ABSTRACT

Title of dissertation:	Spherical Averaged Endpoint Strichartz Estimates for the Two-dimensional Schrödinger Equations with Inverse Square Potential
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In this dissertation, I investigate the two-dimensional Schrödinger equation with repulsive inverse square potential, i.e.,

$$\begin{cases} i\partial_t u + \triangle u - \frac{a^2}{|x|^2}u = 0 \qquad u : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{C}, \\ u(x,0) = u_0(x). \end{cases} \tag{1}$$

I prove the following version of the homogeneous endpoint Strichartz estimate:

$$\|u\|_{L_t^2(L_r^\infty L_\theta)} \leq C\|u_0\|_{L^2}, \tag{2}$$

where the $L_r^\infty L_\theta$ is a norm that takes L^2 average in angular variable first and then supremum norm on radial variable, i.e.,

$$\|f(x,y)\|_{L_r^\infty L_\theta} = \sup_{r>0}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(r\cos\theta, r\sin\theta)|^2 d\theta\right)^{\frac{1}{2}}. \tag{3}$$

The main result is presented in chapter 4. In chapter 2, I give a brief introduction on the equations that inspired my research, namely the Landau-Lifshitz equation and the Schrödinger map equation. In chapter 3, I introduce a geometric concept in order to obtain a gauge system suitable for analysis.

Spherical Averaged Endpoint Strichartz Estimates for The Two-dimensional Schrödinger Equations with Inverse Square Potential

by

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Dedication

For my parents, Chao-Nan Chen and Fang-Yu Chang

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List of Abbreviations

BMO	bounded mean osillation
$\times$	cross product
$\dot{\times}$	hyperbolic cross product
L^p	space of p-power integrable functions, Lebesgue space
$\mathbb{S}^2$	unit sphere in $\mathbb{R}^3$
$\mathbb{H}^2$	hyperbolic 2-space in $\mathbb{R}^3$
$f^{\#}$	Hankel transform of f

Chapter 1

Introduction

In this dissertation, I investigate the two-dimensional Schrödinger equation with repulsive inverse square potential, i.e.,

$$\begin{cases} i\partial_t u + \triangle u - \frac{a^2}{|x|^2}u = 0 \qquad u : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{C}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

I prove the following version of the homogeneous endpoint Strichartz estimate:

$$\|u\|_{L_t^2(L_r^\infty L_\theta)} \leq C\|u_0\|_{L^2}, \tag{1.2}$$

where the $L_r^\infty L_\theta$ is a norm that takes L^2 average in angular variable first and then supremum norm on radial variable, i.e.,

$$\|f(x,y)\|_{L_r^\infty L_\theta} = \sup_{r>0}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(r\cos\theta, r\sin\theta)|^2 d\theta\right)^{\frac{1}{2}}. \tag{1.3}$$

This problem by itself is mathematically interesting . It is also related to problems that are concerned with models of ferromagnetism. I will explain more in detail.

One can model magnetization by spins arranged on a lattice. This is the famous Heisenberg model. The spins interact with each other according to the nature of the material. For ferromagnetism, the spins try to align if they are neighbors.

In 1935, Landau and Lifshitz took the continuum limit and derived an evolution equation for the spin density. Later, the equation became known as Landau-Lifshitz equation.

The Landau-Lifshitz equation has two parts which reflect the effects of precession and damping. When we apply the magnetic field on the spin, the precession effect rotates the spin with the direction of the magnetic field as the axis, on the other hand, the damping effect rotates the spin so that it becomes parallel to the magnetic field. If we consider only the damping effect, we will have the harmonic map heat flow equation, which is well studied. If we consider only the precession effect, we will have the Schrödinger map equation.

For evolution equations, it is natural to ask weather the solution exist for all time or it blows up in finite time. For the Schrödinger map equation with one space dimension, it is well known that the solution has global existence. As a matter of fact it can be reduced to the cubic nonlinear Schrödinger equation which is integrable. It is a conjecture that in space dimension two, the solution for the Schrödinger map equation may blow-up in finite time. There are partial results analyzing a simplified special class of solutions, namely equivariant solutions by Gustafson, Kang, and Tsai [16] [17]. The investigation involves deriving from Schrödinger map equation a nonlinear Schrödinger equation with potential which behaves like inverse square and using Strichartz estimates. In [16], it was mentioned that the endpoint Strichartz estimate for space dimension 2 with inverse square potential is open. Very recently, there is a further result on inhomogeneous endpoint estimate for radial symmetric data [18].

The main result is presented in chapter 4. In chapter 2, I give a brief introduction on the equations that inspired my research, namely the Landau-Lifshitz equation and the Schrödinger map equation. In chapter 3, I introduce a geometric concept in order to obtain a gauge system suitable for analysis.

Chapter 2

The Landau-Lifshitz equation and Schrödinger map equation

The equations that motivate this dissertation are Landau-Lifshitz equation and the related Schrödinger map equation. The Landau-Lifshitz equation is used to model ferromagnetism in physics. The initial-value problem for Landau-Lifshitz equation is as follows.

$$\begin{cases} u_t = au \times \triangle u - \epsilon u \times (u \times \triangle u), \\ u(x,0) = u_0(x), \end{cases} \tag{2.1}$$

where u is a function which maps $\mathbb{R}^n \times \mathbb{R}^+$ to the unit sphere in $\mathbb{R}^3$. $\triangle$ is the Laplacian operator and $\times$ is the cross product in $\mathbb{R}^3$. The first term on the right hand side of equation (2.1) is dispersive and the second term is dissipative. If we consider only the dispersive term, we have the Schrödinger map equation.

$$\begin{cases} u_t = u \times \triangle u, \\ u(x,0) = u_0(x). \end{cases} \tag{2.2}$$

This equation can be generalized. For example, instead of considering the unit sphere, $\mathbb{S}^2$, as the target, we can consider the hyperbolic space $\mathbb{H}^2$.

In this chapter, I will give an introduction on the basic properties, and mathematical questions on Landau-Lifshitz and Schrödinger map equations.

2.1 The Heisenberg Model

In the Heisenberg model, we model the magnetism by spins arranged in a lattice. There are three different interactions between spins depending on the material : paramagnetism, ferromagnetism, and antiferromagnetism. For paramagnetism, the spins interact with an applied magnetic field but do not interact with each other. For ferromagnetic materials, like iron, spins not only interact with an applying field but also with each other. They try to align the neighboring spins so that they are parallel. Unlike ferromagnetism, the spins of antiferromagnetism tend to align their neighboring spins so that they are antiparallel.

In 1935, Landau and Lifshitz used the Heisenberg Model for ferromagnetism and took the continuum limit to derived the equation.(2.1) [25].

The Schrödinger map corresponds to the case $\epsilon = 0$, $a = 1$. The case $\epsilon = 1$, $a = 0$ is the well-studied harmonic map heat flow.

2.2 Conservation Laws

First, we consider the Schrödinger map equation (2.2). The energy is defined as follows:

$$E(t) = \int |\nabla u|^2 dx. \tag{2.3}$$

If we take the time derivative, we obtain

$$
\begin{aligned}
E_t &= \int 2\nabla u_t \cdot \nabla u dx && (2.4)\\
&= \int \nabla(2u_t \cdot \nabla u) - 2u_t \cdot \Delta u = -\int 2u_t \cdot \Delta u && (2.5)\\
&= -\int 2(u \times \Delta u) \cdot \Delta u dx = 0. && (2.6)
\end{aligned}
$$

For the Schrödinger map equation, the energy is formally conserved, i.e. the energy is conserved if the solution is sufficiently smooth and ∇u decays to zero at infinity sufficiently fast.

We can also do the same calculation for the solutions of the Landau-Lifshitz equation. We obtain

$$
E_t = -a \int 2(u \times \Delta u) \cdot \Delta u dx + \epsilon \int 2(u \times (u \times \Delta u)) \cdot \Delta u dx. \tag{2.7}
$$

The first integral on the right hand side is zero. For the second integral, we can apply Lagrange's formula

$$
A \times (B \times C) = B(A \cdot C) - C(A \cdot B), \tag{2.8}
$$

and we obtain,

$$
E_t = 2\epsilon \int |u \cdot \Delta u|^2 - |\Delta u|^2 |u|^2 dx, \tag{2.9}
$$

which is non-positive. Thus, the energy decays, and this is the reason we call the second term in the equation dissipative.

Remark 2.2.1. *Both the Landau-Lifshitz equation and Schrödinger map equation are defined on the unit sphere. It is natural to check that the maps stay on the sphere during the evolution.*

Suppose u satisfies equation (2.1). We want to see how the vector norm of u changes over time. We differentiate $|u|^2$ *directly and the substitute* u_t *by equation 2.1, we have*

$$\begin{aligned} \frac{\partial |u|^2}{\partial t} &= 2 < u, u_t > & (2.10) \\ &= < u, au \times \triangle u - \epsilon u \times (u \times \triangle u) >= 0. \end{aligned}$$

Thus, the solution stays on the sphere.

2.3 Scaling Properties

The equation is nonlinear, and it has some interesting features. If we rescale the space by a ratio s and the time by s^2. i.e. Define $\tilde{u}(t,x) = u(t/s^2, x/s)$, $t' = t/s^2$, and $x' = x/s$. We have

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial u(t/s^2, x/s)}{\partial t} = \frac{1}{s^2}\frac{\partial u(t', x')}{\partial t'}, \tag{2.11}$$

$$\tilde{u} \times \Delta \tilde{u} = \frac{1}{s^2} u(t', x') \times \Delta u(t', x'). \tag{2.12}$$

We can see that the new function still satisfies the same Schrödinger map equation.

The energy after rescaling depends on the space dimension through the Jacobian.

$$E(\tilde{u}) = \int |\nabla \tilde{u}|^2 dx \tag{2.13}$$

$$= \int s^{-2} |\nabla u(t', x'|^2 dx \tag{2.14}$$

$$= \int s^{-2+n} |\nabla u(t', x')|^2 d\acute{x} = s^{-2+n} E(u). \tag{2.15}$$

For space dimension 1, the energy increases after shrinking the function. For space dimension 3, the energy decreases after shrinking. For space dimension 2, the energy does not change under the rescaling. We also describe space dimension 1, 3 and 2 as energy subcritical, supercritical and critical respectively.

For space dimension 1 case, it is well-known that the Schrödinger map equation is equivalent to a cubic nonlinear Schrodinger equation, which is integrable via the inverse scattering method, and it is globally well-posed. For higher space dimension, the local well-posedness for initial data in $H^s(\mathbb{R}^n), s > n/2 + 2, n \geq 2$ is established by Sulem, Sulem, and Bardos[29]. For gauge system derived from Schrödinger map, called the molified Schrödinger map equation, Kato [20], Kenig and Nahmod [22] established local existence result for initial data in $H^{\frac{1}{2}+\epsilon}(\mathbb{R}^2)$. Later Koch and Kato [23] proved the uniqueness for $H^{\frac{3}{4}+\epsilon}(\mathbb{R}^2)$. Since the gauge system involves derivative of the original map, for the origninal Schrödinger map equation local existence is established for data in $H^{1+\frac{1}{2}+\epsilon}(\mathbb{R}^2)$ and uniqueness for data in $H^{1+\frac{3}{4}+\epsilon}(\mathbb{R}^2)$. Recently, there is a result by Bejenaru [1] on local well-posedness for small initial data in $H^{\frac{n}{2}+\varepsilon}(\mathbb{R}^n),\ n \geq 2$ for Schrödinger map equation. For space dimension 3 or higher, one expects that singularities are formed. However, there are no definite results so far. For space dimention greater the 4, Bejenaru, Ionescu, and Kenig proved global existence for small data in some critical norm [2]. For space dimension 2, it is a conjecture that solution with finite energy may blow-up in finite time. This problem attracted some interest and there are some partial results by Gustafson, Kang, and Tsai[16] [17].

Chapter 3

Gauge Transformation

In this chapter the goal is to start from the Schrödinger map equation and derive a gauge invariant system which is more suitable for analysis. The Schrödinger map equation is highly nonlinear. By using gauge systems the nonlinearity can be transformed to quasi-linear. For equivariant case, it can even be transformed to semi-linear. For this case the usual bootstrap strategy can be applied, provided we have good estimates for the linear part of the solution.

We start by introducing quaternions to reformulate the evolution equations. Then, we use the geometric concept of covariant derivative and compatibility relations in order to obtain a systems of equations that can be solved up to gauge fixing. We will discuss different cases in the end.

3.1 Rotation using Quaternions

We will introduce the space of quaternions, which are widely used to describe rotation in 3 or 4 dimensional space. We define the quaternion space

$$\{P = p_0 I + p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3 | p_i \in \mathbb{R}\}, \tag{3.1}$$

where σ_j are Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{3.2}$$

We can check that σ_1, σ_2, and σ_3 have the following properties.

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -I, \tag{3.3}$$

$$\sigma_1\sigma_2 = \sigma_3,\ \sigma_2\sigma_3 = \sigma_1,\ \sigma_3\sigma_2 = \sigma_1, \tag{3.4}$$

$$\sigma_i\sigma_j = -\sigma_j\sigma_i. \tag{3.5}$$

We can map a 3 vector to a quaternion.

$$S(u) = u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3. \tag{3.6}$$

We can check directly that

$$S(u)S(v) = S(u \times v) - (u \cdot v)I. \tag{3.7}$$

Thus, we have

$$S(u \times v) = \frac{1}{2}[S(u), S(v)], \tag{3.8}$$

where [,] is the commutator between two matrices. For the quaternion $P = p_0I + p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3$, this is the reason we name the first term scalar part and the rest vector part. As an analogy of complex numbers, we also call them real part and imaginary part respectively. Suppose S is a quaternion only with vector part correspond to the vector u. We can rewrite the Landau-Lifshitz equation as follows:

$$\partial_t S = \frac{a}{2}[S, \triangle S] - \frac{\epsilon}{4}[S, [S, \triangle S]]. \tag{3.9}$$

We define the conjugate quaternion by

$$P^\dagger = p_0 - p_1\sigma_1 - p_2\sigma_2 - p_3\sigma_3. \tag{3.10}$$

Then, we can check that

$$PP^\dagger = P^\dagger P = (p_0^2 + p_1^2 + p_2^2 + p_3^2). \tag{3.11}$$

We define the norm of P to be $|P| = (PP^\dagger)^{\frac{1}{2}}$. Note that $|P|^2 = \det P$. Suppose $|P| \neq 0$, P has inverse $P^{-1} = |P|^{-2}P^\dagger$.

It is well known that if we conjugate a quaternion having only vector part by a unit quaternion P with real part $\cos\alpha$, the effect corresponds to a rotation with a angle 2α with respect to the vector part of P as axis. Thus, if S is correspond to a unit vector, we can find a unit lenght quaternion, P, such that

$$S = P\sigma_3 P^{-1}. \tag{3.12}$$

3.2 Gauge Transform

I would like to explain the emergence of covariant derivatives under gauge transformation. If we have two similar matrices, S and S_0 which are related via

$$S = PS_0P^{-1}, \tag{3.13}$$

and S, S_0, P are smooth functions of variables x^μ for $\mu = 0, 1, ..., n$, we have the following relations.

$$\begin{aligned}\partial_\mu S &= (\partial_\mu P)S_0P^{-1} + P(\partial_\mu S_0)P^{-1} + PS_0(\partial_\mu P^{-1}) \\ &= P(P^{-1}\partial_\mu PS_0 + \partial_\mu S_0 + S_0\partial_\mu P^{-1}P)P^{-1}.\end{aligned} \tag{3.14}$$

Let

$$a_\mu = P^{-1}\partial_\mu P. \tag{3.15}$$

Differentiating the identity $P^{-1}P = I$, we obtain $P^{-1}(\partial_\mu P) + (\partial_\mu P^{-1})P = 0$, thus $(\partial_\mu P^{-1})P = -a_\mu$. Equation (3.14) becomes

$$\partial_\mu S = P(\partial_\mu S_0 + [a_\mu, S_0])P^{-1}. \tag{3.16}$$

Now, we can define the covariant derivative

$$D_\mu^{a_\mu} S_0 = \partial_\mu S_0 + [a_\mu, S_0] \tag{3.17}$$

and equation (3.14) can be written as

$$\partial_\mu S = P(D_\mu^{a_\mu} S_0)P^{-1}. \tag{3.18}$$

From

$$\partial_\mu \partial_\nu P = \partial_\nu \partial_\mu P, \tag{3.19}$$

we calculate directly.

$$\partial_\mu a_\nu - \partial_\nu a_\mu + [a_\mu, a_\nu] = 0. \tag{3.20}$$

We call these equations compatibility equations.

In our problem, we want to rotate the matrix S to σ_3 by a unit length quaternion, namely we would like to write

$$S = P\sigma_3 P^{-1}, \tag{3.21}$$

where $|P|^2 = 1$. We write a quaternion in matrix form,

$$P = \begin{pmatrix} p_0 + ip_3 & p_1 + ip_2 \\ -p_1 + ip_2 & p_0 - ip_1 \end{pmatrix} = \begin{pmatrix} \psi & \phi \\ -\bar{\phi} & \bar{\psi} \end{pmatrix}, \tag{3.22}$$

where

$$|P|^2 = |\psi|^2 + |\phi|^2 = 1. \tag{3.23}$$

We can compute the inverse $P^{-1} = \begin{pmatrix} \bar{\psi} & -\phi \\ \bar{\phi} & \psi \end{pmatrix}$. We calculate

$$a_\mu = P^{-1}\partial_\mu P = \begin{pmatrix} \bar{\psi}\partial_\mu\psi + \phi\partial_\mu\bar{\phi} & \bar{\psi}\partial_\mu\phi - \phi\partial_\mu\bar{\psi} \\ -\psi\partial_\mu\bar{\phi} + \bar{\phi}\partial_\mu\psi & \psi\partial_\mu\bar{\psi} + \bar{\phi}\partial_\mu\phi \end{pmatrix}. \tag{3.24}$$

We can observe that on the minor diagonal $(a_\mu)_{21} = -(\bar{a}_\mu)_{12}$, and on the major diagonal that $(a_\mu)_{11} = (\bar{a}_\mu)_{22}$. We differentiate the $|\phi|^2 + |\psi|^2 = 1$ to obtain $(a_\mu)_{11} + (a_\mu)_{22} = 0$. Thus, we know that a_μ is of the following form.

$$\begin{pmatrix} \frac{b_\mu i}{2} & \Phi_\mu \\ -\bar{\Phi}_\mu & -\frac{b_\mu i}{2} \end{pmatrix}, \tag{3.25}$$

where b_μ is real and Φ_μ is complex. From equation (3.20), we have

$$\partial_\mu b_\nu - \partial_\nu b_\mu = \frac{2}{i}(\Phi_\mu\bar{\Phi}_\nu - \Phi_\nu\bar{\Phi}_\mu), \tag{3.26}$$

$$\partial_\mu\Phi_\nu + ib_\mu\Phi_\nu = \partial_\nu\Phi_\mu + ib_\nu\Phi_\mu. \tag{3.27}$$

Now, we are going to discuss the evolution equations (3.9). $\partial_0 S$ corresponds to $D_0^{a_0}\sigma_3$, that is,

$$D_0^{a_o}\sigma_3 = \begin{pmatrix} 0 & -2i\Phi_0 \\ -2i\bar{\Phi}_0 & 0 \end{pmatrix}. \tag{3.28}$$

We denote $\triangle^a = \sum_{j=1}^n D_j^{a_j} D_j^{a_j}$. We can check directly that

$$D_1^{a_1} D_1^{a_1} \sigma_3 = \begin{pmatrix} -4i\Phi_1\bar{\Phi}_1 & -2i\partial_1\Phi_1 + 2b_1\Phi_1 \\ -2i\partial_1\bar{\Phi}_1 - 2b_1\bar{\Phi}_1 & 4i\Phi_1\bar{\Phi}_1 \end{pmatrix}, \tag{3.29}$$

$$[\sigma_3, \triangle^a \sigma_3] = \begin{pmatrix} 0 & \sum_{j=1}^n 4\partial_j\Phi_j + 4ib_j\Phi_j \\ \sum_{j=1}^n 4\partial_j\bar{\Phi}_j + 4ib_j\bar{\Phi}_j & 0 \end{pmatrix}, \tag{3.30}$$

$$[\sigma_3, [\sigma_3, \triangle^a \sigma_3]] = \begin{pmatrix} 0 & \sum_{j=1}^n 8i\partial_j\Phi_j - 8b_j\Phi_j \\ \sum_{j=1}^n 8i\partial_j\bar{\Phi}_j + 8b_j\bar{\Phi}_j & 0 \end{pmatrix}. \tag{3.31}$$

From these, we have an evolution equation, namely,

$$\Phi_0 = i(a - i\epsilon)(\sum_{j=1}^n \partial_j\Phi_j + ib_j\Phi_j). \tag{3.32}$$

For the two-dimensional case, we have the following equations.

$$\Phi_0 = i(a - i\epsilon)(\textstyle\sum_{j=1,2} \partial_j\Phi_j + ib_j\Phi_j), \tag{3.33}$$

$$\begin{cases} \partial_1 b_2 - \partial_2 b_1 = -2i(\Phi_1\bar{\Phi}_2 - \bar{\Phi}_1\Phi_2), \\ \partial_t b_1 - \partial_1 b_0 = -2i(\Phi_0\bar{\Phi}_1 - \bar{\Phi}_0\Phi_1), \\ \partial_t b_2 - \partial_2 b_0 = -2i(\Phi_0\bar{\Phi}_2 - \bar{\Phi}_0\Phi_2), \end{cases} \tag{3.34}$$

$$\begin{cases} (\partial_t + ib_0)\Phi_1 = (\partial_1 + ib_1)\Phi_0, \\ (\partial_t + ib_0)\Phi_2 = (\partial_2 + ib_2)\Phi_0, \\ (\partial_1 + ib_1)\Phi_2 = (\partial_2 + ib_2)\Phi_1. \end{cases} \tag{3.35}$$

Remark 3.2.1. *The system above is under determined. The system of equations is invariant under the gauge transformation*

$$\begin{aligned} b_\mu &\longrightarrow b_\mu + \partial_\mu \theta, \\ \Phi_\mu &\longrightarrow e^{i\theta}\Phi_\mu, \end{aligned} \tag{3.36}$$

where θ is a arbitrary smooth function. In order to solve this system, we need to add an extra constrain, called gauge fixing. For example, the Coulomb gauge imposes the extra restriction $\partial_1 b_1 + \partial_2 b_2 = 0$.

If we consider the one-dimensional case for the Schrödinger map equation, we have the evolution equation

$$\Phi_0 = i(\partial_1\Phi_1 + ib_1\Phi_1), \tag{3.37}$$

and the compatibility equations

$$\partial_1 b_0 - \partial_t b_1 = -2i(\Phi_1\bar{\Phi}_0 - \bar{\Phi}_1\Phi_0), \tag{3.38}$$

$$(\partial_t + ib_0)\Phi_1 = (\partial_1 + ib_1)\Phi_0. \tag{3.39}$$

We use (3.37) to substitute Φ_0 in (3.38) and (3.39). We obtain

$$\partial_1 b_0 - \partial_t b_1 = -2\partial_1(\Phi_1\bar{\Phi}_1), \tag{3.40}$$

$$(\partial_t + ib_0)\Phi_1 = i(\partial_1^2\Phi_1 + i(\partial_1 b_1)\Phi_1 + 2ib_1\partial_1\Phi_1 - b_1^2\Phi_1). \tag{3.41}$$

With $b_1 = 0$, which satisfies the Coulomb gauge condition, we can solve $b_0 = -2|\Phi_1|^2$. We substitute this to (3.41), and we obtain

$$\partial_t\Phi_1 = i(\partial_1^2\Phi_1 + 2|\Phi_1|^2\Phi_1). \tag{3.42}$$

This explains that the one-dimensional Schrödinger map equation is equivalent to a cubic nonlinear Schrödinger equation.

3.3 Complexification

We can simplify the equations if we represent $\mathbb{R}^2$ by a complex plane, $\mathbb{C}$. We let

$$z = x_1 + ix_2. \tag{3.43}$$

Then, we define the complex derivatives

$$\partial_z = \partial_1 - i\partial_2,\ \partial_{\bar{z}} = \partial_1 + i\partial_2. \tag{3.44}$$

We also let

$$b = b_1 - ib_2,\ \bar{b} = b_1 + ib_2,$$

$$\Phi_+ = \Phi_1 + i\Phi_2,\ \Phi_- = \Phi_1 - i\Phi_2.$$

We define

$$D_z^b = \partial_z + ib\ ; D_{\bar{z}}^{\bar{b}} = \partial_{\bar{z}} + i\bar{b}. \tag{3.45}$$

Under these changes, equations (3.33) becomes

$$\Phi_0 = i(a + i\epsilon)D_{\bar{z}}^{\bar{b}}\Phi_- = i(a + i\epsilon)D_z^b\Phi_+. \tag{3.46}$$

From equation (3.35), we have

$$(\partial_t + ib_0)\Phi_+ = D_{\bar{z}}^{\bar{b}}\Phi_0 = i(a + i\epsilon)D_{\bar{z}}^{\bar{b}}D_z^b\Phi_+, \tag{3.47}$$

$$(\partial_t + ib_0)\Phi_- = D_z^b\Phi_0 = i(a + i\epsilon)D_z^bD_{\bar{z}}^{\bar{b}}\Phi_-. \tag{3.48}$$

From equation (3.34), we have

$$\partial_z\bar{b} - \partial_{\bar{z}}b = 2i(\bar{\Phi}_+\Phi_+ - \bar{\Phi}_-\Phi_-), \tag{3.49}$$

$$\partial_t b - \partial_z b_0 = -2i(\Phi_0\bar{\Phi}_+ - \bar{\Phi}_0\Phi_-). \tag{3.50}$$

Subsituting Φ_0 in (3.52) and (3.53), we have

$$\partial_t b - \partial_z b_0 = 2a\partial_{\bar{z}}(\bar{\Phi}_+\Phi_-) + 2i\epsilon((\partial_{\bar{z}}\Phi_-)\bar{\Phi}_+ - \Phi_-(\partial_{\bar{z}}\bar{\Phi}_+)) - 4\epsilon\bar{b}\bar{\Phi}_+\Phi_-. \tag{3.51}$$

If we consider the Schrödinger map, namely $\epsilon = 0, a = 1$, we have the system of equations

$$(\partial_t + ib_0)\Phi_+ = iD_{\bar{z}}^{\bar{b}}D_z^b\Phi_+, \tag{3.52}$$

$$(\partial_t + ib_0)\Phi_- = iD_z^b D_{\bar{z}}^{\bar{b}}\Phi_-, \tag{3.53}$$

$$\partial_z\bar{b} - \partial_{\bar{z}}b = 2i(\bar{\Phi}_+\Phi_+ - \bar{\Phi}_-\Phi_-), \tag{3.54}$$

$$\partial_t b - \partial_z b_0 = 2\partial_{\bar{z}}(\bar{\Phi}_+\Phi_-). \tag{3.55}$$

Note that D_z^b and $D_{\bar{z}}^{\bar{b}}$ do not commute. i.e.

$$[D_z^b, D_{\bar{z}}^{\bar{b}}]\phi = i(\partial_z\bar{b} - \partial_{\bar{z}}b)\phi \neq 0. \tag{3.56}$$

3.4 Gauge fixing

We mentioned that the system of equations is under determined. We can pose an extra condition in order to solve the system. A choice is the Coulomb gauge condition, $\partial_1 b_1 + \partial_2 b_2 = 0$. Under this condition we can find a real potential, p, such that

$$b_1 = \partial_2 p \ ; \ b_2 = -\partial_1 p. \tag{3.57}$$

Then,

$$\partial_z p = \partial_1 p - i\partial_2 p = -ib \ ; \ \partial_{\bar{z}} p = \partial_1 p + i\partial_2 p = i\bar{b}. \tag{3.58}$$

Thus, we can rewrite the covariant derivatives in the following manner

$$D_z^b \phi = e^p \partial_z e^{-p} \phi \ ; \ D_{\bar{z}}^{\bar{b}} \phi = e^{-p} \partial_{\bar{z}} e^p \phi. \tag{3.59}$$

The equations for $\Phi_+, \ \Phi_-$ become

$$\begin{cases} \partial_t \Phi_+ + ib_0 \Phi_+ = ie^{-p} \partial_{\bar{z}}^2 e^p \Phi_-, \\ \partial_t \Phi_- + ib_0 \Phi_- = ie^p \partial_z^2 e^{-p} \Phi_+. \end{cases} \tag{3.60}$$

We differentiate equation (3.55) with respect to $\bar{z}$ and differentiate its complex conjugate equation with respect to z and sum them up to obtain an equation for the temporal gauge

$$\partial_z \partial_{\bar{z}} b_0 = -(\partial_{\bar{z}}^2 (\bar{\Phi}_+ \Phi_-) + \partial_z^2 (\Phi_+ \bar{\Phi}_-)). \tag{3.61}$$

From equation (3.49), we obtain an equation for the potential, p:

$$\triangle p = -(|\Phi_+|^2 - |\Phi_-|^2). \tag{3.62}$$

3.5 Stationary Solutions and Radial Symmetric Solutions

We want to consider the stationary solutions of the gauge system derived from the Schrödinger map equation. Under this setting, we assume P is independent of time. This implies $\Phi_0 = 0$, $b_0 = 0$. If we choose the Coulomb gauge, we have

$$0 = \Phi_0 = e^{-p}\partial_{\bar{z}} e^{p}\Phi_- = e^{p}\partial_z e^{-p}\Phi_+. \tag{3.63}$$

We observe that if $e^p\Phi_-$ is analytic,and if $e^{-p}\Phi_+$ is anti-analytic, they satisfy the equations above. If we choose $\Phi_+ = 0$, the temporal gauge equation is automatically satisfied. We let $\Phi_- = e^{-p}f'(z)$, and substitute this into (3.62). We obtain

$$\triangle p = e^{-2p}|f'|^2. \tag{3.64}$$

We rewrite the equation in complex differentiation:

$$\partial_z\partial_{\bar{z}} p = e^{-2p}|f'|^2. \tag{3.65}$$

We can check that $p = \ln(1 + f\bar{f})$ is a solution of the equation above. Thus, we obtain static solutions. Suppose the solution is radial symmetric, then we can write the evolution equations as follows

$$\partial_t S = \frac{1}{2}[S, (\partial_r^2 + \frac{1}{r}\partial_r)S]. \tag{3.66}$$

From this, we can derive an evolution equation for the gauge system:

$$\Phi_0 = i(\partial\Phi_r + ib_r\Phi_r + \frac{1}{r}\Phi_r). \tag{3.67}$$

We have the compatibility equations

$$\begin{aligned} (\partial_t + ib_0)\Phi_r &= (\partial_r + ib_r)\Phi_0, & (3.68)\\ \partial_t b_r - \partial_1 b_0 &= -2i(\Phi_0\bar{\Phi}_r - \bar{\Phi}_0\Phi_r). & (3.69) \end{aligned}$$

If we substitute Φ_0 by evolution equation (3.67) to the equations above, we have

$$\begin{aligned} (\partial_t + ib_0)\Phi_r &= i(\partial_r^2\Phi_r + \frac{1}{r}\partial_r\Phi_r - \frac{1}{r^2}\Phi_r & (3.70)\\ &\quad +i(\partial_r b_r)\Phi_r + 2ib_r\partial_r + i\frac{b_r}{r}\Phi_r), & (3.71)\\ \partial_t b_r - \partial_r b_0 &= 2\partial_r(\Phi_r\bar{\Phi}_r + \frac{4}{r}|\Phi_r|^2). & (3.72) \end{aligned}$$

$b_r = 0$ satisfies the Coulomb gauge condition. We have

$$(\partial_t + ib_0)\Phi_r = i(\partial_r^2\Phi_r + \frac{1}{r}\partial_r\Phi_r - \frac{1}{r^2}\Phi_r), \tag{3.73}$$

$$-\partial_r b_0 = 2\partial_r(\Phi_r\bar{\Phi}_r) + \frac{4}{r}|\Phi_r|^2). \tag{3.74}$$

We can solve (3.74) and obtain

$$b_0 = -2\Phi_r\bar{\Phi}_r - \int_0^r \frac{4}{s}|\Phi_r|^2 ds. \tag{3.75}$$

By substituting back into (3.73), we obtain a Schrödinger-type equation with inverse square potential and cubic nonlinearity.

$$i\partial_t \Phi_r = -(\partial_r^2 \Phi_r + \frac{1}{r}\partial_r \Phi_r - \frac{1}{r^2}\Phi_r + 2|\Phi_r|^2\Phi_r + \int_0^r \frac{4}{s}|\Phi_r|^2 ds). \tag{3.76}$$

3.6 Equivariant Solutions

We consider the polar coordinate on $\mathbb{R}^2$. We call a map ,$v(r, \theta, t)$, from $\mathbb{R}^2$ to unit sphere $\mathbb{S}^2$ is m-equivarient if

$$v(r, \theta, t) = e^{m\theta R} V(r, t), \tag{3.77}$$

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.78}$$

R generate rotation respect to z axis.

If we spell it out, we have

$$v(r, \theta, t) = \begin{pmatrix} \cos m\theta V_1 - \sin m\theta V_2 \\ \sin m\theta V_1 + \cos m\theta V_2 \\ V_3 \end{pmatrix}. \tag{3.79}$$

If we map it to a 2×2 matrix we defined before, we have

$$S = \begin{pmatrix} iV_3 & e^{im\theta}(V_1 + iV_2) \\ e^{-im\theta}(-V_1 + iV_2) & -iV_3 \end{pmatrix} \tag{3.80}$$

$$= \begin{pmatrix} e^{\frac{im\theta}{2}} & 0 \\ 0 & e^{-\frac{im\theta}{2}} \end{pmatrix} \begin{pmatrix} iV_3 & (V_1 + iV_2) \\ (-V_1 + iV_2) & -iV_3 \end{pmatrix} \begin{pmatrix} e^{\frac{-im\theta}{2}} & 0 \\ 0 & e^{\frac{im\theta}{2}} \end{pmatrix}. \tag{3.81}$$

We rename $M(\theta) = \begin{pmatrix} e^{\frac{im\theta}{2}} & 0 \\ 0 & e^{-\frac{im\theta}{2}} \end{pmatrix}$. If we rotate the matrix σ_3 by P to S as before, we have a choice to break P into two matrices,

$$P = M(\theta)Q(r,t). \tag{3.82}$$

Thus we can compute

$$\begin{aligned} a_0 &= Q^{-1}\partial_t Q, \\ a_1 &= Q^{-1}\sigma_3 Q\tfrac{-\sin\theta}{r} + Q^{-1}\partial_r Q\cos\theta, \\ a_2 &= Q^{-1}\sigma_3 Q\tfrac{\cos\theta}{r} + Q^{-1}\partial_r Q\sin\theta. \end{aligned} \tag{3.83}$$

These equations implie

$$(b_1, b_2) = f(r,t)v_\theta + g(r,t)v_r, \tag{3.84}$$

where $v_\theta = (-\sin\theta, \cos\theta)$, $v_r = (\cos\theta, \sin\theta)$. If we impose Coulomb gauge condition, we have

$$\partial_r g + \frac{1}{r} g = 0. \tag{3.85}$$

If we want solution without singularity, we have $g(r) = 0$. Thus, the gauge field has only the component in θ direction. If we compute Φ_+ and Φ_-, we find

$$\Phi_+ = \hat{\Phi}(r,t)e^{i\theta}, \quad \Phi_- = \hat{\Phi}(r,t)e^{-i\theta}. \tag{3.86}$$

Write (3.49) in term of (b_1, b_2), we have

$$\partial_1 b_2 - \partial_2 b_1 = (|\Phi_+|^2 - |\Phi_-|^2). \tag{3.87}$$

Apply Stoke's theorem on a circle of radius r, we have

$$f(r,t) = \frac{1}{r}\int_0^r (|\Phi^+|^2 - |\Phi^-|^2) s ds. \tag{3.88}$$

If we directly compute $f(r,t)$, we have

$$f(r,t) = \frac{m}{2r}(|p_0^2 + p_3^2 - p_1^2 - p_2^2|). \tag{3.89}$$

Since P is a unit quaternion, $|f(r,t)| \leq \frac{m}{2r}$. We let

$$\rho = \frac{2}{m}\int_0^r |\Phi_+|^2 - |\Phi_-|^2, \tag{3.90}$$

then $f(r) = \frac{m\rho}{2r}$. With this gauge and the fact $\Phi_+,\ \Phi_-$ is of special form in (3.86) , we can calculate

$$
\begin{aligned}
&D_{\bar z}^{\bar b} D_z^b \Phi_+ = \partial_{\bar z}\partial_z \Phi_+ + i\bar b \partial_z \Phi_+ + ib\partial_{\bar z}\Phi_+ - \bar b b \Phi_+ + i(\partial_{\bar z b})\Phi_+ \\
&= \partial_{\bar z}\partial_z \Phi_+ - \tfrac{(m^2\rho^2+4m\rho)}{4r^2}\Phi_+ + i(|\Phi_+|^2 - |\Phi_-|^2)\Phi_+.
\end{aligned}
\tag{3.91}
$$

Similarly

$$
D_z^b D_{\bar z}^{\bar b} \Phi_- = \partial_z \partial_{\bar z}\Phi_- - \frac{(m^2\rho^2 - 4m\rho)}{4r^2}\Phi_- + i(|\Phi_+|^2 - |\Phi_-|^2)\Phi_-. \tag{3.92}
$$

Thus, we have

$$
\begin{cases}
-i\partial_t \Phi_+ = \partial_{\bar z}\partial_z \Phi_+ - \tfrac{(m^2\rho^2+4m\rho)}{4r^2}\Phi_+ + i(|\Phi_+|^2 - |\Phi_-|^2)\Phi_+ - b_0\Phi_+, \\
-i\partial_t \Phi_- = \partial_z\partial_{\bar z} \Phi_- - \tfrac{(m^2\rho^2-4m\rho)}{4r^2}\Phi_- + i(|\Phi_+|^2 - |\Phi_-|^2)\Phi_- - b_0\Phi_-, \\
\rho = \tfrac{2}{m}\int_0^r |\Phi_+|^2 - |\Phi_-|^2, \\
\partial_t \tfrac{im\rho}{2\bar z} - \partial_z a_0 = 2\partial_{\bar z}(\bar\Phi_+ \Phi_-).
\end{cases}
\tag{3.93}
$$

3.7 Equations for blow up solution

We observe equation (3.52) and (3.53) are of very similar form. We consider a solution of the following form.

$$
\Phi_+ = \partial_{\bar z} f(s), \Phi_- = \partial_z f(s), \tag{3.94}
$$

where $s = \frac{z\bar{z}}{t}$. By direct calculation, we have

$$\Phi_+ = f'(s)\frac{z}{t}; \Phi_- = f'(s)\frac{\bar{z}}{t}, \tag{3.95}$$

$$\partial_t \Phi_+ = -f''(s)s\frac{z}{t^2} - f'(s)\frac{z}{t^2}, \tag{3.96}$$

$$\partial_z \Phi_+ = f''(s)\frac{s}{t} + f'(s)\frac{1}{t}, \tag{3.97}$$

$$\partial_{\bar{z}} \Phi_+ = f''(s)\frac{z^2}{t^2}, \tag{3.98}$$

$$\partial_{\bar{z}}\partial_z \Phi_+ = f'''(s)s\frac{z}{t^2} + 2f''(s)\frac{z}{t^2}. \tag{3.99}$$

If solutions are of form (3.94), equation (3.50) becomes

$$\partial_z \bar{b} - \partial_{\bar{z}} b = 0. \tag{3.100}$$

We can choose the trivial solution $b = 0$. Thus, we can solve b_0:

$$b_0 = \frac{1}{t}(-2f'\bar{f}'s - 2\int f'\bar{f}'ds). \tag{3.101}$$

Then, from both (3.52) and (3.53), $f(s)$ satisfies the following ODE:

$$sf''' + (2 - is)f'' - (i + l(s))f' = 0, \tag{3.102}$$

where

$$l(s) = -2f'\bar{f}'s - 2\int f'\bar{f}'ds. \tag{3.103}$$

3.8 $\mathbb{H}^2$ Case

In this section, we will consider the hyperbolic space, $\mathbb{H}^2$, instead of unit sphere, $\mathbb{S}^2$, as target for the Schrödinger map equation. The hyperbolic space, $\mathbb{H}^2$, is defined as follows.

$$\mathbb{H}^2 = \{(x, y, z) | z^2 - x^2 - y^2 = 1\}. \tag{3.104}$$

It has two components. We consider u that maps $\mathbb{R}^n \times \mathbb{R}^+$ to the upper component where $z > 0$, and satisfies the following evolution equation.

$$\begin{cases} u_t = u \dot{\times} \triangle u, \\ u(x, 0) = u_0(x), \end{cases} \tag{3.105}$$

where the pseudo-crossproduct $\dot{\times}$ is defined as follows.

Definition 3.8.1. *If $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, we define*

$$a \dot{\times} b = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, -(a_1 b_2 - a_2 b_1)). \tag{3.106}$$

We map $u = (u_1, u_2, u_3)$ to a 2×2 matrix, S, in the following way.

$$S = i u_1 \sigma_1 + i u_2 \sigma_2 + u_3 \sigma_3. \tag{3.107}$$

We can check directly that this is a homomorphism from $(u, \cdot \dot{\times} \cdot)$ to $(s, \frac{1}{2}[\cdot, \cdot])$. We have the evolution equation

$$S_t = \frac{1}{2}[S, \triangle S]. \tag{3.108}$$

We consider a quaternion $P = p_0 I + ip_1\sigma_1 + ip_2\sigma_2 + p_3\sigma_3$. Then,

$$P = \begin{pmatrix} p_0 + ip_3 & -p_2 - ip_1 \\ -p_2 - ip_1 & p_0 - ip_3 \end{pmatrix} = \begin{pmatrix} \psi & \phi \\ \bar{\phi} & \bar{\psi} \end{pmatrix}. \tag{3.109}$$

We note that

$$\det P = |\psi|^2 - |\phi|^2. \tag{3.110}$$

Suppose $\det P \neq 0$, we have $P^{-1} = (\det P)^{-1} \begin{pmatrix} \bar{\psi} & -\phi \\ -\bar{\phi} & \psi \end{pmatrix}$. If we consider

$$S = P\sigma_3 P^{-1}, \tag{3.111}$$

where $\det P = 1$. Similar to the sphere case, we can define covariant derivative,

$$D_\mu^{a_\mu} S_0 = \partial_\mu S_0 + [a_\mu, S_0], \tag{3.112}$$

where $a_\mu = P^{-1}\partial_\mu P$. In this case, a_μ is of the following form:

$$a_\mu = \begin{Bmatrix} \frac{ib_\mu}{2} & \Phi_\mu \\ \bar{\Phi}_\mu & -\frac{ib_\mu}{2} \end{Bmatrix}. \tag{3.113}$$

With the same argument as before, we obtain

$$\Phi_0 = iD_z^b\Phi_+ = iD_{\bar{z}}^{\bar{b}}\Phi_-, \tag{3.114}$$

$$\begin{cases} \partial_1 b_2 - \partial_2 b_1 = 2i(\Phi_1\bar{\Phi}_2 - \bar{\Phi}_1\Phi_2), \\ \partial_t b_1 - \partial_1 b_0 = 2i(\Phi_0\bar{\Phi}_1 - \bar{\Phi}_0\Phi_1), \\ \partial_t b_2 - \partial_2 b_0 = 2i(\Phi_0\bar{\Phi}_2 - \bar{\Phi}_0\Phi_2) \end{cases} \tag{3.115}$$

$$\begin{cases} (\partial_t + ib_0)\Phi_1 = (\partial_1 + ib_1)\Phi_0, \\ (\partial_t + ib_0)\Phi_2 = (\partial_2 + ib_2)\Phi_0, \\ (\partial_1 + ib_1)\Phi_2 = (\partial_2 + ib_2)\Phi_1. \end{cases} \tag{3.116}$$

Note that equations (3.115) are different from the equations which have unit sphere as target only up to a sign. Using the complex notation as we introduced in section 2.2, we obtain

$$\partial_z\bar{b} - \partial_{\bar{z}}b = -2i(\bar{\Phi}_+\Phi_+ - \bar{\Phi}_-\Phi_-), \tag{3.117}$$

$$\partial_t b - \partial_z b_0 = -2\partial_{\bar{z}}(\bar{\Phi}_+\Phi_-), \tag{3.118}$$

$$(\partial_t + ib_0)\Phi_+ = D_{\bar{z}}^{\bar{b}}\Phi_0 = iD_{\bar{z}}^{\bar{b}}D_z^b\Phi_+, \tag{3.119}$$

$$(\partial_t + ib_0)\Phi_- = D_z^b\Phi_0 = iD_z^bD_{\bar{z}}^{\bar{b}}\Phi_-. \tag{3.120}$$

We consider a solution of the following form

$$\Phi_+ = \partial_{\bar{z}}f(\frac{z\bar{z}}{t}), \Phi_- = \partial_z f(\frac{z\bar{z}}{t}). \tag{3.121}$$

Then, $f(s)$ satisfies the following ODE:

$$sf''' + (2 - is)f'' - (i + l(s))f' = 0, \tag{3.122}$$

$$l(s) = 2f'\bar{f}'s + 2\int f'\bar{f}'ds. \tag{3.123}$$

We rearrange the equation terms in (3.122) to obtain

$$(2f'' - if') + s(f''' - if'' - 2f'\bar{f}'f' - \frac{2}{s}\int f'\bar{f}'dsf') = 0. \tag{3.124}$$

If we collect terms without s and set to be 0, we have

$$2f'' - if' = 0. \tag{3.125}$$

We have a general solution

$$f' = \alpha e^{\frac{i}{2}s}. \tag{3.126}$$

Substituting this into equation (3.123), we obtain $l(s) = 4\alpha^2 s$. Equation (3.122) then becomes

$$s(-\frac{1}{4}\alpha + \frac{1}{2}\alpha - 4|\alpha|^2\alpha) = 0. \tag{3.127}$$

We have $|\alpha| = \frac{1}{4}$. This gives us the infinite energy blow-up solution in the paper by Ding [10].

Remark 3.8.2. *If we try the same approach on the* $\mathbb{S}^2$ *case, we have* $|\alpha|^2 = -\frac{1}{16}$*, which is impossible.*

Because we want to keep the coefficients of the equation real, we let $f'(s) = e^{iws}g(s)$. Then, equation (3.122) becomes

$$(sg'' + (2 - is + 2iws)g' + (2iw + sw - i - sw^2 - l(s))g)e^{iws} = 0. \tag{3.128}$$

If we choose $w = \frac{1}{2}$, we obtain

$$g'' + \frac{2}{s}g' + (\frac{1}{4} - 2g^2 - \frac{2}{s}\int g^2 ds)g = 0. \tag{3.129}$$

Immediately, we have the constant solutions $g = 0, \pm\frac{1}{4}$.

Remark 3.8.3. *For the $\mathbb{S}^2$ case, we have the similar equation*

$$g'' + \frac{2}{s}g' + (\frac{1}{4} + 2|g|^2 + \frac{2}{s}\int |g|^2 ds)g = 0. \tag{3.130}$$

The only constant solution is trivial.

Since we are interested in the blow-up solution with finite energy. We want to find a solution which decays quickly to zero at infinity enough so that the energy is finite. Suppose we have such solutions, we can substitute $s = \frac{z\bar{z}}{b-t}$ and obtain blow up solution at time b.

Chapter 4

Spherical Averaged Endpoint Strichartz Estimates for The Two-dimensional Schrödinger Equations with Inverse Square Potential

4.1 Introduction

Strichartz estimates are crucial in handling local and global well-posedness problems for nonlinear dispersive equations (See[5] [11] [35]). For the Schrödinger equation below

$$\begin{cases} i\partial_t u - \triangle u = 0 \qquad u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{C}, \\ u(x,0) = u_0(x), \end{cases} \tag{4.1}$$

one considers estimates in mixed spacetime Lebesque norms of the type

$$\|u(x,t)\|_{L_t^q L_x^r} = (\int \|u(\cdot,t)\|_{L_x^r}^q dt)^{1/q}. \tag{4.2}$$

Let us define the set of admissible exponents.

Definition 4.1.1. *We say that the exponent pair (q,r,n) is admissible if $q,r \geq 2$, $(q,r,n) \neq (2,\infty,2)$ and they satisfy the relation*

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}\ . \tag{4.3}$$

Under this assumption, the following estimates are known.

Theorem 4.1.2. *If (q, r, n) is admissible, we have the estimates*

$$\|u(x,t)\|_{L_t^q L_x^r} \le \|u_0\|_{L_x^2} \ . \tag{4.4}$$

From a scaling argument, or in other words by dimensional analysis, we can see that the relation (4.3) is necessary for inequality (4.4) to hold.

There is a long line of investigation on this problem. The original work was done by Strichartz (see [34][32] [33]). A more general result was done by Ginibre and Velo(See [14]). For dimension $n \ge 3$, the endpoint cases $(q, r, n) = (2, \frac{2n}{n-2}, n)$ was proved by Keel and Tao [21].

The double endpoint $(q, r, n) = (2, \infty, 2)$ is proved not to be true by Montgomery-Smith (see [26]), even when we replace L_∞ norm with BMO norm. However, it can be recovered in some special setting, for example see Stefanov [30] and Tao [36]. In particular, Tao replaces L_x^∞ by a norm that takes the L^2 average over the angular variables then the L^∞ norm over the radial variable.

In the present work, I want to consider the end point estimates for the Schrödinger equation with inverse square potential,

$$\begin{cases} i\partial_t u - \triangle u + \frac{a^2}{|x|^2} u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{4.5}$$

where $x \in \mathbb{R}^n$, and initial data, $u_0 \in L^2$. For $n \ge 2$ the same Strichartz estimates as in Theorem 1 are proved by Planchon, Stalker, and Tahvidar-Zadeh(see [7]). They did not cover the end point cases for $n = 2$.

We use the same norm as Tao in [36]. We define the L_θ norm as follows.

Definition 4.1.3.

$$\|f\|_{L_\theta}^2 := \frac{1}{2\pi}\int_0^{2\pi} |f(r\cos\theta, r\sin\theta)|^2 d\theta. \tag{4.6}$$

The main result in this paper is the following theorem.

Theorem 4.1.4. *For $x \in \mathbb{R}^2,\ a \geq 0$, suppose $u(x,t)$ satisfies the following homogeneous initial value problem,*

$$\begin{cases} i\partial_t u - \triangle u + \frac{a^2}{|x|^2}u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{4.7}$$

then the following a priori estimate holds

$$\|u\|_{L_t^2(L_r^\infty L_\theta)} \leq C\|u_0\|_{L^2(\mathbb{R}^2)}. \tag{4.8}$$

Remark 4.1.5. *The theorem deals with repulsive potential,i.e. the potential and $-\triangle$ are both non-negative operators. The attractive potential has long been studied in the physics literature and it is known to allow for bounded states in the time independent equation[8]. Thus, we have the oscillatory solution*

$$u(x,t) = e^{iEt}\psi(x). \tag{4.9}$$

Then, $\int \|u(\cdot,t)\|^2_{L^\infty_r L_\theta} dt = \int \|\psi(x)\|_{L^\infty_r L_\theta} = \infty$*, while* $\|\psi\|^2_{L^2}$ *is finite. Therefore , the inequality (4.8) does not hold for an attractive potential.*

Let us consider the equation in polar coordinates. Write $v(r,\theta,t) = u(x,t)$ and $f(r,\theta) = u_0(x)$. We have that $v(r,\theta,t)$ satisfies the equation below,

$$\begin{cases} i\partial_t v - \partial_r^2 v - \frac{1}{r}\partial_r v - \frac{1}{r^2}\partial_\theta^2 v + \frac{a^2}{r^2} v = 0, \\ v(r,\theta,0) = f(r,\theta). \end{cases} \tag{4.10}$$

We write the initial data as a superposition as follows

$$f(r,\theta) = \sum_{k\in\mathbb{Z}} f_k(r) e^{ik\theta}.$$

Using separation of variables, we can write v as a superposition,

$$v(r,\theta,t) = \sum_{-\infty}^{\infty} e^{ik\theta} v_k(r,t),$$

where the radial functions, v_k, satisfy the equations below

$$\begin{cases} i\partial_t v_k - \partial_r^2 v_k - \frac{1}{r}\partial_r v_k + \frac{a^2+k^2}{r^2} v_k = 0, \qquad k\in\mathbb{Z} \\ v_k(r,0) = f_k(r). \end{cases} \tag{4.11}$$

Remark 4.1.6. *Combining Tao's result in [36] with the equation (4.11) above, we can conclude that Theorem 4.1.4 is true in the special cases where* $a \in \mathbb{N}$ *and* u *is radially symmetric. However, the analysis in [36] does not apply to general cases.*

For fixed r, we take the L_θ norm and from the orthogonality of spherical harmonics, we have

$$||v(r,\theta,t)||_{L_\theta}^2 = \sum_{k\in\mathbb{Z}} |v_k(r,t)|^2.$$

We will prove the following lemma.

Lemma 4.1.7. *Suppose v_k satisfies (4.11). For every $k \in \mathbb{Z}$, the following a priori estimate holds.*

$$\int |v_k(r,t)|_{L_r^\infty}^2 dt \leq C \int_0^\infty |f_k(r)|^2 r dr, \tag{4.12}$$

where C is a constant independent of k.

The main theorem follows from the Lemma 4.1.7 above because of the following observation:

$$\begin{aligned} ||v(r,\theta,t)||_{L_t^2 L_r^\infty L_\theta}^2 &= \int \left(\sup_{r>0} \{ (\sum_{k\in\mathbb{Z}} |v_k(r,t)|^2)^{\frac{1}{2}} \} \right)^2 dt \\ &\leq \sum_{k\in\mathbb{Z}} \int \sup_{r>0} |v_k(r,t)|^2 dt \\ \leq \int_0^\infty \sum_{k\in\mathbb{Z}} |f_k(r)|^2 r dr &= \int_0^{2\pi} \int_0^\infty |f(r)|^2 r dr d\theta. \end{aligned}$$

The rest of the paper is devoted to the proof of Lemma 4.1.7.

4.2 Hankel Tranform

The main tools will be the Fourier and Hankel transforms. We want to introduce certain well-known properties of the Hankel transform which are necessary for the proof. We consider the kth mode. Let $\nu(k)^2 = a^2 + k^2$. We define the following elliptic operator

$$A_\nu := -\partial_r^2 - \frac{1}{r}\partial_r + \frac{\nu^2}{r^2}. \tag{4.13}$$

For fixed k, we skip the k in the notation for convenience. Equation (4.11) becomes

$$\begin{cases} i\partial_t v + A_\nu v = 0, \\ v(r,0) = f(r). \end{cases} \tag{4.14}$$

Next, we define the Hankel transform as follows.

$$\phi^{\#}(\xi) := \int_0^\infty J_\nu(r\xi)\phi(r) r dr, \tag{4.15}$$

where J_ν is the Bessel function of real order $\nu > -\frac{1}{2}$ defined via

$$J_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu+1/2)\pi^{1/2}} \int_{-1}^{1} e^{irt}(1-t^2)^{\nu-1/2} dt. \tag{4.16}$$

The following properties of the Hankel transform are well known, (see [7])

Proposition 4.2.1.

$$\begin{aligned} &(i) \quad (\phi^{\#})^{\#} = \phi \\ &(ii) \quad (A_\nu\phi)^{\#}(\xi) = |\xi|^2\phi^{\#}(\xi) \\ &(iii) \quad \int_0^\infty |\phi^{\#}(\xi)|^2 \xi d\xi = \int_0^\infty |\phi(r)|^2 r dr. \end{aligned}$$

If we apply Hankel transform to equation (4.14), we obtain

$$\begin{cases} i\partial_t v^{\#}(\xi,t) + |\xi|^2 v^{\#}(\xi,t) = 0, \\ \quad v^{\#}(\xi,0) = f^{\#}(\xi). \end{cases} \tag{4.17}$$

Solving the ODE and inverting the Hankel transform, we have the formula

$$v(r,t) = \int_0^\infty J_\nu(sr) e^{is^2 t} f^{\#}(s) s ds. \tag{4.18}$$

The change of variables $y = s^2$ implies that

$$v(r,t) = \frac{1}{2}\int_0^\infty J_\nu(r\sqrt{y}) e^{iyt} f^{\#}(\sqrt{y}) dy. \tag{4.19}$$

Let us define the function h as follows

$$h(y) := \begin{cases} f^{\#}(\sqrt{y}) & y > 0, \\ 0 & y \leq 0. \end{cases}$$

Then the expression in (4.19) becomes

$$v(r,t) = \frac{1}{2}\int_{\mathbb{R}} J_\nu(|r||y|^{1/2}) h(y) e^{iyt} dy. \tag{4.20}$$

From Proposition 4.2.1, we have

$$\int_{-\infty}^\infty |h(y)|^2 dy = \frac{1}{2}\int_0^\infty |f^{\#}(s)|^2 s ds = \frac{1}{2}\int_0^\infty |f(\eta)|^2 \eta d\eta < \infty. \tag{4.21}$$

So, h is an L^2 function. We will work with $h(y)$ belonging to the Schwartz class. These are C^∞ functions that tend to zero faster than any polynomial at infinity, i.e.,

$$S(\mathbb{R}) = \{f(x) \in C^\infty(x) | \sup_{x\in\mathbb{R}} |\frac{d^\alpha f(x)}{d^\alpha x}| < C^{\alpha\beta} x^{-\beta} \; \forall \; \alpha, \beta \in \; \mathbb{N}\}.$$

The general case of $h \in L_2$ follows by a density argument. We use smooth cut off functions to partition the Bessel function J_ν as follows,

$$J_\nu(\eta) = m_\nu^0(\eta) + m_\nu^1(\eta) + \sum_{j \gg \log \nu} m_\nu^j(\eta), \tag{4.22}$$

where m_ν^0, m_ν^1 and m_ν^j are supported on $\eta < \frac{\nu}{\sqrt{2}}$, $\eta \sim \nu$ and $\eta \sim 2^j$ for $j \gg \log_\nu$ respectively. Let $J_\nu^k = \sum_0^k m_\nu^j$. Equation (4.20) holds in the sense that we can write

$$v(r,t) = \lim_{k\to\infty} \frac{1}{2} \int_{\mathbb{R}} J_\nu^k(|r||y|^{1/2}) h(y) e^{iyt} dy. \tag{4.23}$$

Substituting h by the inverse Fourier formula

$$h(y) = \int_{\mathbb{R}} e^{i(\eta - t)y} \hat{h}(\eta - t) d\eta,$$

and changing the order of integration, we have

$$v(r,t) = \lim_{k\to\infty} \int_{\mathbb{R}} \left(\frac{1}{2} \int_{\mathbb{R}} J_\nu^k(\sqrt{|y|}|r|) e^{i\eta y} dy \right) \hat{h}_k(\eta - t) d\eta. \tag{4.24}$$

Let us define the kernel below

$$K^j_{\nu,r}(\eta) = \frac{1}{2}\int_{\mathbb{R}} m^j_\nu(\sqrt{|y|}|r|)e^{i\eta y}dy. \tag{4.25}$$

For convenience, rename $g(y) = \hat{h}(-y)$ and define an operator

$$T^j_{\nu,r}[g](t) = (K^j_{\nu,r} * g)(t). \tag{4.26}$$

Since it is a convolution, it becomes a multiplication in Fourier space. Thus, this operator has another equivalent expression, namely,

$$T^j_{\nu,r}[g](t) = \frac{1}{\sqrt{2\pi}}\int m^j_\nu(r|\xi|^{\frac{1}{2}})\widehat{g}(\xi)e^{i\xi t}d\xi. \tag{4.27}$$

Notice that both the kernel $K^j_{\nu,r}$ and the operator $T^j_{\nu,r}$ are functions of ν. We can rewrite equation (4.24) in the following form:

$$v(r,t) = \lim_{k\to\infty}\sum_{j\le k} T^j_{\nu,r}[g(\eta)](t). \tag{4.28}$$

The main theorem in this paper will follow from the lemma below.

Lemma 4.2.2. *For $g \in L^2$, $a \ge 0$, C_1, C_2, C_3 independent of $\nu^2(k) = a^2 + k^2$ $k \in \mathbb{N}$, the following estimates hold:*

$$\int_{\mathbb{R}} \sup_{r>0} |T^0_{\nu,r}(g)(t)|^2 dt \le C_1 \int_{\mathbb{R}} |g(y)|^2 dy, \tag{4.29}$$

$$\int_{\mathbb{R}} \sup_{r>0} |T^1_{\nu,r}(g)(t)|^2 dt \leq C_2 \int_{\mathbb{R}} |g(y)|^2 dy, \tag{4.30}$$

$$\int_{\mathbb{R}} \sup_{r>0} |T^j_{\nu,r}(g)(t)|^2 dt \leq C_3 2^{-\frac{1}{2}j} \int_{\mathbb{R}} |g(y)|^2 dy \ for \ j \gg \log \nu. \tag{4.31}$$

Notice $\|v\|^2_{L^2_t(L^\infty_r L_\theta)}$ can be bounded by the sum of the left hand side terms in Lemma 4.2.2 and the right hand side terms are summable. Thus, Lemma 4.1.7 follows.

We will refer to these three cases as low frequency, middle frequency, and high frequency, respectively. We will prove inequalities (4.29), (4.30), and (4.31) in the following sections.

4.3 Estimates for Low Frequency

Our strategy is to estimate the kernel defined in (4.25) and apply Hardy-Littlewood maximal inequality in this case. By changing variable $z := r^2 y$ in (4.25), we can write

$$K^0_{\nu,r}(\eta) = \frac{1}{r^2} K^0_{\nu,1}(\frac{\eta}{r^2}), \tag{4.32}$$

and therefore we have

$$\|K^0_{\nu,r}(\eta)\|_{L_1} = \|K^0_{\nu,1}(\eta)\|_{L_1}. \tag{4.33}$$

We will prove the following estimate.

Lemma 4.3.1. *The kernel $K^0_{\nu,1}(\eta)$ is bounded as follows,*

$$|K^0_{\nu,1}(\eta)| \leq \Phi^0_\nu(\eta), \tag{4.34}$$

where Φ_ν^0 is an even nonnegative decaying L^1 function defined as follows.

$$\Phi_\nu^0 = \begin{cases} c(1+|\eta|)^{-(1+\nu/2)} & \textit{when } 0 < \nu \leq 2, \\ C(\nu)(1+|\eta|)^{-2} & \textit{when } 2 < \nu, \end{cases} \tag{4.35}$$

where $C(\nu)$ is uniformly bounded.

We can see $\|\Phi_\nu^0\|_{L^1}$ is finite for every ν from a direct calculation. Since $C(\nu)$ is uniformly bounded, $\|\Phi_\nu^0\|_{L^1}$ is uniformly bounded when $\nu > 2$. For $0 < \nu \leq 2$, $\|\Phi_\nu^0\|_{L^1} = 4/\nu$. However, since $\nu(k)^2 = a^2 + k^2$ are discrete, we can find a universal L_1 bound for given $a \neq 0$. Since Φ_ν^0 is an even nonnegative decaying function, we can use the property of approximate identity and obtain

$$\sup_{r>0} T_{\nu,r}^0[g](t) \leq \|\Phi_\nu^0\|_{L^1} M[g](t), \tag{4.36}$$

where $M[g](t)$ is Hardy-Littlewood maximal function of g at t, defined as follows:

$$M(g)(t) = \sup_{r>0} \frac{1}{|I(t,r)|} \int_{I(t,r)} |g| dx, \tag{4.37}$$

where $I(t,r) = (t-r, t+r)$. Finally, we apply the Hardy-Littlewood maximal inequality

$$\|M(F)\|_{L_p} \leq C(p)\|F\|_{L_p} \quad 1 < p < \infty \tag{4.38}$$

to finish the proof.

We will prove Lemma (4.3.1) case by case as presented in (4.35).

Proof. We need to prove $K^0_{\nu,1}(\eta)$ is bounded and decays with the the power advertised in (4.35). We first prove the decay of the tail.

Because m^0_ν is even, we have

$$K^0_{\nu,1}(\eta) = \frac{1}{2}\int_{\mathbb{R}} m^0_\nu(\sqrt{|y|})e^{i\eta y}dy = \int_0^\infty m^0_\nu(\sqrt{y})\cos(\eta y)dy. \tag{4.39}$$

Integrate by parts to obtain

$$K^0_{\nu,1}(\eta) = -\frac{1}{2\eta}\int_0^\infty {m^0_\nu}'(y^{1/2})y^{-1/2}\sin(\eta y)dy. \tag{4.40}$$

Differentiating the expression of the Bessel function in (4.16), we can find the following recursive relation for Bessel functions:

$$J'_\nu(r) = \nu r^{-1}J_\nu(r) - J_{\nu+1}(r). \tag{4.41}$$

From the definition of Bessel function (4.16), we can see

$$J_\nu(r) \sim \frac{1}{\Gamma(\nu+1)}(\frac{r}{2})^\nu \ if \ r < \sqrt{\nu+1}. \tag{4.42}$$

Moreover, for all r the following upper bound is true

$$J_\nu(r) \leq \frac{c}{\Gamma(\nu+1)}(\frac{r}{2})^\nu. \tag{4.43}$$

Combining (4.41) and (4.42), the integrant in (4.40) behaves like $\sim \nu y^{\frac{\nu}{2}-1}$, when

$y \ll 1 + \nu$.

We will examine various cases of the parameter ν.

- Case 1: $0 < \nu \leq 2$

We break the integral into two parts, from 0 to $|\eta|^{-\alpha}$ and the rest and integrate by parts the latter, i.e., we write $K^0_{\nu,1}(\eta) = I_1 + B_2 + I_2$, where

$$\begin{aligned}
I_1 &= -\frac{1}{2\eta}\int_0^{|\eta|^{-\alpha}} m_\nu^{0\prime}(y^{1/2})y^{-1/2}\sin(\eta y)dy,\\
B_2 &= -\frac{1}{2\eta^2}\cos(\eta|\eta|^{-\alpha})m_\nu^{0\prime}(|\eta|^{-\alpha/2})|\eta|^{\alpha/2},\\
I_2 &= -\frac{1}{4\eta^2}\int_{|\eta|^{-\alpha}}^{\infty} m_\nu^{0\prime\prime}(y^{1/2})y^{-1}\cos(\eta y) - m_\nu^{0\prime}(y^{1/2})y^{-3/2}\cos(\eta y)dy,
\end{aligned}$$

where α is a parameter to be determined later.

Estimate I_1 using the equation (4.42), we have $|I_1| \sim |\eta|^{-1-\alpha\frac{\nu}{2}}$. Taking the absolute value, we have $|B_2| \sim \nu|\eta|^{-\alpha\frac{\nu}{2}+\alpha-2}$. For I_2, we use the fact that Bessel function is the solution of the following differential equation

$$J''_\nu(r) + \frac{1}{r}J'_\nu(r) + (1 - \frac{\nu^2}{r^2})J_\nu(r) = 0. \tag{4.44}$$

Combining (4.44) with the identity (4.41), we have

$$J''_\nu(r) = \frac{1}{r}J_{\nu+1}(r) - (1 + \frac{\nu}{r^2} - \frac{\nu^2}{r^2})J_\nu(r). \tag{4.45}$$

Using (4.43), we can estimate the integrant in I_2 by $c\nu(\nu-2)y^{\frac{\nu}{2}-2}$. Thus, we have $|I_2| \leq c\nu|\eta|^{-\alpha\frac{\nu}{2}+\alpha-2}$. To balance the contribution from I_1, B_2, and I_2, we choose

$\alpha = 1$. Thus, we have $K^0_{\nu,1}(\eta) < c\nu^{-(1+\frac{\nu}{2})}$.

- Case 2: $\nu < 2$

We do not split the integral in this case. We can integrate by parts twice without introducing boundary terms and obtain

$$K^0_1(\eta) = -\frac{1}{4\eta^2}\int_0^\infty \Big(m^{0\prime\prime}_\nu(y^{1/2})y^{-1}\cos(\eta y) - m^{0\prime}_\nu(y^{1/2})y^{-3/2}\cos(\eta y)\Big)\,dy.$$

Since m^0_ν is supported within $[0, \nu/\sqrt{2})$, the integral is bounded by η^{-2} multiplied by a constant namely $C(\nu) = c(\nu-2)(\Gamma(\nu+1))^{-1}2^{\frac{-3\nu}{2}}P(\nu)$, where P is a polynomial with finite degree. Using Stirling's formula

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z\left(1 + O(\frac{1}{z})\right), \tag{4.46}$$

and observing that $e < 2^{3/2}$, we can see that $C(\nu)$ has a bound independent of ν.

Now, we took care of the tail. The remaining task is to prove that $K^0_{\nu,1}(\eta)$ is bounded. We take absolute value of the integrant in (4.40)

$$|K^0_{\nu,1}(\eta)| \le \int |m^0_\nu(\sqrt{|y|})|dy. \tag{4.47}$$

Since m^0_ν is a bounded function with a compact support, we proved $K^1_{\nu,1}(\eta)$ is bounded for fixed ν. Furthermore if we apply (4.43), we have

$$|K^0_{\nu,1}(\eta)| \le c\frac{\nu^3\nu^\nu}{\Gamma(\nu+1)2^{\frac{3\nu}{2}}}. \tag{4.48}$$

Using Stirling's formula (4.46) again, we can show that there is a bound independent of ν. $\square$

4.4 Estimates for Middle Frequency

The goal is to prove inequality (4.30), namely,

$$\|T^1_{\nu,r}(g)(t)\|_{L^2_t L^\infty_r} \leq C\|g\|_{L^2}.$$

First, we want to estimate the L^∞_r norm for fixed t. Recall the equation (4.27), we have

$$T^1_{\nu,r}(g)(t) = \frac{1}{\sqrt{2\pi}} \int m^1_\nu(r|\xi|^{\frac{1}{2}})\widehat{g}(\xi)e^{i\xi t}d\xi. \tag{4.49}$$

Since composing Fourier transform with inverse Fourier transform will form identity map , we have

$$T^1_{\nu,r_0}(g)(t) = \frac{1}{2\pi^{\frac{3}{2}}} \int\int\int m^1_\nu(r|\xi|^{\frac{1}{2}})\widehat{g}(\xi)e^{i\xi t}d\xi \; e^{ir\rho}dr \; e^{-i\rho r_0}d\rho. \tag{4.50}$$

Using smooth dyadic decomposition, we write $\widehat{g}(\xi) = \sum_{-\infty}^{\infty} \widehat{g_n}(\xi)$ where $\widehat{g_n}$ is supported on $(-2^{n+1}, -2^{n-1})\bigcup(2^{n-1}, 2^{n+1})$. We will prove the following lemma.

Lemma 4.4.1. *For $g_n \in L^2(\mathbb{R})$ such that $\widehat{g_n}$ is supported on $(-2^{n+1}, -2^{n-1})\bigcup(2^{n-1}, 2^{n+1})$, we have the estimate*

$$\|T^1_r(g_n)(t)\|_{L^2_t L^\infty_r} \leq C\|g_n\|_{L^2}, \tag{4.51}$$

where C is independent of n.

Proof. On the right hand side of (4.50), we multiply and divide by $\sqrt{b+\rho^2 b^{-1}}$, where $b>0$ is a parameter to be chosen later. We change the order of integration, and apply Hölder's inequality to obtain

$$|T^1_{r_0}(g_n)(t)| \le C\left(\int \frac{|e^{-i\rho r_0(t)}|^2}{(b+\rho^2 b^{-1})}d\rho\right)^{\frac{1}{2}} \cdot \tag{4.52}$$
$$\left(\int\left|\int\int m^1_\nu(r|\xi|^{\frac{1}{2}})\widehat{g_n}(\xi)e^{i\xi t}e^{ir\rho}d\xi dr\right|^2(b+\rho^2 b^{-1})d\rho\right)^{\frac{1}{2}}.$$

Note that the first integral on the right hand side is π for any $b>0$. Thus, equation (4.52) reduces to

$$\|T^1_{r_0}(g_n)(t)\|_{L^\infty_{r_0}} \le \tag{4.53}$$
$$C\quad\left(\int\left|\int\int m^1_\nu(r|\xi|^{\frac{1}{2}})\widehat{g_n}(\xi)e^{i\xi t}d\xi e^{ir\rho}dr\right|^2(b+\rho^2 b^{-1})d\rho\right)^{\frac{1}{2}}.$$

We name the integral on the right hand side of (4.53) as $l(t)$. We distribute the sum $(b+b^{-1}\rho^2)$ and write $l^2(t)=l_1(t)+l_2(t)$, where

$$l_1(t)=\int\left|\int\int b^{\frac{1}{2}}m^1_\nu(r|\xi|^{\frac{1}{2}})\widehat{g_n}(\xi)e^{i\xi t}d\xi e^{ir\rho}dr\right|^2 d\rho,$$
$$l_2(t)=\int\left|\int\int b^{-\frac{1}{2}}\rho m^1_\nu(r|\xi|^{\frac{1}{2}})\widehat{g_n}(\xi)e^{i\xi t}d\xi e^{ir\rho}dr\right|^2 d\rho.$$

For $l_2(t)$, we integrate by parts with respect to r to remove ρ and obtain,

$$l_2(t)=\int\left|\int\int b^{-\frac{1}{2}}|\xi|^{\frac{1}{2}}(m^1_\nu)'(r|\xi|^{\frac{1}{2}})\widehat{g_n}(\xi)e^{i\xi t}d\xi e^{ir\rho}dr\right|^2 d\rho.$$

Using Plancherel's theorem, we have

$$l_1(t) = \int \left| \int b^{\frac{1}{2}} m_\nu^1(r|\xi|^{\frac{1}{2}}) \widehat{g_n}(\xi) e^{i\xi t} d\xi \right|^2 dr, \tag{4.54}$$

$$l_2(t) = \int \left| \int b^{-\frac{1}{2}} |\xi|^{\frac{1}{2}} (m_\nu^1)'(r|\xi|^{\frac{1}{2}}) \widehat{g_n}(\xi) e^{i\xi t} d\xi \right|^2 dr. \tag{4.55}$$

We square both sides of (4.53) and integrate over t. Then, we change the order of integration with respect to $r,\ t$, and apply Plancherel's theorem again to obtain

$$\begin{aligned} \|T_{r_0}^1(g_n)(t)\|^2_{L_t^2 L_{r_0}^\infty} \leq C \int \int \left| b^{\frac{1}{2}} m_\nu^1(r|\xi|^{\frac{1}{2}}) \widehat{g_n}(\xi) \right|^2 d\xi dr \\ + C \int \int \left| b^{-\frac{1}{2}} |\xi|^{\frac{1}{2}} m_\nu^1(r|\xi|^{\frac{1}{2}}) \widehat{g_n}(\xi) \right|^2 d\xi dr. \end{aligned} \tag{4.56}$$

We change the variable by $y = r|\xi|^{\frac{1}{2}}$. We have

$$\|T_{r_0}^1(g_n)(t)\|^2_{L_t^2 L_{r_0}^\infty} \leq C \int \left(\int \frac{b}{|\xi|^{\frac{1}{2}}} |m_\nu^1(y)|^2 + \frac{|\xi|^{\frac{1}{2}}}{b} |m_1'(y)|^2 dy \right) |\widehat{g_n}(\xi)|^2 d\xi. \tag{4.57}$$

Use Lemma (4.6.1) (see appendix) which implies

$$\int |m_\nu^1(y)|^2 dy < C, \quad | \int (m_\nu^1)'(y)|^2 dy < C. \tag{4.58}$$

Recall that the $\widehat{g_n}$ is supported on $(-2^{n+1}, -2^{n-1}) \bigcup (2^{n-1}, 2^{n+1})$. By choosing $b = 2^{\frac{n}{2}}$, we complete the proof. $\square$

We proved (4.30) for function that has bounded support in the Fourier space described above. Now we are going to discuss the general case without restriction in Fourier space.

Proof. (4.30) Suppose $r_0(t)$ realizes at least half of the supremun at every t for the function g, i.e.,

$$\frac{1}{2}\sup_{r>0}|T^1_{\nu,r}(g)(t)| \leq |T^1_{\nu,r_0(t)}(g)(t)|. \tag{4.59}$$

Then, it is enough to prove the inequality

$$\int |T^1_{\nu,r_0(t)}(g)(t)|^2 dt \leq C\|g\|^2_{L_2}. \tag{4.60}$$

We will prove (4.60) for an arbitrary function $r_0(t)$. We dyadically decompose the range of $r_0(t)$. The corresponding domains are defined as follows:

$$I_k = \{t|\ 2^k < r_0(t) \leq 2^{k+1}\}. \tag{4.61}$$

Since m^1_ν is supported on $(\nu/2, 2\nu)$, we have

$$\frac{\nu}{2} < r_0|\xi|^{\frac{1}{2}} < 2\nu. \tag{4.62}$$

On I_k, by definition we have $2^k < r_0(t) \leq 2^{k+1}$. Combining (4.61) and (4.62), the integrant in expression (4.50) is nonzero only when

$$2\log_2\nu - 2k - 4 < \log_2|\xi| < 2\log_2\nu - 2k + 2. \tag{4.63}$$

As a result, there are only 8 components in the dyadic decomposition in $\{\widehat{g}_n\}$ involved. When $t \in I_k$, we can rewrite (4.50) as

$$T^1_{\nu,r_0(t)}(g)(t) = \frac{1}{\sqrt{2\pi}} \int m^1_\nu(r_0|\xi|^{\frac{1}{2}}) \sum_{n=n_0(k)}^{n_0+7} \widehat{g_n}(\xi) e^{i\xi t} d\xi = \sum_{n=n_0(k)}^{n_0+7} T^1_{\nu,r_0(t)}(g_n)(t), \quad (4.64)$$

where $n_0(k) = \lfloor 2\log_2 \nu - 2k - 4 \rfloor$. Thus, use the Cauchy-Schwartz inequality in the finite sum to obtain

$$|T^1_{\nu,r_0(t)}(g)(t)|^2 = \left| \sum_{n=n_0(k)}^{n_0+7} T^1_{\nu,r_0(t)}(g_n)(t) \right|^2 \leq 8 \sum_{n=n_0(k)}^{n_0+7} \left| T^1_{\nu,r_0(t)}(g_n)(t) \right|^2. \quad (4.65)$$

Combine the above with the Lemma (4.4.1), we have

$$\int_{I_k} |T^1_{\nu,r_0(t)}(g)(t)|^2 dt \leq C \sum_{n=n_0(k)}^{n_0+7} \|g_n\|^2_{L_2}. \quad (4.66)$$

We sum over k to obtain

$$\int |T^1_{\nu,r_0(t)}(g)(t)|^2 dt \leq C \sum_{k\in\mathbb{Z}} \sum_{n=n_0(k)}^{n_0+7} \|g_n\|^2_{L_2}. \quad (4.67)$$

Note that when we increase from k to $k+1$, n_0 increases by 2. As a result, every n only appears four times. Thus

$$\int |T^1_{\nu,r_0(t)}(g)(t)|^2 dt \leq 4C \sum_{n\in\mathbb{Z}} \|g_n\|^2_{L_2} \leq 4C\|g\|^2_2. \quad (4.68)$$

This completes the proof. $\square$

4.5 Estimates for High Frequency

The goal is to prove (4.31), which is equivalent to

$$\left\| \int_{\mathbb{R}} K^j_{\nu,r(t)}(t-\eta)g(\eta)d\eta \right\|_{L^2_t} \leq C2^{-\frac{1}{4}j}\|g(y)\|_{L^2}, \tag{4.69}$$

for an arbitrary function $r(t)$. Using the T^*T argument, we have the following lemma.

Lemma 4.5.1. *The following three inequalities are equivalent.*

$$\left\| \int_{\mathbb{R}} K^j_{\nu,r(t)}(t-\eta)g(\eta)d\eta \right\|_{L^2_t} \leq C2^{-\frac{1}{4}j}\|g(y)\|_{L^2}, \quad \forall g \in L^2(\mathbb{R}^1), \tag{4.70}$$

$$\left\| \int_{\mathbb{R}} K^j_{\nu,r(t)}(t-\eta)F(t)dt \right\|_{L^2} \leq C2^{-\frac{1}{4}j}\|F\|_{L^2} \quad \forall F \in L^2(\mathbb{R}^1), \tag{4.71}$$

$$\begin{aligned} &\left\| \int_{\mathbb{R}} \int_{\mathbb{R}} K^j_{\nu,r(t)}(t-\eta)\overline{K^j_{\nu,r(t')}(t'-\eta)}d\eta F(t')dt' \right\|_{L^2} \\ &\qquad \leq C2^{-\frac{1}{2}j}||F||_{L^2}, \ \forall F \in L^2(\mathbb{R}^1). \end{aligned} \tag{4.72}$$

Proof. Suppose we have (4.70). We want to show it implies (4.71). We multiply the integrand on the left hand side of (4.70) with an arbitrary L^2 function $F(t)$, then integrate over t, η. We apply Hölder's inequality and (4.70) to obtain

$$\int \int_{\mathbb{R}} K^j_{\nu,r(t)}(t-\eta)F(t)dtg(\eta)d\eta \leq Ce^{-\frac{1}{4}j}\|g\|_{L^2}\|F\|_{L^2}. \tag{4.73}$$

Using the property that L^2 is self-dual, i.e.

$$\|h\|_{L^2} = \sup_{f\in L^2} \frac{\int f(t)h(t)dt}{\|f\|_{L^2}}, \tag{4.74}$$

we obtain (4.71). Using the same argument again, we can prove (4.70) $\Longleftrightarrow$ (4.71). Suppose we have (4.71). We will show that (4.72) holds . We multiply the integrant on the left hand side of (4.72) with an arbitrary L^2 function $G(t)$ and integrate over η, t', and t. We change the order of integration and apply Holder's inequality and (4.71) to obtain

$$\begin{aligned}
&\left|\int\int\int K^j_{\nu,r(t)}(t-\eta)\overline{K^j_{\nu,r(t')}(t'-\eta)}d\eta F(t')dt'G(t)dt\right| \\
\leq &\left\|\int K^j_{\nu,r(t)}(t-\eta)G(t)dt\right\|_{L^2}\left\|\int \overline{K^j_{\nu,r(t')}(t'-\eta)}F(t')dt'\right\|_{L^2} \\
&\leq Ce^{-\frac{1}{2}j}\|G\|_{L^2}\|F\|_{L^2},
\end{aligned}$$

which implies (4.72) by duality.

Suppose (4.72) holds. We multiply the integrand on the left hand side of (4.72) with the complex conjugate of $F(t)$, $\overline{F(t)}$, integrate over η, t', and t, apply Hölder's inequality and (4.72), we obtain (4.71). This completes the proof. □

Thus, to prove (4.69), we have to prove (4.72). Inequality (4.72) will follow from the following estimate.

Lemma 4.5.2. *For any $a,b>0$, t, $t'\in\mathbb{R}$ we have*

$$\left|\int K^j_a(t-\eta)\overline{K^j_b(t'-\eta)}d\eta\right| < \frac{1}{a^2}\Phi_j(\frac{|t-t'|}{a^2}), \tag{4.75}$$

where Φ_j are even non-increasing non-negative functions with

$$\|\Phi_j\|_{L^1} = \|\frac{1}{a^2}\Phi_j(\frac{y}{a^2})\|_{L^1} \leq C2^{-\frac{1}{2}j}. \tag{4.76}$$

Estimate (4.75) does not depend on b. Φ_j is even, non-increasing, and non-negative. Thus, we have

$$\begin{aligned}\left|\int_{\mathbb{R}}\int_{\mathbb{R}} K^j_{r(t)}(t-\eta)\overline{K^j_{r(t')}(t'-\eta)}d\eta F(t')dt'\right| &\leq \int_{\mathbb{R}} \frac{1}{r(t)^2}\Phi_j(\frac{|t-t'|}{r(t)^2})|F(t')|dt' \\ \leq \sup_{r>0}\int_{\mathbb{R}} \frac{1}{r^2}\Phi_j(\frac{|t-t'|}{r^2})|F(t')|dt' &\leq \|\Phi_j\|_{L^1} M(F)(t) \leq C2^{-\frac{j}{2}}M(F)(t),\end{aligned}$$

where $M(F)$ is the Hardy-Littlewood maximal function of F. We apply the Hardy-Littlewood maximal inequality (4.38) to finish the proof of (4.31).

Proof. (Lemma 4.5.2) The kernel is in the form of the inverse Fourier transform

$$K^j_{\nu,r}(\eta) = \int_{\mathbb{R}} m^j_\nu(r|y|^{1/2})e^{i\eta y}dy. \tag{4.77}$$

By Plancherel's theorem, we have

$$\int K^j_{\nu,a}(t-\eta)\overline{K^j_{\nu,b}(t'-\eta)}d\eta = \int m^j_\nu(a|y|^{1/2})\overline{m^j_\nu(b|y|^{1/2})}e^{i(t-t')y}dy. \tag{4.78}$$

We introduce a standard asymptotic of Bessel functions for large r (see [37])

$$J_\nu(r) \sim (\frac{\pi r}{2})^{-\frac{1}{2}} \cos(r - \frac{\nu\pi}{2} - \frac{\pi}{4}) \sum_{j=0}^{\infty} a_j r^{-2j} \tag{4.79}$$

$$+(\frac{\pi r}{2})^{-\frac{1}{2}} \sin(r - \frac{\nu\pi}{2} - \frac{\pi}{4}) \sum_{j=0}^{\infty} b_j r^{-2j}, \tag{4.80}$$

where $a_j = (-1)^j(\nu, 2j)2^{-2j}$ and $b_j = (-1)^j(\nu, 2j+1)2^{-2j-1}$ with

$$(\nu, k) = \frac{\Gamma(\frac{1}{2} + \nu + k)}{k! \cdot \Gamma(\frac{1}{2} + \nu - k)}. \tag{4.81}$$

With this asymptotic, we can write

$$m_\nu^j(\xi) = \sum_{\pm} 2^{-j/2} e^{\pm i\xi} \psi_j^\pm(2^{-j}\xi). \tag{4.82}$$

where $\psi_j^\pm(\xi)$ are function supported on $|\xi| \sim 1$ and bounded uniformly in j, ν . We can rewrite the right hand side of (4.77) as a finite number of expressions of the form

$$2^{-j} \left| \int e^{i(\pm a \pm b)|y|^{1/2}} e^{i(t-t')y} \psi_j^\pm(2^{-j}a|y|^{1/2}) \psi_j^\pm(2^{-j}b|y|^{1/2}) dy \right|, \tag{4.83}$$

where the $\pm$ signs need not agree. Since the bump functions are supported on $(\frac{1}{2}, 2)$, this expression is identically zero except when $\frac{1}{4} < \frac{b}{a} < 4$. Let $\alpha = \frac{b}{a}$. The expression in (4.83) becomes

$$2^{-j}\left|\int e^{i(\pm 1\pm\alpha)a|y|^{1/2}}e^{i(t-t')y}\psi_j^{\pm}(2^{-j}a|y|^{1/2})\psi_j^{\pm}(2^{-j}\alpha a|y|^{1/2})dy\right|. \tag{4.84}$$

Let $s = (-t'$. We denote the sum of this expression $\phi^j_{a,\alpha}(s)$. By changing variable to $y' = a^2 y$, we can see that

$$\phi^j_{a,\alpha}(s) = \frac{1}{a^2}\phi^j_{1,\alpha}\left(\frac{s}{a^2}\right). \tag{4.85}$$

By letting $s' = \frac{s}{a^2}$, we have

$$\|\phi^j_{a,\alpha}(s)\|_{L^1} = \|\phi^j_{1,\alpha}(s)\|_{L^1}. \tag{4.86}$$

Now, $\phi^j_{1,\alpha}(s)$ is finite sum of the following expressions

$$2^{-j}\left|\int e^{i(\pm 1\pm\alpha)|y'|^{1/2}}e^{i(s)y'}\psi_j^{\pm}(2^{-j}|y'|^{1/2})\psi_j^{\pm}(2^{-j}\alpha|y'|^{1/2})dy'\right|. \tag{4.87}$$

By changing the variable to $z = 2^{-j}|y'|^{1/2}$, the expression in (4.87) becomes

$$2^{j+2}\left|\int_0^{\infty} e^{i(\pm 1\pm\alpha)2^j z+s2^{2j}z^2}\psi_j^{\pm}(z)\psi_j^{\pm}(\alpha z)zdz\right|. \tag{4.88}$$

We will prove the following lemma.

Lemma 4.5.3. *$\phi^j_{1,\alpha}$ is controlled by the following function*

$$\Phi_j(s) = C \begin{cases} 2^j & \text{when } 0 < s \le 2^{-2j}, \\ s^{-1/2} & \text{when } 2^{-2j} < s < 40\ 2^{-j}, \\ 2^j(2^{2j}s)^{-10} & \text{otherwise.} \end{cases} \tag{4.89}$$

Notice that we can estimate directly and get $\|\Phi_j\|_{L_1} \le C2^{-\frac{j}{2}}$. □

The remaining task is to prove the lemma (4.5.3).

Proof. (Lemma 4.5.3) Take the absolute value of the integrand to obtain the bound 2^j. We will use stationary phase technique to prove the other two estimates. We call the function on the index of exponential phase. If we differentiate the phase, we get $2\ 2^{2j}sz + (\pm 1 \pm \alpha)2^j$. Note α is between 1/4 and 4, and the product of bump functions is supported on $(1/8, 8)$. Suppose $s > 20\ 2^{-j}$, the derivative is never zero on the support. We get the bound $2^j(2^{2j}s)^{-10}$ from non-stationary phase analysis. Otherwise, we note the second derivative of the phase, namely $2^{2j+1}s$, is not zero. By stationary phase analysis we obtain the bound $s^{-1/2}$.

□

4.6 Appendix: Estimates of Bessel Function around ν

Lemma 4.6.1. *For the Bessel function J_ν of positive order ν with $\frac{1}{2}\nu \leq r \leq 2\nu$, we have the following estimates*

$$|J_\nu(r)| \leq C\nu^{-\frac{1}{3}}(1+\nu^{-\frac{1}{3}}|r-\nu|)^{-\frac{1}{4}}, \tag{4.90}$$

$$|J'_\nu(r)| \leq C\nu^{-\frac{1}{2}}. \tag{4.91}$$

Proof. We have the following integral representation for the Bessel function of order $\nu > -\frac{1}{2}$ (see[37]):

$$J_\nu(r) = A_\nu(r) - B_\nu(r), \tag{4.92}$$

where

$$A_\nu(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(r\sin\theta - \nu\theta)}d\theta, \tag{4.93}$$

$$B_\nu(r) = \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu t - r\sinh(t)}dt. \tag{4.94}$$

We see that

$$B_\nu(r) < \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu t}dt < C\nu^{-1}. \tag{4.95}$$

So, we only need to estimate A_ν. We will accomplish this by using stationary phase for two different cases: $r > \nu$ and $r \leq \nu$. Let us consider the case when $r > \nu$ first. Call the phase in (4.93) $\phi(\theta) = r\sin\theta - \nu\theta$. We differentiate the phase, $\phi'(\theta) = (r\cos\theta - \nu)$. We find $\phi' = 0$ at $\pm\theta_0$, where $\theta_0 = \cos^{-1}(\frac{\nu}{r})$. In order to obtain the estimate, we will break the integral into a small neighborhood around these points and the rest, that is cosider the regions

$$N_\varepsilon := \{\theta : \ |\theta \pm \theta_0| < \varepsilon\}, \tag{4.96}$$

$$S_\varepsilon := [-\pi, \pi]/N_\varepsilon. \tag{4.97}$$

Since the integrand in (4.93) is bounded, we have

$$|\int_{N_\varepsilon} e^{-i(r\sin\theta - \nu\theta)} d\theta| < c\varepsilon. \tag{4.98}$$

On S_ε, we integrate by parts to obtain

$$\int_{S_\varepsilon} e^{-i(r\sin\theta-\nu\theta)} d\theta = \frac{e^{i(r\sin\theta-\nu\theta)}}{i(r\cos\theta-\nu)}\Bigg|_{\{-\theta_0\pm\varepsilon,-\pi\}}^{\{\pi,\theta_0\pm\varepsilon\}} + \int_{S_\varepsilon} \frac{e^{i(r\sin\theta-\nu\theta)} r\sin\theta}{i(r\cos\theta-\nu)^2} d\theta.$$

All terms in the expression above are controlled by $c|r\cos(\theta_0 \pm \varepsilon) - \nu|^{-1}$. We want to balance the contribution from N_ε and S_ε by choosing appropriate ε, such that

$$\varepsilon \sim |r\cos(\theta_0 \pm \varepsilon) - \nu|^{-1}. \tag{4.99}$$

By using the trigonometric identities $\cos(\theta_0 \pm \varepsilon) = \cos(\theta_0)\cos(\varepsilon) \mp \sin(\theta_0)\sin(\varepsilon)$, and the definition of θ_0, we have

$$|r\cos(\theta_0 \pm \varepsilon) - \nu| = |\nu\cos\varepsilon - \sqrt{r^2 - \nu^2}\sin\varepsilon - \nu|. \tag{4.100}$$

When ε is small, (4.100) is approximately $\frac{\nu}{2}\varepsilon^2 + \varepsilon\sqrt{r^2-\nu^2}$. Thus, we have the two estimates

$$|r\cos(\theta_0 \pm \varepsilon) - \nu|^{-1} \leq 2\nu^{-1}\varepsilon^{-2}, \tag{4.101}$$

$$|r\cos(\theta_0 \pm \varepsilon) - \nu|^{-1} \leq \varepsilon^{-1}(r^2 - \nu^2)^{-\frac{1}{2}}. \tag{4.102}$$

When $r - \nu$ is small, (4.101) is sharper. We pick $\varepsilon \sim \nu^{-\frac{1}{3}}$. When $r - \nu$ is big, (4.102) is sharper. We pick optimal $\varepsilon \sim (r^2 - \nu^2)^{-\frac{1}{4}}$. Since $r \leq 2\nu$, we have $|(r^2 - \nu^2)^{-\frac{1}{4}}| < (3\nu)^{-\frac{1}{4}}(r - \nu)^{-\frac{1}{4}}$. Thus, we have proven (4.90) for the case $r \geq \nu$.

Now, we will discuss the case when $r \leq \nu$. When $\nu - \nu^{-\frac{1}{3}} < r < \nu$, we follow the analysis above by choosing $\theta_0 = 0$, $\varepsilon = \nu^{-\frac{1}{3}}$. We have an estimate $J_\nu(r) \leq C\nu^{-\frac{1}{3}}$. When $\nu - \nu^{-\frac{1}{3}} > r$, we use non-stationary phase to obtain $J_\nu(r) \leq C\frac{1}{|\nu - r|} \leq C\nu^{-\frac{1}{4}}|\nu - r|^{-\frac{1}{4}}$.

The remaining task is to prove (4.91). Considering the derivative of J_ν, we can show that

$$|B'_\nu(r)| = |\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu t - r\sinh(t)}\sinh(t)dt| \leq c|\int_0^\infty e^{-\nu t}dt| \leq \frac{c}{\nu}. \tag{4.103}$$

So, we only need to estimate

$$A'_\nu(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(r\sin\theta - \nu\theta)} i\sin\theta d\theta. \tag{4.104}$$

When $r > \nu$, break the integral into two parts as we did in the previous case. Take

L^∞ estimate of the integrand of the integral on N_ε, we obtain

$$\left|\frac{1}{2\pi}\int_{N_\varepsilon} e^{i(r\sin\theta-\nu\theta)} i\sin\theta d\theta\right| < C\varepsilon. \tag{4.105}$$

Integrate by parts for the integral on S_ε, and use trigonometric identity and Taylor expansion. Then, we can find it is controlled by

$$\frac{c_1\sqrt{\frac{r-\nu}{\nu}} + c_2\varepsilon}{\nu\frac{\varepsilon^2}{2} + c_3\sqrt{\nu}\sqrt{r-\nu}}. \tag{4.106}$$

If we balance between integral on N_ε and S_ε, we get an optimal $\nu^{-\frac{1}{2}}$ estimate. For the case $r \le \nu$, we can apply similar ideas as in the proof of (4.90). $\square$

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