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Homoclinic orbits for periodic second order Hamiltonian systems with superlinear terms

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Abstract. We obtain the existence of nontrivial homoclinic orbits for nonautonomous second order Hamiltonian systems by using critical point theory under some different superlinear conditions from those previously used in Hamiltonian systems. In particular, an example is given to illustrate our result.

Keywords: second order Hamiltonian systems, homoclinic orbits, superlinear.

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1 Introduction and main result

We consider the following nonautonomous second order Hamiltonian system

$$u''(t) - A(t)u(t) + \nabla H(t, u(t)) = 0, \quad t \in \mathbb{R}, \tag{1.1}$$

where $A(t) \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is T-periodic $N \times N$ symmetric matrix, and is positive definite uniformly for $t \in [0, T]$; $H(t, u) \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is T-periodic in t for each $u \in \mathbb{R}^N$ and $\nabla H(t, u)$ denotes its gradient with respect to the u variable. We say that a solution $u(t)$ of (1.1) is homoclinic (with 0) if $u(t) \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that $u(t) \to 0$ and $u'(t) \to 0$ as $|t| \to \infty$. If $u(t) \not\equiv 0$, then $u(t)$ is called a nontrivial homoclinic solution.

In the past decades, many authors have studied the existence and multiplicity of periodic or homoclinic solutions of (1.1). In this paper, we are interested in the case where the nonlinearity ∇H is superlinear as $|u| \to \infty$. Therefore, here we only state some related results. There are some authors [1–4,7,9–11,13–16] who have obtained homoclinic orbits for (1.1) with ∇H being superlinear as $|u| \to \infty$ by critical point theory under the following A–R condition due to Ambrosetti and Rabinowitz (e.g., [2]): there exists a constant $\mu > 2$ such that

$$0 < \mu H(t, u) \le (\nabla H(t, u), u), \quad u \in \mathbb{R}^N \backslash \{0\}, \tag{1.2}$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^N$, and the corresponding norm is denoted by $|\cdot|$. Roughly speaking the role of (1.2) is to insure that all Palais–Smale sequences for the

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corresponding function of (1.1) at the Mountain-Pass level are bounded. For related papers, we refer the readers to see [5,6] and so on.

Let $G(t,u) := \frac{1}{2}(\nabla H(t,u), u) - H(t,u)$. We weaken the condition (1.2) and obtain the following result.

Theorem 1.1. *Assume that the following conditions hold.*

(H_1) $H(t,u) \geq 0$, $\forall (t,u) \in \mathbb{R} \times \mathbb{R}^N$.

(H_2) $|\nabla H(t,u)| \leq c(1+|u|^{p-1})$ *for some* $p > 2$ *and* $c > 0$, $\forall t \in \mathbb{R}$.

(H_3) $|\nabla H(t,u)| = o(|u|)$ *as* $|u| \to 0$ *uniformly in* $t \in \mathbb{R}$.

(H_4) $\frac{H(t,u)}{|u|^2} \to +\infty$ *as* $|u| \to +\infty$ *uniformly in* $t \in \mathbb{R}$.

(H_5) *If* $|u| \leq |v|$, *then* $G(t,u) \leq DG(t,v)$ *for some* $D \geq 1$, $\forall t \in \mathbb{R}$.

Then there is at least one nontrivial homoclinic orbit of (1.1).

Remark 1.2. Note that (H_5) implies $G(t,u) \geq 0$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$. In fact, the condition (H_5) was used firstly to study Schrödinger equations [12], but as far as we know, the condition was not used by other authors to study the seconder order Hamiltonian system (1.1).

Example 1.3. Let

$$H(t,u) = \frac{1}{2}|u|^2 \ln(1+|u|) - \left(\frac{1}{2}|u|^2 - |u| + \ln(1+|u|)\right).$$

A simple calculation shows that H satisfies (H_1)–(H_5) but does not satisfy the superquadratic condition (1.2).

To prove our main result, we need the following theorem developed by Jeanjean [12].

Theorem A ([12])**.** *Let E be a Banach space equipped with the norm $\|\cdot\|$. Let $J \subset \mathbb{R}^+$ be an interval, and $I_\lambda \in C^1(E,\mathbb{R})$ $(\lambda \in J)$ is defined by*

$$I_\lambda(u) := A(u) - \lambda B(u).$$

If the following conditions hold:

(1) $B(u) \geq 0$ *for all* $u \in E$;

(2) either $A(u) \to +\infty$ *or* $B(u) \to +\infty$ *as* $\|u\| \to +\infty$;

(3) there are two points v_1 *and* v_2 *in* E *such that setting*

$$\Gamma = \{\gamma \in C([0,1],E),\ \gamma(0) = v_1,\ \gamma(1) = v_2\}$$

it holds for all $\lambda \in J$ *that*

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\},$$

then, for almost every $\lambda \in J$, *there is a sequence* $\{u_j\} \subset E$ *such that*

$$\{u_j\} \text{ is bounded in } E, \quad I_\lambda(u_j) \to c_\lambda, \quad I'_\lambda(u_j) \to 0 \text{ in the dual } E^{-1} \text{ of } E.$$

Theorem A means that for a wide class of functionals, having a Mountain-Pass geometry, almost every functionals in this class has a bounded Palais–Smale sequence at the Mountain-Pass level.

The rest of our paper is organized as follows. In Section 2, we give the variational framework of (1.1) and some preliminary lemmas, and then we give the detailed proof of our result.

2 Variational frameworks and the proof of Theorem 1.1

Throughout this paper we denote by $\|\cdot\|_{L^q}$ the usual $L^q(\mathbb{R},\mathbb{R}^N)$ norm and C for generic constants.

Let $E := H^1(\mathbb{R},\mathbb{R}^N)$ under the usual norm

$$\|u\|_E^2 = \int_{-\infty}^{+\infty}(|u|^2+|u'|^2)dt.$$

Thus E is a Hilbert space and it is not difficult to show that $E \subset C^0(\mathbb{R},\mathbb{R}^N)$, the space of continuous functions u on $\mathbb{R}$ such that $u(t) \to 0$ as $|t| \to \infty$ (see, e.g., [15]). We will seek solutions of (1.1) as critical points of the functional I associated with (1.1) and given by

$$I(u) := \frac{1}{2}\int_{-\infty}^{+\infty}\left(|u'|^2+(A(t)u,u)\right)dt - \int_{-\infty}^{+\infty}H(t,u)dt.$$

Let

$$\|u\|^2 := \int_{-\infty}^{+\infty}((A(t)u,u)+|u'|^2)dt,$$

then $\|\cdot\|$ can and will be taken as an equivalent norm on E. Hence I can be written as

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{-\infty}^{+\infty}H(t,u)dt.$$

The assumptions on H imply that $I \in C^1(E,\mathbb{R})$. Moreover, critical points of I are classical solutions of (1.1) satisfying $u'(t) \to 0$ as $|t| \to \infty$. Thus u is a homoclinic solution of (1.1).

In what follows, we always assume that (H_1)–(H_5) hold. Let us show that I has a Mountain-Pass geometry. That is a consequence of the two following results:

Lemma 2.1. $I(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2)$ *as* $u \to 0$.

Proof. By (H_2) and (H_3), we know for any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that

$$|\nabla H(t,u)| \le \varepsilon|u| + C_\varepsilon|u|^{p-1}, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N. \tag{2.1}$$

Note that Remark 1.2 implies that $\frac{1}{2}(\nabla H(t,u),u) \ge H(t,u)$, which together with (2.1) implies that

$$|H(t,u)| \le \frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{2}|u|^p, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N. \tag{2.2}$$

Thus (2.2) follows from the Sobolev embedding theorem that $\int_{-\infty}^{+\infty}|H(t,u)|dt \le \frac{\varepsilon}{2}\|u\|^2 + C\|u\|^p$, that is, $\int_{-\infty}^{+\infty}H(t,u)dt = o(\|u\|^2)$. The proof is finished. □

Lemma 2.2. *There exists a function* $u^0 \in E$ *with* $u^0 \neq 0$ *satisfying* $I(u^0) \le 0$.

Proof. For every $v \in E$ with $v \neq 0$, $|sv| \to +\infty$ as $s \to \infty$. It follows from (H_4) that

$$\lim_{s\to\infty}\frac{H(t,sv)}{s^2} = \lim_{s\to\infty}\frac{H(t,sv)}{s^2|v|^2}|v|^2 = +\infty \quad \text{uniformly in } t \in \mathbb{R}.$$

Thus by (H_1) and Fatou's lemma, we have

$$\lim_{s\to\infty}\int_{-\infty}^{+\infty}\frac{H(t,sv)}{s^2}dt = +\infty.$$

It follows from the definition of I that

$$\lim_{s\to\infty}\frac{I(sv)}{s^2}=\frac{1}{2}\|v\|^2-\lim_{s\to\infty}\int_{-\infty}^{+\infty}\frac{H(t,sv)}{s^2}dt\to-\infty.$$

Thus we can choose $u^0 := sv$ with $|s|$ big enough such that $u^0 \in E$ with $u^0 \neq 0$ satisfying $I(u^0) \leq 0$. □

We define on E the family of functionals

$$I_\lambda(u)=A(u)-\lambda B(u):=\frac{1}{2}\|u\|^2-\lambda\int_{-\infty}^{+\infty}H(t,u)dt,\quad \lambda\in[1,2].$$

Lemma 2.3. *For almost every $\lambda \in [1,2]$, there exists a sequence $\{u_j\} \subset E$ satisfying*

$$\{u_j\}\text{ is bounded in }E,\quad 0<\lim_{j\to\infty}I_\lambda(u_j)=c_\lambda,\quad I'_\lambda(u_j)\to 0.$$

Proof. We will use Theorem A to prove this lemma. Obviously, conditions (1) and (2) in Theorem A hold. Next we prove the condition (3) holds. Let

$$\Gamma:=\{\gamma\in C([0,1],E):\ \gamma(0)=0\text{ and }\gamma(1)=u^0\},\quad u^0\text{ is obtained in Lemma 2.2},$$

$$c_\lambda:=\inf_{\gamma\in\Gamma}\max_{s\in[0,1]}I_\lambda(\gamma(s)),\quad \forall\lambda\in[1,2].$$

Lemma 2.1 implies that $I_\lambda(\gamma(s)) > 0$ ($\forall\lambda \in [1,2]$) for any small enough $|\gamma(s)|$ (i.e., $\gamma(s) \to 0$), and $I_\lambda(0) = I(0) = 0$ ($\forall\lambda \in [1,2]$) by Lemma 2.1, besides, (H_1) and Lemma 2.2 imply that $I_\lambda(u^0) \leq 0$, $\forall\lambda \in [1,2]$. Therefore,

$$c_\lambda:=\inf_{\gamma\in\Gamma}\max_{s\in[0,1]}I_\lambda(\gamma(s))>0=\max\{I_\lambda(0),I_\lambda(u^0)\}.$$

That is the condition (3) of Theorem A holds. An application of Theorem A implies that for almost every $\lambda \in [1,2]$ there exists a sequence $\{u_j\} \subset E$ satisfying

$$\{u_j\}\text{ is bounded in }E,\quad I_\lambda(u_j)\to c_\lambda,\quad I'_\lambda(u_j)\to 0.$$

Obviously, the definition of c_λ and $I_\lambda(u_j) \to c_\lambda$ imply that $0 < \lim_{j\to\infty} I_\lambda(u_j) = c_\lambda$. □

Lemma 2.4. *Let $\lambda \in [1,2]$ is fixed. If $\{u_j\} \subset E$ satisfying*

$$\{u_j\}\text{ is bounded in }E,\quad 0<\lim_{j\to\infty}I_\lambda(u_j)=c_\lambda,\quad I'_\lambda(u_j)\to 0,$$

then up to a subsequence, $u_j \rightharpoonup u_\lambda \neq 0$ with $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) \leq c_\lambda$.

Proof. If $\{u_j\} \subset E$ satisfying

$$\{u_j\}\text{ is bounded in }E,\quad 0<\lim_{j\to\infty}I_\lambda(u_j)=c_\lambda,\quad I'_\lambda(u_j)\to 0,$$

then

$$\lim_{j\to\infty}\int_{-\infty}^{+\infty}G(t,u_j)dt=\lim_{j\to\infty}\left(I_\lambda(u_j)-\frac{1}{2}I'_\lambda(u_j)u_j\right)=\lim_{j\to\infty}I_\lambda(u_j)>0,\tag{2.3}$$

where $G(t,u) := \frac{1}{2}(\nabla H(t,u),u) - H(t,u)$ is defined in Section 1. To continue the proof, we need the following remark:

Remark 2.5. If $\{w_j\} \subset E$ is bounded and vanishing, then $\lim_{j\to\infty}\int_{-\infty}^{+\infty} G(t,w_j)dt = 0$.

Now, we give the proof of Remark 2.5. If $\{w_j\}$ vanishes, then Lion's concentration compactness principle implies $w_j \to 0$ in $L^q(\mathbb{R},\mathbb{R}^N)$ for all $q \in (2,\infty)$, which together with (2.1), (2.2), $\|w_j\| < \infty$ and the Sobolev embedding theorem implies that

$$\int_{-\infty}^{+\infty}(\nabla H(t,w_j),w_j) \le \int_{-\infty}^{+\infty}|\nabla H(t,w_j)||w_j| \le \varepsilon\|w_j\|_{L^2}^2 + C_\varepsilon\|w_j\|_{L^p}^p \le \varepsilon C\|w_j\|^2 + C_\varepsilon\|w_j\|_{L^p}^p \to 0$$

and

$$\int_{-\infty}^{+\infty} H(t,w_j)dt \le \int_{-\infty}^{+\infty}|H(t,w_j)|dt \le \frac{\varepsilon}{2}\|w_j\|_{L^2}^2 + \frac{C_\varepsilon}{2}\|w_j\|_{L^p}^p \le \frac{\varepsilon}{2}C\|w_j\|^2 + \frac{C_\varepsilon}{2}\|w_j\|_{L^p}^p \to 0.$$

It follows from the definition of $G(t,w)$ that $\lim_{j\to\infty}\int_{-\infty}^{+\infty} G(t,w_j)dt = 0$.

By Remark 2.5, (2.3) and the boundedness of $\{u_j\}$ in E, we know $\{u_j\}$ does not vanish, i.e., there exist $r,\delta > 0$ and a sequence $\{s_j\} \subset \mathbb{R}$ such that

$$\lim_{j\to\infty}\int_{B_r(s_j)} u_j^2 dt \ge \delta,$$

where $B_r(s_j) := [s_j - r, s_j + r]$. Note that $\{u_j\}$ is bounded implies that $u_j \rightharpoonup u_\lambda$ in E and $u_j \to u_\lambda$ in $L^2_{loc}(\mathbb{R},\mathbb{R}^N)$ (see [8]) after passing to a subsequence, which together with

$$\lim_{j\to\infty}\int_{B_r(s_j)} u_j^2 dt \ge \delta$$

implies $u_\lambda \ne 0$. By the fact I'_λ is weakly sequentially continuous [17] and $I'_\lambda(u_j) \to 0$, we have $I'_\lambda(u_\lambda)v = \lim_{j\to\infty} I'_\lambda(u_j)v = 0$ for all $v \in E$. Therefore, $I'_\lambda(u_\lambda) = 0$.

Next, we still need to prove $I_\lambda(u_\lambda) \le c_\lambda$. Since (H_5) implies $G(t,u) \ge 0$ for all $(t,u) \in \mathbb{R}\times\mathbb{R}^N$, it follows from Fatou's lemma, $I_\lambda(u_j) \to c_\lambda$, $I'_\lambda(u_j) \to 0$ and $I'_\lambda(u_\lambda) = 0$ that

$$\begin{aligned} c_\lambda = \lim_{j\to\infty}\left(I_\lambda(u_j) - \frac{1}{2}I'_\lambda(u_j)u_j\right) &= \lim_{j\to\infty}\lambda\int_{-\infty}^{+\infty} G(t,u_j)dt \\ &\ge \lambda\int_{-\infty}^{+\infty} G(t,u_\lambda)dt \\ &= I_\lambda(u_\lambda) - \frac{1}{2}I'_\lambda(u_\lambda)u_\lambda = I_\lambda(u_\lambda). \end{aligned}$$

The proof is finished. □

By Lemmas 2.3 and 2.4, we deduce the existence of a sequence $\{(\lambda_j,u_j)\} \subset [1,2]\times E$ such that

- $\lambda_j \to 1$ and $\{\lambda_j\}$ is decreasing.
- $u_j \ne 0$, $I_{\lambda_j}(u_j) \le c_{\lambda_j}$ and $I'_{\lambda_j}(u_j) = 0$.

(2.4)

Lemma 2.6. *The sequence $\{u_j\}$ obtained in (2.4) is bounded.*

Proof. Arguing by contradiction, suppose $\|u_j\| \to \infty$. Let $v_j := \frac{u_j}{\|u_j\|}$, then $\|v_j\| = 1$ and thus $v_j \rightharpoonup v$ and $v_j \to v$ a.e. $t \in \mathbb{R}$, up to a subsequence. So either $\{v_j\}$ vanishes or it does not vanish. Next, we shall prove that the two cases are all impossible.

Part 1. The non-vanishing of $\{v_j\}$ is impossible. By contradiction, if $\{v_j\}$ is non-vanishing, that is, there exist $r, \delta > 0$ and a sequence $\{s_j\} \subset \mathbb{R}$ such that

$$\lim_{j\to\infty} \int_{B_r(s_j)} v_j^2 dt \geq \delta. \tag{2.5}$$

Thus it follows from $v_j \to v$ in $L^2_{loc}(\mathbb{R}; \mathbb{R}^N)$ that $v \neq 0$.

Since $I'_{\lambda_j}(u_j) = 0$ implies $\|u_j\|^2 = \lambda_j \int_{-\infty}^{+\infty} (\nabla H(t, u_j), u_j) dt$, thus it follows from Remark 1.2 that

$$1 = \lambda_j \int_{-\infty}^{+\infty} \frac{(\nabla H(t,u_j), u_j)}{\|u_j\|^2} dt \geq 2 \int_{-\infty}^{+\infty} \frac{H(t,u_j)}{\|u_j\|^2} dt = 2 \int_{-\infty}^{+\infty} \frac{H(t,u_j)}{|u_j|^2} |v_j|^2 dt. \tag{2.6}$$

On the other hand, the facts $v_j \to v$ a.e. $t \in \mathbb{R}$, $v \neq 0$ and $\|u_j\| \to \infty$ imply that $|u_j| = |v_j| \cdot \|u_j\| \to +\infty$, which together with (H_4) implies

$$\frac{H(t,u_j)}{|u_j|^2} |v_j|^2 \to +\infty \quad \text{a.e. } t \in \mathbb{R}.$$

It follows from Fatou's lemma that

$$\int_{-\infty}^{+\infty} \frac{H(t,u_j)}{|u_j|^2} |v_j|^2 dt \to +\infty \quad \text{as } j \to \infty,$$

which contradicts with (2.6).

Part 2. The vanishing of $\{v_j\}$ is impossible. If $\{v_j\}$ is vanishing. We define a sequence $\{z_j\} \subset E$ by $z_j = t_j u_j$ with $0 \leq t_j \leq 1$ satisfying

$$I_{\lambda_j}(z_j) := \max_{0 \leq t \leq 1} I_{\lambda_j}(t u_j). \tag{2.7}$$

(Here, if for a $j \in N$, t_j defined by (2.7) is not unique we choose the smaller possible value). We claim that

$$\lim_{j\to\infty} I_{\lambda_j}(z_j) = +\infty. \tag{2.8}$$

Seeking a contradiction we assume for all $t_j \in [0,1]$ there exists a positive constant M such that

$$\liminf_{j\to\infty} I_{\lambda_j}(z_j) \leq M. \tag{2.9}$$

Let $\{k_j\}$ be defined by $k_j := \frac{\sqrt{4M}}{\|u_j\|} u_j$. With the relationships of $\{v_j\}$ and $\{k_j\}$, we know $\{k_j\}$ is also bounded and vanishing. Hence Remark 2.5 in Lemma 2.4 implies that $\int_{-\infty}^{+\infty} H(t, k_j) dt \to 0$. Thus for j sufficiently large,

$$I_{\lambda_j}(k_j) = 2M - \lambda_j \int_{-\infty}^{+\infty} H(t, k_j) dt \geq \frac{3}{2} M. \tag{2.10}$$

If we let $t_j := \frac{\sqrt{4M}}{\|u_j\|}$ for j sufficiently large, then $t_j \in [0,1]$. Thus (2.10) contradicts with (2.9). Therefore, (2.8) holds. Note that $I'_{\lambda_j}(z_j) z_j = 0$ for all $j \in N$ by (2.7), thus

$$I_{\lambda_j}(z_j) = I_{\lambda_j}(z_j) - \frac{1}{2} I'_{\lambda_j}(z_j) z_j = \lambda_j \int_{-\infty}^{+\infty} G(t, z_j) dt,$$

which together with (2.8) implies that

$$\int_{-\infty}^{+\infty} G(t, z_j)dt \to +\infty. \tag{2.11}$$

Note that conditions $I_{\lambda_j}(u_j) \leq c_{\lambda_j}$ and $I'_{\lambda_j}(u_j) = 0$ in (2.4) imply that

$$\frac{1}{2}\|u_j\|^2 - \lambda_j \int_{-\infty}^{+\infty} H(t, u_j)dt \leq c_{\lambda_j}, \quad \|u_j\|^2 - \lambda_j \int_{-\infty}^{+\infty} (\nabla H(t, u_j), u_j)dt = 0.$$

It follows from the definition of G that $\int_{-\infty}^{+\infty} G(t, u_j)dt \leq \frac{c_{\lambda_j}}{\lambda_j}$. Clearly, $\frac{c_{\lambda_j}}{\lambda_j}$ is increasing and bounded by $c = c_1$, thus we have

$$\int_{-\infty}^{+\infty} G(t, u_j)dt \leq c, \quad \forall j \in N.$$

It follows from (H_5) that $\int_{-\infty}^{+\infty} G(t, z_j)dt \leq D \int_{-\infty}^{+\infty} G(t, u_j)dt \leq C$, which contradicts with (2.11).

Therefore, the proof is finished by Part 1 and Part 2. □

Proof of Theorem 1.1. Since Lemma 2.6 implies that $\{u_j\}$ is bounded in E, we can assume $u_j \rightharpoonup u$ in E and $u_j \to u$ a.e. $t \in \mathbb{R}$, up to a subsequence. Obviously,

$$I(u_j) = I_{\lambda_j}(u_j) + (\lambda_j - 1) \int_{-\infty}^{+\infty} H(t, u_j)dt. \tag{2.12}$$

We distinguish two cases: either $\limsup_{j\to\infty} I_{\lambda_j}(u_j) > 0$ or $\limsup_{j\to\infty} I_{\lambda_j}(u_j) \leq 0$.

Case 1. If $\limsup_{j\to\infty} I_{\lambda_j}(u_j) > 0$, then (2.12) implies that $\limsup_{j\to\infty} I(u_j) > 0$, besides, the facts $\lambda_j \to 1$ and $I'_{\lambda_j}(u_j) = 0$ (see (2.4)) imply that $I'(u_j) \to 0$, by the similar proof of Lemma 2.4, we can get $u_j \rightharpoonup u \neq 0$ with $I'(u) = 0$, up to a subsequence.

Case 2. If $\limsup_{j\to\infty} I_{\lambda_j}(u_j) \leq 0$, we use the sequence $\{z_j\}$ defined in (2.7). Since $\{u_j\}$ is bounded, $\{z_j\}$ is also bounded. Note that $I'_{\lambda_j}(z_j)z_j = 0$ for all $j \in N$ by (2.7), thus

$$\lambda_j \int_{-\infty}^{+\infty} G(t, z_j)dt = I_{\lambda_j}(z_j) - \frac{1}{2} I'_{\lambda_j}(z_j)z_j = I_{\lambda_j}(z_j). \tag{2.13}$$

Similarly to Lemma 2.1, we have

$$I'_{\lambda_j}(u_j)u_j = \|u_j\|^2 + o(\|u_j\|^2) \quad \text{as } u_j \to 0,$$

uniformly in $j \in N$. Note that $I'_{\lambda_j}(u_j) = 0$, thus there is $\theta > 0$ such that $\|u_j\| \geq \theta$, $\forall j \in N$. Similarly to Lemma 2.1, we also get

$$I_{\lambda_j}(tu_j) = \frac{1}{2} t^2 \|u_j\|^2 + o(t^2\|u_j\|^2) \quad \text{as } t \to 0, \quad t \in [0, 1],$$

uniformly in $j \in N$, thus $I_{\lambda_j}(tu_j) > 0$ for small enough t. It follows from $\limsup_{j\to\infty} I_{\lambda_j}(u_j) \leq 0$ that the maximum $I_{\lambda_j}(z_j) := \max_{0\leq t\leq 1} I_{\lambda_j}(tu_j)$ (see (2.7)) can not be obtained at $t = 1$, and there holds $\liminf_{j\to\infty} I_{\lambda_j}(z_j) > 0$. It follows from (2.13) and $\lambda_j \to 1$ that

$$\liminf_{j\to\infty} \int_{-\infty}^{+\infty} G(t, z_j)dt = \liminf_{j\to\infty} I_{\lambda_j}(z_j) > 0,$$

it follows from the fact $\{z_j\}$ is bounded and the Remark 2.5 in Lemma 2.4 that $\{z_j\}$ does not vanish. Therefore, $\{u_j\}$ does not vanish. Moreover, (2.4) implies that

$$I'(u_j)\varphi = I'_{\lambda_j}(u_j)\varphi + (\lambda_j - 1) \int_{-\infty}^{+\infty} (\nabla H(t, u_j), \varphi)\, dt \to 0 \quad \text{as } j \to \infty, \quad \forall \varphi \in E.$$

Therefore, similar to the proof of Lemma 2.4, we can easily get $u \neq 0$ and $I'(u) = 0$. □

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