ABSTRACT

Title of dissertation: ON THE MODEL THEORY OF RANDOM GRAPHS

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Hrushovski's amalgamation construction can be used to join a collection of finite graphs to produce a "generic" of this collection. The choice of the collection and the way they are joined are determined by a real-valued parameter α. Classical results have shown that for α irrational in $(0, 1)$, the model theory of the resulting structure is very well-behaved.

This dissertation examines analogous constructions for rational r. Depending on the way in which the parameter's control of the construction is defined, the model theory of the resulting generic will be either very well-behaved or very wild. We characterize when each of these situations occurs.

ON THE MODEL THEORY OF RANDOM GRAPHS

by

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Dedication

For my family - my wife, parents, sister, and nieces. They've all worked for my well-being, and all brought me a great deal of happiness.

Acknowledgments

Despite the modest contribution to mathematics which this thesis represents, it was possible only with a great deal of help. I could rather easily start with Plato and spend hundreds of pages in gushing gratitude, but will limit my thanks to more immediate sources.

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My wife Jessica has been a source of tremendous encouragement and support. While not a mathematician, she has a deep understanding of the creative process and provided me with much needed perspective on the ups and downs of doing research. She has constantly reminded me to do math because I enjoy it, and not to gratify my ego. As with the rest of my life, her presence has been invaluable.

I have had the great privilege of growing up in a house where serious ideas were deemed an appropriate topic of conversation, and there is no question that without this I would never have seen the pursuit of a purely academic degree as valuable. I am therefore very grateful to all the members of my family for providing me with such a rich and stimulating start to life.

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Chapter 1

Introduction

1.1 Introduction

The study of random graphs, initiated by Erdös and Renyi, has more recently been examined from a logical viewpoint, notably in papers of Shelah, Spencer, and Baldwin ([11], [1]). In particular, for the graphs $G(n, n^{-\alpha})$, which are graphs of size n with the the probability that any two vertices form an edge being given by $n^{-\alpha}$, Shelah and Spencer proved the following 0-1 law: If α is irrational in $(0, 1)$ then for σ any sentence in the language of graphs, $\lim_{n\to\infty} Pr[G(n, n^{-\alpha}) \models \sigma]$ is 0 or 1. Thus, for a fixed such α the *almost sure theory*, denoted T^α, is complete. More recently, Laskowksi has given a Π_2 axiomatization for T^α (see [10]).

It was later noticed by Baldwin and Shelah [1] that models of the resulting theory could be obtained via Hrushovski's amalgamation construction. This proceeds by amalgamating a class of finite structures in a way which is determined by a notion of "strong substructure". The latter is in turn often determined by a *pre-dimension* function, which in the current context limits the proportion of new edges to new vertices in a strong extension.

Arguably, the crucial observation in the connection between the probabilistic and model-theoretic approaches is that the probability of extensions of a given graph occurring is determined by precisely such a function. Specifically, if a given graph A

almost surely occurs as a subgraph of $G(n, n^{-\alpha})$ in the limit; then A almost surely extends to a copy of the extension B in the limit if and only if $|B \setminus A| - \alpha e(B/A) \geq 0$, where $e(B/A)$ denotes the number of edges in AB that aren't in A.

This paper examines that case that α is rational from a model-theoretic perspective. We note that there is no $0-1$ law in this case, but the model theoretic construction can be generalized to the rational case. Also note that for α irrational the expression $|B \setminus A| - \alpha e(B/a)$ is always strictly positive or strictly negative, while for rational α this expression can be 0. In effect, we are left with two ways to generalize the irrational case - we can either demand that the expression be strictly positive or else we can merely require that it be non-negative. We will see that each approach leads to vastly different model-theoretic properties - in the latter case we will have a single well behaved theory while the former gives rise to uncountably many undecidable theories.

We will examine these approaches by looking at different kinds of limits. It will turn out that these two approach largely suffice to characterize the behavior that can result from taking any ultraproduct $\prod_{\mathcal{U}} M_{\alpha_n}$ with $\mathcal{U}$ any ultrafilter, $\alpha_n \in (0, 1)$ irrational, and M_{α_n} a model of T^{α_n}.

We will also examine the analogues of Laskowksi's Π_2 axioms for the rational case and will show that their completeness is equivalent to the model theory of the appropriate structure being "tame".

1.2 Notation

For the purposes of this thesis, we will restrict our attention to classes of graphs. In particular, we work in the language of graphs, although the results should easily generalize to arbitrary relational structures. We will denote the single binary relation of our language by $E(x, y)$.

We will denote $A \cup B$ simply by AB, and will write $A \subseteq_\omega M$ to indicate that A is a *finite* substructure of M. For any finite graph A, we will implicitly fix an enumeration of A and denote it's quantifier free type by $\Delta_A(\bar{x})$.

1.3 Hrushovski Constructions

Hrushovski's amalgamation construction proceeds by joining together a collection of finite structures $\mathbf{K}$ in accordance with some notion of *strong substructure* $\leq$. It was introduced by Hrushovski in [6, 7] to create stable structures with "exotic" geometries. Good expositions of the construction can be found in [2], [14], and [9]. Our notion of strong substructure will be based on a *predimension* function:

Definition 1.3.1. For a class of finite structures $\mathbf{K}$, closed under substructure and isomorphism, a predimension function on $\mathbf{K}$ is a real-valued function $\delta : \mathbf{K} \to \mathbf{R}^{\geq 0}$ satisfying:

1. δ is total on $\mathbf{K}$, and if $A, A' \in \mathbf{K}$ satisfy $A \simeq A'$, then $\delta(A) = \delta(A')$

2. $\delta(\varnothing) = 0$

3. (**Submodularity**) For A, B elements of $\mathbf{K}$ embedded in a common structure,

we have that $\delta(AB) \leq \delta(A) + \delta(B) - \delta(A \cap B)$.

Given δ a predimension on $\mathbf{K}$, for any $A, B \in \mathbf{K}$ we define the *relative predimension* of A over B as $\delta(A/B) := \delta(AB) - \delta(B)$.

Convention 1.3.2. *We will work throughout with pairs* $(\mathbf{K}, \leq)$ *where* $\mathbf{K}$ *is a class of finite structures and* $\leq$ *is a strong substructure relation on* $\mathbf{K} \times \mathbf{K}$*, satisfying:*

1. $\mathbf{K}$ *is closed under substructures and isomorphisms*

2. *For* $A \in \mathbf{K}$*,* $\varnothing \leq A$

3. *For* $A \leq B$ *from* $\mathbf{K}$*, we have* $A \cap C \leq B \cap C$ *for every* $C \in \mathbf{K}$*.*

4. $\leq$ *is preserved under isomorphisms: for* $A \leq B$ *and* $A \simeq A', B \simeq B'$ *we have* $A' \leq B'$*.*

For $A, B \in \mathbf{K}$*, if* $A \leq B$ *then we will say that* A *is strong (or closed) in* B*.*

We will want to talk about structures whose finite substructures are members of $\mathbf{K}$. The basic definition is:

Definition 1.3.3. For any structure M, it's age is the set of all finite structures which are embeddable in M. We will denote it by $\mathrm{Age}(M)$.

We then extend the notion of strong substructure to apply to potentially infinite structures.

Definition 1.3.4. For $A \subseteq M$ with $\mathrm{Age}(M) \subseteq \mathbf{K}$, we say $A \leq M$ just in case $A \leq AX$ for every $X \subseteq_\omega M$.

The axioms (1) - (3) guarantee the existence of a well-defined closure operation. In classical contexts, the predimension function was chosen to determine certain properties of the geometry generated by this closure (e.g., non-trivial, non-locally-modular). An analysis of this geometry generally yields good information about the model theory of the associated generic.

Definition 1.3.5. For any M with $\mathrm{Age}(M) \subseteq \mathbf{K}$, taking the intersection of all $A' \subseteq M$ satisfying $A \subseteq A' \leq M$ yields a unique minimal superset of A which is strong in M. This will be denoted by $\mathrm{cl}_M(A)$.

An embedding $f : A \hookrightarrow B$ so that $f(A) \leq B$ is called a *strong embedding*. In order to proceed with the construction, we must be able to amalgamate finite structures in a coherent way. The basic definition is:

Definition 1.3.6. For $A, B, C \in \mathbf{K}$, if $A \leq B$ and $f : A \hookrightarrow C$ is strong implies that there is a $D \in \mathbf{K}$ so that $C \leq D$ and a strong embedding $g : B \hookrightarrow D$ so that $g(A) = f(A)$, then we will call $(\mathbf{K}, \leq)$ an *amalgamation class*. We will call D an *amalgam* of B and C over A. Diagramatically, we want the following to commute:

$$\begin{tikzcd} & B \arrow[dr, hook, "\leq", "g"'] & \\ A \arrow[ur, hook, "\leq", "i"'] \arrow[dr, hook, "f", "\leq"'] & & D \\ & C \arrow[ur, hook, "i", "\leq"'] & \end{tikzcd}$$

A special kind of amalgamation involves no extraneous relations between the amalgamated structures:

Definition 1.3.7. Suppose A, B, C are elements of $\mathbf{K}$ with $A = B \cap C$, and let D

be the structure whose universe is BC and whose relations are precisely those of B and those of C. Then we will denote D by $B \oplus_A C$.

If $(\mathbf{K}, \leq)$ is an amalgamation class in which $B \oplus_A C$ is an amalgam of B and C over A, then we will call $B \oplus_A C$ the *free amalgam* of B and C over A and say that $(\mathbf{K}, \leq)$ is a *free amalgamation class.*

Such classes often have nice combinatorial properties. A stronger form of amalgamation occurs when A is not necessarily closed in C. This will play a crucial role in what follows.

Definition 1.3.8. Suppose that $(\mathbf{K}, \leq)$ is a free amalgamation class and for A, B, C with $B \cap C = A$, and $A \leq B$ we have $B \oplus_A C \in \mathbf{K}$ and $C \leq B \oplus_A C$, then we say that $(\mathbf{K}, \leq)$ is a *full amalgamation class.* Full amalgamation is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd}
& B \arrow[dr, hook, "\subseteq", "i"'] & \\
A \arrow[ur, hook, "\leq", "i"'] \arrow[dr, hook, "i", "\subseteq"'] & & B \oplus_A C \\
& C \arrow[ur, hook, "i", "\leq"'] &
\end{tikzcd}$$

For any amalgamation class satisfying (1) - (3) of 1.3.2, we can inductively amalgamate all finite structures in $\mathbf{K}$ together in imitation of the Fraïssé construction (see [5]; the joint embedding property comes from amalgamation and having $\varnothing \leq A$ for $A \in \mathbf{K}$). The resulting structure is called the $(\mathbf{K}, \leq)$-generic; it is unique up to isomorphism and is characterized by three properties.

Definition 1.3.9. The $(\mathbf{K}, \leq)$-generic G is the unique (up to isomorphism) structure satisfying:

1. Age(G) $\subseteq \mathbf{K}$

2. For $A, B \in \mathbf{K}$ with $A \leq B$ and $f : A \hookrightarrow G$ a strong embedding, f extends to a strong embedding $g : B \hookrightarrow G$.

3. For $A \subseteq_\omega G$, $\mathrm{cl}_G(A)$ is finite.

1.3.1 Shelah-Spencer Graphs

The specific classes of finite graphs we will be concerned with will be generated by predimension functions which force strong extensions to be relatively sparse - i.e. the ratio of new edges to new vertices will be bounded. Specifically, for a graph A, let $e(A)$ denote the number of edges in A. For A, B, finite graphs contained in a common extension, let $e(A, B)$ denote the number of edges from vertices in A to vertices in $B \setminus A$, and let $e(B/A)$ denote $e(B \setminus A) + e(B \setminus A, A)$. Then for $\alpha \in [0, 2]$, let $\delta_\alpha(A) = |A| - \alpha e(A)$ and let $\delta_\alpha(B/A) = \delta_\alpha(AB) - \delta_\alpha(A)$; note that $\delta_\alpha(B/A) = |B \setminus A| - \alpha e(B/A)$. We then say $A \leq_\alpha B$ if and only $\delta(B'/A) \geq 0$ for every $B' \subseteq_\omega B$. Then define $\mathbf{K}_\alpha$ as the class $\{ A : \varnothing \leq_\alpha A, |A| < \aleph_0 \}$.

It is shown in [2] that $(\mathbf{K}_\alpha, \leq_\alpha)$ so defined is a full amalgamation class. When α is irrational in $(0, 1)$ the $(\mathbf{K}_\alpha, \leq_\alpha)$-generic will be called the Shelah-Spencer graph of weight α - these graphs have been extensively studied in [11, 12, 1, 2, 10]. These graphs display good model-theoretic behavior. In particular, they are all stable and axiomatized by the $\forall\exists$ schemes S_α defined below. They are also models of the almost-sure theories T^α studied by Shelah and Spencer and discussed in the introduction.

Our study of analogues of this construction will proceed by studying various forms of limits of the irrationally weighted graphs. We note some basic facts about relations between the notions of sparsity.

Lemma 1.3.10. *For any $A \subseteq B$ and $\alpha_0 \geq \alpha_1$:*

1. *If $\delta_{\alpha_0}(B/A) \geq 0$ then $\delta_{\alpha_1}(B/A) \geq 0$.*

2. *If $A \leq_{\alpha_0} B$ then $A \leq_{\alpha_1} B$.*

Proof. We have $\delta_{\alpha_0}(B/A) = |B \setminus A| - \alpha_0 e(B/A)$ which is clearly at most $|B \setminus A| - \alpha_1 e(B/A) = \delta_{\alpha_1}(B/A)$. Both statements follow immediately. □

Thus, for finite $A \subseteq B$, the set $\{\alpha : A \leq_\alpha B\}$ is a sub-interval of $[0, \beta]$ for some β. We determine this interval with:

Lemma 1.3.11. *For finite graphs $A \subseteq B$ define $h^*(A, B)$ to be $\sup\{\alpha : \delta_\alpha(B/A) \geq 0, \alpha \leq 2\}$. Also let $h(A, B) := \sup\{\alpha : A \leq_\alpha B, \alpha \leq 2\}$. Note that if $e(B/A) = 0$ then $h^*(A, B) = h(A, B) = 2$. Otherwise*

1. *$h^*(A, B) = \frac{|B \setminus A|}{e(B/A)}$. In particular, $h^*(A, B)$ is rational and is $\max\{\alpha : \delta_\alpha(B/A) \geq 0\}$*

2. *$h(A, B) = \min_{H : A \subseteq H \subseteq B}\{h^*(A, H)\}$.*

In particular, $h(A, B)$ is rational and for any $\alpha \in [0, 2]$, we have $A \leq_\alpha B$ if and only $\alpha \in [0, h(A, B)]$

Proof. We have $\delta_\alpha(B/A) \geq 0$ iff $|B \setminus A| - \alpha(e(B/A)) \geq 0$ iff $\alpha \leq \frac{|B \setminus A|}{e(B/A)}$. Both statements follow immediately. □

We will work frequently with the class defined in the following lemma.

Definition 1.3.12. For $r \in [0,2]$, let $(\mathbf{K}_r^+, \preccurlyeq_r)$ be defined by

1. For finite $A \subseteq B$, $A \preccurlyeq_r B$ iff $A \leq_\beta B$ for some $\beta > r$.

2. $\mathbf{K}_r^+ = \{H | \varnothing \preccurlyeq_r H\}$

Note that $\mathbf{K}_r^+ = \{\, H : H \in \mathbf{K}_r \wedge (\delta_r(H) > 0 \vee H = \varnothing)\,\}$ and for $H \neq \varnothing$, $H \in \mathbf{K}_r^+$ if and only if $h(H) > r$.

Lemma 1.3.13. *$(\mathbf{K}_r^+, \preccurlyeq_r)$ is a full amalgamation class.*

Proof. Let $A \leq B_0, B_1$ for $A, B_i \in \mathbf{K}$. Then there exist β_0, β_1 greater than r so that $A \leq_{\beta_0} B_0$ and $A \leq_{\beta_1} B_1$. Let $\beta = \min(\beta_0, \beta_1)$; then we have $A \leq_\beta B_0, B_1$ and by free amalgamation in $\leq_\beta$ we have $C = B_1 \oplus_A B_2$ witnesses the amalgamation property. The same reasoning establishes fullness.

If $H \in \mathbf{K}_r^+$, we have that $\varnothing \leq_\beta H$ for some $\beta > r$; this implies that $r < h(H)$. Conversely, if $r < h(H)$, then $\varnothing \leq_\beta H$ for some $\beta > r$, so that $\varnothing \preccurlyeq_r H$. $\square$

It is worth noting that for α irrational, $\preccurlyeq_\alpha$ is the same as $\leq_\alpha$, and $\mathbf{K}_\alpha^+ = \mathbf{K}_\alpha$. The following properties of δ_α and $(\mathbf{K}_\alpha, \leq_\alpha)$ are well-known and discussed in, e.g., [14], [8], [2], [10]:

Lemma 1.3.14. *Fix α in $[0,2]$ Then for $A, A_i, B, B_i, C \in \mathbf{K}_\alpha$:*

1. *If $B \cap C = A_0$ with $A \subseteq A_0$, then $\delta_\alpha(B \oplus_{A_0} C/B) \leq \delta_\alpha(C/A)$. Furthermore, equality holds when $A = A_0$.*

2. *(**Linearity**)If $D = B_1 \oplus_A B_2 \oplus_A \ldots \oplus_A B_n$ then $\delta_\alpha(D/A) = \sum_{1 \leq i \leq n} \delta_\alpha(B_i/A)$.*

3. (**Submodularity**) *If A and B are embedded in a common superstructure, we have that $\delta(AB) \leq \delta(A) + \delta(B) - \delta(A \cap B)$.*

4. $A \leq_\alpha A$

5. *If $A \leq_\alpha B$, then $A \subseteq B$*

6. *If $A \leq_\alpha B$ and $B \leq_\alpha C$, then $A \leq_\alpha C$*

7. $\varnothing \leq_\alpha A$

8. $\mathbf{K}_\alpha$ *is closed under substructure and isomorphism*

9. *If $A \leq_\alpha B$ then $A \cap C \leq_\alpha B \cap C$.*

Furthermore, it is clear that (1) - (8) *hold for structures taken from $\mathbf{K}_\alpha^+$ and $\preccurlyeq_\alpha$ as well. In particular, both $(\mathbf{K}, \leq_\alpha)$ and $(\mathbf{K}_\alpha^+, \preccurlyeq_\alpha)$ satisfy the conditions in Convention 1.3.2.*

We show that (9) also holds for $(\mathbf{K}_r^+, \preccurlyeq_r)$:

Lemma 1.3.15. *Fix r rational in $[0, 2]$. If $A \preccurlyeq_r B$ for $A, B \in \mathbf{K}_r^+$, then for $C \in \mathbf{K}_r^+$ we have $A \cap C \preccurlyeq_r B \cap C$.*

Proof. Let X be any subset of $B \cap C$ containing $A \cap C$. By submodularity, $\delta_r(X/A \cap C) \geq \delta_r(X/A)$. Since $A \preccurlyeq_r B$, we have that $A \preccurlyeq_r AX$, so that the latter term is > 0 unless $AX = A$, in which case $X = A \cap C$. $\square$

The following definition is from [14]:

Definition 1.3.16. Let A, B be any two sub-graphs of a common extension. Then the base of B over A, denoted B^A, is the set of all vertices in A which have an edge to some vertex in $B \setminus A$.

Definition 1.3.17. A pair of structures (A, B) is said to be *minimal* if $A \not\leq B$ but $A \leq AX$ for any proper subset $X \subsetneq B$. If (A, B) is a minimal pair such that $B^A = A$ then we say that (A, B) is *biminimal.*

Notation 1.3.18. *If $\{ B_i : i \in I \}$ is a set of structures which are pairwise disjoint over some A, then $\oplus_{i \in I}(B_i/A)$ denotes the free amalgam of all the B_i over A. If each B_i has base $X_i \subseteq A$, then we write $\oplus_{i \in I}(B_i/A^{X_i}))$ to denote this.*

Lemma 1.3.19. *For any $\beta \in [0, 1]$: $\delta_\beta(B/A) = \delta_\beta(B/B^A)$*

Proof. Note that $\delta_\beta(B/A) = |B \setminus A| - \beta(e(B \setminus A) + e(B \setminus A, A)) = |B \setminus A| - \beta(e(B \setminus A) + e(B, B^A)) = \delta(B/B^A)$. $\square$

Definition 1.3.20. The amalgamation class $(\mathbf{K}, \leq)$ has the *granularity property* if for any positive $m \in \omega$ there is some positive real number $Gr(m)$ so that for any $A \in \mathbf{K}$, if B is an extension of A with $|B \setminus A| < m$ and $\delta(B/A) < 0$, we have $\delta(B/A) \leq -Gr(m)$.

For $\alpha \in (0, 1)$ irrational, it is shown in [10] and [2] that $(\mathbf{K}_\alpha, \leq_\alpha)$ has the granularity property. It is clear for rational $\alpha = \frac{p}{q}$. For $m \in \omega$ let $Gr(m) := 1/q$ where $\alpha = \frac{p}{q}$.

Definition 1.3.21. For any amalgamation class $(\mathbf{K}, \leq)$, the sentences $S_{(\mathbf{K},\leq)}$ say that for $M \models S_{(\mathbf{K},\leq)}$:

- Existential axioms stating that $\mathrm{Age}(M) \subseteq \mathbf{K}$
- $\forall\exists$ axioms stating that for $A \subseteq_\omega M$ and $A \leq B$, A extends to an embedding of B into M

When $\mathbf{K} = \{ A : \varnothing \leq A \}$ then we will write $S_{(\mathbf{K},\leq)}$ simply as $S_\leq$. When we further have that $\leq$ is $\leq_\alpha$ as defined above, we will denote $S_{\leq_\alpha}$ by S_α.

It is shown in both [10] and [8] that for irrational α the complete theory of the $(\mathbf{K}_\alpha, \leq_\alpha)$ is axiomatized by S_α.

It will often be useful to talk about locally closed sets and local closures:

Definition 1.3.22. For $m \in \omega, \alpha \in [0, 2]$:

- $A \leq_\alpha^m B$ means that for any $X \subseteq B$ with $|X| < m$, $A \leq_\alpha AX$.
- If there is a unique minimal superset A' of A so that $A' \leq_\alpha^m B$, then we will denote A' by $\mathrm{cl}_B^m(A)$.

It is shown in Lemma 3.17 of [2] that $\mathrm{cl}^m(A)$ is well defined for structures M with $\mathrm{Age}(M) \subseteq \mathbf{K}_\alpha$; the same argument also applies to M with $\mathrm{Age}(M) \subseteq \mathbf{K}_\alpha^+$

For $r \in [0, 2]$ rational, we will also want to talk about the notion of semi-genericity, as introduced in [1]. This is a local approximation of genericity, and is defined by: For every $m \in \omega$ and finite $A \preccurlyeq_r B$, if $A \subseteq M$ then A can be extended in M to B', a copy of B over A satisfying $\mathrm{cl}_M^m(B') = B' \oplus_A \mathrm{cl}_M^m(A)$. Unlike genericity, this is a first-order notion:

Definition 1.3.23. Let r be any rational in $[0, 1]$. For $A, B \in \mathbf{K}_r^+$ with $A \preccurlyeq_r B$ and $m \in \omega$, let $\psi_{A,B}^m(\bar{x}, \bar{y})$ be the formula $\forall z_1 \dots z_m \bigvee_C \Delta_C(\bar{x}\bar{y}\bar{z})$ where C ranges over m-ary extensions of B which satisfy either $B \preccurlyeq_r C$ or $C^B \subseteq A$, and the enumeration is chosen so that $\Delta_C(\bar{x}\bar{y}\bar{z}) \implies \bar{x} \subseteq A$. Then we define Σ_r to be the extension of $S_{\preccurlyeq_r}$ obtained by adding, for each $A \preccurlyeq_r B$ and $m \in \omega$, the sentence: $\forall \bar{x}[\Delta_A(\bar{x}) \to \exists \bar{y} \left(\Delta_B(\bar{x}\bar{y}) \wedge \psi^m_{A,B}(\bar{x}\bar{y})\right)]$.

We note that Σ_r is Π_3 and will be satisfied by the $(\mathbf{K}_r^+, \preccurlyeq_r)$-generic (by full amalgamation: $\mathrm{cl}(A) \preccurlyeq_r \mathrm{cl}(A) \oplus_{\mathrm{cl}^m(A)} B$, so the latter embeds strongly into the generic). We also note that the equivalent axioms are satisfied by the $(\mathbf{K}_\alpha, \leq_\alpha)$-generics for irrational α, but since these are axiomatized by S_α, we simply define Σ_α to be S_α (as a notational convenience).

1.3.2 0-Extensions

Our main results will be that the model theory of the rationally weighted analogues of the Shelah-Spencer graphs is wild. This wildness is introduced by non-trivial extensions of relative pre-dimension 0. Such extensions will be part of the base set's closure - the possible types of these closures are thereby greatly increased, to the point that the resulting structure will often be undecidable.

The existence of such extensions will be based on the following notion. The term was coined in [8] and reflects a similar idea in [10].

Definition 1.3.24. An amalgamation class $(\mathbf{K}, \leq)$ which is defined by a delta function δ is said to have the *approximating extension property*, or **AEP**, if for any $A \in \mathbf{K}$ and $A \leq B$, given $m \in \omega$ and $\epsilon > 0$ there is some $C \in \mathbf{K}$ which extends B and satisfies:

1. $A \leq C$
2. $\delta(C/A) < \epsilon$
3. $B \leq^m C$

Ikeda et al. show that $(\mathbf{K}_\beta, \leq_\beta)$ has **AEP** for any $\beta \in (0, 1]$. We note that for rational r, **AEP** gives us that for any structure $A \leq B$ as above, there is a C as above with pre-dimension 0. We call such an extension a 0-extension; if (A, C) is additionally a minimal pair we will call C a minimal 0-extension. Similarly, if (A, C) is biminimal we call C a biminimal 0-extension. The remainder of this section will be occupied with showing that such extensions exist. In doing so, we make heavy use of the machinery developed in [8].

Definition 1.3.25. Fix $r \in (0, 1]$ rational. For s a real number with $0 \leq s \leq 2$ we say that (E, a, b) is an s-component if $E \in \mathbf{K}_r$, $a, b \in E$ and for non-empty $X \subseteq E$:

- $\delta_r(X) \geq 1$ if $\{ a, b \} \not\subseteq X$
- $\delta_r(X) \geq s$ if $\{ a, b \} \subseteq X$
- $\delta_r(E) = s$

If, in addition, we have that $\delta(X) = s$ implies $X = E$ whenever $\{ a, b \} \subseteq X$, then we say that (E, a, b) is a *minimal* s-component. Any s-component contains a minimal s-component.

We will adopt the convention in this paper that all components are *proper* - that is there is no edge between a and b. It is shown in [8] that proper components exist. Specifically, we have:

Lemma 1.3.26. *Let $r = \frac{p}{q}$, and let $t = \frac{1}{q}$.*

- *There exist minimal $1+t$ and $1-t$ components*
- *For $1 \leq s, t \leq 2$ and $s+t-1 \leq 2$, if (E_0, a, b) and (E_1, b, c) are respectively s- and t-components, then $(E_0 \oplus_b E_1)$ is an $s+t-1$-component.*

Given a minimal u-component (D, d_0, d_1), we define a chain of m copies of D as (D_{m-1}, d_0, d_m), where D_0 is isomorphic to D, and given D_i we let $D_{i+1} := D_i \oplus_{d_{i+1}} D_{i+1}$ where $(D_{i+1}, d_{i+1}, d_{i+2})$ is isomorphic to (D, d_0, d_1).

Lemma 1.3.27. *Let (E, e_0, e_k) be a chain of k copies of a minimal $(1-t)$-component, let (F^l, f^l, g^l) be a chain of $\lceil \frac{q}{2} \rceil$ copies of a minimal $(1+t)$-component, and let (F^r, f^r, g^r) be a chain of $\lfloor \frac{q}{2} \rfloor$ copies of a minimal $(1+t)$-component. Then:*

1. *(F^l, f^l, g^l) is a minimal $(1 + \lceil \frac{q}{2} \rceil t)$-component.*
2. *(F^r, f^r, g^r) is a minimal $(1 + \lfloor \frac{q}{2} \rfloor t)$-component.*
3. *For any subset $X \subseteq E$, $\delta(X) \geq 1 - kt$ and equality holds if and only if $X = E$.*

Proof. That the first two are components of the required pre-dimension follows from Lemma 1.3.26. Let (D, d_0, d_1) be a minimal $(1+t)$ or $(1-t)$ componenet; we show by indcution that if (D_k, d_0, d_k) is a chain of k copies of D, then for $X \subseteq D$, $\delta(X) \geq 1 \pm kt$ with equality holding exactly when $X = D$. For $k = 1$ this is just the definition of a minimal component. Otherwise, let $D_{k+1} = D_k \oplus_{d_k} D'$ where (D', d_k, d_{k+1}) is isomorphic to (D, d_0, d_1). If $X \subseteq D_k$ or $X \subseteq D'$, then the result is immediate from induction. Otherwise, let $X_k = X \cap D_k$ and let $X' = X \cap D'$. Then

$X = X_k \oplus_{d_k} X'$, and $\delta(X) = \delta(X_k) + \delta(X') - 1 \geq (1 \pm kt) + (1 \pm t) - 1 = 1 \pm (k+1)t$ as desired. □

We would like to use the above to construct biminimal 0-extensions. We first note the following special case:

Remark 1.3.28. If $A \in \mathbf{K}_r$ is a singleton , then let $B = Ab$ for some b with an edge from A to b. Let C be an extension of B which is a 0-extension of A. In their proof of **AEP** for this class, Ikeda, Kikyo, and Tsuboi gave a construction of C which satisfied $C^A = C^B$. Therefore taking a subset C' of C so that (A, C') is minimal gives a biminimal pair.

Proposition 1.3.29. *Let $r \in (0,1)$ be rational. Then for $A \in \mathbf{K}_r$ there is some $C \in \mathbf{K}_r$ so that (A, C) is a biminimal 0-extension*[1].

Proof. Let $|A| = n$; by the previous remark we may assume that $n > 1$. Extend A to a structure B which consists of n unrelated points, each with an edge to a unique vertex of A. For any $b \in B$, $\delta_r(b/A) = 1 - r$; then $1 - r = kt$ for some $k \in \omega$, and $\delta_r(B/A) = nkt$.

Let (E, e_0, e_k), (F^l, f^l, g^l), and (F^r, f^r, g^r) be as defined in Lemma 1.3.27. We define C as follows. For $i < n$, let (E_i, x_i, y_i) be a copy of (E, e_0, e_k), let (F^l_i, y_i, b_i) be a copy of (F^l, f^l, g^l), and let (F^r_i, b_i, x_{i+1}) be a copy of (F^r_i, f^r, g^r). We adopt the convention that for $i \leq n-1, i' = i+1$ if $i < n-1$ and 0 otherwise. Then let $C_0 := E_1 \oplus_{y_1} F^l_1$, and for $i < n-1$, let $C_{i'} := C_i \oplus_{b_{i'}} F^r_{i'} \oplus_{x_{i''}} E_{i''} \oplus_{y_{i''}} F^l_{i''}$. Finally,

[1]It would probably be fairly straightforward to strengthen this result to obtain a proper strengthening of the original **AEP**

we let $C := C_{n-1} \oplus_{b_0,x_1} F^r_0$. The idea is that C forms a circle of structures with the "negative" copies of E "buffered" by the "positive" copies of F^l and F^r.

We need to show that $C \in \mathbf{K}_r$ and that (A, C) is a biminimal 0-extension. Let X be an arbitrary subset of $C \setminus A$, and let $B_0 = B^X$. Then we show that $\delta_r(X) > 0$ and that $\delta_r(X/A) \geq 0$ with equality exactly when $X = C \setminus A$. Note that by the linearity of δ_r, we may assume that X is connected.

We will calculate $\delta(X/X \cap B)$. Let $m = |X \cap B|$ and define $W_i := (X \cap F^r_i) \oplus_{x_i} (X \cap E_i) \oplus_{y_i} \left(X \cap F^l_{i'}\right)$. For each i, $\delta(W_i/b_i b_{i'})$ is given by

$$\delta(X \cap F^r_i/b_i) + \delta(X \cap E_i/x_i y_i) + \delta(X \cap F^l_{i'}/b_{i'})$$

The bounds on each term depend on $\{\, x_i, y_i \,\} \cap X$. The first term is at least $\left\lfloor \frac{q}{2} \right\rfloor t$ if $x_i \in X$ and strictly greater than 0 otherwise. Similarly the last term is either at least $\left\lceil \frac{q}{2} \right\rceil t$ for $y_i \in X$ or else strictly greater than 0. We also have that the middle term is at least $-1 - kt$. Combining this information (and using connectedness), we have

$$\delta(W_i/b_i b_{i'}) \geq \begin{cases} -1 - kt & \text{if } X \cap \{\, x_i, y_i \,\} = \varnothing \text{ and } X \cap \{\, b_i, b_{i'} \,\} = \varnothing \\ 0 & \text{if } X \cap \{\, x_i, y_i \,\} = \varnothing \text{ and } X \cap \{\, b_i, b_{i'} \,\} \neq \varnothing \\ \left\lfloor \frac{q}{2} \right\rfloor t - kt & \text{if } X \cap \{\, x_i, y_i \,\} = \{\, x_i \,\} \\ \left\lceil \frac{q}{2} \right\rceil t - kt & \text{if } X \cap \{\, x_i, y_i \,\} = \{\, y_i \,\} \\ -kt & \text{if } X \cap \{\, x_i, y_i \,\} = \{\, x_i, y_i \,\} \end{cases} \tag{$\star$}$$

with equality holding in the last case exactly when $W_i = F^r_i \oplus_{x_i} E_i \oplus_{y_i} F^l_{i'}$. We also have that $\delta(W_i/b_i) \geq \left\lceil \frac{q}{2} \right\rceil t - kt$ and $\delta(W_i/b_{i'}) \geq \left\lfloor \frac{q}{2} \right\rfloor t - kt$.

For $m = 0$, we then have that $\delta(X) \geq \delta(W_0/b_0b_1) + \delta(b_0b_1) \geq 1 - kt = r > 0$. We also have that $\delta(X/A) = \delta(X) > 0$.

For $0 < m < n$, if $\left\lfloor \frac{q}{2} \right\rfloor t - kt < 0$, we have:

$$\begin{aligned}
\delta(X/X \cap B) &= \delta(W_{n-1}/b_0) + \delta(W_{m-1}/b_{m-1}) + \sum_{i=0}^{m-2} \delta(W_i/b_ib_{i'}) \\
&\geq (\left\lceil \frac{q}{2} \right\rceil t - kt) + (\left\lfloor \frac{q}{2} \right\rfloor t - kt) - (m-1)kt \\
&= 1 - 2kt - mkt + kt \\
&= 1 - mkt - kt \\
&= r - mkt
\end{aligned}$$

We have $\delta(X) = \delta(X/X \cap B) + \delta(X \cap B) \geq (r - mkt) + m = r + mr > 0$. Also, $\delta(X/A) = \delta(AX/X^A) = \delta(X/X \cap B) + \delta(X \cap B/A) \geq (r - mkt) + mkt = r > 0$. If $\left\lfloor \frac{q}{2} \right\rfloor t - kt > 0$, then $\delta(X) \geq m - (m-1)kt > mr > 0$ and $\delta(X/A) \geq -(m-1)kt + mkt > mr > 0$.

For $m = n$, we have $\delta(X/X \cap B) = \sum_{i=0}^{n-1} \delta(W_i/b_ib_{i'}) = \sum_{i=0}^{n-2} \delta(W_i/b_ib_{i'}) + \delta(W_{n-1}/b_{n-1}b_0)$. Applying $(\star)$, we have that $\delta(X/X \cap B) \geq (n-1)(-kt) - kt = -nkt$. Then $\delta(X) \geq -nkt + n = nr > 0$ and $\delta(X/A) \geq -nkt + nkt = 0$, with equality holding only if every W_i is $F_i^r \oplus_{x_i} E_i \oplus_{y_i} F_{i'}^l$. Thus (A, C) is a biminimal pair, as desired. $\square$

Chapter 2

Up and Down

In this chapter we will study ultraproducts $\prod_{\mathcal{U}} M_{\alpha_n}$ for $\{\alpha_n\}$ a sequence converging to a rational $r \in (0,1)$ and M_{α_n} a model of Σ_{α_n}. We will see that the theory of the ultraproduct is either ω-stable or undecidable, depending on whether the sequence can be thought of as converging upward or downward.

2.1 Going Up

Let $\{\alpha_n\}$ be a sequence converging to some rational $r \in (0,1)$ which is bounded above by r. Let M_{α_n} be any model of Σ_{α_n}, let $\mathcal{U}$ be any non-principal ultrafilter and let $\mathbf{M}_r$ be the ultraproduct $\prod_{\mathcal{U}} M_{\alpha_n}$. Then we will show that $\mathbf{M}_r$ is elementarily equivalent to the $(\mathbf{K}_r, \leq_r)$ generic.

Lemma 2.1.1. *The following statements hold of* $\mathbf{M}_r$*:*

1. *The age of* $\mathbf{M}_r$ *is precisely the set of finite graphs* G *satisfying* $\delta_r(H) \geq 0$ *for every* $H \subseteq G$*. That is,* $\mathrm{Age}(\mathbf{M}_r) = \mathbf{K}_r$*.*

2. *For every* $A \subseteq_\omega \mathbf{M}_r$*, if* $A \leq_r \mathbf{M}_r$ *and* $A \leq_r B$*, then* B *embeds strongly into* $\mathbf{M}_r$ *over* A*.*

3. *For* $A \subseteq_\omega \mathbf{M}_r$ *and* $A \leq_r B$*,* A *extends to an embedding of* B *into* $\mathbf{M}_r$

In particular, $\mathbf{M}_r \models S_r$

Proof. For all three items, we note that for $A \subseteq B$, if $h = h(A, B)$ then $A \leq_r B$ iff $r \in [0, h]$ iff $\alpha_n \in [0, h]$ for every n.

To show (1) we note that, by Łoś, $\mathbf{M}_r \models \exists \bar{x} \Delta_A(\bar{x})$ iff $M_{\alpha_n} \models \exists \bar{x} \Delta_A(\bar{x})$ for cofinitely many n. For a given n, $M_{\alpha_n} \models \exists \bar{x} \Delta_A(\bar{x})$ iff $\alpha_n \in [0, h(A)]$, so $\mathbf{M}_r \models \exists \bar{x} \Delta_A(\bar{x})$ iff there is some N so that $\alpha_n \in [0, h]$ for $n > N$. Since $\{ \alpha_n \}$ is bounded above by r, we must have $r \in [0, h]$

For (2) consider the type $p(\bar{y})$ over A consisting of the following schema:

- $\Delta_B(A\bar{y})$
- For each $k \in \omega$, the formula e_k:

$$\forall z_1 \dots z_k u(\bar{z}) \to \bigvee_i \Delta_{C_i^k}(A\bar{y}\bar{z})$$

Where the $\Delta_{C_i^k}$ enumerate the diagrams of strong extensions of B of size k and $u(\bar{z})$ states that all the z_i are distinct.

We show that p is finitely satisfiable and hence consistent; the ω_1-saturation of $\mathbf{M}_r$ will then guarantee that it is realized. Since $A \leq_r B$, we have that $A \leq_{\alpha_n} B$ for all n. Since $A \leq_r \mathbf{M}_r$, we have that $A \restriction M_{\alpha_n} \leq_r^k M_{\alpha_n}$ for cofinitely many n. Then, by our choice of M_{α_n}, $A \restriction M_{\alpha_n}$ extends to a k-strong copy of B in M_{α_n}. This copy will witness any finite subset of p that contains no e_l for $l > k$.

To show (3), we apply Łoś's theorem. Suppose $A \subseteq_\omega \mathbf{M}_r$ and $A \leq_r B$; let $h = h(A, B)$. Since $A \leq_r B$, we must have $r \in (0, h]$. Therefore $A \leq_{\alpha_n} B$ for every $n \in \omega$, so M_{α_n} models that every copy of A extends to a copy of B; thus the ultraproduct does as well by Łoś. $\square$

We will need the following:

Definition 2.1.2. For $m \in \omega$:

- An m-chain over a finite subset A is a sequence of extensions B_i with $B_0 = A$, $B_i \subseteq B_{i+1}$, $B_i \not\leq B_{i+1}$, and $|B_{i+1} \setminus B_i| < m$.

- $X_m(A)$ is the set of all B such that B is the final element of some m-chain over A. It is worth noting that the relation $B \in X_m(A)$ is equivalent to the notion of A being *intrinsic* in B, used in [2] and [1].

- The class $(\mathbf{K}, \leq)$ has *bounded m-closures* if there is a function $t : \omega \times \omega \to \omega$ which is monotone increasing in both arguments and such that: for M any model with $\mathrm{Age}(M) \subseteq \mathbf{K}$, if $A \subseteq_\omega M$, then for any $m \in \omega$ and $B \in X_m(A)$, there are at most $t(|A|, |B|)$ copies of B which embed into M over A.

A theory T is said to be *near model complete* if every formula is equivalent to a boolean combination of existential formulae mod T. A model is near model complete if it's theory is.

Definition 2.1.3. An amalgamation class $(\mathbf{K}, \leq)$ is *good* if it satisfies all of the following:

- $(\mathbf{K}, \leq)$ is a full amalgamation class

- There is a predimension function δ so that $A \leq B$ is given by $\delta(B'/A) \geq 0$ for every $B' \subseteq_\omega B$

- $(\mathbf{K}, \leq)$ has the granularity property, satisfies **AEP**, and has bounded m-closures.

We extract the following theorem from [10]:

Main Theorem 2.1.1. *Let* $(\mathbf{K}, \leq)$ *be a good amalgamation class, and let* $M \models S_{\leq}$*. Then the theory of* M *is nearly model complete and is axiomatized by* $S_{\leq}$*.*

We will make use of the following theorem, paraphrased from [1]:

Theorem 2.1.4. *Let* $(\mathbf{K}, \leq)$ *be any full amalgamation class which satisfies the conclusions of Lemma 1.3.14 and has bounded* m*-closures. Then the* $(\mathbf{K}, \leq)$*-generic is near model complete.*

(Proof of Main Theorem). It is shown in [8] that for any good amalgamation class, $S_{\leq}$ is complete. By Theorem 2.1.4 we have that $S_{\leq}$ is near model complete, since it is the theory of the generic. $\square$

Note that this is slightly stronger than the individual results in either Baldwin-Shelah ([1]) or Ikeda, Kikyo, and Tsuboi ([8]): Baldwin Shelah show near model completeness of a Π_3 theory, while the latter authors show the axiomatization by $S_{\leq}$ but don't show near model completeness.

Corollary 2.1.5. *Let* $T = \mathrm{Th}(\mathbf{M}_r)$*, then* T *is nearly model complete and is axiomatized by* S_r*.*

Proof. We need only show that $(\mathbf{K}_r, \leq_r)$ is a good amalgamation class. It has free amalgamations: for $A \leq_r B_1, A \leq_r B_2$, let $D = B_1 \oplus_A B_2$. For any subset $D' = B_1' \oplus_A B_2'$, we have $\delta_r(D'/A) = \delta_r(B_1'/A) + \delta_r(B_2'/A)$, and both terms are non-negative by hypothesis. For full amalgamation, we note that $\delta_r(B \oplus_A C) = \delta_r(B) + \delta_r(C) - \delta_r(A) = \delta_r(B/A) + \delta_r(C)$ which must be positive by the hypotheses.

AEP for $(\mathbf{K}, \leq_r)$ is shown in Proposition 3.11 of [8]

Boundedness of m-closures comes from the rationality of r: if $r = p/q$, then any extension of B of A which is not strong satisfies $\delta_r(B/A) \leq -\frac{1}{q}$. Therefore A can have at most $\delta(A)q$ copies of B which embed over it in M, so we simply let $t(|A|, |B|) := \max_{\{A' : |A'| = |A|\}} \delta_r(A')q$. $\square$

The following is Theorem 3.34 of [2]:

Theorem 2.1.6. *The theory of the* $(\mathbf{K}_r, \leq_r)$*-generic is* ω*-stable.*

2.2 Coming Down

In this section we consider a decreasing sequence $\{ a_n \}$ which converges to some rational $r \in (0, 1)$. We want to examine the theory of the ultraproduct $\mathbf{M}_r := \prod_{\mathcal{U}} M_{\alpha_n}$, where $\mathcal{U}$ is any non-principal ultrafilter and M_{α_n} is a model of Σ_{α_n}. We will see that any such ultraproduct satisfies Σ_r, but that Σ_r is far from complete. In fact, we will see that it has continuum many completions, and that the theory of the ultraproduct is not even recursively axiomatizable.

Fix r throughout the rest of this section; we will work with the class $(\mathbf{K}_r^+, \preccurlyeq_r)$. We will show that any model of Σ_r interprets Robinson's R and is thus essentially undecidable. We will also show that the $(\mathbf{K}_r^+, \preccurlyeq_r)$-generic and any $\mathbf{M}_r$ (independently of the sequence chosen or the ultrafilter) are models of Σ_r.

Our key proposition states that relative to a finite subset of a model of Σ_r, finite relations are definable; this generalizes a similar result in [12]. Recall that $[M]^n$ denotes the subsets of M with cardinality precisely n.

Proposition 2.2.1 (Definability of Finite Relations)**.** *Let $M \models \Sigma_r$. For any $n \in \omega$, there is a predicate $R(x_0, \ldots, x_{n-1}; v)$ and an $m \in \omega$ so that for any $R_0 \subseteq_\omega [M]^n$ and S with $\cup R_0 \subseteq S \subseteq_\omega M$, there is some $v \in M$ so that $R(S; v) = R_0$; that is, for $\bar{a} \in S^n$, $M \models R(\bar{a}; v)$ if and only if $\bar{a} \in R_0$ (where we view $\bar{a}$ as a set rather than a tuple). We will denote the relation $R(\cdot; v)$ by R_v.*

Proof. Enumerate R_0 as $\{ \bar{a}'_i : i < N \}$. Let $z\bar{a}$ be a graph with $n+1$ vertices and no edges. Fix U a biminimal extension of $z\bar{a}$, and note that for every $i < N$ there is a graph $z\bar{a}_i U_i$, in which $\bar{a}_i$ is isomorphic to $\bar{a}'_i$ and U_i is a copy of U over $z\bar{a}_i$ (that is, the internal structure of $\bar{a}'_i$ is irrelevant). Let S' be an isomorphic copy of S in which the image of each $\bar{a}'_i$ is $\bar{a}_i$, let z be a new vertex with no edge to any $\bar{a}_i$, and let W be $\bigoplus\limits_{i<N}(U_i/\bar{a}_i z)$ with U_i chosen as in the previous sentence.

We will show that W embeds strongly into M over $\cup R_0$. Let $\beta(\bar{u}, \bar{u}')$ state that $\bar{u}$ is a permutation of $\bar{u}'$ or else that $\bar{u} \cap \bar{u}' = \varnothing$. Then let $R(\bar{x}; v)$ be $\exists \bar{u} \bigvee_\sigma \Delta_U(v\sigma(\bar{x})\bar{u}) \wedge [\forall \bar{u}, \bar{u}' \Delta_U(v\sigma(\bar{x})\bar{u}) \rightarrow \beta(\bar{u}, \bar{u}')]$ where σ runs over permutations of $\bar{x}$, we will see that R_v is precisely as required.

We show that $\bigcup_{i<N} \bar{a}_i \preccurlyeq_r W$, via the following sequence of calculations, which hold for every $i < N$:

$$\bar{a}_i \preccurlyeq_r z \tag{2.1}$$

$$\bar{a}_i \preccurlyeq_r U_i \text{ and } z \preccurlyeq U_i \tag{2.2}$$

$$\bar{a}_i \preccurlyeq_r zU_i \tag{2.3}$$

$$\cup_{i<N}\bar{a}_i \preccurlyeq_r W \tag{2.4}$$

(2.1) is immediate since v is unrelated to $\bar{a}_i$ - it has relative pre-dimension 1.

To prove (2.2), note that for any $A \subseteq U_i \setminus \bar{a}_i$, with $z \in A$, we have $\delta(A/\bar{a}_i) = \delta(A/\bar{a}_i z) + \delta(\bar{a}_i z/\bar{a}_i)$. By the definition of U, $\delta(A/\bar{a}_i z) \geq 0$; we also have $\delta(\bar{a}_i z/\bar{a}_i) = 1$, so $\delta(A/\bar{a}_i) > 0$. If $z \notin A$, then by submodularity we have that $\delta(A/\bar{a}_i) \geq \delta(A/z\bar{a}_i)$, and we just showed the latter to be positive.

Similarly, if $\bar{a}_i \subseteq A$ then $\delta(A/z) = \delta(A/\bar{a}_i z) + \delta(\bar{a}_i z/z)$. The first term is non-negative since U_i is a 0-extension and the second term is equal to $\delta(\bar{a}_i) > 0$ since z is unrelated and $\bar{a} \subseteq M$. If $\bar{a}_i \not\subseteq A$ then $\delta(A/z) \geq \delta(A/\bar{a}_i z)$ by sub-modularity, and the latter is positive by the previous sentence.

To show (2.3), let A be any subset of zU_i. Then if $z \notin A$ the result is immediate by (2.2); if $z \in A$ we have $\delta(A/\bar{a}_i) = \delta(A/\bar{a}_i z) + \delta(\bar{a}_i z/\bar{a}_i)$ which must be positive since $\bar{a}_i z \leq_r A$ and $\bar{a}_i \preccurlyeq \bar{a}_i z$.

For (2.4), we first note that for fixed j, $\bar{a}_j \preccurlyeq W$. In fact, for $A \subseteq W \setminus \bar{a}_j$, we have

$$\begin{aligned}
\delta(A/\bar{a}_j) &= |A \setminus \bar{a}_j| - re(A/\bar{a}_j) \\
&= |A \setminus \bar{a}_j| - r\left(e(U_j z \cap A/\bar{a}_j) + \sum_{i \neq j} e(U_j z \cap A)\right) \\
&= |U_j z \cap A \setminus \bar{a}_j| - re(U_j z \cap A/\bar{a}_j) + \sum_{i \neq j} |U_i \cap A| - re(U_i z \cap A) \\
&= |U_j z \cap A \setminus \bar{a}_j| - re(U_j z \cap A/\bar{a}_j) + \sum_{i \neq j} |U_i z \cap A \setminus z| - re(U_i z \cap A) \\
&= |U_j z \cap A \setminus \bar{a}_j| - re(U_j z \cap A/\bar{a}_j) + \\
&\qquad \sum_{i \neq j}\Big(|U_i z \cap A \setminus z| - r(e(U_i z \cap A \setminus z) + e(U_i z \cap A, z))\Big) \\
&= \delta(A \cap U_j z/\bar{a}_j) + \sum_{i \neq j} \delta(A \cap U_i z/z)
\end{aligned}$$

Each term is positive by (2.2).

Let $X = \cup_{i<N} \bar{a}_i$. Then note that for any $U' \subsetneq U_i \backslash Xz$, $Xz \preccurlyeq U'$. We calculate:

$$\begin{aligned} \delta(U'/Xz) &= |U'| - r[e(U') + e(U', z) + e(U', X)] \\ &= |U'| - r[e(U') + e(U', z) + e(U', \bar{a}_i)] \\ &= \delta(U'/\bar{a}_i z) \end{aligned}$$

Since $\bar{a}_i z \preccurlyeq U'$, we have $Xz \preccurlyeq U'$.

Finally, we show that $X \preccurlyeq A$ for A any subset of W. We consider two cases. If $z \in A$, then

$$\delta(A/X) = \delta(A/Xz) + \delta(Xz/X)$$

Since z is unrelated to X, we have $\delta(Xz/X) = 1$. Note that $\delta(A/Xz) = \delta(A \setminus \{ z \}/Xz) = \sum_i \delta(A \cap U_i/Xz)$, which is positive by the previous paragraph unless $A \setminus \{ z \} = \bigcup_{i<N} U_i$, in which case it is zero. In either case $\delta(A/X) \geq 1$.

If $z \notin A$, then

$$\begin{aligned} \delta(A/X) &= \delta(\oplus_{i<N} A \cap U_i/X) \\ &= \sum_{i<N} \delta(A \cap U_i/X) \\ &= \sum_{i<N} \delta(A \cap U_i/(A \cap U_i)^X) \\ &= \sum_{i<N} \delta(A \cap U_i/\bar{a}_i) \end{aligned}$$

Since $\bar{a}_i \preccurlyeq U_i$, each term of the displayed sum is positive, and $X \preccurlyeq A$.

Let $m > |U \setminus \bar{a}|$. By the semi-genericity of M, there is a W' which is an embedding of W into M over $\cup R_0$ satisfying $\mathrm{cl}^m(W') = \mathrm{cl}^m(\cup R_0) \oplus_{\cup R_0} W'$. Letting

z' be the image of z in W', it is clear that $M \models R_z(\bar{a})$ for any $\bar{a} \in R_0$. Conversely, suppose that $M \models R_z(\bar{a})$ for $\bar{a} \in S^n$. Then for some permutation σ and some tuple $\bar{u}'$ in M, $z\sigma(\bar{a})\bar{u}'$ satisfies Δ_U. Thus $\delta_r(u'/z\sigma(\bar{a})) = 0$, so that $\bar{u}' \subseteq \mathrm{cl}^m(W')$. We also have that $M \models E(z, u_0)$ for some $u_0 \in \bar{u}'$ by biminimality. Semigenericity yields that $u_0 \in \mathrm{cl}^m(\cup R_0)$ or $u_0 \in W'$. In the former case, we contradict that $\mathrm{cl}^m(W') = \mathrm{cl}^m(\cup R_0) \oplus_{\cup R_0} W'$ since $z \in W' \setminus \cup R_0$. Thus the latter case holds, and u_0 is part of some realization of U already in W'. Since such realizations are pairwise disjoint over zS, we must have $\bar{u}' \subseteq W'$ and $\bar{a} \in R_0$. $\square$

Corollary 2.2.2. *Let M be as above and consider definable $S, T \subseteq M$. Suppose $D(x, y; \bar{a})$ and $E(x, y; \bar{b})$ are definable classes of equivalence relations on $S(M)$ and $T(M)$ respectively. Let $D_{\bar{a}}$ and $E_{\bar{b}}$ respectively denote the equivalence relations $D(\cdot, \cdot; \bar{a})$ and $E(\cdot, \cdot; \bar{b})$. Also, $n(D_{\bar{a}}), n(E_{\bar{b}})$ will denote the number of $D_{\bar{a}}$ (respectively $E_{\bar{b}}$) equivalence classes in S (respectively T). If $n(D_{\bar{a}})$ and $n(E_{\bar{b}})$ are both finite, then for any v and $R_v(x_0, x_1)$ as in the representation lemma, we can define the following sentences on v, uniformly in $\bar{a}$ and $\bar{b}$:*

1. *$F_{D_{\bar{a}}}(v)$ states that R_v represents a function with domain $S/D_{\bar{a}}$.*
2. *$I(v)$ states that R_v represents an injection.*
3. *$S_{E_{\bar{b}}}(v)$ states that R_v represents a surjection on $E_{\bar{b}}$ classes.*
4. *$J_{D_{\bar{a}}, E_{\bar{b}}}(v)$ states that R_v represents a relation between $D_{\bar{a}}$ classes and $E_{\bar{b}}$ classes.*

As a consequence, if each $D_{\bar{a}}$-class and each $E_{\bar{b}}$-class is finite, we get:

- *$n(D_{\bar{a}}) < n(E_{\bar{b}})$ is first order*

- *$n(D_{\bar a}) = n(E_{\bar b})$ is first order*

- *Given a definable $R \subseteq M$ and a definable equivalence relation $C(x, y, \bar c)$ on R, the relation $n(D_{\bar a})n(E_{\bar b}) = n(C_{\bar c})$ is first order definable (uniformly in $\bar a, \bar b, \bar c$)*

Proof.

$$
\begin{aligned}
F_{D_{\bar a}}(v) :=& \forall x \in S\, \exists x' \in S\ [D_{\bar a}(x, x') \wedge \exists! y \in T\, R_v(x', y)] \wedge \\
& \forall x' \in S[D_{\bar a}(x, x') \wedge \exists y R_v(x', y) \to x = x'] \\
I(v) :=& \forall x_0, x_1 \in S\ [\exists y \in T R_v(x_0, y) \wedge R_v(x_1, y) \to x_0 = x_1] \\
S_{E_{\bar b}}(v) :=& \forall y \in T\, \exists y' \in T\ [\exists x \in S\ R_v(x, y') \wedge E_{\bar b}(y, y')] \\
J_{D_{\bar a}, E_{\bar b}} :=& \forall x_0 x_1 [\exists y_0 y_1 R_v(x_0, y_0) \wedge R_v(x_1, y_1) \wedge D_{\bar a}(x_0, x_1) \to x_0 = x_1] \wedge \\
& \forall y_0 y_1 [\exists x_0 x_1 R_v(x_0, y_0) \wedge R_v(x_1, y_1) \wedge E_{\bar b}(y_0, y_1) \to y_0 = y_1]
\end{aligned}
$$

Given these, $n(D_{\bar a}) < n(E_{\bar b})$ can be written as saying that there is some v so that R_v is defined on $D_{\bar a}$ and $E_{\bar b}$ classes, and is an injective function which is not surjective. That is, we write $\exists v J_{D_{\bar a}, E_{\bar b}}(v) \wedge F_{D_{\bar a}}(v) \wedge I(v) \wedge \neg S_{E_{\bar b}}(v)$. Such a v will exist for finite $D_{\bar a}, E_{\bar b}$-classes by the previous proposition.

Saying $n(D_{\bar a}) = n(E_{\bar b})$ can be accomplished by writing $\exists v J_{D_{\bar a}, E_{\bar b}}(v) \wedge F_{D_{\bar a}}(v) \wedge I(v) \wedge S_{E_{\bar b}}(v)$.

To encode that $n(D_{\bar a})n(E_{\bar b}) = n(C_{\bar c})$, we create an equivalence relation with exactly $n(D_{\bar a})n(E_{\bar b})$ and apply the previous paragraph. Thus, we let $\pi(v)$ be

$$
J_{D_{\bar a}, E_{\bar b}}(v) \wedge \forall x \in S\, \forall y \in T\, [\exists x' \in S\ D_{\bar a}(x, x') \wedge \exists y' \in T E_{\bar b}(y, y') \wedge R_v(x', y')]
$$

(i.e., π says that R_v relates every $D_{\bar a}$ class to every $E_{\bar b}$ class). Let $E_v(u_0, u_1) :=$

$\exists \bar{u}[\exists xy \Delta_U(vxy\bar{u})] \wedge u_0 \in \bar{u} \wedge u_1 \in \bar{u}$ (i.e. u_0 and u_1 are in the same copy of U over some $vxy\bar{u}$). Then we write $n(D_{\bar{a}})n(E_{\bar{b}}) = n(C_{\bar{c}})$ as $\exists v \pi(v) \wedge$ "$n(E_v) = n(C_{\bar{c}})$".

□

This will be enough to show that any model of Σ_r interprets R, which we now define.

Definition 2.2.3. Let L_R, the language of arithmetic, be given by $L_R^{\mathrm{nl}} = \{+, \cdot, \leq, 0, 1\}$. Let η_s represent the term $\underbrace{\text{‘}1 + \cdots + 1\text{'}}_{s \text{ times}}$. Then Robinson's R is given by the following axiom schemes, for every $s, t \in \omega$:

1. $\eta_s + \eta_t = \eta_{s+t}$
2. $(\eta_s) \cdot (\eta_t) = \eta_{st}$
3. $\eta_s \neq \eta_t$ for $s \neq t$
4. $\forall x, x \leq \eta_s \rightarrow x = \eta_0 \vee \ldots \vee x = \eta_s$
5. $\forall x, x \leq \eta_s \vee \eta_s \leq x$

Theorem 2.2.4. *Let $M \models \Sigma_r$. Then M recursively interprets a model $(\omega', +, \cdot, \leq, 0, 1)$ of Robinson's R.*

Proof. Fix $A \in \mathbf{K}$ and choose any B so that (A, B) is a 0-extension. We will equate natural numbers with the number of disjoint copies of B over A. Define $\omega'(\bar{x})$ as $\Delta_A(\bar{x}) \wedge \forall \bar{y}_1 \forall \bar{y}_2 \left[\left(\Delta_B(\bar{x}\bar{y}_1) \wedge \Delta_B(\bar{x}\bar{y}_2) \right) \rightarrow \left(\bar{y}_1 \cap \bar{y}_2 = \varnothing \vee \bigvee_\sigma \sigma(\bar{y}_1) = \bar{y}_2 \right) \right]$ where σ ranges over all permutations of $\bar{y}_1$ and $=$ is interpreted in the obvious way.

Different representations of natural numbers will be equated if they represent the same number. To define this, we will define an equivalence relation which equates elements of the same realization of B over A. We then equate elements of ω' which have the same number of classes under this relation. Specifically, we define $E(u_0, u_1; \bar{x})$ (alternatively, $E_{\bar{x}}(u_0, u_1)$) to be

$$\omega'(\bar{x}) \wedge \exists \bar{b} \Delta_B(\bar{x}\bar{b}) \wedge u_0 \in \bar{b} \wedge u_1 \in \bar{b}$$

. We then define $=_\omega' (\bar{x}, \bar{y})$ as $\omega'(\bar{x}) \wedge \omega'(\bar{y}) \wedge$ "$n(E_{\bar{x}}) = n(E_{\bar{y}})$".

To define addition, first let $E^A(u, v; \bar{x}, \bar{y})$ be

$$\omega'(\bar{x}) \wedge \omega'(\bar{y}) \wedge \exists \bar{b}(\Delta_B(\bar{x}\bar{b}) \vee \Delta_B(\bar{y}\bar{b})) \wedge u \in \bar{b} \wedge v \in \bar{b}$$

We then define addition as $A(\bar{x}, \bar{y}, \bar{z}) :=$ "$n(E^A_{\bar{x}\bar{y}}) = n(E_{\bar{z}})$". Similar, we the graph of multiplication $M(\bar{x}, \bar{y}, \bar{z})$ will be given by "$n(E_{\bar{x}})n(E_{\bar{y}}) = n(E_{\bar{z}})$.

Zero and one are defined in the obvious way: zero is $\omega'(\bar{x}) \wedge \neg \exists \bar{w} \Delta_B(\bar{x}\bar{w})$ while one is $\omega'(\bar{x}) \wedge \exists ! \bar{w} \Delta_B(\bar{x}\bar{w})$. The order $\leq$ is definable within the interpretation: $x \leq y$ is interpreted as $\exists z[\omega'(z) \wedge x + z = y]$

It is clear that this defines a recursive interpretation of R

□

Corollary 2.2.5. Σ_r *is essentially undecidable.*

Proof. It is shown in Part II, Theorem 9 of [13] that R is essentially undecidable. Tarski shows that essentially undecidability is transferred by interpretations in Part I, Theorem 7. Although his notion of an interpretation is syntactic, the same argument goes through: let $M \models \Sigma_r$ and let $(\omega', +\cdot, \leq, 0, 1)$ interpret R as guaranteed by the theorem. Let f be a recursive map from L_R sentences to L sentences which is determined by the interpretation. Then, given an L_R sentence σ, $\omega' \models \sigma$ if and only if $M \models f(\sigma)$. Thus a decision procedure for $\mathrm{Th}(M)$ would decide $\mathrm{Th}(\omega', +\cdot, \leq, 0, 1)$ as well, contradicting the essential undecidability of Σ_r. $\square$

Remark 2.2.6. It is worth noting that while the interpreted model will satisfy Robinson's Q when M is the generic, this is not generally true. In particular, for the ultraproducts $\prod_{\mathcal{U}} M_{\alpha_n}$ with $\{\alpha_n\}$ a sequence of decreasing irrationals converging to r and M_{α_n} the Shelah-Spencer graph of weight α_n, it will be definable in each M_{α_n} that there is a maximal number of realizations of B over A. This definition will carry over to the ultraproduct, and the order type of the interpreted $(\omega', \leq)$ will have a copy of ω^* (ω reversed) as a tail.

Corollary 2.2.7. *Let $\mathbf{M}_r$ denote the ultraproduct $\prod_{\mathcal{U}} M_{\alpha_n}$, where $\mathcal{U}$ is any non-principal ultrafilter, $\{\alpha_n\}$ converges to $r \in (0,1)$ and is bounded below by r, and M_{α_n} is a model of Σ_{α_n}. Then both the $(\mathbf{K}_r^+, \preccurlyeq_r)$-generic and $\mathbf{M}_r$ model Σ_r; thus they have essentially undecidable theories.*

Proof. That the $(\mathbf{K}_r^+, \preccurlyeq_r)$-generic models Σ_r was shown in the first section of this paper, we thus restrict our attention to $\mathbf{M}_r$. Note that for any finite graph A, $\mathbf{M}_r \models \exists \bar{x} \Delta_A(\bar{x})$ if and only if $M_{\alpha_n} \models \exists \bar{x} \Delta_A(\bar{x})$ for cofinitely many n. For any given α_n, $M_{\alpha_n} \models \exists \bar{x} \Delta_A(\bar{x})$ exactly when $\alpha_n < h(A)$. If $h(A) > r$, then by the convergence of $\{a_n\}$, cofinitely many M_{α_n} will model $\exists \bar{x} \Delta_A(\bar{x})$ - thus $\mathbf{M}_r$ will as well.

If $A \preccurlyeq_r B$, then by definition we have that $h(A, B) > r$. Then for $\alpha_n < h(A, B)$, we have $G_{\alpha_n} \models \forall \bar{x} \Delta_A(\bar{x}) \to \exists \bar{y} \Delta_B(\bar{x}\bar{y})$. By convergence, this is true for cofinitely many α_n, so that it is true in the ultraproduct as well.

Finally, we show that $\mathbf{M}_r$ is semi-generic. Let $A \subseteq_\omega M$ with $A \preccurlyeq_r B$, let $\bar{a}$ enumerate A. Recall that we defined a formula $\psi^m_{A,B}(\bar{a}\bar{y})$ which states that $\bar{y}$ is a copy of B over A and $\mathrm{cl}^m(\bar{y}) = \bar{y} \oplus_{\bar{a}} \mathrm{cl}^m(\bar{a})$ (see Definition 1.3.23). We will show that $\mathbf{M}_r \models \exists \bar{y} \Delta_B(\bar{a}\bar{y}) \wedge \psi^m_{A,B}(\bar{a}\bar{y})$ by an appeal to Łoś' Theorem. We note that if $X \preccurlyeq_r Y$, then $X \preccurlyeq_{r+\epsilon} Y$ for ϵ sufficiently small. Thus we can choose ϵ so that $A \preccurlyeq_{r+\epsilon} B$ and the C which appear in $\psi^m_{A,B}$ are also in the corresponding formula for $r+\epsilon$ semi-genericity (because there are only finitely many possible candidates for C - any minimal such will have negative predimension at $r+\epsilon$ and will thus appear in the appropriate formula; if a minimal C appears for all $r+\epsilon$ with ϵ sufficiently small, then it must have relative pre-dimension at most 0 and thus appears.).

□

2.2.1 Approximations

In this subsection, we establish an approximate version of the representation theorem for α close to r and use it to fully represent some relations.

Lemma 2.2.8. *For $A \subseteq B$, $\delta_{r+\epsilon}(B/A) = \delta_r(B/A) - \epsilon e(B/A)$*

Proof.

$$\begin{aligned} \delta_{r+\epsilon}(B/A) &= |B \setminus A| - (r+\epsilon)(e(B/A) \\ &= \delta_r(B/A) - \epsilon e(B/A) \end{aligned}$$

□

Recall that in the proof of Lemma **??** we made use of a structure U which was a minimal 0-extension of $\{\bar{s}v\}$ where $\bar{s}$ is an n-tuple and v is an unrelated point.

For $k \in \omega$, we let $S_k := \oplus_{1\leq i\leq k}\bar{s}_i$ and $W_k := v \cup \bigcup_{1\leq i\leq k} U_i$ where each $\bar{s}_i$ is isomorphic to $\bar{s}$, each U_i is isomorphic to U and extends $\bar{s}_i$.

Lemma 2.2.9. *There is an ϵ' so that for $0 < \epsilon < \epsilon'$ we have that for $k \leq \left\lfloor \frac{1}{e(U/\bar{s}v)} \frac{1}{\epsilon} \right\rfloor$, $S_k \leq_{r+\epsilon} W_k$*

Proof. We first calculate:

$$\begin{aligned}\delta_{r+\epsilon}(W_k/S_k) &= \delta_{r+\epsilon}(W_k/S_k v) + \delta_{r+\epsilon}(S_k v/S_k) \\ &= \delta_r(W_k/S_k v) - \epsilon e(W_k/S_k v) + 1 \\ &= 1 - \epsilon k e(U/\bar{s}v)\end{aligned}$$

This is non-negative for $k \leq \left\lfloor \frac{1}{e(U/\bar{s}v)} \frac{1}{\epsilon} \right\rfloor$; we need to show we can make it hereditarily non-negative. We choose ϵ' so that for $0 < \epsilon < \epsilon'$, we have:

1. $\bar{s} \leq_{r+\epsilon} v$

2. $\bar{s} \leq_{r+\epsilon} U$

3. $\bar{s} \leq_{r+\epsilon} vU$

4. $(\bar{s}v, U)$ is a $\leq_{r+\epsilon}$-minimal pair

(We can do this by the equivalent statements in the proof of Lemma **??** since for any A, B if $A \preccurlyeq_r B$ then there is some ϵ' so that $A \leq_{r+\epsilon} B$ for $0 < \epsilon < \epsilon'$). Let W' be a proper subset of W_k; we show that $\delta_{r+\epsilon}(W'/S_k) \geq 0$

If $v \notin W'$, then this is clear from 2) above since $\delta_{r+\epsilon}(W'/S_k) = \sum_{1 \le i \le k} \delta_{r+\epsilon}(W' \cap U_i/\bar{s}_i)$.

If $v \in W'$, we have $\delta_{r+\epsilon}(W'/S_k) = \delta_{r+\epsilon}(W'/S_k v) + \delta_{r+\epsilon}(S_k v/S_k) = 1 + \delta_{r+\epsilon}(W'/S_k v)$. Then $\delta_{r+\epsilon}(W'/S_k v) = \sum_{1 \le i \le k} \delta_{r+\epsilon}(W' \cap U_i/\bar{s}_i v)$. Since $(\bar{s}v, U)$ is a minimal pair, this sum is at least $\delta_{r+\epsilon}(W_k/S_k v)$. Therefore $\delta_{r+\epsilon}(W'/S_k) \ge \delta_{r+\epsilon}(W_k/S_k) \ge 0$ $\square$

For $0 < \epsilon < \epsilon'$, let k_ϵ denote $\left\lfloor \frac{1}{e(U/\bar{s}v)} \frac{1}{\epsilon} \right\rfloor$ and let $G_{r+\epsilon}$ denote the $(K_{r+\epsilon}, \le_{r+\epsilon})$-generic. The following theorem and it's corollary generalize results in [12].

Theorem 2.2.10 (Approximate Representation)**.** *For any $n \in \omega$, let $R(\bar{x}; v)$ denote the formula $\exists \bar{u} \bigvee_\sigma \Delta_U(v\sigma(\bar{x})\bar{u})$ (where $|\bar{x}| = n$ and σ enumerates the permutations of $\bar{x}$). Then for $0 < \epsilon < \epsilon'$ and R_0 any symmetric irreflexive n-ary relation on $G_{r+\epsilon}$ with at most k_ϵ realizations in $G_{r+\epsilon}$, there is some $v \in G_{r+\epsilon}$ so that $G_{r+\epsilon} \models R_v(\bar{x})$ if and only $R_0(\bar{x})$ holds.*

Proof. Let $\bar{X} = \cup_{i<N} \bar{x}_i$ where the $\bar{x}_i$ enumerate the $\bar{x}$ on which R_0 holds. Then for some $k \le k_\epsilon$, $\bar{X} \simeq S_k$, and by the previous lemma $\bar{X} \le_{r+\epsilon} W_k$. Let $Q' = \mathrm{cl}_{G_{r+\epsilon}}(\bar{X})$; then $W_k^{Q'} = \bar{X}$, so that $Q' \le_{r+\epsilon} W_k$. So we must have that W_k embeds strongly into $G_{r+\epsilon}$ over Q', the image of v, say v', under this embedding satisfies $G_{r+\epsilon} \models R_{v'}(\bar{x})$ if and only $R_0(\bar{x})$ holds. $\square$

Corollary 2.2.11. *Consider definable $S, T \subseteq G_{r+\epsilon}$ for $0 < \epsilon < \epsilon'$. Suppose $D(x, y; \bar{a})$ and $E(x, y; \bar{b})$ are definable classes of equivalence relations on $S(G_{r+\epsilon})$ and $T(G_{r+\epsilon})$ respectively. Let $D_{\bar{a}}$ and $E_{\bar{b}}$ respectively denote the equivalence relations $D(\cdot, \cdot; \bar{a})$ and $E(\cdot, \cdot; \bar{b})$. Also, $n(D_{\bar{a}}), n(E_{\bar{b}})$ will denote the number of $D_{\bar{a}}$*

(respectively $E_{\bar{b}}$) equivalence classes in S (respectively T). If $n(D_{\bar{a}})$ and $n(E_{\bar{b}})$ are both less than k_ϵ, then for any v and $R_v(x_0, x_1)$ as in the representation lemma, we can define the following sentences on v, uniformly in $\bar{a}$ and $\bar{b}$:

1. *$F_{D_{\bar{a}}}(v)$ states that R_v is a function with domain $S/D_{\bar{a}}$.*

2. *$I(v)$ states that R_v is injective.*

3. *$S_{E_{\bar{b}}}(v)$ states that R_v is surjective on $E_{\bar{b}}$ classes.*

4. *$J_{D_{\bar{a}}, E_{\bar{b}}}(v)$ states that R_v is relation between $D_{\bar{a}}$ classes and $E_{\bar{b}}$ classes.*

Let $T(\bar{y}; \bar{z})$ be any formula and let $n_\epsilon(T_{\bar{z}})$ denote the number of distinct $\bar{y} \subseteq G_{r+\epsilon}$ such that $G_{r+\epsilon} \models T(\bar{y}; \bar{z})$. Suppose that there is some real number m so that for every ϵ with $0 < \epsilon < \epsilon'$ and every $\bar{z} \subseteq G_{r+\epsilon}$ we have $n_\epsilon(T_{\bar{z}}) \leq m(\frac{1}{\epsilon})$. Then the following sentences on $\bar{z}$ are uniformly definable in $G_{r+\epsilon}$ for such ϵ:

- *$n_\epsilon(T_{\bar{z}})$ is even.*

- *$n_\epsilon(T_{\bar{z}})$ is maximal over all $\bar{z}$*

Proof. The numbered formulae have the same definitions as before; the approximate representation theorem guarentees that they're valid for the prescribed ϵ.

Let $E_{\bar{z}}(y_1, y_2)$ be $\exists \bar{y} T(\bar{y}; \bar{z}) \wedge y_1 \in \bar{y} \wedge y_2 \in \bar{y}$. Let $l = e(U/\bar{s}v)m + 1$. Then we have that for ϵ sufficiently small, $lk_\epsilon \geq m\frac{1}{\epsilon} \geq n_\epsilon(T_{\bar{z}})$. Replacing ϵ' if necessary, we may assume that this holds.

Note that $n_\epsilon(T_{\bar{z}})$ will be even exactly when we can partition it into two equicardinal sets, which will happen exactly when we can find l disjoint subsets of $T_{\bar{z}}$ which can each be partitioned into two equicardinal sets (some of the subsets can be empty). Therefore we code that $n_\epsilon(T_{\bar{z}})$ is even by saying there exists $v_1, \ldots, v_l$ so that each R_{v_i} is defined on $E_{\bar{z}}$-classes, the unions of the domain and range of each R_{v_i} partition $T_{\bar{z}}$, and for each i, that R_{v_i} is a bijection.

To say that $n_\epsilon(T_{\bar{z}})$ is maximal, we want to encode that for any other $\bar{z}'$ there is a surjection $T_{\bar{z}} \to T_{\bar{z}'}$. We again break this up into l different functions, and say that there exist $v_1, \ldots, v_l$ so that each R_{v_i} is a function on $T_{\bar{z}}$ and the union of the ranges of the R_{v_i} is all of $T_{\bar{z}'}$. Literally, we use the following sentence:

$$\begin{aligned}
&\exists v_0 \ldots \exists v_{l-1} \forall \bar{y} T_{\bar{z}}(\bar{y}) \to \exists y_1 \in \bar{y} \exists y_2, \bar{y}' \\
&\qquad\qquad \left[T_{\bar{z}'}(\bar{y}') \wedge y_2 \in \bar{y} \wedge \bigvee_{i<l} R_{v_i}(y_1, y_2)\right] \wedge \\
&\qquad\qquad \left[\forall y_3 y_3 \in \bar{y} \wedge \exists y_4, \bar{y}' \left(T_{\bar{z}'}(\bar{y}) \wedge \bigvee_{i<l} R_{v_i}(y_3, y_4) \to y_3 = y_1\right)\right] \\
&\qquad \bigwedge \forall \bar{y}' T_{\bar{z}'}(\bar{y}') \to \exists y_2 \in \bar{y}' \exists y_1, \bar{y}' \left[T_{\bar{z}}(\bar{y}) \wedge y_1 \in \bar{y} \wedge \bigvee_{i<l} R_{v_i}(y_1, y_2)\right]
\end{aligned}$$

□

2.2.2 Completions

We show in this subsection that Σ_r has continuum many completions and specify a set of formulae on which these differ. We first note that the number of completions comes from very quickly from essential undecidability.

Theorem 2.2.12. *Σ_r has $2^{\aleph_0}$ completions.*

Proof. We will define tree of completions T_η of Σ_r for $\eta \in 2^\omega$ such that each T_η is incomplete and essentially undecidable, and $T_{\eta\wedge 0}, T_{\eta\wedge 1}$ are pairwise inconsistent extension of T_η. Let $T_\varnothing = \Sigma_r$. Having defined T_η, we note that by incompleteness there is a sentence σ so that $T_\eta \cup \{\sigma\}$ and $T_\eta \cup \{\neg\sigma\}$ are both consistent, hence essentially undecidable. We let $T_{\eta\wedge 0}$ denote the former and $T_{\eta\wedge 1}$ denote the latter.

□

The remainder of this subsection will be spent finding explicit families of sentences on which the completions differ. Throughout, G_α will denote the $(\mathbf{K}_\alpha, \leq_\alpha)$-generic for α irrational and the $(\mathbf{K}_\alpha^+, \preccurlyeq_\alpha)$-generic otherwise. Let A be a pair of unrelated points and choose B so that (A, B) is a proper biminimal 0-extension. For $n \in \omega \setminus \{0, 1\}$, let A_n denote a set of n unrelated points, labeled $a_0 \ldots a_{n-1}$. We want to count the total number of copies of B over A_n for various n; this will enable us to define a countable set of sentences whose truth values can be indpendently specified in $G_{r+\epsilon}$ for ϵ close to 0.

As a first step, we determine the maximal number of pariwise disjoint (over A_n) copies of B which will result in a structure with non-negative predimention $\delta_{r+\epsilon}$. If a structure D consists of A_n with a total of N disjoint over A_n copies of B then we compute:

$$\begin{aligned}\delta_{r+\epsilon}(D) &= \delta_{r+\epsilon}(A_n) + N\delta_{r+\epsilon}(B/A)\\ &= n + N(\delta_r(B/A) - \epsilon e(B/A))\\ &= n - N\epsilon e(B/A)\end{aligned}$$

Thus we have that $\delta_{r+\epsilon}(D) \geq 0$ for $N \leq \frac{n}{\epsilon e(B/A)}$. Substituting $y = \frac{1}{\epsilon e(B/A)}$, we have that A_n can have no more than $\lfloor ny \rfloor$ copies of B over any model of Σ_r. We want to show that for y sufficiently large, A_n will have $\lfloor ny \rfloor$ copies of B in $G_{r+\epsilon}$.

Lemma 2.2.13. *Fix $n \in \omega$, let $N := \lfloor ny \rfloor$. For $m = \binom{n}{2}$, write $N = qm + r$ where $q = \left\lfloor \frac{N}{m} \right\rfloor$. Fix a bijection f from m to $\{ (i,j) : i < j < n \}$, and for any l let B^l be $\oplus_{j<l}(B_j/A)$ with B_j isomorphic to B. Finally, we define D_n by*

$$D_n := \bigoplus_{k<m} \left(B^{l_k} / A_n^{a(k)} \right)$$

where $a_{(k)} := \{ a_i, a_j : f(k) = (i,j) \}$ and $l_k := q + 1$ if $k < r$ and is q otherwise.

Then there is a y' so that for $y > y'$, $\varnothing \leq_{r+\epsilon} D_n$, where $y = \frac{1}{\epsilon e(B/A)}$.

Proof. Choose y' so that $AB \in \mathbf{K}_{r+\epsilon}$ for $y \geq y'$; we then induct on n, increasing y' as necessary.

For $n = 2$, let X be any subset of D_2. We have that D_2 is $\bigoplus_{i<\lfloor 2y \rfloor}(B_i/A)$ where B_i is isomorphic to B. If X contains A then $X = \bigoplus_{i<\lfloor 2y \rfloor}(X \cap B_i/A)$, and since (A, B) is a minimal 0-extension, we have $\delta_{r+\epsilon}(X \cap B_i/A) \geq \delta_{r+\epsilon}(B/A)$. Therefore $\delta_{r+\epsilon}(X) \geq \delta_{r+\epsilon}(D_2) \geq 0$.

If not, we consider two subcases. If $X \cap A = \varnothing$, then we have that $\varnothing \preccurlyeq_r X$ so that $\delta_r(X) > 0$. Since $\delta_{r+\epsilon}(X) = \delta_r(X) - \epsilon e(X)$, this quantity is also positive for ϵ sufficiently small. If necessary, increase y' so that $\delta_{r+\epsilon}(X)$ is positive for all $y > y'$ and every $X_0 \subseteq B \setminus A$. Then for such ϵ, any $X \subseteq D_2 \setminus A$ will have positive predimension in $\delta_{r+\epsilon}$.

In the final case, $X \cap A = \{ a \}$ for some point in A. Let $B_0 = B \setminus A$, and note that:

$$\delta_{r+\epsilon}(B/a) = \delta_r(B_0/a) - \epsilon(e(B_0) + e(B_0, a)) \tag{2.5}$$

Letting $A = \{ a, a' \}$, we then calculate:

$$\begin{aligned}
\delta_r(B_0/a) &= \delta_r(B_0a) - \delta_r(a) = \delta_r(B_0a) - 1 \\
\delta_r(B_0a) &= |B_0a| - r(e(B_0) + e(B_0, a)) \\
\delta_r(B_0A) &= |B_0aa'| - r(e(B_0) + e(B_0, a) + e(B_0, a')) \\
&= (|B_0a| + 1) - r(e(B_0) + e(B_0, a)) - re(B_0, a') \\
&= 2
\end{aligned}$$

We thus have $2 - \delta_r(B_0a) = 1 - re(B_0, a')$ so that $\delta_r(B_0a) = 1 + re(B_0, a')$ and $\delta_r(B_0/a) = re(B_0, a')$. Thus (2.5) becomes:

$$\delta_{r+\epsilon}(B/a) = re(B_0, a') - \epsilon(e(B_0) + e(B_0, a))$$

And this is clearly positive for ϵ sufficiently small. We can increase y' large enough to guarentee this.

We have that $X = \oplus_i(X_i/a)$ where $X_i = X \cap B_i$. Therefore $\delta_{r+\epsilon}(X) = \delta_{r+\epsilon}(a)+\sum_i \delta_{r+\epsilon}(X_i/a) = 1+\sum_i \delta_{r+\epsilon}(X_i/a)$. We showed above that for y sufficiently large, $\delta_{r+\epsilon}(B_i/a)$ is positive. If $X_i \subsetneq B_i$, we note that $\delta_r(X_i/a) \geq \delta_r(X_i/A) > 0$ since (A, B) is a $\leq_r$-minimal pair. Thus we have $\delta_{r+\epsilon}(X_i/a) > 0$ for ϵ sufficiently small. Choose y' large enough so that this is true for every X_i properly contained in B_i with $X_i \cap A = \{ a \}$. Then for ϵ determined by $y \geq y'$, we will have $\delta_{r+\epsilon}(X) \geq 0$. Thus we have that $\varnothing \leq_\epsilon D_2$ for y greater than y'.

For the inductive step, let X be any subset of D_n, and consider two cases. If $A_n \subseteq X$, let $C := D_n \setminus X$ and for each $k < m$, let $C_k := C \cap B_k^l$ where l is q or $q + 1$ depending on k. Then $\delta_{r+\epsilon}(D_n/A_n) = \delta_{r+\epsilon}(D_n/X) + \delta_{r+\epsilon}(X/A_n)$ so that $\delta_{r+\epsilon}(X/A_n) = \delta_{r+\epsilon}(D_n/A_n) - \delta_{r+\epsilon}(D_n/X)$. I claim that $\delta_{r+\epsilon}(D_n/X)$ is non-positive - since $\delta_{r+\epsilon}(D_n/A_n) \geq 0$ this will show that $\delta_{r+\epsilon}(X/A_n) \geq 0$ and hence that $\delta_{r+\epsilon}(X) \geq n$. Note that $D_n := \oplus_k(C_k/X)$, so that $\delta_{r+\epsilon}(D_n/X) = \sum_k \delta_{r+\epsilon}(C_k/X)$. Also note that for each k, $(C_k)^X = X \cap a_{(k)}B_k$, so that

$$\begin{aligned}
\delta_{r+\epsilon}(C_k/X) &= \delta_{r+\epsilon}\left(B_k^l/(B_k^l a_{(k)} \setminus C_k)\right) \\
&= \delta_{r+\epsilon}(\oplus_{i<l}(B_i \setminus C_k \cap B_i)/A) \\
&= \sum_{i<l} \delta_{r+\epsilon}\left((B_i \setminus C_k \cap B_i)/A\right)
\end{aligned}$$

Since (A, B) is a $\leq_{r+\epsilon}$-minimal pair, each term of the sum is at most 0, which shows what we want.

If $A_n \not\subseteq X$, then let k be maximal such that $A_k \subseteq X$. We may assume without loss that X is connected. Note that X will then contain at most $\binom{k}{2}$ pairs of vertices with copies of B over them, each of which will have at most $\left\lceil \frac{\lfloor ny \rfloor}{\binom{n}{2}} \right\rceil$ copies of B over it. By the inductive hypothesis, for y sufficiently large, we have that $\varnothing \leq_{r+\epsilon} D_k$. In D_k, each pair of vertices has at least $\left\lfloor \frac{\lfloor ky \rfloor}{\binom{k}{2}} \right\rfloor$ copies of B over it. So we want to find y sufficiently large that $\left\lfloor \frac{\lfloor ky \rfloor}{\binom{k}{2}} \right\rfloor \geq \left\lceil \frac{\lfloor ny \rfloor}{\binom{n}{2}} \right\rceil$. We calculate:

$$\begin{aligned}
\left\lfloor \frac{\lfloor ky \rfloor}{\binom{k}{2}} \right\rfloor &\geq \left\lfloor \frac{2(ky-1)}{k(k-1)} \right\rfloor \\
&= \left\lfloor \frac{2ky}{k(k-1)} - \frac{1}{k(k-1)} \right\rfloor \\
&= \left\lfloor \frac{2y}{(k-1)} - \frac{1}{k(k-1)} \right\rfloor
\end{aligned}$$

Also:

$$\left\lceil \frac{\lfloor ny \rfloor}{\binom{n}{2}} \right\rceil \leq \left\lceil \frac{2ny}{n(n-1)} \right\rceil$$
$$= \left\lceil \frac{2y}{(n-1)} \right\rceil$$

We note that $\frac{2y}{(k-1)} - \frac{1}{k(k-1)}$ is linear in y with slope $\frac{2}{k-1}$ and that $\frac{2y}{(n-1)}$ is linear in y with slope $\frac{2}{n-1}$. Since $k < n$, the slope of the former is greater and it will eventually be large enough so that it's floor is always greater than the ceiling of the latter. $\square$

Corollary 2.2.14. *Fix $n \in \omega$ and let y denote $\frac{1}{\epsilon\ e(B/A)}$. Then there is some y' so that for $y > y'$, there are exactly $\lfloor ny \rfloor$ pairwise disjoint copies of B over A_n in $G_{r+\epsilon}$.*

For $|\bar{x}| = n, |\bar{y}| = |\bar{y}'| = |B \setminus A|$; for $k \in \omega$ let $\bar{x}_k$ denote the pair (x_i, x_j) where $f(k) = (i, j)$. Then let $\Psi_n(\bar{y}; \bar{x})$ be $\Delta_{A_n}(\bar{x}) \wedge \bigvee_{k<\binom{n}{2}} \Delta_B(\bar{x}_k\bar{y}) \wedge \forall \bar{y}'[\bigvee_{k<\binom{n}{2}} \Delta_B(\bar{x}_k\bar{y}') \rightarrow \bar{y} \cap \bar{y}' = \varnothing \vee \bigvee_{k<\binom{n}{2}} \bar{y} \cap \bar{y}' = \bar{x}_k \vee \bigvee_\sigma \bar{y}' = \sigma(\bar{y})$ where σ ranges over permuations of $\bar{y}$. By the results of the previous subsection, there is a sentence σ_m which holds in $G_{r+\epsilon}$ exactly when the maximal number of realizations of Ψ_{2^m} is even for ϵ sufficiently close to 0.

Fix $\eta \in 2^\omega$ such that $\sum_m \eta(m)2^{-m}$ is irrational; we will define a sequence $\{\, \alpha_i : i \in \omega \,\}$ of irrationals converging down to r so that, eventually, $G_{\alpha_i} \models \sigma_m$ if and only if $\eta(m) = 0$. Let I_0 be the interval $(0, 1)$ if $\eta(0) = 0$ or the interval $(1, 2)$ otherwise. Having defined I_m as (c, d), let I_{m+1} be defined as $(c, \frac{d}{2})$ if $\eta(m + 1) = 0$ or the interval $(\frac{d}{2}, d)$ otherwise. If we let y_0 be $\sum_m \eta(m)2^{-m}$ then y_0 is in every interval I_m; and $\cap_m I_m$ must equal $\{\, y_0 \,\}$. Having defined y_i, let $y_{i+1} := y_i + 2$. Then for y sufficiently large and any m, we have that in $G_{r+\epsilon}$, the maximal number of realization of Ψ_{2^m} is $\lfloor 2^m y \rfloor$.

For $m, k \in \omega$ we note that $\lfloor 2^m y \rfloor = k$ on the interval $[2^{-m}k, 2^{-m}(k+1)]$.Therefore $\lfloor 2^m y \rfloor$ is even on the interval $(k2^{-m}, (k+1)2^{-m})$ exactly when k is even. Thus $\lfloor 2^m y \rfloor$ is even on I_m exactly when $\eta(m) = 0$. Furthermore, this remains true in the intervals $2l + I_m$. Since $\lfloor 2^m y \rfloor$ eventually represents the number of realizations of Ψ_{2^m}, we have that $G_{r+\epsilon}$ eventually models σ_m exactly when $\eta(m) = 0$, for y defined as before.

We let $x_i = r+\epsilon_i$, where $y_i = \frac{1}{\epsilon_i \ e(B/A)}$ and let $\mathcal{U}$ be any non-principal ultrafilter, we then have that $\prod_{\mathcal{U}} G_{x_i} \models \sigma_m$ for every m. Since we have uncountably many choices for η, and since each such ultraproduct models Σ, we have that the latter has uncountably many completions.

2.3 General Ultraproducts

In previous sections we analyzed ultraproducts $\prod_{\mathcal{U}} M_{\alpha_n}$ for $\mathcal{U}$ non-principal and $\{ \alpha_n \}$ converging to r, bounded either above or below by r. In this section we work with arbitrary sequences and ultraproducts, and show that no new cases are introduced for the resulting model theory. The basic point is that given any sequence $\{ \alpha_n \}$ on an interval and any ultrafilter $\mathcal{U}$, up to elementary equivalence the ultraproduct $\prod_{\mathcal{U}} M_{\alpha_n}$ looks like an ultraproduct taken over a sequence which is monotonic or constant.

We begin with the following:

Notation 2.3.1. *Let $\mathcal{U}$ be an ultrafilter on ω, let A be in $\mathcal{U}$. Fix a sequence $\{ a_n : n \in \omega \}$*

1. *$\mathcal{U}_A$ will denote the set of subsets of A which are in $\mathcal{U}$.*

2. *$\{ a_n \}^A$ will denote the subsequence $\{ a_n : n \in A \}$*

Remark 2.3.2. Note that for any $A \subseteq \omega$, we have $A \in \mathcal{U}$ if and only if $A \cap B \in \mathcal{U}$ for every $B \in U$: If $A \in \mathcal{U}$ and $B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$ since ultrafilters are closed under finite intersections. On the other hand, if $A \cap B \in \mathcal{U}$ for every $B \in \mathcal{U}$, then $A \in \mathcal{U}$ since $B = \omega$ is an element of $\mathcal{U}$.

There is a sense in which an ultrafilter will choose a unique limit in the current context:

Lemma 2.3.3. *Let I be the interval $[b, c]$, let $m \in \omega$ and let $\{ a_n \}$ be a sequence on I and let $\mathcal{U}$ be an ultrafilter. Let $p_0 = b$, and for $i \leq 2^m$ let $p_i = b + i(\frac{c-b}{2^m})$. Then one of the following two statements holds:*

- *There is some $i \leq 2^m$ and some $A \in \mathcal{U}$ so that $\{ a_n \}^A$ is constantly p_i*

- *There is some $i < 2^m$ and some $A \in \mathcal{U}$ so that $\{ a_n \}^A$ is contained in $[p_i, p_{i+1}]$*

Proof. For $i < 2^m$ let I_i denote the open interval (p_i, p_{i+1}). Let $C_i := \{ n : a_n \in I_i \}$ and let $D_i := \{ n : a_n = p_i \}$. Then the C_i and D_i form a finite partition of ω, so that exactly one of them is an element of $\mathcal{U}$. □

Corollary 2.3.4. *There is a unique $\alpha \in I$ so that for every $A \in \mathcal{U}$, α is an accumulation point for $\{ a_n \}^A$. Furthermore, for M_{α_n} chosen arbitrarily, there is* *some subsequence $\{ b_n \}$ of $\{ a_n \}$ for which $\prod_{\mathcal{U}} M_{a_n} \equiv \prod_{\mathcal{U}} M_{b_n}$ and $\{ b_n \}$ converges monotonically to α.*

Proof. If there is some $m \in \omega$ such that for some $A \in \mathcal{U}$ we have $\{ a_n \}^A$ is constantly p_i for one of the p_i associated with m, then let $\alpha = p_i$ and let the sequence be the constant sequence on p_i.

Otherwise, for each m let choose an interval C_m and an $A_m \in \mathcal{U}$ as in the second clause of the conclusion of the last lemma. Let r_m denote the right endpoint of this interval, then $\{ r_m \}$ is a Cauchy sequence and must converge to some $\alpha \in I$. Letting $B_\epsilon(\alpha)$ denote the ϵ-ball around α, we note that if there is some $B \in \mathcal{U}$ and $\epsilon > 0$ so that $\{ a_n \}^B$ is disjoint from $B_\epsilon(\alpha)$, then choosing m so that $2^{-m} < \frac{\epsilon}{2}$ gives $C_m \subseteq B_\epsilon(\alpha)$ and $B \cap A_m = \varnothing$, a contradiction.

We partition ω with the three sets $P_l := \{ n : a_n < \alpha \}, P_c := \{ n : a_n = \alpha \}$, and $P_r := \{ n : a_n > \alpha \}$. Exactly one of these sets, call it P, is in $\mathcal{U}$. If $P = P_c$ we're done as before, otherwise enumerate $\mathrm{Th}(\prod_{\mathcal{U}} M_{a_n})$ as $\{ \sigma_i : i \in \omega \}$. Then for each i, let Q_i be the intersection of $\{ n : M_{a_n} \models \sigma_i \}$ with P. By Łoś, $Q_i \in \mathcal{U}$. For any i, define R_i to be $A_i \cap Q_i$, an element of $\mathcal{U}$. Then R_i will define a set of indices so that for $n \in R_i$, $|a_n - \alpha| < 2^{-i}$ and $M_{a_n} \models \sigma_i$. Also, for $i \in \omega$, let $D_i = R_1 \cap \ldots \cap R_i$, so that D_i defines a set of indices n so that $n \in D_i$ implies that $|a_n - \alpha| < 2^{-i}$ and for every $m \leq n$, $M_{a_n} \models \sigma_m$. Choose b_1 to be any element of D_1. If b_n has been defined, let m be the least number for which $b_n \notin D_m$, and pick b_{n+1} to an arbitrary element of D_m. Then $\{ b_n \}$ clearly converges monotonically to α. Note that for any i, $b_n \in Q_i$ for $n \geq i$. Thus cofinitely many $M_{b_n} \models \sigma_i$, which shows that any ultraproduct of the M_{b_n} over an ultrafilter is elementarily equivalent to $\prod_{\mathcal{U}} M_{a_n}$, since $\cup_i\{\,\sigma_i\,\}$ is complete. $\square$

Notation 2.3.5. *The point α in the above theorem will be denoted by $\{\,a_n\,\}^{\mathcal{U}}$*

Theorem 2.3.6. *Let $\{\,\alpha_n : n \in \omega\,\}$ be any sequence in $(0,1)$. Let $M_{\alpha_n} \models \Sigma_{\alpha_n}$. Let $\mathcal{U}$ be any ultrafilter on ω. Then the following are equivalent:*

1. *$\prod_{\mathcal{U}} M_{\alpha_n}$ has a decidable theory.*

2. *There is some $\{\,b_n\,\}$ a monotonically increasing subsequence of $\{\,\alpha_n\,\}$ converging on r such that $\prod_{\mathcal{U}} M_{\alpha_n} \equiv \prod_{\mathcal{U}'} M_{b_n}$*

3. *$\mathrm{Th}(\prod_{\mathcal{U}} M_{\alpha_n})$ is the theory of the $(\mathbf{K}_r, \leq_r)$-generic.*

Proof. Choose $\{\,b_n\,\}$ and α from Corollary 2.3.4 so that $\{\,b_n\,\}$ converges monotonically to α and $\prod_{\mathcal{U}} M_{\alpha_n} \equiv \prod_{\mathcal{U}} M_{b_n}$. Then by the results of the previous section, if $\{\,b_n\,\}$ is not strictly increasing the resulting theory is an extension of Σ_r, which is undecidable. The equivalence of the three conditions is now immediate. $\square$

Chapter 3

Tirthikas [1]

So far, we have focused on rationals in $(0,1)$. This chapter will examine the behaviour of $S_{\preccurlyeq_r}$ for other rational values of r. The case $r = 0$ is familiar:

Remark 3.0.7. The $(\mathbf{K}_0^+, \preccurlyeq_0)$-generic is precisely the Rado graph. Indeed, I claim that for any $A \subsetneq B$, $\delta_\beta(B/A) > 0$ for some $\beta > 0$. If $e(B) = e(A)$ then this is obvious, otherwise to ensure that $|B - A| - \beta(e(B) - e(A)) > 0$, we can choose any $\beta < \frac{|B-A|}{e(B)-e(A)}$, since the right hand side is always positive. Since there are only finitely many subgraphs between A and B, it follows that $A \preccurlyeq_0 B$ (choose β to be the minimum of the βs that work for each subset.) Thus the amalgamation class is simply the set of finite graphs and $\preccurlyeq_0$ is simply substructure, so that the limit gives the random graph.

Intuitively, we can think of $(\mathbf{K}_0^+, \preccurlyeq_0)$ as trivializing the notion of $\preccurlyeq$ in that it reduces to $\subseteq$. At the other extreme, for $r \geq 1$, the relation $A \preccurlyeq_r B$ is trivialized in a different way - it expresses that A and $B \setminus A$ are in different components:

Lemma 3.0.8. *For $r \geq 1$ and $A, B \in \mathbf{K}_r^+$, we have that $A \preccurlyeq_r B$ if and only if $e(B, A) = \varnothing$*

Proof. Suppose $e(B, A) = \varnothing$. Then for $X \subseteq B \setminus A$, we have $\delta_r(X/A) = |X| -$

[1] In Buddhist philosophy, a *tirthika* is someone with extreme beliefs. In this section, we examine structures with extreme beliefs about the meaning of $\varnothing \leq A$ and $A \leq B$.

$re(X, A) = |X|$ which is positive for non-empty X. If $e(B, A) \neq \varnothing$ choose $b \in B$ with an edge to some vertex in A. Then $\delta_r(Ab/A) = 1 - re(b, A) \leq 0$, which shows that $A \not\preccurlyeq_r B$ □

Corollary 3.0.9. *For $r \geq 1$ and $M \models S_{\preccurlyeq_r}$, for any $A \subseteq_\omega M$, $\mathrm{cl}_M(A) = \mathrm{cl}_M^m(A)$ (for $m \geq 1$) and both are given by the union of the connected components of M containing a non-trivial subset of A.*

Proof. Let A_0 be a connected subset of A; we show that it's closure is the connected component of M containing A_0, which we denote by A_1. Choose $b \in A_1$ and let n denote the length of the shortest path from b to some element of A. If $n = 0$ we have $b \in A_0 \subseteq \mathrm{cl}(A_0)$. If $n = 1$, we have by the above lemma that $A_0 \not\preccurlyeq_r A_0 b$, so that $b \in \mathrm{cl}(A_0)$. Inducting on n, we get $A_1 \subseteq \mathrm{cl}(A_0)$. The previous lemma give us that $A_1 \preccurlyeq_r M$ if A_1 is finite. For infinite A_1, choose finite $X \subseteq A_1$ and finite Y such that (X, Y) is a minimal pair. We have that $e(Y, X) \neq \varnothing$, so that some vertex in $Y \setminus X$ is in A_1. An induction on $|Y \setminus X|$ shows that $Y \subseteq A_1$. □

Proposition 3.0.10. *For $r \geq 1$, let $\{ A_i : i \in \omega \}$ enumerate the connected elements of $\mathbf{K}_r^+$. Then the $(\mathbf{K}_r^+, \preccurlyeq_r)$-generic is equal to $\bigoplus_{i \in \omega} \oplus_{j < \omega} A_i$.*

Proof. Let $G = \bigoplus_{i \in \omega} \oplus_{j < \omega} A_i$. It suffices to show that G satisfies the properties of a generic. It is clearly countable with age contained in $\mathbf{K}_r^+$. We show that if $A \preccurlyeq_r G$ and $A \preccurlyeq_r B$, then there is a strong embedding of B into G over A, for $A, B \in \mathbf{K}_r^+$.

Since $A \preccurlyeq_r B$, we have $e(A, B) = 0$, and $B \setminus A$ is an element of $\mathbf{K}_r^+$ since B is. Each connected component of $B \setminus A$ must embed strongly into G (i.e., as a connected component), hence $B \setminus A$ does as well. □

Remark 3.0.11. It is worth noting that no cycles can appear in any of the generics: since an n-cycle has n vertices and n edges, it's pre-dimension with respect to δ_r for $r \geq 1$ will be non-positive.

The remainder of this chapter will be devoted to an analysis of the cases that $r > 1$ and $r = 1$

3.1 Behavior for $r > 1$

For $1 \leq r < 2$, the $(\mathbf{K}_r^+, \preccurlyeq_r)$-generic will be a countable forest of finite trees, while for $r \geq 2$ the generics becomes simply a countable collection of isolated points.

Lemma 3.1.1. *For $r > 1$, the connected elements of $\mathbf{K}_r^+$ are finite trees with fewer than $\frac{r}{r-1}$ elements. In particular, for $r \geq 2$, the only connected element of $\mathbf{K}_r^+$ is a singleton.*

Proof. A tree A with n vertices will have $n-1$ edges, and thus $\delta_r(A) = n - r(n-1) = n(1-r) + r$. This will be strictly positive when $n < \frac{r}{r-1}$. Since $n(1-r)+r$ is clearly strictly decreasing in n, we have that any tree satisfying $\delta_r(A) > 0$ will be in $\mathbf{K}_r^+$ and that these are precisely the connected components of $\mathbf{K}_r^+$ $\square$

Proposition 3.1.2. *For $r > 1$, $S_{\preccurlyeq_r}$ is complete and the $(\mathbf{K}_r^+, r)$-generic is ω-categorical.*

Proof. I claim that any model of $S_{\preccurlyeq_r}$ has finite closures. If not, then there is some $M \models S_{\preccurlyeq_r}$ and $a \in M$ which is contained in an infinite connected component. This contradicts that $\mathrm{Age}(M) = \mathbf{K}_r^+$. It is clear that for $A \preccurlyeq_r B$, any embedding of B into M over A will be strong. It follows that M is the generic. $\square$

3.2 Behavior for $r = 1$

To analyze the model-theory of $(\mathbf{K}_1^+, \preccurlyeq_1)$-generic, we first recall that each component consists of cycle-free graphs (i.e. trees without a named root) and introduce some ancillary definitions. Throughout, $\mathbf{M}$ will denote a monster model of the theory of the $(\mathbf{K}_1^+, \preccurlyeq_1)$-generic.

Definition 3.2.1.

- For $A \subseteq \mathbf{M}$, $\mathrm{comp}(A)$ is the set of connected components of elements of A.
- For a, b in the same component, $\mathrm{path}(a, b)$ denotes the shortest path from a to b. Because $\mathbf{M}$ is cycle-free, this is uniquely defined.
- For a, b as before, $\mathrm{dist}(a, b)$ represents the length of $\mathrm{path}(a, b)$.

We want to analyze dividing in $\mathbf{M}$ and show that the theory of the generic is simple. The crux of the argument is to understand the action of the automorphism group of $\mathbf{M}$, the main case of which will involve automorphisms in a fixed component.

Definition 3.2.2. For a, b, c in the same component:

- $\mathbf{\Delta}_c$ is the tree which consists of $\mathrm{comp}(c)$ with c as the root.
- $\mathbf{\Delta}_c^a$ is the maximal subtree of $\mathbf{\Delta}_c$ which is rooted at a child of c (i.e. vertex of distance 1) and contains a. The root of this tree will be denoted as $\mathrm{root}(\mathbf{\Delta}_c^a)$.
- For any set of vertices $\{\, b_i : i < N \,\}$ $(N > 1)$ in the same component as c, define $\mathrm{meet}_c(b_0, \ldots, b_{N-1})$ to be the element of $\mathbf{\Delta}_c$ which is a common ancestor

of every element of $\{ b_i \}$ and has the maximal distance from c among such common ancestors. Note that for tuples $\bar{a}, \bar{b}$ with $\operatorname{comp}(\bar{a}) = \operatorname{comp}(\bar{b}) = \operatorname{comp}(c)$, we have that $\operatorname{meet}_c(\bar{a}, \bar{b}) = \operatorname{meet}_c(\operatorname{meet}_c(\bar{a}), \operatorname{meet}_c(\bar{b}))$.

- For $\{ b_i \}$ as above, $\mathbf{\Delta}_c^{b_0,\ldots,b_{N-1}}$ is the subtree of $\mathbf{\Delta}_c$ which is rooted at $\operatorname{meet}_c(b_0, \ldots, b_{N-1})$.

Dividing over the empty set is easily characterized:

Lemma 3.2.3. *For $\bar{a}, \bar{b} \subseteq_\omega \mathbf{M}$, $\bar{a} \mathop{\smile\!\!\!\!\!\!\,|}\, \bar{b}$ if and only if* $\operatorname{comp}(\bar{a}) \cap \operatorname{comp}(\bar{b}) = \varnothing$

Proof. Suppose that there are $a \in \bar{a}, b \in \bar{b}$ so that $\operatorname{comp}(a) = \operatorname{comp}(b)$. Let $\phi(x, b)$ state that $\operatorname{dist}(x, b) = \operatorname{dist}(a, b)$, and let $\{ b_i : i \in \omega \}$ be of $\operatorname{tp}(b)$ with each b_i in a different component. Then $\{ b_i \}$ witnesses that ϕ 2-divides. Define a sequence $\{ \bar{b}_i \}$ by letting $\bar{b}_i$ be the image of $\bar{b}$ under an automorphism $b \mapsto b_i$. Then, if b is the kth element of $\bar{b}$, letting $\psi(\bar{x}, \bar{b})$ be $\phi(\bar{x}^{(k)}, b)$ (where $\bar{x}^{(k)}$ denotes the kth element of $\bar{x}$), we have that $\{ \bar{b}_i \}$ witnesses the dividing of ψ over $\varnothing$

If $\operatorname{comp}(\bar{a}) \cap \operatorname{comp}(\bar{b}) = \varnothing$, then for any $\{ \bar{b}_i : i \in \omega \}$ of $\operatorname{tp}(\bar{b})$, there is some infinite I so that $\{ b_i : i \in \omega \}$ are all in different components or all in the same component. Without loss, we may assume that $I = \omega$. In the first case, we can choose automorphisms $\bar{b} \mapsto \bar{b}_i$ which fix $\bar{a}$, so that $\operatorname{tp}(\bar{a}/\bar{b})$ does not divide over $\varnothing$. In the second case, we apply saturation to choose $\bar{a}'$ of $\operatorname{tp}(\bar{a})$ in a different component than every $\bar{b}_i$; this allows a choice of automorphisms $\bar{b} \mapsto \bar{b}_i$ which fix $\bar{a}'$, so that $\bar{a}' \models \cup_{i \in \omega, \phi(\bar{x}, \bar{b}) \in \operatorname{tp}(\bar{a}/\bar{b})} \phi(\bar{x}, \bar{b}_i)$ and $\operatorname{tp}(\bar{a}/\bar{b})$ does not divide. □

Characterizing dividing over a non-empty base will rest on the following lemma:

Lemma 3.2.4. *Fix $\sigma : \bar{x} \mapsto \bar{y}$ an automorphism over c, and let $z := \mathrm{meet}_c(\bar{x}\bar{y})$, $z_{\bar{x}} := \mathrm{meet}_c(\bar{x})$, $z_{\bar{y}} := \mathrm{meet}_c(\bar{y})$. Then $\sigma(\mathbf{\Delta}_z^{z_{\bar{x}}}) \subseteq \mathbf{\Delta}_z^{z_{\bar{y}}}$.*

Further, for any $\bar{x}, \bar{y}$ with the same type over c, there is some $\sigma \in \mathrm{Aut}(\mathbf{M}/c)$ which maps $\mathbf{\Delta}_z^{z_{\bar{x}}}$ isomorphically onto $\mathbf{\Delta}_z^{z_{\bar{y}}}$ and fixes everything else (for $z, z_{\bar{x}}, z_{\bar{y}}$ defined as above).

Proof. Let z_0 be the root of $\mathbf{\Delta}_z^{z_{\bar{x}}}$ so that $\mathbf{\Delta}_z^{z_{\bar{x}}} = \mathbf{\Delta}_{z_0}$; similarly let z_1 be the root of $\mathbf{\Delta}_c^{z_{\bar{y}}}$. Note that z_0 is definable over $\bar{x}c$, and that it's image under σ must be z_1. Let $k = \mathrm{dist}(c, z_1)$ and let $\phi(x, c)$ be the formula which says $\mathrm{dist}(x, c) > k$. Note that for $i \in 2$ we can define $\Delta_z^{z_i}$ as the set of all w for which the kth element of $\mathrm{path}(c, w)$ is z_i and $\mathrm{dist}(c, w) \geq \mathrm{dist}(c, z_i)$. Thus we must have $\sigma(\mathbf{\Delta}_{z_0}) \subseteq \mathbf{\Delta}_{z_1}$

For the second statement, we know that for some $\sigma \in \mathrm{Aut}(\mathbf{M}/c)$, $\sigma : \bar{x} \mapsto \bar{y}$ since $\mathbf{M}$ is homogeneous. Letting z_0, z_1 be as before, a compactness argument shows that this can be chosen as an isomorphism from $\mathbf{\Delta}_{z_0}$ to $\mathbf{\Delta}_{z_1}$. Let τ be σ on $\mathbf{\Delta}_{z_0}$, σ^{-1} on $\mathbf{\Delta}_{z_1}$, and the identity everywhere else. Then τ is an automorphism: fixing a, b, we show that $E(a, b)$ if and only if $E(\tau(a), \tau(b))$. If a, b are both outside of $\mathbf{\Delta}_{z_0} \cup \mathbf{\Delta}_{z_1}$ or both in one subtree, this is clear. If a is outside the sub-trees but b is in one of them, then $E(a, b)$ implies that a is z (thus fixed by τ) and b is one of z_0, z_1, so that $\tau(b)$ is the other and $E(\tau(a), \tau(b))$ holds. □

Lemma 3.2.5. *For $\mathrm{comp}(a) = \mathrm{comp}(b) = \mathrm{comp}(c)$, we have $a \mathop{\smile\!\!\!\!\!\!\;|}_c b$ if and only if the tree $\mathbf{\Delta}_c^{ab}$ has finitely many conjugates over c.*

Proof. If $\mathbf{\Delta}_c^{ab}$ has only finitely many conjugates over c, then for $\{ b_i : i \in \omega \}$ elements of $\mathrm{tp}(b/c)$, there is some conjugate $\mathbf{\Delta}'$ of $\mathbf{\Delta}_c^{ab}$ which contains $\{ b_i : i \in I \}$ for I an infinite subset of ω. Let σ be an automorphism over c which maps $\mathbf{\Delta}_c^{ab}$ to $\mathbf{\Delta}'$. Letting a' be $\sigma(a)$, b' be $\sigma(b)$ and $z = \mathrm{meet}(b_i : i \in \omega)$, Lemma 3.2.4 implies that there is an automorphism $b' \mapsto b_i$ over c which fixes everything outside $\mathbf{\Delta}_z^{b'}$. If a' is in this tree, then we must have that $b = \mathrm{meet}_c\, a, b$, so that all $b_i = b$. In any case, for every i there is an automorphism $b' \mapsto b_i$ which fixes a'. Thus for any $\phi(x, bc) \in \mathrm{tp}(a/bc)$, $\models \phi(a', b_i c)$, so that ϕ does not divide over c

If $\mathbf{\Delta}_c^{a,b}$ has infinitely many conjugates over c, let $\{\, b_i : i \in \omega \,\}$ be defined by choosing images of b in pairwise disjoint conjugates of $\mathbf{\Delta}_c^{a,b}$. We want $\phi(x, bc)$ to guarantee that any realization is in $\mathbf{\Delta}_c^{a,b}$. Let $d = \mathrm{dist}(c, \mathrm{root}(\mathbf{\Delta}_c^{a,b}))$ and define $\phi(x, bc)$ as the conjunction of formulae asserting that $\mathrm{dist}(c, x) = \mathrm{dist}(c, a)$ and that the dth element of $\mathrm{path}(c, b)$ is also the dth element of $\mathrm{path}(c, x)$. Then ϕ 2-divides, since any realization of $\phi(x, b_i c) \wedge \phi(x, b_j c)$ would have to be an element of both disjoint conjugates of $\mathbf{\Delta}_c^{a,b}$. $\square$

Lemma 3.2.6. *Let $\bar{c}$ be a finite tuple whose elements are in a single component. Let $\bar{b}_c$ be a finite tuple satisfying $\mathrm{comp}(\bar{b}_c) = \mathrm{comp}(\bar{c})$ and that further there is a $c \in \bar{c}$ such that c is the closest element of $\mathrm{dcl}(\bar{c})$ to every $b \in \bar{b}$. For $\{\, \bar{b}_i : i \in \omega \,\}$ of $\mathrm{tp}(\bar{b}_c/c)$, there exist automorphisms $\sigma_i : \bar{b}_c \mapsto \bar{b}_i$ which are over $\bar{c}$ (hence over $\mathrm{dcl}(\bar{c})$).*

Proof. Let $z_1 = \mathrm{meet}_c(\bar{b}_c)$, $z_2 = \mathrm{meet}_c(\bar{b}_i)$, and $z = \mathrm{meet}_c(z_1, z_2)$. By Lemma (3.2.4) we can choose σ_i to map $\mathbf{\Delta}_z^{z_1} \to \mathbf{\Delta}_z^{z_2}$ over c, . If there is some $c' \in \mathrm{dcl}(\bar{c})$ so that $c' \in \mathbf{\Delta}_z^{z_1}$, then for some $b \in \bar{b}_c$ we either have $c' \in \mathrm{path}(bc)$ or $b \in \mathrm{path}(cc')$. The first case contradicts that c is the closest element of $\mathrm{dcl}(\bar{c})$ to b; in the second we have that $b \in \mathrm{dcl}(\bar{c})$ so that $c = b$. Also every σ_i must fix b, so that the σ_i can be chosen to map $\mathbf{\Delta}_w^{w_1} \to \mathbf{\Delta}_w^{w_2}$ where $w_1 = \mathrm{meet}_c(\bar{b}_c \setminus \{b\}), w_2 = \mathrm{meet}_c(\bar{b}_i \setminus \{b\}), w = \mathrm{meet}_c(w_1, w_2)$. If there is some $c'' \in \mathbf{\Delta}_w^{w_1} \cap \mathrm{dcl}(\bar{c})$ then there is some $b' \in \bar{b}_c \setminus \{ b \}$ for which $b' \in \mathrm{path}(b, c'')$ or $c'' \in \mathrm{path}(b, b')$. In the first case, $b' \in \mathrm{dcl}(\bar{c})$, so that b' is the closest element of $\mathrm{dcl}(\bar{c})$ to itself. In the second case, c'' is the closest element to b' of $\mathrm{dcl}(\bar{c})$. Either way, we contradict that $b = c$ is the closest element of $\mathrm{dcl}(\bar{c})$ to b'. Thus we may choose the σ_i over $\bar{c}$. □

Lemma 3.2.7. *Fix $\bar{a}, \bar{b}, \bar{c}$, and for $c \in \mathrm{dcl}(\bar{c})$, let $\bar{a}_c, \bar{b}_c$ denote the subset of $\bar{a}$ (respectively $\bar{b}$) which is closer to c than any other element of $\mathrm{dcl}(\bar{c})$. Similarly, let $\bar{a}_\varnothing, \bar{b}_\varnothing$ denote the subset of $\bar{a}$ (resp. $\bar{b}$) satisfying $\mathrm{comp}(\bar{a}_\varnothing) \cap \mathrm{comp}(\bar{c}) = \varnothing$. Then $\bar{a} \perp_{\bar{c}} \bar{b}$ if and only for every $c \in \mathrm{dcl}(\bar{c}) \cup \{\varnothing\}$, $\bar{a}_c \perp_c \bar{b}_c$.*

Proof. First suppose that $\bar{a}_c \not\perp_c \bar{b}_c$ for some $c \in \mathrm{dcl}(\bar{c})$. Choose $\{ \bar{b}_i : i \in \omega \}$ of $\mathrm{tp}(\bar{b}_c/c)$ and $\phi(\bar{x}_c, \bar{b}_c c) \in \mathrm{tp}(\bar{a}_c/\bar{b}_c c)$ witnessing the dividing. Using Lemma 3.2.6, let $\bar{d}_i := \sigma_i(\bar{b})$ for $\sigma_i : \bar{b}_c \mapsto \bar{b}_i$ over $\bar{c}$. Letting $\psi(\bar{x}, \bar{b}\bar{c}) := \phi(\bar{x}_c, \bar{b}_c c)$, where $\bar{x}_c$ is given the obvious interpretation, we have that $\{ \bar{b}_i \}$ and ψ witness that $\mathrm{tp}(\bar{a}/\bar{b}\bar{c})$ divides over $\bar{c}$.

For the other direction, let $\{ \bar{b}_i \}$ be of $\mathrm{tp}(\bar{b}/\bar{c})$. Then, for each $c \in \bar{c}$, $\bar{b}_i^{(c)}$ is of $\mathrm{tp}(\bar{b}_c/c)$, and we showed that without loss of generality we can find automorphisms $\bar{b}_c \mapsto \bar{b}_i^{(c)}$ which fix some conjugate of $\bar{a}_c$ and also fix $\mathrm{dcl}(\bar{c})$. Lemma 3.2.4 implies that for $c \in \mathrm{dcl}\,\bar{c}$, these can be chosen to fix all elements which do not have c as the closest element of $\mathrm{dcl}(\bar{c})$. Thus composing these maps will map $\bar{b} \mapsto \bar{b}_i$ and fix some conjugate of $\bar{a}$, showing that $\bar{a} \perp_c \bar{b}$. □

Theorem 3.2.8. *For any $\bar{a}, \bar{b}, \bar{c}$, $\bar{a} \perp_{\bar{c}} \bar{b}$ if and only if $\bar{b} \perp_{\bar{c}} \bar{a}$. Thus, T is simple.*

Proof. The previous lemma shows that $\bar{a} \downarrow_{\bar{c}} \bar{b}$ if and only $\bar{a}_c \downarrow_c \bar{b}_c$ for every $c \in \mathrm{dcl}(\bar{c}) \cup \{ \varnothing \}$ which happens if and only if $a \downarrow_c b$ for $c \in \bar{c}, a \in \bar{a}, b \in \bar{b}$; i.e. if and only if $\mathbf{\Delta}_c^{a,b}$ has finitely many conjugates over c. This is clearly a symmetric condition, so that it implies that $b \downarrow_c a$ for all a, b, c and $\bar{b} \downarrow_{\bar{c}} \bar{a}$. $\square$

Counting types, we can show that T is actually stable (but not ω-stable).

Lemma 3.2.9. *For any tree τ with root ρ and depth d, there is a tree $T^0 \subseteq \omega^d$ which is elementarily equivalent to τ (where we interpret $E(\eta_0, \eta_1)$ in ω^d as holding exactly when $\eta_1 = \eta_0 \wedge k$ for some $k \in \omega$ or $\eta_0 = \eta_1 \wedge k$ for such a k).*

Proof. Induct on d; the statement is clear if $d = 0$. Let τ be a tree of depth $d + 1$ with root ρ and consider the set $\{ \mu_\alpha \}$ of subtrees rooted at children of ρ. By the inductive hypothesis, each of these is elementarily equivalent to a subtree of ω^d, say m_α. For $\eta \in \omega^{\leq d}$, let

$$\kappa(\eta) = \begin{cases} |\{ \alpha : m_\alpha = \eta \}| & \text{if } |\{ \alpha : m_\alpha = \eta \}| < \omega \\ \aleph_0 & \text{otherwise} \end{cases}$$

. Then define T_0 as $0 \wedge \bigwedge_{\eta \in \omega^{\leq d}} \eta^{\kappa(\eta)}$.

Fix $k \in \omega$ and consider a k-round Ehrenfeucht-Fraïssé game on T and T^0. If the spoiler plays the root of either structure, the duplicator responds with the root of the other structure. Otherwise the spoiler plays in some m_α or an equivalent subtree η of ω^d. If either have already been chosen from, the duplicator continues with the strategy established by the inductive hypothesis. Otherwise, the duplicator initiates play in an un-played copy of the other structure, using the inductive hypothesis.

There will always be enough copies in either structure to do this by our choice of $\kappa(\eta)$. $\square$

Lemma 3.2.10. *If T is rooted at ρ of depth ω, let T_d denote the maximal subtree of T rooted at ρ with depth d. Let T_d^0 denote an elementarily equivalent tree in ω^d as guaranteed by the previous lemma. Noting that $T_d^0 \subseteq T_{d+1}^0$, we let $T^0 := \cup_{d<\omega} T_d^0$. Then T^0 is a tree in ω^ω which is elementarily equivalent to T.*

Proof. Fix $k \in \omega$ and consider a k-round Ehrenfeucht-Fraïssé game on T and T^0.

If the spoiler picks from level d for $d \in \omega$, the duplicator plays by the strategy witnessing that $T_d \equiv T_d^0$ - by elementary equivalence this strategy can be chosen in a way that is compatible with any previous play. $\square$

Noting the any single connected component can be viewed as a tree of depth at most ω, we immediately get that the theory of the generic is small:

Corollary 3.2.11. *There are at most $2^{\aleph_0}$ 1-types over $\varnothing$ consistent with the theory of the $(\mathbf{K}_1^+, \preccurlyeq_1)$-generic.*

This allows to show that the theory of the $(\mathbf{K}_1^+, \preccurlyeq_1)$ is stable. We note that the $\varnothing$-type of any given connected component will be the type of a tree, and hence will be one of $2^{\aleph_0}$ possibilities.

Theorem 3.2.12. *The theory of the $(\mathbf{K}_1^+, \preccurlyeq_1)$-generic is $2^{\aleph_0}$-stable.*

Proof. Let M be a model of cardinality of $2^{\aleph_0}$; then M clearly realizes at most $2^{\aleph_0}$ types. We show that there are $2^{\aleph_0}$ 1-types over M. Let $a \in \mathbf{M} \setminus M$, and consider

$\mathrm{tp}(a/M)$. If $\mathrm{comp}(a) \cap M = \varnothing$, then $\mathrm{tp}(a/M)$ is determined by $\mathrm{tp}(a/\varnothing)$, and hence is one of $2^{\aleph_0}$ possibilities.

If $\mathrm{comp}(a) \cap M \neq \varnothing$, define $\mathrm{dist}(a, M)$ to be $\inf\{\, \mathrm{dist}(a, m) : m \in M \,\}$ - this is clearly well-defined. I claim that there is a unique element $m \in M$ satisfying $\mathrm{dist}(a, M) = \mathrm{dist}(a, m)$. If $m \neq m'$ satisfy $\mathrm{dist}(a, m) = \mathrm{dist}(a, m')$, let $n = \mathrm{meet}_a(m, m')$. Then n is definable over mm', and is hence an element of M. This implies that $\mathrm{dist}(a, n) < \mathrm{dist}(a, m)$, a contradiction.

Let $m \in M$ satisfy $\mathrm{dist}(a, m) = \mathrm{dist}(a, M)$. It is clear that $\mathrm{tp}(a/M)$ is determined by the type of $\mathbf{\Delta}^a_m$ and $\mathrm{tp}(m/M)$. Since there are $2^{\aleph_0}$ possibilities for such types, we have what we want. □

3.3 Summary

In contrast to our results for $r \in (0, 1)$, we have:

Theorem 3.3.1. *For $r \geq 1$, the theory $S_{\preccurlyeq_r}$ is complete. In particular, the theory of the $(\mathbf{K}^+_r, \preccurlyeq_r)$-generic is decidable.*

Proof. For $r > 1$, we already showed this in Lemma (3.1.2). For $r = 1$, let $M \models S_{\preccurlyeq_1}$. Then we know that M consists of countably many copies of various trees. Then Lemma 3.2.10 shows that M is elementarily equivalent to the $(\mathbf{K}^+_1, \preccurlyeq_1)$-generic. Thus $S_{\preccurlyeq_1}$ is complete, and since it is clearly decidable we have what we want. □

Corollary 3.3.2. *Let $\{\, \alpha_n : n \in \omega \,\}$ be any sequence in $[0, 2]$ and let $M_{\alpha_n} \models \Sigma_{\alpha_n}$. Let $\mathcal{U}$ be any ultrafilter on ω. Then exactly one of the following holds of $\mathbf{M} := \prod_{\mathcal{U}} M_{\alpha_n}$*

1. *$\{\, \alpha_n \,\}^{\mathcal{U}} = 0$ and $\mathbf{M}$ is elementarily equivalent to the Rado graph*

2. $\{ \alpha_n \}^{\mathcal{U}} = \alpha$ *for irrational* α*, and* $\mathbf{M}$ *is elementarily equivalent to the Shelah-Spencer graph of weight* α*.*

3. $\{ \alpha_n \}^{\mathcal{U}} = r$ *for rational* $r \in (0, 1]$*,* $\mathbf{M}$ *is equivalent to the* $(\mathbf{K}_r, \leq_r)$ *generic and has a decidable,* ω*-stable theory.*

4. $\{ \alpha_n \}^{\mathcal{U}} = r$ *for rational* $r \in (0, 1)$*,* $\mathbf{M}$ *models* Σ_r *and has an undecidable theory.*

5. $\{ \alpha_n \}^{\mathcal{U}} = r$ *for rational* $r \in (1, \infty)$*,* $\mathbf{M}$ *is equivalent to the* $(\mathbf{K}_r, \leq_r)$ *generic and has a decidable,* ω*-categorical theory.*

6. $\{ \alpha_n \}^{\mathcal{U}} = 1$ *,* $\mathbf{M}$ *is equivalent to the* $(\mathbf{K}_r^+, \preccurlyeq_r)$ *generic and has a decidable, strictly stable theory.*

Chapter 4

Other Properties

4.1 Connectedness

We show in this section that for $\alpha \in (0,1)$, both the $(\mathbf{K}_\alpha, \leq_\alpha)$-generic and the $(\mathbf{K}_\alpha^+, \preccurlyeq_\alpha)$ generic are connected. Choose any rational $r < \alpha$, and let C be a biminimal 0-extension of a two point graph $\{a, b\}$. By biminimality, abC is connected. Also, for $X \subseteq C$, $\delta_\alpha(X/ab) = |abX| - 2 - \alpha e(X/ab) > |abX| - 2 - re(X/ab) \geq 0$. Letting G be either the $(\mathbf{K}_\alpha, \leq_\alpha)$-generic or the $(\mathbf{K}_\alpha^+, \preccurlyeq_\alpha)$-generic, for any $a', b' \in G$, there is a partial isomorphism $f : ab \mapsto a'b'$. By full amalgamation, f extends to an embedding of C into G over ab, which shows that any two points in G are in the same component.

Note that this is in stark contrast to the case for $r \geq 1$, where there are infinitely many components.

4.2 Other Models of S_r

We show in this section that for rational $r \in (0,1)$ it is easy to extend certain countable graphs G to models of S_r. In particular, we will use this to show that the $(\mathbf{K}_r^+, \preccurlyeq_r)$ generic is not AE axiomatizable, and that $S_r \nvdash \Sigma_r$.

Throughout, fix a rational $r \in (0,1)$.

Proposition 4.2.1. *Let C be any countable graph with $\varnothing \preccurlyeq_r C$ and with finite closures (i.e. for finite $A \subset C$ there is a unique finite $\bar{A}$ such that $A \subseteq \bar{A} \preccurlyeq_r C$). Then C can be extended to a graph M which is $(\mathbf{K}_r^+, \preccurlyeq_r)$-generic; furthermore, if $A \subseteq C$, then $\mathrm{cl}_M(A) = \mathrm{cl}_C(A)$.*

Proof. We proceed by a modification of the construction of a generic structure. Enumerate the class $\mathbf{K}_R^+$ as $\{G_i : i \in \omega\}$ and a decomposition of C as $C = \bigcup_i C_i$ where $C_0 = \varnothing$ and $C_i \preccurlyeq_r C_{i+1}$ for every i. Also fix a bijection $\eta : \omega \times \omega \to \omega$. We will inductively construct a sequence of finite structures $M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n$ with the following properties:

- $M_n \preccurlyeq_r M_{n+1}$
- $|M_{n+1} \setminus M_n| < \omega$
- $C_n \preccurlyeq_r M_n$
- If $G_i \preccurlyeq_r M_n$ and $G_i \preccurlyeq_r H_j$ with $\eta(i,j) < n$ then G_i extends to a copy of $H_j \preccurlyeq_r M_{n+1}$

Begin by setting $M_0 = C_0 = \varnothing$. Given M_n, we show how to construct M_{n+1}. We begin by setting $D_0 = C_{n+1} \oplus_{C_n} M_n$. Note that the amalgamation property gives us $C_{n+1} \preccurlyeq_r D_0$ and $M_n \preccurlyeq_r D_0$.

We now enumerate as $(A_0, B_0), \ldots, (A_m, B_m)$ all pairs (A_i, B_i) such that $A_i \preccurlyeq_r D_0$, $A_i \simeq G_k, B_i \simeq G_l$ with $G_k \preccurlyeq_r G_l$ and $\eta(k,l) < n$. For each $0 \leq i < m$ let $D_{i+1} = D_i \oplus_{A_i} B_j$, so that $D_i \preccurlyeq_r D_{i+1}$. We let $M_{n+1} = D_m$, and it is clear by induction that $M_n \preccurlyeq_r M_{n+1}$ and $C_{n+1} \preccurlyeq_r M_{n+1}$.

We show that $M = \cup_i M_n$ meets the required properties. First note that each D_0 is an amalgamation over some C_i, so that $C \subseteq M$. It is clear that M has finite closures - if $A \subseteq_\omega M$ there is some n so that $A \subseteq M_n$; but each M_n is closed in M. Consider an embedding $f : A \hookrightarrow M$ and $A \preccurlyeq_r B$, again choose n so that $f(A) \subseteq M_n$. Then if $A \simeq G_i$ and $B \simeq G_j$, for $m = \max(n, \eta(i, j))$ we will have that some D_{i+1} is the amalgam of D_i with B over $f(A)$. Since each such $D_i \preccurlyeq_r M$, this copy of B is strong in M.

Finally, we show that for $A \subseteq C$, $\mathrm{cl}_M(A) = \mathrm{cl}_C(A)$. Fix such an A, and fix $n \in \omega$ minimal such that $A \subseteq C_n$.

if $A \subseteq C$, we can choose some n for which $A \subseteq C_n$. Then $A \preccurlyeq_r C_n \preccurlyeq_r M_n \preccurlyeq_r D_0$ as shown above; since $D_0 \preccurlyeq_r M$ the result follows.

$\square$

Lemma 4.2.2. *Fix any graph C which satisfies $\varnothing \preccurlyeq_r C$. Then for any minimal pair (A, B) with $\delta_r(B/A) = 0$ the structure C' obtained from C by replacing finitely many instances of A with instances of B satisfies $\varnothing \preccurlyeq_r C'$. Furthermore, if C has finite closures then so does C'.*

Proof. Let $A_1 \dots A_n$ be the copies of A which extend to copies of B, so that $C' = C \oplus_{A_1} B' \oplus_{A_2} B' \dots \oplus_{A_n} B'$, where $B' = B \setminus A$ and we abuse notation and write $A \oplus_C B$ to indicate the free join of A and B over C, without requiring the $C \preccurlyeq_r A$ and $C \preccurlyeq_r B$. Let X be any finite subset of C'; then $X = C_0 \oplus B_1 \oplus \dots \oplus B_n$, where $C_0 \subset_\omega C$, and $B_i \subset A_i \oplus B'$. Then $\delta_r(X) = \delta_r(C_0) + \sum_{1 \leq i \leq n} \delta_r(B_i)$. If $C_0 \neq \varnothing$, then $\delta_r(C_0) > 0$. Since each $B_i \preccurlyeq_r B$ and $\varnothing \preccurlyeq_r B$, we have $\delta(B_i) > 0$ as desired.

We also show that X has a finite closure. Let $X' = \mathrm{cl}_C(C_0)$. Then I claim that $Y = X' \cup \oplus_{i<n,A_i} B' \preccurlyeq_r C'$. Let $X' \subseteq Z \subseteq_\omega C'$ - then $Z = C_1 \cup \oplus B$ for some finite $C_1 \subseteq C$. Then $\delta(Z/Y) = \delta(C_1/X') + \sum \delta(B/B) = \delta(C_1/X') > 0$ since $X' \preccurlyeq_r C$.

□

From this, we easily get:

Theorem 4.2.3. *The theory of the $(\mathbf{K}_r^+, \preccurlyeq_r)$-generic is not AE-axiomatizable.*

Proof. We fix a $(\mathbf{K}_r^+, \preccurlyeq_r)$-minimal pair (A, B) with $\delta(B/A) = 0$ and a bijection $\eta : \omega \times \omega \to \omega$. Let M_0 denote any generic structure and enumerate as $\{A_{0,i} : i \in \omega\}$ all distinct instances of A that occur in M_0. For any M_n, we define M_n' by replacing each $A_{i,j}$ with a copy of B for $\eta(i,j) < n$. We then let M_{n+1} be a generic structure containing M_n' as above. Since each M_n is generic, they all have the same theory. In particular, for every n, $M_n \models \neg[\forall \bar{x} \Delta_A(\bar{x}) \to \exists \bar{y} \Delta_B(\bar{x}\bar{y})]$. However, if we let $M = \cup_n M_n$ then $M \models \forall \bar{x} \Delta_A(\bar{x}) \to \exists \bar{y} \Delta_B(\bar{x}\bar{y})$. Therefore M has a different theory from the theory of the generic, showing that the latter theory does not have models closed under unions of chains.

□

Noting that M constructed in the proof of the previous theorem satisfies S_r but is not semi-generic (since $\varnothing \preccurlyeq_r A$ but for $m > |B \setminus A|$, no embedding A' of A into M will satisfy the required condition since $B \not\subseteq \mathrm{cl}^m(\varnothing)$ but $B \subseteq \mathrm{cl}^m(A')$)

4.3 Direct Limits

We will show in this section that for rational $r \in (0,1)$, we can obtain the $(\mathbf{K}, \preccurlyeq_r)$-generic as a direct limit of a sequence of graphs M_{α_n} with α_n converging down to r.

We begin by noting that the irrational generics are totally ordered under strong embedding:

Lemma 4.3.1. *Let $\alpha < \beta$ be irrational in $(0,1)$. Then $M_\beta \hookrightarrow M_\alpha$, and the image of the embedding is $\leq_\alpha$ strong. Furthermore, for $f_0 : \bar{a} \mapsto \bar{b}$ a partial isomorphism of closed sets ($\bar{a} \leq_\beta M_\beta, \bar{b} \leq_\alpha M_\alpha$), the embedding can be taken over f_0.*

Proof. Write $M_\beta = \bigcup_i C_i$ with $C_i \leq_\beta C_{i+1}$ and $C_0 = \bar{a}$ (if f_0 is not specified, let $C_0 = \varnothing$). We will show by induction on i that C_i embeds strongly into M_α via some f_i. If f_{i-1} has been defined, we have $\mathrm{im}(f_{i-1}) \leq_\alpha M_\alpha$, so by genericity f_{i-1} extends to a strong embedding of C_i into M_α over C_{i-1}. Our embedding will be $f = \bigcup_i f_i$.

We show that f is strong. Suppose (X, Y) is a minimal pair in M_α with $X \subseteq \mathrm{im}\, f$. Then choose n so that $X \subseteq \mathrm{im}\, f_n$: because f_n is a strong embedding, we must have $Y \subseteq \mathrm{im}\, f_n$. $\square$

Given a sequence $\{\alpha_n\}$ of irrationals which montonically converge down to a rational r, we can then define a limiting structure as follows. We essentially want N_α to look like a union of all the structures G_{α_n}. The technical obstacle to writing this is that while each generic embeds in the next, that next one won't necessarily be *contained* in the succeeding generic; there is no obvious way of of taking a union of embeddings. We get the same idea via compactness.

Consider a language $\mathcal{L}'$ which adds a countable set of new constants indexed by $\omega \times \omega$ - $C := \{ c_{(i,j)} : i, j \in \omega \}$. For each $n \in \omega$, let $\{ a_m : m \in \omega \}$ enumerate the elements of G_{α_n}, fix an embedding $f_n : G_{\alpha_n} \hookrightarrow G_{\alpha_{n+1}}$, and define Δ_n inductively as follows. For $n = 0$, fix an enumeration $\{ a_i : i \in \omega \}$ of G_{α_0} and let Δ_0 be the set of sentences $\{ R(c_{(0,i)}, c_{(0,j)}) : G_{\alpha_0} \models R(a_i, a_j) \} \cup \{ \neg R(x_{(0,i)}, x_{(0,j)}) : G_{\alpha_0} \models \neg R(a_i, a_j)$. Having defined Δ_n, we fix an enumeration of $G_{\alpha_{n+1}}$ as $\{ a_{(m,i)} : m \leq n+1, i < \omega \}$ so that $\mathrm{im}(f_n) = \{ a_{(m,i)} : m \leq n+1, i < \omega \}$. Letting Δ_{n+1} be $\{ R(c_{(n+1,i)}, c_{(n+1,j)}) : G_{\alpha_{n+1}} \models R(a_{(n+1,i)}, a_{(n+1,j)}) \} \cup \{ \neg R(c_{(n+1,i)}, c_{(n+1,j)}) : G_{\alpha_0} \models \neg R(a_{(n+1,i)}, a_{(n+1,j)})$, it is clear that $T := \cup_n \Delta_n$ is consistent - let $N_r \models T$.

We will frequently abuse notation and conflate M_{α_n} with it's image in N_r. We first note that the resulting structure does not depend on the sequence used:

Lemma 4.3.2. *Let $\{ \alpha_n : n \in \omega \}$ and $\{ \beta_n : n \in \omega \}$ be sequences which converge monotonically down to r. Let N_r^0 and N_r^1 denote the respective limits. Then $N_r^0 \simeq N_r^1$*

Proof. We construct a back-and-forth system of partial isomorphisms $\{ f_i : i \in \omega \}$ such that:

- $f_0 : \varnothing \to \varnothing$
- For every i, $f_i : \bar{a} \to \bar{b}$ and there is an m_i so that $\bar{a} \leq_{\alpha_{m_i}} M_{\alpha_{m_i}}$ and $\bar{b} \leq_{\beta_{m_i}} M_{\beta_{m_i}}$
- If $f = \cup_i f_i$, then f is an isomorphism from N_r^0 to N_r^1.

Suppose that $f_i : \bar{a} \to \bar{b}$ has been defined and the i is even. Choose any $a \in N_r^0 \setminus \bar{a}$, and choose m so that $\bar{a}a \subseteq M_{\alpha_m}$ and choose m_{i+1} so that $\alpha_{m_{i+1}} < \beta_m$

and $\bar{b} \subseteq M_{\beta_{m_i}}$. Let $A = \mathrm{cl}_{M_{\alpha_{m_i}}}$, then $\bar{a} \leq_{\alpha_{m_i}} M_{\alpha_{m_i}}$ implies that $\bar{a} \leq_{\alpha_{m_{i+1}}} M_{\alpha_{m_i}}$, so that $\bar{a} \leq_{\alpha_{m_{i+1}}} A$, and the latter embeds strongly into $M_{\beta_{m_{i+1}}}$ over $\bar{b}$ as desired.

The case for i odd is handled in exactly the same manner, except that the closure of B in the appropriate generic is taken. □

Lemma 4.3.3. *The following hold of N_r:*

1. $\mathrm{Age}(N_r) = \{ A : \delta_r(A') > 0$ *for* $A' \subseteq A \} = \mathbf{K}_r^+$
2. *N_r has finite closures with respect to $\preccurlyeq_r$.*

Proof. Let $A \subseteq_\omega N_r$; then there is some n some that $A \subseteq M_{\alpha_n}$. Therefore $\delta_{\alpha_n}(A') \geq 0$ for all $A' \subseteq A$, so that $\delta_r(A') > 0$ for such A'. Conversely, if some finite A satisfies $\delta_r(A') > 0$ for every $A' \subseteq A$, then there is some $\beta > \alpha$ so that $\delta_\beta(A') \geq 0$ for such A'. Therefore A will be in the age of M_{α_n} for all $\alpha_n < \beta$, so that A will be in the age of N_r.

For finite closures, let $A \subseteq M_{\alpha_n}$ as before. Then A is contained in a finite closed set in M_{α_n}; since $M_{\alpha_n} \preccurlyeq_r N_r$, we have $\mathrm{cl}_{M_{\alpha_n}}(A) \preccurlyeq_r N_r$. □

Remark 4.3.4. We note that although $M_{\alpha_n} \leq_{\alpha_{n+1}} M_{\alpha_{n+1}}$, it is not the case that $M_{\alpha_n} \leq_{\alpha_n} M_{\alpha_{n+1}}$. Choose any $A_0 \in \mathbf{K}_{\alpha_0}$ closed in M_{α_0} and for each $n \in \omega$ define A_n as follows. Choose q_n rational in (α_{n+1}, α_n), and let X_n be a minimal 0-extension of A_n with respect to q_n. Then $\delta_{\alpha_n}(X_n/A_n) < 0$, while by minimality we have that $A \leq_{\alpha_{n+1}} X_n$. Letting $A_{n+1} = A_n X_n$, we have that $A_n \leq_{\alpha_{n+1}} A_{n+1}$ but $A_n \not\leq_{\alpha_n} A_{n+1}$. Thus, A_{n+1} embeds strongly into $M_{\alpha_{n+1}}$ over A_n (but not in M_{α_n} since A_n is closed in M_{α_n}), and $A_n \not\leq_{\alpha_n} A_{n+1}$.

The following lemma will imply that N_r is the $(\mathbf{K}_r^+, \preccurlyeq_r)$-generic.

Lemma 4.3.5.

1. *For any* $A \subseteq_\omega M_{\alpha_n}$ *and* $m > n$, $\mathrm{cl}_{M_{\alpha_m}}(A) \subseteq \mathrm{cl}_{M_{\alpha_n}}(A)$

2. *For any* $A \subseteq_\omega M_{\alpha_n}$, $\mathrm{cl}_{N_r}(A) \subseteq \mathrm{cl}_{M_{\alpha_n}}(A)$

3. *For* $A \subseteq_\omega N_r$, *there is some* n *so that* $\mathrm{cl}_{N_r}(A) = \mathrm{cl}_{M_{a_n}}(A)$

Proof. For 1, we note that $\mathrm{cl}_{G_{\alpha_n}}(A) \leq_{\alpha_n} M_{\alpha_n}$ implies that $\mathrm{cl}_{G_{\alpha_n}}(A) \leq_{\alpha_m} M_{\alpha_n}$. Since $M_{\alpha_n} \leq_{\alpha_m} M_{\alpha_m}$, we have that $\mathrm{cl}_{G_{\alpha_n}}(A)$ is $\leq_{\alpha_m}$-closed in M_{α_m}; thus $\mathrm{cl}_{G_{\alpha_m}}(A)$ must be contained in it.

The exact same argument, replacing M_{α_m} with N_r and $\leq_{\alpha_m}$ with $\preccurlyeq_r$ gives 2.

For 3, we note that by 1 the sequence $\{\,\mathrm{cl}_{M_{\alpha_n}}(A) : n \in \omega\,\}$ is a descending sequence of finite structures and must thus eventually be constant.

□

Corollary 4.3.6. *N_r is isomorphic to the $(\mathbf{K}_r^+, \preccurlyeq_r)$-generic*

Proof. We know from Lemma 4.3.3 that that N_r has the age of the generic, and has finite closures. It thus suffices to show that for any $A \preccurlyeq_r N_r$, if $A \preccurlyeq_r B$ then B embeds $\preccurlyeq_r$-strongly into N_r over A. Fixing such an A, we have by the previous lemma that $A \leq_{\alpha_n} M_{\alpha_n}$ for n sufficiently large. If $A \preccurlyeq_r B$, then for n sufficiently large, $A \leq_{\alpha_n} B$. Thus choosing n sufficiently large guarantees that B embeds $\leq_{\alpha_n}$ strongly into M_{α_n} over A. Since $M_{\alpha_n} \preccurlyeq_r N_r$, this copy of B will also be $\preccurlyeq_r$-strong in N_r. □

4.4 Independence

There is an intrinsic notion of dimension and independence associated with generic structures generated by a pre-dimension function. In this section, we will look at the behavior of this function in the context of $(\mathbf{K}_r^+, \preccurlyeq_r)$-generics. We will see that once again, the existence of 0-extensions will complicate the picture.

Fixing a generic model G, for any finite $A \subseteq_\omega G$ we define the dimension of A by $d(A) = \delta(\mathrm{cl}(A)) = \inf\{\,\delta(X) : A \subseteq X \subseteq_\omega G\,\}$. This gives rise to a well-defined notion of independence as follows (see [2, 14]):

Definition 4.4.1.

- For A, B closed finite sets, $A \perp^d_{A\cap B} B$ if $d(A/B) = d(A/A\cap B)$ and $A \cap B \subseteq \mathrm{cl}(A \cap B)$
- For A, B arbitrary finite sets and C any set, $A \perp^d_C B$ if $d(A/BC) = d(A/B)$ and $\mathrm{cl}(AC) \cap \mathrm{cl}(BC) \subseteq \mathrm{cl}(C)$.
- For A, B, C arbitrary sets, $A \perp^d_C B$ if $A' \perp^d_C B'$ for every $A' \subseteq_\omega A, B' \subseteq_\omega B$.

For $(\mathbf{K}, \leq_r)$, we know that for closed A, B we have $A \perp^d_{A\cap B} B$ if and only if $AB = A \oplus_{A\cap B} B$ and AB is closed [2]. We get one and a half directions of this equivalence for $(\mathbf{K}_r^+, \preccurlyeq_r)$:

Lemma 4.4.2. *Let A, B be finite closed subsets of the $(\mathbf{K}_r^+, \preccurlyeq_r)$-generic. If $A \perp^d_{A\cap B} B$, then $AB = A \oplus_{A\cap B} B$. If $AB = A \oplus_{A\cap B} B$ and AB is closed, then $A \perp^d_{A\cap B} B$.*

Proof. If $A \perp^d_{A\cap B} B$, we have $d(AB/B) = d(A/B) = d(A/A \cap B) = \delta(A) - \delta(A \cap B) = \delta(A/A \cap B)$. If there is some $a \in A \setminus B$ and $b \in B \setminus A$ which are joined by an edge, then $\delta(AB/B) \leq \delta(A/A \cap B) - r$, so that $\delta(A/A \cap B) \geq \delta(AB/B) + r \geq d(AB/B) + r$, contradicting that $A \not\perp_{A \cap B} B$.

If $AB = A \oplus_{A \cap B} B$ and AB is closed, then $d(AB) = \delta(AB)$ and $d(A/B) = \delta(AB/B) = \delta(A/A \cap B) = d(A/A \cap B)$ as desired. $\square$

Remark 4.4.3. Note that the $(\mathbf{K}_r^+, \preccurlyeq_r)$ generics will not in general satisfy that AB is closed when $A \perp^d_{A \cap B} B$. For $r < 1$, choose A_0, B_0 with $A_0 \cap B_0 \preccurlyeq_r A_0 \oplus_{A_0 \cap B_0} B_0$. Choose C to be a bi-minimal 0-extension of A_0B_0, then C embeds strongly into the generic, denote the respective images of A_0, B_0 by A, B. Then it is clear that AB is not strong in the generic, since the image of C ensures a non-trivial closure. I claim that A and B are strong in the generic, however. It suffices to show that $A \preccurlyeq_r C$ - the argument for B is the same. For $A \subseteq X \subseteq C$, we have that $\delta_r(X/A) = \delta_r(X/X \cap AB) + \delta_r(X \cap AB/A)$. Then the former term is at least 0 by choice of C, while the latter term is greater than 0 since $A \preccurlyeq_r AB$.

On the other hand, for $r \geq 1$ being closed means being connected, and we will have AB closed whenever $A \perp^d_{A \cap B} B$.

Finally, we note that for $r = 1$, $\perp^d$-independence is coarser than forking-independence. This again contrasts with the situation for the Shelah-Spencer graphs, in which the two independence notions coincide [2].

Lemma 4.4.4. *For $r = 1$, we have:*

a) For any $A \subseteq \mathbf{M}$, $\mathrm{cl}(A)$ is the union of the components of $\mathbf{M}$ which intersect A.

b) For finite $A \subseteq \mathbf{M}$, $d(A)$ is the number of components of $\mathrm{cl}(A)$.

c) The pseudo-geometry $(\mathbf{M}, \mathrm{cl})$ *is* trivial - *i.e. for any* $A \subseteq \mathbf{M}$*,* $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(\{ a \})$.

Proof. **a)** Consider $a \in A$, and let b be any element of the component of $\mathbf{M}$ containing a. Then there is a finite path from a to b; this verices in this path give an extension of non-positive pre-dimension, so that we must have $b \in \mathrm{cl}(a) \subseteq \mathrm{cl}(A)$. Conversely, let A' be the union of the components of elements in A; we want to show that A' is closed. For finite $X \subseteq A'$, suppose that (X, Y) is a minimal pair. Then Y must be singleton and with at least one edge from Y to some vertex in X. Therefore Y is in A' as desired. **b)** We first note that for any finite $A_0 \subseteq \mathbf{M}$, $\delta_r(A_0)$ is r times the number of components of A_0. To see this observe that a fixed component of A_0 cannot have any cycles since $r = 1$. Such a component must then be a tree, which can be viewed as a 0-extension of it's root, so that each component has pre-dimension 1.

Let $C_1 \dots C_l$ be the distinct components of $\mathrm{cl}(A)$, and let $A_i := C_i \cap A$. Note that if A_i has two components X_0, X_1 then then there is some finite X' containing $X_0 \cup X_1$ which is connected and contained in C_i. Let X_i be a finite graph connected all the components of A_i, and let $X = \cup X_i$. Then $\delta(X) = l$, so that we must have $d(A) \leq l$. It is clear from a) that any finite graph containing A must have at least l components, so that $d(A) = l$

c) For any A, $\mathrm{cl}(A) = \bigcup C_i$ where the C_i enumerate the components that have an element of A in them. □

In [2], Baldwin and Shi ask whether or not finitely based theories without finite closures exist. We answer in the affirmative:

Lemma 4.4.5. *The class* $(\mathbf{K}_1^+, \preccurlyeq_1)$ *is finitely based. That is, for every* $a \in \mathbf{M}$ *and* $B \subseteq \mathbf{M}$*, there is a finite* $C \subseteq B$ *so that* $a \mathop{\smile\!\!\!\!\!\!\;|}^{d}_{C} B$.

Proof. Let C consist of a single representative of every component in $A \cap B$. Then we have to show that for finite $B_0 \subseteq B$, $d(A/B_0C) = d(A/C)$ and $\overline{AB_0} \cap \overline{AC} \subseteq \bar{C}$. Both of these are immediate from the previous lemma $\square$

4.5 Quantifier Elimination

In this section, we prove the following.

Theorem 4.5.1. *For* r *rational in* $(0, 1]$*, there is no* $k \in \omega$ *so that the theory of the* $(\mathbf{K}_r^+, \preccurlyeq_r)$ *generic eliminates quantifiers to the level of* $\mathbf{\Sigma}_k$ *formulae.*

We will show this by providing explicit counterexamples. In particular, for a fixed k we will find two complete types p_0 and p_1 which will be consistent with T, the theory of the generic. These will differ on a $\mathbf{\Sigma}_{k+1}$ formula but will be the same when restricted to $\mathbf{\Sigma}_k$ formulae. We will construct the p_i as complete types of a certain graph - each type will say that this graph is embedded in a larger graph which will witness the equivalence up to $\mathbf{\Sigma}_k$ formula and inequivalence on a $\mathbf{\Sigma}_{k+1}$ formula.

We fix k for the remainder of this section.

We first define the graphs $B_1 \dots B_N$ which will form the components of our larger graphs. The crucial property of these is that for any n, B_n can extend to the following *distinctly*:

- A copy of B_{n-1} that itself extends to B_{n-2}

- A copy of B_{n-1} that omits B_{n-2}

Definition 4.5.2. Fix any $N \in \omega$. For $n < N$, we define B_n as follows:

- We choose B_N to be any graph in $\mathbf{K}_r^+$.

- Let A_N be a 0-extension of B_N

- Let B_{N-1} and A_{N-1} be disjoint (over A_N) 0-extensions of A_N with distinct diagrams. We can ensure by defining B_{N-1}, letting A'_N be the free join of of B_{N-1} with $|B_{N-1} \setminus A_N| + 1$ isolated vertices; A_N will then be defined by using **AEP** to get a 0-extension of A_N containing A'_N.

- For $n > 2$, given A_{n-1}, we let B_{n-2} and A_{n-2} be disjoint (over A_{n-1}) 0-extensions of A_N. As above, we can guarantee that they have distinct diagrams.

In what follows we fix some odd $N > 2k + 1$. We will have occasion to speak of *attaching* some graph G to some B_n contained in a graph H. This just means generating the free amalgam $G \oplus_{B_n} H$.

Lemma 4.5.3. *Let $\bar{x}_N$ be a tuple with length $|B_N|$. For $n < N$, let $\bar{x}_n$ be a tuple of length $|B_n \setminus B_{n+1}|$.*

We inductively define the formula $\gamma_N(\bar{x}_1\bar{x}_2 \dots \bar{x}_N)$ as follows:

- *Let $\gamma_1(\bar{x}_2 \dots \bar{x}_N)$ denote $\Delta_{B_2}(\bar{x}_2 \dots \bar{x}_N) \wedge \exists \bar{x}_1 \Delta_{B_1}(\bar{x}_1 \dots \bar{x}_N)$.*

- *Given γ_n, define $\gamma_{n+2}(\bar{x}_{n+3}, \dots, x_N)$ as $\Delta_{B_{n+3}}(\bar{x}_{n+3} \dots \bar{x}_N) \wedge \exists \bar{x}_{n+2} \Delta_{B_{n+2}}(\bar{x}_{n+2} \dots \bar{x}_N) \wedge \neg \exists \bar{x}_{n+1} \gamma_n(\bar{x}_{n+1})$.*

Then γ_{2k+1} so defined is a $\mathbf{\Sigma}_k$ formula.

Proof. We proceed by induction on k. For $k = 0$, it is clear that γ_1 is $\mathbf{\Sigma}_1$. Let $n = 2k+1$; then changing the negative existential quantifier to a universal quantifier in γ_{n+2} yields:

$$\Delta_{B_{n+3}}(\cdots) \wedge \exists\bar{x}_{n+2}\Delta_{B_{n+2}}(\cdots) \wedge \forall\bar{x}_{n+1}\neg\gamma_n(\cdots)$$

By induction, γ_n is $\mathbf{\Sigma}_k$, so $\neg\gamma_n$ is Π_k as is $\forall\bar{x}_{n+1}\neg\gamma_n(\cdots)$; thus $\exists\bar{x}_{n+2}\Delta_{B_{n+2}}(\cdots)\wedge \forall\bar{x}_{n+1}\neg\gamma_n(\cdots)$ is $\mathbf{\Sigma}_{k+1}$

□

Definition 4.5.4. For **odd** $n < N$, define the graphs C_n, D_n, G^0_n and G^1_n by the following recursion:

- C_1 and D_1 are both the empty graph.
- G^0_1 consists of the graph of B_2 attached to the graph of B_1; while G^1_1 consists of the graph of B_2.
- C_n consists of a copy of B_n which extends to $\aleph_0$ copies of B_{n-1} attached to G^1_{n-2}.
- D_n consists of a copy of B_n to which are attached:
  - $\aleph_0$ copies of B_{n-1} attached to G^0_{n-2}.
  - $\aleph_0$ copies of B_{n-1} attached to G^1_{n-2}.
- G^0_n and G^1_n are defined by:

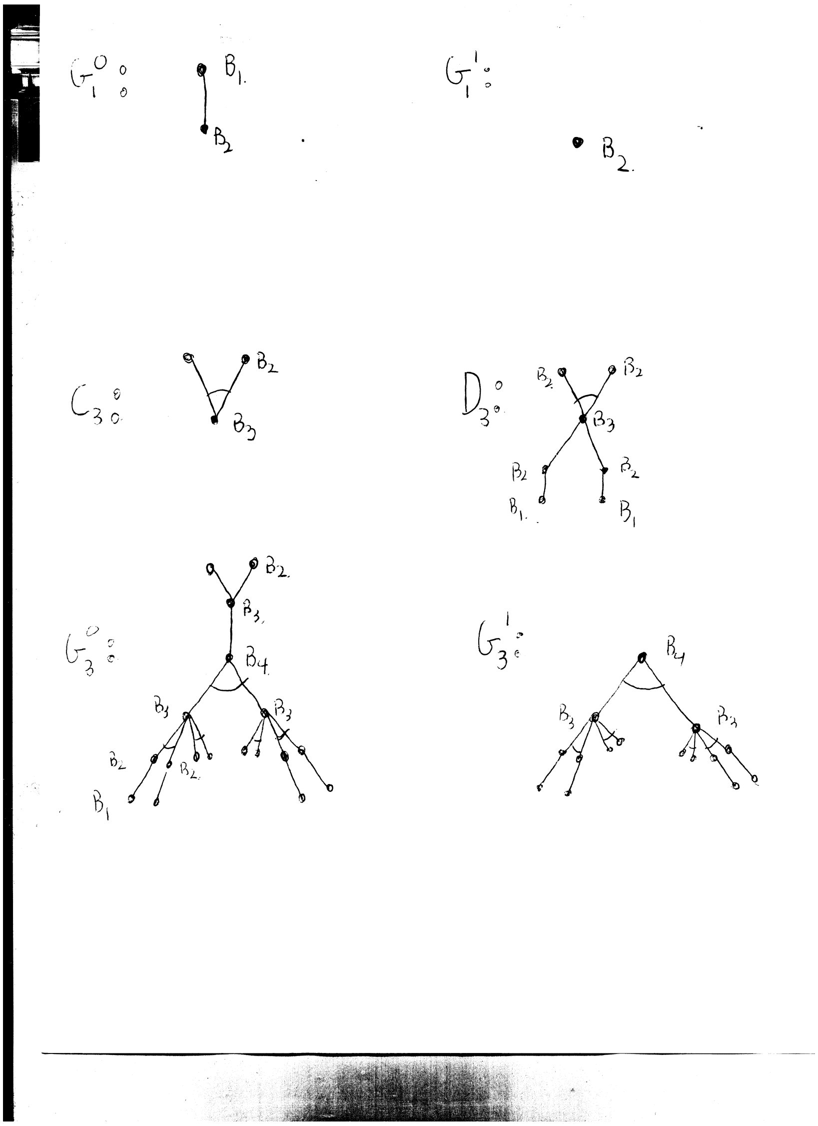

Figure 4.1: Construction of G_3^i

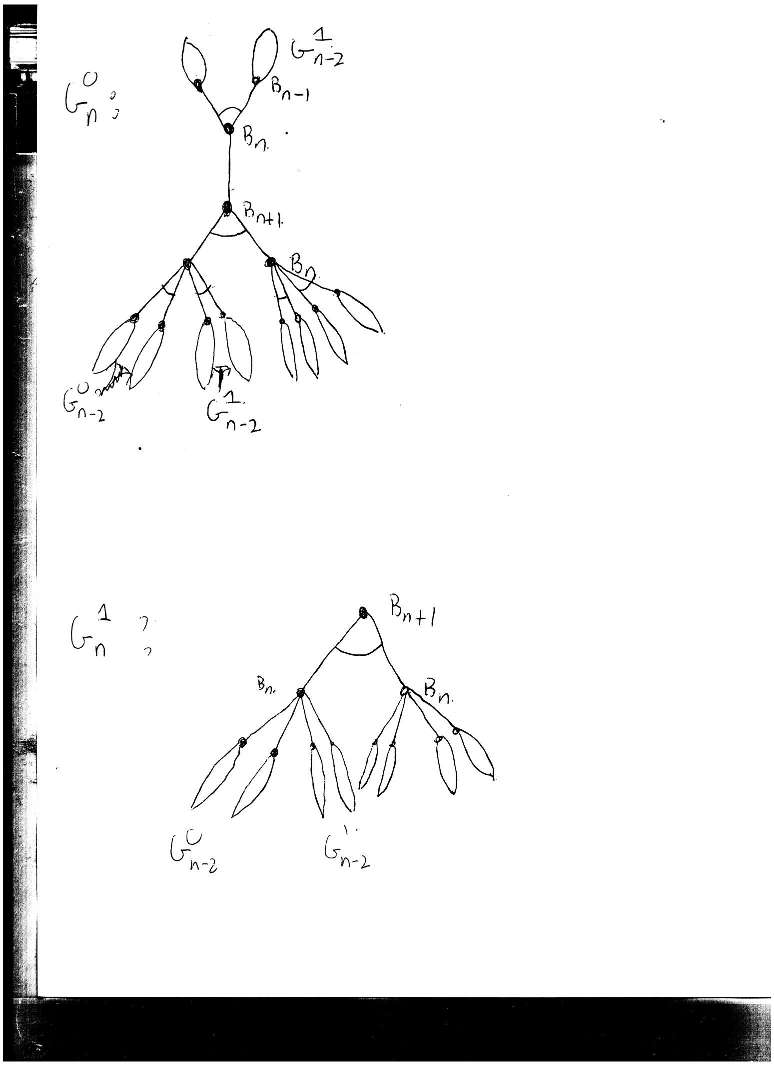

Figure 4.2: Construction of G_n^i

- G_n^0 consists of the graph of B_{n+1}, to which is attached a single copy of B_n attached to C_n and $\aleph_0$ copies of B_n attached to D_n.
- G_n^1 consists of the graph of B_{n+1} to which are attached $\aleph_0$ copies of B_n attached to D_n

For any finite m and $i \in 2$, we will denote by $G_{n,m}^i$ the subgraph which is obtained by replacing $\aleph_0$ by m in the above construction.

We note that for any n, there is a natural embedding $C_n \hookrightarrow D_n$ which is surjective onto the copies of G_{n-2}^1 in D_n.

Proposition 4.5.5. *Let $n = 2k+1$ for $k \in \omega$. Let $\bar{b}$ denote the vertices of B_{n+1} in G_n^0 and G_n^1; then $G_n^0 \models \gamma_n(\bar{b})$ and $G_n^1 \models \neg\gamma_n(\bar{b})$.*

We proceed by induction on k; the case $k = 0$ is clear. For $k > 1$, note that $\gamma_n(\bar{b})$ says that $\bar{b}$ has the diagram of B_{n+1} and extends to a copy of B_n which extends to a copy of B_{n-1} which does not extend to any model of γ_{n-2}. By induction, $G_{n-2}^1 \models \neg\gamma_{n-2}$, so C_n witnesses that $G_n^0 \models \gamma_n$. Also, $\neg\gamma_n$ says that every copy of B_n extends to some B_{n-1} which does admit a realization of γ_{n-2}. Since every B_n is attached to D_n, and each D_n has an extension to G_{n-2}^0, the inductive hypothesis shows that $G_n^1 \models \neg\gamma_n$.

We finish showing that these are the graphs we want in the next lemma.

Notation 4.5.6. *For two structures A, B of the same signatures, we say $A \approx_k^l B$ if the duplicator has a winning strategy for the l-round Ehrenfeucht-Fraisse game where the spoiler is only allowed to change structures k times. Any omitted parameter will be assumed to be $|A|$.*

If $\mathrm{gcl}_A(\cdot)$ *and* $\mathrm{gcl}_B(\cdot)$ *are closure operators on* A, B *respectively, then we will say that* $A \approx_k^l B$ *preserving closures if the partial isomorphism* f *constructed by the duplicator at any stage can be taken to satisfy* $\mathrm{gcl}_A(\mathrm{dom}(f)) = \mathrm{dom}(f)$ *and* $\mathrm{gcl}_B(\mathrm{rng}(f)) = \mathrm{rng}(f)$*. We will omit reference to the specific closure operators if they clear.*

Lemma 4.5.7. *If* $\mathrm{gcl}(\cdot)$ *refers to the natural closure for* $(\mathbf{K}_r, \leq_r)$*, then* $G^0_{2k+1} \approx_{k-1} G^1_{2k+1}$ *preserving* gcl

Proof. We play a modified Ehrenfeucht-Fraïssé game in which the spoiler can alternate structures at most $k-1$ times - we must show that the duplicator always has a winning strategy in this case. When $k = 1$, we must show that G^0_3 and G^1_3 have the same existential diagram. Note that the difference between them is that in G^0_3, B_4 has an extension to B_3 which only extends to copies of G^1_1; whereas in G^1_3 all copies of B_3 extend to both G^1_1 and G^0_1. In either case, the possible extensions are the same.

To ease notation, let $n = 2k+1$; we will construct a partial isomorphism $\sigma : G^0_n \to G^1_n$ between 0-trees that the duplicator will use as her strategy. We begin with the case in which the spoiler picks G^0_n first.

Start by setting $\sigma : B_{n+1} \mapsto B_{n+1}$. For any play of the spoiler's in an "unused" copy of D_n in G^0_n, the duplicator chooses an unused copy of D_n in G^1_n and extends σ by mapping the first copy to the second.

When a play is made in C_n, we pick an unused copy D' of D_n in G^1_n and extend σ by the natural embedding $C_n \hookrightarrow D'$. We then further extend σ by mapping all unused copies of D_n in G_n^0 onto all unused copies of D_n in G_n^1. When this is done, we have an isomorphism $(G_n^0 \setminus C_n) \mapsto (G_n^1 \setminus D')$. The spoiler must now show that C_n and D' are different. As long as he plays from C_n, he must pick some copy of G_{n-2}^1 to play from - these can be answered with the copies of G_{n-2}^1 in D'. His only hope then is to switch structures and choose from copies of G_{n-2}^0 in D'. However, he now has only $k-2$ alternations left, and by the inductive hypothesis the duplicator has a strategy by playing from new copies of G_{n-2}^1.

If the spoiler starts by playing from G_n^1, fix an embedding $\tau : G_n^1 \hookrightarrow G_n^0$ and let the duplicator play according to τ. The spoiler will have to switch structures; but now he has $k-2$ alternations to show that $G_n^0 \setminus \mathrm{rng}(\tau)$ is different from $G_n^1 \setminus \mathrm{dom}(\tau)$. Since these are respectively isomorphic to G_n^0 and G_n^1, we are reduced to the previous case.

$\square$

For the remainder of this section, we fix $n = 2k+1$ and denote G_n^0 and G_n^1 simply by G_0 and G_1. We want to use these graphs to show that T does not eliminate quantifiers to the level of $\mathbf{\Sigma}_k$ formulae. There are two approaches we can take here. The first is to use the following lemma.

Lemma 4.5.8. *For $i \in 2$ and $m \in \omega$, $G_m^i \approx^m G^i$, preserving closures.*

Proof. We build a strategy that plays a copy of some B_k at a time. Here are the possibilities:

- If the spoiler plays an element of some unplayed B_k in G_m^i, then let l be the maximal such that B_k is an extension of a copy of B_l that is already part of the strategy. Extend whatever embedding of B_l into G^i is given to an embedding of B_k into G_i. If there already is such an extension defined, extend to a new embedding - this can be done since there are infinitely many extensions of B_l to B_k in G^i.

- If the spoiler plays an element of some unplayed B_k in G^i, then let l be the maximal such that B_k is an extension of a copy of B_l that is already part of the strategy. Extend whatever embedding of B_l into G^i is given to an embedding of B_k into G^i. If there already is such an extension defined, extend to a new embedding - this can be done since there are m extensions in G^i_m.

$\square$

At this point, we have enough to prove our main theorem. For a fixed m, we embed each G^i_m as a closed substructure of a generic M_i. Then we will have $(M_0, B_N) \approx^m_{k-1} (M_1, B_N)$ and we let $p_0 = \mathrm{tp}_{M_0}(B_N)$ and $p_1 = \mathrm{tp}_{M_1}(B_N)$.

We also develop another approach which involves working with the entirety of the graphs in a suitable model of T. This gives rise to the following:

Definition 4.5.9. Let M be a graph with distinguished subgraph $G \preccurlyeq_r M$; let $\mathrm{gcl}_G(\cdot)$ be a closure operator on G. For finite $A \subseteq M$, We define the pseudo-closure (relative to G, gcl_G) as follows:

- Let $A_M = \mathrm{cl}(A) \setminus G$
- Let A_G be the minimal C satisfying:
  - $A \cap G \subseteq C \subseteq G$

- For all finite $X \subseteq M$ containing $A_M C$ with $X \cap G = C$, $A_M C \preccurlyeq_r X$

- $\mathrm{gcl}_G(C) = C$

- Let $\mathrm{pcl}(A) = A_M A_G$.

For any $N \subseteq M$ with $A \subseteq N$, we define the relative pseudo-closure $\mathrm{pcl}_N(A)$ similarly:

- Let $A_N = \mathrm{cl}_N(A) \setminus G$

- Let A_H be the minimal C satisfying:

  - $A \cap G \subseteq C \subseteq G \cap N$

  - For all finite $X \subseteq N$ containing $A_N C$ with $X \cap G = C$, $A_N C \preccurlyeq_r X$

  - $\mathrm{gcl}_G(C) = C$

- Let $\mathrm{pcl}_N(A) = A_N A_H$.

In what follows, we will take gcl_G to be the closure in G with respect to $(\mathbf{K}_r, \leq_r)$.

We will write $A \leq_p M$ to indicate that $A = \mathrm{pcl}(A)$ and similarly for $A \leq_p N$.

Given this, we will say that (M, G) is $(\mathbf{K}_r^+, \preccurlyeq_r)$ pseudo-generic if:

1. For every finite $A \subseteq M$, $A \in \mathbf{K}$

2. For every finite $A \subseteq M$, if $A \leq_p M$ and $A \preccurlyeq_r B'$, then there is an extension of A in M to B, a copy of B' satisfying $B \leq_p M$.

3. For every finite $A \subseteq M$, $\mathrm{pcl}(A)$ exists and is finite (we will say that M has *finite pseudo-closures*). The pseudo-closure is here taken with respect to G

Remark 4.5.10. Fix a finite $A \subseteq M$, with M pseudo generic. Then $A \leq_p M$ iff $A_M = A \setminus G, A_G = A \cap G$. Also, for $N \subseteq M$, we have $A \leq_p M$ iff $A_N = \mathrm{cl}_N(A) \setminus G$ and $A_H = A \cap (G \cap N)$.

Proof. Suppose $A = \mathrm{pcl}(A) = A_M A_G$. By definition, A_M is disjoint from G and A_G is contained in G; therefore $A \setminus G = A_M A_G \setminus G = A_M \setminus G = \mathrm{cl}(A) \setminus G = A_M$ and $A_G = A_M A_G \cap G$.

Suppose $A_M = A \setminus G$ and $A_G = A \cap G$. Then $A_M A_G = (A \setminus G)(A \cap G) = A$.

For the relativised version, we note that as above A_N is disjoint from $G \cap N$ and that A_H is contained in $G \cap N$, so the same argument works. □

Remark 4.5.11. If $A \leq_p M$ and $A \subseteq N \leq_p M$, then $A \leq_p N$

Proof. It suffices to show that $A_N = A \setminus (G \cap N)$ and $A_H = A \cap G \cap N$. By definition, $A_N = \mathrm{cl}_N(A) \setminus G$. Since $A \subseteq \mathrm{cl}_N(A) \subseteq \mathrm{cl}_M(A)$, we have $A \setminus G \subseteq \mathrm{cl}_N(A) \setminus G \subseteq \mathrm{cl}_M(A) \setminus G$. Since the outer terms are equal, we have $A_N = \mathrm{cl}_N(A) \setminus G$ as desired. In fact, we have more strongly that $A_N = A_M$

To show that $A_H = A \cap G \cap N$, we first note that $A \cap G \cap N = A \cap G$ since $A \subseteq N$. Then we want to show that $A_G = A_H$. We have $A_M = A_N$, so we know that for any $X \supseteq A$ with $X \cap G = A \cap G$, $A_M A_G \preccurlyeq_r X$. Then $A_N A_G \preccurlyeq_r X$; and since this is the minimal possible A_H satisfying $A_G = A \cap G \subseteq A_H$, we must have $A_G = A_H$. □

Remark 4.5.12. In a pseudo generic M, given $A \leq_p M$ and $B = \mathrm{pcl}(Am)$ for $m \notin G$, we have $A \preccurlyeq_r B$.

Proof. I claim that $B_G = A_G$ - given this the definition of pseudo closures and it's finiteness will provide what we want. It suffices to show that for any finite $X \supseteq B$, $B_M A_G \preccurlyeq_r X$ (since $A_G \subseteq B_G$, this will show that $A_G = B_G$). We have that $B_M = \mathrm{cl}(B) \setminus G$, so we have that $B_M G \preccurlyeq_r XG$ by definition of closure. We set $V = (X \setminus G) \cap A_G$ - then intersection with V gives $B_M A_G \preccurlyeq_r X$ as desired. □

Theorem 4.5.13. *Suppose (M_0, G_0) and (M_1, G_1) are pseudo-generic structures and that $G_0 \approx_k G_1$ in a way that preserves pseudo closures. Then $M_0 \approx_k M_1$, preserving pseudo-closures.*

Proof. Throughout, we construct a partial isomorphism $M_0 \to M_1$, which we write as $f : (\bar{g}_0, \bar{m}_0) \mapsto (\bar{g}_1, \bar{m}_1)$ where the $\bar{g}_i$ represent the part of the structure in G_i and the $\bar{m}_i$ represent the rest of the partial isomorphism. We will further require throughout that $\mathrm{dom}(f) \leq_p M_0$ and $\mathrm{rng}(f) \leq_p M_1$. At any stage, if the spoiler plays from $\mathrm{dom}(f)$ or $\mathrm{rng}(f)$, the duplicator responds according to f. Otherwise:

- If the spoiler chooses an element of G_0 or G_1, then the duplicator responds by extending f in accordance with the pseudo closure preserving strategy witnessing $G_0 \approx_k G_1$.

- If the spoiler chooses some $m \in M_0 \setminus G_0$, let $C = \mathrm{dom}(f)$ and let $C' = \mathrm{pcl}(Cm)$. Then $C \leq_p M$, so by finite pseudo-closures and the previous remark we have that $C \preccurlyeq_r C'$. Then by pseudo-genericity, C' embeds into M_1 as a pcl-closed extension. We extend f by this embedding.

- If the spoiler chooses $m \in M_1 \setminus G_1$, the strategy is the symmetric version of the previous case.

□

We construct the appropriate pseudo-generics:

Theorem 4.5.14. *Let G be a countable graph which can be written as $G = \cup G_i$ with $G_i \leq_r G_{i+1}$ and $G_0 = \varnothing$. Then G extends to an M so that (M, G) is pseudo-generic.*

Proof. We proceed by a modification of the construction of a generic structure. Enumerate the class $\mathbf{K}$ as $\{H_i : i \in \omega\}$ and a decomposition of G as $G = \bigcup_n G_n$ where $G_0 = \varnothing$, and for every n, $G_n \leq_r F_{n+1}$ and (G_n, G_{n+1}) is a $\leq_r$-minimal pair. We will inductively construct a sequence of finite structures $\{M_n\}$ with the following properties:

- $M_n \leq_r M_{n+1}$
- $C_n \leq_r M_n$
- If $A \leq_r M_n$ and $A \preccurlyeq_r H_j$ with $j < n$ then A extends to a copy of $H_j \subseteq M_{n+1}$

Begin by setting $M_0 = C_0 = \varnothing$. Given M_n, we show how to construct M_{n+1}. We begin by setting $D_0^{n+1} = C_{n+1} \oplus_{C_n} M_n$. Note that the amalgamation property gives us $C_{n+1} \leq_r D_0^{n+1}$ and $M_n \leq_r D_0^{n+1}$.

We now enumerate as $(A_0, B_0), \ldots, (A_m, B_m)$ all pairs (A_i, B_i) such that $A_i \leq_r D_0^{n+1}$, $A_i \preccurlyeq_r B_i$, and $B_i \simeq H_l$ with $l < n$. For each $0 \preccurlyeq_r i < m$ let $D_{i+1} = D_i \oplus_{A_i} B_i$, so that $D_i \preccurlyeq_r D_{i+1}$. We let $M_{n+1} = D_m$, and it is clear by induction that $M_n \leq_r M_{n+1}$ and $C_{n+1} \leq_r M_{n+1}$.

We let $M = \cup M_n$ and claim that (M, G) is then pseudo generic. We need to show:

1. $G \preccurlyeq_r M$

2. Pseudo-closures exist and are finite.

3. If $A \leq_p M$ and $A \preccurlyeq_r B$, then there is some B' isomorphic to B such that $A \subseteq B' \leq_p M$.

The bulk of the work is is contained in the following:

Claim 4.5.15. *For any i, m such that D_i^m is defined, $D_i^m \leq_p M$*

Proof of claim: Let $A = D_i^m$; we want to show that $A_M = A \setminus G$ and $A_G = A \cap G$. Since this is clear for $i = 0$, we assume without loss that $i > 0$. For the first part, we show that AG is closed; it will follow immediately that $\mathrm{cl}(A) \subseteq AG$; so that $A \setminus G \subseteq \mathrm{cl}(A) \setminus G \subseteq (AG) \setminus G = A \setminus G$ and thus that $A_M = A \setminus G$. To show AG is closed, we fix a minimal pair (X, Y) with $X \subseteq AG$; then we must show that $Y \subseteq AG$. Fix $n \in \omega$ minimal such that $Y \subseteq M_n$. Then we have:

$$X \subseteq D_i^m \preccurlyeq_r D_{i+1}^m \preccurlyeq_r \cdots \preccurlyeq_r M_m \leq_r D_0^{m+1} \preccurlyeq_r \cdots \preccurlyeq_r M_{m+1} \leq_r \cdots D_0^n \preccurlyeq_r \cdots \preccurlyeq_r M_n$$

Since (X, Y) is minimal, we have $X \preccurlyeq_r D_i^m \cap Y$ unless $Y \subseteq D_i^m$. So without loss of generality, $X \preccurlyeq_r D_i^m \cap Y$ and intersecting with Y gives:

$$X \preccurlyeq_r D_i^m \cap Y \preccurlyeq_r \cdots \preccurlyeq_r M_m \cap Y \leq_r D_0^{m+1} \cap Y \preccurlyeq_r \cdots \preccurlyeq_r M_{m+1} \cap Y \leq_r \cdots \preccurlyeq_r Y$$

Each step has pre-dimension non-decreasing. If it is ever increasing, we contradict that $\delta(Y/X) \leq 0$; so at each step we must must have constant pre-dimension. In particular, each instance of $\preccurlyeq_r$ must be an equality, and each instance of $\leq_r$ must be a 0-extension. Thus for every $m \leq l < n$, we have $D_*^l \cap Y = M_l \cap Y$ and

$M_l \cap Y \leq_r D_0^{l+1} \cap Y$ where $*$ is i for $l = m$ and 0 otherwise. We show by induction on $n - m$ that $Y \subseteq AG$. This is obvious for $n - m = 0$; if $n - m = 1$ we have $Y = M_n \cap Y = D_0^n \cap Y = (Y \cap C_n) \oplus_{Y \cap C_m} (Y \cap M_m) = (Y \cap C_n) \oplus_{Y \cap C_m} (Y \cap D_i^m)$ Since $(Y \cap D_i^m) \subseteq A$ and $(Y \cap C_n), (Y \cap C_m) \subseteq G$, we have $Y \subseteq AG$ as desired. For the inductive step, we have $Y = M_n \cap Y = D_0^n \cap Y = (Y \cap C_n) \oplus_{Y \cap C_{n-1}} (Y \cap M_{n-1})$. By induction, $(Y \cap M_{n-1}) \subseteq AG$ and $(Y \cap C_n), (Y \cap C_{n-1}) \subseteq G$ so $Y \subseteq AG$.

We must also show that for any finite X containing A with $X \cap G = A \cap G = C_m$, $A \preccurlyeq_r X$. Choose n so that $X \subseteq M_n$, then write

$$A \subseteq D_i^m \preccurlyeq_r D_{i+1}^m \preccurlyeq_r \cdots \preccurlyeq_r M_m \leq_r D_0^{m+1} \preccurlyeq_r \cdots \preccurlyeq_r M_{m+1} \leq_r \cdots D_0^n \preccurlyeq_r \cdots \preccurlyeq_r M_n$$

Intersecting with X gives:

$$A \subseteq D_i^m \cap X \preccurlyeq_r D_{i+1}^m \cap X \preccurlyeq_r \cdots \preccurlyeq_r M_m \cap X \leq_r D_0^{m+1} \cap X \preccurlyeq_r \cdots \preccurlyeq_r M_{m+1} \cap X \cdots \preccurlyeq_r X$$

Note that for each $l \geq m$, we have $D^{l+1} \cap X = (C_{l+1} \cap X) \oplus_{C_l \cap X} (M_l \cap X)$. By our assumption $C_{l+1} \cap X = C_l \cap X$, so that $M_l \cap X = D^{l+1} \cap X$. Therefore we can replace every instance of $\leq_r$ in the above sequence with $\preccurlyeq_r$.

□

Pseudo-genericity follows quickly from this.

Proof of (1)*:* This is immediate from the proof of our claim: taking $A = C_m$ for any m we showed that $C_m G = G$ is closed. □

Proof of (2)*.* For any $X \subseteq M_m$; $\mathrm{pcl}(X) \subseteq M_m$ since M_m is pcl-closed. □

Proof of (3)*:* Fix $A \leq_p M$ and choose n so that $A \subseteq D_0^n$. Then by remark (4.5.11), we have $A \leq_p D_0^m$ for every $m \geq n$, which implies $A \leq_r D_0^m$ by remark (4.5.12), so that eventually one of the D_j^m will be precisely the amalgamation of B' over A. Since each such is pcl closed, we are done. □

□

Corollary 4.5.16. *We have the following:*

1. *Let* G *be a countable graph which consists of a union of* 0*-extensions over some finite* $G_0 \subseteq G$*. Then each* G *extends to an* M *so that* (M, G) *is pseudo-generic.*
2. *Let* G_0, G_1 *be as defined in* (4.5.4)*. Then each* G_i *extends to an* M_i *so that* (M_i, G_i) *is pseudo-generic.*

Our final step is to show that the pseudo-generics are models of T. We do so with the following:

Theorem 4.5.17. *For either pseduo-generic* (M_i, G_i) *constructed in* (4.5.14) *and* N *a generic and* $l \in \omega$*, we have* $M_i \approx^l N$*.*

Proof. Fix l, the number of rounds. At each stage we construct a partial isomorphism $f : (\bar{g}, \bar{m}) \mapsto (\bar{g}', \bar{m}')$ where $\mathrm{dom}(f) \leq_p M_i$ and $\mathrm{rng}(f) \preccurlyeq_r N$. If the spoiler plays from the domain or range of f, then the duplicator plays according to f. Otherwise:

- If the spoiler chooses some $h \in G_i$, then we fix a closed copy of $G_{i,l}$ in N; call it H. For this and any future play in G_i, the duplicator plays according to the strategy guaranteed in lemma (4.5.8).

- If the spoiler chooses $m \in M_i \setminus G_i$, let $C = \mathrm{pcl}(\mathrm{dom}(f)m)$. Then by pseudo genericity, $\mathrm{dom}(f) \preccurlyeq_r C$ and C embeds strongly into N over f. We extend f by this embedding.

- For $n \in N \setminus \mathrm{rng}(f)$, we let $C = \mathrm{cl}(\mathrm{rng}(f)n$. Then $\mathrm{rng}(f) \preccurlyeq_r C$, so by pseudo-genericity C embeds into M over f^{-1} with a pseudo-closed image. We then extend f by (the inverse of) this embedding.

$\square$

Chapter 5

Summary

We summarize some of the model-theoretic properties of various generics investigated in this thesis and in [1, 10, 2, 12, 11]. Throughout, r^+ refers to the $(\mathbf{K}_r^+, \preccurlyeq_r)$-generic, QE refers to the level of quantifier elimination, and AE refers to the existence of a Π_2 set of axioms.

For the QE column, "$\mathbf{\Sigma}_0$" refers to complete quantifier elimination, "NMC" refers to "near model completeness", and "N" means that there is no elimination to the level of $\mathbf{\Sigma}_k$ formulae for any $k \in \omega$.

Generic Weight	ω-stable	Stable	Simple	Decidable	ω-Categorical	Q.E.	AE
0	N	N	Y	Y	Y	$\mathbf{\Sigma}_0$	Y
Irrational $\alpha \in (0, 2)$	N	Y	Y	Y	N	NMC	Y
Rational $r \in (0, 1)$	Y	Y	Y	Y	N	NMC	Y
Rational $r^+ \in (0, 1)$	N	N	N	N	N	N	N
Rational $r \in [1, 2)$	Y	Y	Y	Y	Y	NMC	Y
Rational $r^+ = 1$	N	Y	Y	Y	N	N	Y
Rational $r^+ \in (1, 2)$	Y	Y	Y	Y	Y	Y	Y
Arbitrary $\alpha \geq 2$	Y	Y	Y	Y	Y	Y	Y

Table 5.1: Summary of results

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