - 1 = x'_{\hat{l}}$ for all $k < k(x)$, which implies x' and x should accept the same set of requests (this is obtained by comparing the outcome vectors in LBH order, as used in the proof of Theorem 11), so $R(\acute{q}^k, x) = R(\acute{q}^k, x'), k < k(x)$.

(ii) $x_{\eta(k(x))} \leq L_{\eta(k(x))}$. Then any class-$\eta(k(x))$ fare request above the lower bound will be rejected, thus reducing one class-$\eta(k(x))$ request above the lower bound will not affect the set of accepted requests, so $w(\acute{q}^{k(x)}; x) = w(\acute{q}^{k(x)-1}; x)$, which gives $s_{k(x)-1}(x) = s_{k(x)}(x)$ and $R(\acute{q}^{k(x)-1}, x) = R(\acute{q}^{k(x)}, x)$. Then we also have

$f'_{\eta(k(x)-1)} \leq \Delta v(s_{k(x)-1}(x))$. Now we must have $\eta(k(x)-1) > u(x)$ or the contradiction of $k(x)-1 \in K(x)$, thus $k < k(x)$ implies $\eta(k) > u(x)$. That means all class-$\eta(k)$ requests are rejected by x, so $R(\acute{q}^k, x) = R(\acute{q}^{k(x)}, x), k < k(x)$. By similar arguments, the same can be obtained for x' with $k < k(x)$, as we have $x'_{\eta(k(x))} \leq L_{\eta(k(x))}$ and $\eta(k) > u(x')$ (since $\hat{l} = u(x)$ and $u(x) \geq u(x')$ in this case). Then we obtain the same result as in case (i).

So in summary, the revenue by x' is no less than that by x for all extreme inputs, so x' is a new LBA optimal policy.

Let $x^{(0)} = x$, and if $x^{(k)}, k \geq 0$ is not light loaded, we can recursively construct $x^{(k+1)}$ from $x^{(k)}$ in the same fashion above, obtaining a series of LBA optimal policies. We claim that such a series is finite, because the maximal number of accepted requests $\sum_{i=1}^{m} x_i^{(k)}$ reduces by one each time, and in the worst case we end up with a policy with all bucket sizes being zero, which is obviously a light loaded policy as $\Delta v(y) = 0, y \leq n$. •

From the proof above we see that light loaded policies provide not only more on-line revenues but also better service levels by having less total bookings. And it also becomes evident that the on-line revenues of light loaded policies strictly increases with $s_k(x)$. However, it should be emphasized that these advantages may not exist for any input stream, but only exist for the LBH inputs with the extreme profiles. From the policy designer's perspective, we see that we can restrict the attention to light loaded policies, this fact in turn allows us to further investigate the role an extreme profile plays.

Theorem 13 *If $v(\cdot)$ is convex and x is a light loaded policy, then for any input stream I, let $k = \|q(I)\|_+$, we have $R(I;x) \geq R(\acute{q}^k;x)$.*

Proof We only need to consider LBH streams. It is sufficient to prove that for any LBH input stream $\acute{q}$ with $L \leq q \leq U$ and $\|q\|_+ = k$, we have $R(\acute{q};x) \geq R(\acute{q}^k;x)$. If $q = q^k$, we are done, otherwise we show the dominance by constructing a series of intermediate profiles.

Let the LBH stream $\acute{q}$ belongs to category $c+1$, so $\sum_{i=j}^{c} q_i < \sum_{i=j}^{c} x_i, j \leq c$. Consider the difference $D = q - q^k$, then $\|D\|_+ = 0$. As $q \neq q^k$, there exist a $\hat{i}$ and a $\hat{j}$ such that $D_{\hat{i}} < 0$ and $D_{\hat{j}} > 0$. Note that $\|q\|_+ = k = \|q^k\|_+$, $q_i \leq U_i = q_i^k, i > \eta(k)$, and $q_i \geq L_i = q_i^k, i < \eta(k)$, we obviously have $\hat{i} \geq \eta(k)$, $\hat{j} \leq \eta(k)$, and thus $\hat{i} > \hat{j}$. Construct a new profile $q' = q + e_{\hat{i}} - e_{\hat{j}}$. We show q' dominates q by two cases. The first case is when $\hat{i} \leq c$, by similar arguments as in the proof of Theorem 10, we know no class-$i (i \leq c)$ requests are rejected by x for either $\acute{q}$ or $\acute{q}'$, thus $R(\acute{q};x) - R(\acute{q}';x) = f'_{\hat{j}} - f'_{\hat{i}} \geq 0$, so q' dominates q.

The second case is when $\hat{i} > c$. Choose $\hat{j}$ more carefully: $\hat{j} = \min\{j : D_j > 0\}$. Let $\hat{k} = \min\{t : q^t \geq q\}$, then $s_{\hat{k}}(x) \geq \|w(\acute{q};x)\|_+$ and $\hat{j} = \eta(\hat{k})$ (as $q_i = q_i^k = L_i$ for $i < \hat{j}$ and $q_{\hat{j}} > q_{\hat{j}}^k \geq L_{\hat{j}}$). As x is light loaded, we have

$$f'_{\hat{j}} \geq f'_{\eta(\hat{k})} > \Delta v(s_{\hat{k}}(x)) \geq \Delta v(\|w(\acute{q};x)\|_+)$$

by Definition 5 and convexity of $v(\cdot)$. The change from q to q' can be achieved by three steps: (1) append to the end a class $\hat{i}$ request; (2) make adjustments to arrange it into a LBH stream; (3) take out a class $\hat{j}$ request. The on-line revenue changes are described below for each step: (1) As $\hat{i} > c$, the appended class $\hat{i}$ request will be rejected, so there is no change in on-line revenue. (2) This step will never increase the on-line revenue, by Theorem 9. (3) If there was already a class $\hat{j}$ request rejected by x, then taking out one will not affect the revenue. If no class $\hat{j}$ request was rejected, then we must have $u(x) \geq \hat{j}$, thus $x_i \geq L_i$ for all $i < \hat{j}$. Since $D_i = 0$ for $i < \hat{j} \leq \eta(k)$ (recall how we carefully choose $\hat{j}$ in this case), so $q_i = L_i \leq x_i$ for $i < \hat{j}$, thus all those class $i(< \hat{j})$ requests were already accepted, and taking out a class $\hat{j}$ request will not help accepting more higher fare requests, so the net effect on the on-line revenue is

$$\Delta v(\|w(\acute{q}; x)\|_+) - f'_{\hat{j}} \leq \Delta v(s_{\hat{k}}(x)) - f'_{\hat{j}} < 0.$$

So the on-line revenue can not increase in this case either.

Finally, we can repeat such adjustment procedure recursively for finite times until we obtain q^k, then the theorem becomes evident. •

Thus far we have shown the existence of a light loaded optimal nested policy, and further characterized the extreme profiles. It is still an open question whether an optimal nested policy is *the* optimal policy for our problem. We will leave the analysis of that for future research. For now we turn our attention to the design of an optimization procedure to find the optimal nested policy. The optimization method involves sampling on the policy space and evaluating the objective function value of the randomly chosen policies. In the computations, we will utilize the results developed in the previous section, those results enable us to efficiently compute the objective function value. Our numerical tests show that the optimization procedure (to be specified in the next section) always produces light loaded policies. Meanwhile, in case the optimization procedure produces a policy that is not light loaded, which is very unlikely from our numerical experiments, a light loaded policy can still be obtained by the construction steps used in the proof of the existence theorems for LBA and light loaded optimal nested policies, i.e., from Theorems 11 and 12, respectively. The construction steps in the proofs of these theorems can be readily incorporated into an efficient procedure that will take an arbitrary optimal nested policy, and transform it into a light loaded optimal nested policy. However, as we observed, such a procedure is not needed in general.

5.4 Optimizing with Model-Based Procedures

Model-based methods have found its popularity due to their simplicity and effectiveness as demonstrated by benchmarking against other well-known methods over a variety of global optimization problems; two of the most effective model-based methods are the cross-entropy (CE) method (Boer et al., 2008) and the model reference adaptive search (MRAS) method (Hu et al., 2006). In many optimization methods, the search for new candidate solutions depend directly on previously obtained solutions, such as simulated annealing (SA) (S. Kirkpatrick, C.D. Gelatt, and M.P. Vecchi 1983), genetic algorithms (GAs) (D.E. Goldberg 1989), tabu search (F.W. Glover 1990). Those methods can be called instance-based methods, and unlike these instance-based methods, model-based methods start with a distribution model of the solution space, which can be viewed as an initial guess of the likelihood of the location of the optimal solution. In order to improve the distribution model, the model-based methods repeat these two phrases to generate new candidate models instead of candidate instances, until a stopping criterion is met:

(1) take random samples according to a distribution model;

(2) the samples are evaluated based on the objective function, so that the distribution model can be updated for a new round of random sampling.

We develop a model-based procedure for the continuous version of the problem described by equation (5.2). Notice that the booking limits and bucket sizes can take discrete or continuous values in our CR or MAR problems. However, some no-show models (such as Binomial) only permit discrete values, limiting the policy parameters b and x to be discrete, too. In this section, we drop the integrality constraint on x. One detail pertaining this continuous relaxation is to define a continuous expected service denial cost function $\tilde{v}(\cdot)$ which extends the discrete version $v(\cdot)$. We suggest using the linear interpolation of $v(\cdot)$, which gives $\tilde{v}(y) = (1 + \lfloor y \rfloor - y)v(\lfloor y \rfloor) + (y - \lfloor y \rfloor)v(1 + \lfloor y \rfloor)$. Note that the expected denial penalty cost $v(\cdot)$ is nonlinear. Further, even if the cost function $v(\cdot)$ were convex, $v(\|w(\acute{q}^k; x)\|_+)$ is not necessarily convex in x. Thus a general global optimization procedure is needed to solve both CR and MAR problems, and we consider model-based methods.

Designing an effective model-based optimization procedure is not always straight forward, and insights on the special problem structure can help develop a good procedure. What kind of distribution models to work with, how to efficiently sample from the current model, and how to generate a new candidate model from the samples are three key issues in designing a model-based optimization procedure. We choose the multivariate normal distribution model for x, with each x_i being statistically independent from others. Such a model belongs to the natural exponential family (NEF), and has been shown to have global convergence with MRAS and CE methods (Hu et al., 2006). Also, it is simple to work with, since there are only $2m$ parameters: a mean μ_i and a standard deviation σ_i for each x_i (no covariance between x_i's.

Recall that we assume $0 \leq x_i \leq U_i$, while multivariate normal can have $-\infty < x_i < \infty$. So we extend the objective function to the range of the distribution, while ensuring the same optimal solution. We propose the following extension: $\mathrm{CR}'(x) = \mathrm{CR}(x') - \sum_{i=1}^{m} |x_i - x_i'|$ for $\mathrm{CR}(x)$, and $\mathrm{MAR}'(x) = \mathrm{MAR}(x') + \sum_{i=1}^{m} |x_i - x_i'|$ for $\mathrm{MAR}(x)$, where $x_i' = \min(x_i^+, U_i), x_i^+ = \max(0, x_i)$.

In the CE method, the random samples are sorted by the objective value, and the best ρ percent (typically about 1% for large sample sizes) of the samples are selected as the elite set. Comparing and selecting the random samples involves the evaluation of the objective function (either CR or MAR) at the random samples. With our scenario reduction results, the objective function becomes computationally efficient, and this makes the design of a model-based procedure possible. The procedure then produces the next candidate model, which is the distribution obtained by applying maximum likelihood estimation (MLE) to the elite set. The MLE is easily computed for multivariate normal distribution. The theory behind this is that the

MLE minimizes the cross entropy.

Let N be the random sample size, T the maximum number of iterations. An outline our optimization procedure is given below:

1. Initialize: $t \leftarrow 0$, the iteration count;

   $\mu_i^{(0)} \leftarrow (L_i + U_i)/2$ for $i = 1, \cdots, m$;

   $\sigma_i^{(0)} \leftarrow U_i$ for $i = 1, \cdots, m$.

2. Generate N random nested booking policies x from the multivariate normal model with $\mu^{(t)}, \sigma^{(t)}$.

3. Evaluate the policies (compute CR or MAR) and select the best ρ percent as the elite set.

4. Keep track of the best policy found so far.

5. Compute the MLE from the elite set to update $\mu^{(t+1)}, \sigma^{(t+1)}$.

6. $t \leftarrow t + 1$, if $t = T$ then STOP, else GOTO step 2.

While there are more sophisticated stopping criteria, we use a fixed number of iterations above, mainly for illustration purposes. An alternative is to stop when the best solution does not improve after a given number of consecutive iterations. In our experiments, we use sample size $N = 500$, maximum number of iterations $N = 20$, and elite threshold $\rho = 2\%$.

We tested the procedure on a three class problem with the following parameter settings: $n = 124, \beta = 0.2, f = (\$1050, \$647, \$350), L = (20, 30, 0), U =$

$(64, 120, 39)$, and service denial cost function $V(Z) = \$1400[Z - 124]^+$. For the no-shows, we use the Binomial given a no-show probability, which is drawn from a uniform distribution: $p \sim \text{Binomial}(\hat{p})$ with $\hat{p} \sim U[0.0, 0.2]$. This particular choice of distribution satisfies our assumptions on no-show random variables. In Table 5.1 below we report the best solution found at each iteration. We truncated the results at iteration 6 as the remaining iterations no longer improve the best solution. Note that the mean of the multivariate normal model converges to the best solution, while the standard deviation converges to zero. At iteration 20, the vector of mean is $(46.9, 81.5, 18.6)$, and the vector of standard deviation is $(3.2e-5, 3.2e-5, 3.2e-5)$ (reached the precision threshold). The procedure is tested several times and converges to the same solution. Note that the policy is LBA, and we show it is also light loaded next.

Figure 5.4 depicts the on-line revenue obtained by the optimal policy as bounded by the offline revenue and the guaranteed revenue (offline revenue multiplied by the obtained competitive ratio, 0.868 in this case). This figure is produced by applying the policy to the LBH inputs with the extreme profiles, and the x-axis gives the total number of requests in the extreme profiles, as there is a one-to-one correspondence. We see in the figure that the on-line revenue of the policy never decreases, indicating that the policy produced by CE method is already light loaded.

Itr.	CR(x)	x_1	x_2	x_3
0	0.803	47.3	83.2	8.6
1	0.846	117.6	81.9	15.2
2	0.858	47.1	80.7	20.7
3	0.867	45.7	81.7	18.4
4	0.868	48.2	81.4	18.7
5	0.868	46.8	81.5	18.7
6	0.868	46.9	81.5	18.6

Table 5.1: The best solution obtained by CE iterations.

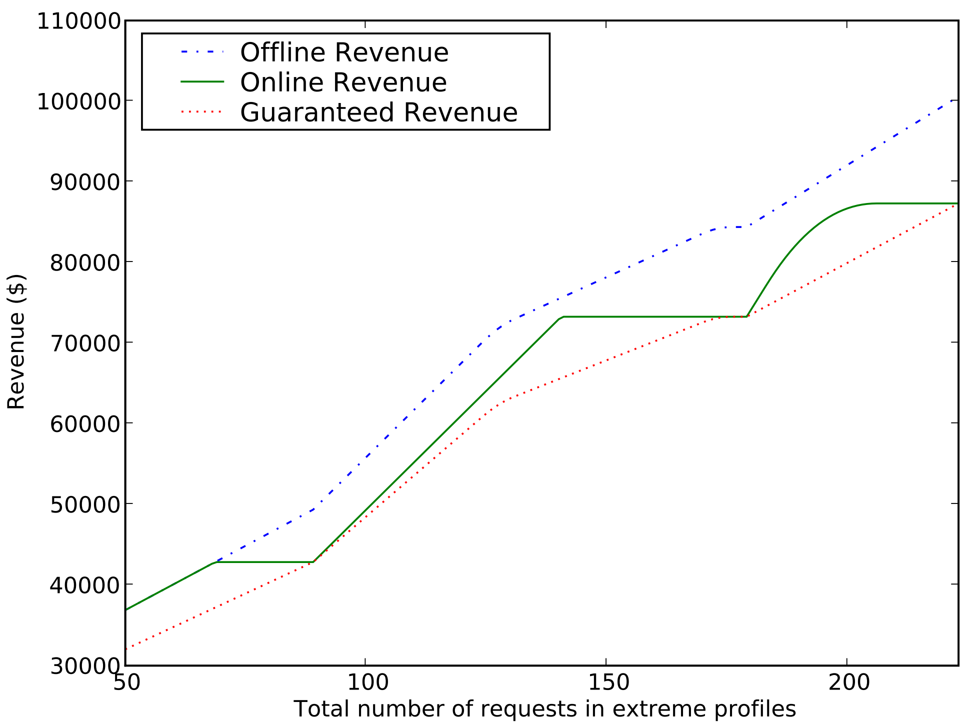

Figure 5.1: Revenue plot for the policy found by CE.

Chapter 6

Numerical Experiments

In this chapter we conduct computational experiments (i) to quantify the performance of our policies in practical settings, and (ii) to compare our policies to other well-known procedures in single-resource RM. We report the result of experiments that include no-shows below. A comparison of fare-class allocation methods in the absence of cancellations and no-shows, including the ones introduced in Chapter 3, is available in Gao (2008).

6.1 Experiment Design

In the experiments, we test the following methods: (i) OBSA/CR (is based on our method in Chapter 4) uses the solution to BON to process the bookings. (ii) EMSR/CR is a hybrid method where virtual capacity is determined using BON and seat inventory control policy is the nested booking limits provided by the expected marginal seat revenue (EMSR) heuristic (see Talluri and van Ryzin, 2004a). (iii) EMSR/NV is a hybrid method where virtual capacity is determined using the cost-based (news-vendor type) model described in Chapter 9 of Phillips (2005) for the multi-fare problem: This cost-based model balances the expected 'spoilage' cost, i.e., loss of revenue on each unit of unsold inventory with the expected 'spill' cost, i.e. cost of overbooking. The weighted average (weighted by mean fare-class demand) of fares is used as the estimate of the unit spoilage cost. (iv) CRSA/NV is a hybrid method where virtual capacity is computed by the news-vendor type model that balances expected spoilage and expected spill, and this virtual capacity is used to determine nested booking limits of the linear program GBM which was developed in Chapter 3 (Note that GBM is equivalent to BON when $p_0 = p_1 = 0$). (v) DP/LBH solves a stochastic dynamic program to determine the optimal fare class allocation and total number of units to be sold given probabilistic information on demand and no-shows. (vi) EMSR/SL determines the virtual capacity using a target type-I service level of 0.1%, i.e., probability of oversales is 10 in 10,000. Given the virtual capacity, nested booking limits are determined using EMSR. See Talluri and van Ryzin (2004a) on service-level based models of overbooking. (vii) CRE/OSA utilizes full no-show information and limited demand information to determine overbooking and fare-class allocation simultaneously. (iix) EMSR/NO involves no overbooking, and determines nested booking limits by EMSR using the actual capacity. This latter method serves as a point of reference for the economic benefit of overbooking.

Our numerical experiments test the effect of one or more of the problem parameters and modeling assumptions on the performance of the policies derived using the above mentioned methods. Each experiment involves 5000 simulation runs, guaranteeing tight confidence intervals around the mean value of the performance measures. The base-case used in our experiments has the following parameter values: $m = 2$, $f_1 = 100$, $f_2 = 40$, $\beta = 0.2$, $V = 500$, and $n = 100$. The demand bounds are $U_1 = 70, L_1 = 40$ and $U_2 = 80$, $L_2 = 50$. The demand is uniformly distributed over $[L_i, U_i]$ for $i = 1, 2$. The requests arrive in LBH order. The simulation set up is explained in detail for each experiment below.

6.2 Use of accurate no-show information

In this section, the methods are tested using a simulation model that is compatible with the models' assumptions on no-shows. The no-show rate is a random variable distributed uniformly over the range $[p_0, p_1]$. The (random) no-show rate is independent of the number of reservations on-hand and each unit reservation contributes the same amount to no-shows given the no-show rate. All the methods that use stochastic models in overbooking-related decisions (EMSR/NV, CRSA/NV, DP/LBH, EMSR/SL) choose their booking control parameters based on this probabilistic information. OBSA/CR uses the information that no-show rate lies in the range $[p_0, p_1]$, while CRE/OSA is able to utilize the probabilistic distribution of the no-show rate.

Example-1. Effect of no-show range. In this example, we vary the *spread* of the no-show rate. We fix the mean value of the no-show rate distribution to 0.15 and vary the spread, S, from 0.0 to 0.3, where $p_0 = 0.15 - S/2$ and $p_1 = 0.15 + S/2$. The average revenues, average number of unused units at the time of service, and the average number of service denials are presented in Figure 6.1. DP/LBH provides the optimal risk-neutral policy and has the highest net revenues. Net revenues of all the methods, except EMSR/NO, decrease as the spread increases, showing the effect of variability in no-shows. Spoilage (unused units) increases and spill (denied service) decreases as spread increases. When the no-show behavior is highly volatile, the economic advantage of overbooking is lower (see the difference in net revenues of the methods compared to EMSR/NO). The important observation here is that distribution-free method based on seller' s regret is as effective as any other stochastic method from an economic perspective and is superior in service quality. Both OBSA/CR and EMSR/CR have lower average denied service compared to CRE/OSA, DP/LBH, EMSR/NV and CRSA/NV. Therefore, the distribution-free methods of overbooking achieve a good trade-off in net revenues vs. denied service, while, for e.g., DP/LBH fails to do so.

It is interesting to observe that in this example CRE/OSA and DP/LBH perform almost the same in all the three measures, and indeed both produce very similar booking limits, although CRE/OSA uses less information compared to DP/LBH. However, the differences between CRE/OSA and DP/LBH are visible in other examples.

Example-2. Effect of mean no-show rate. In this example, the support of the no-show rate distribution shifts: we fix the spread at 0.2, and vary the mean no-show rate from 0.1 to 0.4. The rest of the problem parameters are the same as Example-1. The average revenues, average number of unused units at the time of service, and the average number of service denials are presented in Figure 6.2. In this example, a shift in the range of the no-show rate affects the demand factor, which is the ratio of expected demand (net of cancellations) to the capacity. The differences among the performances of the policies are reduced when demand factor is low. In this example, there is significant economic benefit to overbooking. Distribution-free methods in overbooking again are effective, with high net revenues and good quality

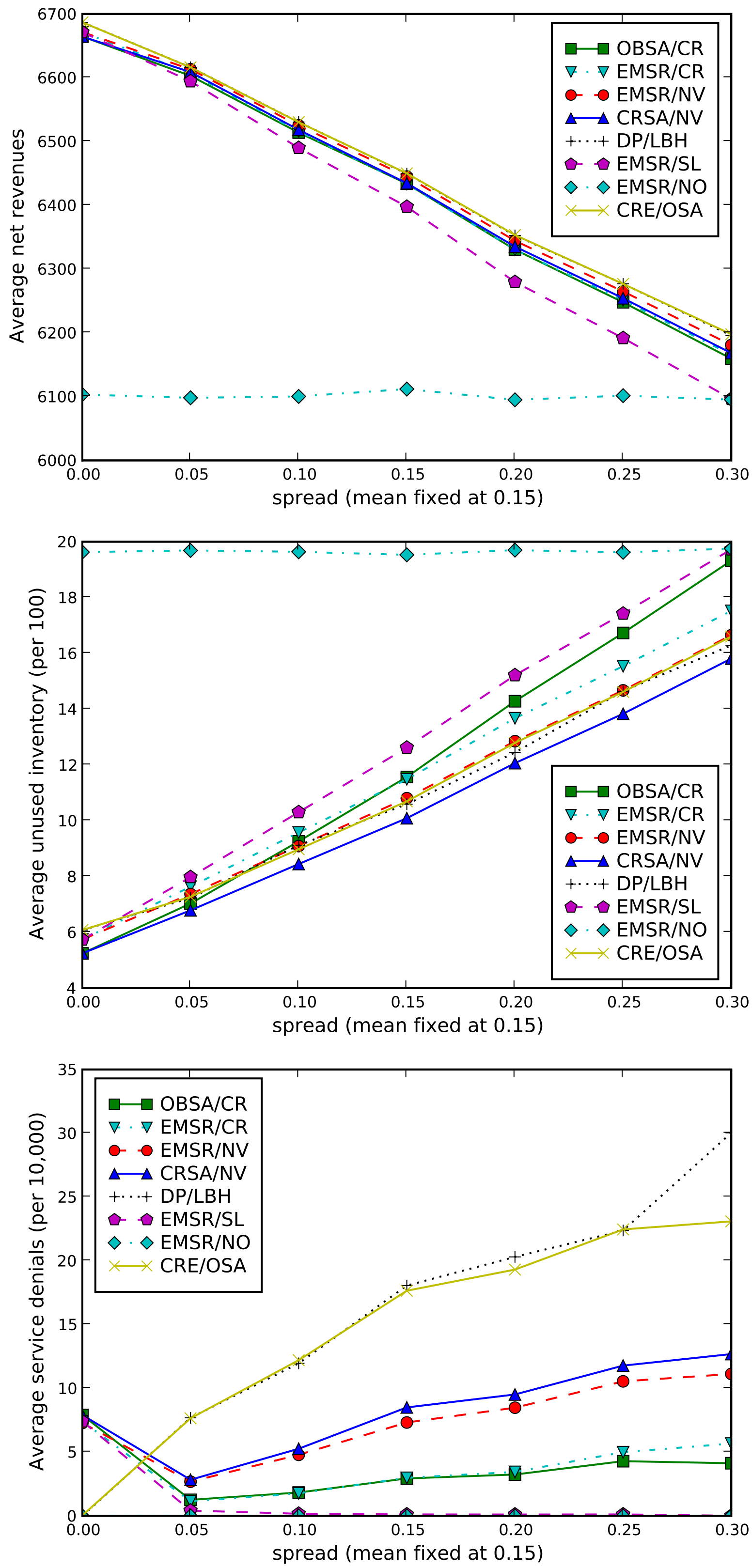

Figure 6.1: Net revenues, amount of unused units and amount of denied service in Example-1.

of service compared to other stochastic models. And the CRE/OSA has the highest level of service denial next to DP/LBH, we think this is mainly due to the uniform no-show distribution, as will be explained in the next experiment, where the no-show distribution is set up in a different way.

We conducted more experiments, testing the effect of fares, demand factor, and overbooking costs (see Appendix B.6). Our observations on the advantage of distribution-free methods in overbooking are valid in those experiments, too.

6.3 Effect of Limited No-Show Information

In this section, the number of no-shows is characterized differently in the simulation: After the reservation requests are processed, each of the accepted reservations may independently be cancelled with probability $\hat{p}$, which is a random variable uniformly distributed over the interval $[\hat{p}_0, \hat{p}_1]$. We call this the *conditional Binomial distribution*: The number of no-shows is Binomial distributed with parameters $\hat{p}$ and ξ where ξ is the total number of on-hand reservations and $\hat{p}$ is the random no-show probability. We use this setup in two experiments to see the effect of limited/inaccurate no-show information on effectiveness of the methods.

Notice that the no-show distribution varies with the actual number of reservations in this case. However, no-show information is independent of number of reservations in our model. Therefore, we use crude estimates of the no-show range to determine the overbooking levels in OBSA/CR and EMSR/CR. Given parameters $\hat{p}_0$ and $\hat{p}_1$ of the conditional Binomial distribution, we calculate 5^{th} and 95^{th}

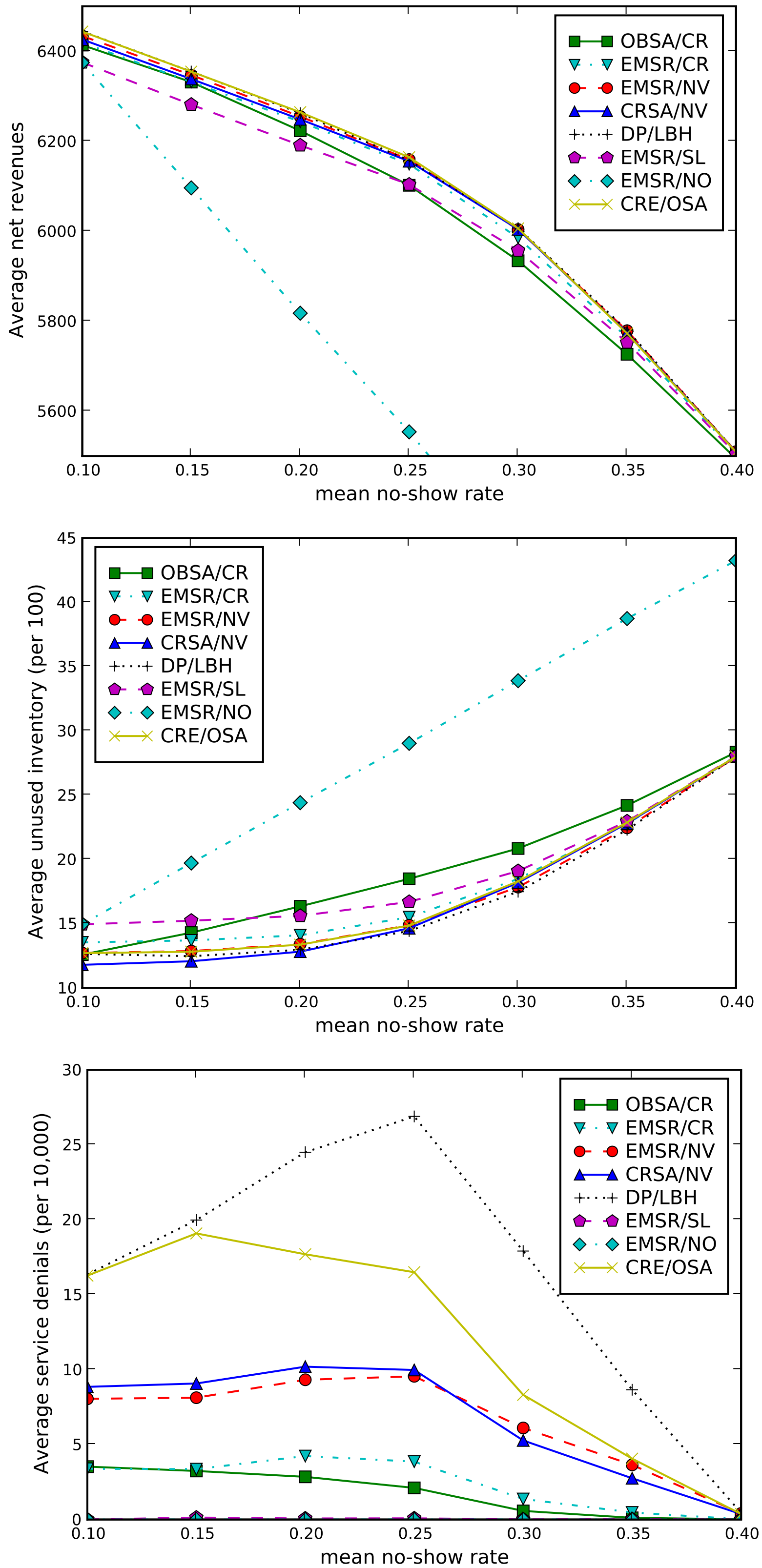

Figure 6.2: Net revenues, amount of unused units and denied service in Example-2.

percentiles of the no-show distribution when there are only 100 reservations. These quantiles estimates are used for p_0 and p_1.

Example-3. Limited no-show information. Here, all the stochastic models have perfect probabilistic information on demand and the conditional Binomial distribution of no-shows. We use the default parameter values. The experiment designed is similar to that of Example-1 where the spread varies, except that the spread in this case effects the range of no-show probability $[\hat{p}_0, \hat{p}_1]$. The spread parameter S varies from 0 to 0.3, while the mid-point $(\hat{p}_0 + \hat{p}_1)/2 = 0.15$ is fixed. The experiment results are reported in Figure 6.3. The performance of the OBSA/CR and EMSR/CR methods are again very good, despite using limited information on no-shows. These methods achieve slightly lower revenues compared to DP/LBH when spread is zero but that economic loss comes with a significant gain in service quality.

In this particular setting of no-show distribution, we begin to see the difference between CRE/OSA and DP/LBH in service level, while CRE/OSA still obtains high average revenues comparable to DP/LBH. A closer look at the service level by CRE/OSA reveals a transition from the conservative extreme with OBSA/CR to the aggressive extreme with DP/LBH as the spread S increase from 0 to 0.3. As the spread increases, the variance of no-shows also increases. Since CRE/OSA uses expected revenue penalty according to the given no-show distribution, it should be applied with caution when the variance is too high. We also remark that it is easy to incorporate variance awareness into CRE/OSA, simply by including a variance related term into the no-show penalty. Our model-based optimization methodology readily applies to this kind of modifications.

Example-4. Inaccurate no-show information. We repeat the experiment in Example-3 here. The no-shows are simulated using the conditional Binomial distribution. However, the methods evaluated fail to take into account the variability in no-show probabilities of the conditional Binomial distribution: EMSR/NV, CRSA/NV, DP/LBH, EMSR/SL, and CRE/OSA all assume that no-shows are binomial distributed with exact probability 0.15. We vary the actual spread of the conditional binomial distribution and test the effect of inaccurate information on the effectiveness of all the models. This time, the no-show bounds used in OBSA/CR and EMSR/CR are chosen as the 5^{th} and 95^{th} quantile of the binomial distribution with no-show probability 0.15 and 100 reservations.

Note that none of the methods correctly portray no-shows in making the overbooking decisions in this experiment. We illustrate the actual simulated distribution of show-ups vs. the *assumed* distributions of show-ups in Figure 6.4. In this graph, the probability mass function (PMF) of the correct conditional Binomial distribution is plotted when the no-show probability is distributed uniformly between 0.10 and 0.20 (i.e., show-up probability is Uniform[0.8,0.9]) and there are 120 reservations. The assumed binomial distribution (denoted NV on Figure 6.4) and the range used by our overbooking methods (denoted CR) are also plotted on the same graph.

This particular experiment shows the effect of modeling assumptions on estimating no-shows. The experiment results are reported in Figure 6.5. Notice that when spread is zero, the stochastic models of overbooking have perfect probabilistic information and achieve higher net revenues compared to EMSR/CR and

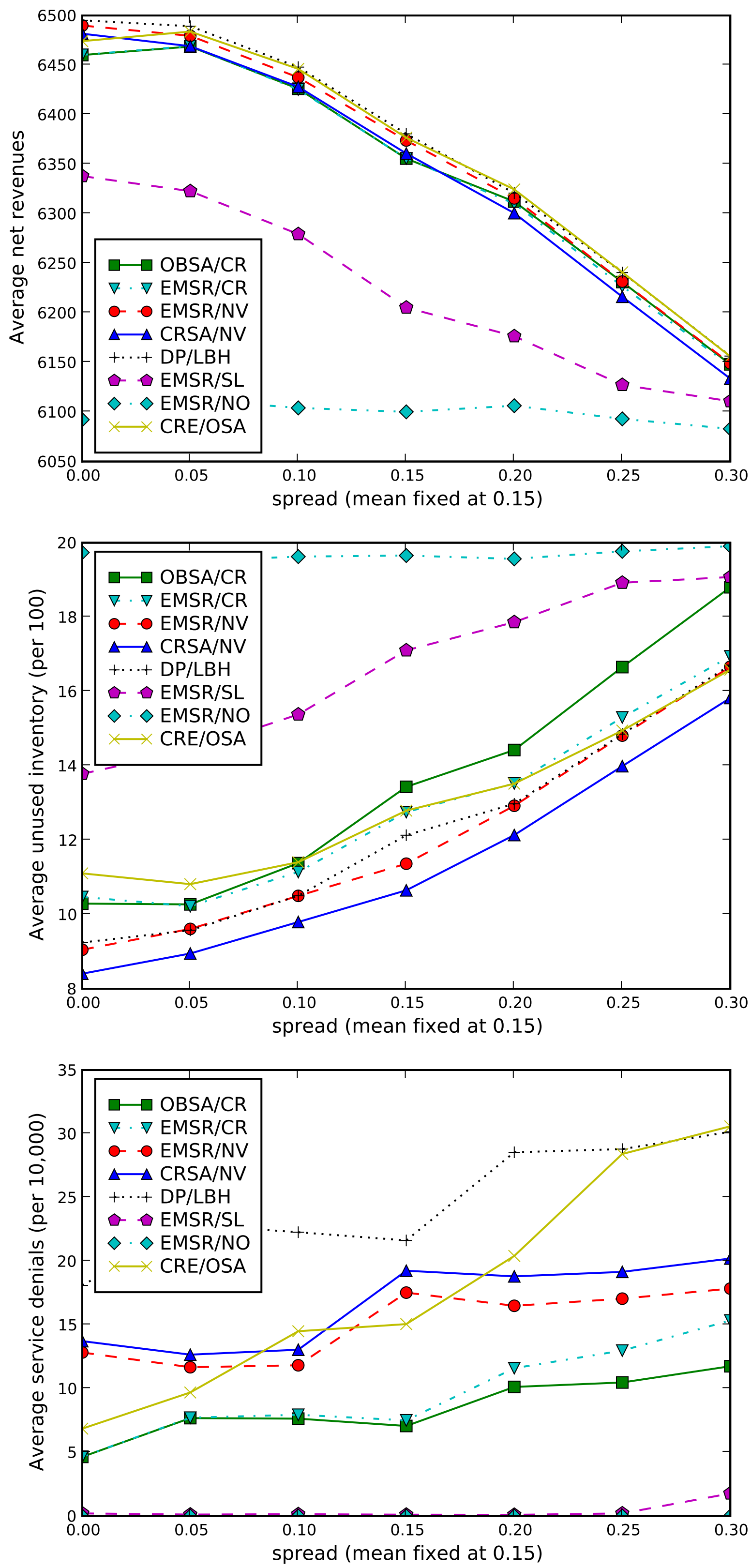

Figure 6.3: Net revenues, amount of unused units and denied service in Example-3.

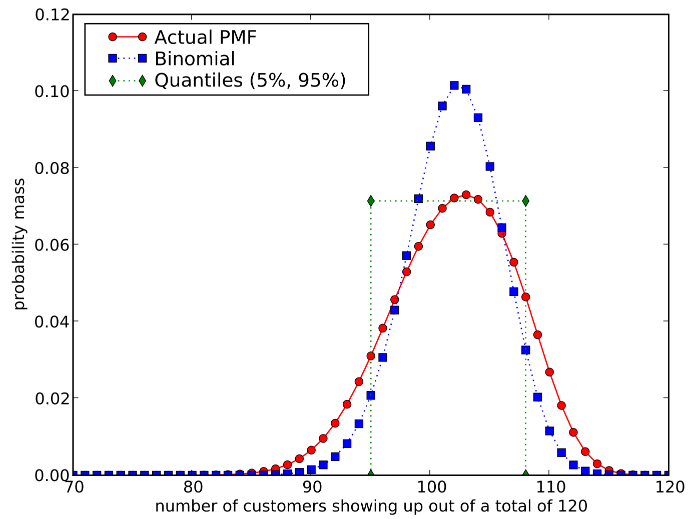

Figure 6.4: Probability mass functions (PMF) of the assumed and actual distributions of show-ups in Example-4 when no-show probability is distributed uniformly between 0.10 and 0.20 and there are 120 reservations.

OBSA/CR. Also note that CRE/OSA is conservative in this low variance with a zero spread. However, as the spread increases, EMSR/SL and EMSR/NO become better economic options compared to the other methods. EMSR/CR, OBSA/CR and CRE/OSA emerge as being more robust compared to DP/LBH and EMSR/NV. It is particularly interesting to see that the benefit of conservativeness from competitive analysis in CRE/OSA even offsets its wrong assumption about the no-show distribution in this example.

This last experiment is illustrative of what might go wrong with the stochastic, news-vendor type models if no-show estimation is not done correctly. If one blindly accepts that no-shows are Binomial distributed and chooses the no-show probability based on a point estimate, the resulting policy is effective as long as the standard error in forecasting the no-show probability is very small. However, as the forecast error increases, the distribution-free models and our hybrid model behind CRE/OSA (even though they are subject to the same bias in these stochastic models) provide better results.

Overall, these experiments demonstrate the effectiveness of our distribution-free methods in fare-class allocation and overbooking. These methods not only achieve net revenues as high as others when given accurate information, they also provide better service quality (lower number of service denials). In addition, the overbooking level determined using our distribution-free model is a viable alternative for other fare-class allocation methods. These experiments also show that our hybrid approach indeed offers robust solutions and provides higher revenues, and at the same time yield better revenues than the other overbooking and allocation policies studied in the second part of this dissertation.

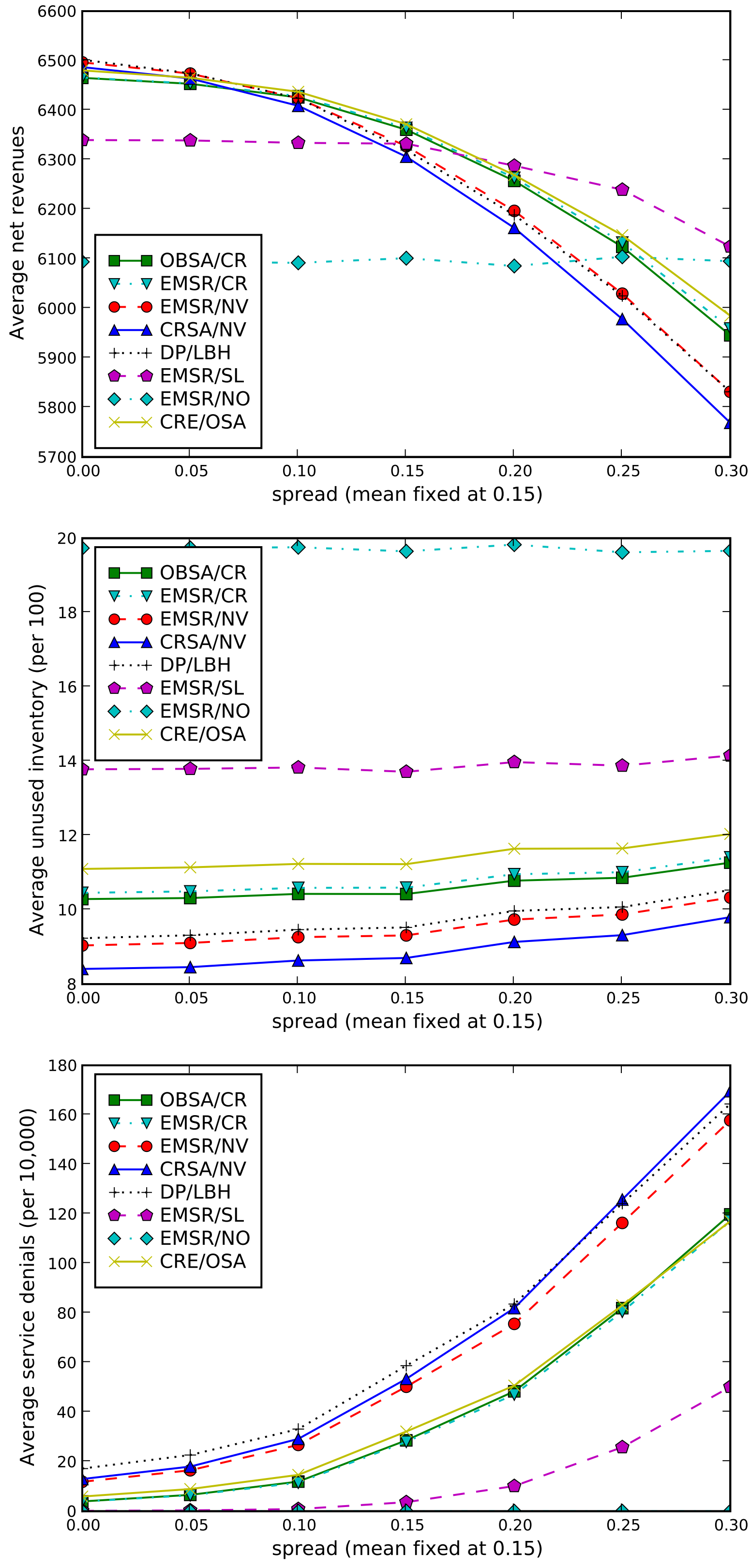

Figure 6.5: Average performance in Example-4.

Chapter 7

Future Work and Conclusions

In this dissertation, we analyzed the single-leg fare-class allocation problem from the perspective of competitive analysis of on-line algorithms. In Chapter 3, we make use of limited demand information and we derive static and dynamic policies for both competitive ratio and absolute regret criterion. Our optimal booking control policies have significant practical advantages: the nesting property of booking limits is preserved while the need for information is reduced, and the optimal policy parameters are obtained in closed-form, hence the computational burden is minimal.

We analyzed the traditional single-resource RM problem in the presence of cancellations and no-shows in Chapter 4. We developed a joint overbooking and fare-class allocation model under limited demand and no-show information. Using competitive analysis of on-line algorithms, we proposed methods for two different criteria: maximization of minimum competitive ratio and minimization of maximum (absolute) regret. We showed that the overbooking and seat inventory control problem in either case can be solved by developing an appropriate linear programming model, for which closed-form solutions exist. We also proved that nesting by revenue-order is the optimal static booking control policy in either case. we bench-marked our methods to other static overbooking and/or fare-class allocation policies. Our joint approach provides effective results in many cases, and our policies are robust as they are able hedge against inaccuracies in information. The overbooking level computed using our methods can be used in conjunction with other seat inventory control methods that assume no cancellations; our computational experiments indicate that the virtual capacities determined using our methods increase the effectiveness of the commonly used seat inventory control and overbooking methods.

We have developed a hybrid model for the simultaneous overbooking and fare-class allocation problem in Chapter 5. This hybrid approach enables us to take advantage of the often faced practical situation where there are both partial (on the demand) and full (on the no-show) information. Our competitive analysis of this problem greatly reduced the complexity to compute the relative performance measure, so that the competitive objective function become computationally tractable. However, the optimization problem is non-linear in nature, and we adopted the CE/MRAS methodology and designed our procedure to numerically solve our problem to optimality with very high probability.

We believe our model and approach is a first attempt in addressing a very important problem in RM. We next discuss a list of future research directions.

7.1 Future Research Directions

From a research perspective, competitive analysis of on-line algorithms approach is very promising in RM. In this dissertation, we took a first step in using limited demand information to increase practical effectiveness of on-line algorithms in RM. Adding more demand information to the single-leg problem (e.g., use of time-varying bounds on demand in each fare class as opposed to static, aggregate bounds in our model) and application of the competitive analysis ideas to the network RM problem remain as challenging future research topics. Analysis was carried out assuming the demand information is given and static. However, it is worth investigating how demand bounds can be estimated, what types of estimation procedures and/or choice of demand bounds make our policies most effective, and how robust methods can be adapted to changes in demand information.

The research community has seen an increased interest in robust decision-making in recent years, and ours is the first to use this framework in overbooking decisions. There are several future research directions in this context. First, the difficulties associated with class-dependent refunds discussed above indicate the need for investigating booking control methods other than nesting to address this problem. Second, we assumed range of no-show rates was given; further research is needed to analyze (i) the appropriate choice of bounds given historical information, and (ii) how one can obtain/learn these bounds if there is no information. Third, analysis of the problem using mean-variance information - as opposed to range information - can provide alternative robust policies for the single-resource RM problem. Finally, developing distribution-free methods for the dynamic booking control problem with cancellations lies as an intriguing future research direction.

Last but not least, extending our approach to the network RM problem and relaxing the independent demand model assumption to incorporate choice behavior of customers are two interesting and challenging avenues for future research.

Appendix A

Appendices for Chapter 3

A.1 Proposition 1

Proposition 1 *Relative to all input sequences with the same profile, a nested booking limit policy b generates the least revenue when applied to the unique LBH sequence with that profile. This is true for both the continuous and discrete problems.*

Proof We prove this first for the case of a *discrete* (integer) booking limit policy b. The main idea behind the proof is that, starting with any initial input sequence I that is not LBH, we can iteratively swap the order of requests in this sequence (eventually to reach an LBH sequence) such that the iterations/swaps will never lead to on-line revenue gains for the nested policy b.

Let $I(k) \in \{1, ..., m\}$ to be the fare class of the k^{th} request in sequence I. If an input sequence, I, is not LBH, then we can find some index k such that $I(k) < I(k+1)$; we construct a new sequence I' such that $I'(j) = I(j)$ for $j \leq k-1$ and $j \geq k+2$, $I'(k) = I(k+1)$, and $I'(k+1) = I(k)$. We start with the following observation regarding (standard) nested policies: If any two consecutive requests are both accepted or rejected by a nested policy b, then a swap in their arrival order will not affect the decision on any request in the entire sequence. For policy b, if $I(k)$ and $I(k+1)$ were both rejected/accepted in I, then the same would happen in I', and the same on-line revenue is generated.

Since the application of a static policy does not allow re-opening of closed fare classes, it is impossible that $I(k)$ is rejected and $I(k+1)$ is accepted in I under (standard) nested policy b. So the only remaining case is when $I(k)$ is accepted and $I(k+1)$ is rejected in I. There are two possibilities for I': The first one is to reject $I'(k) = I(k+1)$ then accept $I'(k+1) = I(k)$, which is obviously identical to the decisions made for I, and so we will have the same on-line revenue for both I and I'. The other one is to accept $I'(k)$ and reject $I'(k+1)$. We show below that policy b would generate *less* revenue executing sequence I' in this case.

We keep track of the number of accepted reservations while processing an input: We denote the *length* of a sequence (i.e., the total number of requests in a sequence) by $|I|$. By definition, $|I| = \sum_{i=1}^{m} I[i]$. Let $B_j(t), t = 0, 1, \cdots, |I|$ be the total number of accepted reservations of classes j to m by policy b *after* the t^{th} request of sequence I has been processed, and $B'_j(t)$ be the similar quantity for the sequence I'. We refer to $B_j(t)$ and $B'_j(t)$ as the *booking record* of class j under sequences I and I', respectively. Let us study how a nested booking policy b works. If a request of fare class j is accepted, it increases the booking record of each of the classes 1 to j. A request of class j is rejected if booking record is already equal to the booking limit for class j. Notice that the first $k-1$ requests of sequences I and I' are identical and we have $B_j(k-1) = B'_j(k-1), j = 1, ..., m$.

The case we are investigating is the one where both $I(k)$ and $I'(k)$ are accepted, which gives $B_j(k) = B_j(k-1)+1$ for $j \le I(k)$ and $B'_j(k) = B'_j(k-1)+1$ for $j \le I'(k)$. As $I(k) < I'(k) = I(k+1)$ and $B_j(k-1) = B'_j(k-1)$, we have $B_j(k) = B'_j(k)$ for $j \le I(k)$. Notice that the next request $I'(k+1) = I(k)$ is rejected, which means for some $i \le I(k)$, the booking limit has been reached: $B'_i(k) = b_i$. As $i \le I(k)$, we must have $B_i(k) = B'_i(k) = b_i$. Thus only requests with fares strictly higher than f_i may be accepted in the future for both sequences, and so only the identical booking records $B_j(k) = B'_j(k)$ for $j \le i$ matters in the future. As $I(k+1) \ne I'(k+1)$ are both rejected, resulting in no change in the booking records, we know that all the future decisions would be the same for both I and I'. So, the only difference in decisions occur while processing requests $I(k)$ and $I(k+1)$, and a lower on-line revenue is generated by processing sequence I'.

We conclude from the analysis above that a swap of two consecutive fare requests into LBH order will never lead to a gain in revenue. From any input I, after finite number of swaps, we will eventually end up with a LBH sequence. Since each swap gains no revenue, that means no other inputs can have lower revenue than the LBH input which is the desired result for the discrete case. For a continuous b, the same logic applies. The only technical detail that differs involves the swap operation. The swap must be applied to consecutive requests of arbitrary sizes; the details of this generalization are omitted.

•

A.2 Proof of Proposition 2

Proposition 2 *(Dominance of CAST) Consider a nested booking limit policy b and all input sequences in some category A^b_j. $CAST_j \in A^b_j$ dominates the other* *sequences in category* A_j^b*, i.e.,* $\frac{R(I;b)}{R^*(I)} \geq \frac{R(CAST_j;b)}{R^*(CAST_j)}$ *for all* $I \in A_j^b$*.*

Proof The framework we use here is similar to the one in the proof of Proposition 1. We show that from any given sequence I in the desired category A_j^b, a finite series of input sequences can be constructed, with non-increasing values of ratio of the on-line to offline revenues, leading to the sequence satisfying the proposition. We present the proof for the discrete case; the proof for the continuous case is similar and is omitted.

Remember that CASTs have the following properties by definition: They are LBH sequences with a particular profile. In this proof, we focus on LBH sequences with arbitrary profiles only and show what happens when iterative adjustments are made to those sequences. This is without loss of generality: If there is a sequence that is not LBH, it can be transformed into a new sequence that is LBH using the techniques in the proof of Proposition 1 in Appendix A.1. The transformation does not change the profile of the sequence (hence offline revenues are unaffected) and the resulting sequence has on-line revenues that are no more than the revenues of the original sequence (see Proposition 1 above).

Consider a LBH sequence $I \in A_j^b$ whose profile does not match that of a CAST. First we will make adjustments to I to ensure $I[k] = U_k$ for all $k \geq j$. Suppose there is a $k \geq j$ such that $I[k] < U_k$. Consider the new sequence I' where a class k request is appended to the end of I. The sequence I' belongs to the same category as I by definition of A_j^b and it follows that the additional request in I' will be rejected by policy b (because $k \geq j$). Therefore, policy b obtains the same on-line revenues on sequences I and I'. However, offline revenues for I' will be no less than that of I.

Therefore the resultant ratio of on-line revenues to offline revenues cannot increase if sequence I is adjusted to I'. One can continue appending requests to sequence I until $I[k] = U_k$ for all $k \geq j$. The resulting sequence may not be LBH, but can be converted to one without loss of generality; see Proposition 1 and our discussion above.

Now consider $I[k] = L_k$ with $k < j$. Suppose there is a $k < j$ such that $I[k] > L_k$, and consider removing a class k request from I, resulting in a sequence I'. Since I is an LBH stream in category A_j^b where $j > k$, the resultant on-line revenue is simply reduced by f_k: $R(I';b) = R(I;b) - f_k$. As for the offline optimal revenue, we obviously have $R^*(I') \geq R^*(I) - f_k$. Given that $R^*(I) \geq R(I;b) \geq f_k > 0$, the resultant ratios satisfy

$$\frac{R(I';b)}{R^*(I')} \leq \frac{R(I;b) - f_k}{R^*(I) - f_k} \leq \frac{R(I;b)}{R^*(I)}.$$

We continue creating new sequences based on these adjustments until we obtain $I[k] = L_k$ for all $k < j$. Thus, after finite number of adjustments without ever increasing the CR, the desired LBH sequence is reached from any sequence within the given category. •

A.3 Parameter Relationships in the GBM and Proof of Proposition 3

The coefficients in the constraints of GBM are not arbitrary, but have some useful relationships to make the GBM solvable in closed-form. These relationships reflect the structures of the offline optimal revenues from all the CASTs.

For the sake of mathematical completeness in investigating the relationship among the problem parameters, we utilize a virtual fare class $m+1$ with $f_{m+1} = 0$. So, instead of looking at a m-fare problem, we are now looking at an $m+1$-fare problem. We refer to the GBM model that includes the virtual class $m+1$ as *the virtual extension of GBM*. We first clarify the notation for the virtual extension of GBM and introduce the shorthand notation used in this section.

We set $L_{m+1} = 0$ for the virtual class. Let $N_j = n - \sum_{i=1}^{j-1} L_i$, $j = 1, 2, \cdots, m+1$. Remember that $\sum_{i=1}^{m-1} L_i < n$ (see the discussion in Section 3.2.2). Therefore, we have $N_m > 0$. Note that L_m does not affect any parameters in the original GBM, so we can set L_m to $L_m = \min(U_m, N_m) > 0$, so that $N_{m+1} \geq 0$. Also, we set $U_{m+1} = n$, so we will always have at least n requests in $CAST_j$, for all $j = 1, 2, \cdots, m+1$. Let $R_j^* = R^*(CAST_j)$, and $R_j^+ = \sum_{i=1}^{j-1} L_i f_i$ for all $j = 1, 2, \cdots, m+1$. We have $R_{m+1}^* = R_{m+1}^+ > 0$ because $N_m > 0$, $L_m > 0$ and $f_{m+1} = 0$. With this shorthand notation, we can rewrite $g_i = (R_i^* - R_{i+1}^*)/f_i \geq 0, i = 1, ..., m$ from (3.9). For any input stream I, let $I(k)$ denote the fare class of the k^{th} request in I, and $I(-k)$ denote the fare class of the k^{th} request *counting backwards* in I, for all $k = 1, 2, ..., |I|$ where $|I| = \sum_{j=1}^{m} I[j]$ is the total number of requests in sequence I.

Our initial goal is to derive the following relationship:

$$N_{m+1} = n - \sum_{i=1}^{m} L_i \leq \sum_{i=1}^{m} g_i. \tag{A.1}$$

We will also use later some of the inequalities derived along the way.

The offline optimal revenue from an input I is simply the sum of the fares from the n highest requests in I. As any $CAST_j$ is LBH, we have

$$R_j^* = \sum_{k=1}^{n} f_{CAST_j(-k)}. \tag{A.2}$$

We first seek to derive a relationship between R_j^* and R_{j+1}^*. Observe that $CAST_{j+1}$ can be obtained from $CAST_j$ by taking out $U_j - L_j$ of fare class $j \leq m$ requests. As $\sum_{i=1}^{j} L_i < n$, we know that some of those $U_j - L_j$ requests belong to the n highest requests in $CAST_j$. Thus, some of the n highest requests in $CAST_j$ are "missing" from $CAST_{j+1}$, and some of the succeeding requests must take the place of these missing requests. Depending on the value of $U_j + \sum_{i=1}^{j-1} L_i$, we have two cases:

- **Case 1:** If $U_j + \sum_{i=1}^{j-1} L_i \leq n$, then all of the $U_j - L_j$ belong to the highest n requests in $CAST_j$. Thus, exactly $U_j - L_j$ of the highest n requests in $CAST_j$ are missing from $CAST_{j+1}$, and the next $U_j - L_j$ highest requests are used in determining R_{j+1}^*. So we have:

$$R_{j+1}^* = R_j^* - (U_j - L_j) f_j + \sum_{k=n+1}^{n+U_j-L_j} f_{CAST_j(-k)}. \tag{A.3}$$

- **Case 2:** If $U_j + \sum_{i=1}^{j-1} L_i > n$, then $U_j + (\sum_{i=1}^{j-1} L_i) - n$ of the $U_j - L_j$ are not in the highest n requests in $CAST_j$. Thus, only $U_j - L_j - [U_j + (\sum_{i=1}^{j-1} L_i) - n] = N_{j+1}$ of the highest n requests from $CAST_j$ are missing from $CAST_{j+1}$. These are N_{j+1} highest requests following the first $U_j + \sum_{i=1}^{j-1} L_i$ highest requests in $CAST_j$. But note that, in $CAST_j$, the highest requests ranked between $n + 1$ and $U_j + \sum_{i=1}^{j-1} L_i$ all belong to fare class j. This gives us a total of

$U_j + (\sum_{i=1}^{j-1} L_i) - n = U_j - L_j - N_{j+1}$ requests of class j. So we have:

$$R^*_{j+1} = R^*_j - N_{j+1} f_j + \sum_{k=n+1}^{n+U_j-L_j} f_{CAST_j(-k)} - (U_j - L_j - N_{j+1}) f_j, \qquad \text{(A.4)}$$

which turns out to be identical to (A.3).

For $j \leq m$, let Q_j denote the number of requests of *non-virtual* classes in $CAST_j$ among the $U_j - L_j$ request that follow the first n requests in high to low order. Also let $\bar{f}_j$ denote the average fare of these Q_j requests, so we have

$$\sum_{k=n+1}^{n+U_j-L_j} f_{CAST_j(-k)} = Q_j \bar{f}_j. \qquad \text{(A.5)}$$

As none of the Q_j requests has a fare higher than f_j, we have $\bar{f}_j \leq f_j$ and

$$g_i = (R^*_i - R^*_{i+1})/f_i = U_i - L_i - Q_i \bar{f}_i / f_i \leq U_i - L_i. \qquad \text{(A.6)}$$

As we build the relationships between two consecutive CASTs in this way, starting with $CAST_1$ and ending with $CAST_{m+1}$, all the requests not among the n highest of $CAST_1$ can contribute to a Q_j for some j at most once, so we have

$$\sum_{j=1}^{m} Q_j \leq \sum_{i=1}^{m} U_i - n. \qquad \text{(A.7)}$$

The preceding analysis implies:

$$\begin{aligned}
\sum_{i=1}^{m} g_i &= \textstyle\sum_{i=1}^{m} (U_i - L_i) - \sum_{i=1}^{m} Q_i \bar{f}_i / f_i \\
&\geq \textstyle\sum_{i=1}^{m} (U_i - L_i) - \sum_{i=1}^{m} Q_i \\
&\geq \textstyle\sum_{i=1}^{m} (U_i - L_i) - (\sum_{i=1}^{m} U_i - n) \\
&= n - \textstyle\sum_{i=1}^{m} L_i = N_{m+1}.
\end{aligned}$$

We have now proved relationship (A.1), which plays an important role in our presentation of closed-form solutions to the GBM below. We use the shorthand defined in this appendix, instead of the explicit notation of the main text, for convenience of discussion. We leave it for the reader to verify that the following statements are the same as in the main text.

Proposition 3 *(a) The optimal solution of GBM is*

$$\begin{aligned}
\bar{z}^{CR} &= \frac{R_u^+/f_u + N_u}{R_u^*/f_u + \sum_{i=1}^{u-1} g_i} \\
x_j^{CR} &= \begin{cases} g_j z^{CR} + L_j & j < u \\ (R_u^* z^{CR} - R_u^+)/f_u & j = u \\ 0 & j > u \end{cases} \\
u &= \max\{j \leq m : R_j^+ \sum_{i=1}^{j-1} g_i < N_j R_j^*\}
\end{aligned}$$

where the index u denotes the critical fare class such that all classes $k > u$ are closed.

(b) The nested booking limits defined by

$$b_j^{CR} = \sum_{i=j}^{m} x_i^{CR} \text{ for } j = 1, \cdots, m \tag{A.8}$$

maximizes the CR in problem (3.1) and the optimal CR is $z^{CR} = \bar{z}^{CR}$.

Proof (a) Using condition (A.1), we see that the u obtained in (3.8) is the same as

$$u = \max\{j \leq m : R_j^+ \sum_{i=1}^{j-1} g_i < N_j R_j^*\}.$$

This fact together with $R_{m+1}^* = R_{m+1}^+$ immediately imply $u \leq m$. We also see that $u \geq 1$. Observe that $R_j^+ \sum_{i=1}^{j-1} g_i$ is non-decreasing in j, starting from 0 for $j = 1$;

and that $N_j R_j^*$ is decreasing in j, starting from $nR_1^* > 0$ for $j = 1$. So u is well defined in the range $1 \leq u \leq m$.

Consider the system of the linear equations below:

$$R_j^* z = R_j^+ + \sum_{i=j}^{u} f_i x_i \quad j = 1, \cdots, u$$

$$\sum_{i=1}^{u} x_i = n.$$

The closed-form solution to this system, denoted z^*, x_i^*, is given by

$$\begin{aligned} z^* &= (R_u^+/f_u + N_u)/(R_u^*/f_u + \sum_{i=1}^{u-1} g_i) \\ x_u^* &= (R_u^* z^* - R_u^+)/f_u \\ x_j^* &= g_j z^* + L_j, \quad 1 \leq j < u \end{aligned}$$

This turns out to be the optimal solution to GBM, after we set $x_i^* = 0, i > u$. Some of the constraints were not part of the equation system used to obtain the above solution; we now need to show that they are all satisfied. With $z^* > 0$, it becomes obvious that $x_i^* \geq 0, i \neq u$. We still need to show $x_u \geq 0$. By the definition of u, we have

$$R_u^+ \textstyle\sum_{i=1}^{u-1} g_i < N_u R_u^*.$$

By adding $R_u^+ R_u^*/f_u$ to both sides of the inequality and regrouping the terms, we get

$$\begin{aligned} \Rightarrow \quad R_u^+ \textstyle\sum_{i=1}^{u-1} g_i + R_u^+ R_u^*/f_u &< N_u R_u^* + R_u^+ R_u^*/f_u \\ \Rightarrow \quad R_u^+ (R_u^*/f_u + \textstyle\sum_{i=1}^{u-1} g_i) &< (N_u + R_u^+/f_u) R_u^*. \end{aligned}$$

By definition of z^* from our analysis above:

$$\Rightarrow \quad R_u^+ \quad < z^* R_u^*,$$

an we get

$$\Rightarrow \quad x_u^* = (R_u^* z^* - R_u^+)/f_u \quad > 0.$$

Next we show that the constraints of $R_j^* z \leq R_j^+ + \sum_{i=j}^m f_i x_i, j > u$ are satisfied. Since $x_i^* = 0, i > u$, these constraints reduce to $zR_j^* \leq R_j^+, j > u$. Observing the monotonicity of R_j^+/R_j^* in j, we see that we only need to show $z^* R_{u+1}^* \leq R_{u+1}^+$. By the definition of u, we have:

$$R_{u+1}^+ \textstyle\sum_{i=1}^u g_i \quad \geq N_{u+1} R_{u+1}^*.$$

We divide both sides by f_u, add $N_{u+1} \sum_{i=1}^u g_i$ to both sides:

$$\Rightarrow \quad (R_{u+1}^+/f_u + N_{u+1}) \textstyle\sum_{i=1}^u g_i \quad \geq N_{u+1}(R_{u+1}^*/f_u + \displaystyle\sum_{i=1}^u g_i).$$

Regrouping the terms and by replacing z^* with the appropriate terms, we get:

$$\begin{aligned} \Rightarrow \quad (R_u^+/f_u + N_u) \textstyle\sum_{i=1}^u g_i \quad &\geq N_{u+1}(R_u^*/f_u + \displaystyle\sum_{i=1}^{u-1} g_i) \\ \Rightarrow \quad z^* \textstyle\sum_{i=1}^u g_i \quad &\geq N_{u+1} = N_u - L_u \end{aligned}$$

Finally, rearranging the terms and adding R_u^+/f_u to both sides of the inequality:

$$\Rightarrow \quad R_u^+/f_u + N_u - z^* \textstyle\sum_{i=1}^u g_i \quad \leq R_u^+/f_u + L_u = R_{u+1}^+/f_u.$$

Since $R_u^+/f_u + N_u = z^*(R_u^*/f_u + \sum_{i=1}^{u-1} g_i)$, we have: $z^*(R_u^*/f_u - g_u) = z^* R_{u+1}^*/f_u \leq R_{u+1}^+/f_u$. The proof is valid for $j = m+1$, so we have $z^* \leq R_{m+1}^+/R_{m+1}^* = 1$.

We now show that the upper bound constraints $x_i \le U_i, i = 1, \cdots, m$ also hold. First we show that $x_u^* \le z^* g_u + L_u$:

$$
\begin{aligned}
& z^* R_{u+1}^* && \le R_{u+1}^+ \\
\Rightarrow \quad & z^*(R_u^* - g_u f_u) && \le R_u^+ + L_u f_u \\
\Rightarrow \quad & x_u^* = (z^* R_u^* - R_u^+)/f_u && \le z^* g_u + L_u
\end{aligned}
$$

Given $g_i \le U_i - L_i$ and $z^* \le 1$, we have $x_i \le z^* g_i + L_i \le U_i$ for $i \le u$, and $x_i^* = 0 \le U_i$ for $i > u$. This concludes the proof of feasibility of the solution.

To prove optimality, we consider the dual of GBM, with variables v, y_j, w_j:

$$
\begin{aligned}
\min \quad & nv + \textstyle\sum_{j=1}^m (R_j^+ y_j + U_j w_j) \\
s.t. \quad & \textstyle\sum_{j=1}^m R_j^* y_j \ge 1 \\
& w_j + v - \textstyle\sum_{i=1}^j f_j y_i \ge 0 \qquad j = 1, ..., m \\
& v \ge 0, y_j, w_j \ge 0 \qquad j = 1, ..., m
\end{aligned}
$$

A dual feasible solution is constructed as

$$
\begin{aligned}
y_j &= \begin{cases} v(1/f_j - 1/f_{j-1}) & j \le u \\ 0 & j > u \end{cases} \\
v &= 1/(R_u^*/f_u + \sum_{i=1}^{u-1} g_i) \\
w_j &= 0 \qquad \forall j
\end{aligned}
$$

where $f_0 = \infty$ for convenience. So the solution is optimal for GBM, and we let $x^{CR} = x^*$, $\bar{z}^{CR} = z^*$.

(b) We show optimality of the policy b^{CR} to the CR problem. As discussed in the main text, GBM provides an upper bound on the true objective function, z^{CR}, of the CR problem. To prove that b^{CR} is the optimal nested booking limit policy that maximizes CR, we need to show that $\bar{z}^{CR} = z^{CR}$. Obviously, when $k \leq u$, R_k^+ correctly calculates the revenues generated by x^{CR} from the requests in classes from 1 through $k-1$ in $CAST_k$. A discrepancy occurs only when $k > u$. Noting that $x_k^{CR} = 0$, we have

$$R(CAST_k; b^{CR}) = R(CAST_{u+1}; b^{CR}) = f_u \min(x_u^{CR}, L_u) + R_u^+ = \min(R(CAST_u; b^{CR}), R_{u+1}^+).$$

The monotonicity of R_k^* with respect to k allows that

$$\frac{R(CAST_k; b^{CR})}{R_k^*} \geq \min\left(\frac{R(CAST_u; b^{CR})}{R_u^*}, \frac{R_{u+1}^+}{R_{u+1}^*}\right) \geq \bar{z}^{CR}.$$

So, $\bar{z}^{CR} = z^{CR}$. •

A.4 Proof of Theorem 1

Theorem 1 *For the continuous m-fare problem with demand bounds, the nested booking control policy with booking limit vector* b^{CR} *defined by (3.11) has a CR of* $\bar{z}^{CR}$ *given by (3.6) and this is the best possible among deterministic policies.*

Proof Let P be an arbitrary policy, the only requirement being that it should accept/reject any portion of a request upon its arrival. We generate an input sequence by specifying the actions of an adversary that can observe each of P's "decisions" and immediately react by manipulating future input. When specifying the adversary's actions, we characterize the policy's effect by defining an effective booking limit vector, b^P, and bucket size vector, x^P. Consider the following adversary strategy:

step 0: let the current fare index $\hat{i} = m + 1$ and its effective booking limit $b^P_{\hat{i}} = 0$;

step 1: set $\hat{i} = \hat{i} - 1$, send in $U_{\hat{i}}$ of class $\hat{i}$ requests;

step 2: set $x_{\hat{i}}$ to the number of class $\hat{i}$ requests accepted by P;

step 3: let effective booking limit be $b^P_{\hat{i}} = b^P_{\hat{i}+1} + x^P_{\hat{i}}$;

step 4: if $b^P_{\hat{i}} \geq b^{CR}_{\hat{i}}$, go to step 1) if $\hat{i} > 1$;

step 5: if $b^P_{\hat{i}} < b^{CR}_{\hat{i}}$, send in the rest of $CAST_{\hat{i}}$.

The execution of this strategy will terminate having generated the input stream $CAST_{\hat{i}}$. Define the vector b^{CR} as the optimal nested booking limit vector obtained using GBM, i.e., $b^{CR}_j = \sum_{i=j}^{m} x^{CR}_i$ for all $j = 1, .., m$. The conditions in steps 4 and 5 above imply that $b^P_j \geq b^{CR}_j$ for $j > \hat{i}$ and $b^P_{\hat{i}} \leq b^{CR}_{\hat{i}}$, which, in turn, imply that the revenue of policy P based on classes $k > \hat{i}$ is no more than the corresponding revenue of the optimal nested policy b^{CR}. The revenue from classes $k \leq \hat{i} - 1$ would be at most $\sum_{i=1}^{\hat{i}-1} L_i f_i$, which is the revenue obtained by the optimal nested policy b^{CR}. Combining these two revenue portions, we have that the revenue of P cannot be higher than that of the optimal nested policy b^{CR}. Hence, policy b^{CR} is better (or no-worse) than any other arbitrary policy. This then implies the CR is at most $\bar{z}^{CR}$, and the nested booking control policy b^{CR} that achieves $\bar{z}^{CR}$ is the best possible among all deterministic policies. •

A.5 Proof of Corollary 1

Corollary 1 *For the continuous m-fare problem with all lower bounds equal to zero, the nested booking control policy with booking limits defined by:*

$$b_i^{CR} = n\frac{\sum_{j=i}^m g_i}{\sum_{j=1}^m g_j} \text{ for } i = 2, ..., m$$

has a CR of $n/(\sum_{j=1}^m g_j)$ *and this is the best possible among deterministic policies.*

Proof We use the shorthand notation introduced in Appendix A.3 above in this proof. When the lower bounds are all zero ($L_i = 0, i = 1, \cdots, m$), it can easily be seen that $R_i^+ = 0$, $N_i = n$ for all $i = 1, \cdots, m$ and the GBM reduces to the *Upper Bound Model (UBM)* below:

$$\begin{aligned}
\textbf{UBM}: \quad & \max \quad z \\
& \text{s.t.} \quad R_j^* z \leq \sum_{i=j}^m f_i x_i, \quad j = 1, \cdots, m && \text{(A.9)} \\
& \sum_{i=1}^m x_i \leq n && \text{(A.10)} \\
& 0 \leq x_j \leq U_j, \quad j = 1, ..., m. && \text{(A.11)}
\end{aligned}$$

To derive the optimal solution to UBM, the upper bound constraints in (A.11) are dropped, and we assume the remaining constraints in (A.9) and (A.10) are binding. We solve a linear system of equations with $m + 1$ variables. The solution z^*, x_i^* to this system is:

$$z^* = n/\sum_{i=1}^m g_i \quad \text{and} \quad x_i^* = g_i \bar{z}^{CR}, \ i = 1, \cdots, m; \tag{A.12}$$

where

$$g_i = \frac{R_i^* - R_{i+1}^*}{f_i} \quad i = 1, \cdots, m \tag{A.13}$$

with $R_{m+1}^* = 0$. Note that $g_i \le U_i$ and $0 \le z^* \le 1$, hence $x_i^* = g_i z^* \le U_i$, which implies that constraint (A.11) is satisfied. Hence, the solution provided in (A.12) is feasible for UBM. We consider the dual formulation of UBM in order to show that this solution is optimal:

$$\begin{aligned} \min \quad & nv \\ s.t. \quad & \sum_{j=1}^{m} R_j^* y_j \ge 1 \\ & v \ge f_j \sum_{i=1}^{j} y_i \quad j = 1, ..., m \\ & v \ge 0, y_j \ge 0 \quad j = 1, ..., m. \end{aligned}$$

By treating the first two constraint sets as equalities and solving this linear system, we obtain the following dual solution: $v^* = z^*/n, y_j^* = v^*/f_j - v^*/f_{j-1}$, where we define $f_0 = +\infty$ for convenience. It is easy to verify that this dual solution is both feasible and has the same objective function value as the primal solution obtained in (A.12). This proves that $x^{CR} = x^*$ is the optimal solution to UBM and the optimal CR is $\bar{z}^{CR} = z^*$. •

A.6 Proof of Theorem 2

Theorem 2 *For the continuous m-fare problem with demand bounds, no randomized booking policy has a CR larger than $\bar{z}^{CR}$ given in (3.6). Therefore, the* *deterministic nested booking control policy with booking limits by b^{CR} in (3.11) is the best possible among* all *policies.*

Proof (Some of the shorthand notation introduced in Appendix A.3 is used in this proof.) Let $\tilde{D}$ denote the set of all deterministic on-line policies for this problem. Let Θ be the set of all probability distributions on $\tilde{D}$. Any randomized algorithm may be viewed as a random choice $\tilde{D}(\tilde{p})$ among deterministic algorithms, defined by some probability distribution $\tilde{p} \in \Theta$. On the other hand, the adversary makes a choice among the input streams $I \in \Omega(L, U)$. Using as a payoff function the expected CR, we define a zero-sum two-person game between a seller choosing a randomized policy to maximize her expected CR and an adversary choosing a distribution of input streams to minimize this expected ratio. The von Neuman/Yao principle (e.g., see Seiden, 2000) implies that the best possible CR of any randomized policy satisfies

$$z^* = \sup_{\tilde{p}\in\Theta} \inf_{I\in\Omega} E_{\tilde{p}} \left[\frac{R(I; \tilde{D}(\tilde{p}))}{R^*(I)} \right] = \inf_{q\in\Psi} \sup_{\tilde{d}\in\tilde{D}} E_q \left[\frac{R(I(q); \tilde{d})}{R^*(I(q))} \right]$$

where Ψ denotes the set of all probability distributions on set Ω, and $I(q)$ is a random instance chosen according to probability distribution $q \in \Psi$. The right hand side of this equality may be interpreted as the adversary's problem of choosing a probability distribution of input instances to force every deterministic algorithm to experience an expected competitive ratio at most z^*. We note that, since we have shown that $\bar{z}^{CR}$ is the best possible CR for a deterministic algorithm and a deterministic algorithm is a special case of a randomized one, it is clear that $\bar{z}^{CR} \le z^*$. To prove the result we will show that $z^* \leq \bar{z}^{CR}$. This will be accomplished by showing for a particular $q^* \in \Psi$,

$$\sup_{\tilde{d}\in\tilde{D}} E_{q^*}\left[\frac{R(I(q^*);\tilde{d})}{R^*(I(q^*))}\right] \leq \bar{z}^{CR}$$

If this inequality holds for a particular q^*, it must hold for the infimum over all $q \in \Psi$.

We define the q^* as follows. Set $\bar{q} = 1/(\sum_{i\leq q}(f_i^{-1} - f_{i-1}^{-1})R_i^*)$ and $q_i^* = \bar{q}(f_i^{-1} - f_{i-1}^{-1})R_i^*, i \leq u$. We can then choose the input instance $CAST_i$ with probability of q_i^*. Let $X_i^k(\tilde{d})$ denote the number of accepted requests in class i when deterministic on-line policy $\tilde{d} \in \tilde{D}$ is applied to $CAST_k$, and $R(I;\tilde{d})$ denote the on-line revenue generated by $\tilde{d}$ from input stream I. Since $CAST_k$ and $CAST_1$ are LBH streams and they are identical before any $k-1$ class request is seen, we have $X_j^k(\tilde{d}) = X_j^1(\tilde{d}), j \geq k$. On the other hand, $X_j^k(\tilde{d}) \leq CAST_k[j] = L_j, j < k$. So:

$$R(CAST_k;\tilde{d}) = \sum_{j=1}^{m} f_j X_j^k(\tilde{d}) \tag{A.14}$$

$$\leq R_k^+ + \sum_{j=k}^{u} f_j X_j^1(\tilde{d}) + f_u \sum_{j=u+1}^{m} X_j^1(\tilde{d}), k \leq u. \tag{A.15}$$

The above expression now enables us to derive a relationship between the performance under an arbitrary $\tilde{d}$ and the performance of the best nested booking policy defined by x^{CR} in (3.7):

$$E_{q^*}\left[\frac{R(I(q^*);\tilde{d})}{R^*(I(q^*))}\right] = \sum_{k=1}^{u} q_k^* \frac{R(CAST_k;\tilde{d})}{R^*(CAST_k)}$$

$$= \sum_{k=1}^{u} \bar{q}(f_k^{-1} - f_{k-1}^{-1})R(CAST_k;\tilde{d})$$

Using the relationship in (A.15),

$$\leq \sum_{k=1}^{u} \bar{q}(f_k^{-1} - f_{k-1}^{-1})(R_k^+ + \sum_{j=k}^{u} f_j X_j^1(\tilde{d}) + f_u \sum_{j=u+1}^{m} X_j^1(\tilde{d})) \quad \text{(A.16)}$$

$$= \sum_{k=1}^{u} \bar{q}(f_k^{-1} - f_{k-1}^{-1})R_k^+ + \bar{q}\sum_{k=1}^{m} X_k^1 \quad \text{(A.17)}$$

$$\leq \sum_{k=1}^{u} \bar{q}(f_k^{-1} - f_{k-1}^{-1})R_k^+ + \bar{q}\sum_{k=1}^{m} x_k^* \quad \text{(A.18)}$$

$$= \bar{z}^{CR} \quad \text{(A.19)}$$

The term in (A.17) is obtained by algebraic manipulation. The inequality in (A.18) results since $\sum_{k=1}^{m} X_k^1 \leq n = \sum_{k=1}^{m} x_k^{CR}$. The final equation follows because the CR achieved by x^{CR} equals $\bar{z}^{CR}$ for $CAST_k, k \leq u$. Therefore no randomized policy has a CR greater than $\bar{z}^{CR}$; this completes the proof. •

A.7 Randomized Policies for the Discrete CR Problem

A randomized policy consists of a policy set (or a set of deterministic algorithms) and a discrete probability distribution over the policy set. The policy set is a finite set of deterministic, discrete (integral) policies. We will first use the solution x^{CR} given in equation (3.7) to construct a policy set. Then we prove that there always exists a discrete probability distribution over that policy set such that the randomized policy is optimal.

Note that $\sum_{i=1}^{u} x_i^{CR} = n$ because we exclude the trivial cases from our analysis. Each of the deterministic policies will be defined by rounding up or down each fractional x_i^{CR}, and any integral x_i^{CR} can be simply left unchanged. For convenience, we assume that x_i^{CR}, $i = 1, \cdots, u$ are all fractional. Let $p_i = x_i^{CR} - \lfloor x_i^{CR} \rfloor$ denote the fractional part of x_i^{CR}, and $\tilde{r} = \sum_{i=1}^{u} p_i < u$ denote the sum of the fractional parts. Note that $\tilde{r} = n - \sum_{i=1}^{u} \lfloor x_i^{CR} \rfloor$ must be integral and satisfy $u > \tilde{r} > 0$. For each $s \subset \{1, \cdots, u\}$, with $|s| = \tilde{r}$, we define a deterministic policy based on x^s given by:

$$x_i^s = \lfloor x_i^{CR} \rfloor + \mathbf{1}\{i \in s\}, i = 1, \cdots, u \tag{A.20}$$

where $\mathbf{1}\{\cdot\}$ is the indicator function. Notice each policy defined in this way satisfies $x_i^s \geq L_i, i < u$, $U_i \geq x_i^s, i \leq u$, and $\sum_{i=1}^{u} x_i^s = n$. All of these deterministic integral policies are feasible for GBM and there are $\binom{u}{\tilde{r}}$ in total. The set of these policies is the policy set for our random policy. We now need to find a probability q_s for each policy x^s, subject to

$$\sum_{s:i \in s} q_s = p_i, \; i = 1, \cdots, u \tag{A.21}$$

$$q_s \geq 0 \quad \text{and} \quad \sum_{s} q_s = 1. \tag{A.22}$$

Note that (A.22) ensures q_s represents a probability distribution, and (A.21) implies $E[x_i^s] = x_i^{CR}, i = 1, \cdots, u$. Let us assume for now that there exists such a set of q_s, which we will prove shortly in Lemma 1 (see below).

To show that this randomized policy achieves the optimal performance, we examine its expected performance on the CAST streams. The analysis provided in Section 3.2.2 shows that the performance of the policy x^s upon $CAST_j$ is exactly $\sum_{i=1}^{j-1} f_i L_i + \sum_{i=j}^{u} f_i x_i^s$. For all $j \leq u$, the expected performance can then be obtained as follows:

$$\begin{aligned}
E\left[\sum_{i=1}^{j-1} f_i L_i + \sum_{i=j}^{u} f_i x_i^s\right] &= \sum_{i=1}^{j-1} f_i L_i + \sum_{i=j}^{u} f_i E[x_i^s] \\
&= R_j^+ + \sum_{i=j}^{m} f_i x_i^{CR} \\
&= \bar{z}^{CR} R_j^*
\end{aligned}$$

where the final equality in the equation is implied by the properties of the optimal solution of GBM (see Appendix A.3 above). For $j > u$, the final equality above is replaced by the inequality '$\geq$'.

Therefore, the expected performance of the randomized policy on each of the CAST streams is exactly the same as the performance of the best deterministic policy for the continuous problem, which proves that the randomized algorithm on the discrete problem achieves the same CR as the optimal deterministic policy applied to the continuous problem. It remains to show that the probabilities of q_s indeed exist; see the next result.

Lemma 1 *Given $0 \leq p_i \leq 1$ for $i = 1, \cdots, u$ and $0 < \sum_{i=1}^{u} p_i = \tilde{r} < u$, there always exists a solution q_s to the system of (A.21) and (A.22).*

Proof We prove by induction on the pair of $(\tilde{r}, u)$. Notice that when $\tilde{r} = 1$, the solution is found by $q_{\{i\}} = p_i, i = 1, \cdots, u$. Also, when $u - \tilde{r} = 1$ the solution is obviously $q_{\{1,\cdots,u\}\backslash i} = \bar{p}_i, i = 1, \cdots, u$, where $\bar{p}_i = 1 - p_i > 0, i = 1, \cdots, u$. It remains to show that for any $(\tilde{r}, u)$, with $\tilde{r} > 1$ and $u - \tilde{r} > 1$, a solution can be found if any system of $(u', \tilde{r}')$, with $u' < u$ and $\tilde{r}' \leq \tilde{r}$, has a solution.

As indicated earlier, without loss of generality, we assume $0 < p_i < 1$ for all

i. Let $t_i \geq 0, i = 1, \cdots, u-1$ satisfy $\sum_{i=1}^{u-1} t_i = p_u\bar{p}_u$ and $0 \leq t_i \leq \min(p_u p_i, \bar{p}_u\bar{p}_i)$. Obviously, such t exists if

$$p_u\bar{p}_u \leq \sum_{i=1}^{u-1} \min(p_u p_i, \bar{p}_u\bar{p}_i). \tag{A.23}$$

We will not prove that this inequality holds. First note that $\min(X, Y) = X - [X-Y]^+$, so we have

$$\min(p_u p_i, \bar{p}_u\bar{p}_i) = p_u p_i - [p_u p_i - \bar{p}_u\bar{p}_i]^+ = p_u p_i - [p_u + p_i - 1]^+.$$

We define set K such that $i \in K$ if and only if $p_u + p_i - 1 > 0$. The following sequence of inequalities are all equivalent to (A.23):

$$\begin{aligned} p_u\bar{p}_u &\leq \sum_{i=1}^{u-1}(p_u p_i - [p_u + p_i - 1]^+) \\ \sum_{i\in K}[p_u + p_i - 1]^+ &\leq \sum_{i=1}^{u-1} p_u p_i - p_u\bar{p}_u \\ \sum_{i\in K}(p_u + p_i - 1) &\leq p_u(\tilde{r} - 1) \\ |K|p_u - |K| + \sum_{i\in K} p_i &\leq p_u(\tilde{r} - 1) \end{aligned} \tag{A.24}$$

Thus, we can now show that (A.23) holds by proving (A.24). We consider two cases:

- Case 1: $|K| \geq \tilde{r}$. Noting that $\sum_{i\in K} p_i \leq \tilde{r} - p_u$, it follows that

$$\begin{aligned} & |K|p_u - |K| + \sum_{i\in K} p_i \\ \leq\ & |K|p_u - |K| + \tilde{r} - p_u \\ =\ & p_u(\tilde{r} - 1) - \bar{p}_u(|K| - \tilde{r}). \end{aligned}$$

Now inequality (A.24) follows since the condition of Case 1 implies $\bar{p}_u(|K| - \tilde{r}) \geq 0$.

- Case 2: $|K| < \tilde{r}$. We have $\sum_{i \in K} p_i \leq |K|$, so

$$|K|p_u - |K| + \sum_{i \in K} p_i \leq |K|p_u \leq p_u(\tilde{r} - 1).$$

and inequality (A.24) follows.

For all $i = 1, \cdots, u-1$, let $p_i^0 = p_i + t_i/\bar{p}_u$, and $p_i^1 = p_i - t_i/p_u$, so we can have $p_i = p_i^0 \bar{p}_u + p_i^1 p_u$. From the constraints put on t, we obviously have $0 \leq p_i^0 \leq 1, 0 \leq p_i^1 \leq 1$ and $\sum_{i=1}^{u-1} p_i^0 = \tilde{r}, \sum_{i=1}^{u-1} p_i^1 = \tilde{r} - 1$. By the assumption of induction, we know both the systems have a solution, and let them be q_s^0, q_s^1 respectively. Note that the subscripts for q_s^0 are sets with $|s| = \tilde{r}$, while those for q_s^1 satisfy $|s| = \tilde{r} - 1$. Let q_s be defined as

$$q_s = \begin{cases} \bar{p}_u q_s^0, & u \notin s \\ p_u q_{s \setminus u}^1, & u \in s. \end{cases} \tag{A.25}$$

We can directly verify that q_s is a solution to the system with $(\tilde{r}, u)$, which is omitted here. •

Based on the proof of Lemma 1, one can develop an efficient algorithm to generate the randomized policies. The algorithm chooses the non-zero probability values without the need for complete enumeration of the policy set; details are omitted.

A.8 Dynamic Policies

Given the adjusted bounds $\hat{L}$, $\hat{U}$, remaining capacity $\hat{n}$, number of accepted requests h_i, $i = 1, ..., m$, and the partial input sequence I_0, the new values of nested booking limits $\hat{b}$ and bucket sizes $\hat{x}$ that improve the minimum CR and maximum MAR are obtained by solving the problems in (3.18) and (3.19), respectively.

The structure of the optimal policy is the same as in GBM (GBM-AR), except that some parameters are replaced. We use $\hat{n}$ for n, $\hat{L}_i$ for L_i. The CAST for the set of partial input sequences are now defined over the set $\Omega(\hat{L}, \hat{U})$. We denote the new CAST by CAST′. By definition, $CAST'_j[k] = \hat{U}_k$ for $k \geq j$, and $CAST'_j[k] = \hat{L}_j$ for $k < j$.

The new bucket sizes $\hat{x}^{CR}$ that improve minimum CR are expressed as:

$$\hat{z}^{CR} = \frac{(1/f_{\hat{u}})(\sum_{i=1}^{m} h_i f_i + \sum_{i=1}^{\hat{u}-1} f_i \hat{L}_i) + (\hat{n} - \sum_{i=1}^{\hat{u}-1} \hat{L}_i)^+}{R^*(I_0 CAST'_{\hat{u}})/f_{\hat{u}} + \sum_{i=1}^{\hat{u}-1} \hat{g}_i} \quad \text{(A.26)}$$

$$\hat{x}_j^{CR} = \begin{cases} \hat{g}_j \hat{z}^{CR} + \hat{L}_j & j < \hat{u} \\ (1/f_{\hat{u}})(R^*(I_0 CAST'_{\hat{u}})\hat{z}^{CR} - \sum_{i=1}^{m} h_i f_i - \sum_{i=1}^{\hat{u}-1} f_i \hat{L}_i) & j = \hat{u} \\ 0 & j > \hat{u} \end{cases} \quad \text{(A.27)}$$

$$\hat{u} = \max\{j \leq m : \sum_{i=1}^{j-1} \hat{g}_i < \frac{R^*(I_0 CAST'_j)(\hat{n} - \sum_{i=1}^{j-1} \hat{L}_i)^+}{\sum_{i=1}^{m} h_i f_i + \sum_{i=1}^{j-1} f_i \hat{L}_i}\} \quad \text{(A.28)}$$

where the index $\hat{u}$ denotes the critical fare-class such that all classes $k > \hat{u}$ are closed, and $\hat{g}_i$ is an auxiliary parameter defined as

$$\hat{g}_m = R * (I_0 CAST_m)/f_m, \quad \hat{g}_i = \frac{R^*(I_0 CAST'_i) - R^*(I_0 CAST'_{i+1})}{f_i}, \quad i = 1, .., m-1 \quad \text{(A.29)}$$

Likewise, the new bucket sizes $\hat{x}^{AR}$ that improve MAR are expressed as:

$$\hat{z}^{AR} = R^*(I_0 CAST'_{\tilde{u}'}) - \sum_{i=1}^{m} h_i f_i - \sum_{i=1}^{\tilde{u}'-1} f_i \hat{L}_i - f_{\tilde{u}'} \hat{x}^{AR}_{\tilde{u}'} \tag{A.30}$$

$$\hat{x}^{AR}_j = \begin{cases} \hat{g}_j + \hat{L}_j & j < \tilde{u}' \\ (\hat{n} - \sum_{i=1}^{\tilde{u}'-1} \hat{L}_i)^+ - \sum_{i<\tilde{u}'} \hat{g}_i & j = \tilde{u}' \\ 0 & j > \tilde{u}' \end{cases} \tag{A.31}$$

$$\tilde{u}' = \max\{j \leq m : \sum_{i<j} \hat{g}_i < (\hat{n} - \sum_{i=1}^{j-1} \hat{L}_i)^+ .\} \tag{A.32}$$

One key observation pertaining to our dynamic policies is the following: Static and dynamic policies are identical under LBH input sequences when $m = 2$ but not when $m > 2$. This is because the only 'mistake' the adversary can make in LBH sequences is in the total amount of requests of each class (not the sequence of arrivals because the sequence is LBH). When $m = 2$, the only inputs of interest, CASTs, call for the adversary to send U_2 class-2 requests first. The 'mistake' implies the adversary switches to sending class-1 requests earlier than expected. Resolving the problem and re-allocating the seats between classes does not improve the worst-case performance in this case, because there cannot be any more class-2 requests in an LBH input after the arrival of the first class-1 request.

Appendix B

Appendices for Chapter 4

B.1 Proof of Proposition 7

We use a shorthand notation in this section to simplify the mathematical expressions. Given an m-vector w, let $\|\cdot\|_+$ be the summation operator, i.e. $\|w\|_+ = \sum_{i=1}^m w_i$ and $\bar{f}(w)$ be the weighted-average fare, i.e., $\bar{f}(w) = \frac{\sum_{i=1}^m w_i f_i}{\|w\|_+}$. We also define the shorthand notation $\chi(p)$ for the CR given a scenario (I, p) and on-line policy b, i.e., $\chi(p) = \frac{R(I,p;b)}{R^*(I,p)}$.

The net revenue function can be rewritten using the shorthand notation for a vector w and a no-show rate p as

$$NR(w|p) = [(1-p) + p\beta]\|w\|_+ \bar{f}(w) - V[(1-p)\|w\|_+ - n]^+.$$

We use the following property of the offline optimal solution in proving Proposition 7.

Corollary 3 *Given input sequence I, the weighted average fare of the offline optimal booking profile is non-increasing in p, that is*

$$\frac{1-p'}{1-p} \leq \frac{\bar{f}(z^*(I,p'))}{\bar{f}(z^*(I,p))} \leq 1 \quad for \quad p' > p. \tag{B.1}$$

Proof Consider the offline optimal booking profile in (4.5) which is obtained by solving the continuous knapsack problem in (4.7). For any $p' > p$, the knapsack capacity is higher. Let Δ be the change in the knapsack capacity, i.e., $\Delta = n/(1 - p') - n/(1-p) > 0$. The solution to the knapsack problem for no-show rate p' is such that $z_i^*(I,p') = z_i^*(I,p)$ for $i = 1,..,k'$ where $k' \geq 1$ is the lowest-fare class included in the optimal knapsack solution for scenario (I,p), i.e.

$$k' = argmin_k \ \{\sum_{i=1}^{k} z_i^*(I,p) = min(\sum_{i=1}^{m} q_i(I), n/(1-p)) \ , 1 \leq k \leq m\}$$

and $z_i^*(I,p) = 0$ for all $i > k'$. Notice that the optimal knapsack solution $z^*(I,p')$ allocates the additional Δ capacity all to classes j with fares $f_j \leq f_{k'} \leq \bar{f}(z^*(I,p))$. Therefore, $\bar{f}(z^*(I,p) \geq \bar{f}(z^*(I,p'))$, which gives the upper bound. Notice that $f_j > 0$, then force these additional fare request at a price of zero will give us the lower bound. •

Proof of Proposition 7 is provided next.

Proposition 2 *Given an input sequence I and a policy b with profile z^b, scenario (I,p) where $p \in (p_0,p_1)$ is dominated by either (I,p_0) or (I,p_1), that is,*

$$\frac{R(I,p;b)}{R^*(I,p)} \geq min(\frac{R(I,p_0;b)}{R^*(I,p_0)}, \frac{R(I,p_1;b)}{R^*(I,p_1)}).$$

<u>Proof</u> By definition, the booking profile of policy b for input I is the same, regardless of the no-show rate p. Let $w^b = z^b(I,p)$ for any $p \in [p_0,p_1]$ be the booking profile of the on-line policy in the remainder of the proof. Notice that $R(I,p;b) = NR(w^b|p)$ for any p.

Recall from Proposition 6 that the offline optimal booking profile satisfies

$$\|z^*(I,p)\|_+ = min(\|q(I)\|_+, \frac{n}{1-p}) \tag{B.2}$$

and that no overbooking penalties are incurred by the offline optimal, i.e.,

$$R^*(I,p) = NR(z^*(I,p)|p) = [(1-p)+p\beta]\|z^*(I,p)\|_+ \bar{f}(z^*(I,p)) \tag{B.3}$$

given a scenario (I,p).

Consider the following two cases regarding w_b of policy b given a scenario: (i) $\|w^b\|_+ \le \|z^*(I,p)\|_+$ and (ii) $\|w^b\|_+ > \|z^*(I,p)\|_+$.

(i) When $\|w^b\|_+ \le \|z^*(I,p)\|_+$, all show-ups are served by both on-line and offline policies, hence *no overbooking penalties are incurred* when no-show rate is p. Since $p \le p_1$, we can see from property (B.2) that $\|z^*(I,p_1)\|_+ \ge \|z^*(I,p)\|_+$. By definition of offline optimality, $R^*(I,p_1) = NR(z^*(I,p_1)|p_1) \ge NR(z^*(I,p)|p_1)$. Combining all these observations, we get

$$\begin{aligned} \chi(p) &= \frac{R(I,p;b)}{R^*(z^*(I,p))} = \frac{NR(w^b|p)}{NR(z^*(I,p)|p)} && \text{(B.4)} \\ &= \frac{\|w^b\|_+ \bar{f}(w^b)}{\|z^*(I,p)\|_+ \bar{f}((z^*(I,p))} = \frac{NR(w^b|p_1)}{NR(z^*(I,p)|p_1)} && \text{(B.5)} \\ &\ge \frac{NR(w^b|p_1)}{R^*(I,p_1)} = \frac{R(I,p_1;b)}{R^*(I,p_1)} = \chi(p_1) && \text{(B.6)} \end{aligned}$$

The relations in (B.5) follow from the fact that neither the on-line nor the offline policies incur overbooking penalties for no-show rates p and p_1. Thus, the theorem is proved for case (i).

(ii) The case of $\|w^b\|_+ > \|z^*(I,p)\|_+$. Since $w^b \le q(I)$, we have $\|q(I)\|_+ \ge \|w^b\|_+ > \|z^*(I,p)\|_+$. Recall property (B.2), we must have $\|w^b\|_+ > \|z^*(I,p)\|_+ =$

$C/(1-p)$. So the online policy b now *incurs service denials and overbooking penalties* at scenario (I,p).

Let $h(\theta,p) = (1-p+p\beta)\bar{f}(w^b)\theta - V[(1-p)\theta - C]$, a function of scalars θ and p. By condition (4.4) and $f_1 \geq \bar{f}(w^b)$, we have $V(1-p)-(1-p+p\beta)\bar{f}(w^b) > 0$. Let $h(\theta,p)=0$ and solve for θ, denote the solution by $\theta(p)$,

$$\theta(p) = \frac{VC}{V(1-p)-(1-p+p\beta)\bar{f}(w^b)}.$$

As $h(\theta,p)$ is linear in θ, and $h(\theta(p),p)=0$, we have

$$h(\theta,p) = (\theta(p)-\theta)[V(1-p)-(1-p+p\beta)\bar{f}(w^b)].$$

Since $\|w^b\|_+ > C/(1-p)$, we have $NR(w^b|p) = h(\|w^b\|_+, p)$. Note that $\|z^*(I,p)\|_+ = C/(1-p)$, we will also have

$$\begin{aligned} R^*(I,p) &= NR(z^*(I,p)|p) \\ &= (1-p+p\beta)\bar{f}(z^*(I,p))\|z^*(I,p)\|_+ \\ &= h(\|z^*(I,p)\|_+, p)\bar{f}(z^*(I,p))/\bar{f}(w^b). \end{aligned}$$

So we can write down $\chi(p) = R(I,p;b)/R^*(I,p)$ as:

$$\begin{aligned} \chi(p) = \frac{R(I,p;b)}{R*(I,p)} &= \frac{h(\|w^b\|_+,p)}{h(\|z^*(I,p)\|_+,p)} \cdot \frac{\bar{f}(w^b)}{\bar{f}(z^*(I,p))} \\ &= \frac{\theta(p)-\|w^b\|_+}{\theta(p)-\|z^*(I,p)\|_+} \cdot \frac{\bar{f}(w^b)}{\bar{f}(z^*(I,p))}. \end{aligned}$$

And similarly, for $p_0 < p$, we can also rewrite $\chi(p_0)$ as:

$$\chi(p_0) = \frac{\theta(p_0)-\|w^b\|_+}{\theta(p_0)-\|z^*(I,p_0)\|_+} \cdot \frac{\bar{f}(w^b)}{\bar{f}(z^*(I,p_0))}.$$

Now let us consider two sub-cases on the sign of $NR(w^b|p)$.

(ii-A) $NR(w^b|p) \geq 0$. So $h(\|w^b\|_+, p) \geq 0$, or equivalently, $\|w^b\|_+ \leq \theta(p)$ in this sub-case. We have $\|z^*(I,p)\|_+ < \|w^b\|_+ \leq \theta(p)$, $\theta(p_0) < \theta(p)$ by monotonicity of $\theta(\dot{})$, and $\|z^*(I,p_0)\|_+ < \|z^*(I,p)\|_+$ from property (B.2). And as $R^*(I,p) > 0$ and $R^*(I,p_0) > 0$, we also have $\theta(p) > \|z^*(I,p)\|_+$ and $\theta(p_0) > \|z^*(I,p)\|_+$. So we can work out as follows:

$$\frac{\theta(p) - \|w^b\|_+}{\theta(p) - \|z^*(I,p)\|_+} > \frac{\theta(p) - \|w^b\|_+}{\theta(p) - \|z^*(I,p_0)\|_+} > \frac{\theta(p_0) - \|w^b\|_+}{\theta(p_0) - \|z^*(I,p_0)\|_+}.$$

As $p_0 \leq p$, from property (B.1) we have $\bar{f}(z^*(I,p)) \leq \bar{f}(z^*(I,p_0))$, so

$$\frac{\bar{f}(w^b)}{\bar{f}(z^*(I,p))} \geq \frac{\bar{f}(w^b)}{\bar{f}(z^*(I,p_0))} \geq 0.$$

These together give us the result of $\chi(p) \geq \chi(p_0)$.

(ii-B) $NR(w^b|p) < 0$. So $h(\|w^b\|_+, p) < 0$, or equivalently, $\|w^b\|_+ > \theta(p)$ in this sub-case. As $\theta(p_0) < \theta(p)$, we also have $h(\|w^b\|_+, p_0) < 0$. So $\chi(p) < 0, \chi(p_0) < 0$, and the ratio between them is

$$\begin{aligned}\frac{\chi(p_0)}{\chi(p)} &= \frac{\theta(p_0) - \|w^b\|_+}{\theta(p_0) - \|z^*(I,p_0)\|_+} \cdot \frac{\bar{f}(z^*(I,p))}{\bar{f}(z^*(I,p_0))} \cdot \frac{\theta(p) - \|z^*(I,p)\|_+}{\theta(p) - \|w^b\|_+} \\ &= \frac{\theta(p_0) - \|w^b\|_+}{\theta(p) - \|w^b\|_+} \cdot \frac{\bar{f}(z^*(I,p))}{\bar{f}(z^*(I,p_0))} \cdot \frac{\theta(p) - \|z^*(I,p)\|_+}{\theta(p_0) - \|z^*(I,p_0)\|_+}.\end{aligned}$$

We only need to show that this ratio is greater than one. It is clear that

$$\frac{\theta(p_0) - \|w^b\|_+}{\theta(p) - \|w^b\|_+} > 1,$$

since $\|w^b\|_+ > \theta(p) > \theta(p_0)$. We rewrite $\theta(p) - \|z^*(I,p)\|_+$ as

$$\theta(p) - \|z^*(I,p)\|_+ = \frac{h(\|z^*(I,p)\|_+, p)}{(V - \bar{f}(w^b))(1-p) - p\beta\bar{f}(w^b)},$$

and similarly rewrite $\theta(p_0) - \|z^*(I,p_0)\|_+$ as

$$\theta(p_0) - \|z^*(I,p_0)\|_+ = \frac{h(\|z^*(I,p_0)\|_+, p_0)}{(V - \bar{f}(w^b))(1-p_0) - p_0\beta\bar{f}(w^b)}.$$

With $p_0 < p$, $\|z^*(I,p)\|_+ = C/(1-p)$, and $\|z^*(I,p_0)\|_+ = C/(1-p_0)$, it is clear from the definition of $h(\cdot)$ that $h(\|z^*(I,p)\|_+, p) > h(\|z^*(I,p_0)\|_+, p_0)$, so:

$$\begin{aligned}
&\frac{\theta(p) - \|z^*(I,p)\|_+}{\theta(p_0) - \|z^*(I,p_0)\|_+} \\
=&\frac{h(\|z^*(I,p)\|_+, p)}{h(\|z^*(I,p_0)\|_+, p_0)} \cdot \frac{(V - \bar{f}(w^b))(1-p_0) - p_0\beta\bar{f}(w^b)}{(V - \bar{f}(w^b))(1-p) - p\beta\bar{f}(w^b)} \\
\geq&\frac{h(\|z^*(I,p)\|_+, p)}{h(\|z^*(I,p_0)\|_+, p_0)} \cdot \frac{1-p_0}{1-p} > \frac{1-p_0}{1-p}.
\end{aligned}$$

By Corollary 3, we have

$$\frac{\bar{f}(z^*(I,p))}{\bar{f}(z^*(I,p_0))} \geq \frac{1-p}{1-p_0}.$$

Putting all these inequalities together, we find that

$$\frac{\chi(p_0)}{\chi(p)} > 1 \cdot \frac{1-p_0}{1-p} \cdot \frac{1-p}{1-p_0} = 1.$$

Thus, $0 > \chi(p) \geq \chi(p_0)$ in this sub-case.

Combining the analysis of the two cases, we prove $\chi(p) \geq min(\chi(p_1), \chi(p_0))$.

•

B.2 Proof of Proposition 9

Proposition 4 *Consider no-show rate p, nested booking limit policy b, and a A_j^b for some $j = 1, ..., m$. Any sequence I in A_j^b is dominated by either $CAST_1$ or* *by* $CAST_j$*, i.e.,*

$$\frac{R(I,p;b)}{R^*(I,p)} \geq \min\left(\frac{R(CAST_j,p;b)}{R^*(CAST_j,p)}, \frac{R(CAST_1,p;b))}{R^*(CAST_1,p)}\right). \tag{B.7}$$

Proof Given policy b and a sequence $I \in A_j^b$, consider the booking profile $z^b(I,p)$. Given the scenario and the booking profile, the policy either incurs overbooking penalties or not:

(1) No overbooking charges: In this case the net policy revenue and the offline optimal revenue are equal to their respective gross revenues multiplied by $(1-p+p\beta)$. So the ratio $\frac{R(I,p;b)}{R^*(I,p)}$ equals the ratio between the respective gross revenues, which is the criteria considered in the proof of Proposition A.2. We proved that any $I \in A_j^b$ is dominated by $CAST_j$ for $p=0$, i.e., $\frac{R(I,0;b)}{R^*(I,0)} \geq \frac{R(CAST_j,0;b)}{R^*(CAST_j,0)}$. That proof is based on generating a set of input sequences in A_j^b by iteratively adjusting I. The adjustments in their procedure do not increase the total number of accepted reservations for the policy. Therefore, overbooking penalties are never incurred and their proof remains valid for any $p \geq 0$.

(2) Overbooking charges are incurred: Based on ROC, the overbooking cost is so high that the adversary prefers to send in the highest amount of requests possible to increase the overbooking cost and decrease the net policy revenue, while potentially increasing the offline optimal. Thus $CAST_1$ dominates I in this case.

•

B.3 Proof of Theorem 4

Theorem 1 $(CAST_1, p_0), (CAST_1, p_1)$, $(CAST_2, p_1), \cdots, (CAST_m, p_1)$ *are the only non-dominated sequences.*

Proof Based on the previous results, there are $2m$ non-dominated scenarios: $(CAST_j, p_t)$ for $j = 1, ..., m$ and $t = 0, 1$. We show that scenario $(CAST_k, p_0), k > 1$ is either dominated by $(CAST_1, p_0)$ or $(CAST_k, p_1)$. Given $(CAST_k, p_0)$, a policy b, and the booking profile $z^b(CAST_k, p_0)$, consider two cases:

(1) There are no overbooking charges for policy b given $z^b(CAST_k, p_0)$: In this case $(CAST_k, p_0)$ must be dominated by $(CAST_k, p_1)$, by the first case in the proof of Theorem 7.

(2) Overbooking cost is positive for policy b given $z^b(CAST_k, p_0)$: In this case $(CAST_k, p_0)$ must be dominated by $(CAST_1, p_0)$ from the argument given in the second case in the proof of Theorem 9.

•

B.4 Proof of Theorem 5

Theorem 2 *(a) The optimal solution to BON is*

$$
\begin{aligned}
\gamma^{BON} &= \frac{R_u^+/f_u^1 + N_u}{R_u^*/f_u^1 + \sum_{i=0}^{u-1} g_i} \\
x_j^{BON} &= \begin{cases} g_j\gamma^{BON} + L_j & j < u \\ (R_u^*\gamma^{BON} - R_u^+)/f_u^1 & j = u \\ 0 & j > u \end{cases} \\
y^{BON} &= -(1-p_0)g_0\gamma^{BON} \\
u &= \max\{j \le m : R_j^+ \sum_{i=0}^{j-1} g_i < N_j R_j^*\}
\end{aligned}
$$

where the index u denotes the critical fare-class such that all classes $k > u$ are closed, and R_j^*, R_j^+, g_i, N_j are auxiliary parameters defined as

$$
\begin{aligned}
R_j^* &= R^*(CAST_j, p_1), \\
R_j^+ &= \sum_{i=1}^{j-1} f_i^1 L_i, \\
g_0 &= \frac{R^*(CAST_1, p_0)/f_1^0 - R_1^*/f_1^1}{(1-p_0)V/f_1^0}, \\
g_i &= \frac{R_i^* - R_{i+1}^*}{f_i^1}, i = 1, \cdots, m-1, \\
N_j &= \min\left(\sum_{i=1}^{m} U_i, \frac{n}{1-p_0}\right) - \sum_{i=1}^{j-1} L_i, j = 1, .., m+1.
\end{aligned}
$$

(b) The nested booking limits defined by

$$
b_j^{BON} = \sum_{i=j}^{m} x_i^{BON} \text{ for } j = 1, \cdots, m
$$

maximizes the CR in problem (4.1) and the optimal CR is $\gamma^{CR} = \gamma^{BON}$.

Proof To prove that x^{BON} is the optimal solution, we have to prove that it is a feasible solution to both BON and its dual. This can be done by studying the relationship among the problem parameters. We investigated the parameter relations for $p = 0$ in Appendix A. Replacing the capacity n with $n/(1-p)$ in their work, we know the following hold: :

$$g_i \leq U_i - L_i \tag{B.8}$$

$$\min\left(\sum_{i=1}^{m} U_i, \frac{n}{1-p_1}\right) - \sum_{i=1}^{m} L_i \leq \sum_{i=1}^{m} g_i. \tag{B.9}$$

Using (4.6), note that $f_{m+1} = 0, q_j(CAST_1) = U_j$ and $\min(x, \frac{n}{1-p_1}) - \min(x, \frac{n}{1-p_0})$ is non-decreasing in x, we proceed as follows:

$$\begin{aligned}
& R^*(CAST_1, p_1)/f_1^1 - R^*(CAST_1, p_0)/f_1^0 \\
= & \sum_{i=1}^{m} \frac{f_i - f_{i+1}}{f_1} \min\left(\sum_{j=1}^{i} q_j(CAST_1), \frac{n}{1-p_1}\right) - \\
& \sum_{i=1}^{m} \frac{f_i - f_{i+1}}{f_1} \min\left(\sum_{j=1}^{i} q_j(CAST_1), \frac{n}{1-p_0}\right) \\
\leq & \sum_{i=1}^{m} \frac{f_i - f_{i+1}}{f_1} \left[\min\left(\sum_{j=1}^{m} U_j, \frac{n}{1-p_1}\right) - \min\left(\sum_{j=1}^{m} U_j, \frac{n}{1-p_0}\right)\right] \\
= & \min\left(\sum_{j=1}^{m} U_j, \frac{n}{1-p_1}\right) - \min\left(\sum_{j=1}^{m} U_j, \frac{n}{1-p_0}\right).
\end{aligned}$$

Also, from the ROC condition, we have

$$V > f_1(1 - p_0 + p_0\beta)/(1 - p_0) = f_1^0/(1 - p_0).$$

Given these inequalities and by definition of g_0 in (4.21), we get the following relation:

$$\min\left(\sum_{j=1}^{m} U_j, \frac{n}{1-p_1}\right) - \min\left(\sum_{j=1}^{m} U_j, \frac{n}{1-p_0}\right) \geq -g_0. \tag{B.10}$$

Note that $g_0 \leq 0$ in this problem. Combining these observations, we get

$$\begin{aligned}
N_{m+1} &= \min\left(\sum_{j=1}^{m} U_j, \frac{n}{1-p_0}\right) - \sum_{i=1}^{m} L_i \\
&= \left[\min\left(\sum_{j=1}^{m} U_j, \frac{n}{1-p_0}\right) - \min\left(\sum_{j=1}^{m} U_j, \frac{n}{1-p_1}\right)\right] + \\
&\quad \left[\min\left(\sum_{j=1}^{m} U_j, \frac{n}{1-p_1}\right) - \sum_{i=1}^{m} L_i\right] \\
&\leq \sum_{i=0}^{m} g_i.
\end{aligned}$$

Given all these properties of the problem parameters, the solution x^{BON} can be shown to satisfy constraints (4.11) for $j = 1, \cdots, u$, (4.12) and (4.13) at equality. Following the steps in Appendix A, x^{BON} can be shown to be feasible for BON and its dual; details are omitted. •

B.5 Proof of Theorem 6

Theorem 3 *The nested booking control policy with booking limit vector b^{BON} defined by (4.24) has a CR of γ^{BON} given by (4.15) and this is the best possible among all deterministic, static policies.*

Proof We prove this result by showing how an adversary would choose a scenario to minimize the CR, if he could observe how the seller was processing the input (making accept/reject decisions) during the booking horizon. Let π be an arbitrary static policy, the only requirement being that it should accept/reject any portion of a request upon its arrival. Suppose the adversary observes each of π's "decisions" and immediately reacts by adjusting the remaining order and amount of requests in the input. Once the input sequence is processed, the adversary decides on a no-show rate to complete the scenario. When specifying the adversary's actions, we characterize the policy's effect by defining an effective booking limit vector, b^π, and bucket size vector, x^π. Consider the following algorithm for the adversary:

STEP 0 Initialize $x^\pi = b^\pi = 0$ and let the current fare index $\hat{\imath} = m + 1$;

STEP 1 Set $\hat{\imath} = \hat{\imath} - 1$, send in $U_{\hat{\imath}}$ of class $\hat{\imath}$ requests;

STEP 2 Set $x^\pi_{\hat{\imath}}$ to the number of class $\hat{\imath}$ requests accepted by π;

STEP 3 Let $b^\pi_{\hat{\imath}} = b^\pi_{\hat{\imath}+1} + x^\pi_{\hat{\imath}}$;

STEP 4 If $b^\pi_{\hat{\imath}} \geq b^{BON}_{\hat{\imath}}$, go to STEP 1 if $\hat{\imath} > 1$;

STEP 5 If $b^\pi_{\hat{\imath}} < b^{BON}_{\hat{\imath}}$, send in the rest of $CAST_{\hat{\imath}}$ and update b^π;

STEP 6 If $b^\pi_1 \leq b^{BON}_1$, let $\hat{p} = p_1$; otherwise, let $\hat{p} = p_0$.

Adversary's algorithm terminates by generating the scenario $(CAST_{\hat{\imath}}, \hat{p})$. The conditions in steps 4 and 5 above imply that $b^\pi_j \geq b^{BON}_j$ for $j > \hat{\imath}$. Consider two cases based on the choice of $\hat{p}$.

- $\hat{p} = p_1$, implying $b^\pi_1 \leq b^{BON}_1$ by step 6. Thus we have $b^\pi_{\hat{\imath}} \leq b^{BON}_{\hat{\imath}}$ whether $\hat{\imath} > 1$ or $\hat{\imath} = 1$. Then the revenue of policy π based on classes $k > \hat{\imath}$ is no more than the corresponding revenue of the optimal nested policy b^{BON}. The revenue from classes $k < \hat{\imath}$ would be at most $\sum_{i=1}^{\hat{\imath}-1} L_i f_i^1$, which is the revenue obtained by the optimal nested policy b^{BON}. Note that in such a case there would be no overbooking penalties, so we only need to consider revenues. Combining these observations, the revenue of π cannot be higher than that of the optimal nested policy b^{BON}.

- $\hat{p} = p_0$, implying $b_1^{\pi} > b_1^{BON}$ by step 6, which further implies $\hat{i} = 1$ by step 4, 5 and the fact that $x_i^{BON} \geq L_i, i < m$. So we have $b_j^{\pi} \geq b_j^{BON}, \forall j$. Note that in this case overbooking costs of b^BON are positive, and the net revenue per request accepted beyond the point of b_1^{BON} is negative because of ROC. So it is sufficient to show that the total revenues generated by the requests accepted within the limit of b_1^{BON} is less than the that of the optimal policy. That holds because $\min(b_j^{\pi}, b_1^{BON}) \geq b_j^{BON}$, for all $j = 1, .., m$. So again the revenue of π cannot be higher than that of the optimal nested policy b^{BON}.

Hence, policy b^{BON} is better (or no-worse) than any other arbitrary static policy. This then implies the CR is at most γ^{BON}, and the nested booking control policy b^{BON} that achieves γ^{BON} is the best possible among all deterministic policies.

•

B.6 Addendum to Computational Experiments

We present an additional example to show the effect of no-show rates and the demand factor.

Example-5. Effect of no-shows when demand factor is constant. Here, no-show distribution shifts as in Example-2, but the demand factor is fixed at 1.08.

That is, the demand bounds vary so that the effect of the shift of the mean no-show rate on the demand factor is counter-acted. The average revenues, average number of unused units at the time of service, and the average number of service denials are presented in Figure B.1.

Example-6. Effect of overbooking cost. The no-show distribution is Uniform[0.1,0.2] in this experiment and all the methods have accurate no-show information. The other problem parameters are kept at their default values while the overbooking cost V varies from 200 to 600. Figure B.2 displays the average performance. The observations are similar to that of Example-1 and Example-2.

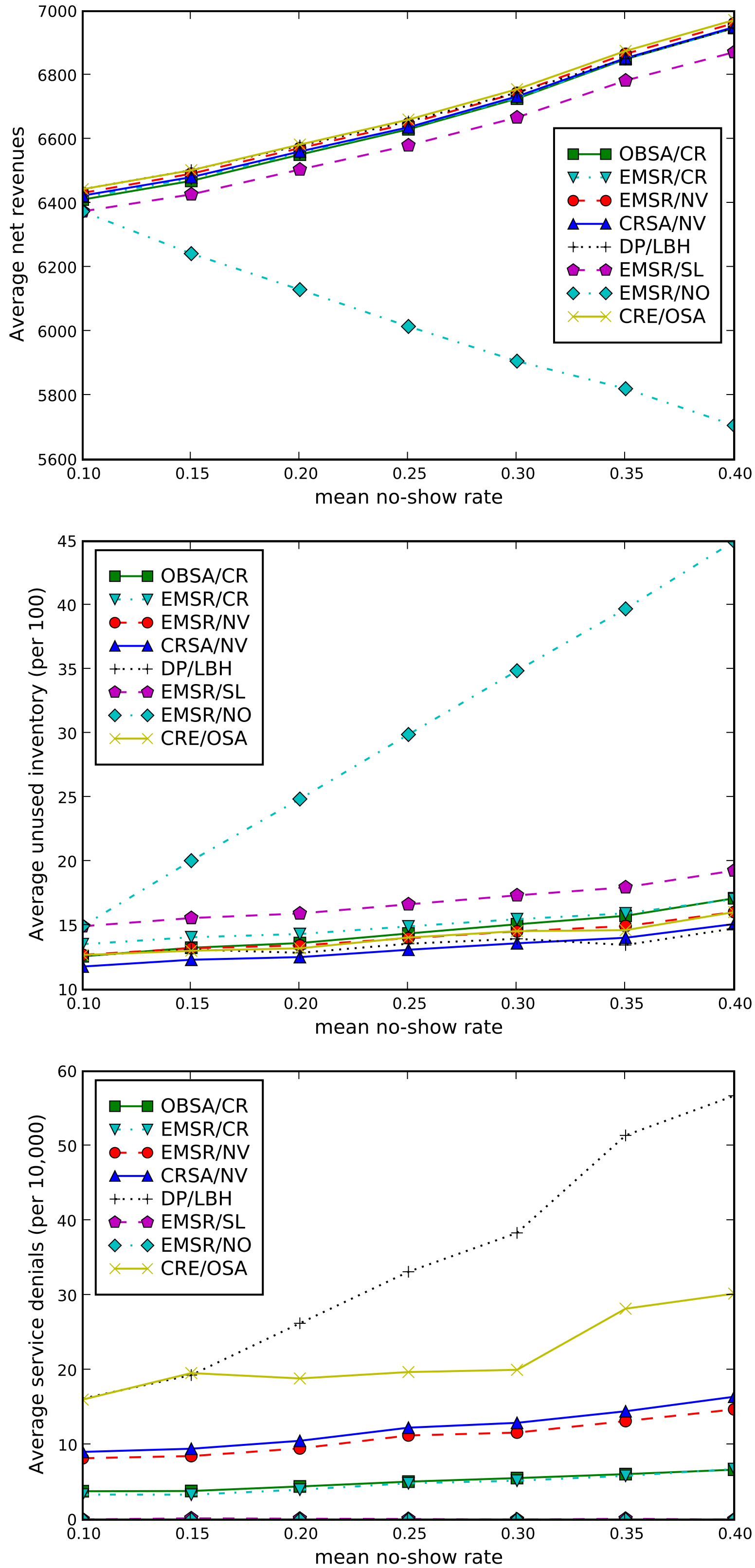

Figure B.1: Average performance in Example-5.

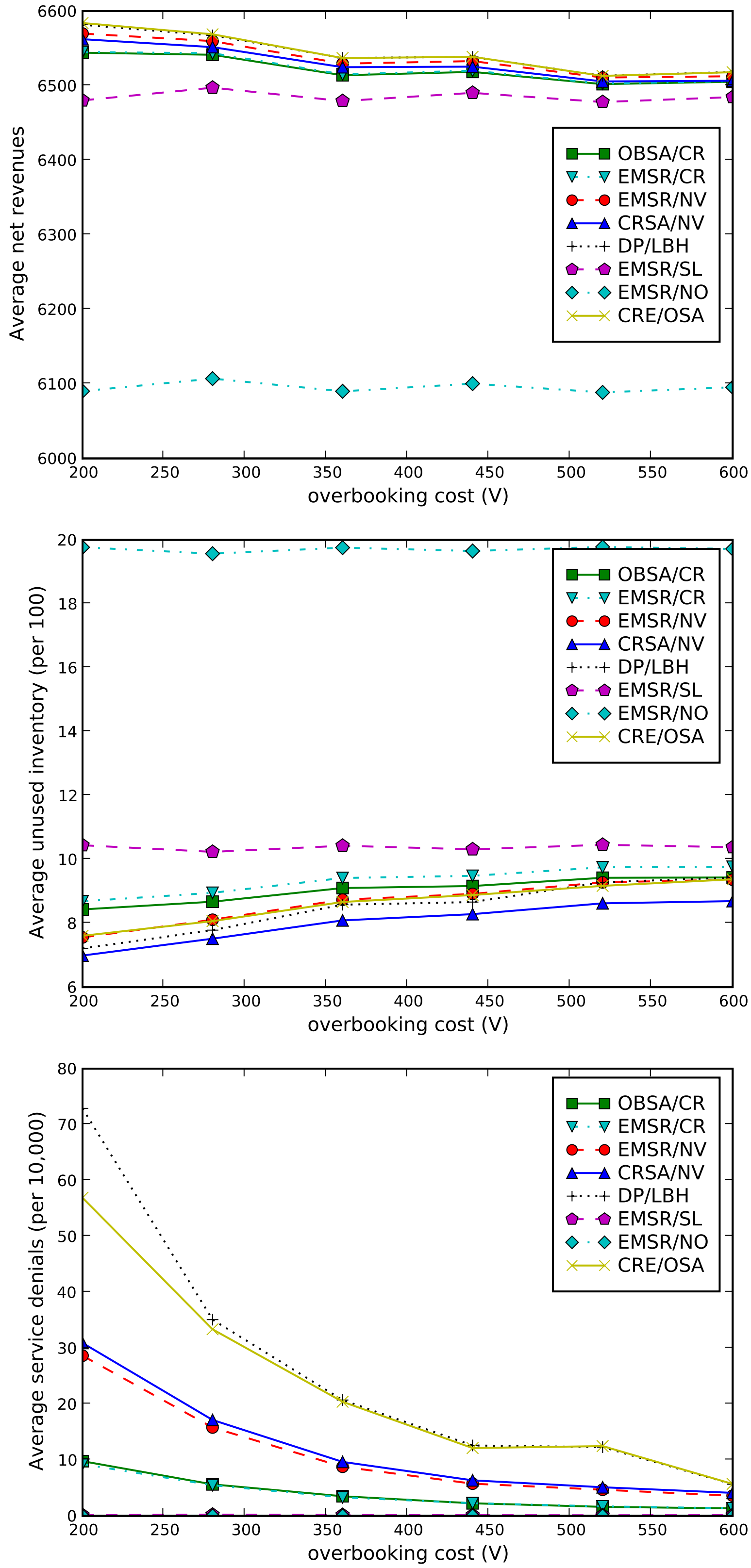

Figure B.2: Average performance in Example-6.

Bibliography

Albers, S. 2003. Online algorithms: A Survey. *Math. Programming, Ser. B,* 97, 3-26.

Alstrup, J., S. Boas, O.B.G. Madsen and R.V.V. Vidal. 1986. Booking Policy for Flights with Two Types of Passengers. *European Journal of Operational Research* **27** 274-288.

Alstrup, J., S. Andersson, S. Boas, O.B.G. Madsen and R.V.V. Vidal. 1989. Booking Control Increases Profit at Scandinavian Airlines. *Interfaces* **19** 10-19.

Bailey, J. 2007. Bumped Fliers and No Plan B. *The New York Times,* May 30, 2007.

Ball, M. O. and M. Queyranne. 2006. Toward Robust Revenue Management: Competitive Analysis of Online Booking. Working paper, University of Maryland, College Park, MD; available at SSRN: http://ssrn.com/abstract=896547.

Beckmann, M.J. 1958. Decision and Team Problems in Airline Reservations. *Econometrica,* **26** 134-145.

Belobaba, P.P. 1987. Airline Yield Management: An Overview of Seat Inventory Control. *Transportation Science* 21, 63-73.

Belobaba, P.P. 1989. Application of a Probabilistic Decision Model to Airline Seat Inventory Control. *Operations Research,* 37, 183-197.

Belobaba, P. 2006. Flight Overbooking: Models and Practice. Lecture notes. Massachusettes Institute of Technology, Boston, MA, *http : //ocw.mit.edu/NR/ rdonlyres/Aeronautics−and−Astronautics*/16−75*JSpring*−2006/ *E*2*A*09*BF*6− *E*275 − 431*B* − 9*C*07 − 2*C*98*BEA*50541/0/*lect*19*.pdf*

Birbil, S. I., J.B.G. Frenk, J. A.S. Gromicho and S. Zhang. 2006. An Integrated Approach to Single-Leg Airline Revenue Management: The Role of Robust Optimization. Working paper, Erasmus University, Erasmus Research University Institute of Management (ERIM), Rotterdam, The Netherlands.

Bitran, G.R. and S. Gilbert. 1996. Managing Hotel Reservations with Uncertain Arrivals. *Operations Research* **44** 35-49.

Boer, P.D., Kroese, D., Mannor, S., and Rubinstein, R. 2005. A Tutorial on the Cross-Entropy Method. *Annals of Operations Research*, 134:19-67.

Brumelle, S.L. and J.I. McGill. 1993. Airline Seat Allocation with Multiple Nested Fare Classes. *Operations Research* **41** 127-137.

Brumelle, S. and D. Walczak. 2003. Dynamic Airline Revenue Management with Multipe Semi-Markov Demand. *Oper. Res.* 51, 137-148.

Chatwin, R.E. 1992. Optimal Airline Overbooking. Ph.D. Thesis, Stanford University, Palo Alto.

Chatwin, R.E. 1999. Continuous-Time Airline Overbooking with Time Dependent Fares and Refunds. *Transportation Science* **33** 182-191.

Couglan, J. 1999. Airline overbooking in the multi-class case. *Journal of the Operational Research Society,* 50, 1098-1103.

Curry, R.E. 1990. Optimal Airline Seat Allocation with Fare Classes Nested by Origins and Destinations. *Transportation Science* **24** 193-204.

D.E. Goldberg. 1989. Genetic Algorithms in Search, Optimization, and Machine Learning. *Addison Wesley*.

Eren, S. and C. Maglaras. 2006. Revenue management heuristics under limited market information: A maximum entropy approach. presented at The 6th Annual INFORMS Revenue Management Conference, June 5-6, Columbia University.

F.W. Glover. 1990. Tabu Search: A Tutorial. *Interfaces*, Vol.20, 74-94.

Gao 2008. Analysis of Distribution-Free Methods for Revenue Management. Dissertation.

Geraghty, M.K. and E. Johnson. 1997. Revenue Management Saves National Car Rental. *Interfaces,* v.27, pp.107127.

Hu, J., Fu, M., and Marcus, S. 2006. A Model Reference Adaptive Search Algorithm for Global Optimization. Forthcoming in *Operations Research*.

Huh, T. and Rusmevichientong, P. 2006. An Adaptive Algorithm for Multiple-Fare-Class Capacity Control Problems. Presented at the 6th Annual INFORMS Revenue Management Conference, June 5-6, Columbia University.

Hadjinicola, G.C. and C. Panayi. 1997. The overbooking problem in hotels with multiple tour-operators. *International Journal of Operations and Production Management,* 17, 874-.

Karaesmen, I. and G. van Ryzin. 2004. Overbooking with Substitutable Inventory Classes. *Operations Research* **52** 83-104.

Kimes, S. and L. Schruben. 2002. Golf course revenue management: A study of tee time intervals. *Journal of Revenue and Pricing Management,* 1, 111-120.

Kunnumkal, S. and H. Topaloglu. 2007. *A stochastic approximation method for the revenue management problem on a single flight leg with discrete demand distributions.* Working paper, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY.

Lahoti, A. 2002. Why CEOs Should Care About Revenue Management. *OR/MS Today*, February 2002, http://www.lionhrtpub.com/orms-2-02/rm.html.

Lan, Y., H. Gao, M. Ball and I. Karaesmen. 2007. Revenue Management with Limited Information. Working paper, University of Maryland, Robert H. Smith School of Business, College Park, MD.

Lawrence, R.D., S.J. Hong and J. Cherrier. 2003. Passenger-based predictive modeling of airline no-show rates. *KDD '03: Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining,* pp. 397-406.

Lee, T.C. and M. Hersh. 1993. A Model for Dynamic Airline Seat Inventory Control with Multiple Seat Bookings. *Trans. Sci.* 27, 252-265.

Lennon, J.J. 2004. Revenue management and customer forecasts: A bridge too far for the UK visitor attractions sector? *J. Revenue and Pricing Manag.* 2, 338-352.

Liberman, V. and U. Yechiali. 1978. On the Hotel Overbooking Problem - An Inventory System with Stochastic Cancellations. *Management Science* **24** 1117-1126.

Lim, A.E.B. and J.G. Shanthikumar. 2006. Relative entropy, exponential utility, and robust dynamic pricing. forthcoming in *Operations Research.*

Littlewood, K. 1972. Forecasting and Control of Passengers. *12th AGIFORS Symposium Proceedings,* 95-128.

Matheu, B. 2007. Keynote Speech, 7th Annual INFORMS Revenue Management and Pricing Section Conference, June 28-29, 2007, University of Pompeu Fabra, Barcelona, Spain.

Maynor, J. (University of Maryland Golf Course, Head Professional and Director) 2006. Personal communication.

Perakis, G. and G. Roels. 2006. Robust Controls for Network Revenue Management. Working paper, UCLA, Anderson School of Management, Los Angeles, CA.

Phillips, R. L. 2005. *Pricing and Revenue Optimization.* Stanford University Press, Stanford.

Ratliff, R.M. 1998. Ideas on Overbooking. Presentation at *AGIFORS Reservation and Yield Management Study Group Meeting,* Melbourne, Australia.

Robinson, L.W. 1995. Optimal and Approximate Control Policies for Airline Booking with Sequential Nonmonotonic Fare Classes. *Operations Research* **43** 252-263.

Rothstein, M. and A.W. Stone. 1967. Passenger Booking Levels. *Proceedings of the Seventh AGIFORS Symposium*, American Airlines, New York.

Rothstein, M. 1971. An Airline Overbooking Model. *Transportation Science* **5** 180-192.

Rothstein, M. 1974. Hotel Overbooking as a Markovian Sequential Decision Process. *Decision Sciences* **5** 389-404.

Rothstein, M. 1985. OR and the Airline Overbooking Problem. *Operations Research* **33** 237-248.

Rusmevichientong, P., B. Van Roy, and P. W. Glynn. 2006. A Non-Parametric Approach to Multi-Product Pricing. *Oper. Res.* 54, 82-98.

S. Kirkpatrick, C.D. Gelatt, and M.P. Vecchi. 1983. Optimization by Simulated Annealing. *Science*, Vol.220, 671-680.

Shlifer, E. and Y. Vardi. 1975. An Airline Overbooking Policy. *Transportation Science* **9** 101-114.

Simon, J.L. 1968. An Almost-Practical Solution to Airline Overbooking. *Journal of Transport Economics and Policy* **II** 201-202.

Smith,B., J. Leimkuhler, R. Darrow and J. Samuels. 1992. Yield Management at American Airlines. *Interfaces* **22** 8-31.

Subramanian, J., S. Stidham Jr.and C. Lautenbacher. 1999. Airline Yield Management with Overbooking, Cancellations and No-Shows. *Transportation Science* **33** 147-167.

Talluri, K. and G. van Ryzin. 2004a. *The Theory and Practice of Revenue Management*, Kluwer Academic Publishers, Boston, 2004.

Talluri, K. and G. van Ryzin. 2004b. Revenue Management under a general discrete choice model of customer behavior. *Manag. Sci.* 50, 15-33.

Taylor, C.J. 1962. The Determination of Passenger Booking Levels. *Proceedings of the Second AGIFORS Symposium*, American Airlines, New York.

Thompson, H.R. 1961. Statistical Problems in Airline Reservation Control. *Operations Research Quarterly* **12** 167-185.

van Ryzin, G. J. and J. McGill. 2000. Revenue Management without Forecasting or Optimization: An Adaptive Algorithm for Determining Airline Seat Protection Levels. *Manag. Sci.* 46, 760-775.

Wade, B. 1996. PRACTICAL TRAVELER;No-Show Fees: Well, Yes and No. *The New York Times,* March 17, 1996.

Weatherford, L.R. and S.E. Bodily. 1992. A Taxonomy and Research Overview of Perishable-Asset Revenue Management: Yield Management, Overbooking and Pricing. *Operations Research* **40** 831-844.

Yeoman, I. and A. Ingold. 1997. *Yield Management Strategies for the Service Industries.* Cassell, London.